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Approaches to the Erdős–Straus Conjecture

Ivan V. Morozov
CUNY New York City College of Technology

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A Problem of Primes and Residue Classes

Ivan V. Morozov

New York City College of Technology, Department of Mathematics

Professor Satyanand Singh, Mentor

Abstract

The Erdős–Straus conjecture, initially proposed in 1948 by Paul Erdős and Ernst G. Straus, asks whether the equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is solvable for all $n \in \mathbb{N}$ and some $x, y, z \in \mathbb{N}$. This problem touches on properties of Egyptian fractions, which had been used in ancient Egyptian mathematics. There exist many partial solutions, mainly in the form of arithmetic progressions and therefore residue classes. In this work we explore partial solutions and aim to expand them.

Introduction

At first glance, it seems rather simple, but as with many problems in mathematics, the superficial simplicity is what creates the allure. The conjecture is unsolved in the general case, but numerous partial solutions exist. For instance, if $n = 2k$ for $k \in \mathbb{N}$ is an even number, then the equation is easily satisfied by

$$\frac{4}{2k} = \frac{1}{2k} + \frac{1}{2k} + \frac{1}{k}. \tag{1}$$

Note that two of the fractions are identical. Some expressions of this conjecture require all fractions to be distinct, but it is not a concern as two identical fractions can be transformed into distinct ones using

$$\frac{1}{2k} + \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k(k+1)} \tag{2}$$

for even denominators and

$$\frac{1}{2k+1} + \frac{1}{2k+1} = \frac{1}{k+1} + \frac{1}{(k+1)(2k+1)} \tag{3}$$

for odd ones. However, the only exception here is where $n = 2$, for which the solution

$$\frac{4}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{1} \tag{4}$$

with two identical fractions is the only possible solution.

First Approach – A Dense Sieve of Residue Classes

The most promising method to shed light on this problem seems to consist of two steps: 1) find a pattern, and 2) extend the pattern to cover as many natural numbers as possible. In my first approach, I noticed that all initial examples followed the form

$$\frac{4}{n} = \frac{1}{abn/c} + \frac{1}{an} + \frac{1}{b}, \quad a, b, c \in \mathbb{N}. \tag{5}$$

This is merely a guess, but a fortunate one, as it easily yields many arithmetic progressions of solutions. One way to find these progressions is to plot all of the solutions for each n, select solutions with a common parameter (for which I chose c), and seek arithmetic progressions for a and b. This method requires a lot of picking from an assortment of solutions by human judgement, but picking the tightest arithmetic progressions narrows the scope of potential counterexamples by a wide margin. Manually, I found solutions for the following:

$$\frac{4}{4n-1} = \frac{1}{n(n+1)(4n-1)} + \frac{1}{(n+1)(4n-1)} + \frac{1}{n} \tag{6}$$

$$\frac{4}{12n-7} = \frac{1}{n(3n-1)(12n-7)} + \frac{1}{n(12n-7)} + \frac{1}{3n-1} \tag{7}$$

$$\frac{4}{12n-3} = \frac{1}{3n(n+1)(12n-3)/3} + \frac{1}{(n+1)(12n-3)} + \frac{1}{3n} \tag{8}$$

$$\frac{4}{24n-11} = \frac{1}{2n(6n-2)(24n-11)/2} + \frac{1}{2n(24n-11)} + \frac{1}{6n-2} \tag{9}$$

$$\frac{4}{120n-95} = \frac{1}{(40n-30)(120n-95)/5} + \frac{1}{120n-95} + \frac{1}{40n-30} \tag{10}$$

$$\frac{4}{120n-47} = \frac{1}{4(32n-12)(120n-47)/8} + \frac{1}{4(120n-47)} + \frac{1}{32n-12} \tag{11}$$

$$\frac{4}{120n-23} = \frac{1}{10n(30n-5)(120n-23)/5} + \frac{1}{10n(120n-23)} + \frac{1}{30n-5} \tag{12}$$

This leaves out solutions congruent to $1 \mod 120$ and $49 \mod 120$. This sieve is "dense" in the sense that it aims to fill every positive number greater than 2, in contrast to the third method presented later. As the number of solutions for each n grows as n increases, it becomes laborious to find these arithmetic progressions manually. Under automation or even a generating formula for these progressions, the pool of counterexamples could be reduced greatly at the very least, and research continues into the generality of this approach.

Second Approach – The Three Cases

The second approach deals with the non-triviality in a different way. Similarly to the first approach, we use the format $\frac{4}{p} = \frac{1}{abp/c} + \frac{1}{ap} + \frac{1}{b}$, but in this approach we solve for p to obtain

$$p = 4b - \frac{b+c}{a}. \tag{13}$$

here I consider this equation to have three solution cases. The first case is for $p \equiv 0 \mod 2$, the second is for $p \equiv 3 \mod 4$, and the third is for $p \equiv 1 \mod 4$. The first case has a very trivial solution,

$$\frac{4}{p} = \frac{1}{p} + \frac{1}{p} + \frac{1}{p/2}. \tag{14}$$

For the second case, there exists an explicit solution (6), but for all intents and purposes, let us use (13). Now, if we pick the smallest b that satisfies $4b > p$, in this case $4b = p + 1$, we can set $c = 1$ and $a = b + 1$. Divisibility by c is obviously not a concern, and we are done with case two.

The third case is more complicated, as we cannot generate a solution for every p by setting $c = 1$. If $c = 1$ and $4b = p + 3$, then b must be 1 less than a multiple of 3, which is not guaranteed. b can be 2 or 3 less than a multiple of 3, but in those cases we would have to guarantee that abp is divisible by 2 or 3, respectively. This is also not always true, and in those cases we would have to repeat the same procedure with $4b = p + 7$, $4b = p + 11$, and so on. To proceed with the idea, let us designate $b_1 = \frac{p+3}{4}$ and $b_n = b_1 + n - 1$.

We can then construct a table of statements about each b_n as shown,

Statement	Condition
$b_1 \equiv 0 \mod 3$	$abp \equiv 0 \mod 3$
$b_1 \equiv 1 \mod 3$	$abp \equiv 1 \mod 3$
$b_1 \equiv 2 \mod 3$	We are done
$b_2 \equiv 0 \mod 7$	$abp \equiv 0 \mod 7$
$b_2 \equiv 1 \mod 7$	$abp \equiv 1 \mod 7$
$b_2 \equiv 2 \mod 7$	$abp \equiv 2 \mod 7$
$b_2 \equiv 3 \mod 7$	$abp \equiv 3 \mod 7$
$b_2 \equiv 4 \mod 7$	$abp \equiv 4 \mod 7$
$b_2 \equiv 5 \mod 7$	$abp \equiv 5 \mod 7$
$b_2 \equiv 6 \mod 7$	We are done
$\vdots$	$\vdots$
$b_k \equiv 0 \mod (4k-1)$	$abp \equiv 0 \mod (4k-1)$
$\vdots$	$\vdots$
$b_k \equiv (4k-2) \mod (4k-1)$	We are done
$\vdots$	$\vdots$

Exactly one statement from each box must be true, and if two statements from the same row are true at any point in the infinite table, then p satisfies the Erdős–Straus equation. It should be noted that if for some k, $b_k \equiv (4k-2) \mod (4k-1)$, we are immediately done, as that simply requires setting $c = 1$ and $a = b + 1$, in which case the divisibility condition is always satisfied. Every number p has a distinct set of true statements, but there is not complete freedom over which statements could be true, as the divisibility conditions influence each other. The exact way of how the possibilities are narrowed down and whether a same-row statement and condition are guaranteed to be true are subjects of research.

Third Approach – A Non-Dense Sieve of Residue Classes

Observing the numerical solutions for small p, I noticed that many contained a solution of the form

$$\frac{4}{p} = \frac{1}{ap} + \frac{1}{bp} + \frac{1}{c}, \tag{15}$$

where a, b, c are some elements of $\mathbb{N}$. While this form may not necessarily be true for all $p \in \mathbb{N}$, it was ubiquitous enough among the critical solutions to be hypothesised. If every prime p solves this equation, the conjecture is true. This allows the equation to be rearranged to

$$\frac{p}{c} = \frac{4ab - a - b}{ab}. \tag{16}$$

Clearly, if $c = ab$, then $p = (4b-1)a - b$ and we have our first collection of residue classes $P_1^1 := \{(3n-1) \mod (4n-1) : n \in \mathbb{N}\}$. However, P_1^1 does not cover many primes, including $3, 7, 19, 31, 37, 43, 61, 67, 73, 79, 97, \ldots$.

However, $c = ab$ only implies that the numerators of the fractions match, without taking into account that the fraction on the right could simplify. When it does, the denominator would reduce by a factor k. Thus in a more general case we designate $c = ab/k$. However, that implies ab is divisible by k, and to ensure that, we set $a = km$ and $b = kn$. This gives

$$\frac{p}{ab/k} = \frac{4(km)(kn) - km - kn}{ab} \Rightarrow p = (4kn-1)m - n, \tag{17}$$

and infinitely many more collections of residue classes are yielded. Specifically, if $P_k^1 := \{(4kn - n - 1) \mod (4kn-1) : n, k \in \mathbb{N}\}$, we define the union of all our collections so far as

$$P^1 := \{P_k^1 : k \in \mathbb{N}\} = \{(4kn - n - 1) \mod (4kn-1) : n, k \in \mathbb{N}\}. \tag{18}$$

Despite the infinitude of residue classes, many numbers are still skipped due to significant overlap, as shown in the diagram below,

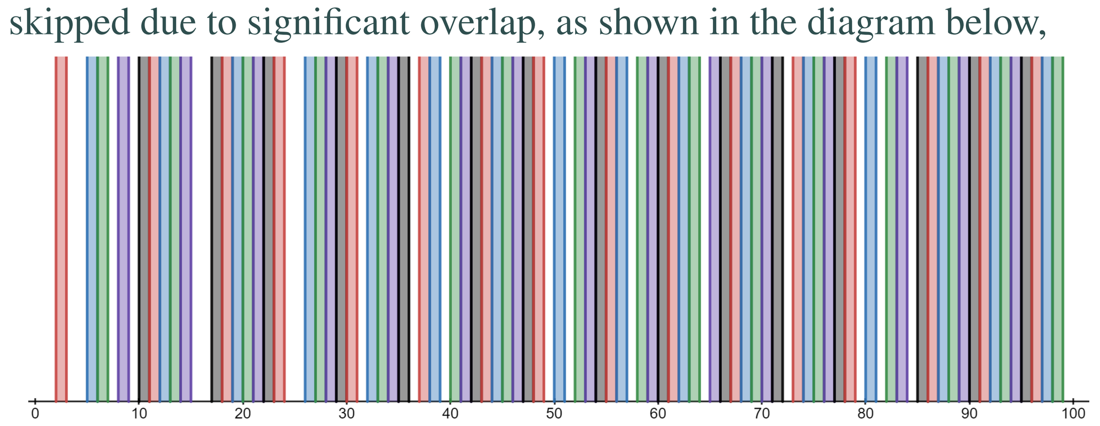

Figure 1: The image above shows the numbers P^1 covers. Gaps are clearly visible, but their density reduces as the range is increased.

Many numbers are omitted because in the setup, both a and b are divisible by k. This does not have to be the case – only ab as a product has to be divisible by k, but the rigorous findings on that are subject of ongoing research. We hope that by finding the correct forms for such products will fill in the necessary gaps.

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References

[1] P. Erdős. On a diophantine equation. *Matimatikoi Lapok*, 1:192–210, 1950.

[2] R.K. Guy. *Unsolved Problems in Number Theory*. Springer, New York, 1981.

[3] L.J. Mordell. *Diophantine Equations*. Academic Press, London-New York, 1969.

[4] A. Schinzel. Sur quelques propriétés des nombres 3/n et 4/n, où n est un nombre impair. *Mathesis*, 65:219–222, 1956.

[5] W. Sierpiński. Sur les décompositions de nombres rationnels en fractions primaires. *Mathesis*, 65:16–32, 1956.