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On String Topology Operations and Algebraic Structures on Hochschild Complexes

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Abstract

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The field of string topology is concerned with the algebraic structure of spaces of paths and loops on a manifold. It was born with Chas and Sullivan's observation of the fact that the intersection product on the homology of a smooth manifold M can be combined with the concatenation product on the homology of the based loop space on M to obtain a new product on the homology of LM, the space of free loops on M. Since then, a vast family of operations on the homology of LM have been discovered.

In this thesis we focus our attention on a non trivial coproduct of degree $1-\dim(M)$ on the homology of LM modulo constant loops. This coproduct was described by Sullivan on chains on general position and by Goresky and Hingston in a Morse theory context. We give a Thom-Pontryagin type description for the coproduct. Using this description we show that the resulting coalgebra is an invariant on the oriented homotopy type of the underlying manifold. The coproduct together with the loop product induce an involutive Lie bialgebra structure on the S^1-equivariant homology of LM modulo constant loops. It follows from our argument that this structure is an oriented homotopy invariant as well.

There is also an algebraic theory of string topology which is concerned with the structure of Hochschild complexes of DG Frobenius algebras and their homotopy versions. We make several observations about the algebraic theory around products, coproducts and their compatibilities. In particular, we describe a BV-coalgebra structure on the coHochschild complex of a DG cocommutative Frobenius coalgebra. Some conjectures and partial results regarding

homotopy versions of this structure are discussed.

Finally, we explain how Poincaré duality may be incorporated into Chen's theory of iterated integrals to relate the geometrically constructed string topology operations to algebraic structures on Hochschild complexes.

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Contents

Introduction

The history of string topology starts with Goldman and Turaev's constructions of operations on the vector space of free homotopy classes of non trivial curves on a surface. By intersecting and then concatenating pairs of curves Goldman constructed a Lie bracket. Turaev constructed a Lie cobracket by considering self intersections on a single curve and then splitting into two curves. These two operations satisfy certain compatibilities yielding a highly nontrivial structure of the name involutive Lie bialgebra. Chas and Sullivan generalized this structure to families of curves on general manifolds. This was the starting point of the discovery of a vast family of algebraic structures on $H_*(LM)$, the homology of the free loop space LM of a manifold M, and on its S^1-equivariant version $H_*^{S^1}(LM)$.

The first operation of this type, discovered by Chas and Sullivan, is a commutative and associative product on $H_*(LM)$ that combines the intersection product on the homology of the underlying manifold M with the concatenation product on the homology of the space of based loops on M. The free loop space LM admits a circle action given by rotating loops and such action induces a degree $+1$ rotation operator $H_*(LM) \to H_{*+1}(LM)$ which squares to zero. The loop product is compatible with the rotation operator in the following way: the failure of the rotation operator of being a derivation of the loop product defines a binary operations which is skew symmetric, satisfies the Jacobi identity, and it is a derivation of the loop product on each variable. This induces an algebraic structure known as a BV-algebra.

Moreover, the loop product induces a Lie bracket on the S^1-equivariant homology of LM, generalizing Goldman's bracket. This was the tipping point of the discovery of a big family of operations that combine intersecting and then concatenating or splitting families of (marked or unmarked) curves on manifolds. Using homotopy theoretic methods, Godin introduced operations $H_*(LM)^{\otimes k} \to H_*(LM)^{\otimes l}$ parametrized by the homology of the open moduli space of Riemann surfaces with k input and l output boundary circles.

Turaev's cobracket for curves on surfaces also generalizes to families of curves on a manifold. Among the operations parametrized by the homology of the open moduli space of Riemann surfaces, there is a coproduct $H^*(LM) \to H^*(LM)^{\otimes 2}$ of degree $-\dim(M)$ associated to the pair of pants or the "figure 8" ribbon graph. This coproduct, defined on each loop of a chain of loops by intersecting the point at a fixed time t with the base point of the loop, is trivial when we work modulo constant loops. In fact, at the chain level there are two null homotopies each corresponding to collapsing one of the lobes of the figure 8 to a point. Putting these two nullhomotopies together we obtain a coproduct $H_*(LM,M) \to H_*(LM,M)^{\otimes 2}$ of degree $1-\dim(M)$. This coproduct is part of a new family of operations associated associated to the compactification of the moduli space of Riemann surfaces and it is quite non trivial. It was also discovered by Goresky and Hingston in a different context. As in the case of the loop product, the coproduct is compatible with the rotation operator and it induces a Lie cobracket on $H_*^{S^1}(LM,M)$. Moreover, the cobracket and the bracket define on $H_*^{S^1}(LM,M)$ the structure of an involutive Lie bialgebra.

String topology has an analogue algebraic theory. This part of the story starts with F. Adams who described an algebraic construction on the chains on a simply connected space that yields a homological model for the *based* loop space. E. Brown, K.T. Chen, J. Jones, and others "twisted" Adams construction to obtain an algebraic model for the *free* loop space fibration. F. Adams' construction is known as the cobar construction of a differential graded coalgebra while Chen and Jones' is known as the (co)Hochschild complex of a differential graded (co)algebra; a construction also present in the deformation theory of algebras. To recover string topology operations in the algebraic theory we need to incorporate some form of Poincaré duality at the chain level. Zeinalian and Tradler have described the algebraic structure of the coHochschild homology of DG cocommutative Frobenius coalgebras (a dual theory to Hochschild homology), also organized by the homology of a moduli space, and is similar to the string topology structure on the homology of the free loop space of a manifold. One can also incorporate the S^1 action algebraically and obtain an S^1-equivariant version of the theory.

The main difficulty to relate the algebraic theory with the geometric theory is lifting Poincaré duality (or the commutative Frobenius algebra structure on cohomology of a manifold) to the chain level. At the present we are aware of several ways to do this but none seem completely satisfactory. We can lift Poincaré duality using rational homotopy methods and making choices to obtain a finite dimensional commutative Frobenius algebra model for a (simply connected) manifold; we can also relax Poincaré duality up to homotopy (there are several ways of doing this); or we can use the cap product between differential forms and currents together with the integration pairing to obtain a partially defined intersection product on the chain complex of currents.

In this thesis we focus our study on the product and the coproduct described above. Chapter 1 is concerned with geometric and topological constructions. It begins with an overview of the operations on the curves on surfaces and their generalization to families of curves making transversality assumptions. We proceed by giving Thom-Pontryagin formulations for some string topology operations following Cohen-Jones description of the loop product. In particualr, we describe the coproduct of degree $1-d$ on homology without making any transversality assumptions. Using this formulation we prove that the coproduct on $H_*(LM,M)$ is an invariant of the oriented homotopy type of the underlying manifold, settling a question posed by

Sullivan.

In Chapter 2 we present a self contained exposition of the algebraic theory of coHochschild complexes of coalgebras with structure. We prefer the viewpoint of coHochschild complexes of coalgebras as opposed to Hochschild complexes of algebras because of its resemblance with string topology constructions at the level of transversal chains on the free loop space and because we believe it is the most convenient setting to relate to string topology through explicit maps. In certain sense, this viewpoint goes back to F. Adams who constructed an explicit geometric map from the cobar construction on the singular chain complex of a space X to the singular chain complex of the based loop space on X, and to E. Brown who constructed maps from the singular chain complex of the total space of a fibration to a twisted tensor product of the base and the fiber and vice versa. We discuss symmetry properties and the Gerstenhaber and BV algebra and coalgebra structures on the coHochcshild complex of a DG cocommutative Frobenius coalgebra. We give precise formulas for products, coproducts, and chain homotopies, that resemble those discussed in Chapter 1.

Chapter 3 is concerned with the link between the geometric and algebraic theories through Chen's iterated integrals. We review Chen original construction which describes a way to obtain a differential form on the path space of a manifold from a iterated integration procedure of an ordered sequence of differential forms on a manifold. We introduce a way of incorporating Poincaré duality into the iterated integral map and describe how it can be used to relate string topology operations to operations on coHochschild complexes of Frobenius coalgebras.

Finally, in Chapter 4 we describe several conjectures and partial results about homotopy versions of our algebraic constructions. In particular, we conjecture that the reduced coHochschild homology of the cyclic C_∞-model of a manifold M (as provided by a theorem of Hamilton and Lazarev) has the structure of an A_∞-coalgebra of degree $1-d$ quasi-isomorphic to the A_∞-

coalgebra on the singular chains on (LM, M) extending the chain level coproduct of degree $1-d$. Moreover, a similar conjecture concerning the BV-coalgebra structure is discussed.

Chapter 1

Products and Coproducts in String Topology

1.1 The involutive Lie bialgebra of curves on surfaces

We start by recalling a construction due to Goldman that gave birth to string topology: a Lie bracket on the vector space generated by free homotopy classes of closed curves on an oriented surface Σ.

Let Σ be an oriented smooth surface and V the $\mathbb{Q}$-vector space generated by all free homotopy classes of loops on Σ. Given two free homotopy classes of loops $\alpha, \beta \in V$ choose two unparametrized closed curves f_α, f_β representing them respectively, such that f_α and f_β are in general position, meaning that the intersection of the two curves is transversal and it consists only of a finite number of double points $p_1, ..., p_m \in \Sigma$. Denote by $f_\alpha \cap f_\beta$ the set of such intersection points. To each point p_i we can associate a new curve $f_\alpha \bullet_{p_i} f_\beta$ which starts at p_i runs around f_α and then around f_β. For each p_i we also have an associated sign $\epsilon_{p_i}(f_\alpha, f_\beta)$, where $\epsilon_{p_i}(f_\alpha, f_\beta) = 1$ if the orientation at p_i given by the branches of f_α and f_β coincides with the orientation of Σ, and $\epsilon_{p_i}(f_\alpha, f_\beta) = -1$ otherwise. Define $\{,\} : V \otimes V \to V$ by

$$\{\alpha, \beta\} := \sum_{p_i \in f_\alpha \cap f_\beta} \epsilon_{p_i}(f_\alpha, f_\beta)[f_\alpha \bullet_{p_i} f_\beta]$$

where for a curve $f : S^1 \to \Sigma$, we have denoted by $[f] \in V$ its free homotopy class.

Since we have made choices to define this operation, we must argue why it is well defined. The space of pairs of closed curves in Σ in general position is open and dense in the C^∞ topology, so given a pair $(\alpha, \beta) \in V \times V$ we can always choose $(f_\alpha, f_\beta) \in L\Sigma \times L\Sigma$ satisfying the general position condition described above. We can furthermore assume that, individually, each f_α and f_β are in general position as well, meaning that the general position condition is satisfied at the self intersection points. Of course, the choice of the pair (f_α, f_β) is not unique. However, for any two choices (f_α, f_β) and (g_α, g_β) satisfying the general position conditions there is a homotopy between them obtained by composing a finite sequence of three types of homotopies corresponding to Redeimeister moves I, II, or III. The type I homotopies involve, at some time, a cusp, type II a tangency, and type III a triple point. Moreover, observe that the definition of the above bracket is invariant with respect to such moves.

Proposition 1 (Goldman) $(V, \{,\})$ *is a Lie algebra.*

Proof. The skew symmetry of $\{,\}$ is straightforward from the definition so we check the Jacobi identity. Given $\alpha, \beta, \gamma \in V$ we will denote by f, g, and h curves representing the respective free homotopy classes satisfying the desired general position conditions. We have

$$\{\{\alpha, \beta\}, \gamma\} = \sum_{p \in f \cap g} \epsilon_p(f, g)\{[f \bullet_p g], \gamma\} = \sum_{p \in f \cap g, q \in (f \bullet_p g) \cap h} \epsilon_p(f, g)\epsilon_q((f \bullet_p g), h)[(f \bullet_p g) \bullet_q h]$$

Since there a no triple points, the index set of the sum above is actually $(f\cap g)\bigcup\Big((f\cap h)\cup(g\cap h)\Big)$. Hence we can split the sum as

$$\{\{\alpha,\beta\},\gamma\}=$$

$$\sum_{p\in(f\cap g),q\in(f\cap h)}\epsilon_p(f,g)\epsilon_q(f\bullet_p g,h)[(f\bullet_p g)\bullet_q h]+\sum_{p\in(f\cap g),q\in(g\cap h)}\epsilon_p(f,g)\epsilon_q(f\bullet_p g,h)[(f\bullet_p g)\bullet_q h].$$

Similarly, we write

$$\{\{\beta,\gamma\},\alpha\}=$$

$$\sum_{p\in(g\cap h),q\in(g\cap f)}\epsilon_p(g,h)\epsilon_q(g\bullet_p h,f)[(g\bullet_p h)\bullet_q f]+\sum_{p\in(g\cap h),q\in(g\cap h)}\epsilon_p(g,h)\epsilon_q(g\bullet_p h,f)[(g\bullet_p h)\bullet_q f],$$

and

$$\{\{\gamma,\alpha\},\beta\}=$$

$$\sum_{p\in(h\cap f),q\in(h\cap g)}\epsilon_p(h,f)\epsilon_q(h\bullet_p f,g)[(h\bullet_p f)\bullet_q g]+\sum_{p\in(h\cap f),q\in(f\cap g)}\epsilon_p(h,f)\epsilon_q(h\bullet_p f,g)[(h\bullet_p f)\bullet_q g].$$

Notice that $(f\bullet_p g)\bullet_q h$ is freely homotopic to $(g\bullet_q h)\bullet_p f$, so every term in the second sum of $\{\{\alpha,\beta\},\gamma\}$ also appears in the first sum of $\{\{\beta,\gamma\},\alpha\}$ after reindexing, which we can do since the index sets for these two sums are the same. Moreover, in these two sums, each of these pairs have opposite signs since $\epsilon_p(f,g)=\epsilon_p(g\bullet_q h,f)$ and $\epsilon_q(g,h)=-\epsilon_q(f\bullet_q g,h)$. Similarly, we can pair the first and second sums of $\{\{\beta,\gamma\},\alpha\}$ with the second and first sums of $\{\{\gamma,\alpha\},\beta\}$, respectively.

Some years later, Turaev constructed a cobracket as follows. Let f be a closed curve in Σ which self intersects in a finite number of double points $p_1,\dots,p_n\in\Sigma$. Denote by $\cap f$ denote the set of the self intersection points. At each $p\in\cap f$ there are two outgoing arcs of f. Order these two arcs coming out of p according to the orientation of the surface. Then the two arcs define an ordered (f_p^1, f_p^2) of curves; f_p^1 starts at p and follows the first arc until it reaches back to p, and f_p^2 is defined similar along the second arc. Define a map $\nu : L\Sigma \to V \otimes V$ by $\nu(f) := \sum_{p\in\cap f}[f_p^1]\otimes[f_p^2]-[f_p^2]\otimes[f_p^1]$. The map ν is *not* well defined on V because is not invariant with respect to Reidemeister type I moves. More precisely, suppose we have a closed curve f with a exactly one self intersection at a point p such that one of the resulting two curves, say f_p^1, is homotopic to a constant loop, but f_p^2 is not; so $[f_p^1] = [*]$ and $[f] = [f_p^2]$. We have $\nu(f) = [*]\otimes[f_p^2]-[f_p^2]\otimes[*] \neq 0$ but $\nu(f_p^2) = 0$ since f_p^2 does not have any self intersections, hence ν is not well defined on free homotopy classes. However, note that we do obtain a well defined map if we work modulo constant curves; in other words, there is a well defined map on the quotient vector space V/V_0 where V_0 is the sub vector space of V generated by the single class $[*]$ of curves which are constant at a point. We still denote the map by $\nu : V/V_0 \to V/V_0 \otimes V/V_0$.

The Lie bracket defined above is well defined on the quotient V/V_0 since for any $\alpha \in V$, it follows that $\{\alpha, [*]\} = 0$ because we can choose a curve f in general position as a representative for α and a constant curve at a point b disjoint from f as a representative for $[*]$. Denote $\tilde{V} = V/V_0$. Turaev explained the compatibility between the bracket and the cobracket. Chas showed that the cobracket followed by the bracket is zero.

Theorem 1 (Turaev, Chas) *$(\tilde{V}, \{,\}, \nu)$ is an involutive Lie bialgebra.*

Proof. We omit the proof of this theorem and we refer to Turaev's original paper. We remind a picture is worth a thousand words. □

Let us make several algebraic remarks about involutive Lie bialgebras. We will recall an elegant construction that encodes the identities defining an involutive Lie bialgebra into a single equation $D^2 = 0$.

First, recall the definition of a involutive Lie bialgebra

Definition 1 A *Lie bialgebra* is a vector space W equipped with two linear maps $\{,\}: W^{\otimes 2} \to W$ and $\nu: W \to W^{\otimes 2}$ such that the following hold

(i) $\{,\}: W \otimes W \to W$ is a Lie bracket, i.e. we have skew symmetry $\{,\} \circ \tau = -\{,\}$ and the Jacobi identity $\{,\} \circ (1_W \otimes \{,\}) \circ (1_W + \sigma + \sigma^2) = 0$, where $\sigma(x \otimes y \otimes z) = x \otimes z \otimes y$

(ii) $\nu: W \to W \otimes W$ is a Lie ccobracket, i.e. we have skew symmetry $\tau \circ \nu = -\nu$ and the coJacobi identity $(\sigma^2 + \sigma + 1_W) \circ (1_W \otimes \nu) \circ \nu = 0$

(iii) Drinfeld compatibility: $\nu\{\alpha, \beta\} = \alpha \nu(\beta) - \beta \nu(\alpha)$, where W acts on $W \otimes W$ by the rule $x(y \otimes z) := \{x, y\} \otimes z + y \otimes \{x, z\}$

(iv) Involutivity: $\{,\} \circ \nu = 0$

Denote by ΛW the free graded commutative associative algebra generated by a basis of W, where each basis element is declared to have degree 1. The bracket $\{,\}: W \otimes W \to W$ extends to a bracket on ΛW defined for any two monomials $A = a_1 \wedge \ldots \wedge a_m, B = b_1 \wedge \ldots \wedge b_n \in \Lambda W$ by the formula

$$\{A, B\} := \sum_{i,j} (-1)^{i+j} \{a_i, b_j\} \wedge a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_m \wedge b_1 \wedge \ldots \wedge \hat{b}_j \wedge \ldots b_n$$

and extending by multilinearity to all of ΛW. Note that this bracket has degree -1. This extended bracket is obviously graded commutative (graded skew symmetric) and it is an easy calculation to check that the Jacobi identity holds as well. Also observe that by construction this extension of the bracket is a graded derivation of the product $\wedge$ of ΛW.

Define a map $D_0 : \Lambda W \to \Lambda W$ by the formula

$$D_0(A) = \sum_{i<j} (-1)^{i+j} \{a_i, a_j\} \wedge a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_m,$$

and extending by multilinearity to all of ΛW. It is an easy calculation to check that $D_0^2 = 0$ is equivalent to the Jacobi identity for $\{,\}$. Also observe that by construction D_0 is a graded coderivation of the deconcatenation coproduct of ΛW. Moreover, it can be verified that D_0 is a second order operator with respect to the product $\wedge$. This means that D_0 is not a derivation of $\wedge$, however, the failure of being a derivation is a derivation. In fact, this failure is precisely the above extended bracket.

The cobracket $\nu: W \to W \otimes W$ extends to a map $D_1 : \Lambda W \to \Lambda W$ defined by the formula by

$$D_1(A) := \sum_{i=1}^{m} (-1)^{i-1} a_1 \wedge \ldots \wedge a_{i-1} \wedge \nu(a_i) \wedge a_{i+1} \wedge \ldots \wedge a_m$$

and extending by multilinearity to all of ΛW. Note that this map has degree $+1$. It is an easy calculation to check that that the coJacobi identity for ν is equivalent to $D_1^2 = 0$. Also observe that by construction D_1 is a derivation of the product $\wedge$ of ΛW.

Define $D := D_0 + D_1 : \Lambda W \to \Lambda W$. Then $D^2 = D_0^2 + D_1^2 + D_0 D_1 + D_1 D_0$. We have already remarked above that $D_0^2 = 0 = D_1^2$ and this correspond to the Jacobi and coJacobi identities. Note that a term in $D_0 D_1 + D_1 D_0$ arises by applying ν followed by $\{,\}$ or vice versa, so the terms of $D_0 D_1 + D_1 D_0$ can be split into two sums. One of these sums is zero if and only if $\{,\} \circ \nu = 0$. The other sum being zero is equivalent to the second order operator $D_0 D_1 + D_1 D_0$ having vanishing highest order term which is equivalent to the Drindfeld compatibility. It follows that $(W, \nu, \{,\})$ is an involutive Lie bialgebra if and only if $D^2 = 0$.

1.2 The involutive Lie bialgebra on families of curves on manifolds: a geometric description

Inspired by the involutive Lie bialgebra of Goldman and Turaev on non contractible free homotopy classes of curves on a surface, Chas and Sullivan introduced analogue constructions for *families* of unmarked curves on general smooth manifolds. In this section, we give a geometric description of their construction making certain transversality assumptions at the chain level. Later on, we present a construction of such structure at the homology level without these assumptions using Thom-Pontryagin theory.

Let M be an oriented smooth d-manifold and define LM, the *free loop space* of M, to the be the space of piecewise smooth $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1)$. Define the *space of closed strings* on M to be $\Sigma M = \mathrm{Emb}(S^1, \mathbb{R}^\infty) \times_{\mathrm{Diff}^+(S^1)} LM$, where $\mathrm{Emb}(S^1, \mathbb{R}^\infty)$ is the space of embeddings of S^1 in $\mathbb{R}^\infty$, which admits a free action of the space $\mathrm{Diff}^+(S^1)$ of orientation preserving diffeomorphisms of S^1. Thus a point in ΣM is given by a pair (S, f) where $S \subset \mathbb{R}^\infty$ is a closed, oriented, connected submanifold of $\mathbb{R}^\infty$ of dimension 1 and $f : S \to M$ a continuous map. Note that ΣM is homotopy equivalent to $ES^1 \times_{S^1} LM$.

Consider the $\mathbb{Q}$-vector space generated by tuples $(K; \sigma_1, ..., \sigma_k)$ where K is an m-dimensional compact connected oriented manifold with corners, and each σ_i denotes a smooth map $\sigma_i : P_i \to M$ where P_i is the total space of a circle bundle over K. We identify $(-K, \sigma_1, ..., \sigma_k)$ with $-(K, \sigma_1, ..., \sigma_k)$ (where $-K$ is K with the opposite orientation) in such vector space and denote the resulting quotient by $C_m(\Sigma M^k)$. Finally, define a boundary operator $\partial : C_m(\Sigma M^k) \to C_{m-1}(\Sigma M^k)$ to be the map that sends $(K; \sigma_1, ..., \sigma_k)$ to $(\partial K; \sigma_1|_{\partial_K}, ..., \sigma_k|_{\partial_K})$ where we have restricted the σ_i to the total space of the restriction of each circle bundle P_i restricted to the geometric boundary of K. We think of $(K; \sigma_1, ..., \sigma_k)$ as a family of (unparametrized) curves in the cartesian product M^k parametrized by K. We call $(C_*(\Sigma M^k), \partial)$ the chain complex of *smooth chains on* ΣM^k.

Remark 1 The homology of $(C_*(\Sigma M), \partial)$ is isomorphic to the rational S^1-equivariant singular homology of LM defined by $H_*^{S^1}(LM;\mathbb{Q}) := H_*(LM \times_{S^1} ES^1;\mathbb{Q})$.

Let us define a product $\mu : C_m(\Sigma M^2) \to C_{m+2-d}(\Sigma M)$. Let $\alpha = (K, \sigma_1, \sigma_2) \in C_m(\Sigma M^2)$, so we have circle bundles $P_j \to K$ and smooth maps $\sigma_j : P_j \to M$ for $j = 1, 2$. First, we consider the tautological lift of the m-chain α to a $m+2$-chain $M(\alpha)$ in $LM \times LM$. This lift is also known as applying the "mark" map, since it marks in all possible ways every curve in a family of unmarked curves; it is defined more precisely as follows. For each $j = 1, 2$ we have a pullback diagram

$$\begin{tikzcd} \tilde{P}_j \arrow[r] \arrow[d] & P_j \arrow[d] \\ P_1 \times_K P_2 \arrow[r] & K \end{tikzcd}$$

where $P_1 \times_K P_2$ is the fiber product of the circle bundles. The tautological sections $s_j : P_1 \times_K P_2 \to \tilde{P}_j$ defined by $s_j(p_1, p_2) = (p_1, p_2, p_j)$ induce a smooth map $M(\sigma_j) : P_1 \times_K P_2 \to LM$ since for each $p \in P_1 \times_K P_2$, σ_j defines a smooth map from the fiber above p, which is now a circle with marked point $s_i(p)$, to M. This gives an $m+2$-chain $M(\alpha) := M(\sigma_1) \times M(\sigma_2) : P_1 \times_K P_2 \to LM \times LM$.

Let $Q_\alpha := (M(\alpha) \circ (e_0 \times e_0))^{-1}(\Delta(M)) \subset P_1 \times_K P_2$, where $e_0 : LM \to M$ is defined by $e_0(\gamma) = \gamma(0)$ and $\Delta : M \to M \times M$ is the diagonal map. Assuming that the smooth map $(e_0 \times e_0) \circ M(\alpha) : P_1 \times_K P_2 \to M \times M$ is transversal to the diagonal, we have that $Q_\alpha \subset P_1 \times_K P_2$ is a submanifold with corners of dimension $m + 2 - d$. We now have a smooth chain $Q_\alpha \to LM$ that sends a point $q \in Q_\alpha$ to the concatenation of loops $M(\sigma_2)(q) * M(\sigma_1)(q) \in LM$. To obtain an element

$\mu(\alpha)$ in $C_{m+2}(\Sigma M^2)$ we apply the "erase" map to $Q_\alpha \to LM$; that is, the circle bundle associated to $\mu(\alpha)$ is a trivial circle bundle over Q_α and the maps from the total space to M are given by $Q_\alpha \to LM$. The map $\mu : C_m(\Sigma M^2) \to C_{m+2-d}(\Sigma M)$ commutes with the boundary operator.

Similarly, we define a coproduct $\delta : C_m(\Sigma M) \to C_{m+2-d}(\Sigma M^2)$. Given $\beta = (K, \sigma) \in C_m(\Sigma M)$, we first consider a lift $M(\beta) : P \to LM$, as explained above, in this case $P \to K$ is the circle bundle associated to β and $M(\beta)$ is defined by sending a point $p \in P$ to the σ applied to the fiber containing p, which can now be regarded as a circle with marked point p. Note that $\dim(P) = \dim(K) + 1$.

Let $e : LM \times (0,1) \to M \times M$ be the map $e(\gamma, t) = (\gamma(0), \gamma(t))$ and $R^\circ_\beta := ((M(\beta) \times id_{(0,1)}) \circ e^{-1})(\Delta(M)) \subset P \times [0,1]$ and denote by R_β the closure of R°_β inside $P \times [0,1]$. Assuming that $e \circ (M(\beta) \times \mathrm{id}_{(0,1)}) : P \times (0,1) \to M \times M$ is transversal to the diagonal, $R_\beta \subset P \times [0,1]$ is a submanifold with corners of dimension $i + 2 - d$. We now have a smooth chain $R_\beta \to LM \times LM$ that sends a point $(p, t) \in R_\alpha$ to $(M(\beta)(p)|_{[0,t]}, M(\beta)(p)|_{[t,1]})$. Finally, as above, we obtain an element of $\delta(\beta) \in C_{m+2-d}(\Sigma M^2)$ by considering the trivial circle bundle over R_β together with the map from the total space to M induced by $R_\beta \to LM \times LM$.

However, $\delta : C_m(\Sigma M) \to C_{m+2-d}(\Sigma M^2)$ does not commute with the boundary operator. In fact, the picture is similar to the argument explaining why the coproduct is not well defined on free homotopy classes of curves on a surface. For example, suppose we have a 1-parameter family β of curves on a surface such that at $t = 0$ we have a single self intersection creating two closed curves $\beta'(0)$ and $\beta''(0)$ and one of them, say $\beta'(0)$ is homotopic to a constant curve at a point. Assume that throughout the 1-parameter family $\beta(t) = \beta'(t), \beta''(t)$ only the loop $\beta'(0)$ is being deformed to a constant curve at the point of intersection of both curves, so that $\beta'(1)$ is a constant curve. Then we can see that $\partial \delta(\beta) \neq \delta(\partial \beta)$.

The anomaly $\partial\delta(\beta)-\delta(\partial\beta)$ is zero when we pass to the relative complex $C_*(\Sigma M^2, M)$, the quotient of the smooth chains by those which map to constant curves. Note that here $C_*(\Sigma M^2, M)$ really means $C_*(\Sigma M^2, (M\times\Sigma M)\cup(\Sigma M\times M))$. The coproduct δ induces a chain map on $C_*(\Sigma M, M)$. Moreover, the product μ is also well defined in this relative chain complex, since applying μ to a chain in $\Sigma M\times\Sigma M$ in which one of the coordinates is a family of constant curves yields a geometrically degenerate chain (a chain which has lower dimensional image than its domain dimension) and we can go back and define $C_*(\Sigma M)$ to be the normalized chain complex by taking a quotient by the degenerate chains. In fact, we assume we have taken this quotient from now on.

We call μ the bracket and δ the cobracket because of the following theorem of Chas and Sullivan which generalizes Goldman and Turaev's result for curves on surfaces to families of curves on manifolds.

Theorem 2 (Chas-Sullivan) *The product μ and the coproduct δ induce the structure of a involutive graded Lie bialgebra of degree $2-d$ on $H_*^{S^1}(LM, M;\mathbb{Q})$, the rational S^1-equivariant homology of LM relative constant loops.*

We will come back to this result later on in this chapter.

1.3 Operations on $H_*(LM)$ via Thom-Pontryagin type constructions

In the above construction of the bracket and the cobracket we started with a family of unparametrized curves, lifted it to a family of parametrized curves of one dimension higher by marking in all possible ways, preformed a transversal intersection followed by cocatenation or splitting of loops, and finally considered the resulting family as one of unparametrized curves once again. This construction suggests that there are more basic operations for families of *parametrized* curves in M yielding operations in the ordinary homology of LM that induce the operations in the S^1-equivariant homology. In this section, we will discuss the loop product and coproduct on ordinary homology that induce the above bracket and cobracket, among other operations that interact with these. We will also take a different viewpoint from the above discussion by using Thom-Pontryagin type constructions to avoid the transversality assumptions in the intersections. We begin by recalling the formulation of the intersection product on an oriented closed manifold via Thom-Pontryagin theory. We assume that homology is taken with integer coefficients throughout this section.

1.3.1 Intersection product

Let M be an oriented smooth closed manifold of dimension d and let $\Delta : M \to M \times M$ be the diagonal embedding. By the tubular neighborhood theorem of differential topology there exists a tubular neighborhood of $\Delta(M)$ inside $M \times M$. By definition, a tubular neighborhood of $\Delta(M)$ is an open neighborhood $N \subseteq M \times M$ of $\Delta(M)$ such that there is a smooth vector bundle $\eta : E \to M$ of finite rank and a diffeomorphism $\phi : N \to E$ such that $\eta \circ \phi \circ \Delta = \mathrm{id}_M$.

Construct a rank d vector bundle $\eta : E \to M$ by choosing a metric on M and splitting the restriction of the tangent bundle of $M \times M$ to $\Delta(M)$ into the tangent bundle of $\Delta(M)$ and its orthogonal complement. The vector bundle $\eta : E \to M$ is defined to be such orthogonal complement and it is called the *normal bundle* of $\Delta(M)$ in $M \times M$. Its geometric picture is the set of vectors in $M \times M$ based at $\Delta(M)$ pointing orthogonal to $\Delta(M)$ with respect to the metric; in other words, a vector in E based at $(x, x) \in \Delta(M)$ is given by a pair of vectors in M based at x pointing in opposite directions. Moreover, this identification defines an isomorphism between the normal bundle $\eta : E \to M$ and the tangent bundle $\tau : TM \to M$. Through exponential flow we can construct a neighborhood N of $\Delta(M)$ in $M \times M$ together with a diffeomorphism $\phi : N \to E$ sending $\Delta(M)$ to the zero section via the identity and geodesics in N normal to $\Delta(M)$ to straight line segments in the fibers of $\eta : E \to M$.

The *intersection product* on $H_*(M)$ is defined by the composition

$$H_*(M \times M) \to H_*(M \times M/(M \times M - N)) \to H_*(Th(\eta)) \to H_{*-d}(M).$$

The first map is induced by the collapse map $M \times M \to M \times M/(M \times M - N)$ where $M \times M - N$ denotes the complement of N inside $M \times M$, so $M \times M/(M \times M - N)$ is homeomorphic to the one point compactification of N. The space $Th(\eta)$ is the *Thom space* of the vector bundle $\eta : E \to M$ defined by the quotient space $D(E)/S(E)$, where $D(E) = \{v \in E : ||v|| \leq 1\}$ and $S(E) = \{v \in E : ||v|| = 1\}$ with respect to the chosen metric. Note if M is compact, $Th(\eta)$ is the one-point compactification of E. We have a diffeomorphism $M \times M/(M \times M - N) \cong Th(\eta)$ induced by ϕ and this induces the second map in the composition. Finally, the last map is the *Thom isomorphism* given by the cap product with the *Thom class*, i.e. the unique cohomology class in $H^d(Th(\eta))$ that evaluates to 1 on the chosen generator of the top degree homology of each fiber (which are spheres of dimension d in $Th(\eta)$). Intuitively, the cap product of the Thom class and a chain counts the intersections of the chain with the zero section of E, thus the Thom class is the Poincaré dual to the zero section of E.

1.3.2 Thom class and pullback fibrations

We make several general comments about the Thom class and pullback fibrations before we go into the context of loop spaces. Let $i : M \to Q$ be a smooth embedding of smooth oriented closed manifold of dimensions d and q. Let $\eta : E \to M$ be the normal bundle of the embedding oriented as $E \oplus i_*(TM) \cong TQ|_{i(M)}$. This is a vector bundle of rank $q-d$. Let $u \in H^{q-d}(Th(E))$ be the Thom class of $\eta : E \to M$ and let $u_M \in H^{q-d}(M)$ be the corresponding class under the composition $H^{q-d}(Th(E)) \to H^{q-d}(E) \cong H^{q-d}(M)$, where the first map is induced by the collapse $E \to Th(E)$ onto the Thom space. The class u_M is characterized by $u_M \frown [Q] = i_*[M]$; i.e. u_M is Poincaré dual to the fundamental class of M.

Let $e : L \to Q$ be a fibration. We can pullback such fibration along $i : M \to Q$ to obtain a fibration denoted by $e_i^* : i^*L \to M$. We can think of this fibration as the restriction of $e : L \to Q$ to $i(M)$. Let $j : e^*L \hookrightarrow L$ be the inclusion map.

Let N be a tubular neighborhood of $i(M)$ inside Q, so we have a diffeomorphism $\phi : N \to E$ sending $i(M)$ to the zero section. We will be interested in the case when we have a map $\tilde{\phi} : \tilde{N} \to e_i^*E$ lifting ϕ, where $e_i^*E \to i^*L$ is the pullback vector bundle of the normal bundle $\eta : E \to M$ along $e_i : i^*L \to M$.

We then define $j_! : H_*(L) \to H_{*+q-d}(i^*(L))$ as the composition

$$j_! : H_*(L) \to H_*(L/(L-\tilde{N})) \to H_*(Th(e_i^*(E))) \to H_{*+q-d}(e_i^*E) \cong H_{*+q-d}(i^*(L))$$

where the first map is induced by the collapse map, the second by the lift $\tilde{\phi}$, the third by cap product with the pullback of $u \in H^{q-d}(Th(E))$ along $Th(e_i^*E) \to Th(E)$, and the last map is the isomorphism induced by the vector bundle projection.

From the naturality properties of the cap product we have the formula $j_* j_!(\alpha) = \alpha \frown e^* u_M$.

1.3.3 The Chas-Sullivan loop product

We now present the construction of a intersection-type product on the homology of LM that extends the intersection product described above.

Let $LM \times_M LM := \{(\gamma_1, \gamma_2) \in LM \times LM : \gamma_1(0) = \gamma_2(0)\}$, so we have a pullback diagram

$$\begin{tikzcd} LM \times_M LM \arrow[r, hook] \arrow[d, "e_0"'] & LM \times LM \arrow[d, "e_0 \times e_0"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

where $e_0 : LM \to M$ is the evaluation at 0 map $e_0(\gamma) = \gamma(0)$.

As defined above, let $(N \subset M \times M, \psi : E \to N)$ be a tubular neighborhood of $\Delta(M) \subset M \times M$, where $\eta : E \to M$ is the normal bundle of $M \cong \Delta(M)$ and ψ a diffeomorphism sending the zero section $s_0(M) \subset E$ to $\Delta(M) \subset N$. Consider the inverse image $\tilde{N} := (e_0 \times e_0)^{-1}(N) \subset LM \times LM$. Note that $\tilde{N}$ is a neighborhood of LM consisting of pairs of loops whose base points are close in M. It turns out that this $\tilde{N}$ is homeomorphic to the total space of a vector bundle as shown in the following proposition. The result can interpreted as stating that $LM \times_M LM \hookrightarrow LM \times LM$ is an embedding of codimension d.

Proposition 2 *There exists a homeomorphism $\tilde{\psi} : e_0^*(E) \to \tilde{N}$ lifting $\psi : E \to N$, where $\tilde{N} := (e_0 \times e_0)^{-1}(N)$ and $e_0^*(\eta) : e_0^*(E) \to LM \times_M LM$ is the pullback bundle of the the normal bundle $\eta : E \to M$ along the evaluation map $e_0 : LM \times_M LM \to M$.*

Proof. Recall that by the tubular neighborhood theorem we may assume $\psi : E \to N$ is given by exponential flow, so a straight line segment from $(x, 0)$ to (x, v) in the fiber E_x is mapped by ψ to the geodesic segment from $(x, x) \in \Delta(M)$ to the point $\psi(x, v) \in N$.

For any tubular neighborhood $(N, \psi : E \to N)$ we can associate a *propagating flow*. A propagating flow for the bundle $\eta : E \to M$ is a continuous map $F : E \to \chi_c^{\text{vert}}(E)$, where the latter is the space of compactly supported vertical vector fields on E, with the property that for all $v \in E$ we have $F(v)_x = v$ if $x = tv$ for $t \in [0,1]$ and $F(0) = 0$. For any bundle E one can construct an associated propagating flow using partitions of unity. Choose a propagating flow F_E for E. F_E yields a map $X : E \to \chi_c(M \times M)$ by pushing forward vector fields along the embedding $\psi : E \to N \hookrightarrow M \times M$. The map X has the property that for any $v \in E$ we have $X(v)(\exp_{\Delta\eta(v)}(tv)) = v$ for $t \in [0,1]$. Furthermore, $X : E \to \chi_c(M \times M)$ gives rise to a map $\theta : E \times \mathbb{R} \to \mathrm{Diff}_c(M \times M)$ by flowing along each vector field. Then it follows that $\psi(v) = \theta(v,1)(\eta(v), \eta(v))$.

We now define $\tilde{\psi} : e_0^*(E) \to \tilde{N}$ using the map θ obtained from a propagating flow associated to $\eta : E \to M$. Let $\theta_i : E \to \mathrm{Diff}(M)$ be $\theta_i(v)(x) := \pi_i(\theta(v,1)(x,x))$ where $\pi_i : M \times M \to M$ is the projection to the i-th component for $i = 1,2$. A point in $e_0^*(E)$ is given by a triple (γ_1, γ_2, v) where $(\gamma_1, \gamma_2) \in LM \times_M LM$ and v is a tangent vector in $M \times M$ at $\gamma_1(0) = \gamma_2(0)$ normal to $\Delta(M)$. Define

$$\tilde{\psi}(\gamma_1, \gamma_2, v) := (\theta_1(v) \circ \gamma_1, \theta_2(v) \circ \gamma_2).$$

The map $\tilde{\psi}$ is clearly a homeomorphism by construction, since $\psi : E \to N$ is a homeomorphism. In fact, we can think of the inverse of $\tilde{\psi}$ as the map that moves two loops with base points lying in the tubular neighborhood N to two new loops which actually intersect at their base points and records the original lack of intersection with a tangent vector in $M \times M$ normal to $\Delta(M)$. $\square$

Now that we know $LM \times_M LM \hookrightarrow LM \times LM$ has a tubular neighborhood we will define the loop product by "intersecting" the base points of two chains in LM and concatenating loops in the locus of intersection, as the following definition makes precise.

Definition 2 The *loop product* on $H_*(LM)$ is defined as the composition

$$\bullet : H_*(LM \times LM) \to H_*((LM \times LM)/(LM \times_M LM - \tilde{N})) \to H_*(Th(e_0^*(\eta)) \to H_{*-d}(LM \times_M LM) \to H_{*-d}(LM)$$

where the first map is induced by the collapse map, the second by $\tilde{\psi}^{-1} : \tilde{N} \to e^*(E)$, the third is the Thom isomorphism applied to the pullback bundle $e_0^*(\eta) : e_0^*(E) \to LM \times_M LM$ which is the cap product with the pullback of the Thom class of $E \to M$, and the last map is induced by the concatenation of loops $LM \times_M LM \to LM$. We will also denote by $\bullet$ the map obtained by precomposing the above composition with $H_*(LM) \otimes H_*(LM) \to H_*(LM \times LM)$. Using the notation introduced in section 3.2, the loop product is the composition $c_* \circ i_! : H_*(LM \times LM) \to H_{*-d}(LM)$ where $i : LM \times_M LM \hookrightarrow LM \times LM$ is the inclusion and $c : LM \times_M LM \to LM$ is the concatenation map.

Remark 2 The definition of the loop product (and of the intersection product as well) on the homology of LM is independent of the choices made. First we chose a metric on M which determines a smooth map by geodesic flow from a small neighborhood U of E_0 (the zero section of the normal bundle $\eta : E \to M$) to $M \times M$ which is a diffeomorphism onto its image (this image is what we called N). The space of metrics on a smooth manifold is contractible. Moreover, the space of neighborhoods $U \subset E$ of the zero section E_0 which can be continuously deformed into E_0 is also contractible. This follows since fixing U we can first retract the space of such neighborhoods to the subspace of neighborhoods contained in U and then we can expand this subspace onto U.

For the loop product there was one more choice involved: in order to identify $\tilde{N}$ with $e_0^*(E)$, we chose propagating flow for E. The space of propagating flows is contractible as well, in fact it is convex.

Remark 3 By construction the following diagram commutes

$$\begin{tikzcd} H_*(LM \times LM) \arrow[r] \arrow[d] & H_{*-d}(LM \times LM) \arrow[d] \\ H_*(M \times M) \arrow[r] & H_{*-d}(M) \end{tikzcd}$$

where the horizontal maps are the loop product and the intersection product and the vertical maps are induced by evaluation at 0.

Chas and Sullivan observed that the loop product is associative and commutative.

Theorem 3 (Chas-Sullivan) *Let M be a closed oriented manifold of dimension d. Then $(H_*(LM), \bullet)$ is a unital associative, commutative algebra of degree $-d$.*

Proof. The associativity follows from the fact that the concatenation product $c : LM \times_M LM \to LM$ extends the concatenation product on the based loop space $\Omega M \times \Omega M \to \Omega M$ which is associative up to a homotopy determined by reparametrization; in fact ΩM is an A_∞-space.

The commutativity follows from the fact that $c : LM \times_M LM \to LM$ is homotopy commutative. An explicit homotopy $H : (LM \times_M LM) \times I \to LM$ is given by $H(\gamma_1, \gamma_2, t)(s) = \gamma_2(2s - t)$ for $0 \leq s \leq t/2$, $H(\gamma_1, \gamma_2, t)(s) = \gamma_2(2s - t)$ for $t/2 \leq s \leq (t+1)/2$ and $H(\gamma_1, \gamma_2, t)(s) = \gamma_2(2s - t)$ for $(t+1)/2 \leq s \leq 1$. Thus, $H(\gamma_1, \gamma_2, t)$ is the loop that starts at $\gamma_2(-t)$ and runs along γ_2 to reach $\gamma_2(0) = \gamma_1(0)$, then goes once around the loop γ_1 and finally runs along γ_2 back to $\gamma_2(-t)$.

The unit is given by $s_*[M]$ where $[M] \in H_d(M)$ is the fundamental class of M and $s : M \hookrightarrow LM$ is the inclusion of M as constant loops in LM. □

We will shift the homology of LM by d to make the loop product of degree 0. Denoting $\mathbb{H}_*(LM) = H_{*+d}(LM)$, we have that $(\mathbb{H}_*(LM), \bullet)$ is a unital commutative associative algebra in the usual sense (the product and the unit have degree 0).

1.3.4 Gerstenhaber and BV-algebra structures

Rotation operator. The free loop space LM admits a circle action $\rho : S^1 \times LM \to LM$ given by rotation of loops: $\rho(t,\gamma)(s) = \gamma(t+s)$. This circle action induces a degree $+1$ map in homology $\Delta : H_*(LM) \to H_{*+1}(LM)$ by $\alpha \mapsto \rho_*([S^1]\times\alpha)$ where $[S^1] \in H_1(S^1)$ is the fundamental class. Note $\Delta^2 = 0$ since for $\alpha \in H_*(LM)$ we have $\Delta^2(\alpha) = \rho_*\big(([S^1]*[S^1])\otimes\alpha\big)$, where $*$ is the product in $H_*(S^1)$ induced by the group structure of S^1, but of course $[S^1]*[S^1] = 0$. *Warning*: do not confuse the rotation operator Δ with the diagonal map also denoted by Δ. We will change the notation for this section only and denote by $diag$ the diagonal map.

Loop bracket. We now describe another product on $\mathbb{H}_*(LM)$, called the loop bracket, related to the above rotation operator. Chas and Sullivan described this operation on transversal chains as the commutator of the chain homotopy for the commutativity of the loop product. Here we give a Thom-Pontryagin collapse description based on the ideas of Cohen-Jones.

Let $e : LM \times LM \times S^1 \to M \times M$ be the fibration $e(\gamma_1,\gamma_2,t) = (\gamma_1(0),\gamma_2(t))$. Define $G \subset LM \times LM \times S^1$ by $G := \{(\gamma_1,\gamma_2,t) : \gamma_1(0) = \gamma_2(t)\}$. Note that G fits in a pullback diagram of fibrations

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "e"'] & LM \times LM \times S^1 \arrow[d, "e"] \\ M \arrow[r, "diag"'] & M \times M \end{tikzcd}$$

As in the case of the loop product, the top horizontal map is a codimension d embedding.

This can be shown precisely in a similar manner: by constructing a homeomorphism from $e^{-1}(N)$, where N is a tubular neighborhood of the diagonal, to the pullback $e^*(E)$ of the normal bundle $\eta : E \to M$ along the map $e : G \to M$ using a propagating flow. As before, we have a composition of maps

$$g : H_*(LM \times LM \times S^1) \to H_*((LM \times LM \times S^1)/(LM \times LM \times S^1 - e^{-1}(N))) \to H_*(Th(e^*(\eta))) \to H_{*-d}(G)$$

Both $LM \times LM$ and S^1 admit a $\mathbb{Z}/2$ action by switching factors and by sending a point to its antipode, respectively. Thus, $(LM \times LM) \times S^1$ admits a diagonal $\mathbb{Z}/2$ action. Observe that $(LM \times_M LM) \times S^1$ is homeomorphic to G, with homeomorphism given by $(\gamma_1, \gamma_2, t) \mapsto (\gamma_1, \gamma_2^{-t}, t)$ where $\gamma_2^{-t}(s) := \gamma_2(s-t)$. Moreover, this homeomorphism is $\mathbb{Z}/2$-equivariant. Hence, post-composing g with the map in homology induced by the inverse of such homeomorphism and using its $\mathbb{Z}/2$-equivariance we obtain a map

$$\tilde{g} : H_*((LM \times LM) \times_{\mathbb{Z}/2} S^1) \to H_{*-d}((LM \times_M LM) \times_{\mathbb{Z}/2} S^1).$$

We now precompose $\tilde{g}$ with the map $\iota : H_*(LM \times LM) \to H_{*+1}((LM \times LM) \times_{\mathbb{Z}/2} S^1)$ defined by $\iota(\alpha \otimes \beta) = [S^1] \times (\alpha \otimes \beta - \tau(\alpha \otimes \beta))$, that is, crossing the fundamental class of S^1 with the commutator of $\alpha \otimes \beta$. Denote

$$h := \tilde{g} \circ \iota : H_*(LM \times LM) \to H_{*+1-d}((LM \times_M LM) \times_{\mathbb{Z}/2} S^1).$$

Finally, recall that we have a homotopy $H : (LM \times_M LM) \times I \to LM$ which was defined above in the proof of the commutativity of the loop product at the level of homology. The homotopy H induces a map $\tilde{H} : (LM \times_M LM) \times_{\mathbb{Z}/2} S^1 \to LM$ by identifying $[0,1]$ with the upper semicircle of S^1.

Definition 3 The *loop bracket* is the map of degree $1-d$ defined by the composition

$$\{,\}: H_i(LM)\otimes H_j(LM) \xrightarrow{\times} H_{i+j}(LM\times LM) \xrightarrow{h_*} H_{i+j+1-d}((LM\times_M LM)\times_{\mathbb{Z}/2} S^1) \xrightarrow{\tilde{H}_*} H_{i+j+1-d}(LM).$$

Remark 4 $(\mathbb{H}_*(LM),\{,\})$ is an algebra of degree $+1$.

We now explain the relationship between the rotation operator and the loop bracket. First, Chas and Sullivan showed that the rotation operator and the loop product define a BV structure.

Theorem 4 *The loop product* $\bullet$ *and the rotation operator* Δ *define a BV-algebra structure on* $\mathbb{H}_*(LM)$*, namely*
(i) $(\mathbb{H}_*(LM),\bullet)$ *is a graded commutative associative algebra.*
(ii) $\Delta^2=0$
(iii) The binary operator $u(\alpha,\beta):=(-1)^{deg(\alpha)}\Delta(\alpha\bullet\beta)-(-1)^{deg(\alpha)}\Delta(\alpha)\bullet\beta-\alpha\bullet\Delta(\beta)$ *is a derivation of the loop product* $\bullet$ *on each variable.*

Then it was observed that the binary operator u is actually the loop bracket, as we state in the following

Theorem 5 *If* $\{,\}$ *is the loop bracket and* u *is the binary operator defined in the previous theorem, i.e. the deviation of* Δ *being a derivation of the loop product* $\bullet$*, then* $\{\alpha,\beta\}=u(\alpha,\beta)$ *for any* $\alpha,\beta\in\mathbb{H}_*(LM)$.

As a formal consequence of Theorem 4 and Theorem 5 we obtain

Corollary 1 $(\mathbb{H}_*(LM),\{,\},\bullet)$ *is a Gerstenhaber algebra, namely*
(i) $(\mathbb{H}_*(LM),\bullet)$ *is a graded commutative associative algebra.*

(ii) $\{,\}$ *is a Lie bracket of degree* $+1$*, which means that* $\{,\}$ *satisfies graded skew symmetry and the graded Jacobi identity.*

(iii) $\{\alpha, \beta \bullet \gamma\} = \{\alpha, \beta\} \bullet \gamma + (-1)^{deg(\beta)(deg(\alpha)-1)} \beta \bullet \{\alpha, \gamma\}$

Remark 5 In their original paper Chas and Sullivan define the loop bracket $\{,\}$ is as the commutator of the chain homotopy for the commutativity of the loop product, then it is proven that skew symmetry, Jacobi identity, and the derivation property hold, namely that the loop bracket and the loop product define a Gerstenhaber algebra structure. Finally, the loop bracket is identified with the deviation of the rotation operator Δ from being a derivation of the loop product $\bullet$. This proves that Δ and $\bullet$ define a BV-algebra structure.

Remark 6 Voronov arrived to an equivalent BV-algebra structure on the homology of $H_*(LM)$ by defining an action of certain topological operad, called the "cacti operad" on the free loop space. The cacti operad is homotopy equivalent to the framed little disks operad, which was shown to be the operad describing BV-algebras by Getzler.

Remark 7 We can also arrive to the BV-algebra structure by identifying the loop bracket, rotation operator, and the loop product with algebraic operations on Hochschild and coHochschild complexes and then proving the identities in the algebraic context. This is similar to the original arguments of Chas an Sullivan, since the algebraic operations are analogue to partially defined operations on transversal chains. Several aspects of the relation between algebraic operations and geometric string topology operations are discussed in the next chapter.

1.3.5 The loop coproduct $\vee_t$ of degree $-d$

We now define our first example of a coproduct $H_*(LM)$, also constructed by Sullivan on transversal chains. We give a Thom collapse description. We will see that this coproduct will not be as interesting as other string topology operations, as it was shown by Tamanoi. We follow Tamanoi's exposition when verifying the main properties of such coproduct.

Fix a time $t \in (0,1)$ and define $F_t \subset LM$ to be the subspace of loops with a self intersection at time t; i.e. $F_t = \{\gamma \in LM : \gamma(0) = \gamma(t)\}$. Thus F_t fits in a pullback diagram of fibrations

$$\begin{tikzcd} F_t \arrow[r, hook] \arrow[d, "e_0"'] & LM \arrow[d, "e_{0,t}"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

where $e_0 : F_t \to M$ is the evaluation at 0 map, $e_{0,t} : LM \to M \times M$ is $e_{0,t}(\gamma) = (\gamma(0), \gamma(t))$, and Δ is the diagonal map.

As before, let $N \subseteq M \times M$ be a tubular neighborhood of $\Delta(M)$ and $\eta : E \to M$ the normal bundle of $\Delta(M)$ inside $M \times M$. Denote by $\tilde{N}_t := e_{0,t}^{-1}(N)$, so $\tilde{N}_t$ is a neighborhood of F_t. The neighborhood $\tilde{N}_t$ is actually a tubular neighborhood. In fact, a similar argument as the one in the proof of Proposition 2 shows that we have a diffeomorphism $\tilde{N}_t \to e_0^* E$ where $e_0^*(\eta) : e_0^* E \to F_t$ is the pullback of the normal bundle $\eta : E \to M$ along the map $e_0 : F_t \to M$. Thus $F_t \to LM$ is a codimension d embedding. A diffeomorphism $\tilde{N}_t \to e_0^* E$ is intuitively described as follows. Given $\gamma \in \tilde{N}_t$ we have that $\gamma(0)$ and $\gamma(t)$ lie inside N so we can assume they are sufficiently close. Use a propagating flow for the normal bundle to continuously deform γ to a new loop which brings $\gamma(t)$ and $\gamma(0)$ to intersect at the middle point of the geodesic between them. This new loop will have a self intersection at some time t', we now reparametrize linearly so that the intersection occurs at time t and the resulting loop is an element of F_t. We also record the normal vector to $\Delta(M)$ in $M \times M$ measuring the lack of intersection between $\gamma(0)$ and $\gamma(t)$. The new loop together with the normal vector define an element of $e_0^* E$.

A rigorous definition of the map above involves several technical details. However, we do not need a to show there exists a tubular neighborhood to define a wrong way map. We just need lift of the diffeomorphism $\phi : N \to E$ to a continuous map $\tilde{N}_t \to e_0^*(E)$. In particular, instead of the propagating flow argument we can use the following map. Given $\gamma \in N_t$ define $\tilde{\phi}_t : N_t \to e_0^* E$ by

$$\tilde{\phi}_t(\gamma) := \left(\beta_1^\gamma * \gamma|_{[0,t]} * (\beta_2^\gamma)^{-1} * \beta_2^\gamma * \gamma|_{[t,1]} * (\beta_1^\gamma)^{-1}, (\beta^\gamma)'(0)\right)$$

where $\beta^\gamma(s) = (\beta_1^\gamma(s), \beta_2^\gamma(s))$ is the geodesic in $N \subset M \times M$ from $(\eta \circ \phi)(\gamma(0), \gamma(t)) \in \Delta(M)$ to $(\gamma(0), \gamma(t)) \in N$, and $(\beta^\gamma)'(0)$ is the tangent vector to β^γ at 0 which is a vector in the normal bundle $\eta : E \to M \cong \Delta(M)$ based at the midpoint between $\gamma(0)$ and $\gamma(t)$. We assume that we have linearly reparametrized $\beta_1^\gamma * \gamma|_{[0,t]} * (\beta_2^\gamma)^{-1} * \beta_2^\gamma * \gamma|_{[t,1]} * (\beta_1^\gamma)^{-1}$ so that the intersection we have created in $\tilde{\phi}_t(\gamma)$ at the midpoint of $\gamma(0)$ and $\gamma(t)$, happens at time t.

Definition 4 The *loop coproduct of degree -d* is defined by the composition

$$\vee_t : H_*(LM) \to H_*(LM/(LM - \tilde{N}_t)) \to H_*(Th(e_0^*(\eta))) \to H_{*-d}(F_t) \to H_{*-d}(LM \times LM)$$

where the first map is induced by the collapse map, the second is induced by $\tilde{\phi}_t : \tilde{N}_t \to e_0^*(E)$, the third is the Thom isomorphism, and the last one is induced by the "cutting" map $F_t \to LM \times LM$ defined by $\gamma \mapsto (\gamma|_{[0,t]}, \gamma|_{[t,1]})$ and then reparametrizing to obtain an element of $LM \times LM$.

Remark 8 The loop coproduct $\vee_t$ is essentially given by capping with the pullback of the Euler class of M along the evaluation map $e_0 : LM \to M$ and then applying the cutting map. Denote the collapse maps by $q : M \times M \to (M \times M)/(M \times M - N)$ and $\tilde{q} : LM \to LM/(LM - \tilde{N}_t)$, $i : F_t \hookrightarrow LM$ the inclusion, and $u \in H^d((M \times M)/(M \times M - N))$ the Thom class. Consider the composition $f = i_* \circ i_! : H_*(LM) \to H_*(LM/(LM - \tilde{N}_t)) \to H_*(Th(e_0^*(\eta))) \to H_{*-d}(F_t) \to H_*(LM)$.

For any $\alpha \in H_*(LM)$ we then have $f(\alpha) = \tilde{q}_*\alpha \frown e_{0,t}^* u$, where we have also denoted by $e_{0,t}$: $LM/(LM - \tilde{N}_t) \to (M \times M)/(M \times M - N)$ the map induced by $\gamma \mapsto (\gamma(0), \gamma(t))$, and $\frown$ is the cap product. But note

$$f(\alpha) = \tilde{q}_*\alpha \frown e_{0,t}^* u = \alpha \frown \tilde{q}^* e_{0,t}^* u = \alpha \frown e_{0,t}^* q^* u = \alpha \frown \Delta^*(e_0 \times e_t)^* q^* u.$$

Since the map $e_0 \times e_t : LM \times LM \to M \times M$ is homotopic to $e_0 \times e_0 : LM \times LM \to M \times M$ we have

$$f(\alpha) = \alpha \frown \Delta^*(e_0 \times e_0)^* q^* u = \alpha \frown e_0^* e_M$$

where $e_M \in H^d(M)$ is the Euler class of M, which by definition is the pullback of the Thom class along the zero section. Notice that f is independent of t. Observe also that $f(s_*[M]) = \chi(M)[c_b]$ where $[M]$ is the fundamental class of the oriented closed manifold M, $s : M \to LM$ is the inclusion as constant loops, $\chi(M)$ the Euler characteristic, and $[c_b] \in H_0(LM)$ is $[c_b] = s_*[b]$, i.e. the zero degree homology class of the constant loop at a base point $b \in M$. This calculation can be obtained from the more general formula stated at the end of section 3.2.

The loop product $\bullet$ defines a left action of $H_*(LM)$ on $H_*(LM \times LM)$ by the composition

$$H_*(LM) \otimes H_*(LM \times LM) \to H_*(LM \times LM \times LM) \xrightarrow{(j\times 1)_!} H_*(LM \times_M LM) \times LM) \xrightarrow{(c\times 1)_*} H_*(LM \times LM).$$

Here $j : LM \times_M LM \to LM \times LM$ is the inclusion and $c : LM \times_M LM \to LM$ is the concatenation map. We have denoted by a lower shriek "!" the associated wrong way map defined in a similar manner as we have been doing earlier: a Thom collapse followed by a Thom isomorphism. By 1 we just mean the identity map. A right action is defined analogously.

With respect to this $H_*(LM)$-bimodule structure on $H_*(LM \times LM)$, the coproduct $\vee_t$ satisfies the following Frobenius compatibility with the loop product.

Proposition 3 *For any $\alpha, \beta \in H_*(LM)$ we have* $\vee_t(\alpha \bullet \beta) = (-1)^{d(deg(\alpha)-d)} \alpha \bullet \vee_t(\beta) = \vee_t(\alpha) \bullet \beta$.

Proof. By reparametrizing we may identify F_t with $LM \times_M LM$. Note that under this identification the inclusion j and concatenation c defined above correspond to the "cutting" map $F_t \to LM \times LM$ and to the inclusion $F_t \hookrightarrow LM$, respectively.

We will also consider the inclusion $1 \times_M j : LM \times_M LM \times_M LM \to LM \times_M LM \times LM$ and the concatenation of the last two loops $1 \times_M c : LM \times_M LM \times_M LM \to LM \times_M LM$. Define $j \times_M 1$ and $c \times_M 1$ similarly. Observe that all these maps have associated wrong way maps in homology. We will denote the wrong way maps with a lower shriek, for example $(1 \times_M j)_! : H_*(LM \times_M LM \times LM) \to H_{*-d}(LM \times_M LM \times_M LM)$.

The first equality of the Frobenius compatibility follows from the following diagram which commutes up to a sign:

$$\begin{tikzcd}
H_*(LM \times LM) \arrow[r, "(1\times c)_!"] \arrow[d, "j_!"] & H_{*-d}(LM \times LM \times_M LM) \arrow[r, "(1\times j)_*"] \arrow[d, "(j\times_M 1)!"] & H_{*-d}(LM \times LM \times LM) \arrow[d, "(j\times 1)_!"] \\
H_{*-d}(LM \times_M LM) \arrow[r, "(1\times_M c)_!"] \arrow[d, "c_*"] & H_{*-2d}(LM \times_M LM \times_M LM) \arrow[r, "(1\times_M j)_*"] \arrow[d, "(c\times_M 1)_*"] & H_{*-2d}(LM \times_M LM \times LM) \arrow[d, "(c\times 1)_*"] \\
H_{*-d}(LM) \arrow[r, "c_!"] & H_{*-2d}(LM \times_M LM) \arrow[r, "j_*"] & H_{*-2d}(LM \times LM)
\end{tikzcd} \tag{1.1}$$

Applying the top horizontal row by the right most vertical column to a class $\alpha \times \beta \in H_*(LM \times LM)$ gives us $\pm\alpha \bullet \vee_t(\beta)$. Applying the left most vertical column by the bottom horizontal row yields $\vee_t(\alpha \bullet \beta)$. Finally, note that the diagram commutes: the bottom right square commutes because of functoriality of pushforward maps, the top left also commutes because of the functoriality of wrong way maps, and the other two squares commute because the corresponding squares of fibrations (over products of M) commute and because of the naturality properties of the cap product. The second equality of the Frobenius compatibility follows from a similar diagram. $\square$

From the Frobenius compatibility we can conclude that the product $\vee_t$ behaves quite trivially.

Proposition 4 *For any $\alpha \in H_*(LM)$, we have $\vee_t(\alpha) = \pm\chi(M)(\alpha \bullet [c_b]) \otimes [c_b] = \chi(M)[c_b] \otimes ([c_b] \bullet \alpha)$, where $[c_b]$ is the generator of $H_0(L_c M)$, the zeroth homology group of the connected component of LM consisting of contractible loops.*

Proof. We first compute $\vee_t(s_*[M])$. By Remark 8 we have that $i_* i_!(s_*[M]) = \chi(M)[c_b]$ for $i : F_t \hookrightarrow LM$. But note that i_* defines an isomorphism $H_0(L_c M \times L_c M) \to H_0(L_c)$ where $L_c M$ denotes the connected component of LM consisting of contractible loops. Moreover, this isomorphism is given by $i_*[c_b, c_b] = [c_b]$. This forces $i_!(s_*[M]) = \chi(M)[c_b, c_b]$. Therefore the coproduct is given by $\vee_t(s_*[M]) = j_*(\chi(M)[c_b, c_b]) = \chi(M)[c_b] \otimes [c_b]$. The desired formula now follows since $s_*[M]$ is the unit for the loop product $\bullet$ and from the Frobenius compatibility proved in the above proposition. $\square$

Proposition 4 has the following consequences:

(1) $\vee_t : H_*(LM) \to H_{*-d}(LM \times LM)$ is independent of t.

(2) The coproduct $\vee_t$ is non zero only on $H_d(LM)$ because of degree reasons. Moreover, the formula of Proposition 4 implies that it is non zero only on $H_d(L_c M)$.

(3) The coproduct $\vee_t$ always lands in $H_0(L_c M \times L_c M) \cong H_0(L_c M) \otimes H_0(L_c M) \cong \mathbb{Z}[c_b] \otimes [c_b]$. In particular $\vee_t$ vanishes if we work modulo constant loops.

(4) The coproduct $\vee_t$ is cocommutative.

(5) $\vee_t$ vanishes if $\chi(M) = 0$.

1.3.6 The coproduct $\vee$ of degree $1-d$

We would like to construct a coproduct showing more interesting properties than those of $\vee_t$. Sullivan's original idea was to construct a map at the level of transversal chains from the one parameter family of chain level coproducts $\vee_t$, which considers all self intersections of points in a loop with its marked point. The resulting coproduct has degree $1-d$. Goresky and Hingston also give a construction of this coproduct using different methods while studying closed geodesics on a Riemannian manifold. We give a more homotopy theoretic description using Thom-Pontryagin type constructions.

There are certain subtleties we have to take into account. For example, in the above construction of $\vee_t : H_*(LM) \to H_*(LM \times LM)$ we required $t \in (0,1)$. We did this because of the codimension change at $t = 0$ and $t = 1$: the space $F_t = \{\gamma \in LM : \gamma(0) = \gamma(t)\}$ is a codimension d subspace of LM for $t \in (0,1)$, while $F_0 = F_1 = LM$. This codimension change implies that the union $\bigcup_{t\in(0,1)} \tilde{N}_t$ of the tubular neighborhoods of $F_t \subset \tilde{N}_t \subset LM$ for each $t \in (0,1)$, which is a tubular neighborhood of $F_{(0,1)} = \{(\gamma, t) \in LM \times (0,1) : \gamma(0) = \gamma(t)\}$ inside $LM \times (0,1)$, does not extend to a tubular neighborhood of $F := F_I = \{(\gamma, t) \in LM \times [0,1] : \gamma(0) = \gamma(t)\}$ inside $LM \times [0,1]$.

However, this does not mean a tubular neighborhood does not exist. In fact, there exists a tubular neighborhood of F but we do not discuss the proof here since we will not need this result in full generality to construct the coproduct. The construction of such tubular neighborhood was suggested by Nathalie Wahl and will be discussed in [Rivera- Summer 2015 Arxiv] in more detail. We will just provide a continuous map $\tilde{\phi}$ from the neighborhood $\tilde{N}_I$ of F, where $\tilde{N}_I$ is inverse image of a small tubular neighborhood N of the diagonal in $M \times M$, to a (pullback) vector bundle of rank d, and this map will lift the given diffeomorphism ϕ between N and the normal bundle of the diagonal embedding $\Delta : M \hookrightarrow M \times M$. Using $\tilde{\phi}$ we construct a wrong way map $H_*(LM \times [0,1]) \to H_{*-d}(F)$ (lifting the intersection product on M) that yields a coproduct $\vee$ on $H_*(LM, M)$. Moreover, we show that the map $\tilde{\phi}$ is actually a homotopy equivalence and we describe a homotopy inverse.

For this section it will be convenient to fix a Riemannian metric on M and to take as a model for LM the space of piecewise smooth loops $\gamma : [0,1] \to M$ parametrized *proportional to arc length*. The constructions we make at the level of homology will be independent of the choice of metric.

Denote by $e : LM \times [0,1] \to M \times M$ the map $e(\gamma, t) = (\gamma(0), \gamma(t))$. We have a pullback diagram of fibrations

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, "e_0"'] & LM \times [0,1] \arrow[d, "e"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd} \tag{1.2}$$

where $e_0 : F \to M$ is the evaluation at 0 map .

As before, let $(N \subset M \times M, \psi : E \to N)$ be a tubular neighborhood of $\Delta(M) \subset M \times M$, so N is a neighborhood of $\Delta(M)$ inside $M \times M$, $\eta : E \to M$ is the normal bundle of $M \cong \Delta(M)$ in $M \times M$, and ψ is a diffeomorphism sending the zero section $s_0(M) \subset E$ to $\Delta(M) \subset M \times M$. We will denote the inverse of ψ by $\phi : N \to E$. We may assume that if $(x, y) \in N$ then the distance between x and y is smaller than the injectivity radius of M and ϕ sends geodesics in N normal to $\Delta(M)$ to straight line segments in the fibers of E. Let $\tilde{N}_I := e^{-1}(N) \subset LM \times I$, so $\tilde{N}_I$ is a neighborhood of F. We construct a lift $\tilde{\phi} : \tilde{N}_I \to e_0^*(E)$ of $\phi : N \to E$. We use a different method from the earlier case of $LM \times_M LM$: instead of using propagating flows and compactly supported diffeomorphisms to deform loops to create an intersection we use a short geodesic to connect nearby points.

Proposition 5 *There exists a homotopy equivalence* $\tilde{\phi} : \tilde{N}_I \to e_0^*(E)$ *lifting* $\phi : N \to E$.

Proof. For any $(\gamma, t) \in \tilde{N}_I$ we let $\beta^{\gamma,t}(s) = (\beta_1^{\gamma,t}(s), \beta_2^{\gamma,t}(s))$ be the unique geodesic path in $N \subset M \times M$ from
$(\eta \circ \phi)(\gamma(0), \gamma(t)) \in \Delta(M)$ to $(\gamma(0), \gamma(t)) \in N$. Define $\tilde{\phi} : \tilde{N}_I \to e_0^*(E)$ by

$$\tilde{\phi}(\gamma, t) := \left(\beta_1^{\gamma,t} * \gamma|_{[0,t]} * (\beta_2^{\gamma,t})^{-1} * \beta_2^{\gamma,t} * \gamma|_{[t,1]} * (\beta_1^{\gamma,t})^{-1}, t_\beta, (\beta^{\gamma,t})'(0)\right)$$

where $*$ denotes concatenation of paths (read from left to right), the time parameter t_β is given by the proportion of the length of $\beta_1^{\gamma,t} * \gamma|_{[0,t]} * (\beta_2^{\gamma,t})^{-1}$ to the entire loop $\beta_1^{\gamma,t} * \gamma|_{[0,t]} * (\beta_2^{\gamma,t})^{-1} * \beta_2^{\gamma,t} * \gamma|_{[t,1]} * (\beta_1^{\gamma,t})^{-1}$ (assuming our paths are parametrized proportional to arc length), and $(\beta^{\gamma,t})'(0)$ is the tangent vector to $\beta^{\gamma,t}$ at 0, which is a vector in the normal bundle $E \to M \cong \Delta(M)$ based at the midpoint between $\gamma(0)$ and $\gamma(t)$. We can think of $(\beta^{\gamma,t})'(0)$ as a pair of tangent vectors in M measuring the lack of intersection between $\gamma(0)$ and $\gamma(t)$.

Note that in the case when the initial segment $\gamma|_{[0,t]}$ is a geodesic the map $\tilde{\phi}$ is given by

$$\tilde{\phi}(\gamma, t) = (\gamma^{-1}|_{[t/2,0]} * \gamma|_{[0,t]} * \gamma^{-1}|_{[t,t/2]} * \gamma|_{[t/2,t]} * \gamma|_{[t,1]} * \gamma|_{[0,t/2]}, t', v)$$

where t' and v are as before. In this case $\tilde{\phi}(\gamma, t)$ is ("thinly") homotopic, inside $e_0^*(E)$, to an element of the form $(\gamma', 0, v)$.

The map $\tilde{\phi}$ is continuous by construction. We also have that $\tilde{\phi}$ is a lift of ϕ meaning that the following diagram commutes.

$$\begin{tikzcd} \tilde{N}_I \arrow[r, "\tilde{\phi}"] \arrow[d, "e"'] & e_0^*(E) \arrow[d, "\pi"] \\ N \arrow[r, "\phi"'] & E \end{tikzcd} \tag{1.3}$$

We show that $\tilde{\phi}$ is a homotopy equivalence. Its homotopy inverse, lets call it $\tilde{\rho}$, is given by

$$\tilde{\rho}(\gamma, t, v) = \left((\beta_1^v)^{-1} * \gamma|_{[0,t]} * \beta_2^v * (\beta_2^v)^{-1} * \gamma|_{[t,1]} * \beta_1^v, t'\right)$$

where $\exp_v(s) = (\beta_1^v(s), \beta_2^v(s))$ is the geodesic in $M \times M$ with tangent vector v based at $(\gamma(0), \gamma(t)) \in \Delta(M)$, and t' is given by the proportion of the length of $(\beta_1^v)^{-1} * \gamma|_{[0,t]} * \beta_2^v$ to the length of the entire loop $(\beta_1^v)^{-1} * \gamma|_{[0,t]} * \beta_2^v * (\beta_2^v)^{-1} * \gamma|_{[t,1]} * \beta_1^v$. It follows that $\tilde{\rho} \circ \tilde{\phi} \simeq \mathrm{id}_{\tilde{N}_I}$ and $\tilde{\phi} \circ \tilde{\rho} \simeq \mathrm{id}_{e_0^*(E)}$.
$\square$

Using the above map define a wrong way map associated to the inclusion $j : F \hookrightarrow LM \times I$. It will be useful to introduce the wrong way map as a chain map at the level of singular chains. Fix a cocycle representative $u \in C^d(Th(E))$ of the Thom class of the normal bundle $\eta : E \to M$. Denote by $\tilde{u} = \pi^*(u) \in C^d(Th(e_0^*(E)))$ the pullback cocycle along the map of Thom spaces induced by $\pi : e_0^*(E) \to E$, and the corresponding cohomology classes by $[u]$ and $[\tilde{u}]$. Define a chain map

$$j_{\#!} : C_*(LM \times I) \to C_*(LM \times I / (LM \times I - \tilde{N}_I)) \to C_*(Th(e_0^*(E))) \to C_{*-d}(F)$$

where the first map is induced by the collapse map, the second is induced by $\tilde{\phi} : \tilde{N}_I \to e_0^*(E)$, and the third is given by the cap product with the cocycle $\tilde{u}$. The wrong way map induced in homology will be denoted by $j_!$ as before. We can follow this composition with the chain map induced by the cutting map $w : F \to LM \times LM$ to obtain a chain map $h := w_\# \circ j_{\#!}$:

$C_*(LM \times I) \to C_{*-d}(LM \times LM)$.

Consider the prism operator associated to h, namely, the map $P^h : C_*(LM) \to C_{*+1-d}(LM \times LM)$ defined on a singular simplex $\sigma \in C_i(LM)$ by applying h to the prism $\sigma \times I \in C_{i+1}(LM \times I)$, where $\sigma \times I$ is the $i+1$ chain obtained by subdividing $\Delta^i \times I$ into $i+i$-simplices and taking the signed sum. Of course, P^h is not a chain map; it satisfies the chain homotopy equation

$$\partial P^h + P^h \partial = h_1 - h_0.$$

Here h_0 is the composition $h_0 : C_*(LM) \to C_*(LM \times I) \xrightarrow{h} C_{*-d}(LM \times LM)$ where the first map is induced by $LM \cong LM \times \{0\} \hookrightarrow LM \times I$, and h_1 is defined similarly using the inclusion at the other endpoint $LM \cong LM \times \{1\} \hookrightarrow LM \times I$. Moreover, from the construction h, we see that h_0 and h_1 factor through $C_{*-d}(M \times LM) \to C_{*-d}(LM \times LM)$ and $C_{*-d}(LM \times M) \to C_{*-d}(LM \times LM)$, respectively. It follows that both h_0 and h_1 are zero if we postcompose them with the quotient map $C_{*-d}(LM \times LM, M \times LM \cup LM \times M)$ (in fact, h_0 and h_1 are chain homotopic chain maps inducing in homology the coproduct $\vee_t$ defined in the earlier section). Thus P^h induces a chain map between relative chain complexes $C_*(LM, M) \to C_{*-d}(LM \times LM, M \times LM \cup LM \times M)$. We summarize this construction in the following definition.

Definition 5 The *loop coproduct of degree 1-d* is defined to be the map

$$\vee : H_*(LM, M) \to H_{*+1-d}(LM \times LM, M \times LM \cup LM \times M)$$

induced by the composition

$$C_*(LM, M) \xrightarrow{\times I} C_{*+1}(LM \times I, M \times I) \xrightarrow{j_{\#!}} C_{*+1-d}(F, M \times I) \xrightarrow{w_\#} C_{*+1-d}(LM \times LM, M \times LM \cup LM \times M).$$

Notice that the first map is not a chain map but its boundary terms will be mapped to 0 in the composition.

Equivalently, the coproduct $\vee$ is given by the following composition of maps between relative homology groups

$$\begin{aligned}
H_*(LM, M) &\cong H_{*+1}(LM \times I, (LM \times \partial I) \cup (M \times I)) \\
&\xrightarrow{\text{collapse}} H_{*+1}((LM \times I)/((LM \times I - \tilde{N}_I) \cup (M \times I) \cup (LM \times \partial I))) \\
&\xrightarrow{\tilde{\phi}} H_{*+1}(e_0^*(E), e_0^*(E)^c \cup e_0^*(E)|_{LM \times \partial I \cup M \times I}) \\
&\xrightarrow{\text{relative Thom iso.}} H_{*+1-d}(F, LM \times \partial I \cup M \times I) \xrightarrow{w_*} H_{*+1-d}(LM \times LM, M \times LM \cup LM \times M)
\end{aligned}$$

where $e_0^*(E)^c$ denotes the complement of the zero section in the pullback bundle $e_0^*(E)$ and $e_0^*(E)|_{LM \times \partial I \cup M \times I}$ the restriction of the bundle $e_0^*(E) \to F$ to $(LM \times \partial I) \cup (M \times I) \hookrightarrow F$.

In Sullivan's description of the coproduct at the level of transversal chains, working modulo constant loops was the natural step to take to obtain a well defined coproduct in homology. However, from the above description we see it is possible to lift $\vee$ to a map $\tilde{\vee} : H_*(LM) \to H_{*+1-d}(LM \times LM)$ defined in absolute homology without modding out by constant loops. In fact, consider $e : LM \times S^1 \to M \times M$ where $e(\gamma, t) = (\gamma(0), \gamma(t))$ and let $F_{S^1} := e^{-1}(\Delta(M))$ and $\tilde{N}_{S^1} := e^{-1}(N)$ where N is the tubular neighborhood of $\Delta(M)$. We have an evaluation at 0 map $e_{S^1} : F_{S^1} \to M$ through which we can pullback the normal bundle of M to obtain a bundle $e_{S^1}^* E \to F_{S^1}$. There exists a continuous map $\tilde{\phi}_{S^1} : \tilde{N}_{S^1} \to e_{S^1}^* E$ given by the same formula that defined $\tilde{\phi} : \tilde{N}_I \to e_0^* E$. We can use the same formula since $\tilde{N}_{S^1}$ is essentially obtained from $\tilde{N}_I$ by identifying points $(\gamma, 0)$ and $(\gamma, 1)$ (which are already in $F \subset \tilde{N}_I$) and both $\tilde{\psi}(\gamma, 0)$ and $\tilde{\psi}(\gamma, 1)$ (which are in the zero section of $e_0^*(E)$) go to the same point under the map $e_0^*(E) \to e_{S^1}^*(E)$.

However, we run into trouble if we try to define a homotopy inverse to $\tilde{\phi}_{S^1}$ in the same way we did for $\tilde{\phi}$. The problem arises since $\tilde{\rho}(\gamma, 0, v)$ and $\tilde{\rho}(\gamma, 1, v)$ differ by the location of the base point, so the map $\tilde{\rho} : e_0^* E \to \tilde{N}_I$ defined above does not descend to a map $e_{S^1}^* E \to \tilde{N}_{S^1}$. Another way to see this is as follows. Consider $x = (\gamma, t) \in \tilde{N}_I$ such that $\gamma|_{[0,t]}$ is a geodesic and let $x' = (\gamma^t, 1-t) \in \tilde{N}_I$ where $\gamma^t(s) = \gamma(s+t)$. Then $\tilde{\phi}(x)$ and $\tilde{\phi}(x')$ might not be homotopic in $e_0^* E$ (for instance if γ is not homotopic to constant loop) but indeed both go to the same ("thin") homotopy class when applying the map $e_0^* E \to e_{S^1}^* E$.

This phenomenon is a reflection of the fact that $\tilde{N}_{S^1}$ is not a tubular neighborhood of F_{S^1}. We do not discuss this further, since the continuous map $\tilde{\phi}_{S^1} : \tilde{N}_{S^1} \to e_{S^1}^* E$ allows us to define a lift of $\vee$ as follows.

Definition 6 Define the *lifted coproduct of degree 1-d* to be the following composition

$$\tilde{\vee} : H_*(LM) \to H_{*+1}(LM \times S^1) \to H_{*+1}(LM \times S^1/(LM \times S^1 - \tilde{N}_{S^1})) \to H_{*+1}(Th(e_{S^1}^* E))$$
$$\to H_{*+1-d}(F_{S^1}) \to H_{*+1-d}(LM \times LM)$$

where the sequence of maps in the composition is as in the earlier cases: first cross with the fundamental class of S^1, then the collapse map, followed by the map induced by $\tilde{\phi}_{S^1}$, then Thom isomorphism, and finally the cutting map.

Remark 9 Notice that $\vee$ and $\tilde{\vee}$ are not coproducts in the usual sense (maps of the form $V \to V \otimes V$). They are if we consider homology with field coefficients and post compose with the isomorphism from the homology group of a product to the tensor product of homology groups given by Kunneth theorem. However, there is no need to take field coefficients and use Kunneth theorem for the diagrams that express cocommutativity and coassociativity to make sense. We still have such diagrams for the coproducts having as a target the homology group of a product of spaces. In fact these diagrams commute as shown in the following theorem.

Theorem 6 $(H_*(LM,M),\vee)$ *is a graded coassociative cocommutative coalgebra of degree* $1-d$.

Proof. The cocommutativity of $\vee$ will follow from the commutativity of the diagram

$$\begin{tikzcd} H_*(LM\times I) \arrow[r, "j_!"] \arrow[dr, "j_!"'] & H_{*-d}(F) \arrow[r] \arrow[d, "\tau'_*"] & H_{*-d}(LM\times LM, M\times LM\cup LM\times M) \arrow[d, "\tau_*"] \\ & H_{*-d}(F) \arrow[r] & H_{*-d}(LM\times LM, M\times LM\cup LM\times M) \end{tikzcd} \tag{1.4}$$

where $\tau' : F\to F$ is $\tau'(\gamma,t) := (\gamma|_{[t,1]} * \gamma|_{[0,t]}, 1-t)$, the two middle horizontal maps are induced by the cutting map followed by the quotient map to relative homology, and τ is the switching map. The commutativity of the left triangle in the above diagram follows from a chain homotopy induced by the map $J : LM\times I\times I \to LM\times I$ defined by $J(\gamma,t,s) := J_s(\gamma,t) := (\gamma^{-s(1-t)}, (1-t)s + t(1-s))$ where $\gamma^{-s(1-t)}(r) := \gamma((r - s(1-t)) \bmod 1)$. Notice J_0 is the identity map on $LM\times I$, and as s goes to 1, J_s rotates the loop γ going backwards until it reaches the loop $\gamma^{-(1-t)}$ which has $\gamma(t)$ as base point and the new time parameter is now $(1-t)$, so $\gamma^{-(1-t)}(1-t) = \gamma(0)$. We have that $P^{j_{\#!}\circ J_\#} : C_*(LM\times I) \to C_{*+1-d}(F)$, the prism operator associated to $j_{\#!}\circ J_\# : C_*(LM\times I\times I) \to C_*(LM\times I) \to C_{*-d}(F)$, is a chain homotopy for the commutativity of the left triangle. The right square obviously commutes. Finally, the cocommutativity of $\vee$ follows by noting that J sends $M\times I\times I$ to $M\times I$. The proof of associativity uses several reparametrizations and we refer to [Rivera- Summer 2015 Arxiv] for the details. $\square$

A similar argument can be used to show that the lifted coproduct $\tilde{\vee} : H_*(LM) \to H_*(LM\times LM)$ is cocommutative and coassociative.

1.3.7 Gerstenhaber and BV-coalgebra structures

As in the case of the loop product, we construct a cobracket associated to the coproduct $\vee : H_*(LM,M) \to H_{*+1-d}(LM\times LM, M\times LM \cup LM\times M)$ arising from the chain homotopy defined above to show its cocommutativity.

By identifying the interval with the upper semi circle in S^1, the map $J : LM\times I\times I \to LM\times I$ induces a map $\tilde{J} : (LM\times I)\times_{\mathbb{Z}/2} S^1 \to LM\times I$ where $\mathbb{Z}/2$ acts on $LM\times I$ by sending (γ,t) to $J_1(\gamma,t)$ and on S^1 by sending a point to its antipode. We consider the following composition of chain maps

$$f_\# : C_*((LM\times I)\times_{\mathbb{Z}/2} S^1) \xrightarrow{\tilde{J}_\#} C_*(LM\times I) \xrightarrow{j_{\#!}} C_{*-d}(F) \xrightarrow{\text{cut}} C_{*-d}(LM\times LM) \xrightarrow{(1-\tau)_\#} C_{*-d}(LM\times LM)$$

We can precompose the above composition with the map $\kappa : C_*(LM) \to C_*((LM\times I)\times_{\mathbb{Z}/2} S^1)$ of degree +2 that crosses a chain in $C_*(LM)$ with I and with the fundamental class of S^1. The map κ is not a chain map, but its boundary terms are mapped to in $C_*(M\times LM\cup LM\times M)$, so after modding out by this complex in the range we obtain a chain map

$$C_*(LM) \xrightarrow{\kappa} C_*((LM\times I)\times_{\mathbb{Z}/2} S^1) \xrightarrow{f_\#} C_{*+2-d}(LM\times LM) \xrightarrow{q} C_{*+2-d}(LM\times LM, (M\times LM)\cup(LM\times M))$$

where q is the quotient map. Finally notice that the above composition passes to $C_*(LM,M)$ since chains of constant loops are sent to chains on $(M\times LM)\cup(LM\times M)$ under the composition. We summarize the construction in the following definition.

Definition 7 The *loop cobracket of degree 2-d* is defined to be the map

$$\theta : H_*(LM,M) \to H_{*+2-d}(LM\times M, (M\times LM)\cup(LM\times M))$$

induced by the composition

$$C_*(LM,M) \xrightarrow{\times I\times S^1} C_{*+2}((LM\times I)\times_{\mathbb{Z}/2} S^1, M\times I\times S^1) \xrightarrow{\tilde{J}_\#} C_{*+2}(LM\times I, M\times I) \xrightarrow{j_{\#!}} C_{*+2-d}(F, M\times I)$$
$$\xrightarrow{\text{cut}} C_{*+2-d}(LM\times LM, (M\times LM)\cup(LM\times M))$$

The two following theorems, which are analogue to the corresponding statements for the loop product, are proved in [Rivera- Summer 2015 Arxiv].

Theorem 7 *The shifted graded abelian group $H_*(LM,M)[1-d]$ is a Gerstenhaber coalgebra with coproduct $\vee$ and cobracket θ, namely*

(i) $(H_*(LM,M)[1-d],\vee)$ is a graded cocommutative coassociative coalgebra

(ii) θ is a Lie coalgebra of degree $+1$, which means that θ satisfies graded skew symmetry and the graded coJacobi identity.

(iii) θ is a graded coderivation of the coproduct $\vee$; i.e. the following diagram commutes

$$\begin{tikzcd}
H_*(LM\times LM\times LM, c(LM^3)) & H_*(LM\times LM, c(LM^2)) \arrow[l, "\vee\times 1"'] \\
H_*(LM\times LM, c(LM^2)) \arrow[u, "(1\times\tau)\circ(\theta\times 1)+1\times\theta"] & H_*(LM,M) \arrow[l, "\vee"] \arrow[u, "\theta"']
\end{tikzcd}$$

where $c(LM^3)=(M\times LM\times LM)\cup(LM\times M\times LM)\cup(LM\times LM\times M)$ and $c(LM^2)=(M\times LM)\cup(LM\times M)$.

Theorem 8 *The failure of the rotation operator $\Delta : H_*(LM,M)\to H_{*+1}(LM,M)$ from being a coderivation of the coproduct $\vee$ is precisely given by the loop cobracket $\theta : H_*(LM,M)\to H_{*+2-d}(LM\times LM,(M\times LM)\cup(LM\times M))$.*

Combining the two theorems above we obtain that the rotation operator $\Delta : H_*(LM,M)\to$

$H_{*+1}(LM,M)$ together with the loop cobracket $\theta : H_*(LM,M) \to H_{*+2-d}(LM \times LM, (M \times LM) \cup (LM \times M))$ define a BV-coalgebra structure on $H_*(LM,M)$. A similar result holds for the lifted coproduct $\tilde{\vee}$.

1.4 Back to S^1-equivariant homology

In section 2 we gave a geometric description of the involutive Lie bialgebra on smooth chains on the closed string space making transversality assumptions. That construction suggested that the bracket and the cobracket are induced by operations on the ordinary homology groups $H_*(LM)$ and $H_*(LM,M)$ respectively. In this section, we outline how the loop product $\bullet$ and the loop coproduct $\vee$ induce operations on the S^1-equivariant homology of LM which satisfy Lie versions of their properties. We also obtain the bracket and cobracket in the S^1-equivariant satisfy the Drinfeld compatibility condition.

The principal S^1-bundle

$$\begin{tikzcd} S^1 \arrow[r] & ES^1 \times LM \arrow[d] \\ & ES^1 \times_{S^1} LM \end{tikzcd}$$

gives rise to the Gysin exact sequence in homology

$$\ldots \to H_i(LM) \xrightarrow{E} H_i^{S^1}(LM) \xrightarrow{D} H_{i-2}^{S^1}(LM) \xrightarrow{M} H_{i-1}(LM) \to \ldots$$

where, following Chas and Sullivan, we have called E the map of degree 0 that "erases" the marked point of each loop in a chain on the free loop space, M the map of degree 1 that "marks" each string of a chain on the string space in all possible ways, and D is the cap product with the Euler class of the circle bundle above. The rotation operator Δ is then the composition $M \circ E$.

We now consider homology with $\mathbb{Q}$ coefficients. In this context have an induced loop product $\bullet : H_*(LM;\mathbb{Q}) \otimes H_*(LM;\mathbb{Q}) \to H_*(LM;\mathbb{Q})$ of degree $-d$ and loop coproduct $\vee : H_*(LM,M;\mathbb{Q}) \to H_*(LM,M;\mathbb{Q}) \otimes H_*(LM,M;\mathbb{Q})$ of degree $1-d$.

Definition 8 The *string bracket* $[,] : H_*^{S^1}(LM;\mathbb{Q}) \otimes H_*^{S^1}(LM;\mathbb{Q}) \to H_*^{S^1}(LM;\mathbb{Q})$ is the map of degree $2-d$ defined by the formula

$$[\alpha,\beta] = (-1)^{\deg(\alpha)} E(M(\alpha) \bullet M(\beta)).$$

Definition 9 The *string cobracket* $\nu : H_*^{S^1}(LM,M;\mathbb{Q}) \to H_*^{S^1}(LM,M;\mathbb{Q}) \otimes H^{S^1}(LM,M;\mathbb{Q})$ is the map of degree $2-d$ defined by the formula

$$\nu(\alpha) = (E \otimes E)(\vee(M(\alpha)))$$

Notice that the loop product $\bullet$ is not defined on $H_*(LM,M)$, since the image of $C_*(M)$ in $C_*(LM)$ under the map induced by the inclusion of constant loops $M \hookrightarrow LM$ is not an ideal under $\bullet$ (in fact $C_*(M)$ contains the fundamental class of M which is the unit for the loop product). However, in the S^1-equivariant case, we have the same phenomenon that was described in the geometric context: marking a chain on the string space in all possible ways (applying M) we obtain a chain of one degree higher in the free loop space, when we take the loop product of a lifted chain of constant strings together with any other chain in the free loop space the resulting chain will be degenerate. Assuming that we have been working with the normalized chain complex from the beginning this implies that the string bracket passes to $H_*^{S^1}(LM,M)$. We arrive again to the following theorem originally proven by Sullivan in the level of transversal chains.

Theorem 9 *The string bracket* $[,]$ *and cobracket* ν *induce an involutive Lie bialgebra structure of degree* $2-d$ *on* $H_*^{S^1}(LM, M;\mathbb{Q})$.

We refer the reader to [Rivera- Summer 2015 Arxiv] for a detailed proof of the above theorem (at the level of homology) using the Thom-Pontryagin formulation discussed in this section.

1.5 Homotopy invariance

The operations we have defined on the homology of LM rely on intersections of chains and homology classes. We would like to understand to what extent these operations are sensitive to the diffeomorphism and homeomorphism type of the underlying manifold. Cohen, Klein, and Sullivan showed that the loop product and the string bracket are oriented homotopy invariants of the underlying manifold. More precisely, they showed the following.

Theorem 10 *Let* M_1 *and* M_2 *be closed oriented manifolds of dimension* d. *Let* $f: M_1 \to M_2$ *be an orientation preserving homotopy equivalence. Then the induced homology equivalence of free loop spaces* $Lf: LM_1 \to LM_2$ *induces an isomorphism of algebras* $(Lf)_*: (H_*(LM_1), \bullet) \to (H_*(LM_2), \bullet)$, *where* $\bullet$ *is the loop product. Moreover, the induced map on* S^1*-equivariant homology is an isomorphism of Lie algebras with respect to the string bracket.*

The proof of the above theorem uses the Thom-Pontryagin formulation of the loop product and a theorem of Klein which essentially states that for a sufficiently large k the complement of the embedding $\Delta_k: M \to M \times M \times D^k$, $\Delta_k(x) = (x, x, 0)$ is a homotopy invariant when considered as a space over $M \times M$, where D^k is the k-dimensional disk. However, it is important to remark that the complement of the diagonal $\Delta : M \to M \times M$, also known as the configuration space of two points in M, is not a homotopy invariant in general. In fact, Longoni and Salvatore showed that the homotopy equivalent lens spaces $L(7,1)$ and $L(7,2)$ have non homotopy equivalent configuration spaces of two points. Somnath Basu used this result in his PhD thesis to show that *transversal string topology*, a geometric modification of the string topology construction, is not a homotopy invariant of the underlying manifold.

In this section we present an outline of an adaptation of the argument of Cohen, Klein, and Sullivan in order to show that the coproduct $\vee : H_*(LM, M) \to H_{*+1-d}(LM, (M \times LM) \cup (LM \times M))$ and the induced cobracket on $H_*^{S^1}(LM, M)$ are invariant of the homotopy type of M. We use the Thom-Pontryagin formulation of the coproduct to reduce the question of homotopy invariance to the *stable* homotopy invariance of the complement of $F = \{(\gamma, t) \in LM \times I : \gamma(0) = \gamma(t)\}$ inside $LM \times I$. This last fact is shown by applying certain pullback properties to Klein's theorem. Then we use the formulation of the coproduct given in section 3.6 which uses the wrong way map $H_*(LM \times I) \to H_{*-d}(F)$ to conclude the following.

Theorem 11 *Let M_1 and M_2 be closed oriented smooth manifolds of dimension d. Let $f : M_1 \to M_2$ be an orientation preserving homotopy equivalence. Then the induced homology equivalence of pairs of spaces $Lf : (LM_1, M_1) \to (LM_2, M_2)$ induces an isomorphism of coalgebras $(Lf)_* : (H_*(LM_1, M_1), \vee) \to (H_*(LM_2, M_2), \vee)$, where $\vee$ is the loop coproduct of degree $1-d$, namely the following diagram commutes*

$$\begin{array}{ccc} H_*(LM_1, M_1) & \xrightarrow{\vee} & H_{*+1-d}(LM_1 \times LM_1, (M_1 \times LM_1) \cup (LM_1 \cup M_1)) \\ \downarrow{\scriptstyle (Lf)_*} & & \downarrow{\scriptstyle (Lf)_*} \\ H_*(LM_2, M_2) & \xrightarrow[\vee]{} & H_{*+1-d}(LM_2 \times LM_2, (M_2 \times LM_2) \cup (LM_2 \cup M_2)). \end{array}$$

Moreover, the induced map on S^1-equivariant homology is an isomorphism of Lie coalgebras with respect to the string cobracket.

We give an outline of the proof. We start with four remarks about Klein's theorem.

1) Let D^k be a closed unit disk of dimension k. The normal bundle of the embedding $\Delta_k : M \to M \times M \times D^k$ is isomorphic to the stabilized normal bundle $E \oplus \epsilon^k$ where E is the total space of the normal bundle of $M \cong \Delta(M)$ in $M \times M$ and ϵ^k is the trivial k-dimensional bundle over M. We can then identify the disk bundle $D(E \oplus \epsilon^k)$ with a closed tubular neighborhood of $\Delta_k(M)$ and the sphere bundle $S(E \oplus \epsilon^k)$ with the boundary of this tubular neighborhood. Denote the closure of the complement of the embedding $D(E \oplus \epsilon^k) \to M \times M \times D^k$ by $F_k(M)$.

2) We consider the commutative diagram of embeddings

$$\begin{tikzcd} S(E \oplus \epsilon^k) \arrow[r] \arrow[d] & F_k(M) \arrow[d] & M \times M \times S^{k-1} \arrow[l] \\ D(E \oplus \epsilon^k) \arrow[r] & M \times M \times D^k & \end{tikzcd} \tag{1.5}$$

Let us call the above commutative diagram $\mathscr{M}(k)$. Note that the square in the above diagram is a pushout square.

3) Let M_1 and M_2 be two closed oriented manifolds of dimension d. Call E_i the normal bundle of $M_i \cong \Delta(M_i)$ in $M_i \times M_i$. For each M_i and each k we have a corresponding commutative diagram $\mathscr{M}_i(k)$ as the one defined above. Using this notation we state Klein's theorem.

Theorem 12 (Klein) *Let $f : M_1 \to M_2$ be an oriented homotopy equivalence of smooth oriented closed manifolds of dimension d. Then for a sufficiently large k there exists a commutative diagram $\mathscr{T}(k)$ of the form*

$$\begin{tikzcd} T_\emptyset \arrow[r] \arrow[d] & T_1 \arrow[d] & T_b \arrow[l] \\ T_0 \arrow[r] & T_{01} & \end{tikzcd} \tag{1.6}$$

together with morphisms of diagrams $\mathscr{M}_1(k) \xrightarrow{\phi_1} \mathscr{T}(k) \xleftarrow{\phi_2} \mathscr{M}_2(k)$ (meaning continuous maps between each of the corresponding vertices of the diagrams such that every square commutes) satisfying the following properties.

(i) Each space in the diagram $\mathscr{T}(k)$ has the homotopy type of a CW complex.

(ii) Each morphism given by ϕ_1 and ϕ_2 is a weak equivalence.

(iii) $T_{01} = \big((M_2 \times M_2)\bigcup_{f\times f}(M_1 \times M_1) \times I\big) \times D^k$ and the weak equivalences $M_1 \times M_1 \times D^k \xrightarrow{\phi_1} T_{01} \xleftarrow{\phi_2} M_2 \times M_2 \times D^k$ are given by the obvious inclusions as the two the ends of the mapping cylinder times the identity on D^k.

(iv) The weak equivalence $D(E_1 \oplus \epsilon^k) \xrightarrow{\phi_1} T_0 \xleftarrow{\phi_2} D(E_2 \oplus \epsilon^k)$ is homotopic to the weak equivalence given by the composition

$D(E_1 \oplus \epsilon^k) \xrightarrow{\textit{project}} M_1 \xrightarrow{f} M_2 \xrightarrow{\textit{zero section}} D(E_2 \oplus \epsilon^k)$.

4) The above theorem implies the existence of a homotopy commutative square of iterated suspensions

$$\begin{array}{ccc} \Sigma^k((M_1 \times M_1)_+) & \longrightarrow & \Sigma^k(Th(E_1)) \\ \downarrow & & \downarrow \\ \Sigma^k((M_2 \times M_2)_+) & \longrightarrow & \Sigma^k(Th(E_2)) \end{array} \tag{1.7}$$

such that the vertical maps are weak equivalences. Property (iii) of Klein's theorem implies that the left vertical map is homotopic to the k-fold suspension of the map $f \times f : M_1 \times M_1 \to M_2 \times M_2$. Property (iv) implies that the isomorphism in cohomology $H^*(\Sigma^k(Th(E_2))) \to H^*(\Sigma^k(Th(E_1)))$ induced by the right vertical map preserves the Thom classes.

Now we outline how these results can be applied to show the homotopy invariance of the loop coproduct.

5) We construct a commutative diagram $L^I\mathscr{T}(k)$ of the form

$$\begin{tikzcd} L^I T_\emptyset \arrow[r] \arrow[d] & L^I T_1 \arrow[d] & L^I T_b \arrow[l] \\ L^I T_0 \arrow[r] & L^I T_{01} & \end{tikzcd} \tag{1.8}$$

lifting the commutative diagram $\mathscr{T}(k)$ provided by Klein's theorem. We now explain how to obtain each space in the above diagram and what we mean by "lifting". First let $T_f := M_2 \bigcup_f M_1 \times I$ and $T_{f\times f} := M_2 \times M_2 \bigcup_{f\times f} M_1 \times M_1 \times I$ be the mapping cylinders of $f : M_1 \to M_2$ and $f \times f : M_1 \times M_1 \to M_2 \times M_2$, respectively. Note $T_{01} = T_{f\times f} \times D^k$. Consider the fibration $e \times 1 : LT_f \times I \times D^k \to T_f \times T_f \times D^k$ given by $(e \times 1)(\gamma, t, x) = (\gamma(0), \gamma(t), x)$. We pullback this fibration along the map $d \times 1 : T_{f\times f} \times D^k \to T_f \times T_f \times D^k$ where $d : T_{f\times f} \to T_f \times T_f$ is defined by $d(x, y, t) = (x, t, y, t)$ for $(x, y, t) \in M_1 \times M_1 \times I$ and by $d(x, y) = (x, y)$ for $(x, y) \in M_2 \times M_2$. We denote the resulting pullback fibration by

$$L^I T_{01} \xrightarrow{e} T_{01}.$$

This defines the space in the lower right of the diagram $L^I\mathscr{T}(k)$ together with a map to the lower right of $\mathscr{T}(k)$. To define the other four spaces and maps just pullback the above fibration along the maps in the diagram $\mathscr{T}(k)$. Hence, we obtain a commutative diagram $L^I\mathscr{T}(k)$ of the form (43) together with a map of commutative diagrams

$$L^I\mathscr{T}(k) \xrightarrow{e} \mathscr{T}(k)$$

in which each of the five maps given by e is a fibration by construction.

6) We define similarly a diagram $L^I\mathscr{M}_i(k)$ lifting $\mathscr{M}_i(k)$. The lower right corner of $L^I\mathscr{M}_i(k)$ is $LM_i \times I \times D^k$ and the other four spaces are obtain by pulling back the fibration $(e \times 1)$:

$LM_i \times I \times D^k \to M_i \times M_i \times D^k$ along the maps of the diagram $\mathscr{M}_i(k)$. We also have weak equivalences $L^I\mathscr{M}_1(k) \xrightarrow{L^I\phi_1} L^I\mathscr{T}(k) \xleftarrow{L^I\phi_2} L^I\mathscr{M}_2(k)$ induced by ϕ_1 and ϕ_2.

These constructions yield the following commutative diagram of commutative diagrams where the vertical arrows are fibrations and the horizontal arrows are weak equivalences. Note each arrow actually represents five maps of spaces:

$$\begin{tikzcd} L^I\mathscr{M}_1(k) \arrow[r, "L^I\phi_1"] \arrow[d, "e"'] & L^I\mathscr{T}(k) \arrow[d, "e"'] & L^I\mathscr{M}_2(k) \arrow[l, "L^I\phi_2"'] \arrow[d, "e"'] \\ \mathscr{M}_1(k) \arrow[r, "\phi_1"'] & \mathscr{T}(k) & \mathscr{M}_2(k) \arrow[l, "\phi_2"] \end{tikzcd} \tag{1.9}$$

7) The top row of the above diagram induces a homotopy commutative diagram of iterated suspensions

$$\begin{tikzcd} \Sigma^k((LM_1 \times I)_+) \arrow[r] \arrow[d] & \Sigma^k(Th(e_0^*E_1)) \arrow[d] \\ \Sigma^k((LM_2 \times I)_+) \arrow[r] & \Sigma^k(Th(e_0^*E_2)) \end{tikzcd} \tag{1.10}$$

where for $i = 1,2$, e_0 is the evaluation map from $F_i = \{(\gamma, t) \in LM_i \times I : \gamma(0) = \gamma(t)\}$ to M_i and $E_i \to M_i$ is the normal bundle of the diagonal embedding $\Delta : M_i \to M_i \times M_i$, so $e_0^*(E_i)$ is the corresponding pullback bundle. Each space in this homotopy commutative square is obtained by taking a quotient of the corresponding spaces in $L^I\mathscr{M}_1(k)$ and $L^I\mathscr{M}_2(k)$ and the maps in such square are induced by the weak equivalences $L^I\phi_1$ and $L^I\phi_2$. More precisely, $\Sigma^k((LM_i \times I)_+)$ is the mapping cone of the map $LM_i \times LM_i \times S^{k-1} \to LM_i \times LM_i \times D^k$ (which is a composition of two maps in $L^I\mathscr{M}_i(k)$) and $\Sigma^k(Th(e_0^*E_i))$ is the mapping cone of the map left vertical map in $L^I\mathscr{M}_i(k)$.

Notice that to get the horizontal maps in the above diagram we have used the map $(\tilde{N}_i)_I \to e_0^*(E_i)$ defined in Proposition 5 of section 3.6 where $(\tilde{N}_i)_I$ is the inverse image of a tubular neighborhood N_i of the diagonal in $M_i \times M_i$ along the map $e : LM_i \times I \to M_i \times M_i$. Moreover, the vertical maps of (47) are weak equivalences, the left vertical arrow is homotopic to the k-fold suspension of $Lf \times 1 : LM_1 \times I \to LM_2 \times I$, and the isomorprhism in cohomology $H^*(\Sigma^k(Th(e_0^* E_2))) \to H^*(\Sigma^k(Th(e_0^* E_1)))$ induced by the right vertical arrow preserves Thom classes. Also, when applying the Thom isomorphism to both sides of $H^*(\Sigma^k(Th(e_0^* E_2))) \to H^*(\Sigma^k(Th(e_0^* E_1)))$, the resulting isomorphism is $(Lf \times 1)^* : H^*(F_2) \to H^*(F_1)$.

8) From the diagram of step (7) we obtain a commutative diagram in homology by composing with the Thom isomorphism.

$$\begin{tikzcd}
H_*(LM_1 \times I) \arrow[r, "j_!"] \arrow[d, "(Lf\times 1)_*"'] & H_{*-d}(F_1) \arrow[d, "(Lf\times 1)_*"] \\
H_*(LM_2 \times I) \arrow[r, "j_!"] & H_{*-d}(F_2)
\end{tikzcd} \tag{1.11}$$

To obtain the diagram for the coproduct, as in Theorem 11, we post compose the above diagram with the natural map induced by the "cutting" map $F_i \to LM_i \times LM_i$ and then check that the above diagrams still commute when we work modulo constant loops. We refer to [Rivera-Summer 2015 Arxiv] for a detailed proof of each of the steps and claims above.

Chapter 2

Products and Coproducts on CoHochschild Complexes

Given a vector space W equipped with two linear maps $\wedge : W \otimes W \to W$ and $\vee : W \to W \otimes W$, there are three basic compatibilities between $\wedge$ and $\vee$ that one can consider:
(i) Hopf compatibility: $\vee$ is an algebra map, where the algebra structure on W is given by $\wedge$ and on $W \otimes W$ by $\wedge \otimes \wedge$
(ii) Frobenius compatibility: $\vee$ is a bimodule map, where the bimodule structure on W is given by $\wedge$, and on $W \otimes W$ by $v \cdot (w \otimes x) = (v \wedge w) \otimes x$ and $(w \otimes x) \cdot v = w \otimes (x \wedge v)$, and
(iii) infinitesimal compatibility: $\vee$ is a derivation of $\wedge$, i.e. $\vee(v \wedge w) = \vee(v) \cdot w + v \cdot \vee(w)$, where $\cdot$ is defined as in (ii).

In this chapter we give examples of each of the above compatibilities: we encounter (i) in the cobar construction of a DG cocommuativave coalgebra, and (ii) and (iii) on the coHochschild complex of a DG (open) cocommutative Frobenius coalgebra.

Other properties of these algebraic structures, in our specific context of coHochschild complexes, will be discussed: symmetry properties which hold up to homotopy, Gerstenhaber structures, and the compatibilities with Connes' boundary operator, i.e. BV-structures. This chapter arose from a careful study of T. Tradler and M. Zeinalian's action of Sullivan diagrams on the Hochschild cochain complex of a DG Frobenius algebra and making certain modifications at the chain level.

2.1 The CoHochschild Complex

Throughout this section $\mathbb{K}$ will be a field of characteristic 0, $(V = \overline{V} \oplus \mathbb{K}, d, \Delta : V \to V \otimes V)$ a differential graded (DG) coassociative coalgebra with a coaugmentation $\eta_V : \mathbb{K} \to V$, and $(N, d_N, {}_V\rho, \rho_V)$ a differential graded V-bicomodule, where ${}_V\rho : N \to V \otimes N$ and $\rho_V : N \to N \otimes V$ denote the left and right coactions, respectively. We use Einstein summation notation to write $\Delta(x) = \sum_j x^j \otimes x_j = x^j \otimes x_j$, ${}_V\rho(y) = \sum_k \rho(y)^k \otimes y_k = \rho(y)^k \otimes y_k$, and $\rho_V(y) = \sum_m y^m \otimes \rho(y)_m = y^m \otimes \rho(y)_m$. For computations it will sometimes be convenient to use Sweedler's notation:
$\Delta(x) = \sum_{(x)} x' \otimes x'' = x' \otimes x''$.

Definition 10 The *coHochschild chain complex of V with coefficients on N* is the graded vector space

$$C_*(N, V) := N \otimes \bigoplus_{n=0}^{\infty} (s^{-1}\overline{V})^{\otimes n}$$

with differential $D : C_*(N,V) \to C_{*-1}(N,V)$ defined by

$$D(y \otimes [x_1|...|x_n]) = d_N y \otimes [x_1|...x_n] + \sum_{i=1}^{n} (-1)^{\epsilon_i} y \otimes [x_1|...|d x_i|...|x_n] + \sum_{i=1}^{n} (-1)^{\epsilon_{i,j}} y \otimes [x_1|...|x_i^j|x_{ij}|...|x_n]$$
$$+(-1)^{\lambda_m} y^m \otimes [\rho(y)_m|x_1|...|x_n] + (-1)^{\lambda'_k} y_k \otimes [x_1|...|x_n|\rho(y)^k]$$

where $\epsilon_i = \deg(y) + \deg[x_1|...|x_{i-1}]$ and $\epsilon_{i,j} = \deg(y) + \deg[x_1|...|x_{i-1}|x_i^j]$ for $i \geq 1$, $\lambda_m = \deg(y^m)$, and $\lambda'_k = (\deg(\rho(y)^k) - 1)(\deg[x_1|...|x_n] + \deg(y_k))$. These signs are obtained by the Koszul sign convention and taking into account that we have written $[x_1|...|x_n]$ for $s^{-1}x_1 \otimes ... \otimes s^{-1}x_n$. It is easy to check that $D^2 = 0$.

Remark 10 In order to have a chain complex dual to the Hochschild complex of a differential graded (finite dimensional) algebra we must define the underlying vector space of the co-Hochschild complex to be the direct product $\prod_{n=0}^{\infty} N \otimes (s^{-1}\overline{V})^{\otimes n}$. For simplicity, in this chapter we do not use this convention, however, we do use it in the next chapter.

Given a degree 0 map $\phi : V \to N$ of DG V-bicomodules and a degree 0 linear map $\epsilon_N : N \to \mathbb{K}$ such that $\epsilon_N \circ \phi|_{\overline{V}} = 0$, we can define a degree $+1$ map $B_{\phi,\epsilon_N} : C_*(N,V) \to C_{*+1}(N,V)$ by

$$B_{\phi,\epsilon_N}(y \otimes [x_1|...|x_n]) = \sum_{i=1}^{n} (-1)^{\deg[x_i|...|x_n]\deg[x_1|...|x_{i-1}]} \epsilon_N(y)\phi(x_i) \otimes [x_{i+1}|...|x_n|x_1|...|x_{i-1}].$$

We write $B = B_{\phi,\epsilon_N}$ when ϕ and ϵ_N are clear from the context. Notice that $B^2 = 0$ and $DB + BD = 0$.

Example 1 We can regard V as a DG V-bicomodule by defining both the left and right coactions via the coproduct Δ. Then $(C_*(V,V), D)$ is called the *coHochschild chain complex* of the DG coalgebra V. If V has a counit $\epsilon_V : V \to \mathbb{K}$ then $B = B_{\mathrm{id}_V,\epsilon_V}$ is known as *Connes' coboundary operator*.

Example 2 Consider $\mathbb{K}$ as a DG V-bicomodule with the trivial V-coactions and trivial differential. Then the associated coHochschild chain complex, which has $\Omega V := C_*(\mathbb{K}, V) = \bigoplus_{n=0}^{\infty} s^{-1}\overline{V}^{\otimes n}$ as underlying vector space, is the *cobar construction* of the DG coassociative coalgebra V. In this case we denote the differential by D_Ω.

Example 3 We can also regard V as a DG V-bicomodule by defining the left coaction via the coaugmentation, i.e. $\eta_V \otimes 1_V : V \cong \mathbb{K} \otimes V \to V \otimes V$, and the right coaction via the coproduct Δ; call such V-bicomodule ${}_\eta V$. Then, the associated coHochschild complex $(C_*({}_\eta V, V), D)$ is acyclic.

Remark 11 We can write the differential $D : C_*(V, V) \to C_{*-1}(V, V)$ of the coHochschild complex as $D = d \otimes 1_{\Omega V} + 1_V \otimes D_{\Omega V} + \delta$ where $D_{\Omega V}$ is the differential of the cobar construction on (V, d, Δ) and $\delta(x_0 \otimes [x_1|...|x_n]) := x_0' \otimes [x_0''|x_1|...|x_n] \pm x_0'' \otimes [x_1|....|x_n|x_0']$.

We recall some algebraic structures on coHochschild complexes of DG *cocommutative* coassociative coalgebras all of which have topological interpretations, which will be discussed later on.

Proposition 6 *If (V, d, Δ) is a DG cocommutative coassociative coalgebra, then Δ induces a DG left V-comodule structure on the coHochschild complex $(C_*(V, V), D)$.*

Proof. We check $\Delta \otimes 1_{\Omega V} : (V \otimes \Omega V, D) \to (V \otimes V \otimes \Omega V, d \otimes 1_V \otimes 1_{\Omega V} + 1_V \otimes D)$ is a chain map:

$$(\Delta \otimes 1_{\Omega V}) \circ D = \tag{2.1}$$

$$(\Delta \otimes 1_{\Omega V}) \circ (d \otimes 1_{\Omega V} + 1_V \otimes D_{\Omega V} + \delta) = \tag{2.2}$$

$$(\Delta \circ d) \otimes 1_{\Omega V} + \Delta \otimes D_{\Omega V} + (\Delta \otimes 1_{\Omega V}) \circ \delta \tag{2.3}$$

and

$$(d \otimes 1_V \otimes 1_{\Omega V} + 1_V \otimes D) \circ (\Delta \otimes 1_{\Omega V}) = \tag{2.4}$$

$$\Big((d \otimes 1_V + 1_V \otimes d) \circ \Delta\Big) \otimes 1_{\Omega V} + \Delta \otimes D_{\Omega V} + (1_V \otimes \delta) \circ (\Delta \otimes 1_{\Omega V}). \tag{2.5}$$

Since d is a coderivation with respect to Δ, to conclude (3)=(5) we just have to show that $(\Delta \otimes 1_{\Omega V}) \circ \delta = (1_V \otimes \delta) \circ (\Delta \otimes 1_{\Omega V})$, which we check directly:

$$(\Delta \otimes 1_{\Omega V}) \circ \delta(x_0 \otimes [x_1|...|x_n]) = \tag{2.6}$$

$$(\Delta \otimes 1_{\Omega V})(x_0' \otimes [x_0''|x_1|...|x_n] \pm x_0'' \otimes [x_1|...|x_n|x_0']) = \tag{2.7}$$

$$(\Delta \otimes 1_{\Omega V})(x_0' \otimes [x_0''|x_1|...|x_n] \pm x_0' \otimes [x_1|...|x_n|x_0'']) = \tag{2.8}$$

$$(x_0')' \otimes (x_0')'' \otimes [x_0''|x_1|...|x_n] \pm (x_0')' \otimes (x_0')'' \otimes [x_1|...|x_n|x_0''] = \tag{2.9}$$

$$x_0' \otimes (x_0'')' \otimes [(x_0'')''|x_1|...|x_n] \pm x_0' \otimes (x_0'')' \otimes [x_1|...|x_n|(x_0'')''] = \tag{2.10}$$

$$x_0' \otimes (x_0'')' \otimes [(x_0'')''|x_1|...|x_n] \pm x_0' \otimes (x_0'')'' \otimes [x_1|...|x_n|(x_0'')'] = \tag{2.11}$$

$$(1_V \otimes \delta) \circ (\Delta \otimes 1_{\Omega V})(x_0 \otimes [x_1|....|x_n]) \tag{2.12}$$

where we have used cocommutativity from (7) to (8), coassociativity from (9) to (10), and cocommutativity again from (10) to (11). The comodule compatibility is straightforward.□

The cobar construction of V has as underlying vector space the free associative algebra $\Omega V = \bigoplus_{n=0}^{\infty} s^{-1}\overline{V}^{\otimes n}$. Denote by $\mu : \Omega V \otimes \Omega V \to \Omega V$ the free product on ΩV, called the *concatenation product*. Define $\Delta_{sh} : \Omega V \to \Omega V \otimes \Omega V$ to be the unique extension of the map $s^{-1}\overline{V} \to \Omega V \otimes \Omega V$ given by $[x] \mapsto [x] \otimes 1 + 1 \otimes [x]$ to an algebra map defined on all of ΩV. By direct inspection we obtain the formula

$$\Delta_{sh}([x_1|...|x_n]) = \sum_{\sigma\in Sh_n} \pm[x_{\sigma(1)}|...|x_{\sigma(p)}]\bigotimes[x_{\sigma(p+1)}|...|x_{\sigma(n)}] \tag{2.13}$$

where Sh_n is the set of all $(p, n-p)$-shuffles for all $1 \leq p \leq n$ and the sign is given by the permutation of graded elements. The coproduct Δ_{sh}, called the *unshuffle coproduct,* is clearly coassociative and cocommutative.

On the coHochschild complex we also have a coproduct $\Delta\otimes\Delta_{sh} : C_*(V,V) \to C_*(V,V)\bigotimes C_*(V,V)$ defined by

$$\Delta\otimes\Delta_{sh}(x_0\otimes[x_1|...|x_n]) := \sum_{\sigma\in Sh_n} \pm x_0'\otimes[x_{\sigma(1)}|...|x_{\sigma(p)}]\bigotimes x_0''\otimes[x_{\sigma(p+1)}|...|x_{\sigma(n)}]$$

where we have written $\Delta(x) = x'\otimes x''$. If Δ is cocommutative then $\Delta\otimes\Delta_{sh}$ is obviously cocommutative as well.

In the next two propositions we discuss the compatibilities between the maps Δ_{sh}, μ, and $D_{\Omega V}$ and between $\Delta\otimes\Delta_{sh}$ and D.

Proposition 7 *If* (V,d,Δ) *is a DG coassociative coalgebra then* $(\Omega V, D_{\Omega V}, \mu)$ *is a DG associative algebra. Moreover, if* Δ *is cocommutative then* $(\Omega V, D_{\Omega V}, \mu, \Delta_{sh})$ *is a DG cocommutative Hopf algebra.*

Proof. Write $D_{\Omega V} = b_0 + b_1$ where $b_0[x_1|...|x_n] := \sum_{i=1}^n \pm[x_1|...|dx_i|...|x_n]$ and $b_1[x_1|...|x_n] := \sum_{i=1}^n \pm[x_1|...|x_i'|x_i''|...x_n]$. It follows directly from such formula that $D_{\Omega V}$ is a derivation of the concatenation product μ, so $(\Omega V, D_{\Omega V}, \mu)$ is a DG associative algebra. Now assume Δ is cocommutative and lets first check Δ_{sh} commutes with differentials for monomials of length

1:

$$\Delta_{sh}(D_{\Omega V}[x]) = \Delta_{sh}(b_0[x] + b_1[x]) = \tag{2.14}$$

$$\Delta_{sh}(d[x]) + \Delta_{sh}([x'|x'']) = \tag{2.15}$$

$$d[x]\otimes 1 + 1\otimes d[x] + [x'|x'']\otimes 1 \pm [x']\otimes[x''] \pm [x'']\otimes[x'] + 1\otimes[x'|x''''] = \tag{2.16}$$

$$d[x]\otimes 1 + 1\otimes d[x] + [x'|x'']\otimes 1 + 1\otimes[x'|x''''] = \tag{2.17}$$

$$(b_0\otimes 1_{\Omega V} + 1_{\Omega V}\otimes b_0 + b_1\otimes 1_{\Omega V} + 1_{\Omega V}\otimes b_1)([x]\otimes 1 + 1\otimes[x]) = \tag{2.18}$$

$$(D_{\Omega V}\otimes 1_{\Omega V} + 1_{\Omega V}\otimes D_{\Omega V})\Delta_{sh}[x]. \tag{2.19}$$

where we used cocommutativity to get from (16) to (17). Using the fact that Δ_{sh} is an algebra map and that $D_{\Omega V}$ is a derivation of the concatenation product it follows by induction on the length of monomials that $D_{\Omega V}$ is a coderivation of Δ_{sh}. The antipode for the Hopf algebra structure is given by $S[x_1|...|x_n] = \pm[x_n|...|x_1]$ with sign given by permutation of graded elements, as usual. □

Proposition 8 *If* (V, d, Δ) *is a DG cocommutative coassociative coalgebra then* $(C_*(V, V), D, \Delta\otimes\Delta_{sh})$ *is a DG cocommutative coassociative coalgebra as well.*

Proof. We must show that $D = d\otimes 1_{\Omega V} + 1_V\otimes D_{\Omega V} + \delta$ is a coderivation of $\Delta\otimes\Delta_{sh}$. Since d is a coderivation of Δ and $D_{\Omega V}$ is a coderivation of Δ_{sh} (Proposition 2) we just have to check that

$(\delta\otimes 1_{V\otimes\Omega V}+1_{V\otimes\Omega V}\otimes\delta)\circ(\Delta\otimes\Delta_{sh})=(\Delta\otimes\Delta_{sh})\circ\delta$. On one hand we have

$$(\delta\otimes 1_{V\otimes\Omega V}+1_{V\otimes\Omega V}\otimes\delta)\circ(\Delta\otimes\Delta_{sh})(x_0\otimes[x_1|...|x_n])= \tag{2.20}$$

$$\sum_{\sigma\in Sh_n}\Big(\pm(x_0')'\otimes[(x_0')''|x_{\sigma(1)}|...|x_{\sigma(p)}]\pm(x_0')''\otimes[x_{\sigma(1)}|...|x_{\sigma(p)}|(x_0')']\Big)\bigotimes x_0''\otimes[x_{\sigma(p+1)}|...|x_{\sigma(n)}] \tag{2.21}$$

$$+\sum_{\sigma\in Sh_n}\pm x_0'\otimes[x_{\sigma(1)}|...|x_{\sigma(p)}]\bigotimes\Big((x_0'')'\otimes[(x_0'')''|x_{\sigma(p+1)}|...|x_{\sigma(n)}]\pm(x_0'')''\otimes[x_{\sigma(p+1)}|...|x_{\sigma(n)}|(x_0'')']\Big). \tag{2.22}$$

On the other hand,

$$(\Delta\otimes\Delta_{sh})\delta(x_0\otimes[x_1|...|x_n])= \tag{2.23}$$

$$(\Delta\otimes\Delta_{sh})(\pm x_0'\otimes[x_0''|x_1|...|x_n]\pm x_0''\otimes[x_1|...|x_n|x_0'])= \tag{2.24}$$

$$\sum_{\alpha\in Sh_{n+1}}\pm(x_0')'\otimes[x''_{\alpha(0)}|x_{\alpha(1)}|...|x_{\alpha(p)}]\bigotimes(x_0')''\otimes[x_{\alpha(p+1)}|...|x_{\alpha(n)}] \tag{2.25}$$

$$+\sum_{\beta\in Sh_{n+1}}\pm(x_0'')'\otimes[x_{\beta(1)}|...|x_{\beta(p)}]\bigotimes(x_0'')''\otimes[x_{\beta(p+1)}|...|x_{\beta(n)}|x'_{\beta(0)}] \tag{2.26}$$

where we have abused notation: each shuffle α and β moves the prime superscripts with them, in other words, in all the above sums, a tensor has 0 as a subscript if and only if it also has primes as superscripts. Note that in all the sums the tensors with primes must always be next to a bracket [or], by the definition of a shuffle. Lets explain why (21)+(22)= (25)+(26). Consider the sum (26), by cocommutativity and coassociativity we have that (26)=

$$\sum_{\beta\in Sh_{n+1}}\pm x_0'\otimes[x_{\beta(1)}|...|x_{\beta(p)}]\bigotimes(x_0'')''\otimes[x_{\beta(p+1)}|...|x_{\beta(n)}|(x''_{\beta(0)})']. \tag{2.27}$$

Note that (27) contains the sum

$$\sum_{\sigma \in Sh_n} \pm x_0' \otimes [x_{\sigma(1)}|...|x_{\sigma(p)}] \bigotimes (x_0'')'' \otimes [x_{\sigma(p+1)}|...|x_{\sigma(n)}|(x_0'')'] \tag{2.28}$$

which is precisely one of the two sums of (22). The remaining terms in (27), i.e. (27)-(28), is one of the sums of (21). A similar argument applies to cancel (25) with the remaining terms in (21) and (22). □

2.2 The CoHocschild Complex of a DG Frobenius Coalgebra

We analyze certain structures on the coHochschild complex of a differential graded coalgebra with an additional product operation satisfying a module compatibility with the coproduct.

Definition 11 A *differential graded (DG) open Frobenius coalgebra of degree* $-d$ is a quadruple $(V, d, \Delta, \cdot)$ such that

(i) (V, d, Δ) is a differential graded coassociative coalgebra, where $\Delta : V \to V \otimes V$ is of degree 0,

(ii) $(V, d, \cdot)$ is a differential graded associative algebra, where $\cdot : V \otimes V \to V$ is of degree $-d$, and

(iii) $\Delta : V \to V \otimes V$ is a map of V-bimodules, where the bimodule structure on V and $V \otimes V$ are the usual ones induced by the product $\cdot$.

If we write $\Delta(x) = \sum_{(x)} x' \otimes x''$ and $x \cdot y = xy$ then the compatibility of (iii) may be expressed as

$$\Delta(xy) = \sum_{(xy)} (xy)' \otimes (xy)'' = \sum_{(y)} xy' \otimes y'' = \sum_{(x)} (-1)^{d|x|} x' \otimes x'' y. \tag{2.29}$$

Remark 12 We call such a structure an DG open Frobenius *coalgebra* since we will consider it as a DG coalgebra with the extra structure of a product of degree $-d$. An example of a DG open Frobenius *coalgebra* is the rational homology of a closed manifold with the diagonal coproduct, intersection product, and trivial differential. Throughout the rest of the chapter, we shall think of $(V, d, \Delta, \cdot)$ as a chain model for a d-dimensional smooth manifold M, $d : V \to V$ as the boundary operator on chains, Δ as a model for the diagonal coproduct at the chain level, and $\cdot$ as a model for the intersection product on chains, and we think of the coHochschild complex of V as a chain model for the space of free loops on M. Under some hypotheses, this association can be made mathematically rigorous, as we see in the next chapter.

Throughout this section $(V, d, \Delta, \cdot)$ will denote a DG open Frobenius coalgebra of degree $-d$ with counit $\epsilon : V \to \mathbb{K}$ and moreover we assume $\cdot : V \otimes V \to V$ is commutative and $\Delta : V \to V \otimes V$ is cocommutative.

2.2.1 Product

Definition 12 Given $a = x_0 \otimes [x_1|...|x_m], b = y_0 \otimes [y_1|...|y_n] \in C_*(V, V)$ define a product $\bullet : C_*(V, V) \bigotimes C_*(V, V) \to C_*(V, V)$ of degree $-d$ by

$$a \bullet b = (-1)^{\deg(y_0)\deg([x_1|...|x_n])}(x_0 \cdot y_0) \otimes [x_1|...|x_m|y_1|...|y_n]. \tag{2.30}$$

Proposition 9 $(C_*(V, V), D, \bullet)$ *is a differential graded associative algebra of degree* $-d$.

Proof. The associativity $\bullet$ is straightforward. We check that D is a derivation of $\bullet$. Since d is a derivation of $\cdot$ and $D_{\Omega V}$ is a derivation of the concatenation product μ, we only have to check that δ is a derivation of $\bullet$. We have

$$\delta\Big((x_0 \otimes x) \bullet (y_0 \otimes y)\Big) = \tag{2.31}$$

$$\delta\Big((x_0 \cdot y_0) \otimes [x|y]\Big) = \tag{2.32}$$

$$\pm (x_0 \cdot y_0)' \otimes [(x_0 \cdot y_0)''|x|y] \tag{2.33}$$

$$\pm (x_0 \cdot y_0)'' \otimes [x|y|(x_0 \cdot y_0)'] \tag{2.34}$$

and

$$\delta(x_0 \otimes x) \bullet (y_0 \otimes y) \pm (x_0 \otimes x) \bullet \delta(y_0 \otimes y) = \tag{2.35}$$

$$(x_0' \cdot y_0) \otimes [x_0''|x|y] \tag{2.36}$$

$$\pm (x_0'' \cdot y_0) \otimes [x|x_0'|y] \pm (x_0 \cdot y_0') \otimes [x|y_0''|y] \tag{2.37}$$

$$\pm (x_0 \cdot y_0'') \otimes [x|y|y_0']. \tag{2.38}$$

By the Frobenius compatibility and the commutativity of $\cdot$ we have that (33)=(36) and (34)=(38). Finally note that, using Frobenius compatibility again and cocommutativity of Δ, (37)=0. $\square$

Moreover, the product $\bullet$ is commutative up to a chain homotopy and such homotopy resembles Gerstenhaber's pre-Lie algebra, as we see below.

Definition 13 Let $a = x_0 \otimes [x_1|...x_m]$, $b = y_0 \otimes [y_1|...|y_n], \in C_*(V,V)$. Define $*_i : C_*(V,V) \otimes C_*(V,V) \to C_*(V,V)$ by

$$a *_i b = (-1)^{\deg(y_0)+(\deg(y)-1)(\deg([x_{i+1}|...|x_n]))} \epsilon(x_i \cdot y_0) x_0 \otimes [x_1|...|x_{i-1}|y_1|...|y_n|x_{i+1}|...|x_m] \tag{2.39}$$

and $*: C_*(V,V)\otimes C_*(V,V)\to C_*(V,V)$ by

$$a*b=\sum_{i=1}^{m} a*_i b \tag{2.40}$$

Note that $*$ is a map of degree $1-d$.

Proposition 10 *The product* $*: C_*(V,V)\bigotimes C_*(V,V)\to C_*(V,V)$ *is a chain homotopy for the commutativity of* $\bullet$*, i.e. the following equation holds*

$$D(a*b)-(Da*b+(-1)^{deg(a)+1}a*Db)=a\bullet b-(-1)^{deg(a)deg(b)}b\bullet a. \tag{2.41}$$

Proof. The proof is an adaptation of an argument of Gerstenhaber to our context of open Frobenius coalgebras. Write $D=D_0+D_1$, and $D_1=b_\Omega+\delta$ where D_0 is the tensor differential on $V\otimes\Omega V$ with respect to d, b_Ω is the term of the cobar differential defined by applying Δ consecutively, and δ is as in Remark 1. From the definition of $*$ it is clear that D_0 is a derivation of $*$. So we prove the above equation for $D_1=b_\Omega+\delta$. We write $a=x_0\otimes[x_1|...|x_m]=x_0\otimes x$ and $b=y_0\otimes[y_1|...|y_n]=y_0\otimes y$, so using this notation $a*b=\sum\pm\epsilon(x_i\cdot y_0)x_0\otimes[x_1|...|x_{-1}|y|x_{i+1}|...|x_m]$. The structure of the computation is as follows: first we compute $D_1(a)*b$ where we identify $a\bullet b$ and $\pm b\bullet a$, then we compute $a*D_1(b)$ and cancel terms among the two computations.

Finally we show the remaining terms are $\delta(a*b)+b_\Omega(a*b)=D_1(a*b)$.

$$D_1(a)*b\Big(x_0'\otimes[x_0''|x]\pm x_0''\otimes[x|x_0']+\sum_{j=1}^{m}\pm x_0\otimes[x_1|...|x_j'|x_j''|...|x_m]\Big)*(y_0\otimes y)= \tag{2.42}$$

$$\pm\epsilon(x_0''\cdot y_0)x_0'\otimes[y|x]+\sum_{i=1}^{m}\pm\epsilon(x_i\cdot y_0)x_0'\otimes[x_0''|x_1|...|x_{i-1}|y|x_{i+1}|...|x_m] \tag{2.43}$$

$$+\sum_{i=1}^{m}\pm\epsilon(x_i\cdot y_0)x_0''\otimes[x_1|...|x_{i-1}|y|x_{i+1}|...|x_m|x_0']\pm\epsilon(x_0'\cdot y_0)x_0''\otimes[x|y] \tag{2.44}$$

$$+\sum_{i<j}\pm\epsilon(x_i\cdot y_0)x_0\otimes[x_1|...|x_{i-1}|y|x_{i+1}|...|x_j'|x_j''|...|x_m] \tag{2.45}$$

$$+\sum_{j<i}\pm\epsilon(x_i\cdot y_0)x_0\otimes[x_1|...|x_j'|x_j''|...|x_{i-1}|y|x_{i+1}|...|x_m] \tag{2.46}$$

$$+\sum_{i}\pm\epsilon(x_i'\cdot y_0)x_0\otimes[x_1|...|x_{i-1}|y|x_i''|...|x_m] \tag{2.47}$$

$$+\sum_{i}\pm\epsilon(x_i''\cdot y_0)x_0\otimes[x_1|...|x_i'|y|x_{i+1}|...|x_m]. \tag{2.48}$$

In the above computation the first two lines is are $\delta(a)*b$ and the last four are $b_\Omega(a)*b$. Note that the terms without summation symbols in front (which appear in the first and second line) are precisely $a\bullet b$ and $\pm b\bullet a$ since ϵ is the counit of the coalgebra (V,Δ) meaning $\sum\epsilon(x_0')x_0''=x=\sum x_0'\epsilon(x_0'')$. Also note that the first two summations (which appear (43) and (44)) are precisely $\delta(a*b)$. We now compute $a*D_1(b)$:

$$a*D_1(b)=a*(\delta(b)+b_\Omega(b))= \tag{2.49}$$

$$\sum_{i=1}^{m}\pm\epsilon(x_i\cdot y_0')x_0\otimes[x_1|...|x_{i-1}|y_0''|y|x_{i+!}|...|x_m]+\sum_{i=1}^{m}\pm\epsilon(x_i\cdot y_0'')x_0\otimes[x_1|...|x_{i-1}|y|y_0'|x_{i+1}|...|x_m] \tag{2.50}$$

$$\pm\sum_{(i,j)}\epsilon(x_i\cdot y_0)x_0\otimes[x_1|...|x_{i-1}|y_1|...|y_j'|y_j''|...|y_n|x_{i+1}|...|x_m]. \tag{2.51}$$

Observe that by Frobenius compatibility (50) is exactly (47)+(48). Finally, note that (51) + (45) + (46) = $b_\Omega(a*b)$. Putting all together we have the desired chain homotopy equation. □

The above proposition implies the commutativity of $\bullet$ at the level of homology, as we summarize in the following

Theorem 13 *Let $(V,d,\Delta,\cdot)$ be a cocommutative commutative DG open Frobenius coalgebra. Then the product on coHochschild homology $\bullet: coHH_*(V,V)\otimes coHH_*(V,V)\to coHH_*(V,V)$ is commutative, associative, and of degree $-d$.*

We call the product $\bullet: coHH_*(V,V)\otimes coHH_*(V,V)\to coHH_*(V,V)$ the *algebraic loop product* on the coHochshild homology of a cocommutative commutative DG open Frobenius coalgebra of degree $-d$.

2.2.2 Coproducts

Definition 14 Given $x_0\otimes x\in C_*(V,V)$ define a coproduct $\vee_0: C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ of degree $-d$ by

$$\vee_0(x_0\otimes x)=\pm(x_0'\cdot x_0'')'\otimes 1\bigotimes(x_0'\cdot x_0'')''\otimes x \tag{2.52}$$

Similarly, define $\vee_1: C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ by

$$\vee_1(x_0\otimes x)=\pm(x_0''\cdot x_0')'\otimes x\bigotimes(x_0'\cdot x_0'')''\otimes 1 \tag{2.53}$$

Proposition 11 $(C_*(V,V),D,\vee_i)$ *is DG coassociative coalgebra of degree $-d$, for both $i=0,1$.*

Proof. Each $\vee_i$ is clearly coassociative. Lets check $\vee_0$ commutes with the differential, the proof for $\vee_1$ is similar. It is straight forward to verify that $\vee_0$ commutes with D_0 and with $1_V\otimes D_{\Omega V}$ so we check it commutes with δ. Note that computation only uses the Frobenius compatibility and coassocitivity. We have:

$$\vee_0\delta(x_0\otimes x)=\pm\vee_0(x_0'\otimes[x_0''|x]\pm x_0''\otimes[x|x_0']) \tag{2.54}$$

$$=\pm\Big((x_0')'\cdot(x_0')''\Big)'\otimes 1\bigotimes\Big((x_0')'\cdot(x_0')''\Big)''\otimes[x_0''|x] \tag{2.55}$$

$$\pm\Big((x_0'')'\cdot(x_0'')''\Big)'\otimes 1\bigotimes\Big((x_0'')'\cdot(x_0'')''\Big)''\otimes[x|x_0']. \tag{2.56}$$

and, using the fact that for any $z_0\in V$ we have $\delta(z_0\otimes 1)=0$ by cocommutativity of Δ, we obtain:

$$(\delta\otimes 1+1\otimes\delta)\vee_0(x_0\otimes x)=(\delta\otimes 1+1\otimes\delta)\Big(\pm(x_0'\cdot x_0'')'\otimes 1\bigotimes(x_0'\cdot x_0'')''\otimes x\Big)= \tag{2.57}$$

$$(x_0'\cdot x_0'')'\otimes 1\bigotimes\Big(\pm((x_0'\cdot x_0'')'')'\otimes[((x_0'\cdot x_0'')'')''|x]\pm((x_0'\cdot x_0'')'')''\otimes[x|((x_0'\cdot x_0'')'')']\Big)= \tag{2.58}$$

$$\pm(x_0'\cdot x_0'')'\otimes 1\bigotimes((x_0'\cdot x_0'')'')'\otimes[((x_0'\cdot x_0'')'')''|x] \tag{2.59}$$

$$\pm(x_0'\cdot x_0'')'\otimes 1\bigotimes((x_0'\cdot x_0'')'')''\otimes[x|((x_0'\cdot x_0'')'')'] \tag{2.60}$$

To see that (second line above)= (third line below) we use the following equality of tensors obtained using Frobenius compatibility in the first equality and coassociativity in the last two:

$$(x_0'\cdot x_0'')'\otimes((x_0'\cdot x_0'')'')'\otimes((x_0'\cdot x_0'')'')''=$$

$$\pm x_0'\cdot(x_0'')'\otimes((x_0'')'')'\otimes((x_0'')'')''=$$

$$\pm(x_0')'\cdot(x_0')''\otimes(x_0'')'\otimes(x_0'')''=$$

$$\pm((x_0')'\cdot(x_0')'')'\otimes((x_0')'\cdot(x_0')'')''\otimes x_0''.$$

With a similar identity we show (third line above)=(fourth line below). □

Proposition 12 *The coproduct* $\vee_0: C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ *is a map of right DG* $C_*(V,V)$*-bimodules, where the right bimodule structures are given by the product* $\bullet$ *in the usual way.*

Proof. We check the bimodule compatibility:

$$\vee_0\Big((x_0 \otimes x) \bullet (y_0 \otimes y)\Big) = \vee_0\Big(\pm(x_0 \cdot y_0 \otimes [x|y]\Big) = \Big((x_0 \cdot y_0)' \cdot (x_0 \cdot y_0)''\Big)' \otimes 1 \bigotimes \Big((x_0 \cdot y_0)' \cdot (x_0 \cdot y_0)''\Big)'' \otimes [x|y] = \tag{2.61}$$

$$\pm(x_0' \cdot x_0'' \cdot y_0)' \otimes 1 \bigotimes (x_0' \cdot x_0'' \cdot y_0)'' \otimes [x\, y] = \tag{2.62}$$

$$\pm(x_0' \cdot x_0'')' \otimes 1 \bigotimes (x_0' \cdot x_0'')'' \cdot y_0 \otimes [x|y] = \tag{2.63}$$

$$\pm \vee_0 (x_0 \otimes x) \bullet (y_0 \otimes y), \tag{2.64}$$

where we have used the Frobenius compatibility twice. The compatibility with differentials is similar to the above proofs. □

We also have the analogue compatibility for $\vee_1$.

Proposition 13 *The coproduct* $\vee_1 : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ *is a map of left DG* $C_*(V,V)$*-bimodules, where the left bimodule structures are given by the product* $\bullet$ *in the usual way.*

Proof. The proof is exactly analogue to the above proof. □.

Moreover, the coproducts $\vee_0$ and $\vee_1$ are chain homotopic. We now construct such chain homotopy.

Definition 15 Define a coproduct $\vee : C_*(V,V) \to C_*(V,V) \bigotimes C_*(V,V)$ of degree $1-d$ by

$$\vee(x_0 \otimes [x_1|...|x_n]) = \sum_{i=1}^{n} (-1)^{\lambda_i} (x_0 \cdot x_i)' \otimes [x_1|...|x_{i-1}] \bigotimes (x_0 \cdot x_i)'' \otimes [x_{i+1}|...|x_n] \tag{2.65}$$

where the sign is given by Koszul sign convention and taking into account implicit suspensions, i.e. $\lambda_i = \deg(x_0) + \big(1 + \deg(x_i) + \deg((x_0 \cdot x_i)'')\big)(\deg[x_1|...x_{i-1}])$

Proposition 14 *The coproduct* $\vee : C_*(V,V) \to C_*(V,V) \bigotimes C_*(V,V)$ *is a chain homotopy between*

$\vee_0$ *and* $\vee_1$*, i.e. the following equation holds*

$$(D \otimes 1 + 1 \otimes D) \vee - \vee D = \vee_1 - \vee_0. \tag{2.66}$$

Proof. Write $D = D_0 + b_\Omega + \delta$ where D_0 is the internal differential (the tensor differential with respect to d), $b_\Omega = 1_V \otimes b_1$ and b_1 is the external differential of the cobar construction (applying the coproduct consecutively on ΩV), and δ is as before. It is clear that $\vee$ commutes with D_0. We compute $(\delta \otimes 1 + 1 \otimes \delta)\vee$ and $\vee\delta$. We have

$$(\delta \otimes 1 + 1 \otimes \delta) \vee (x_0 \otimes [x_1|...|x_n]) = (\delta \otimes 1 + 1 \otimes \delta)\Big(\sum_{i=1}^{n} (x_0 \cdot x_i)' \otimes [x_1|...|x_{i-1}] \bigotimes (x_0 \cdot x_i)'' \otimes [x_{i+1}|...|x_n]\Big) = \tag{2.67}$$

$$\sum_{i=1}^{n} \pm ((x_0 \cdot x_i)')' \otimes [((x_0 \cdot x_i)')''|x_1...|x_{i-1}] \bigotimes (x_0 \cdot x_i)'' \otimes [x_{i+1}|...|x_n] \tag{2.68}$$

$$+ \sum_{i=1}^{n} \pm ((x_0 \cdot x_i)')'' \otimes [x_1|...|x_{i-1}|((x_0 \cdot x_i)')'] \bigotimes (x_0 \cdot x_i)'' \otimes [x_{i+1}|...|x_n] \tag{2.69}$$

$$+ \sum_{i=1}^{n} \pm (x_0 \cdot x_i)' \otimes [x_1|...|x_i] \bigotimes ((x_0 \cdot x_i)'')' \otimes [((x_0 \cdot x_i)'')''|x_{i+1}|...|x_n] \tag{2.70}$$

$$+ \sum_{i=1}^{n} \pm (x_0 \cdot x_i)' \otimes [x_1|...|x_{i-1}] \bigotimes ((x_0 \cdot x_i)'')'' \otimes [x_{i+1}|...|x_n|((x_0 \cdot x_i)'')'] \tag{2.71}$$

and

$$\vee\delta(x_0\otimes[x_1|...|x_n]) = \pm\vee(x_0'\otimes[x_0''|x_1|...|x_n] \pm x_0''\otimes[x_1|...|x_n|x_0']) = \tag{2.72}$$

$$\pm(x_0'\cdot x_0'')'\otimes 1 \bigotimes (x_0'\cdot x_0'')''\otimes[x_1|...|x_n] \tag{2.73}$$

$$+\sum_{i=1}^{n}\pm(x_0'\cdot x_i)'\otimes[x_0''|x_1|...|x_{i-1}] \bigotimes (x_0'\cdot x_i)''\otimes[x_{i+1}|...|x_n] \tag{2.74}$$

$$+\sum_{i=1}^{n}\pm(x_0''\cdot x_i)'\otimes[x_1|...|x_{i-1}] \bigotimes (x_0''\cdot x_i)''\otimes[x_{i+1}|...|x_n] \tag{2.75}$$

$$\pm(x_0''\cdot x_0')'\otimes[x_1|...|x_n] \bigotimes (x_0'\cdot x_0'')''\otimes 1. \tag{2.76}$$

By commutativity of $\cdot$, coassociativity of Δ, and Frobenius compatibility between both operations we have that (second line above) and (fifth line above) cancel with (third line below) and (fourth line below), respectively. Also note that (second line below)$=\vee_0(x_0\otimes[x_1|...|x_n])$ and (fifth line below)$=\vee_1(x_0\otimes[x_1|...|x_n])$. Hence it follows that

$$\Big((\delta\otimes 1+1\otimes\delta)\vee-\vee\,\delta\Big)(x_0\otimes[x_1|...|x_n]) = (\vee_1-\vee_0)(x_0\otimes[x_1|...|x_n]) \tag{2.77}$$

$$+\sum_{i=1}^{n}((x_0\cdot x_i)')''\otimes[x_1|...|x_{i-1}|((x_0\cdot x_i)')'] \bigotimes (x_0\cdot x_i)''\otimes[x_{i+1}|...|x_n] \tag{2.78}$$

$$+\sum_{i=1}^{n}\pm(x_0\cdot x_i)'\otimes[x_1|...|x_i] \bigotimes ((x_0\cdot x_i)'')'\otimes[((x_0\cdot x_i)'')''|x_{i+1}|...|x_n] \tag{2.79}$$

These last two sums above cancel with $\Big((b\otimes 1+1\otimes b)\vee-\vee b\Big)(x_0\otimes[x_1|...|x_n])$ as we can conclude from the two computations below:

$$(b\otimes 1+1\otimes b)\vee(x_0\otimes[x_1|...|x_n])=(b\otimes 1+1\otimes b)\Big(\sum_{i=1}^{n}(x_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\otimes[x_{i+1}|...|x_n]\Big)= \tag{2.80}$$

$$\sum_{j<i}\pm(x_0\cdot x_i)''\otimes[x_1|...|x_j'|x_j''|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\otimes[x_{i+1}|...|x_n]+ \tag{2.81}$$

$$\sum_{j>i}\pm(x_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\otimes[x_{i+1}|...|x_j'|x_j''|...|x_n], \tag{2.82}$$

and

$$\vee b(x_0\otimes[x_1|...|x_n])=\vee(\sum_{i=1}^{n}\pm x_0\otimes[x_1|...|x_i'|x_i''|...|x_n])= \tag{2.83}$$

$$\sum_{j<i}\pm(x_0\cdot x_j)'\otimes[x_1|...|x_{j-1}]\bigotimes(x_0\cdot x_j)''\otimes[x_{j+1}|...|x_i'|x_i''|...|x_n] \tag{2.84}$$

$$+\sum_{i=1}^{n}\pm(x_0\cdot x_i')'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i')''\otimes[x_i''|...|x_n] \tag{2.85}$$

$$+\sum_{i=1}^{n}\pm(x_0\cdot x_i'')'\otimes[x_1|...|x_i']\bigotimes(x_0\cdot x_i'')''\otimes[x_{i+1}|...|x_n] \tag{2.86}$$

$$+\sum_{j>i}\pm(x_0\cdot x_j)'\otimes[x_1|...|x_i'|x_i''|...|x_{j-1}]\bigotimes(x_0\cdot x_j)''\otimes[x_{j+1}|...|x_n]. \tag{2.87}$$

The sum indexed with $j<i$ above cancels with the sum indexed with $j>i$ below and analogously the sum indexed with $j>i$ above cancels with the sum indexed with $j<i$ below. The two remaining terms, i.e. (third below) +(fourth below), are precisely (second from top of page)+(third from top of page). Hence we have shown that

$$(b_\Omega\otimes 1+1\otimes b_\Omega+\delta\otimes 1+1\otimes\delta)\vee-\vee(b_\Omega+\delta)=\vee_1-\vee_0. \tag{2.88}$$

Since D_0 commutes with $\vee$, it follows that $\vee : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ is a chain homotopy between $\vee_1$ and $\vee_0$. $\square$

Corollary 2 *The coproducts $\vee_0$ and $\vee_1$ induce the same map in coHochschild homology.*

Note that since Δ is cocommutative we have that the coproduct $(\vee_0)_* = (\vee_1)_*$ on coHochschild homology is cocommutative as well. In fact, the chain homotopy for the cocommutativity of $\vee_0$ is given by $\vee$ since, by the cocommutativity of Δ, $\tau \circ \vee_0 = \vee_1$. Finally, combining propositions 7, 8, and corollary 1 we obtain the following

Theorem 14 *Let $(V, d, \Delta, \cdot)$ be a cocommutative commutative DG open Frobenius coalgebra of degree $-d$. Then the commutative product $\bullet : coHH_*(V,V) \otimes coHH_*(V,V) \to coHH_*(V,V)$ and the cocommutative coproduct $(\vee_0)_* = (\vee_1)_* : coHH_*(V,V) \to coHH(V,V)$ are Frobenius compatible; i.e. the latter is a map of $coHH(V,V)$-bimodules.*

We call the coproduct $(\vee_0)_* = (\vee_1)_* : coHH_*(V,V) \to coHH(V,V)$ the *algebraic coproduct of degree -d* on the coHochschild homology of a commutative cocommutative DG open Frobenius coalgebra of degree $-d$.

Even if $\vee$ does not pass to homology, as it has boundary terms $\vee_0$ and $\vee_1$, we highlight the following compatibility with the product $\bullet$.

Proposition 15 *The coproduct $\vee : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ and the product $\bullet : C_*(V,V) \otimes C_*(V,V) \to C_*(V,V)$ are infinitesimally compatible, i.e. $\vee(a \bullet b) = \vee(a) \bullet b \pm a \bullet \vee(b)$.*

Proof. Note that using commutativity of $\cdot$ and the Frobenius compatibility we have

$$\vee\Big((x_0\otimes[x_1|...|x_m])\bullet(y_0\otimes[y_1|...|y_n])\Big)=\vee\Big(\pm(x_0\cdot y_0)\otimes[x_1|...|x_m|y_1|...|y_n]\Big)= \tag{2.89}$$

$$\sum_{i=1}^{m}\pm(x_0\cdot y_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot y_0\cdot x_i)''\otimes[x_{i+1}|...|x_m|y_1|...|y_n] \tag{2.90}$$

$$+\sum_{j=1}^{n}\pm(x_0\cdot y_0\cdot y_j)'\otimes[x_1|...|x_m|y_1|...|y_{j-1}]\bigotimes(x_0\cdot y_0\cdot y_j)''[y_{j+1}|...|y_n]= \tag{2.91}$$

$$\sum_{i=1}^{m}\pm(x_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\cdot y_0\otimes[x_{i+1}|...|x_m|y_1|...|y_n] \tag{2.92}$$

$$+\sum_{j=1}^{n}\pm x_0\cdot(y_0\cdot y_j)'\otimes[x_1|...|x_m|y_1|...|y_{j-1}]\bigotimes(y_0\cdot y_j)''[y_{j+1}|...|y_n]= \tag{2.93}$$

$$\vee\Big(x_0\otimes[x_1|...|x_m]\Big)\bullet y_0\otimes[y_1|...|y_n]\pm x_0\otimes[x_1|...|x_n]\bullet\vee\Big(y_0\otimes[y_1|...|y_n]\Big).\square \tag{2.94}$$

Remark 13 There are two ways in which we can obtain a chain map from $\vee$:

1) Reduce $C_*(V,V)$: Consider the sub vector space $V=V\otimes\mathbb{K}\hookrightarrow V\otimes\Omega V=C_*(V,V)$. Assuming Δ is cocommutative, it follows that V is a subcomplex of the coHochschild complex $(C_*(V,V),D)$. Define the *relative coHochschild complex* of (V,d,Δ) by $(\tilde{C}_*(V,V),D):=(C_*(V,V)/V,D)$. Note that the underlying vector space of $\tilde{C}_*(V,V)$ is isomorphic to $V\otimes T^{>0}s^{-1}\overline{V}$.

The coproducts $\vee_0$ and $\vee_1$ factor as maps $C_*(V,V)\to V\otimes C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ and $C_*(V,V)\to C_*(V,V)\otimes V\to C_*(V,V)\otimes C_*(V,V)$, respectively. Therefore both $\vee_0$ and $\vee_1$ are identically 0 on $\tilde{C}_*(V,V)$. Since $\vee_0$ and $\vee_1$ are the boundary terms of $\vee: C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ it follows that the induced map $\vee:\tilde{C}_*(V,V)\to\tilde{C}_*(V,V)\otimes\tilde{C}_*(V,V)$ is a chain map. We call this coproduct the *algebraic coproduct of degree 1-d on the relative coHochschild complex* of a DG cocommutative commutative open Frobenius coalgebra.

2) Reduce $\vee$: Recall that $\vee$ is defined by $\vee(x_0\otimes[x_1|...|x_n])=\sum_{i=1}^{n}(-1)^{\lambda_i}(x_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\otimes[x_{i+1}|...|x_n]$ and note that in the above proof of the identity $(D\otimes 1+1\otimes D)\vee-\vee D=\vee_1-\vee_0$

the boundary terms are created precisely by $(x_0\cdot x_1)'\otimes 1\bigotimes(x_0\cdot x_1)''\otimes[x_2|...|x_n]$ and $(x_0\cdot x_n)'\otimes[x_1|...|x_{n-1}]\bigotimes(x_0\cdot x_n)''\otimes 1$, i.e. the first and last terms of the sum defining $\vee$. Therefore we can define a new coproduct $\tilde{\vee}: C_*(V,V)\to C_*(V,V)\otimes C_*(V,V)$ by $\tilde{\vee}(x_0\otimes[x_1|...|x_n]) := \sum_{i=2}^{n-1}(-1)^{\lambda_i}(x_0\cdot x_i)'\otimes[x_1|...|x_{i-1}]\bigotimes(x_0\cdot x_i)''\otimes[x_{i+1}|...|x_n]$ for $n>2$ and $\tilde{\vee}(x_0\otimes[x_1|...|x_n]) := 0$ for $n\leq 2$. We call this coproduct the *reduced algebraic coproduct of degree 1-d on the coHochschild complex* of a DG cocommutative commutative open Frobenius coalgebra. Moreover note that $\tilde{\vee}$ is a lift of the coproduct $\vee$ on the relative complex $\tilde{C}_*(V,V)$, in other words, the following diagram commutes

$$\begin{array}{ccc} C_*(V,V) & \xrightarrow{\tilde{\vee}} & C_*(V,V)^{\otimes 2} \\ \downarrow & & \downarrow \\ \tilde{C}_*(V,V) & \xrightarrow{\vee} & \tilde{C}_*(V,V)^{\otimes 2} \end{array}$$

where vertical arrows are projections. We would like to have some kind of compatibility between $\bullet$ and any of the two coproducts described above. At the present we do not know if $\tilde{\vee}$ and $\bullet$ satisfy a significant compatibility: note that the product is not sable under the coproduct, for example if $a,b\in V\otimes s^{-1}\overline{V}^{\otimes 2}\subset C_*(V,V)$ then $\tilde{\vee}(a)=0=\tilde{\vee}(b)$ while $\tilde{\vee}(a\bullet b)$ is non zero. However, we do have the desired stability when we consider cyclic chains.

Remark 14 For topological applications it will be useful to get rid of an acyclic sub complex of the coHochschild complex consisting of iterated coproducts. Denote by V_Δ the sub vector space of $C_*(V,V)$ generated by iterated coproducts of V, i.e. elements of the form $\Delta^1(x)=x'\otimes[x''], \Delta^2(x)=x'\otimes[(x'')'|(x'')''],...$ for some $x\in V$. It follows that V_Δ is a sub complex of the coHochschild complex $(C_*(V,V),D)$ since $D_0\Delta^k(x)=\Delta^k(D_0x)$ and, by coassociativity of Δ we have $D_1\Delta^k(x)=\Delta^{k+1}(x)$ if k is odd and $D_1\Delta^k(x)=0$ if k is even.

In fact, V_Δ is an acyclic sub complex of $(C_*(V,V),D)$; a chain contraction $s: V_\Delta\to V_\Delta$ can be defined by $s\Delta^k(x)=\Delta^{k-1}(x)$ if k is even and $s\Delta^k(x)=0$ if k is odd. Since s clearly commutes with D_0 and $(D_1 s + sD_1)\Delta^k(x) = \Delta^k(x)$ it follows that $Ds + sD = 1_{V_\Delta}$.

Consider the the subspace $W = V \oplus V_\Delta$ of $C_*(V,V)$. W is clearly a sub complex of $(C_*(V,V),D)$, and moreover, by cocommutativity, $D(V) = d(V) \subset V$. Since V_Δ is contractible, (W,D) is chain equivalent to (V,d). Hence $C_*(V,V)/W$ is chain equivalent to $\tilde{C}_*(V,V)$. The coproduct $\vee$ is well defined on $C_*(V,V)/W$. Later on we will relate the homology of the quotient complex $C_*(V,V)/W$ to the homology of the space of free loops on a space relative to constant loops.

In any case, we have the following involutivity condition between $\tilde{\vee}$ and $\bullet$ at the level of co-Hochschild homology.

Proposition 16 *The composition* $C_*(V,V) \xrightarrow{\tilde{\vee}} C_*(V,V) \otimes C_*(V,V) \xrightarrow{\bullet} C_*(V,V)$ *is chain homotopic to the trivial map.*

Proof. The chain homotopy is given by

$$h(x_0 \otimes [x_1|\ldots|x_n]) = \sum_{i=1}^{n-1} \pm (x_0 \cdot x_i \cdot x_{i+1}) \otimes [x_1|\ldots|x_{i-1}|x_{i+2}|\ldots|x_n] \tag{2.95}$$

for $n > 2$ and $h := 0$ otherwise. We will check $D \circ h + h \circ D = \bullet \circ \tilde{\vee}$. Since h clearly commutes with the tensor differential D_0 we will verify the equation for D_1. Note that $\bullet \circ \tilde{\vee}(x_0 \otimes [x_1|\ldots|x_n]) = \sum_{i=2}^{n-1} \pm (x_0 \cdot x_i' \cdot x_i'') \otimes [x_1|\ldots|x_{i-1}|x_{i+1}|\ldots|x_n]$. We first compute $h \circ D_1 = h \circ (b_\Omega + \delta)$. Note that, in the computation below, the first four terms (second and third line) correspond to $h \circ \delta$ and the rest to $h \circ b_\Omega$:

$$h \circ D_1\Big(x_0 \otimes [x_1|...|x_n]\Big) = h\Big(\pm x_0' \otimes [x_0''|x_1...|x_n] \pm x_0'' \otimes [x_1|...|x_n|x_0'] + \sum_{i=1}^{n} x_0 \otimes [x_1|...|x_i'|x_i''|...|x_n]\Big) =$$

$$\pm(x_0' \cdot x_0'' \cdot x_1) \otimes [x_2|...|x_n] + \sum_{i=1}^{n-1} \pm(x_0' \cdot x_i \cdot x_{i+1}) \otimes [x_0''|...|x_{i-1}|x_{i+1}|...|x_n]$$

$$\pm(x_0'' \cdot x_n \cdot x_0') \otimes [x_1|...|x_{n-1}] + \sum_{i=1}^{n-1} \pm(x_0'' \cdot x_i \cdot x_{i+1}) \otimes [x_1|...|x_{i-1}|x_{i+1}|...|x_n|x_0']$$

$$\pm(x_0 \cdot x_1' \cdot x_1'') \otimes [x_2|...|x_n] \pm (x_0 \cdot x_n' \cdot x_n'') \otimes [x_1|...x_{n-1}]$$

$$\sum_{i=2}^{n-1} \pm(x_0 \cdot x_i' \cdot x_i'') \otimes [x_1|...|x_{i-1}|x_{i+1}|...|x_n]$$

$$+ \sum_{i=1}^{n-1} \pm(x_0 \cdot x_i'' \cdot x_{i+1}) \otimes [x_1|...|x_i'|x_{i+2}|...|x_n] + \sum_{i=1}^{n-1} \pm(x_0 \cdot x_i \cdot x_{i+1}') \otimes [x_1|...|x_{i-1}|x_{i+1}''|...|x_n]$$

$$+ \sum_{j+1<i} \pm(x_0 \cdot x_j \cdot x_{j+1}) \otimes [x_1|...|x_{j-1}|x_{j+2}|...|x_i'|x_i''|...|x_n] + \sum_{j>i} \pm(x_0 \cdot x_j \cdot x_{j+1}) \otimes [x_1|...|x_i'|x_i''|...|x_{j-1}|x_{j+1}|...|x_n].$$

Observe that all the terms which do not have a summation sign cancel between them by Frobenius compatibility and we are left with seven sums. The third of these sums is precisely $\bullet \circ \tilde{\vee}$, the next two (fifth and sixth sum) cancel each other by Frobenius compatibility and cocommutativity of Δ. Finally, the first two sums together with the last two sums are precisely $D_1 \circ h(x_0 \otimes [x_1|...|x_n])$ (the first two are equal to $\delta \circ h$ and the last two to $b_\Omega \circ h$). $\square$

We now check the homotopy cocommutativity of $\tilde{\vee}$ by describing an explicit chain homotopy κ. The chain homotopy κ also induces a chain homotopy for the cocommutativity of $\vee : \tilde{C}_*(V,V) \to \tilde{C}_*(V,V) \otimes \tilde{C}_*(V,V)$.

Definition 16 Define $\kappa_{i,j} : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ by

$$\kappa_{i,j}(x_0 \otimes [x_1|...|x_n]) := (-1)^{\lambda_{ij}} x_0 \otimes [x_1|...|x_{i-1}|x_{j+1}|...|x_n] \bigotimes (x_i \cdot x_j) \otimes [x_{i+1}|...|x_{j-1}] \qquad (2.96)$$

and

$$\kappa(x_0 \otimes [x_1|...|x_n]) := \sum_{(i,j)\in R_n} \kappa_{i,j}(x_0 \otimes [x_1|...|x_n]) \tag{2.97}$$

for $n > 3$,where $R_n = \{(i,j) : i,j \in \{1,...,n\}, i < j-1, (i,j) \neq (1,n)\}$, and $\kappa(x_0 \otimes [x_1|...|x_n) := 0$ for $n \leq 3$. Note that we have defined κ in such way that we do not have elements of the form $z_0 \otimes 1 \bigotimes a$ or $a \bigotimes z_0 \otimes 1$ in its image. Note that κ is a map of degree $2-d$ and signs are given by $\lambda_{ij} = \deg[x_{j+1}|...|x_n]\deg[x_i|...|x_j] + \deg[x_j]\deg[x_{i+1}|...|x_{j-1}] + \deg(x_0) + \deg[x_1|...|x_i|x_{j+1}|...|x_n](d+1)$.

Proposition 17 *$\kappa : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ is a chain homotopy for the cocommutativity of $\tilde{\vee}$, i.e. the following equation holds*

$$(D \otimes 1 + 1 \otimes D) \circ \kappa + \kappa \circ D = \tau \circ \tilde{\vee} - \tilde{\vee}. \tag{2.98}$$

Proof. As usual write the coHochschild differential as $D = D_0 + D_1$ where D_0 is the tensor differential and $D_1 = b_\Omega + \delta$. From the formula for κ it is clear that κ commutes with the tensor differential D_0, so we check that the above equation holds for D_1. We compute $(\delta \otimes 1 + 1 \otimes \delta) \circ \kappa + \kappa \circ \delta$ and among its terms we recognize $(b \otimes 1 + 1 \otimes b) \circ \kappa + \kappa$ and $\tau \circ \tilde{\vee} - \tilde{\vee}$. Note that in the calculation below we split $\kappa \circ \delta$ into four sums, which are the last four lines:

$$\Big((\delta\otimes 1+1\otimes\delta)\circ\kappa+\kappa\circ\delta\Big)\Big(x_0\otimes[x_1|\ldots|x_n]\Big)=$$

$$(\delta\otimes 1+1\otimes\delta)\Big(\sum_{(i,j)\in R_n}\pm x_0\otimes[x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n]\bigotimes(x_i\cdot x_j)\otimes[x_{i+1}|\ldots|x_{j+1}]\Big)$$

$$+\kappa\Big(\pm x_0'\otimes[x_0''|x_1|\ldots|x_n]\pm x_0''\otimes[x_1|\ldots|x_n|x_0']\Big)=$$

$$\sum_{(i,j)\in R_n}\Big(x_0'\otimes[x_0''|x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n]\pm x_0''\otimes[x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n|x_0']\Big)\bigotimes(x_i\cdot x_j)\otimes[x_{i+1}|\ldots|x_{j+1}]\Big)$$

$$+\sum_{(i,j)\in R_n}\pm x_0\otimes[x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n]\bigotimes\Big((x_i\cdot x_j)'\otimes[(x_i\cdot x_j)''|x_{i+1}|\ldots|x_{j+1}]\pm(x_i\cdot x_j)''\otimes[x_{i+1}|\ldots|x_{j+1}|(x_i\cdot x_j)']\Big)$$

$$+\sum_{j=2}^{n-1}\pm x_0'\otimes[x_{j+1}|\ldots|x_n]\bigotimes(x_0''\cdot x_j)\otimes[x_1|\ldots|x_{j-1}]$$

$$+\sum_{(i,j)\in R_n}\pm x_0'\otimes[x_0''|x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n]\bigotimes(x_i\cdot x_j)\otimes[x_{i+1}|\ldots|x_{j+1}]$$

$$+\sum_{(i,j)\in R_n}\pm x_0''\otimes[x_1|\ldots|x_{i-1}|x_{j+1}|\ldots|x_n|x_0']\bigotimes(x_i\cdot x_j)\otimes[x_{i+1}|\ldots|x_n]$$

$$+\sum_{i=2}^{n-1}\pm x_0''\otimes[x_1|\ldots|x_{i-1}]\bigotimes(x_i\cdot x_0')\otimes[x_{i+1}|\ldots|x_{j+1}].$$

Note that by Frobenius compatibility, commutativity of $\cdot$ and cocommutativity of Δ we have that (last line)$=\tilde{\nabla}(x_0\otimes[x_1|\ldots|x_n])$ and (sixth line)$=\tau\circ\tilde{\nabla}(x_0\otimes[x_1|\ldots|x_n])$. Also note that (fourth line) cancels with (seventh line) + (eigth line). So the remaining term is the (fifith line). However, by Frobenius compatibility, these remaining terms are exactly $\Big((b_\Omega\otimes 1+1\otimes b_\Omega)\circ\kappa+\kappa\circ b_\Omega\Big)(x_0\otimes[x_1|\ldots|x_n])$. We note that the above proof works for $n=3$ (some of the terms that have canceled above are zero in this case) and for $n<3$ there is nothing to check, because of how we have defined $\tilde{\nabla}$and κ. Thus we have shown the desired equation. □

As a corollary, we obtain the cocommutativity of the reduced coproduct $\tilde{\nabla}$ at the level of homology. We summarize our results in the following

Theorem 15 *Let $(V, d, \Delta, \cdot)$ be a cocommutative commutative DG open Frobenius coalgebra of degree $-d$. Then the coproduct $\tilde{\vee}$ induces the structure of a coassociative cocommutative coalgebra of degree $1-d$ on $coHH_*(V, V)$, the coHochschild homology of V.*

2.2.3 Gerstenhaber and BV structures

There is more structure in the coHochschild homology of an DG open Frobenius algebra obtained from the chain homotopies for the commutativity of $\bullet$ and for cocommutativity of $\vee_0$ and $\tilde{\vee}$. The structures that arise are known as Gerstenhaber algebras and coalgebras. We start with the definition of these two structures.

Definition 17 A *Gerstenhaber algebra* is a triple $(V, \bullet, \{,\})$ where
i) $(V, \bullet)$ is a (graded) commutative and associative algebra
ii) $(V, \{,\})$ is a (graded) Lie algebra of degree $+1$
iii) For any c, $\{, c\}$ is a derivation of degree $1 + \deg(c)$ of the product $\bullet$, i.e.

$$\{a \bullet b, c\} = \{a, c\} \bullet b + (-1)^{\deg(a)(\deg(c)-1)} a \bullet \{b, c\} \tag{2.99}$$

The above derivation condition is equivalent to requiring the following diagram to commute:

$$\begin{tikzcd} V \otimes V \otimes V \arrow[r, "\bullet\otimes 1"] \arrow[d, "1\otimes\{,\}+(\{,\}\otimes 1)\circ(1\otimes\tau)"'] & V \otimes V \arrow[d, "\{,\}"] \\ V \otimes V \arrow[r, "\bullet"] & V \end{tikzcd}$$

We also have the dual notion.

Definition 18 A *Gerstenhaber coalgebra* is a triple (V, ν, θ) where

i) (V,ν) is a (graded) cocommutative and coassociative coalgebra

ii) (V,θ) is a (graded) Lie coalgebra of degree $+1$

iii) The cobracket θ is a graded coderivation of the coproduct ν, i.e. writing $\nu(a) = a^i \otimes a_i$ and $\theta(a) = a^k \otimes a_k$ the following equation holds

$$(a^k)^i \otimes (a_k)_i \otimes a_k = (a^i)^k \otimes a_i \otimes (a^i)_k \pm a^i \otimes (a_i)^k \otimes (a_i)_k \tag{2.100}$$

The commutative diagram for the above coderivation condition is the dual of the one above, i.e.

$$\begin{tikzcd} V \otimes V \otimes V & V \otimes V \arrow[l, "\nu\otimes 1"'] \\ V \otimes V \arrow[u, "(1\otimes\tau)\circ(\theta\otimes 1)+1\otimes\theta"] & V \arrow[l, "\nu"] \arrow[u, "\theta"'] \end{tikzcd}$$

In each statement of the three statements below, $(V, d, \Delta, \cdot)$ is a commutative, cocommutative, DG open Frobenius coalgebra of degree $-d$ with counit $\epsilon : V \to \mathbb{K}$. For the computations and proofs below we denote $w_0 \otimes [w_1|...|w_n]$ by $w_0 \otimes w$ and a monomial $[w_r|...|w_s]$ by $[w_r^s]$.

Recall that $*: C_*(V,V) \otimes C_*(V,V) \to C_*(V,V)$ is the chain homotopy for the commutativity of $\bullet$, and $\vee : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ and $\kappa : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ are the chain homotopies for the cocommutativity of $\vee_0$ and $\tilde{\vee}$ respectively.

Theorem 16 *Let $\{,\} : C_*(V,V) \otimes C_*(V,V) \to C_*(V,V)$ be defined as the graded commutator of $*$, i.e. $\{,\} := * - (* \circ \tau)$. Then $\{,\}$ passes to coHochschild homology, where it defines the structure of a (shifted) Gerstenhaber algebra together with product $\bullet$.*

Proof. The skew symmetry, graded Jacobi identity, and compatibility of $\{,\}$ with the coHochschild differential are easy calculations. We show the derivation equation $\{a \bullet b, c\} = \{a, c\} \bullet b \pm a \bullet$

$\{b,c\}$ holds at the level of homology. Using the definition of $\{,\}$ this equation is equivalent to

$$(a\bullet b)*c \pm c*(a\bullet b) = (a*c)\bullet b \pm (c*a)\bullet b \pm a\bullet(b*c) \pm a\bullet(c*b).$$

It follows by an easy calculation to check that for any c the map $w \mapsto w*c$ is a derivation of $\bullet$ *at the chain level.* We show that for any c the map $w \mapsto c*w$ is a derivation, *up to a chain homotopy,* $f : C_*(V,V)^{\otimes 3} \to C_*(V,V)$ given by

$$f(a\otimes b\otimes c) := \sum_{1=i<j=n} \epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0 \otimes [z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^{n}]$$

where $a = x_0\otimes x$, $b = y_0\otimes y$, and $z = z_0\otimes z = z_0\otimes[z_1|...|z_n]$. Thus, the equation which we will show, which says that the map $w\mapsto c*w$ is a derivation of $\bullet$ up to chain homotopy, is

$$D\circ f(a\otimes b\otimes c) + f\circ D(a\otimes b\otimes c) = c*(a\bullet b) \pm (c*a)\bullet b \pm a\bullet(c*b) = \tag{2.101}$$

$$\sum_{i=1}^{n} \pm\epsilon(z_i\cdot x_0\cdot y_0)z_0\otimes[z_1^{i-1}|x|y|z_{i+1}^{n}] \tag{2.102}$$

$$+\sum_{i=1}^{n} \pm\epsilon(z_i\cdot x_0)z_0\cdot y_0\otimes[z_1^{i-1}|x|z_{i+1}^{n}|y] \tag{2.103}$$

$$+\sum_{i=1}^{n} \pm\epsilon(z_i\cdot y_0)x_0\cdot z_0\otimes[x|z_1^{i-1}|y|z_{i+1}^{n}]. \tag{2.104}$$

where we have used the definition of $*$ and $\bullet$ in the last equality.

Clearly f commutes with D_0 so we check the above equation for $D_1 = b_{\Omega V} + \delta$. On one hand we have:

$$D_1 \circ f(a \otimes b \otimes c) = \tag{2.105}$$

$$\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0) z_0' \otimes [z_0''|z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.106}$$

$$\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0) z_0 \otimes b_{\Omega V}[z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.107}$$

$$+\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0) z_0'' \otimes [z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^n|z_0']. \tag{2.108}$$

On the other hand, we compute $f \circ D_1(a \otimes b \otimes c)$ in two steps: first $f(D_1(a \otimes b) \otimes c)$ and then $f(a \otimes b \otimes D_1(c))$.

Step 1:

$$f(D_1(a \otimes b) \otimes c) = f(b_{\Omega V}(a \otimes b) \otimes c) + f(\delta(a \otimes b) \otimes c) = \tag{2.109}$$

$$\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0) z_0 \otimes [z_1^{i-1}|b_{\Omega V}(x)|z_{i+1}^{j-1}|y|z_{j+1}^n] + \sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0) z_0 \otimes [z_1^{i-1}|x|z_{i+1}^{j-1}|b_{\Omega V}(y)|z_{j+1}^n] \tag{2.110}$$

$$+\sum_{i<j} \pm\epsilon(z_i \cdot x_0')\epsilon(z_j \cdot y_0) z_0 \otimes [z_1^{i-1}|x_0''|x|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.111}$$

$$+\sum_{i<j} \pm\epsilon(z_i \cdot x_0'')\epsilon(z_j \cdot y_0) z_0 \otimes [z_1^{i-1}|x|x_0'|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.112}$$

$$+\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0') z_0 \otimes [z_1^{i-1}|x|z_{i+1}^{j-1}|y_0''|y|z_{j+1}^n] \tag{2.113}$$

$$+\sum_{i<j} \pm\epsilon(z_i \cdot x_0)\epsilon(z_j \cdot y_0'') z_0 \otimes [z_1^{i-1}|x|z_{i+1}^{j-1}|y|y_0'|z_{j+1}^n]. \tag{2.114}$$

We split Step 2, the calculation of $f(a \otimes b \otimes D_1(c))$, into two; first $f(a \otimes b \otimes \delta(c))$ and then $f(a \otimes b \otimes b_{\Omega V}(c))$:

$$f(a\otimes b\otimes\delta(c))= \tag{2.115}$$

$$+\sum_{j=1}^{n}\pm\epsilon(z_0''\cdot x_0)\epsilon(z_j\cdot y_0)z_0'\otimes[x|z_1^{j-1}|y|z_{j+1}^n] \tag{2.116}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0'\otimes[z_0''|z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.117}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0''\otimes[z_1^{i-1}|x|z_{i+1}^{j-1}|y|z_{j+1}^n|z_0'] \tag{2.118}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_0'\cdot y_0)z_0''\otimes[z_1^{i-1}|x|z_{i+1}^n|y] \tag{2.119}$$

and

$$f(a\otimes b\otimes b_{\Omega V}(c))= \tag{2.120}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0\otimes[b_{\Omega V}(z_1^{i-1})|x|z_{i+1}^{j-1}|y|z_{j+1}^n] \tag{2.121}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0\otimes[z_1^{i-1}|x|b_{\Omega V}(z_{i+1}^{j-1})|y|z_{j+1}^n] \tag{2.122}$$

$$+\sum_{i<j}\pm\epsilon(z_i\cdot x_0)\epsilon(z_j\cdot y_0)z_0\otimes[z_1^{i-1}|x|z_{i+1}^{j-1}|y|b_{\Omega V}(z_{j+1}^n)] \tag{2.123}$$

$$+\sum_{i=1}^{n}\epsilon(z_i'\cdot x_0)\epsilon(z_i''\cdot y_0)z_0\otimes[z_1^{i-1}|x|y|z_{i+1}^n] \tag{2.124}$$

$$+A. \tag{2.125}$$

Note that in the above calculation we have not expanded certain terms and have called them A. Using the Frobenius compatibility, one can see that A consists exactly of the last four sums in step 1, i.e. (118)+(119)+(120)+(121). We have the following cancelations: (114) and (116) cancel with (124) and (125) respectively, and (114) cancels with (117) + (128) +(129) +(130). Finally, using the fact that ϵ is the counit for Δ and the Frobenius compatibility, we recognize that the remaining terms (123), (126), and (131) are precisely (111), (110), and (109), respectively. □

Theorem 17 *Let $\theta : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ be defined as the graded cocommutator of κ, i.e. $\theta := \kappa - \tau \circ \kappa$. Then θ passes to coHochchild homology, where it defines the structure of a (shifted) Gerstenhaber coalgebra together with coproduct $\tilde{\nabla}$.*

Proof. The skew symmetry, graded coJacobi identity, and compatibility of θ with the coHochschild differential are easy calculations. We show the coderivation equation $(\tilde{\nabla} \otimes 1) \circ \theta = (1 \otimes \tau) \circ (\theta \otimes 1) \circ \tilde{\nabla} \pm (1 \otimes \theta) \circ \tilde{\nabla}$ holds at the level of homology. The structure of the proof is the same as above: κ is a coderivation of $\tilde{\nabla}$ *at the chain level* and $\tau \circ \kappa$ is a coderivation $\tilde{\nabla}$ *up to chain homotopy.* These two facts imply the desired result since $\theta = \kappa - \tau \circ \kappa$. It is easy to check the first fact, so we will show the second, and a chain homotopy $g : C_*(V,V) \to C_*(V,V)^{\otimes 3}$ is given by

$$g(x_0 \otimes [x_1|...|x_n]) := \sum_{1=i<j<k<l=n} (x_i \cdot x_j) \otimes [x_{i+1}^{j-1}] \bigotimes (x_k \cdot x_l) \otimes [x_{k+1}^{l-1}] \bigotimes x_0 \otimes [x_1^{i-1}|x_{j+1}^{k-1}|x_{k+1}^{n}].$$

Thus the equation we will show, which says that $\tau \circ \kappa$ is a coderivation of $\tilde{\nabla}$ up to chain homotopy, is

$$(D \circ g + g \circ D)(x_0 \otimes [x_1|...|x_n]) = \Big((\tilde{\nabla} \otimes 1) \circ (\tau \circ \kappa) - (1 \otimes \tau) \circ ((\tau \circ \kappa) \otimes 1) \circ \tilde{\nabla} \pm (1 \otimes (\tau \circ \kappa)) \circ \tilde{\nabla}\Big)(x_0 \otimes [x_1|...|x_n]) \tag{2.126}$$

$$= \sum_{i<k<j} \pm (x_i \cdot x_j \cdot x_k)' \otimes [x_{i+1}^{k-1}] \bigotimes (x_i \cdot x_j \cdot x_k)'' \otimes [x_{k+1}^{j-1}] \bigotimes x_0 \otimes [x_1^{i-1}|x_{j+1}^{n}] \tag{2.127}$$

$$- \sum_{i<j<k} \pm (x_i \cdot x_j) \otimes [x_{i+1}^{j-1}] \bigotimes (x_0 \cdot x_k)'' \otimes [x_{k+1}^{n}] \bigotimes (x_0 \cdot x_k)' \otimes [x_1^{i-1}|x_{j+1}^{k-1}] \tag{2.128}$$

$$+ \sum_{k<i<j} \pm (x_0 \cdot x_k)' \otimes [x_1^{k-1}] \bigotimes (x_i \cdot x_j) \otimes [x_{i+1}^{j-1}] \bigotimes (x_0 \cdot x_k)'' \otimes [x_{k+1}^{i-1}|x_{j+1}^{n}]. \tag{2.129}$$

Since g clearly commutes with D_0, we will outline the verification of the above equation for D_1. First, look at:

$$g \circ D_1(x_0 \otimes x) = g(x_0' \otimes [x_0''|x]) \pm g(x_0'' \otimes [x|x_0']) \pm g(x_0 \otimes b_{\Omega V}[x]) \tag{2.130}$$

The first term $g(x_0' \otimes [x_0''|x])$ contains the sum

$$\sum_{j<k<l} \pm (x_0'' \cdot x_j) \otimes [x_1^{j-1}] \bigotimes (x_k \cdot x_l) \otimes [x_{k+1}^{l-1}] \bigotimes x_0' \otimes [x_{j+1}^{k-1}|x_{k+1}^n] \tag{2.131}$$

among other terms, which we call A_1. By Frobeninus compatibility, cocommutativity, and reindexing, the sum above is precisely (136). The second term $g(x_0'' \otimes [x|x_0'])$ contains the sum

$$\sum_{i<j<k} \pm (x_i \cdot x_j) \otimes [x_{i+1}^{j-1}] \bigotimes (x_k \cdot x_0') \otimes [x_{k+1}^n] \bigotimes x_0'' \otimes [x_1^{i-1}|x_{j+1}^{k-1}] \tag{2.132}$$

among other terms, which we call A_2. By Frobenius compatibility, the above sum is precisely (135). Similarly, the third term $g(x_0 \otimes b_{\Omega V}[x])$ contains the sum

$$\sum_{i<j<l} (x_i \cdot x_j') \otimes [x_{i+1}^{j-1}] \bigotimes (x_j'' \cdot x_l) \otimes [x_{j+1}^{l-1}] \bigotimes x_0 \otimes [x_1^{i-1}|x_{l+1}^n] \tag{2.133}$$

among other terms, which we call A_3. By Frobenius compatibility and reindexing, the sum above precisely (134). Finally, one can easily check that $A_1 + A_2 + A_3 = D_1 \circ g(x_0 \otimes x)$. □

Theorem 18 *Let $\phi : C_*(V,V) \to C_*(V,V) \otimes C_*(V,V)$ be defined as the graded cocommutator of $\vee$, i.e. $\phi := \vee - \tau \circ \vee$. Then ϕ passes to coHochchild homology, where it defines the structure of a (shifted) Gerstenhaber coalgebra together with coproduct $\vee_0$.*

Proof. The calculation for the proof has a similar structure to the two above, so we omit it. □

These three Gerstenhaber structures (the Gerstenhaber algebra of Theorem 4 and the Gerstenhaber coalgebras of Theorems 5 and 6) are actually induced by BV-structures, with Connes operator as the BV-operator in all three cases. Moreover, the product of degree $-d$ and the coproduct of degree $1-d$ induce an involutive Lie bialgebra on the cyclic coHochschild homology of a cocommutative DG Frobenius coalgebra. The disucssion of the cyclic theory will appear elsewhere.

Chapter 3

Iterated Integrals: A Bridge Between Geometry and Algebra

3.1 Chen Iterated Integrals of Differential Forms

Let M be a smooth manifold and let $(\mathscr{A}^*(M), d, \wedge)$ denote the commutative differential graded algebra of differential forms on M.

Let $\gamma : [0,1] \to M$ be a piecewise smooth path on M and $\omega \in \mathscr{A}^1(M)$ a 1-form on M. We can write $\gamma^*(\omega) = f(t)dt$ for some real valued piecewise smooth function f on $[0,1]$. The path integral of ω over γ is defined by

$$\int_\gamma \omega = \int_0^1 f(t)dt. \tag{3.1}$$

Thus we can consider $\int \omega$ as a real valued function on the space PM of piecewise smooth paths.

We iterate path integration of 1-forms as follows. Given an ordered set of smooth functions $f_1,...,f_r$ on $[a,b]$ and $r>1$ define their iterated integral inductively by

$$\int_a^b f_1(t)dt...f_r(t)dt := \int_a^b (\int_a^t f_1(s)ds...f_{r-1}(s)ds)f_r(t)dt \tag{3.2}$$

The integral in the right hand side above can also be written as

$$\int_{\Delta^r} f_1(t_1)dt_1...f_r(t_r)dt_r \tag{3.3}$$

where $\Delta^r = \{(t_1,...,t_r)\in\mathbb{R}^r : 0\le t_1\le ...\le t_r\le 1\}$.

Now, given an ordered set of 1-forms $\omega_1,...,\omega_r\in\mathscr{A}^1(M)$ and $\gamma\in PM$ with $\gamma^*(\omega_i)=f_i(t)dt$ we define

$$\int_\gamma \omega_1...\omega_r = \int_0^1 f_1(t)dt...f_r(t)dt \tag{3.4}$$

This process yields a vector space map $\int : T\mathscr{A}^1(M)\to\mathscr{A}^0(PM)$ where $T\mathscr{A}^1(M)$ is the vector space generated by non commutative monomials of 1-forms (the tensor algebra on $\mathscr{A}^1(M)$) and $\mathscr{A}^0(PM)$ is the vector space of continuous functions on PM.

Chen generalized this idea to differential forms of arbitrary degree, i.e. given $\omega_i\in\mathscr{A}^{p_i}(M)$ for $i=1,...,r$ he defined $\int\omega_1...\omega_r\in\mathscr{A}^{p_1+...+p_r-r}(PM)$, with an appropriate notion of differential forms on the path space. This defines a vector space map $\int : T(\mathscr{A}(M))\to\mathscr{A}(PM)$ where $T(\mathscr{A}(M))$ is the tensor algebra on $\mathscr{A}(M)$. Moreover, by computing the coboundary of $\int\omega_1...\omega_r$ Chen defined a differential on $T(\mathscr{A}(M))$ such that $\int$ became a chain map. This differential is exactly the differential of the two sided bar construction of a differential graded algebra. In this section present a self contained exposition of Chen's original construction.

3.1.1 Differential forms on path spaces

We define differential forms on path spaces through Chen's notion of plots. Let M be smooth manifold and let PM the space of piecewise smooth paths on M with the compact open topology.

Definition 19 Let N be a smooth manifold possibly with boundary. A continuous map $\alpha : N \to PM$ is said to be *smooth* if the adjoint map $\alpha_\# : N \times [0,1] \to M$ defined by $\alpha_\#(x,t) = \alpha(x)(t)$ is smooth in the usual sense. We say α is a *plot* if it is a smooth map and its domain N is a closed convex subset of $\mathbb{R}^n$ for some n.

Definition 20 A *differential p-form ω on PM* is a rule which assigns to each plot $\alpha : U \to PM$ a differential p-form $\omega_\alpha \in \mathscr{A}^p(U)$ such that if $\alpha' : U' \to PM$ is another plot and $\phi : U' \to U$ a smooth map such that $\alpha \circ \phi = \alpha'$ we have $\omega_{\alpha'} = \phi^*(\omega_\alpha)$. Denote by $\mathscr{A}^p(PM)$ the vector space of differential p-forms on PM and $\mathscr{A}(PM) = \bigoplus_{i=0}^{\infty} \mathscr{A}^i(PM)$

The graded vector space $\mathscr{A}(PM)$ can be given the structure of a commutative differential graded algebra as follows: if $\omega, \omega' \in \mathscr{A}(PM)$ then $(\omega \wedge \omega')_\alpha = \omega_\alpha \wedge \omega'_\alpha$ and $(d\omega)_\alpha = d\omega_\alpha$. Similarly, define $\mathscr{A}(\Omega M)$ and $\mathscr{A}(LM)$ where ΩM and LM are the based loop space and free loop space, respectively.

3.1.2 The iterated integral of a monomial of differential forms

Given $\omega \in \mathscr{A}^p(M)$ and a plot $\alpha : U \to PM$ where $U \subseteq \mathbb{R}^n$, we may write $\alpha_\#^*(\omega) \in \mathscr{A}^p(U \times [0,1])$ as

$$\alpha_\#^*(\omega) = dt \wedge \omega'(t) + \omega''(t) \tag{3.5}$$

where $\omega'(t) \in \mathscr{A}^{p-1}(U \times [0,1])$ and $\omega''(t) \in \mathscr{A}^{p}(U \times [0,1])$ are of the form

$$\omega'(t) = \sum f_I(x,t)dx^I \tag{3.6}$$

$$\omega''(t) = \sum g_J(x,t)dx^J. \tag{3.7}$$

In the above expressions $\sum f_I(x,t)$ runs through all multi-indices $I = (i_1, ..., i_{p-1})$ and $\sum g_J(x,t)dx^J$ runs through all multi-indices $J = (i_1, ..., i_p)$. Also, $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_{p-1}}$, $dx^J = dx^{i_1} \wedge ... \wedge dx^{i_p}$, and $f_I(x,t) = f_I(x_1, ..., x_n, t)$ and $g_J(x,t) = g_J(x_1, ..., x_n, t)$ are smooth functions on $U \times [0,1]$. Thus $\omega'(t)$ and $\omega''(t)$ are forms in $\mathscr{A}(U \times [0,1])$ not involving dt. We call these $\mathscr{A}(U)$-valued functions on $[0,1]$.

For any $a, b \in [0,1]$ define

$$\int_a^b \omega'(t)dt = \sum (\int_a^b f_I(x,t)dt)dx^I \tag{3.8}$$

where the dt is just the usual calculus notation indicating the variable we are integrating. This process is also known as integrating $\alpha_{\#}^*(w)$ along the fiber of the trivial bundle $U \times [0,1] \to U$.

Definition 21 Let $\omega_j \in \mathscr{A}^{p_j}(M)$ for $j = 1, ..., r$. The *iterated integral* of the monomial $\omega_1 \otimes ... \otimes \omega_r$ is the differential form $\int \omega_1 ... \omega_r \in \mathscr{A}^{p_1 + ... + p_r - r}(PM)$ defined on a plot $\alpha : U \to PM$ by the inductive formula

$$(\int \omega_1 ... \omega_r)_\alpha = \int_0^1 (\int_0^t \omega_1'(s)ds ... \omega_{r-1}'(s)ds) \wedge \omega_r(t)dt \in \mathscr{A}^{p_1 + ... + p_r - r}(M). \tag{3.9}$$

where if $r=1$ we have $(\int \omega_1)_\alpha = \int_0^1 \omega_1'(t)dt$. Unraveling the induction we have

$$(\int \omega_1...\omega_r)_\alpha = \int_0^1 (\int_0^{t_r} ...(\int_0^{t_3} (\int_0^{t_2} \omega_1'(t_1)dt_1) \wedge \omega_2'(t_2)dt_2) \wedge ...) \wedge \omega_r'(t_r)dt_r \tag{3.10}$$

$$= \int_0^1 \int_0^{t_r} ... \int_0^{t_3} \int_0^{t_2} \omega_1'(t_1) \wedge ... \wedge \omega_r'(t_r) dt_1...dt_r \tag{3.11}$$

$$= \int_{\Delta^r} \omega_1'(t_1) \wedge ... \wedge \omega_r'(t_r) dt_1...dt_r \tag{3.12}$$

It is straightforward to verify that if $\alpha' : U' \to PM$ is another plot and $\phi : U' \to U$ a smooth map such that $\alpha \circ \phi = \alpha'$ then $(\int \omega_1...\omega_r)_{\alpha'} = \phi^*(\int \omega_1...\omega_r)_\alpha$.

Equivalently, we can define the iterated integral on a plot $\alpha : U \to PM$ by

$$(\int \omega_1...\omega_r)_\alpha = \int_{\Delta^r} \phi_{\alpha,1}^*(\omega_1) \wedge ... \wedge \phi_{\alpha,r}^*(\omega_r) \tag{3.13}$$

where $\phi_{\alpha,j} = \alpha_\# \circ (id_U \times p_j) : U \times \Delta^r \to U \times [0,1] \to M$, $p_j : \Delta^r \to [0,1]$ is the projection onto the j-th coordinate, and $\int_{\Delta^r} : \mathscr{A}^p(U \times \Delta^r) \to \mathscr{A}^{p-r}(U)$ denotes integration along the fiber of the trivial bundle $U \times \Delta^r \to U$.

3.1.3 Coboundary of an iterated integral

We have defined the iterated integral of a monomial of differential forms on M. Hence, this yields a map $\int : T(\mathscr{A}(M)) \to \mathscr{A}(PM)$ where $T(\mathscr{A}(M)) = \bigoplus_{i=0}^{\infty} \mathscr{A}(M)^{\otimes i}$. We would like to equip $T(\mathscr{A}(M))$ with a differential such that the map $\int$ becomes a chain map, so the first step is to compute the coboundary of $\int$ in $\mathscr{A}(PM)$.

Proposition 18 *Let $\omega_i \in \mathscr{A}^{p_i}(M)$ for $i = 1, ..., r$. We have*

$$d \int \omega_1 ... \omega_r = \sum_{i=1}^{r} (-1)^{\epsilon_{i-1}-i} \int \omega_1 ... d\omega_i ... \omega_r \tag{3.14}$$

$$- \sum_{i=1}^{r-1} (-1)^{\epsilon_i - i} \int \omega_1 ... (\omega_i \wedge \omega_{i+1}) ... \omega_r \tag{3.15}$$

$$-e_0^* \omega_1 \wedge \int \omega_2 ... \omega_r + (-1)^{\epsilon_{r-1}-r+1} \int \omega_1 ... \omega_{r-1} \wedge e_1^* \omega_r \tag{3.16}$$

where $\epsilon_i = p_1 + ... + p_i$ and for any $\omega \in \mathscr{A}^p(M)$ define $e_j^* \omega \in \mathscr{A}^p(PM)$ in a plot $\alpha : U \to PM$ by $(e_j^* \omega)_\alpha = \omega''(j)$ for $j = 0, 1$.

Proof. The above formula essentially follows by applying Stoke's theorem for integration along the fiber. However, here we give a complete proof only using the fundamental theorem of calculus.

The proof is by induction on r. Given $\omega \in \mathscr{A}^p(M)$ and a plot $\alpha : U \to PM$ we use the same notation as above to write $\alpha_\#^*(\omega) = dt \wedge \omega' + \omega''$ with $\omega' = \sum f_I(x,t) dx^I \in \mathscr{A}^{p-1}(U \times [0,1])$ and $\omega'' = \sum g_J(x,t) dx^J \in \mathscr{A}^p(U \times [0,1])$ where $\alpha_\# : U \times [0,1] \to M$ is the adjoint map. Define $d_x \omega'$ and $\partial_t \omega'$ by

$$d_x \omega' = \sum_I (\sum_i \frac{\partial f_I(x,t)}{\partial x_i} dx_i) \wedge dx^I \tag{3.17}$$

$$\partial_t \omega' = \sum_I (\frac{\partial f_I(x,t)}{\partial t}) \wedge dx^I \tag{3.18}$$

Similarly define d_x and ∂_t on any $\mathscr{A}(U)$-valued function on $[0,1]$. We have $d\omega' = d_x\omega' + dt \wedge \partial_t\omega'$ and similarly for $d\omega''$, so

$$\alpha_{\#}^{*}(d\omega) = d\alpha_{\#}^{*}(\omega) = d(dt \wedge \omega') + d\omega'' = -dt \wedge d\omega' + d\omega'' \tag{3.19}$$

$$= -dt \wedge (d_x\omega' + dt \wedge \partial_t\omega') + d_x\omega'' + dt \wedge \partial_t\omega'' \tag{3.20}$$

$$= dt \wedge (-d_x\omega' + \partial_t\omega'') + d_x\omega''. \tag{3.21}$$

Therefore, in the plot α, $(d\omega)' = -d_x\omega' + \partial_t\omega''$. Now, since

$$(d\int \omega)_\alpha = d(\int \omega)_\alpha = \int_0^1 d_x\omega'(t)dt \tag{3.22}$$

we have that

$$(\int d\omega)_\alpha = -\int_0^1 d_x\omega'(t)dt + \int_0^1 \partial_t\omega''(t)dt = -(d\int \omega)_\alpha + \omega''(1) - \omega''(0) \tag{3.23}$$

where we have used the fundamental theorem of calculus in the last equality. This proves our Proposition for $r = 1$, the base case of the induction.

Before we continue with the induction, we introduce some notation. Given a plot $\alpha : U \to PM$ and $0 \le t \le 1$ we define another plot $\alpha^t : U \to PM$ by $\alpha^t(x)(s) = \alpha(x)(ts)$. Thus $\alpha^t_{\#} = (id \times t) \circ \alpha_{\#}$ where $id \times t : U \times [0,1] \to U \times [0,1]$ is defined by $(id \times t)(x,s) = (x, ts)$. By a change of variables argument we have that

$$(\int \omega_1 ... \omega_r)_{\alpha^t} = \int_0^t \omega_1'(s)ds ... \omega_r'(s)ds. \tag{3.24}$$

Let us show this for $r=1$, i.e. $(\int\omega)_{\alpha^t}=\int_0^t\omega'(s)ds$. Writing $\alpha_{\#}^*(\omega)=ds\wedge\omega'(s)+\omega''(s)$ we have

$$(\alpha_{\#}^t)^*(\omega)=(id\times t)^*(ds\wedge\omega'(s)+\omega''(s))=d(ts)\wedge\omega'(ts)+\omega''(ts)=t\,ds\wedge\omega'(ts)+\omega''(ts). \tag{3.25}$$

Hence, by the definition of the iterated integral map and by the change of variables theorem

$$(\int\omega)_{\alpha^t}=\int_0^1\omega'(ts)t\,ds=\int_0^t\omega'(s)ds \tag{3.26}$$

as desired. It is easy to generalize the argument for any r. Therefore, by the inductive definition of the iterated integral map we may write $(\int\omega_1...\omega_r)_\alpha=\int_0^1(\int\omega_1...\omega_{r-1})_{\alpha^t}\wedge\omega_r'(t)dt$.

We now continue with the inductive step: assuming the formula for $k<r$ we compute $d\int\omega_1...\omega_r$. First note that

$$d(\int\omega_1...\omega_r)_\alpha=d(\int_0^1(\int_0^t\omega_1'(s)ds...\omega_{r-1}'(s)ds)\wedge\omega_r'(t)dt) \tag{3.27}$$

$$=\int_0^1 d_x(\int_0^t\omega_1'(s)ds...\omega_{r-1}'(s)ds)\wedge\omega_r(t)dt+(-1)^{\epsilon_{r-1}-r+1}\int_0^1(\int_0^t\omega_1'(s)ds...\omega_{r-1}'(s)ds)\wedge d_x\omega_r'(t)dt. \tag{3.28}$$

Call $A=\int_0^1 d_x(\int_0^t\omega_1'(s)ds...\omega_{r-1}'(s)ds)\wedge\omega_r(t)dt$ and $B=(-1)^{\epsilon_{r-1}-r+1}\int_0^1(\int_0^t\omega_1'(s)ds...\omega_{r-1}'(s)ds)\wedge d_x\omega_r'(t)dt$ so that $d(\int\omega_1...\omega_r)_\alpha=A+B$. We compute A and B separately.

In the following computation we use the induction hypothesis and the fact $(e_k^*\omega_r)_{\alpha^t} = \omega''(kt)$ for $k = 0, 1$:

$$A = \int_0^1 (d\int \omega_1 \ldots \omega_{r-1})_{\alpha^t} \wedge \omega_r'(t)dt \tag{3.29}$$

$$= \int_0^1 \Big(\sum_{i=1}^{r-1}(-1)^{\epsilon_{i-1}-i}(\int \omega_1 \ldots d\omega_i \ldots \omega_{r-1})_{\alpha^t} - \sum_{i=1}^{r-2}(-1)^{\epsilon_i - i}(\int \omega_1 \ldots (\omega_i \wedge \omega_{i+1}) \ldots \omega_{r-1})_{\alpha^t} \tag{3.30}$$

$$-\omega_1''(0) \wedge (\int \omega_2 \ldots \omega_{r-1})_{\alpha^t} + (-1)^{\epsilon_{r-2}-r}(\int \omega_1 \ldots \omega_{r-2})_{\alpha^t} \wedge \omega_{r-1}''(t)\Big) \wedge \omega_r'(t)dt \tag{3.31}$$

$$= \sum_{i=1}^{r-1}(-1)^{\epsilon_{i-1}-i}(\int \omega_1 \ldots d\omega_i \ldots \omega_r)_\alpha - \sum_{i=1}^{r-2}(-1)^{\epsilon_i - i}(\int \omega_1 \ldots (\omega_i \wedge \omega_{i+1}) \ldots \omega_r)_\alpha \tag{3.32}$$

$$-\omega_1''(0) \wedge (\int \omega_2 \ldots \omega_r)_\alpha + (-1)^{\epsilon_{r-2}-r}\int_0^1 (\int \omega_1 \ldots \omega_{r-2})_{\alpha^t} \wedge \omega_{r-1}''(t) \wedge \omega_r'(t)dt. \tag{3.33}$$

Also, since $(d\omega_r)' = -d_x\omega_r' + \partial_t \omega''$

$$B = (-1)^{\epsilon_{r-1}-r+1}\int_0^1 (\int \omega_1 \ldots \omega_{r-1})_{\alpha^t} \wedge d_x\omega_r'(t)dt \tag{3.34}$$

$$= (-1)^{\epsilon_{r-1}-r+1}\int_0^1 (\int \omega_1 \ldots \omega_{r-1})_{\alpha^t} \wedge (-(d\omega_r)' + \partial_t \omega_r'')dt \tag{3.35}$$

$$= (-1)^{\epsilon_{r-1}-r}(\int \omega_1 \ldots \omega_{r-1} d\omega_r)_\alpha \tag{3.36}$$

$$+(-1)^{\epsilon_{r-1}-r+1}\int_0^1 (\int \omega_1 \ldots \omega_{r-1})_{\alpha^t} \wedge \partial_t \omega_r'' dt \tag{3.37}$$

Finally we compute the last term (37) above using integration by parts and the fundamental theorem of calculus:

$$\int_0^1 (\int \omega_1 ... \omega_{r-1})_{\alpha^t} \wedge \partial_t \omega_r''(t) dt \tag{3.38}$$

$$= (\int \omega_1 ... \omega_r)_{\alpha^t} \wedge \omega_r''(t) \Big|_0^1 - \int_0^1 \partial_t (\int \omega_1 ... \omega_{r-1})_{\alpha^t} \wedge \omega_r''(t) dt \tag{3.39}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - \int_0^1 \partial_t (\int_0^t \omega_1'(s) ds ... \omega_{r-1}'(s) ds) \wedge \omega_r''(t) dt \tag{3.40}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - \int_0^1 \partial_t \int_0^t (\int_0^s (\omega_1'(\tau) d\tau ... \omega_{r-2}(\tau) d\tau) \wedge \omega_{r-1}'(s) ds) \wedge \omega_r''(r) dt \tag{3.41}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - \int_0^1 \int_0^t (\omega_1'(\tau) d\tau ... \omega_{r-2}(\tau) d\tau) \wedge \omega_{r-1}'(t) \wedge \omega_r''(t) dt \tag{3.42}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - \int_0^1 (\int \omega_1 ... \omega_{r-2})_{\alpha^t} \wedge \omega_{r-1}'(t) \wedge \omega_r''(t) dt \tag{3.43}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - \int_0^1 (\int \omega_1 ... \omega_{r-2})_{\alpha^t} \wedge (\omega_{r-1} \wedge \omega_r)'(t) dt - \int_0^1 (\int \omega_1 ... \omega_{r-2})_{\alpha^t} \wedge \omega_{r-1}''(t) \wedge \omega_r'(t) dt \tag{3.44}$$

$$= (\int \omega_1 ... \omega_{r-1})_\alpha \wedge \omega_r''(1) - (\int \omega_1 ... \omega_{r-2}(\omega_{r-1} \wedge \omega_r))_\alpha - \int_0^1 (\int \omega_1 ... \omega_{r-2})_{\alpha^t} \wedge \omega_{r-1}''(t) \wedge \omega_r'(t) dt \tag{3.45}$$

From (43) to (44) we have used the fact $(\omega_{r-1} \wedge \omega_r)' = \omega_{r-1}' \wedge \omega_r'' + \omega_{r-1}'' \wedge \omega_r'$. This computations shows that (37)=(45) (with the right signs). Summarizing, we have shown $d(\int \omega_1 ... \omega_r) = A + B = (32) + (33) + (36) + (45)$, which (after canceling two terms) is the desired formula. □

3.2 Iterated integrals, bar constructions, and Hochschild complexes

In the first section we give cohomological models for path spaces on a manifold based on algebraic constructions on the DGA of differential forms on the underlying manifold. In the second section we identify cohomology cup product and the coproduct induced by concatenation of paths with algebraic operations in the bar and Hochschild complex.

3.2.1 Algebraic constructions as models for path spaces

In the coboundary formula for an iterated integral shown in the previous section we recognize the formula for the differential of the two sided bar construction of a differential graded algebra. Let us recall this construction.

Let $(A, d, \cdot)$ be a differential graded associative algebra (DGA) with differential of degree $+1$ over a field $\mathbb{K}$. Moreover, assume A has an augmentation $\mu : A \to \mathbb{K}$ and let $\overline{A} := \mathrm{Ker}\mu$. Denote by sA the shifted DGA so that $(sA)^i = A^{i-1}$. The *two sided bar construction* of A is the graded vector space $B(A, A, A) := A \otimes T(s\overline{A}) \otimes A$ with differential D_P given by

$$D_P(a_0[a_1|...|a_n]a_{n+1}) = \sum_{i=0}^{n+1}(-1)^{\epsilon_{i-1}-i} a_0[a_1|...|da_i|...|a_n]a_{n+1}$$

$$+(-1)^{\epsilon_0}(a_0 \cdot a_1)[a_2|...|a_i|...|a_n]a_{n+1} + \sum_{i=1}^{n-1}(-1)^{\epsilon_i - i} a_0[a_1|...|a_i \cdot a_{i+1}|...|a_n]a_{n+1}$$

$$+(-1)^{\epsilon_n - n} a_0[a_1|...|a_i|...|a_n - 1](a_n \cdot a_{n+1})$$

where $\epsilon_i = \deg(a_0) + ... + \deg(a_i)$.

Let (M,b) be a manifold with base point and $(\mathscr{A}(M)=\mathscr{A},d,\wedge)$ the DGA of differential forms. The augmentation $\mu:\mathscr{A}\to\mathbb{R}$ is given by evaluating a 0-form on the base point b and defined to be 0 on forms of higher degree. Define $\psi_P : B(\mathscr{A},\mathscr{A},\mathscr{A})\to\mathscr{A}(PM)$ to be the map

$$\psi_P(\omega_0[\omega_1|...|\omega_n]\omega_{n+1}) = e_0^*\omega_0\wedge\int\omega_1...\omega_n\wedge e_1^*\omega_{n+1} \tag{3.46}$$

where $e_j : PM\to M$, $e_j(\gamma)=\gamma(j)$ for $j=0,1$. It follows, almost tautologically from the coboundary formula of an iterated integral, that ψ commutes with the differentials.

Proposition 19 $\psi_P : (B(\mathscr{A},\mathscr{A},\mathscr{A}),D_P)\to(\mathscr{A}(PM),d)$ *is a map of differential graded vector spaces.*

Proof. On one hand

$$\begin{gathered}\psi_P D_P(\omega_0[\omega_1|...|\omega_n]\omega_{n+1}) = \\ e_0^*(d\omega_0)\wedge\int\omega_1...\omega_n\wedge e_1^*\omega_{n+1}+\sum_{i=1}^{n}\pm e_0^*\omega_0\wedge\int\omega_1...d\omega_i...\omega_n\wedge e_1^*\omega_{n+1}\pm e_0^*(\omega_0)\wedge\int\omega_1...\omega_n\wedge e_1^*(d\omega_{n+1}) \\ \pm e_0^*(\omega_0\wedge\omega_1)\wedge\int\omega_2...\omega_n\wedge e^*\omega_{n+1}+\sum_{i=1}^{n-1}\pm e_0^*\omega_0\wedge\int\omega_1...(\omega_i\wedge\omega_{i+1})...\omega_n\wedge e_1^*\omega_{n+1} \\ \pm e_0^*\omega_0\int\omega_1...\omega_{n-1}\wedge e_1^*(\omega_n\wedge\omega_{n+1}).\end{gathered}$$

On the other, using that d is a derivation of $\wedge$ and the coboundary formula for an iterated integral, we have

$$
\begin{aligned}
d\psi_P(\omega_0[\omega_1|...|\omega_n]\omega_{n+1}) &= d e_0^*\omega_0 \wedge \int \omega_1...\omega_n \wedge e_1^*\omega_{n+1} \\
&+ e_0^*\omega_0 \wedge \Big(\sum_{i=1}^{n} \pm \int \omega_1...d\omega_i...\omega_n + \sum_{i=1}^{n-1} \pm \int \omega_1...(\omega_i \wedge \omega_{i+1})...\omega_n \\
&- e_0^*\omega_1 \wedge \int \omega_2...\omega_n \pm \int \omega_1...\omega_{n-1} \wedge e_1^*\omega_n\Big) \wedge e_1^*\omega_{n+1} \pm e_0^*\omega_0 \wedge \int \omega_1...\omega_n \wedge d e_1^*\omega_{n+1}.
\end{aligned}
$$

The two computations are clearly equal to each other. □

If we restrict the coboundary formula for an iterated integral to differential forms on the based loop space ΩM the terms $e_0^*\omega_1 \wedge \int \omega_2...\omega_n$ and $\int \omega_1...\omega_{n-1} \wedge e_1^*\omega_n$ vanish, since the evaluation maps e_0 and e_1 are now constant maps and in the resulting formula we recognize the differential of the bar construction of a DGA, which we now recall.

Let A be an augmented DGA as above. The *bar construction* of A is the graded vector space $B(A) := T(s\overline{A})$ with differential D_Ω given by

$$D_\Omega([a_1|...|a_n]) = \sum_{i=1}^{n} (-1)^{\epsilon_{i-1}-i}[a_1|...|da_i|...|a_n] + \sum_{i=1}^{n-1} (-1)^{\epsilon_i - i}[a_1|...|a_i \cdot a_{i+1}|...|a_n]$$

Moreover, $(B(A), D_\Omega)$ is a DG coassociative coalgebra with coproduct given by deconcatenation of monomials. More about this product and its topological significance will be discussed in the next section.

Let (M, b) and $\mathscr{A}$ be as above and let ΩM be the space of loops based at b. Define $\psi_\Omega : B(\mathscr{A}) \to \mathscr{A}(\Omega M)$ to be the map

$$\psi_\Omega([\omega_1|...|\omega_n]) = \int \omega_1...\omega_n. \tag{3.47}$$

The coboundary formula implies the following.

Proposition 20 $\psi_\Omega : (B(\mathscr{A}), D_\Omega) \to (\mathscr{A}(\Omega M), d)$ *is a map of differential graded vector spaces.*

Proof. The proof is similar to the proof of Proposition 2. $\square$

We now consider the free loop space LM. In this case the evaluation maps e_0 and e_1 are equal, so if we restrict the coboundary formula of an iterated integral to differential forms on LM we may write the last term $\int \omega_1 ... \omega_{n-1} \wedge e_1^* \omega_n$ as $\pm e_0^* \omega_n \wedge \int \omega_1 ... \omega_{n-1}$ by using the graded commutativity of $\wedge$. This suggests that the Hochschild chain complex is related to differential forms on the free loop space. We recall the definition of such complex.

Let A be an augmented DGA as above. The *Hochschild chain complex of A with values on A* is the graded vector space $CH(A, A) := A \otimes T(s\overline{A})$ with differential D_L given by

$$D_L(a_0[a_1|...|a_n]) = \sum_{i=0}^{n} (-1)^{\epsilon_{i-1}-i} a_0[a_1|...|da_i|...|a_n] \tag{3.48}$$

$$+(-1)^{\epsilon_0}(a_0 \cdot a_1)[a_2|...|a_i|...|a_n] + \sum_{i=1}^{n-1} (-1)^{\epsilon_i - i}[a_1|...|a_i \cdot a_{i+1}|...|a_n] - (-1)^{\epsilon_n - n}(a_n \cdot a_0)[a_1|...|a_{n-1}]. \tag{3.49}$$

Finally we define a map $\psi_L : CH(\mathscr{A}, \mathscr{A}) \to \mathscr{A}(LM)$ by

$$\psi_L(\omega_0[\omega_1|...|\omega_n]) = e_0^* \omega_0 \wedge \int \omega_1 ... \omega_n. \tag{3.50}$$

As in the above cases, we have the following

Proposition 21 $\psi_L : CH(\mathscr{A}, \mathscr{A}), D_L) \to (\mathscr{A}(LM), d)$ *is a map of differential graded vector spaces.*

Proof. Again, the proof is similar to the proof of Proposition 2. □

Let $C^*_{sm}(PM;\mathbb{R})$ be the vector space of real smooth singular cochains on PM. By Stokes' theorem we have a de Rham cochain map $\rho : \mathscr{A}(PM) \to C^*_{sm}(PM;\mathbb{R})$ given by

$$\rho : \omega \mapsto \Big((\alpha : \Delta^n \to PM) \mapsto \int_{\Delta^n} (\omega)_\alpha \Big) \tag{3.51}$$

where $\omega \in \mathscr{A}(PM)$ and $(\omega)_\alpha$ is the associated differential form on the plot $\alpha : \Delta^n \to PM$. We have a similar map for ΩM and LM by restricting differential forms to these subspaces of PM. Define $\Psi_P = \rho \circ \psi_P$ and Ψ_Ω and Ψ_L similarly.

Chen proved that $\Psi_{\Omega M} : (B(\mathscr{A}), D_\Omega) \to (\mathscr{A}(\Omega M), d) \to C^*_{sm}(\Omega M)$ induces an isomorphism in cohomology if M is simply connected by comparing bar construction on $\mathscr{A}$ with Adams' cobar construction through the de Rham pairing. Adams provided an explicit map

$$\Phi : \mathrm{Cobar}(C_*(M,b))) \to C_*(\Omega M)$$

where Cobar is a functor from coaugmented DG coassociative coalgebras to augmented DG associative algebras dual to the bar construction, and $C_*(M,b)$ the singular chain complex of generated by singular simplices which have all their vertices at the base point b. The map Φ is induced by the map $C_*(M,b) \to C_{*-1}(\Omega M)$ which takes a singular simplex on M with all of its vertices on the base point b, foliates it into paths starting and ending at b, and this family of paths defines a singular simplex of 1 dimension lower in ΩM. Moreover, Φ is a quasi isomorphism if M is simply connected.

Adams' construction works for any space M and in the manifold case it can be adapted to smooth singular chains. The de Rham pairing together with the smoothened Adams' map induce a map $B(\mathscr{A}) \to \mathrm{Hom}(\mathrm{Cobar}(C_*(M), \mathbb{R})$ which respects certain convergent filtrations.

This map induces an isomorphism in the first page of the associated spectral seqeuence. Then a standard argument yields that Ψ_Ω induces an isomorphism in cohomology if M is simply connected. The maps Ψ_P are Ψ_L also induce an isomorphism in cohomology if M is simply connected; this follows from comparing spectral sequences once we know the result for ψ_Ω.

In the following section we will show that the iterated integral map carries a commutative producct (shuffle product) in the bar and Hochschild side corresponds to the wedge product of differential forms. Hence, Chen's results can be interpreted as a de Rham theorem for path spaces on simply connected manifolds: the sub cochain complex of $(\mathscr{A}(\Omega M), d)$ generated by iterated integrals of differential forms computes the singular cohomology of ΩM and the quasi isomorphism is induced by integrating these iterated integrals over smooth chains on ΩM and it induces an algebra isomorphism in cohomology.

We also have the adjoint chain map $\Psi_P^\# : C_*^{sm}(PM) \to \mathrm{Hom}(B(\mathscr{A}, \mathscr{A}, \mathscr{A}), \mathbb{R})$ which on a smooth singular simplex $\alpha : \Delta^n \to PM$ is defined by

$$\Psi_P^\#(\alpha)(\omega_0[\omega_1|\ldots|\omega_r]\omega_{r+1}) = \int_{\Delta^n} (\psi_P(\omega_0[\omega_1|\ldots|\omega_r]\omega_{r+1})_\alpha \tag{3.52}$$

Unraveling the definitions of the iterated integral and of the map ψ_P, the above map can also be written as

$$\Psi_P^\#(\alpha)(\omega_0[\omega_1|\ldots|\omega_r]\omega_{r+1}) = \int_{\Delta^n \times \Delta^r} (e_0 \circ \alpha)^* \omega_0 \wedge \alpha_1^* \omega_1 \wedge \ldots \wedge \alpha_r^* \omega_r \wedge (e_0 \circ \alpha)^* \omega_{r+1} \tag{3.53}$$

where $\alpha_j : \Delta^n \times \Delta^r \to M$ is defined by $\alpha_j(x, t_1, \ldots, t_r) = \alpha(x)(t_j)$. The maps $\Psi_\Omega^\# : C_*^{sm}(\Omega M) \to \mathrm{Hom}(B(\mathscr{A}), \mathbb{R})$ and $\Psi_L^\# : C_*^{sm}(LM) \to \mathrm{Hom}(CH(\mathscr{A}, \mathscr{A}), \mathbb{R})$ are defined analogously. Notice that we do not need to make sense of a notion of differential forms on path spaces in order to define the maps Ψ and $\Psi^\#$.

3.2.2 Basic operations: wedge product, concatenation of paths, rotation of loops

Recall that the cup product $\smile: H^*(\Omega M;\mathbb{R})\otimes H^*(\Omega M;\mathbb{R}) \to H^*(\Omega M;\mathbb{R})$ together with the coproduct $c^*: H^*(\Omega M;\mathbb{R}) \to H^*(\Omega M;\mathbb{R})\otimes H^*(\Omega M;\mathbb{R})$ induced by the concatenation map $c: \Omega M \times \Omega M \to \Omega M$ give $H^*(\Omega M;\mathbb{R})$ the structure of a commutative Hopf algebra, namely $(H^*(\Omega M;\mathbb{R}), \smile, c^*)$ is a unital commutative algebra and counital coalgebra such that the coproduct c^* is an algebra map. Moreover, there is an antipode map induced by $\gamma \mapsto \gamma^{-1}$. We model this structure on the bar construction $(B(\mathscr{A}(M)), D_\Omega)$.

Given a DGA $(A, d, \cdot)$, the underlying vector space of the bar construction $B(A) = Ts\overline{A}$ has a coproduct Δ making it a free coassociative coalgebra. Δ is called the *deconcatenation coproduct* and it is given by

$$\Delta([a_1|...|a_n] = \sum_{i=0}^{n} [a_1|...|a_i] \otimes [a_{i+1}|...|a_n]. \tag{3.54}$$

A straightforward calculation reveals that D_Ω is a coderivation of Δ, so $B(A)$ has the structure of a DG coassociative counital coalgebra. The counit $\epsilon: B(A) \to \mathbb{K}$ is given by the projection $B(A) \to s\overline{A}^{\otimes 0} = \mathbb{K}$.

If we assume the product in A is graded commutative we have that $(B(A), D_\Omega)$ has a unital CDGA algebra structure with the *shuffle product* $sh: B(A)\otimes B(A) \to B(A)$ defined by

$$sh([a_1|...|a_r]\otimes[a_{r+1}|...|a_{r+s}]) = \sum_{\sigma\in \mathrm{Sh}(r,s)} (-1)^{\epsilon_\sigma}[a_{\sigma(1)}|...|a_{\sigma(r+s)}] \tag{3.55}$$

where $\mathrm{Sh}(r,s)$ is the set of (r,s) shuffles, i.e. permutations $\sigma \in S_{r+s}$ such that $\sigma^{-1}(1) < ... < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < ... < \sigma^{-1}(r+s)$, and $(-1)^{\epsilon_\sigma}$ is determined by the sign of the permutation taking into account the grading of the a_i. The unit is given by $\mathbb{K} = s\overline{A}^{\otimes 0} \hookrightarrow B(A)$. The proof that D_Ω is a derivation of the shuffle product is a calculation using the graded commutativity of the product in A. Moreover, $\Delta : B(A) \to B(A) \otimes B(A)$ is an algebra map with respect to the shuffle product. $B(A)$ also has an antipode map $[a_1|...|a_n] \mapsto (-1)^n[a_n|...|a_1]$. Hence, if A is a CDGA then $(B(A), D_\Omega, \Delta, sh)$ is a DG commutative Hopf algebra. When A is the CDGA of differential forms on a simply connected manifold, $(B(A), D_\Omega, \Delta, sh)$ is a cochain level model for the commutative Hopf algebra $H^*(\Omega M;\mathbb{R})$ as the next theorem shows.

Theorem 19 *Let M be a manifold. The cochain map $\Psi_\Omega : B(\mathscr{A}(M)) \to C^*_{sm}(\Omega M;\mathbb{R})$ induces a map of Hopf algebras in cohomology. If M is simply conencted, Ψ induces an isomorphism of Hopf algebras.*

Proof. Under the iterated integral map, the shuffle product in $B(\mathscr{A})$ corresponds to the wedge product of differential forms in ΩM. More precisely, we have the following identity

$$\sum_{\sigma \in \mathrm{Sh}(r,s)} (-1)^{\epsilon_\sigma} \int \omega_{\sigma(1)}...\omega_{\sigma(r+s)} = \int \omega_1...\omega_r \wedge \int \omega_{r+1}...\omega_{r+s}. \tag{3.56}$$

The iterated integral of the left hand side of the above equation is given by a sum of integrals over the simplex Δ^{r+s}, while the right hand side is given by a single integral over $\Delta^r \times \Delta^s$. We can decompose $\Delta^r \times \Delta^s$ into a union $r+s$ simplices each corresponding to an (r,s)-shuffle of the coordinates $t_1,...,t_r$ and $t_{r+1},...,t_{r+s}$ of $\Delta^r \times \Delta^s$. An integral in the sum in left hand side corresponding to a shuffle σ is equal to restricting the integral on the right hand side to the embedded $r+s$ simplex in $\Delta^r \times \Delta^s$ given by the shuffle σ. This identity shows that the shuffle product corresponds to the cup product in cohomology.

Now we show that under $\Psi_\Omega : B(\mathscr{A}) \to C^*_{sm}(\Omega M;\mathbb{R})$, the deconcatenation coproduct corresponds to the map induced by concatenation of loops. More precisely, we have the following identity

$$\int_{\alpha*\beta}\int \omega_1...\omega_r = \sum_{i=0}^{r}\int_{\alpha}\int \omega_1...\omega_i \int_{\beta}\int \omega_{i+1}...\omega_r \tag{3.57}$$

for any smooth singular simplices $\alpha : \Delta^m \to \Omega M$ and $\beta : \Delta^n \to \Omega M$ where $\alpha*\beta : \Delta^m \times \Delta^n \to \Omega M$ takes a point $(x, y) \in \Delta^m \times \Delta^n$ to the concatenation of based loops $\alpha(x) * \beta(y)$. The left hand side of the equation can be computed as a single integral over $\Delta^m \times \Delta^n \times \Delta^r$, while the right hand side as a sum of $r+1$ integrals each over $(\Delta^m \times \Delta^i) \times (\Delta^n \times \Delta^{r-i})$ for $i = 0, ..., r$. We can decompose Δ^r into a union $\bigcup_{i=0}^{r} K_i$ where each K_i is a product $\Delta^i \times \Delta^{r-i}$ of scaled down simplices, i.e. $K_i = \{(t_1, ..., t_r) \in \Delta^r : 0 \leq t_1 \leq ... \leq t_i \leq 1/2 \leq t_{i+1} \leq ... \leq t_r\}$. The i-th term in the sum of the right hand side corresponds to integrating over K_i in the left hand side. This follows since concatenation of two loops is defined so that from time 0 to 1/2 we run around the first loop and from time 1/2 to 1 around the second. □

Remark 15 The cochain complexes $B(\mathscr{A}, \mathscr{A}, \mathscr{A})$ and $CH(\mathscr{A}, \mathscr{A})$ are also CDGA's. The product in $B(\mathscr{A}, \mathscr{A}, \mathscr{A})$ is given by $\wedge \otimes sh \otimes \wedge$ and in $CH(\mathscr{A}, \mathscr{A})$ by $\wedge \otimes sh$. The maps $\Psi_P : B(\mathscr{A}, \mathscr{A}, \mathscr{A}) \to C^*_{sm}(PM; \mathbb{R})$ and $\Psi_L : CH(\mathscr{A}, \mathscr{A}) \to C^*_{sm}(LM; \mathbb{R})$ also induce maps of commutative algebras in cohomology by the same argument of the above proof.

We have an S^1-action $\theta : S^1 \times LM \to LM$ given by rotation of loops $\theta(s,\gamma)(t) = \gamma(t+s)$, where $t+s$ is taken mod 1 of course. The action θ induces a chain map which we denote by $J_\# : C^{sm}_*(LM) \to C^{sm}_{*+1}(LM)$ by crossing with the fundamental class of S^1 and applying the map induced by θ, i.e. $J_\#(\sigma) = \theta_\theta(\sigma \times [S^1])$. At the level of cochains, we have a map $J^\# : C^*_{sm}(LM) \to C^{*-1}_{sm}(LM)$. This is called the rotation operator.

On the other hand, on the Hochschild chain complex $CH(A, A)$ of a unital DGA A there is a operator $B : CH(A,A) \to CH(A,A)$ of degree -1, known as *Connes' operator*, defined by

$$B(a_0[a_1|...|a_n]) = \sum_{i=0}^{n} (-1)^{(\epsilon_{i-1}+1)(\epsilon_n - \epsilon_{i-1})} 1 \otimes [a_i|...|a_n|a_0|...|a_{i-1}]. \tag{3.58}$$

B (anti)commutes with the Hochschild differential D_L. Moreover, B corresponds to the rotation operator as the following proposition shows.

Proposition 22 *Let M be a manifold. The map $\Psi_L : CH(\mathscr{A}(M), \mathscr{A}(M)) \to C^*_{sm}(LM)$ satisfies $J^\# \circ \Psi_L = \Psi_L \circ B$.*

Let $\alpha : \Delta^n \to LM$ be a smooth singular simplex on LM. First note

$$J^\# \circ \Psi_L(\omega_0[\omega_1|...|\omega_r])(\alpha) = \Psi_L(\omega_0[\omega_1|...|\omega_r])(J_\#\alpha) = \int_{\Delta^n \times [0,1] \times \Delta^r} (e_0 \circ J_\#\alpha)^* \omega_0 \wedge (J_\#\alpha)^*_1 \omega_1 \wedge ... \wedge (J_\#\alpha)^*_r \omega_r \tag{3.59}$$

$$= \int_{[0,1]\times\Delta^r} \omega'_0(s) \wedge \omega'_1(t_1 + s) \wedge ... \wedge \omega'_r(t_r + s) ds\, dt_1 ... dt_r \tag{3.60}$$

where $(J_\#\alpha)_j : \Delta^n \times [0,1] \to M$ is the map $(t_1, ..., t_n, s) \mapsto \alpha(t_j + s)$ and in the last equality we have just used the notation introduced at the beginning of the chapter when defining an iterated integral, i.e. ω'_j is obtained by pulling back ω_j by along the map $\Delta^n \times [0,1] \to M$ given by $(x,t) \mapsto \alpha(x)(t)$ and then taking the interior product of this pullback with $\partial/\partial t$.

We can decompose $[0,1] \times \Delta^r$ into a union $\bigcup_{i=0}^r R_i$ where each R_i is the $(k+1)$-simplex given by $R_i = \{(s, t_1, ..., t_r) \in [0,1] \times \Delta^r : 0 \le s + t_i \le ... \le s + t_r \le s \le s + t_1 \le ... \le s + t_{i-1} \le 1\}$, taking

$s+t_i$ modulo 1. Under a change of variables, we obtain that

$$\int_{R_i} \omega_0'(s)\wedge\omega_1'(t_1+s)\wedge\ldots\wedge\omega_r'(t_r+s)ds\,dt_1\ldots dt_r$$
$$=\pm\int_{\Delta^{r+1}} 1\wedge\omega_i'(\tau_1)\wedge\ldots\wedge\omega_r'(\tau_{r+1-i})\wedge\omega_0'(\tau_{r+2-i})\wedge\ldots\wedge\omega_{i-1}'(\tau_{r+1})d\tau_1 d\tau_2\ldots d\tau_{r+1}$$

This last integral is the i-th term in the sum $\Psi_L\circ B(\omega_0[\omega_1|\ldots|\omega_r])$. Hence, the proposition follows since each term in $\Psi_L\circ B(\omega_0[\omega_1|\ldots|\omega_r])$ corresponds to the integral $J^\#\circ\Psi_L(\omega_0[\omega_1|\ldots|\omega_r])$ over R_i. $\Box$

3.3 Polynomial differential forms, currents, and string topology operations

We have described in Chapter 1 some of the rich algebraic structure of the homology of the free loop space of a manifold given by string topology constructions. We have also seen in Chapter 2 that the structure of the coHochschild homology of a CDG Frobenius coalgebra is analogue to string topology. In this current chapter we have seen how certain algebraic constructions, when applied to differential forms, are related to the free path space, the based loop space, and the free loop space. However, we have not made use yet of the Poincaré duality of the underlying manifold.

The main difficulty is that we do not have a strict open Frobenius algebra structure. In this section, we go around this difficulty by considering the CDGA of rational polynomial differential forms (on a fixed triangulation of the underlying manifold) and its algebraic dual, which we call polynomial currents. Under Poincaré duality, the "cap product" of a differential form and a current models the intersection product. We think of this cap product as a partially defined intersection product at the level of currents. Xiaojun Chen, in his Stony Brook Phd thesis, proposed this setting as a convenient one to model string topology operations.

We will describe some of the algebraic operations in the coHochschild complex of a CDG Frobenius coalgebra in the setting of polynomial differential forms and currents. We recover both the Chas -Sullivan loop product and the loop coproduct of degree $1-d$. We also provide explicit maps, using iterated integrals and a choice of Thom form supported in a neighborhood of the diagonal in $M \times M$, to relate the algebraic constructions that use the cap product and the geometric intersection type constructions of string topology. However, we should mention that this approach is somewhat limited since we are not able to recover the full structure of chain level string topology. In order to recover the full structure it seems that one must use homotopy transfer methods from homotopical algebra. More about this is discussed in the next section on conjectures and future work.

3.3.1 Rational polynomial differential forms and currents

Throughout this section M will be a smooth closed oriented manifold of dimension d. By a celebrated theorem of Whitehead every smooth manifold admits a triangulation. We fix a triangulation of M and an ordering of the vertices of the triangulation. Since we are assuming M is a closed manifold the triangulation contains a finite number of simplices.

Definition 22 Let K be a simplicial complex. A *polynomial p-form ω* on K is a family $\{\omega_\sigma\}$ running through all the simplices σ of K such that
(i) if σ is a k-simplex of K then ω_σ is a differential p-form on Δ^k such that on the barycentric

coordinates $(t_1, ..., t_k)$ of Δ^k we have that

$$\omega_\sigma = \sum \phi_{i_1 \dots i_p} d t_{i_1} \wedge \dots \wedge d t_{i_p} \tag{3.61}$$

where $\phi_{i_1 \dots i_p}$ is a polynomial on $t_1, ..., t_k$ with $\mathbb{Q}$-coefficients, and

(ii) if τ is another simplex in K such that we have a face inclusion $i : \tau \to \sigma$, then $i^*\omega_\sigma = \omega_\tau$.

Denote by $A(K)$ to be the collection of polynomial forms on K. This has an induced CDGA structure by defining d and $\wedge$ simplexwise.

The following de Rham theorem for polynomial forms was shown by Sullivan.

Theorem 20 *Let $C^*(K;\mathbb{Q})$ denote the simplicial cochain complex of K. The map $\rho : A(K) \to C^*(K;\mathbb{Q})$ defined by*

$$\rho(\omega)(\alpha) = \int_\alpha \omega \tag{3.62}$$

is a cochain map inducing an isomorphism of algebras in cohomology.

We consider the CDGA of rational polynomial forms $A(M)$ on a fixed (ordered) triangulation of a smooth closed oriented manifold M. Define, the chain complex of *polynomial currents* by $C(M) := \mathrm{Hom}(A(M), \mathbb{Q})$ with the differential ∂_C dual to the differential d of $A(M)$. We use homological grading, so that both $A(M)$ and $C(M)$ have differentials of degree -1. We list several useful properties about $A(M)$ and $C(M)$.

1) $A(M)$ has a canonical countable ordered basis $\{e_i\}_{i=1}^\infty$.

2) Let $A^j(M) \subset A(M)$ be the subspace spanned by polynomial forms ω such that $\deg(\omega)$ +

(degree of polynomial coefficient of ω) is at most j. Since M is closed we have a finite number of simplices, this implies that each $A^j(M)$ is a finite dimensinal dimensional vector space. Thus $A(M)$ has a canonical countable ascending filtration $A^0(M) \subset A^1(M) \subset \ldots$ by finite dimensional vector spaces. Hence, we may write $A(M) = \varinjlim_j A^j(M)$.

3) Let $C^j(M) = \mathrm{Hom}(A^j(M), \mathbb{Q})$, then by the properties of limits we have $C(M) = \varprojlim_j C^j(M)$. Thus $C(M)$ is an inverse limit of a canonical countable family of finite dimensional vector spaces.

4) $C(M)$ is a differential graded cocommutative *completed* coalgebra, namely there is a map

$$\Delta : C(M) \to C(M) \hat{\otimes} C(M) = \varprojlim_{i,j} C^i(M) \otimes C^j(M) \tag{3.63}$$

compatible with the differential (the dual of the exterior derivative) and the (completed) switching map. The completed coproduct Δ is defined by dualizing the wedge product, i.e.

$$\Delta : C(M) = \mathrm{Hom}(A(M), \mathbb{Q}) \xrightarrow{\text{dual of} \wedge} \mathrm{Hom}(A(M) \otimes A(M), \mathbb{Q}) = \mathrm{Hom}(\varinjlim_i A^i(M) \otimes \varinjlim_j A^j(M), \mathbb{Q}) \tag{3.64}$$

$$\cong \mathrm{Hom}(\varinjlim_{i,j} A^i(M) \otimes A^j(M); \mathbb{Q}) \cong \varprojlim_{i,j} \mathrm{Hom}(A^i(M) \otimes A^j(M), \mathbb{Q}) \tag{3.65}$$

$$\cong \varprojlim_{i,j} \mathrm{Hom}(A^i(M); \mathbb{Q}) \otimes \mathrm{Hom}(A^j(M); \mathbb{Q}) = \varprojlim_{i,j} C^i(M) \otimes C^j(M) = C(M) \hat{\otimes} C(M). \tag{3.66}$$

Moreover, the dual of the unit and augmentation of $A(M)$ give $C(M)$ a counit and a coaugmentation.

5) Since $A(M)$ has a countable basis, we may write elements in $C(M) \otimes C(M)$ as a formal in-

finite sum. Thus, we use Sweedler's notation to write $\Delta(x) = \sum_{(x)} x'\hat{\otimes}x'' = x'\hat{\otimes}x''$, where this expression should be interpreted as a formal infinite sums.

6) $C(M)$ is a DG $A(M)$-bimodule defined by $(\eta \cdot g \cdot \eta')(\omega) = g(\eta \wedge \omega \wedge \eta')$ for $g \in C(M)$ and $\eta, \eta', \omega \in A(M)$. We call this action the *cap product of a form and a current.*

7) The injective map $\rho : A(M) \to C(M)$ of degree $-d$ defined by

$$\rho(\eta)(\omega) = \int_M \eta \wedge \omega \tag{3.67}$$

is a quasi isomorphism of chain complexes. This is essentially Poincaré duality for the manifold M. Note that the injectivity of ρ follows since the differential forms in the domain have polynomial coefficients. Thus we can consider a polynomial form as a current.

8) The cap product and the completed coproduct on $C(M)$ satisfy a formal Frobenius compatibility via ρ. We make this precise in the following proposition.

Proposition 23 *Let $\eta, \omega \in A(M)$ then*

$$\Delta(\rho(\eta \wedge \omega)) = \sum_{(\rho(\eta)} \rho(\eta)'\hat{\otimes}(\rho(\eta)'' \cdot \omega) = (-1)^{deg(\eta)d} \sum_{\rho(\omega)} (\eta \cdot \rho(\omega)')\hat{\otimes}\rho(\omega)''. \tag{3.68}$$

Proof. By definition we have $\Delta(\rho(\eta \wedge \omega))(x \otimes y) = \int_M \eta \wedge \omega \wedge x \wedge y$ for any $x, y \in A(M)$. Note

$$\Big(\sum_{\rho(\eta)} \rho(\eta)'\hat{\otimes}(\rho(\eta)'' \cdot \omega)\Big)(x \otimes y) = \sum_{\rho(\eta)} \rho(\eta)'(x)(\rho(\eta)'' \cdot \omega)(y) = \sum_{\rho(\eta)} \rho(\eta)'(x)\rho(\eta)''(y \wedge \omega) \tag{3.69}$$

$$= \Big(\sum_{\rho(\eta)} \rho(\eta)'\hat{\otimes}\rho(\eta)''\Big)(x \otimes (y \wedge \omega)) = \Delta(\rho(\eta))(x \otimes (y \wedge \omega)) = \int_M \eta \wedge x \wedge y \wedge \omega. \tag{3.70}$$

The second equality of (68) is obtained similarly. $\square$

9) Since the cohomology of $A(M)$ is isomorphic to the singular cohomology of M, by the universal coefficient theorem, it follows that $C(M)$ is a chain model for M. Moreover, the cap product of a form and a current is a model for the intersection product; i.e the cap product is Poincaré dual to the wedge product in the sense of the following commutative diagram

$$\begin{tikzcd} A\otimes C \arrow[r, "\mathrm{cap}"] & C \\ A\otimes A \arrow[u, "\mathrm{id}_A\times\rho"] \arrow[r, "\wedge"'] & A \arrow[u, "\rho"'] \end{tikzcd} \tag{3.71}$$

10) The maps ρ and Δ define on $A(M)$ the structure of a completed $DG\ C(M)$-bicomodule as the following proposition states precisely.

Proposition 24 *Denote by $\{e_i\}_{i=0}^{\infty}$ the canonical basis of $A(M)$. We have the following formal expression for*
$A(M)\xrightarrow{\rho} C(M)\xrightarrow{\Delta} C(M)\hat{\otimes}C(M)$. For any $\omega\in A(M)$,

$$\Delta(\rho(\omega))=\sum_{i=0}^{\infty}\rho(\omega\wedge e_i)\otimes e_i^* \tag{3.72}$$

where e_i^* denotes the dual element to e_i.

Proof. By definition $\Delta(\rho(\omega))(x\otimes y)=\rho(\omega)(x\wedge y)=\int_M \omega\wedge x\wedge y$ for any $x, y\in A(M)$. On the other hand,

$$\Big(\sum_{i=0}^{\infty}\rho(\omega\wedge e_i)\otimes e_i^*\Big)(x\otimes y)=\sum_{i=0}^{\infty}\rho(\omega\wedge e_i)(x)e_i^*(y)=\sum_{i=0}^{\infty}\Big(\int_M \omega\wedge e_i\wedge x\Big)e_i^*(y)=\int_M \omega\wedge x\wedge y.\ \square \tag{3.73}$$

The above proposition can be interpreted as saying that $A(M) \xrightarrow{\rho} C(M) \xrightarrow{\Delta} C(M)\hat{\otimes}C(M)$ factors through

$A(M)\hat{\otimes}C(M) \xrightarrow{\rho\times \mathrm{id}_C} C(M)\hat{\otimes}C(M)$, defining a DG right $C(M)$-comodule structure on $A(M)$. A left $C(M)$-comodule structure is defined similarly.

3.3.2 Completed coHochschild complexes as models for the free loop space

We apply some of the algebraic constructions described on Chapter 2 to the completed DG coalgebra $(C(M), \partial_C, \Delta)$.

Consider the completed cobar construction $(\widehat{\Omega}C(M), \partial_\Omega)$. The underlying vector space is $\widehat{T}s^{-1}\overline{C}(M) = \prod_{i=0}^{\infty} C(M)^{\hat{\otimes} i}$ the completed tensor algebra on the shifted vector space $s^{-1}\overline{C}(M)$ (where $\overline{C}(M)$ is the coaugmentation cokernel). The differential ∂_C is defined by extending the usual cobar differential to the case of completed coalgebras. We also have a cocommutative completed coproduct $\widehat{\Delta}_{sh} : \widehat{\Omega}C(M) \to \widehat{\Omega}C(M)\hat{\otimes}\widehat{\Omega}C(M)$ defined by extending the unshuffle coproduct to the completion. These operations make $\widehat{\Omega}C(M)$ into a completed DG cocommutative Hopf algebra. Moreover, its homology is a cocommutative Hopf algebra but not in the complete sense.

The cobar construction $(\widehat{\Omega}(C(M), \partial_\Omega, \widehat{\Delta}_{sh})$ is a homological model for the based loop space ΩM. This follows from Hain's adaptation of the iterated integral map to the case of polynomial forms and then taking the adjoint of such map. In other words, there is a chain map

$$\Phi_\Omega : (C_*(\Omega M;\mathbb{Q}), \partial) \to (\widehat{\Omega}C(M), \partial_\Omega) = (\mathrm{Hom}(B(A(M)), \mathbb{Q}), D^*_\Omega) \tag{3.74}$$

inducing an isomorphism of Hopf algebras in homology if M is simply connected. Φ is the adjoint of the iterated integral map, as discussed earlier.

Similarly, there is a completed coHochschild complex $(\widehat{C_*}(C(M), C(M)), \partial_L)$ as defined in chapter 2. The underlying vector space is $C_*(C(M), C(M)) = \prod_{i=0}^{\infty} C(M)\hat{\otimes} C(M)^{\hat{\otimes} i} = C(M)\hat{\otimes}\widehat{\Omega}C(M)$. This is also a DG cocommutative completed coalgebra with coproduct given by the completed tensor product of the unshuffle coproduct with the coproduct of the underlying coalgebra. The differential ∂_L is defined by extending the coHochschild differential to the case of completed coalgebras. There is a chain map

$$\Phi_L : (C_*(LM;\mathbb{Q}), \partial) \to (\widehat{C_*}(C(M), C(M)), \partial_L) = (\mathrm{Hom}(CH(A(M), A(M)), \mathbb{Q}), D_{\Omega}^*) \tag{3.75}$$

inducing an isomorphism of coalgebras in homology if M is simply connected. For any $(\alpha : \Delta^n \to LM) \in C_n(LM;\mathbb{Q})$ the map Φ_L is given by the formula

$$\Phi_L(\alpha)(\omega_0[\omega_1|...|\omega_t]) = \int_{\Delta^n \times \Delta^r} (e_0 \circ \alpha)^* \omega_0 \wedge \alpha_1^* \omega_1 \wedge ... \wedge \alpha_t^* \omega_t \tag{3.76}$$

where $\alpha_j : \Delta^n \times \Delta^r \to M$ is defined as $\alpha_j(x, t_1, ..., t_r) = \alpha(x)(t_j)$. Note that we have assumed that the adjoint map $\alpha_\# : \Delta^n \times [0,1] \to M$ is a smooth simplex in the triangulation of M.

We now describe a new model for the free loop space. Consider the completed tensor product $A(M)\hat{\otimes}\widehat{\Omega}C(M)$. Here we think of $A(M)$ as currents through the injective quasi isomorphism $\rho : A(M) \to C(M)$. Define a differential ∂ on $A(M)\hat{\otimes}\widehat{\Omega}C(M)$ using the $C(M)$-comodule structure of $A(M)$. More precisely, let $\partial = d_A \otimes 1 + 1 \otimes \partial_\Omega + b$ where

$$b(\omega[x_1|...|x_n]) = \sum (\omega \wedge e_i)[e_i^*|x_1|...|x_n] - (-1)^{(\deg(e_i)-1)\deg[x_1|...|x_n]}(\omega \wedge e_i)[x_1|...|x_n|e_i^*] \tag{3.77}$$

where the above sums is a formal infinite sum in $A(M)\hat{\otimes}\widehat{\Omega}C(M)$.It is a straightforward check that $\partial^2 = 0$. Denote the resulting complex by $(\widehat{C_*}(A(M), C(M)), \partial)$. In fact, this is the *completed coHochschild complex of* $C(M)$ *with coefficients on* $A(M)$. There is a quasi-isomorphism of chain complexes between $\widehat{C}_*(A(M), C(M))$ and $\widehat{C}_*(C(M), C(M))$, given by

$$\rho\hat{\otimes}\mathrm{id}_A : A(M)\hat{\otimes}\widehat{\Omega}C(M) \to C(M)\hat{\otimes}\widehat{\Omega}C(M). \tag{3.78}$$

The above quasi-isomorphism is of degree $-d$, so we may shift the grading of $\widehat{\Omega}C(M) \to C(M)\hat{\otimes}\widehat{\Omega}C(M)$ up by d so that $\rho\hat{\otimes}\mathrm{id}_A$ is of degree 0. Thus, we may think of $(\widehat{C}_*(A(M), C(M)), \partial) = A(M)\hat{\otimes}\widehat{\Omega}C(M)$ as a chain model for $\mathbb{H}_*(LM) = H_*(LM)[d]$.

We may think of $A(M)\hat{\otimes}\widehat{\Omega}C(M)$ (completed) twisted tensor product in the sense of E.Brown, where $A(M)$ is a chain model for the base space M, $\widehat{\Omega}C(M)$ is a chain model for the based loop space ΩM, and the tensor differential is "twisted" by b.

Notice that we have an isomorphism of vector spaces $A(M)\hat{\otimes}\widehat{\Omega}C(M) \cong \mathrm{Hom}(BA(M), A(M))$. The latter is the undelying vector space of the Hochschild *cochain* complex of $A(M)$ with values on $A(M)$. We adopt the viewpoint of completed tensor products for several reasons. Firstly, because of its resemblance with Brown's twisted tensor product models for fibrations and Adams' cobar construction for the based loop space. These chain models can be related to singular chains on the total space of the fibration in consideration through explicit geometric maps. Later on, we would like to generalize this model for other fibrations, in particular, to model pullbacks algebraically. Secondly, the algebraic string operations in the case of finite dimensional DG Frobenius algebras are defined in the context of coHochschild complexes. This can be seen in Zeinalian and Tradler's finite dimensional package of algebraic operations on Hochschild cochains, where first an isomorphism to the coHochschild complex is used, then they define the operations there, and finally apply the inverse isomorphism to get back to Hochschild cochains. Also not all of their operations are well defined in the context of differential forms. Lastly, the analogy with string topology operations defined in transversal chains, as in Chas and Sullivan's original construction, is very explicit with intersections being modeled by the cap product of a form and a current.

3.3.3 String topology operations in the context of differential forms and currents

We now explain how to obtain some of the algebraic operations described in chapter 2 in the context of polynomial differential forms and currents, and we describe explicit maps relating the algebraic constructions to chains on the free loop space. The key is the construction of a lift of the iterated integral map via Poincaré duality using a Thom form. The definition of such map is outline at the end of the section.

For ease of notation we fix our triangulated manifold M and let $A = A(M)$ and $C = C(M)$.

First we recall the model for the Chas-Sullivan loop product as defined originally by X.Chen. Compare the following definition with Definition 3 of Chapter 2.

Definition 23 Define a product

$$\bullet : A\hat{\otimes}\widehat{\Omega}C \bigotimes A\hat{\otimes}\widehat{\Omega}C \to A\hat{\otimes}\widehat{\Omega}C \tag{3.79}$$

by

$$(\omega[x_1|...|x_m])\bullet(\eta[y_1|...|y_n]) := (-1)^{\deg(\eta)\deg[x_1|...|x_m]}(\omega\wedge\eta)[x_1|...|x_m|y_1|...|y_n]. \tag{3.80}$$

Proposition 25 *$(\widehat{C}_*(A,C), \partial, \bullet)$ is a differential graded associative algebra. Moreover, $\bullet$ is homotopy commutatative, so $H_*(A,C)$ is a commutative associative algebra.*

Proposition 8 is a generalization of Proposition 4 of Chapter 2 to our context. Its proof is a formal analogue substituting the Frobenius compatibility by the formal compatibility of Proposition 6 above.

The product $\bullet$ is defined by applying the wedge product on A the model for the base space M and concatenation product on $\widehat{\Omega}C$. The wedge product is Poincaré dual to intersection product in homology and the concatenation product is a model for the concatenation of loops. Thus we have avoided using a commutative intersection product on the currents C Frobenius compatible with the completed coproduct (which we do not have, nor a cocommutative coproduct on A Frobenius compatible with the wedge product) by using a Poincaré dual model for the base points.

Furthermore, the homology $H_*(A,C)$ has the structure of a Gerstenhaber algebra. The bracket is defined in an analogue manner to the case of DG cocommutative Frobenius coalgebras of Chapter 2: as the commutator of the chain homotopy for the commutativity of $\bullet$. This can be thought of as Gerstenhaber's original construction but now in the context of (formally) Frobenius algebras. In the next section we explain how to directly relate the structures of $H_*(A,C)$ to the free loop space, in fact $\bullet$ corresponds to the Chas- Sullivan loop product.

We now turn our attention to model the string topology coproduct of degree $1-d$. We adapt the the algebraic coproduct of degree $1-d$ in the coHochschild complex of a DG cocommutative Frobenius coalgebra described in Chapter 2 to our context. To define $\bullet$ above we used the wedge product to model intersections, now we use the cap product.

We use a prime symbol to denote $\widehat{\Omega}'C := \prod_{i=1}^{\infty} s^{-1}\overline{C}^{\hat{\otimes} i}$ (note the direct product starts with $i=1$).

Define $\widehat{C_*'}(C,C) := C\hat{\otimes}\widehat{\Omega}'C$. This is a chain complex with the coHochschild differential and we call it the *reduced completed coHochschild complex of* C. Similarly define $\widehat{C_*'}(A,C)$. These are chain models for $H_*(LM,M)$, where we consider M inside LM as the constant loops.

Definition 24 Define a map

$$\nu : A\hat{\otimes}\widehat{\Omega}'C \to (C\hat{\otimes}\widehat{\Omega}'C)\widehat{\bigotimes}(C\hat{\otimes}\widehat{\Omega}'C) \tag{3.81}$$

by

$$\nu(\omega[x_1|...|x_n]) = \sum_{i=1}^{n}\sum_{(\omega\cdot x_i)} \pm(\omega\cdot x_i)'[x_1|...|x_{i-1}]\widehat{\bigotimes}(\omega\cdot x_i)''[x_{i+1}|...|x_n]. \tag{3.82}$$

Remark 16 Recall that $\omega\cdot x_i \in C$ denotes the current obtained as the cap product of the form ω with the current x_i. We have use Sweedler's notation to denote the completed coproduct of C, so the sum above should be interpreted as a formal infinite sum. We remind the reader that we can write formal infinite sums when applying coproduct of C since A has a countable basis.

Proposition 26 *The map* $\nu : \widehat{C}'_*(A,C) \to \widehat{C}'_*(C,C)\hat{\otimes}\widehat{C}'_*(C,C)$ *defined above is a chain map. Moreover, it induces a coassociative coproduct of degree* $1-d$ *on* $H'_*(A,C)[-d] \cong H'_*(C,C)$ *(or a coassociative coproduct of degree* $+1$ *on* $H'_*(A,C) \cong H'_*(C,C)[d]$.

Notice that ν is not a coproduct in the usual sense since the domain and range are different. However $\widehat{C}'_*(A,C)[-d]$ and $\widehat{C}'_*(C,C)$ are quasi-isomorphic, in fact their homologies are isomorphic to $\mathbb{H}_*(LM,M;\mathbb{Q})$. Hence, ν defines a coassociative coproduct on $H_*(LM,M)$ of degree $1-d$ (or a coproduct of degree $+1$ on $\mathbb{H}_*(LM,M)$). The proof is the same as in Chapter 2 where we showed $\vee$ is a chain map on the reduced coHochschild complex of a DG cocommutative coalgebra, but adapted to our context of completed coalgebras and using the formal Frobenius compatibility of Proposition 6.

Finally, it follows from the next two results that these structures agree with those geometrically defined string topology operations on the homology of the free loop space.

Theorem 21 *There exists a chain map $\theta : C_*(LM;\mathbb{Q}) \to \widehat{C}_*(A,C)$ making the following diagram commute*

$$\begin{tikzcd} & H_*(A,C) \\ H_*(LM;\mathbb{Q}) \arrow[ur, "\theta_*"] \arrow[r, "\Phi_*"'] & H_*(C,C) \arrow[u, "(\rho\otimes id_A)_*"'] \end{tikzcd} \tag{3.83}$$

Theorem 22 *Let M be simply connected.*

(i) The induced map $\theta_* : \mathbb{H}_*(LM;\mathbb{Q}) \to H_*(A,C)$ is an isomorphism of BV-algebras.

(ii) The map θ induces an isomorphism $(H_*(LM,M),\vee) \to (H'_*(A,C)[-d],\upsilon)$ of coassociative coalgebras of degree $1-d$.

We make a few comments about the map $\theta : C_*(LM;\mathbb{Q}) \to \widehat{C}_*(A,C)$.

Let N be a simplicial tubular neighborhood of the diagonal $M \hookrightarrow M\times M$ and let $u \in A^d(M\times M)$ be a rational polynomial form supported in N representing the Thom class.

Let $W \subset M\times LM$ be $(\mathrm{id}_M \times e_0)^{-1}(N)$ where $e_0 : LM \to M$ is the evaluation at 0 map. Thus a point in W consists of a point $x \in M$ and a loop $\gamma \in LM$ such that x and $\gamma(0)$ are in N. We assume M has a fixed Riemmanian metric. We also assume the triangulation in M is subdivided finely enough and that N is small enough so that if $(x,y) \in N$ then there is a unique simplicial geodesic $\beta(x,y)$ between x and y. Define a map $g : W \to LM$ by $g(x,\gamma(0)) = \beta(x,\gamma(0)) * \gamma * \beta(x,\gamma(0))^{-1}$. Hence, g takes a loop and a point close to the base point of the loop and produces a free loop.

We define θ as a map from $C_*(LM;\mathbb{Q})$ to $\mathrm{Hom}(BA,A) \cong A\hat{\otimes}\widehat{\Omega}(C) = \widehat{C}_*(A,C)$ by

$$\theta(\alpha)([\omega_1|\ldots|\omega_r] := \int_{\Delta^n} (\mathrm{id}_M\times\alpha)^*\Big((\mathrm{id}_M\times e_0)^* u \wedge g^* \int \omega_1\ldots\omega_r\Big) \in A. \tag{3.84}$$

for any simplex $\alpha : \Delta^n \to LM$.

In the above formula we consider $\int \omega_1 \ldots \omega_r$ as a form on LM, so $g^* \int \omega_1 \ldots \omega_r$ is a form on W. We pullback the Thom form u along $\mathrm{id}_M \times e_0 : M \times LM \to M \times M$ to obtain a form in $M \times LM$ supported in W. Thus we can consider $(\mathrm{id}_M \times e_0)^* u \wedge g^* \int \omega_1 \ldots \omega_r$ as a form on $M \times LM$. Finally, we integrate this form along the simplex $\alpha : \Delta^n \to LM$ to obtain a form in M.

Recall the chain map $\Phi : C_*(LM; \mathbb{Q}) \to \mathrm{Hom}(A \otimes BA, \mathbb{Q}) \cong \widehat{C}_*(C, C)$ is given by

$$\Phi(\alpha)(\omega_0[\omega_1|\ldots|\omega_r]) = \int_\alpha \Big(e_0^* \omega_0 \wedge \int \omega_1 \ldots \omega_r \Big). \tag{3.85}$$

Theorem 3 says that $\theta_* : H_*(LM; \mathbb{Q}) \to H_*(A, C)$ lifts the map $\Phi_* : H_*(LM; \mathbb{Q}) \to H_*(C, C)$ via Poincaré duality.

For a detailed proof of the calculation showing that θ preserves the BV-algebra structure and the coproduct of degree $1 - d$ we refer the reader to [Rivera- Arxiv Summer 2015].

Chapter 4

Conjectures and Work in Progress

We describe homotopy versions of some of our results.

Let M be a closed oriented manifold. Choose a homotopy retract for the real cohomology $H^*(M)$ and the differential forms $\mathscr{A}(M)$. By definition, a homotopy retract is given by the data of two quasi isomorphisms of chain complexes $\iota : (H^*(M), 0) \to (\mathscr{A}(M), d)$ and $p : (\mathscr{A}(M), d) \to (H^*(M), 0)$, together with a degree $+1$ map $G : \mathscr{A}(M) \to \mathscr{A}(M)$ such that $dG + Gd = \mathrm{id}_{\mathscr{A}} - p \circ \iota$.

By the Homotopy Transfer Theorem for C_∞-algebra structures along homotopy retracts, we may transfer the C_∞-algebra structure on $(\mathscr{A}(M), d, \wedge)$ to a quasi isomorphic C_∞-algebra structure on $H^*(M)$ extending the 0 differential. By definition, a C_∞-algebra is a coderivation $D_{H^*} : T^c H^*(M) \to T^c H^*(M)$ of the tensor coalgebra $T^c H^*(M)$ (with deconcatenation coproduct) such that $D^2 = 0$ and D vanishes on the shuffle product of monomials. D_{H^*} is determined by the projections $(D_{H^*})_k : H^*(M)^{\otimes k} \to H^*(M)$. In this special case, we have that $(D_{H^*})_1 = 0$ and $(D_{H^*})_2$ is the cohomology cup product.

By a theorem of A. Hamilton and A. Lazarev we may assume that D_{H^*} is cyclically compatible with the Poincaré duality pairing $<,>: H^*(M) \otimes H^*(M) \to \mathbb{R}$. More precisely, Hamilton and Lazarev show that given C_∞-algebra $(V, m_k : V^{\otimes k} \to V)$ such that $m_1 = 0$ and an inner product $<,>$ on V cyclically compatible with m_2, there exists a cyclic C_∞ structure $\{m'_k : V^{\otimes k} \to V\}$ such that $m'_1 = 0$, $m'_2 = m_2$, and (V, m) and (V, m') are C_∞-quasi isomorphic. (V, m') is called a a cyclic lift of (V, m). Moreover, any two cyclic lifts of (V, m) are cyclic C_∞-isomorphic. The hypotheses are satisfied in our case where $V = H^*(M)$, m is the transferred C_∞-algebra structure, and $<,>$ is Poincaré duality. Hence we may assume that our C_∞-structure D_{H^*} is cyclically compatible with $<,>$.

Dualize D_{H^*} to obtain a cyclic C_∞-coalgebra structure $D_{H_*} : TH_*(M) \to TH_*(M)$. Then D_{H_*} is a derivation of the concatenation product in the tensor algebra $TH_*(M)$ such that $D_{H_*}^2 = 0$ and D_{H_*} preserves the free Lie algebra $LH_*(M)$ which sits inside $TH_*(M)$ as a sub vector space. The cyclic compatibility is the now the property dual to the cyclic compatibility of a C_∞-algebra with a non degenerate bilinear pairing.

Conjecture 1 *(i) Let V be a finite dimensional cyclic C_∞-coalgebra with non degenerate bilinear pairing $<,>$ of degree $-d$. The reduced coHochschild complex of the C_∞-coalgebra V has the structure of an A_∞-coalgebra of degree $1-d$ generalizing the coalgebra structure in the case of DG cocommutative Frobenius coalgebras.*

(ii) Let M be a simply connected closed manifold and let $\tilde{C}_*(H_*(M), H_*(M))$ be the reduced coHochschild complex of the cyclic C_∞-coalgebra $(H_*(M), D_{H_*})$. There exists a quasi-isomorphism $\Psi : C_*(LM, M) \to \tilde{C}_*(H_*(M), H_*(M))$ inducing an isomorphism of coalgebras in homology.

Part (i) is an algebraic generalization of the result for DG cocommutative Frobenius coalgebras. The analogue of (i) for the product is well known, since the Hochschild cochains of an A_∞-algebra has an A_∞-algebra structure (the generalization of Gerstenhaber's cup product) which we can transfer to coHochschild chains using the isomorphism given by the non degenerate bilinear pairing.

The quasi-isomoprhism of part (ii) in the above conjecture should be a quasi-isomorphism of A_∞-coalgebras at the chain level. This requires a careful understanding of the chain level coproduct on $C_*(LM,M)$. The construction of Ψ should involve iterated integrals of Harmonic forms and a compatible specific choice of chain homotopy G for the contraction between forms and differential forms.

Moreover, we also expect to be able to relate BV-coalgebra structures:

Conjecture 2 *i) Let V be a finite dimensional cyclic C_∞-coalgebra with non degenerate bilinear pairing $<,>$ of degree $-d$. The reduced coHochschild homology of the C_∞-algebra V has the structure of a BV-coalgebra generalizing the BV-coalgebra structure in the case of DG cocommutative Frobenius coalgebras.*

ii) If M is a simplhy connected closed manifold then $\Psi : C_*(LM,M) \to \tilde{C}_*(H_*,H_*)$ induces an isomorphism of BV-coalgebras in homology.

Of course, (i) is a BV-coalgebra version of Tradler's BV-algebra structure on the Hochschild homology of a cyclic A_∞-algebra. Tradler actually proved this result for A_∞-aglebras with ∞-inner products, a more general class of objects of which cyclic A_∞-algebras are a particular example. At the chain level the map Ψ should be a quasi-isomorphism of ∞-structures for some suitable notion of BV_∞-algebra.

There is also a similar conjecture for the involutive Lie bialgebra structure of the S^1-equivariant homology of (LM,M). This case is harder to analyze since now we have higher homotopies for Drinfeld compatibility and for involutivity. It is being studied by Fukaya, Cieliebak, and Latschev using different methods.

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