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The Moduli Space of Rational Maps

Lloyd William West

Graduate Center, City University of New York

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The Moduli Space of Rational Maps

by

Lloyd William West

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2015

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

Lucien Szpiro

Date

Chair of Examining Committee

Linda Keen

Date

Executive Officer

Lucien Szpiro (Chair)

Raymond Hoobler

Kenneth Kramer

Liang-Chung Hsia

Supervisory Committee

Abstract

The Moduli Space of Rational Maps

by

Lloyd William West

Advisor: Lucien Szpiro

We construct the moduli space, M_d , of degree d rational maps on \mathbb{P}^1 in terms of invariants of binary forms. We apply this construction to give explicit invariants and equations for M_3 .

Using this construction, we give a method for solving the following problems: (1) explicitly construct, from a moduli point $P \in M_d(k)$, a rational map ϕ with the given moduli; (2) find the field of definition of a rational map ϕ . We work out the method in detail for the case $d = 3$.

Dedicated to my grandparents

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Chapter 1

Introduction

In this thesis, we study the dynamics of rational maps from a number theoretic perspective. Let k be a field. By a *rational map over k of degree d* we shall mean a rational function of one variable over k of degree d : this can be thought of as an element ϕ in the function field $k(z)$; geometrically it is a morphism, $\phi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$, from the projective line to itself. In complex dynamics, one studies the iteration of such maps over the field of complex numbers. In arithmetic dynamics one studies rational maps over non-algebraically closed fields, such as \mathbb{Q} or \mathbb{Q}_p (see the article [ST06] or [Sil07] for a book length introduction).

Rational maps are in some ways analogous to abelian varieties and elliptic curves; for example, torsion points of an elliptic curve are analogous to preperiodic points of a rational map. These analogies have inspired conjectures and theorems about the arithmetic of rational maps that correspond

to famous conjectures and theorems in the world of abelian varieties, such as the Uniform Boundedness, André-Oort and Shafarevich conjectures (see, for example, [Sil07], [GKN14], [ST08]).

In this thesis, we address two issues that play an important role in arithmetic dynamics – as their corresponding analogues do in the world of abelian varieties: the moduli space and the field of definition.

1.1 The Moduli Space M_d

The dynamical behavior of a rational map is unchanged after performing the same change of coordinates on the source and target spaces. This leads to the following notion of isomorphism for rational maps.

Definition 1. Let S be a scheme and let $\psi, \phi : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1$ be two rational maps. We say that ψ and ϕ are S -isomorphic, written $\psi \simeq_S \phi$, if there exists an automorphism $\gamma \in \text{Aut}(\mathbb{P}_S^1)$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{P}_S^1 & \xrightarrow{\psi} & \mathbb{P}_S^1 \\ \downarrow \gamma & & \downarrow \gamma \\ \mathbb{P}_S^1 & \xrightarrow{\phi} & \mathbb{P}_S^1 \end{array}$$

In the case that S is the spectrum of a field k , we shall say that ψ and ϕ are k -isomorphic. We shall say that ψ and ϕ are isomorphic, if they become \bar{k} -isomorphic after base change to \bar{k} .

We shall refer to an \mathcal{S} -isomorphism equivalence class of rational maps as an \mathcal{S} -*dynamical system*. We denote the class of a map ϕ by $[\phi]_{\mathcal{S}}$. For maps over a field k , denote $[\phi]_{\bar{k}}$ by $[\phi]$.

The coarse moduli space, denoted M_d , of degree- d rational maps up to isomorphism exists as a scheme over \mathbb{Z} [Sil98]. For M_2 there is an explicit description, due to Milnor [Mil93]: namely, there is an isomorphism

$$(\sigma_1, \sigma_2) : M_2 \xrightarrow{\sim} \mathbb{A}_{\mathbb{C}}^2$$

This isomorphism is given explicitly by the invariants σ_1 and σ_2 , which are the first two symmetric functions in the multipliers of the fixed points of a map. Moreover Silverman [Sil98] proved that one has an isomorphism $M_2 \xrightarrow{\sim} \text{Spec} \mathbb{Z}[\sigma_1, \sigma_2] \simeq \mathbb{A}_{\mathbb{Z}}^2$ as schemes over \mathbb{Z} . In other words, over any field the ring of absolute invariants of quadratic rational maps is generated by σ_1 and σ_2 . It has been shown that M_d is a rational variety for any d [Lev11], but there were no specific results on the equations or geometry of M_d for d greater than two.

In this thesis we construct M_d as an SL_2 -quotient of a space of pairs of binary forms. Such quotients are well studied and there are methods for explicitly computing the ring of invariants. As an example of such computations, we give an explicit description of the defining invariants and equations

for M_3 (Theorem 12).

1.2 Field of Moduli and Fields of Definition

Fix a base field k . For a rational map ϕ of degree d with coefficients in \bar{k} , write $\phi^\gamma \stackrel{\text{def}}{=} \gamma \circ \phi \circ \gamma^{-1}$ for the conjugation action of $\gamma \in \text{PGL}_2(\bar{k})$. Write $[\phi]$ for the point of $M_d(\bar{k})$ corresponding to the equivalence class of maps conjugate to ϕ . Given a k -point $P \in M_d(k)$, there always exists a rational map ϕ with coefficients in \bar{k} such that $[\phi] = P$; we say that ϕ is a *model* for P . In sections 5 and 6 we present a method for explicitly constructing such a model from the coordinates of the point P .

However, when k is not algebraically closed, a k -point $P \in M_d(k)$ may fail to have a model with coefficients in k . Recall the following standard definitions.

Definition 2. (*Field of Definition and Field of Moduli*)

1. One says that a field L is a *field of definition* (FOD) for ϕ if there exists $\gamma \in \text{PGL}_2(\bar{k})$ such that the coefficients of ϕ^γ are in L . We shall also say that L is a field of definition for a moduli point P if there exist a rational map ϕ defined over L such that $[\phi] = P$.

2. Define

$$G_\phi := \{\sigma \in \text{Gal}(\bar{k}/k) : \sigma(\phi) = \phi^{\gamma_\sigma} \text{ for some } \gamma_\sigma \in \text{PGL}_2(\bar{k})\}$$

Then the *field of moduli* (FOM) of ϕ , denoted k_ϕ , is the fixed field \bar{k}^{G_ϕ} of G_ϕ . One has $[\phi] \in M_d(k_\phi)$.

For d even, Silverman [Sil95] has shown that the FOM is always equal to the FOD. Given a point $P \in M_d(k)$ (d even), one should therefore be able to find a map ϕ defined over k such that $[\phi] = P$. In section 6 we give a method for finding such a model that works for a generic odd degree map, and illustrate it explicitly in the case of quadratic maps. (Note that Manes and Yasufuku [MY11] have already given an explicit description of models, as well as twists, in the quadratic case).

When d is odd, the FOD may in general be larger than the FOM. For example, Silverman [Sil95] notes that

$$\psi(x) = \sqrt{-1} \cdot \left(\frac{x-1}{x+1} \right)^3,$$

has field of definition $\mathbb{Q}(\sqrt{-1})$, but field of moduli \mathbb{Q} .

The obstruction to equality of FOM and FOD can be described cohomologically. This approach was taken in [Sil95]. The details are covered in section 2.2, but it essentially corresponds to a certain conic curve \mathcal{C}_P defined

over k ; the obstruction is trivial precisely when the conic has a k -point. In section 5 we give a construction, in terms of the coordinates of the point P , of the conic \mathcal{C}_P . In the case where the obstruction vanishes (that is when the conic has a k -point), the construction furnishes an explicit model ϕ defined over k . On a nonempty open set of M_d , the conic can be constructed using identities in the ring of covariants associated to the SL_2 action, due to Clebsch [Cle72]. Such identities have been applied by Mestre and others (see [Mes91], [LR12]) to construct hyperelliptic curves of genus 2 and 3 from their moduli. We shall call this the *covariants method*.

The covariants method fails on the closed subset of M_3^{Aut} of points with nontrivial automorphism group; here one must apply alternative *ad hoc* constructions. In this thesis we give explicit formulas only in the case $d = 3$, since for larger d generators for the ring of invariants are not known. However, the covariants method can be applied for any odd d , once the invariants are known.

Chapter 2

Preliminary results on moduli spaces and fields of definition of rational maps

2.1 GIT construction of the moduli space

In this section we recall the construction of M_d using geometric invariant theory (originally given in [Sil98]).

The space of rational maps

The first step is to find a scheme that parametrizes rational maps of degree d .

Let $\underline{\text{Rat}}_d$ denote the functor from schemes to sets given by

$$\underline{\text{Rat}}_d(S) = \left\{ \begin{array}{l} \text{S-morphisms } \phi : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1 \\ \text{with } \phi^* \mathcal{O}_{\mathbb{P}_S^1}(1) \simeq \mathcal{O}_{\mathbb{P}_S^1}(d) \end{array} \right\}$$

The problem of this subsection is to find a scheme representing this functor.

Let $\phi : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ be a rational map of degree \mathbf{d} . We can write ϕ in terms of coordinates X_0, X_1 on \mathbb{P}^1 as

$$\phi(X_0, X_1) = [F_0(X_0, X_1) : F_1(X_0, X_1)]$$

where $F_0(X_0, X_1), F_1(X_0, X_1)$ are homogenous forms of degree \mathbf{d} in $k[X_0, X_1]$ with no common factor in $\bar{k}[X_0, X_1]$. The forms F_0 and F_1 are specified by their coefficients:

$$\begin{aligned} F_0(X_0, X_1) &= c_1 X_0^{\mathbf{d}} + c_2 X_0^{\mathbf{d}-1} X_1 + \cdots + c_{\mathbf{d}} X_0 X_1^{\mathbf{d}-1} + c_{\mathbf{d}+1} X_1^{\mathbf{d}} \\ F_1(X_0, X_1) &= c_{\mathbf{d}+2} X_0^{\mathbf{d}} + c_{\mathbf{d}+3} X_0^{\mathbf{d}-1} X_1 + \cdots + c_{2\mathbf{d}+1} X_0 X_1^{\mathbf{d}-1} + c_{2\mathbf{d}+2} X_1^{\mathbf{d}} \end{aligned}$$

Such expressions can be parametrized by k -points of the projective space of the $2\mathbf{d} + 2$ coefficients, which we shall denote by $\mathbf{P}_{\mathbf{d}} = \text{Proj } \mathcal{A}_{\mathbf{d}}$, where $\mathcal{A}_{\mathbf{d}} = \mathbb{Z}[c_1, \dots, c_{2\mathbf{d}+2}]$.

For a point of $\mathbf{P}_{\mathbf{d}}$ to represent a rational map of degree \mathbf{d} , the polynomials F_0 and F_1 must have no common factor. This condition can be expressed by the non-vanishing of the resultant, $\text{Res}(F_0, F_1)$, a polynomial in $\mathcal{A}_{\mathbf{d}}$ of degree $2\mathbf{d}$, which we shall denote by $\rho_{\mathbf{d}}$.

Proposition 1. *The space $\text{Rat}_{\mathbf{d}} = \mathbf{P}_{\mathbf{d}} - \{\rho_{\mathbf{d}} = 0\}$ – henceforth the space of rational maps – is a fine moduli space for rational maps of degree \mathbf{d} in given*

coordinates. This holds over \mathbb{Z} ; i.e. for any scheme S , there is a bijection

$$\mathrm{Rat}_d(S) \xrightarrow{\sim} \left\{ \begin{array}{l} S\text{-morphisms } \phi : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^1 \\ \text{with } \phi^* \mathcal{O}_{\mathbb{P}_S^1}(1) \simeq \mathcal{O}_{\mathbb{P}_S^1}(d) \end{array} \right\}$$

Proof. This is Theorem 3.1 in [Sil98]. \square

Note that the coordinate ring on Rat_d is $A_d[\rho_d^{-1}]_{(0)}$, where $R_{(0)}$ stands for elements of degree zero in a graded ring R .

The moduli functor

Let \underline{M}_d denote the functor from schemes to sets given by

$$\underline{M}_d(S) = \underline{\mathrm{Rat}}_d(S) / \simeq_S$$

Proposition 2. *There exists a coarse moduli space for this functor. This means that there is a scheme, denoted M_d , and a natural map of functors*

$$\underline{M}_d(\cdot) \longrightarrow \mathrm{Hom}(\cdot, M_d)$$

that induce bijections

$$\underline{M}_d(\Omega) \xrightarrow{\sim} \mathrm{Hom}(\Omega, M_d)$$

for every algebraically close field Ω .

Proof. This is Theorem 3.2 in [Sil98]. \square

We shall presently explain how to construct \mathcal{M}_d . Before that, let us note that there is no *fine* moduli scheme for $\underline{\mathcal{M}}_d$. The functor $\underline{\mathcal{M}}_d$ fails to be representable because for a non-algebraically closed field k , the map

$$\underline{\mathcal{M}}_d(k) \longrightarrow \text{Hom}(k, \mathcal{M}_d)$$

may fail to be bijective. The failure of injectivity is due to the fact that rational maps can have non-trivial automorphism groups, leading to the existence of *twists* (see [Sil07], Chapter 4). The failure of surjectivity is the problem of FOD versus FOM; an explicit solution to this problem is given in Chapter 5 of this dissertation.

General plan for the construction of \mathcal{M}_d

The functor $\underline{\mathcal{M}}_d$ is a quotient of $\underline{\text{Rat}}_d$ by an equivalence relation. The strategy for constructing \mathcal{M}_d is to find an action of an algebraic group G on Rat_d such that the orbits of the group are equal to equivalence classes of maps – at least over an algebraically closed field. Then \mathcal{M}_d can be constructed as a GIT quotient of Rat_d by G (see [MFK94] for a thorough account of Geometric Invariant Theory and applications to moduli spaces).

The action of SL_2

The group of automorphisms of \mathbb{P}^1 is isomorphic to the projective general linear group: $\text{Aut}(\mathbb{P}^1) \simeq \text{PGL}_2$. The action on \mathbb{P}^1 induces a conjugation

action on the space Rat_d of rational maps on \mathbb{P}^1 :

$$\begin{aligned} \theta : \text{PGL}_2 \times \text{Rat}_d &\rightarrow \text{Rat}_d \\ (\gamma, \phi) &\mapsto \phi^\gamma \end{aligned}$$

where $\phi^\gamma = \gamma \circ \phi \circ \gamma^{-1}$ for any $\gamma \in \text{PGL}_2$. If ϕ is written as $[F_0(X_0, X_1), F_1(X_0, X_1)]$

then an element $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ acts as

$$\begin{aligned} [F_0(X_0, X_1) : F_1(X_0, X_1)]^\gamma = \\ [pF_0(sX_0 - qX_1, -rX_1 + pX_1) + qF_1(sX_0 - qX_1, -rX_1 + pX_1) : \\ rF_0(sX_0 - qX_1, -rX_1 + pX_1) + sF_1(sX_0 - qX_1, -rX_1 + pX_1)] \end{aligned}$$

Note that this action extends to all of \mathbb{P}_d , not just Rat_d .

Over an algebraically closed field, orbits of this action on Rat_d correspond to isomorphism classes of rational maps.

To construct the GIT quotient of Rat_d by this action, we must give a linearization in the sense of [MFK94]. Since linearizing the PGL_2 -action is a little inconvenient, we instead lift to an action of SL_2 , which has a natural linearization (cf. [MFK94] 1.§3).

To that end, let V be a two dimensional k vector space with the standard representation of SL_2 . Let $\mathbb{P}V = \text{Proj}(\text{Sym}V^*)$ – note that, for psychological reasons, we follow the classical, not the Grothendieck convention. Then the

space $V_d = \Gamma(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(d))$ is the space of degree d binary forms on V , with the induced representation of SL_2 . Note that $V_d = S^d(V^*)$. (See Section 6 for more on the representation theory of SL_2).

An homogenous binary form of degree d is an element of $V_d = V_d$. A pair of homogenous forms can be thought of as an element of $W_d = V_d \otimes V$, by writing $[F_0, F_1]$ as $\mathbf{y}_0 F_0(X_0, X_1) + \mathbf{y}_1 F_1(X_0, X_1)$, where $\mathbf{y}_0, \mathbf{y}_1$ are a basis for V . In this way we can identify P_d with $\mathbb{P}W_d$.

We have the representation of SL_2 on $W_d = V_d \otimes V$ and an associated SL_2 -action on $\mathbb{P}W_d$. On the other hand, via the canonical isogeny $\bar{\omega} : SL_2 \rightarrow PGL_2$ we get an SL_2 -action $\tilde{\theta} = \theta \circ (\bar{\omega} \times \text{id}_{P_d})$ on P_d . The following proposition is immediate from the preceding discussion.

Proposition 3. *Identifying $\mathbb{P}W_d$ with P_d , identifies the action of SL_2 on $\mathbb{P}W_d$ with $\tilde{\theta}$. The linear action of SL_2 on W_d gives a natural linearization of the action on P_d .*

The calculation of the Hilbert-Mumford criterion in [Sil98] shows that Rat_d is contained in the stable locus of the linearized action of SL_2 on P_d , therefore we can form a geometric quotient

$$M_d = \text{Rat}_d / SL_2$$

Concretely, the quotient is given by

$$\mathrm{Spec} \mathbf{R}_d$$

where $\mathbf{R}_d \stackrel{\mathrm{def}}{=} H^0(\mathrm{Rat}_d, \mathcal{O}_{\mathrm{Rat}_d})^{\mathrm{SL}_2} = ((\mathbf{A}_d[\frac{1}{\rho}])_{(0)})^{\mathrm{SL}_2}$. Since ρ is itself an SL_2 -invariant, we have $\mathbf{R}_d \simeq (\mathbf{A}_d)^{\mathrm{SL}_2[\frac{1}{\rho}]_{(0)}}$. The ring \mathbf{R}_d is the subring of \mathbf{A}_d consisting of all functions in the coefficients of ϕ that remain unchanged under the conjugation action.

We have a compactification of M_d by the natural embedding into

$$M_d^{ss} = \mathrm{Proj} (\mathbf{A}_d^{\mathrm{SL}_2}).$$

It remains to find generators and relations for the ring $\mathbf{A}_d^{\mathrm{SL}_2}$. This problem can be reduced to classical invariant theory. We shall take this up again in Chapter 4 after first reviewing some background from Classical Invariant Theory.

2.2 Cohomological obstruction to the equality of FOD and FOM

In this section we recall the cohomological obstruction to equality of FOD and FOM that was first described by [Sil95]. Similar methods were applied by Cadoret in [Cad08] to analyze the related question for Hurwitz moduli

spaces. We first need some results on the automorphism groups of rational maps.

An *automorphism* of a rational map ϕ of degree $d \geq 2$ is an element $\gamma \in \mathrm{PGL}_2(\bar{k})$ such that $\phi = \phi^\gamma$.

The group of all automorphisms of the rational map ϕ shall be denoted $\mathrm{Aut}(\phi)$. This group is finite and its order is bounded in terms of d ([Sil07] Prop. 4.65).

The finite subgroups of PGL_2 are well known:

Proposition 4. (1) Any finite subgroup of $\mathrm{PGL}_2(\bar{k})$ is isomorphic to one of the following: a cyclic group \mathfrak{C}_n of order n , a Klein Viergruppe \mathfrak{V}_4 , a dihedral group \mathfrak{D}_{2n} of order $2n$, the alternating group \mathfrak{A}_4 , the symmetric group \mathfrak{S}_4 , or the alternating group \mathfrak{A}_5 .

(2) Any two finite subgroups of $\mathrm{PGL}_2(\bar{k})$ that are abstractly isomorphic are conjugate in $\mathrm{PGL}_2(\bar{k})$.

(3) Any finite subgroup $A < \mathrm{PGL}_2(\bar{k})$ is conjugate to a group A_0 for which both A_0 and its normalizer $N(A_0)$ are G_k -stable; hence these groups have the structure of a G_k -module, as does $Q(A_0) = N(A_0)/A_0$.

(4) The groups A_0 can be taken to be

$$\begin{aligned} \triangleright \mathfrak{C}_n &= \left\{ \begin{pmatrix} \zeta_n^r & 0 \\ 0 & 1 \end{pmatrix} : r = 0, \dots, n-1 \right\} \text{ where } \zeta_n \text{ is a primitive } n\text{-th root of unity, } n \geq 1 \\ \triangleright \mathfrak{D}_{2n} &= \left\{ \begin{pmatrix} \zeta_n^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_n^r \\ 1 & 0 \end{pmatrix} : r = 0, \dots, n-1 \right\} \text{ where } \zeta_n \text{ is a primitive } n\text{-th root of unity,} \end{aligned}$$

$$\begin{aligned}
& n \geq 3 \\
\triangleright \mathfrak{A}_4 &= \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \right\} \\
\triangleright \mathfrak{A}_4 &= \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i^\nu & i^\nu \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} i^\nu & -i^\nu \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i^\nu \\ 1 & -i^\nu \end{pmatrix}, \begin{pmatrix} -1 & -i^\nu \\ 1 & -i^\nu \end{pmatrix} : \nu = 1, 3 \right\} \\
\triangleright \mathfrak{S}_4 &= \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i^\nu \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i^\nu & -i^{\nu+\nu'} \\ 1 & i^{\nu'} \end{pmatrix} : \nu, \nu' = 0, 1, 2, 3 \right\} \\
\triangleright \mathfrak{S}_4 &= \left\{ \begin{pmatrix} \zeta^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta^r \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta^r \omega & \zeta^{r-s} \\ 1 & -\zeta^s \omega \end{pmatrix}, \begin{pmatrix} \zeta^r \bar{\omega} & \zeta^{r-s} \\ 1 & -\zeta^{-s} \bar{\omega} \end{pmatrix} : r, s = 0, 1, 2, 3, 4 \right\} \text{ where } \zeta_n \\
& \text{is a primitive 5-th root of unity, } \omega = \frac{-1+\sqrt{5}}{2} \text{ and } \bar{\omega} = \frac{-1-\sqrt{5}}{2}.
\end{aligned}$$

(5) Recall the definition of the infinite Dihedral group

$$\mathfrak{D}_\infty = \mathbb{G}_m \rtimes \mu_2$$

and consider the copy in $\mathrm{PGL}_2(\bar{\mathbf{k}})$ given by

$$\mathfrak{D}_\infty(\bar{\mathbf{k}}) = \left\{ \begin{pmatrix} \mathbf{a} & 0 \\ 0 & 1 \end{pmatrix} : \mathbf{a} \in \bar{\mathbf{k}}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & \mathbf{b} \\ 1 & 0 \end{pmatrix} : \mathbf{b} \in \bar{\mathbf{k}}^\times \right\}$$

Then we can list the normalizers of the groups A_0 from (4) as follows.

$$\begin{aligned}
\triangleright \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{C}_n) &= \mathfrak{D}_\infty(\bar{\mathbf{k}}) \\
\triangleright \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{D}_{2n}) &= \mathfrak{D}_{4n}, \quad n \geq 3 \\
\triangleright \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{A}_4) &= \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{A}_4) = \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{A}_4) = \mathfrak{S}_4 \\
\triangleright \mathrm{Nor}_{\mathrm{PGL}_2(\bar{\mathbf{k}})}(\mathfrak{A}_5) &= \mathfrak{A}_5
\end{aligned}$$

(6) The corresponding quotients with \mathbb{G}_k -galois module structure are

- ▷ $Q(\mathfrak{C}_n) = \mathfrak{D}_\infty(\bar{k})$
- ▷ $Q(\mathfrak{D}_2n) = \mathbb{Z}/2\mathbb{Z}$, $n \geq 3$
- ▷ $Q(\mathfrak{A}_4) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- ▷ $Q(\mathfrak{A}_4) = \mathbb{Z}/2\mathbb{Z}$
- ▷ $Q(\mathfrak{S}_4) = 1$
- ▷ $Q(\mathfrak{A}_5) = 1$

Proof. [Cad08] Lemma 2.1 □

Let ϕ be a model over \bar{k} for $P \in M_d(k)$. After conjugation we may assume that $\mathcal{A} = \text{Aut}(\phi)$ is G_k -stable. From definition 2, for each $\sigma \in G_k := \text{Gal}(\bar{k}/k)$ we have an element $\gamma_\sigma \in \text{PGL}_2(\bar{k})$ such that

$$\sigma(\phi) = \phi^{\gamma_\sigma}$$

The assumption that \mathcal{A} is G_k -stable has the following consequence:

Lemma 3. ([Sil95] Lemma 4.2)

- ▷ $\gamma_\sigma \in N(\mathcal{A})$
- ▷ *The map*

$$G_k \longrightarrow Q(\mathcal{A})$$

$$\sigma \longmapsto \gamma_\sigma$$

gives a well defined element of $H^1(\mathbf{G}_k, \mathbf{Q}(\mathcal{A}))$, which we shall denote \mathbf{c}_ϕ .

From the quotient and inclusion morphisms $\mathbf{N}(\mathcal{A}) \rightarrow \mathbf{Q}(\mathcal{A})$ and $\mathbf{N}(\mathcal{A}) \hookrightarrow \mathrm{PGL}_2(\bar{k})$, we get maps

$$\begin{array}{ccc} H^1(\mathbf{G}_k, \mathbf{N}(\mathcal{A})) & \xrightarrow{i} & H^1(\mathbf{G}_k, \mathrm{PGL}_2(\bar{k})) \\ \downarrow p & & \\ H^1(\mathbf{G}_k, \mathbf{Q}(\mathcal{A})) & & \end{array}$$

The *cohomological obstruction* is defined to be the (possibly empty) set $I_k(\phi) := i(p^{-1}(\mathbf{c}_\phi))$.

Proposition 5. *The field k is a field of definition for ϕ if and only if $I_k(\phi)$ contains the trivial class.*

Proof. (Compare [Cad08] Prop. 2.4 and [Sil95] Prop. 4.5) Suppose ϕ has a model ψ defined over k . Then there exists $\eta \in \mathrm{PGL}_2(\bar{k})$ such that $\psi = \phi^\eta$. From $\sigma(\psi) = \psi$, we have $\sigma(\phi) = \phi^{\eta\sigma(\eta)^{-1}}$ for all $\sigma \in \mathbf{G}_k$. So the class \mathbf{c}_ϕ can be represented by the cocycle $\eta\sigma(\eta)^{-1}$, which clearly lifts to a true cocycle in $H^1(\mathbf{G}_k, \mathbf{N}(\mathcal{A}))$ and is a $\mathrm{PGL}_2(\bar{k})$ -coboundary.

The converse is covered by [Sil95] Prop. 4.5. □

In Chapter 5 we take up the problem of explicitly constructing the classes in $I_k(\phi)$.

Chapter 3

Results from classical invariant theory

We briefly recall the basics of classical invariant theory. For a fuller account see the classic books [Cle72], [GY10] and [Gor87]. The references [Dol03], [Bri96], [Olv99] are excellent modern accounts.

3.1 Representations of SL_2

Let k be an algebraically closed field of characteristic zero. The following proposition summarizes the representation theory of SL_2 over k .

- Proposition 6.**
1. Let V be a two dimensional k vector space with the standard action of SL_2 . Let $V_1 = V^*$ be the space of linear binary forms over k . Then V_1 is an irreducible representation.
 2. As an SL_2 -module V_1 is self-dual: i.e. $V_1 \simeq (V_1)^*$

3. Up to isomorphism the irreducible representations of SL_2 over \mathbf{k} are

$$\{\mathrm{S}^n \mathbf{V}_1 : \mathbf{n} \in \mathbb{Z}_{\geq 0}\}$$

4. Every representation of SL_2 over \mathbf{k} is decomposable into a direct sum of irreducible representations.

We shall write \mathbf{V}_n for symmetric power $\mathrm{S}^n \mathbf{V}_1$; it is identified with the space of binary forms of degree \mathbf{n} and the space $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d}))$.

3.2 Invariants and Covariants

Definition 4. Given an SL_2 -variety, W , an SL_2 -equivariant morphism $H : W \rightarrow \mathbf{V}_e$ is called a *covariant* of order e .

When W is a linear representation of SL_2 , by Proposition 6 we can write $W \cong \bigoplus_{1 \leq \ell \leq \mathbf{n}} \mathbf{V}_{d_\ell}$ for some \mathbf{n} and $d_\ell \in \mathbb{Z}_{\geq 0}$. So a point of W may be given as system of \mathbf{n} binary forms, $\{f_\ell ; 1 \leq \ell \leq \mathbf{n}\}$, where

$$f_\ell = \sum_{i=0}^{d_\ell} c_i^{(\ell)} X_0^{d_\ell-i} X_1^i.$$

Let us fix a notation: if $f(X_0, X_1)$ is a form of degree \mathbf{d} with coefficients $\mathbf{c} = (c_1, \dots, c_d)$, the result of substituting $X_0 = sX'_0 - qX'_1$ and $X_1 = -rX'_0 + pX'_1$ is a new form in X'_0 and X'_1 , whose coefficients we shall denote by $\mathbf{c}' = (c'_1, \dots, c'_d)$.

With this notation, a *covariant* of W of order \mathbf{e} is given by a form H of degree \mathbf{e} in variables X_0, X_1 , with coefficients that are polynomials in $(\mathbf{c}^{(\ell)})_{1 \leq \ell \leq n}$ satisfying the following transformation identity

$$H((\mathbf{c}'^{(\ell)})_{1 \leq \ell \leq n}, X'_0, X'_1) = H((\mathbf{c}^{(\ell)})_{1 \leq \ell \leq n}, X_0, X_1)$$

Note that the forms f_ℓ are themselves covariants.

The covariants of order \mathbf{e} form a \mathbb{N}^n -graded k -algebra $\text{Cov}(W)_{\mathbf{e}}$, where the grading is given by the degrees with respect to the variables $(\mathbf{c}^{(\ell)})_{1 \leq \ell \leq n}$. The total degree of an homogenous element H in the variables $(\mathbf{c}^{(\ell)})_{1 \leq \ell \leq n}$ shall be called simply the *degree* of H . A covariant of order 0 is an *invariant*. The quotient of two invariants of the same degree is called an *absolute invariant*.

The SL_2 -covariants form a graded k -algebra

$$\text{Cov}(W) = \bigoplus_{\mathbf{e}=0}^{\infty} \text{Cov}(W)_{\mathbf{e}}.$$

We write $\text{Inv}(W)$ for the sub-algebra of invariants.

We note that the algebra of covariants is isomorphic to a ring of invariants of the space $W \times V$; that is,

$$\text{Cov}(W) \simeq k[W \times V]^{\text{SL}_2}$$

The following result is one of the high-points of the classical theory.

Proposition 7. (Gordon, Hilbert) *The algebras $\text{Inv}(W)$ and $\text{Cov}(W)$ are of finite type over k . A set of generators is called a set of basic invariants or basic covariants respectively.*

The original method of proof, due to Gordon, involved an explicit method of construction the generators. In the subsequent sections we will recall some of the classical techniques for constructing invariants and covariants.

3.3 Symbolic Notation

A key tool of the classical theory is the so-called *symbolic notation*. We give a brief summary of this technique here. For more details on classical symbolic notation see the article [Osg92] or the books [GY10], [Cle72], [Gor87]; for a modern treatment see [Dol03].

Write a binary form with binomial coefficients as

$$f = \sum_{i=0}^n \frac{n!}{(n-i)!i!} a_i X_0^{n-i} X_1^i \quad (3.1)$$

Were f a pure power of the linear form $\alpha_X = \alpha_0 X_0 + \alpha_1 X_1$, then, comparing the expansion of

$$f = \alpha_X^n$$

with (3.1), one would have identities

$$\begin{aligned} \mathbf{a}_0 &= \alpha_0^n, \\ \mathbf{a}_1 &= \alpha_0^{n-1} \alpha_1, \\ \mathbf{a}_2 &= \alpha_0^{n-2} \alpha_1^2, \\ &\dots \\ \mathbf{a}_n &= \alpha_1^n; \end{aligned}$$

The symbols α_0 and α_1 are called symbolic quantities. They have meaning in terms of the original coefficients only when they occur as monomials of degree n . The technique of symbolic notation applies these identities to write any homogenous polynomial in \mathbf{a}_i and X_0, X_1 in terms of symbolic quantities – regardless of whether the original form was in fact a pure power – the result is called a *symbolic expression* for the polynomial.

The procedure to obtain the symbolic expression is as follows. To symbolically express a polynomial P of degree d in the coefficients \mathbf{a}_i , one introduces d sets of symbolic quantities $(\underline{\alpha}^{(j)})_{j=1}^d$, where $\underline{\alpha}^{(j)} = (\alpha_0^{(j)}, \alpha_1^{(j)})$. Classically one writes

$$f = (\alpha_X^{(1)})^n = (\alpha_X^{(2)})^n = \dots = (\alpha_X^{(d)})^n.$$

Using the above identities between the original coefficients and degree n monomials in the symbolic quantities, one find a expression for the given polynomial terms of the $\alpha^{(j)}$; for each degree d monomial in the \mathbf{a}_i one replaces each \mathbf{a}_i by the corresponding symbolic expression, using a different set of symbolic quantities for each \mathbf{a}_i in the the monomial. Any permutation

of the indices j yields another expression for P , one defines *the* symbolic expression to be the symmetrization.

Proposition 8. *The process of forming symbolic expressions yields an SL_2 -equivariant isomorphism*

$$\text{symb} : S^{\mathbf{d}}[V_n^*] \rightarrow \text{Sym}_{(n)^{\mathbf{d}}}(V^*)$$

between polynomials of degree \mathbf{d} in the coefficients of f and the space $\text{Sym}_{(n)^{\mathbf{d}}}(V^)$ of symmetric multi-homogenous expressions in the variables $(\underline{\alpha}^{(j)})_{j=1}^{\mathbf{d}}$.*

Proof. See [Dol03], Chapter 1. □

Example 5. *Let f be the quadratic form $f = \mathbf{a}_0 X_0^2 + 2\mathbf{a}_1 X_0 X_1 + \mathbf{a}_2 X_1^2$. Then the symbolic expression for the monomial $\mathbf{a}_0 \mathbf{a}_2$ will be*

$$\text{symb}(\mathbf{a}_0 \mathbf{a}_2) = \frac{1}{2} \{ (\alpha_0^{(1)})^2 (\alpha_1^{(2)})^2 + (\alpha_0^{(2)})^2 (\alpha_1^{(1)})^2 \}$$

Classically one uses different letters for each set of symbolic quantities, so in this example one would write instead $\frac{1}{2} \{ \alpha_0^2 \beta_1^2 + \beta_0^2 \alpha_1^2 \}$. For the discriminant $\mathbf{a}_0 \mathbf{a}_2 - \mathbf{a}_1^2$, one has the symbolic expression

$$\text{symb}(\mathbf{a}_0 \mathbf{a}_2 - \mathbf{a}_1^2) = \frac{1}{2} (\alpha_0 \beta_1 - \beta_0 \alpha_1)^2$$

3.4 The Fundamental Theorems

In the preceding example we saw that the discriminant of a quadratic form could be written in symbolic notation as

$$\frac{1}{2}(\alpha_0\beta_1 - \beta_0\alpha_1)^2$$

The notation $(\alpha\beta)$ is used for the expression $\alpha_0\beta_1 - \beta_0\alpha_1$, which is referred to a *bracket functions of the first kind*. The bracket functions are easily seen to be invariant under the SL_2 -action. Moreover, the symbolic expressions $\alpha_X = \alpha_0X_0 + \alpha_1X_1$ are clearly covariant; these are called *bracket functions of the second kind*. From the proceeding section we know that any symbolic expression with d symbols in which each symbol appears to degree n will be the symbolic expression of a genuine polynomial in the coefficients \mathbf{a}_i of a form of degree n . Accordingly we can easily write down many invariants and covariants, such as

$$(\alpha\beta)^n, \quad (\alpha\beta)^2(\beta\gamma)^2(\alpha\gamma)^2\alpha_X^{n-4}\beta_X^{n-4}\gamma_X^{n-4}, \quad \text{and so on } \dots$$

Let us write $\text{Cov}^{\text{bracket}}(V_n)$ for the sub-algebra generated by covariants formed from bracket functions.

We can now state the following key classical theorem.

Theorem 6. (*First Fundamental Theorem of Classical Invariant Theory*)

$$\text{Cov}^{\text{bracket}}(\mathbf{V}_n) = \text{Cov}(\mathbf{V}_n)$$

There are three basic syzygies in the algebra of bracket functions. The first is

$$(I) \quad (\alpha\beta) = -(\beta\alpha)$$

By expanding the determinant

$$\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_X & \beta_X & \gamma_X \end{vmatrix}$$

and noticing that the third row is the sum of the second and third rows, one obtains the second syzygy:

$$(II) \quad \alpha_X(\beta\gamma) + \beta_X(\gamma\alpha) + \gamma_X(\alpha\beta) = 0$$

By substituting δ_1 for X_0 and $-\delta_0$ for X_0 , one obtains

$$(III) \quad (\alpha\delta)(\beta\gamma) + (\beta\delta)(\gamma\alpha) + (\gamma\delta)(\alpha\beta) = 0$$

We note in passing a third useful relation obtained by substituting Y_1 for γ_0 and $-Y_0$ for γ_0 in (I):

$$\alpha_X\beta_Y + \beta_X\alpha_Y = (XY)(\alpha\beta) = 0$$

We can now state a second key classical theorem.

Theorem 7. (*Second Fundamental Theorem of Classical Invariant Theory*)

Every polynomial identity between the bracket functions is obtained as a linear combination of the three basic syzygies.

3.5 Transvectants

The *omega process* with respect to variables $\mathbf{Z}^{(p)} = (Z_0^{(p)}, Z_1^{(p)})$ and $\mathbf{Z}^{(q)} = (Z_0^{(q)}, Z_1^{(q)})$ is defined as

$$\Omega_{pq} = \frac{\partial^2}{\partial Z_0^{(p)} \partial Z_1^{(q)}} - \frac{\partial^2}{\partial Z_0^{(q)} \partial Z_1^{(p)}}$$

Given two binary forms F and G in X_0, X_1 of orders m and n respectively, one defines the r -th *transvectant* (also called the r -th *überschiebung*) as

$$(F, G)_r = \frac{(n-r)!(m-r)!}{n!m!} \left[(\Omega_{12})^r \left\{ F(Z_0^{(1)}, Z_1^{(1)}) \cdot G(Z_0^{(2)}, Z_1^{(2)}) \right\} \right]_{X_0=Z_0^{(1)}=Z_0^{(2)}, X_1=Z_1^{(1)}=Z_1^{(2)}}$$

If F and G are covariants of degree d and e respectively, then $(F, G)_k$ is a covariant of order $m+n-2k$ and degree $e+d$. Transvection allows us to write the decomposition of tensor products of SL_2 -modules into irreducibles as follows.

Proposition 9. (*Clebsch-Gordon formula*) The map

$$\begin{aligned} S^m V \otimes S^n V &\longrightarrow \bigoplus_{r=0}^{\min(m,n)} S^{m+n-2r} V \\ f, g &\longmapsto \sum (f, g)_r \end{aligned}$$

is an isomorphism of SL_2 modules,.

Note the symbolic expression for the transvection of two forms $F = \alpha_X^n$ and $G = \beta_X^m$ is

$$(F, G)_r = (\alpha\beta)^r \alpha_X^{n-r} \beta_X^{m-r}$$

The First Fundamental Theorem can be restated as follows.

Proposition 10. (*First Fundamental Theorem of Invariant Theory*) Any covariant of a system of binary forms can be expressed as a linear combination of iterated transvectants of the groundforms.

We shall also use the *generalized transvectant* (see [Olv99] and [GY10] § 81). Given a sequence of distinct pairs $(\mathbf{p}_1, \mathbf{q}_1), \dots, (\mathbf{p}_k, \mathbf{q}_k)$, where $\mathbf{p}_i, \mathbf{q}_i \in \{1, \dots, m\}$ and $\mathbf{p}_i \neq \mathbf{q}_i$, define

$$\kappa_\ell \stackrel{\text{def}}{=} \sum_{i: \ell \in \{\mathbf{p}_i, \mathbf{q}_i\}} r_i$$

Then one can define the generalized transvectant of the m forms G_1, \dots, G_m of orders s_1, \dots, s_m as

$$(\mathbf{p}_1, \mathbf{q}_1)_{r_1} (\mathbf{p}_2, \mathbf{q}_2)_{r_2} \cdots (\mathbf{p}_k, \mathbf{q}_k)_{r_k} [G_1, G_1, \dots, G_m] = \prod_{\ell=1}^m \frac{(s_\ell - \kappa_\ell)!}{s_\ell!} \cdot \left[\prod_{i=1}^k (\Omega_{\mathbf{p}_i \mathbf{q}_i})^{r_i} \cdot \prod_{\ell=1}^m G(Z_0^{(\ell)}, Z_1^{(\ell)}) \right]_{X_0=Z_0^{(\ell)}, X_1=Z_1^{(\ell)} \quad 1 \leq \ell \leq m}$$

3.6 Invariants of Quadratic Forms

In the sequel we will make repeated use of facts about the algebra of covariants of three quadratic forms, $\text{Cov}(V_2 \oplus V_2 \oplus V_2)$, which we record here.

Let

$$u_1 = a_0X_0^2 + 2a_1X_0X_1 + a_2X_1^2,$$

$$u_2 = b_0X_0^2 + 2b_1X_0X_1 + b_2X_1^2,$$

$$u_3 = c_0X_0^2 + 2c_1X_0X_1 + c_2X_1^2$$

be three quadratic binary forms. We shall denote them in symbolic notation as

$$u_1 = \alpha_x^2, \quad u_2 = \beta_x^2, \quad u_3 = \gamma_x^2$$

Proposition 11. *The k -algebra $\text{Cov}(V_2 \oplus V_2 \oplus V_2)$ is generated by the following elements*

- ▷ *The three covariants of order 2 defined by*

$$\xi_i = (u_j, u_k)_1 \quad \text{for cyclic permutations } (ijk) \text{ of } (123)$$

- ▷ *The six quadratic invariants defined by*

$$C_{ij} = (u_j, u_k)_2 \quad \text{for } 1 \leq i, j \leq 3$$

▷ *The cubic invariant*

$$\mathbf{R}_{123} = (\mathbf{u}_1, \mathbf{u}_2)_1 (\mathbf{u}_2, \mathbf{u}_3)_1 (\mathbf{u}_1, \mathbf{u}_3)_1$$

Proof. See [Gor87] No. 123

□

Note the form of the symbolic expressions; for example

$$\begin{aligned} \mathbf{C}_{12} &= (\alpha\beta)^2 \\ \xi_1 &= (\beta\gamma)\beta_x\gamma_x \\ \mathbf{R}_{12,3} &= (\alpha\beta)(\beta\gamma)(\alpha\gamma) = \begin{vmatrix} \mathbf{a}_0 & \mathbf{b}_0 & \mathbf{c}_0 \\ \mathbf{a}_1 & \mathbf{b}_1 & \mathbf{c}_1 \\ \mathbf{a}_2 & \mathbf{b}_2 & \mathbf{c}_2 \end{vmatrix} \end{aligned}$$

Proposition 12. *The following relations hold between the covariants in*

$\text{Cov}(\mathbf{V}_2 \oplus \mathbf{V}_2 \oplus \mathbf{V}_2)$

1. $2\mathbf{R}_{123}^2 = \det \mathbf{C}_{ij}$
2. $\mathbf{u}_1\xi_1 + \mathbf{u}_2\xi_2 + \mathbf{u}_3\xi_3 = 0$
3. $\mathbf{R}_{123}\mathbf{u}_i = \sum_j \mathbf{C}_{ij}\xi_j$ for $i = 1, 2, 3$
4. $\sum_{ij} \mathbf{C}_{ij}\xi_i\xi_j = 0$

Proof. See [Gor87] No. 123, or [Cle72] (§58). Modern proofs are given in

[LR12].

□

3.7 Typical Presentations

Our methods in the sequel depend on Clebsch's construction of 'typical presentations' ('*typische Darstellungen*') for systems of binary forms ([Cle72] §81).

Systems where at least one of the forms is of odd order

Let f_1, \dots, f_n be a system of binary forms that contains at least one form of odd order.

Proposition 13. *In the ring of covariants of any system of forms containing at least one form of odd order there exists a pair of quadratic covariants of f_1, \dots, f_n , denoted \mathbf{u}_1 and \mathbf{u}_2 say, whose resultant does not vanish identically.*

Proof. [Cle72] §90

□

We shall now give a formula to express any form f of the system as a binary form in new variables \mathbf{u}_0 and \mathbf{u}_1 such that the coefficients are invariants and the change of variables is given by covariants; this is what is meant by 'typical presentation' of the form f .

To this end, denote \mathbf{u}_i symbolically as $\mathbf{u}_0 = \alpha_X$ and $\mathbf{u}_1 = \beta_X$. Now apply the second fundamental syzygy of section 3.4 to the forms \mathbf{u}_i and a linear

form $f = \eta_x$ to get

$$(\alpha \beta) \cdot \eta_x = (\eta \beta) \cdot \mathbf{u}_0 - (\eta \alpha) \cdot \mathbf{u}_1 \quad (3.2)$$

On taking the n -th power (introducing new symbolic quantities $\alpha^{(j)}, \beta^{(j)}$ as needed) we get an expression for any form $f = \eta_x^n$ in the covariant ring.

$$\prod_{j=1}^n \{(\alpha^{(j)} \beta^{(j)})\} \cdot f = \prod_{j=1}^n \{(\eta \beta^{(j)}) \cdot \mathbf{u}_0 - (\eta \alpha^{(j)}) \cdot \mathbf{u}_1\} \quad (3.3)$$

This is the typical presentation of a generic form of odd order.

Systems of forms of even order

In the case that all of the original forms f_ℓ are of even order, one does not dispose of independent linear covariants. One can however prove the following:

Proposition 14. *In the ring of covariants of any system of forms of even order there exists a pair of quadratic covariants, \mathbf{u}_1 and \mathbf{u}_2 say, whose resultant does not vanish identically. Given these one can always find a third quadratic covariant \mathbf{u}_3 such that all three are generically linearly independent; i.e. R_{123} is not identically zero. For example, one can take the first transvectant of \mathbf{u}_1 and \mathbf{u}_2 .*

Proof. [Cle72] §102

□

The three covariants $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ can be treated as a system of quadratic forms as in section 3.6, whence one has the invariants and covariants $\xi_i, C_{ij}, \mathbf{R}_{123}$.

Following [Cle72] (§103), we use the syzygies given in Proposition 12 to express any form of degree $2\mathbf{n}$ in terms of its invariants and covariants, as follows.

Note that the equation

$$\mathbf{R} \mathbf{u}_1 = \sum_j C_{ij} \xi_j$$

can be written symbolically as

$$\mathbf{R}_{123} \eta_X^2 = (\eta \alpha)^2 \xi_1 + (\eta \beta)^2 \xi_2 + (\eta \gamma)^2 \xi_3$$

where $f = \eta_X^2$ is any quadratic form. If we take the \mathbf{n} -th power of both sides (introducing new symbolic quantities $\alpha^{(j)}, \beta^{(j)}, \gamma^{(j)}$ as needed), we get an expression for any form $f = \eta_X^{2\mathbf{n}}$ of degree $2\mathbf{n}$ in the covariant ring.

$$\mathbf{R}_{123}^{\mathbf{n}} f = \prod_{j=1}^{\mathbf{n}} \left\{ (\eta \alpha^{(j)})^2 \xi_1 + (\eta \beta^{(j)})^2 \xi_2 + (\eta \gamma^{(j)})^2 \xi_3 \right\} \quad (3.4)$$

This is the typical presentation of a generic form of even degree.

Chapter 4

Invariants and equations for the moduli space

4.1 Expressing rational maps in terms of binary forms

In this section we show how to find the ring of invariants $(A_d)^{\mathrm{SL}_2}$ – and hence find equations for the variety M_d – by recasting the problem into the classical invariant theory question of finding the simultaneous invariants of two binary forms.

Proposition 15. *There is an isomorphism of SL_2 -modules*

$$\alpha : W_d \xrightarrow{\sim} S^{d+1}V \oplus S^{d-1}V$$

A rational map ϕ , given by homogenous polynomials $F_0(X_0, X_1)$ and $F_1(X_0, X_1)$, corresponds, under α , to a pair of binary forms (f, g) , where g is the fixed

point polynomial

$$X_1 F_0(X_0, X_1) - X_0 F_1(X_0, X_1);$$

and f is the divergence

$$\frac{\partial F_0}{\partial X_0} + \frac{\partial F_1}{\partial X_1}$$

of the map $\mathbb{A}^2 \rightarrow \mathbb{A}^2$ defined by (F_0, F_1) .

Proof. We apply the Clebsch-Gordan isomorphism of Proposition 9. This is classical, but to obtain the explicit forms, f and g , with fairly transparent dynamical significance, we shall modify the usual definitions of the operators in the Clebsch-Gordan series.

To explicitly realize the isomorphism, define the following operators

$$\begin{pmatrix} \Delta_{xx} & \Delta_{xy} \\ \Delta_{yx} & \Delta_{yy} \end{pmatrix} = \begin{pmatrix} X_0 & -X_1 \\ Y_1 & Y_0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X_0} & \frac{\partial}{\partial Y_1} \\ -\frac{\partial}{\partial X_1} & \frac{\partial}{\partial Y_0} \end{pmatrix}$$

$$\tilde{\Omega} = \det \begin{pmatrix} \frac{\partial}{\partial X_0} & \frac{\partial}{\partial Y_1} \\ -\frac{\partial}{\partial X_1} & \frac{\partial}{\partial Y_0} \end{pmatrix}$$

and write

$$(x, y) = \det \begin{pmatrix} X_0 & -X_1 \\ Y_1 & Y_0 \end{pmatrix}$$

Then one has the following identity

$$(x, y)\tilde{\Omega} = (\Delta_{xx} + 1)\Delta_{yy} - \Delta_{yx}\Delta_{xy} \tag{4.1}$$

If h is a homogenous form of degree m in X_0 and X_1 , and degree n in Y_0 and Y_1 , then

$$\Delta_{xx}h = mh$$

$$\Delta_{yy}h = nh$$

So applying the identity (4.1) one has

$$h = \frac{1}{n(m+1)}[(x, y)\tilde{\Omega} + \Delta_{yx}\Delta_{xy}]h$$

This shows that the maps

$$\alpha : S^dV \otimes V \rightarrow S^{d+1}V \oplus S^{d-1}V$$

$$h \mapsto (\Delta_{xy}h, \tilde{\Omega}h)$$

and

$$\beta : S^{d+1}V \oplus S^{d-1}V \rightarrow S^dV \otimes V$$

$$(g, f) \mapsto \frac{1}{d+1}[\Delta_{yx}g + (x, y)f]$$

are mutually inverse.

Given a rational map of degree d , written as $h = Y_0F_0(X_0, X_1) + Y_1F_1(X_0, X_1)$, we obtain a form f of degree $d - 1$ and a form g of degree $d + 1$:

$$f = \tilde{\Omega}h$$

$$g = \Delta_{xy}h$$

(4.2)

It is a simple calculation, using the above-described explicit Clebsch-Gordon maps, to check that these correspond to the divergence and fixed point polynomials as claimed in the theorem statement. \square

From now on we shall refer to a rational map of degree d as being specified either by the pair (F_0, F_1) of homogenous forms of degree d or by the pair (f, g) of forms of degree $d - 1$ and $d + 1$. We shall refer to the moduli point in M_d corresponding to a pair (f, g) with the notation $[f, g]$.

Remark 8. In [McM87] section 5, Doyle and McMullen observe that a rational map on \mathbb{P}^1 determines a homogenous 1-form on \mathbb{P}^1 . They note that such a form can be given in terms of two homogenous forms, as in the preceding proposition (for example, f would be obtained as the exterior derivative of the 1-form), but they work only in the context of finding maps with given automorphism group; they fail to draw out the consequences for the construction of M_d in terms of classical invariant theory.

Remark 9. Let $P \in \mathbb{P}^1(\bar{k})$ be a fixed point and choose coordinates ξ_0, ξ_1 for P such that $F_i(\xi_0, \xi_1) = \xi_i$ for $i = 0, 1$. Then the multiplier of the fixed point P is given by $f(\xi_0, \xi_1) - d$.

Remark 10. By work of Katsylo [Kat84], it is known that SL_2 -quotients of spaces of binary forms are rational varieties. This gives an alternative means

– apart from the direct method of [Lev11] – for proving the rationality of \mathcal{M}_d .

4.2 Invariants, Covariants and Relations

The generators of the algebra of covariants for systems of forms of low degree were calculated in the nineteenth century. We now list the basic covariants in the case relevant to the moduli space of cubic maps; namely the system of one quadratic form f and one quartic form g . In this case there are 18 basic covariants, of which six are invariants (see [GY10] § 143, [Cle72] §60). For ease of notation, define $H = (g, g)_2$ and $T = (g, H)_1$.

Deg. Ord.	1	2	3	4	5	6
0	–	$(g, g)_4$ $(f, f)_2$	$(H, g)_4$ $(g, f^2)_4$	$(H, f^2)_4$	–	$(T, f^3)_6$
2	f	$(g, f)_2$	$(H, f)_2$ $(g, f^2)_3$	$(H, f^2)_3$	$(T, f^2)_4$	–
4	g	$(g, f)_1$ $(g, g)_2$	$(H, f)_1$	$(T, f)_2$	–	–
6	–	–	$(g, H)_1$	–	–	–

Table 4.1: Covariants and Invariants for quadratic and quartic (f, g)

The six basic invariants – denoted i, j, a, b, c, d – are bihomogenous in the coefficients of f and g respectively. The multidegrees are recorded in Table 4.2.

Invariants	Degree	Multidegree
$\mathbf{d} = (\mathbf{f}, \mathbf{f})_2$	2	(2,0)
$\mathbf{i} = (\mathbf{g}, \mathbf{g})_4$	2	(0,2)
$\mathbf{j} = (\mathbf{H}, \mathbf{g})_4$	3	(0,3)
$\mathbf{a} = (\mathbf{g}, \mathbf{f}^2)_4$	3	(2,1)
$\mathbf{b} = (\mathbf{H}, \mathbf{f}^2)_4$	4	(2,2)
$\mathbf{c} = (\mathbf{T}, \mathbf{f}^3)_6$	6	(3,3)

Table 4.2: Invariants for quadratic and quartic (\mathbf{f}, \mathbf{g})

For later use we define the following invariant

$$\tilde{\mathbf{c}} = \frac{1}{6}\mathbf{d}^2\mathbf{i} - \frac{1}{2}\mathbf{a}^2 + \frac{1}{2}\mathbf{d}\mathbf{b} \quad (4.3)$$

The resultant ρ of the forms F_0 and F_1 can be expressed in terms of the basic invariants as

$$\rho = \frac{1}{8}\mathbf{i}^3 + \frac{1}{384}\mathbf{i}\mathbf{d}^2 - \frac{3}{4}\mathbf{j}^2 - \frac{3}{16}\mathbf{j}\mathbf{a} + \frac{1}{256}\mathbf{a}^2 + \frac{3}{16}\mathbf{i}\mathbf{b} - \frac{1}{64}\mathbf{d}\mathbf{b} - \frac{1}{8}\mathbf{c} \quad (4.4)$$

Remark 11. Multiplying all coefficients of a rational map ϕ by a nonzero factor α does not change the rational map; however the invariants of degree \mathbf{n} change by a factor of $\alpha^{\mathbf{n}}$. Conversely any two rational maps ϕ and ψ are conjugate over $\bar{\mathbf{k}}$ if and only if there is an element $\alpha \in \bar{\mathbf{k}}$ such that

$$(\mathbf{d}_\phi, \mathbf{i}_\phi, \mathbf{j}_\phi, \mathbf{a}_\phi, \mathbf{b}_\phi, \mathbf{c}_\phi) = (\alpha^2\mathbf{d}_\psi, \alpha^2\mathbf{i}_\psi, \alpha^3\mathbf{j}_\psi, \alpha^3\mathbf{a}_\psi, \alpha^4\mathbf{b}_\psi, \alpha^6\mathbf{c}_\psi);$$

or equivalently if and only if $(\mathbf{d}_\phi, \mathbf{i}_\phi, \mathbf{j}_\phi, \mathbf{a}_\phi, \mathbf{b}_\phi, \mathbf{c}_\phi)$ and $(\mathbf{d}_\psi, \mathbf{i}_\psi, \mathbf{j}_\psi, \mathbf{a}_\psi, \mathbf{b}_\psi, \mathbf{c}_\psi)$ represent the same point of the weighted projective space $\mathbb{P}(2, 2, 3, 3, 4, 6)$,

where, for example, d_ϕ means the value of the invariant d evaluated at the coefficients of ϕ .

In the next section we shall use various covariants of f and g . They are listed in Table 4.3.

	Covariants	Order	Degree
H	$= (g, g)_2$	4	2
T	$= (g, H)_1$	6	3
u_1	$= f$	2	1
u_2	$= (g, f)_2$	2	2
u_3	$= (H, f)_2$	2	3
ξ_1	$= (u_2, u_3)_1$	2	9
ξ_2	$= (u_3, u_1)_1$	2	8
ξ_3	$= (u_1, u_2)_1$	2	7
C_{ij}	$= (u_i, u_j)_2$	0	$i + j$ for $1 \leq i, j \leq 3$
A_i	$= (f, u_i)_2$	0	$1 + i$ for $1 \leq i \leq 3$
B_{ij}	$= (g, u_i)_2(g, u_j)_2$	0	$1 + i + j$ for $1 \leq i, j \leq 3$
r	$= -(u_1, u_2)_1(u_1, u_3)_1(u_2, u_3)_1$	0	6

Table 4.3: Covariants for cubic rational map

4.3 Relations

There is a single relation among the six basic invariants of the cubic rational map; namely the relation

$$2r^2 = \det(C_{ij}) \tag{4.5}$$

coming from the relation (1) in Proposition 12. One can calculate the C_{ij} in terms of the basic invariants (see [Cle72] § 60):

$$\begin{aligned} C_{11} &= \mathbf{d}, & C_{12} &= \mathbf{a}, & C_{13} &= \mathbf{b} \\ C_{22} &= \mathbf{b} + \frac{1}{3}\mathbf{id}, & C_{23} &= \frac{1}{6}\mathbf{ia} + \frac{1}{3}\mathbf{j}\mathbf{d}, & C_{33} &= \frac{1}{3}\mathbf{ja} - \frac{1}{6}\mathbf{ib} + \frac{1}{18}\mathbf{i}^2\mathbf{d} \end{aligned}$$

Furthermore, one can check that $\mathbf{r} = \mathbf{c}$; so the relation (4.5) becomes

$$2\mathbf{c}^2 = \frac{1}{54}\mathbf{d}^3\mathbf{i}^3 - \frac{1}{9}\mathbf{d}^3\mathbf{j}^2 - \frac{1}{12}\mathbf{d}\mathbf{i}^2\mathbf{a}^2 - \frac{1}{3}\mathbf{j}\mathbf{a}^3 + \mathbf{d}\mathbf{j}\mathbf{a}\mathbf{b} + \frac{1}{2}\mathbf{ia}^2\mathbf{b} - \frac{1}{2}\mathbf{d}\mathbf{i}\mathbf{b}^2 - \mathbf{b}^3 \quad (4.6)$$

As a corollary of these classical invariant theory computations, we have the following description of \mathbf{M}_3 .

Theorem 12. *The space \mathbf{M}_3 is isomorphic to the 4-dimensional variety in $\mathbb{P}(2, 2, 3, 3, 4, 6)$ determined by the relation (4.6) and the non-vanishing of the resultant ρ .*

4.4 Automorphisms

In this section we determine the locus of maps having automorphism group isomorphic to \mathbf{A} , for each of the possible groups \mathbf{A} listed in Proposition 4.

If a rational map ϕ corresponds to the pair of binary forms \mathbf{f} , \mathbf{g} then, for each $\gamma \in \text{Aut}(\phi)$, we must have

$$\mathbf{f} \circ \gamma = \chi(\gamma)\mathbf{f}$$

$$\mathbf{g} \circ \gamma = \chi(\gamma)\mathbf{g}$$

for some $\chi(\gamma) \in \bar{k}^\times$. It is easy to see that $\chi : G \rightarrow \bar{k}^\times$ must be a character of G . In other words f and g must be *relative invariants* with the same character. Therefore, to find representatives of the \bar{k} -conjugacy classes of maps with a given automorphism group it is enough to find relative invariants. Any relative invariant is a polynomial in the so-called *Grundformen*; that is forms whose set of zeros is equal to an exceptional orbit of the action of G on $\mathbb{P}^1(\bar{k})$ (i.e. an orbit with non-trivial stabilizer). (This approach is used in [McM87] to find maps with icosahedral automorphism group.)

By calculating Grundformen for each finite subgroup $A_0 < \mathrm{PGL}_2(\bar{k})$ from Proposition 4, we find normal forms for the maps whose automorphism group contains A_0 ; they are listed in Table 4.4. Each such normal form corresponds to a locus in the moduli space M_3 . We find a defining ideal for closure of this locus in M_3 by computing the invariants of the normal form in terms of the parameters p_i and then eliminating the p_i . In fact the loci are themselves closed; this is a corollary of the construction of rational maps from their moduli given in section 5. In this way, we have a stratification of the moduli space by automorphism group; the organization of the strata is given in Figure 4.1. Note that we do not list normal forms of binary forms that do not correspond to rational maps, i.e. those for which the resultant vanishes. There are no cubic rational maps with automorphism group \mathfrak{S}_4 or \mathfrak{S}_5 since

the Grundformen for these groups have degree at least 6 and 11 respectively.

The singular locus is related to nontrivial automorphisms as follows.

Theorem 13. *The variety M_3 is singular. The locus of points in M_3 corresponding to maps with non-trivial automorphism group, which we shall denote M_3^{Aut} , is equal to the singular locus M_3^{sing} . The locus $C_2^{(1)}$ is precisely the singular locus of the affine quasi-cone over M_3 .*

Proof. This is easy to verify computationally from the explicit description of the invariant ring in terms of generators and relations. □

Group	Name	Normal Form		Ideal	Dimension
		f	g		
$\mathbb{Z}/2\mathbb{Z}$	$C_2^{(1)}$	sX_0X_1	$tX_0^4 + uX_0^2X_1^2 + vX_1^4$	(c, \tilde{c})	2
	$C_2^{(2)}$	$sX_0^2 + tX_1^2$	$X_0X_1(uX_0^2 + vX_1^2)$	(a, j)	2
$\mathbb{Z}/3\mathbb{Z}$	C_3	sX_0^2	$X_1(tX_0^3 + uX_1^3)$	(d, i, b)	1
$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/4\mathbb{Z}) \leq \text{Aut}(\phi) \implies \text{Aut}(\phi) = \mathfrak{D}_8$; see \mathfrak{D}_8 below				
\mathfrak{D}_4	$D_4^{(1)}$	sX_0X_1	$t(X_0^4 - X_1^4)$	(a, j, c, \tilde{c})	1
	$D_4^{(2)}$	0	$s(X_0^4 + X_1^4) + tX_0^2X_1^2$	(d, a, b, c)	1
\mathfrak{D}_8	D_8	0	$s(X_0^4 + X_1^4)$	(d, j, a, b, c)	0
\mathfrak{A}_4	A_4	0	$X_0^4 - 2\sqrt{-3}X_0^2X_1^2 + X_1^4$	(c, b, a, i, d)	0

Table 4.4: Loci with non-trivial automorphism group in M_3

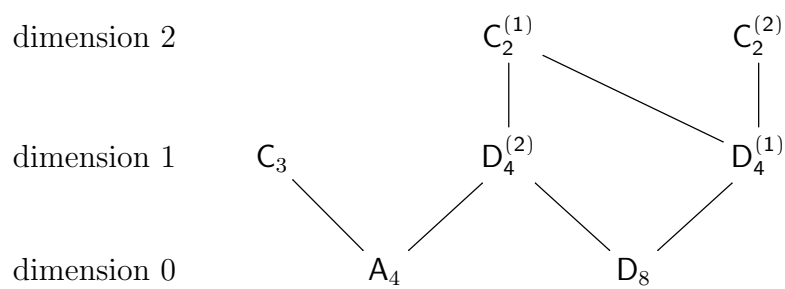


Figure 4.1: Organization of the strata

Chapter 5

FOD and FOM: Constructing Rational Maps from their Moduli

Given a point $P \in M_3(k)$ there is always a rational map ϕ defined over \bar{k} corresponding to the given point. In this section we explicitly construct such a model ϕ and investigate when it can be defined over k .

We shall see that, so long as P is not in the locus $C_2^{(1)}$, any model of P has two independent quadratic covariants. In this case, the covariants formulas of section 3.7 afford a method for constructing a model ϕ with $[\phi] = P$. In section 5.2 we explain this construction in full.

On the locus $C_2^{(1)}$, one cannot use the covariants method. In section 5.3 we give a different construction that works in this case.

5.1 Weighted projective coordinates

A point $P \in M_3(\bar{k})$ is defined over k if all the absolute invariants are in k , or equivalently if $P^\sigma = P$ for all $\sigma \in \text{Gal}(\bar{k}/k)$. By the construction of M_3 in terms of relative invariants, we have an embedding $M_3 \hookrightarrow \mathbb{P}(2, 2, 3, 3, 4, 6)$. Recall that a \bar{k} -point P of a weighted projective space $\mathbb{P}(w_0, \dots, w_n)$ can be represented by tuples $(x_0, \dots, x_n) \in \bar{k}^{n+1}$; another such tuple (x'_0, \dots, x'_n) represents the same point if and only if there exists $\alpha \in \mathbb{G}_m(\bar{k})$ such that $\alpha^{w_i} x_i = x'_i$ for $0 \leq i \leq n$. Such an $n+1$ -tuple will be called a set of *weighted projective coordinates* for the point P . We note that by Hilbert's Theorem 90, one can choose weighted projective coordinates with values in k to represent any point $P \in \mathbb{P}(w_0, \dots, w_n)$. In particular this applies to points $P \in M_3(k)$.

Lemma 14. *If $P \in M_3(k)$ is a point defined over k , then there exist values $\mathbf{d}, \mathbf{i}, \mathbf{j}, \mathbf{a}, \mathbf{b}, \mathbf{c} \in k$ of the invariants such that $P = [\mathbf{d} : \mathbf{i} : \mathbf{j} : \mathbf{a} : \mathbf{b} : \mathbf{c}]$.*

Proof. This is Hilbert's Theorem 90. In general, if $P \in \mathbb{P}(w_0, \dots, w_n)$ is a point defined over k , say represented by (x_0, \dots, x_n) , then for each $\sigma \in \text{Gal}(\bar{k}/k)$ we have $x_i^\sigma = \alpha_\sigma^{w_i} x_i$ for some $\alpha_\sigma \in \bar{k}$. Let $W_P = \{w_i : x_i \neq 0\}$, and set $\ell = \text{gcd}(W_P)$. Then the map $\sigma \mapsto \alpha_\sigma^\ell$ is well defined. It is a cocycle in $H^1(\text{Gal}(\bar{k}/k), \mathbb{G}_m(\bar{k}))$, so by Hilbert's Theorem 90 it must be a coboundary; i.e. there exists $\beta \in \mathbb{G}_m(\bar{k})$ such that $\alpha_\sigma^\ell = \beta^\sigma / \beta$. Now the coordinates

$\beta^{-w_i/\ell} \chi_i$ are defined over k . □

5.2 The covariants method

We start with the following observation.

Proposition 16. *A pair (f, g) of forms of degree 2 and 4 respectively fails to have two independent quadratic covariants if and only if both c and \tilde{c} vanish.*

Proof. [Cle72] §§107-108. □

In the case $c \neq 0$ the relevant quadratic covariants were given in Table 4.3. The associated invariants, A_i, B_{ij} and C_{ij} , were defined in section 4.3 and the expressions for C_{ij} in terms of the basic invariants were listed. For completeness, we list here the expressions for A_i and B_{ij} .

$$\begin{array}{lll} A_1 = d, & A_2 = a, & A_3 = b \\ B_{11} = a, & B_{12} = b + \frac{1}{3}id, & B_{13} = \frac{1}{6}ia + \frac{1}{3}jd \\ B_{22} = \frac{1}{2}ia + \frac{1}{3}jd, & B_{23} = \frac{1}{3}ja - \frac{1}{6}ib + \frac{1}{18}i^2d, & B_{33} = \frac{1}{3}jb - \frac{1}{36}i^2a + \frac{1}{18}dij \end{array}$$

In the following theorem we use the identities from section 3.7 to construct a model ϕ corresponding to a given moduli point P . The theorem also explicitly and effectively determines when FOM=FOD.

Theorem 15. *Let $P \in M_3(k)$ be a moduli point outside of the locus $\{c = 0\}$.*

Choose weighted projective coordinates $P = [d : i : j : a : b : c]$ with values in

\mathbf{k} ; whence we also have values $\mathbf{A}_i, \mathbf{B}_{ij}, \mathbf{C}_{ij} \in \mathbf{k}$.

Let \mathcal{C}_P be the conic in \mathbb{P}^2 with equation

$$\sum_{1 \leq i, j \leq 3} \mathbf{C}_{ij} x_i x_j = 0$$

Then

$$\mathcal{C}_P(\mathbf{k}) \neq \emptyset \implies \text{there exists a model } (f, g) \text{ of } P \text{ defined over } \mathbf{k}.$$

Explicitly, given a point in $\mathcal{C}_P(\mathbf{k})$, let

$$\begin{aligned} \theta : \quad \mathbb{P}^1 &\rightarrow \mathcal{C}_P \\ (X_0, X_1) &\mapsto (\vartheta_1, \vartheta_2, \vartheta_3) \end{aligned} \quad (5.1)$$

be the parametrization corresponding to that \mathbf{k} -point. Then

$$f = \frac{1}{\beta \mathbf{c}} \sum_{i=1}^3 \mathbf{A}_i \vartheta_i \quad (5.2)$$

and

$$g = \frac{1}{\beta^2 \mathbf{c}^2} \sum_{i=1}^3 \mathbf{B}_{ij} \vartheta_i \vartheta_j \quad (5.3)$$

are forms defined over \mathbf{k} corresponding to the point P , i.e. with $[f, g] = P$.

The factor β is defined as follows: at least one of the values

$$\frac{R_{\vartheta_1 \vartheta_2 \vartheta_3}}{R_{\xi_1 \xi_2 \xi_3}} \cdot \frac{(\xi_i \xi_j)_2}{(\vartheta_i \vartheta_j)_2}$$

is defined; those that are defined are equal, and the factor β is given by this value. Note that $(\xi_i \xi_j)_2$ and $R_{\xi_1 \xi_2 \xi_3}$ are invariants that can be written as

expressions in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{i}, \mathbf{j}$; in the above, $(\xi_i \xi_j)_2$ and $\mathbf{R}_{\xi_1 \xi_2 \xi_3}$ stand for the value of these expressions when evaluated at the point $\mathbf{P} = [\mathbf{d} : \mathbf{i} : \mathbf{j} : \mathbf{a} : \mathbf{b} : \mathbf{c}]$

Proof. We apply the quadratic invariants \mathbf{u}_i listed in Table 4.3 to obtain typical presentations of the pair (f, g) .

First we show that, in the case that $\mathcal{C}_{\mathbf{P}}(\mathbf{k})$ is non-empty, the formulas do give a model over \mathbf{k} . There exists a pair (f_0, g_0) of forms with coefficients in $\bar{\mathbf{k}}$ whose moduli point is \mathbf{P} . Moreover (f_0, g_0) can be chosen to have invariants equal to $\mathbf{d}, \mathbf{i}, \mathbf{j}, \mathbf{a}, \mathbf{b}, \mathbf{c}$. Writing $\xi_{i,0}$ for the three quadratic covariants of the pair (f_0, g_0) , and applying equation (3.4) to these forms, one obtains

$$\mathbf{c}^{-1} \sum_{i=1}^3 \mathbf{A}_i \xi_{i,0} = f_0 \quad (5.4)$$

$$\mathbf{c}^{-2} \sum_{i=1}^3 \mathbf{B}_{ij} \xi_{i,0} \xi_{j,0} = g_0 \quad (5.5)$$

Both ϑ_i and $\xi_{i,0}$ give parametrizations of $\mathcal{C}_{\mathbf{P}}$. Any two such parametrizations differ by an automorphism of \mathbb{P}^1 ; so on substituting ϑ_i for $\xi_{i,0}$ in the lefthand side of (5.4) and (5.5) one obtains a new pair of forms f_1 and g_1 , which have coefficients in \mathbf{k} and which differ from f_0, g_0 only by an automorphism of \mathbb{P}^1 . That is, there exists $M \in \mathrm{GL}_2(\bar{\mathbf{k}})$ such that $\theta_i = \xi_i \circ M$; hence also $f_1 = f_0 \circ M$ and $g_1 = g_0 \circ M$. Write $M = \alpha N$, for some choice of $N \in \mathrm{SL}_2(\bar{\mathbf{k}})$ and $\alpha \in \bar{\mathbf{k}}$. Set $\beta = \alpha^2$. Then we have $\theta_i = \beta \xi_i \circ N$; hence

also $f_1 = \beta f_0 \circ N$ and $g_1 = \beta^2 g_0 \circ N$. With this definition of β , the pair of forms (f, g) from the theorem statement are $SL_2(\bar{k})$ -equivalent to (f_0, g_0) , as claimed. It remains to find β explicitly, and to show that $\beta \in k$; then we have the desired model (f, g) , defined over k .

We first verify the claim in the theorem statement that at least one of the values

$$\frac{R_{\vartheta_1 \vartheta_2 \vartheta_3}}{R_{\xi_1 \xi_2 \xi_3}} \cdot \frac{(\xi_i \xi_j)_2}{(\vartheta_i \vartheta_j)_2} \quad (5.6)$$

is defined. Note that $R_{\xi_1 \xi_2 \xi_3}$ is non-zero by the hypothesis that the conic \mathcal{C}_P is not degenerate. On the other hand, one can calculate that the invariant $(\xi_i \xi_j)_2$ is equal to the i, j -cofactor of the matrix (C_{ij}) . The determinant of C_{ij} is $2c^2$, which is non-zero, by hypothesis; so at least one of $(\xi_i \xi_j)_2$ is non-zero, say for $i = \mu$ and $j = \nu$. We also know that $\theta_i = \beta \xi_i \circ N$; so $(\theta_i \theta_j)_2 = \beta^2 (\xi_i \xi_j)_2$ and $R_{\vartheta_1 \vartheta_2 \vartheta_3} = \beta^3 R_{\xi_1 \xi_2 \xi_3}$. From this we see that $R_{\vartheta_1 \vartheta_2 \vartheta_3}$ and $(\theta_\mu \theta_\nu)_2$ are non-zero; so the expression (5.6) is defined and equal to β , as claimed. Moreover, we have $\beta \in k$, since the values

$$R_{\vartheta_1 \vartheta_2 \vartheta_3}, \quad R_{\xi_1 \xi_2 \xi_3}, \quad (\xi_i \xi_j)_2, \quad \text{and} \quad (\vartheta_i \vartheta_j)_2$$

are all in k .

We remark that since the invariants in the cubic case are bihomogenous, one has an alternative recipe for β as follows. Write $\mathbf{d}_1, \mathbf{i}_1, \dots$ for the invariants

of the pair (f_1, g_1) . Then from the bi-degrees of the invariants (see Table 4.2) we have

$$\mathbf{d}_1 = \beta^2 \mathbf{d}, \mathbf{i}_1 = \beta^4 \mathbf{i}, \mathbf{j}_1 = \beta^6 \mathbf{j}, \mathbf{a}_1 = \beta^4 \mathbf{a}, \mathbf{b}_1 = \beta^6 \mathbf{b}, \mathbf{c}_1 = \beta^9 \mathbf{c} \quad (5.7)$$

By (4.6) and the hypothesis $\mathbf{c} \neq 0$, at least one other invariant, together with \mathbf{c} , does not vanish. So according to (5.7) we can obtain β explicitly as a quotient of the appropriate powers of invariants of (f_1, g_1) and (f, g) . In particular, β is in k^\times , since all the invariants are in k . \square

Theorem 16. *If \mathbf{P} does not belong to the locus M_3^{Aut} , the converse to Theorem 15 also holds:*

$$\mathcal{C}_{\mathbf{P}}(k) \neq \emptyset \iff \text{there exists a model } (f, g) \text{ of } \mathbf{P} \text{ defined over } k.$$

Proof. For the converse, suppose $\mathbf{P} \notin M_3^{\text{Aut}}$ and let (f', g') be a model over k corresponding to the point \mathbf{P} . Note that the invariants of (f', g') are in k . We must show $\mathcal{C}_{\mathbf{P}}(k) \neq \emptyset$. Write \mathbf{d}', \mathbf{i}' , etc. for the values of the invariants of the pair (f', g') . Since

$$[\mathbf{d}' : \mathbf{i}' : \mathbf{j}' : \mathbf{a}' : \mathbf{b}' : \mathbf{c}'] = [\mathbf{d} : \mathbf{i} : \mathbf{j} : \mathbf{a} : \mathbf{b} : \mathbf{c}]$$

as points of $\mathbb{P}(2, 2, 3, 3, 4, 6)$, there exists $\alpha \in \mathbb{G}_m(\bar{k})$ such that

$$\mathbf{I}' = \alpha^{\deg(\mathbf{I})} \mathbf{I}, \quad (5.8)$$

for any non-zero invariant I . Since $P \notin M_3^{\text{Aut}}$, at least one of \mathbf{a} or \mathbf{j} does not vanish (else the automorphism group contains $\mathbb{Z}/2\mathbb{Z}$); also, at least one of \mathbf{d} , \mathbf{i} or \mathbf{b} does not vanish (else the automorphism group contains $\mathbb{Z}/3\mathbb{Z}$). Therefore we can find two non-zero invariants with coprime degrees; so (5.8) implies α is in k . Replacing (f', g') with $(\alpha^{-1}f', \alpha^{-1}g')$, we may take (f', g') to have invariants exactly equal to $\mathbf{d}, \mathbf{i}, \mathbf{j}, \mathbf{a}, \mathbf{b}, \mathbf{c}$. The covariants ξ_i of this new pair (f', g') give a parametrization of \mathcal{C}_P defined over k ; so the conic has a k -point. \square

If $\mathbf{c} = 0$, the conic \mathcal{C}_P will be singular. But when $\tilde{\mathbf{c}} \neq 0$, one can choose a different pair of independent quadratic covariants: set $\tilde{\mathbf{u}}_1 = f$, $\tilde{\mathbf{u}}_2 = (f, g)_2$ and $\tilde{\mathbf{u}}_3 = (\mathbf{u}_2, f)_1$, then proceed to form the other covariants and invariants as in Table 4.3 but with $\tilde{\mathbf{u}}_i$ in place of \mathbf{u}_i . One has

$$\begin{aligned} \tilde{\mathbf{C}}_{11} &= \mathbf{d}, & \tilde{\mathbf{C}}_{12} &= \mathbf{a}, & \tilde{\mathbf{C}}_{13} &= 0 \\ \tilde{\mathbf{C}}_{22} &= \mathbf{b} + \frac{1}{3}\mathbf{id}, & \tilde{\mathbf{C}}_{23} &= 0, & \tilde{\mathbf{C}}_{33} &= \tilde{\mathbf{c}} \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{B}}_{11} &= \mathbf{a}, & \tilde{\mathbf{B}}_{12} &= \mathbf{b} + \frac{1}{3}\mathbf{id}, & \tilde{\mathbf{B}}_{13} &= 0 \\ \tilde{\mathbf{B}}_{22} &= \frac{1}{2}\mathbf{ia} + \frac{1}{3}\mathbf{jd}, & \tilde{\mathbf{B}}_{23} &= -\mathbf{c}, & \tilde{\mathbf{B}}_{33} &= \frac{1}{2}\mathbf{ab} - \frac{1}{12}\mathbf{iad} - \frac{1}{6}\mathbf{jd}^2 \end{aligned} \tag{5.9}$$

Applying these invariants and the construction of Theorem 15, we get an explicit solution to the FOM/FOD question that applies when $\mathbf{c} = 0$.

Theorem 17. *Let $P \in M_3(k)$ be a moduli point outside of the locus $\{\tilde{\mathbf{c}} = 0\}$. The formulas (5.1), (5.2) and (5.2) – with $\mathbf{c}, \mathbf{A}_i, \mathbf{B}_{ij}, \mathbf{C}_{ij}$ everywhere*

replaced with $\tilde{\mathbf{c}}$, $\tilde{\mathbf{A}}_i$, $\tilde{\mathbf{B}}_{ij}$, $\tilde{\mathbf{C}}_{ij}$ – yield a model (\mathbf{f}, \mathbf{g}) of \mathbf{P} . One has the following implications

▷ $\tilde{\mathcal{C}}_{\mathbf{P}}(\mathbf{k}) \neq \emptyset \implies$ there exists a model (\mathbf{f}, \mathbf{g}) of \mathbf{P} defined over \mathbf{k} .

▷ If moreover \mathbf{P} does not belong to the locus $\mathbf{M}_3^{\text{Aut}}$, then

$\tilde{\mathcal{C}}_{\mathbf{P}}(\mathbf{k}) \neq \emptyset \iff$ there exists a model (\mathbf{f}, \mathbf{g}) of \mathbf{P} defined over \mathbf{k} .

Proof. The proof is the same as that of Theorem 15 and 16.

Note that the alternative method of computing β by exploiting the bihomogeneity of the invariants does not apply in the case that $\mathbf{c} = 0$: in that case, the factor β in (5.7) appears to even powers only.

We also note that in the case $\mathbf{c} = 0$, one can obtain a simplified conic and a particularly simple parametrization as follows. From the covariant theory of quadratic forms ([Cle72] §57), we have the following relations:

$$\tilde{\xi}_1 = (\tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3)_1 = \frac{1}{2}(\tilde{\mathbf{C}}_{22}\tilde{\mathbf{u}}_1 - \tilde{\mathbf{C}}_{12}\tilde{\mathbf{u}}_2) \quad (5.10)$$

$$\tilde{\xi}_2 = (\tilde{\mathbf{u}}_3, \tilde{\mathbf{u}}_1)_1 = \frac{1}{2}(\tilde{\mathbf{C}}_{11}\tilde{\mathbf{u}}_2 - \tilde{\mathbf{C}}_{12}\tilde{\mathbf{u}}_1) \quad (5.11)$$

$$\tilde{\xi}_3 = (\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)_1 = -\tilde{\mathbf{u}}_3 \quad (5.12)$$

Moreover

$$\tilde{\xi}_3^2 = \frac{1}{2}(\tilde{\mathbf{C}}_{11}\tilde{\mathbf{u}}_2^2 - 2\tilde{\mathbf{C}}_{12}\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_2 + \tilde{\mathbf{C}}_{22}\tilde{\mathbf{u}}_1)$$

Notice from (5.9) that when $\mathbf{c} = 0$, the typical presentation of \mathfrak{g} (i.e. the analogue of (5.5) for $\tilde{\xi}_i$) contains no odd power of $\tilde{\xi}_3$; so we can use the above expressions to write a typical presentation of \mathfrak{g} entirely in terms of the invariants \tilde{B}_{ij} together with the covariants $\tilde{u}_1 = \mathbf{f}$ and \tilde{u}_2 .

Now, the substitution

$$X_1 \mapsto \tilde{C}_{12}x_2 - \tilde{C}_{22}x_1, \quad x_2 \mapsto \tilde{C}_{11}x_2 - \tilde{C}_{12}x_1, \quad x_3 \mapsto x_3,$$

corresponding to (5.10), gives a k -isomorphism of $\tilde{\mathcal{C}}_{\mathcal{P}}$ with the conic

$$\tilde{\mathcal{D}}_{\mathcal{P}} : \quad \tilde{C}_{11}x_2^2 - 2\tilde{C}_{12}x_1x_2 + \tilde{C}_{22}x_1^2 + 2x_3^2$$

As before, there exists a pair (f_0, g_0) of forms with coefficients in \bar{k} and invariants equal to $\mathbf{d}, \mathbf{i}, \mathbf{j}, \mathbf{a}, \mathbf{b}, \mathbf{c}$. Write $\tilde{u}_{i,0}$ for the quadratic covariants of the pair (f_0, g_0) . Then the map $(X_0, X_1) \mapsto (\tilde{u}_{i,0}(X_0, X_1))_i$ gives a parameterization of $\tilde{\mathcal{D}}_{\mathcal{P}}$. The system of quadratic forms $\{\tilde{u}_{i,0}\}$ has invariants $(\tilde{u}_{i,0}, \tilde{u}_{j,0})_2 = \tilde{C}_{ij}$.

Given a k -point of $\tilde{\mathcal{D}}_{\mathcal{P}}$, let $(t_1, t_2, t_3) \in k^3$ be a choice of coordinates for the point. Then one has a k -parameterization $(X_0, X_1) \mapsto (\tau_i(X_0, X_1))_i$ of $\tilde{\mathcal{D}}_{\mathcal{P}}$, where

$$\tau_1(X_0, X_1) = (t_1\tilde{C}_{11} + t_2\tilde{C}_{12})X_0^2 + 2t_2\tilde{C}_{22}X_0X_1 - t_1\tilde{C}_{22}X_1^2 \quad (5.13)$$

$$\tau_2(X_0, X_1) = -t_2\tilde{C}_{11}X_0^2 + 2t_1\tilde{C}_{11}X_0X_1 + (t_1\tilde{C}_{12} + t_2\tilde{C}_{22})X_1^2 \quad (5.14)$$

$$\tau_3(X_0, X_1) = -t_3(\tilde{C}_{11}X_0^2 + \tilde{C}_{12}X_0X_1 + \tilde{C}_{22}X_1^2) \quad (5.15)$$

Since both τ_i and $\tilde{u}_{i,0}$ are parameterizations of the same conic, there exist $N \in \mathrm{SL}_2(\bar{k})$ and $\beta \in \bar{k}$, such that $\tau_i = \beta \tilde{u}_{i,0} \circ N$. Accordingly the invariants are related by a factor of β^2 ; that is, $(\tau_i, \tau_j)_2 = \beta^2 \tilde{C}_{ij}$. On the other hand, a computation using the explicit formulas (5.13) for τ_i show that $\beta^2 = 4t_3^2$.

□

Remark 18. Given a conic \mathcal{C} over a global field k , the problem of determining whether \mathcal{C} has a k -point and finding a point, in the case that it exists, can be solved effectively. Given a k -point, it is elementary to construct a parametrization of \mathcal{C} over k . Accordingly, the above theorems are completely effective, as we demonstrate in the following example.

Example

Consider the map

$$\psi(x) = i \left(\frac{x-1}{x+1} \right)^3$$

from [Sil95]. The 6-tuple of invariants is $(72i, 10i, 3-3i, -72+72i, -48, 864i)$, which is equivalent to the point $(144, 20, -12, 288, 192, -6912) \in \mathcal{M}_3(\mathbb{Q})$. In particular, the field of moduli is \mathbb{Q} . From the invariants it is easy to check that $\mathrm{Aut}(\psi) = 1$. Applying the method of covariants as in Theorem 15 to these coordinates, one obtains the conic

$$\mathcal{C}_\psi : 144x_1^2 + 576x_1x_2 + 1152x_2^2 + 384x_1x_3 + 768x_2x_3 + 1408x_3^2 = 0.$$

This can be diagonalized to

$$144X_1^2 + 576X_2^2 + 1152X_3^2 = 0.$$

Since this clearly has no points over \mathbb{R} , the conic \mathcal{C}_ψ has no \mathbb{Q} -rational point; therefore Theorem 15 implies that ψ cannot be defined over \mathbb{Q} .

Remark 19. Whilst theorems 15, 16 and 17 were stated only for cubic maps, the analogous formulas can be used to explicitly and effectively solve the FOM/FOD problem for rational maps of any odd degree, once the invariants are known. By Proposition 14, the formulas are guaranteed to work on some dense open set of the moduli space.

5.3 Constructing maps with non-trivial automorphism group

If $\mathbf{P} \in M_3^{\text{Aut}}(\mathbf{k})$, the covariants construction in the previous section can fail in two different ways:

1. If $\mathbf{P} \in C_2^{(1)}(\mathbf{k})$, both $\mathbf{c} = 0$ and $\tilde{\mathbf{c}} = 0$, so both conics $\mathcal{C}_\mathbf{P}$ and $\tilde{\mathcal{C}}_\mathbf{P}$ are singular.
2. If $\mathbf{P} \in C_2^{(2)}(\mathbf{k})$ or $\mathbf{P} \in C_3(\mathbf{k})$, then one of the conics $\mathcal{C}_\mathbf{P}$ or $\tilde{\mathcal{C}}_\mathbf{P}$ may be non-singular, but the proofs of Theorems 15 and 17 do not give a necessary condition for the existence of a model for \mathbf{P} defined over \mathbf{k}

We shall show in sections 5.3 and 5.3 that FOM=FOD for any point $P \in C_2^{(2)}$ or $P \in C_3$ for which at least one of c or \tilde{c} does not vanish. That leaves case (1), which is dealt with in section 5.3.

Note that in this section we refer to models by the 8-tuple of coefficients of the corresponding pair of binary forms.

The locus $C_2^{(2)}$

If $P \in C_2^{(2)}(\mathbf{k})$ and \tilde{c} vanishes at P , then c also vanishes at P , by (4.6); so $P \in D_4^{(1)}(\mathbf{k})$. This case is dealt with in section 5.3 below, so we may assume $\tilde{c} \neq 0$. A point $P \in C_2^{(2)}(\mathbf{k})$ can be represented by coordinates $[\mathbf{d} : \mathbf{i} : 0 : 0 : \mathbf{b} : \mathbf{c}]$ with values in \mathbf{k} and with $\mathbf{d} = -2\lambda^2$, for some $\lambda \in \mathbf{k}$ (to see this, take arbitrary coordinates for P with values in \mathbf{k} , then act by $\alpha = \sqrt{-2\mathbf{d}} \in \mathbb{G}_m(\bar{\mathbf{k}})$).

Using these values, the conic \tilde{C}_P has equation

$$-2\lambda^2 X_1^2 + \left(\frac{1}{3}\mathbf{d}\mathbf{i} + \mathbf{b}\right) X_2^2 - \lambda^2 \left(\frac{1}{3}\mathbf{d}\mathbf{i} + \mathbf{b}\right) X_3^2 = 0$$

This has the \mathbf{k} -point $[0 : \lambda : 1]$. Therefore, when $\mathbf{c} \neq 0$, the point P always has a model defined over the field of moduli, by the construction of Theorem 17.

In the case $\mathbf{c} = 0$, one can check that the model

$$\left[-2\lambda X_0 X_1, \lambda^{-3} \left(\frac{1}{3}\mathbf{d}\mathbf{i} + \mathbf{b}\right) + 2\lambda X_0 X_1^3\right]$$

corresponds to P .

The locus C_3

If $P \in C_3(k)$, and at least one of \mathbf{a} or \mathbf{j} does not vanish, the conic \mathcal{C}_P is

$$\mathbf{a}X_0X_1 - \frac{1}{3}\mathbf{j}aX_3^2,$$

which has the k -point $[1 : \frac{1}{3}\mathbf{j} : 1]$. From this one obtains a model (f, g) defined over k . Explicitly, (f, g) has coefficients

$$\left[\frac{-\mathbf{j}^2\mathbf{a}^2}{9\mathbf{c}}, \frac{-2\mathbf{j}\mathbf{a}^2}{3\mathbf{c}}, \frac{-\mathbf{a}^2}{\mathbf{c}}, \frac{2\mathbf{j}^4\mathbf{a}^4 + 9\mathbf{j}^2\mathbf{a}^3}{81\mathbf{c}^2}, \frac{2\mathbf{j}^3\mathbf{a}^3}{9\mathbf{c}^2}, \frac{2\mathbf{j}^2\mathbf{a}^3}{3\mathbf{c}^2}, \frac{2\mathbf{j}\mathbf{a}^3}{3\mathbf{c}^2}, 0 \right]$$

If $\mathbf{j} = 0$, i.e. $P = [0 : 0 : 0 : 1 : 0 : 0]$, one has a model with the following coefficients

$$[0, 0, 1, 0, 1, 0, 0, 1]$$

If $\mathbf{a} = 0$ then $P \in A_4(k)$ – see below.

The locus $C_2^{(1)}$: the case where there is no pair of independent quadratic covariants

In this case $\mathbf{c} = \tilde{\mathbf{c}} = 0$ and the covariants method yields no information. Nonetheless, in this case one can use Gröbner bases to find the reconstruction from the invariants and the obstruction to ‘FOM=FOD’, as detailed in the following proposition.

Proposition 17. *Let $P \in C_2^{(1)}(\mathbf{k})$ with weighted projective coordinates*

$$P = [\mathbf{d} : \mathbf{i} : \mathbf{j} : \mathbf{a} : \mathbf{b} : 0].$$

1. *If $\mathbf{d} \neq 0$, then P has a model over \mathbf{k} if and only if the conic defined by*

$$9\mathbf{d}^3X^2 + 8\mathbf{d}^2Y^2 - 24\mathbf{d}\mathbf{a}YZ + (-36\mathbf{d}^2\mathbf{i} + 72\mathbf{a}^2)Z^2 = 0 \quad (5.16)$$

has a point, say (x, y, z) , over \mathbf{k} . In that case set

$$\begin{aligned} \mathbf{c}_5 &= x/z & ; & & \mathbf{c}_6 &= y/z \\ \mathbf{c}_3 &= \mathbf{d}/2 & ; & & \mathbf{c}_4 &= \frac{2\mathbf{a}}{\mathbf{d}^2} - \frac{1}{3\mathbf{d}}\mathbf{c}_6. \end{aligned} \quad (5.17)$$

Then the model over \mathbf{k} is $[1, 0, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6, -\mathbf{c}_3\mathbf{c}_5, \mathbf{c}_3^2\mathbf{c}_4]$.

2. *If $\mathbf{d} = 0$, then $P \in D_4^{(2)}(\mathbf{k})$. Assume neither \mathbf{i} nor \mathbf{j} vanishes. In this case, P always has a model over \mathbf{k} , given by*

$$[0, 0, 0, -27\mathbf{i}^3, -27\mathbf{i}^3, 0, 24\mathbf{j}^2, 0].$$

3. *If $\mathbf{d} = \mathbf{j} = 0$, then $P \in D_8(\mathbf{k})$, and P always has a model over \mathbf{k} , given by $[0, 0, 0, 1, 0, 0, 0, 1]$.*

4. *If $\mathbf{d} = \mathbf{i} = 0$, then $P \in A_4(\mathbf{k})$. Over $\bar{\mathbf{k}}$ one has the model ψ with coefficients $[0, 0, 0, 1, 0, 2\sqrt{-3}, 0, 1]$. The obstruction $I_{\mathbf{k}}(\psi)$ contains a class corresponding to the conic*

$$X^2 + 3Y^2 - 2Z^2 = 0$$

We need the following two lemmas.

Lemma 20. [Bea10] Let $B \leq \mathrm{PGL}_2(\mathbf{k})$ be a cyclic subgroup of order two.

Then B is conjugate by an element of $\mathrm{PGL}_2(\mathbf{k})$ to $A_\alpha := \langle z \mapsto \alpha/z \rangle$ for some $\alpha \in \mathbf{k}^\times / (\mathbf{k}^\times)^2$.

Lemma 21. If $\phi = (f, g)$ is a model defined over \mathbf{k} for a point $P \in \mathbb{C}_2^{(1)}(\mathbf{k})$ with $\mathbf{d} \neq 0$, then each element of $\mathrm{Aut}(\phi)$ is defined over \mathbf{k} .

Proof. From the normal form in Table 4.4, one can see that fixed points of the involution $\gamma \in \mathrm{Aut}(\phi)$ are precisely the roots of f . Since f has coefficients in \mathbf{k} , the element γ must also be defined over \mathbf{k} . \square

Proof. (of Proposition 17) For part (1): suppose that $\mathbf{d} \neq 0$ and that P has a model ϕ over \mathbf{k} . Then by the preceding two lemmas, we may assume that $\mathrm{Aut}(\phi)$ is generated by $\langle z \mapsto \alpha/z \rangle$ for some $\alpha \in \mathbf{k}^\times / (\mathbf{k}^\times)^2$. Computing relative invariants for this group as in Section 4.4, one sees that ϕ must have the form

$$(X_0^2 - \alpha X_1^2, c_4(X_0^4 + \alpha^2 X_1^4) + c_5(X_0^3 X_1 + \alpha X_0 X_1^3) + c_6 X_0^2 X_1^2)$$

for some $c_4, c_5, c_6 \in \mathbf{k}$. Set $c_3 = -\alpha$. By computing a Gröbner basis, one can see that the only non-trivial relations between the coefficients c_i and the invariants are those given in equations (5.16) and (5.17). This establishes the assertion of the proposition.

One can also construct the conic (5.16) by explicit cohomology computations, without using the two lemmas. Start with the model ψ defined over $L = k(\sqrt{-2\mathbf{d}})$ with coefficients

$$[0, \sqrt{-2\mathbf{d}}, 0, 1, 0, -3\mathbf{a}/\mathbf{d}, 0, \mathbf{i}/2 - 3\mathbf{a}^2/4\mathbf{d}^2];$$

this can easily be found using Gröbner bases. Set $\mathbf{e} = \mathbf{i}/2 - 3\mathbf{a}^2/4\mathbf{d}^2$. The model has $\mathbf{A} = \text{Aut}(\psi) = \langle z \mapsto -z \rangle$. And $\mathbf{N}(\mathbf{A}) = \mathfrak{D}_\infty = \mathbb{G}_m \rtimes \mu_2$. The class \mathbf{c}_ψ is represented by the cocycle

$$\gamma_\tau = \begin{cases} 1 & \text{if } \tau \in \mathbf{G}_L \\ (z \mapsto r/z) & \text{otherwise} \end{cases}$$

Write $\mathbf{G}_{L/k} = \langle \sigma \rangle$. By inflation-restriction we can reduce from \mathbf{G}_k to $\mathbf{G}_{L/k}$, where we have the following situation (see [Sil95]: proof of Theorem 3.2):

$$\begin{array}{ccc} \begin{pmatrix} 1 & \mapsto & (1,1) \\ \sigma & \mapsto & (x,-1) \end{pmatrix} & \mathbf{H}^1(\mathbf{G}_{L/k}, \mathbf{L}^\times \rtimes \mu_2(\mathbf{L})) & \xrightarrow{\mathbf{p}} & \mathbf{H}^1(\mathbf{G}_{L/k}, \mathbf{L}^\times \rtimes \mu_2(\mathbf{L})) \\ \uparrow & \downarrow \mathcal{I} & & \downarrow \mathcal{I} \\ \frac{1}{x} & \frac{\mathbf{k}^\times / \text{Nm}_{L/k}(\mathbf{L}^\times)}{x = x^{-1}} & \xrightarrow{x \mapsto x^2} & \frac{\mathbf{k}^\times / \text{Nm}_{L/k}(\mathbf{L}^\times)}{x = x^{-1}} \end{array}$$

Since $\frac{\mathbf{k}^\times / \text{Nm}_{L/k}(\mathbf{L}^\times)}{x = x^{-1}}$ is a group of exponent 2, the element \mathbf{c}_ψ is in the image of \mathbf{p} if and only if it is trivial; that is, if and only if either r is the norm of an element of L or $\tau = 1$; that is, if and only if there is a solution in k to $X^2 + 2\mathbf{d}Y^2 - rZ^2$. One can easily check that this conic is isomorphic over k to the one in the theorem statement.

For part (4): The model ψ was computed in Section 4.4: it is the normal form of a map with automorphism group $\mathbf{A} = \langle z \mapsto -z, z \mapsto \frac{1}{z}, z \mapsto i\frac{z+1}{z-1} \rangle$; we have $\mathbf{N}(\mathbf{A}) = \mathbf{A} \rtimes \langle \frac{z+1}{z-1} \rangle \simeq \mathfrak{S}_4$, and $\mathbf{Q} \simeq \{\pm 1\}$. Let $\mathbf{L} = \mathbf{k}(\sqrt{-3})$ and write $\mathbf{G}_{\mathbf{L}/\mathbf{k}} = \langle \sigma \rangle$. From the model ψ , we compute that

$$\gamma_\tau = \begin{cases} 1 & \text{if } \tau \in \mathbf{G}_{\mathbf{L}} \\ -1 & \text{otherwise} \end{cases}$$

is a cocycle representing \mathbf{c}_ψ . Using the splitting coming from the presentation of \mathbf{A} as a semidirect product we can find an inverse image for \mathbf{c}_ψ under \mathbf{p} ; namely

$$\eta_\tau = \begin{cases} 1 & \text{if } \tau \in \mathbf{G}_{\mathbf{L}} \\ z \mapsto \frac{z+1}{z-1} & \text{otherwise} \end{cases}$$

Computing the twist corresponding to this PGL_2 -cocycle, we obtain the conic from the statement.

The other cases are easily verified by computing the invariants of the claimed models, which come from the normal forms given in Section 4.4.

□

Chapter 6

Even degrees: Quadratic Maps

Covariants can be used to explicitly construct generic maps of any even degree from their moduli.

In this section we work through the covariants method in the case of quadratic maps in order to illustrate the general method for even degree maps and to make clear how the invariant theory approach to M_d tallies with the description in [Sil98] and [MY11].

6.1 The invariants and covariants

Given a quadratic rational map $\phi = F_0/F_1$, one obtains a pair of binary forms – one cubic form g and one linear form f – using the Clebsch-Gordan isomorphism. Then one can construct the covariants by transvection as described in [GY10] (§ 138). The covariants that we shall need are given in Table 6.1. The ring of invariants is generated by s_1, s_2, s_3 and R . Since R is the only

	Covariants	Order	Degree	
H	= $(g, g)_2$	2	2	
t	= $(g, H)_1$	3	3	
s ₁	= $(g, f^3)_3$	0	4	
s ₂	= $(H, f^2)_2$	0	4	
s ₃	= $(t, g)_3$	0	4	
R	= $(t, f^3)_3$	0	6	
V ₀	= $(H, f)_1$	1	3	
V ₁	= $(g, f^2)_2$	1	3	
b ₀	= $(V_1, f)_1$	0	4	
b ₁	= $(V_0, f)_1$	0	4	
r	= $(b_0, b_1)_1$	0	6	
a _{ijk}	= $(g, V_i)_1(g, V_j)_1(g, V_k)_1$	0	10	for $0 \leq i, j, k \leq 1$

Table 6.1: Covariants for quadratic maps

invariant of degree 6, r must be a multiple of R ; in fact $R = r$.

One usually uses σ_1 and σ_2 – the first and second elementary symmetric functions in the multipliers of the map – as the coordinates on M_2 . If one writes

$$\sigma_i = \tau_i \cdot \rho^{-1} \quad \text{for } i = 1, 2$$

where ρ is the resultant of F_0 and F_1 , then one can relate the invariants from

Table 6.1 to σ_1 and σ_2 as follows:

$$\begin{aligned} s_1 &= 5\tau_1 + 2\tau_2 + 6\rho \\ s_2 &= \frac{2}{3}\tau_1 + \frac{2}{3}\tau_2 - 4\rho \\ s_3 &= \frac{-4}{27}\tau_1 + \frac{2}{27}\tau_2 + \frac{2}{9}\rho \end{aligned}$$

There is a single relation among the four basic invariants:

$$r^2 - \frac{1}{2}s_1^2s_3 + \frac{1}{2}s_2^3 = 0 \quad (6.1)$$

Let \mathbf{A} be the ring of invariants graded by degree. Then the fourth Veronese subring $\mathbf{A}^{[4]}$ is generated by s_1, s_2, s_3 and r^2 . But (6.1) expresses r^2 as a polynomial in the other invariants, so

$$\text{Proj } \mathbf{A} \cong \text{Proj } \mathbf{A}^{[4]} = \text{Proj } k[s_1, s_2, s_3] = \text{Proj } k[\tau_1, \tau_2, \rho] = \mathbb{P}(4, 4, 4) \cong \mathbb{P}^2$$

Thus one recovers the usual description of $\mathcal{M}_2 \cong \mathbb{A}^2$ as the complement of the line $\{\rho = 0\}$ inside $\text{Proj } k[\tau_1, \tau_2, \rho] = \mathbb{P}^2$.

Let us write the relation (6.1) in terms of τ_1, τ_2 and ρ :

$$r^2 = -2\tau_1^3 - \tau_1^2\tau_2 + \tau_1^2\rho + 8\tau_1\tau_2\rho - 12\tau_1\rho^2 + 4\tau_2^2\rho - 12\tau_2\rho^2 + 36\rho^3$$

Comparing with [Sil12] Prop. 4.15, one can see that the locus $\{r = 0\}$ is precisely the locus in \mathcal{M}_2 of maps with non-trivial automorphism group.

6.2 Constructing a model from the moduli

Remark 22. The method of this section will work for maps of any even degree.

For concreteness, we illustrate it only in the quadratic case.

We first note that α_{ijk} are invariant under permutations of the indices.

In terms of the basic invariants they are

$$\alpha_{000} = \frac{1}{9}s_3r; \quad \alpha_{100} = 0; \quad \alpha_{110} = \frac{-1}{9}s_2r; \quad \alpha_{111} = \frac{-2}{9}s_1r$$

Theorem 23. *Let $P \in M_2(k)$ be a moduli point corresponding to a \bar{k} -conjugacy class of maps with trivial automorphism group. Let $s_1, s_2, s_3, r \in k$ be weighted projective coordinates with values in k corresponding to P (see Lemma 14). Let $W_0, W_1 \in k[X_0, X_1]$ be any pair of linear forms for which the corresponding coordinate transformation*

$$X_0 \mapsto W_0(X_0, X_1) \quad ; \quad X_1 \mapsto W_1(X_0, X_1)$$

is in $SL_2(k)$. Then the linear and cubic forms

$$\begin{aligned} f &= b_0W_0 - b_1W_1 \\ g &= \frac{9}{2r} \sum_{i,j,k \in \mathbb{F}_2} (-1)^{i+j+k} \alpha_{ijk} W_{i+1} W_{j+1} W_{k+1} \end{aligned}$$

have coefficients in k and correspond to the moduli point P , i.e. $[f, g] = P$.

Proof. We apply the typical presentation for systems of odd order forms from section 3.7.

Note that $\mathbf{b}_i, \mathbf{a}_{ijk} \in \mathbf{k}$, since $s_1, s_2, s_3, r \in \mathbf{k}$. There exists a pair (f_0, g_0) of forms with coefficients in $\bar{\mathbf{k}}$ whose invariants are equal to the s_1, s_2, s_3 and r from the theorem statement. Applying (3.2) and (3.3) to the linear form f_0 and cubic form g_0 , and using the notation of Table 6.1, one has

$$\begin{aligned} rf_0 &= \mathbf{b}_0 V_0 - \mathbf{b}_1 V_1 \\ 2r^3 g_0 &= 9 \sum_{i,j,k \in \mathbb{F}_2} (-1)^{i+j+k} \mathbf{a}_{ijk} V_{i+1} V_{j+1} V_{k+1} \end{aligned}$$

Note that V_i are covariants of the pair (f_0, g_0) . Since the point P is outside of the locus of maps with nontrivial automorphism group, r is nonzero.

Given any pair $W_0, W_1 \in \mathbf{k}[X_0, X_1]$ of linearly independent linear forms, let $M \in \mathrm{GL}_2(\bar{\mathbf{k}})$ be the coordinate transformation from V_0, V_1 , to W_0, W_1 . Write $M = \alpha N$ for $N \in \mathrm{SL}_2(\bar{\mathbf{k}})$ and $\alpha \in \bar{\mathbf{k}}$. Note that $r = \det[V_0, V_1] = \alpha^{-2}$. Then

$$\begin{aligned} rf_1 &= \mathbf{b}_0 W_0 - \mathbf{b}_1 W_1 \\ 2r^3 g_1 &= 9 \sum_{i,j,k \in \mathbb{F}_2} (-1)^{i+j+k} \mathbf{a}_{ijk} W_{i+1} W_{j+1} W_{k+1} \end{aligned}$$

On the other hand, $f_1 = \alpha^{-1}f_0 \circ N$ and $g_1 = \alpha^{-3}g_0 \circ N$. Setting $f = f_1$ and $g = r^{-1}g_1$ one obtains the forms a pair of forms projectively equivalent to those from the theorem statement; their invariants are $(r^2s_i)_{i=1}^3$, so they correspond to the moduli point \mathcal{P} , as required. \square

Appendices

Appendix A

Invariants for cubic rational maps

The expressions for the invariants terms of the coefficients c_i of the pair

$$f = c_1 X_0^2 + c_2 X_0 X_1 + c_3 X_1^2$$

$$g = c_4 X_0^4 + c_5 X_0^3 X_1 + c_6 X_0^2 X_1^2 + c_7 X_0 X_1^3 + c_8 X_1^4$$

of quadratic and quartic forms are

$$\begin{aligned}
\mathbf{d} &= -\frac{1}{2}\mathbf{c}_2^2 + 2\mathbf{c}_1\mathbf{c}_3 \\
\mathbf{i} &= \frac{1}{6}\mathbf{c}_6^2 - \frac{1}{2}\mathbf{c}_5\mathbf{c}_7 + 2\mathbf{c}_4\mathbf{c}_8 \\
\mathbf{j} &= -\frac{1}{36}\mathbf{c}_6^3 + \frac{1}{8}\mathbf{c}_5\mathbf{c}_6\mathbf{c}_7 - \frac{3}{8}\mathbf{c}_4\mathbf{c}_7^2 - \frac{3}{8}\mathbf{c}_5^2\mathbf{c}_8 + \mathbf{c}_4\mathbf{c}_6\mathbf{c}_8 \\
\mathbf{a} &= \mathbf{c}_3^2\mathbf{c}_4 - \frac{1}{2}\mathbf{c}_2\mathbf{c}_3\mathbf{c}_5 + \frac{1}{6}\mathbf{c}_2^2\mathbf{c}_6 + \frac{1}{3}\mathbf{c}_1\mathbf{c}_3\mathbf{c}_6 - \frac{1}{2}\mathbf{c}_1\mathbf{c}_2\mathbf{c}_7 + \mathbf{c}_1^2\mathbf{c}_8 \\
\mathbf{b} &= -\frac{1}{8}\mathbf{c}_3^2\mathbf{c}_5^2 + \frac{1}{3}\mathbf{c}_3^2\mathbf{c}_4\mathbf{c}_6 + \frac{1}{12}\mathbf{c}_2\mathbf{c}_3\mathbf{c}_5\mathbf{c}_6 - \frac{1}{36}\mathbf{c}_2^2\mathbf{c}_6^2 - \frac{1}{18}\mathbf{c}_1\mathbf{c}_3\mathbf{c}_6^2 - \frac{1}{2}\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_7 \\
&\quad + \frac{1}{24}\mathbf{c}_2^2\mathbf{c}_5\mathbf{c}_7 + \frac{1}{12}\mathbf{c}_1\mathbf{c}_3\mathbf{c}_5\mathbf{c}_7 + \frac{1}{12}\mathbf{c}_1\mathbf{c}_2\mathbf{c}_6\mathbf{c}_7 \\
&\quad - \frac{1}{8}\mathbf{c}_1^2\mathbf{c}_7^2 + \frac{1}{3}\mathbf{c}_2^2\mathbf{c}_4\mathbf{c}_8 + \frac{2}{3}\mathbf{c}_1\mathbf{c}_3\mathbf{c}_4\mathbf{c}_8 - \frac{1}{2}\mathbf{c}_1\mathbf{c}_2\mathbf{c}_5\mathbf{c}_8 + \frac{1}{3}\mathbf{c}_1^2\mathbf{c}_6\mathbf{c}_8 \\
\mathbf{c} &= \frac{1}{32}\mathbf{c}_3^3\mathbf{c}_5^3 - \frac{1}{8}\mathbf{c}_3^3\mathbf{c}_4\mathbf{c}_5\mathbf{c}_6 - \frac{1}{32}\mathbf{c}_2\mathbf{c}_3^2\mathbf{c}_5^2\mathbf{c}_6 + \frac{1}{8}\mathbf{c}_2\mathbf{c}_3^2\mathbf{c}_4\mathbf{c}_6^2 + \frac{1}{4}\mathbf{c}_3^3\mathbf{c}_4^2\mathbf{c}_7 \\
&\quad - \frac{1}{16}\mathbf{c}_2\mathbf{c}_3^2\mathbf{c}_4\mathbf{c}_5\mathbf{c}_7 + \frac{1}{32}\mathbf{c}_2^2\mathbf{c}_3\mathbf{c}_5^2\mathbf{c}_7 + \frac{1}{32}\mathbf{c}_1\mathbf{c}_3^2\mathbf{c}_5^2\mathbf{c}_7 - \frac{1}{8}\mathbf{c}_2^2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_6\mathbf{c}_7 - \frac{1}{8}\mathbf{c}_1\mathbf{c}_3^2\mathbf{c}_4\mathbf{c}_6\mathbf{c}_7 \\
&\quad + \frac{1}{32}\mathbf{c}_2^3\mathbf{c}_4\mathbf{c}_7^2 + \frac{3}{16}\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_7^2 - \frac{1}{32}\mathbf{c}_1\mathbf{c}_2^2\mathbf{c}_5\mathbf{c}_7^2 - \frac{1}{32}\mathbf{c}_1^2\mathbf{c}_3\mathbf{c}_5\mathbf{c}_7^2 + \frac{1}{32}\mathbf{c}_1^2\mathbf{c}_2\mathbf{c}_6\mathbf{c}_7^2 \\
&\quad - \frac{1}{32}\mathbf{c}_1^3\mathbf{c}_7^3 - \frac{1}{2}\mathbf{c}_2\mathbf{c}_3^2\mathbf{c}_4^2\mathbf{c}_8 + \frac{1}{4}\mathbf{c}_2^2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_5\mathbf{c}_8 + \frac{1}{4}\mathbf{c}_1\mathbf{c}_3^2\mathbf{c}_4\mathbf{c}_5\mathbf{c}_8 - \frac{1}{32}\mathbf{c}_2^3\mathbf{c}_5^2\mathbf{c}_8 \\
&\quad - \frac{3}{16}\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_5^2\mathbf{c}_8 + \frac{1}{8}\mathbf{c}_1\mathbf{c}_2^2\mathbf{c}_5\mathbf{c}_6\mathbf{c}_8 + \frac{1}{8}\mathbf{c}_1^2\mathbf{c}_3\mathbf{c}_5\mathbf{c}_6\mathbf{c}_8 - \frac{1}{8}\mathbf{c}_1^2\mathbf{c}_2\mathbf{c}_6^2\mathbf{c}_8 - \frac{1}{4}\mathbf{c}_1\mathbf{c}_2^2\mathbf{c}_4\mathbf{c}_7\mathbf{c}_8 \\
&\quad - \frac{1}{4}\mathbf{c}_1^2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_7\mathbf{c}_8 + \frac{1}{16}\mathbf{c}_1^2\mathbf{c}_2\mathbf{c}_5\mathbf{c}_7\mathbf{c}_8 + \frac{1}{8}\mathbf{c}_1^3\mathbf{c}_6\mathbf{c}_7\mathbf{c}_8 + \frac{1}{2}\mathbf{c}_1^2\mathbf{c}_2\mathbf{c}_4\mathbf{c}_8^2 - \frac{1}{4}\mathbf{c}_1^3\mathbf{c}_5\mathbf{c}_8^2
\end{aligned}$$

Appendix B

Magma Scripts

Electronic copies of a ‘magma’ computer algebra package containing routines to carry out the reconstruction algorithm detailed in this thesis will be made available.

Bibliography

- [Bea10] Arnaud Beauville, *Finite subgroups of $\mathrm{PGL}_2(\mathbf{K})$* , Vector bundles and complex geometry, Contemp. Math., vol. 522, Amer. Math. Soc., Providence, RI, 2010, pp. 23–29. MR 2681719 (2011h:20096)
- [Bri96] Michel Brion, *Invariants et covariants des groupes algébriques réductifs*, Online (1996).
- [Cad08] Anna Cadoret, *Lifting results for rational points on Hurwitz moduli spaces*, Israel J. Math. **164** (2008), 19–59. MR 2391139 (2009d:14026)
- [Cle72] A. Clebsch, *Theorie der binären algebraischen formen*, Teubner, 1872.
- [Dol03] Igor Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003. MR 2004511 (2004g:14051)

- [GKN14] Dragos Ghioca, Holly Krieger, and Khoa Nguyen, *A case of the dynamical andre-oort conjecture*, arXiv preprint arXiv:1404.6011 (2014).
- [Gor87] Paul Gordan, *Vorlesungen über Invariantentheorie*, second ed., Chelsea Publishing Co., New York, 1987, Erster Band: Determinanten. [Vol. I: Determinants], Zweiter Band: Binäre Formen. [Vol. II: Binary forms], Edited by Georg Kerschensteiner. MR 917266 (89g:01034)
- [GY10] John Hilton Grace and Alfred Young, *The algebra of invariants*, Cambridge Library Collection, Cambridge University Press, Cambridge, 2010, Reprint of the 1903 original. MR 2850282 (2012g:01022)
- [Kat84] P. I. Katsylo, *Rationality of fields of invariants of reducible representations of the group SL_2* , Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1984), no. 5, 77–79. MR 764040 (86c:14009)
- [Lev11] Alon Levy, *The space of morphisms on projective space*, Acta Arith. **146** (2011), no. 1, 13–31. MR 2741188 (2012d:37211)

- [LR12] R. Lercier and C. Ritzenthaler, *Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects*, J. Algebra **372** (2012), 595–636.
- [McM87] Curt McMullen, *Families of rational maps and iterative root-finding algorithms*, Ann. of Math. (2) **125** (1987), no. 3, 467–493.
MR 890160 (88i:58082)
- [Mes91] Jean-François Mestre, *Construction de courbes de genre 2 à partir de leurs modules*, Effective methods in algebraic geometry (Castiglioncello, 1990), Progr. Math., vol. 94, Birkhäuser Boston, Boston, MA, 1991, pp. 313–334. MR 1106431 (92g:14022)
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906 (95m:14012)
- [Mil93] John Milnor, *Geometry and dynamics of quadratic rational maps*, Experiment. Math. **2** (1993), no. 1, 37–83, With an appendix by the author and Lei Tan. MR 1246482 (96b:58094)

- [MY11] Michelle Manes and Yu Yasufuku, *Explicit descriptions of quadratic maps on \mathbb{P}^1 defined over a field K* , Acta Arith. **148** (2011), no. 3, 257–267. MR 2794930 (2012d:37212)
- [Olv99] Peter J. Olver, *Classical invariant theory*, London Mathematical Society Student Texts, vol. 44, Cambridge University Press, Cambridge, 1999. MR 1694364 (2001g:13009)
- [Osg92] William F. Osgood, *The Symbolic Notation of Aronhold and Clebsch*, Amer. J. Math. **14** (1892), no. 3, 251–261. MR 1505594
- [Sil95] Joseph H. Silverman, *The field of definition for dynamical systems on \mathbf{P}^1* , Compositio Math. **98** (1995), no. 3, 269–304. MR 1351830 (96j:11090)
- [Sil98] ———, *The space of rational maps on \mathbf{P}^1* , Duke Math. J. **94** (1998), no. 1, 41–77. MR 1635900 (2000m:14010)
- [Sil07] ———, *The arithmetic of dynamical systems*, Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007. MR 2316407 (2008c:11002)

- [Sil12] ———, *Moduli spaces and arithmetic dynamics*, CRM Monograph Series, vol. 30, American Mathematical Society, Providence, RI, 2012. MR 2884382
- [ST06] Lucien Szpiro and Thomas J. Tucker, *Algebraic dynamics*, Colloquium De Giorgi 2006, Colloquia, vol. 1, Ed. Norm., Pisa, 2006, pp. 51–57. MR 2252829 (2008m:14045)
- [ST08] ———, *A Shafarevich-Faltings theorem for rational functions*, Pure Appl. Math. Q. 4 (2008), no. 3, part 2, 715–728. MR 2435841 (2009g:14027)