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Some Bernstein Type Results of Graphical  
Self-Shrinkers with High Codimension in  
Euclidean Space

by

Hengyu Zhou

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2015

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

Some Bernstein type results  
of graphical self-shrinkers with high codimension  
in Euclidean space

by

Hengyu Zhou

Advisors: Professor Zheng Huang, Professor Yunping Jiang

A self-shrinker characterizes the type I singularity of the mean curvature flow. In this thesis we concern about some Bernstein type results of graphical self-shrinkers with high codimension in Euclidean space.

There are two main tools in our work. The first one is structure equations of graphical self-shrinkers in terms of parallel forms (Theorem 2.3.6). This is motivated by M.T.Wang's work ([Wan02]) on graphical mean curvature flows with arbitrary codimension in product manifolds. The second one is an integration technique (Lemma 2.4.5) based on the fact that every graphical self-shrinker has the polynomial volume growth property (Corollary 2.4.4). Because of it the derivations of all results are independent of the maximal

principle of elliptic equations.

A general process we attack the rigidity of graphical self-shrinkers mainly consists of the following two steps:

- a) derive the structure equation of graphical self-shrinkers under certain geometric conditions;
- b) apply the integration technique to establish the minimality of the graphical self-shrinkers.

The rigidity follows from the well-known fact that every minimal, complete, smooth self-shrinker is a plane through 0 (Theorem 2.2.4). An example is given in §2.5 to illustrate the above process.

This thesis is organized as follows. Chapter 1 is devoted to our main results and some geometric background. We also discuss some Bernstein type results on minimal submanifolds in Euclidean space and the long time existence of graphical mean curvature flows in product manifolds.

In Chapter 2, we construct main tools to explore graphical self-shrinkers. They includes structure equations of self-shrinkers in term of parallel forms, the integration technique and other technique results. In Chapter 3 we discuss the rigidity of graphical self-shrinker surfaces in  $\mathbb{R}^4$  with codimension two. In Chapter 4 we investigate the rigidity of graphical self-shrinkers with

arbitrary codimension under certain geometric conditions. In Chapter 5, we give a new proof of the fact that a Lagrangian self-shrinker of zero Maslov class is a plane through 0 if its Lagrangian angle has a upper bound or lower bound. Here we use the structure equation of Lagrangian angles (Lemma 5.1.3.)

# Acknowledgements

First of all, I would like to express my deepest gratitude to my advisors Professor Zheng Huang and Professor Yunping Jiang for all their guidance and helps for years during my Ph.D. study in the Department of Mathematics at CUNY, Graduate Center. Without their supports and encouragements, I never have been able to finish my dissertation.

It is Professor Yunping Jiang who helped me to apply a fellowship at US for pursuing my Ph.D. degree. He generously shared his research experiences with me which directed me to find the appropriate research direction. His patience and encouragement are very appreciated when my research interests were transferred from complex dynamics into differential geometry.

Professor Zheng Huang is very insightful and considerate. His profound knowledge and passion for mathematics always driven me for excellence. Not only he encouraged me to study mean curvature flows but also gave me an model to do research with a professional manner.



I am also highly indebted to Professor Jun Hu and Professor Olympia Hadjiliadis for their helps when I worked at Brooklyn College. With the helps from Professor Olympia Hadjiliadis I got a teaching adjunct position to support my life of the last year at Graduate Center. The talks with Professor Jun Hu helped me to go back the correct way when I lost the direction about my thesis study.

I am very appreciated for helps from Professor Linda Keen and Professor Józef Dodziuk. Their suggestions helped me to avoid various mistakes in my Ph.D life and focus on the research.

I would also like to thank Professor Longzhi Lin and Professor Dan Lee for their useful comments, suggestions during the writing of this dissertation. I want to express my thanks to Professor Mao-Pei Tsui and Professor Mu-Tao Wang for very inspired discussions.

I am very grateful to express my appreciations for helps and suggestions from Professor Jiaqiang Mei, Professor Gaofei Zhang on the job search.

Also many thanks to my friends and classmates: Haifeng Chu, Chong Wu, Tao Chen, Zhe Wang and Liang Zhao.

Finally, my thanks go to my family. This thesis would not have been possible without their supports over years.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Main results . . . . .	1
1.2	Bernstein type results of minimal submanifolds in Euclidean space . . . . .	6
1.3	Hypersurfaces v.s. high codimension . . . . .	10
<b>2</b>	<b>Some properties of self-shrinkers</b>	<b>15</b>
2.1	Minimality of self-shrinkers . . . . .	16
2.2	Vanishing normal projection . . . . .	18
2.3	Parallel forms and structure equations . . . . .	22
2.4	Volume growth for self-shrinkers . . . . .	29
2.5	A proof of Theorem 1.2.4 . . . . .	32
<b>3</b>	<b>Graphical self-shrinkers in <math>\mathbb{R}^4</math></b>	<b>34</b>
3.1	Structure equations for graphical self-shrinkers in $\mathbb{R}^4$ . . . . .	35

<i>CONTENTS</i>	x
3.2 The proof of Theorem A . . . . .	40
<b>4 Graphical self-shrinkers in <math>\mathbb{R}^{n+l}</math></b>	<b>44</b>
4.1 Structure equations of graphical self-shrinkers in $\mathbb{R}^{n+l}$ . . . . .	44
4.2 The proof of Theorem B . . . . .	48
<b>5 Rigidity of Lagrangian self-shrinkers</b>	<b>52</b>
5.1 Structure equations of Lagrangian angles . . . . .	53
5.2 The proof of Theorem C . . . . .	57
<b>Bibliography</b>	<b>59</b>

# Chapter 1

## Introduction

In this thesis we concern about Bernstein type results of graphical self-shrinkers in Euclidean space.

The purpose of this chapter is to introduce our main results in this thesis and develop some geometric background. §1.1 contains preliminary definitions and main results in this thesis. In §1.2, we review some Bernstein type results about minimal submanifolds in Euclidean space. In §1.3 we discuss the difference between hypersurfaces and submanifolds with high codimension. Some results about the long time existence of graphical mean curvature flows in product manifolds is presented.

### 1.1 Main results

Let us define the second fundamental form, the mean curvature vector and the mean curvature flow first.

**Definition 1.1.1.** Suppose  $N$  is a smooth  $n$ -dimensional submanifold of a Riemannian manifold  $M$  with dimension  $n + k$ . We denote by  $\{e_i\}_{i=1}^n$  an orthonormal frame in the tangent bundle of  $N$ . The second fundamental form  $A$  and the mean curvature vector  $\vec{H}$  of  $N$  are defined by

$$A(e_i, e_j) = (\bar{\nabla}_{e_i} e_j)^\perp \quad (1.1.1)$$

$$\vec{H} = \sum_{i=1}^n (\bar{\nabla}_{e_i} e_i)^\perp \quad (1.1.2)$$

where ‘ $\perp$ ’ is the projection into the normal bundle of  $N$  and  $\bar{\nabla}$  is the covariant derivative of the ambient manifold  $M$ .

**Definition 1.1.2.** A mean curvature flow  $F_t(N)$  with the initial data  $N$  is the solution of the following quasilinear equation

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = \vec{H}(x, t) , \\ F(x, 0) = x , \quad \text{for } x \in N \end{cases} \quad (1.1.3)$$

where  $\vec{H}(x, t)$  is the mean curvature vector of  $F_t(N)$  in  $M$  when  $t$  is fixed.

When  $N$  is a closed smooth submanifold, it is easy to see that the mean curvature flow  $F_t(N)$  exists on  $[0, T)$  for some  $T$ .

**Definition 1.1.3.** A smooth submanifold  $\Sigma^n$  in  $\mathbb{R}^{n+k}$  is a self-shrinker if the equation

$$\vec{H} + \frac{1}{2} \vec{F}^\perp = 0 \quad (1.1.4)$$

holds for any position vector  $\vec{F}$  on  $\Sigma^n$ . Here  $\vec{H}$  is the mean curvature vector of  $\Sigma^n$  and  $\vec{F}^\perp$  is the projection of the position vector  $\vec{F}$  on  $\Sigma^n$ .

Self-shrinkers are important in the study of the mean curvature flow for at least two reasons. First, if  $\Sigma$  is a self-shrinker, one easily verifies that

$$\Sigma_t = \sqrt{-t}\Sigma,$$

for  $-\infty < t < 0$ , is a solution to the mean curvature flow. On the other hand, by Huisken ([Hui90]) the blow-ups around a type I singularity converge weakly to self-shrinkers after rescaling and choosing subsequences. Since there are no closed minimal submanifolds in Euclidean space, the finite time singularity of the mean curvature flow with initial compact smooth hypersurface is unavoidable. Therefore it is desirable to classify self-shrinkers under various geometric conditions.

Throughout the thesis, a plane is an  $n(n \geq 2)$  dimensional totally geodesic subspace in  $\mathbb{R}^{n+k}$  isometric to  $\mathbb{R}^n$ .

Our main results are stated as follows. First we study graphical self-shrinker surfaces in  $\mathbb{R}^4$  with codimension two.

**Theorem A.** *Let  $f = (f_1(x_1, x_2), f_2(x_1, x_2))$  be a smooth map from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  with its Jacobian  $J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$  satisfying one of the following*

conditions:

$$(1) J_f + 1 > 0 \text{ for all } x \in \mathbb{R}^2,$$

$$(2) 1 - J_f > 0 \text{ for all } x \in \mathbb{R}^2.$$

If  $\Sigma = (x, f(x))$  is a self-shrinker in  $\mathbb{R}^4$ , then  $\Sigma$  is a plane through 0.

Chapter 3 is devoted to the proof of Theorem A. We are not sure that whether the above result is optimal. This result is similar to the Bernstein theorem for minimal two dimensional graphs ([CO67]) in  $\mathbb{R}^4$ .

For the arbitrary codimension case, we get a Bernstein type result as follows.

**Theorem B.** *Suppose  $f$  is a smooth map from  $\mathbb{R}^n$  to  $\mathbb{R}^l$ ,  $df : T_x\mathbb{R}^n \rightarrow T_{f(x)}\mathbb{R}^l$  is the differential of  $f$  and  $r$  is the rank of  $df$ . Let  $\{\lambda_i\}_{i=1}^r$  denote the eigenvalue of  $df$ . If  $\Sigma = (x, f(x))$  is a self-shrinker in  $\mathbb{R}^{n+l}$  and  $|\lambda_i\lambda_j| \leq 1$  for  $i \neq j$ , then  $\Sigma$  is a plane through 0.*

The proof of Theorem B is given in Chapter 4.

**Remark 1.1.4.** The proof of Theorem B is given in Chapter 4. It was first obtained by Ding-Wang in [DW10] as an application of the maximal principle for harmonic functions. Here our proof does not rely on the maximal principle, it may be of independent interests.

At last we consider Lagrangian self-shrinkers in  $\mathbb{C}^n$ , which is Euclidean space  $\mathbb{R}^{2n}$  with the complex structure. Let  $J$  and  $\omega$  stand for the standard complex structure and the standard symplectic form of  $\mathbb{C}^n$ . The closed complex  $n$ -form is given by

$$\Omega = dz_1 \wedge \cdots \wedge dz_n$$

and the symplectic form is

$$\omega = \sum_i dx_i \wedge dy_i$$

where  $z_j = x_j + iy_j$  for  $j = 1, \dots, n$  are complex coordinates of  $\mathbb{C}^n$ .

A smooth  $n$ -dimensional submanifold  $L$  in  $\mathbb{C}^n$  is said to be *Lagrangian* if  $\omega_L \equiv 0$ . This means that  $\omega(X, Y) = g(JX, Y) = 0$  for any tangent vectors  $X, Y$  of  $L$ . A simple computation shows that

$$\Omega_L = e^{i\theta} vol_L$$

where  $vol_L$  denotes the volume form of  $L$  and  $\theta$  is a multivalued function called the *Lagrangian angle*. When  $\theta$  is a single valued function, the Lagrangian is of *zero-Maslov*.  $L$  is said to have the *polynomial volume growth property* if

$$\int_{L \cap B_r(0)} \leq Cr^n;$$

for any  $r \geq 1$  and  $B_r(0)$  is the ball centered at 0 with radius  $r$  in  $\mathbb{C}^n$ .



**Theorem C.** *Assume  $L$  is a smooth, complete zero-Maslov Lagrangian self-shrinker with the polynomial volume growth property and its Lagrangian angle  $\theta$  satisfying one of the followings:*

(1)  $\theta + C_1 > 0$  for all points on  $L$ ;

(2)  $\theta + C_2 < 0$  for all points on  $L$ ;

where  $C_1, C_2$  are some constants. Then  $L$  is a plane through  $0$ .

We prove Theorem C in Chapter 5.

**Remark 1.1.5.** In [Nev11], Neves established the above result under the condition that the Lagrangian angle  $\theta$  is uniformly bounded via applying the Gaussian density of the mean curvature flow. Here we require that  $\theta$  has a finite upper bound or a finite lower bound. Our proof depends on the structure equation of the Lagrangian angle (Lemma 5.1.3).

## 1.2 Bernstein type results of minimal submanifolds in Euclidean space

The Bernstein theorem of minimal graphs in Euclidean space is one of the fundamental results of minimal manifolds in Euclidean space. We recall some related results in this direction. Our work are motivated by the comparison between the Bernstein theorem of graphical self-shrinkers and minimal

graphs. In particular, we can compare Theorem 1.2.1 and Theorem 1.2.4.

First let us see the case of minimal graphs with codimension one.

**Theorem 1.2.1** ([Sim68]). *Assume  $2 \leq n \leq 7$ . Let  $f$  be a  $C^2$  function on  $\mathbb{R}^n$ . If  $\Sigma = (x, f(x))$  is minimal in  $\mathbb{R}^{n+1}$ , then  $f$  is a linear function and  $\Sigma$  is a plane.*

**Remark 1.2.2.** When  $n \geq 8$ , Bombieri, De Giorgi and Giusti gave an example called the Simon cone that  $\Sigma$  is minimal but not planar in [BDGG69].

Let  $Df$  denote the gradient of  $f$ . With some restrictions on  $Df$ , Ecker-Huisken established that

**Theorem 1.2.3** ([EH90]). *Assume  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function with  $|Df| = o(\sqrt{|x|^2 + |f|^2})$  as  $|x| \rightarrow \infty$ . If  $\Sigma = (x, f(x))$  is minimal, then  $f$  is a linear function and  $\Sigma$  is a plane.*

We digress minimal surfaces for a while. For graphical self-shrinkers with codimension one, its Bernstein type result takes a more stronger version.

**Theorem 1.2.4** ([EH89],[Wan11]). *Let  $f$  be a  $C^2$  function on  $\mathbb{R}^n$ . If  $\Sigma = (x, f(x))$  is a self-shrinker in  $\mathbb{R}^{n+1}$ , then  $f$  is a linear function and  $\Sigma$  is a plane through 0.*

The underlying reason that Theorem 1.2.4 is stronger than Theorem 1.2.1 is that a graphical self-shrinker has the polynomial volume growth property (Corollary 2.4.4). In fact, In ([EH89]) Ecker-Huisken showed Theorem 1.2.4 under the assumption that  $\Sigma$  is of polynomial volume growth (Definition 2.4.1). L. Wang removed this assumption in ([Wan11]).

Now we continue the review on the Bernstein theorem of minimal submanifolds. The high codimension case of the Bernstein theorem for minimal surfaces was established earlier in 1960s by Chern-Osserman ([CO67]). Here is an abridged version of their results.

**Theorem 1.2.5** ([CO67]). *Assume  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^l (l \geq 2)$  is a smooth map with  $|Df| \leq C$ . If  $\Sigma = (x, f(x))$  is minimal in  $\mathbb{R}^{2+l}$ , then  $f$  is a linear function and  $\Sigma$  is a plane.*

Here  $Df$  is the gradient of  $f$ . More surprisingly, Fischer-Colbrie ([FC80]) obtained the following rigidity in the case of  $n = 3$  when she investigated the rigidity of minimal surfaces in the sphere.

**Theorem 1.2.6** ([FC80]). *Assume  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^l (l \geq 2)$  is a smooth map with  $|Df| \leq C$ . If  $\Sigma = (x, f(x))$  is minimal in  $\mathbb{R}^{3+l}$ , then  $f$  is a linear function and  $\Sigma$  is a plane.*

**Remark 1.2.7.** Fischer-Colbrie applied the blow-down technique to convert

the Bernstein problem in the above setting into the rigidity of minimal graph on the sphere  $S^{2+l}$ . In this conversion, the monotonicity formula of minimal surface ( §17 in [Sim83]) plays an essential role. For self-shrinkers with our knowledge there is no similar monotonicity formula.

For the case of the arbitrary dimension and codimension, we need the concept of the Hodge star of a parallel form.

Assume  $M$  is a Riemannian manifold and  $N$  is an  $n$  dimensional smooth submanifold. Let  $\Omega$  be a parallel  $n$  form on  $M$  if  $\bar{\nabla}\Omega = 0$  where  $\bar{\nabla}$  is the covariant derivative of  $M$ . The Hodge star of  $\Omega$  on  $N$  is given by

$$*\Omega = \frac{\Omega(X_1, \dots, X_n)}{\sqrt{\det(g_{ij})}} \quad (1.2.1)$$

where  $\{X_1, \dots, X_n\}$  is a local frame on  $N$  and  $g_{ij} = \langle X_i, X_j \rangle$ . The definition of  $*\Omega$  only dependent on the orientation of the frame  $\{X_1, \dots, X_n\}$ .

With this concept, M.T. Wang ([Wan03]) obtained a Bernstein type result as follows.

**Theorem 1.2.8.** ([Wan03]) *Assume  $f : \mathbb{R}^n \rightarrow \mathbb{R}^l (l \geq 2)$  is a smooth map and  $\{\lambda_i\}_{i=1}^r$  are the eigenvalue of  $df : \mathbb{R}^n \rightarrow \mathbb{R}^l$  with the rank  $r$ . Let  $\Omega = dx_1 \wedge \dots \wedge dx_n$ . If there are two constants  $K > 0, 0 < \delta < 1$  such that  $\Sigma = (x, f(x))$  is minimal in  $\mathbb{R}^{n+l}$  with  $|\lambda_i \lambda_j| < 1 - \delta$  for any  $i \neq j$  and  $*\Omega \geq K$ , then then  $f$  is a linear map and  $\Sigma$  is a plane.*

**Remark 1.2.9.** For similar Bernstein type results, we refer to Jost-Xin and Yang ([JX99], [JXY13]). While the results in ([Wan03]) are motivated by graphical mean curvature flows with arbitrary codimension (see [Wan02]), the results of ([JX99], [JXY13]) are based on the rigidity of harmonic functions. In this thesis, we prefer to viewing the tools in [Wan02] as one of our motivations since  $\sqrt{-t}\Sigma$  is a graphical mean curvature flow when  $\Sigma$  is a graphical self-shrinker .

### 1.3 Hypersurfaces v.s. high codimension

In the recent study of self-shrinkers, most of important progresses are for self-shrinker hypersurfaces. Since Lagrangian mean curvature flows and Lagrangian self-shrinkers are always of high codimension, it is desirable to study self-shrinkers with high codimension.

To illustrate the motivation of our work in this thesis more clearly, it is necessary to discuss some difficulties in the study of self-shrinkers with high codimension. We also present related results about the long time existence of graphical mean curvature flows with arbitrary codimension. The reader will find a weak correspondence between the rigidity results of graphical self-shrinkers in this thesis and the long time existence of graphical mean curvature flows. It may be of independent interests for the future study.

The contrast between hypersurfaces and higher codimension submanifolds is the main obstacle to investigate the self-shrinker with high codimension. It can be seen at least two points.

Firstly, the normal bundle of hypersurfaces is always trivial, but that of submanifolds with higher codimension could be very complicated. The mean curvature vector of a hypersurface is a scalar function but that of a submanifold with high codimension is a vector. Because of this, the computations about the second fundamental form and the mean curvature vector could be very involved in the study of self-shrinkers. A famous example is about the self-shrinker with non-zero mean curvature vector. In the case of hypersurfaces, Huisken ([Hui90]) established that a self-shrinker with non-zero mean curvature vector must be a sphere. In the case of self-shrinkers with high codimension, Smoczyk ([Smo05]) showed that a self-shrinker with non-zero mean curvature vector must be a sphere with an additional requirement that  $\bar{\nabla}^\perp \vec{v}$  is vanishing in the normal bundle for any normal vector  $\vec{v}$ .

Secondly, there are few tools to attack the problems about self-shrinkers with arbitrary codimension. One of the recent remarkable progresses in the study of self-shrinkers is the generic singularity of self-shrinkers with codimension one by Colding-Minicozzi ([CM12a]). With the concept of the stability of a self-shrinker, they showed that a stable, complete self-shrinker

(codimension one) with polynomial volume growth is a sphere, a cylinder and a plane. This stability concept is also valid for self-shrinkers with high codimension (Lee-Lee ([LL15])). It is not clear whether there is a corresponding version for Colding-Minicozzi's generic singularity results in the situation of high codimension.

Next we see how to deal with high codimension problems of mean curvature flows. This is necessary since a graphical self-shrinker corresponds to a graphical mean curvature flow in Euclidean space. It is tempting to deal with graphical self-shrinkers with the tools used to explore graphical mean curvature flows.

A classical way to investigate graphical mean curvature flows is M.T. Wang's seminal work in [Wan02]. His main idea is to investigate  $*\Omega$  along graphical mean curvature flow in product manifolds where  $\Omega$  is a parallel form. This idea is generalized into various settings by Wang and his coauthors ([Wan01b], [SW02], [TW04], [MW11]).

We start with the long time existence of entire graphs obtained by Ecker-Huisken [EH91]. First we view  $\mathbb{R}^{n+1}$  as the product manifold of  $\mathbb{R}^n \times \mathbb{R}^1$  with the canonical coordinate  $(x_1, \dots, x_n, x_{n+1})$ . After necessary rotations and translations, an entire graph is a graphical hypersurface over  $\mathbb{R}^n$  in  $\mathbb{R}^{n+1}$ . Notice that the mean curvature flow of entire graphs is a special case of

graphical mean curvature flows with arbitrary codimension.

**Theorem 1.3.1** (Theorem 4.6, [EH91]). *Let  $f$  be a smooth function on  $\mathbb{R}^n$  and let  $N = (x, f(x))$  be a smooth graph over  $\mathbb{R}^n$  in  $\mathbb{R}^{n+1}$ . Denote by  $\vec{v}$  the normal vector of  $N$ . If the angle function  $\theta = \langle \vec{v}, \frac{\partial}{\partial x_{n+1}} \rangle \geq c$  for a positive constant  $c$ , then the mean curvature flow  $F_t(N)$  exists smoothly for all time with  $\theta \geq c$ .*

**Remark 1.3.2.** The corresponding version of self-shrinkers for the above result is Theorem 1.2.4.

In [TW04], Tsui-Wang obtained the long time existence of mean curvature flows of the graph of area decreasing maps. A map  $f$  between Riemannian manifold  $N_1$  and Riemannian manifold  $N_2$  is *area-decreasing* if  $|\lambda_i \lambda_j| \leq 1 - \delta$  for  $i \neq j$  and  $1 > \delta > 0$ . Here  $\{\lambda_i\}_{i=1}^n$  is the eigenvalue of the differential  $df : T_x N_1 \rightarrow T_{f(x)} N_2$ .

**Theorem 1.3.3** ([TW04]). *Let  $N_1$  and  $N_2$  be compact Riemannian manifolds of constant section curvatures  $k_1$  and  $k_2$ , respectively. Suppose  $k_1 \geq |k_2|$  and  $k_1 + k_2 > 0$ . Suppose  $f$  is a smooth area decreasing map. We denote by  $\Sigma$  the graph of  $f$ . Then the mean curvature flow  $F_t(\Sigma)$  exists for all time.*

**Remark 1.3.4.** The corresponding version of self-shrinkers for the above result is Theorem B.



When  $N_1, N_2$  are two dimensional surfaces, Wang established the following result.

**Theorem 1.3.5** ([Wan01a]). *Let  $N_1$  and  $N_2$  be compact Riemann surface with the same constant curvature. Suppose  $f$  is a smooth area preserving map. That is  $\lambda_1\lambda_2 = 1$  where  $\lambda_1, \lambda_2$  are the eigenvalue of  $df : T_xN_1 \rightarrow T_{f(x)}N_2$ . We denote by  $\Sigma$  the graph of  $f$ . Then the mean curvature flow  $F_t(\Sigma)$  exists for all time.*

**Remark 1.3.6.** The corresponding version of self-shrinkers for the above result is Theorem A.

Although there is an interesting correspondence between our main results in this thesis and the long time existence of graphical mean curvature flows, their proofs are based on totally different techniques. Their underlying relationships are still expected to be clarified.

## Chapter 2

# Some properties of self-shrinkers

In this chapter, we prove some technique results about self-shrinkers. In §2.1 we present the fact that self-shrinker is minimal in Euclidean space with a weighted metric. In §2.2 we give a direct proof that in Euclidean space a smooth, complete manifold with the vanishing normal projection is totally geodesic (See Theorem 2.2.1). §2.3 is devoted to the parallel form theory of self-shrinkers. We derive the general structure equations of self-shrinkers in terms of parallel forms. In §2.4 we discuss the polynomial volume growth property of complete, proper self-shrinkers. A key integration result (Lemma 2.4.5) for self-shrinkers with a well-behaved structure equation is established. At last, we use the tools developed in this chapter to give a very short proof of Theorem 1.2.4. This reveals a general process to establish the main results in this thesis.

## 2.1 Minimality of self-shrinkers

In fact the self-shrinker in Euclidean space is minimal in Euclidean space with a certain weighted metric. This idea is very helpful when one considers the compactness of self-shrinkers ([CM12b],[CM14]). It is a well-known result (For example, see [CM12b]). For the sake of completeness, we give its proof here.

**Theorem 2.1.1.** *Let  $\Sigma$  be an  $n$  dimensional smooth manifold in  $\mathbb{R}^{n+n_1}$ . We denote by  $dx^2$  the standard Euclidean metric. If  $\Sigma$  is a self-shrinker in  $(\mathbb{R}^{n+n_1}, dx^2)$ , then  $\Sigma$  is minimal in  $(\mathbb{R}^{n+n_1}, e^{-\frac{|x|^2}{2n}} dx^2)$ .*

*Proof.* Let  $\{e_k\}_{k=1}^n$  be a local frame on  $\Sigma$ . Let  $(\mathbb{R}_1, g_1)$ ,  $(\mathbb{R}_2, g_2)$  stand for  $(\mathbb{R}^{n+k}, dx^2)$ ,  $(\mathbb{R}^{n+k}, e^{-\frac{|x|^2}{2n}} dx^2)$ , respectively. For  $i = 1, 2$ , we set

$$g_{i,kl} = g_i(e_k, e_l), \quad (g_i^{kl}) = (g_{i,kl})^{-1}.$$

$\bar{\nabla}^i$  denotes covariant derivative of  $\mathbb{R}_i$ . Assume that  $\{n_\alpha\}_{\alpha=1}^k$  is an orthonormal frame in the normal bundle of  $\Sigma$  in  $\mathbb{R}_1$ .

Let  $\vec{F}$  be the position vector of  $\Sigma$ . The mean curvature vectors  $\vec{H}_1$  ( $\vec{H}_2$ ) of  $\Sigma$  with respect to  $\mathbb{R}_1$  ( $\mathbb{R}_2$ ) can be written as follows:

$$\vec{H}_1 = g_1^{kl} g_1(\bar{\nabla}_{e_k}^1 e_l, n_\alpha) n_\alpha; \quad (2.1.1)$$

$$\vec{H}_2 = e^{\frac{|\vec{F}|^2}{2n}} g_2^{kl} g_2(\bar{\nabla}_{e_k}^2 e_l, n_\alpha) n_\alpha; \quad (2.1.2)$$

By the Koszul formula, we have

$$\begin{aligned}
2g_1(\bar{\nabla}_{e_k}^1 e_l, n_\alpha) &= e_k g_1(e_l, n_\alpha) + e_l g(e_k, n_\alpha) - n_\alpha g_1(e_k, e_l) + g_1([e_k, e_l], n_\alpha) \\
&\quad - g_1([e_k, n_\alpha], e_l) - g_1([e_l, n_\alpha], e_k); \\
&= -n_\alpha g_1(e_k, e_l) + g_1([e_k, e_l], n_\alpha) - g_1([e_k, n_\alpha], e_l) - g_1([e_l, n_\alpha], e_k);
\end{aligned}$$

Using  $g_2(X, Y) = e^{-\frac{|\bar{F}|^2}{2n}} g_1(X, Y)$  and the Koszul formula, we get

$$\begin{aligned}
2g_2(\bar{\nabla}_{e_k}^2 e_l, n_\alpha) &= e_k g_2(e_l, n_\alpha) + e_l g(e_k, n_\alpha) - n_\alpha g_2(e_k, e_l) + g_2([e_k, e_l], n_\alpha) \\
&\quad - g_2([e_k, n_\alpha], e_l) - g_2([e_l, n_\alpha], e_k); \\
&= -n_\alpha g_2(e_k, e_l) + g_2([e_k, e_l], n_\alpha) - g_2([e_k, n_\alpha], e_l) - g_2([e_l, n_\alpha], e_k); \\
&= e^{-\frac{|\bar{F}|^2}{2n}} \frac{g_1(\vec{x}, n_\alpha)}{n} g_2(e_k, e_l) + \\
&\quad e^{-\frac{|\bar{F}|^2}{2n}} (-n_\alpha g_1(e_k, e_l) + g_1([e_k, e_l], n_\alpha) - g_1([e_k, n_\alpha], e_l) - g_1([e_l, n_\alpha], e_k)); \\
&= e^{-\frac{|\bar{F}|^2}{2n}} \left( \frac{g_1(\vec{x}, n_\alpha)}{n} g_1(e_k, e_l) + 2g_1(\bar{\nabla}_{e_k}^1 e_l, n_\alpha) \right)
\end{aligned}$$

Together with (2.1.2), the above computations indicate that

$$\begin{aligned}
\vec{H}_2 &= g_2^{kl} g_2(\bar{\nabla}_{e_k}^2 e_l, n_\alpha) n_\alpha; \\
&= e^{-\frac{|\bar{F}|^2}{2n}} \left( \frac{g_1(\vec{F}, n_\alpha) n_\alpha}{2} + \vec{H}_1 \right); \\
&= e^{-\frac{|\bar{F}|^2}{2n}} \left( \frac{\vec{F}^\perp}{2} + \vec{H}_1 \right).
\end{aligned}$$

Here we use  $\vec{F}^\perp = g_1(\vec{F}, n_\alpha) n_\alpha$ . Since  $\Sigma$  is a self-shrinker in  $\mathbb{R}_1$ , we have

$$\frac{\vec{F}^\perp}{2} + \vec{H}_1 \equiv 0$$

Therefore  $\Sigma$  is minimal in  $\mathbb{R}_2$ . We complete the proof.  $\square$

**Remark 2.1.2.** From this result, the Bernstein problem for  $n$  dimensional graphical self-shrinkers is the Bernstein problem for  $n$  dimensional minimal graphs in the weighted Euclidean space  $(\mathbb{R}^{n+n_1}, e^{-\frac{|x|^2}{2n}} dx^2)$ .

## 2.2 Vanishing normal projection

In this section, we prove the following result. A smooth submanifold  $N$  in Euclidean space has the *vanishing normal projection* if  $\vec{F}^\perp \equiv 0$  where  $\vec{F}$  is the position vector of  $N$  and  $\vec{F}^\perp$  is the projection of  $\vec{F}$  into the normal bundle of  $N$ . The following result seems to be well-known. With our knowledge we can not find its proof in the literature. For the sake of the completeness we write a proof here.

**Theorem 2.2.1.** *Let  $U$  be an open set in  $\mathbb{R}^{n+n_1}$ . If  $\Sigma$  is a smooth, complete  $n$ -dimensional submanifold in  $U$  such that its point  $\vec{F}$  verifies  $\vec{F}^\perp \equiv 0$ , then  $\Sigma$  is totally geodesic in  $U$ .*

**Remark 2.2.2.** An advantage of our proof is that it works locally. A proof that a smooth, minimal, complete self-shrinker  $\Sigma$  is totally geodesic can be briefly stated as follows (See Corollary 2.8, [CM12a]). If  $\Sigma$  is minimal, then  $\sqrt{-t}\Sigma = \Sigma$  since  $\sqrt{-t}\Sigma$  is a solution of the mean curvature flow. Therefore

$\Sigma$  is a smooth minimal cone, the rigidity of  $\Sigma$  follows from that a smooth minimal cone is a totally geodesic plane. If a self-shrinker is minimal only in its open subset, then the above derivations are invalid. In this case, we can not claim the self-shrinker is a smooth minimal cone, which is a global property.

*Proof.* Choose any point  $\vec{F}_1$  on  $\Sigma$ . We denote by  $\{e_i\}_{i=1}^n$  the orthonormal frame of the tangent bundle and denote by  $\{n_\alpha\}_{\alpha=1}^k$  the normal bundle of  $\Sigma$  in a neighborhood of  $\vec{F}_1$ . Without confusion, we assume this neighborhood is still  $U \cap \Sigma$ . We define a set  $V$  in  $U \cap \Sigma$  as follows:

$$V = \{\vec{F} : \vec{F} \in U \cap \Sigma, \quad \langle \vec{F}, e_i \rangle \neq 0 \quad \text{for } i = 1, \dots, n\}$$

We claim that  $V$  is an open dense set in  $U \cap \Sigma$ .

It is obvious that  $V$  is open. If  $V$  is not dense, without loss of generality, we can suppose that there is an open set  $W$  in  $U \cap \Sigma$  such that  $\langle \vec{F}, e_1 \rangle \equiv 0$  in  $W$ . This gives a representation for the position vector  $\vec{F}$  of  $\Sigma$  in  $W$  as follows:

$$\begin{aligned} \vec{F} &= \vec{F}^\perp + \sum_{i=1}^n \langle \vec{F}, e_i \rangle e_i; \\ &= \sum_{i=2}^n \langle \vec{F}, e_i \rangle e_i; \end{aligned} \tag{2.2.1}$$

Then we can take  $\{\langle \vec{F}, e_2 \rangle, \dots, \langle \vec{F}, e_n \rangle\}$  as a coordinate of  $W$ . This leads

to a contradiction since such coordinates implies we have a diffeomorphism from  $W$  to an open set of  $\mathbb{R}^{n-1}$ . However  $W$  is an  $n$  dimensional open set. Hence  $W$  is empty and  $V$  is an open dense set of  $U \cap \Sigma$ .

For any normal vector  $n_\alpha$ ,  $A^\alpha$  denotes the second fundamental matrix  $(h_{ij}^\alpha)$  where  $h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} e_j, n_\alpha \rangle$  and  $\bar{\nabla}$  is the covariant derivative of  $\mathbb{R}^{n+k}$ . We define the following sets:

$$V_k = \{\vec{F} : \vec{F} \in V, \text{rank}(A^\alpha) = k\}; \quad \text{for } k = 0, 1, \dots, n$$

It is easy to see that  $V = V_0 \cup \cup_{k=1} V_k$  and  $A^\alpha \equiv 0$  on  $V_0$ . Assume we can show that  $V_0$  is dense in  $V$ . By the continuity of  $A^\alpha$ , we will obtain that  $A^\alpha \equiv 0$  in  $V$ . Since  $V$  is dense in  $U \cap \Sigma$ ,  $A^\alpha \equiv 0$  in  $U \cap \Sigma$  for any normal vector  $n_\alpha$ . This gives that  $\Sigma$  is totally geodesic in  $U$ .

Next we show that  $V_0$  is dense in  $V$ . Let  $V_k^0$  be the interior of  $V_k$  for  $k \geq 1$ . To prove the denseness of  $V_0$  in  $V$ , it is suffice to prove that  $V_k^0$  is empty for  $k \geq 1$ .

Fix a  $k$  where  $k \geq 1$ . Suppose  $V_k^0$  is not empty. Assume  $\{X_1, \dots, X_{n-k}\}$  with  $X_i \in \mathbb{R}^n$  is an basis of the linear vector space consisting of all solutions for the linear equation

$$XA^\alpha \equiv 0$$

on  $V_k^0$ . Such basis exists because the rank of  $A^\alpha$  is  $k$  in  $V_k^0$  and  $V_k^0$  is an open

set in  $\Sigma$  (If necessary, we can take  $U$  small enough to be close to  $\vec{F}_1$ ).

Since  $\langle \vec{F}, n_\alpha \rangle \equiv 0$ , taking the derivatives with respect to  $\{e_i\}_{i=1}^n$  gives that

$$\sum_k \langle \vec{F}, e_k \rangle h_{ik}^\alpha \equiv 0; \quad (2.2.2)$$

Here we use that  $\bar{\nabla}_{e_i} n_\alpha = -h_{ik}^\alpha e_k + \sum_\beta \langle \bar{\nabla}_{e_i} n_\alpha, n_\beta \rangle n_\beta$  and  $e_i \langle \vec{F}, n_\alpha \rangle \equiv 0$ . Let  $Y_{\vec{F}}$  be  $\langle \langle \vec{F}, e_1 \rangle, \dots, \langle \vec{F}, e_n \rangle \rangle$ . By (2.2.2), on  $V_k^0$

$$Y_{\vec{F}} A^\alpha \equiv 0$$

Therefore,  $Y_{\vec{F}} = a_{1, \vec{F}} X_1 + \dots + a_{n-k, \vec{F}} X_{n-k}$  on  $V_k^0$ . Here  $a_{1, \vec{F}}$  is a function on  $V_k^0$ . Again, we can choose  $\{a_{1, \vec{F}}, \dots, a_{n-k, \vec{F}}\}$  as a local coordinate of  $V_k^0$ . However,  $V_k^0$  is an open set of  $\Sigma$  with dimension  $n$ . This leads to a contradiction. Hence  $V_k^0$  is empty for  $k \geq 1$ . Finally  $\Sigma$  is totally geodesic in  $U$ .

It completes the proof.  $\square$

**Remark 2.2.3.** The central topological fact in the proof above is that there is no diffeomorphism from  $\mathbb{R}^n$  into  $\mathbb{R}^k$  for  $k < n$ .

The following result for self-shrinkers is very useful.

**Theorem 2.2.4.** *If  $\Sigma$  is an  $n$  dimensional smooth, minimal, complete self-shrinker in  $\mathbb{R}^{n+1}$ , then  $\Sigma$  is a plane through 0.*



*Proof.* Since  $\vec{H} \equiv 0$ , then  $\Sigma$  satisfies that  $\vec{F}^\perp \equiv 0$  for any point  $\vec{F}$  on  $\Sigma$ . Theorem 2.2.1 implies that  $\Sigma$  is a plane. If  $0$  is not contained by  $\Sigma$ . Then there exists a point  $\vec{F}_0 \neq 0$  on  $\Sigma$  such that  $d(\vec{F}_0, 0) = \min_{\vec{F} \in \Sigma} d(\vec{F}, 0)$ . Let  $X$  be a tangent vector in  $T_{\vec{F}_0}\Sigma$ . Therefore  $X\langle \vec{F}, \vec{F} \rangle(\vec{F}_0) = 0$ . We conclude that  $\langle \vec{F}_0, X \rangle(\vec{F}_0) = 0$ . Let  $\vec{F}_0^T$  be the component of  $\vec{F}$  in  $T\Sigma$ , then  $\vec{F}_0^T = 0$ . This is a contradiction because

$$\vec{F}_0 = \vec{F}_0^T + \vec{F}_0^\perp = 0;$$

□

## 2.3 Parallel forms and structure equations

The parallel forms in Euclidean space play the fundamental role in this thesis. A self-shrinker naturally satisfies some structure equations in terms of parallel forms. Those equations are particularly useful when the parallel forms contain some geometric information of the self-shrinker. For the application of parallel forms into the mean curvature flow, we refer to [Wan02]. All materials in this section are self-contained.

We will adapt the notation in [Wan08] and [LL11]. Assume that  $N$  is a smooth  $n$ -dimensional submanifold in a Riemannian manifold  $M$  of dimension  $n+k$ . We denote by  $\{e_i\}_{i=1}^n$  the orthonormal frame of the tangent bundle

of  $N$  and let  $\{e_\alpha\}_{\alpha=n+1}^{n+k}$  stand for the orthonormal frame of the normal bundle of  $N$ . The Riemann curvature tensor of  $M$  is defined by

$$R(X, Y)Z = -\bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_Y \bar{\nabla}_X Z + \bar{\nabla}_{[X, Y]} Z,$$

for smooth vector fields  $X, Y$  and  $Z$ . The second fundamental form  $A$  and the mean curvature vector  $\vec{H}$  are defined as

$$A(e_i, e_j) = (\bar{\nabla}_{e_i} e_j)^\perp = h_{ij}^\alpha e_\alpha \quad (2.3.1)$$

$$\vec{H} = (\bar{\nabla}_{e_i} e_i)^\perp = h_{ii}^\alpha e_\alpha = h^\alpha e_\alpha. \quad (2.3.2)$$

Here we used Einstein notation and  $h^\alpha = h_{ii}^\alpha$ .

Let  $\nabla$  be the covariant derivative of  $\Sigma$  with respect to the induced metric.

Then  $\nabla^\perp A$  can be written as follows:

$$\nabla_{e_k}^\perp A(e_i, e_j) = h_{ij,k}^\alpha e_\alpha. \quad (2.3.3)$$

Note that  $h_{ij,k}^\alpha$  is not equal to  $e_k(h_{ij}^\alpha)$  unless  $\Sigma$  is a hypersurface. In fact, we have:

**Lemma 2.3.1.**  $h_{ij,k}^\alpha$  takes the following form:

$$h_{ij,k}^\alpha = e_k(h_{ij}^\alpha) + h_{ij}^\beta \langle e_\alpha, \bar{\nabla}_{e_k} e_\beta \rangle - C_{ki}^l h_{lj}^\alpha - C_{kj}^l h_{li}^\alpha, \quad (2.3.4)$$

where  $\nabla_{e_i} e_j = C_{ij}^k e_k$ .

*Proof.* By its definition,

$$h_{ij,k}^\alpha = \langle \nabla_{e_k}^\perp A(e_i, e_j), e_\alpha \rangle.$$

The conclusion follows from expanding  $\nabla_{e_k}^\perp A(e_i, e_j)$ :

$$\begin{aligned} h_{ij,k}^\alpha &= \langle \bar{\nabla}_{e_k}(A(e_i, e_j)), e_\alpha \rangle - \langle A(\nabla_{e_k} e_i, e_j), e_\alpha \rangle - \langle A(e_i, \nabla_{e_k} e_j), e_\alpha \rangle \\ &= \langle \bar{\nabla}_{e_k}(h_{ij}^\beta e_\beta), e_\alpha \rangle - C_{ki}^l h_{lj}^\alpha - C_{kj}^l h_{li}^\alpha \\ &= e_k(h_{ij}^\alpha) + h_{ij}^\beta \langle e_\alpha, \bar{\nabla}_{e_k} e_\beta \rangle - C_{ki}^l h_{lj}^\alpha - C_{kj}^l h_{li}^\alpha. \end{aligned}$$

□

For later calculation, we recall that the Codazzi equation is

$$R_{\alpha ikj} = h_{ij,k}^\alpha - h_{ik,j}^\alpha, \quad (2.3.5)$$

where  $R_{\alpha ikj} = R(e_\alpha, e_i, e_k, e_j)$ .

**Definition 2.3.2.** An  $n$ -form  $\Omega$  is called parallel if  $\bar{\nabla}\Omega = 0$ , where  $\bar{\nabla}$  is the covariant derivative of  $M$ .

The Hodge star  $*\Omega$  on  $N$  is defined by

$$*\Omega = \frac{\Omega(X_1, \dots, X_n)}{\sqrt{\det(g_{ij})}} \quad (2.3.6)$$

where  $\{X_1, \dots, X_n\}$  is a local frame on  $N$  and  $g_{ij} = \langle X_i, X_j \rangle$ .

**Remark 2.3.3.** We denote by  $M$  the product manifold  $N_1 \times N_2$ , and denote by  $\Omega$  the volume form of  $N_1$ . Then  $\Omega$  is a parallel form in  $M$ . If  $N$  is a graphical manifold over  $N_1$ . Then  $*\Omega > 0$  on  $N$  for an appropriate orientation. For example, the graphical self-shrinker  $\Sigma$  in §1.2 satisfies that  $*\Omega > 0$  on  $\Sigma$  where  $\Omega$  is  $dx_1 \wedge \cdots \wedge dx_n$ .

$*\Omega$  is independent of the frame  $\{X_1, \cdots, X_n\}$ , up to a fixed orientation. This fact greatly simplifies our calculation. When  $\{X_1, \cdots, X_n\}$  is the orthonormal frame  $\{e_1, \cdots, e_n\}$ ,  $*\Omega = \Omega(e_1, \cdots, e_n)$ .

The evolution equation of  $*\Omega$  along mean curvature flows is the key ingredient in ([Wan02]).

The following equation (2.3.7) is first appeared as equation (3.4) in [Wan02] in the proof of the evolution equation of  $*\Omega$  along the mean curvature flow. We provide a proof for the sake of completeness.

**Proposition 2.3.4.** *Let  $N^n$  be a smooth submanifold of  $M^{n+k}$ . Suppose  $\Omega$  is a parallel  $n$ -form and  $R$  is the Riemann curvature tensor of  $M$ . Then  $*\Omega = \Omega(e_1, \cdots, e_n)$  satisfies the following equation:*

$$\Delta(*\Omega) = - \sum_{i,k} (h_{ik}^\alpha)^2 * \Omega + \sum_i (h_{,i}^\alpha + \sum_k R_{\alpha k i k}) \Omega_{i\alpha} + 2 \sum_{i < j, k} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha, j\beta}. \quad (2.3.7)$$

Here  $\Delta$  denotes the Laplacian on  $N$  with respect to the induced metric, and  $h_{,k}^\alpha$  denotes  $\sum_{i=1}^n h_{ii,k}^\alpha$ . In the second group of terms,  $\Omega_{i\alpha} = \Omega(\hat{e}_1, \cdots, \hat{e}_n)$

with  $\hat{e}_s = e_s$  for  $s \neq i$  and  $\hat{e}_s = e_\alpha$  for  $s = i$ . In the last group of terms,  $\Omega_{i\alpha,j\beta} = \Omega(\hat{e}_1, \dots, \hat{e}_n)$  with  $\hat{e}_s = e_s$  for  $s \neq i, j$ ,  $\hat{e}_s = e_\alpha$  for  $s = i$  and  $\hat{e}_s = e_\beta$  for  $s = j$ .

*Proof.* Recall that  $\nabla$  and  $\bar{\nabla}$  are the covariant derivatives of  $N$  and  $M$ , respectively. Fixing a point  $p$  on  $\Sigma$ , we assume that  $\{e_1, \dots, e_n\}$  is normal at  $p$  with respect to  $\nabla$ . Lemma 2.3.1 implies that

$$\nabla_{e_i} e_j(p) = 0, \quad h_{ij,k}^\alpha(p) = e_k(h_{ij}^\alpha)(p) + h_{ij}^\beta \langle e_\alpha, \bar{\nabla}_{e_k} e_\beta \rangle(p). \quad (2.3.8)$$

Since  $\bar{\nabla}\Omega = 0$ , we have

$$\begin{aligned} \nabla_{e_k}(*\Omega) &= \Omega(\bar{\nabla}_{e_k} e_1, \dots, e_n) + \dots + \Omega(e_1, \dots, \bar{\nabla}_{e_k} e_n) \\ &= \sum_i h_{ik}^\alpha \Omega_{i\alpha}; \end{aligned} \quad (2.3.9)$$

For  $\nabla_{e_k} \nabla_{e_k}(*\Omega)$ , we get

$$\nabla_{e_k} \nabla_{e_k}(*\Omega) = \sum_i e_k(h_{ik}^\alpha) \Omega_{i\alpha} + \sum_i h_{ik}^\alpha e_k(\Omega_{i\alpha}). \quad (2.3.10)$$

The second term in (2.3.10) can be computed as

$$\begin{aligned} \sum_i h_{ik}^\alpha e_k(\Omega_{i\alpha}) &= \sum_i h_{ik}^\alpha \Omega(e_1, \dots, \bar{\nabla}_{e_k} e_\alpha, \dots, e_n) + 2 \sum_{i < j} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha, j\beta} \\ &= \sum_{i, \alpha} -(h_{ik}^\alpha)^2 * \Omega + h_{ik}^\beta \langle e_\alpha, \bar{\nabla}_{e_k} e_\beta \rangle \Omega_{i\alpha} + 2 \sum_{i < j} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha, j\beta}. \end{aligned}$$

Plugging this into (2.3.10) yields that

$$\begin{aligned}\nabla_{e_k} \nabla_{e_k} (*\Omega) &= - \sum_{i,\alpha} (h_{ik}^\alpha)^2 * \Omega + 2 \sum_{i < j} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha,j\beta} + \sum_i h_{ki,k}^\alpha \Omega_{i\alpha} \\ &= - \sum_{i,\alpha} (h_{ik}^\alpha)^2 * \Omega + 2 \sum_{i < j} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha,j\beta} + \sum_i (h_{kk,i}^\alpha + R_{\alpha k i k}) \Omega_{i\alpha}.\end{aligned}$$

in view of (2.3.8) and (2.3.5), we can finally conclude that

$$\begin{aligned}\Delta(*\Omega(p)) &= \nabla_{e_k} \nabla_{e_k} (*\Omega)(p) - \nabla_{\nabla_{e_k} e_k} (*\Omega)(p) \\ &= - \sum_{i,k,\alpha} (h_{ik}^\alpha)^2 * \Omega + 2 \sum_{i < j,k} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha,j\beta} + \sum_i (h_{,i}^\alpha + \sum_k R_{\alpha k i k}) \Omega_{i\alpha}.\end{aligned}$$

The proposition follows.  $\square$

Next, we work in the situation that  $M^{n+k}$  is the Euclidean space and  $N^n$  is a self-shrinker.

**Lemma 2.3.5.** *Let  $\Omega$  be a parallel  $n$ -form in  $\mathbb{R}^{n+k}$ . Suppose  $N^n$  is an  $n$ -dimensional self-shrinker in  $\mathbb{R}^{n+k}$ . Using the notation in Proposition 2.3.4, we have:*

$$\sum_i \Omega_{i\alpha} h_{,i}^\alpha = \frac{1}{2} \langle \vec{F}, \nabla(*\Omega) \rangle \quad (2.3.11)$$

where  $\vec{F}$  is any point on  $N$ .

*Proof.* As in (2.3.8), we assume that  $\{e_1, \dots, e_n\}$  is normal at  $p$ . From (2.3.9), we compute  $\nabla(*\Omega)$  as follows:

$$\nabla(*\Omega) = \nabla_{e_k} (*\Omega) e_k = \left( \sum_i h_{ik}^\alpha \Omega_{i\alpha} \right) e_k.$$

This leads to

$$\frac{1}{2}\langle \vec{F}, \nabla(*\Omega) \rangle = \frac{1}{2}\langle \vec{F}, e_k \rangle \left( \sum_i h_{ki}^\alpha \Omega_{i\alpha} \right). \quad (2.3.12)$$

Recalling that  $\vec{H} = h^\alpha e_\alpha$ , we have  $h^\alpha = -\frac{1}{2}\langle \vec{F}, e_\alpha \rangle$  since  $\vec{H} + \frac{1}{2}\vec{F}^\perp = 0$ .

Taking the derivative of  $h^\alpha$  with respect to  $e_i$ , we get

$$\begin{aligned} e_i(h^\alpha) &= \frac{1}{2}h_{ik}^\alpha \langle \vec{F}, e_k \rangle - \frac{1}{2}\langle \vec{F}, e_\beta \rangle \langle \bar{\nabla}_{e_i} e_\alpha, e_\beta \rangle \\ &= \frac{1}{2}h_{ik}^\alpha \langle \vec{F}, e_k \rangle - h^\beta \langle \bar{\nabla}_{e_i} e_\beta, e_\alpha \rangle. \end{aligned} \quad (2.3.13)$$

Since we assume that  $\nabla_{e_i} e_j(p) = 0$ , (2.3.4) yields that  $h_{kk,i}^\alpha(p) = e_i(h_{kk}^\alpha)(p) + h_{kk}^\beta \langle \bar{\nabla}_{e_i} e_\beta, e_\alpha \rangle(p)$ . Then we conclude that

$$h_{,i}^\alpha(p) = e_i(h^\alpha)(p) + h^\beta \langle \bar{\nabla}_{e_i} e_\beta, e_\alpha \rangle(p).$$

Comparing above with (2.3.13), we get  $h_{,i}^\alpha(p) = \frac{1}{2}h_{ik}^\alpha \langle \vec{F}, e_k \rangle(p)$ . The lemma follows from combining this with (2.3.12).  $\square$

Using Proposition 2.3.4 and Lemma 2.3.5, we obtain the following structure equation of self-shrinkers in terms of the parallel form.

**Theorem 2.3.6** (Structure Equation). *In  $\mathbb{R}^{n+k}$ , suppose  $\Sigma$  is an  $n$ -dimensional self-shrinker. Let  $\Omega$  be a parallel  $n$ -form, then  $*\Omega = \Omega(e_1, \dots, e_n)$  satisfies that*

$$\Delta(*\Omega) + \sum_{i,k,\alpha} (h_{ik}^\alpha)^2 * \Omega - 2 \sum_{i < j} \Omega_{i\alpha,j\beta} h_{ik}^\alpha h_{jk}^\beta - \frac{1}{2}\langle \vec{F}, \nabla(*\Omega) \rangle = 0, \quad (2.3.14)$$

where  $\vec{F}$  is the coordinate of the point on  $\Sigma$  and  $\Omega_{i\alpha,j\beta} = \Omega(\hat{e}_1, \dots, \hat{e}_n)$  with  $\hat{e}_s = e_s$  for  $s \neq i, j$ ,  $\hat{e}_s = e_\alpha$  for  $s = i$  and  $\hat{e}_s = e_\beta$  for  $s = j$ .

This theorem enables us to obtain various information of self-shrinkers for different parallel forms. We will apply it in Chapter 3 and Chapter 4 to derive structure equations in the case that self-shrinkers are the graphs of smooth maps. We refer the readers to compare Theorem 3.1.5 and Theorem 4.1.2 with the above result.

## 2.4 Volume growth for self-shrinkers

Another special property for self-shrinkers is that a proper, smooth self-shrinker has the polynomial volume growth property (Theorem 2.4.2). This makes the integration technique possible, which is summarized in Lemma 2.4.5. Notice that not all self-shrinkers have the polynomial volume growth property. Some examples appeared in the recent work of Cheng-Ogata ([CO15]).

**Definition 2.4.1.** *Let  $N^n$  be a complete, immersed  $n$ -dimensional submanifold in  $\mathbb{R}^{n+k}$ , we say  $N$  has the polynomial volume growth property, if for any  $r \geq 1$ ,*

$$\int_{N \cap B_r(0)} dvol \leq Cr^n,$$



where  $B_r(0)$  is the ball in  $\mathbb{R}^{n+k}$  centered at 0 with radius  $r$ .

Recently Cheng-Zhou [CZ13] and Ding-Xin [DX13] showed the polynomial volume growth property is automatic under the following condition without any restriction of dimension and codimension.

**Theorem 2.4.2** ([CZ13, DX13]). *If  $N^n$  is a  $n$ -dimensional complete, immersed, proper self-shrinker in  $\mathbb{R}^{n+k}$ , then it satisfies the polynomial volume growth property.*

**Remark 2.4.3.** The properness assumption can not be removed. See for example Remark 4.1 in [CZ13].

Note that any graphical self-shrinker in Euclidean space is embedded, complete and proper. Therefore we state:

**Corollary 2.4.4.** *Let  $\Sigma = (x, f(x))$  be a smooth graphical self-shrinker in  $\mathbb{R}^{n+k}$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is a smooth map. Then  $\Sigma$  has the polynomial volume growth property.*

The following lemma is crucial for our main results in this thesis.

**Lemma 2.4.5.** *Let  $N^n \subset \mathbb{R}^{n+k}$  be a complete, immersed smooth  $n$ -dimensional submanifold with at most polynomial volume growth. Assume there are a positive functions  $g$  and a nonnegative function  $K$  on  $N^n$  satisfying the following*

inequality:

$$0 \geq \Delta g - \frac{1}{2} \langle \vec{F}, \nabla g \rangle + Kg, \quad (2.4.1)$$

Here  $\Delta(\nabla)$  is the Laplacian (covariant derivative) of  $N^n$ , and  $\vec{F}$  is the position vector on  $N^n$ . Then  $g$  is a (positive) constant and  $K \equiv 0$ .

*Proof.* Fixing  $r \geq 1$ , we denote by  $\phi$  a compactly supported smooth function in  $\mathbb{R}^{n+k}$  such that  $\phi \equiv 1$  on  $B_r(0)$  and  $\phi \equiv 0$  outside of  $B_{r+1}(0)$  with  $|\nabla \phi| \leq |D\phi| \leq 2$ . Here  $D\phi$  and  $\nabla \phi$  are the gradient of  $\phi$  in  $\mathbb{R}^{n+k}$  and  $N^n$  respectively.

Since  $g$  is positive, let  $u = \log g$ . Then the inequality (2.4.1) becomes

$$0 \geq \Delta u - \frac{1}{2} \langle \vec{F}, \nabla u \rangle + (K + (\nabla u)^2).$$

Multiplying  $\phi e^{-\frac{|\vec{F}|^2}{4}}$  to the righthand side of the above equation and integrating on  $N^n$ , we get

$$\begin{aligned} 0 &\geq \int_N \phi^2 \operatorname{div}_N (e^{-\frac{|\vec{F}|^2}{4}} \nabla u) + \int_N \phi^2 e^{-\frac{|\vec{F}|^2}{4}} (K + |\nabla u|^2) \\ &= - \int_N 2\phi \langle \nabla \phi, \nabla u \rangle e^{-\frac{|\vec{F}|^2}{4}} + \int_N \phi^2 e^{-\frac{|\vec{F}|^2}{4}} (K + |\nabla u|^2) \\ &\geq - \int_N 2|\nabla \phi|^2 e^{-\frac{|\vec{F}|^2}{4}} + \int_N \phi^2 e^{-\frac{|\vec{F}|^2}{4}} \left( K + \frac{|\nabla u|^2}{2} \right) \end{aligned} \quad (2.4.2)$$

In (2.4.2), we used the inequality

$$|2\phi \langle \nabla \phi, \nabla u \rangle| \leq \frac{\phi^2 |\nabla u|^2}{2} + 2|\nabla \phi|^2.$$

Now we estimate, using the construction that  $|\nabla\phi| \leq |D\phi| \leq 2$ ,

$$\begin{aligned} \int_{N \cap B_r(0)} e^{-\frac{|\bar{F}|^2}{4}} \left( K + \frac{|\nabla u|^2}{2} \right) &\leq \int_N \phi^2 e^{-\frac{|\bar{F}|^2}{4}} \left( K + \frac{|\nabla u|^2}{2} \right) \\ &\leq \int_N 2|\nabla\phi|^2 e^{-\frac{|\bar{F}|^2}{4}} \quad \text{by (2.4.2)} \\ &\leq 8 \int_{N \cap (B_{r+1}(0) \setminus B_r(0))} e^{-\frac{|\bar{F}|^2}{4}} \\ &\leq 8C(r+1)^n e^{-\frac{r^2}{4}}. \end{aligned}$$

In the last line we use the fact that the submanifold  $N^n$  has the polynomial volume growth property.

Letting  $r$  go to infinity, we obtain that

$$\int_N e^{-\frac{|\bar{F}|^2}{4}} \left( K + \frac{(\nabla u)^2}{2} \right) \leq 0.$$

Since  $K$  is nonnegative, we have  $K \equiv \nabla u \equiv 0$ . Therefore  $g$  is a positive constant.  $\square$

## 2.5 A proof of Theorem 1.2.4

Now we use the tools in this Chapter to give a short proof of Theorem 1.2.4.

Let  $f$  be a smooth function in  $\mathbb{R}^n$ . If  $\Sigma = (x, f(x))$  is a smooth self-shrinker in  $\mathbb{R}^{n+1}$ , we have to show that  $\Sigma$  is a plane through 0.

Let  $\Omega$  be  $dx_1 \wedge \cdots \wedge dx_n$ . It is easy to see that  $*\Omega > 0$  on  $\Sigma$  and  $\Omega$  is parallel. Notice that the normal bundle of  $\Sigma$  is trivial. We have  $\Omega_{i\alpha, j\beta} \equiv 0$  in Theorem 2.3.6.

1. By Theorem 2.3.6 the structure equation of  $\Sigma$  is given as follows:

$$\Delta(*\Omega) + \sum_{i,k,\alpha} (h_{ik}^\alpha)^2 * \Omega - \frac{1}{2} \langle \vec{F}, \nabla(*\Omega) \rangle = 0 \quad (2.5.1)$$

2. By Corollary 2.4.4,  $\Sigma$  has the polynomial volume growth property.

Applying Lemma 2.4.5, (2.5.1) indicates that  $\Sigma$  is totally geodesic.

Therefore,  $\vec{F}^\perp \equiv 0$  for any  $\vec{F}$  on  $\Sigma$ .

Theorem 2.2.1 implies that  $\Sigma$  is a plane through 0. We obtain Theorem 1.2.4.

**Remark 2.5.1.** The above proof indicates a general process to show rigidity results of graphical self-shrinkers that we will follow in the next three chapters.

1. derive structure equations of self-shrinkers under some geometric setting,
2. apply Lemma 2.4.5 to show that self-shrinkers are minimal,

The rigidity follows from Theorem 2.2.1.

# Chapter 3

## Graphical self-shrinkers in $\mathbb{R}^4$

In this chapter we prove Theorem A. Recall that it is stated as follows.

**Theorem A.** *Let  $f = (f_1(x_1, x_2), f_2(x_1, x_2))$  be a smooth map from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  with its Jacobian  $J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$  satisfying one of the following conditions:*

(1)  $J_f + 1 > 0$  for all  $x \in \mathbb{R}^2$ ; or

(2)  $1 - J_f > 0$  for all  $x \in \mathbb{R}^2$ .

*If  $\Sigma = (x, f(x))$  is a self-shrinker surface in  $\mathbb{R}^4$ , then  $\Sigma$  is a plane through 0.*

In §3.1 we derive structure equations of graphical self-shrinkers in Theorem 3.1.5. In §3.2 we finish the proof.

### 3.1 Structure equations for graphical self-shrinkers in $\mathbb{R}^4$

We consider the following four different parallel 2-forms in  $\mathbb{R}^4$ :

$$\begin{aligned}\eta_1 &= dx_1 \wedge dx_2, & \eta' &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \\ \eta_2 &= dx_3 \wedge dx_4, & \eta'' &= dx_1 \wedge dx_2 - dx_3 \wedge dx_4\end{aligned}\tag{3.1.1}$$

**Lemma 3.1.1.** *Suppose  $\Sigma$  denotes  $(x, f(x))$  where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a smooth map. Then on  $\Sigma$*

$$*\eta_2 = J_f * \eta_1;$$

*Proof.* Notice that  $*\eta_1$  and  $*\eta_2$  are independent of the choice of the local frame. Denote by  $X_1 = \frac{\partial}{\partial x_1} + \frac{\partial f_1}{\partial x_1} \frac{\partial}{\partial x_3} + \frac{\partial f_2}{\partial x_1} \frac{\partial}{\partial x_4}$ ,  $X_2 = \frac{\partial}{\partial x_2} + \frac{\partial f_1}{\partial x_2} \frac{\partial}{\partial x_3} + \frac{\partial f_2}{\partial x_2} \frac{\partial}{\partial x_4}$  and  $g_{ij} = \langle X_i, X_j \rangle$ . Then

$$\begin{aligned}*\eta_2 &= \frac{dx_3 \wedge dx_4(X_1, X_2)}{\sqrt{\det(g_{ij})}} \\ &= \frac{J_f}{\sqrt{\det(g_{ij})}} \\ &= J_f * \eta_1;\end{aligned}$$

□

The above lemma is not enough to explore structure equations in Theorem 2.3.6. We need further information about the microstructure of a point on  $\Sigma$ .

**Lemma 3.1.2.** *Assume  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a smooth map. Denote by  $df$  the differential of  $f$ . Then for any point  $x$ ,*

- (1) *There exist oriented orthonormal bases  $\{a_1, a_2\}$  and  $\{a_3, a_4\}$  in  $T_x\mathbb{R}^2$  and  $T_{f(x)}\mathbb{R}^2$  respectively: such that*

$$df(a_1) = \lambda_1 a_3, \quad df(a_2) = \lambda_2 a_4; \quad (3.1.2)$$

*Here ‘oriented’ means  $dx_i \wedge dx_{i+1}(a_i, a_{i+1}) = 1$  for  $i = 1, 3$ .*

- (2) *Moreover, we have  $\lambda_1 \lambda_2 = J_f$ .*

*Proof.* Fix a point  $x$ . First, we prove the existence of (1). By the Singular Value Decomposition Theorem (P291,[Sho07]) there exist two  $2 \times 2$  orthogonal matrices  $Q_1, Q_2$  such that

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = Q_1 \begin{pmatrix} \lambda'_1 & 0 \\ 0 & \lambda'_2 \end{pmatrix} Q_2$$

with  $\lambda'_1, \lambda'_2 \geq 0$ .

Letting  $\lambda_1 = \det(Q_1)\lambda'_1\det(Q_2)$ ,  $\lambda_2 = \lambda'_2$ ,  $A = \det(Q_1)Q_1$ ,  $B = \det(Q_2)Q_2$ , we find that  $\det(A) = \det(B) = 1$ . And we have

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = A \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} B \quad (3.1.3)$$

We consider the new basis  $(a_1, a_2)^T = A^T(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$ ,  $(a_3, a_4)^T = B(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})^T$ , then  $dx_1 \wedge dx_2(a_1, a_2) = 1$  and  $dx_3 \wedge dx_4(a_3, a_4) = 1$  ( $A^T$  is the transpose of

A). Moreover, (3.1.3) implies that

$$df(a_1, a_2)^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (a_3, a_4)^T$$

Now we obtain (1).

Next we prove (2). Since  $dx_1 \wedge dx_2(a_1, a_2) = 1$  and  $dx_3 \wedge dx_4(a_3, a_4) = 1$ , there exist two  $2 \times 2$  orthogonal matrices  $C, D$  with  $\det(C) = \det(D) = 1$  such that  $(a_1, a_2)^T = C(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$ ,  $D(a_3, a_4)^T = (\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})^T$ . Then

$$\begin{aligned} df(a_1, a_2)^T &= C \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \left( \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right)^T; \\ &= C \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} D(a_3, a_4)^T; \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (a_3, a_4)^T \end{aligned}$$

therefore  $\lambda_1 \lambda_2 = \det(C) J_f \det(D) = J_f$ . We obtain (2). The proof is completed.  $\square$

**Remark 3.1.3.** The conclusion (2) is independent of the special choice of  $\{a_i\}_{i=1}^4$  satisfying (1).

With these bases, we construct the following local frame for a fixed point  $p = (x, f(x))$  on  $\Sigma$ .

**Definition 3.1.4.** Fixing a point  $p = (x, f(x))$  on  $\Sigma$ , we construct a special orthonormal basis  $\{e_1, e_2\}$  of the tangent bundle  $T\Sigma$  and  $\{e_3, e_4\}$  of the



normal bundle  $N\Sigma$  such that at the point  $p$  we have for  $i = 1, 2$ :

$$e_i = \frac{1}{\sqrt{1 + \lambda_i^2}}(a_i + \lambda_i a_{2+i}); \quad e_{2+i} = \frac{1}{\sqrt{1 + \lambda_i^2}}(a_{2+i} - \lambda_i a_i); \quad (3.1.4)$$

where  $\{a_1, a_2, a_3, a_4\}$  are from (3.1.2).

For a parallel 2-form  $\Omega$ , we have  $*\Omega = \Omega(e_1, e_2)$ . Applying (3.1.4) and  $\lambda_1 \lambda_2 = J_f$ , direct computations show that  $*\eta_1, *\eta_2, *\eta'$  and  $*\eta''$  take the following forms:

$$*\eta_1 = \frac{1}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}} > 0, \quad (3.1.5)$$

$$*\eta_2 = \frac{\lambda_1 \lambda_2}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}}, \quad (3.1.6)$$

$$*\eta' = (1 + J_f)(* \eta_1), \quad (3.1.7)$$

$$*\eta'' = (1 - J_f)(* \eta_1). \quad (3.1.8)$$

There is a symmetric relation between  $*\eta_1$  and  $*\eta_2$ . More precisely, we have the structure equations for graphical self-shrinkers in  $\mathbb{R}^4$  as follows:

**Theorem 3.1.5.** *Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a smooth map, and  $\Sigma = (x, f(x))$  is a graphical self-shrinker in  $\mathbb{R}^4$ . Using the notations in Definition 3.1.4,*

we have

$$\Delta(*\eta_1) + *\eta_1(h_{ik}^\alpha)^2 - 2*\eta_2(h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta_1) \rangle = 0; \quad (3.1.9)$$

$$\Delta(*\eta_2) + *\eta_2(h_{ik}^\alpha)^2 - 2*\eta_1(h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta_2) \rangle = 0; \quad (3.1.10)$$

$$\Delta(*\eta') + *\eta'((h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta') \rangle = 0; \quad (3.1.11)$$

$$\Delta(*\eta'') + *\eta''((h_{1k}^3 + h_{2k}^4)^2 + (h_{1k}^4 - h_{2k}^3)^2) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta'') \rangle = 0, \quad (3.1.12)$$

where  $h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} e_j, e_\alpha \rangle$  are the second fundamental form of  $\Sigma$ ,  $\Delta$  and  $\nabla$  are the Laplacian and the covariant derivative of  $\Sigma$ , respectively. Here  $\vec{F}$  is the position vector of  $\Sigma$ .

*Proof.* It suffices to show the first two equations (3.1.9) and (3.1.10), since the other two (3.1.11) and (3.1.12) follow from combining (3.1.9) and (3.1.10) together.

First we consider the equation (3.1.9). Using the frame in (3.1.4), the third term in Theorem 2.3.6 becomes:

$$\begin{aligned} 2(\eta_1)_{i\alpha, j\beta} h_{ik}^\alpha h_{jk}^\beta &= 2dx_1 \wedge dx_2(e_3, e_4)(h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4) \\ &= 2 \frac{\lambda_1 \lambda_2}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}} (h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3) \\ &= 2*\eta_2(h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3). \end{aligned}$$

Here in the second line we used the fact that  $dx_1 \wedge dx_2(a_1, a_2) = 1$ . Plugging this into (2.3.14), we obtain (3.1.9).

Similarly we obtain that

$$\begin{aligned}
2(\eta_2)_{i\alpha,j\beta}h_{ik}^\alpha h_{jk}^\beta &= 2dx_3 \wedge dx_4(e_3, e_4)(h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4) \\
&= 2 \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}(h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3) \\
&= 2 * \eta_1(h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3). \tag{3.1.13}
\end{aligned}$$

Here in the second line we used the fact that  $dx_3 \wedge dx_4(a_3, a_4) = 1$ . Then (3.1.10) follows from plugging (3.1.13) into (2.3.14).  $\square$

## 3.2 The proof of Theorem A

Adapting to our case of graphical self-shrinker surfaces in  $\mathbb{R}^4$ , we are ready to prove Theorem A.

*Proof.* We claim that  $\Sigma$  is minimal under the assumptions. We prove this case by case.

**Assuming Condition (1):**

the equations (3.1.5) and (3.1.7) imply that the parallel form  $*\eta'$  has the same sign as  $1 + J_f$ , hence  $*\eta'$  is a positive function. Moreover from (3.1.11)  $*\eta'$  satisfies

$$\Delta(*\eta') + (*\eta')((h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta') \rangle = 0.$$

Here  $\vec{F} = (x, f(x))$ . Since  $\Sigma$  has the polynomial volume growth property, using Lemma 2.4.5, we conclude that

$$(h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2 \equiv 0.$$

We then obtain

$$h_{11}^3 = h_{21}^4, \quad h_{22}^3 = -h_{12}^4, \quad (3.2.1)$$

$$h_{11}^4 = -h_{21}^3, \quad h_{22}^4 = h_{12}^3. \quad (3.2.2)$$

Then  $\vec{H} = (h_{11}^3 + h_{22}^3)e_3 + (h_{11}^4 + h_{22}^4)e_4 \equiv 0$ . So  $\Sigma$  is a minimal surface.

**Assuming Condition (2):**

this is similar to the above case. (3.1.5) and (3.1.8) imply that  $*\eta''$  has the same sign as  $1 - J_f$ , hence  $*\eta''$  is positive. From (3.1.12), it also satisfies

$$\Delta(*\eta'') + (*\eta'')((h_{1k}^3 + h_{2k}^4)^2 + (h_{1k}^4 - h_{2k}^3)^2) - \frac{1}{2}\langle \vec{F}, \nabla(*\eta'') \rangle = 0$$

Again we apply Lemma 2.4.5 to find that

$$(h_{1k}^3 + h_{2k}^4)^2 + (h_{1k}^4 - h_{2k}^3)^2 \equiv 0.$$

Then we have

$$h_{11}^3 = -h_{21}^4, \quad h_{22}^3 = h_{12}^4, \quad (3.2.3)$$

$$h_{11}^4 = h_{21}^3, \quad h_{22}^4 = -h_{12}^3. \quad (3.2.4)$$

Therefore we arrive at:

$$\vec{H} = (h_{11}^3 + h_{22}^3)e_3 + (h_{11}^4 + h_{22}^4)e_4 \equiv 0,$$

which means  $\Sigma$  is minimal.

Now  $\Sigma$  is a graphical self-shrinker and minimal. From (1.1.4), we have  $\vec{F}^\perp \equiv 0$  for any point  $\vec{F}$  on  $\Sigma$ . For any normal unit vector  $e_\alpha$  in the normal bundle of  $\Sigma$ , we have

$$\langle \vec{F}, e_\alpha \rangle \equiv 0. \quad (3.2.5)$$

Taking derivative with respect to  $e_i$  for  $i = 1, 2$  from (3.2.5), we get

$$\langle \vec{F}, e_1 \rangle h_{11}^\alpha + \langle \vec{F}, e_2 \rangle h_{12}^\alpha = 0,$$

$$\langle \vec{F}, e_1 \rangle h_{21}^\alpha + \langle \vec{F}, e_2 \rangle h_{22}^\alpha = 0,$$

Now assume  $\vec{F} \neq 0$ . Since  $\vec{F}^\perp = 0$ ,  $(\langle \vec{F}, e_1 \rangle, \langle \vec{F}, e_2 \rangle) \neq (0, 0)$ . According to the basic linear algebra, we conclude that

$$h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 = 0 \quad (3.2.6)$$

The minimality implies  $h_{11}^\alpha = -h_{22}^\alpha$ . Hence (3.2.6) becomes  $-(h_{11}^\alpha)^2 = (h_{12}^\alpha)^2$ .

We find that  $h_{ij}^\alpha = 0$  for  $i, j = 1, 2$ . Therefore  $\Sigma$  is totally geodesic except  $\vec{F} = 0$ . Since  $\Sigma$  is a graph, there is at most one point on  $\Sigma$  such that  $\vec{F} = 0$ .

By the continuity of the second fundamental form,  $\Sigma$  is totally geodesic

everywhere.

Now  $\Sigma$  is a plane. If  $0$  is not on the plane, then we can find a point  $\vec{F}_0$  in this plane which is nearest to  $0$ . It is easy to see that  $\vec{F}_0 = \vec{F}_0^\perp \neq 0$ . This is contradict that  $\vec{F}^\perp = -\frac{\vec{H}}{2} \equiv 0$ .

We complete the proof. □

# Chapter 4

## Graphical self-shrinkers in $\mathbb{R}^{n+l}$

In this chapter, we prove Theorem B which is stated as follows.

**Theorem B.** *Suppose  $f$  is a smooth map from  $\mathbb{R}^n$  to  $\mathbb{R}^l$ ,  $df : T_x\mathbb{R}^n \rightarrow T_{f(x)}\mathbb{R}^l$  is the differential of  $f$  and  $r$  is the rank of  $df$ . Let  $\{\lambda_i\}_{i=1}^r$  denote the eigenvalue of  $df$ . If  $\Sigma = (x, f(x))$  is a self-shrinker in  $\mathbb{R}^{n+l}$  and  $|\lambda_i\lambda_j| \leq 1$  for  $i \neq j$ , then  $\Sigma$  is a plane through 0.*

In §4.1 we derive the general structure equations of graphical self-shrinkers in Theorem 4.1.2. In §4.2 we give its proof.

### 4.1 Structure equations of graphical self-shrinkers in $\mathbb{R}^{n+l}$

First, we examine the microstructure of a smooth map  $f$  at a fixed point.

With the same proof of Lemma 3.1.2, it is easy to see that we can obtain the following special orthonormal basis.

**Lemma 4.1.1.** *Fix  $x \in \mathbb{R}^n$ . Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^l$  be a smooth map and  $r$  be the rank of  $df$ . Then there exist oriented orthonormal bases  $\{a_1, \dots, a_n\}$  at  $T_x \mathbb{R}^n$  and  $\{a_{n+1}, \dots, a_{n+l}\}$  at  $T_{f(x)} \mathbb{R}^n$  such that at  $x$*

$$df(a_i) = \lambda_i a_{n+i}, \quad 1 \leq i \leq r;$$

$$dx_1 \wedge \dots \wedge dx_n(a_1, \dots, a_n) = 1;$$

$$dx_{n+1} \wedge \dots \wedge dx_{n+l}(a_{n+1}, \dots, a_{n+l}) = 1;$$

With to this local bases, we can construct a special orthonormal frame  $\{e_i\}_{i=1}^n$  in the tangent bundle of  $\Sigma$  and  $\{e_\alpha\}_{\alpha=n+1}^{n+l}$  in the normal bundle of  $\Sigma$  near  $p = (x, f(x))$  such that at  $p$ ,

$$e_i = \begin{cases} \frac{1}{\sqrt{1+\lambda^2}}(a_i + \lambda_i a_{n+i}) & , \quad 1 \leq i \leq r; \\ a_i, & r+1 \leq i \leq n; \end{cases} \quad (4.1.1)$$

$$e_{n+i} = \begin{cases} \frac{1}{\sqrt{1+\lambda^2}}(a_{n+i} - \lambda_i a_i) & , \quad 1 \leq i \leq r. \\ a_{n+i}, & r+1 \leq i \leq n. \end{cases} \quad (4.1.2)$$

In term of the orthonormal frame above and Theorem 2.3.6, we give the following structure equations of self-shrinkers.

**Theorem 4.1.2.** *Let  $\Omega$  be  $dx_1 \wedge \dots \wedge dx_n$ . Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}^l$  is a smooth map, and  $\Sigma = (x, f(x))$  is a self-shrinker. With the orthonormal frame in*



(4.1.1) and (4.1.2),  $*\Omega$  satisfies that

$$\begin{aligned} \Delta * \Omega - \frac{1}{2} \langle \vec{F}, *\Omega \rangle - \frac{|\nabla * \Omega|^2}{*\Omega} + \sum_{i,k} \lambda_i^2 (h_{ik}^{n+i})^2 * \Omega \\ + \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 * \Omega + 2 \sum_{i<j,k} \lambda_i \lambda_j (h_{ik}^{n+j} h_{jk}^{n+i}) * \Omega = 0. \end{aligned} \quad (4.1.3)$$

Here  $h_{ij}^\alpha = \langle \bar{\nabla}_{e_i} e_j, e_\alpha \rangle$ ,  $\bar{\nabla}(\nabla)$  is the covariant derivative of  $\mathbb{R}^{n+l}(\Sigma)$ ,  $\Delta$  is the Laplacian of  $\Sigma$  and  $\vec{F}$  is the position vector of  $\Sigma$ .

**Remark 4.1.3.** For the completeness of notation, we always assume that  $h_{sk}^{n+i} = 0$  for  $i > l$ .

*Proof.* We use the notation in Theorem 2.3.6 and fix a point  $p = (x, f(x))$ .

Then at  $p$ , for  $i < j$  plugging the frame in (4.1.2) gives that

$$\begin{aligned} \sum_{\alpha,\beta,k} \Omega_{i\alpha} \Omega_{j\beta} h_{ik}^\alpha h_{jk}^\beta &= \sum_{\alpha,\beta,k} \Omega(e_1, \dots, e_\alpha, \dots, e_\beta, \dots, e_n) h_{ik}^\alpha h_{jk}^\beta; \\ &= \sum_k (\Omega(e_1, \dots, e_{n+i}, \dots, e_{n+j}, \dots, e_n) h_{ik}^{n+i} h_{jk}^{n+j} \\ &\quad + \Omega(e_1, \dots, e_{n+j}, \dots, e_{n+i}, \dots, e_n) h_{ik}^{n+j} h_{jk}^{n+i}); \\ &= \lambda_i \lambda_j * \Omega \sum_k (h_{ik}^{n+i} h_{jk}^{n+j} - h_{ik}^{n+j} h_{jk}^{n+i}); \end{aligned} \quad (4.1.4)$$

Here  $*\Omega(p) = \Omega(e_1, \dots, e_n)(p) = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i) \dots (1+\lambda_n)}}$ . On the other hand,

$$\begin{aligned} |\nabla * \Omega|^2 &= \sum_i |\nabla_{e_i} \Omega|^2; \\ &= \sum_i \left( \sum_k \Omega(e_1, \dots, \bar{\nabla}_{e_i} e_k, \dots, e_n) \right)^2; \\ &= \sum_i \left( \sum_{k, \alpha} h_{ik}^\alpha \Omega(e_1, \dots, e_\alpha, \dots, e_n) \right)^2; \end{aligned} \quad (4.1.5)$$

$$= \sum_i \left( \sum_k \lambda_i h_{ik}^{n+k} \right)^2 (*\Omega)^2; \quad (4.1.6)$$

In (4.1.5), we use that  $\bar{\nabla}_{e_i} e_k = \sum_{l \neq k} \langle \nabla_{e_i} e_k, e_l \rangle e_l + \sum_\alpha h_{ik}^\alpha e_\alpha$ ,  $\Omega(e_1, \dots, e_l, \dots, e_n) = 0$  for  $l \neq k$  and  $\langle \nabla_{e_i} e_k, e_k \rangle \equiv 0$ .

Plugging (4.1.6) and (4.1.4) into Theorem 2.3.6, we obtain that

$$\begin{aligned} 0 &= \Delta * \Omega - \frac{1}{2} \langle \vec{F}, * \Omega \rangle + \sum_{i, k, \alpha} (h_{ik}^\alpha)^2 * \Omega + 2 \sum_{i, j} \lambda_i \lambda_j * \Omega \sum_k (h_{ik}^{n+j} h_{jk}^{n+i} - h_{ik}^{n+i} h_{jk}^{n+j}) \\ &= \Delta * \Omega - \frac{1}{2} \langle \vec{F}, * \Omega \rangle + \sum_{i, k, \alpha} (h_{ik}^\alpha)^2 * \Omega - \frac{|\nabla * \Omega|^2}{*\Omega} \\ &\quad + \frac{|\nabla * \Omega|^2}{*\Omega} + 2 \sum_{i, j} \lambda_i \lambda_j * \Omega \sum_k (h_{ik}^{n+j} h_{jk}^{n+i} - h_{ik}^{n+i} h_{jk}^{n+j}); \\ &= \Delta * \Omega - \frac{1}{2} \langle \vec{F}, * \Omega \rangle - \frac{|\nabla * \Omega|^2}{*\Omega} + \sum_{i, k} \lambda_i^2 (h_{ik}^{n+i})^2 * \Omega \\ &\quad + \sum_{i, j, \alpha} (h_{ij}^\alpha)^2 * \Omega + 2 \sum_{i < j, k} \lambda_i \lambda_j (h_{ik}^{n+j} h_{jk}^{n+i}) * \Omega; \end{aligned}$$

We obtained the proof.  $\square$

## 4.2 The proof of Theorem B

Now we are ready to show Theorem B. In the following proof, the key condition is  $|\lambda_i \lambda_j| \leq 1$  for any  $i \neq j$ .

*Proof.* First, we denote by  $C$

$$\sum_{i,k} \lambda_i^2 (h_{ik}^{n+i})^2 + \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + 2 \sum_{i < j,k} \lambda_i \lambda_j (h_{jk}^{n+i} h_{ik}^{n+j});$$

and  $g \log(*\Omega)$ . Then (4.1.3) is rewritten as

$$\Delta g - \frac{1}{2} \langle \vec{F}, \nabla g \rangle + C \equiv 0; \quad (4.2.1)$$

Notice that  $\nabla g = \frac{\nabla * \Omega}{* \Omega}$ . Moreover, from (4.1.6) we have

$$\begin{aligned} C - \frac{1}{2n} |\nabla g|^2 &\geq C - \frac{1}{2n} \sum_i \left( \sum_k \lambda_i h_{ik}^{n+i} \right)^2; \\ &\geq C - \frac{1}{2} \sum_{i,k} (\lambda_i h_{ik}^{n+i})^2 \\ &\geq \frac{1}{2} \sum_{i,k} \lambda_i^2 (h_{ik}^{n+i})^2 + \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + 2 \sum_{i < j,k} \lambda_i \lambda_j (h_{jk}^{n+i} h_{ik}^{n+j}); \\ &\geq \frac{1}{2} \sum_{i,k} \lambda_i^2 (h_{ik}^{n+i})^2 + \sum_i (h_{ii}^\alpha)^2 \\ &\quad + \sum_{i < j,k} ((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2 + 2\lambda_i \lambda_j h_{jk}^{n+i} h_{ik}^{n+j}); \\ &\geq 0; \end{aligned} \quad (4.2.2)$$

In the first inequality, we apply the Cauchy inequality  $(\sum_k \lambda_i h_{ik}^{n+i})^2 \leq n(\sum_k \lambda_i^2 (h_{ik}^{n+i})^2)$ .

In the last inequality, we use the fact that  $|\lambda_i \lambda_j| \leq 1$ .

Now we use the similar technique in Lemma 2.4.5. Fix  $r_1 \geq 1$ , we denote by  $\phi$  a compactly supported smooth function in  $\mathbb{R}^{n+k}$  such that  $\phi \equiv 1$  on  $B_{r_1}(0)$  and  $\phi \equiv 0$  outside of  $B_{r_1+1}(0)$  with  $|\nabla\phi| \leq |D\phi| \leq 2$ . Here  $D\phi$  and  $\nabla\phi$  are the gradient of  $\phi$  in  $\mathbb{R}^{n+k}$  and  $\Sigma$  respectively. Multiplying  $\phi^2 e^{-\frac{|\bar{F}|^2}{4}}$  in (4.2.1) and integrating on  $\Sigma$  gives that

$$\begin{aligned} 0 &= \int_{\Sigma} \phi^2 \operatorname{div}_{\Sigma}(e^{-\frac{|\bar{F}|^2}{4}} \nabla g) + \int_{\Sigma} \phi^2 e^{-\frac{|\bar{F}|^2}{4}} C \\ &= - \int_N 2\phi \langle \nabla\phi, \nabla g \rangle e^{-\frac{|\bar{F}|^2}{4}} + \int_{\Sigma} \phi^2 e^{-\frac{|\bar{F}|^2}{4}} C \\ &\geq -2n \int_{\Sigma} |\nabla\phi|^2 e^{-\frac{|\bar{F}|^2}{4}} + \int_{\Sigma} \phi^2 e^{-\frac{|\bar{F}|^2}{4}} \left( C - \frac{|\nabla g|^2}{2n} \right) \end{aligned} \quad (4.2.3)$$

In the last step above, we use that  $2|\phi \langle \nabla\phi, \nabla g \rangle| \leq 2n|\nabla\phi|^2 + \phi^2 \frac{|\nabla g|^2}{2n}$ . By our assumption of  $\phi$ , (4.2.3) implies that

$$\begin{aligned} \int_{\Sigma \cap B_{r_1}(0)} e^{-\frac{|\bar{F}|^2}{4}} \left( C - \frac{|\nabla g|^2}{2n} \right) &\leq 2n \int_{\Sigma \cap B_{r_1+1}(0)} |\nabla\phi|^2 e^{-\frac{|\bar{F}|^2}{4}} \\ &\leq 2Mr_1^n e^{-\frac{r_1^2}{4}} \end{aligned}$$

Here we use that  $\Sigma$  has the polynomial growth property according to Corollary 2.4.4. Taking  $r_1$  to infinity, (4.2.2) indicates that

$$\begin{aligned} 0 &= C - \frac{|\nabla g|^2}{2n} \\ &= \frac{1}{2} \sum_{i,k} \lambda_i^2 (h_{ik}^{n+i})^2 + \sum_{i,k} (h_{ik}^{n+i})^2 + \sum_{i < j, k} ((h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2 + 2\lambda_i \lambda_j h_{jk}^{n+i} h_{ik}^{n+j}) \end{aligned}$$

Since  $|\lambda_i \lambda_j| \leq 1$  for any  $i \neq j$ , we conclude that

$$\lambda_i h_{ik}^{n+i} = 0 \quad \text{for any } i, k \quad (4.2.4)$$

$$h_{ik}^{n+i} = 0 \quad \text{for any } k, i \quad (4.2.5)$$

$$(h_{jk}^{n+i})^2 + (h_{ik}^{n+j})^2 + 2\lambda_i \lambda_j h_{jk}^{n+i} h_{ik}^{n+j} = 0 \quad \text{for any } i, j, k \quad (4.2.6)$$

Next, we claim that

$$\vec{H} = \sum_{i=1}^k (\sum_j h_{jj}^{n+i}) e_{n+i} \equiv 0 \quad (4.2.7)$$

If we have this claim, Theorem 2.2.4 implies that  $\Sigma$  must be a plane through 0.

Fix  $i_0$  for a while, we proceed as followings:

1. By (4.2.5),  $h_{i_0 i_0}^{n+i_0} = 0$ .

2. Fix  $j \neq i_0$ .

(a) If  $|\lambda_{i_0} \lambda_j| = 1 - \delta < 1$ , (4.2.6) gives that

$$\begin{aligned} 0 &= \delta((h_{jk}^{n+i_0})^2 + (h_{i_0 k}^{n+j})^2) + (1 - \delta)((h_{jk}^{n+i_0})^2 + (h_{i_0 k}^{n+j})^2) \\ &\quad + 2\text{sign}(\lambda_{i_0} \lambda_j) h_{jk}^{n+i_0} h_{i_0 k}^{n+j} \end{aligned}$$

Then  $h_{jk}^{n+i_0} = 0$ . Let  $k = j$ , we get  $h_{jj}^{n+i_0} = 0$ .

(b) Now we assume  $|\lambda_{i_0} \lambda_j| = 1$ . Then  $\lambda_j \neq 0$ , (4.2.4) implies that

$$h_{jk}^{n+j} = 0 \text{ for any } k. \text{ In particular, } h_{j i_0}^{n+j} = 0. \text{ With this fact,}$$

(4.2.6) indicates that

$$0 = (h_{jj}^{n+i_0})^2 + (h_{i_0j}^{n+j})^2 + 2\lambda_i\lambda_j h_{jj}^{n+i_0} h_{i_0j}^{n+j} = (h_{jj}^{n+i_0})^2$$

Hence  $h_{jj}^{n+i_0} = 0$ .

3. We conclude that  $h_{jj}^{n+i_0} = 0$  for any  $j \neq i_0$ . Finally,  $\sum_j h_{jj}^{n+i_0} = 0$ .

Since we choose  $i_0$  arbitrarily, (4.2.7) holds true. Theorem B follows from Theorem 2.2.4. □

# Chapter 5

## Rigidity of Lagrangian self-shrinkers

In this chapter we prove Theorem C which is stated as follows.

**Theorem C.** *Assume  $L$  is a smooth, complete zero-Maslov Lagrangian self-shrinker with the polynomial volume growth property and its Lagrangian angle  $\theta$  satisfying one of the followings:*

(1)  $\theta + C_1 > 0$  for all points on  $L$ ;

(2)  $\theta + C_2 < 0$  for all points on  $L$ ,

where  $C_1, C_2$  are some constants. Then  $L$  is a plane through 0.

**Remark 5.0.1.** A special case for the result above is that there is no smooth compact, zero-Maslov Lagrangian self-shrinker in  $\mathbb{C}^n$ . This is first obtained by Smoczyk [Smo00].

In §5.1 we derive structure equations of Lagrangian angles for Lagrangian self-shrinkers. We give the proof of Theorem C in §5.2. All materials in this section are self-contained.

## 5.1 Structure equations of Lagrangian angles

Let  $J$  and  $\omega$  denote the standard complex structure and the standard symplectic form of  $\mathbb{C}^n$ . We consider the closed complex  $n$ -form given by

$$\Omega = dz_1 \wedge \cdots \wedge dz_n$$

and the symplectic form

$$\omega = \sum_i dx_i \wedge dy_i$$

where  $z_j = x_j + iy_j$  for  $j = 1, \dots, n$  are complex coordinates of  $\mathbb{C}^n$ .

A smooth  $n$ -dimensional submanifold  $L$  in  $\mathbb{C}^n$  is said to be *Lagrangian* if  $\omega_L \equiv 0$ . This means that  $\omega(X, Y) = g(JX, Y)$  for any tangent vectors  $X, Y$  of  $L$ . A simple computation shows that

$$\Omega_L = e^{i\theta} vol_L$$

where  $vol_L$  denotes the volume form of  $L$  and  $\theta$  is a multivalued function called the Lagrangian angle. When  $\theta$  is a single valued function, the Lagrangian is of *zero-Maslov*.

There is a remarkable property of the Lagrangian angle given as follows.



**Lemma 5.1.1.** *Assume  $L$  is a smooth Lagrangian in  $\mathbb{C}^n$ . Let  $\vec{H}$  be the mean curvature vector and  $\theta$  be the Lagrangian angle. Then*

$$\vec{H} = J\nabla\theta$$

where  $\nabla$  is the covariant derivative of  $L$ .

**Remark 5.1.2.** For the proof of general cases, we refer to Thomas-Yau ([TY02]). In particular, when  $\theta$  is a constant,  $L$  is minimal.

*Proof.* Fix a point  $p \in L$  and let  $g$  denote the conical metric of  $\mathbb{C}^n$ . Assume  $\{e_1, \dots, e_n\}$  is an orthonormal frame on  $L$  such that  $\nabla_{e_i} e_j(p) = 0$  for any  $i, j$ . Since  $L$  is Lagrangian,  $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$  is an orthonormal frame of  $\mathbb{C}^n$  on  $L$ . Here we use that  $g(Je_j, e_i) = \omega(e_j, e_i) = 0$ .

Let  $e_i^*$  be the cotangent vector dual to  $e_i$ . In fact, it is not hard to see that  $Je_i^* = (Je_i)^*$ . With these notation,  $\Omega = dz_1 \wedge \dots \wedge dz_n$  can be rewritten as

$$\Omega = e^{i\theta} \wedge_{i=1}^n (e_i^* + iJe_i^*);$$

Since  $\Omega$  is parallel, then  $\bar{\nabla}_X \Omega \equiv 0$  for any tangent vector of  $L$ . Here  $\bar{\nabla}$  is the covariant derivative of  $\mathbb{C}^n$ . Expanding  $\bar{\nabla}_X \Omega$  we obtain that

$$\begin{aligned} 0 &= id\theta(X)\Omega + \sum_k (e^{i\theta}(e_1^* + iJe_1^*) \wedge \dots \wedge (\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) \wedge \dots \wedge (e_n^* + iJe_n^*)); \\ &= id\theta(X)\Omega + \sum_k (\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) \frac{(e_k - iJe_k)}{2} \Omega; \end{aligned} \quad (5.1.1)$$

The last step is from that

$$(\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) = \sum_i (\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) \frac{(e_i - iJe_i)}{2} (e_i^* + iJe_i^*)$$

Since  $(e_k^* + iJe_k^*) \frac{(e_k - iJe_k)}{2} \equiv 1$ ,

$$(\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) \frac{(e_k - iJe_k)}{2} = -(e_k^* + iJe_k^*) \frac{(\bar{\nabla}_X e_k - i\bar{\nabla}_X Je_k)}{2} \quad (5.1.2)$$

Since is also parallel,  $\bar{\nabla}_X(Je_k) = J\bar{\nabla}_X e_k$  and  $\nabla_{e_i} e_j(p) = 0$ , we have at the point  $p$

$$\bar{\nabla}_X e_k(p) = \sum_{i=1}^n g(\bar{\nabla}_X e_k, Je_i) Je_i(p)$$

This implies that

$$Je_k^*(\bar{\nabla}_X e_k)(p) = g(\bar{\nabla}_X e_k, Je_k) Je_k^*(Je_k)(p) = -g(\bar{\nabla}_X e_k, Je_k)(p); \quad (5.1.3)$$

and

$$\bar{\nabla}_X Je_k(p) = J\bar{\nabla}_X e_k(p) = -\sum_{i=1}^n g(\bar{\nabla}_X e_k, Je_i) e_i(p)$$

Therefore,  $e_k^*(\bar{\nabla}_X Je_k(p)) = -g(\bar{\nabla}_X e_k, Je_k)(p)$ . Combining this with (5.1.3),

(5.1.2) gives that

$$(\bar{\nabla}_X(e_k^*) + i\bar{\nabla}_X(Je_k^*)) \frac{(e_k - iJe_k)}{2} = -ig(\bar{\nabla}_X e_k, Je_k)(p);$$

Plugging above into (5.1.1), we see that

$$0 = (id\theta(X) - i \sum_k g(\bar{\nabla}_X e_k, Je_k) \Omega(p)$$

We obtain that

$$\begin{aligned}
 d\theta(X)(p) &= \sum_k g(\bar{\nabla}_X e_k, J e_k)(p); \\
 &= \sum_k g(\bar{\nabla}_{e_k} X, J e_k)(p); \\
 &= - \sum_k g(X, J \bar{\nabla}_{e_k} e_k)(p);
 \end{aligned}$$

In the second step, we use that  $\bar{\nabla}_X e_k(p) = \bar{\nabla}_X e_k(p)$  since  $\nabla_{e_i} e_j(p) = 0$ . Therefore,  $\nabla\theta = -J\vec{H}$ . Taking  $J$  on both sides, we obtain the lemma.  $\square$

The structure equation of the Lagrangian angle is given as follow.

**Lemma 5.1.3.** *Assume  $L$  is a smooth Lagrangian self-shrinker, then its Lagrangian angle  $\theta$  satisfies that*

$$\Delta\theta - \frac{1}{2}\langle \vec{F}, \nabla\theta \rangle = 0 \tag{5.1.4}$$

where  $\vec{F}$  is the position vector of  $L$ ,  $\Delta(\nabla)$  is the Lagrangian (covariant derivative) of  $L$ .

*Proof.* Fix any point  $p$  on  $L$ . Let  $\{e_1, \dots, e_n\}$  be an orthonormal frame on  $L$  such that  $\nabla_{e_i} e_j(p) = 0$  for any  $i, j$ . Then from Lemma 5.1.1 using

$\nabla\theta = -J\vec{H}$ , we compute

$$\begin{aligned}\Delta\theta(p) &= \operatorname{div}_L(\nabla\theta) = -\operatorname{div}_L(J\vec{H}) \\ &= \frac{1}{2}\operatorname{div}_L((J\vec{F})^T) = \frac{1}{2}\sum_i e_i\langle J\vec{F}, e_i\rangle \\ &= -\frac{1}{2}\sum_i e_i\langle \vec{F}, Je_i\rangle(p)\end{aligned}$$

Notice that  $-e_i\langle \vec{F}, Je_i\rangle(p) = -\langle e_i, Je_i\rangle(p) - \langle \vec{F}, J\bar{\nabla}_{e_i}e_i\rangle(p)$ . Hence

$$\begin{aligned}\Delta\theta(p) &= -\frac{1}{2}\langle \vec{F}, J\vec{H}\rangle(p); \\ &= \frac{1}{2}\langle \vec{F}, \nabla\theta\rangle(p)\end{aligned}$$

We finish the proof. □

## 5.2 The proof of Theorem C

Now we are ready to prove Theorem C.

*Proof.* Define a function  $\bar{\theta}$  as

$$\bar{\theta} = \begin{cases} \theta + C_1 & \text{if } \theta + C_1 > 0 \text{ on } L; \\ -C_2 - \theta & \text{if } \theta + C_2 < 0 \text{ on } L. \end{cases}$$

According to the conditions of Theorem C,  $\bar{\theta}$  is a always singular valued, positive function. By Lemma 5.1.3,  $\bar{\theta}$  also satisfies that

$$\Delta\bar{\theta} - \frac{1}{2}\langle \vec{F}, \nabla\bar{\theta}\rangle = 0;$$

Since  $L$  has the polynomial volume growth property, Lemma 2.4.5 implies that  $\bar{\theta}$  is a positive constant. By the definition of  $\bar{\theta}$ ,  $\theta$  has to be a constant on  $L$ . Since  $\vec{H} = J\nabla\theta$ , then  $L$  is minimal. This means  $L$  is a smooth, complete, self-shrinker, we have  $\vec{F}^\perp \equiv 0$  for any point  $\vec{F}$  on  $L$ . The theorem follows from Theorem 2.2.1. □

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