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Explicit Reciprocity Laws for Higher Local Fields

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Abstract

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In this thesis we generalize to higher dimensional local fields the explicit reciprocity laws of Kolyvagin [14] for the Kummer pairing associated to a formal group. The formulas obtained describe the values of the pairing in terms of multidimensional p-adic differentiation, the logarithm of the formal group, the generalized trace and the norm on Milnor K-groups.

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Contents

0.1 Terminology and notation

For the convenience of the reader we will list some terminology and notation that we will use through out the paper. The letters K, L, M, will be used for one dimensional local fields and the corresponding script capital Latin symbols $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ will denote higher local fields containing the corresponding local fields.

- p denotes a fixed prime number.

- Let $K/\mathbb{Q}_p$ denote a local field with ring of integers $\mathcal{O}_K$.

- $F = F(X,Y) \in \mathcal{O}_K[[X,Y]]$ will denote a formal group of finite height h. Let $\mathrm{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of F and $c : \mathrm{End}_{\mathcal{O}_K}(F) \to \mathcal{O}_K$ the embedding $g \mapsto c(g) = g'(0)$.

- Let S be a local field with ring of integers C such that $C \subset c(\mathrm{End}_{\mathcal{O}_K}(F))$. We will fix a uniformizer π of C. Given $a \in C$, the endomorphism $aX + \cdots$ will be denoted by $[a]_F$. For convenience, $[\pi]_F$ will be denoted by f. Thus $f^{(n)}$, the nth fold of f, corresponds to $[\pi^n]_F$.

- Let $\kappa_n \simeq (C/\pi^n C)^h$ be the π^nth torsion group of F and let $\kappa = \varprojlim \kappa_n \simeq C^h$. Let $\{e^i\}_{i=1}^h$ be a basis for κ and $\{e_n^i\}_{i=1}^h$ be the corresponding reductions to the group κ_n.

- Let α denote the ramification index of S over $\mathbb{Q}_p$. We say that a pair (n, t) is admissible if there exist an integer k such that $t - 1 - n \geq \alpha k \geq n$. For example, the pair $(n, 2n + \alpha + 1)$ is admissible with $k = [(n + \alpha/\alpha)]$.

- If R is a discrete valuation ring, the symbols v_R, $\mathcal{O}_R$, μ_R and π_R will always denote the valuation, ring of integers, maximal ideal, and some fix uniformizer of R, respectively. Moreover, if the characteristic of the residue field is p, then we define $\mu_{R,1} = \{x \in R : v_R(x) > v_R(p)/(p-1)\}$. If $R \supset K$, then $F(\mu_R)$ and $F(\mu_{R,1})$ will denote the group of μ_R-points and $\mu_{R,1}$-points of F, respectively.

- $\mathcal{L}$ is a d-dimensional complete field of characteristic 0, i.e., a field for which there are fields $\mathcal{L}_d = \mathcal{L}$, $\mathcal{L}_{d-1}$, . . . , $\mathcal{L}_0$ such that $\mathcal{L}_{i+1}$ is a complete discrete valuation field with residue field $\mathcal{L}_i$, $0 \leq i \leq d-1$, and $\mathcal{L}_0$ is a finite field of characteristic p.

- In what follows we assume that $\mathcal{L}$ is a field of mixed characteristic, i.e., $\mathrm{char}(\mathcal{L}) = 0$ and $\mathrm{char}(\mathcal{L}_{d-1}) = p$. Moreover, we also assume that $\mathcal{L} \supset K(\kappa_n)$.

- Let $t_1, \ldots, t_d$ be a fixed system of local parameters for $\mathcal{L}$, i.e, t_d is a uniformizer for $\mathcal{L} = \mathcal{L}_d$, t_{d-1} is a unit in $\mathcal{O}_{\mathcal{L}}$ but its residue in $\mathcal{L}_{d-1}$ is a uniformizer element of $\mathcal{L}_{d-1}$ and so on.

- Let $\mathsf{v}_{\mathcal{L}} = (v_1, \ldots, v_d) : \mathcal{L}^* \to \mathbb{Z}^d$ be the valuation of rank d corresponding to the system of parameters $t_1, \ldots, t_d$, i.e., $v_d = v_{\mathcal{L}_d}$, $v_{d-1}(\alpha) = v_{\mathcal{L}_{d-1}}(\alpha_{d-1})$

where α_{d-1} is the residue of $\alpha/t_d^{-v_d(\alpha)}$ in $\mathcal{L}_{d-1}$, and so on. Here the group $\mathbb{Z}^d$ is ordered lexicographically as follows: $\bar{a} = (a_1, \ldots, a_d) < (b_1, \ldots, a_d)$ if $a_l < b_l$ and $a_{l+1} = b_{l+1}$, ..., $a_d = b_d$ for some $l \leq d$. We denote by $\mathbb{Z}^d_+ = \{\ (a_1, \ldots, a_d) \in \mathbb{Z}^d\ :\ (a_1, \ldots, a_d) > (0, \ldots, 0)\ \}$.

- $O_{\mathcal{L}} = \{x \in \mathcal{L} : \bar{v}_{\mathcal{L}}(x) \geq \bar{0}\}$ is the valuation ring of rank d of $\mathcal{L}$, with maximal ideal $M_{\mathcal{L}} = \{x \in \mathcal{L} : \bar{v}_{\mathcal{L}}(x) > \bar{0}\}$. Notice that $\mu_{\mathcal{L}} \subset M_{\mathcal{L}} \subset O_{\mathcal{L}} \subset \mathcal{O}_{\mathcal{L}}$.

- $\mathcal{V}_{\mathcal{L}} = 1 + M_{\mathcal{L}}$ is the group of principal units.

- The field $\mathcal{L}$ comes equipped with a special topology, the Parshin topology (cf. §2.4)

- Observe that $\mathcal{L}^* \cong V_{\mathcal{L}} \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$, where $\mathcal{R}$ is a multiplicative closed system of representatives of the last residue field $\mathcal{L}_0$ and $\mathcal{R}^* = \mathcal{R} \diagdown \{0\}$. Then we can endow $\mathcal{L}^* \cong V_{\mathcal{L}} \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$ with the product topology, where the group of principal units $\mathcal{V}_{\mathcal{L}}$ has the induced topology from $\mathcal{L}$, and $\langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$ with the discrete topology.

- $K_d(\mathcal{L})$ is the dth Milnor K-group of the field $\mathcal{L}$, $d \geq 1$.

- Let L be a complete discrete valuation field and define

$$\mathcal{L} = L\{\{T\}\} = \left\{ \sum_{-\infty}^{\infty} a_i T^i :\ a_i \in L,\ \inf v_L(a_i) > -\infty,\ \lim_{i \to -\infty} v_L(a_i) = +\infty \right\}.$$

Let $v_{\mathcal{L}}(\sum a_i T^i) = \min_{i\in\mathbb{Z}} v_L(a_i)$, so $\mathcal{O}_{\mathcal{L}} = \mathcal{O}_L\{\{T\}\}$ and $\mu_{\mathcal{L}} = \mu_L\{\{T\}\}$. Observe that $\mathcal{O}_{\mathcal{L}}/\mu_{\mathcal{L}} = k_L((\overline{T}))$, where k_L is the residue field of L.

- We define

$$L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\},$$

  inductively by $E_{d-1}\{\{T\}\}$ where $E_{d-1} = L\{\{T_1\}\}\cdots\{\{T_{d-2}\}\}$.

- $D(L/K)$ will denote the different of the the extension of local fields L/K.

Chapter 1

Introduction

1.1 Background

The history for finding explicit formulations of local class field theory dates back to Kummer [15] in 1858 when he tried to describe the Hilbert symbol, associated to the cyclotomic field $\mathbb{Q}_p(\zeta_p)$, in terms of logarithmic derivatives. Seventy years later, when class field theory was acquiring its final shape, Artin and Hasse [1] provided an explicit formula for the Hilbert symbol associated to $\mathbb{Q}_p(\zeta_{p^m})$; p being an odd prime.

The formulas of Artin-Hasse would find an application in the work of Iwasawa [10] when he studied the units of the cyclotomic field $\mathbb{Q}(\zeta_{p^m})$. Iwasawa extended the formulas of Artin-Hasse to describe more values of the Hilbert symbol (cf. [9]), in terms of p-adic logarithmic derivatives. This would be the first manifestation of a deep connection, via explicit reciprocity laws, between arithmetical objects and values of zeta functions. The work of Iwasawa inspired Coates and Wiles [3] to do fundamental work on the Birch and Swinnerton Dyer conjecture. One of the main tools in their paper [27] was to extend Iwasawa's reciprocity laws to Kummer pairings associated to Lubin-Tate formal groups, which are analogues of the Hilbert symbol for formal groups.

Soon after, Kolyvagin [14] extended the formulas of Wiles from Lubin-Tate formal groups to arbitrary formal groups (of finite height) and described the Kummer pairing in terms of p-adic derivations. Moreover, Kolyvagin also showed that the pairing is completely characterized by its values at the torsion points of the formal group.

Parallel to this, the theory of higher local fields was developing and along with it was the class field theory for such fields. Different formulations for a such theory were proposed by many people but it would be Kato's [11] (see also [22]), who by means of Milnor K-groups, developed the most commonly used theory. Kato's formulation shows that the finite abelian extensions of a d-dimensional higher local field correspond to norm subgroups of its Milnor dth K-group. This correspondence was established via a reciprocity map between the dth Milnor K-group and the galois group of the maximal abelian extension of the higher local field. I. Fesenko [4] nicely contributed to the class field theory for higher local fields also.

After Kato developed this class field theory, the program to make this theory more explicit began. Some of the higher dimensional explicit reciprocity laws that can be found in the literature are those of Kurihara [16], Vostokov [25], [26], Zinoviev [28] and Kato [12]. These explicit laws describe the analogue of Hilbert symbol for higher local fields, in terms of higher dimensional p-adic logarithmic differentiation.

Using his explicit reciprocity laws, Kato made fundamental progress on Iwasawa main conjecture for modular forms. Kato's explicit laws provide a link between the Euler system of Beilinson elements and some special values of the L-function attached to a modular form. This allowed Kato [13] to prove a divisibility statement related to the Iwasawa main conjecture for modular forms.

In this thesis, we will derive explicit reciprocity laws for the higher dimensional analogue of the Kummer pairing associated to a formal group F, or generalized Kummer pairing. The construction of these laws involve higher dimensional p-adic derivations, the logarithm of the formal group F, the generalized trace and the norm of Milnor K-groups. These formulas constitute a generalization, to higher local fields, of the explicit reciprocity laws of Kolyvagin [14]. In particular, the formulas apply to the generalized Hilbert symbol, like those of Kurihara [16] and Zinoviev [28]. It is still a work in progress to determine if the formulas of these two authors can be derived from the explicit reciprocity laws constructed in this thesis.

Finally, it is worth mentioning that the techniques that are used to obtain the explicit reciprocity laws are inspired by the work of Kolyvagin in [14]. In particular, this allows for a more classical and conceptual approach to the higher dimensional reciprocity laws.

1.2 Description of the problem and results

Let $K/\mathbb{Q}_p$ be a local field and F a one-dimensional formal group of finite height with coefficients in the ring of integers of K. Let $\mathrm{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of F and $c : \mathrm{End}_{\mathcal{O}_K}(F) \to \mathcal{O}_K$ the embedding $f \mapsto c(f) = f'(0)$. Let $S/\mathbb{Q}_p$ be a local field with ring of integers C such that $C \subset c(\mathrm{End}_{\mathcal{O}_K}(F)) \subset \mathcal{O}_K$. Let us fix a uniformizer π for S and denote by $f = [\pi]_F$ the endomorpshism such that $c(f) = \pi$.

Let h be the height of the endomorphism $[\pi]_F$ with respect to C, i.e, $[\pi]_F(X) = g(X^{q^h}) \pmod{\pi C}$ where $g'(0) \neq 0 \pmod{\pi C}$ and $q = |k_S|$. Denote by $f^{(n)} = [\pi^n]_F$ the n-fold of f. Let κ_n be the π^nth torsion group, i.e., the set of those $v \in \overline{K}$ such that $f^{(n)}(v) = 0$. Let $\{e_n^i\}_{i=1}^h$ be a $C/\pi^n C$-basis for the group $\kappa_n \simeq (C/\pi^n C)^h$.

Let $L \supset K$ be a local field containing κ_n. Let $\mathcal{O}_L$ be the ring of integers of L, μ_L its maximal ideal, v_L its valuation and π_L be a uniformizer for L. Define

$$\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\},$$

inductively. The ring on integers and maximal ideal are, respectively,

$$\mathcal{O}_{\mathcal{L}} = \mathcal{O}_L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad \mu_{\mathcal{L}} = \mu_L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.$$

We define the canonical pairing (cf. § 4.1)

$$(\,,\,)_{\mathcal{L},n} : K_d(\mathcal{L}) \times F(\mu_{\mathcal{L}}) \to \kappa_n,$$

by

$$(\alpha, x) \mapsto \Upsilon_{\mathcal{L}}(\alpha)(z) -_F z,$$

where $K_d(\mathcal{L})$ is the dth Milnor K-group of $\mathcal{L}$, $\Upsilon_{\mathcal{L}} : K_d(\mathcal{L}) \to G_{\mathcal{L}}^{ab}$ is the Kato's reciprocity map for $\mathcal{L}$, $f^{(n)}(z) = x$ and $-_F$ is the subtraction in the formal group F, cf. §4.1. Denote by $(,)^i_{\mathcal{L},n}$ the ith coordinate of $(,)_{\mathcal{L},n}$ with respect to the basis $\{e^i_n\}$.

In this paper we will use the technique of Kolyvagin in [14] to derive explicit formulas for the symbol $(,)^i_{\mathcal{L},n}$ in terms of the generalized trace, the norm of Milnor K-groups, d-dimensional derivations and the logarithm of the formal group.

More specifically, let E be a complete discrete valuation field. We define a map

$$c_{E\{\{T\}\}/E} : E\{\{T\}\} \to E$$

by $c_{E\{\{T\}\}/E}(\sum_{i\in\mathbb{Z}} a_i T^i) = a_0$. Then we can define $c_{\mathcal{L}/L}$ by the composition

$$c_{L\{\{T_1\}\}/L} \circ \cdots \circ c_{\mathcal{L}/L\{\{T_1\}\}\ldots\{\{T_{d-2}\}}$$

We define the *generalized trace* by $\mathbb{T}_{\mathcal{L}/S} = \mathrm{Tr}_{L/S} \circ c_{\mathcal{L}/L}$.

The main result in this paper is the following (cf. Theorem 7.3.1). Let $M = L(\kappa_t)$, $t \geq m \geq n$, for t large enough with respect to m and put $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, then

$$(N_{\mathcal{M}/\mathcal{L}}\{a_1, \ldots, a_d\},\ x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{M}/S}\left(\frac{D^i_{\mathcal{M},m}(a_1, \ldots, a_d)}{a_1 \cdots a_d}\ l_F(x)\right)$$

for all $a_1, \ldots, a_d \in \mathcal{M}^*$ (see (7.1) where cases when a_i are units or not are considered) and all $x \in F(\mu_{\mathcal{L}})$. Here l_F is the formal logarithm, $N_{\mathcal{M}/\mathcal{L}}$ is the norm on Milnor K-groups and $D^i_{\mathcal{M},m}$ is a d-dimensional derivation of $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$, i.e.,

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to W$$

is a map, with W an $\mathcal{O}_{\mathcal{M}}$-module, satisfying the properties in definition 6.2.1, that is, for all $a_1, \ldots, a_d$ and $a'_1, \ldots, a'_d$ in $\mathcal{O}_{\mathcal{M}}$ we have:

1. Leibniz rule:

   $$D(a_1, \ldots, a_i a'_i, \ldots, a_d) = a'_i D(a_1, \ldots, a_i, \ldots, a_d) + a_i D(a_1, \ldots, a'_i, \ldots, a_d),$$

   for any $1 \leq i \leq d$.

2. Linearity:

   $$D(a_1, \ldots, a_i + a'_i, \ldots, a_d) = D(a_1, \ldots, a_i, \ldots, a_d) + D(a_1, \ldots, a'_i, \ldots, a_d),$$

   for any $1 \leq i \leq d$.

3. Alternate: $D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0$, if $a_i = a_j$ for $i \neq j$.

4. $D(a_1, \ldots, a_d) = 0$ if some $a_i \in \mathcal{O}_K$.

Notice that D is a 1-dimensional derivation in each component and moreover, it can be parametrized by 1-dimensional derivations. Indeed, just as one dimensional derivations are parametrized by $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$, the $\mathcal{O}_{\mathcal{L}}$-module of Khaler differentials of

$\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$, then it will also be shown in §6.2 Proposition 6.2.1 that all such d-dimensional derivations are parametrized by the $\mathcal{O}_{\mathcal{L}}$-module $\bigwedge^d \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$, i.e., the d-th exterior product of $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$. This is the $\mathcal{O}_{\mathcal{L}}$-module $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}) \otimes \cdots \otimes \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$ divided out by the $\mathcal{O}_{\mathcal{L}}$-submodule generated by the elements

$$x_1 \otimes \cdots \otimes x_d$$

where $x_i = x_j$ for some $i \neq j$, where $x_1, \ldots, x_d \in \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$.

Moreover, notice that from properties (2), (3) and (4) it easily follows that D satisfies the relations

$$D(a_1, \ldots, a_i, \ldots, a_j \ldots, a_d) = 0, \tag{1.1}$$

if $a_i + a_j = 1$, for $i \neq j$, which are called the Steinberg relations. Therefore, following Kolyvagin [14], we define the map

$$\psi : (\mathcal{O}_{\mathcal{M}}^{\times})^d \to W$$

by

$$\{a_1, \ldots, a_d\} \to \frac{D^i_{\mathcal{M},m}(a_1, \ldots, a_d)}{a_1 \ldots a_d},$$

This map can be extended to all of $K_d(\mathcal{M}^*)$ by equation (7.1) from Section 7.1.

As in the work of Kolyvagin [14], these derivations will be normalized in terms of the invariants (7.7) attached to the Galois representation on the Tate-module (7.5).

More specifically, we will build the d-dimensional derivation $D^i_{\mathcal{M},m}$ out of the condition

$$D^i_{\mathcal{M},m}(T_1,\ldots,T_{d-1},e^j_t) = -\frac{T_1\cdots T_{d-1}\ c_{i,j}}{l'(e^j_t)},$$

cf. Proposition 7.2.3, where $c_{i,j}$ is defined in equation (7.7).

The deduction of the formula starts by proving that the pairing $(,)_{\mathcal{L},n}$ is sequentially continuous in the second argument with respect to the Parshin topology, i.e., if $\alpha \in K_d(\mathcal{L})$ and $\{x_k\}$ is a sequence in $F(\mu_{\mathcal{L}})$ such that $x_k \to 0$ then $(\alpha, x_k)_{\mathcal{L},n} \to 0$, cf. Proposition 4.2.2, which is vital for the rest of the proof and was inspired by the proof of Proposition 4.1 in [14]. From this, we can prove the existence of the so called Iwasawa maps, $\psi^i_{\mathcal{L},n} : K_d(\mathcal{L}) \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ (cf. Proposition 5.2.1), such that

$$(\alpha, x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}(\ \psi^i_{\mathcal{L},n}(\alpha)\ l_F(x)\ ) \quad \forall x \in F(\mu_{\mathcal{L},1}),$$

where $F(\mu_{\mathcal{L},1})$ is the set $\{x \in \mathcal{L} :\ v_{\mathcal{L}}(x) \geq [v_{\mathcal{L}}(p)/(p-1)] + 1\}$ considered with the operation defined by F. We then define the maps $D^i_{\mathcal{L},n}$ out of $\psi^i_{\mathcal{L},n}$ as in equation (5.17). The main goal is to prove that under certain conditions the map $D^i_{\mathcal{M},m}$ is a d-dimensional derivation over $\mathcal{O}_K$, cf. Proposition 7.1.1. A vital role in this paper is given by the *norm series* relations

$$(\{r(x), a_2, \ldots, a_d\}, x)_{\mathcal{L},n} = 0, \quad \forall x \in F(\mu_{\mathcal{L}}),\ \forall a_2, \ldots, a_d \in \mathcal{L}^*,$$

cf. Proposition 4.3.1. These series allow us to obtain the representation (5.8) in Proposition 5.2.3 which together with the continuity of the pairing we obtain the identity (5.18) from Proposition 5.4.2. But Proposition 5.4.4 tells us that every element in $\mu_{\mathcal{L}}$ has a representation of the form $\eta(T_1, \ldots, T_{d-1}, \pi_L)$, where $\eta(X_1, \ldots, X_d)$ is a power series in the variables $X_1, \ldots, X_d$ with coefficients in $\mathcal{O}_L$, given by equation (5.4.5).

Then, we will conclude from Proposition 5.4.5 and corollary 5.4.4 that

$$D^i_{\mathcal{L},m}(\alpha_1, \ldots, \alpha_d) = \det \left[\frac{\partial \eta_i}{\partial X_j}\right]_{i,j} \Bigg|_{\substack{X_k = T_k,\\ k=1,\ldots,d}} D^i_{\mathcal{L},n}(T_1, \ldots, \pi_L),$$

where $\alpha_i = \eta_i(T_1, \ldots, \pi_L)$, $i = 1, \ldots, d$ and $T_d = \pi_L$. This last equation tells us that $D^i_{\mathcal{L},n}$ behaves like a d-dimensional derivation when restricted to $\mu_{\mathcal{L}}$.

This will be used in the proof of Proposition 7.1.1 to conclude that $D^i_{\mathcal{M},m}$, for $\mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and M a finite extension of L containing κ_t for large enough t with respect to m, is a d-dimensional derivation when restricted to certain quotient of $\mathcal{M}$ as an $\mathcal{O}_{\mathcal{M}}$-module, namely $R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$. Moreover, Proposition 7.2.3 gives us a way of constructing explicitly this derivation out of the condition

$$D^i_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, e^j_t) = -T_1 \cdots T_{d-1}\; c_{i,j}/l'(e^j_t).$$

We can express back the Iwasawa map $\psi^i_{\mathcal{M},m} : K_d(\mathcal{M}) \to R_{\mathcal{M},m}/\pi^m R_{\mathcal{M},m}$ in terms of the derivation $D^i_{\mathcal{M},m}$ as

$$\psi^i_{\mathcal{M},m}(\{a_1, \ldots, a_d\}) = \frac{D^i_{\mathcal{M},m}(a_1, \ldots, a_d)}{a_1 \cdots a_d}$$

cf. 7.1. Finally, the return to the field $\mathcal{L}$ will be guaranteed by Proposition 7.1.2. The final formulas are contained in Theorem 7.3.1.

In chapter 8 we specialize to the case of a Lubin-tate formal group and refine Theorem 7.3.1 for this case. In particular, we give an explicit computation of the invariants $c_{i,j}$, namely

$$c_{i,j} = -\frac{1}{\pi^t}.$$

Chapter 2

Higher-dimensional local fields

2.1 2-dimensional local fields

We will start defining 2-dimensional local fields since most of the proofs will be reduced to this case. In the next sections all the concepts will be generalized to higher dimensions.

Definition 2.1.1. *We say that $\mathcal{K}$ is a 2-dimensional local field if there are fields $\mathcal{K}_2 = \mathcal{K}$, $\mathcal{K}_1$, and $\mathcal{K}_0$ such that $\mathcal{K}_{i+1}$ is a complete discrete valuation ring with residue field $\mathcal{K}_i$, $0 \leq i \leq 1$, and $\mathcal{K}_0$ is a finite field of characteristic p.*

We have the following examples of 2-dimensional local fields:

1. $\mathbb{F}_q((T_1))((T_2))$, where q is a power of the prime p. In this case $\mathcal{K}_2 = \mathbb{F}_q((T_1))((T_2))$, $\mathcal{K}_1 = \mathbb{F}_q((T_1))$ and $K_0 = \mathbb{F}_q$. Recall that if k is a field then $k((T))$ is the fraction field of the ring $k[[T]]$.

2. $E((T))$, where E is a finite extension of $\mathbb{Q}_p$. Here $\mathcal{K}_1 = E$ and $\mathcal{K}_0 = k_E$

3. For a complete discrete valuation field E we define

$$\mathcal{K} = E\{\{T\}\} = \left\{ \sum_{-\infty}^{\infty} a_i T^i : \; a_i \in E, \; \inf v_E(a_i) > -\infty, \; \lim_{i \to -\infty} v_E(a_i) = +\infty \right\}$$

Let $v_{\mathcal{K}}(\sum a_i T^i) = \min v_E(a_i)$. Then $\mathcal{K}$ is a complete discrete valuation field with residue field $k_F((t))$ and ring of integers

$$\mathcal{O}_{E\{\{T\}\}} = \left\{ \sum_{-\infty}^{\infty} a_i T^i \in F\{\{T\}\} : \; a_i \in \mathcal{O}_F, \; \forall i \in \mathbb{Z} \right\}$$

Thus, if E is a local field then $E\{\{T\}\}$ is a 2-dimensional local field with $\mathcal{K}_1 = k_E((t))$ and $\mathcal{K}_0 = k_E$. These fields are called *standard* 2-dimensional local fields (cf. [7] §1.1).

We are interested only in the case where $\mathcal{K}$ is of mixed characteristic, i.e., $\mathrm{char}(\mathcal{K}_1) = p$. The following Theorem classifies such fields.

Theorem 2.1.1 (Classification Theorem)**.** *Let $\mathcal{K}$ be a 2-dimensional local field of mixed characteristic. Then $\mathcal{K}$ is a finite extension of a 2-dimensional standard field $E\{\{T_1\}\}$, where E is a finite extension of $\mathbb{Q}_p$, and there is a finite extension of $\mathcal{K}$ which is a standard 2-dimensional local field.*

Proof. cf. [7] § 1.1 *Classification Theorem.* □

We define the ring of integers of $\mathcal{K}$ to be the set

$$O_{\mathcal{K}} = \{ \; x \in \mathcal{O}_{\mathcal{K}} : \; \bar{x} \in \mathcal{O}_{K_1} \}$$

and the maximal ideal

$$M_{\mathcal{K}} = \{\, x \in \mathcal{O}_{\mathcal{K}} : \ \bar{x} \in \mu_{K_1}\}$$

where $\bar{x}$ denotes the reduction of x in $\mathcal{K}_1$. Notice that

$$\mu_{\mathcal{K}} \subset M_{\mathcal{K}} \subset O_{\mathcal{K}} \subset \mathcal{O}_{\mathcal{K}}.$$

In the examples above we can compute these two sets explicitly:

1. If $\mathcal{K} = \mathbb{F}_q((T_1))((T_2))$: $O_{\mathcal{K}} = \mathbb{F}_q[[T_1]] + T_2\mathbb{F}_q((T_1))[[T_2]]$ and $M_{\mathcal{K}} = T_1\mathbb{F}_q[[T_1]] + T_2\mathbb{F}_q((T_1))[[T_2]]$

2. If $\mathcal{K} = E((T))$: $O_{\mathcal{K}} = \mathcal{O}_E + TE[[T]]$ and $M_{\mathcal{K}} = \pi_E\mathcal{O}_E + TE[[T]]$.

3. If $\mathcal{K} = E\{\{T\}\}$: $O_{\mathcal{K}} = \left\{ \sum_{-\infty}^{\infty} a_iT^i \in \mathcal{K} : a_i \in \mathcal{O}_E,\ \forall i \geq 0 \text{ and } a_i \in \mu_E\ \forall i < 0 \right\}$ and $M_{\mathcal{K}} = \left\{ \sum_{-\infty}^{\infty} a_iT^i \in F\{\{T\}\} :\ a_i \in \mathcal{O}_E,\ \forall i > 0 \text{ and } a_i \in \mu_E\ \forall i \leq 0 \right\}$

In particular, if $\mathcal{K} = \mathbb{Q}_p\{\{T\}\}$ we have

$$O_{\mathcal{K}} = \{ \sum_{-\infty}^{\infty} a_iT^i \in \mathcal{K} :\ a_i \in \mathbb{Z}_p,\ \forall i \geq 0 \text{ and } a_i \in p\mathbb{Z}_p\ \forall i < 0\},$$

and

$$M_{\mathcal{K}} = \{ \sum_{-\infty}^{\infty} a_iT^i \in \mathcal{K} :\ a_i \in \mathbb{Z}_p,\ \forall i > 0 \text{ and } a_i \in p\mathbb{Z}_p\ \forall i \leq 0\}.$$

Topology on $\mathcal{K}$: The topology on $\mathcal{K} = E\{\{T\}\}$ is called the Parshin topology or Higher-dimensional topology (cf. [7] §1.3). A basis of neighborhoods of 0 for this topology is given by the following: Let $\{U_i\}_{i\in\mathbb{Z}}$ be a sequence of neighborhoods of 0 in the local field E, i.e., $U_i = \pi_E^{n_i}\mathcal{O}_E$, where $n_i \in \mathbb{Z}$, such that

1. There exist a $c \in \mathbb{Z}$ such that $\pi_E^c\mathcal{O}_E \subset U_i$ for all $i \in \mathbb{Z}$, i.e., $\cap U_i \supset \pi_E^c\mathcal{O}_E$. This says that the neighborhoods U_i's cannot be arbitrarily small.

2. For every $l \in \mathbb{Z}$ we have $\pi_E^l\mathcal{O}_E \subset U_i$ for sufficiently large i. This implies that the U_i's get bigger as $i \to \infty$.

We put

$$\mathcal{U}_{\{U_i\}} = \{ \sum_{-\infty}^{\infty} a_iT^i \in \mathcal{K} : \ a_i \in U_i\}$$

These sets $\mathcal{U}_{\{U_i\}}$ form a base of neighborhoods of 0 in $\mathcal{K}$.

Remark 2.1.1. Observe that in this topology an element $x = \sum_{-\infty}^{\infty} a_iT^i \in \mathcal{K} = E\{\{T\}\}$ converges as an infinite sum since both tails $\sum_{i>N} a_iT^i$ and $\sum_{i<-N} a_iT^i$ approach to 0 as $N \to \infty$. To see this take a basis element for zero, $\mathcal{U}_{\{U_i\}}$, and let c be given by condition 1 above, and for $l = v_K(x) = \inf v_E(a_i)$ let N be large enough to satisfy condition 2 above.

Since by definition $v_E(a_i) \to \infty$ as $i \to -\infty$ we can find an N such that $v_E(a_i) > c$ for $i < -N$. Then $a_i \in \pi_E^c\mathcal{O}_E \subset U_i$ for $i < -N$, i.e., $\sum_{i<-N} a_iT^i \in \mathcal{U}_{\{U_i\}}$.

Now $v_E(a_i) \geq l$ for all i, then $a_i \in \pi_E^l \mathcal{O}_E \subset U_i$ for $i > N$. Again, this implies $\sum_{i>N} a_i T^i \in \mathcal{U}_{\{U_i\}}$

Remark 2.1.2. Observe also that a sequence $x_n = \sum_{-\infty}^{\infty} a_i^{(n)} T^i$ converging to zero in this topology does not necessarily satisfy that $v_{\mathcal{K}}(x_n) \to \infty$, i.e., the coefficients $a_i^{(n)}$ do not necessarily converge uniformly on i to zero. For instance, the sequence $x_n = T^n$ converges to zero in this topology and $v_{\mathcal{K}}(x_n) = 0$ for all n. But the following Proposition is true:

Proposition 2.1.1. *The sequence $x_n = \sum_{-\infty}^{\infty} a_i^{(n)} T^i$ converges to zero if and only if the following two conditions are satisfied*

(i) $\inf v_{\mathcal{K}}(x_n) > -\infty$*;*

(ii) for a given $m \in \mathbb{Z}_+$ the sequence $a_i^{(n)} \overset{n\to\infty}{\to} 0$ uniformly for every $i \leq m$.

That is, the coefficients are uniformly bounded and converge to zero uniformly for $i \leq m$, for every m.

Proof. $\to$) Suppose the sequence x_n converges to zero. For a fix $m \in \mathbb{Z}$ and given $M > 0$ define

$$U_i^M = \begin{cases} \pi_E^M \mathcal{O}_E, & i \leq m \\ \pi_E^{-i} \mathcal{O}_E, & i > m \end{cases}$$

Then $\mathcal{V}_M = \mathcal{V}_{U_i^M}$ is a basis element for zero, so there exists an N such that $x_n \in \mathcal{V}_M$ for $n > N$. That is $v_E(a_i^{(n)}) \geq M$ for $n > N$ and all $i \leq m$, then $a_i^{(n)} \overset{n\to\infty}{\to} 0$ uniformly on every $i \leq m$.

Let us show now that $\inf v_K(x_n) > -\infty$. For $i > 1$ define $m_i = \inf_n v_E(a_i^{(n)})$ and $M_i = \min_{1\le k\le i}\{m_k\}$. Then $M_i \ge M_{i+1}$ and moreover, if $\inf v_K(x_n) = -\infty$, then $M_i \overset{i\to\infty}{\to} -\infty$ which follows from the just proven condition (ii). Consider

$$U_i = \begin{cases} \pi_E \mathcal{O}_E, & i \le 0 \\ \pi_E^{M_i+1}\mathcal{O}_E, & i > 0 \end{cases}$$

Then $\mathcal{U}_{\{U_i\}}$ is a base element for 0. Again, since we are assuming $\inf v_K(x_n) = -\infty$, for certain large enough i we have the strict inequality $M_{i-1} > M_i$, i.e., $M_i = m_i$ and so we can find a subsequence $x_{n_i} = \sum_j a_j^{(n_i)} T^j$ such that $v_K(a_i^{(n_i)}) = M_i$, thus $v_K(x_{n_i}) \le v_E(a_i^{(n_i)}) = M_i$. Then $a_i^{(n_i)} \notin U_i$, which implies $x_{n_i} \notin \mathcal{U}_{\{U_i\}}$. Finally, since $n_i \to \infty$ as $i \to \infty$ this contradicts the convergence of x_n to zero.

$\leftarrow$) Let x_n be a sequence in K satisfying (i) and (ii). Let $\mathcal{U}_{\{U_i\}}$ is a base element for 0. Then there exist a $c \in \mathbb{Z}$ such that $\pi_E^c \mathcal{O}_E \subset U_i$ for all $i \in \mathbb{Z}$. Also for $l = \inf v_K x_n$ there exist an i_0 such that $\pi_E^l \mathcal{O}_E \subset U_i$ for all $i > i_0$.

By condition (ii) applied to $m = i_0$ there exist an N such that $v_E(a_i^{(n)}) > c$ for all $n \ge N$ and all $i \le i_0$. Then $a_i^{(n)} \in \pi_E^c \mathcal{O}_E \subset U_i$ for all $n \ge N$ and all $i \le i_0$. On the other hand, $v_E(a_i^{(n)}) \ge \inf v_K(x_n) = l$, so $a_i^{(n)} \in \pi_E^l \mathcal{O}_E \subset U_i$ for all $i > i_0$. Summarizing we have that $a_i^{(n)} \in U_i$ for all $n > N$ and all $i \in \mathbb{Z}$. Therefore $x_n \in \mathcal{U}_{\{U_i\}}$ and so it converges to zero in the Parshin topology.

$\square$

Topology on $\mathcal{K}^*$:

Let $\mathcal{R} \subset \mathcal{K} = \mathcal{K}_2$ be a set of representatives of the last residue field $\mathcal{K}_0$. Let t_1 and t_2 be a fixed system of local parameters for $\mathcal{K}$, i.e, t_2 is a uniformizer for $\mathcal{K}$ and t_1 is a unit in $\mathcal{O}_\mathcal{K}$ but its residue in $\mathcal{K}_1$ is a uniformizer element of $\mathcal{K}_1$. Then

$$\mathcal{K}^* = \mathcal{V}_\mathcal{K} \times \langle t_1 \rangle \times \langle t_2 \rangle \times \mathcal{R}^*,$$

where the group of principal units $V_\mathcal{K} = 1 + M_\mathcal{K}$ and $\mathcal{R}^* = \mathcal{R} \diagdown \{0\}$. From this observation we have the following,

Proposition 2.1.2. *We can endow $\mathcal{K}^*$ with the product of the induced topology from $\mathcal{K}$ on the group $\mathcal{V}_\mathcal{K}$ and the discrete topology on $\langle t_1 \rangle \times \langle t_2 \rangle \times \mathcal{R}^*$. Moreover, every Cauchy sequence with respect to this topology converges in $\mathcal{K}^*$.*

Proof. cf [7] Chapter 1: *Higher dimensional local fields.* □

2.2 d-dimensional local fields

We will now generalize all the concepts introduce before to any dimension. In the rest of this section E will denote a local field, k_E its residue field and π_E a uniformizer for E.

Definition 2.2.1. *$\mathcal{K}$ is an d-dimensional local field, i.e., a field for which there is a chain of fields $\mathcal{K}_d = \mathcal{K}$, $\mathcal{K}_{d-1}$, ..., $\mathcal{K}_0$ such that $\mathcal{K}_{i+1}$ is a complete discrete valuation ring with residue field $\mathcal{K}_i$, $0 \leq i \leq d-1$, and $\mathcal{K}_0$ is a finite field of characteristic p.*

If k is a finite field then $\mathcal{K} = k((T_1))\dots((T_d))$ is a d-dimensional local field with

$$\mathcal{K}_i = k((T_1))\dots((T_i)) \quad 1 \le i \le d.$$

If E is a local field, then $\mathcal{K} = E\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$ is defined inductively as

$$E_{d-1}\{\{T_{d-1}\}\},$$

where $E_{d-1} = E\{\{T_1\}\}\dots\{\{T_{d-2}\}\}$. We have that $\mathcal{K}$ is a d-dimensional local field with residue field $\mathcal{K}_{d-1} = k_{E_{d-1}}((T_{d-1}))$, and by induction

$$k_{E_{d-1}} = k_E((T_1))\dots((T_{d-2})).$$

Therefore $\mathcal{K}_{d-1} = k_E((T_1))\dots((T_{d-1}))$. These fields are called the *standard* fields.

From now on we will assume $\mathcal{K}$ has *mixed* characteristic, i.e., char($\mathcal{K}$)=0 and char($\mathcal{K}_{d-1}$)= p. The following theorem classifies all such fields.

Theorem 2.2.1 (Classification Theorem)**.** *Let $\mathcal{K}$ be an d-dimensional local field of mixed characteristic. Then $\mathcal{K}$ is a finite extension of a standard field*

$$E\{\{T_1\}\}\dots\{\{T_{d-1}\}\},$$

where E is a local field, and there is a finite extension of $\mathcal{K}$ which is a standard field.

Proof. cf. [7] § 1.1 *Classification Theorem.* □

Definition 2.2.2. *An d-tuple of elements $t_1, \dots, t_d \in \mathcal{K}$ is called a system of local* *parameters of* $\mathcal{K}$*, if* t_d *is a prime in* $\mathcal{K}_d$*,* t_{d-1} *is a unit in* $\mathcal{O}_{\mathcal{K}}$ *but its residue in* $\mathcal{K}_{d-1}$ *is a prime element of* $\mathcal{K}_{d-1}$*, and so on.*

For the standard field $E\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$ we can take as a system of local parameters $t_d = \pi_E$, $t_{d-1} = T_{d-1}, \dots, t_1 = T_1$.

Definition 2.2.3. *We define a discrete valuation of rank* d *to be the map* $\mathsf{v} = (v_1, \dots, v_d) : \mathcal{K}^* \to \mathbb{Z}^d$*,* $v_d = v_{\mathcal{K}_d}$*,* $v_{d-1}(x) = v_{\mathcal{K}_{d-1}}(x_{d-1})$ *where* x_{d-1} *is the residue in* $\mathcal{K}_{d-1}$ *of* $xt_d^{-v_n(x)}$*, and so on.*

Although the valuation depends, for $n > 1$, on the choice of $t_2, \dots, t_d$, it is independent in the class of equivalent valuations.

2.3 Extensions of $\mathcal{K}$

Let $\mathcal{L}/\mathcal{K}$ be a finite extension of the d-dimensional local field $\mathcal{K}$. Then $\mathcal{L}$ is also a d-dimensional local field.

Definition 2.3.1. *Let* $t_1, \dots, t_d$ *and* $t'_1, \dots, t'_d$ *be a system of local parameters for* K *and* L*, respectively, with associated valuations* v *and* v'*, respectively. Put*

$$E(\mathcal{L}/\mathcal{K}) = (v'_j(t_i))_{i,j} = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ \dots & e_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & e_d \end{pmatrix}$$

where $e_i = e(\mathcal{L}_i/\mathcal{K}_i)$*,* $i = 1, \dots, n$*. Then* e_i *does not depend on the choice of the* *parameters, and* $[\mathcal{L}/\mathcal{K}] = f(\mathcal{L}/\mathcal{K})\prod_{i=1}^{n} e_i(\mathcal{L}/\mathcal{K})$, *where* $f(\mathcal{L}/\mathcal{K}) = [\mathcal{L}_0/\mathcal{K}_0]$ *(cf. [7] §1.2).*

2.4 Topology on $\mathcal{K}$

We define the topology on $E\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$ by induction on d. For $d = 1$ we define the topology to be the topology of a one-dimensional local field. Suppose we have defined the topology on a standard d-dimensional local field E_d and let $\mathcal{K} = E_d\{\{T\}\}$. Denote by $P_{E_d}(c)$ the set $\{x \in E_d : v_{E_d}(x) \geq c\}$. Let $\{V_i\}_{i\in\mathbb{Z}}$ be a sequence of neighborhoods of zero in E_d such that

$$\begin{cases} \text{1. there is a } c \in \mathbb{Z} \text{ such that } P_{E_d}(c) \subset V_i \text{ for all } i \in \mathbb{Z}. \\ \\ \text{2. for every } l \in \mathbb{Z} \text{ we have } P_{E_d}(l) \subset V_i \text{ for all sufficiently large } i. \end{cases} \tag{2.1}$$

and put $\mathcal{V}_{\{V_i\}} = \{\sum b_i T^i \ : b_i \in V_i\}$. These sets form a basis of neighborhoods of 0 for a topology on $\mathcal{K}$. For an arbitrary d-dimensional local field L of mixed characteristic we can find, by the Classification Theorem, a standard field that is a finite extension of L and we can give L the topology induced by the standard field.

Proposition 2.4.1. *Let* $\mathcal{L}$ *be a d-dimensional local field of mixed characteristic with the topology defined above.*

1. $\mathcal{L}$ *is complete with this topology. Addition is a continuous operation and multiplication by a fixed* $a \in \mathcal{L}$ *is a continuous map.*

2. *Multiplication is a sequentially continuous map, i.e., if* $x \in \mathcal{L}$ *and* $y_k \to y$ *in* $\mathcal{L}$ *then* $xy_k \to xy$.

3. *This topology is independent of the choice of the standard field above* $\mathcal{L}$.

4. *If* $\mathcal{K}$ *is a standard field and* $\mathcal{L}/\mathcal{K}$ *is finite, then the topology above coincides with the natural vector space topology as a vector space over* $\mathcal{K}$.

5. *The reduction map* $\mathcal{O}_{\mathcal{L}} \to k_{\mathcal{L}} = \mathcal{L}_{d-1}$ *is continuous and open (where* $\mathcal{O}_{\mathcal{L}}$ *is given the subspace topology from* $\mathcal{L}$*, and* $k_{\mathcal{L}} = \mathcal{L}_{d-1}$ *the* $(d-1)$*-dimensional topology).*

Proof. All the proofs can be found in [20] Theorem 4.10.

□

2.5 Topology on $\mathcal{K}^*$

Let $\mathcal{R} \subset \mathcal{K} = \mathcal{K}_d$ be a set of representatives of the last residue field $\mathcal{K}_0$. Let $t_1, \ldots, t_d$ be a fixed system of local parameters for $\mathcal{K}$, i.e., t_d is a uniformizer for $\mathcal{K}$, t_{d-1} is a unit in $\mathcal{O}_{\mathcal{K}}$ but its residue in $\mathcal{K}_{d-1}$ is a uniformizer element of $\mathcal{K}_{d-1}$, and so on. Then

$$\mathcal{K}^* = \mathcal{V}_{\mathcal{K}} \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*,$$

where the group of principal units $V_{\mathcal{K}} = 1 + M_{\mathcal{K}}$ and $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. From this observation we have the following,

Proposition 2.5.1. *We can endow $\mathcal{K}^*$ with the product of the induced topology from $\mathcal{K}$ on the group $\mathcal{V}_{\mathcal{K}}$ and the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$. In this topology we have,*

1. *Multiplication is sequentially continuous, i.e., if $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

2. *Every Cauchy sequence with respect to this topology converges in $\mathcal{K}^*$.*

Proof. cf [7] Chapter 1 §1.4.2. □

Chapter 3

Formal groups

In the rest of this paper F will be a formal group over the ring of integers, $\mathcal{O}_K$, of a local field $K/\mathbb{Q}_p$. Let $\mathrm{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of F. For $t \in \mathrm{End}_{\mathcal{O}_K}(F)$ we denote $t'(0)$ by $c(t)$. This induces an embedding

$$c : \mathrm{End}_{\mathcal{O}_K}(F) \to \mathcal{O}_K.$$

We have that that $c(\mathrm{End}_{\mathcal{O}_K}(F))$ is a closed subring of $\mathcal{O}_K$ (cf. [14] §2.3). In particular $\mathbb{Z}_p \subset c(\mathrm{End}_{\mathcal{O}_K}(F))$. Let $S/\mathbb{Q}_p$ be a local field with ring of integers C such that $C \subset c(\mathrm{End}_{\mathcal{O}_K}(F))$. Let us fix once and for all a uniformizer π for C. Let j be the degree of inertia of $S/\mathbb{Q}_p$, i.e., $|k_S| = p^j$. In what follows f will denote the element in $\mathrm{End}_{\mathcal{O}_K}(F)$ such that $c(f) = \pi$.

3.1 The Weierstrass lemma

Let E be a discrete valuation field of zero characteristic with integer ring $\mathcal{O}_E$ and maximal ideal μ_E.

Lemma 3.1.1 (Weierstrass lemma)**.** *Let $g = a_0 + a_1X + \cdots \in \mathcal{O}_E[[X]]$ be such that $a_0, \ldots, a_{n-1} \in \mu_E$, $n \geq 1$, and $a_n \notin \mu_E$. Then there exist a unique monic polynomial $c_0 + \cdots + X^n$ with coefficients in μ_E and a series $b_0 + b_1X \cdots$ with coefficients in $\mathcal{O}_E$ and b_0 a unit, i.e., $b_0 \neq \mu_E$, such that*

$$g = (c_0 + \cdots + X^n)(b_0 + b_1X \cdots).$$

Proof. See [18] IV. §9 Theorem 9.2.

□

3.2 The group $F(\mu_{\mathcal{M}})$

Let $\mathcal{K}$ be a d-dimensional local field containing the local field K. For example, we may consider $\mathcal{K}$ to be $K\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Denote by $F(\mu_{\mathcal{K}})$ the group with underlying set $\mu_{\mathcal{K}}$ and operation defined by the formal group F. More generally, if $\mathcal{M}$ is an algebraic extension of $\mathcal{K}$ we define

$$F(\mu_{\mathcal{M}}) := \bigcup_{\mathcal{M}\supset\mathcal{L}\supset\mathcal{K}|[\mathcal{L}/\mathcal{K}]<\infty} F(\mu_{\mathcal{L}}).$$

An element $f \in \mathrm{End}(F)$ is said to be an isogeny if the map $f : F(\mu_{\bar{\mathcal{K}}}) \to F(\mu_{\bar{\mathcal{K}}})$ induced by it is surjective with finite kernel.

If the reduction of f in $k_K[[X]]$, k_K the residue field of K, is not zero then it is of the form $f_1(X^{p^h})$ with $f_1'(0) \in \mathcal{O}_K^*$, cf. [14] Proposition 1.1. In this case we say that f has finite height. If on the other hand the reduction of f is zero we say it has infinite height.

Proposition 3.2.1. *f is an isogeny if and only if f has finite height. Moreover, in this situation $|\ker f| = p^h$.*

Proof. If the height is infinite, the coefficients of f are divisible by a uniformizer of the local field K, so f cannot be surjective. Let $h < \infty$ and $x \in \mu_{\mathcal{L}}$ where $\mathcal{L}$ is a finite extension of $\mathcal{K}$. Consider the series $f - x$ and apply Lemma 3.1.1 with $E = \mathcal{L}$, i.e.,

$$f - x = (c_0 + \cdots + X^{p^h})(b_0 + b_1 X + \cdots),$$

where c's $\in \mu_{\mathcal{L}}$, b's $\in \mathcal{O}_{\mathcal{L}}$ and $b_0 \in \mathcal{O}_{\mathcal{L}}^*$. Therefore the equation $f(X) = x$ is equivalent to the equation $c_0 + \cdots + X^{p^h} = 0$ and since the c's $\in \mu_{\mathcal{L}}$ every root belongs to $\mu_{\overline{\mathcal{K}}}$.

Moreover, the polynomial $P(X) = c_0 + \cdots + X^{p^h}$ is separable because $f'(X) = \pi t(X)$, $t(X) = 1 + \ldots$ is an invertible series and $f' = P'(b_0 + b_1X + \cdots) + P(b_0 + b_1X + \cdots)'$ so P' can not vanish at a zero of P. We conclude that P has p^h roots, i.e., $|\ker\ f| = p^h$.

□

Proposition 3.2.2. *Denote by j the degree of inertia of $S/\mathbb{Q}_p$ and by h_1 the height of $f = [\pi]_F$. Then j divides h_1, namely $h_1 = jh$. Let κ_n be the kernel of $f^{(n)}$. Then*

$$\kappa_n \simeq (C/\pi^n C)^h \quad \text{and} \quad \varprojlim \kappa_n \simeq C^h,$$

as C-modules. This h is called the height of the formal group with respect to $C = \mathcal{O}_S$.

Proof. cf. [14] Proposition 2.3. □

Remark 3.2.1. Notice that since the coefficients of F are in the local field K then $\kappa_n \subset \overline{K}$ for all $n \geq 1$.

Let us fix once and for all a basis $\{e^i\}_{i=1}^h$ for $\varprojlim \kappa_n$. Denote by e_n^i the reduction of e^i to κ_n. Clearly $\{e_n^i\}$ is a basis for κ_n.

Throughout K_n will denote the one dimensional local field $K(\kappa_n)$. Suppose M/K is a finite extension, then M_n will denote the local field $M(\kappa_n)$. Let $\mathcal{M}$ and $\mathcal{M}_n$ denote, respectively,

$$M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\} \quad \text{and} \quad M_n\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}.$$

3.3 The logarithm of the formal group

We define the logarithm of the formal group F to be the series

$$l_F = \int_0^X \frac{dX}{F_X(0,X)}$$

Observe that since $F_X(0,X) = 1 + \cdots \in \mathcal{O}_K[[X]]^*$ then l_F has the form

$$X + \frac{a_2}{2}X^2 + \cdots + \frac{a_n}{n}X^n + \cdots$$

where $a_i \in \mathcal{O}_K$.

Proposition 3.3.1. *Let E be a field of characteristic 0 that is complete with respect to a discrete valuation, $\mathcal{O}_E$ the valuation ring of E with maximal ideal μ_E and valuation v_E. Consider a formal group F over $\mathcal{O}_E$, then*

1. *The formal logarithm induces a homomorphism*

$$l_F : F(\mu_E) \to E$$

with the additive group law on E.

2. *The formal logarithm induces the isomorphism*

$$l_F : F(\mu_E^r) \xrightarrow{\sim} \mu_E^r$$

for all $r \geq [v_E(p)/(p-1)] + 1$ *and*

$$v_E(l(x)) = v(x) \quad (\ \forall x \in \mu_E^r\ ).$$

In particular, this holds for $\mu_{E,1} = \{x \in E :\ v_E(x) > v_E(p)/(p-1) + 1\ \}$.

Proof. [24] IV Theorem 6.4 and Lemma 6.3. □

Lemma 3.3.1. *Let* E *and* v_E *as in the previous proposition. Then*

$$v_E(n!) \leq \frac{(n-1)v_E(p)}{p-1},$$

and $v_E(x^n/n!) \to \infty$ *as* $n \to \infty$ *for* $x \in \mu_{E,1}$.

Proof. The first assertion can be found in [24] IV. Lemma 6.2. For the second one notice that

$$\begin{aligned} v_E(x^n/n!) &\geq nv_E(x) - v(n!) \\ &\geq nv_E(x) - (n-1)\frac{v_E(p)}{p-1} \\ &= v_E(x) + (n-1)\left(v_E(x) - \frac{v_E(p)}{p-1}\right). \end{aligned}$$

Since we are assuming that $x \in \mu_{E,1}$, i.e., $v_E(x) > v_E(p)/(p-1)$, then $v_E(x^n/n!) \to \infty$ as $n \to \infty$. □

Lemma 3.3.2. *Let* $\mathcal{L}$ *be a* d*-dimensional local field containing the local field* K, $g(X) = a_1X + \frac{a_2}{2}X^2 + \cdots + \frac{a_n}{n}X^n + \cdots$ *and* $h(X) = a_1X + \frac{a_2}{2!}X^2 + \cdots + \frac{a_n}{n!}X^n + \cdots$ *with*

$a_i \in \mathcal{O}_K$. Then g and h define, respectively, maps $g : \mu_{\mathcal{L}} \to \mu_{\mathcal{L}}$ and $h : \mu_{\mathcal{L},1} \to \mu_{\mathcal{L},1}$ that are sequentially continuous in the Parshin topology.

Proof. We may assume $\mathcal{L}$ is a standard d-dimensional local field. Let $\mathcal{V}_{\{V_i\}}$ be a basic neighborhood of zero that we can consider to be a subgroup of $\mathcal{L}$, and let $c > 0$ such that $P_{\mathcal{L}}(c) \subset \mathcal{V}_{\{V_i\}}$. If $x_n \in \mu_{\mathcal{L}}$ for all n, then there exists an $N_1 > 0$ such that $v_{\mathcal{L}}(x_n^i/i), v_{\mathcal{L}}(x^i/i) > c$ for all $i > N_1$ and all n; because $iv_{\mathcal{L}}(x_n) - v(i) \geq iv_{\mathcal{L}}(x_n) - \log_p(i) \geq i - \log_p(i) \to \infty$ as $i \to \infty$. On the other hand, if $x_n \in \mu_{\mathcal{L},1}$ for all n, then there exists an $N_2 > 0$ such that $v_{\mathcal{L}}(x_n^i/i!), v_{\mathcal{L}}(x^i/i!) > c$ for all $i > N_2$ and all n by Lemma 3.3.1. Then, for $N = \max\{N_1, N_2\}$, we have

$$\sum_{i=N+1}^{\infty} a_i \frac{x_n^i - x^i}{i}, \sum_{i=N+1}^{\infty} a_i \frac{x_n^i - x^i}{i!} \in P_{\mathcal{L}}(c) \subset \mathcal{V}_{\{V_j\}}.$$

Now, since multiplication is sequentially continuous and $x_n \to x$ then

$$\sum_{i=1}^{N} a_i \frac{x_n^i - x^i}{i} \to 0, \ \sum_{i=1}^{N} a_i \frac{x_n^i - x^i}{i!} \to 0 \quad \text{as } n \to \infty$$

Thus for n large enough we have that

$$g(x_n) - g(x), \ h(x_n) - h(x) = \sum_{i=1}^{N} + \sum_{i=N+1}^{\infty} \in \mathcal{V}_{\{V_i\}}.$$

□

Remark 3.3.1. In particular, $\log : \mu_{\mathcal{L}} \to \mu_{\mathcal{L}}$, $l_F : \mu_{\mathcal{L}} \to \mu_{\mathcal{L}}$ and $\exp_F = l_F^{-1} : \mu_{\mathcal{L},1} \to \mu_{\mathcal{L},1}$ are sequentially continuous.

Chapter 4

The Kummer Pairing

4.1 Higher-dimensional local class field theory

4.1.1 Milnor-K-groups and norms

Definition 4.1.1. *Let R be a ring and $m \geq 0$. We denote by $K_m(R)$ the group*

$$\underbrace{R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times}_{m-times} / I$$

where I is the subgroup of $(R^\times)^{\otimes m}$ generated by

$$\{a_1 \otimes \cdots \otimes a_m \ : \ a_1, \ldots, a_m \in R^\times \text{ such that } a_i + a_j = 1 \text{ for some } i \neq j\}$$

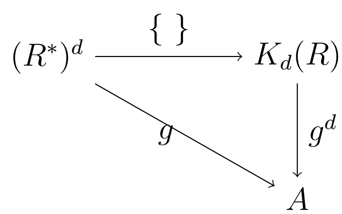
$K_m(R)$ is called the m^{th} Milnor-K-group of R. The element $a_1 \otimes \cdots \otimes a_m$ is denoted by $\{a_1, \ldots, a_m\}$.

We define the symbol map $\{,\}$ to be the map

$$R^\times \times \cdots \times R^\times \to K_d(R) : \quad (a_1, \ldots, a_d) \to \{a_1, \ldots, a_d\}.$$

If A is any abelian group, a map $g : (R^\times)^d \to A$ which is multilinear and satisfies $g(a_1, \ldots, a_d) = 0$, whenever $a_i + a_j = 1$ for some $i \neq j$ is called an *Steinberg map.*

Remark 4.1.1. It follows from the very definition that any Steinberg map $g : (R^\times)^d \to A$ can be factored through the symbol map, i.e., there exist a homomorphism $g^d : K_d(R) \to A$ such that



Proposition 4.1.1. *The elements of the Milnor K-group satisfy the relations*

1. $\{a_1, \ldots, a_i, \ldots, -a_i, \ldots, a_m\} = 1$

2. $\{a_1, \ldots, a_i, \ldots, a_j, \ldots, a_m\} = \{a_1, \ldots, a_j, \ldots, a_i, \ldots, a_m\}^{-1}$

Proof. To simplify the notation we will assume $m = 2$.

1. Noticing that $(1 - a)/(1 - 1/a) = -a$ it follows that

$$\begin{aligned}\{a, -a\} &= \{a, 1 - a\}\{a, 1 - 1/a\}^{-1} \\ &= \{a, 1 - 1/a\}^{-1} \\ &= \{1/a, 1 - 1/a\} \\ &= 1\end{aligned}$$

2. This follows immedeately from the previous item

$$\begin{aligned}\{a, b\}\{b, a\} &= \{-b, b\}\{a, b\}\{b, a\}\{-a, a\} \\ &= \{-ab, b\}\{-ab, a\} \\ &= \{-ab, ab\} \\ &= 1\end{aligned}$$

□

From the definition we have $K_1(R) = R^\times$ and we define $K_0(R) := \mathbb{Z}$. We also have a product

$$K_n(R) \times K_m(R) \to K_{n+m}(R)$$

where $\{a_1, \ldots, a_n\} \times \{a_{n+1}, \ldots, a_{n+m}\} \mapsto \{a_1, \ldots, a_{n+m}\}$.

It is possible to define a norm on the Milnor-K-groups:

Proposition 4.1.2. *For each finite extension of fields L/E there is a group homomorphism*

$$N_{L/E} : K_m(L) \to K_m(E),$$

satisfying

1. *When $m = 1$ this maps coincides with the usual norm.*

2. *For the tower* $L/E_1/E_2$ *of finite extensions we have* $N_{L/E_2} = N_{E_1/E_2} \circ N_{L/E_1}$.

3. *The composition* $K_m(E) \to K_m(L) \xrightarrow{N_{L/E}} K_m(E)$ *coincides with multiplication by* $[L/E]$.

4. *If* $\{a_1, \dots, a_m\} \in K_m(L)$ *with* $a_1, \dots, a_i \in L^\times$ *and* $a_{i+1}, \dots, a_m \in E^\times$, *then*

$$N_{L/E}(\{a_1, \dots, a_m\}) = N_{L/E}(\{a_1, \dots, a_i\})\{a_{i+1}, \dots, a_m\} \in K_m(E),$$

the right hand side is the product of a norm in $K_i(L)$ *and a symbol in* $K_{m-i}(E)$.

Proof. [6] IV and [8] 7.3. □

Note in particular that if $a_1 \in L^\times$ and $a_2, \dots, a_m \in E^\times$, (1) and (4) imply

$$N_{L/E}(\{a_1, \dots, a_m\}) = \{N_{L/E}(a_1), a_2, \dots, a_m\}.$$

In the case where L is a discrete valuation field there exist a boundary map between the Milnor K-group of L and its residue field k_L given in the following

Proposition 4.1.3. *Suppose* L *is a discrete valuation field with residue field* k_L. *There is a unique group homomorphism*

$$\partial : K_m(L) \to K_{m-1}(k_L)$$

which satisfies

$$\partial\{u_1, \dots, u_{m-1}, \pi_L\} = \{\overline{u}_1, \dots, \overline{u}_{m-1}\}$$

for all uniformizers $\pi_L \in L^*$ and all $u_i \in \mathcal{O}_L^*$.

Proof. cf. [19] §2 Lemma 2.1. □

Remark 4.1.2. Note that in the case $m = 1$ the boundary map $\partial : L^* = K_1(L) \to K_0(k_L) = \mathbb{Z}$ is clearly the valuation map $v : L^* \to \mathbb{Z}$.

Remark 4.1.3. In the case $\mathcal{L} = L\{\{T\}\}$, the composition of boundary maps

$$K_2(L\{\{T\}\}) \xrightarrow{\partial} K_1(k_L((T))) \xrightarrow{\partial} K_0(k_L),$$

sends the element $\{T, u\}$, $u \in L^*$, to $v_L(u)$. Indeed, let π_L be any uniformizer for L, then since T is a unit for the valuation field $L\{\{T\}\}$ we have $\partial(\{T, \pi_L\}) = T$. By remark 4.1.2 we have that $\partial(T) = v_{k_L((T))}(T) = 1$. Thus $\partial(\partial(\{T, \pi_L\})) = 1 = v_L(\pi_L)$. Since this is true for all uniformizers π_L of L and $\partial \circ \partial$ is a homomorphisms then the claim follows.

Definition 4.1.2. *We endow $K_d(\mathcal{L})$ with the finest topology λ_d for which the map*

$$(\mathcal{L}^*)^{\otimes d} \to K_d(\mathcal{L}) : \quad (a_1, \dots, a_d) \mapsto \{a_1, \dots, a_d\}$$

is sequentially continuous in each component with respect to the product topology on $\mathcal{L}^*$ and for which subtraction in $K_d(\mathcal{L})$ is sequentially continuous. Define

$$K_d^{top}(\mathcal{L}) = K_d(\mathcal{L})/\Lambda_d(\mathcal{L}),$$

with the quotient topology where $\Lambda_d(\mathcal{L})$ denotes the intersection of all neighborhoods of 0 with respect to λ_d (and so is a subgroup).

In [7] Chapter 6 Theorem 3, Fesenko proved that

$$\Lambda_d(\mathcal{L}) = \cap_{l \geq 1} l K_d(\mathcal{L}). \tag{4.1}$$

Proposition 4.1.4. *Let $\mathcal{M}/\mathcal{L}$ be a finite extention, then norm $N_{\mathcal{M}/\mathcal{L}} : K_d(\mathcal{M}) \to K_d(\mathcal{L})$ induces a norm*

$$N_{\mathcal{M}/\mathcal{L}} : K_d^{top}(\mathcal{M}) \to K_d^{top}(\mathcal{L}).$$

For this norm we have $N_{\mathcal{M}/\mathcal{L}}$(open subgroup) is open in $K_d^{top}(\mathcal{L})$. In particular, $N_{\mathcal{M}/\mathcal{L}}(K_d^{top}(\mathcal{M}))$ is open in $K_d^{top}(\mathcal{L})$.

Proof. Section 4.8, claims (1) and (2) of page 15 of [5]. □

4.1.2 The reciprocity map

Theorem 4.1.1 (A. Parshin, K. Kato)**.** *Let $\mathcal{L}$ be a d-dimensional local field. Then there exist a reciprocity map*

$$\Upsilon_{\mathcal{L}} : K_d(\mathcal{L}) \to \mathrm{Gal}(\mathcal{L}^{\mathrm{ab}}/\mathcal{L}),$$

satisfying the properties

1. *If $\mathcal{M}/\mathcal{L}$ is a finite extension of d-dimensional local fields then the following diagrams commute:*

$$\begin{array}{ccc} K_d(\mathcal{M}) & \xrightarrow{\Upsilon_{\mathcal{M}}} & \mathrm{Gal}(\mathcal{M}^{\mathrm{ab}}/\mathcal{M}) \\ {\scriptstyle N_{\mathcal{M}/\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle restriction} \\ K_d(\mathcal{L}) & \xrightarrow[\Upsilon_{\mathcal{L}}]{} & \mathrm{Gal}(\mathcal{L}^{\mathrm{ab}}/\mathcal{L}) \end{array} \qquad \begin{array}{ccc} K_d(\mathcal{M}) & \xrightarrow{\Upsilon_{\mathcal{M}}} & \mathrm{Gal}(\mathcal{M}^{\mathrm{ab}}/\mathcal{M}) \\ \uparrow & & \uparrow{\scriptstyle transfer} \\ K_d(\mathcal{L}) & \xrightarrow[\Upsilon_{\mathcal{L}}]{} & \mathrm{Gal}(\mathcal{L}^{\mathrm{ab}}/\mathcal{L}) \end{array}$$

If moreover $\mathcal{M}/\mathcal{L}$ is abelian, then $\Upsilon_{\mathcal{L}}$ induces an isomorphism

$$K_d(\mathcal{L})/N_{\mathcal{M}/\mathcal{L}}(K_d(\mathcal{M})) \xrightarrow{\Upsilon_{\mathcal{L}}} \mathrm{Gal}(\mathcal{M}/\mathcal{L})$$

2. *The map is compatible with the residue field $\bar{\mathcal{L}}$ of $\mathcal{L}$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} K_d(\mathcal{L}) & \xrightarrow{\Upsilon_{\mathcal{L}}} & \mathrm{Gal}(\mathcal{L}^{\mathrm{ab}}/\mathcal{L}) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \sigma\to\bar{\sigma}} \\ K_{d-1}(\bar{\mathcal{L}}) & \xrightarrow[\Upsilon_{\bar{\mathcal{L}}}]{} & \mathrm{Gal}(\bar{\mathcal{L}}^{\mathrm{ab}}/\bar{\mathcal{L}}) \end{array}$$

Here ∂ is the boundary map defined in Proposition 4.1.3.

3. *The reciprocity map $\Upsilon_{\mathcal{L}}$ is sequentially continuous if we endow $K_d(\mathcal{L})$ with the topology λ_d from definition 4.1.2.*

Proof. The first two assertions can be found in [11] § 1 Theorem 2. For the third one, let $\mathcal{M}$ be a finite abelian extension of $\mathcal{L}$, thus $\mathrm{Gal}(\mathcal{L}^{ab}/\mathcal{M})$ is an open neighborhood of $G_{\mathcal{L}}^{ab}$. Let x_n be a convergent sequence to the zero element of $K_d(\mathcal{L})$. Since $N_{\mathcal{M}/\mathcal{L}}(K_d^{top}(\mathcal{M}))$ is a open subgroup of $K_d^{top}(\mathcal{L})$ by Proposition 4.1.4, then

$$\overline{x_n} \in N_{\mathcal{M}/\mathcal{L}}(K_d^{top}(\mathcal{M})) \quad (n >> 0),$$

where $\overline{x_n}$ is the image of x_n in $K_d^{top}(\mathcal{L})$. Thus, there exist $y_n \in K_d(\mathcal{M})$ and $\beta_n \in \Lambda_m(\mathcal{L})$ such that

$$x_n = \beta_n N_{\mathcal{M}/\mathcal{L}}(y_n) \quad (n >> 0).$$

From equation (4.1) we have that $\beta_n \in \cap_{l\geq 1} lK_d(\mathcal{L})$ which implies that $\Upsilon_{\mathcal{L}}(\beta_n)$ is the identity element in $G_{\mathcal{L}}^{ab}$ (because $G_{\mathcal{L}}^{ab}$ is a profinite group). Therefore

$$\Upsilon_{\mathcal{L}}(x_n) = \Upsilon_{\mathcal{L}}(N_{\mathcal{M}/\mathcal{L}}(y_n)) \quad (n >> 0),$$

but the element on the right hand side of this equality is the identity on $\mathrm{Gal}(\mathcal{L}^{ab}/\mathcal{M})$ by the second item of this Theorem. It follows that $\Upsilon_{\mathcal{L}}(x_n)$ converges to the identity elemenet of $G_{\mathcal{L}}^{ab}$. □

4.2 The pairing $(,)_{\mathcal{L},n}$

Let $\mathcal{L}$ be a d-dimensional local field containing K and the group κ_n. We have the pairing

$$(,)_{\mathcal{L},n} : K_d(\mathcal{L}) \times F(\mu_{\mathcal{L}}) \to \kappa_n$$

defined by $(\{a_1, \dots, a_d\}, x)_{\mathcal{L},n} = \Upsilon_{\mathcal{L}}(\{a_1, \dots, a_d\})(Z) -_F Z$, where $f^{(n)}(Z) = x$ and $-_F$ is the subtraction in the formal group F.

Proposition 4.2.1. *The pairing just defined satisfies the following:*

1. $(,)_{\mathcal{L},n}$ *is bilinear and C-linear on the right.*

2. *The kernel on the right is* $f^{(n)}(F(\mu_{\mathcal{L}}))$.

3. $(a,x)_{\mathcal{L},n} = 0$ *if and only if* $a \in N_{\mathcal{L}(z)/\mathcal{L}}(K_d(\mathcal{L}(z)))$, *where* $f^{(n)}(z) = x$.

4. *If* $\mathcal{M}/\mathcal{L}$ *is finite,* $x \in F(\mu_{\mathcal{L}})$ *and* $b \in K_d(\mathcal{M})$. *Then*

$$(b,x)_{\mathcal{M},n} = (N_{\mathcal{M}/\mathcal{L}}(b), x)_{\mathcal{L},n}.$$

5. *Let* $L \supset \kappa_m$, $m \geq n$. *Then*

$$(a,x)_{\mathcal{L},n} = f^{(m-n)}((a,x)_{\mathcal{L},m}) = (a, f^{(m-n)}(x))_{\mathcal{L},m}$$

6. *For a given* $x \in K_d(\mathcal{L})$, *the map*

$$K_d(\mathcal{L}) \to \kappa_n : a \mapsto (a,x)_{\mathcal{L},n}$$

   is sequentially continuous.

7. *Let* M *be a finite extension of* L, $a \in K_d(\mathcal{L})$ *and* $y \in F(\mu_{\mathcal{M}})$. *Then*

$$(a,y)_{\mathcal{M},n} = (a, N^F_{\mathcal{M}/\mathcal{L}}(y)\ )_{\mathcal{M},n},$$

   where $N^F_{\mathcal{M}/\mathcal{L}}(y) = \oplus_\sigma y^\sigma$, *where* σ *ranges over all embeddings of* M *in* $\overline{K}$ *over* L. *Notice that such an embedding extends uniquely to an embedding from* $\mathcal{M} = M\{\{T\}\}$ *over* $\mathcal{L} = L\{\{T\}\}$.

8. Let $t : F \to \tilde{F}$ be a isomorphism. Then

$$(a, t(x))^{\tilde{F}}_{\mathcal{L},n} = t((a, x)^{F}_{\mathcal{L},n})$$

for all $a \in K_d(\mathcal{L})$, $x \in F(\mu_{\mathcal{L}})$.

Proof. The first 5 properties and the last one follow from the definition of the pairing and Theorem 4.1.1.

The property 6 follows from the fact that the reciprocity map $\Upsilon_{\mathcal{L}} : K_m(\mathcal{L}) \to \mathrm{Gal}(\mathcal{L}^{\mathrm{ab}}/\mathcal{L})$ is sequentially continuous, here $K_d(\mathcal{L})$ is endowed with the finest topology for which the map

$$\phi : \mathcal{L}^{*\oplus d} \to K_d(\mathcal{L})$$

is sequentially continuous in each component with respect to the product topology on $\mathcal{L}^*$ and for which subtraction in $K_d(\mathcal{L})$ is sequentially continuous. Then for z such that $f^{(n)}(z) = x$ consider the extension $\mathcal{L}(z)/\mathcal{L}$. The group $\mathrm{Gal}(\mathcal{L}^{ab}/\mathcal{L}(z))$ is a neighborhood of $G^{ab}_{\mathcal{L}}$, so for any sequence $\{a_m\}$ coverging to zero in $K_d(\mathcal{L})$ we can take m large enough such that $\Upsilon_{\mathcal{L}}(a_m) \in \mathrm{Gal}(\mathcal{L}^{ab}/\mathcal{L}(z))$, that is $\Upsilon_{\mathcal{L}}(a_m)(z) = z$, so $(a_m, x)_{\mathcal{L},n} = 0$ for large enough m.

Finally, let us prove property 7. Let $f^{(n)}(z) = y$ an take a finite Galois extension $\mathcal{N} \supset \mathcal{M}(z)$ over $\mathcal{L}$. Let $G = G(\mathcal{N}/\mathcal{L})$ and $H = G(\mathcal{N}/\mathcal{M})$, $w = [G : H]$, and $V : G/G' \to H/H'$ the transfer homomorphism. Let $g = \Upsilon_{\mathcal{L}}(a)$, then by Theorem 4.1.1 we have $V(\Upsilon_{\mathcal{L}}(a)) = \Upsilon_{\mathcal{M}}(a)$. The explicit computation of V at $g \in G$ proceeds as follows (cf. [23] § 3.5). Let $\{c_i\}$ be a set of representatives of for the right cosets of H in G, i.e., $G = \sqcup\, Hc_i$. Then for each c_i, $i = 1, \dots w$ there exist a c_j such that $c_i g c_j^{-1} = h_i \in H$ and no two c_j's are equal; this is because $c_i g$ belongs to one and only one of the right cosets Hc_j. Then $V(g) = \prod_{i=1}^{w} h_i$.

Also, notice that since $gc_j^{-1} = c_i^{-1}h_i$ then

$$h_i(z) \ominus z = c_i^{-1}(h_i(z) \ominus z) = g(c_j^{-1}(z)) \ominus c_i^{-1}(z).$$

So we have

$$\begin{aligned}
(a, y)_{\mathcal{M},n} &= \Upsilon_{\mathcal{M}}(a)(z) \ominus z \\
&= V(g)(z) \ominus z \\
&= (\prod_{i=1}^{w} h_i)z \ominus z \\
&= \oplus_{i=1}^{w}(h_i(z) \ominus z) \\
&= \oplus_{i=1}^{w}(g(c_j^{-1}(z)) \ominus c_i^{-1}(z)) \\
&= g(\oplus_{i=1}^{w} c_j^{-1}(z)) \ominus (\oplus_{i=1}^{w} c_j^{-1}(z)) \\
&= (a, N^F_{\mathcal{M}/\mathcal{L}}(y))_{\mathcal{L},n}
\end{aligned}$$

the last equality being true since $g = \Upsilon_{\mathcal{L}}(a)$ and $f^{(n)}(\oplus_{i=1}^{w} c_j^{-1}(z)) = \oplus_{i=1}^{w} c_j^{-1}(y) = N^F_{\mathcal{M}/\mathcal{L}}(y)$.

□

Let α denote the ramification index of S over $\mathbb{Q}_p$. We say that a pair (n, t) is admissible if there exist an integer k such that $t - 1 - n \geq \alpha k \geq n$. For example, the pair $(n, 2n + \alpha + 1)$ is admissible with $k = [(n + \alpha/\alpha)]$. In the special case where $\alpha = 1$, then $t - 1 - n \geq \alpha k \geq n$ becomes $t - 1 - n \geq k \geq n$. So in this case, the pair $(n, 2n + 1)$ is admissible with $k = n$.

The following proposition is vital for the main results in this paper. The idea of the proof was inspired by the proof of Proposition 4.1 in [14].

Proposition 4.2.2. *Let $L \supset K_t = K(\kappa_t)$ with (n, t) an admissible pair and let $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. For a given $\alpha \in K_d(\mathcal{L})$, the map*

$$F(\mu_{\mathcal{L},1}) \to \kappa_n : x \mapsto (\alpha, x)_{\mathcal{L},n},$$

is sequentially continuous, i.e., if $x_k \to x$ then $(\alpha, x_k)_{\mathcal{L},n} \to (\alpha, x)_{\mathcal{L},n}$. Here $F(\mu_{\mathcal{L},1})$ is the set

$$\left\{ x \in \mathcal{L} :\ v_{\mathcal{L}}(x) \geq \left[\frac{v_L(p)}{p-1} \right] + 1 \right\}.$$

considered with the operation induced by the formal group F.

Remark: We will make the following two assumptions. First notice that it is enough to prove the result for $a \in \mathcal{L}^*$ such that $v_{\mathcal{L}}(a) = 1$, because then it will be true for π_L and $\pi_L u$ with $v_{\mathcal{L}}(u) = 0$. Here π_L a uniformizer for L.

Let r be a t-normalized series, cf. §4.3. Notice also that we may assume that $r(X) = X$ is a t-normalized, otherwise we go to the isomorphic group law

$r(F(r^{-1}(X), r^{-1}(Y)))$.

Proof. We will drop the subscript $\mathcal{L}$ from the pairing notation and will make the two assumptions in the remark above, i.e., let $u \in \mathcal{L}^*$ with $v_{\mathcal{L}}(u) = 1$, $b_2 \ldots, b_d \in \mathcal{L}^*$ and $x \in F(\mu_{\mathcal{L},1})$. Then

$$(\{u, b_2 \ldots, b_d\}, x)_n = \left(\left\{ \frac{u}{u \oplus f^{(\alpha k+1)}(x)}, b_2 \ldots, b_d \right\},\ x \right)_n \oplus \left(\left\{ u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d \right\}, x \right)_n . \tag{4.2}$$

We will show that the first term on the right hand side is zero and that the second side goes to zero when we take a sequence $\{x_k\}$ converging to zero. This would complete the proof.

Let us start with the second term. Let $m = n + \alpha k + 1$. By (5) in the Proposition 4.2.1 and the fact that $r(X) = X$, i.e., $(\{a, b_2 \ldots, b_d\}, a) = 0\ \forall a \in F(\mu_{\mathcal{L}})$, it follows that

$$\left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\}, x\right)_n =$$
$$= \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\}, f^{(m-n)}(x)\right)_m$$
$$= \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\},\ u \oplus f^{(\alpha k+1)}(x) \ominus u\right)_m$$
$$= \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\}, u \oplus f^{(\alpha k+1)}(x)\right)_m \oplus \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\},\ \ominus u\right)_m$$
$$= \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\},\ \ominus u\right)_m$$
$$= \left(\left\{u^{-1}, b_2 \ldots, b_d\right\},\ \ominus u\right)_m \oplus \left(\left\{u \oplus f^{(\alpha k+1)}(x), b_2 \ldots, b_d\right\},\ \ominus u\right)_m$$
$$= \left(\left\{\frac{u \oplus f^{(\alpha k+1)}(x)}{u}, b_2 \ldots, b_d\right\},\ \ominus u\right)_m$$

Since $F(X,Y) \equiv X+Y \pmod{XY}$, then $u \oplus f^{(\alpha k+1)}(x) \equiv u + f^{(\alpha k+1)}(x) \pmod{u\, f^{(\alpha k+1)}(x)}$ and so

$$\frac{u \oplus f^{(\alpha k+1)}(x)}{u} \equiv 1 + \frac{f^{(\alpha k+1)}(x)}{u} \pmod{f^{(\alpha k+1)}(x)} \tag{4.3}$$

But $v_{\mathcal{L}}(u) = 1$, so this element is a principal unit and if we take a sequence $\{x_i\}_{i\geq 1}$ converging to zero in the Parshin topology then, as $f : \mu_{\mathcal{L},1} \to \mu_{\mathcal{L},1}$ is sequentially continuous in the Parshin topology by Lemma 3.3.2, this element goes to 1 and hence

$$\left\{\frac{u \oplus f^{(\alpha k+1)}(x_i)}{u}, b_2 \ldots, b_d\right\} \to \{1, b_2 \ldots, b_d\}$$

in the topology of $K_2(\mathcal{L})$. Notice that $\{1, b_2 \ldots, b_d\} = \{1-b_2, b_2 \ldots, b_d\}\{1, b_2 \ldots, b_d\} = \{1-b_2, b_2 \ldots, b_d\} = \mathbf{1}$, where $\mathbf{1}$ is the unit element in $K_2(\mathcal{L})$. Then by (6) in the

Proposition 4.2.1

$$\left(\left\{ \frac{u \oplus f^{(\alpha k+1)}(x_i)}{u}, b_2 \ldots, b_d \right\},\ \ominus u \right)_m \xrightarrow{i\to\infty} (\mathbf{1}, \ominus u)_m = 0$$

Now we will show that first term on the right hand side of equation (4.2) is zero by showing that $(u \oplus f^{(\alpha k+1)}(x))/u$ is a p^kth power in $\mathcal{L}^*$, which is enough since π^n divides p^k, because $n \leq \alpha k$. Indeed, since $x \in F(\mu_{\mathcal{L},1})$ then by Proposition 3.3.1

$$f^{(\alpha k+1)}(x) = l_F^{-1} \circ l_F(f^{(\alpha k+1)}(x)) = l_F^{-1}(\pi^{\alpha k+1} l_F(x)) = \pi^{\alpha k+1} w \tag{4.4}$$

for some $w \in \mu_{\mathcal{L},1}$. Then equation (4.3) implies

$$\frac{u \oplus f^{(\alpha k+1)}(x)}{u} = 1 + p^k w_2$$

for $w_2 \in \mu_{\mathcal{L},1}$, since $\pi^{\alpha k} = \epsilon p^k$ for some unit ϵ. Then

$$\log\left(\frac{u \oplus f^{(\alpha k+1)}(x)}{u} \right) = \log(1 + p^k w_2) = p^k w_3$$

where $w_3 \in \mu_{\mathcal{L},1}$, so there exist a $w_4 \in \mu_{\mathcal{L},1}$ such that $\log(1 + w_4) = w_3$, thus

$$\frac{u \oplus f^{(\alpha k+1)}(x)}{u} = (1 + w_4)^{p^k}$$

and so

$$\left(\left\{ \frac{u}{u \oplus f^{(\alpha k+1)}(x)}, b_2 \ldots, b_d \right\},\ x \right)_n = 0.$$

This proves the theorem. □

4.3 Norm Series

A power series $r \in \mathcal{O}_K[[X]]$, $r(0) = 0$, $c(r) \in \mathcal{O}_K^*$ is called $n-$*normalized* if for every local field $L \supset \kappa_n$, $\mathcal{L} = L\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$, and all $1 \le i \le d$

$$(\{a_1, \dots, a_{i-1}, r(x), a_{i+1}, \dots, a_d\}, x)_{\mathcal{L},n} = 0,$$

for all $x \in F(\mu_{\mathcal{L}})$ and all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d \in \mathcal{L}^*$.

The following proposition will provide a way of constructing norm series.

Proposition 4.3.1. *Let $g \in \mathcal{O}_K[[X]]$, $g(0) = 0$ and $c(g) \in \mathcal{O}_K^*$. The series $s = \prod_{v \in \kappa_n} g(F(X, v))$ belongs to $\mathcal{O}_K[[X]]$ and has the form $r_g(f^{(n)})$, where $r_g \in \mathcal{O}_K[[X]]$. Then, the series r_g is $n-$normalized and*

$$r_g'(0) = \frac{\prod_{v \neq 0 \in \kappa_n} g(v)}{\pi^n} g'(0).$$

Proof. The proof follows closely [14] Proposition 3.1.

First, the coefficients of s are in $\mathcal{O}_K$ because $s^\sigma = s$ for every $\sigma \in G_K = Gal(\overline{K}/K)$. Now, applying Lemma 3.1.1 to s and $f^{(n)}$, we get $s = Ps_1$ and $f^{(n)} = Qf_1$, where P and Q is a monic polynomials and $s_1, f_1 \in \mathcal{O}_K[[X]]^*$. Since $s(F(X, v)) = s(X)$ for all $v \in \kappa_n$, then $P(v) = 0$ for all $v \in \kappa_n$ and so $Q = \prod_{v \in \kappa_n}(X - v)$ divides P. This implies that s is divisible by $f^{(n)}$, i.e., $s = f^{(n)}(a_0 + a_1 X + \cdots)$.

In particular,

$$s - f^{(n)}.a_0 = f^{(n)}.(\ a_1 X + \cdots).$$

But from $s(F(X,v)) = s(X)$ we see that $a_1X + \cdots$ must satisfy the same property and so $a_1v + \cdots = 0$, for all $v \in \kappa_n$. Therefore this series is also divisible by $f^{(n)}$ and repeating the process we get $s = r_g(f^{(n)})$. Let us compute now $c(r_g)$. Taking the logarithmic derivative on s and then multiplying by X we get

$$\frac{s'(X)}{s(X)}X = \sum_{v\in\kappa_n} \frac{g'(F(X,v))F_X(X,v)X}{g(F(X,v))},$$

which implies

$$\frac{s'(0)}{\prod_{0\neq v\in\kappa_n} g(v)} = g'(0),$$

From $s' = r_g'(f^{(n)})f^{(n)'}$ we obtain

$$r_g'(0) = \frac{s'(0)}{f^{(n)'}(0)} = \frac{c(g)\prod_{0\neq v\in\kappa_n} g(v)}{\pi^n}.$$

Each $g(v)$ is associated to v, $0 \neq v \in \kappa_n$, then $\prod_{0\neq v\in\kappa_n} g(v)$ is associated to $\prod_{0\neq v\in\kappa_n} v$, but the latter is associated to π^n from the equation $f = Pf_1$. Then $c(r_g) \in \mathcal{O}_K^*$. Finally, we will show that $(\{a_1, \ldots, a_{i-1}, r_g(x), a_{i+1}, \ldots, a_d\}, x) = 0$. Let L be a local field containing κ_n, $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, $x \in F(\mu_{\mathcal{L}})$ and z such that $f^{(n)}(z) = x$. Then

$$r_g(x) = \prod_{v\in\kappa_n} g(z +_F v) = \prod_i N_{\mathcal{L}(z)/\mathcal{L}}(g(z_i)),$$

where the z_i are pairwise non-conjugate over $\mathcal{L}$ distinct roots of $f^{(n)}(X) = x$, so

$$\begin{aligned}\{a_1, \ldots, a_{i-1}, r_g(x), a_{i+1}, \ldots, a_d\} &= \{a_1, \ldots, a_{i-1}, N_{\mathcal{L}(z)/\mathcal{L}}(\prod_i g(z_i)), a_{i+1}, \ldots, a_d\} \\ &= N_{\mathcal{L}(z)/\mathcal{L}}(\{a_1, \ldots, a_{i-1}, \prod_i g(z_i), a_{i+1}, \ldots, a_d\}),\end{aligned}$$

The last equality follows from Proposition 4.1.2 (1) and (4). The result now follows from Proposition 4.2.1.

□

Chapter 5

The maps ψ and ρ

5.1 The generalized trace

Let E be a complete discrete valuation field. We define a map

$$c_{E\{\{T\}\}/E} : E\{\{T\}\} \to E,$$

by $c_{E\{\{T\}\}/E}(\sum_{i\in\mathbb{Z}} a_i T^i) = a_0$. Let $\mathcal{E} = E\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$, we can define $c_{\mathcal{E}/E}$ by the composition

$$c_{E\{\{T_1\}\}/E} \circ \cdots \circ c_{\mathcal{E}/E\{\{T_1\}\}\ldots\{\{T_{d-2}\}}.$$

Lemma 5.1.1. *This map satisfies the following properties*

1. $c_{\mathcal{E}/E}$ *is* E*-linear.*

2. $c_{\mathcal{E}/E}(a) = a$*, for all* $a \in E$*.*

3. $c_{\mathcal{E}/E}$ *is continuous with respect to the the Parshin topology on* $\mathcal{E}$ *and the discrete valuation topology on* E*.*

Proof. cf. [28] Lemma 2.1. □

Suppose L/S is a finite extension of local fields. Let π be a uniformizer for $C = \mathcal{O}_S$ and let $\mathcal{L} = L\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$. The generalized trace $\mathbb{T}_{\mathcal{L}/S}$ is defined by the composition $\mathrm{Tr}_{L/S} \circ c_{\mathcal{L}/L}$ and this gives us the pairing

$$\langle , \rangle : \mathcal{L} \times \mathcal{L} \to C, \tag{5.1}$$

defined by $\langle x, y\rangle = \mathbb{T}_{\mathcal{L}/S}(xy)$. Let $\mathrm{Hom}^c_C(\mathcal{L}, S)$ and $\mathrm{Hom}^{seq}_C(\mathcal{L}, S)$ be the group of continuous and sequentially continuous, respectively, C-homomorphisms with respect to the Parshin topology on $\mathcal{L}$.

Proposition 5.1.1. *We have an isomorphism of C-modules*

$$\mathcal{L} \xrightarrow{\sim} \mathrm{Hom}^{seq}_C(\mathcal{L}, S) : \ \alpha \mapsto (x \mapsto \mathbb{T}_{\mathcal{L}/S}(\alpha x)).$$

In particular, $\mathrm{Hom}^{seq}_C(\mathcal{L}, S) = \mathrm{Hom}^c_C(\mathcal{L}, S)$ *since the generalized trace is continuous.*

Proof. Here we prove the case where the dimension d is equal to 2, i.e., $\mathcal{L} = L\{\{T\}\}$. The general case is proved in the remark 5.1.1 after this proposition.

Let $\phi : \mathcal{L} \to S$ be a sequentially continuous C-linear map and define, for each $i \in \mathbb{Z}$, the map $\phi_i(x) = \phi(xT^i)$ for all $x \in L$. Then clearly $\phi_i \in \mathrm{Hom}_C(L, S)$ and this corresponds to the case $d = 1$ for which we know that it exists an $a_{-i} \in L$ such that $\phi(xT^i) = \mathrm{Tr}_{L/S}(a_{-i}x)$ for all $x \in L$. Let $\alpha = \sum a_i T^i$, we must show that

I. $\min\{v_L(a_i)\} > -\infty$.

II. $v_L(a_{-i}) \to \infty$ as $i \to \infty$ (i.e., conditions (I) and (II) imply that $\alpha \in \mathcal{L}$).

III. $\phi(x) = \mathbb{T}_{\mathcal{L}/S}(\alpha x)$, $\forall x \in \mathcal{L}$.

For any $x = \sum x_i T^i \in \mathcal{L}$ we have, by the sequential continuity of ϕ that

$$\phi(x) = \sum_{i\in\mathbb{Z}} \phi(x_i T^i) = \sum_{i\in\mathbb{Z}} \mathrm{Tr}_{L/S}(a_{-i}x_i). \tag{5.2}$$

Suppose (I) was not true, then there exist a subsequence $\{a_{n_k}\}$ such that $v_L(a_{n_k}) \to -\infty$ as $n_k \to \infty$ or as $n_k \to -\infty$. In the first case we take an $x = \sum x_i T^i \in \mathcal{L}$ such that x_i is equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i}x_i$=1 if $i = -n_k$ and 0 if $i \neq -n_k$. Then the sum on the right of (5.2) would not converge. In the second case we take x_i to be equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i}x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and again the sum on the right would not converge.

Suppose (II) was not true. Then $v_E(a_{n_k}) < M$ for some positive integer M and a of negative integers $n_k \to -\infty$. Then take $x = \sum x_i T^i \in \mathcal{L}$ such that x_i is equal to $1/a_{n_k}$ for $i = -n_k$ and 0 for $i \neq -n_k$. So $a_{-i}x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and the sum on the right of (5.2) would not converge.

Finally, (III) follows by noticing that by (I) and (II) the sum $\sum_{i\in\mathbb{Z}} a_{-i}x_i$ converges and

$$\sum_{i\in\mathbb{Z}} \mathbb{T}_{E/S}(a_{-i}x_i) = \mathbb{T}_{E/S}(\sum_{i\in\mathbb{Z}} a_{-i}x_i) = \mathbb{T}_{\mathcal{L}/S}(x\alpha),$$

since $\mathbb{T}_{E/S} \circ c_{\mathcal{L}/E} = \mathbb{T}_{\mathcal{L}/S}$. $\square$

Remark 5.1.1. We include the proof of the general case of the proposition above.

Proof. The proof is done by induction in d. If $d = 1$ the result is known. Suppose the result is true for $d \geq 1$ and let $\mathcal{L} = E\{\{T_d\}\}$ where $E = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$.

Let $\phi : \mathcal{L} \to S$ be a sequentially continuous C-linear map and define, for each $i \in \mathbb{Z}$, the sequentially continuous map $\phi_i(x) = \phi(xT_d^i)$ for all $x \in E$. Then clearly $\phi_i \in \mathrm{Hom}_C(E, S)$ and by the induction hypothesis we know that there exists an $a_{-i} \in E$ such that $\phi(xT_d^i) = \mathbb{T}_{E/S}(a_{-i}x)$ for all $x \in E$. Let $\alpha = \sum a_i T_d^i$, we must show that

I. $\min\{v_E(a_i)\} > -\infty$.

II. $v_E(a_{-i}) \to \infty$ as $i \to \infty$ (i.e., conditions (I) and (II) imply that $\alpha \in \mathcal{L}$).

III. $\phi(x) = \mathbb{T}_{\mathcal{L}/S}(\alpha x)$, $\forall x \in \mathcal{L}$.

For any $x = \sum x_i T^i \in \mathcal{L}$ we have, by the sequential continuity of ϕ that

$$\phi(x) = \sum_{i \in \mathbb{Z}} \phi(x_i T_d^i) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E/S}(a_{-i}x_i). \tag{5.3}$$

Suppose (I) was not true, then there exist a subsequence $\{a_{n_k}\}$ such that $v_E(a_{n_k}) \to -\infty$ as $n_k \to \infty$ or as $n_k \to -\infty$. In the first case we take an $x = \sum x_i T_d^i \in \mathcal{L}$ such that x_i is equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i}x_i$=1 if $i = -n_k$ and 0 if $i \neq -n_k$. Then the sum on the right of (5.3) would not converge. In the second case we take x_i to be equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i}x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and again the sum on the right would not converge.

Suppose (II) was not true. Then $v_L(a_{n_k}) < M$ for some positive integer M and a of negative integers $n_k \to -\infty$. Then take $x = \sum x_i T_d^i \in \mathcal{L}$ such that x_i is equal to $1/a_{n_k}$ for $i = -n_k$ and 0 for $i \neq -n_k$. So $a_{-i}x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and the sum on the right of (5.3) would not converge.

Finally, (III) follows by noticing that by (I) and (II) the sum $\sum_{i\in\mathbb{Z}} a_{-i}x_i$ converges and

$$\sum_{i\in\mathbb{Z}} \mathrm{Tr}_{L/S}(a_{-i}x_i) = \mathrm{Tr}_{L/S}(\sum_{i\in\mathbb{Z}} a_{-i}x_i) = \mathbb{T}_{\mathcal{L}/S}(x\alpha).$$

$\square$

Let $\mathcal{L} = L\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$ where L is a local field. Let v_L and $v_{\mathcal{L}}$ denote the valuations for L and $\mathcal{L}$ respectively. Consider

$$\mu_{\mathcal{L},1} := \left\{ x \in \mathcal{L} : \; v_{\mathcal{L}}(x) \geq \left[\frac{v_L(p)}{p-1}\right] + 1 \right\},$$

and

$$\mu_{L,1} := \mu_{\mathcal{L},1} \cap L = \left\{ x \in L : \; v_L(x) \geq \left[\frac{v_L(p)}{p-1}\right] + 1 \right\},$$

both with the additive structure, and denote by $R_{\mathcal{L},1}$ the dual of $\mu_{\mathcal{L},1}$ with respect to the pairing (5.1), i.e.,

$$R_{\mathcal{L},1} := \left\{ x \in \mathcal{L} : \ \mathbb{T}_{\mathcal{L}/S}(\ x \ \mu_{\mathcal{L},1}) \subset C \ \right\}.$$

Let us show that

$$R_{\mathcal{L},1} = \left\{ x \in \mathcal{L} : \ v_{\mathcal{L}}(x) \geq \ -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1} \right] - 1 \ \right\}, \tag{5.4}$$

where $D(L/S)$ is the different of the extension L/S. Indeed, for $d = 1$ this is proven in [14] §4.1. Suppose it is true for $d \geq 1$, and let $\mathcal{L} = E_d\{\{T_d\}\}$, where $E_d = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. If $x = \sum_{i \in \mathbb{Z}} x_i T_d^i \in R_{\mathcal{L},1}$, then since $\mu_{E_d,1} \subset \mu_{\mathcal{L},1}$ we have that also $\mu_{E_d,1} T_d^{-i} \subset \mu_{\mathcal{L},1}$ and

$$\mathbb{T}_{\mathcal{L}/S}(x \ \mu_{E_d,1} T_d^{-i}) \subset C,$$

which implies $\mathbb{T}_{E_d/S}(x_i \mu_{E_d,1}) \subset C$, since $\mathbb{T}_{\mathcal{L}/S} = \mathbb{T}_{E_d/S} \circ c_{\mathcal{L}/E_d}$. By induction hypothesis we have $v_{E_d}(x_i) \geq \ -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1} \right] - 1$ for all $i \in \mathbb{Z}$, therefore

$$v_{\mathcal{L}}(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1} \right] - 1.$$

Conversely, if $v_{\mathcal{L}}(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1} \right] - 1$, then $v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1} \right] - 1$ for all $i \in \mathbb{Z}$. Then, by the induction hypothesis $\mathbb{T}_{E_d/S}(x_i \mu_{E_d,1}) \subset C$ for all $i \in Z$, and therefore

$$\mathbb{T}_{\mathcal{L}/S}(x \mu_{\mathcal{L},1}) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{\mathcal{L}/S}(x_i T_d^i \ \mu_{\mathcal{L},1}) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E_d/S}(x_i \ \mu_{E_d,1}) \subset C.$$

Thus, identity (5.4) holds.

Let

$$R_{L,1} := R_{\mathcal{L},1} \cap L = \left\{ x \in L : \; v_L(x) \geq \; -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1}\right] - 1 \; \right\}.$$

Lemma 5.1.2. *We have the isomorphism*

$$R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1} \;\stackrel{\sim}{\longrightarrow}\; \mathrm{Hom}_C^{seq}(\mu_{\mathcal{L},1}, C/\pi^n C),$$

defined by

$$\alpha \mapsto (\; x \mapsto \mathbb{T}_{\mathcal{L}/S}(\alpha x) \;).$$

In particular, $\mathrm{Hom}_C^{seq}(\mu_{\mathcal{L},1}, C/\pi^n C) = \mathrm{Hom}_C^{c}(\mu_{\mathcal{L},1}, C/\pi^n C)$.

Proof. The proof is similar to that of Proposition 5.1.1. As in the proof of that proposition we will consider the case $d = 2$, i.e., $\mathcal{L} = L\{\{T\}\}$, and leave the general case as a remark 5.1.2 after the proof of this proposition. Take $\Phi \in \mathrm{Hom}_C^{seq}(\mu_{\mathcal{L},1}, C/\pi^n C)$ and let $\Phi_i(x_i) = \Phi(x_i T^i)$ for all $x_i \in \mu_{L,1}$. Then $\Phi_i \in \mathrm{Hom}_C^{c}(\mu_{L,1}, C/\pi^n C)$ and so there exist $\overline{a}_{-i} \in R_{L,1}/\pi^n R_{L,1}$ such that

$$\Phi_i(x_i) = \mathrm{Tr}_{L/S}(\; \overline{a_{-i}} \; x_i \;).$$

Let $a_{-i} \in R_{L,1}$ be a representative of $\overline{a}_{-i}$. Thus for $x = \sum x_i T^i \in \mu_{\mathcal{L},1}$, the sequential continuity of Φ implies

$$\Phi(x) = \sum_{i\in\mathbb{Z}} \Phi(x_i T^i) = \sum_{i\in\mathbb{Z}} \mathrm{Tr}_{L/S}(a_{-i}x_i) \qquad (\mathrm{mod}\ \pi^n C). \tag{5.5}$$

Let $\alpha = \sum a_i T^i$ and denote by u_i the unit $a_i/\pi_L^{v_L(a_i)}$. We must show that

I. $\min\{v_L(a_i)\} > -\infty$.

II. $v_L(a_{-i}) \geq v_L(\pi^n R_{L,1})$ as $i \to \infty$ (i.e., conditions 1 and 2 imply that $\alpha \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$).

III. $\Phi(x) = \mathbb{T}_{\mathcal{L}/S}(\alpha x)\ (\mathrm{mod}\ \pi^n C)$, $\forall x \in \mu_{\mathcal{L},1}$.

Condition (I) follows immediately since $a_{-i} \in R_{L,1}$, i.e., by equation (5.4)

$$v_L(a_{-i}) \geq -\left(\left[\frac{v_L(p)}{p-1}\right] + 1 + v_L(D(L/S)) \right) \quad \forall i \in \mathbb{Z}.$$

Suppose condition (II) was not true. Instead of passing to a subsequence we may assume for simplicity that $v_L(a_{-i}) < v_L(\pi^n R_{L,1})$ for all $i \geq 0$. Let

$$x = y(\pi_L^{\delta_0} u_0^{-1} + \pi_L^{\delta_1} u_1^{-1} T + \pi_L^{\delta_2} u_2^{-1} T^2 + \cdots),$$

where $u_i = a_i/\pi_L^{v_L(a_i)}$ and

$$\delta_i = v_L(\pi^n R_{L,1}) - v_L(a_{-i}) + [v_L(p)/(p-1)] \quad (\geq [v_L(p)/(p-1)] + 1),$$

for $i \geq 0$ and $y \in \mathcal{O}_L$ is arbitrary. Then $x \in \mu_{\mathcal{L},1}$ and $a_{-i}x_i = \pi_L^w y$ for $i \geq 0$, where $w = v_L(\pi^n R_{L,1}) + [v_L(p)/(p-1)]$. The convergence of the right hand side of (5.5) would imply that

$$\mathrm{Tr}_{L/S}(\pi_L^w y) \in \pi^n C \quad \forall y \in \mathcal{O}_L,$$

Thus $\pi_L^w/\pi^n \in D(L/S)^{-1}$, which implies $w \geq v_L(\pi^n) - v_L(D(L/S))$, that is,

$$v_L(R_{L,1}) \geq -v_L(D(L/S)) - [v_L(p)/(p-1)],$$

which is a contradiction since $v_L(R_{L,1}) = -v_L(D(L/S)) - [v_L(p)/(p-1)] - 1$. Finally, condition (III) immediately follows from equation (5.5). □

Remark 5.1.2. Next we will include the proof in the general case of the proposition above.

Proof. The proof is done by induction in d. If $d = 1$ the result is known. Suppose the result is true for $d \geq 1$ and let $\mathcal{L} = E\{\{T_d\}\}$ where $E = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$.

Take $\Phi \in \mathrm{Hom}_C^{seq}(\mu_{\mathcal{L},1}, C/\pi^n C)$ and let $\Phi_i(x_i) = \Phi(x_i T_d^i)$ for all $x_i \in \mu_{E,1}$. Then $\Phi_i \in \mathrm{Hom}_C^c(\mu_{E,1}, C/\pi^n C)$ and so by the induction hypothesis there exists $\overline{a}_{-i} \in R_{E,1}/\pi^n R_{E,1}$ such that

$$\Phi_i(x_i) = \mathbb{T}_{E/S}(\ \overline{a_{-i}}\ x_i\ ).$$

Let $a_{-i} \in R_{E,1}$ be a representative of $\overline{a}_{-i}$. Thus for $x = \sum x_i T_d^i \in \mu_{\mathcal{L},1}$, the sequential continuity of Φ implies

$$\Phi(x) = \sum_{i \in \mathbb{Z}} \Phi(x_i T_d^i) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E/S}(a_{-i} x_i) \pmod{\pi^n C}. \tag{5.6}$$

Let $\alpha = \sum a_i T_d^i$ and denote by u_i the unit $a_i/\pi_L^{v_E(a_i)}$. We must show that

I. $\min\{v_E(a_i)\} > -\infty$.

II. $v_E(a_{-i}) \geq v_E(\pi^n R_{E,1})$ as $i \to \infty$ (i.e., conditions 1 and 2 imply that $\alpha \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$).

III. $\Phi(x) = \mathbb{T}_{\mathcal{L}/S}(\alpha x) \pmod{\pi^n C}$, $\forall x \in \mu_{\mathcal{L},1}$.

Condition (I) follows immediately since $a_{-i} \in R_{E,1}$, i.e., by equation (5.4)

$$v_E(a_{-i}) \geq -\left(\left[\frac{v_L(p)}{p-1}\right] + 1 + v_L(D(L/S)) \right) \quad \forall i \in \mathbb{Z}.$$

Suppose condition (II) was not true. Instead of passing to a subsequence we may assume for simplicity that $v_E(a_{-i}) < v_E(\pi^n R_{E,1})$ for all $i \geq 0$. Let

$$x = y(\pi_L^{\delta_0} u_0^{-1} + \pi_L^{\delta_1} u_1^{-1} T_d + \pi_L^{\delta_2} u_2^{-1} T_d^2 + \cdots),$$

where $u_i = a_i/\pi_L^{v_E(a_i)}$ and

$$\delta_i = v_E(\pi^n R_{E,1}) - v_E(a_{-i}) + [v_L(p)/(p-1)] \quad (\geq [v_L(p)/(p-1)] + 1 \ ),$$

for $i \geq 0$ and $y \in \mathcal{O}_L$ is arbitrary. Then $x \in \mu_{\mathcal{L},1}$ and $a_{-i} x_i = \pi_L^w y$ for $i \geq 0$, where $w = v_E(\pi^n R_{E,1}) + [v_L(p)/(p-1)]$. The convergence of the right hand side of (5.6) and the fact that $\mathbb{T}_{E/S}(\pi_L^w) = \mathrm{Tr}_{L/S}(\pi_L^w y)$ would imply that

$$\mathrm{Tr}_{L/S}(\pi_L^w y) \in \pi^n C \quad \forall y \in \mathcal{O}_L,$$

Thus $\pi_L^w/\pi^n \in D(L/S)^{-1}$, which implies $w \geq v_L(\pi^n) - v_L(D(L/S))$, that is,

$$v_E(R_{E,1}) \geq -v_L(D(L/S)) - [v_L(p)/(p-1)],$$

which is a contradiction since $v_E(R_{E,1}) = -v_L(D(L/S)) - [v_L(p)/(p-1)] - 1$ by (5.4). Finally, condition (III) immediately follows from equation (5.3).

□

Let $T_{\mathcal{L}}$ be the image of $\mu_{\mathcal{L}}$ under the formal logarithm. This is a C-submodule of $\mathcal{L}$ such that $T_{\mathcal{L}}S = \mathcal{L}$. Indeed, let $x \in \mathcal{L}$ and take n large enough such that $\pi^n x \in \mu_{\mathcal{L},1}$, then by Proposition 3.3.1 there exist a $y \in F(\mu_{\mathcal{L},1})$ such that $\pi^n x = l_F(y)$, thus $x \in T_{\mathcal{L}}S$. Let $R_{\mathcal{L}}$ be the dual of $T_{\mathcal{L}}$ with respect to the trace pairing $\mathbb{T}_{\mathcal{L}/S}$, then by Proposition 5.1.1 and by $\mathcal{L} = T_{\mathcal{L}}S$ we have the isomorphism

$$R_{\mathcal{L}} \simeq \mathrm{Hom}_C^{seq}(T_{\mathcal{L}}, C).$$

We also have the following

Lemma 5.1.3. *The generalized trace induces an injective homomorphism*

$$R_{\mathcal{L}}/\pi^n R_{\mathcal{L}} \to \mathrm{Hom}_C^{seq}(T_{\mathcal{L}}, C/\pi^n C)$$

$$\alpha \mapsto (x \mapsto \mathbb{T}_{\mathcal{L}/S}(\alpha x)).$$

Proof. Immediate from the very definition of $R_{\mathcal{L}}$. □

Let M/L be a finite extension of local fields containing the local field S. Let $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}$ the trace of the finite extension $\mathcal{M}/\mathcal{L}$. Then

Lemma 5.1.4. $\mathbb{T}_{\mathcal{M}/S} = \mathbb{T}_{\mathcal{L}/S} \circ \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}$.

Proof. By induction on d. Let $\mathcal{M} = \mathcal{M}'\{\{T_d\}\}$ and $\mathcal{L} = \mathcal{L}'\{\{T_d\}\}$, where

$$\mathcal{M}' = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\} \quad \text{and} \quad \mathcal{L}' = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}.$$

Let $\sum_i a_i T_d^i \in \mathcal{M}$, then since $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}$ is continuous we have

$$\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\sum_i a_i T_d^i) = \sum_i \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(a_i) T_d^i = \sum_i \mathrm{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_i) T_d^i.$$

Thus, since $\mathbb{T}_{\mathcal{L}/S} = \mathrm{Tr}_{L/S} \circ c_{\mathcal{L}/\mathcal{L}'}$, then

$$\mathbb{T}_{\mathcal{L}/S} \circ \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\sum_i a_i T_d^i) = \mathbb{T}_{\mathcal{L}'/S} \circ c_{\mathcal{L}/\mathcal{L}'}(\sum_i \mathrm{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_i) T_d^i) = \mathbb{T}_{\mathcal{L}'/S}(\mathrm{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_0)).$$

On the other hand

$$\mathbb{T}_{\mathcal{M}/S}(\sum_i a_i T_d^i) = \mathbb{T}_{M'/S} \circ c_{\mathcal{M}/\mathcal{M}'}(\sum_i a_i T_d^i) = \mathbb{T}_{\mathcal{M}'/S}(a_0).$$

The equality follows by the induction hypothesis.

□

5.2 The map $\psi^i_{\mathcal{L},n}$

We will denote by $(\ ,\ )^i_{\mathcal{L},n}$ the ith coordinate of the paring $(\ ,\ )_{\mathcal{L},n}$ with respect to the base $\{e_i\}$ of κ_n. Recall that a pair (n,t) is said to be admissible if there exist a k such that $t-1-n \geq \alpha k \geq n$, where α denotes the ramification index of S over $\mathbb{Q}_p$.

Proposition 5.2.1. *Let $L \supset K_t$, (n,t) admissible and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. For a given $\alpha \in K_d(\mathcal{L})$ there exist a unique element $\psi^i_{\mathcal{L},n}(\alpha) \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$, such that*

$$(\alpha, x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}(\ \psi^i_{\mathcal{L},n}(\alpha)\ l_F(x)\ ) \quad \forall x \in F(\mu_{\mathcal{L},1}), \tag{5.7}$$

and the map $\psi^i_{\mathcal{L},n} : K_d(\mathcal{L}) \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ is a homomorphism.

Proof. Let us first take α to be an element of the form $\{a_1, \ldots, a_d\}$ and consider the map

$$\omega :\ \mu_{\mathcal{L},1} \to C/\pi^n C,$$

defined by

$$x \mapsto (\alpha, l_F^{-1}(x))^i_{\mathcal{L},n}.$$

By Proposition 4.2.2 and Remark 3.3.1 this map is sequentially continuous and so by Lemma 5.1.2 there exist and element $\psi^i_{\mathcal{L},n}(\alpha) \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ satisfying (5.7). This defines a map $\psi^i_{\mathcal{L},n} : \mathcal{L}^{*\oplus d} \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ satisfying the Steinberg relation, therefore it induces a map on $K_d(\mathcal{L})$.

$\square$

Proposition 5.2.2. *Let $M/L/K_t$ be a finite tower such that (n,t) is admissible. Let $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Then*

1. $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}) \subset R_{\mathcal{L},1}$ *and we have the commutative diagram*

$$\begin{array}{ccc} K_d(\mathcal{M}) & \xrightarrow{\psi^i_{\mathcal{M},n}} & R_{\mathcal{M},1}/\pi^n R_{\mathcal{M},1} \\ {\scriptstyle N_{\mathcal{M}/\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}} \\ K_d(\mathcal{L}) & \xrightarrow{\psi^i_{\mathcal{L},n}} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1} \end{array}$$

2. $R_{\mathcal{L},1} \subset R_{\mathcal{M},1}$ *and we have the commutative diagram*

$$\begin{array}{ccc} K_d(\mathcal{L}) & \xrightarrow{\psi^i_{\mathcal{L},n}} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1} \\ {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow \\ K_d(\mathcal{M}) & \xrightarrow{\psi^i_{\mathcal{M},n}} & R_{\mathcal{M},1}/\pi^n R_{\mathcal{M},1} \end{array}$$

   The right-hand vertical map is induced by the embedding of $R_{\mathcal{L},1}$ in $R_{\mathcal{M},1}$.

3. *Let $L \supset K_m$, (m,t) admissible and $m \geq n$. Then for $\alpha \in K_d(\mathcal{L})$, $\psi^i_{\mathcal{L},n}(\alpha)$ is the reduction of $\psi^i_{\mathcal{L},m}(\alpha)$ from $R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1}$ to $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$, i.e., the diagram*

$$\begin{array}{ccc} K_d(\mathcal{L}) & \xrightarrow{\psi^i_{\mathcal{L},m}} & R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1} \\ & {\scriptstyle \psi^i_{\mathcal{L},n}}\searrow & \downarrow{\scriptstyle \text{reduction}} \\ & & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1} \end{array}$$

   commutes.

4. *If $t : (F, e_i) \to (\tilde{F}, \tilde{e}_i)$ is an isomorphism, then*

$$\tilde{\psi}^i_{\mathcal{L},n} = \frac{1}{t'(0)}\psi^i_{\mathcal{L},n}.$$

Proof. 1. By Lemma 5.1.4 and the fact that $\mu_{\mathcal{L},1} \subset \mu_{\mathcal{M},1}$ we obtain

$$\mathbb{T}_{\mathcal{L}/S}(\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1})\mu_{\mathcal{L},1}) = \mathbb{T}_{\mathcal{L}/S}(\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}\mu_{\mathcal{L},1})) \subset \mathbb{T}_{\mathcal{M}/S}(R_{\mathcal{M},1}\mu_{\mathcal{M},1}) \subset C,$$

from which follows that $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}) \subset R_{\mathcal{L},1}$. Now by Proposition 4.2.1 (4) we have, for $b \in K_d(\mathcal{M})$ and $x \in F(\mu_{\mathcal{L},1})$, that

$$(N_{\mathcal{M}/\mathcal{L}}(b), x)_{\mathcal{L},n} = (b, x)_{\mathcal{M},n} = \mathbb{T}_{\mathcal{M}/S}(\psi^i_{\mathcal{M},m}(b)l(x)) = \mathbb{T}_{\mathcal{L}/S}(\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(b))l(x)).$$

It follows from Proposition 5.2.1 that

$$\psi^i_{\mathcal{L},m}(N_{\mathcal{M}/\mathcal{L}}(b)) = \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(b)).$$

2. This is proved in a similar fashion to the previous property but this time using Proposition 4.2.1 (7).

3. This follows from Proposition 5.2.1 and Proposition 4.2.1 (5). Indeed, since $e^i_n = f^{(m-n)}(e^i_m)$ and $(a, x)_n = (a, f^{(m-n)}(x))_m$ we get

$$\pi^{m-n}(a, x)^i_n = (a, f^{(m-n)}(x))^i_m \pmod{\pi^m C}$$

That is

$$\pi^{m-n}\ \mathbb{T}_{\mathcal{L}/S}(\psi^i_{\mathcal{L},n}(a)\ l_F(x)) = \mathbb{T}_{\mathcal{L}/S}(\psi^i_{\mathcal{L},m}(a)\ l_F(f^{(m-n)}(x)))$$
$$= \pi^{m-n}\ \mathbb{T}_{\mathcal{L}/S}(\psi^i_{\mathcal{L},m}(a)\ l_F(x)) \pmod{\pi^m C}$$

Upon dividing by π^{m-n} the result follows.

4. This property follows from Proposition 4.2.1 (8) and $l_{\tilde{F}}(t) = t'(0)l_F$.

□

Proposition 5.2.3. *Let L/K_t be finite extension, (n,t) an admissible pair, and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let r be a t-normalized series. Then*

$$(\{a_1,\ldots,a_{d-1},u\},x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}\left(\log(u)\left[\ \frac{-\psi^i_{L,n}(\{a_1,\ldots,a_{d-1},r(x)\})\ r(x)\ l'_F(x)}{r'(x)}\ \right]\right), \qquad (5.8)$$

for $a_1,\ldots,a_{d-1} \in \mathcal{L}^*$, $u \in V_{\mathcal{L},1} = 1+\mu_{\mathcal{L},1}$ and $x \in F(\mu_{\mathcal{L}})$.

Proof. The proof follows the proof of Proposition 4.1 of [14]. To simplify the notation in this proof we will denote the normalized valuation $v_{\mathcal{L}}/v_{\mathcal{L}}(p)$ by v. We will also omit the subscript $\mathcal{L}$ in the pairing notation. Observe that we may assume that $r(X) = X$ since we can go to the isomorphic formal group $\tilde{F} = r(F(r^{-1}(X), r^{-1}(Y)))$ with torsion points $\tilde{e} = r(e_i)$ through the isomorphism $r : F \to \tilde{F}$. The series $\tilde{r} = X$

is t-normalized for $\tilde{F}$ and if the result were true for $\tilde{F}$ and $\tilde{r}$ then

$$(\{a_1,\dots,a_{d-1},u\},x)^i_{F,n} = (\{a_1,\dots,a_{d-1},u\},r(x))^i_{\tilde{F},n}$$
$$= \mathbb{T}_{\mathcal{L}/S}(\ \log(u)(-\tilde{\psi}^i_n(a_1,\dots,a_{d-1},r(x))l'_{\tilde{F}}(r(x))r(x))\ ),$$

and since $\tilde{\psi}^i_n(a_1,\dots,a_{d-1},r(x)) = r'(0)^{-1}\psi^i_n(a_1,\dots,a_{d-1},r(x))$ and

$$l_{\tilde{F}}(r(X)) = r'(0)l_F(X) \implies l'_{\tilde{F}}(r(X)) = r'(0)l'_F(x)/r'(x),$$

the result follows.

We assume therefore that $r(X) = X$. Since the pair (n,t) is admissible, let k be an integer such that $t-1-n \geq k\alpha \geq n$, α is the ramification index of $S/\mathbb{Q}_p$, and denote by ϵ the unit $\pi^{\alpha k}/p^k$. Let $u \in V_{\mathcal{L},1} = \{x \in \mathcal{L}\ :\ v(x-1) > 1/(p-1)\}$ and $x \in \mu_{\mathcal{L}}$. By bilinearity of the pairing

$$(\{a_1,\dots,a_{d-1},u\},x)_n = (\{a_1,\dots,a_{d-1},u\},x \ominus (x \times u^{p^k}))_n \oplus (\{a_1,\dots,a_{d-1},u\},x \times u^{p^k})_n. \tag{5.9}$$

Now let $m = n + \alpha k$ and $y = x \times u^{p^k}$. Then by (5) of Proposition 4.2.1

$$\begin{aligned}(\{a_1, \ldots, a_{d-1}, u\}, y)_n &= f^{(m-n)}(\ (\{a_1, \ldots, a_{d-1}, u\}, y)_m) \\ &= \pi^{\alpha k}(\{a_1, \ldots, a_{d-1}, u\}, y)_m \\ &= \epsilon(\{a_1, \ldots, a_{d-1}, u^{p^k}\}, y)_m \\ &= \epsilon(\{a_1, \ldots, a_{d-1}, \frac{y}{x}\}, y)_m \\ &= \epsilon(\{a_1, \ldots, a_{d-1}, y\}, y)_m \ominus \epsilon(\{a_1, \ldots, a_{d-1}, x\}, y)_m.\end{aligned}$$

But $r = X$ is t-normalized, hence we may replace $(\{a_1, \ldots, a_{d-1}, y\}, y)_m = 0$ by the expression $(\{a_1, \ldots, a_{d-1}, x\}, x)_m = 0$ and obtain

$$(\{a_1, \ldots, a_{d-1}, u\}, y)_n = \epsilon(\{a_1, \ldots, a_{d-1}, x\}, x)_m \ominus \epsilon(\{a_1, \ldots, a_{d-1}, x\}, y)_m \quad (5.10)$$

$$= \epsilon(\{a_1, \ldots, a_{d-1}, x\}, x \ominus y)_m. \quad (5.11)$$

By the properties of the logarithm in Proposition 3.3.1 we can express

$$u^{p^k} = \exp(\log(u^{p^k})) = \exp(p^k \log(u)) = 1 + p^k \log(u) + p^{2k} w,$$

where $w = \frac{z^2}{2!} + p^k \frac{z^3}{3!} + \cdots$, with $z = \log(u)$. Since $v(z) > 1/(p-1)$ then $v(\frac{z^i}{i!}) > 1/(p-1)$ and so $v(w) > 1/(p-1)$. This follows, for example, by Proposition 2.4 of [14].

Since $x \ominus y \equiv x - y \pmod{xy}$ and $y = xu^{p^k}$ with $v(x) > 0$, then $v(x \ominus y) = v(x - y) = v(x(u^{p^k} - 1)) > 1/(p-1)$. Thus, using the Taylor expansion of $l = l_F$

around $X = x$ we obtain

$$l(y) = l(x + xp^k z + xp^{2k} w) = l(x) + l'(x)(xp^k z + xp^{2k} w) + p^{2k} w_1,$$

where $w_1 = l''(x)\frac{\delta^2}{2!} + p^k l^{(3)}(x)\frac{\delta^3}{3!} + \cdots$ with $\delta = xz + xp^k w$. Since $v(\delta) > 1/(p-1)$ then $v(w_1) > 1/(p-1)$. Moreover

$$l(y) = l(x) + l'(x)xp^k z + p^{2k} w_2,$$

with $v(w_2) > 1/(p-1)$. Then

$$l(x \ominus y) = -l'(x)xp^k z - p^{2k} w_2.$$

Observing that $-l'(x)xz - p^k w_2 \in \mu_{\mathcal{L},1}$ we have by the isomorphism given in 3.3.1 that there is an $\eta \in F(\mu_{\mathcal{L},1})$ such that $l(x \ominus y) = p^k l(\eta) = -l'(x)xp^k z - p^{2k} w_2$. Thus

$$x \ominus y = [p^k](\eta) = [\pi^{\alpha k}](\tilde{\eta}) = f^{(\alpha k)}(\tilde{\eta}),$$

for $\tilde{\eta} = [\epsilon^{-1}]_F(\eta)$. Since $n \geq \alpha k$ then π^n divides $\pi^{\alpha k}$ and we have that $x \ominus y \in f^{(n)}(F(\mu_{\mathcal{L},1}))$. Thus the first term on the right hand side of equation (5.9) is zero.

By equation (5.10) and (5) of Proposition 4.2.1 we have

$$(\{a_1,\ldots,a_{d-1},u\},x)_n = \epsilon(\{a_1,\ldots,a_{d-1},x\},x\ominus y)_m \tag{5.12}$$

$$= \epsilon(\{a_1,\ldots,a_{d-1},x\},f^{(\alpha k)}(\tilde{\eta}))_m \tag{5.13}$$

$$= \epsilon(\{a_1,\ldots,a_{d-1},x\},\tilde{\eta})_n \tag{5.14}$$

$$= (\{a_1,\ldots,a_{d-1},x\},\eta)_n \tag{5.15}$$

Since $v(\eta) > 1/(p-1)$ and (n,t) is admissible we can use Proposition 5.2.1,

$$(\{a_1,\ldots,a_{d-1},x\},\eta)_n = \mathbb{T}_{\mathcal{L}/S}(\ \psi^i_n(a_1,\ldots,a_{d-1},x)\ l_F(\eta)\ )$$

$$= \mathbb{T}_{\mathcal{L}/S}(\ \psi^i_n(a_1,\ldots,a_{d-1},x)\ (-l'(x)xz - p^k w_2)\ ).$$

Since $\psi^i_{L,n}(a_1,\ldots,a_{d-1},x) \in R_{L,1}$, $w_2 \in T_{L,1}$, $\pi^n | p^k$ and $\mathbb{T}_{\mathcal{L}/S}(R_{L,1}T_{L,1}) \subset C$, then we can write the above as

$$\mathbb{T}_{\mathcal{L}/S}(\ \psi^i_n(a_1,\ldots,a_{d-1},x)\ (-l'(x)xz)\ ).$$

From equation 5.12 we get

$$(\{a_1,\ldots,a_{d-1},u\},x)^i_n = \mathbb{T}_{\mathcal{L}/S}(\ \psi^i_n(a_1,\ldots,a_{d-1},x)\ (-l'(x)xz)\ ).$$

Keeping in mind that $z = \log(u)$, the proposition follows.

$\square$

5.3 The map $\rho^i_{\mathcal{L},n}$

We can define a C-linear structure on $V_{\mathcal{L},1} = 1 + \mu_{\mathcal{L},1}$ by using the isomorphism $\log : V_{\mathcal{L},1} \to T_{\mathcal{L},1}$, i.e., $cu := \log^{-1}(c\log(u))$. Let $x \in F(\mu_{\mathcal{L}})$ and $a_1, \ldots, a_{d-1} \in \mathcal{L}^*$ be fixed. Consider the mapping

$$V_{\mathcal{L},1} \to C/\pi^n C,$$

defined by

$$u \mapsto (\{a_1, \ldots, a_{d-1}, u\}, x)^i_{\mathcal{L},n}.$$

According to Proposition 5.2.3 this is a continuous C-linear map and we have the following

Proposition 5.3.1. *There exist a unique element* $\rho^i_{\mathcal{L},n}(\{a_1, \ldots, a_{d-1}\}, x) \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ *such that*

$$(\{a_1, \ldots, a_{d-1}, u\}, x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}\left(\log(u)\ \rho^i_{\mathcal{L},n}(\{a_1, \ldots, a_{d-1}\}, x) \right), \tag{5.16}$$

for all $u \in V_{\mathcal{L},1}$*, and the map* $\rho^i_{\mathcal{L},n} : K_{d-1}(\mathcal{L}) \otimes F(\mu_{\mathcal{L}}) \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ *is a homomorphism.*

From Proposition 5.3.1 and Proposition 5.2.3 it follows the next proposition.

Proposition 5.3.2. *Let* L/K_t *be a finite extension,* $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$*, with*

(n,t) admissible, and let r be a t-normalized series. Then

$$\frac{\psi^i_{\mathcal{L},n}(\{a_1,\dots,a_{d-1},r(x)\})\ r(x)}{r'(x)} = -\frac{\rho^i_{\mathcal{L},n}(\{a_1,\dots,a_{d-1}\},x)}{l'_F(x)}.$$

for all $a_1,\dots,a_{d-1}\in\mathcal{L}^*$ and all $x\in F(\mu_{\mathcal{L}})$.

5.4 The maps $D^i_{\mathcal{L},n}$

Assume $L\supset K(\kappa_t)$ for (n,t) admissible and let $\mathcal{L}=L\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$. Define $D^i_{\mathcal{L},n}:\mathcal{O}^d_{\mathcal{L}}\to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ by

$$D^i_{\mathcal{L},n}(a_1,\dots,a_d)=\psi^i_{\mathcal{L},n}(\{a_1\dots,a_d\})a_1\cdots a_d, \tag{5.17}$$

and consider $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ as an $\mathcal{O}_{\mathcal{L}}$-module. This map satisfies

Proposition 5.4.1. *1. Leibniz Rule:*

$$D^i_{\mathcal{L},n}(a_1,\dots,a_ia'_i,\dots,a_d)=a_iD^i_{\mathcal{L},n}(a_1,\dots,a'_i,\dots,a_d)+a'_iD^i_{\mathcal{L},n}(a_1,\dots,a_i,\dots,a_d).$$

2. Steinberg relation:

$$D^i_{\mathcal{L},n}(a_1,\dots,a_i,\dots,1-a_i,\dots,a_d)=0.$$

3. Skew-symmetric:

$$D^i_{\mathcal{L},n}(a_1,\dots,a_i,\dots,a_j,\dots,a_d)=-D^i_{\mathcal{L},n}(a_1,\dots,a_j,\dots,a_i,\dots,a_d).$$

4. $D^i_{\mathcal{L},n}(a_1,\ldots,a_i^{p^k},\ldots,a_d)=0$ *if* $\pi^n|p^k$.

Proof. Property (1) follows from the fact that $\psi^i_{\mathcal{L},n}$ is a homomorphism. Property (2) follows from the Steinberg relation $\{a_1,\ldots,a_i,\ldots,1-a_i,\ldots,a_d\}=1$ for elements in the Milnor K-group $K_d(\mathcal{L})$ and property (3) follows from the fact that

$$\{a_1,\ldots,a_i,\ldots,a_j,\ldots,a_d\}=\{a_1,\ldots,a_j,\ldots,a_i,\ldots,a_d\}^{-1},$$

in $K_d(\mathcal{L})$ (cf. Proposition 4.1.1).

□

Proposition 5.4.2. *Let $\mathcal{L}$ be as in Proposition 5.3.2, then*

$$\frac{D^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},r(x\oplus y))}{r'(x\oplus y)}=F_X(x,y)\ \frac{D^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},r(x))}{r'(x)}+F_Y(x,y)\ \frac{D^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},r(y))}{r'(y)}, \quad (5.18)$$

for all $a_1,\ldots,a_{d-1}\in\mathcal{L}^*$ and $x,y\in F(\mu_{\mathcal{L}})$.

Proof. This follows from the fact that

$$\rho^i_{\mathcal{L},n}(\{a_1,\ldots,a_{d-1}\},x\oplus_F y)=\rho^i_{\mathcal{L},n}(\{a_1,\ldots,a_{d-1}\},x)+\rho^i_{\mathcal{L},}(\{a_1,\ldots,a_{d-1}\},y),$$

and from differentiating $l_F(F(X,Y))=l_F(X)+l_F(Y)$ with respect to X and Y. □

Let $\tilde{F}$ be the formal group $r(F(r^{-1}(X),r^{-1}(Y)))$. The series $r(X)=X$ is t-normalized for $\tilde{F}$. Denote by $\tilde{\oplus}$ the sum according to this formal group and $\tilde{D}^i_{\mathcal{L},n}$,

$\tilde{\psi}^i_{\mathcal{L},n}$ the corresponding maps. According to Proposition 5.2.2 (4) we have that

$$\tilde{D}^i_{\mathcal{L},n} = \frac{1}{r'(0)} D^i_{\mathcal{L},n}. \tag{5.19}$$

Therefore, Proposition 5.4.2 in terms of $\tilde{\oplus}$ and $D^i_{\mathcal{L},n}$ reads as

Corollary 5.4.1. *Let $\mathcal{L}$ be as in Proposition 5.3.2, then*

$$D^i_{\mathcal{L},n}(a_1,\dots,a_{d-1},x\tilde{\oplus}y) = \tilde{F}_X(x,y)D^i_{\mathcal{L},n}(a_1,\dots,a_{d-1},x) + \tilde{F}_Y(x,y)D^i_{\mathcal{L},n}(a_1,\dots,a_{d-1},y). \tag{5.20}$$

for all $a_1,\dots,a_{d-1} \in \mathcal{L}^*$ and $x,y \in F(\mu_{\mathcal{L}})$.

Let A be the set of all series in $X_d\mathcal{O}_{\mathcal{L}}[[X_1,\dots,X_d]]$ of the form

$$\bigoplus_{k=1}^{\infty}\left(\bigoplus_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,k}^{p^n}\, X_1^{i_1}\cdots X_{d-1}^{i_{d-1}}X_d^k\right), \tag{5.21}$$

where $\gamma_{i,k} \in \mathcal{O}_{\mathcal{L}}$. For $d=2$, the series looks like

$$\bigoplus_{k=1}^{\infty}\left(\bigoplus_{0\le i\le p^n-1} \gamma_{i,k}^{p^n}\, X_1^{i}X_2^k\right),$$

where $\gamma_{i,k} \in \mathcal{O}_{\mathcal{L}}$.

Proposition 5.4.3. *For $x \in \mathcal{O}_{\mathcal{L}}$, there exist elements $\gamma_i \in \mathcal{O}_{\mathcal{L}}$, $i=(i_1,\dots i_{d-1})$, such that*

$$x \equiv \sum_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_i^{p^n}\, T_1^{i_1}\cdots T_{d-1}^{i_{d-1}} \pmod{\pi_L}.$$

Proof. By induction on d. For $d = 1$, the result follows since k_L is perfect. Suppose it is proved for $R = \mathcal{O}_{\mathcal{L}_d}$, where $\mathcal{L}_d = L\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$. Let $\mathcal{L} = \mathcal{L}_d\{\{T_d\}\}$ and $x \in \mathcal{O}_{\mathcal{L}}$. Then $x = \sum_{j\geq m} a_j T_d^j \pmod{\pi_L}$, $a_j \in R$. Then

$$\begin{aligned} x &\equiv \sum_{0\leq i_d} T_d^{i_d}\left(\sum_{m\leq i_d+kp^n} a_{i_d+kp^n} T_d^{kp^n}\right) \pmod{\pi_L} \\ &\equiv \sum_{0\leq i_d} T_d^{i_d}\left(\sum_{m\leq i_d+kp^n}\left(\sum_{0\leq i_1,\ldots,i_{d-1}\leq p^n-1} \gamma_{i_1,\ldots,i_{d-1},i_d;k}^{p^n}\, T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\right)T_d^{kp^n}\right) \pmod{\pi_L} \\ &\equiv \sum_{0\leq i_d} T_1^{i_1}\cdots T_{d-1}^{i_{d-1}} T_d^{i_d}\left(\sum_k\left(\sum_{0\leq i_1,\ldots,i_{d-1}\leq p^n-1} \gamma_{i_1,\ldots,i_{d-1},i_d;k}^{p^n}\, T_d^{kp^n}\right)\right) \pmod{\pi_L} \\ &\equiv \sum_{0\leq i_1,\ldots,i_d\leq p^n-1} \gamma_{i_1,\ldots,i_d}^{p^n}\, T_1^{i_1}\cdots T_d^{i_d}, \end{aligned}$$

where $\gamma_{i_1,\ldots,i_d} = \sum_k \gamma_{i_1,\ldots,i_{d-1},i_d;k}\, T_d^k$ and regrouping terms is valid since the series are absolutely convergent in the Parshin topology. Also by noticing that the congruence

$$\textstyle\sum_{c\leq k} b_k^{p^n} T^{kp^n} \equiv \left(\sum_{c\leq k} b_k T_d^k\right)^{p^n} \pmod{\pi_L},$$

holds in $k_{\mathcal{L}_{d-1}}((T_d))$, where $k_{\mathcal{L}_{d-1}}$ is the residue field of $\mathcal{L}_{d-1}$. □

Remark 5.4.1. This proposition is equivalent to the following two facts:

1. $k_L((T_1))\ldots((T_{d-1}))$ is a finite extension of $k_L((T_1^{p^n}))\ldots((T_{d-1}^{p^n}))$ of degree $(d-1)p^n$ and generated by the elements $T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}$ for $0 \leq i_1,\ldots,i_{d-1} \leq p^n - 1$.

2. $k_L((T_1^{p^n}))\ldots((T_{d-1}^{p^n}))$ is the image of $k_L((T_1))\ldots((T_{d-1}))$ under the Frobenius homomorphsim

$$\sigma_p : k_L((T_1))\ldots((T_{d-1})) \to k_L((T_1))\ldots((T_{d-1})),$$

i.e., every element of $k_L((T_1^{p^n}))\dots((T_{d-1}^{p^n}))$ has a p^nth root in $k_L((T_1))\dots((T_{d-1}))$.

Both facts are easily proven by induction from the fact that, for a field k of characteristic p, the extension $[k((T)) : k((T^p))]$ has degree p and $\sigma_d(k)((T^p))$ is the image of $k((T))$ under the Frobenius homomorphism $\sigma_p : k((T)) \to k((T))$.

Proposition 5.4.4. *Let $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. For $y \in \mu_{\mathcal{L}}$, there exist an $\eta \in A$ such that $y = \eta(T_1, \dots, \pi_L)$.*

Proof. Let $y \in \mu_{\mathcal{L}}$. Then by Proposition 5.4.3

$$\frac{y}{\pi_L} \equiv \sum_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1\le p^n-1}}} \gamma_{i,1}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}} \pmod{\pi_L}$$
$$\equiv \frac{\bigoplus_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1\le p^n-1}}} \gamma_{i,1}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L}{\pi_L} \pmod{\pi_L},$$

this is,

$$y \equiv \bigoplus_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1\le p^n-1}}} \gamma_{i,1}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L \pmod{\pi_L^2}.$$

Denote $\bigoplus_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1\le p^n-1}}} \gamma_{i,1}^{p^n} T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L$ by y_1. Suppose we have defined elements $y_k = \bigoplus_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1\le p^n-1}}} \gamma_{i,k}^{p^n} T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^k$, $1 \le k \le m-1$, such that

$$y \ominus (\oplus_{k=1}^{m-1} y_k) \equiv 0 \pmod{\pi_L^m}.$$

Then

$$\frac{y \ominus (\oplus_{k=1}^{m-1} y_k)}{\pi_L^m} \equiv \sum_{\substack{i=(i_1,\dots,i_{d-1}) \\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,m}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}} \pmod{\pi_L}$$

$$\equiv \frac{\bigoplus_{\substack{i=(i_1,\dots,i_{d-1}) \\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,m}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^m}{\pi_L^m} \pmod{\pi_L}.$$

Denote $\bigoplus_{\substack{i=(i_1,\dots,i_{d-1}) \\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,m}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^m$ by y_m. Then

$$y \ominus (\oplus_{k=1}^{m-1} y_k) \equiv y_m \pmod{\pi_L^{m+1}},$$

or

$$y \ominus (\oplus_{k=1}^{m-1} y_k) \equiv 0 \pmod{\pi_L^{m+1}}.$$

Therefore $y = \oplus_{k=1}^{\infty} y_k$, which is what we wanted to prove. □

Corollary 5.4.2. *For every $x \in \mathcal{O}_{\mathcal{L}}$, there exists $\gamma_{i,k} \in \mathcal{O}_{\mathcal{L}}$ such that*

$$\textstyle\sum_{k=0}^{\infty} \left(\sum_{\substack{i=(i_1,\dots,i_{d-1}) \\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,k}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^k \right).$$

Proof. Take F to be the additive formal group $X+Y$ in Proposition 5.4.4. □

Corollary 5.4.3. *For every $x \in V_{\mathcal{L}} = 1 + \mu_{\mathcal{L}}$, there exist $\gamma_{i,k} \in \mathcal{O}_{\mathcal{L}}$ such that*

$$\textstyle\prod_{k=1}^{\infty} \left(\prod_{\substack{i=(i_1,\dots,i_{d-1}) \\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} (\ 1 + \gamma_{i,k}^{p^n}\ T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^k\ ) \right).$$

Proof. This can be proven either by using Proposition 5.4.3 and induction on k; just as in the proof of Proposition 5.4.4, or by applying Proposition 5.4.4 to the multiplicative formal group $F(X,Y) = X + Y + XY$. Noticing also, that the homomorphism

$$F(\mu_{\mathcal{L}}) \longrightarrow V_{\mathcal{L}},$$

defined by $x \mapsto 1 + x$, is continuous in the topology induced by the valuation $v_{\mathcal{L}}$ of $\mathcal{L}$. □

Let r be a t-normalized series for F and (n,t) an admissible pair. Let $\tilde{F}$ be the formal group $r(F(r^{-1}(X), r^{-1}(Y)))$. Let $\mathcal{L} = L\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$. For $y \in \mu_{\mathcal{L}}$ we will denote by $\tilde{\eta}_y(X_1,\dots,X_d)$ the multivariable series

$$\tilde{\bigoplus}_{k=1}^{\infty}\left(\tilde{\bigoplus}_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,k}^{p^n} X_1^{i_1}\cdots X_{d-1}^{i_{d-1}} X_d^k\right), \quad (\gamma_{i,k}\in\mathcal{O}_{\mathcal{L}}) \tag{5.22}$$

with respect to $\tilde{F}$, such that $y = \tilde{\eta}_y(T_1,\dots,\pi_L)$, whose existence is guaranteed by Proposition 5.4.4.

Proposition 5.4.5. *Let L/K_t be a finite extension with (n,t) admissible and set $\mathcal{L} = L\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$. For $y = \tilde{\eta}_y(T_1,\dots,\pi_L) \in \mu_{\mathcal{L}}$, $\tilde{\eta}_y$ as in (5.22), we have*

$$D^i_{\mathcal{L},n}(a_1,\dots,a_{d-1},y) = \sum_{i=1}^{d} \left.\frac{\partial\tilde{\eta}_y}{\partial X_i}\right|_{\substack{X_k=T_k,\\ k=1,\dots,d}} D^i_{\mathcal{L},n}(a_1,\dots,a_{d-1},T_i),$$

for all $a_1,\dots,a_{d-1}\in\mathcal{O}_{\mathcal{L}}$, where $T_d = \pi_L$.

Proof. Let $y = \oplus_{k=1}^{\infty} y_k$, where

$$y_k = \tilde{\bigoplus}_{\substack{i=(i_1,\dots,i_{d-1})\\ 0\le i_1,\dots,i_{d-1}\le p^n-1}} \gamma_{i,k}^{p^n} T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_L^k.$$

Thus, $\tilde{\eta}_y = \tilde{\oplus}_{m=1}^{\infty}\eta_m$, where

$$\eta_m(X_1,\ldots,X_d) = \tilde{\oplus}_{\substack{i=(i_1,\ldots,i_{d-1})\\ 0\le i_1,\ldots,i_{d-1}\le p^n-1}} \gamma_{i,k}^{p^n} X_1^{i_1}\cdots X_{d-1}^{i_{d-1}} X_d^m.$$

Let us fix $a_1,\ldots,a_{d-1}\in\mathcal{L}^*$ and denote by $D(x)$ the element $D^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},x)$ to simplify notation.

First notice that

$$v_{\mathcal{L}}(y_k) \xrightarrow{k\to\infty} \infty \quad\Longrightarrow\quad D(y_k)=0 \quad \text{for } k \text{ large enough.} \tag{5.23}$$

This follows from $D^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},x) = \psi^i_{\mathcal{L},n}(a_1,\ldots,a_{d-1},x)a_1\cdots a_{d-1}x$ and that $\psi^i_{\mathcal{L},n}$ has values in $R_{\mathcal{L}}/\pi^n R_{\mathcal{L},n}$; $\pi_L^n\psi^i_{\mathcal{L},n}=0$.

Thus, from $\tilde{\eta}_y = \tilde{\oplus}_{m=1}^{k-1}\eta_m \pmod{\pi^k}$ and equation (5.23), it is enough to consider the finite formal sum $\tilde{\oplus}_{m=1}^{k-1}\eta_m$. The proposition follows now from the fact that $D(\gamma^{p^n}) = p^n\gamma^{p^n-1}D(\gamma)=0$ ($\pi^n|p^n$), $D(xy)=yD(x)+xD(y)$, and corollary 5.4.1.

□

Corollary 5.4.4. *Let L/K_t be a finite extension with (n,t) admissible and set $\mathcal{L} = L\{\{T_1\}\}\ldots\{\{T_{d-1}\}\}$. Let $y_i = \tilde{\eta}_{y_i}(T_1\ldots,\pi_L)\in\mu_{\mathcal{L}}$, for $1\le i\le d$, where η_{y_i} is a multivariable series of the form (5.21). Then*

$$D_{\mathcal{L},n}(y_1,\ldots,y_d) = \det\left[\frac{\partial\tilde{\eta}_{y_i}}{\partial X_j}\right]_{i,j}\Bigg|_{\substack{X_k=T_k,\\ k=1,\ldots,d}} D_{\mathcal{L},n}(T_1,\ldots,\pi_L),$$

where $T_d=\pi_L$.

Proof. This follows from Proposition 5.4.5 and Proposition 5.4.1 (3). Indeed, let us illustrate the proof in the case $d = 2$, i.e., $\mathcal{L} = L\{T_1\}\}$ and $T_2 = \pi_L$. To simplify the notation we will denote $D^i_{\mathcal{L},n}$ by D. From Proposition 5.4.5 we have

$$\begin{aligned} D(\eta_1(T_1,T_2),\eta_2(T_1,T_2)) = & \frac{\partial\eta_1}{\partial X_1}\frac{\partial\eta_2}{\partial X_1}\bigg|_{\substack{X_i=T_i,\\ i=1,2}} D(T_1,T_1) + \frac{\partial\eta_1}{\partial X_1}\frac{\partial\eta_2}{\partial X_2}\bigg|_{\substack{X_i=T_i,\\ i=1,2}} D(T_1,T_2) \\ & + \frac{\partial\eta_1}{\partial X_2}\frac{\partial\eta_2}{\partial X_1}\bigg|_{\substack{X_i=T_i,\\ i=1,2}} D(T_2,T_1) + \frac{\partial\eta_1}{\partial X_2}\frac{\partial\eta_2}{\partial X_2}\bigg|_{\substack{X_i=T_i,\\ i=1,2}} D(T_2,T_2) \end{aligned} \tag{5.24}$$

But $D(T_1,T_1) = D(T_2,T_2) = 0$, $D(T_2,T_1) = -D(T_1,T_2)$ from Proposition 5.4.1 (3), therefore

$$D(\eta_1(T_1,T_2),\eta_2(T_1,T_2)) = \left(\frac{\partial\eta_1}{\partial X_1}\frac{\partial\eta_2}{\partial X_2} - \frac{\partial\eta_1}{\partial X_2}\frac{\partial\eta_2}{\partial X_1}\right)\bigg|_{\substack{X_i=T_i,\\ i=1,2}} D(T_2,T_1)). \tag{5.25}$$

The corollary follows. □

Chapter 6

Multidimensional derivations

Let $\mathcal{L}$ be the standard d-dimensional local field $L\{\{T_1\}\}\dots\{\{T_{d-1}\}\}$ and $L/K/\mathbb{Q}_p$ be a tower of finite extensions. Let W be an $\mathcal{O}_{\mathcal{L}}$-module that is p-adically complete, i.e,

$$W \cong \varprojlim W/p^n W.$$

For example, if $p^n W = 0$ for some n, then W is p-adically complete. Actually, this is going to be our situation, since W will be the $\mathcal{O}_{\mathcal{L}}$-module $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$.

6.1 Derivations and the module of differentials

Definition 6.1.1. *A derivation of $\mathcal{O}_{\mathcal{L}}$ into W over $\mathcal{O}_K$ is a map $D : \mathcal{O}_{\mathcal{L}} \to W$ such that for all $a, b \in \mathcal{O}_{\mathcal{L}}$ we have*

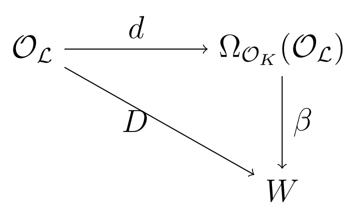
1. $D(ab) = aD(b) + bD(a)$.

2. $D(a + b) = D(a) + D(b)$.

3. $D(a) = 0$ *if* $a \in \mathcal{O}_K$.

We denote by $D_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}, W)$ the $\mathcal{O}_{\mathcal{L}}$-module of all derivations $D : \mathcal{O}_{\mathcal{L}} \to W$. The universal object in the category of derivations of $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$ is the $\mathcal{O}_{\mathcal{L}}$-module of Khaler differentials, denoted by $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$. This is the $\mathcal{O}_{\mathcal{L}}$-module generated by finite linear combinations of the symbols da, for all $a \in \mathcal{O}_{\mathcal{L}}$, divided out by the submodule generated by all the expressions of the form $dab - adb - bda$ and $d(a+b) - da - db$ for all $a, b \in \mathcal{O}_{\mathcal{L}}$ and da for all $a \in \mathcal{O}_K$. The derivation $d : \mathcal{O}_{\mathcal{L}} \to \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$ is defined by sending a to da.

If $D : \mathcal{O}_{\mathcal{L}} \to W$ is a derivation, then $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$ is universal in the following way. There exist a unique homomorphism $\beta : \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}) \to W$ of $\mathcal{O}_{\mathcal{L}}$-modules such that the diagram



is commutative.

Let $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$ be the p-adic completion of $\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$, i.e.,

$$\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}) = \varprojlim \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})/p^n\Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}).$$

Since we are assuming that W is p-adically complete, the homomorphism β induces the homomorphism

$$\beta : \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}) \to W.$$

Proposition 6.1.1. *We have the isomorphism of $\mathcal{O}_{\mathcal{L}}$-modules*

$$\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}) \cong \mathcal{O}_{\mathcal{L}} dT_1 \oplus \cdots \oplus \mathcal{O}_{\mathcal{L}} dT_{d-1} \oplus (\mathcal{O}_{\mathcal{L}}/D(L/K)\mathcal{O}_{\mathcal{L}})d\pi_L,$$

where π_L is a uniformizer for L and $D(L/K)$ is the different of the extension L/K. That is, $dT_1, \ldots, dT_{d-1}$ and $d\pi_L$ generate $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$.

Proof. To simplify the notation let us denote $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$ by $\hat{\Omega}$, where $\Omega = \Omega_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}})$. We will show that

$$\frac{\Omega}{\pi_L^n \Omega} \simeq \frac{\mathcal{O}_{\mathcal{L}}}{\pi^n \mathcal{O}_{\mathcal{L}}} \oplus \cdots \oplus \frac{\mathcal{O}_{\mathcal{L}}}{\pi^n \mathcal{O}_{\mathcal{L}}} \oplus \frac{\mathcal{O}_{\mathcal{L}}}{\pi^n \mathcal{O}_{\mathcal{L}} + D(L/K)\mathcal{O}_{\mathcal{L}}} \tag{6.1}$$

for all $n \geq 1$. These isomorphisms are compatible: $\simeq_{n+1} \equiv \simeq_n \pmod{p^n}$, then we can take the projective limit $\varprojlim$ to obtain the result. To prove (6.1), we will start by showing that $\Omega/\pi_L^n \Omega$ is generated by $d\pi_L$ and $dT_1, \ldots, dT_{d-1}$ for all n.

Let $x \in \mathcal{O}_{\mathcal{L}}$, then by corollary 5.4.2, we have that

$$x = \sum_{k=0}^{\infty} \left(\sum_{0 \leq i_1, \ldots, i_{d-1} \leq p^n - 1} \gamma_{i,k}^{p^n} \, T_1^{i_1} \cdots T_{i_{d-1}} \pi_L^k \right).$$

Therefore, in $\Omega/p^n\Omega$, we can consider the truncated sum

$$\sum_{k=0}^{m} \left(\sum_{0 \leq i_1, \ldots, i_{d-1} \leq p^n - 1} \gamma_{i,k}^{p^n} \, T^i Y^k \right),$$

where m is such that $p^n|\pi^{m+1}$. Thus, dx is generated by $d\pi_L$ and dT_i, $i = 1, \ldots, d-1$ in $\Omega/p^n\Omega$.

Let us construct now the isomorphism (6.1). We will define derivations D_k of $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$ for $k = 0, \ldots, d-1$ as follows. Let $\mathcal{L}_0 = L$ and $\mathcal{L}_k = \mathcal{L}_{k-1}\{\{T_k\}\}$, $k = 1, \ldots, d-1$. For $k = 0$ define

$$D_0 : \mathcal{O}_L \to \mathcal{O}_L/D(L/K)\mathcal{O}_L$$

by $D_0(g(\pi_L)) = g'(\pi_L)$, where $g(X)$ is a polynomial with coefficients in the ring of integers of the maximal subextension of L unramified over K. This is a well defined derivation of $\mathcal{O}_L$ over $\mathcal{O}_K$ by Corollary 5.2 of [14]. For $1 \leq k \leq d-1$, we define the derivation of $\mathcal{O}_{\mathcal{L}_k}$ over $\mathcal{O}_K$

$$D_k : \mathcal{O}_{\mathcal{L}_k} \to \mathcal{O}_{\mathcal{L}_k}, \tag{6.2}$$

by $D_k(\sum a_i T_k^i) = \sum a_i \; i T_k^{i-1}$, $a_i \in \mathcal{O}_{\mathcal{L}_{k-1}}$.

We now lift these derivations to derivations of $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$, by induction, in the following way. Suppose $D : \mathcal{O}_{\mathcal{L}_k} \to W$ is a derivation of $\mathcal{O}_{\mathcal{L}_{k-1}}$ over $\mathcal{O}_K$, where $1 \leq k \leq d$ and the $\mathcal{O}_{\mathcal{L}_k}$-module W is either $\mathcal{O}_{\mathcal{L}_{k-1}}$ or $\mathcal{O}_{\mathcal{L}_{k-1}}/D(L/K)\mathcal{O}_{\mathcal{L}_{k-1}}$. Then D extends to a derivation of $\mathcal{O}_{\mathcal{L}_k}$ over $\mathcal{O}_K$

$$D : \mathcal{O}_{\mathcal{L}_k} \to \mathcal{O}_{\mathcal{L}_k} \otimes W,$$

by

$$D(\sum_i a_i \ T_k^i \ ) := \sum_i D(a_i) \ T_k^i.$$

This derivation is well defined since D is continuous with respect to the valuation topology of $\mathcal{O}_{\mathcal{L}_{k-1}}$.

Let us now define the map

$$\partial : \mathcal{O}_{\mathcal{L}} \to \frac{\mathcal{O}_{\mathcal{L}}}{p^n\mathcal{O}_{\mathcal{L}} + \mathcal{D}(L/K)\mathcal{O}_{\mathcal{L}}} \oplus \frac{\mathcal{O}_{\mathcal{L}}}{p^n\mathcal{O}_{\mathcal{L}}} \oplus \cdots \oplus \frac{\mathcal{O}_{\mathcal{L}}}{p^n\mathcal{O}_{\mathcal{L}}}$$

by

$$a \to (\overline{D_0}(a), \ldots, \overline{D_{d-1}}(a))$$

where $\overline{D_k}$ is the reduction of D_k. This is a well-defined derivation of $\mathcal{O}_{\mathcal{L}}$ over $\mathcal{O}_K$ and by the universality of Ω, this induces a homomorphism of $\mathcal{O}_{\mathcal{L}}$-modules

$$\partial : \frac{\Omega}{p^n\Omega} \to \frac{\mathcal{O}_{\mathcal{L}}}{p^n\mathcal{O}_{\mathcal{L}} + D(L/K)\mathcal{O}_{\mathcal{L}}} \oplus \cdots \oplus \frac{\mathcal{O}_{\mathcal{L}}}{p^n\mathcal{O}_{\mathcal{L}}}.$$

Let us show that ∂ is an isomorphism. Indeed, it is clearly surjective since for $(a_0, \ldots, a_{d-1}) \in \mathcal{O}_{\mathcal{L}} \oplus \cdots \oplus \mathcal{O}_{\mathcal{L}} \oplus (\mathcal{O}_{\mathcal{L}}/\mathcal{D}(L/K)\mathcal{O}_{\mathcal{L}})$ we have that

$$\partial(a_0 d\pi_L + a_1 dT_1 + \cdots a_{d-1} dT_{d-1}) = (a_0, \ldots, a_{d-1}).$$

since

$$\overline{D_k}(d\pi_L) = \begin{cases} 0, & k = 0, \\ 1, & 1 \leq k \leq d-1, \end{cases} \qquad \overline{D_k}(dT_i) = \begin{cases} 0, & k \neq i, \\ 1, & k = i, \end{cases}$$

Also, ∂ is injective for if $a = a_0 d\pi_L + a_1 dT_1 + \cdots a_{d-1} dT_{d-1} \in \Omega/p^n\Omega$ is such that

$\partial(a) = 0$, then $\overline{D_k}(a) = a_k = 0$ in $\mathcal{O}_{\mathcal{L}}/p^n\mathcal{O}_{\mathcal{L}}$, for $1 \le k \le d-1$, and $\overline{D_0}(a) = a_0 = 0$ (mod $p^n\mathcal{O}_{\mathcal{L}} + \mathcal{D}(L/K)\mathcal{O}_{\mathcal{L}}$). But then $a_0 d\pi_L = 0$, since $\mathcal{D}(L/K)d\pi_L = 0$, and therefore $a = 0 \mod p^n\Omega$. This concludes the proof. □

Definition 6.1.2. *The derivations $D_k : \mathcal{O}_{\mathcal{L}} \to \mathcal{O}_{\mathcal{L}}$, $1 \le k \le d-1$, which are obtained by the lifts of the derivations given in equation (6.2) described in Proposition 6.1.1, will be denoted by $\frac{\partial}{\partial T_k}$ for $1 \le k \le d-1$. Moreover, the derivation $D_0 : \mathcal{O}_{\mathcal{L}} \to \mathcal{O}_{\mathcal{L}}/D(L/K)\mathcal{O}_{\mathcal{L}}$, also defined in the above proposition, will be denoted by $\frac{\partial}{\partial T_d}$.*

Remark 6.1.1. The derivations $\frac{\partial}{\partial T_k}$ for $1 \le k \le d-1$ coincide, informally, with the partial derivative with respect to T_k of an element α in $\mathcal{O}_{\mathcal{L}} = \mathcal{O}_L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. On the other hand, the derivation $\frac{\partial}{\partial T_d}$ has the following interpretation. If $\alpha \in \mathcal{O}_{\mathcal{L}}$, then there exist a polynomial $g(X) \in \mathcal{O}_{\tilde{\mathcal{L}}}[X]$ such that $\alpha = g(\pi_L)$. Then

$$\frac{\partial \alpha}{\partial T_d} = g'(\pi_L).$$

Here $\tilde{\mathcal{L}} = \tilde{L}\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, where $\tilde{L}$ is the maximal subextension of L unramified over K.

Proposition 6.1.2. *Let $D : \mathcal{O}_{\mathcal{L}} \to W$ be a derivation and let $\frac{\partial}{\partial T_k}$, $1 \le k \le d$, be the derivations form Definition 6.1.2. Then $D(L/K)D(\pi_{\mathcal{L}}) = 0$ and*

$$D(\beta) = \sum_{k=1}^{d-1} \frac{\partial \beta}{\partial T_k} D(T_k) \ + \ \frac{\partial \beta}{\partial T_d} D(\pi_L). \qquad (6.3)$$

Moreover, given $w_1, \ldots, w_d \in W$ such that $D(L/K)w_d = 0$, the map

$$D(\beta) := \sum_{k=1}^{d} \frac{\partial \beta}{\partial T_k} \, w_k \tag{6.4}$$

is a well-defined derivation from $\mathcal{O}_{\mathcal{L}}$ into W over $\mathcal{O}_K$. In other words, the map

$$D \mapsto (D(T_1), \ldots, D(\pi_L))$$

defines an isomorphism

$$D_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}, W) \tilde{\to} \underbrace{W \oplus \cdots \oplus W}_{(d-1)-times} \oplus W_{D(L/K)},$$

where $W_{\mathcal{D}(L/K)}$ is the submodule of elements killed by $D(L/K)$.

Proof. The proof follows from the proof of Proposition 6.1.1 and the fact that $D_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}, W) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{L}}}(\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_{\mathcal{L}}), W)$. □

If L is a finite extension of the local field K, then we denote by $\Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K)$ the $\mathcal{O}_K$-module of differentials of $\mathcal{O}_L$ over $\mathcal{O}_K$.

Proposition 6.1.3. *1. $\Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K)$ as $\mathcal{O}_L$-modules. Moreover, the element $d\pi_L$ generates $\Omega_{\mathcal{O}_K}(\mathcal{O}_L)$.*

2. If M is a finite extension of L, the homomorphism $\Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to \Omega_{\mathcal{O}_K}(\mathcal{O}_M)$ is an embedding.

Proof. cf. [14] Proposition 5.1. □

We will denote by K_m (resp. L_m) the field obtained by adjoining the m-th torsion points to K, i.e., $K_m = K(\kappa_m)$. Let v denote the normalized valuation $v_M/v_M(p)$, for every finite extension M of $\mathbb{Q}_p$.

Proposition 6.1.4. *There are positive constants c_1, $c_2 \in \mathbb{R}$, depending on (F, π), such that*

1. $v(D(L_m/L)) \leq m/\alpha + \log_2(m)/(p-1) + c_2$ *and* $v(D(K_m/K)) \geq m/\alpha - c_1$. *Where α is the ramification index of S over $\mathbb{Q}_p$.*

2. *Let $1 \leq j \leq h$ be fixed. Let p_m be the period (i.e., the generator of the annihalator ideal) of the $\mathcal{O}_{K_m}$-submodule of $\Omega_{\mathcal{O}_K}(\mathcal{O}_{K_m})$ generated by de_m^j. Then there exists a j for which $v(p_m) \geq m/\alpha - c_1$.*

Proof. cf. [14] Proposition 5.3. □

Proposition 6.1.5. *Suppose K/S is an unramified extension and let $q = |k_S|$. Let h be the height of F with respect to $C = \mathcal{O}_S$, cf. Proposition 3.2.2. Then*

1. $v(D(K_m/K)) \geq m/\alpha - 1/\alpha(q^h - 1)$.

2. $K_m = K(e_m^i)$ *is totally unramified over K and e_m^i is a uniformizer for K_m and*

$$v(\text{ period of } de_m^i) = m/\alpha - 1/\alpha(q^h - 1).$$

Proof. cf. [14] Proposition 5.6. □

6.2 Multidimensional derivations

Let $\mathcal{L}$, $\tilde{\mathcal{L}}$, L, $\tilde{L}$ K and W be as in the beginning of the previous section.

Definition 6.2.1. *A d dimensional derivation of $\mathcal{O}_L^d$ into W over $\mathcal{O}_K$ is map $D : \mathcal{O}_{\mathcal{L}}^d \to W$ such that for all $a_1 \dots, a_d$ and all $a'_1, \dots, a'_d$ in $\mathcal{O}_{\mathcal{L}}$ it satisfies*

1. *Leibniz rule:*

$$D(a_1, \dots, a_i a'_i, \dots, a_d) = a'_i D(a_1, \dots, a_i, \dots, a_d) + a_i D(a_1, \dots, a'_i, \dots, a_d),$$

for any $1 \leq i \leq d$.

2. *Linearity:*

$$D(a_1, \dots, a_i + a'_i, \dots, a_d) = D(a_1, \dots, a_i, \dots, a_d) + D(a_1, \dots, a'_i, \dots, a_d),$$

for any $1 \leq i \leq d$.

3. *Alternate: $D(a_1, \dots, a_i, \dots, a_j, \dots, a_d) = 0$, if $a_i = a_j$ for $i \neq j$.*

4. *$D(a_1, \dots, a_d) = 0$ if some $a_i \in \mathcal{O}_K$.*

We denote by $D^d_{\mathcal{O}_\mathcal{K}}(\mathcal{O}^d_{\mathcal{L}}, W)$ the $\mathcal{O}_{\mathcal{L}}$-module of all d-dimensional derivations $D : \mathcal{O}^d_{\mathcal{L}} \to W$.

Consider the dth exterior product $\bigwedge^d_{\mathcal{O}_\mathcal{L}} \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L})$ (cf. [18] Chapter 19 §1). This is the $\mathcal{O}_\mathcal{L}$-module $\Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}) \otimes \cdots \otimes \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L})$ divided out by the $\mathcal{O}_\mathcal{L}$-submodule generated by the elements

$$x_1 \otimes \cdots \otimes x_d, \tag{6.5}$$

where $x_i = x_j$ for some $i \neq j$. The symbols $x_1 \otimes \cdots \otimes x_d$ will be denoted by

$$x_1 \wedge \cdots \wedge x_d,$$

instead. For $\bigwedge^d_{\mathcal{O}_\mathcal{L}} \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L})$ we consider the d-dimensional derivation $\mathsf{d} : (\mathcal{O}^d_\mathcal{L}) \to \bigwedge^d_{\mathcal{O}_\mathcal{L}} \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L})$ that sends $(a_1, \ldots, a_d)$ to the wedge product $a_1 \wedge \cdots \wedge a_d$. This $\mathcal{O}_L$-module is the universal object in the category of d-dimensional derivations of $\mathcal{O}_L$ over $\mathcal{O}_K$, i.e,

Proposition 6.2.1. *If $D : (\mathcal{O}_L)^d \to W$ is a d-dimensonal derivation over $\mathcal{O}_K$ then there exist a homomorphism $\beta : \bigwedge^d_{\mathcal{O}_\mathcal{L}} \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}) \to W$ of $\mathcal{O}_L$-modules such that the diagram*

$$\begin{array}{ccc} (\mathcal{O}_\mathcal{L})^d & \xrightarrow{\mathsf{d}} & \bigwedge^d_{\mathcal{O}_\mathcal{L}} \Omega_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}) \\ & \searrow^{D} & \downarrow \beta \\ & & W \end{array}$$

is commutative.

Proof. The homomorhpsim β is clearly the homomorphism defined by $\beta(da_1 \wedge \cdots \wedge$

$da_d) = D(a_1, \ldots, a_d)$. □

Proposition 6.2.2. *$\bigwedge^d_{\mathcal{O}_\mathcal{L}} \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}) \cong (\mathcal{O}_\mathcal{L}/\mathcal{D}(L/K)\mathcal{O}_\mathcal{L})\ dT_1 \wedge \cdots \wedge dT_{d-1} \wedge d\pi_L$ as $\mathcal{O}_\mathcal{L}$-modules, where π_L is a uniformizer for L and $\mathcal{D}(L/K)$ is the different of the extension L/K. That is, $dT_1 \wedge \cdots \wedge dT_{d-1} \wedge d\pi_L$ generates $\bigwedge^d_{\mathcal{O}_\mathcal{L}} \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L})$ as an $\mathcal{O}_\mathcal{L}$-module.*

Proof. This follows immediately from Lemma 6.1.1. □

Proposition 6.2.3. *Let $D \in D^d_K(\mathcal{O}^d_\mathcal{L}, W)$, then $\mathcal{D}(L/K)D(T_1, \ldots, \pi_L) = 0$. Let $\beta_1, \ldots, \beta_d \in \mathcal{O}_\mathcal{L}$, and $g_i(T_1, \ldots, T_{d-1}, T_d) \in \mathcal{O}_{\tilde{\mathcal{L}}}[T_d]$ $i = 1, \ldots, d$, such that $\beta_i = g_i(T_1, \ldots, \pi_L)$. Then*

$$D(\beta_1, \ldots, \beta_d) = \det \left[\frac{\partial \beta_i}{\partial T_j}\right]_{i,j} D(T_1, \ldots, \pi_L), \tag{6.6}$$

where $\frac{\partial}{\partial T_k}$, $k = 1, \ldots, d$ are the derivations from Definition 6.1.2. Moreover, let $w \in W$ such that $\mathcal{D}(L/K)w = 0$, then the map

$$D(\beta_1, \ldots, \beta_d) := \det \left[\frac{\partial \beta_i}{\partial T_j}\right]_{i,j} w,$$

is well-defined and belongs to $D^d_K(\mathcal{O}^d_\mathcal{L}, W)$. In other words, the map

$$D \mapsto D(T_1, \ldots, \pi_L),$$

defines an isomorphism from $D^d_K(\mathcal{O}^d_\mathcal{L}, W)$ to the $\mathcal{D}(L/\tilde{L})$-torsion subgroup of W.

Proof. This follows Proposition 6.2.2 and the fact that

$$D^d_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}, W) \cong \mathrm{Hom}_{\mathcal{O}_K}(\bigwedge^d_{\mathcal{O}_\mathcal{L}} \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_\mathcal{L}), W).$$

□

Chapter 7

Deduction of the formulas

In this section M will denote a local field containing $K_t = K(\kappa_t)$, π_M a unifromizer for M and π_t a uniformizer for K_t.

7.1 Description of the map $\psi^i_{\mathcal{M},m}$ in terms of derivations

Proposition 7.1.1. *Let $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, $M \supset K_t$, with (m,t) admissible. Suppose π^m divides $D(M/K)$; Proposition 6.1.4 guarantees that this happens when $(t-m)/\alpha \geq c_1$. Then the reduction*

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to \frac{R_{\mathcal{M},1}}{\frac{\pi^m}{\pi_M} R_{\mathcal{M},1}}$$

of $D^i_{\mathcal{M},m}$ to $R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$ is a d-dimensional derivation over $\mathcal{O}_K$.

Proof. Let us fix an $a_1, \ldots, a_{d-1} \in \mathcal{O}^*_{\mathcal{M}}$. From Proposition 6.1.2 and the fact

$$D(M/K)D^i_{M,m}(a_1, \ldots, a_{d-1}, \pi_M) = 0 \pmod{\pi^m R_{\mathcal{M},1}},$$

we can to construct, by Proposition 6.1.2, a derivation

$$D : \mathcal{O}_M \to R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1},$$

such that $D(\pi_M) = D^i_{\mathcal{M},m}(a_1, \ldots, a_{d-1}, \pi_M)$ and $D(T_k) = D^i_{\mathcal{M},m}(a_1, \ldots, a_{d-1}, T_k)$, $k = 1, \ldots, d-1$, in the following way

$$D(\alpha) = \sum_{k=1}^{d-1} \frac{\partial \alpha}{\partial T_k} D(T_k) + \frac{\partial \alpha}{\partial T_d} D(\pi_M),$$

where $\alpha \in \mathcal{O}_{\mathcal{M}}$.

According to Proposition 5.4.5 both D and $D^i_{\mathcal{M},m}(a_1, \ldots, a_{d-1}, \cdot)$ coincide in $\mu_{\mathcal{M}}$. But from the Leibniz rule it follows, by comparing $D(\pi_M x)$ and $D^i_{\mathcal{M},m}(a_1, \ldots, a_{d-1}, \pi_M x)$ when $x \in \mathcal{O}_{\mathcal{M}}$, that they coincide $\pmod{(\pi^m/\pi_M) R_{\mathcal{M},1}}$ in all $\mathcal{O}_{\mathcal{M}}$.

It follows now that

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to \frac{R_{\mathcal{M},1}}{\frac{\pi^m}{\pi_M} R_{\mathcal{M},1}}$$

satisfies all conditions from definition 6.2.1 and so by Proposition 6.2.3 we have that it is a d-dimensional derivation such that

$$D^i_{\mathcal{M},m}(\beta_1, \ldots, \beta_d) = \det \left[\frac{\partial \beta_i}{\partial T_j} \right]_{i,j} D_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, \pi_M),$$

where $\beta_1, \ldots, \beta_d \in \mathcal{O}_{\mathcal{M}}$.

□

We can express the map $\psi^i_{\mathcal{M},m}$ out of $D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$ in the following way

$$\begin{cases} \psi^i_{\mathcal{M},m}(u_1, \ldots, u_{d-1}, \pi_M) = \dfrac{D^i_{\mathcal{M},m}(u_1, \ldots, u_{d-1}, \pi_M)}{u_1 \cdots u_{d-1}\pi_M} \pmod{\dfrac{\pi^m}{\pi_M^2} R_{\mathcal{M},1}} \\ \psi^i_{\mathcal{M},m}(u_1, \ldots, u_d) = \dfrac{D^i_{\mathcal{M},m}(u_1, \ldots, u_d)}{u_1 \cdots u_d} \pmod{\dfrac{\pi^m}{\pi_M} R_{\mathcal{M},1}} \\ \psi^i_{\mathcal{M},m}(u_1, \ldots, \pi_M^k u_d) = k\psi^i_{\mathcal{M},m}(u_1, \ldots, \pi_M) + \psi^i_{\mathcal{M},m}(u_1, \ldots, u_d),\ k \in \mathbb{Z} \\ \psi^i_{\mathcal{M},m}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0, \text{whenever } a_i = a_j \text{ for } i \neq j,\ a_1 \ldots, a_d \in \mathcal{M}^* \end{cases} \tag{7.1}$$

where $u_1, \ldots, u_d$ are in $\mathcal{O}^*_{\mathcal{M}} = \{x \in \mathcal{O}_{\mathcal{M}} : v_{\mathcal{M}}(x) = 0\}$. It is clear form the definition that this is independent from the choice of a uniformizer π_M of M. Notice that the fourth property says that $\psi^i_{\mathcal{M},m}$ is alternate, in particular it is skew-symmetric, i.e.,

$$\psi^i_{\mathcal{M},m}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -\psi^i_{\mathcal{M},m}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d).$$

whenever $i \neq j$.

Let $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and v denote the normalized valuation $v_{\mathcal{L}}/v_{\mathcal{L}}(p)$. Let $\kappa_L = \kappa \cap L$, i.e., the subgroup of torsion points contained in $\mathcal{L}$. We denoted by $T_{\mathcal{L}}$ be the image of $F(\mu_{\mathcal{L}})$ under the logarithm of the formal group F. Then we have a continuous isomorphism $l_F : F(\mu_{\mathcal{L}})/\kappa_L \to T_{\mathcal{L}}$ and by Lemma 5.1.3 we have the embedding

$$R_{\mathcal{L}}/\pi^n R_{\mathcal{L}} \to \mathrm{Hom}_C^{seq}(T_{\mathcal{L}}, C/\pi^n C) \tilde{\to} \mathrm{Hom}_C^{seq}(F(\mu_{\mathcal{L}})/\kappa_L, C/\pi^n C). \qquad (7.2)$$

Proposition 7.1.2. *Let m such that $v(f^{(m-n)}(x)) > 1/(p-1)$ for all $x \in F(\mu_{\mathcal{L}})$. Let (m,t) be admissible and put $\mathcal{M} = L_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Then $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}$ induces a homomorphism from $R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1}$ to $\mathcal{L}/\pi^n R_{\mathcal{L}}$ and we have the representation*

$$(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}\left(\ Tr_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha)) l_F(x)\ \right), \quad \forall \alpha \in K_d(\mathcal{M}), \forall x \in F(\mu_{\mathcal{L}}). \qquad (7.3)$$

In particular, $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha))$ belongs to $R_{\mathcal{L}}/\pi^n R_{\mathcal{L}}$ and it is the unique element satisfying (7.3).

Proof. Since $e^i_n = f^{(m-n)}(e^i_m)$ and $f^{(m-n)}(x) \in \mu_{\mathcal{L},1} \subset \mu_{\mathcal{M},1}$, then

$$\begin{aligned}(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L},n} &= \frac{1}{\pi^{m-n}}(\alpha, f^{(m-n)}(x))^i_{\mathcal{M},m} \\ &= \frac{1}{\pi^{m-n}}\mathbb{T}_{\mathcal{M}/S}\left(\psi^i_{\mathcal{M},m}(\alpha)\ l(f^{(m-n)}(x))\right) \\ &= \mathbb{T}_{\mathcal{L}/S}\left(\ \left[\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha))\right]\ l(x)\ \right)\end{aligned}$$

From the condition on m we have that $\pi^{m-n}T_{\mathcal{L}} \subset T_{\mathcal{L},1}$. Thus, after taking duals with respect to $\mathbb{T}_{\mathcal{L}/S}$ we obtain

$$\frac{1}{\pi^{m-n}}R_{\mathcal{L}} \supset R_{\mathcal{L},1}$$

or,

$$\pi^n R_{\mathcal{L}} \supset \pi^m R_{\mathcal{L},1}$$

Then from $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}) \subset R_{\mathcal{L},1}$, cf. Proposition 5.2.2 (1), it follows that

$$\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\pi^m R_{\mathcal{M},1}) \subset \pi^m R_{\mathcal{L},1} \subset \pi^n R_{\mathcal{L}}. \tag{7.4}$$

It then follows that $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha)) \in L/\pi^n R_{\mathcal{L}}$. Moreover, since $(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L},n}$ belongs to $C/\pi^n C$ then $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha)) \in R_{\mathcal{L}}/\pi^n R_{\mathcal{L}}$. The uniqueness follows from Lemma 5.1.3.

□

Remark 7.1.1. For m in Proposition 7.1.2 we can take any

$$m > n + \alpha \log_p(\frac{v_L(p)}{p-1}) + \alpha \frac{1}{p-1}.$$

This follows the same proof of Proposition 6.2 in [14]. Indeed, let $k = m - n$. Then $f^{(k)\prime}$ is divisible by π^k, which implies that every term $a_i X^i$ of the series $f^{(k)}$ satisfies $v(a_i) + v(i) \geq k/\alpha$. If $v(a_i) > 1/(p-1)$ then there is nothing to prove. If on the other hand $v(a_i) \leq 1/(p-1)$ then $v(i) \geq k/\alpha - 1/(p-1)$. In this case

$$v(x^i) \geq \frac{i}{v_L(p)} \geq \frac{p^{\frac{k}{\alpha} - \frac{1}{p-1}}}{v_L(p)} > \frac{1}{p-1}$$

for all $x \in \mu_{\mathcal{L}}$, since $k/\alpha - 1/(p-1) > \log_p(v_L(p)/(p-1))$. Then

$$v(f^{(k)}(x)) > \frac{1}{p-1}$$

for all $x \in \mu_{\mathcal{L}}$.

7.2 Invariants of the representation τ

Recall that we have fixed a basis $\{e^i\}_{i=1}^h$ for $\varprojlim \kappa_n$. We also denoted by e^i_n the reduction of e^i to κ_n. Clearly $\{e^i_n\}$ is a basis for κ_n.

Let M be a finite extension of K and let $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Denote by $T(\kappa)$ be the Tate-module $\varprojlim \kappa_n$. The action of $G_{\mathcal{M}} = Gal(\overline{\mathcal{M}}/\mathcal{M})$ on $T(\kappa)$ defines a continuous representation

$$\tau : G_{\mathcal{M}} \to GL_h(C) \tag{7.5}$$

Let $\mathcal{M}_n = M_n\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. The reduction of τ to $GL_h(C/\pi^n C)$ is the analogous representation of $G_{\mathcal{M}}$ on κ_n and will be denoted by τ_n. This clearly induces an embedding $\tau_n : G(\mathcal{M}_n/\mathcal{M}) \to GL_h(C/\pi^n C)$.

Assume $M \supset K_t$ where $\beta = (m,t)$ is admissible. If $a \in \mathcal{M}^*$, then

$$\tau_{m+t}(\Upsilon_{\mathcal{M}}(\{T_1, \ldots, T_{d-1}, a\}))$$

is congruent to the identity matrix $I \pmod{\pi^t}$ because the Galois group $G(\mathcal{M}_{m+t}/\mathcal{M})$

fixes κ_t and so correspond in $GL_h(C/\pi^{m+t}C)$ to matrices $\equiv I \pmod{\pi^t}$, i.e., there exist characters $\chi_{\mathcal{M},\beta:i,j} : \mathcal{M}^* \to C/\pi^m C$ such that

$$\tau_{m+t}(\Upsilon_{\mathcal{M}}(\{T_1,\dots,T_{d-1},a\})) = I + \pi^t(\chi_{\mathcal{M},\beta:i,j}(a)) \in GL_h(C/\pi^{m+t}C).$$

For $M = K_t$ we simply write $\chi_{\beta:i,j}$. By Proposition 4.1.2 (4) we have that

$$N_{\mathcal{M}/\mathcal{K}_t}\{T_1,\dots,T_{d-1},a\} = \{T_1,\dots,T_{d-1},N_{\mathcal{M}/\mathcal{K}_t}(a)\},$$

where $\mathcal{K}_t = K_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, and Proposition 4.2.1 (4) implies

$$\chi_{\mathcal{M},\beta:i,j}(a) = \chi_{\beta:i,j}(N_{\mathcal{M}/\mathcal{K}_t}(a)) \tag{7.6}$$

The definition of the pairing $(,)_{\mathcal{M},m}$ implies, for $v \in \kappa_t$, that

$$(\ (\{T_1,\dots,T_{d-1},a\},v)^i_{\mathcal{M},m}\ ) = (\chi_{\mathcal{M},\beta:i,j}(a))\ (v^j),$$

here (v^j) are the coordinates of v. In particular, for $v = e_t^j$ we have

$$(\{T_1,\dots,T_{d-1},a\},e_t^j)^i_{\mathcal{M},m} = \chi_{\mathcal{M},\beta:i,j}(a).$$

According to Proposition 5.3.1 we see that $\chi_{\mathcal{M}:i,j}$ uniquely determines on $V_{\mathcal{M},1}$ a constant $c_{\mathcal{M},\beta:i,j} \in R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1}$ such that

$$\chi_{\mathcal{M},\beta:i,j}(u) = \mathbb{T}_{\mathcal{M}/S}(\log(u)c_{\mathcal{M},\beta:i,j}) \quad \forall u \in V_{\mathcal{M},1}. \tag{7.7}$$

Namely, $\rho^i_{\mathcal{M},m}(T_1,\dots,T_{d-1},e_t^j) = c_{\mathcal{M},\beta:i,j}$. Observe that by equation (7.6) $c_{\mathcal{M},\beta:i,j}$ is the image of $c_{\beta,i,j} := c_{\mathcal{K}_t,\beta:i,j}$, where $\mathcal{K}_t = K_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, under the map

$$R_{\mathcal{K}_t,1}/\pi^m R_{\mathcal{K}_t,1} \to R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1}$$

$(R_{\mathcal{K}_t,1} \subset R_{\mathcal{M},1})$. So we wil denote $c_{\mathcal{M},\beta:i,j}$ by $\overline{c}_{\beta:i,j}$.

Finally, observe that $c_{\beta:i,j}$ is an invariant of the isomorphism class of (F, e_j) because if $g : (F, e_j) \to (\tilde{F}, \tilde{e}_j)$ is such isomorphism then $\tilde{\rho}^i_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, g(x)) = \rho^i_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, x)$.

From Proposition 5.3.2 we conclude that

Proposition 7.2.1. *Let $M \supset K_t$, (m, t) admissible and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. If $r(X)$ is a t-normalized series for F, then*

$$\begin{aligned} D^i_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, r(e^j_t)) &= \psi^i_{\mathcal{M},m}(T_1, \ldots, T_{d-1}, r(e^j_t))\ r(e^j_t)\ T_1 \cdots T_{d-1} \\ &= -r'(e^j_t)\ T_1 \cdots T_{d-1}\ \frac{\overline{c}_{\beta:i,j}}{l'(e^j_t)}. \end{aligned} \tag{7.8}$$

Let L be a local field and let $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Define

$$R'_{\mathcal{L},1} := \left\{ x \in \mathcal{L}\ :\ v_{\mathcal{L}}(x) \geq -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1}\right] \right\}. \tag{7.9}$$

Note that

$$v_{\mathcal{L}}(x) \geq -v_L(D(L/S)) - \left[\frac{v_L(p)}{p-1}\right],$$

if and only if

$$v_{\mathcal{L}}(x) \geq -v_L(D(L/S)) - \left(\frac{v_L(p)}{p-1}\right).$$

This holds since $v_{\mathcal{L}}(x)$ and $v_L(D(L/S))$ are integers, therefore the conditions

$$\left[\frac{v_L(p)}{p-1}\right] \geq -v_L(D(L/S)) - v_{\mathcal{L}}(x)$$

and

$$\left(\frac{v_L(p)}{p-1}\right) \geq -v_L(D(L/S)) - v_{\mathcal{L}}(x)$$

are equivalent by the very definition of the integral part of a real number.

Comparing with equations (7.9) and (5.4) we see that $R'_{\mathcal{L},1} = \pi_L R_{\mathcal{L},1}$.

If M/L is a finite extension of local fields and we put $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, then clearly

$$R'_{\mathcal{M},1} = (1/D(M/L))R'_{\mathcal{L},1}. \tag{7.10}$$

Proposition 7.2.2. *Let $\beta = (k,t)$ be an admissible pair such that $\pi^k | D(K_t/K)$; this happens for example when $(t-k)/\alpha \geq c_1$; this c_1 is the constant from Proposition 6.1.4. Let $a_j \in \mathcal{O}_{K_t}$ such that $de_t^j = a_j d\pi_t$ in $\Omega_{\mathcal{O}_K}(\mathcal{O}_{K_t})$; π_t is a uniformizer for K_t. Then*

$$c_{\beta:i,j} \in \frac{a_j R'_{\mathcal{K}_t,1} + \frac{\pi^k}{\pi_t} R'_{\mathcal{K}_t,1} + \pi^k R_{\mathcal{K}_t,1}}{\pi^k R_{\mathcal{K}_t,1}},$$

where $\mathcal{K}_t = K_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$.

Proof. Following the proof of Proposition 6.5 of [14]. We begin by taking a representative $\lambda_{i,j}$ of $c_{\beta:i,j}$ in $R_{\mathcal{K}_t,1}$. We have to show that

$$\lambda_{i,j} \in a_j R'_{\mathcal{K}_t,1} + \left(\frac{\pi^k}{\pi_t}\right) R'_{\mathcal{K}_t,1}. \tag{7.11}$$

Let $M \supset K_t$, π_M and π_t uniformizers for M and K_t, respectively, and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, $\mathcal{K}_t = \mathcal{K}_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let $b \in \mathcal{O}_M$ such that $d\pi_t = bd\pi_M$; this exist by Proposition 6.1.3. Then $D(M/K_t) = b\mathcal{O}_M$. Set $\beta_j = ba_j \in \mathcal{O}_M$. Clearly, $de_t^j = \beta_j d\pi_M$. By Proposition 7.1.1,

$$D^i_{\mathcal{M},k} : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$$

is a d-dimensional derivation over $\mathcal{O}_K$, which together with Proposition 7.2.1 implies

$$\begin{aligned} r'(e_t^j)\ \beta_j\ D^i_{\mathcal{M},k}(T_1,\ldots,T_{d-1},\pi_M) &= D^i_{\mathcal{M},k}(T_1,\ldots,T_{d-1},r(e_t^j)) \\ &= -r'(e_t^j)\ T_1\cdots T_{d-1}\ \frac{\overline{c}_{\beta:i,j}}{l'(e_t^j)} \pmod{(\pi^k/\pi_M)R_{\mathcal{M},1}}. \end{aligned}$$

Recall that $\overline{c}_{\beta:i,j}$ is the image of $c_{\beta:i,j}$ under the map $R_{\mathcal{K}_t,1}/\pi^k R_{\mathcal{K}_t,1} \to R_{\mathcal{M},1}/\pi^k R_{\mathcal{M},1}$; $R_{\mathcal{K}_t,1} \subset R_{\mathcal{M},1}$. This identity implies

$$\lambda_{i,j} \in \beta_j R_{\mathcal{M},1} + \left(\frac{\pi^k}{\pi_M}\right) R_{\mathcal{M},1}. \tag{7.12}$$

Then

$$v_{\mathcal{M}}(\lambda_{i,j}) \geq \min\{\ v_{\mathcal{M}}(\beta_j R_{\mathcal{M},1})\ ,\ v_{\mathcal{M}}\big(\ (\pi^k/\pi_M) R_{\mathcal{M},1}\ \big)\ \}$$
$$\geq \min\left\{ v_{\mathcal{M}}(\beta_j) - v_{\mathcal{M}}(D(M/S)) - \frac{e(M)}{p-1} - 1\ ,\right.$$
$$\left. v_{\mathcal{M}}(\frac{\pi^k}{\pi_M}) - v_{\mathcal{M}}(D(M/S)) - \frac{e(M)}{p-1} - 1\ \right\}$$

We will further assume that M is the local field obtained by adjoining to K_t the roots of the Eisenstein polynomial $X^n - \pi_t$, $(n,p) = 1$. Then $e(M) = ne(K_t)$ and $D(M/K_t) = n\pi_M^{n-1} = \pi_t/\pi_M$.

$$v_{\mathcal{M}}(\lambda_{i,j}) \geq \min\{\ v_{\mathcal{M}}(a_j) - v_{\mathcal{M}}(D(K_t/S)) - \frac{e(M)}{p-1} - 1\ ,$$
$$v_{\mathcal{M}}(\frac{\pi^k}{\pi_t}) - v_{\mathcal{M}}(D(K_t/S)) - \frac{e(M)}{p-1} - 1\ \}$$

Dividing everything by $e(M/K_t) = n$ and noticing that $v_{\mathcal{M}}(x) = e(M/K_t)v_{\mathcal{K}_t}(x)$ for $x \in \mathcal{K}_t$ we obtain

$$v_{\mathcal{K}_t}(\lambda_{i,j}) \geq \min\{\ v_{\mathcal{K}_t}(a_j) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n}\ ,$$
$$v_{\mathcal{K}_t}(\frac{\pi^k}{\pi_{K_t}}) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n}\ \}$$

Letting $n \to \infty$ we obtain

$$v_{\mathcal{K}_t}(\lambda_{i,j}) \geq \min\{\ v_{\mathcal{K}_t}(a_j) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1}\ ,$$
$$v_{\mathcal{K}_t}(\frac{\pi^k}{\pi_t}) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1}\ \}$$

which implies (7.11). □

Lemma 7.2.1. *Assume $\pi^k | D(K_t/K)$, $\beta = (k,t)$ admissible, and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ with $M \supset K_t$. Moreover, assume that j is such that the inequality in Proposition 6.1.5 (2) holds. Then there exist a d-dimensional derivation over $\mathcal{O}_K$*

$$D : \mathcal{O}_{\mathcal{M}}^d \to \frac{R_{\mathcal{M},1}}{\frac{\pi^k}{\pi_M} R_{\mathcal{M},1}}, \tag{7.13}$$

such that $D(T_1, \ldots, T_{d-1}, e_t^j) = -T_1 \cdots T_{d-1}\ \bar{c}_{\beta:i,j}/l'(e_t^j)$. Moreover, all such derivations coincide when reduced to

$$R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1},$$

for $m \le k$, if $m/\alpha \le k/\alpha - v(D(M/K)) + t/\alpha - c_1$ (c_1 is the constant from Proposition 6.1.4). Here v denotes normalized valuation $v_{\mathcal{M}}/v_{\mathcal{M}}(p)$. In particular, they coincide with the reduction of $D^i_{\mathcal{M},k} : \mathcal{O}_{\mathcal{M}}^d \to R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$ to $R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$.

Proof. Let $\beta_j \in \mathcal{O}_M$ such that $de_t^j = \beta_j d\pi_M$ in $\Omega_{\mathcal{O}_K}(\mathcal{O}_M)$; this β_i exists by Proposition 6.1.3. We shall prove that there exist a $\gamma_j \in R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$ such that $-\bar{c}_{\beta:i,j}/l'(e_t^j) = \beta_j\gamma_j$, in particular $-T_1 \cdots T_{d-1}\ \bar{c}_{i,j}/l'(e_j^t) = \beta_j\ T_1 \cdots T_{d-1}\ \gamma_j$. Then using Proposition 6.2.3 we define

$$D(\alpha_1, \ldots, \alpha_d) := \det\left[\frac{\partial \alpha_i}{\partial T_j}\right]_{1 \le i,j \le d} T_1 \cdots T_{d-1}\ \gamma_j,$$

where $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_{\mathcal{M}}$, which is the required multidimensional derivation since

$D(T_1,\ldots,T_{d-1},e_t^j) = \beta_j T_1\cdots T_{d-1}\gamma_j = -T_1\cdots T_{d-1}\ \bar{c}_{\beta:i,j}/l'(e_t^j)$.

Let λ_j be a representative for $c_{\beta:i,j}$ in $R_{\mathcal{K}_t,1}$. By Proposition 7.2.2, $\lambda_j \in a_j R'_{\mathcal{K}_t,1} + (\pi^k/\pi_t)R'_{\mathcal{K}_t,1}$ where $de_j^t = a_j d\pi_t$. Let $d\pi_t = bd\pi_M$, in particular $D(M/K_t) = b\mathcal{O}_M$, and set $\beta_j = a_j b$. We have by (7.10) that

$$\frac{1}{b}R'_{\mathcal{K}_t,1} = (1/D(M/K_t))R'_{\mathcal{K}_t,1} = R'_{\mathcal{M},1} \subset R_{\mathcal{M},1},$$

$$\frac{\pi^k}{\pi_t}R'_{\mathcal{K}_t,1} \subset \frac{\pi^k}{\pi_M}\frac{D(M/K_t)}{\pi_t/\pi_M}R'_{\mathcal{M},1} \subset \frac{\pi^k}{\pi_M}R_{\mathcal{M},1},$$

where the last inclusion follows since $D(M/K_t)$ is divisible by π_t/π_M; this follows from the general inequality $v_M(D(M/K_t)) \geq e(M/K_t) - 1$ (cf. [2] chap 1 prop 5.4). It thus follows

$$-\frac{c_{\beta:i,j}}{l'(e_t^j)} = -\frac{\lambda_j}{l'(e_t^j)} \in \beta_j \frac{R_{\mathcal{M},1}}{(\pi^k/\pi_M)R_{\mathcal{M},1}}.$$

Now let us prove the second assertion. Since $-c_{\beta:i,j}/l'(e_t^j) = b_j\gamma_j$, then γ_j is uniquely defined $\pmod{(\pi^k/\pi_M\beta_j)R_{\mathcal{M},1}}$. Let $m \leq k$, then the d-dimensional derivation over $\mathcal{O}_K$

$$D : \mathcal{O}_{\mathcal{M}}^d \to \frac{R_{\mathcal{M},1}}{\frac{\pi^m}{\pi_M}R_{\mathcal{M},1}}$$

such that $D(T_1,\ldots,T_{d-1},\pi_M) = T_1\cdots T_{d-1}\ \overline{\gamma_j}$ is uniquely determined if $\pi^m|(\pi^k/\beta_j)$. Under this condition it follows now that the reduction of all the derivations (7.13) to $R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$ coincide.

Finally, the condition $\pi^m|(\pi^k/\beta_j)$ is fulfill when

$$m/\alpha \leq k/\alpha - v(D(M/K)) + t/\alpha - c_1.$$

Indeed, $de_t^j = \beta_j d\pi_M$ implies that $v(p_t) + v(\beta_j) = v(D(M/K))$, with p_t as in Proposition 6.1.4 (2), thus by this same proposition

$$\begin{aligned} m/\alpha &\leq k/\alpha - v(D(M/K)) + t/\alpha - c_1 \\ &\leq k/\alpha - v(D(M/K)) + v(p_t) \\ &= k/\alpha - v(\beta_j) \end{aligned}$$

which is precisely the condition $\pi^m|(\pi^k/\beta_j)$. □

Proposition 7.2.3. *Let $M \supset K_t$ and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, with $\beta = (k,t)$ admissible and $(t-k)/\alpha \geq c_1$; c_1 the constant from Proposition 6.1.4. Let $1 \leq j \leq h$ be as in Lemma 7.2.1. Let $m \leq k$ such that*

$$m/\alpha \leq k/\alpha + t/\alpha - v(D(M/K)) - c_1.$$

Then the reduction of

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1}$$

to $R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$ is a d-dimensional derivation over $\mathcal{O}_K$. Moreover, this d-dimensional derivation can be explicitly constructed as follows. Let $de_t^j = \beta_1 d\pi_M$

in $\Omega_{\mathcal{O}_K}(\mathcal{O}_M)$*, so that there exists a uniquely determined* $\overline{\gamma}_i \in R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$ *such that that there is a lifting* γ_i *of* $\overline{\gamma}_i$ *in* $R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$ *for which* $\beta_1\gamma_i = -\overline{c}_{\beta:i,j}/l'(e_t^j)$*. Then*

$$D^i_{\mathcal{M},m}(T_1,\ldots,T_{d-1},\pi_M) = T_1\cdots T_{d-1}\ \overline{\gamma}_i$$

and

$$D^i_{\mathcal{M},m}(\alpha_1,\ldots,\alpha_d) := \det\left[\frac{\partial\alpha_i}{\partial T_j}\right]_{1\le i,j\le d} T_1\cdots T_{d-1}\ \overline{\gamma}_i,$$

where $\alpha_1,\ldots,\alpha_d \in \mathcal{O}_{\mathcal{M}}$ *and* $\frac{\partial}{\partial T_k}$, $1 \le k \le d$*, are the derivations from Definition 6.1.2. In particular,*

$$D^i_{\mathcal{M},m}(T_1,\ldots,T_{d-1},e_t^a) = -T_1\cdots T_{d-1}\overline{c}_{\beta:i,j}/l'(e_t^a), \quad \textit{for all } 1\le a\le h.$$

Proof. By Proposition 5.2.2 (3),

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}$$

is the reduction of

$$D^i_{\mathcal{M},k} : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$$

and the latter is a d-dimensional derivation over $\mathcal{O}_K$ according to Proposition 7.1.1 and

$$D^i_{\mathcal{M},k}(T_1,\ldots,T_{d-1},e_t^j) = -T_1\cdots T_{d-1}\ \overline{c}_{\beta:i,j}/l'(e_t^j)$$

according Proposition 7.2.1. Finally, by Proposition 6.1.4 the condition $(t-k)/\alpha \ge c_1$

implies that $\pi^k | D(K_t/K)$, and Lemma 7.2.1 provides a way of constructing explicitly a d-dimensional derivation

$$D : \mathcal{O}^d_{\mathcal{M}} \to R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$$

with $D(T_1, \dots, T_{d-1}, e^j_t) = -T_1 \cdots T_{d-1}\overline{c}_{\beta:i,j}/l'(e^j_t)$; there exist a $\gamma_j \in R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}$, by the proof of Lemma 7.2.1, such that $-\overline{c}_{\beta:i,j}/l'(e^j_t) = \beta_j\gamma_j$ and we define

$$D(\alpha_1 \dots, \alpha_d) := \det \left[\frac{\partial \alpha_i}{\partial T_j}\right]_{1\le i,j\le d} T_1 \cdots T_{d-1}\ \gamma_j,$$

where $\alpha_1, \dots, \alpha_d \in \mathcal{O}_{\mathcal{M}}$. By this same lemma both D and $D^i_{\mathcal{M},k}$ coincide with $D^i_{\mathcal{M},m}$ when reduced to

$$R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}.$$

The last statement holds because $D^i_{\mathcal{M},k}$ is a multidimensional derivation and the formula for $D^i_{\mathcal{M},k}(T_1, \dots, T_{d-1}, r(e^a_t))$ in equation (7.8). □

7.3 Main formulas

Theorem 7.3.1. *Let $L \supset K_n$ and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let*

$$m = n + 2 + \left[\ \alpha \log_p\left(\frac{v_L(p)}{p-1}\right) + \frac{\alpha}{p-1}\ \right], \tag{7.14}$$

Take k large enough such that

$$t/\alpha + k/\alpha \geq m/\alpha + c_1 + v(D(M/K)) \tag{7.15}$$

$$k + \alpha + 1 \geq c_1\alpha, \tag{7.16}$$

where $t = 2k + \alpha + 1$, $M = L_t$ and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. This happens for example when

$$k/\alpha \geq m/\alpha + \log_2(2k + \alpha + 1)/(p-1) + c_1 + c_2 + v(D(L/K)), \tag{7.17}$$

where c_1 and c_2 are the constants from Proposition 6.1.4. Then $\mathcal{M}$, t, k, and m satisfy Proposition 7.2.3. Let

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \to \frac{R_{\mathcal{M},1}}{\frac{\pi^m}{\pi_M} R_{\mathcal{M},1}} \tag{7.18}$$

be the d-dimensional derivation constructed as in Proposition 7.2.3, and let

$$\psi^i_{\mathcal{M},m} : K_d(\mathcal{M}) \to \frac{R_{\mathcal{M},1}}{\frac{\pi^m}{\pi_M^2} R_{\mathcal{M},1}} \tag{7.19}$$

be the multidimensional logarithmic derivative built out of $D^i_{\mathcal{M},m}$ by equation (7.1). Then

$$(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}\left(\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha))\ l(x)\right) = \mathbb{T}_{\mathcal{M}/S}\left(\psi^i_{\mathcal{M},m}(\alpha) l(x)\right), \tag{7.20}$$

for all $\alpha \in K_d(\mathcal{M})$ and all $x \in F(\mu_{\mathcal{L}})$.

Proof. By considering the tower $L_t \subset L \subset K$ and the upperbound in Proposition 6.1.4 (1), we get

$$v(D(M/K)) \leq v(L_t/L) + v(D(L/K)) \leq t/\alpha + \frac{\log_2(t)}{p-1} + c_2 + v(D(L/K)).$$

Adding $m/\alpha + c_1$ we obtain, by (7.17), the inequality (7.15). The definition of t clearly implies that (k, t) is admissible and condition (7.16) implies $(t-k)/\alpha \geq c_1$, thus M, $\mathcal{M}$, t, k and m satisfy the hypothesis of Proposition 7.2.3. The result now follows from Proposition 7.1.2, the Remark 7.1.1 and equation (7.1). It remains only to check that $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}((\pi^m/\pi_M^2)R_{\mathcal{M},1}) \subset \pi^n R_{\mathcal{L}}$, so that $\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha))$ is well defined in $R_{\mathcal{L}}/\pi^n R_{\mathcal{L}}$. To do this notice that condition (7.14) implies

$$m - 1 > n + \alpha \log_p(\frac{v_L(p)}{p-1}) + \frac{\alpha}{p-1}$$

and we can apply Remark 7.1.1 to $m-1$ and get, by equation (7.4), that

$$\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}\left(\pi^{m-1}\frac{\pi}{\pi_M^2}R_{\mathcal{M},1}\right) \subset \mathrm{Tr}_{\mathcal{M}\mathcal{L}}(\pi^{m-1}R_{\mathcal{M},1}) \subset \pi^n R_{\mathcal{L}}$$

bearing in mind that $\pi_M^2|\pi$, since $\pi^k|D(M/K)$ implies $e(M/K) > 1$ (i.e., M/K is not unramified). $\square$

We state a simplified version of the theorem above

Theorem 7.3.2. *Let $L \supset K_n$ and $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let $\mathcal{M}$ and m be* *defined as in Theorem 7.3.1. Then*

$$(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L},n} = \mathbb{T}_{\mathcal{L}/S}\left(\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(\alpha))\ l(x)\right) = \mathbb{T}_{\mathcal{M}/S}\left(\psi^i_{\mathcal{M},m}(\alpha) l(x)\right),$$

for all $\alpha \in K_d(\mathcal{M})$ *and all* $x \in F(\mu_{\mathcal{L}})$*. Here* $\psi^i_{\mathcal{M},m}$ *is the explicit multidimensional logarithmic derivative constructed in Theorem 7.3.1.*

Chapter 8

Formulas for the Lubin-Tate formal groups

In this section, guided in much by Section 7 in [14], we will give an explicit description of the corresponding formulas for the case where the formal group is Lubin-Tate. This will include an explicit computation of the invariants defined in equation (7.7) and of the lower bounds for t, m and k in Theorem 7.3.1.

Let $K/\mathbb{Q}_p$ be a local field with ring of integers $\mathcal{O}_K$, π a uniformizer for K, k_K its residue field and $q = |k_K|$. Let Λ_π be the subset of $\mathcal{O}_K[[X]]$ consiting of the series f such that

1. $f(X) \equiv \pi X \pmod{\deg 2}$.
2. $f(X) \equiv X^q \pmod{\pi}$.

Let F_f be the Lubin-Tate formal group such that $[\pi]_{F_f} = f$. In this case we will take $S = K$ and $C = \mathcal{O}_K$. Evidently f has height equal to 1 with respect to to

$C = \mathcal{O}_K$. Thus $\kappa_n \simeq \mathcal{O}_K/\pi^n\mathcal{O}_K$ and $\varprojlim \kappa_n \simeq \mathcal{O}_K$. Let e be a generator for $\varprojlim \kappa_n$ and e_n its reduction in κ_n.

Let $K_n = K(\kappa_n)$ and $K_\infty = K(\kappa)$, where $\kappa = \cup\kappa_n$. Let $\mathcal{K}_n = K_n\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathcal{K} = K\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. The extension K_n/K is totally ramified and e_n is a uniformizer for K_n. Moreover, $[K_n/K] = q^n - q^{n-1}$, then the imbedding τ_n

$$\tau_n : \mathrm{Gal}(K_n\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}/K\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}) \to (\mathcal{O}_K/\pi^n\mathcal{O}_K)^*,$$

which is the map induced by the representation $\tau : G_K \to \mathcal{O}_K^*$ (cf. § 7.1, equation (7.5)), is an isomorphism since $[K_n\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}/K\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}] = [K_n : K] = q^n - q^{n-1}$, $|(\mathcal{O}_K/\pi^n\mathcal{O}_K)^*| = q^n - q^{n-1}$.

Proposition 8.0.1. *Let $w_n = l'(e_n)\ de_n$, $n \geq 1$. Then w_n generate $\Omega_C(\mathcal{O}_{\overline{K}})$ as $\mathcal{O}_{\overline{K}}$-modules. We also have*

$$w_n = \pi w_{n+1}. \tag{8.1}$$

Proof. cf. [14] Proposition 7.9. □

Proposition 8.0.2. $w_n^g = \tau(g)w_n$ *for all* $g \in G(\overline{K}/K)$.

Proof. cf. [14] § 7.2.3. □

Let M be a finite extension of K_s, π_M a uniformizer for M and let $\mathcal{M}$ denote the field $M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let π_1 be a uniformizer for $K_1 = K(\kappa_1)$. Let

$$P_{\mathcal{M}} = (1/(\pi_1 D(M/K)))\mathcal{O}_{\mathcal{M}} = \{x \in \mathcal{M} :\ v_{\mathcal{M}}(x) \geq -v_{\mathcal{M}}(\pi_1 D(M/K))\}.$$

We define a d-dimensional derivation

$$Q_{\mathcal{M},s} : \mathcal{O}_{\mathcal{M}}^d \to P_{\mathcal{M}}/(\pi^s/\pi_1)P_{\mathcal{M}}, \tag{8.2}$$

over $\mathcal{O}_K$, in the following way. Let $b' \in \mathcal{O}_M$ such that $w_s = l'(e_s)de_s = b'd\pi_M$, then $b'\mathcal{O}_M = D(M/K_s)$. Let us put

$$Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M) = \frac{T_1 \cdots T_{d-1}}{b'\pi^s}. \tag{8.3}$$

Clearly, $T_1 \cdots T_{d-1}/(b'\pi^s) \in P_{\mathcal{M}}$. By Proposition 6.1.5 (2) the period of de_s is generated by π^s/π_1, then $D(K_s/K) = \pi^s/\pi_1\mathcal{O}_s$ and so

$$D(M/K) = D(M/K_s)D(K_s/K) = D(M/K_s)(\pi^s/\pi_1). \tag{8.4}$$

Hence $D(M/K)Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M) \in (\pi^s/\pi_1)P_{\mathcal{M}}$. Therefore, by Proposition 6.2.3, $Q_{\mathcal{M},s}$ defines a d-dimensional derivation as follows

$$Q_{\mathcal{M},s}(\alpha_1, \ldots, \alpha_d) := \det\left[\frac{\partial \alpha_i}{\partial T_j}\right]_{1\le i,j\le d} \frac{T_1 \cdots T_{d-1}}{b'\pi^s},$$

where $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_{\mathcal{M}}$. Note that the definition of $Q_{\mathcal{M},s}$ is independent of the choice of uniformizer π_M of M.

Proposition 8.0.3. *Let $M/L/K_s$, $M \supset K_t$, be a finite tower of local fields and let*

$$\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.$$

Suppose that $D(M/L)|\pi^{t-s}$. Then $(\pi^t/\pi_1)P_{\mathcal{M}}$ is contained in $X = (\pi^s/\pi_1)P_{\mathcal{L}}\mathcal{O}_{\mathcal{M}}$ so

$Q_{\mathcal{M},t} \pmod X$ *is well-defined. Let*

$$\alpha : P_{\mathcal{L}}/(\pi^s/\pi_1)P_{\mathcal{L}} \to P_{\mathcal{M}}/X$$

be the injection induced by the composition $P_{\mathcal{L}} \subset P_{\mathcal{M}}$. *Then for* $y \in \mathcal{O}_{\mathcal{L}}^d$

$$\alpha(Q_{\mathcal{L},s}(y)) = Q_{\mathcal{M},t}(y) \pmod X.$$

Proof. The Proposition 8.0.3 and its proof were suggested by Professor V. Kolyvagin. First,

$$\frac{\pi^t}{\pi_1}P_{\mathcal{M}} = \frac{\pi^t}{\pi_1}\frac{1}{\pi_1 D(M/K)} = \frac{\pi^t}{\pi_1}.\frac{1}{\pi_1 D(M/L)D(L/K)} = \frac{\pi^t}{\pi_1}.\frac{\pi^{t-s}}{D(M/L)}.\frac{1}{\pi_1 D(L/K)} \subset X$$

because $(\pi^{t-s}/D(M/L)) \subset \mathcal{O}_{\mathcal{M}}$ by our assumption.

Let a and c in $\mathcal{O}_M$ and $b \in \mathcal{O}_L$ are such that

$$w_t = cd\pi_M,\ w_s = bd\pi_L \quad \text{and} \quad d\pi_L = ad\pi_M.$$

Because $\pi^{t-s}w_t = w_s$ we have $\pi^{t-s}cd\pi_M = bd\pi_M = bad\pi_M$. So

$$\pi^{t-s}c \equiv ba \pmod{D(M/L)}.$$

Dividing this congruence by $\pi^t cb$ and taking into account that $c\mathcal{O}_M = D(M/K_t)$ and $b\mathcal{O}_L = D(L/K_s)$, we have

$$\frac{1}{\pi^s b} \equiv \frac{a}{\pi^t c} \pmod Z,$$

where

$$Z = \frac{D(M/K)}{\pi^t D(M/K_t) D(L/K_s)} = \frac{\mathcal{O}_{\mathcal{M}}}{\pi_1 D(L/K_s)} = \frac{\pi^s}{\pi_1} \frac{\mathcal{O}_{\mathcal{M}}}{\pi_1 D(L/K)} = X$$

(we are using that $D(M/K) = D(M/K_t)D(K_t/K)$ and $D(K_t/K) = \pi^t/\pi_1$, $D(L/K) = (\pi^s/\pi_1)D(L/K_s)$). So

$$Q_{\mathcal{M},t}(T_1, \ldots, T_{d-1}, \pi_L) = \frac{a}{\pi^t c} = \frac{1}{\pi^s b} \pmod{X} = \alpha(Q_{\mathcal{L},s}(T_1, \ldots, T_{d-1}, \pi_L)).$$

This implies the corresponding equality for arbitrary $y \in \mathcal{O}_{\mathcal{L}}^d$ since right hand and left hand mappings are multidimensional derivations of $\mathcal{O}_{\mathcal{L}}^d$ over $\mathcal{O}_K$. □

Proposition 8.0.4. *$Q^g_{\mathcal{M},s} = \tau(g^{-1})Q_{\mathcal{M},s}$. Here $Q^g_{\mathcal{M},s}$ is the d-dimensional derivation defined by $Q^g_{\mathcal{M},s}(a_1, \ldots, a_{d-1}) = [Q_{\mathcal{M},s}(a_1^{g^{-1}}, \ldots, a_d^{g^{-1}})]^g$.*

Proof. Notice that it is enough to check that

$$Q^g_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M) = \tau(g^{-1})Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M),$$

by equation (6.6) of Proposition 6.2.3.

Let $b' \in \mathcal{O}_M$ such that $w_s = b' d\pi_M$. Then $w_s^g = (b')^g d\pi_M^g$ and, by Proposition 8.0.2, we have that $w_s = \tau(g^{-1})(b')^g d\pi_M^g$. Since π_M^g is also a uniformizer for M and the definition of $Q_{\mathcal{M},s}$ is independent of the uniformizer, then

$$Q_{\mathcal{M},s}(T_1, \ldots, T_{d-1}, \pi_M^g) = \frac{1}{\tau(g^{-1})\ (b')^g\ \pi^s},$$

but

$$\frac{1}{\tau(g^{-1})(b')^g\pi^s} = \tau(g)\left(\frac{1}{b'\pi^s}\right)^g.$$

This last expression is equal to $\tau(g)Q_{M,s}(T_1,\ldots,T_{d-1},\pi_M)^g$ by equation (8.3).

□

Proposition 8.0.5. *Let M be a finite extension of K, $M\cap K_\infty = K_s$ and let $N = M_{s+1}$. Put $\mathcal{N} = N\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Then*

$$\tau_{s+1}(G(\mathcal{N}/\mathcal{M})) = \frac{1+\pi^s C}{1+\pi^{s+1}C} \subset (C/\pi^{s+1}C)^*$$

where $C = \mathcal{O}_K$, and the element

$$\sum_{g\in G(\mathcal{N}/\mathcal{M})} (\tau_{s+1}(g)-1)g$$

takes $(\pi/D(N/M))\mathcal{O}_{\mathcal{N}}$ to $(\pi^{s+1}/\pi_1)\mathcal{O}_{\mathcal{N}}$. Also $D(N/M)|\pi$.

Proof. This follows immediately from the fact that $G(N/M)\cong G(\mathcal{N}/\mathcal{M})$ and Proposition 5.12 in [14] and its proof.

□

Let M be a finite extension of K_s and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. From $Q_{M,s}$ we can define the logarithmic derivative

$$QL_{\mathcal{M},s} : K_d(\mathcal{M}) \to \frac{\frac{1}{\pi_M}P_{\mathcal{M}}}{\frac{\pi^s}{\pi_1\pi_M}P_{\mathcal{M}}}$$

by

$$\begin{cases} QL_{\mathcal{M},s}(u_1,\ldots,u_{d-1},\pi_M) = \dfrac{Q_{\mathcal{M},s}(u_1,\ldots,u_{d-1},\pi_M)}{u_1\cdots u_{d-1}\pi_M} \pmod{\dfrac{\pi^s}{\pi_1\pi_M}P_{\mathcal{M}}}, \\ QL_{\mathcal{M},s}(u_1,\ldots,u_d) = \dfrac{Q_{\mathcal{M},s}(u_1,\ldots,u_d)}{u_1\cdots u_d} \pmod{\dfrac{\pi^s}{\pi_1}P_{\mathcal{M}}}, \\ QL_{\mathcal{M},s}(u_1,\ldots,\pi_M^k u_d) = kQL_{\mathcal{M},s}(u_1,\ldots,\pi_M) + QL_{\mathcal{M},s}(u_1,\ldots,u_d),\ k\in\mathbb{Z} \\ QL_{\mathcal{M},s}(a_1,\ldots,a_{d-1}) = 0, \text{ whenever } a_i = a_j \text{ for } i\neq j \text{ and } a_1,\ldots,a_{d-1}\in\mathcal{M}^*. \end{cases} \tag{8.5}$$

where $u_1,\ldots,u_d$ are in $\mathcal{O}_{\mathcal{M}}^* = \{x\in\mathcal{O}_{\mathcal{M}} : v_{\mathcal{M}}(x) = 0\}$. Notice the forth property says that $QL_{\mathcal{M},s}$ is alternate, in particular it is skew-symmetric, i.e,

$$QL_{\mathcal{M},s}(a_1,\ldots,a_i,\ldots,a_j,\ldots,a_d) = -QL_{\mathcal{M},s}(a_1,\ldots,a_j,\ldots,a_i,\ldots,a_d).$$

whenever $i\neq j$.

Let $L\supset K$ and take the smallest r such that $L\cap K_\pi\subset K_r$. Let γ_m be a uniformizer for $L_m = L(\kappa_m)$ and define $\mathcal{L}_m = L_m\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Then

$$L_s\cap K_\pi = (L\cap K_\pi)K_s = K_s \quad (s\geq r).$$

Let $\mathcal{M} = \mathcal{L}_s$. For abbreviation we will write P_s, Q_s, QL_s, $N_{t/s}$ and $\mathrm{Tr}_{t/s}$ instead of

$P_{\mathcal{L}_s}$, $Q_{\mathcal{M},s}$, $QL_{\mathcal{M},s}$, $N_{\mathcal{L}_s/\mathcal{L}_t}$ and $\mathrm{Tr}_{\mathcal{L}_s/\mathcal{L}_t}$. Moreover,

$$\mathrm{Tr}_{t/s}(P_t) \subset P_s \tag{8.6}$$

and

$$\mathrm{Tr}_{t/s}((1/\gamma_t)P_t) \subset (1/\gamma_s)P_s. \tag{8.7}$$

Indeed, since $D(L_t/K) = D(L_t/L_s)D(L_s/K)$ then $P_t = (1/D(L_t/L_s))P_s$ and so

$$\mathrm{Tr}_{t/s}(P_t) = \mathrm{Tr}_{t/s}((1/D(L_t/L_s))P_s) = P_s \mathrm{Tr}_{t/s}(1/D(L_t/L_s)) \subset P_s,$$

and $\mathrm{Tr}_{t/s}((\gamma_s/\gamma_t)P_t) \subset \mathrm{Tr}_{t/s}(P_t) \subset P_s$.

Proposition 8.0.6. *For $s \geq r+1$ and $t \geq s$ we have*

$$QL_s(N_{t/s}(a_1), a_2, \ldots, a_d) = \mathrm{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)) \quad \left(\in \frac{(1/\gamma_s)P_s}{(\pi^s/\pi_1\gamma_s)P_s} \right)$$

for $a_1 \in \mathcal{L}_t^* = L_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}^*$ and $a_2, \ldots, a_d \in \mathcal{L}_s^* = L_s\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}^*$.

Proof. It will be enough to consider the case $t = s+1$. From Proposition 8.0.4 it follows that

$$QL_t(\prod_{g\in G(\mathcal{L}_t/\mathcal{L}_s)} a_1^g, a_2, \ldots, a_d) = \left(\sum \tau_t(g)g\right) QL_t(a_1, \ldots, a_d) \tag{8.8}$$

$$= \left(\sum g\right) QL_t(a_1, \ldots, a_d) \tag{8.9}$$

$$+ \left(\sum (\tau_t(g) - 1)g\right) QL_t(a_1, \ldots, a_d) \tag{8.10}$$

and by Proposition 8.0.5 we see that $\sum(\delta_t(g) - 1)g$ takes

$$P_t = \frac{1}{D(L_t/L_s)} P_s = \frac{\pi}{D(L_t/L_s)} \left(\frac{1}{\pi} P_s\right)$$

to

$$\left(\frac{\pi^t}{\pi_1}\mathcal{O}_{\mathcal{L}_t}\right) \left(\frac{1}{\pi}P_s\right) = \frac{\pi^s}{\pi_1} P_s \mathcal{O}_{\mathcal{L}_t}$$

Then

$$QL_s(N_{t/s}(a_1), a_2, \ldots, a_d) = QL_t(N_{t/s}(a_1), a_2, \ldots, a_d) \pmod{\frac{\pi^s}{\pi_1\gamma_s} P_s}$$

$$= \mathrm{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)),$$

where the first equality follows from Proposition 8.0.3. □

8.1 Computations of the invariants

We will assume that $p \neq 2$. Let (m,t) be an admissible pair, $M = K_t$ and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. We can take e_t as a uniformizer π_t of K_t. We will show that

$$\begin{aligned}(\{T_1,\ldots,T_{d-1},u\},\ e_t)_{\mathcal{M},m} &= \mathbb{T}_{\mathcal{M}/K}\left(\log u.\left(-\frac{1}{\pi^t}\right)\right) \pmod{\pi^n C}\\ &= \mathrm{Tr}_{K_t/K}\left(c_{\mathcal{M}/K_t}(\log u).\ \left(-\frac{1}{\pi^t}\right)\right) \pmod{\pi^n C}\end{aligned} \tag{8.11}$$

for all $u \in V_{\mathcal{M},1} = 1+\mu_{\mathcal{M},1}$. From equation (7.7) we have

$$c_\beta = -\frac{1}{\pi^t} \pmod{\pi^m R_{t,1}}. \tag{8.12}$$

Furthermore, every $u \in V_{\mathcal{M},1}$ can be expressed as

$$\prod_{\substack{\bar{i}=(i_1,\ldots,i_d)\in S\\ i_d\geq[v_M(p)/(p-1)]+1}} (1+\theta_{\bar{i}}T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d})$$

where $\theta_{\bar{i}} \in \mathcal{R}$, $\mathcal{R}$ is the group of $q-1$th roots of 1 in K_t^*, and $S \subset \mathbb{Z}^d$ is an admissible set (see Corollary from Section 1.4.3 [7]), then it is enough to check (8.11) for

$$u = 1+\theta T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d}, \quad (\theta^{q-1}=1).$$

Case 1) Suppose $(i_1\ldots,i_{d-1}) \neq (0,\ldots,0)$. Then the right hand side of (8.11) is zero since $c_{\mathcal{M}/K_t}(\log u) = 0$. Let us show that the left hand side is also zero as well.

Suppose first $i_k > 0$ for some $1 \leq k \leq d-1$. Consider $\mathcal{N} = K_t\{\{Y_1\}\}\ldots\{\{Y_{d-1}\}\}$ where $Y_k = T_k^{i_k}$ and $Y_r = T_r$, for $r \neq k$. By lemma 8.1.1 below, $\mathcal{M}/\mathcal{N}$ is a finite extension of degree i_k and $N_{\mathcal{M}/\mathcal{N}}(T_k) = \pm Y_k$. Let also $i'_k = 1$ and $i'_r = i_r$ for $r \neq k$. Therefore by proposition 4.2.1 (4)

$$\begin{aligned}
&\left(\{T_1, \dots, T_{d-1}, 1 + \theta_{\bar{i}} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}\ \pi_t^{i_d}\},\ e_t\right)_{\mathcal{M},m} = \\
&= \left(N_{\mathcal{M}/\mathcal{N}}\{T_1, \dots, T_{d-1}, 1 + \theta_{\bar{i}} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}\ \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} \\
&= \left(\{Y_1, \dots, N_{\mathcal{M}/\mathcal{N}}(T_k), \dots, Y_{d-1},\ 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}} \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} \\
&= \left(\{Y_1, \dots, \pm 1, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}}\ \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} \\
&\quad \oplus \left(\{Y_1, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}} \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m}
\end{aligned}$$

Since $p \neq 2$, $\left(\{Y_1, \dots, \pm 1, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}}\ \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} = 0$. On the other hand, since $\theta^{q-1} = 1$ then

$$\begin{aligned}
&\left(\{Y_1, \dots, \theta, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}} \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} \\
&= [1/(q-1)] \left(\{Y_1, \dots, \theta^{q-1}, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}} \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} = 0
\end{aligned}$$

and also $\left(\{Y_1, \dots, \pi_t, \dots, Y_{d-1}, 1 + \theta Y_1^{i'_1} \cdots Y_{d-1}^{i'_{d-1}} \pi_t^{i_d}\},\ e_t\right)_{\mathcal{N},m} = 0$ by the norm series relation for Lubin-Tate formal groups $(\{a_1, \dots, -X, \dots, a_d\},\ X)_{\mathcal{N},m} = 0$ and recalling that $\pi_t = e_t$. Thus

$$\begin{aligned}
&\Big(\{T_1, \ldots, T_{d-1}, 1+\theta T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{M},m}\\
&=\Big(\{Y_1, \ldots, Y_k, \ldots, Y_{d-1}, 1+\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{N},m}\\
&=\Big(\{Y_1, \ldots, \theta Y_1^{i'_1}\cdots Y_k^{i'_k}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}, \ldots, Y_{d-1}, 1+\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{N},m}\\
&=\Big(\{Y_1, \ldots, -\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}, \ldots, Y_{d-1}, 1+\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{N},m}
\end{aligned}$$

The second equality follows from the fact that $\{Y_1, \ldots, Y_r, \ldots, Y_{d-1}, 1+\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}\}$ is trivial, for $r \neq k$, in the Milnor K-group $K_d(\mathcal{M})$. Moreover, the last expression in the chain of equalities is again zero because $\{Y_1, \ldots, -\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}, \ldots, Y_{d-1}, 1+\theta Y_1^{i'_1}\cdots Y_{d-1}^{i'_{d-1}}\pi_t^{i_d}\}$ is the zero element, by the Steinberg property, in the Milnor K-group $K_d(\mathcal{M})$.

Suppose now $i_k < 0$. We take $Y_k = T_k^{-i_k}$ instead and by lemma 8.1.1 we have $N_{\mathcal{M}/\mathcal{N}}(T_k^{-1}) = \pm T_k^{-i_k} = \pm Y_k$. Noticing that

$$\begin{aligned}
&\Big(\{T_1, \ldots, T_k, \ldots, T_{d-1}, 1+\theta T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{M},m}\\
&=-\Big(\{T_1, \ldots, T_k^{-1}, \ldots, T_{d-1}, 1+\theta T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{M},m}
\end{aligned}$$

we can now apply the same argument as before to conclude that

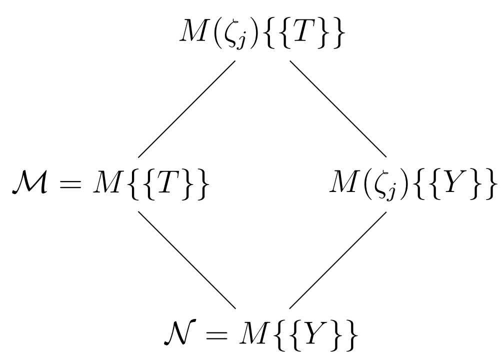
$$\Big(\{T_1, \ldots, T_k, \ldots, T_{d-1}, 1+\theta T_1^{i_1}\cdots T_{d-1}^{i_{d-1}}\pi_t^{i_d}\},\ e_t\Big)_{\mathcal{M},m} = 0.$$

Case 2) Suppose $(i_1 \ldots, i_{d-1}) = (0, \ldots, 0)$. That is, when u is an element of the one dimensional local field K_t. In this case, we will show in lemma 8.1.2 below that the pairing $(\{T_1 \dots, T_{d-1}, u\},\ e_t)_{\mathcal{M},m}$ coincides with the pairing taking values in the one dimensional local field K_t: $(u,\ e_t)_{K_t,m}$. Thus, by [14] section 7.3.1 and the fact that $c_{\mathcal{M}/K}(\log(u)) = \log(u)$ formula (8.11) follows.

Lemma 8.1.1. *Let M be a complete discrete valuation field and $\mathcal{M} = M\{\{T\}\}$. Put $Y = T^j$ for $j > 0$. Define $\mathcal{N} = M\{\{Y\}\}$. Then $\mathcal{M}/\mathcal{N}$ is a finite extension of degree j and $N_{\mathcal{M}/\mathcal{N}}(T) = \pm Y$.*

Remark: Since M is a complete discrete valuation field, the result immedeately generalizes to the d-dimensional case, for if $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, then we can take M to be $L\{\{T_1\}\} \dots \{\{T_{d-2}\}\}$ and apply the result to $\mathcal{M} = M\{\{T_{d-1}\}\}$ and $\mathcal{N} = M\{\{Y_{d-1}\}\}$, where $Y_{d-1} = T_{d-1}^j$.

Proof. We can assume that M contains ζ_j, a primitive jth root of unity of 1. Otherwise we can considering the diagram



we see that $[\mathcal{M}/\mathcal{N}] = [M(\zeta_j)\{\{T\}\} : M(\zeta_j)\{\{Y\}\}]$ since $M\{\{T\}\} \cap (M(e_j)\{\{T\}\}) =$

$M\{\{Y\}\}$.

Note that Y has exact order j in $\mathcal{N}^*/(\mathcal{N}^*)^j$ for if $Y = \alpha^k$, $\alpha \in \mathcal{N}^*$, then $0 = v_{\mathcal{N}}(Y) = kv_{\mathcal{N}}(\alpha)$, thus $\alpha \in \mathcal{O}_{\mathcal{N}}^*$, and we can go to the residue field $k_{\mathcal{N}}$ of $\mathcal{N}$ where we have $1 = v_{k_{\mathcal{N}}}(\overline{Y}) = kv_{k_{\mathcal{N}}}(\overline{\alpha})$, which implies $k = 1$. Then by Kummer theory (cf. [2] Chapter 3 Lemma 2) we have that the polynomial $P(X) = X^j - Y \in \mathcal{O}_{\mathcal{N}}[X]$ is irreducible. Thus $[\mathcal{M}/\mathcal{N}] = j$, and $N_{\mathcal{M}/\mathcal{N}}(T)$ is the product of the roots of the polynomial $P(X)$. These roots are $\zeta_j^k T$, $k = 1, \ldots, j$. Thus

$$N_{\mathcal{M}/\mathcal{N}}(T) = \prod_{k=1}^{j} \zeta_j^k T = \zeta_j^{\frac{j(j+1)}{2}} T^j = \begin{cases} T^j = Y, & \text{if } j \text{ is odd,} \\ -T^j = -Y, & \text{if } j \text{ is even.} \end{cases} = (-1)^{j+1} Y.$$

□

Lemma 8.1.2. *For a local field $L/\mathbb{Q}_p$, let $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and consider the map*

$$f : L^* \to K_d(\mathcal{L}) \to Gal(\mathcal{L}^{ab}/\mathcal{L}) \to Gal(L^{ab}/L), \tag{8.13}$$

defined by

$$a \to \{T_1, \ldots, T_{d-1}, a\} \to \Upsilon_{\mathcal{L}}(\{T_1, \ldots, T_{d-1}, a\}) \to \Upsilon_{\mathcal{L}}(\{T_1, \ldots, T_{d-1}, a\})\Big|_{Gal(L^{ab}/L)}.$$

This coincides with the reciprocity map for L, $\theta_L : L^* \to Gal(L^{ab}/L)$. Thus, for $L = K_t$ and $\mathcal{M} = K_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ we have

$$(\{T_1, \ldots, T_{d-1}, u\},\ e_t)_{\mathcal{M},m} = (u,\ e_t)_{K_t,m}\,,$$

for all $u \in V_{L,1} = \{x \in L : v_L(x-1) > [v_L(p)/(p-1)] + 1\}$.

Proof. It is enough to verify the two conditions of [2] Chapter 5 §2.8 Proposition 6. Let $\mathcal{L}_d$ be $\mathcal{L}$, and $\mathcal{L}_{d-1} = k_{\mathcal{L}((t_1))\cdots((t_{d-1}))}, \ldots, \mathcal{L}_0 = k_{\mathcal{L}}$ the chain of residue fields of $\mathcal{L}$. By (2) of Theorem 4.1.1 we have

$$\begin{array}{ccc}
K_d(\mathcal{L}_d) & \xrightarrow{\Upsilon_{\mathcal{L}_d}} & \mathrm{Gal}(\mathcal{L}_d^{\mathrm{ab}}/\mathcal{L}_d) \\
\partial\downarrow & & \downarrow\sigma\to\overline{\sigma} \\
K_{d-1}(\mathcal{L}_{d-1}) & \xrightarrow[\Upsilon_{\mathcal{L}_{d-1}}]{} & \mathrm{Gal}(\mathcal{L}_{d-1}^{\mathrm{ab}}/\mathcal{L}_{d-1}) \\
\partial\downarrow & & \downarrow\sigma\to\overline{\sigma} \\
\ldots & \longrightarrow & \ldots \\
\partial\downarrow & & \downarrow\sigma\to\overline{\sigma} \\
K_1(\mathcal{L}_1) & \xrightarrow[\Upsilon_{\mathcal{L}_1}]{} & \mathrm{Gal}(\mathcal{L}_1^{\mathrm{ab}}/\mathcal{L}_1) \\
\partial\downarrow & & \downarrow\sigma\to\overline{\sigma} \\
\mathbb{Z} = K_0(\mathcal{L}_0) & \xrightarrow[\Upsilon_{\mathcal{L}_0}]{} & \mathrm{Gal}(\mathcal{L}_0^{\mathrm{ab}}/\mathcal{L}_0)
\end{array}$$

By Remark 4.1.3 the composition of the vertical maps ∂'s we have

$$\partial(\partial\{T_1 \ldots, T_{d-1}, a\}) = v_L(a),$$

thus $f : L^* \to \mathrm{Gal}(L^{ab}/L) \to \mathrm{Gal}(L^{un}/L)$ is the valuation map $v_L : L^* \to \mathbb{Z}$. Thus condition (1) of [2] Chapter 5 §2.8 Proposition 6 is verified.

If $a \in L^*$, L'/L is a finite abelian extension, $\mathcal{L}' = L'\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$, and a is a norm from L'^*, namely $a = N_{L'/L}(\alpha)$, then clearly

$\{T_1, \ldots, T_{d-1}, a\}$ is a norm from $K_d(\mathcal{L}')$, namely

$$\{T_1, \ldots, T_{d-1}, a\} = \{T_1, \ldots, T_{d-1}, N_{L'/L}(\alpha)\} = N_{\mathcal{L}'/\mathcal{L}}\{T_1, \ldots, T_{d-1}, \alpha\},$$

and by (1) of Theorem 4.1.1 we have that $\Upsilon_{L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}}(T_1, \ldots, T_{d-1}, a)$ is trivial on $\mathcal{L}'$ and so $f(a)$ is trivial on L'. Thus condition (2) of [2] Chapter 5 §2.8 Proposition 6 is verified. □

8.2 Explicit formulas for the Lubin-Tate formal groups

In this subsection we will provide a refinement of the formulas given for Theorem 7.3.1 to the case of a Lubin-Tate formal group F_f.

Let us introduce the following simplified notation throughout this subsection. Let L be a finite extension of K_n and put $\mathcal{L} = L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let $M = L_s$ and $\mathcal{M} = L_s\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Then $Q_s = Q_{\mathcal{M},s}$, $QL_s = QL_{\mathcal{M},s}$ and

$$P_s = P_{\mathcal{M}} = 1/(\pi_1 D(M/K))\mathcal{O}_{\mathcal{M}} = \{x \in \mathcal{M} : v_{\mathcal{M}}(x) \geq -v_{\mathcal{M}}(\pi_1) - v_{\mathcal{M}}(D(M/K))\ \},$$

where π_1 is a uniformizer for $K_1 = K(\kappa_1)$. Let $N_s = N_{\mathcal{M}/\mathcal{L}}$ and $\mathrm{Tr}_s = \mathrm{Tr}_{\mathcal{M}/\mathcal{L}}$. Let γ_s denote a uniformizer of $\mathcal{M}$. For $t \geq s$ and $\mathcal{N} = L_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ we define $N_{t/s} = N_{\mathcal{N}/\mathcal{M}}$ and $\mathrm{Tr}_{t/s} = \mathrm{Tr}_{\mathcal{N}/\mathcal{M}}$. Finally, let $\mathcal{L}'_s$ be

$$\bigcap_{t\geq s} N_{t/s}(L_t\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}^*).$$

Theorem 8.2.1. *Let r be maximal such that $L \supset K_r$ and r' minimal such that $L \cap K_\pi \subset K_{r'}$. Let $s \geq \max\{r', n + r + \log_q(e(L/K_r))\}$. Then Tr_s takes $(\pi^s/\pi_1\gamma_s)P_s$ to $\pi^n R_{\mathcal{L}}$ so that it induces a homomorphism*

$$\mathrm{Tr}_s : \frac{\frac{1}{\gamma_s}P_s}{\frac{\pi^s}{\pi_1\gamma_s}P_s} \longrightarrow \mathcal{L}/\pi^n R_{\mathcal{L}}$$

and the following formula holds

$$(\{N_{\mathcal{L}_s/\mathcal{L}}(a_1), a_2, \ldots, a_d\}, x)_{\mathcal{L},n} = \left[\mathbb{T}_{\mathcal{L}/K}\left(\ \mathrm{Tr}_s(QL_s(\{a_1, \ldots, a_d\})\ l_F(x)\ \right)\right]_f(e_n) \tag{8.14}$$

for all $a_1 \in \mathcal{L}'_s = \cap_{t\geq s} N_{t/s}(\mathcal{L}_t)$ and all $a_2, \ldots, a_d \in \mathcal{L}^*$.

Proof. Let $M = L_t$ and $\mathcal{M} = M\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let v be the normalized valuation $v_{\mathcal{M}}/v_{\mathcal{M}}(p)$ and let $R_{t,1} = R_{\mathcal{M},1}$, $P_t = P_{\mathcal{M}}$. Since K_1/K is totally ramified and $[K_1/K] = q-1$ then $\pi_1^{q-1} \sim \gamma_t^{e(M)}$, so π_1 divides $\gamma_t^{e(M)/(p-1)}$ and we have

$$\frac{1}{\gamma_t}P_t = \frac{1}{\gamma_t\pi_1 D(L_t/K)}\mathcal{O}_{\mathcal{L}_t} \subset \frac{1}{\gamma_t^{e(M)/(p-1)+1}D(L_t/K)}\mathcal{O}_{\mathcal{L}_t} = R_{t,1}.$$

If $k < t$, then $\pi_1 | \pi^{t-k}\gamma_t$, which implies $(\pi^k/\gamma_t)|(\pi^t/\pi_1)$ and so

$$\frac{\pi^t}{\pi_1\gamma_t}P_t \subset \frac{\pi^k}{\gamma_t}R_{t,1}.$$

Consider t, k and m as in Theorem 7.3.1. In particular, since $t = 2k + \alpha + 1$ then $k < t$ and therefore we can look at the factorization of $Q_t : (\mathcal{O}^*_{\mathcal{M}})^d \to P_t/(\pi^t/\pi_1))P_t$

into $R_{t,1}/(\pi^k/\gamma_t)R_{t,1}$. This is a d-dimensional derivation such that

$$Q_t(T_1,\ldots,T_{d-1},e_t) = T_1\cdots T_{d-1}/\pi^t l'(e_t) = -T_1\cdots T_{d-1}c_\beta/l'(e_t),$$

by (8.12), where $\beta = (k,t)$ and $c_\beta = c_{\mathcal{M},\beta}$. But from Lemma 7.2.1 we know that all the derivations satisfying such condition must coincide when reduced to $R_{t,1}/(\pi^m/\gamma_t)R_{t,1}$. Therefore the reduction of Q_t to $R_{t,1}/(\pi^m/\gamma_t)R_{t,1}$ coincides with the derivation $D^i_{M,m}$ defined in (7.18) from Theorem 7.3.1. This allows us to replace the logarithmic derivative $\psi_{\mathcal{M},m}$ defined in (7.19) from Theorem 7.3.1 by the logarithmic derivative

$$QL_t : K_d(\mathcal{M}) \to \frac{\frac{1}{\gamma_t}P_t}{\frac{\pi^t}{\pi_1\gamma_t}P_t}$$

defined in (8.5), factored $\pmod{(\pi^m/\pi\gamma_t)P_t}$. That is, for $c \in \mathcal{L}_t^*$ and $a_2,\ldots,a_d \in \mathcal{L}^*$ we have

$$\mathrm{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M},m}(c,a_2,\ldots,a_d)) = \mathrm{Tr}_t(QL_t(c,a_2,\ldots,a_d)) \tag{8.15}$$

But we know by Proposition 8.0.6 that

$$QL_s(N_{t/s}(c),a_2,\ldots,a_d) = \mathrm{Tr}_{t/s}(QL_t(c,a_2,\ldots,a_d) \pmod{\frac{\pi^s}{\pi_1\gamma_s}P_s})$$

Suppose for the moment that $\mathrm{Tr}_s((\pi^s/\pi_1\gamma_s)P_s) \subset \pi^n R_{\mathcal{L}}$, then taking Tr_s we get

$$\begin{aligned}\mathrm{Tr}_t(\ QL_t(c,a_2,\ldots,a_d)\ ) &= \mathrm{Tr}_s(\ \mathrm{Tr}_{t/s}(\ QL_t(c,a_2,\ldots,a_d)\ )\ )\\ &= \mathrm{Tr}_s(\ QL_s(N_{t/s}(c),a_2,\ldots,a_d)\ ) \pmod{\pi^n R_{\mathcal{L}}}\end{aligned}$$

Therefore if $a_1 \in \mathcal{L}'_s$ there exist a $c \in \mathcal{L}^*_t$ such that $N_{t/s}(c) = a_1$ and thus, by (8.15) and Proposition 7.1.2, identity (8.17) follows. It remains to prove that $\mathrm{Tr}_s((\pi^s/\pi_1\gamma_s)P_s) \subset \pi^n R_{\mathcal{L}}$ for $s \geq n + r + \log_q(e(L/K_r))$.

Let $x \in F(\mu_{\mathcal{L}})$. Then $f(x) \equiv x^q \pmod{\pi x}$ implies

$$v(f(x)) \geq \min\{v(x^q),\ v(\pi x)\}. \tag{8.16}$$

Let

$$s' \geq \log_q\left(\frac{e(L/K)}{q-1}\right) + 1 = r + \log_q\left(e(L/K_r)\right).$$

Then $v(f^{(s'-1)}(x)) \geq \min\{v(x^{q^{s'-1}}),\ v(\pi x)\}$. But

$$v(x^{q^{s'-1}}) = q^{s'-1}v(x) = \frac{e(L/K)}{q-1}\frac{v_{\mathcal{L}}(x)}{e(L/K)\alpha} \geq \frac{1}{\alpha(q-1)} = \frac{v(\pi)}{q-1},$$

$$v(\pi x) > v(\pi),$$

so $v(f^{(s'-1)}(x)) \geq v(\pi)/(q-1)$. Thus by equation (8.16) applied to $x = f^{(s'-1)}(x)$ we have

$$v(f^{(s')}(x)) \geq \min\left\{v\left(\ (f^{(s'-1)}(x))^q\ \right),\ v(\pi f^{(s'-1)}(x))\ \right\} \geq \left(1 + \frac{1}{q-1}\right)v(\pi) > \frac{v(\pi)}{q-1}$$

By Lemma 8.2.1 below, we have $v(l_F(f^{(s')}(x))) = v(f^{(s')}(x))$, and since $v(\pi)/(q-1) = v(\pi_1)$, then

$$v(\pi^{s'} l_F(x)) \geq (1 + 1/(q-1))v(\pi) = v(\pi) + v(\pi_1).$$

This implies $(\pi^{s'-1}/\pi_1)T_{\mathcal{L}} \subset \mathcal{O}_{\mathcal{L}}$, where $T_{\mathcal{L}} = l_F(F(\mu_{\mathcal{L}}))$. Taking duals with respect to $\mathbb{T}_{\mathcal{L}/K}$ we get

$$\frac{\pi^{s'-1}}{\pi_1}\frac{1}{D(L/K)}\mathcal{O}_{\mathcal{L}} \subset R_{\mathcal{L}}.$$

Since $\mathrm{Tr}_s(P_s) \subset P_{\mathcal{L}}$, by equation (8.6), then Tr_s takes

$$\frac{\pi^s}{\pi_1\gamma_s}P_s \subset \frac{\pi^s}{\pi_1\pi_L}P_s$$

to

$$\frac{\pi^s}{\pi_1\pi_L}P_{\mathcal{L}} = \frac{\pi^s}{\pi_1^2\pi_L}\frac{1}{D(L/K)}\mathcal{O}_{\mathcal{L}} \subset \frac{\pi^{s-s'+1}}{\pi_1\pi_L}R_{\mathcal{L}}$$

Noticing now that $q \neq 2$, since $p \neq 2$, implies that $\pi_1\pi_L|\pi$, because $v_L(\pi/\pi_1\pi_L)) = e(L/L_1)(q-2)-1$. Thus, if $s \geq n+s'$ then Tr_s takes $(\pi^s/\pi_1\gamma_s)P_s$ to $\pi^n R_{\mathcal{L}}$ and we conclude the theorem. □

Lemma 8.2.1. *For every element $w \in F(\mu_{\mathcal{L}})$ such that*

$$v(w) > v(\pi)/(q-1),$$

we have that

$$v(l_F(w)) = v(w).$$

Proof. The inequality $v(w) > v(\pi)/(q-1)$ is equivalent to $v(w^q) > v(\pi)+v(w)$, and since $f(x) \equiv x^q \pmod{\pi x}$ this implies $v(f(w)) = v(\pi w)$. Thus $v(f^{(n)}(w)) = v(\pi^n w)$ and therefore we can take n large enough such that $v(f^{(n)}(w)) > 1/(p-1)$. By Proposition 3.3.1 we have that $v(l_F(f^{(n)}(w))) = v(f^{(n)}(w))$. Thus $v(\pi^n l_F(w)) =$

$v(\pi^n w)$, and this proves the lemma. $\square$

We restate the theorem in a simplified form.

Theorem 8.2.2. *Let r be minimal such that $L \cap K_\pi \subset K_r$ and $\mathcal{L}$ denote the field $L\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$. Let $s \geq n + r + \log_q(e(L/K_r))$. Then*

$$(\{N_{\mathcal{L}_s/\mathcal{L}}(a_1), a_2, \ldots, a_d\}, x)_{\mathcal{L},n} = \left[\mathbb{T}_{\mathcal{L}/K}\left(\ \mathrm{Tr}_s(QL_s(\{a_1, \ldots, a_d\})\ l_F(x)\ \right)\right]_f (e_n) \tag{8.17}$$

for all $a_1 \in \mathcal{L}'_s = \cap_{t \geq s} N_{t/s}(\mathcal{L}_t)$ and all $a_2, \ldots, a_d \in \mathcal{L}^*$.

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