Explicit Reciprocity Laws for Higher Local Fields

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Explicit Reciprocity Laws for Higher Local Fields

by

Jorge Florez

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

Explicit Reciprocity Laws for Higher Local Fields

by

Jorge Florez

Advisor: Victor Kolyvagin

In this thesis we generalize to higher dimensional local fields the explicit reciprocity laws of Kolyvagin [14] for the Kummer pairing associated to a formal group. The formulas obtained describe the values of the pairing in terms of multidimensional $p$-adic differentiation, the logarithm of the formal group, the generalized trace and the norm on Milnor K-groups.
I would like to start by thanking my adviser Victor Kolyvagin for suggesting the problem for this thesis and directions for future research. I am also thankful to him for sharing his mathematical ideas, and for the time he spent going over the manuscript providing valuable feedback to improve it. Moreover, I thank the committee members, Kenneth Kramer and Carlos Moreno for taking the time to read my paper and providing feedback.

I want to add an especial acknowledgment to Carlos Moreno for his friendship and for the guidance, support and motivation throughout my professional career.

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0.1 Terminology and notation

For the convenience of the reader we will list some terminology and notation that we will use throughout the paper. The letters $K$, $L$, $M$, will be used for one dimensional local fields and the corresponding script capital Latin symbols $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ will denote higher local fields containing the corresponding local fields.

- $p$ denotes a fixed prime number.

- Let $K/\mathbb{Q}_p$ denote a local field with ring of integers $\mathcal{O}_K$.

- $F = F(X,Y) \in \mathcal{O}_K[[X,Y]]$ will denote a formal group of finite height $h$. Let $\text{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of $F$ and $c : \text{End}_{\mathcal{O}_K}(F) \rightarrow \mathcal{O}_K$ the embedding $g \mapsto c(g) = g'(0)$.

- Let $S$ be a local field with ring of integers $C$ such that $C \subset c(\text{End}_{\mathcal{O}_K}(F))$. We will fix a uniformizer $\pi$ of $C$. Given $a \in C$, the endomorphism $aX + \cdots$ will be denoted by $[a]_F$. For convenience, $[\pi]_F$ will be denoted by $f$. Thus $f^{(n)}$, the $n$th fold of $f$, corresponds to $[\pi^n]_F$.

- Let $\kappa_n \simeq (C/\pi^nC)^h$ be the $\pi^n$th torsion group of $F$ and let $\kappa = \varprojlim \kappa_n \simeq C^h$. Let $\{e_i\}_{i=1}^h$ be a basis for $\kappa$ and $\{e_i^n\}_{i=1}^h$ be the corresponding reductions to the group $\kappa_n$. 
• Let $\alpha$ denote the ramification index of $S$ over $\mathbb{Q}_p$. We say that a pair $(n, t)$ is admissible if there exist an integer $k$ such that $t - 1 - n \geq \alpha k \geq n$. For example, the pair $(n, 2n + \alpha + 1)$ is admissible with $k = [(n + \alpha/\alpha)]$.

• If $R$ is a discrete valuation ring, the symbols $v_R$, $\mathcal{O}_R$, $\mu_R$ and $\pi_R$ will always denote the valuation, ring of integers, maximal ideal, and some fix uniformizer of $R$, respectively. Moreover, if the characteristic of the residue field is $p$, then we define $\mu_{R,1} = \{x \in R : v_R(x) > v_R(p)/(p-1)\}$. If $R \supset K$, then $F(\mu_R)$ and $F(\mu_{R,1})$ will denote the group of $\mu_R$-points and $\mu_{R,1}$-points of $F$, respectively.

• $\mathcal{L}$ is a $d$-dimensional complete field of characteristic 0, i.e., a field for which there are fields $\mathcal{L}_d = \mathcal{L}$, $\mathcal{L}_{d-1}$, \ldots, $\mathcal{L}_0$ such that $\mathcal{L}_{i+1}$ is a complete discrete valuation field with residue field $\mathcal{L}_i$, $0 \leq i \leq d - 1$, and $\mathcal{L}_0$ is a finite field of characteristic $p$.

• In what follows we assume that $\mathcal{L}$ is a field of mixed characteristic, i.e., $\text{char}(\mathcal{L}) = 0$ and $\text{char}(\mathcal{L}_{d-1}) = p$. Moreover, we also assume that $\mathcal{L} \supset K(\kappa_n)$.

• Let $t_1, \ldots, t_d$ be a fixed system of local parameters for $\mathcal{L}$, i.e, $t_d$ is a uniformizer for $\mathcal{L} = \mathcal{L}_d$, $t_{d-1}$ is a unit in $\mathcal{O}_\mathcal{L}$ but its residue in $\mathcal{L}_{d-1}$ is a uniformizer element of $\mathcal{L}_{d-1}$ and so on.

• Let $\nu_\mathcal{L} = (v_1, \ldots, v_d) : \mathcal{L}^* \to \mathbb{Z}^d$ be the valuation of rank $d$ corresponding to the system of parameters $t_1, \ldots, t_d$, i.e., $v_d = v_\mathcal{L}$, $v_{d-1}(\alpha) = v_\mathcal{L}_{d-1}(\alpha_{d-1})$.
where $\alpha_{d-1}$ is the residue of $\alpha/t_d^{-v_d(\alpha)}$ in $L_{d-1}$, and so on. Here the group $\mathbb{Z}^d$ is ordered lexicographically as follows: $\bar{a} = (a_1, \ldots, a_d) < (b_1, \ldots, a_d)$ if $a_l < b_l$ and $a_{l+1} = b_{l+1}, \ldots, a_d = b_d$ for some $l \leq d$. We denote by $\mathbb{Z}_+^d = \{ (a_1, \ldots, a_d) \in \mathbb{Z}^d : (a_1, \ldots, a_d) > (0, \ldots, 0) \}.$

- $O_L = \{ x \in L : \bar{v}_L(x) \geq \bar{0} \}$ is the valuation ring of rank $d$ of $L$, with maximal ideal $M_L = \{ x \in L : \bar{v}_L(x) > \bar{0} \}$. Notice that $\mu_L \subset M_L \subset O_L \subset O_L$.

- $V_L = 1 + M_L$ is the group of principal units.

- The field $L$ comes equipped with a special topology, the Parshin topology (cf. §2.4)

- Observe that $L^* \cong V_L \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$, where $\mathcal{R}$ is a multiplicative closed system of representatives of the last residue field $L_0$ and $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. Then we can endow $L^* \cong V_L \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$ with the product topology, where the group of principal units $V_L$ has the induced topology from $L$, and $\langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*$ with the discrete topology.

- $K_d(L)$ is the $d$th Milnor $K$-group of the field $L$, $d \geq 1$.

- Let $L$ be a complete discrete valuation field and define

$$L = L\{\{T\}\} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in L, \inf v_L(a_i) > -\infty, \lim_{i \to -\infty} v_L(a_i) = +\infty \right\}.$$
Let $v_L(\sum a_i T^i) = \min_{i \in \mathbb{Z}} v_L(a_i)$, so $\mathcal{O}_L = \mathcal{O}_L\{\{T\}\}$ and $\mu_L = \mu_L\{\{T\}\}$. Observe that $\mathcal{O}_L/\mu_L = k_L(\overline{T})$, where $k_L$ is the residue field of $L$.

- We define

$$L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\},$$

inductively by $E_{d-1}\{\{T\}\}$ where $E_{d-1} = L\{\{T_1\}\} \cdots \{\{T_{d-2}\}\}$.

- $D(L/K)$ will denote the different of the the extension of local fields $L/K$. 
Chapter 1

Introduction

1.1 Background

The history for finding explicit formulations of local class field theory dates back to Kummer \cite{15} in 1858 when he tried to describe the Hilbert symbol, associated to the cyclotomic field $\mathbb{Q}_p(\zeta_p)$, in terms of logarithmic derivatives. Seventy years later, when class field theory was acquiring its final shape, Artin and Hasse \cite{1} provided an explicit formula for the Hilbert symbol associated to $\mathbb{Q}_p(\zeta_{p^m})$; $p$ being an odd prime.

The formulas of Artin-Hasse would find an application in the work of Iwasawa \cite{10} when he studied the units of the cyclotomic field $\mathbb{Q}(\zeta_{p^m})$. Iwasawa extended the formulas of Artin-Hasse to describe more values of the Hilbert symbol (cf. \cite{9}), in terms of $p$-adic logarithmic derivatives. This would be the first manifestation of a deep connection, via explicit reciprocity laws, between arithmetical objects and values of zeta functions. The work of Iwasawa inspired Coates and Wiles \cite{3} to do fundamental work on the Birch and Swinnerton Dyer conjecture. One of the main
tools in their paper [27] was to extend Iwasawa’s reciprocity laws to Kummer pairings associated to Lubin-Tate formal groups, which are analogues of the Hilbert symbol for formal groups.

Soon after, Kolyvagin [14] extended the formulas of Wiles from Lubin-Tate formal groups to arbitrary formal groups (of finite height) and described the Kummer pairing in terms of $p$-adic derivations. Moreover, Kolyvagin also showed that the pairing is completely characterized by its values at the torsion points of the formal group.

Parallel to this, the theory of higher local fields was developing and along with it was the class field theory for such fields. Different formulations for a such theory were proposed by many people but it would be Kato’s [11] (see also [22]), who by means of Milnor K-groups, developed the most commonly used theory. Kato’s formulation shows that the finite abelian extensions of a $d$-dimensional higher local field correspond to norm subgroups of its Milnor $d$th K-group. This correspondence was established via a reciprocity map between the $d$th Milnor $K$-group and the galois group of the maximal abelian extension of the higher local field. I. Fesenko [4] nicely contributed to the class field theory for higher local fields also.

After Kato developed this class field theory, the program to make this theory more explicit began. Some of the higher dimensional explicit reciprocity laws that can be found in the literature are those of Kurihara [16], Vostokov [25], [26], Zinoviev [28] and Kato [12]. These explicit laws describe the analogue of Hilbert symbol for
higher local fields, in terms of higher dimensional $p$-adic logarithmic differentiation.

Using his explicit reciprocity laws, Kato made fundamental progress on Iwasawa main conjecture for modular forms. Kato’s explicit laws provide a link between the Euler system of Beilinson elements and some special values of the $L$-function attached to a modular form. This allowed Kato [13] to prove a divisibility statement related to the Iwasawa main conjecture for modular forms.

In this thesis, we will derive explicit reciprocity laws for the higher dimensional analogue of the Kummer pairing associated to a formal group $F$, or generalized Kummer pairing. The construction of these laws involve higher dimensional $p$-adic derivations, the logarithm of the formal group $F$, the generalized trace and the norm of Milnor $K$-groups. These formulas constitute a generalization, to higher local fields, of the explicit reciprocity laws of Kolyvagin [14]. In particular, the formulas apply to the generalized Hilbert symbol, like those of Kurihara [16] and Zinoviev [28]. It is still a work in progress to determine if the formulas of these two authors can be derived from the explicit reciprocity laws constructed in this thesis.

Finally, it is worth mentioning that the techniques that are used to obtain the explicit reciprocity laws are inspired by the work of Kolyvagin in [14]. In particular, this allows for a more classical and conceptual approach to the higher dimensional reciprocity laws.
1.2 Description of the problem and results

Let $K/Q_p$ be a local field and $F$ a one-dimensional formal group of finite height with coefficients in the ring of integers of $K$. Let $\text{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of $F$ and $c : \text{End}_{\mathcal{O}_K}(F) \to \mathcal{O}_K$ the embedding $f \mapsto c(f) = f'(0)$. Let $S/Q_p$ be a local field with ring of integers $C$ such that $C \subset c(\text{End}_{\mathcal{O}_K}(F)) \subset \mathcal{O}_K$. Let us fix a uniformizer $\pi$ for $S$ and denote by $f = [\pi]_F$ the endomorphism such that $c(f) = \pi$.

Let $h$ be the height of the endomorphism $[\pi]_F$ with respect to $C$, i.e., $[\pi]_F(X) = g(X^{q^h}) \pmod{\pi C}$ where $g'(0) \neq 0 \pmod{\pi C}$ and $q = |k_S|$. Denote by $f^{(n)} = [\pi^n]_F$ the $n$-fold of $f$. Let $\kappa_n$ be the $\pi^n$th torsion group, i.e., the set of those $v \in K$ such that $f^{(n)}(v) = 0$. Let $\{e^i_n\}_{i=1}^h$ be a $C/\pi^nC$-basis for the group $\kappa_n \simeq (C/\pi^nC)^h$.

Let $L \supset K$ be a local field containing $\kappa_n$. Let $\mathcal{O}_L$ be the ring of integers of $L$, $\mu_L$ its maximal ideal, $v_L$ its valuation and $\pi_L$ be a uniformizer for $L$. Define

$$\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\},$$

inductively. The ring on integers and maximal ideal are, respectively,

$$\mathcal{O}_\mathcal{L} = \mathcal{O}_L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad \mu_\mathcal{L} = \mu_L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.$$

We define the canonical pairing (cf. § 4.1)

$$(\cdot, \cdot)_{\mathcal{L}, n} : K_d(\mathcal{L}) \times F(\mu_\mathcal{L}) \to \kappa_n,$$
by
\[(\alpha, x) \mapsto \Upsilon_L(\alpha)(z) - F z,\]
where \(K_d(L)\) is the \(d\)th Milnor \(K\)-group of \(L\), \(\Upsilon_L: K_d(L) \rightarrow G^\text{ab}_L\) is the Kato's reciprocity map for \(L\), \(f^{(n)}(z) = x\) and \(-F\) is the subtraction in the formal group \(F\), cf. §4.1. Denote by \((,)_L^n\) the \(i\)th coordinate of \((,)_L^n\) with respect to the basis \(\{e^i_n\}\).

In this paper we will use the technique of Kolyvagin in [14] to derive explicit formulas for the symbol \((,)_L^n\) in terms of the generalized trace, the norm of Milnor \(K\)-groups, \(d\)-dimensional derivations and the logarithm of the formal group.

More specifically, let \(E\) be a complete discrete valuation field. We define a map
\[c_{E\{\{T\}\}/E}: E\{\{T\}\} \rightarrow E\]
by \(c_{E\{\{T\}\}/E}(\sum_{i \in \mathbb{Z}} a_i T^i) = a_0\). Then we can define \(c_L/L\) by the composition
\[c_L\{\{T_1\}\}/L \circ \cdots \circ c_L/L\{\{T_1\}\} \cdots \{\{T_{d-2}\}\}.

We define the generalized trace by \(T_L/S = \text{Tr}_{L/S} \circ c_L/L\).

The main result in this paper is the following (cf. Theorem [7,3,1]). Let \(M = L(\kappa_t)\), \(t \geq m \geq n\), for \(t\) large enough with respect to \(m\) and put \(M = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\), then
\[(N_{M/L}\{a_1, \ldots, a_d\}, x)^i_n = T_{M/S} \left( \frac{D^{i}_{M,m}(a_1, \ldots, a_d)}{a_1 \cdots a_d} \right) I_F(x)\]
for all \(a_1, \ldots, a_d \in \mathcal{M}^*\) (see (7,1) where cases when \(a_i\) are units or not are considered).
and all \( x \in F(\mu_L) \). Here \( l_F \) is the formal logarithm, \( N_{M/L} \) is the norm on Milnor \( K \)-groups and \( D^i_{\mathcal{M},m} \) is a \( d \)-dimensional derivation of \( \mathcal{O}_L \) over \( \mathcal{O}_K \), i.e.,

\[
D^i_{\mathcal{M},m} : \mathcal{O}_{\mathcal{M}}^d \to W
\]
is a map, with \( W \) an \( \mathcal{O}_{\mathcal{M}} \)-module, satisfying the properties in definition \([6.2.1]\) that is, for all \( a_1, \ldots, a_d \) and \( a'_1, \ldots, a'_d \) in \( \mathcal{O}_{\mathcal{M}} \) we have:

1. Leibniz rule:

\[
D(a_1, \ldots, a_i a'_i, \ldots, a_d) = a'_i D(a_1, \ldots, a_i, \ldots, a_d) + a_i D(a_1, \ldots, a'_i, \ldots, a_d),
\]
for any \( 1 \leq i \leq d \).

2. Linearity:

\[
D(a_1, \ldots, a_i + a'_i, \ldots, a_d) = D(a_1, \ldots, a_i, \ldots, a_d) + D(a_1, \ldots, a'_i, \ldots, a_d),
\]
for any \( 1 \leq i \leq d \).

3. Alternate: \( D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0 \), if \( a_i = a_j \) for \( i \neq j \).

4. \( D(a_1, \ldots, a_d) = 0 \) if some \( a_i \in \mathcal{O}_K \).

Notice that \( D \) is a 1-dimensional derivation in each component and moreover, it can be parametrized by 1-dimensional derivations. Indeed, just as one dimensional derivations are parametrized by \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \), the \( \mathcal{O}_L \)-module of Khalel differentials of
\( \mathcal{O}_L \) over \( \mathcal{O}_K \), then it will also be shown in §6.2 Proposition \( \text{Proposition 6.2.1} \) that all such \( d \)-dimensional derivations are parametrized by the \( \mathcal{O}_L \)-module \( \bigwedge^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \), i.e., the \( d \)-th exterior product of \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \). This is the \( \mathcal{O}_L \)-module \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \otimes \cdots \otimes \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \) divided out by the \( \mathcal{O}_L \)-submodule generated by the elements

\[
x_1 \otimes \cdots \otimes x_d
\]

where \( x_i = x_j \) for some \( i \neq j \), where \( x_1, \ldots, x_d \in \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \).

Moreover, notice that from properties (2), (3) and (4) it easily follows that \( D \) satisfies the relations

\[
D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0,
\]

if \( a_i + a_j = 1 \), for \( i \neq j \), which are called the Steinberg relations. Therefore, following Kolyvagin \[14\], we define the map

\[
\psi : (\mathcal{O}_\mathcal{M})^d \to W
\]

by

\[
\{a_1, \ldots, a_d\} \to \frac{D_{\mathcal{M},m}(a_1, \ldots, a_d)}{a_1 \cdots a_d},
\]

This map can be extended to all of \( K_d(\mathcal{M}^*) \) by equation (7.1) from Section 7.1.

As in the work of Kolyvagin \[14\], these derivations will be normalized in terms of the invariants (7.7) attached to the Galois representation on the Tate-module (7.5).

More specifically, we will build the \( d \)-dimensional derivation \( D_{\mathcal{M},m}^i \) out of the
CHAPTER 1.

condition

\[ D_{M,m}^i(T_1, \ldots, T_{d-1}, e_i^j) = - \frac{T_1 \cdots T_{d-1}}{l'(c_i^j)} c_{i,j}, \]

cf. Proposition 7.2.3, where \( c_{i,j} \) is defined in equation (7.7).

The deduction of the formula starts by proving that the pairing \((\cdot, \cdot)_{L,n}\) is sequentially continuous in the second argument with respect to the Parshin topology, i.e., if \( \alpha \in K_d(L) \) and \( \{x_k\} \) is a sequence in \( F(\mu_L) \) such that \( x_k \to 0 \) then \( (\alpha, x_k)_{L,n} \to 0 \), cf. Proposition 4.2.2, which is vital for the rest of the proof and was inspired by the proof of Proposition 4.1 in [14]. From this, we can prove the existence of the so called Iwasawa maps, \( \psi_{L,n}^i : K_d(L) \to R_{L,1}/\pi^n R_{L,1} \) (cf. Proposition 5.2.1), such that

\[ (\alpha, x)_{L,n}^i = \mathbb{T}_{L/S}( \psi_{L,n}^i(\alpha) l_F(x) ) \quad \forall x \in F(\mu_{L,1}), \]

where \( F(\mu_{L,1}) \) is the set \( \{ x \in L : v_L(x) \geq [v_L(p)/(p - 1)] + 1 \} \) considered with the operation defined by \( F \). We then define the maps \( D_{L,n}^i \) out of \( \psi_{L,n}^i \) as in equation (5.17). The main goal is to prove that under certain conditions the map \( D_{M,m}^i \) is a \( d \)-dimensional derivation over \( \mathcal{O}_K \), cf. Proposition 7.1.1. A vital role in this paper is given by the norm series relations

\[ (\{r(x), a_2, \ldots, a_d\}, x)_{L,n} = 0, \quad \forall x \in F(\mu_L), \quad \forall a_2, \ldots, a_d \in L^*, \]

cf. Proposition 4.3.1. These series allow us to obtain the representation (5.8) in Proposition 5.2.3 which together with the continuity of the pairing we obtain the
identity (5.18) from Proposition 5.4.2. But Proposition 5.4.4 tells us that every element in $\mu_L$ has a representation of the form $\eta(T_1, \ldots, T_{d-1}, \pi_L)$, where $\eta(X_1, \ldots, X_d)$ is a power series in the variables $X_1, \ldots, X_d$ with coefficients in $O_L$, given by equation (5.4.5).

Then, we will conclude from Proposition 5.4.5 and corollary 5.4.4 that

$$D^i_{L,m}(\alpha_1, \ldots, \alpha_d) = \det \left[ \frac{\partial \eta_i}{\partial X_j} \right]_{i,j} \bigg|_{X_k = T_k, k = 1, \ldots, d} D^i_{L,n}(T_1, \ldots, \pi_L),$$

where $\alpha_i = \eta_i(T_1, \ldots, \pi_L)$, $i = 1, \ldots, d$ and $T_d = \pi_L$. This last equation tells us that $D^i_{L,n}$ behaves like a $d$-dimensional derivation when restricted to $\mu_L$.

This will be used in the proof of Proposition 7.1.1 to conclude that $D^i_{M,m}$, for $M = M \{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and $M$ a finite extension of $L$ containing $\kappa_t$ for large enough $t$ with respect to $m$, is a $d$-dimensional derivation when restricted to certain quotient of $M$ as an $O_M$-module, namely $R_{M,1}/(\pi^m/\pi_M)R_{M,1}$. Moreover, Proposition 7.2.3 gives us a way of constructing explicitly this derivation out of the condition

$$D^i_{M,m}(T_1, \ldots, T_{d-1}, e^j_t) = -T_1 \cdots T_{d-1} c_{i,j}/l^i(e^j_t).$$

We can express back the Iwasawa map $\psi^i_{M,m} : K_d(M) \to R_{M,m}/\pi^m R_{M,m}$ in terms of the derivation $D^i_{M,m}$ as

$$\psi^i_{M,m}\{a_1, \ldots, a_d\} = \frac{D^i_{M,m}(a_1, \ldots, a_d)}{a_1 \cdots a_d}$$
Finally, the return to the field $\mathcal{L}$ will be guaranteed by Proposition 7.1.2.

The final formulas are contained in Theorem 7.3.1.

In chapter 8 we specialize to the case of a Lubin-tate formal group and refine Theorem 7.3.1 for this case. In particular, we give an explicit computation of the invariants $c_{i,j}$, namely

$$c_{i,j} = -\frac{1}{\pi^t}.$$
Chapter 2

Higher-dimensional local fields

2.1 2-dimensional local fields

We will start defining 2-dimensional local fields since most of the proofs will be reduced to this case. In the next sections all the concepts will be generalized to higher dimensions.

Definition 2.1.1. We say that $K$ is a 2-dimensional local field if there are fields $K_2 = K$, $K_1$, and $K_0$ such that $K_{i+1}$ is a complete discrete valuation ring with residue field $K_i$, $0 \leq i \leq 1$, and $K_0$ is a finite field of characteristic $p$.

We have the following examples of 2-dimensional local fields:

1. $\mathbb{F}_q((T_1))((T_2))$, where $q$ is a power of the prime $p$. In this case $K_2 = \mathbb{F}_q((T_1))((T_2))$, $K_1 = \mathbb{F}_q((T_1))$ and $K_0 = \mathbb{F}_q$. Recall that if $k$ is a field then $k((T))$ is the fraction field of the ring $k[[T]]$.

2. $E((T))$, where $E$ is a finite extension of $\mathbb{Q}_p$. Here $K_1 = E$ and $K_0 = k_E$
3. For a complete discrete valuation field $E$ we define

$$\mathcal{K} = E\{(T)\} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in E, \inf v_E(a_i) > -\infty, \lim_{i \to -\infty} v_E(a_i) = +\infty \right\}$$

Let $v_\mathcal{K}(\sum a_i T^i) = \min v_E(a_i)$. Then $\mathcal{K}$ is a complete discrete valuation field with residue field $k_F((t))$ and ring of integers

$$\mathcal{O}_{E\{(T)\}} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i \in F\{(T)\} : a_i \in \mathcal{O}_F, \forall i \in \mathbb{Z} \right\}$$

Thus, if $E$ is a local field then $E\{(T)\}$ is a 2-dimensional local field with $\mathcal{K}_1 = k_E((t))$ and $\mathcal{K}_0 = k_E$. These fields are called standard 2-dimensional local fields (cf. [7] §1.1).

We are interested only in the case where $\mathcal{K}$ is of mixed characteristic, i.e., $\text{char}(\mathcal{K}_1) = p$. The following Theorem classifies such fields.

**Theorem 2.1.1** (Classification Theorem). Let $\mathcal{K}$ be a 2-dimensional local field of mixed characteristic. Then $\mathcal{K}$ is a finite extension of a 2-dimensional standard field $E\{T_1\}$, where $E$ is a finite extension of $\mathbb{Q}_p$, and there is a finite extension of $\mathcal{K}$ which is a standard 2-dimensional local field.


We define the ring of integers of $\mathcal{K}$ to be the set

$$\mathcal{O}_\mathcal{K} = \{ x \in \mathcal{O}_\mathcal{K} : \bar{x} \in \mathcal{O}_{\mathcal{K}_1} \}$$
and the maximal ideal

\[ M_K = \{ x \in \mathcal{O}_K : \bar{x} \in \mu_{K_1} \} \]

where \( \bar{x} \) denotes the reduction of \( x \) in \( K_1 \). Notice that

\[ \mu_K \subset M_K \subset \mathcal{O}_K \subset \mathcal{O}_K. \]

In the examples above we can compute these two sets explicitly:

1. If \( K = \mathbb{F}_q((T_1))((T_2)) \): \( \mathcal{O}_K = \mathbb{F}_q[[T_1]] + T_2\mathbb{F}_q((T_1))[[T_2]] \) and \( M_K = T_1\mathbb{F}_q[[T_1]] + T_2\mathbb{F}_q((T_1))[[T_2]] \)

2. If \( K = E((T)) \): \( \mathcal{O}_K = \mathcal{O}_E + TE[[T]] \) and \( M_K = \pi_E \mathcal{O}_E + TE[[T]] \).

3. If \( K = E\{\{T\}\} \): \( \mathcal{O}_K = \{ \sum_{-\infty}^{\infty} a_i T^i \in \mathcal{K} : a_i \in \mathcal{O}_E, \forall i \geq 0 \text{ and } a_i \in \mu_E \forall i < 0 \} \)
   and \( M_K = \{ \sum_{-\infty}^{\infty} a_i T^i \in F\{\{T\}\} : a_i \in \mathcal{O}_E, \forall i > 0 \text{ and } a_i \in \mu_E \forall i \leq 0 \} \)

In particular, if \( K = \mathbb{Q}_p\{\{T\}\} \) we have

\[ \mathcal{O}_K = \{ \sum_{-\infty}^{\infty} a_i T^i \in \mathcal{K} : a_i \in \mathbb{Z}_p, \forall i \geq 0 \text{ and } a_i \in p\mathbb{Z}_p \forall i < 0 \}, \]

and

\[ M_K = \{ \sum_{-\infty}^{\infty} a_i T^i \in \mathcal{K} : a_i \in \mathbb{Z}_p, \forall i > 0 \text{ and } a_i \in p\mathbb{Z}_p \forall i \leq 0 \}. \]

**Topology on \( K \):** The topology on \( K = E\{\{T\}\} \) is called the Parshin topology
CHAPTER 2. HIGHER-DIMENSIONAL LOCAL FIELDS

or Higher-dimensional topology (cf. [7] §1.3 ). A basis of neighborhoods of 0 for this topology is given by the following: Let \( \{U_i\}_{i \in \mathbb{Z}} \) be a sequence of neighborhoods of 0 in the local field \( E \), i.e., \( U_i = \pi_E^n \mathcal{O}_E \), where \( n_i \in \mathbb{Z} \), such that

1. There exist a \( c \in \mathbb{Z} \) such that \( \pi_E^n \mathcal{O}_E \subset U_i \) for all \( i \in \mathbb{Z} \), i.e., \( \cap U_i \supset \pi_E^n \mathcal{O}_E \). This says that the neighborhoods \( U_i \)'s cannot be arbitrarily small.

2. For every \( l \in \mathbb{Z} \) we have \( \pi_E^l \mathcal{O}_E \subset U_i \) for sufficiently large \( i \). This implies that the \( U_i \)'s get bigger as \( i \to \infty \).

We put

\[
U_{\{U_i\}} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i \in \mathcal{K} : a_i \in U_i \right\}
\]

These sets \( U_{\{U_i\}} \) form a base of neighborhoods of 0 in \( \mathcal{K} \).

**Remark 2.1.1.** Observe that in this topology an element \( x = \sum_{i=-\infty}^{\infty} a_i T^i \in \mathcal{K} = E\{\{T\}\} \) converges as an infinite sum since both tails \( \sum_{i>N} a_i T^i \) and \( \sum_{i<-N} a_i T^i \) approach to 0 as \( N \to \infty \). To see this take a basis element for zero, \( U_{\{U_i\}} \), and let \( c \) be given by condition 1 above, and for \( l = v_K(x) = \inf v_E(a_i) \) let \( N \) be large enough to satisfy condition 2 above.

Since by definition \( v_E(a_i) \to \infty \) as \( i \to -\infty \) we can find an \( N \) such that \( v_E(a_i) > c \) for \( i < -N \). Then \( a_i \in \pi_E^n \mathcal{O}_E \subset U_i \) for \( i < -N \), i.e., \( \sum_{i<-N} a_i T^i \in U_{\{U_i\}} \).
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Now \( v_E(a_i) \geq l \) for all \( i \), then \( a_i \in \pi_E^l \mathcal{O}_E \subset U_i \) for \( i > N \). Again, this implies \( \sum_{i>N} a_i T^i \in \mathcal{U}\{U_i\} \).

Remark 2.1.2. Observe also that a sequence \( x_n = \sum_{i=-\infty}^{\infty} a_i^{(n)} T^i \) converging to zero in this topology does not necessarily satisfy that \( v_K(x_n) \to \infty \), i.e., the coefficients \( a_i^{(n)} \) do not necessarily converge uniformly on \( i \) to zero. For instance, the sequence \( x_n = T^n \) converges to zero in this topology and \( v_K(x_n) = 0 \) for all \( n \). But the following Proposition is true:

Proposition 2.1.1. The sequence \( x_n = \sum_{i=-\infty}^{\infty} a_i^{(n)} T^i \) converges to zero if and only if the following two conditions are satisfied

(i) \( \inf v_K(x_n) > -\infty \);

(ii) for a given \( m \in \mathbb{Z}_+ \) the sequence \( a_i^{(n)} \xrightarrow{n \to \infty} 0 \) uniformly for every \( i \leq m \).

That is, the coefficients are uniformly bounded and converge to zero uniformly for \( i \leq m \), for every \( m \).

Proof. \( \to \) Suppose the sequence \( x_n \) converges to zero. For a fix \( m \in \mathbb{Z} \) and given \( M > 0 \) define

\[
U_i^M = \begin{cases} \pi_E^M \mathcal{O}_E, & i \leq m \\ \pi_E^{-i} \mathcal{O}_E, & i > m \end{cases}
\]

Then \( \mathcal{V}_M = \mathcal{V}_{U_i^M} \) is a basis element for zero, so there exists an \( N \) such that \( x_n \in \mathcal{V}_M \) for \( n > N \). That is \( v_E(a_i^{(n)}) \geq M \) for \( n > N \) and all \( i \leq m \), then \( a_i^{(n)} \xrightarrow{n \to \infty} 0 \) uniformly on every \( i \leq m \).
Let us show now that $\inf_{v} K(x^n) > -\infty$. For $i > 1$ define $m_i = \inf_{E} v_E(a_i^{(n)})$ and $M_i = \min_{1 \leq k \leq i} \{m_k\}$. Then $M_i \geq M_{i+1}$ and moreover, if $\inf_{v} K(x^n) = -\infty$, then $M_i \xrightarrow{i \to \infty} -\infty$ which follows from the just proven condition (ii). Consider

$$U_i = \begin{cases} \pi_E \mathcal{O}_E, & i \leq 0 \\ \pi_E^{M_{i+1}} \mathcal{O}_E, & i > 0 \end{cases}$$

Then $U_{\{U_i\}}$ is a base element for $0$. Again, since we are assuming $\inf_{v} K(x^n) = -\infty$, for certain large enough $i$ we have the strict inequality $M_{i-1} > M_i$, i.e., $M_i = m_i$ and so we can find a subsequence $x_{n_i} = \sum_j a_j^{(n)} T^j$ such that $v_K(a_i^{(n)}) = M_i$, thus $v_K(x_{n_i}) \leq v_E(a_i^{(n)}) = M_i$. Then $a_i^{(n)} \notin U_i$, which implies $x_{n_i} \notin U_{\{U_i\}}$. Finally, since $n_i \to \infty$ as $i \to \infty$ this contradicts the convergence of $x_n$ to zero.

$\leftarrow$) Let $x_n$ be a sequence in $K$ satisfying (i) and (ii). Let $U_{\{U_i\}}$ is a base element for $0$. Then there exist a $c \in \mathbb{Z}$ such that $\pi_E^c \mathcal{O}_E \subset U_i$ for all $i \in \mathbb{Z}$. Also for $l = \inf_{v} K x_n$ there exist an $i_0$ such that $\pi_E^l \mathcal{O}_E \subset U_i$ for all $i > i_0$.

By condition (ii) applied to $m = i_0$ there exist an $N$ such that $v_E(a_i^{(n)}) > c$ for all $n \geq N$ and all $i \leq i_0$. Then $a_i^{(n)} \in \pi_E^c \mathcal{O}_E \subset U_i$ for all $n \geq N$ and all $i \leq i_0$.

On the other hand, $v_E(a_i^{(n)}) \geq \inf_{v} K(x_n) = l$, so $a_i^{(n)} \in \pi_E^l \mathcal{O}_E \subset U_i$ for all $i > i_0$. Summarizing we have that $a_i^{(n)} \in U_i$ for all $n > N$ and all $i \in \mathbb{Z}$. Therefore $x_n \in U_{\{U_i\}}$ and so it converges to zero in the Parshin topology.

$\square$
CHAPTER 2. HIGHER-DIMENSIONAL LOCAL FIELDS

Topology on $\mathcal{K}^*$:

Let $\mathcal{R} \subset \mathcal{K} = \mathcal{K}_2$ be a set of representatives of the last residue field $\mathcal{K}_0$. Let $t_1$ and $t_2$ be a fixed system of local parameters for $\mathcal{K}$, i.e., $t_2$ is a uniformizer for $\mathcal{K}$ and $t_1$ is a unit in $\mathcal{O}_\mathcal{K}$ but its residue in $\mathcal{K}_1$ is a uniformizer element of $\mathcal{K}_1$. Then

$$\mathcal{K}^* = \mathcal{V}_\mathcal{K} \times \langle t_1 \rangle \times \langle t_2 \rangle \times \mathcal{R}^*,$$

where the group of principal units $\mathcal{V}_\mathcal{K} = 1 + M_\mathcal{K}$ and $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. From this observation we have the following,

**Proposition 2.1.2.** We can endow $\mathcal{K}^*$ with the product of the induced topology from $\mathcal{K}$ on the group $\mathcal{V}_\mathcal{K}$ and the discrete topology on $\langle t_1 \rangle \times \langle t_2 \rangle \times \mathcal{R}^*$. Moreover, every Cauchy sequence with respect to this topology converges in $\mathcal{K}^*$.

**Proof.** cf [7] Chapter 1: Higher dimensional local fields. $\square$

### 2.2 $d$-dimensional local fields

We will now generalize all the concepts introduced before to any dimension. In the rest of this section $E$ will denote a local field, $k_E$ its residue field and $\pi_E$ a uniformizer for $E$.

**Definition 2.2.1.** $\mathcal{K}$ is an $d$-dimensional local field, i.e., a field for which there is a chain of fields $\mathcal{K}_d = \mathcal{K}$, $\mathcal{K}_{d-1}$, $\ldots$, $\mathcal{K}_0$ such that $\mathcal{K}_{i+1}$ is a complete discrete valuation ring with residue field $\mathcal{K}_i$, $0 \leq i \leq d - 1$, and $\mathcal{K}_0$ is a finite field of characteristic $p$. 
If $k$ is a finite field then $\mathcal{K} = k((T_1)) \ldots ((T_d))$ is a $d$-dimensional local field with
$$\mathcal{K}_i = k((T_1)) \ldots ((T_i)) \quad 1 \leq i \leq d.$$ If $E$ is a local field, then $\mathcal{K} = E\{\{T_1\}\ldots\{T_{d-1}\}\}$ is defined inductively as
$$E_{d-1}\{\{T_{d-1}\}\},$$
where $E_{d-1} = E\{\{T_1\}\ldots\{T_{d-2}\}\}$. We have that $\mathcal{K}$ is a $d$-dimensional local field with residue field $\mathcal{K}_{d-1} = k_{E_{d-1}}((T_{d-1}))$, and by induction
$$k_{E_{d-1}} = k_E((T_1)) \ldots ((T_{d-2})).$$
Therefore $\mathcal{K}_{d-1} = k_E((T_1)) \ldots ((T_{d-1}))$. These fields are called the standard fields.

From now on we will assume $\mathcal{K}$ has mixed characteristic, i.e., $\text{char}(\mathcal{K})=0$ and $\text{char}(\mathcal{K}_{d-1})= p$. The following theorem classifies all such fields.

**Theorem 2.2.1** (Classification Theorem). *Let $\mathcal{K}$ be an $d$-dimensional local field of mixed characteristic. Then $\mathcal{K}$ is a finite extension of a standard field
$$E\{\{T_1\}\ldots\{T_{d-1}\}\},$$
where $E$ is a local field, and there is a finite extension of $\mathcal{K}$ which is a standard field.*

*Proof. cf. [7] § 1.1 Classification Theorem.*

**Definition 2.2.2.** *An $d$-tuple of elements $t_1, \ldots, t_d \in \mathcal{K}$ is called a system of local*
parameters of \(\mathcal{K}\), if \(t_d\) is a prime in \(\mathcal{K}_d\), \(t_{d-1}\) is a unit in \(\mathcal{O}_\mathcal{K}\) but its residue in \(\mathcal{K}_{d-1}\) is a prime element of \(\mathcal{K}_{d-1}\), and so on.

For the standard field \(E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\}\) we can take as a system of local parameters \(t_d = \pi_E, t_{d-1} = T_{d-1}, \ldots, t_1 = T_1\).

**Definition 2.2.3.** We define a discrete valuation of rank \(d\) to be the map \(\nu = (\nu_1, \ldots, \nu_d) : \mathcal{K}^* \to \mathbb{Z}^d, \nu_d = \nu_{\mathcal{K}_d}, \nu_{d-1}(x) = \nu_{\mathcal{K}_{d-1}}(x_{d-1})\) where \(x_{d-1}\) is the residue in \(\mathcal{K}_{d-1}\) of \(x t_d^{-\nu(x)}\), and so on.

Although the valuation depends, for \(n > 1\), on the choice of \(t_2, \ldots, t_d\), it is independent in the class of equivalent valuations.

### 2.3 Extensions of \(\mathcal{K}\)

Let \(\mathcal{L}/\mathcal{K}\) be a finite extension of the \(d\)-dimensional local field \(\mathcal{K}\). Then \(\mathcal{L}\) is also a \(d\)-dimensional local field.

**Definition 2.3.1.** Let \(t_1, \ldots, t_d\) and \(t'_1, \ldots, t'_d\) be a system of local parameters for \(\mathcal{K}\) and \(\mathcal{L}\), respectively, with associated valuations \(\nu\) and \(\nu'\), respectively. Put

\[
E(\mathcal{L}/\mathcal{K}) = (v'(t_i))_{i,j} = \begin{pmatrix}
e_1 & 0 & \cdots & 0 \\
\cdots & e_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & e_d
\end{pmatrix}
\]

where \(e_i = e(\mathcal{L}_i/\mathcal{K}_i), i = 1, \ldots, n\). Then \(e_i\) does not depend on the choice of the
parameters, and \([\mathcal{L}/\mathcal{K}] = f(\mathcal{L}/\mathcal{K}) \prod_{i=1}^{n} e_i(\mathcal{L}/\mathcal{K})\), where \(f(\mathcal{L}/\mathcal{K}) = [\mathcal{L}_0/\mathcal{K}_0]\) (cf. \(\S\ 1.2\)).

2.4 Topology on \(\mathcal{K}\)

We define the topology on \(E\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\) by induction on \(d\). For \(d = 1\) we define the topology to be the topology of a one-dimensional local field. Suppose we have defined the topology on a standard \(d\)-dimensional local field \(E_d\) and let \(\mathcal{K} = E_d\{\{T\}\}\). Denote by \(P_{E_d}(c)\) the set \(\{x \in E_d : v_{E_d}(x) \geq c\}\). Let \(\{V_i\}_{i \in \mathbb{Z}}\) be a sequence of neighborhoods of zero in \(E_d\) such that

\[
\begin{aligned}
1. \text{ there is a } c \in \mathbb{Z} \text{ such that } P_{E_d}(c) \subset V_i \text{ for all } i \in \mathbb{Z}. \\
2. \text{ for every } l \in \mathbb{Z} \text{ we have } P_{E_d}(l) \subset V_i \text{ for all sufficiently large } i.
\end{aligned}
\]

and put \(\mathcal{V}_{\{V_i\}} = \{\sum b_i T^i : b_i \in V_i\}\). These sets form a basis of neighborhoods of 0 for a topology on \(\mathcal{K}\). For an arbitrary \(d\)-dimensional local field \(L\) of mixed characteristic we can find, by the Classification Theorem, a standard field that is a finite extension of \(L\) and we can give \(L\) the topology induced by the standard field.

Proposition 2.4.1. Let \(\mathcal{L}\) be a \(d\)-dimensional local field of mixed characteristic with the topology defined above.

1. \(\mathcal{L}\) is complete with this topology. Addition is a continuous operation and multiplication by a fixed \(a \in \mathcal{L}\) is a continuous map.
2. Multiplication is a sequentially continuous map, i.e., if \( x \in \mathcal{L} \) and \( y_k \to y \) in \( \mathcal{L} \) then \( xy_k \to xy \).

3. This topology is independent of the choice of the standard field above \( \mathcal{L} \).

4. If \( \mathcal{K} \) is a standard field and \( \mathcal{L}/\mathcal{K} \) is finite, then the topology above coincides with the natural vector space topology as a vector space over \( \mathcal{K} \).

5. The reduction map \( \mathcal{O}_\mathcal{L} \to k_\mathcal{L} = \mathcal{L}_{d-1} \) is continuous and open (where \( \mathcal{O}_\mathcal{L} \) is given the subspace topology from \( \mathcal{L} \), and \( k_\mathcal{L} = \mathcal{L}_{d-1} \) the \( (d-1) \)-dimensional topology).

Proof. All the proofs can be found in [20] Theorem 4.10.

\[ \square \]

### 2.5 Topology on \( \mathcal{K}^* \)

Let \( \mathcal{R} \subset \mathcal{K} = \mathcal{K}_d \) be a set of representatives of the last residue field \( \mathcal{K}_0 \). Let \( t_1, \ldots, t_d \) be a fixed system of local parameters for \( \mathcal{K} \), i.e., \( t_d \) is a uniformizer for \( \mathcal{K} \), \( t_{d-1} \) is a unit in \( \mathcal{O}_\mathcal{K} \) but its residue in \( \mathcal{K}_{d-1} \) is a uniformizer element of \( \mathcal{K}_{d-1} \), and so on. Then

\[
\mathcal{K}^* = V_\mathcal{K} \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathcal{R}^*,
\]

where the group of principal units \( V_\mathcal{K} = 1 + M_\mathcal{K} \) and \( \mathcal{R}^* = \mathcal{R} \setminus \{0\} \). From this observation we have the following,
Proposition 2.5.1. We can endow $K^*$ with the product of the induced topology from $K$ on the group $V_K$ and the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times \mathbb{R}^*$. In this topology we have,

1. Multiplication is sequentially continuous, i.e., if $a_n \to a$ and $b_n \to b$ then $a_nb_n \to ab$.

2. Every Cauchy sequence with respect to this topology converges in $K^*$.

Proof. cf [7] Chapter 1 §1.4.2. \qed
Chapter 3

Formal groups

In the rest of this paper $F$ will be a formal group over the ring of integers, $\mathcal{O}_K$, of a local field $K/\mathbb{Q}_p$. Let $\text{End}_{\mathcal{O}_K}(F)$ be the ring of endomorphisms of $F$. For $t \in \text{End}_{\mathcal{O}_K}(F)$ we denote $t'(0)$ by $c(t)$. This induces an embedding

$$c : \text{End}_{\mathcal{O}_K}(F) \to \mathcal{O}_K.$$  

We have that that $c(\text{End}_{\mathcal{O}_K}(F))$ is a closed subring of $\mathcal{O}_K$ (cf. [14] §2.3). In particular $\mathbb{Z}_p \subset c(\text{End}_{\mathcal{O}_K}(F))$. Let $S/\mathbb{Q}_p$ be a local field with ring of integers $C$ such that $C \subset c(\text{End}_{\mathcal{O}_K}(F))$. Let us fix once and for all a uniformizer $\pi$ for $C$. Let $j$ be the degree of inertia of $S/\mathbb{Q}_p$, i.e., $|k_\pi| = p^j$. In what follows $f$ will denote the element in $\text{End}_{\mathcal{O}_K}(F)$ such that $c(f) = \pi$.
CHAPTER 3. FORMAL GROUPS

3.1 The Weierstrass lemma

Let $E$ be a discrete valuation field of zero characteristic with integer ring $\mathcal{O}_E$ and maximal ideal $\mu_E$.

**Lemma 3.1.1** (Weierstrass lemma). Let $g = a_0 + a_1X + \cdots \in \mathcal{O}_E[[X]]$ be such that $a_0, \ldots, a_{n-1} \in \mu_E$, $n \geq 1$, and $a_n \notin \mu_E$. Then there exist a unique monic polynomial $c_0 + \cdots + X^n$ with coefficients in $\mu_E$ and a series $b_0 + b_1X \cdots$ with coefficients in $\mathcal{O}_E$ and $b_0$ a unit, i.e., $b_0 \neq \mu_E$, such that

$$g = (c_0 + \cdots + X^n)(b_0 + b_1X \cdots).$$

**Proof.** See [18] IV. §9 Theorem 9.2.

\[\square\]

3.2 The group $F(\mu_M)$

Let $\mathcal{K}$ be a $d$-dimensional local field containing the local field $K$. For example, we may consider $\mathcal{K}$ to be $K\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Denote by $F(\mu_\mathcal{K})$ the group with underlying set $\mu_\mathcal{K}$ and operation defined by the formal group $F$. More generally, if $\mathcal{M}$ is an algebraic extension of $\mathcal{K}$ we define
\[ F(\mu_M) := \bigcup_{M \supset L \supset K \mid \lbrack L/K \rbrack < \infty} F(\mu_L). \]

An element \( f \in \text{End}(F) \) is said to be an isogeny if the map \( f : F(\mu_{\bar{K}}) \to F(\mu_{\bar{K}}) \) induced by it is surjective with finite kernel.

If the reduction of \( f \) in \( k_K[[X]] \), \( k_K \) the residue field of \( K \), is not zero then it is of the form \( f_1(X^{p^h}) \) with \( f_1'(0) \in O_K^* \), cf. \([14]\) Proposition 1.1. In this case we say that \( f \) has finite height. If on the other hand the reduction of \( f \) is zero we say it has infinite height.

**Proposition 3.2.1.** \( f \) is an isogeny if and only if \( f \) has finite height. Moreover, in this situation \( |\ker f| = p^h \).

**Proof.** If the height is infinite, the coefficients of \( f \) are divisible by a uniformizer of the local field \( K \), so \( f \) cannot be surjective. Let \( h < \infty \) and \( x \in \mu_L \) where \( L \) is a finite extension of \( K \). Consider the series \( f - x \) and apply Lemma \ref{lem:3.1.1} with \( E = L \), i.e.,

\[
f - x = (c_0 + \cdots + X^{p^h})(b_0 + b_1 X + \cdots),
\]

where \( c \)'s \( \in \mu_L \), \( b \)'s \( \in O_L \) and \( b_0 \in O_L^* \). Therefore the equation \( f(X) = x \) is equivalent to the equation \( c_0 + \cdots + X^{p^h} = 0 \) and since the \( c \)'s \( \in \mu_L \) every root belongs to \( \mu_{\bar{K}} \).
Moreover, the polynomial \( P(X) = c_0 + \cdots + X^{ph} \) is separable because \( f'(X) = \pi t(X), \ t(X) = 1 + \ldots \) is an invertible series and \( f' = P'(b_0 + b_1 X + \cdots) + P(b_0 + b_1 X + \cdots)' \) so \( P' \) can not vanish at a zero of \( P \). We conclude that \( P \) has \( p^h \) roots, i.e., \(|\ker f| = p^h|\).

\[ \]

**Proposition 3.2.2.** Denote by \( j \) the degree of inertia of \( S/\mathbb{Q}_p \) and by \( h_1 \) the height of \( f = [\pi]_F \). Then \( j \) divides \( h_1 \), namely \( h_1 = jh \). Let \( \kappa_n \) be the kernel of \( f^{(n)} \). Then

\[
\kappa_n \simeq (C/\pi^nC)^h \quad \text{and} \quad \varprojlim \kappa_n \simeq C^h,
\]

as \( C \)-modules. This \( h \) is called the height of the formal group with respect to \( C = \mathcal{O}_S \).

**Proof.** cf. [14] Proposition 2.3.

**Remark 3.2.1.** Notice that since the coefficients of \( F \) are in the local field \( K \) then \( \kappa_n \subset \overline{K} \) for all \( n \geq 1 \).

Let us fix once and for all a basis \( \{e^i\}_{i=1}^h \) for \( \varprojlim \kappa_n \). Denote by \( e^i_n \) the reduction of \( e^i \) to \( \kappa_n \). Clearly \( \{e^i_n\} \) is a basis for \( \kappa_n \).

Throughout \( K_n \) will denote the one dimensional local field \( K(\kappa_n) \). Suppose \( M/K \) is a finite extension, then \( M_n \) will denote the local field \( M(\kappa_n) \). Let \( \mathcal{M} \) and \( \mathcal{M}_n \) denote, respectively,

\[
M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad M_n\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.
\]
3.3 The logarithm of the formal group

We define the logarithm of the formal group $F$ to be the series

$$ l_F = \int_0^X \frac{dX}{F_X(0, X)} $$

Observe that since $F_X(0, X) = 1 + \cdots \in \mathcal{O}_K[[X]]^*$ then $l_F$ has the form

$$ X + \frac{a_2}{2} X^2 + \cdots + \frac{a_n}{n} X^n + \cdots $$

where $a_i \in \mathcal{O}_K$.

**Proposition 3.3.1.** Let $E$ be a field of characteristic 0 that is complete with respect to a discrete valuation, $\mathcal{O}_E$ the valuation ring of $E$ with maximal ideal $\mu_E$ and valuation $v_E$. Consider a formal group $F$ over $\mathcal{O}_E$, then

1. The formal logarithm induces a homomorphism

$$ l_F : \mu_E \rightarrow E $$

with the additive group law on $E$.

2. The formal logarithm induces the isomorphism

$$ l_F : \mu_E^r \rightarrow \mu_E^r $$
for all \( r \geq [v_E(p)/(p-1)] + 1 \) and

\[
v_E(l(x)) = v(x) \quad (\forall x \in \mu_E^r).
\]

In particular, this holds for \( \mu_{E,1} = \{ x \in E : v_E(x) > v_E(p)/(p-1) + 1 \} \).


**Lemma 3.3.1.** Let \( E \) and \( v_E \) as in the previous proposition. Then

\[
v_E(n!) \leq \frac{(n-1)v_E(p)}{p-1},
\]

and \( v_E(x^n/n!) \to \infty \) as \( n \to \infty \) for \( x \in \mu_{E,1} \).

**Proof.** The first assertion can be found in [24] IV. Lemma 6.2. For the second one notice that

\[
v_E(x^n/n!) \geq nv_E(x) - v(n!)
\]

\[
\geq nv_E(x) - (n-1)\frac{v_E(p)}{p-1}
\]

\[= v_E(x) + (n-1)\left(v_E(x) - \frac{v_E(p)}{p-1}\right).
\]

Since we are assuming that \( x \in \mu_{E,1} \), i.e., \( v_E(x) > v_E(p)/(p-1) \), then \( v_E(x^n/n!) \to \infty \) as \( n \to \infty \).

**Lemma 3.3.2.** Let \( \mathcal{L} \) be a \( d \)-dimensional local field containing the local field \( K \),

\[g(X) = a_1X + \frac{a_2}{2!}X^2 + \cdots + \frac{a_n}{n!}X^n + \cdots \]

and \( h(X) = a_1X + \frac{a_2}{2!}X^2 + \cdots + \frac{a_n}{n!}X^n + \cdots \) with
$a_i \in \mathcal{O}_K$. Then $g$ and $h$ define, respectively, maps $g : \mu_L \to \mu_L$ and $h : \mu_{L,1} \to \mu_{L,1}$ that are sequentially continuous in the Parshin topology.

**Proof.** We may assume $L$ is a standard $d$-dimensional local field. Let $\mathcal{V}_{\{V_i\}}$ be a basic neighborhood of zero that we can consider to be a subgroup of $L$, and let $c > 0$ such that $P_L(c) \subset \mathcal{V}_{\{V_i\}}$. If $x_n \in \mu_L$ for all $n$, then there exists an $N_1 > 0$ such that $v_L(x_n^i/i), v_L(x^i/i) > c$ for all $i > N_1$ and all $n$; because $iv_L(x_n) - v(i) \geq iv_L(x_n) - \log_p(i) \geq i - \log_p(i) \to \infty$ as $i \to \infty$. On the other hand, if $x_n \in \mu_{L,1}$ for all $n$, then there exists an $N_2 > 0$ such that $v_L(x_n^i/i!), v_L(x^i/i!) > c$ for all $i > N_2$ and all $n$ by Lemma 3.3.1. Then, for $N = \max\{N_1, N_2\}$, we have

$$\sum_{i=N+1}^{\infty} a_i \frac{x_n^i - x^i}{i}, \sum_{i=N+1}^{\infty} a_i \frac{x_n^i - x^i}{i!} \in P_L(c) \subset \mathcal{V}_{\{V_i\}}.$$  

Now, since multiplication is sequentially continuous and $x_n \to x$ then

$$\sum_{i=1}^{N} a_i \frac{x_n^i - x^i}{i} \to 0, \sum_{i=1}^{N} a_i \frac{x_n^i - x^i}{i!} \to 0 \quad \text{as } n \to \infty$$

Thus for $n$ large enough we have that

$$g(x_n) - g(x), \; h(x_n) - h(x) = \sum_{i=1}^{N} + \sum_{i=N+1}^{\infty} \in \mathcal{V}_{\{V_i\}}.$$  

Remark 3.3.1. In particular, $\log : \mu_L \to \mu_L$, $l_F : \mu_L \to \mu_L$ and $\exp_F = l_F^{-1} : \mu_{L,1} \to \mu_{L,1}$ are sequentially continuous.
Chapter 4

The Kummer Pairing

4.1 Higher-dimensional local class field theory

4.1.1 Milnor-$K$-groups and norms

Definition 4.1.1. Let $R$ be a ring and $m \geq 0$. We denote by $K_m(R)$ the group

$$\frac{R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times}{I} \text{m-times}$$

where $I$ is the subgroup of $(R^\times)^{\otimes m}$ generated by

$$\{a_1 \otimes \cdots \otimes a_m : a_1, \ldots, a_m \in R^\times \text{ such that } a_i + a_j = 1 \text{ for some } i \neq j\}$$

$K_m(R)$ is called the $m^{th}$ Milnor-$K$-group of $R$. The element $a_1 \otimes \cdots \otimes a_m$ is denoted by $\{a_1, \ldots, a_m\}$.

We define the symbol map $\{,\}$ to be the map

$$R^\times \times \cdots \times R^\times \to K_d(R) : (a_1, \ldots, a_d) \to \{a_1, \ldots, a_d\}.$$
If $A$ is any abelian group, a map $g : (R^\times)^d \to A$ which is multilinear and satisfies $g(a_1, \ldots, a_d) = 0$, whenever $a_i + a_j = 1$ for some $i \neq j$ is called an Steinberg map.

**Remark 4.1.1.** It follows from the very definition that any Steinberg map $g : (R^\times)^d \to A$ can be factored through the symbol map, i.e., there exist a homomorphism $g^d : K_d(R) \to A$ such that

$$
\begin{array}{ccc}
(R^\times)^d & \{ \} & K_d(R) \\
g & & g^d \\
& A &
\end{array}
$$

**Proposition 4.1.1.** The elements of the Milnor $K$-group satisfy the relations

1. $\{a_1, \ldots, a_i, \ldots, -a_i, \ldots, a_m\} = 1$

2. $\{a_1, \ldots, a_i, \ldots, a_j, \ldots, a_m\} = \{a_1, \ldots, a_j, \ldots, a_i, \ldots, a_m\}^{-1}$

**Proof.** To simplify the notation we will assume $m = 2$.

1. Noticing that $(1 - a)/(1 - 1/a) = -a$ it follows that

$$
\begin{align*}
\{a, -a\} &= \{a, 1 - a\}\{a, 1 - 1/a\}^{-1} \\
&= \{a, 1 - 1/a\}^{-1} \\
&= \{1/a, 1 - 1/a\} \\
&= 1
\end{align*}
$$
2. This follows immediately from the previous item

\[ \{a, b\} \{b, a\} = \{-b, b\} \{a, b\} \{b, a\} \{-a, a\} \]
\[= \{-ab, b\} \{-ab, a\} \]
\[= \{-ab, ab\} \]
\[= 1 \]

\[\square\]

From the definition we have \(K_1(R) = R^x\) and we define \(K_0(R) := \mathbb{Z}\). We also have a product

\[K_n(R) \times K_m(R) \rightarrow K_{n+m}(R)\]

where \(\{a_1, \ldots, a_n\} \times \{a_{n+1}, \ldots, a_{n+m}\} \mapsto \{a_1, \ldots, a_{n+m}\}\).

It is possible to define a norm on the Milnor-\(K\)-groups:

**Proposition 4.1.2.** For each finite extension of fields \(L/E\) there is a group homomorphism

\[N_{L/E} : K_m(L) \rightarrow K_m(E),\]

satisfying

1. When \(m = 1\) this maps coincides with the usual norm.
2. For the tower $L/E_1/E_2$ of finite extensions we have $N_{L/E_1} = N_{E_1/E_2} \circ N_{L/E_1}$.

3. The composition $K_m(E) \to K_m(L) \xrightarrow{N_{L/E}} K_m(E)$ coincides with multiplication by $[L/E]$.

4. If $\{a_1, \ldots, a_m\} \in K_m(L)$ with $a_1, \ldots, a_i \in L^\times$ and $a_{i+1}, \ldots, a_m \in E^\times$, then

$$N_{L/E}(\{a_1, \ldots, a_m\}) = N_{L/E}(\{a_1, \ldots, a_i\})\{a_{i+1}, \ldots, a_m\} \in K_m(E),$$

the right hand side is the product of a norm in $K_i(L)$ and a symbol in $K_{m-1}(E)$.

Proof. \cite{6} IV and \cite{8} 7.3. \qed

Note in particular that if $a_1 \in L^\times$ and $a_2, \ldots, a_m \in E^\times$, (1) and (4) imply

$$N_{L/E}(\{a_1, \ldots, a_m\}) = \{N_{L/E}(a_1), a_2, \ldots, a_m\}.$$

In the case where $L$ is a discrete valuation field there exist a boundary map between the Milnor K-group of $L$ and its residue field $k_L$ given in the following

**Proposition 4.1.3.** Suppose $L$ is a discrete valuation field with residue field $k_L$.

There is a unique group homomorphism

$$\partial : K_m(L) \to K_{m-1}(k_L)$$

which satisfies

$$\partial(u_1, \ldots, u_{m-1}, \pi_L) = \{\pi_1, \ldots, \pi_{m-1}\}.$$
for all uniformizers $\pi_L \in L^*$ and all $u_i \in \mathcal{O}_L^*$.

Proof. cf. [19] §2 Lemma 2.1. \qed

Remark 4.1.2. Note that in the case $m = 1$ the boundary map $\partial : L^* = K_1(L) \to K_0(k_L) = \mathbb{Z}$ is clearly the valuation map $v : L^* \to \mathbb{Z}$.

Remark 4.1.3. In the case $\mathcal{L} = L\{\{T\}\}$, the composition of boundary maps

$$K_2(L\{\{T\}\}) \xrightarrow{\partial} K_1(k_L((T))) \xrightarrow{\partial} K_0(k_L),$$

sends the element $\{T, u\}, u \in L^*$, to $v_L(u)$. Indeed, let $\pi_L$ be any uniformizer for $L$, then since $T$ is a unit for the valuation field $L\{\{T\}\}$ we have $\partial(\{T, \pi_L\}) = T$. By remark 4.1.2 we have that $\partial(T) = v_{k_L((T))}(T) = 1$. Thus $\partial(\{T, \pi_L\}) = 1 = v_L(\pi_L)$. Since this is true for all uniformizers $\pi_L$ of $L$ and $\partial \circ \partial$ is a homomorphisms then the claim follows.

Definition 4.1.2. We endow $K_d(\mathcal{L})$ with the finest topology $\lambda_d$ for which the map

$$(\mathcal{L}^*)^d \to K_d(\mathcal{L}) : (a_1, \ldots, a_d) \mapsto \{a_1, \ldots, a_d\}$$

is sequentially continuous in each component with respect to the product topology on $\mathcal{L}^*$ and for which subtraction in $K_d(\mathcal{L})$ is sequentially continuous. Define

$$K_d^{top}(\mathcal{L}) = K_d(\mathcal{L})/\Lambda_d(\mathcal{L}),$$
with the quotient topology where $\Lambda_d(\mathcal{L})$ denotes the intersection of all neighborhoods of 0 with respect to $\lambda_d$ (and so is a subgroup).

In [7] Chapter 6 Theorem 3, Fesenko proved that

$$\Lambda_d(\mathcal{L}) = \cap_{l \geq 1} lK_d(\mathcal{L}). \quad (4.1)$$

**Proposition 4.1.4.** Let $\mathcal{M}/\mathcal{L}$ be a finite extension, then norm $N_{\mathcal{M}/\mathcal{L}} : K_d(\mathcal{M}) \to K_d(\mathcal{L})$ induces a norm

$$N_{\mathcal{M}/\mathcal{L}} : K_d^{\text{top}}(\mathcal{M}) \to K_d^{\text{top}}(\mathcal{L}).$$

For this norm we have $N_{\mathcal{M}/\mathcal{L}}(\text{open subgroup})$ is open in $K_d^{\text{top}}(\mathcal{L})$. In particular, $N_{\mathcal{M}/\mathcal{L}}(K_d^{\text{top}}(\mathcal{M}))$ is open in $K_d^{\text{top}}(\mathcal{L})$.

**Proof.** Section 4.8, claims (1) and (2) of page 15 of [5]. \qed

### 4.1.2 The reciprocity map

**Theorem 4.1.1** (A. Parshin, K. Kato). Let $\mathcal{L}$ be a $d$-dimensional local field. Then there exist a reciprocity map

$$\Upsilon_\mathcal{L} : K_d(\mathcal{L}) \to \text{Gal}(\mathcal{L}^{ab}/\mathcal{L}),$$

satisfying the properties

1. If $\mathcal{M}/\mathcal{L}$ is a finite extension of $d$-dimensional local fields then the following diagrams commute:
\[
K_d(M) \xrightarrow{\gamma_M} \text{Gal}(M^{ab}/M) \quad K_d(M) \xrightarrow{\gamma_M} \text{Gal}(M^{ab}/M) \\
N_{M/L} \downarrow \text{restriction} \quad \uparrow \text{transfer} \\
K_d(L) \xrightarrow{\gamma_L} \text{Gal}(L^{ab}/L) \quad K_d(L) \xrightarrow{\gamma_L} \text{Gal}(L^{ab}/L)
\]

If moreover $M/L$ is abelian, then $\gamma_L$ induces an isomorphism

\[
K_d(L)/N_{M/L}(K_d(M)) \xrightarrow{\gamma_L} \text{Gal}(M/L)
\]

2. The map is compatible with the residue field $\bar{L}$ of $L$, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
K_d(L) & \xrightarrow{\gamma_L} & \text{Gal}(L^{ab}/L) \\
\downarrow \partial & & \downarrow \sigma \mapsto \sigma \\
K_{d-1}(\bar{L}) & \xrightarrow{\gamma_{\bar{L}}} & \text{Gal}(\bar{L}^{ab}/\bar{L})
\end{array}
\]

Here $\partial$ is the boundary map defined in Proposition 4.1.3.

3. The reciprocity map $\gamma_L$ is sequentially continuous if we endow $K_d(L)$ with the topology $\lambda_d$ from definition 4.1.2.

Proof. The first two assertions can be found in [11] § 1 Theorem 2. For the third one, let $M$ be a finite abelian extension of $L$, thus $\text{Gal}(L^{ab}/M)$ is an open neighborhood of $G_{ab}^{L}$. Let $x_n$ be a convergent sequence to the zero element of $K_d(L)$. Since $N_{M/L}(K_d^{\text{top}}(M))$ is an open subgroup of $K_d^{\text{top}}(L)$ by Proposition 4.1.4, then

\[
\exists n \in N_{M/L}(K_d^{\text{top}}(M)) \quad (n >> 0),
\]
where $x_n$ is the image of $x_n$ in $K_{d}^{top}(\mathcal{L})$. Thus, there exist $y_n \in K_d(\mathcal{M})$ and $\beta_n \in \Lambda_m(\mathcal{L})$ such that

$$x_n = \beta_n N_{\mathcal{M}/\mathcal{L}}(y_n) \quad (n >> 0).$$

From equation (4.1) we have that $\beta_n \in \bigcap_{l \geq 1} lK_d(\mathcal{L})$ which implies that $\Upsilon_{\mathcal{L}}(\beta_n)$ is the identity element in $G^{ab}_\mathcal{L}$ (because $G^{ab}_\mathcal{L}$ is a profinite group). Therefore

$$\Upsilon_{\mathcal{L}}(x_n) = \Upsilon_{\mathcal{L}}(N_{\mathcal{M}/\mathcal{L}}(y_n)) \quad (n >> 0),$$

but the element on the right hand side of this equality is the identity on $\text{Gal}(\mathcal{L}^{ab}/\mathcal{M})$ by the second item of this Theorem. It follows that $\Upsilon_{\mathcal{L}}(x_n)$ converges to the identity element of $G^{ab}_\mathcal{L}$.

\[ \Box \]

4.2 The pairing $(,)_\mathcal{L,n}$

Let $\mathcal{L}$ be a $d$-dimensional local field containing $K$ and the group $\kappa_n$. We have the pairing

$$(,)_\mathcal{L,n} : K_d(\mathcal{L}) \times F(\mu_\mathcal{L}) \to \kappa_n$$

defined by $(\{a_1, \ldots, a_d\}, x)_{\mathcal{L},n} = \Upsilon_{\mathcal{L}}(\{a_1, \ldots, a_d\})(Z) -_F Z$, where $f^{(n)}(Z) = x$ and $-_F$ is the subtraction in the formal group $F$.

**Proposition 4.2.1.** The pairing just defined satisfies the following:

1. $(,)_\mathcal{L,n}$ is bilinear and $C$-linear on the right.
2. The kernel on the right is \( f^{(n)}(F(\mu_\mathcal{L})) \).

3. \( (a, x)_{\mathcal{L}, n} = 0 \) if and only if \( a \in N_{\mathcal{L}/\mathcal{L}}(K_d(\mathcal{L}(z))) \), where \( f^{(n)}(z) = x \).

4. If \( \mathcal{M}/\mathcal{L} \) is finite, \( x \in F(\mu_\mathcal{L}) \) and \( b \in K_d(\mathcal{M}) \). Then

\[
(b, x)_{\mathcal{M}, n} = (N_{\mathcal{M}/\mathcal{L}}(b), x)_{\mathcal{L}, n}.
\]

5. Let \( L \supset \kappa_m, m \geq n \). Then

\[
(a, x)_{\mathcal{L}, n} = f^{(m-n)}((a, x)_{\mathcal{L}, m}) = (a, f^{(m-n)}(x))_{\mathcal{L}, m}
\]

6. For a given \( x \in K_d(\mathcal{L}) \), the map

\[
K_d(\mathcal{L}) \to \kappa_n : a \mapsto (a, x)_{\mathcal{L}, n}
\]

is sequentially continuous.

7. Let \( M \) be a finite extension of \( L \), \( a \in K_d(\mathcal{L}) \) and \( y \in F(\mu_\mathcal{M}) \). Then

\[
(a, y)_{\mathcal{M}, n} = (a, N_{\mathcal{M}/\mathcal{L}}^F(y))_{\mathcal{M}, n},
\]

where \( N_{\mathcal{M}/\mathcal{L}}^F(y) = \oplus_\sigma y^\sigma \), where \( \sigma \) ranges over all embeddings of \( M \) in \( \overline{K} \) over \( L \). Notice that such an embedding extends uniquely to an embedding from \( \mathcal{M} = M\{\{T\}\} \) over \( \mathcal{L} = L\{\{T\}\} \).
8. Let $t : F \to \tilde{F}$ be a isomorphism. Then

$$(a, t(x))_{L, n}^\tilde{F} = t((a, x)_{L, n}^F)$$

for all $a \in K_d(L)$, $x \in F(\mu_L)$.

Proof. The first 5 properties and the last one follow from the definition of the pairing and Theorem 4.1.1.

The property 6 follows from the fact that the reciprocity map $\Upsilon_L : K_m(L) \to \text{Gal}(L^{ab}/L)$ is sequentially continuous, here $K_d(L)$ is endowed with the finest topology for which the map

$$\phi : L^{*d} \to K_d(L)$$

is sequentially continuous in each component with respect to the product topology on $L^*$ and for which subtraction in $K_d(L)$ is sequentially continuous. Then for $z$ such that $f^{(n)}(z) = x$ consider the extension $L(z)/L$. The group $\text{Gal}(L^{ab}/L(z))$ is a neighborhood of $G^{ab}_L$, so for any sequence $\{a_m\}$ converging to zero in $K_d(L)$ we can take $m$ large enough such that $\Upsilon_L(a_m) \in \text{Gal}(L^{ab}/L(z))$, that is $\Upsilon_L(a_m)(z) = z$, so $(a_m, x)_{L, n} = 0$ for large enough $m$.

Finally, let us prove property 7. Let $f^{(n)}(z) = y$ an take a finite Galois extension $N \supset M(z)$ over $L$. Let $G = G(N/L)$ and $H = G(N/M)$, $w = [G : H]$, and $V : G/G' \to H/H'$ the transfer homomorphism. Let $g = \Upsilon_L(a)$, then by Theorem 4.1.1 we have $V(\Upsilon_L(a)) = \Upsilon_M(a)$. The explicit computation of $V$ at $g \in G$ proceeds
as follows (cf. [23 § 3.5 ). Let \( \{ c_i \} \) be a set of representatives of for the right cosets of \( H \) in \( G \), i.e., \( G = \sqcup Hc_i \). Then for each \( c_i, i = 1, \ldots, w \) there exist a \( c_j \) such that 
\[ c_i g c_j^{-1} = h_i \in H \]
and no two \( c_j \)'s are equal; this is because \( c_i g \) belongs to one and only one of the right cosets \( Hc_j \). Then 
\[ V(g) = \prod_{i=1}^{w} h_i. \]

Also, notice that since 
\[ gc_j^{-1} = c_i^{-1} h_i \]
then 
\[ h_i(z) \otimes z = c_i^{-1}(h_i(z) \otimes z) = g(c_j^{-1}(z)) \otimes c_i^{-1}(z). \]

So we have 
\[
(a, y)_{\mathcal{M}, n} = \Upsilon_{\mathcal{M}}(a)(z) \otimes z \\
= V(g)(z) \otimes z \\
= (\prod_{i=1}^{w} h_i)(z) \otimes z \\
= \bigoplus_{i=1}^{w} (h_i(z) \otimes z) \\
= \bigoplus_{i=1}^{w} (g(c_j^{-1}(z)) \otimes c_i^{-1}(z)) \\
= g\left( \bigoplus_{i=1}^{w} c_j^{-1}(z) \right) \otimes \left( \bigoplus_{i=1}^{w} c_j^{-1}(z) \right) \\
= (a, N_{\mathcal{M}/L}^{\mathcal{F}}(y))_{\mathcal{L}, n}
\]
the last equality being true since \( g = \Upsilon_{\mathcal{L}}(a) \) and 
\[ f^{(n)}(\bigoplus_{i=1}^{w} c_j^{-1}(z)) = \bigoplus_{i=1}^{w} c_j^{-1}(y) = N_{\mathcal{M}/L}^{\mathcal{F}}(y). \]
Let \( \alpha \) denote the ramification index of \( S \) over \( \mathbb{Q}_p \). We say that a pair \((n,t)\) is admissible if there exist an integer \( k \) such that \( t - 1 - n \geq \alpha k \geq n \). For example, the pair \((n,2n+\alpha+1)\) is admissible with \( k = \lceil (n+\alpha/\alpha) \rceil \). In the special case where \( \alpha = 1 \), then \( t - 1 - n \geq \alpha k \geq n \) becomes \( t - 1 - n \geq k \geq n \). So in this case, the pair \((n,2n+1)\) is admissible with \( k = n \).

The following proposition is vital for the main results in this paper. The idea of the proof was inspired by the proof of Proposition 4.1 in [14].

**Proposition 4.2.2.** Let \( L \supset K_t = K(\kappa_t) \) with \((n,t)\) an admissible pair and let \( \mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). For a given \( \alpha \in K_d(\mathcal{L}) \), the map

\[
F(\mu_{\mathcal{L},1}) : \kappa_n : x \mapsto (\alpha, x)_{\mathcal{L},n}.
\]

is sequentially continuous, i.e., if \( x_k \to x \) then \( (\alpha, x_k)_{\mathcal{L},n} \to (\alpha, x)_{\mathcal{L},n} \). Here \( F(\mu_{\mathcal{L},1}) \) is the set

\[
\left\{ x \in \mathcal{L} : v_{\mathcal{L}}(x) \geq \left\lceil \frac{v_L(p)}{p-1} \right\rceil + 1 \right\}.
\]

considered with the operation induced by the formal group \( F \).

**Remark:** We will make the following two assumptions. First notice that it is enough to prove the result for \( a \in \mathcal{L}^* \) such that \( v_\mathcal{L}(a) = 1 \), because then it will be true for \( \pi_L \) and \( \pi_L u \) with \( v_\mathcal{L}(u) = 0 \). Here \( \pi_L \) a uniformizer for \( L \).

Let \( r \) be a \( t \)-normalized series, cf. \( \S 4.3 \). Notice also that we may assume that \( r(X) = X \) is a \( t \)-normalized, otherwise we go to the isomorphic group law
CHAPTER 4. THE KUMMER PAIRING

\( r(F(r^{-1}(X), r^{-1}(Y))). \)

**Proof.** We will drop the subscript \( \mathcal{L} \) from the pairing notation and will make the two assumptions in the remark above, i.e., let \( u \in \mathcal{L}^* \) with \( v_{\mathcal{L}}(u) = 1 \), \( b_2 \ldots , b_d \in \mathcal{L}^* \) and \( x \in F(\mu_{\mathcal{L},1}) \). Then

\[
\left( \{u, b_2 \ldots , b_d\}, x \right)_n = \left( \left\{ \frac{u}{u \oplus f^{(\alpha k+1)}(x)}, b_2 \ldots , b_d \right\}, x \right)_n \oplus \left( \left\{ u \oplus f^{(\alpha k+1)}(x), b_2 \ldots , b_d \right\}, x \right)_n. \tag{4.2}
\]

We will show that the first term on the right hand side is zero and that the second side goes to zero when we take a sequence \( \{x_k\} \) converging to zero. This would complete the proof.

Let us start with the second term. Let \( m = n + \alpha k + 1 \). By (5) in the Proposition 4.2.1 and the fact that \( r(X) = X \), i.e., \( \left( \{a, b_2 \ldots , b_d\}, a \right) = 0 \ \forall a \in F(\mu_{\mathcal{L}}) \), it follows
that
\[
\left( \{ u \oplus f^{(\alpha k+1)}_m(x), b_2, \ldots, b_d \}, x \right) 
\]
\[
= \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, f^{(m-n)}(x) \right)_m
\]
\[
= \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, u \oplus f^{(\alpha k+1)}(x) \ominus u \right)_m
\]
\[
= \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, u \oplus f^{(\alpha k+1)}(x) \right)_m \oplus \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, \ominus u \right)_m
\]
\[
= \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, \ominus u \right)_m
\]
\[
= \left( \{ u^{-1}, b_2, \ldots, b_d \}, \ominus u \right)_m \oplus \left( \{ u \oplus f^{(\alpha k+1)}(x), b_2, \ldots, b_d \}, \ominus u \right)_m
\]
\[
= \left( \left\{ \frac{u \oplus f^{(\alpha k+1)}(x)}{u}, b_2, \ldots, b_d \right\}, \ominus u \right)_m
\]

Since \( F(X, Y) \equiv X + Y \pmod{XY} \), then \( u \oplus f^{(\alpha k+1)}(x) \equiv u + f^{(\alpha k+1)}(x) \pmod{u f^{(\alpha k+1)}(x)} \)
and so
\[
\frac{u \oplus f^{(\alpha k+1)}(x)}{u} \equiv 1 + \frac{f^{(\alpha k+1)}(x)}{u} \pmod{f^{(\alpha k+1)}(x)} (4.3)
\]

But \( v_L(u) = 1 \), so this element is a principal unit and if we take a sequence \( \{ x_i \}_{i \geq 1} \) converging to zero in the Parshin topology then, as \( f : \mu_{L,1} \to \mu_{L,1} \) is sequentially continuous in the Parshin topology by Lemma \[3.3.2\] this element goes to 1 and hence
\[
\left\{ \frac{u \oplus f^{(\alpha k+1)}(x_i)}{u}, b_2, \ldots, b_d \right\} \to \{ 1, b_2, \ldots, b_d \}
\]
in the topology of \( K_2(L) \). Notice that \( \{ 1, b_2, \ldots, b_d \} = \{ 1 - b_2, b_2, \ldots, b_d \} \{ 1, b_2, \ldots, b_d \} = \{ 1 - b_2, b_2, \ldots, b_d \} = 1 \), where 1 is the unit element in \( K_2(L) \). Then by (6) in the
Proposition 4.2.1

\[
\left\{ \frac{u \oplus f^{(\alpha k+1)}(x_i)}{u}, b_2, b_3, \ldots, b_d \right\}, \ominus u \xrightarrow{m \to \infty} (1, \ominus u)_m = 0
\]

Now we will show that first term on the right hand side of equation (4.2) is zero by showing that \((u \oplus f^{(\alpha k+1)}(x))/u\) is a \(p^k\)th power in \(\mathcal{L}^*\), which is enough since \(\pi^n\) divides \(p^k\), because \(n \leq \alpha k\). Indeed, since \(x \in F(\mu_{\mathcal{L},1})\) then by Proposition 3.3.1

\[
f^{(\alpha k+1)}(x) = l_F^{-1} \circ l_F(f^{(\alpha k+1)}(x)) = l_F^{-1}(\pi^{\alpha k+1}l_F(x)) = \pi^{\alpha k+1}w
\]

for some \(w \in \mu_{\mathcal{L},1}\). Then equation (4.3) implies

\[
\frac{u \oplus f^{(\alpha k+1)}(x)}{u} = 1 + p^k w_2
\]

for \(w_2 \in \mu_{\mathcal{L},1}\), since \(\pi^{\alpha k} = \epsilon p^k\) for some unit \(\epsilon\). Then

\[
\log \left( \frac{u \oplus f^{(\alpha k+1)}(x)}{u} \right) = \log(1 + p^k w_2) = p^k w_3
\]

where \(w_3 \in \mu_{\mathcal{L},1}\), so there exist a \(w_4 \in \mu_{\mathcal{L},1}\) such that \(\log(1 + w_4) = w_3\), thus

\[
\frac{u \oplus f^{(\alpha k+1)}(x)}{u} = (1 + w_4)^n
\]

and so

\[
\left( \left\{ \frac{u}{u \oplus f^{(\alpha k+1)}(x)}, b_2, \ldots, b_d \right\}, x \right)_n = 0.
\]

This proves the theorem.
4.3 Norm Series

A power series \( r \in \mathcal{O}_K[[X]], r(0) = 0, c(r) \in \mathcal{O}_K^* \) is called \( n \)-normalized if for every local field \( L \supset \kappa_n, \mathcal{L} = L\{T_1\} \ldots \{T_{d-1}\} \), and all \( 1 \leq i \leq d \)

\[
(a_1, \ldots, a_{i-1}, r(x), a_{i+1}, \ldots, a_d, x)_{\mathcal{L}, n} = 0,
\]

for all \( x \in F(\mu_{\mathcal{L}}) \) and all \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d \in \mathcal{L}^* \).

The following proposition will provide a way of constructing norm series.

**Proposition 4.3.1.** Let \( g \in \mathcal{O}_K[[X]], g(0) = 0 \) and \( c(g) \in \mathcal{O}_K^* \). The series \( s = \prod_{v \in \kappa_n} g(F(X, v)) \) belongs to \( \mathcal{O}_K[[X]] \) and has the form \( r_g(f(n)) \), where \( r_g \in \mathcal{O}_K[[X]] \).

Then, the series \( r_g \) is \( n \)-normalized and

\[
r_g'(0) = \frac{\prod_{v \not\in \kappa_n} g(v)}{\pi^n} g'(0).
\]

**Proof.** The proof follows closely [14] Proposition 3.1.

First, the coefficients of \( s \) are in \( \mathcal{O}_K \) because \( s^\sigma = s \) for every \( \sigma \in G_K = Gal(\overline{K}/K) \). Now, applying Lemma 3.1.1 to \( s \) and \( f^{(n)} \), we get \( s = Ps_1 \) and \( f^{(n)} = Qf_1 \), where \( P \) and \( Q \) is a monic polynomials and \( s_1, f_1 \in \mathcal{O}_K[[X]]^* \). Since \( s(F(X, v)) = s(X) \) for all \( v \in \kappa_n \), then \( P(v) = 0 \) for all \( v \in \kappa_n \) and so \( Q = \prod_{v \in \kappa_n} (X - v) \) divides \( P \). This implies that \( s \) is divisible by \( f^{(n)} \), i.e., \( s = f^{(n)}(a_0 + a_1X + \cdots) \).
In particular,

\[ s - f^{(n)}a_0 = f^{(n)}(a_1X + \cdots). \]

But from \( s(F(X, v)) = s(X) \) we see that \( a_1X + \cdots \) must satisfy the same property and so \( a_1v + \cdots = 0 \), for all \( v \in \kappa_n \). Therefore this series is also divisible by \( f^{(n)} \) and repeating the process we get \( s = r_g(f^{(n)}) \). Let us compute now \( c(r_g) \). Taking the logarithmic derivative on \( s \) and then multiplying by \( X \) we get

\[
\frac{s'(X)X}{s(X)} = \sum_{v \in \kappa_n} \frac{g'(F(X, v))F_X(X, v)X}{g(F(X, v))},
\]

which implies

\[
\frac{s'(0)}{\prod_{0 \neq v \in \kappa_n} g(v)} = g'(0),
\]

From \( s' = r'_g(f^{(n)})f^{(n)'} \) we obtain

\[
r'_g(0) = \frac{s'(0)}{f^{(n)'}(0)} = \frac{c(g) \prod_{0 \neq v \in \kappa_n} g(v)}{\pi^n}.
\]

Each \( g(v) \) is associated to \( v \), \( 0 \neq v \in \kappa_n \), then \( \prod_{0 \neq v \in \kappa_n} g(v) \) is associated to \( \prod_{0 \neq v \in \kappa_n} v \), but the latter is associated to \( \pi^n \) from the equation \( f = Pf_1 \). Then \( c(r_g) \in \mathcal{O}_K \).

Finally, we will show that \( (\{a_1, \ldots, a_{i-1}, r_g(x), a_{i+1}, \ldots, a_d\}, x) = 0 \). Let \( L \) be a local field containing \( \kappa_n \), \( \mathcal{L} = L\{T_1\} \cdots \{T_{d-1}\}, x \in F(\mu_\mathcal{L}) \) and \( z \) such that \( f^{(n)}(z) = x \). Then

\[
r_g(x) = \prod_{v \in \kappa_n} g(z + Fv) = \prod_i N_{\mathcal{L}(z)/\mathcal{L}}(g(z_i)),
\]
where the $z_i$ are pairwise non-conjugate over $L$ distinct roots of \( f^{(n)}(X) = x \), so

\[
\{a_1, \ldots, a_{i-1}, r_g(x), a_{i+1}, \ldots, a_d\} = \{a_1, \ldots, a_{i-1}, N_{L(z)/L}(\prod_i g(z_i)), a_{i+1}, \ldots, a_d\}
\]

\[
= N_{L(z)/L}(\{a_1, \ldots, a_{i-1}, \prod_i g(z_i), a_{i+1}, \ldots, a_d\}),
\]

The last equality follows from Proposition 4.1.2 (1) and (4). The result now follows from Proposition 4.2.1.
Chapter 5

The maps \( \psi \) and \( \rho \)

5.1 The generalized trace

Let \( E \) be a complete discrete valuation field. We define a map

\[
c_{E\{\{T\}\}/E} : E\{\{T\}\} \rightarrow E,
\]

by \( c_{E\{\{T\}\}/E}(\sum_{i \in \mathbb{Z}} a_i T^i) = a_0 \). Let \( \mathcal{E} = E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \), we can define \( c_{\mathcal{E}/E} \) by the composition

\[
c_{E\{\{T_1\}\}/E} \circ \cdots \circ c_{\mathcal{E}/E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\}}.
\]

Lemma 5.1.1. This map satisfies the following properties

1. \( c_{\mathcal{E}/E} \) is \( E \)-linear.
2. \( c_{\mathcal{E}/E}(a) = a \), for all \( a \in E \).
3. \( c_{\mathcal{E}/E} \) is continuous with respect to the the Parshin topology on \( \mathcal{E} \) and the discrete valuation topology on \( E \).

Suppose $L/S$ is a finite extension of local fields. Let $\pi$ be a uniformizer for $C = \mathcal{O}_S$ and let $\mathcal{L} = L\{T_1\} \ldots \{T_{d-1}\}$. The generalized trace $\mathbb{T}_{\mathcal{L}/S}$ is defined by the composition $\text{Tr}_{L/S} \circ c_{\mathcal{L}/L}$ and this gives us the pairing

$$\langle \cdot, \cdot \rangle : \mathcal{L} \times \mathcal{L} \to C, \quad (5.1)$$

defined by $\langle x, y \rangle = \mathbb{T}_{\mathcal{L}/S}(xy)$. Let $\text{Hom}_C^C(\mathcal{L}, S)$ and $\text{Hom}_C^{seq}(\mathcal{L}, S)$ be the group of continuous and sequentially continuous, respectively, $C$-homomorphisms with respect to the Parshin topology on $\mathcal{L}$.

**Proposition 5.1.1.** We have an isomorphism of $C$-modules

$$\mathcal{L} \xrightarrow{\sim} \text{Hom}_C^{seq}(\mathcal{L}, S) : \alpha \mapsto (x \mapsto \mathbb{T}_{\mathcal{L}/S}(\alpha x)).$$

In particular, $\text{Hom}_C^{seq}(\mathcal{L}, S) = \text{Hom}_C^C(\mathcal{L}, S)$ since the generalized trace is continuous.

**Proof.** Here we prove the case where the dimension $d$ is equal to 2, i.e., $\mathcal{L} = L\{T\}$. The general case is proved in the remark [5.1.1] after this proposition.

Let $\phi : \mathcal{L} \to S$ be a sequentially continuous $C$-linear map and define, for each $i \in \mathbb{Z}$, the map $\phi_i(x) = \phi(xT^i)$ for all $x \in L$. Then clearly $\phi_i \in \text{Hom}_C(L, S)$ and this corresponds to the case $d = 1$ for which we know that it exists an $a_{-i} \in L$ such that $\phi(xT^i) = \text{Tr}_{L/S}(a_{-i}x)$ for all $x \in L$. Let $\alpha = \sum a_i T^i$, we must show that

1. $\min \{ v_L(a_i) \} > -\infty$. 

II. \( v_L(a_{-i}) \to \infty \) as \( i \to \infty \) (i.e., conditions (I) and (II) imply that \( \alpha \in \mathcal{L} \)).

III. \( \phi(x) = T_{L/S}(\alpha x), \forall x \in \mathcal{L} \).

For any \( x = \sum x_i T^i \in \mathcal{L} \) we have, by the sequential continuity of \( \phi \) that

\[
\phi(x) = \sum_{i \in \mathbb{Z}} \phi(x_i T^i) = \sum_{i \in \mathbb{Z}} \text{Tr}_{L/S}(a_{-i} x_i). \tag{5.2}
\]

Suppose (I) was not true, then there exist a subsequence \( \{a_{n_k}\} \) such that \( v_L(a_{n_k}) \to -\infty \) as \( n_k \to \infty \) or as \( n_k \to -\infty \). In the first case we take \( x = \sum x_i T^i \in \mathcal{L} \) such that \( x_i \) is equal to \( 1/a_{n_k} \) if \( i = -n_k \) and 0 if \( i \neq -n_k \). So \( a_{-i}x_i = 1 \) if \( i = -n_k \) and 0 if \( i \neq -n_k \). Then the sum on the right of (5.2) would not converge. In the second case we take \( x_i \) to be equal to \( 1/a_{n_k} \) if \( i = -n_k \) and 0 if \( i \neq -n_k \). So \( a_{-i}x_i = 1 \) if \( i = -n_k \) and 0 if \( i \neq -n_k \) and again the sum on the right would not converge.

Suppose (II) was not true. Then \( v_E(a_{n_k}) < M \) for some positive integer \( M \) and \( a \) of negative integers \( n_k \to -\infty \). Then take \( x = \sum x_i T^i \in \mathcal{L} \) such that \( x_i \) is equal to \( 1/a_{n_k} \) for \( i = -n_k \) and 0 for \( i \neq -n_k \). So \( a_{-i}x_i = 1 \) if \( i = -n_k \) and 0 if \( i \neq -n_k \) and the sum on the right of (5.2) would not converge.

Finally, (III) follows by noticing that by (I) and (II) the sum \( \sum_{i \in \mathbb{Z}} a_{-i} x_i \) converges and

\[
\sum_{i \in \mathbb{Z}} T_{E/S}(a_{-i} x_i) = T_{E/S}(\sum_{i \in \mathbb{Z}} a_{-i} x_i) = T_{L/S}(x\alpha),
\]

since \( T_{E/S} \circ c_{\mathcal{L}/E} = T_{\mathcal{L}/S} \).
Remark 5.1.1. We include the proof of the general case of the proposition above.

Proof. The proof is done by induction in $d$. If $d = 1$ the result is known. Suppose the result is true for $d \geq 1$ and let $\mathcal{L} = E\{\{T_d\}\}$ where $E = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$.

Let $\phi : \mathcal{L} \rightarrow S$ be a sequentially continuous $C$-linear map and define, for each $i \in \mathbb{Z}$, the sequentially continuous map $\phi_i(x) = \phi(xT^i_d)$ for all $x \in E$. Then clearly $\phi_i \in \text{Hom}_C(E, S)$ and by the induction hypothesis we know that there exists an $a_{-i} \in E$ such that $\phi(xT^i_d) = \mathbb{T}_{E/S}(a_{-i}x)$ for all $x \in E$. Let $\alpha = \sum a_i T^i_d$, we must show that

I. $\min\{v_E(a_i)\} > -\infty$.

II. $v_E(a_{-i}) \rightarrow \infty$ as $i \rightarrow \infty$ (i.e., conditions (I) and (II) imply that $\alpha \in \mathcal{L}$).

III. $\phi(x) = \mathbb{T}_{\mathcal{L}/S}(\alpha x), \forall x \in \mathcal{L}$.

For any $x = \sum x_i T^i_d \in \mathcal{L}$ we have, by the sequential continuity of $\phi$ that

$$\phi(x) = \sum_{i \in \mathbb{Z}} \phi(x_i T^i_d) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E/S}(a_{-i}x_i). \quad (5.3)$$

Suppose (I) was not true, then there exist a subsequence $\{a_{n_k}\}$ such that $v_E(a_{n_k}) \rightarrow -\infty$ as $n_k \rightarrow \infty$ or as $n_k \rightarrow -\infty$. In the first case we take an $x = \sum x_i T^i_d \in \mathcal{L}$ such that $x_i$ is equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i}x_i=1$ if $i = -n_k$ and 0 if $i \neq -n_k$. Then the sum on the right of (5.3) would not converge. In the second case
we take \( x_i \) to be equal to \( 1/a_nk \) if \( i = -nk \) and 0 if \( i \neq -nk \). So \( a_{-i}x_i = 1 \) if \( i = -nk \) and 0 if \( i \neq -nk \) and again the sum on the right would not converge.

Suppose (II) was not true. Then \( v_L(a_nk) < M \) for some positive integer \( M \) and a of negative integers \( n_k \to -\infty \). Then take \( x = \sum x_iT_d^i \in L \) such that \( x_i \) is equal to \( 1/a_nk \) for \( i = -nk \) and 0 for \( i \neq -nk \). So \( a_{-i}x_i = 1 \) if \( i = -nk \) and 0 if \( i \neq -nk \) and the sum on the right of (5.3) would not converge.

Finally, (III) follows by noticing that by (I) and (II) the sum \( \sum_{i \in \mathbb{Z}} a_{-i}x_i \) converges and

\[
\sum_{i \in \mathbb{Z}} \text{Tr}_{L/S}(a_{-i}x_i) = \text{Tr}_{L/S}(\sum_{i \in \mathbb{Z}} a_{-i}x_i) = T_{L/S}(x\alpha).
\]

Let \( L = L\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \) where \( L \) is a local field. Let \( v_L \) and \( v_L \) denote the valuations for \( L \) and \( L \) respectively. Consider

\[
\mu_{L,1} := \left\{ x \in L : v_L(x) \geq \left[ \frac{v_L(p)}{p-1} \right] + 1 \right\},
\]

and

\[
\mu_{L,1} := \mu_{L,1} \cap L = \left\{ x \in L : v_L(x) \geq \left[ \frac{v_L(p)}{p-1} \right] + 1 \right\},
\]

both with the additive structure, and denote by \( R_{L,1} \) the dual of \( \mu_{L,1} \) with respect
to the pairing (5.1), i.e.,

\[ R_{L,1} := \{ x \in L : \mathbb{T}_{L/S}(x \mu_{L,1}) \subset C \}. \]

Let us show that

\[ R_{L,1} = \{ x \in L : v_L(x) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] - 1 \}, \quad (5.4) \]

where \( D(L/S) \) is the different of the extension \( L/S \). Indeed, for \( d = 1 \) this is proven in [14] §4.1. Suppose it is true for \( d \geq 1 \), and let \( L = E_d \{ \{ T_d \} \} \). If \( x = \sum_{i \in \mathbb{Z}} x_i T_i \in R_{L,1} \), then since \( \mu_{E_d,1} \subset \mu_{L,1} \) we have that also \( \mu_{E_d,1} T_{d}^{-i} \subset \mu_{L,1} \) and

\[ \mathbb{T}_{L/S}(x \mu_{E_d,1} T_{d}^{-i}) \subset C, \]

which implies \( \mathbb{T}_{E_d/S}(x_i \mu_{E_d,1}) \subset C \), since \( \mathbb{T}_{L/S} = \mathbb{T}_{E_d/S} \circ c_{L/E_d} \). By induction hypothesis we have \( v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] - 1 \) for all \( i \in \mathbb{Z} \), therefore

\[ v_L(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] - 1. \]

Conversely, if \( v_L(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] - 1 \), then \( v_{E_d}(x_i) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] - 1 \) for all \( i \in \mathbb{Z} \). Then, by the induction hypothesis \( \mathbb{T}_{E_d/S}(x_i \mu_{E_d,1}) \subset C \) for all \( i \in \mathbb{Z} \), and therefore

\[ \mathbb{T}_{L/S}(x \mu_{L,1}) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{L/S}(x_i T_i \mu_{L,1}) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E_d/S}(x_i \mu_{E_d,1}) \subset C. \]
Thus, identity (5.4) holds.

Let
\[ R_{L,1} := R_{L,1} \cap L = \left\{ x \in L : v_L(x) \geq -v_L(D(L/S)) - \frac{v_L(p)}{p - 1} - 1 \right\}. \]

**Lemma 5.1.2.** We have the isomorphism
\[ R_{L,1}/\pi^n R_{L,1} \xrightarrow{\sim} \text{Hom}_C^{\text{seq}}(\mu_{L,1}, C/\pi^n C), \]
defined by
\[ \alpha \mapsto (x \mapsto \mathbb{T}_{L/S}(\alpha x)). \]
In particular, \( \text{Hom}_C^{\text{seq}}(\mu_{L,1}, C/\pi^n C) = \text{Hom}_C^C(\mu_{L,1}, C/\pi^n C). \)

**Proof.** The proof is similar to that of Proposition 5.1.1. As in the proof of that proposition we will consider the case \( d = 2, \) i.e., \( \mathcal{L} = L\{T\}, \) and leave the general case as a remark after the proof of this proposition. Take \( \Phi \in \text{Hom}_C^{\text{seq}}(\mu_{L,1}, C/\pi^n C) \) and let \( \Phi_i(x_i) = \Phi(x_i T^i) \) for all \( x_i \in \mu_{L,1}. \) Then \( \Phi_i \in \text{Hom}_C^C(\mu_{L,1}, C/\pi^n C) \) and so there exist \( \overline{\alpha_{-i}} \in R_{L,1}/\pi^n R_{L,1} \) such that
\[ \Phi_i(x_i) = \text{Tr}_{L/S}(\overline{\alpha_{-i}} x_i). \]

Let \( a_{-i} \in R_{L,1} \) be a representative of \( \overline{\alpha_{-i}}. \) Thus for \( x = \sum x_i T^i \in \mu_{L,1}, \) the sequential continuity of \( \Phi \) implies
\[ \Phi(x) = \sum_{i \in \mathbb{Z}} \Phi(x_i T^i) e^{\sum_{i \in \mathbb{Z}} \text{Tr}_{L/S}(a_{i} x_i)} \pmod{\pi^n C}. \leqno{(5.5)} \]

Let \( \alpha = \sum a_i T^i \) and denote by \( u_i \) the unit \( a_i / \pi_L^{v_L(a_i)} \). We must show that

I. \( \min\{v_L(a_i)\} > -\infty \).

II. \( v_L(a_{-i}) \geq v_L(\pi^n R_L, 1) \) as \( i \to \infty \) (i.e., conditions 1 and 2 imply that \( \alpha \in R_L/\pi^n R_L \)).

III. \( \Phi(x) = T_{L/S}(\alpha x) \pmod{\pi^n C}, \forall x \in \mu_L \).

Condition (I) follows immediately since \( a_{-i} \in R_L, 1 \), i.e., by equation (5.4)

\[ v_L(a_{-i}) \geq -\left( \left[\frac{v_L(p)}{p-1}\right] + 1 + v_L(D(L/S)) \right) \quad \forall i \in \mathbb{Z}. \]

Suppose condition (II) was not true. Instead of passing to a subsequence we may assume for simplicity that \( v_L(a_{-i}) < v_L(\pi^n R_L, 1) \) for all \( i \geq 0 \). Let

\[ x = y(\pi_0^{\delta_0} u_0^{-1} + \pi_1^{\delta_1} u_1^{-1} + \pi_2^{\delta_2} u_2^{-1} + \cdots), \]

where \( u_i = a_i / \pi_L^{v_L(a_i)} \) and

\[ \delta_i = v_L(\pi^n R_L, 1) - v_L(a_{-i}) + [v_L(p)/(p-1)] \quad (\geq [v_L(p)/(p-1)] + 1), \]

for \( i \geq 0 \) and \( y \in O_L \) is arbitrary. Then \( x \in \mu_L \) and \( a_{-i} x_i = \pi_L^{\delta_i} y \) for \( i \geq 0 \), where

\( w = v_L(\pi^n R_L, 1) + [v_L(p)/(p-1)] \). The convergence of the right hand side of (5.5)
would imply that

\[ \text{Tr}_{L/S}(\pi_L^w/y) \in \pi^nC \quad \forall y \in \mathcal{O}_L. \]

Thus \( \pi^w_L/\pi^n \in D(L/S)^{-1} \), which implies \( w \geq v_L(\pi^n) - v_L(D(L/S)) \), that is,

\[ v_L(R_{L,1}) \geq -v_L(D(L/S)) - [v_L(p)/(p-1)], \]

which is a contradiction since \( v_L(R_{L,1}) = -v_L(D(L/S)) - [v_L(p)/(p-1)] - 1 \). Finally, condition (III) immediately follows from equation (5.5).

Remark 5.1.2. Next we will include the proof in the general case of the proposition above.

Proof. The proof is done by induction in \( d \). If \( d = 1 \) the result is known. Suppose the result is true for \( d \geq 1 \) and let \( \mathcal{L} = E\{\{T_d\}\} \) where \( E = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \).

Take \( \Phi \in \text{Hom}^{\text{seq}}(\mu_{L,1}, C/\pi^nC) \) and let \( \Phi_i(x_i) = \Phi(x_i T^i_d) \) for all \( x_i \in \mu_{E,1} \). Then \( \Phi_i \in \text{Hom}^{\text{c}}(\mu_{E,1}, C/\pi^nC) \) and so by the induction hypothesis there exists \( \bar{a}_{-i} \in R_{E,1}/\pi^n R_{E,1} \) such that

\[ \Phi_i(x_i) = T_{E/S}(\bar{a}_{-i} x_i). \]

Let \( a_{-i} \in R_{E,1} \) be a representative of \( \bar{a}_{-i} \). Thus for \( x = \sum x_i T^i_d \in \mu_{L,1} \), the sequential continuity of \( \Phi \) implies

\[ \Phi(x) = \sum_{i \in \mathbb{Z}} \Phi(x_i T^i_d) = \sum_{i \in \mathbb{Z}} T_{E/S}(a_{-i} x_i) \pmod{\pi^nC}. \quad (5.6) \]
Let $\alpha = \sum a_i T_d^i$ and denote by $u_i$ the unit $a_i/\pi_{L}^{v_E(a_i)}$. We must show that

I. $\min\{v_E(a_i)\} > -\infty$.

II. $v_E(a_{-i}) \geq v_E(\pi^n R_{E,1})$ as $i \to \infty$ (i.e., conditions 1 and 2 imply that $\alpha \in R_{L,1}/\pi^n R_{L,1}$).

III. $\Phi(x) = T_{L/S}(\alpha x) \pmod{\pi^n C}$, $\forall x \in \mu_{L,1}$.

Condition (I) follows immediately since $a_{-i} \in R_{E,1}$, i.e., by equation (5.4)

$$v_E(a_{-i}) \geq -\left( \left[ \frac{v_L(p)}{p-1} \right] + 1 + v_L(D(L/S)) \right) \quad \forall i \in \mathbb{Z}.$$ 

Suppose condition (II) was not true. Instead of passing to a subsequence we may assume for simplicity that $v_E(a_{-i}) < v_E(\pi^n R_{E,1})$ for all $i \geq 0$. Let

$$x = y(\pi_{L}^{b_0} u_0^{-1} + \pi_{L}^{b_1} u_1^{-1} T_d + \pi_{L}^{b_2} u_2^{-1} T_d^2 + \cdots),$$

where $u_i = a_i/\pi_{L}^{v_E(a_i)}$ and

$$\delta_i = v_E(\pi^n R_{E,1}) - v_E(a_{-i}) + [v_L(p)/(p-1)] \quad (\geq [v_L(p)/(p-1)] + 1),$$

for $i \geq 0$ and $y \in \mathcal{O}_L$ is arbitrary. Then $x \in \mu_{L,1}$ and $a_{-i} x_i = \pi_{L}^w y$ for $i \geq 0$, where $w = v_E(\pi^n R_{E,1}) + [v_L(p)/(p-1)]$. The convergence of the right hand side of (5.6) and the fact that $T_{E/S}(\pi_{L}^w) = \text{Tr}_{L/S}(\pi_{L}^w y)$ would imply that

$$\text{Tr}_{L/S}(\pi_{L}^w y) \in \pi^n C \quad \forall y \in \mathcal{O}_L.$$
Thus $\pi^w/\pi^n \in D(L/S)^{-1}$, which implies $w \geq v_L(\pi^n) - v_L(D(L/S))$, that is,

$$v_E(R_{E,1}) \geq -v_L(D(L/S)) - [v_L(p)/(p - 1)],$$

which is a contradiction since $v_E(R_{E,1}) = -v_L(D(L/S)) - [v_L(p)/(p - 1)] - 1$ by (5.4).

Finally, condition (III) immediately follows from equation (5.3).

Let $T_L$ be the image of $\mu_L$ under the formal logarithm. This is a $C$-submodule of $L$ such that $T_LS = L$. Indeed, let $x \in L$ and take $n$ large enough such that $\pi^n x \in \mu_L$, then by Proposition 3.3.1 there exist a $y \in F(\mu_L)$ such that $\pi^n x = l_F(y)$, thus $x \in T_LS$. Let $R_L$ be the dual of $T_L$ with respect to the trace pairing $T_{L/S}$, then by Proposition 5.1.1 and by $L = T_LS$ we have the isomorphism

$$R_L \simeq \text{Hom}_{C}^{seq}(T_L, C).$$

We also have the following

**Lemma 5.1.3.** The generalized trace induces an injective homomorphism

$$R_L/\pi^n R_L \to \text{Hom}_{C}^{seq}(T_L, C/\pi^n C)$$

$$\alpha \mapsto (x \mapsto T_{L/S}(\alpha x)).$$

**Proof.** Immediate from the very definition of $R_L$. \qed
Let $M/L$ be a finite extension of local fields containing the local field $S$. Let

$$\mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}, \mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$$

and $\operatorname{Tr}_{\mathcal{M}/\mathcal{L}}$ the trace of the finite extension $\mathcal{M}/\mathcal{L}$. Then

**Lemma 5.1.4.** $\mathcal{T}_{\mathcal{M}/S} = \mathcal{T}_{\mathcal{L}/S} \circ \operatorname{Tr}_{\mathcal{M}/\mathcal{L}}$.

**Proof.** By induction on $d$. Let $\mathcal{M} = \mathcal{M}'\{\{T_d\}\}$ and $\mathcal{L} = \mathcal{L}'\{\{T_d\}\}$, where

$$\mathcal{M}' = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad \mathcal{L}' = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.$$ 

Let $\sum_i a_i T_d^i \in \mathcal{M}$, then since $\operatorname{Tr}_{\mathcal{M}/\mathcal{L}}$ is continuous we have

$$\operatorname{Tr}_{\mathcal{M}/\mathcal{L}}(\sum_i a_i T_d^i) = \sum_i \operatorname{Tr}_{\mathcal{M}/\mathcal{L}}(a_i) T_d^i = \sum_i \operatorname{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_i) T_d^i.$$ 

Thus, since $\mathcal{T}_{\mathcal{L}/S} = \operatorname{Tr}_{\mathcal{L}/S} \circ c_{\mathcal{L}/\mathcal{L}'}$, then

$$\mathcal{T}_{\mathcal{L}/S} \circ \operatorname{Tr}_{\mathcal{M}/\mathcal{L}}(\sum_i a_i T_d^i) = \mathcal{T}_{\mathcal{L}'/S} \circ c_{\mathcal{L}/\mathcal{L}'}(\sum_i \operatorname{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_i) T_d^i) = \mathcal{T}_{\mathcal{L}'/S}(\operatorname{Tr}_{\mathcal{M}'/\mathcal{L}'}(a_0)).$$ 

On the other hand

$$\mathcal{T}_{\mathcal{M}/S}(\sum_i a_i T_d^i) = \mathcal{T}_{\mathcal{M}'/S} \circ c_{\mathcal{M}/\mathcal{M}'}(\sum_i a_i T_d^i) = \mathcal{T}_{\mathcal{M}'/S}(a_0).$$ 

The equality follows by the induction hypothesis. 

\qed
5.2 The map $\psi_{L,n}^i$

We will denote by $(\ , \ )_{L,n}^i$ the $i$th coordinate of the paring $(\ , \ )_{L,n}$ with respect to the base $\{e_i\}$ of $\kappa_n$. Recall that a pair $(n, t)$ is said to be admissible if there exist a $k$ such that $t - 1 - n \geq \alpha k \geq n$, where $\alpha$ denotes the ramification index of $S$ over $\mathbb{Q}_p$.

**Proposition 5.2.1.** Let $L \supset K_t$, $(n, t)$ admissible and $\mathcal{L} = L\{T_1\} \cdots \{T_{d-1}\}$.

For a given $\alpha \in K_d(\mathcal{L})$ there exist a unique element $\psi_{L,n}^i(\alpha) \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$, such that

$$ (\alpha, x)_{L,n}^i = T_{L/S}(\psi_{L,n}^i(\alpha) l_F(x)) \quad \forall x \in F(\mu_{\mathcal{L},1}), \quad (5.7) $$

and the map $\psi_{L,n}^i : K_d(\mathcal{L}) \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ is a homomorphism.

**Proof.** Let us first take $\alpha$ to be an element of the form $\{a_1, \ldots, a_d\}$ and consider the map

$$ \omega : \mu_{\mathcal{L},1} \to \mathbb{C}/\pi^n \mathbb{C}, $$

defined by

$$ x \mapsto (\alpha, l_F^{-1}(x))^i_{L,n}. $$

By Proposition 4.2.2 and Remark 3.3.1 this map is sequentially continuous and so by Lemma 5.1.2 there exist and element $\psi_{L,n}^i(\alpha) \in R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ satisfying (5.7). This defines a map $\psi_{L,n}^i : \mathcal{L}^{n \otimes d} \to R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$ satisfying the Steinberg relation, therefore it induces a map on $K_d(\mathcal{L})$. 
Proposition 5.2.2. Let $M/L/K_i$ be a finite tower such that $(n,t)$ is admissible. Let $\mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Then

1. $\text{Tr}_{M/L}(R_{M,1}) \subset R_{\mathcal{L},1}$ and we have the commutative diagram

$$
\begin{array}{ccc}
K_d(\mathcal{M}) & \xrightarrow{\psi_{\mathcal{M},n}^i} & R_{M,1}/\pi^n R_{M,1} \\
\downarrow N_{M/L} & & \downarrow \text{Tr}_{M/L} \\
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},n}^i} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}
\end{array}
$$

2. $R_{\mathcal{L},1} \subset R_{M,1}$ and we have the commutative diagram

$$
\begin{array}{ccc}
K_d(\mathcal{M}) & \xrightarrow{\psi_{\mathcal{M},n}^i} & R_{M,1}/\pi^n R_{M,1} \\
\downarrow \text{incl} & & \downarrow \\
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},n}^i} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}
\end{array}
$$

The right-hand vertical map is induced by the embedding of $R_{\mathcal{L},1}$ in $R_{M,1}$.

3. Let $L \supset K_m$, $(m,t)$ admissible and $m \geq n$. Then for $\alpha \in K_d(\mathcal{L})$, $\psi_{\mathcal{L},n}^i(\alpha)$ is the reduction of $\psi_{\mathcal{L},m}^i(\alpha)$ from $R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1}$ to $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$, i.e., the diagram

$$
\begin{array}{ccc}
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},m}^i} & R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1} \\
\downarrow \psi_{\mathcal{L},n}^i & & \downarrow \text{reduction} \\
R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}
\end{array}
$$

commutes.
4. If \( t : (F, e_i) \to (\hat{F}, \hat{e}_i) \) is an isomorphism, then
\[
\hat{\psi}_{i}^{\hat{L}, n} = \frac{1}{t'(0)} \psi_{L}^{i, n}.
\]

**Proof.**

1. By Lemma 5.1.4 and the fact that \( \mu_{L, 1} \subset \mu_{M, 1} \) we obtain
\[
T_{L/S}(\text{Tr}_{M/L}(R_{M, 1}\mu_{L, 1})) = T_{L/S}(\text{Tr}_{M/L}(R_{M, 1}\mu_{L, 1})) \subset T_{M/S}(R_{M, 1}\mu_{M, 1}) \subset C,
\]
from which follows that \( \text{Tr}_{M/L}(R_{M, 1}) \subset R_{L, 1} \). Now by Proposition 4.2.1 (4) we have, for \( b \in K_d(M) \) and \( x \in F(\mu_{L, 1}) \), that
\[
(N_{M/L}(b), x)_{L, n} = (b, x)_{M, n} = T_{M/S}(\psi_{M, m}^{i}(b)l(x)) = T_{L/S}(\text{Tr}_{M/L}(\psi_{M, m}^{i}(b))l(x)).
\]
It follows from Proposition 5.2.1 that
\[
\psi_{L, m}^{i}(N_{M/L}(b)) = \text{Tr}_{M/L}(\psi_{M, m}^{i}(b)).
\]

2. This is proved in a similar fashion to the previous property but this time using Proposition 4.2.1 (7).

3. This follows from Proposition 5.2.1 and Proposition 4.2.1 (5). Indeed, since
\[
e^i_n = f^{(m-n)}(e^i_m) \quad \text{and} \quad (a, x)_n = (a, f^{(m-n)}(x))_m
\]
we get
\[
\pi^{m-n}(a, x)^i_n = (a, f^{(m-n)}(x))^i_m \quad (\text{mod} \ \pi^m C)
\]
That is
\[ \pi^{m-n} T_{L/S}(\psi_{L,n}^i(a) l_F(x)) = T_{L/S}(\psi_{L,n}^i(a) l_F(f^{(m-n)}(x))) \]
\[ = \pi^{m-n} T_{L/S}(\psi_{L,n}^i(a) l_F(x)) \quad \text{(mod } \pi^m C) \]

Upon dividing by $\pi^{m-n}$ the result follows.

4. This property follows from Proposition 4.2.1(8) and $l_{\tilde{F}}(t) = t'(0)l_F$.

Proposition 5.2.3. Let $L/K_t$ be finite extension, $(n,t)$ an admissible pair, and $L = L\{\{T_1\}\cdots\{T_{d-1}\}\}$. Let $r$ be a $t$-normalized series. Then
\[ (a_1, \ldots, a_{d-1}, u, x)^i_{L,n} = T_{L/S} \left( \log(u) \left[ \frac{-\psi_{L,n}^i(\{a_1, \ldots, a_{d-1}, r(x)\}) r(x) l_{\tilde{F}}'(x)}{r'(x)} \right] \right) \] (5.8)
for $a_1, \ldots, a_{d-1} \in L^*$, $u \in V_{L,1} = 1 + \mu_{L,1}$ and $x \in F(\mu_L)$.

Proof. The proof follows the proof of Proposition 4.1 of [14]. To simplify the notation in this proof we will denote the normalized valuation $v_L/v_L(p)$ by $v$. We will also omit the subscript $L$ in the pairing notation. Observe that we may assume that $r(X) = X$ since we can go to the isomorphic formal group $\tilde{F} = r(F(r^{-1}(X), r^{-1}(Y)))$ with torsion points $\tilde{e} = r(e_i)$ through the isomorphism $r : F \to \tilde{F}$. The series $\tilde{r} = X$
is \( t \)-normalized for \( \tilde{F} \) and if the result were true for \( \tilde{F} \) and \( \tilde{r} \) then

\[
(a_1, \ldots, a_{d-1}, u, x)^i_{\tilde{F}, n} = (a_1, \ldots, a_{d-1}, u, r(x))^i_{\tilde{F}, n}
\]

\[
= \mathbb{T}_{\mathcal{L}/S}( \log(u)(-\tilde{\psi}_n^i(a_1, \ldots, a_{d-1}, r(x))\tilde{l}'(r(x))r(x)))
\]

and since \( \tilde{\psi}_n^i(a_1, \ldots, a_{d-1}, r(x)) = r'(0)^{-1}\psi_n^i(a_1, \ldots, a_{d-1}, r(x)) \) and

\[
l_{\tilde{F}}(r(X)) = r'(0)l_{F}(X) \implies \tilde{l}'_{\tilde{F}}(r(X)) = r'(0)\tilde{l}_{\tilde{F}}'(x)/r'(x),
\]

the result follows.

We assume therefore that \( r(X) = X \). Since the pair \((n, t)\) is admissible, let \( k \) be an integer such that \( t - 1 - n \geq k\alpha \geq n \), \( \alpha \) is the ramification index of \( S/\mathbb{Q}_p \), and denote by \( \epsilon \) the unit \( \pi^{ak}/p^k \). Let \( u \in V_{\mathcal{L},1} = \{ x \in \mathcal{L} : v(x-1) > 1/(p-1) \} \) and \( x \in \mu_{\mathcal{L}} \). By bilinearity of the pairing

\[
(a_1, \ldots, a_{d-1}, u, x)_n = (a_1, \ldots, a_{d-1}, u, x \odot (x \times u^k))_n \oplus (a_1, \ldots, a_{d-1}, u, x \times u^k)_n.
\]
Now let $m = n + \alpha k$ and $y = x \times w^k$. Then by (5) of Proposition 4.2.1

$$
({a_1, \ldots, a_{d-1}, u}, y)_n = f^{(m-n)}( ({a_1, \ldots, a_{d-1}, u}, y)_m)
= \pi^{\alpha k}({a_1, \ldots, a_{d-1}, u}, y)_m
= \epsilon({a_1, \ldots, a_{d-1}, w^k}, y)_m
= \epsilon({a_1, \ldots, a_{d-1}, \frac{y}{x}}, y)_m
= \epsilon({a_1, \ldots, a_{d-1}, y}, y)_m \ominus \epsilon({a_1, \ldots, a_{d-1}, x}, y)_m.
$$

But $r = X$ is $t$-normalized, hence we may replace $({a_1, \ldots, a_{d-1}, y}, y)_m = 0$ by the expression $({a_1, \ldots, a_{d-1}, x}, x)_m = 0$ and obtain

$$
({a_1, \ldots, a_{d-1}, u}, y)_n = \epsilon({a_1, \ldots, a_{d-1}, x}, x)_m \ominus \epsilon({a_1, \ldots, a_{d-1}, x}, y)_m \quad (5.10)
= \epsilon({a_1, \ldots, a_{d-1}, x}, x \ominus y)_m. \quad (5.11)
$$

By the properties of the logarithm in Proposition 3.3.1 we can express

$$
w^k = \exp(\log(w^k)) = \exp(p^k \log(u)) = 1 + p^k \log(u) + p^{2k}w,
$$

where $w = \frac{z^2}{2!} + p^k \frac{z^3}{3!} + \cdots$, with $z = \log(u)$. Since $v(z) > 1/(p - 1)$ then $v(\frac{z^1}{p}) = 1/(p - 1)$ and so $v(w) > 1/(p - 1)$. This follows, for example, by Proposition 2.4 of [14].

Since $x \ominus y \equiv x - y \pmod{xy}$ and $y = xw^k$ with $v(x) > 0$, then $v(x \ominus y) = v(x) = v(x) > 1/(p - 1)$ and so $v(x \ominus y) = v(xw^k - 1) > 1/(p - 1)$. Thus, using the Taylor expansion of $l = l_F$
around \( X = x \) we obtain

\[
l(y) = l(x + xp^kz + xp^{2k}w) = l(x) + l'(x)(xp^kz + xp^{2k}w) + p^{2k}w_1,
\]

where \( w_1 = l''(x)\frac{\delta^2}{2!} + p^k l^{(3)}(x)\frac{\delta^3}{3!} + \cdots \) with \( \delta = xz + xp^k w \). Since \( v(\delta) > 1/(p-1) \) then \( v(w_1) > 1/(p-1) \). Moreover

\[
l(y) = l(x) + l'(x)xp^kz + p^{2k}w_2,
\]

with \( v(w_2) > 1/(p-1) \). Then

\[
l(x \ominus y) = -l'(x)xp^kz - p^{2k}w_2.
\]

Observing that \( -l'(x)xz - p^k w_2 \in \mu_{*,1} \) we have by the isomorphism given in 3.3.1 that there is an \( \eta \in F(\mu_{*,1}) \) such that \( l(x \ominus y) = l'(x)xp^k z + p^{2k}w_2 \). Thus

\[
x \ominus y = [p^k](\eta) = [\pi^{\alpha k}](\tilde{\eta}) = f^{(\alpha k)}(\tilde{\eta}),
\]

for \( \tilde{\eta} = [\epsilon^{-1}]_F(\eta) \). Since \( n \geq \alpha k \) then \( \pi^n \) divides \( \pi^{\alpha k} \) and we have that \( x \ominus y \in f^{(n)}(F(\mu_{*,1})) \). Thus the first term on the right hand side of equation (5.9) is zero.
By equation (5.10) and (5) of Proposition 4.2.1 we have

\[
(a_1, \ldots, a_{d-1}, u, x)_n = \epsilon((a_1, \ldots, a_{d-1}, x) \otimes y)_m \tag{5.12}
\]

\[
= \epsilon((a_1, \ldots, a_{d-1}, x, f^{(\alpha)}(\tilde{\eta}))_m \tag{5.13}
\]

\[
= \epsilon((a_1, \ldots, a_{d-1}, x, \tilde{\eta})_n \tag{5.14}
\]

\[
= ((a_1, \ldots, a_{d-1}, x, \eta))_n \tag{5.15}
\]

Since \(v(\eta) > 1/(p - 1)\) and \((n,t)\) is admissible we can use Proposition 5.2.1

\[
(a_1, \ldots, a_{d-1}, x, \eta)_n = T_{L/S}(\psi_{L,n}(a_1, \ldots, a_{d-1}, x) l_F(\eta))
\]

\[
= T_{L/S}(\psi_{n}^i(a_1, \ldots, a_{d-1}, x) (-l'(x)xz - p^k w^2)).
\]

Since \(\psi_{L,n}^i(a_1, \ldots, a_{d-1}, x) \in R_{L,1}, w_2 \in T_{L,1}, \pi^n|p^k\) and \(T_{L/S}(R_{L,1}T_{L,1}) \subset C\), then we can write the above as

\[
T_{L/S}(\psi_{n}^i(a_1, \ldots, a_{d-1}, x) (-l'(x)xz)).
\]

From equation 5.12 we get

\[
(a_1, \ldots, a_{d-1}, u, x)_n^i = T_{L/S}(\psi_{n}^i(a_1, \ldots, a_{d-1}, x) (-l'(x)xz)).
\]

Keeping in mind that \(z = \log(u)\), the proposition follows.
5.3 The map $\rho^i_{L,n}$

We can define a $C$-linear structure on $V_{L,1} = 1 + \mu_{L,1}$ by using the isomorphism $\log : V_{L,1} \to T_{L,1}$, i.e., $cu := \log^{-1}(c \log(u))$. Let $x \in F(\mu_L)$ and $a_1, \ldots, a_{d-1} \in \mathcal{L}^*$ be fixed. Consider the mapping

$$V_{L,1} \to C/\pi^n C,$$

defined by

$$u \mapsto (\{a_1, \ldots, a_{d-1}, u\}, x)^i_{L,n}.$$  

According to Proposition 5.2.3 this is a continuous $C$-linear map and we have the following

**Proposition 5.3.1.** There exist a unique element $\rho^i_{L,n}(\{a_1, \ldots, a_{d-1}\}, x) \in R_{L,1}/\pi^n R_{L,1}$ such that

$$\quad (\{a_1, \ldots, a_{d-1}, u\}, x)^i_{L,n} = T_{L,S} \left( \log(u) \rho^i_{L,n}(\{a_1, \ldots, a_{d-1}\}, x) \right), \quad (5.16)$$

for all $u \in V_{L,1}$, and the map $\rho^i_{L,n} : K_{d-1}(\mathcal{L}) \otimes F(\mu_L) \to R_{L,1}/\pi^n R_{L,1}$ is a homomorphism.

From Proposition 5.3.1 and Proposition 5.2.3 it follows the next proposition.

**Proposition 5.3.2.** Let $L/K_t$ be a finite extension, $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, with
(n, t) admissible, and let r be a t-normalized series. Then
\[
\frac{\psi^i_{\mathcal{L},n}(\{a_1, \ldots, a_{d-1}, r(x)\})}{r'(x)} r(x) = -\frac{\rho^i_{\mathcal{L},n}(\{a_1, \ldots, a_{d-1}\}, x)}{l'_r(x)}.
\]
for all \(a_1, \ldots, a_{d-1} \in \mathcal{L}^*\) and all \(x \in F(\mu_{\mathcal{L}})\).

5.4 The maps \(D^i_{\mathcal{L},n}\)

Assume \(L \supset K(\kappa_i)\) for \((n, t)\) admissible and let \(\mathcal{L} = L\{T_1\} \ldots \{T_{d-1}\}\). Define \(D^i_{\mathcal{L},n} : \mathcal{O}_{\mathcal{L}}^{d} \rightarrow R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}\) by
\[
D^i_{\mathcal{L},n}(a_1, \ldots, a_d) = \psi^i_{\mathcal{L},n}(\{a_1, \ldots, a_d\}) a_1 \cdots a_d, \tag{5.17}
\]
and consider \(R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}\) as an \(\mathcal{O}_{\mathcal{L}}\)-module. This map satisfies

**Proposition 5.4.1.**  1. Leibniz Rule:
\[
D^i_{\mathcal{L},n}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_d) = a_i D^i_{\mathcal{L},n}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d) + a'_i D^i_{\mathcal{L},n}(a_1, \ldots, a_i, \ldots, a_d).
\]

2. Steinberg relation:
\[
D^i_{\mathcal{L},n}(a_1, \ldots, a_i, \ldots, 1 - a_i, \ldots, a_d) = 0.
\]

3. Skew-symmetric:
\[
D^i_{\mathcal{L},n}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -D^i_{\mathcal{L},n}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d).
\]
4. \( D_{L,n}^{i}(a_1, \ldots, a_{p_k}^{i}, \ldots, a_d) = 0 \) if \( \pi^n p^k \).

**Proof.** Property (1) follows from the fact that \( \psi^{i}_{L,n} \) is a homomorphism. Property (2) follows from the Steinberg relation \( \{a_1, \ldots, a_i, \ldots, 1 - a_i, \ldots, a_d\} = 1 \) for elements in the Milnor K-group \( K_d(L) \) and property (3) follows from the fact that \( \{a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d\} = \{a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d\}^{-1} \), in \( K_d(L) \) (cf. Proposition 4.1.1).

---

**Proposition 5.4.2.** Let \( L \) be as in Proposition 5.3.2, then

\[
\frac{D_{L,n}^{i}(a_1, \ldots, a_{d-1}, r(x \oplus y))}{r'(x \oplus y)} = F_X(x, y) \frac{D_{L,n}^{i}(a_1, \ldots, a_{d-1}, r(x))}{r'(x)} + F_Y(x, y) \frac{D_{L,n}^{i}(a_1, \ldots, a_{d-1}, r(y))}{r'(y)},
\]

(5.18)

for all \( a_1, \ldots, a_{d-1} \in \mathcal{L}^* \) and \( x, y \in F(\mu_L) \).

**Proof.** This follows from the fact that

\[
\rho_{L,n}^{i}(\{a_1, \ldots, a_{d-1}\}, x \oplus_F y) = \rho_{L,n}^{i}(\{a_1, \ldots, a_{d-1}\}, x) + \rho_{L}^{i}(\{a_1, \ldots, a_{d-1}\}, y),
\]

and from differentiating \( l_F(F(X, Y)) = l_F(X) + l_F(Y) \) with respect to \( X \) and \( Y \). □

Let \( \tilde{F} \) be the formal group \( r(F(r^{-1}(X), r^{-1}(Y))) \). The series \( r(X) = X \) is \( t \)-normalized for \( \tilde{F} \). Denote by \( \oplus \) the sum according to this formal group and \( \tilde{D}_{L,n}^{i} \),
The corresponding maps. According to Proposition 5.2.2 (4) we have that

\[ \tilde{D}^i_{L,n} = \frac{1}{r'(0)} D^i_{L,n}. \]  

Therefore, Proposition 5.4.2 in terms of \( \tilde{\oplus} \) and \( D^i_{L,n} \) reads as

**Corollary 5.4.1.** Let \( L \) be as in Proposition 5.3.2, then

\[ D^i_{L,n}(a_1, \ldots, a_{d-1}, x \tilde{\oplus} y) = \tilde{F}_X(x,y) D^i_{L,n}(a_1, \ldots, a_{d-1}, x) + \tilde{F}_Y(x,y) D^i_{L,n}(a_1, \ldots, a_{d-1}, y). \]  

for all \( a_1, \ldots, a_{d-1} \in L^* \) and \( x, y \in F(\mu_L) \).

Let \( A \) be the set of all series in \( X_d \mathcal{O}_L[[X_1, \ldots, X_d]] \) of the form

\[ \bigoplus_{k=1}^{\infty} \left( \bigoplus_{i=(i_1, \ldots, i_{d-1})}^{0 \leq i_1, \ldots, i_{d-1} \leq p^n-1} \gamma_{i,k}^n X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_2^k \right), \]  

where \( \gamma_{i,k} \in \mathcal{O}_L \). For \( d = 2 \), the series looks like

\[ \bigoplus_{k=1}^{\infty} \left( \bigoplus_{0 \leq i \leq p^n-1} \gamma_{i,k}^n X_1^i X_2^k \right), \]  

where \( \gamma_{i,k} \in \mathcal{O}_L \).

**Proposition 5.4.3.** For \( x \in \mathcal{O}_L \), there exist elements \( \gamma_i \in \mathcal{O}_L \), \( i = (i_1, \ldots, i_{d-1}) \), such that

\[ x \equiv \sum_{i=(i_1, \ldots, i_{d-1})}^{0 \leq i_1, \ldots, i_{d-1} \leq p^n-1} \gamma_{i_1}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pmod{\pi_L}. \]
Proof. By induction on \(d\). For \(d = 1\), the result follows since \(k_L\) is perfect. Suppose it is proved for \(R = \mathcal{O}_d\), where \(\mathcal{L}_d = L\{T_1\} \ldots \{T_{d-1}\}\). Let \(\mathcal{L} = \mathcal{L}_d\{T_d\}\) and \(x \in \mathcal{O}_\mathcal{L}\). Then \(x = \sum_{j \geq m} a_j T_d^j \pmod{\pi_L}\), \(a_j \in R\). Then

\[
x \equiv \sum_{0 \leq i_d} T_d^{i_d} \left( \sum_{m \leq i_d + kp^n} a_{i_d + kp^n} T_d^{kp^n} \right) \pmod{\pi_L}
\]

\[
\equiv \sum_{0 \leq i_d} T_d^{i_d} \left( \sum_{m \leq i_d + kp^n} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} \leq kp^n-1} \gamma_{i_1, \ldots, i_{d-1}, i_d} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \right) T_d^{kp^n} \right) \pmod{\pi_L}
\]

\[
\equiv \sum_{0 \leq i_d} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} T_d^{i_d} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} \leq kp^n-1} \gamma_{i_1, \ldots, i_{d-1}, i_d} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \right) T_d^{kp^n} \pmod{\pi_L}
\]

where \(\gamma_{i_1, \ldots, i_d} = \sum_k \gamma_{i_1, \ldots, i_{d-1}, i_d; k} T_d^k\) and regrouping terms is valid since the series are absolutely convergent in the Parshin topology. Also by noticing that the congruence

\[
\sum_{c \leq k} b_c^p T_c^{kp^n} \equiv \left( \sum_{c \leq k} b_c T_d^k \right)^p \pmod{\pi_L},
\]

holds in \(k_{\mathcal{L}_{d-1}}(T_d)\), where \(k_{\mathcal{L}_{d-1}}\) is the residue field of \(\mathcal{L}_{d-1}\). \(\square\)

Remark 5.4.1. This proposition is equivalent to the following two facts:

1. \(k_L((T_1)) \cdots ((T_{d-1}))\) is a finite extension of \(k_L((T_1^{p^n}) \cdots ((T_{d-1}^{p^n}))\) of degree \((d - 1)p^n\) and generated by the elements \(T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}\) for \(0 \leq i_1, \ldots, i_{d-1} \leq p^n - 1\).

2. \(k_L((T_1^{p^n})) \cdots ((T_{d-1}^{p^n}))\) is the image of \(k_L((T_1)) \cdots ((T_{d-1}))\) under the Frobenius homomorphism

\[
\sigma_p : k_L((T_1)) \cdots ((T_{d-1})) \rightarrow k_L((T_1)) \cdots ((T_{d-1}))\]
i.e., every element of \(k_L((T_1^{p^n})) \cdots (T_{d-1}^{p^n}))\) has a \(p^n\)th root in \(k_L((T_1)) \cdots (T_{d-1}))\).

Both facts are easily proven by induction from the fact that, for a field \(k\) of characteristic \(p\), the extension \([k((T)): k((T^p))]\) has degree \(p\) and \(\sigma_d(k)((T^p))\) is the image of \(k((T))\) under the Frobenius homomorphism \(\sigma_p : k((T)) \to k((T))\).

**Proposition 5.4.4.** Let \(\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\). For \(y \in \mu_L\), there exist an \(\eta \in A\) such that \(y = \eta(T_1, \ldots, \pi_L)\).

**Proof.** Let \(y \in \mu_L\). Then by Proposition 5.4.3

\[
\frac{y}{\pi_L} \equiv \sum_{i=(i_1,\ldots,i_{d-1})}^{0 \leq i_1,\ldots,i_{d-1} \leq p^n-1} \gamma_{i_1}^{p^n} T_{i_1} \cdot \cdot \cdot T_{d-1}^{i_{d-1}} \mod \pi_L
\]

\[
\equiv \bigoplus_{i=(i_1,\ldots,i_{d-1})}^{0 \leq i_1,\ldots,i_{d-1} \leq p^n-1} \frac{\gamma_{i_1}^{p^n} T_{i_1} \cdot \cdot \cdot T_{d-1}^{i_{d-1}} \pi_L}{\pi_L} \mod \pi_L,
\]

this is,

\[
y \equiv \bigoplus_{i=(i_1,\ldots,i_{d-1})}^{0 \leq i_1,\ldots,i_{d-1} \leq p^n-1} \gamma_{i_1}^{p^n} T_{i_1} \cdot \cdot \cdot T_{d-1}^{i_{d-1}} \pi_L \mod \pi_L^2.
\]

Denote \(\bigoplus_{i=(i_1,\ldots,i_{d-1})}^{0 \leq i_1,\ldots,i_{d-1} \leq p^n-1} \gamma_{i_1}^{p^n} T_{i_1} \cdot \cdot \cdot T_{d-1}^{i_{d-1}} \pi_L\) by \(y_1\). Suppose we have defined elements \(y_k = \bigoplus_{i=(i_1,\ldots,i_{d-1})}^{0 \leq i_1,\ldots,i_{d-1} \leq p^n-1} \gamma_{i_k}^{p^n} T_{i_1} \cdot \cdot \cdot T_{d-1}^{i_{d-1}} \pi_L, 1 \leq k \leq m - 1\), such that

\[
y \otimes (\bigoplus_{k=1}^{m-1} y_k) \equiv 0 \mod \pi_L^m.
\]
CHAPTER 5. THE MAPS $\psi$ AND $\rho$

Then

$$\frac{y \otimes \left( \bigoplus_{k=1}^{m-1} y_k \right)}{\pi_L^m} \equiv \sum_{i=(i_1, \ldots, i_{d-1}) \atop 0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \gamma_{i,m}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \mod \pi_L$$

$$\equiv \bigoplus_{i=(i_1, \ldots, i_{d-1}) \atop 0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \frac{\gamma_{i,m}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_L}{\pi_L^m} \mod \pi_L.$$ 

Denote $\bigoplus_{i=(i_1, \ldots, i_{d-1}) \atop 0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \gamma_{i,m}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_L$ by $y_m$. Then

$$y \otimes \left( \bigoplus_{k=1}^{m-1} y_k \right) \equiv y_m \mod \pi_L^{m+1},$$

or

$$y \otimes \left( \bigoplus_{k=1}^{m-1} y_k \right) \equiv 0 \mod \pi_L^{m+1}.$$ 

Therefore $y = \bigoplus_{k=1}^{\infty} y_k$, which is what we wanted to prove. \hfill \Box

**Corollary 5.4.2.** For every $x \in \mathcal{O}_L$, there exists $\gamma_{i,k} \in \mathcal{O}_L$ such that

$$\sum_{k=0}^{\infty} \left( \sum_{i=(i_1, \ldots, i_{d-1}) \atop 0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \gamma_{i,k}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_L \right).$$

**Proof.** Take $F$ to be the additive formal group $X + Y$ in Proposition 5.4.4 \hfill \Box

**Corollary 5.4.3.** For every $x \in V_L = 1 + \mu_L$, there exist $\gamma_{i,k} \in \mathcal{O}_L$ such that

$$\prod_{k=1}^{\infty} \left( \prod_{i=(i_1, \ldots, i_{d-1}) \atop 0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \left( 1 + \gamma_{i,k}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_L \right) \right).$$

**Proof.** This can be proven either by using Proposition 5.4.3 and induction on $k$; just as in the proof of Proposition 5.4.4 or by applying Proposition 5.4.4 to the
multiplicative formal group $F(X,Y) = X + Y + XY$. Noticing also, that the homomorphism

$$F(\mu_L) \to V_L,$$

defined by $x \mapsto 1 + x$, is continuous in the topology induced by the valuation $v_L$ of $L$.

Let $r$ be a $t$-normalized series for $F$ and $(n, t)$ an admissible pair. Let $\tilde{F}$ be the formal group $r(F(r^{-1}(X), r^{-1}(Y)))$. Let $\mathcal{L} = L\{\{T_1\}\ldots\{T_{d-1}\}\}$. For $y \in \mu_L$ we will denote by $\tilde{\eta}_y(X_1, \ldots, X_d)$ the multivariable series

$$\bigoplus_{k=1}^{\infty} \left( \bigoplus_{i=(i_1, \ldots, i_{d-1})}^{0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \gamma_{i,k}^n X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_d^k \right), \quad (\gamma_{i,k} \in \mathcal{O}_L)$$

with respect to $\tilde{F}$, such that $y = \tilde{\eta}_y(T_1, \ldots, \pi_L)$, whose existence is guaranteed by Proposition 5.4.4.

**Proposition 5.4.5.** Let $L/K_t$ be a finite extension with $(n, t)$ admissible and set $\mathcal{L} = L\{\{T_1\}\ldots\{T_{d-1}\}\}$. For $y = \tilde{\eta}_y(T_1, \ldots, \pi_L) \in \mu_L$, $\tilde{\eta}_y$ as in (5.22), we have

$$D_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, y) = \sum_{i=1}^{d} \left. \frac{\partial \tilde{\eta}_y}{\partial X_i} \right|_{X_k = T_k, k=1, \ldots, d} D_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, T_i),$$

for all $a_1, \ldots, a_{d-1} \in \mathcal{O}_L$, where $T_d = \pi_L$.

**Proof.** Let $y = \bigoplus_{k=1}^{\infty} y_k$, where

$$y_k = \bigoplus_{i=(i_1, \ldots, i_{d-1})}^{0 \leq i_1, \ldots, i_{d-1} \leq \rho^{n-1}} \gamma_{i,k}^n T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_L^k.$$
Thus, $\tilde{\eta}_y = \bigoplus_{m=1}^{\infty} \eta_m$, where

$$\eta_m(X_1, \ldots, X_d) = \bigoplus_{0 \leq i_1, \ldots, i_{d-1} \leq p^m - 1} \gamma_{p^m}^{i_1} X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_d^m.$$ 

Let us fix $a_1, \ldots, a_{d-1} \in L^*$ and denote by $D(x)$ the element $D_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x)$ to simplify notation.

First notice that

$$v_L(y_k) \xrightarrow{k \to \infty} \infty \implies D(y_k) = 0 \text{ for } k \text{ large enough.} \quad (5.23)$$

This follows from $D_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x) = \psi_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x)a_1 \cdots a_{d-1}x$ and that $\psi_{\mathcal{L},n}$ has values in $R_L/\pi^n R_L$; $\pi^n \psi_{\mathcal{L},n} = 0$.

Thus, from $\tilde{\eta}_y = \bigoplus_{m=1}^{k-1} \eta_m \pmod{\pi^k}$ and equation (5.23), it is enough to consider the finite formal sum $\bigoplus_{m=1}^{k-1} \eta_m$. The proposition follows now from the fact that $D(\gamma p^n) = p^n \gamma p^{n-1} D(\gamma) = 0 \pmod{\pi^n p^n}$, $D(xy) = yD(x) + xD(y)$, and corollary 5.4.1.

Corollary 5.4.4. Let $L/K_t$ be a finite extension with $(n, t)$ admissible and set $L = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Let $y_i = \tilde{\eta}_{y_i}(T_1, \ldots, \pi_L) \in \mu_L$, for $1 \leq i \leq d$, where $\eta_{y_i}$ is a multivariable series of the form (5.21). Then

$$D_{\mathcal{L},n}(y_1, \ldots, y_d) = \det \left[ \frac{\partial \tilde{\eta}_{y_i}}{\partial X_j} \right]_{i,j} \bigg|_{X_k = T_k, \ k = 1, \ldots, d} D_{\mathcal{L},n}(T_1, \ldots, \pi_L),$$

where $T_d = \pi_L$. 

Proof. This follows from Proposition \ref{prop5.4.5} and Proposition \ref{prop5.4.1} (3). Indeed, let us illustrate the proof in the case $d = 2$, i.e., $\mathcal{L} = L\{T_1\}$ and $T_2 = \pi_L$. To simplify the notation we will denote $D_{\mathcal{L},n}^i$ by $D$. From Proposition \ref{prop5.4.5} we have

$$D(\eta_1(T_1, T_2), \eta_2(T_1, T_2)) = \left. \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_1} \right|_{X_i = T_i, i = 1, 2} D(T_1, T_1) + \left. \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_2} \right|_{X_i = T_i, i = 1, 2} D(T_1, T_2)$$

$$+ \left. \frac{\partial \eta_1}{\partial X_2} \frac{\partial \eta_2}{\partial X_1} \right|_{X_i = T_i, i = 1, 2} D(T_2, T_1) + \left. \frac{\partial \eta_1}{\partial X_2} \frac{\partial \eta_2}{\partial X_2} \right|_{X_i = T_i, i = 1, 2} D(T_2, T_2)$$

(5.24)

But $D(T_1, T_1) = D(T_2, T_2) = 0$, $D(T_2, T_1) = -D(T_1, T_2)$ from Proposition \ref{prop5.4.1} (3), therefore

$$D(\eta_1(T_1, T_2), \eta_2(T_1, T_2)) = \left. \left( \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_2} - \frac{\partial \eta_1}{\partial X_2} \frac{\partial \eta_2}{\partial X_1} \right) \right|_{X_i = T_i, i = 1, 2} D(T_2, T_1).$$

(5.25)

The corollary follows. \hfill \Box
Chapter 6

Multidimensional derivations

Let $\mathcal{L}$ be the standard $d$-dimensional local field $L\{\{T_1\}\ldots\{T_{d-1}\}\}$ and $L/K/\mathbb{Q}_p$ be a tower of finite extensions. Let $W$ be an $\mathcal{O}_\mathcal{L}$-module that is $p$-adically complete, i.e.,

$$W \cong \lim_{\leftarrow} W/p^nW.$$ 

For example, if $p^nW = 0$ for some $n$, then $W$ is $p$-adically complete. Actually, this is going to be our situation, since $W$ will be the $\mathcal{O}_\mathcal{L}$-module $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$.

6.1 Derivations and the module of differentials

Definition 6.1.1. A derivation of $\mathcal{O}_\mathcal{L}$ into $W$ over $\mathcal{O}_K$ is a map $D : \mathcal{O}_\mathcal{L} \to W$ such that for all $a, b \in \mathcal{O}_\mathcal{L}$ we have

1. $D(ab) = aD(b) + bD(a)$.

2. $D(a + b) = D(a) + D(b)$. 

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3. $D(a) = 0$ if $a \in \mathcal{O}_K$.

We denote by $D_{\mathcal{O}_K}(\mathcal{O}_L, W)$ the $\mathcal{O}_L$-module of all derivations $D : \mathcal{O}_L \to W$. The universal object in the category of derivations of $\mathcal{O}_L$ over $\mathcal{O}_K$ is the $\mathcal{O}_L$-module of Kahler differentials, denoted by $\Omega_{\mathcal{O}_K}(\mathcal{O}_L)$. This is the $\mathcal{O}_L$-module generated by finite linear combinations of the symbols $da$, for all $a \in \mathcal{O}_L$, divided out by the submodule generated by all the expressions of the form $dab - adb - bda$ and $d(a + b) - da - db$ for all $a, b \in \mathcal{O}_L$ and $da$ for all $a \in \mathcal{O}_K$. The derivation $d : \mathcal{O}_L \to \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ is defined by sending $a$ to $da$.

If $D : \mathcal{O}_L \to W$ is a derivation, then $\Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ is universal in the following way. There exist a unique homomorphism $\beta : \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to W$ of $\mathcal{O}_L$-modules such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_L & \xrightarrow{d} & \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \\
\downarrow{D} & & \downarrow{\beta} \\
W & & 
\end{array}
\]

is commutative.

Let $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L)$ be the $p$-adic completion of $\Omega_{\mathcal{O}_K}(\mathcal{O}_L)$, i.e.,

\[
\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) = \varprojlim \Omega_{\mathcal{O}_K}(\mathcal{O}_L)/p^n\Omega_{\mathcal{O}_K}(\mathcal{O}_L).
\]

Since we are assuming that $W$ is $p$-adically complete, the homomorphism $\beta$ induces


the homomorphism

$$\beta : \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) \rightarrow W.$$ 

**Proposition 6.1.1.** We have the isomorphism of $\mathcal{O}_L$-modules

$$\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L dT_1 \oplus \cdots \oplus \mathcal{O}_L dT_{d-1} \oplus (\mathcal{O}_L/D(L/K)\mathcal{O}_L)d\pi_L,$$

where $\pi_L$ is a uniformizer for $L$ and $D(L/K)$ is the different of the extension $L/K$. That is, $dT_1, \ldots, dT_{d-1}$ and $d\pi_L$ generate $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L)$.

**Proof.** To simplify the notation let us denote $\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L)$ by $\hat{\Omega}$, where $\Omega = \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$.

We will show that

$$\frac{\Omega}{\pi^n L \Omega} \cong \frac{\mathcal{O}_L}{\pi^n \mathcal{O}_L} \oplus \cdots \oplus \frac{\mathcal{O}_L}{\pi^n \mathcal{O}_L} \oplus \frac{\mathcal{O}_L}{\pi^n \mathcal{O}_L + D(L/K)\mathcal{O}_L}$$

(6.1)

for all $n \geq 1$. These isomorphisms are compatible: $\cong_{n+1} \equiv_{n} (\text{mod } p^n)$, then we can take the projective limit $\lim_{\leftarrow} \cong_n$ to obtain the result. To prove (6.1), we will start by showing that $\Omega/\pi^n L \Omega$ is generated by $d\pi_L$ and $dT_1, \ldots, dT_{d-1}$ for all $n$.

Let $x \in \mathcal{O}_L$, then by corollary 5.4.2 we have that

$$x = \sum_{k=0}^{\infty} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} \leq p^n - 1} \gamma_{i,k}^{p^n} T_{i_1}^{i_1} \cdots T_{i_{d-1}}^{i_{d-1}} \pi_L^k \right).$$

Therefore, in $\Omega/p^n \Omega$, we can consider the truncated sum

$$\sum_{k=0}^{m} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} \leq p^n - 1} \gamma_{i,k}^{p^n} T_{i_1}^{i_1} \cdots T_{i_{d-1}}^{i_{d-1}} \pi_L^k \right),$$
where \( m \) is such that \( p^n | \pi^{m+1} \). Thus, \( dx \) is generated by \( d\pi \) and \(dT_i\), \( i = 1, \ldots, d-1 \) in \( \Omega/p^n\Omega \).

Let us construct now the isomorphism (6.1). We will define derivations \( D_k \) of \( \mathcal{O}_L \) over \( \mathcal{O}_K \) for \( k = 0, \ldots, d-1 \) as follows. Let \( \mathcal{L}_0 = L \) and \( \mathcal{L}_k = \mathcal{L}_{k-1} \{ \{ T_k \} \} \), \( k = 1, \ldots, d-1 \). For \( k = 0 \) define

\[
D_0: \mathcal{O}_L \rightarrow \mathcal{O}_L/D(L/K)\mathcal{O}_L
\]

by \( D_0(g(\pi_L)) = g'(\pi_L) \), where \( g(X) \) is a polynomial with coefficients in the ring of integers of the maximal subextension of \( L \) unramified over \( K \). This is a well defined derivation of \( \mathcal{O}_L \) over \( \mathcal{O}_K \) by Corollary 5.2 of [14]. For \( 1 \leq k \leq d-1 \), we define the derivation of \( \mathcal{O}_{L_k} \) over \( \mathcal{O}_K \)

\[
D_k: \mathcal{O}_{L_k} \rightarrow \mathcal{O}_{L_k}, \tag{6.2}
\]

by \( D_k(\sum a_i T_k^i) = \sum a_i iT_k^{i-1} \), \( a_i \in \mathcal{O}_{L_{k-1}} \).

We now lift these derivations to derivations of \( \mathcal{O}_L \) over \( \mathcal{O}_K \), by induction, in the following way. Suppose \( D: \mathcal{O}_{L_k} \rightarrow W \) is a derivation of \( \mathcal{O}_{L_{k-1}} \) over \( \mathcal{O}_K \), where \( 1 \leq k \leq d \) and the \( \mathcal{O}_{L_k} \)-module \( W \) is either \( \mathcal{O}_{L_{k-1}} \) or \( \mathcal{O}_{L_{k-1}}/D(L/K)\mathcal{O}_{L_{k-1}} \). Then \( D \) extends to a derivation of \( \mathcal{O}_{L_k} \) over \( \mathcal{O}_K \)

\[
D: \mathcal{O}_{L_k} \rightarrow \mathcal{O}_{L_k} \otimes W;
\]
by
\[ D(\sum_i a_i T_k^i) := \sum_i D(a_i) T_k^i. \]
This derivation is well defined since \( D \) is continuous with respect to the valuation topology of \( \mathcal{O}_{L_{k-1}} \).

Let us now define the map
\[ \partial : \mathcal{O}_L \to \mathcal{O}_L / D(L/K) \mathcal{O}_L \oplus \mathcal{O}_L / p^n \mathcal{O}_L \oplus \cdots \oplus \mathcal{O}_L / p^n \mathcal{O}_L \]
by
\[ a \to (\overline{D_0(a)}, \ldots, \overline{D_{d-1}(a)}) \]
where \( \overline{D_k} \) is the reduction of \( D_k \). This is a well-defined derivation of \( \mathcal{O}_L \) over \( \mathcal{O}_K \) and by the universality of \( \Omega \), this induces a homomorphism of \( \mathcal{O}_L \)-modules
\[ \partial : \Omega / p^n \Omega \to \mathcal{O}_L / D(L/K) \mathcal{O}_L \oplus \cdots \oplus \mathcal{O}_L / p^n \mathcal{O}_L. \]
Let us show that \( \partial \) is an isomorphism. Indeed, it is clearly surjective since for \( (a_0, \ldots, a_{d-1}) \in \mathcal{O}_L \oplus \cdots \oplus \mathcal{O}_L \oplus (\mathcal{O}_L / D(L/K) \mathcal{O}_L) \) we have that
\[ \partial(a_0 d\pi_L + a_1 dT_1 + \cdots + a_{d-1} dT_{d-1}) = (a_0, \ldots, a_{d-1}). \]

since
\[ \overline{D_k}(d\pi_L) = \begin{cases} 0, & k = 0, \\ 1, & 1 \leq k \leq d - 1, \end{cases} \quad \overline{D_k}(dT_i) = \begin{cases} 0, & k \neq i, \\ 1, & k = i, \end{cases} \]
Also, \( \partial \) is injective for if \( a = a_0 d\pi_L + a_1 dT_1 + \cdots + a_{d-1} dT_{d-1} \in \Omega / p^n \Omega \) is such that
\[ \partial(a) = 0, \text{ then } D_k(a) = a_k = 0 \text{ in } \mathcal{O}_L/p^n\mathcal{O}_L, \text{ for } 1 \leq k \leq d-1, \text{ and } D_0(a) = a_0 = 0 \] (mod \( p^n\mathcal{O}_L + \mathcal{D}(L/K)\mathcal{O}_L \)). But then \( a_0 d\pi_L = 0 \), since \( \mathcal{D}(L/K) d\pi_L = 0 \), and therefore \( a = 0 \mod p^n\Omega \). This concludes the proof.

\[ \Box \]

**Definition 6.1.2.** The derivations \( D_k : \mathcal{O}_L \to \mathcal{O}_L, 1 \leq k \leq d-1 \), which are obtained by the lifts of the derivations given in equation (6.2) described in Proposition 6.1.1, will be denoted by \( \frac{\partial}{\partial T_k} \) for \( 1 \leq k \leq d-1 \). Moreover, the derivation \( D_0 : \mathcal{O}_L \to \mathcal{O}_L/D(L/K)\mathcal{O}_L \), also defined in the above proposition, will be denoted by \( \frac{\partial}{\partial T_d} \).

**Remark 6.1.1.** The derivations \( \frac{\partial}{\partial T_k} \) for \( 1 \leq k \leq d-1 \) coincide, informally, with the partial derivative with respect to \( T_k \) of an element \( \alpha \) in \( \mathcal{O}_L = \mathcal{O}_L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \).

On the other hand, the derivation \( \frac{\partial}{\partial T_d} \) has the following interpretation. If \( \alpha \in \mathcal{O}_L \), then there exist a polynomial \( g(X) \in \mathcal{O}_{\bar{L}}[X] \) such that \( \alpha = g(\pi_L) \). Then

\[ \frac{\partial \alpha}{\partial T_d} = g'(\pi_L). \]

Here \( \bar{L} = \bar{L}\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \), where \( \bar{L} \) is the maximal subextension of \( L \) unramified over \( K \).

**Proposition 6.1.2.** Let \( D : \mathcal{O}_L \to W \) be a derivation and let \( \frac{\partial}{\partial T_k}, 1 \leq k \leq d, \) be the derivations form Definition 6.1.2. Then \( D(L/K)D(\pi_L) = 0 \) and

\[ D(\beta) = \sum_{k=1}^{d-1} \frac{\partial \beta}{\partial T_k} D(T_k) + \frac{\partial \beta}{\partial T_d} D(\pi_L). \] (6.3)
Moreover, given \( w_1, \ldots, w_d \in W \) such that \( D(L/K)w_d = 0 \), the map
\[
D(\beta) := \sum_{k=1}^{d} \frac{\partial \beta}{\partial T_k} w_k
\]
(6.4)
is a well-defined derivation from \( \mathcal{O}_L \) into \( W \) over \( \mathcal{O}_K \). In other words, the map
\[
D \mapsto (D(T_1), \ldots, D(\pi_L))
\]
defines an isomorphism
\[
D_{\mathcal{O}_K}(\mathcal{O}_L, W) \cong W \oplus \cdots \oplus W \oplus W_{D(L/K)},
\]
where \( W_{D(L/K)} \) is the submodule of elements killed by \( D(L/K) \).

Proof. The proof follows from the proof of Proposition 6.1.1 and the fact that
\[
D_{\mathcal{O}_K}(\mathcal{O}_L, W) \cong \text{Hom}_{\mathcal{O}_L}(\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L), W).
\]

If \( L \) is a finite extension of the local field \( K \), then we denote by \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K) \) the \( \mathcal{O}_K \)-module of differentials of \( \mathcal{O}_L \) over \( \mathcal{O}_K \).

**Proposition 6.1.3.**
1. \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K) \) as \( \mathcal{O}_L \)-modules. Moreover, the element \( d\pi_L \) generates \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \).

2. If \( M \) is a finite extension of \( L \), the homomorphism \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to \Omega_{\mathcal{O}_K}(\mathcal{O}_M) \) is an embedding.
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We will denote by $K_m$ (resp. $L_m$) the field obtained by adjoining the $m$-th torsion points to $K$, i.e., $K_m = K(\kappa_m)$. Let $v$ denote the normalized valuation $v_M/v_M(p)$, for every finite extension $M$ of $\mathbb{Q}_p$.

**Proposition 6.1.4.** There are positive constants $c_1, c_2 \in \mathbb{R}$, depending on $(F, \pi)$, such that

1. $v(D(L_m/L)) \leq \frac{m}{\alpha} + \log_2(m)/(p - 1) + c_2$ and $v(D(K_m/K)) \geq \frac{m}{\alpha} - c_1$.

Where $\alpha$ is the ramification index of $S$ over $\mathbb{Q}_p$.

2. Let $1 \leq j \leq h$ be fixed. Let $p_m$ be the period (i.e., the generator of the annihilator ideal) of the $\mathcal{O}_{K_m}$-submodule of $\Omega(\mathcal{O}_{K_m})$ generated by $de^i_m$. Then there exists a $j$ for which $v(p_m) \geq \frac{m}{\alpha} - c_1$.

Proof. cf. [14] Proposition 5.3.

**Proposition 6.1.5.** Suppose $K/S$ is an unramified extension and let $q = |k_S|$. Let $h$ be the height of $F$ with respect to $C = \mathcal{O}_S$, cf. Proposition 3.2.2. Then

1. $v(D(K_m/K)) \geq \frac{m}{\alpha} - 1/\alpha(q^h - 1)$.

2. $K_m = K(e^i_m)$ is totally unramified over $K$ and $e^i_m$ is a uniformizer for $K_m$ and

\[ v(\text{period of } de^i_m) = \frac{m}{\alpha} - 1/\alpha(q^h - 1). \]

6.2 Multidimensional derivations

Let $\mathcal{L}$, $\tilde{\mathcal{L}}$, $L$, $\tilde{L}$, $K$ and $W$ be as in the beginning of the previous section.

Definition 6.2.1. A $d$ dimensional derivation of $\mathcal{O}_L^d$ into $W$ over $\mathcal{O}_K$ is map $D : \mathcal{O}_L^d \to W$ such that for all $a_1, \ldots, a_d$ and all $a'_1, \ldots, a'_d$ in $\mathcal{O}_L$ it satisfies

1. Leibniz rule:

$$D(a_1, \ldots, a_i a'_i, \ldots, a_d) = a'_i D(a_1, \ldots, a_i, \ldots, a_d) + a_i D(a_1, \ldots, a'_i, \ldots, a_d),$$

for any $1 \leq i \leq d$.

2. Linearity:

$$D(a_1, \ldots, a_i + a'_i, \ldots, a_d) = D(a_1, \ldots, a_i, \ldots, a_d) + D(a_1, \ldots, a'_i, \ldots, a_d),$$

for any $1 \leq i \leq d$.

3. Alternate: $D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0$, if $a_i = a_j$ for $i \neq j$.

4. $D(a_1, \ldots, a_d) = 0$ if some $a_i \in \mathcal{O}_K$.

We denote by $D^d_{\mathcal{O}_K}(\mathcal{O}_L^d, W)$ the $\mathcal{O}_L$-module of all $d$-dimensional derivations $D : \mathcal{O}_L^d \to W$. 
Consider the $d$th exterior product $\bigwedge_{\mathcal{O}_L}^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ (cf. [18] Chapter 19 §1). This is the $\mathcal{O}_L$-module $\Omega_{\mathcal{O}_K}(\mathcal{O}_L) \otimes \cdots \otimes \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ divided out by the $\mathcal{O}_L$-submodule generated by the elements

$$x_1 \otimes \cdots \otimes x_d,$$

(6.5)

where $x_i = x_j$ for some $i \neq j$. The symbols $x_1 \otimes \cdots \otimes x_d$ will be denoted by

$$x_1 \wedge \cdots \wedge x_d,$$

instead. For $\bigwedge_{\mathcal{O}_L}^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ we consider the $d$-dimensional derivation $d : (\mathcal{O}_L^d) \to \bigwedge_{\mathcal{O}_L}^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L)$ that sends $(a_1, \ldots, a_d)$ to the wedge product $a_1 \wedge \cdots \wedge a_d$. This $\mathcal{O}_L$-module is the universal object in the category of $d$-dimensional derivations of $\mathcal{O}_L$ over $\mathcal{O}_K$, i.e.,

**Proposition 6.2.1.** If $D : (\mathcal{O}_L)^d \to W$ is a $d$-dimensional derivation over $\mathcal{O}_K$ then there exist a homomorphism $\beta : \bigwedge_{\mathcal{O}_L}^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to W$ of $\mathcal{O}_L$-modules such that the diagram

$$
\begin{array}{ccc}
(\mathcal{O}_L)^d & \xrightarrow{d} & \bigwedge_{\mathcal{O}_L}^d \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \\
\downarrow{D} & & \downarrow{\beta} \\
W & & 
\end{array}
$$

is commutative.

**Proof.** The homomorphism $\beta$ is clearly the homomorphism defined by $\beta(da_1 \wedge \cdots \wedge
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$da_d = D(a_1, \ldots, a_d)$. □

**Proposition 6.2.2.** $\bigwedge_{\mathcal{O}_L} \hat{\Omega}_{O_K}(\mathcal{O}_L) \cong (\mathcal{O}_E/\mathcal{D}(L/K)\mathcal{O}_E) \, dT_1 \wedge \cdots \wedge dT_{d-1} \wedge d\pi_L$ as $\mathcal{O}_L$-modules, where $\pi_L$ is a uniformizer for $L$ and $\mathcal{D}(L/K)$ is the different of the extension $L/K$. That is, $dT_1 \wedge \cdots \wedge dT_{d-1} \wedge d\pi_L$ generates $\bigwedge_{\mathcal{O}_L} \hat{\Omega}_{O_K}(\mathcal{O}_L)$ as an $\mathcal{O}_L$-module.

**Proof.** This follows immediately from Lemma 6.1.1. □

**Proposition 6.2.3.** Let $D \in D^d_K(\mathcal{O}_L^d, W)$, then $\mathcal{D}(L/K)D(T_1, \ldots, \pi_L) = 0$. Let $\beta_1, \ldots, \beta_d \in \mathcal{O}_L$, and $g_i(T_1, \ldots, T_{d-1}, T_d) \in \mathcal{O}_L[T_d]$ $i = 1, \ldots, d$, such that $\beta_i = g_i(T_1, \ldots, \pi_L)$. Then

$$D(\beta_1, \ldots, \beta_d) = \det \left[ \frac{\partial \beta_i}{\partial T_j} \right]_{i,j} D(T_1, \ldots, \pi_L), \quad (6.6)$$

where $\frac{\partial}{\partial T_k}$, $k = 1, \ldots, d$ are the derivations from Definition 6.1.2. Moreover, let $w \in W$ such that $\mathcal{D}(L/K)w = 0$, then the map

$$D(\beta_1, \ldots, \beta_d) := \det \left[ \frac{\partial \beta_i}{\partial T_j} \right]_{i,j} w,$$

is well-defined and belongs to $D^d_K(\mathcal{O}_L^d, W)$. In other words, the map

$$D \mapsto D(T_1, \ldots, \pi_L),$$

defines an isomorphism from $D^d_K(\mathcal{O}_L^d, W)$ to the $\mathcal{D}(L/\bar{L})$-torsion subgroup of $W$. 
Proof. This follows Proposition \ref{6.2.2} and the fact that

\[ D^d_{\mathcal{O}_K}(\mathcal{O}_L, W) \cong \text{Hom}_{\mathcal{O}_K}(\bigwedge^d \hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L), W). \]
Chapter 7
Deduction of the formulas

In this section $M$ will denote a local field containing $K_t = K(\kappa_t)$, $\pi_M$ a uniformizer for $M$ and $\pi_t$ a uniformizer for $K_t$.

7.1 Description of the map $\psi^i_{M,m}$ in terms of derivations

Proposition 7.1.1. Let $M = M\{\{T_1\}\cdots\{T_{d-1}\}\}$, $M \supset K_t$, with $(m,t)$ admissible. Suppose $\pi^m$ divides $D(M/K)$; Proposition 6.1.4 guarantees that this happens when $(t - m)/\alpha \geq c_1$. Then the reduction

$$D^i_{M,m} : \mathcal{O}_M^d \rightarrow \frac{R_{M,1}}{\pi^m R_{M,1}}$$

of $D^i_{M,m}$ to $R_{M,1}/(\pi^m/\pi_M)R_{M,1}$ is a $d$-dimensional derivation over $\mathcal{O}_K$. 94
Proof. Let us fix an \(a_1, \ldots, a_{d-1} \in \mathcal{O}_M^*\). From Proposition 6.1.2 and the fact

\[
D(M/K)D^i_{M,m}(a_1, \ldots, a_{d-1}, \pi_M) = 0 \pmod{\pi^m R_{M,1}},
\]

we can to construct, by Proposition 6.1.2, a derivation

\[
D : \mathcal{O}_M \to R_{M,1}/\pi^m R_{M,1},
\]

such that \(D(\pi_M) = D^i_{M,m}(a_1, \ldots, a_{d-1}, \pi_M)\) and \(D(T_k) = D^i_{M,m}(a_1, \ldots, a_{d-1}, T_k)\), \(k = 1, \ldots, d-1\), in the following way

\[
D(\alpha) = \sum_{k=1}^{d-1} \frac{\partial \alpha}{\partial T_k} D(T_k) + \frac{\partial \alpha}{\partial T_d} D(\pi_M),
\]

where \(\alpha \in \mathcal{O}_M\).

According to Proposition 5.4.5 both \(D\) and \(D^i_{M,m}(a_1, \ldots, a_{d-1}, \cdot)\) coincide in \(\mu_M\).

But from the Leibniz rule it follows, by comparing \(D(\pi_M x)\) and \(D^i_{M,m}(a_1, \ldots, a_{d-1}, \pi_M x)\) when \(x \in \mathcal{O}_M\), that they coincide \(\pmod{(\pi^m/\pi_M) R_{M,1}}\) in all \(\mathcal{O}_M\).

It follows now that

\[
D^i_{M,m} : \mathcal{O}^d_M \to \frac{R_{M,1}}{\pi^m R_{M,1}}
\]

satisfies all conditions from definition 6.2.1 and so by Proposition 6.2.3 we have that it is a \(d\)-dimensional derivation such that

\[
D^i_{M,m}(\beta_1, \ldots, \beta_d) = \det \left[ \frac{\partial \beta_i}{\partial T_j} \right]_{i,j} D^i_{M,m}(T_1, \ldots, T_{d-1}, \pi_M),
\]
where $\beta_1, \ldots, \beta_d \in \mathcal{O}_M$.

We can express the map $\psi_{i,M,m}$ out of $D^i_{M,m} : \mathcal{O}_M^d \to R_{M,1}/(\pi^m/\pi_M)R_{M,1}$ in the following way

$$
\begin{cases}
\psi_{i,M,m}^i(u_1, \ldots, u_{d-1}, \pi_M) = \frac{D^i_{M,m}(u_1, \ldots, u_{d-1}, \pi_M)}{u_1 \cdots u_{d-1} \pi_M} \pmod{\frac{\pi^m}{\pi_M} R_{M,1}} \\
\psi_{i,M,m}^i(u_1, \ldots, u_d) = \frac{D^i_{M,m}(u_1, \ldots, u_d)}{u_1 \cdots u_d} \pmod{\frac{\pi^m}{\pi_M} R_{M,1}} \\
\psi_{i,M,m}^i(u_1, \ldots, \pi^k_M u_d) = k \psi_{i,M,m}^i(u_1, \ldots, \pi_M) + \psi_{i,M,m}^i(u_1, \ldots, u_d), \ k \in \mathbb{Z} \\
\psi_{i,M,m}^i(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = 0, \text{ whenever } a_i = a_j \text{ for } i \neq j, a_1, \ldots, a_d \in \mathcal{M}^* 
\end{cases}
$$

where $u_1, \ldots, u_d$ are in $\mathcal{O}_M^* = \{ x \in \mathcal{O}_M : v_M(x) = 0 \}$. It is clear from the definition that this is independent from the choice of a uniformizer $\pi_M$ of $M$. Notice that the fourth property says that $\psi_{i,M,m}^i$ is alternate, in particular it is skew-symmetric, i.e.,

$$
\psi_{i,M,m}^i(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -\psi_{i,M,m}^i(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d).
$$

whenever $i \neq j$.

Let $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ and $v$ denote the normalized valuation $v_{\mathcal{L}}/v_{\mathcal{L}}(p)$. Let $\kappa_L = \kappa \cap L$, i.e., the subgroup of torsion points contained in $\mathcal{L}$. We denoted by $T_{\mathcal{L}}$ be the image of $F(\mu_{\mathcal{L}})$ under the logarithm of the formal group $F$. Then we have a continuous isomorphism $l_F : F(\mu_{\mathcal{L}})/\kappa_L \to T_{\mathcal{L}}$ and by Lemma 5.1.3, we have the
embedding

\[ R_L / \pi^n R_L \to \text{Hom}_{\mathcal{C}}^{\text{seq}}(T_L, C / \pi^n C) \to \text{Hom}_{\mathcal{C}}^{\text{seq}}(F(\mu_L) / \kappa_L, C / \pi^n C). \]  

(7.2)

**Proposition 7.1.2.** Let \( m \) such that \( v(f^{(m-n)}(x)) > 1/(p-1) \) for all \( x \in F(\mu_L) \). Let \((m, t)\) be admissible and put \( \mathcal{M} = L_\ell\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \) and \( \mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \).

Then \( \text{Tr}_{\mathcal{M}/\mathcal{L}} \) induces a homomorphism from \( R_{\mathcal{M},1} / \pi^m R_{\mathcal{M},1} \) to \( \mathcal{L}/\pi^n R_{\mathcal{L}} \) and we have the representation

\[(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)_{\mathcal{L},n}^i = \mathbb{T}_{\mathcal{L}/S} \left( \text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) l_f(x) \right), \quad \forall \alpha \in K_d(\mathcal{M}), \forall x \in F(\mu_L). \]

(7.3)

In particular, \( \text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) \) belongs to \( R_L / \pi^n R_L \) and it is the unique element satisfying (7.3).

**Proof.** Since \( e_n^i = f^{(m-n)}(e_m^i) \) and \( f^{(m-n)}(x) \in \mu_{\mathcal{L},1} \subset \mu_{\mathcal{M},1} \), then

\[
(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)_{\mathcal{L},n}^i = \frac{1}{\pi^{m-n}}(\alpha, f^{(m-n)}(x))_{\mathcal{M},m}^i \n
= \frac{1}{\pi^{m-n}} \mathbb{T}_{\mathcal{M}/S} (\psi_{\mathcal{M},m}^i(\alpha) l(f^{(m-n)}(x))) \n
= \mathbb{T}_{\mathcal{L}/S} (\left[ \text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) \right] l(x)) \n
\]

From the condition on \( m \) we have that \( \pi^{m-n}T_L \subset T_{\mathcal{L},1} \). Thus, after taking duals with respect to \( \mathbb{T}_{\mathcal{L}/S} \) we obtain

\[
\frac{1}{\pi^{m-n}} R_L \supset R_{\mathcal{L},1} \]
or,

\[ \pi^n R_L \supset \pi^m R_{L,1} \]

Then from \( \text{Tr}_{M/L}(R_{M,1}) \subset R_{L,1} \), cf. Proposition 5.2.2 (1), it follows that

\[ \text{Tr}_{M/L}(\pi^m R_{M,1}) \subset \pi^n R_{L,1} \subset \pi^n R_L. \] (7.4)

It then follows that \( \text{Tr}_{M/L}(\psi_{M,m}^i(\alpha)) \in L/\pi^n R_L \). Moreover, since \((N_{M/L}(\alpha), x)_{L,n}^i \) belongs to \( C/\pi^n C \) then \( \text{Tr}_{M/L}(\psi_{M,m}^i(\alpha)) \in R_L/\pi^n R_L \). The uniqueness follows from Lemma 5.1.3.

\[ \square \]

Remark 7.1.1. For \( m \) in Proposition 7.1.2 we can take any

\[ m > n + \alpha \log_p \left( \frac{v_L(p)}{p - 1} \right) + \alpha \frac{1}{p - 1}. \]

This follows the same proof of Proposition 6.2 in [14]. Indeed, let \( k = m - n \). Then \( f^{(k)} \) is divisible by \( \pi^k \), which implies that every term \( a_i X^i \) of the series \( f^{(k)} \) satisfies \( v(a_i) + v(i) \geq k/\alpha \). If \( v(a_i) > 1/(p - 1) \) then there is nothing to prove. If on the other hand \( v(a_i) \leq 1/(p - 1) \) then \( v(i) \geq k/\alpha - 1/(p - 1) \). In this case

\[ v(x^i) \geq \frac{i}{v_L(p)} \geq \frac{p^{\frac{k}{\alpha} - \frac{1}{p - 1}}}{v_L(p)} > \frac{1}{p - 1}. \]
for all $x \in \mu_L$, since $k/\alpha - 1/(p-1) > \log_p(v_L(p)/(p-1))$. Then

$$v(f^{(k)}(x)) > \frac{1}{p-1}$$

for all $x \in \mu_L$.

### 7.2 Invariants of the representation $\tau$

Recall that we have fixed a basis $\{e^i\}_{i=1}^h$ for $\lim_\leftarrow \kappa_n$. We also denoted by $e^i_n$ the reduction of $e^i$ to $\kappa_n$. Clearly $\{e^i_n\}$ is a basis for $\kappa_n$.

Let $M$ be a finite extension of $K$ and let $M = M\{\{T_1\}\cdots\{T_{d-1}\}\}$. Denote by $T(\kappa)$ be the Tate-module $\lim_\leftarrow \kappa_n$. The action of $G_M = Gal(\overline{M}/M)$ on $T(\kappa)$ defines a continuous representation

$$\tau : G_M \to GL_h(C)$$

Let $M_n = M\{\{T_1\}\cdots\{T_{d-1}\}\}$. The reduction of $\tau$ to $GL_h(C/\pi^nC)$ is the analogous representation of $G_M$ on $\kappa_n$ and will be denoted by $\tau_n$. This clearly induces an embedding $\tau_n : G(\mathcal{M}_n/\mathcal{M}) \to GL_h(C/\pi^nC)$.

Assume $M \supset K_t$ where $\beta = (m,t)$ is admissible. If $a \in \mathcal{M}^*$, then

$$\tau_{m+t}(\mathcal{Y}_M(\{T_1, \ldots, T_{d-1}, a\}))$$

is congruent to the identity matrix $I \pmod{\pi^t}$ because the Galois group $G(\mathcal{M}_{m+t}/\mathcal{M})$
fixes $\kappa_t$ and so correspond in $GL_h(C/\pi^{m+t}C)$ to matrices $\equiv I \pmod{\pi^t}$, i.e., there exist characters $\chi_{M,\beta;i,j} : \mathcal{M}^* \to C/\pi^mC$ such that

$$\tau_{m+t}(\Upsilon_M(\{T_1, \ldots, T_{d-1}, a\})) = I + \pi^t(\chi_{M,\beta;i,j}(a)) \in GL_h(C/\pi^{m+t}C).$$

For $M = K_t$ we simply write $\chi_{\beta;i,j}$. By Proposition 4.1.2 (4) we have that

$$N_{M/K_t}\{T_1, \ldots, T_{d-1}, a\} = \{T_1, \ldots, T_{d-1}, N_{M/K_t}(a)\},$$

where $K_t = K_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, and Proposition 4.2.1 (4) implies

$$\chi_{M,\beta;i,j}(a) = \chi_{\beta;i,j}(N_{M/K_t}(a)) \quad (7.6)$$

The definition of the pairing $(\cdot, \cdot)_{M,m}$ implies, for $v \in \kappa_t$, that

$$(\{T_1, \ldots, T_{d-1}, a\}, v)^i_{M,m} = (\chi_{M,\beta;i,j}(a)) (v^j),$$

here $(v^j)$ are the coordinates of $v$. In particular, for $v = e^j_t$ we have

$$(\{T_1, \ldots, T_{d-1}, a\}, e^j_t)^i_{M,m} = \chi_{M,\beta;i,j}(a).$$

According to Proposition 5.3.1 we see that $\chi_{M;i,j}$ uniquely determines on $V_{M,1}$ a constant $c_{M,\beta;i,j} \in R_{M,1}/\pi^mR_{M,1}$ such that

$$\chi_{M,\beta;i,j}(u) = T_{M/S}(\log(u)c_{M,\beta;i,j}) \quad \forall u \in V_{M,1}. \quad (7.7)$$

Namely, $\rho^i_{M,m}(T_1, \ldots, T_{d-1}, e^j_t) = c_{M,\beta;i,j}$. Observe that by equation (7.6) $c_{M,\beta;i,j}$ is
the image of $c_{\beta,i,j} := c_{K_t,\beta,i,j}$, where $K_t = K_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, under the map

$$R_{K_t,1}/\pi^m R_{K_t,1} \to R_{M,1}/\pi^m R_{M,1}$$

($R_{K_t,1} \subset R_{M,1}$). So we will denote $c_{M,\beta,i,j}$ by $\tilde{c}_{\beta,i,j}$.

Finally, observe that $c_{\beta,i,j}$ is an invariant of the isomorphism class of $(F,e_j)$ because if $g : (F,e_j) \to (\tilde{F},\tilde{e}_j)$ is such isomorphism then $\tilde{\rho}_{M,m}^i(T_1,\ldots,T_{d-1},g(x)) = \rho_{M,m}^i(T_1,\ldots,T_{d-1},x)$.

From Proposition 5.3.2 we conclude that

**Proposition 7.2.1.** Let $M \supset K_t$, $(m,t)$ admissible and $M = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$.

If $r(X)$ is a $t$-normalized series for $F$, then

$$D_{M,m}^i(T_1,\ldots,T_{d-1},r(e_j^i)) = \psi_{M,m}^i(T_1,\ldots,T_{d-1},r(e_j^i)) \rho_{M,m}^{i}(T_1,\ldots,T_{d-1}) = -r'(e_j^i) T_1 \cdots T_{d-1} \frac{c_{\beta,i,j}}{l'(e_j^i)}. \quad (7.8)$$

Let $L$ be a local field and let $L = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Define

$$R'_{L,1} := \left\{ x \in L : v_L(x) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right] \right\}. \quad (7.9)$$

Note that

$$v_L(x) \geq -v_L(D(L/S)) - \left[ \frac{v_L(p)}{p-1} \right],$$

if and only if

$$v_L(x) \geq -v_L(D(L/S)) - \left( \frac{v_L(p)}{p-1} \right).$$
This holds since $v_{L}(x)$ and $v_{L}(D(L/S))$ are integers, therefore the conditions

$$\left\lfloor \frac{v_{L}(p)}{p-1} \right\rfloor \geq -v_{L}(D(L/S)) - v_{L}(x)$$

and

$$\left( \frac{v_{L}(p)}{p-1} \right) \geq -v_{L}(D(L/S)) - v_{L}(x)$$

are equivalent by the very definition of the integral part of a real number.

Comparing with equations (7.9) and (5.4) we see that $R'_{L,1} = \pi_{L} R_{L,1}$.

If $M/L$ is a finite extension of local fields and we put $M = M\{\{T\}_{1}\} \cdots \{\{T_{d-1}\}\}$ and $L = L\{\{T\}_{1}\} \cdots \{\{T_{d-1}\}\}$, then clearly

$$R'_{M,1} = (1/D(M/L))R'_{L,1}.$$  \hspace{1cm} (7.10)

**Proposition 7.2.2.** Let $\beta = (k,t)$ be an admissible pair such that $\pi^{k}|D(K_{t}/K)$; this happens for example when $(t-k)/\alpha \geq c_{1}$; this $c_{1}$ is the constant from Proposition 6.1.4. Let $a_{j} \in \mathcal{O}_{K_{t}}$ such that $d\epsilon_{j}^{t} = a_{j}d\pi_{t}$ in $\Omega_{\mathcal{O}_{K_{t}}}(\mathcal{O}_{K_{t}})$; $\pi_{t}$ is a uniformizer for $K_{t}$.

Then

$$c_{j,i,j} \in \frac{a_{j}R'_{K_{t},i} + \frac{\pi^{k}_{t}}{\pi} R'_{K_{t},i} + \frac{\pi^{k}_{t}}{\pi} R_{K_{t},i}}{\pi^{k}_{t} R_{K_{t},i}},$$

where $K_{t} = K_{t}\{\{T\}_{1}\} \cdots \{\{T_{d-1}\}\}$.

**Proof.** Following the proof of Proposition 6.5 of [14]. We begin by taking a repre-
sentative $\lambda_{i,j}$ of $c_{\beta_{i,j}}$ in $R_{K_t,1}$. We have to show that

$$\lambda_{i,j} \in a_j R'_{K_t,1} + \left( \frac{\pi^k}{\pi_t} \right) R'_{K_t,1}.$$  \hfill (7.11)

Let $M \supset K_t$, $\pi_M$ and $\pi_t$ uniformizers for $M$ and $K_t$, respectively, and $M = M\{\{T_1\}\cdots\{T_{d-1}\}\}$, $K_t = K_t\{\{T_1\}\cdots\{T_{d-1}\}\}$. Let $b \in \mathcal{O}_M$ such that $d\pi_t = bd\pi_M$; this exist by Proposition 6.1.3. Then $D(M/K_t) = b\mathcal{O}_M$. Set $\beta_j = ba_j \in \mathcal{O}_M$.

Clearly, $de^j_t = \beta_j d\pi_M$. By Proposition 7.1.1,

$$D^i_{M,k} : \mathcal{O}_M^d \rightarrow R_{M,1} / (\pi^k / \pi_M) R_{M,1}$$

is a $d$-dimensional derivation over $\mathcal{O}_K$, which together with Proposition 7.2.1 implies

$$r'(e^j_t) \beta_j D^i_{M,k}(T_1, \ldots, T_{d-1}, \pi_M) = D^i_{M,k}(T_1, \ldots, T_{d-1}, r'(e^j_t))$$

$$= -r'(e^j_t) T_1 \cdots T_{d-1} \frac{\overline{c_{\beta_{i,j}}}}{r'(e^j_t)} \pmod{\pi^k / \pi_M} R_{M,1}.$$  

Recall that $\overline{c_{\beta_{i,j}}}$ is the image of $c_{\beta_{i,j}}$ under the map $R_{K_t,1} / \pi^k R_{K_t,1} \rightarrow R_{M,1} / \pi^k R_{M,1}$; $R_{K_t,1} \subset R_{M,1}$. This identity implies

$$\lambda_{i,j} \in \beta_j R_{M,1} + \left( \frac{\pi^k}{\pi_M} \right) R_{M,1}.$$  \hfill (7.12)
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Then

\[ v_M(\lambda_{i,j}) \geq \min \{ v_M(\beta_jR_{M,1}) , v_M( (\pi^k/\pi_M)R_{M,1} ) \} \]

\[ \geq \min \left\{ v_M(\beta_j) - v_M(D(M/S)) - \frac{e(M)}{p-1} - 1 , v_M(\frac{\pi^k}{\pi_M}) - v_M(D(M/S)) - \frac{e(M)}{p-1} - 1 \right\} \]

We will further assume that \( M \) is the local field obtained by adjoining to \( K_t \) the roots of the Eisenstein polynomial \( X^n - \pi_t \), \( (n,p) = 1 \). Then \( e(M) = ne(K_t) \) and \( D(M/K_t) = n\pi_M^{n-1} = \pi_t/\pi_M \).

\[ v_M(\lambda_{i,j}) \geq \min \{ v_M(a_j) - v_M(D(K_t/S)) - \frac{e(M)}{p-1} - 1 , v_M(\frac{\pi^k}{\pi_t}) - v_M(D(K_t/S)) - \frac{e(M)}{p-1} - 1 \} \]

Dividing everything by \( e(M/K_t) = n \) and noticing that \( v_M(x) = e(M/K_t)v_{\mathcal{K}_t}(x) \) for \( x \in \mathcal{K}_t \) we obtain

\[ v_{\mathcal{K}_t}(\lambda_{i,j}) \geq \min \{ v_{\mathcal{K}_t}(a_j) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n} , v_{\mathcal{K}_t}(\frac{\pi^k}{\pi_t}) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n} \} \]

Letting \( n \to \infty \) we obtain

\[ v_{\mathcal{K}_t}(\lambda_{i,j}) \geq \min \{ v_{\mathcal{K}_t}(a_j) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} , v_{\mathcal{K}_t}(\frac{\pi^k}{\pi_t}) - v_{\mathcal{K}_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} \} \]
which implies \((7.11)\).

**Lemma 7.2.1.** Assume \(\pi^k|D(K_t/K)\), \(\beta = (k, t)\) admissible, and \(\mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}\) with \(M \supset K_t\). Moreover, assume that \(j\) is such that the inequality in Proposition \(6.1.5\) (2) holds. Then there exist a \(d\)-dimensional derivation over \(O_K\)

\[
D : \mathcal{O}_\mathcal{M}^d \to \frac{R_{\mathcal{M},1}}{\pi^k R_{\mathcal{M},1}},
\]

such that \(D(T_1, \ldots, T_{d-1}, e^j_i) = -T_1 \cdots T_{d-1} \varpi_{\beta,i,j}/l'(e^j_i)\). Moreover, all such derivations coincide when reduced to

\[
R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1},
\]

for \(m \leq k\), if \(m/\alpha \leq k/\alpha - v(D(M/K)) + t/\alpha - c_1\) (\(c_1\) is the constant from Proposition \(6.1.4\)). Here \(v\) denotes normalized valuation \(v_M/v_M(p)\). In particular, they coincide with the reduction of \(D_{\mathcal{M},k} : \mathcal{O}_\mathcal{M}^d \to R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}\) to \(R_{\mathcal{M},1}/(\pi^m/\pi_M)R_{\mathcal{M},1}\).

**Proof.** Let \(\beta_j \in \mathcal{O}_M\) such that \(d\alpha_j^i = \beta_j d\pi_M\) in \(\Omega_{\mathcal{O}_M}(\mathcal{O}_M)\); this \(\beta_j\) exists by Proposition \(6.1.3\). We shall prove that there exist a \(\gamma_j \in R_{\mathcal{M},1}/(\pi^k/\pi_M)R_{\mathcal{M},1}\) such that \(-\varpi_{\beta,i,j}/l'(e^j_i) = \beta_j \gamma_j\), in particular \(-T_1 \cdots T_{d-1} \varpi_{i,j}/l'(e^j_i) = \beta_j T_1 \cdots T_{d-1} \gamma_j\). Then using Proposition \(6.2.3\) we define

\[
D(\alpha_1, \ldots, \alpha_d) := \det \left[ \frac{\partial \alpha_i}{\partial T_j} \right]_{1 \leq i, j \leq d} T_1 \cdots T_{d-1} \gamma_j,
\]

where \(\alpha_1, \ldots, \alpha_d \in \mathcal{O}_\mathcal{M}\), which is the required multidimensional derivation since
$D(T_1, \ldots, T_{d-1}, e^j_t) = \beta_j T_1 \cdots T_{d-1} \gamma_j = -T_1 \cdots T_{d-1} \nabla_{\beta i,j} / l'(e^j_t)$.

Let $\lambda_j$ be a representative for $c_{\beta i,j}$ in $R_{K_{t,1}}$. By Proposition 7.2.2, $\lambda_j \in a_j R'_{K_{t,1}} + (\pi^k / \pi_t) R'_{K_{t,1}}$ where $de^j_t = a_j d\pi_t$. Let $d\pi_t = bd\pi_M$, in particular $D(M/K_t) = bO_M$, and set $\beta_j = a_j b$. We have by (7.10) that

$$\frac{1}{b} R'_{K_{t,1}} = (1/D(M/K_t)) R'_{K_{t,1}} = R'_M \subset R_{M,1},$$

where the last inclusion follows since $D(M/K_t)$ is divisible by $\pi_t / \pi_M$; this follows from the general inequality $v_M(D(M/K_t)) \geq e(M/K_t) - 1$ (cf. [2] chap 1 prop 5.4).

It thus follows

$$-c_{\beta i,j} / l'(e^j_t) = -\frac{\lambda_j}{l'(e^j_t)} \in \beta_j \frac{R_{M,1}}{\pi^k / \pi_M R_{M,1}}.$$

Now let us prove the second assertion. Since $-c_{\beta i,j} / l'(e^j_t) = b_j \gamma_j$, then $\gamma_j$ is uniquely defined (mod $(\pi^k / \pi_M \beta_j) R_{M,1}$). Let $m \leq k$, then the $d$-dimensional derivation over $O_K$

$$D : O^d_M \to \frac{R_{M,1}}{\pi^m R_{M,1}},$$

such that $D(T_1, \ldots, T_{d-1}, \pi_M) = T_1 \cdots T_{d-1} \overline{\gamma_j}$ is uniquely determined if $\pi^m | (\pi^k / \beta_j)$.

Under this condition it follows now that the reduction of all the derivations (7.13) to $R_{M,1} / (\pi^m / \pi_M) R_{M,1}$ coincide.
Finally, the condition $\pi^m|(\pi^k/\beta_j)$ is fulfill when

$$m/\alpha \leq k/\alpha - v(D(M/K)) + t/\alpha - c_1.$$ 

Indeed, $de^j = \beta_j d\pi_M$ implies that $v(p_t) + v(\beta_j) = v(D(M/K))$, with $p_t$ as in Proposition 6.1.4 (2), thus by this same proposition

$$m/\alpha \leq k/\alpha - v(D(M/K)) + t/\alpha - c_1$$

$$\leq k/\alpha - v(D(M/K)) + v(p_t)$$

$$= k/\alpha - v(\beta_j)$$

which is precisely the condition $\pi^m|(\pi^k/\beta_j)$. 

\[\square\]

**Proposition 7.2.3.** Let $M \supset K_t$ and $M = M\{\{T_t\}\} \cdots \{\{T_{d-1}\}\}$, with $\beta = (k, t)$ admissible and $(t - k)/\alpha \geq c_1; c_1$ the constant from Proposition 6.1.4. Let $1 \leq j \leq h$ be as in Lemma 7.2.1. Let $m \leq k$ such that

$$m/\alpha \leq k/\alpha + t/\alpha - v(D(M/K)) - c_1.$$ 

Then the reduction of

$$D^i_{\mathcal{M},m} : \mathcal{O}^d_{\mathcal{M}} \rightarrow R_{M,1}/\pi^m R_{M,1}$$

to $R_{M,1}/(\pi^m/\pi_M) R_{M,1}$ is a $d$-dimensional derivation over $\mathcal{O}_K$. Moreover, this $d$-dimensional derivation can be explicitly constructed as follows. Let $de^j = \beta_j d\pi_M$.
in $\Omega_{\mathcal{O}_K}(\mathcal{O}_M)$, so that there exists a uniquely determined $\gamma_i \in R_{M,1}/(\pi^m/\pi_M)R_{M,1}$ such that there is a lifting $\gamma_i$ of $\gamma_i$ in $R_{M,1}/(\pi^k/\pi_M)R_{M,1}$ for which $\beta_1 \gamma_i = -\bar{e}_{\beta,i,j}/l'(e_i)$. Then

$$D^i_{M,m}(T_1, \ldots, T_{d-1}, \pi_M) = T_1 \cdots T_{d-1} \gamma_i$$

and

$$D^i_{M,m}(\alpha_1, \ldots, \alpha_d) := \det \left[ \frac{\partial \alpha_i}{\partial T_j} \right]_{1 \leq i,j \leq d} T_1 \cdots T_{d-1} \gamma_i,$$

where $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_M$ and $\frac{\partial}{\partial T_k}, 1 \leq k \leq d$, are the derivations from Definition 6.1.2. In particular,

$$D^i_{M,m}(T_1, \ldots, T_{d-1}, e^a_t) = -T_1 \cdots T_{d-1} \bar{e}_{\beta,i,j}/l'(e^a_t), \text{ for all } 1 \leq a \leq h.$$

**Proof.** By Proposition 5.2.2 (3),

$$D^i_{M,m} : \mathcal{O}_M^d \rightarrow R_{M,1}/(\pi^m/\pi_M)R_{M,1}$$

is the reduction of

$$D^i_{M,k} : \mathcal{O}_M^d \rightarrow R_{M,1}/(\pi^k/\pi_M)R_{M,1}$$

and the latter is a $d$-dimensional derivation over $\mathcal{O}_K$ according to Proposition 7.1.1 and

$$D^i_{M,k}(T_1, \ldots, T_{d-1}, e^a_t) = -T_1 \cdots T_{d-1} \bar{e}_{\beta,i,j}/l'(e^a_t)$$

according Proposition 7.2.1. Finally, by Proposition 6.1.4 the condition $(t-k)/\alpha \geq c_1$
implies that $\pi^k|D(K_t/K)$, and Lemma 7.2.1 provides a way of constructing explicitly a $d$-dimensional derivation

$$D : \mathcal{O}_M^d \rightarrow R_{M,1}/(\pi^k/\pi_M)R_{M,1}$$

with $D(T_1, \ldots, T_{d-1}, e_t^j) = -T_1 \cdots T_{d-1} \bar{c}_{\beta, i, j}/l'(e_t^j)$; there exist a $\gamma_j \in R_{M,1}/(\pi^k/\pi_M)R_{M,1}$, by the proof of Lemma 7.2.1 such that $-\bar{c}_{\beta, i, j}/l'(e_t^j) = \beta_j \gamma_j$ and we define

$$D(\alpha_1, \ldots, \alpha_d) := \det \left[ \frac{\partial \alpha_i}{\partial T_j} \right]_{1 \leq i, j \leq d} T_1 \cdots T_{d-1} \gamma_j,$$

where $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_M$. By this same lemma both $D$ and $D_{i, M, k}$ coincide with $D_{i, M, m}$ when reduced to

$$R_{M,1}/(\pi^m/\pi_M)R_{M,1}.$$

The last statement holds because $D_{i, M, k}$ is a multidimensional derivation and the formula for $D_{i, M, k}(T_1, \ldots, T_{d-1}, r(e_t^a))$ in equation (7.8).

\( \square \)

### 7.3 Main formulas

**Theorem 7.3.1.** Let $L \supset K_n$ and $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Let

$$m = n + 2 + \left[ \alpha \log_p \left( \frac{v_L(p)}{p - 1} \right) + \frac{\alpha}{p - 1} \right], \quad (7.14)$$

Take $k$ large enough such that
\[
t/\alpha + k/\alpha \geq m/\alpha + c_1 + v(D(M/K))
\]
(7.15)

\[
k + \alpha + 1 \geq c_1 \alpha,
\]
(7.16)

where \( t = 2k + \alpha + 1, \ M = L_t \) and \( \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). This happens for example when

\[
k/\alpha \geq m/\alpha + \log_2(2k + \alpha + 1)/(p - 1) + c_1 + c_2 + v(D(L/K)),
\]
(7.17)

where \( c_1 \) and \( c_2 \) are the constants from Proposition 6.1.4. Then \( \mathcal{M}, \ t, \ k, \) and \( m \) satisfy Proposition 7.2.3. Let

\[
D_{M,m}^i : \mathcal{O}_M^d \rightarrow \frac{R_{M,1}}{\pi_M} R_{M,1}
\]
(7.18)

be the \( d \)-dimensional derivation constructed as in Proposition 7.2.3 and let

\[
\psi_{M,m}^j : K_d(M) \rightarrow \frac{R_{M,1}}{\pi_M} R_{M,1}
\]
(7.19)

be the multidimensional logarithmic derivative built out of \( D_{M,m}^i \) by equation (7.1). Then

\[
(N_{M/L}(\alpha), x)_L^j = \mathbb{T}_{L/S}(\text{Tr}_{M/L}(\psi_{M,m}^j(\alpha)) l(x)) = \mathbb{T}_{M/S}(\psi_{M,m}^j(\alpha) l(x)),
\]
(7.20)

for all \( \alpha \in K_d(M) \) and all \( x \in F(\mu_L) \).
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Proof. By considering the tower $L_t \subset L \subset K$ and the upperbound in Proposition 6.1.4 (1), we get

$$v(D(M/K)) \leq v(L_t/L) + v(D(L/K)) \leq t/\alpha + \frac{\log_2(t)}{p-1} + c_2 + v(D(L/K)).$$

Adding $m/\alpha + c_1$ we obtain, by (7.17), the inequality (7.15). The definition of $t$ clearly implies that $(k,t)$ is admissible and condition (7.16) implies $(t-k)/\alpha \geq c_1$, thus $M, \mathcal{M}, t, k$ and $m$ satisfy the hypothesis of Proposition 7.2.3. The result now follows from Proposition 7.1.2, the Remark 7.1.1 and equation (7.1). It remains only to check that $\text{Tr}_{\mathcal{M}/\mathcal{L}}((\pi^m/\pi^2_M)R_{\mathcal{M},1}) \subset \pi^n R_{\mathcal{L}}$, so that $\text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}(\alpha))$ is well defined in $R_{\mathcal{L}}/\pi^n R_{\mathcal{L}}$. To do this notice that condition (7.14) implies

$$m - 1 > n + \alpha \log_p\left(\frac{v_L(p)}{p-1}\right) + \frac{\alpha}{p-1}$$

and we can apply Remark 7.1.1 to $m - 1$ and get, by equation (7.4), that

$$\text{Tr}_{\mathcal{M}/\mathcal{L}}\left(\pi^{m-1} \frac{\pi}{\pi^2_M} R_{\mathcal{M},1}\right) \subset \text{Tr}_{\mathcal{M}/\mathcal{L}}(\pi^{m-1} R_{\mathcal{M},1}) \subset \pi^n R_{\mathcal{L}}$$

bearing in mind that $\pi^2_M|\pi$, since $\pi^k|D(M/K)$ implies $e(M/K) > 1$ (i.e., $M/K$ is not unramified).

We state a simplified version of the theorem above

**Theorem 7.3.2.** Let $L \supset K_n$ and $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Let $\mathcal{M}$ and $m$ be
defined as in Theorem 7.3.1. Then

\[(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^i_{\mathcal{L}, n} = \mathbb{T}_{\mathcal{L}/S} (\text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi^i_{\mathcal{M}, m}(\alpha)) I(x)) = \mathbb{T}_{\mathcal{M}/S} (\psi^i_{\mathcal{M}, m}(\alpha) I(x)),\]

for all \(\alpha \in K_d(\mathcal{M})\) and all \(x \in F(\mu_{\mathcal{L}})\). Here \(\psi^i_{\mathcal{M}, m}\) is the explicit multidimensional logarithmic derivative constructed in Theorem 7.3.1.
Chapter 8

Formulas for the Lubin-Tate formal groups

In this section, guided in much by Section 7 in [14], we will give an explicit description of the corresponding formulas for the case where the formal group is Lubin-Tate. This will include an explicit computation of the invariants defined in equation (7.7) and of the lower bounds for $t$, $m$ and $k$ in Theorem 7.3.1.

Let $K/\mathbb{Q}_p$ be a local field with ring of integers $\mathcal{O}_K$, $\pi$ a uniformizer for $K$, $k_K$ its residue field and $q = |k_K|$. Let $\Lambda_\pi$ be the subset of $\mathcal{O}_K[[X]]$ consisting of the series $f$ such that

1. $f(X) \equiv \pi X \pmod{\deg 2}$.

2. $f(X) \equiv X^q \pmod{\pi}$.

Let $F_f$ be the Lubin-Tate formal group such that $[\pi]_{F_f} = f$. In this case we will take $S = K$ and $C = \mathcal{O}_K$. Evidently $f$ has height equal to 1 with respect to to
$C = \mathcal{O}_K$. Thus $\kappa_n \simeq \mathcal{O}_K/\pi^n\mathcal{O}_K$ and $\varprojlim \kappa_n \simeq \mathcal{O}_K$. Let $e$ be a generator for $\varprojlim \kappa_n$ and $e_n$ its reduction in $\kappa_n$.

Let $K_n = K(\kappa_n)$ and $K_\infty = K(\kappa)$, where $\kappa = \bigcup \kappa_n$. Let $\mathcal{K}_n = K_n\{T_1\} \cdots \{T_{d-1}\}$ and $\mathcal{K} = K\{T_1\} \cdots \{T_{d-1}\}$. The extension $K_n/K$ is totally ramified and $e_n$ is a uniformizer for $K_n$. Moreover, $[K_n/K] = q^n - q^{n-1}$, then the imbedding $\tau_n$

$$\tau_n : \text{Gal}(K_n\{T_1\} \cdots \{T_{d-1}\}/K\{T_1\} \cdots \{T_{d-1}\}) \to (\mathcal{O}_K/\pi^n\mathcal{O}_K)^*, \tag{8.1}$$

which is the map induced by the representation $\tau : G_K \to \mathcal{O}_K^*$ (cf. § 7.1, equation (7.5)), is an isomorphism since $[K_n\{T_1\} \cdots \{T_{d-1}\}/K\{T_1\} \cdots \{T_{d-1}\}] = [K_n : K] = q^n - q^{n-1}$, $|((\mathcal{O}_K/\pi^n\mathcal{O}_K)^*)| = q^n - q^{n-1}$.

**Proposition 8.0.1.** Let $w_n = l'(e_n)$ de $n \geq 1$. Then $w_n$ generate $\Omega_C(\mathcal{O}_K)$ as $\mathcal{O}_K-$modules. We also have

$$w_n = \pi w_{n+1}. \tag{8.1}$$

**Proof.** cf. [14] Proposition 7.9. \hfill \Box

**Proposition 8.0.2.** $w_n^g = \tau(g)w_n$ for all $g \in G(\overline{K}/K)$.

**Proof.** cf. [14] § 7.2.3. \hfill \Box

Let $M$ be a finite extension of $K_s$, $\pi_M$ a uniformizer for $M$ and let $M$ denote the field $M\{T_1\} \cdots \{T_{d-1}\}$. Let $\pi_1$ be a uniformizer for $K_1 = K(\kappa_1)$. Let

$$P_M = (1/(\pi_1 D(M/K)))\mathcal{O}_M = \{x \in M : v_M(x) \geq -v_M(\pi_1 D(M/K))\}.$$
We define a $d$-dimensional derivation $Q_{M,s} : \mathcal{O}_M^d \to P_M/(\pi^s/\pi_1)P_M$, over $\mathcal{O}_K$, in the following way. Let $b' \in \mathcal{O}_M$ such that $w_s = l'(c_s)de_s = b'd\pi_M$, then $b'\mathcal{O}_M = D(M/K_s)$. Let us put

$$Q_{M,s}(T_1, \ldots, T_{d-1}, \pi_M) = \frac{T_1 \cdots T_{d-1}}{b'\pi^s}. \tag{8.3}$$

Clearly, $T_1 \cdots T_{d-1}/(b'\pi^s) \in P_M$. By Proposition 6.1.5 (2) the period of $de_s$ is generated by $\pi^s/\pi_1$, then $D(K_s/K) = \pi^s/\pi_1\mathcal{O}_s$ and so

$$D(M/K) = D(M/K_s)D(K_s/K) = D(M/K_s)(\pi^s/\pi_1). \tag{8.4}$$

Hence $D(M/K)Q_{M,s}(T_1, \ldots, T_{d-1}, \pi_M) \in (\pi^s/\pi_1)P_M$. Therefore, by Proposition 6.2.3 $Q_{M,s}$ defines a $d$-dimensional derivation as follows

$$Q_{M,s}(\alpha_1, \ldots, \alpha_d) := \det \left[ \frac{\partial \alpha_i}{\partial T_j} \right]_{1 \leq i, j \leq d} \frac{T_1 \cdots T_{d-1}}{b'\pi^s},$$

where $\alpha_1, \ldots, \alpha_d \in \mathcal{O}_M$. Note that the definition of $Q_{M,s}$ is independent of the choice of uniformizer $\pi_M$ of $M$.

**Proposition 8.0.3.** Let $M/L/K_s$, $M \supset K_1$, be a finite tower of local fields and let

$$\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \quad \text{and} \quad \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}.$$

Suppose that $D(M/L)|\pi^{t-s}$. Then $(\pi^t/\pi_1)P_{\mathcal{L}}$ is contained in $X = (\pi^s/\pi_1)P_{\mathcal{L}}\mathcal{O}_{\mathcal{M}}$ so
$Q_{M,t}$ $\pmod{X}$ is well-defined. Let

$$\alpha : P_L/(\pi^s/\pi_1)P_L \to P_M/X$$

be the injection induced by the composition $P_L \subset P_M$. Then for $y \in \mathcal{O}_L^d$

$$\alpha(Q_{L,s}(y)) = Q_{M,t}(y) \pmod{X}.$$

**Proof.** The Proposition 8.0.3 and its proof were suggested by Professor V. Kolyvagin.

First,

$$\frac{\pi^t}{\pi_1}P_M = \frac{\pi^t}{\pi_1} \frac{1}{\pi_1 D(M/K)} = \frac{\pi^t}{\pi_1 \cdot \pi_1 D(M/L) D(L/K)} = \frac{\pi^t}{\pi_1} \frac{\pi^{t-s}}{D(M/L) \cdot \pi_1 D(L/K)} \subset X$$

because $(\pi^{t-s}/D(M/L)) \subset \mathcal{O}_M$ by our assumption.

Let $a$ and $c$ in $\mathcal{O}_M$ and $b \in \mathcal{O}_L$ are such that

$$w_t = cd\pi_M, \quad w_s = bd\pi_L \quad \text{and} \quad d\pi_L = ad\pi_M.$$

Because $\pi^{t-s}w_t = w_s$ we have $\pi^{t-s}cd\pi_M = bd\pi_M = bad\pi_M$. So

$$\pi^{t-s}c \equiv ba \pmod{D(M/L)}.$$

Dividing this congruence by $\pi^tcb$ and taking into account that $c\mathcal{O}_M = D(M/K_t)$ and $b\mathcal{O}_L = D(L/K_s)$, we have

$$\frac{1}{\pi^s b} \equiv \frac{a}{\pi^t c} \pmod{Z},$$
where
\[ Z = \frac{D(M/K)}{\pi^t D(M/K_t) D(L/K_s)} = \frac{\mathcal{O}_M}{\pi_1 D(L/K_s)} = \frac{\pi^s}{\pi_1} \frac{\mathcal{O}_M}{\pi_1 D(L/K)} = X \]

(we are using that \( D(M/K) = D(M/K_t) D(K_t/K) \) and \( D(K_t/K) = \pi^t/\pi_1, D(L/K) = (\pi^s/\pi_1) D(L/K_s) \)). So
\[ Q_{M,t}(T_1, \ldots, T_{d-1}, \pi_M) = \frac{a}{\pi^t c} = \frac{1}{\pi^t b} \pmod{X} = \alpha(Q_{L,s}(T_1, \ldots, T_{d-1}, \pi_L)). \]

This implies the corresponding equality for arbitrary \( y \in \mathcal{O}_L^d \) since right hand and left hand mappings are multidimensional derivations of \( \mathcal{O}_L^d \) over \( \mathcal{O}_K \). \( \square \)

**Proposition 8.0.4.** \( Q_{M,s}^g = \tau(g^{-1})Q_{M,s} \). Here \( Q_{M,s}^g \) is the \( d \)-dimensional derivation defined by \( Q_{M,s}^g(a_1, \ldots, a_{d-1}) = [Q_{M,s}(a_1^{g^{-1}}, \ldots, a_{d-1}^{g^{-1}})]^g \).

**Proof.** Notice that it is enough to check that
\[ Q_{M,s}^g(T_1, \ldots, T_{d-1}, \pi_M) = \tau(g^{-1})Q_{M,s}(T_1, \ldots, T_{d-1}, \pi_M), \]

by equation (6.6) of Proposition 6.2.3.

Let \( b' \in \mathcal{O}_M \) such that \( w_s = b'd\pi_M \). Then \( w^g_s = (b')^g d\pi_M^g \) and, by Proposition 8.0.2 we have that \( w_s = \tau(g^{-1})(b')^g d\pi_M^g \). Since \( \pi_M^g \) is also a uniformizer for \( M \) and the definition of \( Q_{M,s} \) is independent of the uniformizer, then
\[ Q_{M,s}(T_1, \ldots, T_{d-1}, \pi_M^g) = \frac{1}{\tau(g^{-1}) (b')^g \pi_s^g}. \]
but
\begin{equation*}
\frac{1}{\tau(g^{-1})(b')^{g^s}} = \tau(g) \left( \frac{1}{b'^{\pi^s}} \right)^g.
\end{equation*}
This last expression is equal to \(\tau(g)Q_{M,s}(T_1, \ldots, T_{d-1}, \pi_M)^g\) by equation (8.3).

Proposition 8.0.5. Let \(M\) be a finite extension of \(K\), \(M \cap K_\infty = K_s\) and let \(N = M_{s+1}\). Put \(\mathcal{N} = N\{T_1\} \cdots \{T_{d-1}\}\) and \(\mathcal{M} = M\{T_1\} \cdots \{T_{d-1}\}\). Then

\begin{equation*}
\tau_{s+1}(G(N/M)) = \frac{1 + \pi^s C}{1 + \pi^{s+1} C} \subset (C/\pi^{s+1} C)^*\end{equation*}

where \(C = \mathcal{O}_K\), and the element

\begin{equation*}
\sum_{g \in G(N/M)} (\tau_{s+1}(g) - 1)g
\end{equation*}

takes \((\pi/D(N/M))\mathcal{O}_{\mathcal{N}}\) to \((\pi^{s+1}/\pi_1)\mathcal{O}_{\mathcal{N}}\). Also \(D(N/M)|\pi\).

Proof. This follows immediately from the fact that \(G(N/M) \cong G(\mathcal{N}/\mathcal{M})\) and Proposition 5.12 in [14] and its proof.

Let \(M\) be a finite extension of \(K_s\) and \(\mathcal{M} = M\{T_1\} \cdots \{T_{d-1}\}\). From \(Q_{M,s}\) we can define the logarithmic derivative
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\[ QL_{M,s} : K_d(\mathcal{M}) \to \frac{\frac{1}{\pi_M}}{\frac{\pi^s}{\pi_1 \pi_M}} P_M \]

by

\[
\begin{align*}
QL_{M,s}(u_1, \ldots, u_{d-1}, \pi_M) &= \frac{Q_M(u_1, \ldots, u_{d-1}, \pi_M)}{u_1 \cdots u_{d-1} \pi_M} \pmod{\frac{\pi^s}{\pi_1 \pi_M} P_M}, \\
QL_{M,s}(u_1, \ldots, u_d) &= \frac{Q_M(u_1, \ldots, u_d)}{u_1 \cdots u_d} \pmod{\frac{\pi^s}{\pi_1} P_M}, \\
QL_{M,s}(u_1, \ldots, \pi_M^k u_d) &= kQL_{M,s}(u_1, \ldots, \pi_M) + QL_{M,s}(u_1, \ldots, u_d), \quad k \in \mathbb{Z} \\
QL_{M,s}(a_1, \ldots, a_{d-1}) &= 0, \text{ whenever } a_i = a_j \text{ for } i \neq j \text{ and } a_1, \ldots, a_{d-1} \in \mathcal{M}^*. \quad (8.5)
\end{align*}
\]

where \( u_1, \ldots, u_d \) are in \( \mathcal{O}_M^* = \{ x \in \mathcal{O}_M : v_M(x) = 0 \} \). Notice the forth property says that \( QL_{M,s} \) is alternate, in particular it is skew-symmetric, i.e,

\[ QL_{M,s}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -QL_{M,s}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d). \]

whenever \( i \neq j \).

Let \( L \supset K \) and take the smallest \( r \) such that \( L \cap K_\pi \subset K_r \). Let \( \gamma_m \) be a uniformizer for \( L_m = L(\kappa_m) \) and define \( \mathcal{L}_m = L_m\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). Then

\[ L_s \cap K_\pi = (L \cap K_\pi)K_s = K_s \quad (s \geq r). \]

Let \( \mathcal{M} = \mathcal{L}_s \). For abbreviation we will write \( P_s, Q_s, QL_s, N_{t/s} \) and \( Tr_{t/s} \) instead of
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$P_{L_s}, Q_{M,s}, QL_{M,s}, N_{L_s/L_t}$ and $\text{Tr}_{L_s/L_t}$. Moreover,

$$\text{Tr}_{t/s}(P_t) \subset P_s$$

(8.6)

and

$$\text{Tr}_{t/s}((1/\gamma_t)P_t) \subset (1/\gamma_s)P_s.$$  

(8.7)

Indeed, since $D(L_t/K) = D(L_t/L_s)D(L_s/K)$ then $P_t = (1/D(L_t/L_s))P_s$ and so

$$\text{Tr}_{t/s}(P_t) = \text{Tr}_{t/s}((1/D(L_t/L_s))P_s) = P_s\text{Tr}_{t/s}(1/D(L_t/L_s)) \subset P_s,$$

and $\text{Tr}_{t/s}((\gamma_s/\gamma_t)P_t) \subset \text{Tr}_{t/s}(P_t) \subset P_s$.

**Proposition 8.0.6.** For $s \geq r + 1$ and $t \geq s$ we have

$$QL_s(N_{t/s}(a_1, a_2, \ldots, a_d)) = \text{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)) \in \frac{(1/\gamma_s)P_s}{(\pi^s/\pi_1\gamma_s)P_s}$$

for $a_1 \in \mathcal{L}_t^* = L_t\{\langle T_1 \rangle\} \cdots \{\langle T_{d-1} \rangle\}^*$ and $a_2, \ldots, a_d \in \mathcal{L}_s^* = L_s\{\langle T_1 \rangle\} \cdots \{\langle T_{d-1} \rangle\}^*$.

**Proof.** It will be enough to consider the case $t = s + 1$. From Proposition 8.0.4 it follows that
QL_t\left( \prod_{g \in G(L_t/L_s)} a_1^g, a_2, \ldots, a_d \right) = \left( \sum \tau_t(g)g \right) QL_t(a_1, \ldots, a_d) \tag{8.8}
= \left( \sum g \right) QL_t(a_1, \ldots, a_d) \tag{8.9}
+ \left( \sum (\tau_t(g) - 1)g \right) QL_t(a_1, \ldots, a_d) \tag{8.10}

and by Proposition 8.0.5 we see that \( \sum (\delta_t(g) - 1)g \) takes
\[
P_t = \frac{1}{D(L_t/L_s)} P_s = \frac{\pi}{D(L_t/L_s)} \left( \frac{1}{\pi} P_s \right)
\]
to
\[
\left( \frac{\pi^t}{\pi_1 O_{L_t}} \right) \left( \frac{1}{\pi} P_s \right) = \frac{\pi^s}{\pi_1} P_s O_{L_t}
\]
Then
\[
QL_s(N_{t/s}(a_1), a_2, \ldots, a_d) = QL_t(N_{t/s}(a_1), a_2, \ldots, a_d) \pmod{\frac{\pi^s}{\pi_1 \gamma_s} P_s}
= \text{Tr}_{t/s}(QL_t(a_1, \ldots, a_d)),
\]
where the first equality follows from Proposition 8.0.3
\hfill \Box
8.1 Computations of the invariants

We will assume that \( p \neq 2 \). Let \((m, t)\) be an admissible pair, \( M = K_t \) and \( \mathcal{M} = M \{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). We can take \( e_t \) as a uniformizer \( \pi_t \) of \( K_t \). We will show that

\[
\left(\left\{T_1, \ldots, T_{d-1}, u\right\}, e_t\right)_{\mathcal{M},m} = T_{\mathcal{M}/K_t} \left( \log u \cdot \left( -\frac{1}{\pi_t^t} \right) \right) \pmod{\pi^n C}
\]

(8.11)

for all \( u \in V_{\mathcal{M},1} = 1 + \mu_{\mathcal{M},1} \). From equation (7.7) we have

\[
c_\beta = -\frac{1}{\pi_t^t} \pmod{\pi^m R_{t,1}}.
\]

(8.12)

Furthermore, every \( u \in V_{\mathcal{M},1} \) can be expressed as

\[
\prod_{\tilde{i}=(i_1, \ldots, i_d) \in S, i_d \geq v_{\mathcal{M}/(p)} / (p-1) + 1} (1 + \theta_\tilde{i} T_{i_1}^{i_1} \cdots T_{i_{d-1}}^{i_{d-1}} \pi_t^{i_d})
\]

where \( \theta_\tilde{i} \in \mathcal{R} \), \( \mathcal{R} \) is the group of \( q-1 \)th roots of 1 in \( K_t^* \), and \( S \subset \mathbb{Z}^d \) is an admissible set (see Corollary from Section 1.4.3 [7]), then it is enough to check (8.11) for

\[
u = 1 + \theta T_{i_1}^{i_1} \cdots T_{i_{d-1}}^{i_{d-1}} \pi_t^{i_d}, \quad (\theta^{q-1} = 1).
\]

Case 1) Suppose \((i_1, \ldots, i_{d-1}) \neq (0, \ldots, 0)\). Then the right hand side of (8.11) is zero since \( c_{\mathcal{M}/K_t}(\log u) = 0 \). Let us show that the left hand side is also zero as well.

Suppose first \( i_k > 0 \) for some \( 1 \leq k \leq d-1 \). Consider \( \mathcal{N} = K_t \{\{Y_1\}\} \cdots \{\{Y_{d-1}\}\} \)

where \( Y_k = T_k^{i_k} \) and \( Y_r = T_r \), for \( r \neq k \). By lemma 8.1.1 below, \( \mathcal{M}/\mathcal{N} \) is a finite
extension of degree $i_k$ and $N_{M/N}(T_k) = \pm Y_k$. Let also $i'_k = 1$ and $i'_r = i_r$ for $r \neq k$.

Therefore by proposition 4.2.1

\[
\left( \{ T_1, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{M,m} = \\
= \left( N_{M/N} \{ T_1, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} \\
= \left( \{ Y_1, \ldots, N_{M/N}(T_k), \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} \\
= \left( \{ Y_1, \ldots, \pm 1, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} \\
\oplus \left( \{ Y_1, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m}
\]

Since $p \neq 2$, \( \left( \{ Y_1, \ldots, \pm 1, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} = 0 \). On the other hand, since $\theta^{q-1} = 1$ then

\[
\left( \{ Y_1, \ldots, \theta, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} \\
= \left[ 1/(q - 1) \right] \left( \{ Y_1, \ldots, \theta^{q-1}, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} = 0
\]

and also \( \left( \{ Y_1, \ldots, \pi_t, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{i_d} \}, \ e_t \right)_{N,m} = 0 \) by the norm series relation for Lubin-Tate formal groups \( \left( \{ a_1, \ldots, -X, \ldots, a_d \}, \ X \right)_{N,m} = 0 \) and
recalling that $\pi_t = e_t$. Thus

\[
\left( \{T_1, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{M}, m}
= \left( \{Y_1, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{N}, m}
= \left( \{Y_1, \ldots, -\theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j}, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{N}, m}
\]

The second equality follows from the fact that \(\{Y_1, \ldots, Y_{d-1}, 1+\theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j}\}\) is trivial, for \(r \neq k\), in the Milnor K-group \(K_d(\mathcal{M})\). Moreover, the last expression in the chain of equalities is again zero because \(\{Y_1, \ldots, -\theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j}, \ldots, Y_{d-1}, 1 + \theta Y_1^{i_1} \cdots Y_{d-1}^{i_{d-1}} \pi_t^{-j} \}\) is the zero element, by the Steinberg property, in the Milnor K-group \(K_d(\mathcal{M})\).

Suppose now \(i_k < 0\). We take \(Y_k = T_k^{-i_k}\) instead and by lemma 8.1.1 we have \(N_{\mathcal{M}/\mathcal{N}}(T_k^{-1}) = \pm T_k^{-i_k} = \pm Y_k\). Noticing that

\[
\left( \{T_1, \ldots, T_{k-1}, T_k, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{M}, m}
= - \left( \{T_1, \ldots, T_k^{-1}, T_{k+1}, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{M}, m}
\]

we can now apply the same argument as before to conclude that

\[
\left( \{T_1, \ldots, T_{k-1}, T_k, \ldots, T_{d-1}, 1 + \theta T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi_t^{-j} \}, e_t \right)_{\mathcal{M}, m} = 0.
\]

Case 2) Suppose \((i_1, \ldots, i_{d-1}) = (0, \ldots, 0)\). That is, when \(u\) is an element of the
one dimensional local field $K_t$. In this case, we will show in lemma 8.1.2 below that
the pairing $\langle \{ T_1, \ldots, T_{d-1}, u \}, e_t \rangle_{M,m}$ coincides with the pairing taking values in the
one dimensional local field $K_t$: $\langle u, e_t \rangle_{K_t,m}$. Thus, by [14] section 7.3.1 and the fact
that $c_{M/K}(\log(u)) = \log(u)$ formula (8.11) follows.

**Lemma 8.1.1.** Let $M$ be a complete discrete valuation field and $\mathcal{M} = M\{\{T\}\}$.

Put $Y = T^j$ for $j > 0$. Define $\mathcal{N} = M\{\{Y\}\}$. Then $\mathcal{M}/\mathcal{N}$ is a finite extension of
degree $j$ and $N_{\mathcal{M}/\mathcal{N}}(T) = \pm Y$.

**Remark:** Since $M$ is a complete discrete valuation field, the result immediately
generalizes to the $d$-dimensional case, for if $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, then we can
take $M$ to be $L\{\{T_1\}\} \cdots \{\{T_{d-2}\}\}$ and apply the result to $\mathcal{M} = M\{\{T_{d-1}\}\}$ and
$\mathcal{N} = M\{\{Y_{d-1}\}\}$, where $Y_{d-1} = T^j_{d-1}$.

**Proof.** We can assume that $M$ contains $\zeta_j$, a primitive $j$th root of unity of 1. Oth-
erwise we can considering the diagram

\[
\begin{array}{ccc}
M(\zeta_j)\{\{T\}\} & \rightarrow & M(\zeta_j)\{\{Y\}\} \\
\downarrow & & \downarrow \\
\mathcal{M} = M\{\{T\}\} & \rightarrow & M(\zeta_j)\{\{Y\}\} \\
\downarrow & & \downarrow \\
\mathcal{N} = M\{\{Y\}\} & \rightarrow & M(\zeta_j)\{\{Y\}\}
\end{array}
\]

we see that $[\mathcal{M}/\mathcal{N}] = [M(\zeta_j)\{\{T\}\} : M(\zeta_j)\{\{Y\}\}]$ since $M\{\{T\}\} \cap (M(e_j)\{\{T\}\}) = \ldots$
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\[ M\{\{Y\}\}. \]

Note that \( Y \) has exact order \( j \) in \( \mathcal{N}^*/(\mathcal{N}^*)^j \) for if \( Y = \alpha^k, \alpha \in \mathcal{N}^* \), then \( 0 = v_{\mathcal{N}}(Y) = kv_{\mathcal{N}}(\alpha) \), thus \( \alpha \in \mathcal{O}_{\mathcal{N}}^* \), and we can go to the residue field \( k_\mathcal{N} \) of \( \mathcal{N} \) where we have 1 = \( v_{k_\mathcal{N}}(Y) = kv_{k_\mathcal{N}}(\alpha) \), which implies \( k = 1 \). Then by Kummer theory (cf. 2 Chapter 3 Lemma 2) we have that the polynomial \( P(X) = X^j - Y \in \mathcal{O}_{\mathcal{N}}[X] \) is irreducible. Thus \( [\mathcal{M}/\mathcal{N}] = j, \) and \( N_{\mathcal{M}/\mathcal{N}}(T) \) is the product of the roots of the polynomial \( P(X) \). These roots are \( \zeta_j^k T, \ k = 1, \ldots, j \). Thus

\[
N_{\mathcal{M}/\mathcal{N}}(T) = \prod_{k=1}^{j} \zeta_j^k T = \zeta_j^{(j+1)/2} T^j = \begin{cases} T^j = Y, & \text{if } j \text{ is odd}, \\ -T^j = -Y, & \text{if } j \text{ is even}. \end{cases} = (-1)^{j+1} Y.
\]

\[ \square \]

**Lemma 8.1.2.** For a local field \( L/\mathbb{Q}_p \), let \( \mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \) and consider the map

\[
f : L^* \to K_d(\mathcal{L}) \to \text{Gal}(\mathcal{L}^{ab}/\mathcal{L}) \to \text{Gal}(L^{ab}/L), \quad (8.13)
\]

defined by

\[
a \to \{T_1, \ldots, T_{d-1}, a\} \to \gamma_{\mathcal{L}}(\{T_1, \ldots, T_{d-1}, a\}) \to \gamma_{\mathcal{L}}(\{T_1, \ldots, T_{d-1}, a\}) \bigg|_{\text{Gal}(L^{ab}/L)}.
\]

This coincides with the reciprocity map for \( L, \theta_L : L^* \to \text{Gal}(L^{ab}/L) \). Thus, for \( L = K_t \) and \( \mathcal{M} = K_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \) we have

\[
(\{T_1, \ldots, T_{d-1}, u\}, e_t)_{\mathcal{M},m} = (u, e_t)_{K_t,m}.
\]
for all \( u \in V_{L,1} = \{ x \in L : v_L(x - 1) > [v_L(p)/(p - 1)] + 1 \} \).

**Proof.** It is enough to verify the two conditions of [2] Chapter 5 §2.8 Proposition 6.

Let \( \mathcal{L}_d \) be \( \mathcal{L} \), and \( \mathcal{L}_{d-1} = k_{\mathcal{L}(t_1)} \cdots (t_{d-1}) \). The chain of residue fields of \( \mathcal{L} \).

By (2) of Theorem 4.1.1 we have

\[
\begin{array}{ccc}
K_d (\mathcal{L}_d) & \xrightarrow{\gamma_{\mathcal{L}_d}} & \text{Gal}(\mathcal{L}^{ab}_d / \mathcal{L}_d) \\
\partial & \downarrow & \sigma \rightarrow \pi \\
K_{d-1} (\mathcal{L}_{d-1}) & \xrightarrow{\gamma_{\mathcal{L}_{d-1}}} & \text{Gal}(\mathcal{L}^{ab}_{d-1} / \mathcal{L}_{d-1}) \\
\partial & \downarrow & \sigma \rightarrow \pi \\
\ldots & \rightarrow & \ldots \\
\partial & \downarrow & \sigma \rightarrow \pi \\
K_1 (\mathcal{L}_1) & \xrightarrow{\gamma_{\mathcal{L}_1}} & \text{Gal}(\mathcal{L}^{ab}_1 / \mathcal{L}_1) \\
\partial & \downarrow & \sigma \rightarrow \pi \\
\mathbb{Z} = K_0 (\mathcal{L}_0) & \xrightarrow{\gamma_{\mathcal{L}_0}} & \text{Gal}(\mathcal{L}^{ab}_0 / \mathcal{L}_0)
\end{array}
\]

By Remark 4.1.3 the composition of the vertical maps \( \partial \)'s we have

\[
\partial(\partial\{T_1, \ldots, T_{d-1}, a\}) = v_L(a),
\]

thus \( f : L^* \rightarrow \text{Gal}(L^{ab}/L) \rightarrow \text{Gal}(L^{un}/L) \) is the valuation map \( v_L : L^* \rightarrow \mathbb{Z} \). Thus condition (1) of [2] Chapter 5 §2.8 Proposition 6 is verified.

If \( a \in L^* \), \( L'/L \) is a finite abelian extension, \( \mathcal{L}' = L'\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \), \( \mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \), and \( a \) is a norm from \( L'^* \), namely \( a = N_{L'/L}(\alpha) \), then clearly
$\{T_1, \ldots, T_{d-1}, a\}$ is a norm from $K_d(L')$, namely

$$\{T_1, \ldots, T_{d-1}, a\} = \{T_1, \ldots, T_{d-1}, N_{L'/L}(\alpha)\} = N_{L'/\mathcal{L}}\{T_1, \ldots, T_{d-1}, \alpha\},$$

and by (1) of Theorem 4.1.1 we have that $\Upsilon_{L,\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}}(T_1, \ldots, T_{d-1}, a)$ is trivial on $\mathcal{L}'$ and so $f(a)$ is trivial on $L'$. Thus condition (2) of §2.8 Proposition 6 is verified. \qed

### 8.2 Explicit formulas for the Lubin-Tate formal groups

In this subsection we will provide a refinement of the formulas given for Theorem 7.3.1 to the case of a Lubin-Tate formal group $F_f$.

Let us introduce the following simplified notation throughout this subsection. Let $L$ be a finite extension of $K_n$ and put $\mathcal{L} = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Let $M = L_s$ and $\mathcal{M} = L_s\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Then $Q_s = Q_{\mathcal{M},s}$, $QL_s = QL_{\mathcal{M},s}$ and

$$P_s = P_{\mathcal{M}} = 1/(\pi_1D(M/K))\mathcal{O}_\mathcal{M} = \{x \in \mathcal{M} : v_\mathcal{M}(x) \geq -v_\mathcal{M}(\pi_1) - v_\mathcal{M}(D(M/K))\},$$

where $\pi_1$ is a uniformizer for $K_1 = K(\kappa_1)$. Let $N_s = N_{\mathcal{M}/\mathcal{L}}$ and $\text{Tr}_s = \text{Tr}_{\mathcal{M}/\mathcal{L}}$. Let $\gamma_s$ denote a uniformizer of $\mathcal{M}$. For $t \geq s$ and $\mathcal{N} = L_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$ we define $N_{t/s} = N_{\mathcal{N}/\mathcal{M}}$ and $\text{Tr}_{t/s} = \text{Tr}_{\mathcal{N}/\mathcal{M}}$. Finally, let $\mathcal{L}'_s$ be

$$\bigcap_{t \geq s} N_{t/s}(L_t\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}^*).$$
Theorem 8.2.1. Let $r$ be maximal such that $L \supset K_r$ and $r'$ minimal such that $L \cap K_\pi \subset K_{r'}$. Let $s \geq \max\{r', n + r + \log_q(e(L/K_r))\}$. Then $\text{Tr}_s$ takes $(\pi^s/\pi_1 \gamma_s)P_s$ to $\pi^n R_L$ so that it induces a homomorphism

$$\text{Tr}_s : \frac{1}{\pi_1 \gamma_s} P_s \to \mathcal{L}/\pi^n R_L$$

and the following formula holds

$$(\{N_{\mathcal{L}_s/\mathcal{L}}(a_1), a_2, \ldots, a_d\}, x)_{\mathcal{L}, n} = [\mathbb{T}_{\mathcal{L}/K} (\text{Tr}_s(QL_s(\{a_1, \ldots, a_d\}) I_{F'}(x)) ]_f (e_n)$$

(8.14)

for all $a_1 \in \mathcal{L}_s = \bigcap_{t \geq s} N_{t/s}(\mathcal{L}_t)$ and all $a_2, \ldots, a_d \in \mathcal{L}^*$.

Proof. Let $M = L_t$ and $M = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$. Let $v$ be the normalized valuation $v_{\mathcal{M}}/v_M(p)$ and let $R_{t,1} = R_{M,1}$, $P_t = P_M$. Since $K_1/K$ is totally ramified and $[K_1/K] = q - 1$ then $\pi_1^{q-1} \sim \gamma_t^{e(M)}$, so $\pi_1$ divides $\gamma_t^{e(M)/(p-1)}$ and we have

$$\frac{1}{\gamma_t} P_t = \frac{1}{\gamma_t \pi_1 D(L_t/K)} \mathcal{O}_{\mathcal{L}_t} \subset \frac{1}{\gamma_t^{e(M)/(p-1)+1} D(L_t/K) \mathcal{O}_{\mathcal{L}_t} = R_{t,1}}.$$

If $k < t$, then $\pi_1 | \pi^{t-k} \gamma_t$, which implies $(\pi^k/\gamma_t)(\pi^t/\pi_1)$ and so

$$\frac{\pi^t}{\pi_1 \gamma_t} P_t \subset \frac{\pi^k}{\gamma_t} R_{t,1}.$$

Consider $t, k$ and $m$ as in Theorem 7.3.1. In particular, since $t = 2k + \alpha + 1$ then $k < t$ and therefore we can look at the factorization of $Q_t : (\mathcal{O}_{\mathcal{M}})^d \to P_t/(\pi^t/\pi_1))P_t$.
into $R_{t,1}/(\pi^k/\gamma_t)R_{t,1}$. This is a $d$-dimensional derivation such that

$$Q_t(T_1,\ldots,T_{d-1},e_t) = T_1 \cdots T_{d-1}/\pi^t l'(e_t) = -T_1 \cdots T_{d-1}c_\beta/l'(e_t),$$

by (8.12), where $\beta = (k,t)$ and $c_\beta = c_{M,\beta}$. But from Lemma 7.2.1 we know that all the derivations satisfying such condition must coincide when reduced to $R_{t,1}/(\pi^m/\gamma_t)R_{t,1}$. Therefore the reduction of $Q_t$ to $R_{t,1}/(\pi^m/\gamma_t)R_{t,1}$ coincides with the derivation $D^i_{M,m}$ defined in (7.18) from Theorem 7.3.1. This allows us to replace the logarithmic derivative $\psi_{M,m}$ defined in (7.19) from Theorem 7.3.1 by the logarithmic derivative

$$QL_t: K_d(M) \to \frac{1}{\gamma_t} P_t \frac{\pi^t}{\pi_{1\gamma_t} P_t}$$

defined in (8.5), factored (mod $(\pi^m/\pi_{1\gamma_t})P_t$). That is, for $c \in \mathcal{L}_t^*$ and $a_2,\ldots,a_d \in \mathcal{L}^*$ we have

$$\text{Tr}_{M/L}(\psi_{M,m}^i(c,a_2,\ldots,a_d)) = \text{Tr}_t(QL_t(c,a_2,\ldots,a_d)) \quad (8.15)$$

But we know by Proposition 8.0.6 that

$$QL_s(N_{t/s}(c),a_2,\ldots,a_d) = \text{Tr}_{t/s}(QL_t(c,a_2,\ldots,a_d)) \pmod{\pi^s P_s}$$

Suppose for the moment that $\text{Tr}_s((\pi^s/\pi_{1\gamma_s})P_s) \subset \pi^n R_L$, then taking $\text{Tr}_s$ we get

$$\text{Tr}_t(QL_t(c,a_2,\ldots,a_d)) = \text{Tr}_s(\text{Tr}_{t/s}(QL_t(c,a_2,\ldots,a_d)))$$

$$= \text{Tr}_s(QL_s(N_{t/s}(c),a_2,\ldots,a_d)) \pmod{\pi^n R_L}$$
Therefore if \( a_1 \in \mathcal{L}'_s \) there exist a \( c \in \mathcal{L}^*_t \) such that \( N_{t/s}(c) = a_1 \) and thus, by (8.15) and Proposition 7.1.2 identity (8.17) follows. It remains to prove that \( \text{Tr}_s((\pi^s/\pi_1\gamma_s)P_s) \subset \pi^nR_L \) for \( s \geq n + r + \log_q(e(L/K_r)) \).

Let \( x \in F(\mu_L) \). Then \( f(x) \equiv x^q \pmod{\pi x} \) implies

\[
v(f(x)) \geq \min\{v(x^q), v(\pi x)\}.
\]

Let

\[
s' \geq \log_q \left( \frac{e(L/K)}{q-1} \right) + 1 = r + \log_q(e(L/K_r)).
\]

Then \( v(f^{(s'-1)}(x)) \geq \min\{v(x^{q^{s'-1}}), v(\pi x)\} \). But

\[
v(x^{q^{s'-1}}) = q^{s'-1} v(x) = \frac{e(L/K)}{q-1} \frac{v_L(x)}{e(L/K)\alpha} \geq \frac{1}{\alpha(q-1)} = \frac{v(\pi)}{q-1},
\]

\[
v(\pi x) > v(\pi),
\]

so \( v(f^{(s'-1)}(x)) \geq v(\pi)/(q-1) \). Thus by equation (8.16) applied to \( x = f^{(s'-1)}(x) \) we have

\[
v(f^{(s')}(x)) \geq \min \left\{ v \left( (f^{(s'-1)}(x))^q \right), v(\pi f^{(s'-1)}(x)) \right\} \geq (1 + \frac{1}{q-1}) v(\pi) > \frac{v(\pi)}{q-1}
\]

By Lemma 8.2.1 below, we have \( v(l_F(f^{(s')}(x))) = v(f^{(s')}(x)) \), and since \( v(\pi)/(q-1) = v(\pi_1) \), then

\[
v(\pi^s l_F(x)) \geq (1 + 1/(q-1)) v(\pi) = v(\pi) + v(\pi_1).
\]

This implies \( (\pi^{s-1}/\pi_1)T_L \subset \mathcal{O}_L \), where \( T_L = l_F(F(\mu_L)) \). Taking duals with respect
to $\mathbb{T}_{L/K}$ we get
\[
\frac{\pi'^{-1}}{\pi_1} \frac{1}{D(L/K)} \mathcal{O}_L \subset R_L.
\]
Since $\text{Tr}_s(P_s) \subset P_L$, by equation (8.6), then $\text{Tr}_s$ takes
\[
\frac{\pi^s}{\pi_1^{s'} \gamma_s} P_s \subset \frac{\pi^s}{\pi_1 P_s}
\]
to
\[
\frac{\pi^s}{\pi_1^{s'} \pi_L} P_L = \frac{\pi^s}{\pi_1^{s'} \pi L} \frac{1}{D(L/K)} \mathcal{O}_L \subset \frac{\pi^{s'-1}}{\pi_1^{s'} R_L}
\]
noticing now that $q \neq 2$, since $p \neq 2$, implies that $\pi_1 \pi_L \pi$, because $v_L(\pi/\pi_1 \pi L) = e(L/L_1)(q - 2) - 1$. Thus, if $s \geq n + s'$ then $\text{Tr}_s$ takes $(\pi^s/\pi_1^{s'}) P_s$ to $\pi^n R_L$ and we conclude the theorem.  

**Lemma 8.2.1.** For every element $w \in F(\mu_L)$ such that
\[
v(w) > v(\pi)/(q - 1),
\]
we have that
\[
v(l_F(w)) = v(w).
\]

**Proof.** The inequality $v(w) > v(\pi)/(q - 1)$ is equivalent to $v(w^q) > v(\pi) + v(w)$, and since $f(x) \equiv x^q \pmod{\pi x}$ this implies $v(f(w)) = v(\pi w)$. Thus $v(f(n)(w)) = v(\pi^n w)$ and therefore we can take $n$ large enough such that $v(f(n)(w)) > 1/(p - 1)$. By Proposition 3.3.1 we have that $v(l_F(f(n)(w))) = v(f(n)(w))$. Thus $v(\pi^n l_F(w)) = v(\pi^n l_F(w)) = v(w)$. 

\[
= v(\pi^n l_F(w)) = v(w).
\]
$v(\pi^n w)$, and this proves the lemma. \qed

We restate the theorem in a simplified form.

**Theorem 8.2.2.** Let $r$ be minimal such that $L \cap K_\pi \subset K_r$ and $\mathcal{L}$ denote the field $L\{\{T_1\}\cdots\{T_{d-1}\}\}$. Let $s \geq n + r + \log_q(e(L/K_r))$. Then

$$
\left(\{N_{\mathcal{L}_s/\mathcal{L}}(a_1), a_2, \ldots, a_d\}, x\right)_{\mathcal{L}, n} = \left[\mathbb{T}_{\mathcal{L}/K} \left(\text{Tr}_s(QL_s(\{a_1, \ldots, a_d\}) l_F(x))\right)\right]_f (e_n)
$$

(8.17)

for all $a_1 \in \mathcal{L}'_s = \cap_{t \geq s} N_{t/s}(\mathcal{L}_t)$ and all $a_2, \ldots, a_d \in \mathcal{L}^*.$
Bibliography


