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Cohomology of Certain Polyhedral Product Spaces

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Abstract

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The study of torus actions led to the discovery of moment-angle complexes and their generalization, polyhedral product spaces. Polyhedral products are constructed from a simplicial complex. This thesis focuses on computing the cohomology of polyhedral products given by two different classes of simplicial complexes: polyhedral joins (composed simplicial complexes [1]) and n-gons. A homological decomposition of a polyhedral product developed by Bahri, Bendersky, Cohen and Gitler in [5] is used to derive a formula for the case of polyhedral joins. Moreover, methods from [5] and results by Cai in [11] will be used to give a full description of the non-trivial cup products in a real moment-angle manifold over a n-gon in terms of the combinatorial generators.

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(Shout out to Guy and Papa!)

Contents

List of Figures

Chapter 1

Introduction

The objects studied in this dissertation are polyhedral product spaces, which are featured in toric topology. Toric topology is a relatively new field which is currently under active development. It is difficult to pinpoint its exact beginning, but many consider Davis and Januszkiewicz's 1991 paper on quasi-toric manifolds, a topological analogue of smooth toric varieties, and their relationship to moment-angle manifolds via torus actions to be a seminal work in the field [15]. Their approach reproduced results about the cohomology of toric varieties using topological tools instead of algebraic geometry. However, it was in [7–9] where the term moment-angle complex and its generalization were introduced, including its decomposition into a union of products of disks and circles. The study of toric topology has been particularly active and fruitful in the last 15 years, as many important applications and fundamental connections to other areas of mathematics have been found.

In particular, the results of toric topology have uncovered interdisciplinary links with many other fields of mathematics such as algebraic geometry, commutative algebra, and combinatorics. The moment-angle complex provides a formal link between combinatorics and torus actions. As an example, its equivariant cohomology ring is the Stanley-Reisner ring. It has also been found that complements of subspace arrangements deformation retract onto a moment-angle complex [9].

As the field continues to grow and the truly universal nature of moment-angle complexes has become apparent, its generalization, the polyhedral product has also garned much attention. For instance, Lopez de Medrano applies this construction to the study of intersections of quadrics [17]. Interesting in their own right, polyhedral products have been investigated from a homotopy theoretical point of view [2, 18, 19].

Since the field is rather new, there is not yet a concensus on terminology. Since 2000, the terms generalized moment-angle complex, polyhedral product space, K- product, and partial product space have all been used to mean a similar type of space: a topological space constructed from a family of topological spaces (X, A) using a simplicial complex K as an instruction manual. It is also not a surprise that there is no consensus on the notation that should be used; $Z(K; (\underline{X}, \underline{A}))$, $Z_K(X, A)$, and $(X, A)^K$ have all been used interchangeably. In this thesis, we will use the term polyhedral product space and the notation $Z_K(X, A)$.

A polyhedral product space is a subspace of a product of spaces. It is therefore unsurprising that they have existed long before the emergence of their name. For example, in 1967, Gerald J. Porter published the prominent paper “Higher Order Whitehead Products and Postnikov Systems”, in which a polyhedral product played a key role [25]. He constructs a space T, the subset of the product of spheres $\prod S^{n_i}$, consisting of those points with at least one coordinate at a basepoint. T may be interpreted as the polyhedral product space with the family of topological spaces given by spheres with basepoint and the simplicial complex given by the $(k-2)$-skeleton of a $(k-1)$-simplex. Ultimately, he was able to generalize the Hilton-Milnor Theorem, producing a powerful result that continues to be cited today. This example shows the underlying principle that many aspects of algebraic topology can be studied in terms of spaces that decompose as a union of simpler subspaces, indexed by a simplicial complex. Hence, toric topology and its wide-reaching results serve not only to solve problems within its own mathematical domain, but codify and generalize results that can be applied across the mathematical spectrum.

Toric topology has been able to contribute to significant advancements in engineering. For instance, Haynes, Cohen and Koditschek have used results from toric topology to answer concrete questions in robotics [23]. Using flows on polyhedral products, they were able to provide a practical language for the motion of robotic legs - a problem that had proven difficult for more established mathematical approaches. As a proof of concept, they have actually created a robot that walks on many terrains using their methods.

Two open questions will be addressed in this thesis. Both address the issue of finding the cohomology of certain polyhedral product spaces in terms of the cohomology of its constituent parts.

1.1 Real moment-angle complexes

The two motivating examples of polyhedral products in toric topology are the moment-angle complex and the real moment-angle complex. Let (X, A) be the pair (D^2, S^1), the disk and its boundary. The polyhedral product associated to (D^2, S^1) is called a *moment-angle complex*, and often denoted Z_K in the literature. When the pair is (D^1, S^0), the unit interval and its boundary, the polyhedral product is called a real moment-angle complex. Whereas the cohomology ring of the moment-angle complex is known to be $Tor_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K], \mathbb{Z})$, where $\mathbb{Z}[K]$ is the face ring (or Stanley-Reisner ring) of K (see chapter 2), the cohomology ring of the real moment-angle complex is not completely understood.

We know from the splitting theorem [2] and the wedge lemma [29] that the cohomology of real moment-angle complexes can be found by examining only the simplicial complex. Given any subset, I, of integers between 1 and n, denoted $[n]$, one can construct the corresponding *full subcomplex* of K in I, or the restriction of K to I. It is denoted K_I (or sometimes $K|_I$) and is defined by having all the simplices in K which are contained in I. The generators of the cohomology ring are given by the subsets of $[n]$ that yield a noncontractible full subcomplex of K after suspension. Specifically, $H^*(Z_K(D^1, S^0))$ is isomorphic to the direct sum of $H^*(\Sigma\mathcal{K}_I)$ for all subsets I of $[n]$, where $\mathcal{K}$ is the geometric realization of K.

We will consider the case when the simplicial complex K is the boundary of an n-gon. Although this restriction may seem strong, the genus of the associated polyhedral product grows exponentially as n grows. It follows from [15] that the resulting real moment-angle complex is a closed orientable surface of genus $g = 1+(n-4)2^{n-3}$. Of course the ring structure of a genus g surface is given in terms of the symplectic basis. The splitting mentioned in the previous paragraph provides a different set of generators. Fred Cohen posed the question to describe the ring structure in terms of the combinatorial generators. Therefore, calculating $H^1(Z_K(D^1, S^0))$ means finding the $2g$ subsets of $[n]$ whose associated full subcomplexes are homotopy equivalent to S^0. Similarly, the full subcomplex of K that results from the set $[n]$ corresponds to the second degree generator of $H^2(Z_K(D^1, S^0))$. Drawing on these results, we will study the problem:

> *When do two subsets of $[n]$ correspond to degree one generators that multiply to a non-zero class in the cohomology ring of a real moment-angle complex over a n-gon?*

Using a chain complex from [5] and results on cup products in real moment-angle complexes in [11], we will give a full description of the ring structure. Two examples will be discussed. First, all necessary full subcomplexes in the case of the pentagon will be given, as well as the more interesting part, the complete ring structure. Since the multiplication in this case is simple, we will also relate this graded ring, which is described in terms of the full subcomplexes of K, to the classical simplectic basis of an orientable surface of genus five. Additionally, it will be instructive to discuss the example of the hexagon, as the complexity of the computation grows exponentially. The hexagon gives an orientable surface of genus 17; an example that will be helpful in illustrating the approach to deriving the general statement for the n-gon.

Using this chain complex, one can determine when two degree one generators multiply to the degree two generator. Given two subsets of $[n]$, I and J, satisfying certain requirements there is a generator of the first cohomology group associated to I and one associated to J. A combinatorial condition will be given which will guarantee that the corresponding 1 dimensional generators have a non-trivial product.

1.2 Polyhedral Joins

The second class of polyhedral products of interest are the ones produced from a simplicial complex called a "polyhedral join", which are an analogue of polyhedral products for simplicial complexes. Bahri, Bendersky, Cohen and Gitler (BBCG) noticed that there is a way to make the simplicial complex larger and the topological pairs smaller so that the polyhedral product space stays the same. They called this process of making an arbitrary simplicial complex K larger the J-Construction, $K(J)$, which is an iteration of the simplicial wedge construction. Among many fascinating results, they showed that $Z_K(D^2, S^1) = Z_{K(J)}(D^1, S^0)$ [3]. This means that problems of moment-angle complexes can be reduced to problems of real moment-angle complexes. The J-construction has since been used by many studying toric topology such as [12, 16, 19, 25, 27, 28].

Ayzenberg generalized BBCG's J-construction even further, which he called polyhedral joins [1] and are denoted $Z^*_K(\underline{L}, \underline{K})$. The pairs $(\underline{L}, \underline{K}) := \{(L_i, K_i)\}_{i\in[m]}$ are defined so that K_i is a subsimplicial complex of L_i. Using this construction he proved theorems analogous to BBCGs and extended their results from [3]. It was not immediately evident that the simplicial wedge construction used by BBCG was a construction similar to that of the polyhedral product space. In fact, it is the same principle, but one replaces the product of spaces X_i or A_i with the join of simplicial complexes L_i or K_i. Using Ayzenberg's polyhedral join construction and tools from [5], we will address the following question:

> *What is the Hilbert-Poincaré series of the cohomology of the polyhedral product space over a polyhedral join, $Z_{Z^*_K(\underline{L},\underline{K})}(\underline{X},\underline{A})$, in terms of the simplicial complexes K, K_i, L_i and the cohomology of the spaces $(\underline{X},\underline{A})$?*

This generalizes results in [1] where Ayzenberg answers this question for $(X,A) = (D^2,S^1)$. The answer to this question depends on the map induced in cohomology by the inclusion of the link of a simplex in K_i restricted to a subset of the indexing set into the link of the same simplex in L_i restricted to the same subset. Hence, we will give a more specific answer to this question by restricting to a few cases of pairs. We will start by considering the situation when the pair is a simplex and any subsimplicial complex. Ayzenberg calls this polyhedral join a composed simplicial complex and denotes it $K(L_1,\ldots,L_m)$.

In his paper, Ayzenberg gives an equivalent formulation of $Z_{K(L_1,\ldots,L_m)}(\underline{X},\underline{A})$ involving another polyhedral product with simplicial complex K. Moreover, BBCG give a spectral sequence to find the cohomology of any polyhedral product space in terms of the cohomology of the pairs [5]. This spectral sequence leads to a Kunneth-like formula. Using this formula and Ayzenberg's reformulation of $Z_{K(L_1,\ldots,L_m)}(\underline{X},\underline{A})$, we can compute the desired cohomology groups. Unexpectedly, the Stanley-Reisner ring appears in the computation. The Stanley-

Reisner ring of a simplicial complex is studied in combinatorics. This result is indicative of how frequently connections between toric topology and other areas of math arise.

Ayzenberg's reinterpretation of the polyhedral product space over a composed simplicial complex can be generalized to polyhedral joins. Consequently, we will give formulas for the cohomology groups of polyhedral products over other cases of polyhedral joins.

Chapter 2

Polyhedral Product Spaces

Let $[m] = \{1, 2, \ldots, m\}$ denote the set of integers from 1 to m. An *abstract simplicial complex*, K, on $[m]$ is a subset of the power set of $[m]$, such that :

1. $\emptyset \in K$.

2. If $\sigma \in K$ with $\tau \subset \sigma$, then $\tau \in K$.

The sets in K are called (abstract) *simplices*. A n-simplex is the full power set of $[n+1]$ and is denoted Δ^n. The 0-simplex is called a *vertex*.

We do not assume m is minimal, i.e. there may exist $[n] \subsetneq [m]$ such that K is contained in the power set of $[n]$. In particular, we allow *ghost vertices* $\{i\} \subset [m]$ such that $\{i\} \notin K$.

Associated to an abstract simplicial complex is its *geometric realization*, denoted $\mathcal{K}$ or $|K|$ (also called a geometric simplicial complex). A (geometric) n-simplex, Δ^n, is the convex hull of $n+1$ points. For example, points, line segments, triangles, and tetrahedra are 0, 1, 2, and 3 -simplices, respectively. In particular,

$$\Delta^n = \{(t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i\in[n+1]} t_i = 1 \text{ and } t_i > 0\}$$

is the *standard n-simplex*. A (geometric) simplicial complex is a topological space obtained by appropriately gluing together simplices. The geometric realization of K is a (geometric) simplicial complex $|K|$ such that there is a bijection between the vertices with a one-to-one correspondence between simplices. See [10, 22] for further details.

Let K and L be simplicial complexes on sets $[m]$ and $[n]$, respectively. A map from $[m]$ to $[n]$ induces a *simplicial map* from K to L if it satisfies the property that simplices map to simplices.

Definition 2.0.1. *Let I be a subset of $[m]$. The* full subcomplex of K in I *is denoted K_I. It is a simplicial complex on the set I and defined*

$$K_I := \{\sigma \in K | \sigma \subset I\}$$

It will sometimes be helpful to use the notation $K|_I$ and it is often called the restriction of K to I in the literature.

Given an abstract simplicial complex K, let $\mathcal{S}_K$ be the category with simplices of K as the objects and inclusions as the morphisms. In particular, for $\sigma, \tau \in ob(\mathcal{S}_K)$, there is a morphism $\sigma \to \tau$ whenever $\sigma \subset \tau$. Let $\mathcal{CW}$ to be the usual category of CW-complexes. Define $(\underline{X}, \underline{A})$ to be a collection of pairs of CW-complexes $\{(X_i, A_i)\}_{i=1}^m$, where A_i is a subspace of X_i for all i.

Definition 2.0.2. *Given an abstract simplicial complex K on $[m]$, simplices σ, τ of K and a collection of pairs of CW-complexes $(\underline{X}, \underline{A})$, define a functor $D : \mathcal{S}_K \to \mathcal{CW}$, given by*

$$D(\sigma) = \prod_{i \in [m]} Y_i \qquad \text{where } Y_i = \begin{cases} X_i & i \in \sigma \\ \\ A_i & i \in [m] \backslash \sigma \end{cases}$$

For a morphism $f : \sigma \to \tau$, the functor D maps f to $i : D(\sigma) \to D(\tau)$ where i is the canonical injection.

A partial ordering of a set is reflexive, antisymmetric and transitive. A set that has a partial ordering is called a *poset*.

Associated to K is a poset $\bar{K}$. A point in $\bar{K}$ corresponds to a simplex in K, and points in $\bar{K}$ are ordered by inclusion. The polyhedral product $Z_K(\underline{X}, \underline{A})$ can be defined as the colimit of the spaces $D(\sigma)$ for σ a point in the poset, $\bar{K}$. In other words, $\sigma \leq \tau$ in $\bar{K}$ whenever $\sigma \subset \tau$ in K. Then the map $d_{\sigma\tau} : D(\sigma) \to D(\tau)$ is the natural inclusion and the colimit is defined

$$\bigsqcup_{\sigma \in \bar{K}} D(\sigma) \big/ \sim$$

where $x \in D(\sigma)$ we have that $x \sim d_{\sigma\tau}(x)$ for every $\sigma \leq \tau$

Definition 2.0.3. *The associated generalized moment-angle complex, or* polyhedral product space, *$Z_K(\underline{X}, \underline{A}) \subset \prod_{i \in [m]} X_i$ is*

$$Z_K(\underline{X}, \underline{A}) := \underset{\sigma \in K}{colim} D(\sigma) = \bigcup_{\sigma \in K} D(\sigma)$$

Notice that it suffices to take the colimit over the maximal simplices of K. In fact, simplicial complexes can be defined by their maximal simplices and this description will be used throughout.

If X_i has a basepoint and A_i is the basepoint of X_i for all i, we use the notation $Z_K(\underline{X})$. In the case where $(X_i, A_i) = (X, A)$ for all i, we write $Z_K(X, A)$.

The example that brought (real) moment-angle complexes into the spotlight are Davis and Januszkiewicz's *quasitoric manifolds* (and *small covers*), $2n$-dimensional manifolds M^{2n} with an action of a n-dimensional torus T^n which meet certain conditions [15]. In addition to showing connections to algebraic geometry and combinatorics, it was shown that any cobordism class can be represented by a quasi-toric manifold.

Quasi-toric manifolds are characterized by the property that the quotient space M^{2n}/T^n is a simple n-dimensional polytope P^n. A n-polytope is *simple* if exactly n facets meet at each vertex of P. The manifold M^{2n} is called a quasitoric manifold over P^n. Davis and Januszkiewicz simultaneously studied the real analogue, an n-dimensional manifold M^n with an action of the real torus Z_2^n such that $M^n/Z_2^n = P^n$, and called these small covers. They introduced the following construction for a quasitoric manifold from the polytope P^n.

Let $\mathscr{F}$ be the set of codimension-one faces (maximal simplices) of P^n and R^n be the free $\mathbb{Z}$-module of rank n. Since P^n is simple, any codimension-l face F is uniquely given by an intersection of l codimension-one faces $F_1, \ldots, F_l$. Take a map $\lambda : \mathscr{F} \to R^n$ that satisfies the condition that for any codimension-l face F of P^n, the vectors $\lambda(F_1), \ldots, \lambda(F_l)$ span a submodule of rank l which is a summand of R^n. In particular, this determines a rank-one subgroup of T^n. For any face F of P^n, this defines a subgroup $T_{\lambda(F)}$ of the n-torus T^n. Now, for each point $p \in P^n$ take the unique face containing p in its relative interior and denote it $F(p)$. Define an equivalence relation on the space $T^n \times P^n$ by $(g,p) \sim (h,q)$ if and only if $p = q$ and $g^{-1}h \in T_{\lambda(F)}$ where p is in the interior of $F = F(p)$. The manifold

$$T^n \times P^n \big/ \sim$$

is a quasitoric manifold M^{2n} over P^n. The map λ is called the characteristic function. This construction can be reversed so that a characteristic map is obtained from a quasitoric manifold.

They also give a general construction for what is now called the *moment-angle complex*. Let P^n have m codimension-one faces, and $\{e_1, e_2, \ldots, e_m\}$ be the standard basis for R^m. Define a map $\theta : \mathscr{F} \to R^m$ by $\theta(F_i) = e_i$ and a relation on the space $T^m \times P^n$ by $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1^{-1}t_2 \in T_{\theta(F)}$ where p is in the interior of F. The moment-angle complex, Z_P, is defined as

$$T^m \times P^n \big/ \sim$$

They show that every quasitoric manifold is a quotient of a moment-angle complex by the free action of the real torus T^{m-n}. Davis and Januszkiewicz also do this in the real case by replacing $\mathbb{Z}$ with $\mathbb{Z}_2$ and T^n with Z_2^n.

From the simple polytope P^n, one can define an associated simplicial complex K_P. The vertex set of K_P are the m codimension-one faces, $\overline{\mathscr{F}}$. A set of vertices in $\overline{\mathscr{F}}$ span a simplex in K_P if and only if the intersection of their associated faces is non-empty. The simplicial complex K_P is the dual of the boundary of P^n. The following theorem, due to Neil Strickland, describes a moment-angle complex as a polyhedral product.

Theorem 2.0.4 ([26]). *Given a simple polytope P, the moment-angle complex over P is*

$$Z_P = Z_K(D^2, S^1)$$

where K is the simplicial complex dual to the boundary of P.

This makes it especially easy to see that the m-torus T^m acts on the moment-angle complex $Z_K(D^2, S^1)$ since there is a T^m-action on $D(\sigma)$ induced by the natural S^1-action on the pair (D^2, S^1) and $Z_K(D^2, S^1)$ is the colimit obtained from the spaces $D(\sigma)$. Davis and Januszkiewicz introduced a space now called the Davis-Januszkiewicz space

$$\mathcal{DJ}(\mathcal{K}) := ET^m \times_{T^m} Z_K$$

the homotopy quotient (or borel construction) of the torus action on a moment-angle complex.

Definition 2.0.5. *Let K be a simplicial complex on m vertices and k be a ring. Consider $k[m] := k[v_1, \ldots, v_m]$, the ring of polynomials in m indeterminants. The* generalized Stanley-Reisner ideal *of K, $I(K)$, is generated by square-free monomials indexed by the non-simplices of K*

$$I(K) = < v_{i_1} \ldots v_{i_n} \mid \{i_1, \ldots, i_n\} \notin K >$$

The Stanley-Reisner (or face) ring of a simplicial complex, K, is denoted $k[K]$ and is defined *as*

$$k[K] = {}^{k[m]}\!\big/_{I(K)}$$

Strickland showed that the Davis-Januszkievicz space is homotopy equivalent to the polyhedral product of pairs complex projective space and a basepoint, $Z_K(CP^\infty)$ [26]. It was shown in [15] that the cohomology ring of $\mathcal{DJ}(\mathcal{K})$ is the Stanley-Reisner ring of K. In particular, let $x_1, \ldots x_m$ be generators of degree two.

$$H^*(\mathcal{DJ}(\mathcal{K})) = {}^{\mathbb{Z}[x_1, \ldots, x_m]}\!\big/_{I(K)}$$

Likewise, there is a real analogue

$$DJ^{\mathbb{R}}(K) := \mathbb{Z}_2^m \times_{\mathbb{Z}_2^m} Z_K(D^1, S^0)$$

A simplicial complex is called *flag* when n vertices span a simplex if and only if they are all pairwise adjacent. In the case of a flag complex K, $DJ^{\mathbb{R}}(K)$ is the classifying space of the free group on m generators with the relations that the square of any generator is the identity and two generators commute if their indices are an edge in K. This group is called the *right-angled Coxeter group* of K^1, the 1-skeleton of K. A *right-angled Artin group* is a free group on m generators with the relation that two generators commute if their indexes are an edge in K. The polyhedral product $Z_K(S^1)$ is the classifying space of the right-angled Artin group associated to K^1, when K is flag. See [13, 15, 24].

Recall Definitions 2.0.2 and 2.0.3. Below are some examples of polyhedral product spaces.

Example 2.0.6. *Let K be the boundary of a* 1*-simplex,* $\{\{1\}, \{2\}\}$*, and the two pairs be the* *closed interval and its two endpoints* (D^1, S^0). *Then*

$$\begin{aligned} Z_K(D^1, S^0) &= D^1 \times S^0 \cup D^1 \times S^0 \\ &= \partial(D^1 \times D^1) \\ &\cong S^1 \end{aligned}$$

This example will reappear in Section 3.3.

Example 2.0.7. *Now let* K *be the boundary of a* 2*-simplex:*

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$$

The maximal simplices are the edges. Therefore,

$$\begin{aligned} Z_K(D^1, S^0) &= D^1 \times D^1 \times S^0 \cup D^1 \times S^0 \times D^1 \cup S^0 \times D^1 \times D^1 \\ &= \partial(D^1 \times D^1 \times D^1) \\ &\cong S^2 \end{aligned}$$

Recall that $Z_K(X, A)$ *is a subspace of the solid cube. Each edge of the triangle yields a pair of opposite squares of the exterior of the cube. After taking the union over all simplices, we get that* $Z_K(X, A)$ *is the boundary of the cube.*

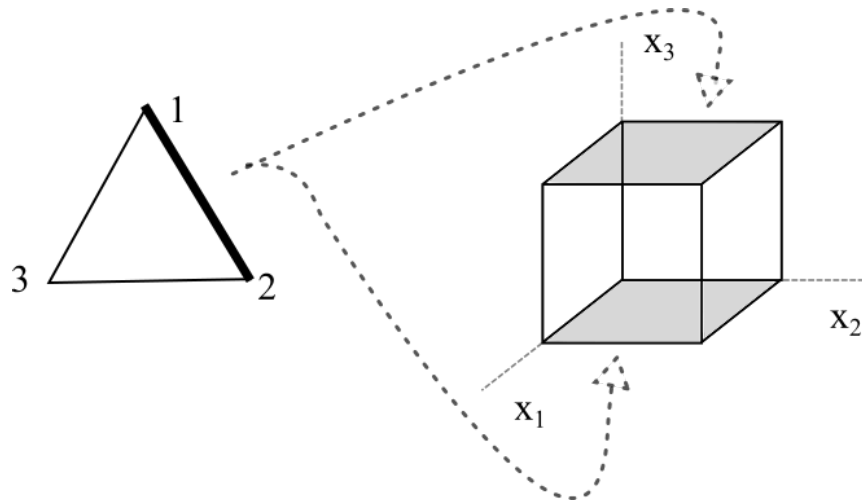

Figure 2.1: Example of a real moment-angle complex

In Figure 2.1, the simplex $\{1, 2\}$ *corresponds to* $D^1 \times D^1 \times S^0$*, which is the top and bottom squares in the real moment-angle complex.*

In general, $Z_{\partial\Delta^m}(D^1, S^0) \cong S^m$ *[10].*

Example 2.0.8. *Let* $K = \{\{v_1\}, \{v_2\}\}$*, so that* $Z_K(D^2, S^1) = D^2 \times S^1 \bigcup S^1 \times D^2$*. Notice that* $S^3 = \partial(D^4) = \partial(D^2 \times D^2) = Z_K(D^2, S^1)$*.*

Consider the pair (D^2, S^1) *as a subspace of* $\mathbb{C}$*, and the standard* S^1*-action on* (D^2, S^1)*.* $\Delta(T^2) = S^1$ *acts on* $Z_K(D^2, S^1) = S^3$*, where* $\Delta(T^2)$ *is the diagonal subgroup of* $S^1 \times S^1 = T^2$*. Given* $(x_1, x_2) \in Z_K(D^2, S^1)$*, if* (x_1, x_2) *is a fixed point, then* x_1 *is the fixed point of the action* $S^1 \curvearrowright D^2$ *and so is* x_2*. This would mean that* x_1 *and* x_2 *are the centers of their respective disks, but this would not correspond to a point in* $Z_K(D^2, S^1)$*. Therefore,* $\Delta(T^2)$ *acts freely. Then,* $\mathbb{CP}^1 \cong Z_K(D^2, S^1) / \Delta(T^2)$*. This is the well-known Hopf Fibration* $S^1 \hookrightarrow S^3 \rightarrow S^2$*.*

The polyhedral product interpolates between the wedge of pointed spaces X_i and the product of spaces X_i.

Definition 2.0.9. *Denote the n-skeleton of an m-simplex by Δ^m_n. It is all the simplices of Δ^m with $n+1$ or less vertices.*

The complex Δ^m_n leads to a filtration of K, which in turn gives a filtration of the associated polyhedral product.

Example 2.0.10. *Note Δ^{m-1}_0 is m disjoint points. Assume X_i has a basepoint, x_i.*

$$\bigvee_{i\in[m]} X_i = Z_{\Delta^{m-1}_0}(\underline{X}) \subset Z_{\Delta^{m-1}_1}(\underline{X}) \subset \ldots \subset Z_{\Delta^{m-1}}(\underline{X}) = \prod_{i\in[m]} X_i$$

Note that $\partial\Delta^{m-1} = \Delta^{m-1}_{m-2}$ and $Z_{\partial\Delta^{m-1}}(\underline{X})$ is called the fat wedge*, which is seen to be*

$$\{(y_1,\ldots,y_m) \in \prod X_i \mid \textit{at least one of the } y_i \textit{ is the basepoint } x_i\}$$

This filtration used was used in [25]. Porter obtained a decomposition of its loop spaces into a wedge when X_i is a suspension, and generalized the Hilton-Milnor theorem.

Definition 2.0.11. *Let K and L be simplicial complexes on the sets $[n]$ and $[m]$ respectively. The (simplicial) join of K and L, $K * L$, is a simplicial complex on the set $[n+m]$ defined by $K * L = \{\sigma \cup \tau \mid \sigma \in K,\ \tau \in L\}$.*

It is worth noting some other nice properties about polyhedral products [10]. They behave nicely with simplicial joins. For simplicial complexes K and L, it is not hard to see that

$$Z_{K*L}(X,A) = Z_K(\underline{X},\underline{A}) \times Z_L(\underline{X},\underline{A})$$

As a corollary, let the cone on K, CK, be the simplicial complex obtained by joining K with a vertex, $\{v\} * K$. Then $Z_{CK}(\underline{X},\underline{A}) = X \times Z_K(\underline{X},\underline{A})$. Additionally, polyhedral products have useful functorial properties:

- It follows from properties of colimits that cellular maps $f_i : (X_i, A_i) \to (Y_i, B_i)$ induce a cellular map $f' : Z_K(\underline{X}, \underline{A}) \to Z_K(\underline{Y}, \underline{B})$
  - if f_i are componentwise homotopic, then $Z_K(\underline{X}, \underline{A}) \simeq Z_K(Y, B)$
- If L and K are simplicial complexes on the same set (i.e. K may have ghost vertices) or if all the A_i are pointed, then the inclusion $f : K \hookrightarrow L$ induces an inclusion $f' : Z_K(\underline{X}, \underline{A}) \hookrightarrow Z_L(X, A)$
  - if L and K are defined on the same set, then $D(\sigma)$ maps to $D(\sigma)$
  - if A_i has basepoint x_i, K is defined on the set S_K and L is defined on the set S_L, then $D(\sigma)$ maps to itself with the addition that x_i is in the coordinates i such that $i \in S_L \backslash S_K$
- $Z_K(D^2, S^1)$ is a functor from the category of simplicial complexes and simplicial maps to the category of topological spaces with torus actions and equivariant maps

Next we will define the polyhedral smash product, a space analogous to the polyhedral smash product with the smash product operation in place of the cartesian product. Recall the definition of a smash product. Given two pointed spaces X and Y, the smash product of X and Y is the quotient of the cartesian product and the wedge sum

$$X \wedge Y := {}^{X \times Y}\!\big/_{X \vee Y}$$

For the smash product of arbitrarily many spaces,

$$X_1 \wedge X_2 \wedge \ldots \wedge X_m := {X_1 \times X_2 \times \ldots \times X_m}\big/{Z_{\partial\Delta^{m-1}}(\underline{X})}$$

As an example, the reduced suspension of a space X is defined to be

$$\Sigma X := X \wedge S^1$$

Let $\mathcal{CW}_*$ be the category of pointed CW-spaces.

Definition 2.0.12. *Let the CW-pairs $(\underline{X}, \underline{A})$ be pointed. Likewise, define a functor $\widehat{D}(\sigma) : \mathcal{S}_K \to \mathcal{CW}_*$ similarly to D by:*

$$\widehat{D}(\sigma) = \wedge Y_i \qquad \textit{where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma \end{cases}$$

Then the polyhedral smash product *is*

$$\widehat{Z}_K(\underline{X}, \underline{A}) = \bigcup \widehat{D}(\sigma)$$

The polyhedral smash product can also be defined as the image of $Z_K(\underline{X}, \underline{A})$ in $X_1 \wedge \ldots \wedge X_m$.

The *(topological) join* of topological spaces X and Y, $X * Y$ is defined as the product of X, Y, and the interval I, where at one end of the interval X is collapsed, and at the other end Y is collapsed.

$$X * Y := {X \times Y \times I}\big/{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

for all $x, x_1, x_2 \in X$ and all $y, y_1, y_2 \in Y$.

In [2], the authors apply the wedge lemma from [29] to polyhedral smash products and obtain the following:

Theorem 2.0.13 (Wedge Lemma, BBCG 2008)**.** *If X_i is contractible for all i, then*

$$\widehat{Z}_K(\underline{X}, \underline{A}) \simeq \Sigma|K| \wedge A^{\wedge[m]} \simeq |K| * A^{\wedge[m]}$$

where $A^{\wedge[m]} = A_1 \wedge \ldots \wedge A_m$

Recall definition 2.0.1 of the full subcomplex $K_I := \{\sigma \in K | \sigma \subset I\}$. The following theorem of BBCG gives a decomposition of a suspension of a polyhedral product space.

Theorem 2.0.14 (Splitting Theorem, BBCG 2008)**.** *Let* $(\underline{X_I}, \underline{A_I}) = \{(X_i, A_i)\}_{i\in I}$. *Then*

$$\Sigma Z_K(\underline{X}, \underline{A}) \simeq \Sigma \bigvee_{I\subset[m]} \widehat{Z}_{K_I}(\underline{X_I}, \underline{A_I}))$$

where Σ denotes the reduced suspension. The theorem is not true without the suspensions, as the following example illustrates.

Example 2.0.15. *Let K be a square with four vertices and four edges. Then $Z_K(D^2, S^1) = S^3 \times S^3$. This can be easily deduced from example 2.0.8 and the fact that K is the join $S^0 * S^0$. From the wedge lemma, we can also easily compute the polyhedral smash product and find that $\bigvee_{I\subset[m]} \widehat{Z}_{K_I}(\underline{D^2_I}, \underline{S^1_I}) \simeq S^3 \bigvee S^3 \bigvee S^6$.*

Grbić, Panov, Theriault and Wu have partial answers to the question of when is the splitting theorem true without suspensions [18, 19]. Grbić and Theriault focus on the case where the pairs (X_i, A_i) are the pairs (CX_i, X_i), the space and its cone. Then if K is shifted (if there is an specific way of ordering the vertices) or a simplicial wedge (see Section 3.1) of a shifted simplicial complex $K(v_i)$, then the splitting theorem can be desuspended. A simplicial complex is called *Golod* if the multiplication and all higher Massey products in $Tor_{k[m]}(k[K], k)$ are trivial. Grbić, Panov, Theriault and Wu show that if K is a flag complex (see the paragraph before Example 2.0.6), then the associated moment-angle complex is a wedge of spheres if and only if K is Golod. This question has also been studied in [20, 21]

Notation: Throughout this thesis, K is an abstract simplicial complex on $[m]$, $\mathcal{K}$ or $|K|$ denotes its geometric realization and $(\underline{X}, \underline{A})$ denotes a collection of m CW-pairs, unless otherwise noted.

Chapter 3

Polyhedral Joins

3.1 Background

Given any simplicial complex, the following procedure allows for the construction of an infinite family of associated simplicial complexes. Let $\mathcal{SC}$ be the category with simplicial complexes as the objects. Recall $\mathcal{S}_K$ is the category of simplices of K.

Definition 3.1.1 (Ayzenberg [1]). *Let K be a simplicial complex on m vertices and σ a simplex of K. Let $(\underline{L}, \underline{K}) = \{L_i, K_i\}_{i\in[m]}$ be m pairs of simplicial complexes, where K_i is a subsimplicial complex of L_i and both are defined on the index set $[l_i]$. Consider a functor $D^* : \mathcal{S}_K \to \mathcal{SC}$ defined in the following way*

$$D^*(\sigma) = \mathop{*}_{i\in[m]} Y_i \ , \ \textit{where } Y_i = \begin{cases} L_i & i \in \sigma \\ K_i & i \notin \sigma \end{cases}$$

The associated polyhedral join *is the colimit of the diagram*

$$Z^*_K(\underline{L}, \underline{K}) := \underset{\sigma\in K}{colim} D^*(\sigma) = \bigcup_{\sigma\in K} D(\sigma)$$

Note that $Z^*_K(\underline{L}, \underline{K})$ is a subsimplicial complex of $\underset{i\in[m]}{*} L_i$, which is a simplicial complex on the set $[\sum_{i\in[m]} l_i]$. In particular, $D(\sigma)$ is the join of simplicial complexes L_i for $i \in \sigma$ and simplicial complexes K_j for $j \in [m]\backslash\sigma$.

Definition 3.1.2. *Let K be a simplicial complex on the set $[m]$ and $\{L_i\}_{i\in[m]}$ be simplicial complexes on the sets $[l_i]$.*

The composition of K with L_i, *denoted $K(L_1, \dots, L_m)$, is defined to be*

$$K(L_1, \dots, L_m) := Z^*_K(\Delta^{l_i-1}, L_i)$$

The composition $K(L_1, \dots, L_m)$ may also be defined by the following condition: for subsets $\sigma_i \subset [l_i]$, the set $\sigma = \sigma_1 \sqcup \dots \sqcup \sigma_m$ is a simplex of $K(L_1, \dots, L_m)$ whenever the set $\{i \in [m] \mid \sigma_i \notin L_i\}$ is a simplex of K.

The composition is a generalization of the J-construction and the simplicial wedge construction [3], which we now describe.

The *link* of $\sigma \in K$, denoted $lk_K(\sigma)$, is a simplicial complex on the set $[m]\backslash\sigma$ defined by $\tau \in lk_K(\sigma)$ if and only if $\sigma \cup \tau \in K$. This indexing set is used to be consistent with the definition in [1]. Given the indexing set of this complex, the link may have ghost vertices that are not ghost vertices of K.

Definition 3.1.3. *Given a vertex $v \in K$, the* simplicial wedge construction, *$K(v)$ is a* *simplicial complex on the vertex set* $([m]\backslash v) \cup \{v_1, v_2\}$ *given by*

$$\{\{v_1, v_2\}\} * lk_K(v) \cup \{\{v_1\}, \{v_2\}\} * K\backslash v$$

Example 3.1.4. *This is an example of how the simplicial wedge construction works. It shows the doubling of a vertex in a the boundary of a triangle.*

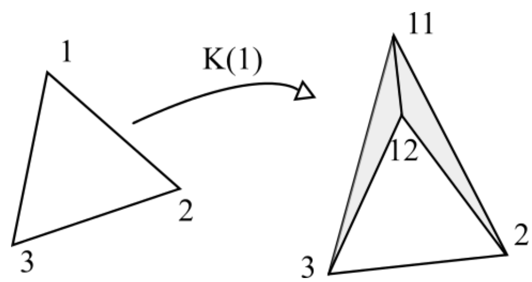

Figure 3.1: The simplicial wedge construction

A set of vertices is a *minimal non-face* of K if it is not a simplex of K and every proper subset is a simplex of K. The set of minimal non-faces generates the Stanley-Reisner ideal (see Definition 2.0.5). Moreover, a simplicial complex may be defined by its minimal non-faces.

Definition 3.1.5. *Let* K *be a simplicial complex on* $v_1, \ldots, v_m$. *For a simplicial complex* K *and a sequence of natural numbers* $J = \{j_1, \ldots, j_m\}$, *a simplicial complex* $K(J)$ *is now constructed. The complex* $K(J)$ *is a simplicial complex on the vertices*

$$(v_{11}, v_{12}, \ldots, v_{1j_1}, v_{21}, v_{22}, \ldots, v_{2j_2}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mj_m})$$

We define the simplicial complex $K(J)$ *by defining the minimal non-faces of* $K(J)$. *A subset* *of the vertex set is a minimal face of $K(J)$ if and only if it is of the form*

$$(v_{i_1 1}, v_{i_1 2}, \ldots, v_{i_1 j_{i_1}}, v_{i_2 1}, v_{i_2 2}, \ldots, v_{i_2 j_{i_2}}, v_{i_k 1}, v_{i_k 2}, \ldots, v_{i_k j_{i_k}})$$

for a minimal face $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ of K.

The simplicial wedge construction is a special case of the J-construction. For $i \in [m]$, $K(v_i) = K(J)$ for $J = \{1, \ldots, 1, 2, 1, \ldots, 1\}$ where the 2 is in the i-th spot.

The simplicial wedge construction and J-constructions are examples of a composition of simplicial complexes L_i. When L_i is the boundary of a 1-simplex and all other simplicial complexes are a ghost vertex, the composition is a simplicial wedge construction, $K(v_i)$ [3]. Let $J = \{j_1, \ldots, j_m\}$, then the J-construction is a composition [1]:

$$K(J) = K(\Delta^{j_1-1}, \Delta^{j_2-1}, \ldots, \Delta^{j_m-1})$$

Example 3.1.6. *After applying the simplicial wedge construction to a boundary of a simplex, the resulting complex remains a boundary of a simplex. For example, if $K = \partial\Delta^1 = \{\{v_1\}, \{v_2\}\}$, then*

$$\begin{aligned} K(v_1) &= \{v_{11}, v_{12}\} * \emptyset \cup \{\{v_{11}\}, \{v_{12}\}\} * v_2 \\ &= \{\{v_{11}, v_{12}\}, \{v_{11}, v_2\}, \{v_{12}, v_2\}\} \\ &= \partial\Delta^2 \end{aligned}$$

Example 3.1.7. *This is an example of a composition of simplicial complexes $K(L_1, \ldots, L_m)$. Let $m = 3$ and $K = \{\{1\}, \{2, 3\}\}$.*

$l_1 = 1$ *and* $L_1 = \{\emptyset\}$

$l_2 = 1$ *and* $L_2 = \{\{21\}\}$

$l_3 = 2$ *and* $L_3 = \{\{31\}, \{32\}\}$

Then

$$\begin{array}{rcccc} K(L_1, L_2, L_3) & = & D^*(\{2,3\}) & \cup & D^*(\{1\}) \\ & = & L_1 * \Delta^0 * \Delta^1 & \cup & \Delta^0 * L_2 * L_3 \\ & = & \{\emptyset\} * \{21\} * \{31, 32\} & \cup & \{11\} * \{21\} * \{31\}, \{32\} \\ & = & \{21, 31, 32\}, & & \{11, 21, 31\}, \{11, 21, 32\} \end{array}$$

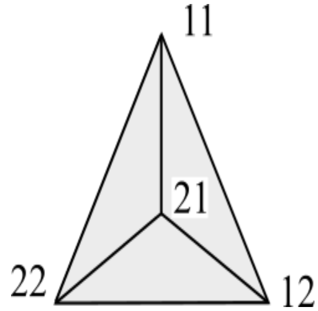

Figure 3.2: Example of a composed simplicial complex

Composed simplicial complexes have a nice relationship with polyhedral products. Note that in the following proposition the indexing set of $(\underline{X}, \underline{A})$ is different than it has been up to this point, and so it will be explicitly labeled. The notation used will be $Z_K(X_i, A_i)_{i \in I}$ for some indexing set I.

Proposition 3.1.8 (Ayzenberg [1, Proposition 5.1])**.** *Let K be a simplicial complex on m vertices and $\{L_i\}_{i \in [m]}$ be simplicial complexes with l_i vertices. We have $\sum\limits_{i \in [m]} l_i$ pairs (X_{ij}, A_{ij}) with $i \in [m]$ and $j \in l_i$. Then*

$$Z_{K(L_1, \ldots, L_m)}(X_{ij}, A_{ij})_{i \in [m], j \in [l_i]} = Z_K(\prod_{j \in [l_i]} X_{ij}, Z_{L_i}(X_{ij}, A_{ij})_{j \in [l_i]})_{i \in [m]}$$

These spaces are equal, not just homeomorphic. The proof inolves a shuffling of the spaces X_i and A_i as in the proof of proposition 3.1.10. In particular, we can see how a moment-angle complex can be expressed as a real moment-angle complex.

Corollary 3.1.9 (Ayzenberg [3]). *Real moment-angle complexes over a composed simplicial complex are homeomorphic to moment-angle complexes:*

$$Z_{K(\partial\Delta_1,\ldots,\partial\Delta_1)}(D^1, S^0) = Z_K(D^2, S^1)$$

This corollary is a special case of work in [3], where it was shown that

$$Z_{K(J)}(D^1, S^0) = Z_K(D^{2|J|}, S^{2|J|-1})$$

Using methods of [1] we prove the analogous result for the polyhedral smash product.

Proposition 3.1.10. *Let K be a simplicial complex on m vertices and $\{L_i\}_{i\in[m]}$ be simplicial complexes with l_i vertices. Then*

$$\widehat{Z}_{K(L_1,\ldots,L_m)}(X_{ij}, A_{ij})_{i\in[m],j\in[l_i]} = \widehat{Z}_K(\bigwedge_{j\in[l_i]} X_{ij}, \widehat{Z}_{L_i}(X_{ij}, A_{ij})_{j\in[l_i]})_{i\in[m]}$$

Proof. Suppose we have a simplices $\sigma \in K$ and $\tau_i \in L_i$. Then

$$\begin{aligned}
&\widehat{Z}_K(\wedge_{j\in[l_i]} X_{ij}, \widehat{Z}_{L_i}(X_{ij}, A_{ij})_{j\in[l_i]})_{i\in[m]} \\
&\quad= \bigcup_{\sigma}\big(\bigwedge_{i\in\sigma}(\bigwedge_{j\in[l_i]} X_{ij}) \wedge \bigwedge_{i\in[m]\setminus\sigma}(\bigcup_{\tau_i}(\bigwedge_{j\in\tau_i} X_{ij}) \wedge (\bigwedge_{j\in[l_i]\setminus\tau_i} A_{ij}))\big) \\
&\quad= \bigcup_{\sigma,\tau_i}\big(\bigwedge_{i\in\sigma}(\bigwedge_{j\in[l_i]} X_{ij}) \wedge \bigwedge_{i\in[m]\setminus\sigma}((\bigwedge_{j\in\tau_i} X_{ij}) \wedge (\bigwedge_{j\in[l_i]\setminus\tau_i} A_{ij}))\big)
\end{aligned}$$

Since simplices in $\sigma' \in K(L_1,\ldots,L_m)$ are of the form $\sigma' = (\bigcup_{i\in[m]\setminus\sigma} \tau_i) \cup (\bigcup_{i\in\sigma} \Delta^{l_i-1})$,

$$\bigwedge_{i\in\sigma}(\bigwedge_{j\in[l_i]} X_{ij}) \wedge \bigwedge_{i\in[m]\setminus\sigma}(\bigwedge_{j\in\tau_i} X_{ij}) = \bigwedge_{ij\in\sigma'} X_{ij}$$

Note that $\bigcup\limits_{i\in[m]\backslash\sigma}[l_i]\backslash\tau_i = [\Sigma l_i]\backslash\sigma'$, and thus

$$\bigwedge_{i\in[m]\backslash\sigma}\Big(\bigwedge_{j\in[l_i]\backslash\tau_i}A_{ij}\Big) = \bigwedge_{ij\in[\Sigma l_i]\backslash\sigma'}A_{ij}$$

And finally,

$$\bigcup_{\sigma'}\Big(\bigwedge_{ij\in\sigma'}X_{ij}\wedge\bigwedge_{ij\in[\Sigma l_i]\backslash\sigma'}A_{ij}\Big) = \widehat{Z}_{K(L_1,\ldots,L_m)}(X_{ij},A_{ij})_{i\in[m],j\in[l_i]}$$

□

Similarly, we can make the same type of argument for the polyhedral join in place of the composed simplicial complex.

Theorem 3.1.11. *Given m pairs of simplicial complexes $(\underline{L},\underline{K})$ where L_i and K_i are simplicial complexes on the vertex set $[l_i]$ (L_i may have ghost vertices). Taking $\sum\limits_{i\in[m]} l_i$ pairs (X_{ij},A_{ij}), we have*

$$Z_{Z^*_K(\underline{L},\underline{K})}(X_{ij},A_{ij})_{i\in[m],j\in[l_i]} = Z_K(Z_{L_i}(X_{ij},A_{ij})_{j\in[l_i]},Z_{K_i}(X_{ij},A_{ij})_{j\in[l_i]})_{i\in[m]}$$

The complexes K_i must have an indexing set of the same cardinality of the indexing set of L_i; otherwise, the statement is not true. In particular, L_i may have ghost vertices. Keep in mind that including ghost vertices does change the polyhedral product by multiplying by A_i where i is a ghost vertex. To illustrate this, consider the polyhedral products $Z_{\{pt\}}(D^2,S^1) = D^2$ and $Z_{\{pt,\text{ ghost vertex}\}}(D^2,S^1) = D^2\times S^1$.

The following example illustrates why Theorem 3.1.11 requires the indexing sets to have the same cardinality for each i.

Example 3.1.12. *Suppose we have the following simplicial complexes: K is two points,*

L_1 is two points, K_1 is one point, L_2 is one point, and K_2 is a ghost vertex. Labeling the vertices, we have $K = \{\{1\},\{2\}\}$, and

$$\begin{aligned}(L_1, K_1) = &\quad (\{\{11\},\{12\}\},\{\{11\}\})\\ (L_2, K_2) = &\qquad (\{\{21\}\},\{\emptyset\})\end{aligned}$$

The subscripts in these calculations will indicate the coordinate. First we will calculate $Z_K(Z_{L_i}(D^1,S^0), Z_{K_i}(D^1,S^0))$.

$Z_{L_1}(D^1,S^0) = D^1_{11} \times S^0_{12} \cup S^0_{11} \times D^1_{12}$

$Z_{K_1}(D^1,S^0) = D^1_{11}$

$Z_{L_2}(D^1,S^0) = D^1_{21}$

$Z_{K_2}(D^1,S^0) = S^0_{21}$

Therefore, $Z_K(Z_{L_i}(D^1,S^0), Z_{K_i}(D^1,S^0)) = \big((D^1_{11} \times S^0_{12} \cup S^0_{11} \times D^1_{12}) \times S^0_{21}\big) \cup (D^1_{11} \times D^1_{21})$,

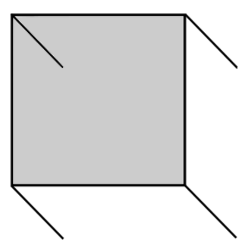

Figure 3.3: $Z_K(Z_{L_i}(D^1,S^0), Z_{K_i}(D^1,S^0))$

whereas $Z^*_K(\underline{L},\underline{K}) = L_1 * K_2 \cup K_1 * L_2 = \{\{12\},\{11,21\}\}$ *and hence* $Z_{Z^*_K(L_i,K_i)}(D^1,S^0) = D^1_{11} \times S^0_{12} \times D^1_{21} \cup S^0_1 1 \times D^1_{12} \times S^0_{21}$.

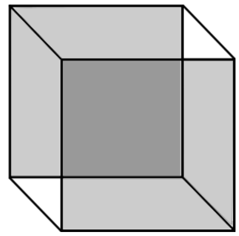

Figure 3.4: $Z_{Z^*_K(L_i,K_i)}(D^1, S^0)$

Clearly, we do not have equality in this example, i.e. $Z_K(Z_{L_i}(D^1, S^0), Z_{K_i}(D^1, S^0)) \neq Z_{Z^*_K(L_i,K_i)}(D^1, S^0)$. *They are not even homotopy equivalent. However, if we change* K_1 *to have one point and one ghost vertex, then* $Z_{K_1}(D^1, S^0) = D^1_{11} \times S^0_{12}$ *and*

$$\begin{aligned} Z_K(Z_{L_i}(D^1, S^0), Z_{K_i}(D^1, S^0)) &= \big((D^1_{11} \times S^0_{12} \cup S^0_{11} \times D^1_{12}) \times S^0_{21}\big) \cup (D^1_{11} \times S^0_{12} \times D^1_{21}) \\ &= Z_{Z^*_K(L_i,K_i)}(D^1, S^0) \end{aligned}$$

3.1.1 The BBCG spectral sequence

Recall that our goal is to compute the cohomology of $Z_{Z^*_K(\underline{L},\underline{K})}(\underline{X}, \underline{A})$ in terms of the K, K_i, L_i, $H^*(X_{ij})$ and $H^*(A_{ij})$. To do so, we will use a spectral sequence developed by BBCG [5]. It gives a Kunneth-like formula for the cohomology of a polyhedral product as long as the pairs $(\underline{X}, \underline{A})$ satisfy the following freeness condition.

Definition 3.1.13. *Given the pair* (X_i, A_i)*, the associated long exact sequence is given by*

$$\ldots \xrightarrow{\delta} \widetilde{H}^*(X_i/A_i) \xrightarrow{g} H^*(X_i) \xrightarrow{f} H^*(A_i) \xrightarrow{\delta} \widetilde{H}^{*+1}(X_i/A_i) \xrightarrow{g} \ldots$$

Assume that the cohomology groups of the pair have the following decomposition

$$\begin{aligned} H^*(A_i) &= B_i \oplus E_i \\ H^*(X_i) &= B_i \oplus C_i \\ \widetilde{H}^*(X_i/A_i) &= C_i \oplus W_i \end{aligned}$$

where W_i *is* sE_i*, the suspension of* E_i*. Additionally, assume* $1 \in B_i$*, and for* $b \in B_i, c \in C_i, e \in E_i, w \in W_i = sE_i$*, we have*

$$b \overset{f}{\mapsto} b \overset{\delta}{\mapsto} 0, \qquad c \overset{g}{\mapsto} c \overset{f}{\mapsto} 0, \qquad e \overset{\delta}{\mapsto} w \overset{g}{\mapsto} 0$$

Before defining the spectral sequence, we will give some notation and recall the definition of a half smash product:

1. for $\sigma = \{i_1, \ldots, i_k\}$, define $\widehat{X}^\sigma := X_{i_1} \wedge \ldots \wedge X_{i_k}$

2. the complement of a set $\sigma \subset [m]$ is $\sigma^c = [m] \backslash \sigma$ or $\sigma^c = [m] - \sigma$

3. for any set $\sigma = \{i_1, \ldots, i_n\}$, we use the notation $A^\sigma = A_{i_1} \times \ldots \times A_{i_n}$

4. for $x_0 \in X$, the right half smash product $X \rtimes Y = (X \times Y)/(x_0 \times Y) = X \wedge (Y_+)$ where Y_+ is the space Y with a disjoint base-point

5. given a subset I and a simplex σ such that $\sigma \subset I$, define $Y^{I,\sigma} := \bigotimes_{i \in \sigma} C_i \otimes \bigotimes_{i \in I - \sigma} B_i$

Choosing a lexicographical ordering for the simplices of K gives a filtration of the associated polyhedral product space and polyhedral smash product, which in turn leads to a spectral sequence converging to the reduced cohomology of $Z_K(\underline{X}, \underline{A})$ and a spectral sequence converging to the reduced cohomology of $\widehat{Z}_K(\underline{X},\underline{A})$. The $E_1^{s,t}$ term for $Z_K(\underline{X},\underline{A})$ has the following description.

Theorem 3.1.14 (Bahri, Bendersky, Cohen and Gitler [5]). *There exist spectral sequences*

$$E_r^{s,t} \to H^*(Z_K(\underline{X},\underline{A}))$$

$$\widetilde{E}_r^{s,t} \to H^*(\widetilde{Z}_K(X,A))$$

with $E_1^{s,t} = \widetilde{H}^t((\widehat{X/A})^\sigma \rtimes A^{\sigma^c})$ *and* $\widetilde{E}_1^{s,t} = \widehat{Z}_K(\underline{X},\underline{A}) = \widetilde{H}^t((\widehat{X/A})^\sigma \wedge \widehat{A}^{\sigma^c})$ *where* s *is index of* σ *in the lexicographical ordering and the differential* $d_r : E_r^{s,t} \to E_r^{s+r,t+1}$ *is induced by the coboundary map* $\delta : E \to W = sE$. *Moreover, the spectral sequence is natural for embeddings of simplicial maps with the same number of vertices and with respect to maps of pairs. The natural quotient map*

$$Z_K(\underline{X},\underline{A}) \to \widehat{Z}_K(\underline{X},\underline{A})$$

induces a morphism of spectral sequences and the stable splitting of Theorem 2.0.14 induces a morphism of spectral sequences.

Following [5], Definition 3.1.13 and the Künneth theorem imply that the E_1 terms of the spectral sequence for $\widehat{Z}_K(\underline{X},\underline{A})$ decompose as a direct sum of spaces $W^N \otimes C^S \otimes B^T \otimes E^J$ such that $N \cup S = \sigma$, $T \cup J = \sigma^c$ and N, S, J, T are disjoint. We have that S is a simplex in K as $N \cup S$ is a simplex in K. Since the differential is induced by the coboundary $\delta : E \to W$, consider all the possible summands $W^N \otimes C^S \otimes B^T \otimes E^J$ for S and T are fixed. It must be the case that N is a simplex in K and that N is a subset of $[m]\backslash(S \cup T)$. Therefore all such N correspond to simplices in the link of S in K restricted to the vertex set $[m]\backslash(S \cup T)$.

Theorem 3.1.15 (Bahri, Bendersky, Cohen and Gitler [5]). *Let $(\underline{X},\underline{A})$ satisfy the decomposition described in Definition 3.1.13*

$$\begin{aligned} H^*(A_i) &= B_i \oplus E_i \\ H^*(X_i) &= B_i \oplus C_i \end{aligned}$$

Then

$$H^*(Z_K(\underline{X},\underline{A})) = \bigoplus_{I\subset[m],\sigma\subset I} E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|)$$

where:

1. σ *is a simplex in* K,
2. $lk(\sigma)_{I^c} = \{\tau \subset [m]\backslash I \mid \tau \cup \sigma \in K\}$ *is the link of* σ *in* K *restricted to the set* $[m]\backslash I$,
3. $Y^{I,\sigma} = \bigotimes_{i\in\sigma} C_i \otimes \bigotimes_{i\in I-\sigma} B_i$, *and*
4. $\widetilde{H}^*(\Sigma\emptyset) = 1$.

Theorem 3.1.16 (Bahri, Bendersky, Cohen and Gitler [5]). *Let*

$$\begin{aligned} \widetilde{H}^*(A_i) &= \widetilde{B}_i \oplus E_i \\ \widetilde{H}^*(X_i) &= \widetilde{B}_i \oplus C_i \end{aligned}$$

Then

$$H^*(\widehat{Z}_K(\underline{X},\underline{A})) = \bigoplus_{I\subset[m],\sigma\subset I} E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|)$$

where:

1. σ *is a simplex in* K,

2. $lk(\sigma)_{I^c} = \{\tau \subset [m]\backslash I \mid \tau \cup \sigma \in K\}$ *is the link of* σ *in* K *restricted to the set* $[m]\backslash I$,

3. $Y^{I,\sigma} = \bigotimes_{i\in\sigma} C_i \otimes \bigotimes_{i\in I-\sigma} \widetilde{B}_i$ *where* $\widetilde{B}_i = B_i\backslash\{1\}$, *and*

4. $\widetilde{H}^*(\Sigma\emptyset) = 1$.

Consequently, assuming that the cohomology of the pairs (X_{ij}, A_{ij}) satisfy the freeness condition, then once we find the E's, B's and C's of the pair $(Z_{L_i}(\underline{X}, \underline{A}), Z_{K_i}(\underline{X}, \underline{A}))$, we can use this theorem.

3.2 General case

In this section we aim to understand the cohomology of the space $Z_{Z^*_K(\underline{L},\underline{K})}(\underline{X}, \underline{A})$. To do so we will use Theorem 3.5.1, and therefore we will be applying Theorem 3.1.15 to the space $Z_K(Z_{L_i}(X_{ij}, A_{ij})_{j\in[l_i]}, Z_{K_i}(X_{ij}, A_{ij})_{j\in[l_i]})_{i\in[m]}$. Fix an i. The inclusion $K_i \hookrightarrow L_i$ induces the obvious inclusion $Z_{K_i}(\underline{X}, \underline{A}) \hookrightarrow Z_{L_i}(\underline{X}, \underline{A})$, which in turn induces a map in cohomology

$$H^*(Z_{L_i}(\underline{X}, \underline{A})) \xrightarrow{\phi} H^*(Z_{K_i}(\underline{X}, \underline{A}))$$

By theorem 3.1.15, we have that

$$\begin{aligned} H^*(Z_{L_i}(\underline{X}, \underline{A})) &= \bigoplus_{\sigma\in L_i, \sigma\subset I\subset[l_i]} E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk_{L_i}(\sigma)_{I^c}|) \\ &= \Big(\bigoplus_{\tau\in K_i, \tau\subset I\subset[l_i]} E^{I^c} \otimes Y^{I,\tau} \otimes \widetilde{H}^*(\Sigma|lk_{L_i}(\tau)_{I^c}|)\Big) \\ &\qquad\qquad \oplus \\ &\quad \Big(\bigoplus_{\sigma\notin K_i, \sigma\subset I\subset[l_i]} E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk_{L_i}(\sigma)_{I^c}|)\Big) \end{aligned}$$

and

$$H^*(Z_{K_i}(\underline{X},\underline{A})) = \bigoplus_{\tau\in K_i,\tau\subset I\subset[l_i]} E^{I^c}\otimes Y^{I,\tau}\otimes \widetilde{H}^*(\Sigma|lk_{K_i}(\tau)_{I^c}|)$$

where $lk_{L_i}(\sigma)_{I^c}$ is the link of σ in L_i restricted to the vertex set $[l_i]\backslash I$. Therefore, understanding the map ϕ is reduced to finding the image of the factor $\alpha\in\widetilde{H}^*(\Sigma|lk_{L_i}(\tau)_{I^c}|)$ for $\tau\in K_i$. Recall (from the discussion before theorem 3.1.15) that α corresponds to the exponent of the Ws in a summand $E^J\otimes W^N\otimes C^\tau\otimes B^{I\backslash\tau}$ of

$$\widetilde{H}^*(\widehat{X/A}^{(N\cup\tau)})\otimes\widetilde{H}^*(\widehat{A}^{(N\cup\tau)^c})$$

in the E_1 page of the BBCG spectral sequence. If $N\cup\tau$ is a simplex in K_i, then $\widetilde{H}^*(\widehat{X/A}^{(N\cup\tau)})\otimes\widetilde{H}^*(\widehat{A}^{(N\cup\tau)^c})$ in the spectral sequence of K_i maps to $\widetilde{H}^*(\widehat{X/A}^{(N\cup\tau)})\otimes\widetilde{H}^*(\widehat{A}^{(N\cup\tau)^c})$ in the spectral sequence of $Z_{L_i}(\underline{X},\underline{A})$ by the naturality of the spectral sequence for embeddings of simplicial maps. In particular, if $N\cup\tau\in K_i$, then

$$\phi: E^J\otimes W^N\otimes C^\tau\otimes B^{I\backslash\tau}\mapsto E^J\otimes W^N\otimes C^\tau\otimes B^{I\backslash\tau}$$

and if $N\cup\tau\notin K_i$, then

$$\phi: E^J\otimes W^N\otimes C^\tau\otimes B^{I\backslash\tau}\mapsto 0$$

This is the dual of the inclusion

$$lk_{K_i}(\sigma)_{I^c}\hookrightarrow lk_{L_i}(\sigma)_{I^c}$$

In other words, the map ϕ is induced by the inclusion $lk_{K_i}(\sigma)_{I^c}\hookrightarrow lk_{L_i}(\sigma)_{I^c}$

3.3 Composed simplicial complexes

Recall that a composed simplicial complex is the polyhedral join $Z^*_K(\Delta^{l_i-1}, L_i)$, denoted $K(L_1, \ldots, L_m)$. In this case, the link of any simplex in Δ^{l_i-1} is a simplex, and hence the geometric realization of the link is contractible. This means the cohomology of the suspension of the link is trivial, and therefore so is a summand $E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|)$ unless $I = [m]$. In particular, the image, kernel and cokernel of the map

$$H^*(\prod_{j\in[l_i]} X_{ij}) \to H^*(Z_{L_i}(X_{ij}, A_{ij})) \tag{3.3.1}$$

can be computed. The following proposition gives the cohomology of the polyhedral product over a composed simplicial complex in terms of the cohomology of the pairs and the simplicial complexes (using notation described after Definition 3.1.13).

Proposition 3.3.1. *Given m simplicial complexes $\{L_i\}_{i\in[m]}$, where L_i is a complex on the set $[l_i]$. For each $i \in [m]$, let there be a family of pairs $(X_{ij}, A_{ij})_{j\in[l_i]}$ satisfying Definition 3.1.13.*

$$\begin{aligned} H^*(A_{ij}) &= B_{ij} \oplus E_{ij} \\ H^*(X_{ij}) &= B_{ij} \oplus C_{ij} \end{aligned}$$

Then

$H^*(Z_{K(L_1,\ldots,L_m)}(X, A)) =$

$$\bigoplus_{I,\sigma,J,\tau,\rho,\rho'} \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|) \otimes \Big(\bigotimes_{k\in[m]\setminus I} E^{J^c} \otimes \widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|) \otimes Y^{J,\tau} \Big) \otimes \Big(\bigotimes_{s\in\sigma} Y^{[l_s],\rho'} \otimes \bigotimes_{s'\in I\setminus\sigma} Y^{[l_{s'}],\rho} \Big)$$

where:

1. *$I \subset [m]$ and σ is a simplex in K, with $\sigma \subset I$,*

2. *$J \subsetneq [l_k]$ and τ is a simplex in L_k, with $\tau \subset J$,*

3. *ρ' is a nonsimplex in L_s and ρ is a simplex in $L_{s'}$,*

Proof. Recall Ayzenberg's Theorem,

$$Z_{K(L_1,\ldots,L_m)}(X_{ij}, A_{ij}) = Z_K(\prod_{j\in[l_i]} X_{ij}, Z_{L_i}(X_{ij}, A_{ij}))$$

and that we want to find the kernel, C_i, image, B_i, and cokernel, E_i, of the induced map $H^*(\prod_{j\in[l_i]} X_{ij}) \to H^*(Z_{L_i}(X_{ij}, A_{ij}))$.

$$\begin{aligned} \text{Since } H^*(\prod_{j=1}^{l_i} X_{ij}) &= \bigotimes_{j=1}^{l_i}(B_{ij} \oplus C_{ij}) \\ &= \bigoplus_{\rho\in\Delta^{l_i-1}} Y^{[l_i],\rho} \\ &= \bigoplus_{\rho\in L_i} Y^{[l_i],\rho} \oplus \bigoplus_{\rho'\notin L_i} Y^{[l_i],\rho'} \end{aligned}$$

and by theorem 2.0.14

$$H^*(Z_{L_i}(X_{ij}, A_{ij})_{j\in[l_i]}) = \bigoplus_{\rho\in L_i} Y^{[l_i],\rho} \oplus \bigoplus_{J\subsetneq[l_i],\tau\subset J} E^{[l_i]-J} \otimes \widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|) \otimes Y^{J,\tau}$$

we have

$$B_i = \bigoplus_{\rho\in L_i} Y^{[l_i],\rho} \tag{3.3.2}$$

$$C_i = \bigoplus_{\rho'\notin L_i} Y^{[l_i],\rho'} \tag{3.3.3}$$

$$E_i = \bigoplus_{\tau\in L_i,\tau\subset J\subsetneq[l_i]} E^{[l_i]-J} \otimes \widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|) \otimes Y^{J,\tau} \tag{3.3.4}$$

Notice that C_i is the Stanley Reisner ideal of L_i, $I(L_i)$, and B_i is the Stanley-Reisner ring of L_i, $SR(L_i)$ (see Definition 2.0.5). Now applying theorem 2.0.14 again,

$$H^*(Z_K(\prod_{j\in[l_i]} X_{ij}, Z_{L_i}(X_{ij}, A_{ij}))) = \bigoplus_{I\subset[m],\sigma\subset I} E^{I^c} \otimes Y^{I,\sigma} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|)$$

Substituting (3.3.4) and expanding,

$$\begin{aligned}
E^{[m]-I} &= \bigotimes_{k\in[m]-I} E_k \\
&= \bigotimes_{k\in[m]-I} \bigoplus_{J,\tau} E^{[l_k]-J} \otimes \widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|) \otimes Y^{J,\tau} \\
&= \bigoplus_{J,\tau} \Big(\bigotimes_{k\in[m]-I} E^{[l_k]-J} \otimes \widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|) \otimes Y^{J,\tau} \Big)
\end{aligned}$$

Next, substituting (3.3.2) and (3.3.3) and expanding,

$$\begin{aligned}
Y^{I,\sigma} &= \bigotimes_{s\in\sigma} C_s \otimes \bigotimes_{s'\in I-\sigma} B_{s'} \\
&= \bigotimes_{s\in\sigma} (\bigoplus_{\rho'\notin L_s} Y^{[l_s],\rho'}) \otimes \bigotimes_{s'\in I-\sigma} (\bigoplus_{\rho\in L_{s'}} Y^{[l_{s'}],\rho}) \\
&= \bigoplus_{\rho'\notin L_s} (\bigotimes_{s\in\sigma} Y^{[l_s],\rho'}) \otimes \bigoplus_{\rho\in L_{s'}} (\bigotimes_{s'\in I-\sigma} Y^{[l_{s'}],\rho}) \\
&= \bigoplus_{\rho',\rho} \Big(\bigotimes_{s\in\sigma} Y^{[l_s],\rho'} \otimes \bigotimes_{s'\in I-\sigma} Y^{[l_{s'}],\rho} \Big)
\end{aligned}$$

The proposition follows.

□

Recall that the decomposition of $H^*(Z_K(\underline{X}, \underline{A}))$ in Theorem 3.1.14 differs from the decomposition for $H^*(\widehat{Z}_K(\underline{X}, \underline{A}))$ by the presence of $1 \in B_{ij}$. As a consequence, the same proof provides a decomposition for $H^*(\widehat{Z}_{K(L_1,\ldots,L_m)}(\underline{X}, \underline{A}))$. Below is an example of how Proposition 3.3.1 can be used to compute the Poincaré series for the cohomology of a polyhedral product over a composed simplicial complex.

Example 3.3.2. *Suppose we have that K is two ghost vertices, L_1 is a point and L_2 is two* *points. Namely,* $K = \{\emptyset\}$ *is indexed by* $[2]$, $L_1 = \{\{11\}\}$, $L_2 = \{\{21\}, \{22\}\}$. *Given a pair of spaces such that* $\widetilde{H}^*(X) =< b_4, c_6 >$ *and* $\widetilde{H}^*(A) =< e_2, b_4 >$, *we will compute the Poincaré series for* $H^*(\widehat{Z}_{K(L_1,L_2)}(X, A))$.

1. *For* $I = \emptyset$, *the only possible simplex,* σ, *is the empty set. Since* $J_1 \neq [11]$, *the only choice for* J_1 *and* τ_1 *is the empty set, then* $lk(\tau_1)_{J_1^c} = L_1$ *is contractible. There is no contribution to the Poincare series. This means we will only consider* I *such that* $1 \in I$.

2. *In the case* $I = \{1\}$ *and* $\sigma = \emptyset$, *we consider subsets,* J_k, *of* L_k *for* $k \in [m]\backslash I$, *and simplices* $\rho'_s \in L_s$ *and* $\rho_{s'} \in L_{s'}$ *for* $s \in \sigma$ *and* $s' \in I\backslash\sigma$.

   (a) *For* $J_2 = \emptyset$ *and* $\tau_2 = \emptyset$, $lk(\tau_2)_{J_2^c} = L_2$ *and* $\widetilde{H}^*(\Sigma|lk(\tau_2)_{J_2^c}|) =< \iota_1 >$

      i. *If* $\rho_1 = \emptyset$, *then* $\overline{P}(E_{21} \otimes E_{22} \otimes \iota_1 \otimes B_{11}) = t^9$ *is contributed to the Poincare series*

      ii. *If* $\rho_1 = \{11\}$, *then* $\overline{P}(E_{21} \otimes E_{22} \otimes \iota_1 \otimes C_{11}) = t^{11}$ *is contributed*

   (b) *Similarly, with* $J_2 = \{21\}$ *and* $\tau_2 = \emptyset$, $lk(\tau_2)_{J_2^c} = \{\{22\}\}$ *is contractible*

   (c) *If* $J_2 = \{21\}$ *and* $\tau_2 = \{21\}$, *then* $lk(\tau_2)_{J_2^c} = \emptyset$

      i. *If* $\rho_1 = \emptyset$, *then* $\overline{P}(C_{21} \otimes E_{22} \otimes 1 \otimes B_{11}) = t^{12}$

      ii. *If* $\rho_1 = \{11\}$, *then* $\overline{P}(C_{21} \otimes E_{22} \otimes 1 \otimes C_{11}) = t^{14}$

   (d) *If* $J_2 = \{22\}, \tau_2 = \{22\}$ *then* $lk(\tau_2)_{J_2^c} = \emptyset$

      i. *If* $\rho_1 = \emptyset$, *then* $\overline{P}(C_{22} \otimes E_{21} \otimes 1 \otimes B_{11}) = t^{12}$

      ii. *If* $\rho_1 = \{11\}$, *then* $\overline{P}(C_{22} \otimes E_{21} \otimes 1 \otimes C_{11}) = t^{14}$

3. *For* $I = \{2,1\}$ *and* $\sigma = \emptyset$, $lk(\sigma)_I = \emptyset$

   (a) *For* $\rho_1 = \emptyset$,

      i. *If* $\rho_2 = \emptyset$, *then* $\overline{P}(B_{11} \otimes B_{21} \otimes B_{22}) = t^{12}$

      ii. *If* $\rho_2 = \{21\}$, *then* $\overline{P}(B_{11} \otimes C_{21} \otimes B_{22}) = t^{14}$

      iii. *If* $\rho_2 = \{22\}$, *then* $\overline{P}(B_{11} \otimes B_{21} \otimes C_{22}) = t^{14}$

   (b) *For* $\rho_1 = \{11\}$

      i. *If* $\rho_2 = \emptyset$, *then* $\overline{P}(C_{11} \otimes B_{21} \otimes B_{22}) = t^{14}$

      ii. *If* $\rho_2 = \{21\}$, *then* $\overline{P}(C_{11} \otimes C_{21} \otimes B_{22}) = t^{16}$

      iii. *If* $\rho_2 = \{22\}$, *then* $\overline{P}(C_{11} \otimes B_{21} \otimes C_{22}) = t^{16}$

In conclusion,

$$\overline{P}(H^*(\widehat{Z}_{K(L_1,L_2)}(X,A))) = t^9 + t^{11} + 3t^{12} + 5t^{14} + 2t^{16}$$

Since

$$\begin{aligned} K(L_1,L_2) &= L_1 * L_2 \\ &= \{\{11,21\},\{11,22\}\} \end{aligned}$$

a simplicial complex with three vertices and two edges, we can see that this is consistent with Example 5.8 in [5]. Their example computes the Poincare series for the polyhedral product over a simplicial complex with three vertices and two edges, and spaces with equivalent cohomology. We obtained the same answer using a different method.

The following is another version of the result in Proposition 3.3.1 which highlights the role of the Stanley-Reisner ring (see Definition 2.0.5).

Corollary 3.3.3. *Following Definition 3.1.13, suppose we have a decomposition*

$$\begin{aligned} H^*(A_{ij}) &= B_{ij} \oplus E_{ij} \\ H^*(X_{ij}) &= B_{ij} \oplus C_{ij} \end{aligned}$$

with E_{ij} the cokernel of $H^*(X_{ij}) \to H^*(A_{ij})$, B_{ij} the image, and C_{ij} the kernel. Then

$H^*(Z_{K(L_1,\ldots,L_m)}(X,A)) =$

$$\bigoplus_{I,\sigma,J,\tau} \widetilde{H}^*(\Sigma|lk(\sigma)_{I^c}|)\otimes\Big(\bigotimes_{k\in[m]\backslash I} E^{[l_k]-J}\otimes\widetilde{H}^*(\Sigma|lk(\tau)_{J^c}|)\otimes Y^{J,\tau}\Big)\otimes\Big(\bigotimes_{s\in\sigma} I(L_s)\otimes\bigotimes_{s'\in I\backslash\sigma} SR(L_{s'})\Big)$$

where

1. *$I \subset [m]$ and σ is a simplex in K, with $\sigma \subset I$*
2. *$J \subsetneq [l_k]$. τ is a simplex in L_k, with $\tau \subset J$*
3. *ρ' is a nonsimplex in L_s and ρ is a simplex in $L_{s'}$..*

Corollary 3.3.4. *Let $I' = I'_1 \sqcup \ldots \sqcup I'_m \subset [\Sigma l_i]$ and $\sigma' = \sigma'_1 \sqcup \ldots \sqcup \sigma'_m$ a simplex in $K(L_1,\ldots,L_m)$ with $\sigma' \subset [\sum l_i]\backslash I'$. Then*

$$\widetilde{H}^*(\Sigma|lk(\sigma')_{I'}|) = \bigotimes_{k\in I}\widetilde{H}^*(\Sigma|lk(\sigma'_k)_{I'_k}|)\otimes\widetilde{H}^*(\Sigma|lk(\sigma)_I|)$$

where $I = \{i \in [m] |\, I'_i \neq \emptyset\}$ and $\sigma = \{i \in [m]\backslash I | \sigma'_i \notin L_i\}$.

Proof. By theorem 2.0.14, $H^*(Z_{K(L_1,\ldots,L_m)}(X,A)) = \bigoplus_{\sigma',I'} E^{[\sum l_i]-I'}\otimes\widetilde{H}^*(\Sigma|lk(\sigma')_{I'^c}|)\otimes Y^{I',\sigma'}$. Moreover, by proposition 3.3.1, we have that $E^{[\sum l_i]-I'} = \bigotimes_{k\in[m]-I} E^{[l_k]-J_k}$ whenever $I'_i = [l_i]$ for all $i \in I$ and $I'_k = J_k$ for all $k \in [m]-I$. We also have that $Y^{I',\sigma'} = \bigotimes_{k\in[m]-I} Y^{J_k,\tau}\otimes(\bigotimes_{s\in\sigma} Y^{[l_s],\rho'})\otimes(\bigotimes_{s'\in I-\sigma} Y^{[l_{s'}],\rho})$ whenever $\sigma' = (\cup\tau)\cup(\cup\rho')\cup(\cup\rho)$. Thus for $k \in [m]-I$, $\sigma'_k = \tau$.

Similarly, for $s \in \sigma$, $\sigma'_s = \rho' \notin L_s$. In other words $s \in \sigma$ if and only if $\sigma'_s \notin L_s$. Lastly, for $s' \in I - \sigma$, $\sigma'_{s'} = \rho$. A change of notation is used so that the proposition is not stated in terms of complements of sets. □

Example 3.3.5. *Refer to Example 3.1.2 of $K(L_1, L_2, L_3)$ for details of the construction. We will find $\widetilde{H}^*(\Sigma|lk_{K(L_1,L_2,L_3)}(\sigma')_{I'}|)$ in terms of the links in K, L_1, L_2, L_3 for several examples $\sigma' \in K(L_1, L_2, L_3)$ and $I' \subset [\sum l_i]$.*

1. *Suppose $\sigma' = \{32\}$. Then $\sigma'_1 = \emptyset, \sigma'_2 = \emptyset, \sigma'_3 = \{32\}$*

   (a) *If $I' = \{11, 31\}$, then $I'_1 = \{11\}, I'_2 = \emptyset, I'_3 = \{31\}$. This means $I = \{1, 3\}$ and $\sigma = \emptyset$. Thus*

   $$\begin{aligned}\widetilde{H}^*(\Sigma|lk(\sigma')_{I'}|) &= \widetilde{H}^*(\Sigma|lk(\sigma'_1)_{I'_1}|) \otimes \widetilde{H}^*(\Sigma|lk(\sigma'_3)_{I'_3}|) \otimes \widetilde{H}^*(\Sigma|lk(\sigma)_I|)\\ &= \widetilde{H}^*(\Sigma\emptyset) \otimes \widetilde{H}^*(\Sigma\emptyset) \otimes \widetilde{H}^*(\Sigma|\{\{1\},\{3\}\}|)\\ &= 1 \otimes 1 \otimes 1 \otimes \widetilde{H}^*(S^1)\end{aligned}$$

   This is consistent with $|lk(\sigma')_{I'}| = |\{\{11, 21\}, \{21, 31\}\}_{I'}| = |\{\{11\}, \{31\}\}| \simeq S^0$

   (b) *If $I' = \{31, 21\}$, then $I = \{2, 3\}$ and $\sigma = \emptyset$. Thus $\widetilde{H}^*(\Sigma|lk(\sigma')_{I'}|) = 1 \otimes 1 \otimes 0$. This is consistent with $|lk(\sigma')_{I'}| = |\{\{31, 21\}\}| = \Delta^1$, which is contactible*

2. *Suppose $\sigma' = \{11, 32\}$ and $I' = \{31\}$. Then $I = \{3\}$ and $\sigma = \{1\}$. Moreover, $\widetilde{H}^*(\Sigma|lk(\sigma')_{I'}|) = 1 \otimes 1$, which is consistent with $lk(\sigma')_{I'} = \emptyset$.*

Recall that Definition 2.0.1 of the full subcomplex, K_I or $K|_I$, and that the notation $\mathcal{K}$ denotes the geometric realization of K. Specializing Proposition 3.3.1 to the case $(\underline{X}, \underline{A}) = (\underline{CA}, \underline{A})$ we have the following corollary.

Corollary 3.3.6.

$$H^*(Z_{K(L_1,\ldots,L_m)}(\underline{CA},\underline{A}))$$

$$= \bigoplus_{I\subset[m],J_k\subset[l_k],J_k\neq\emptyset} \widetilde{H}^*(\Sigma\mathcal{K}_I)\otimes\Big(\bigotimes_{k\in I}\widetilde{H}^*(\Sigma\mathcal{L}_k|_{J_k})\otimes\widetilde{H}^*(A^{\wedge J_k})\Big)$$

where $A^{\wedge J_k} := \underset{j\in[J]}{\wedge} A_j$

Proof. Recall L_k is a simplicial complex on the set $[l_k]$. For the pair (CA_i, A_i), $B_i = 1$, $C_i = 0$ and $E_i = \widetilde{H}^*(A)$. We have that $Y^{J,\tau} \neq 0$ whenever $\tau = \emptyset$, so that $lk(\tau_k)_{J_k} = L_k|_{J_k}$. Also, if $\sigma \neq \emptyset$, then $Y^{[l_i],\rho'} = 0$ because the emptyset is not a nonsimplex of any simplicial complex. Since $\sigma = \emptyset$, it follows that $lk(\sigma)_{I^c} = K_{I^c}$. Recall that $J_k \neq [l_k]$ and hence $J_k^c \neq \emptyset$. $\square$

An immediate corollary of Corollary 3.3.6 is a computation of the Poincaré series of

$$\widetilde{H}^*(Z_{K(L_1,\ldots,L_m)}(CA, A)).$$

Corollary 3.3.7.

$$\bar{P}(\widetilde{H}^*(Z_{K(L_1,\ldots,L_m)}(CA, A)))$$

$$= \sum_{I\subset[m],J_k\subset[l_k],J_k\neq\emptyset} \bar{P}(\widetilde{H}^*(\Sigma\mathcal{K}_I))\prod_{i\in I}\bar{P}(\widetilde{H}^*(\Sigma\mathcal{L}_k|_{J_k}))\bar{P}(\widetilde{H}^*(A^{\wedge J_k}))$$

Remark 3.3.8. The Poincaré series

$$\bar{P}(\widetilde{H}^*(Z_{K(L_1,\ldots,L_m)}(CA, A)))$$

$$= \sum_{B\subset[m]} \bar{P}(\widetilde{H}^*(\Sigma\mathcal{K}_B)\prod_{b\in B}\bar{P}(\widetilde{H}^*(Z_{L_b}(CA, A))))$$

This generalizes the computation for (D^2, S^1) in Ayzenberg to the case $(\underline{CA},\underline{A})$. In fact, the above forumlas can be proven by our methods without appealing to Ayzenberg's theorem.

Alternate proof of Corollary 3.3.7. Ayzenberg's lemma 7.5 states that for $J = \cup J_i \subset [\sum l_i]$, $K(L_1,\ldots,L_m)_J = K_B(L_{b_1}|_{J_{b_1}},\ldots,L_{b_k}|_{J_{b_k}})$ where $B = \{b_i | J_i \neq \emptyset\}$ and corollary 6.2 states that $K(L_1,\ldots,L_m) \simeq K * L_1 * \ldots * L_m$. Now, using the splitting theorem, the wedge lemma and that the join of spaces is the suspension of the smash product of the spaces, we have the following

$$\begin{aligned}
\widetilde{H}^*(Z_{K(L_1,\ldots,L_m)}(CA,A))) &= \bigoplus_J \widetilde{H}^*(\Sigma K(L_1,\ldots,L_m)_J \wedge \widehat{A}^J) \\
&= \bigoplus_{B,J_{b_1},\ldots,J_{b_k}} \widetilde{H}^*(\Sigma K_B(L_{b_1}|_{J_{b_1}},\ldots,L_{b_k}|_{J_{b_k}}) \wedge \widehat{A}^{J_{b_1}} \wedge \ldots \wedge \widehat{A}^{J_{b_k}}) \\
&= \bigoplus_{B,J_{b_1},\ldots,J_{b_k}} \widetilde{H}^*(\Sigma K_B \wedge \Sigma L_{b_1}|_{J_{b_1}} \wedge \ldots \wedge \Sigma L_{b_k}|_{J_{b_k}} \wedge \widehat{A}^{J_{b_1}} \wedge \ldots \wedge \widehat{A}^{J_{b_k}}) \\
&= \bigoplus_{B,J_{b_1},\ldots,J_{b_k}} \widetilde{H}^*(\Sigma K_B) \otimes \bigotimes_{b_i \in B} \widetilde{H}^*(\Sigma L_{b_i}|_{J_{b_i}} \wedge \widehat{A}^{J_{b_i}}) \\
&= \bigoplus_B \widetilde{H}^*(\Sigma K_B) \otimes \bigotimes_{b_i \in B} \bigoplus_{J_{b_i}} \widetilde{H}^*(\Sigma L_{b_i}|_{J_{b_i}} \wedge \widehat{A}^{J_{b_i}}) \\
&= \bigoplus_B \widetilde{H}^*(\Sigma K_B) \otimes \bigotimes_{b_i \in B} \widetilde{H}^*(Z_{L_{b_i}}(CA,A))
\end{aligned}$$

□

In [1], Ayzenberg defines the multigraded betti numbers of a simplicial complex in terms of the Tor-module, which is the cohomology of the moment-angle complex. Let $\Bbbk$ be the ground field and $\Bbbk[m] = \Bbbk[v_1,\ldots,v_m]$ be the ring of polynomials in m indeterminates. The ring $\Bbbk[m]$ has a $\mathbb{Z}^m$-grading defined by $\deg(v_i) = (0,\ldots,2,\ldots,0)$ with 2 in the i-th place. Given a free resolution $\ldots \to R^{-i} \to \ldots \to \Bbbk[K]$ by $\mathbb{Z}^m$-graded $\Bbbk[m]$ modules, we have the Tor-module

$$\mathrm{Tor}_{\Bbbk[m]}(\Bbbk[K],\Bbbk) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}, \overline{j} \in \mathbb{Z}^m} Tor_{\Bbbk[m]}^{-i,2\overline{j}}(\Bbbk[K],\Bbbk)$$

The $(-i,2\overline{j})$-th betti number of K is the dimension of $Tor_{\Bbbk[m]}^{-i,2\overline{j}}(\Bbbk[K],\Bbbk)$ over $\Bbbk$. Since the cohomology ring of the moment-angle complex is $Tor_{\Bbbk[m]}(\Bbbk[K],\Bbbk)$, we adapt the definition to an arbitrary polyhedral product of pairs $(\underline{CA}, \underline{A})$.

The splitting theorem (2.0.14) and wedge lemma (2.0.13) enable us to define the multi-graded Betti numbers of a polyhedral product space.

Definition 3.3.9. *For $i \in \mathbb{Z}$ and $J \subset [m]$, the* multigraded Betti numbers *of $Z_K(\underline{CA}, \underline{A})$, denoted $\beta^{i,J}(Z_K(\underline{CA}, \underline{A}))$, are defined as*

$$\beta^{i,J}(Z_K(\underline{CA}, \underline{A})) := \dim \widetilde{H}^i(\Sigma \mathcal{K}_J \wedge \widehat{A}^J)$$

Let s and $t_1, \ldots, t_m$ be indeterminates such that $\bar{t}^J = t_1^{j_1} \ldots t_m^{j_m}$ where $j_i = 1$ if $i \in J$ and $j_i = 0$ otherwise. The beta-polynomial *of $Z_K(\underline{CA}, \underline{A})$ is defined as*

$$\beta_{Z_K(\underline{CA},\underline{A})}(s, \bar{t}) := \sum_{i \in \mathbb{Z}, J \subset [m]} \beta^{i,J}(Z_K(\underline{CA}, \underline{A})) s^i \bar{t}^J$$

and the reduced beta-polynomial

$$\widetilde{\beta}_{Z_K(\underline{CA},\underline{A})}(s, \bar{t}) := \beta_{Z_K(\underline{CA},\underline{A})}(s, \bar{t}) - 1 = \sum_{i \in \mathbb{Z}, \emptyset \neq J \subset [m]} \beta^{i,J}(Z_K(\underline{CA}, \underline{A})) s^i \bar{t}^J$$

Proposition 3.3.10. *The multigraded Betti numbers of $Z_{K(L_1,\ldots,L_m)}(\underline{CA}, \underline{A})$ can be expressed in terms of the multigraded Betti numbers of the polyhedral products associated to each of the simplicial complexes $K, L_1, \ldots, L_m$. Their beta-polynomials have the following relationship.*

$$\beta_{Z_{K(L_1,\ldots L_m)}(\underline{CA},\underline{A})}(s, \bar{t}) = \beta_{Z_K(D^1,S^0)}(s, \widetilde{\beta}_{Z_{L_1}(CA,A)}(s, \bar{t}), \ldots, \widetilde{\beta}_{Z_{L_m}(CA,A)}(s, \bar{t}))$$

Proof. Let $i' \in \mathbb{Z}$, $J = \cup J_i \subset [\sum l_i]$, $B = \{b_i \in [m] | J_i \neq \emptyset\} = \{b_1, \ldots, b_k\}$, $n + p = i'$, $r + \sum r_s = n$, $\sum c_s = p$. Using Corollary 3.3.6, we have

$\beta_{Z_{K(L_1,\ldots,L_m)}(\underline{CA},\underline{A})}(s, \bar{t}) = \sum_{i',J} \dim \widetilde{H}^{i'}(\Sigma K(L_1, \ldots, L_m)_J \wedge \widehat{A}^J) s^{i'} \bar{t}^J$

$$= \sum_{\substack{i',B,\\ J_{b_1},\ldots,J_{b_k}}} \sum_{n,p} \dim \widetilde{H}^n(\Sigma K(L_1,\ldots,L_m)_J) \dim \widetilde{H}^p(\widehat{A}^{J_{b_1}} \wedge \ldots \wedge \widehat{A}^{J_{b_k}}) s^{i'} \bar{t}^J$$

$$= \sum_{\substack{i',B,\\ J_{b_1},\ldots,J_{b_k}}} \sum_{n,p} (\sum_{r,r_1,\ldots,r_k} \dim \widetilde{H}^r(\Sigma K_B) \prod_{i=1}^{k} \dim \widetilde{H}^{r_i}(\Sigma L_{b_i}|_{J_{b_i}}))(\sum_{c_1,\ldots,c_k} \prod_{i=1}^{k} \dim \widetilde{H}^{c_i}(\widehat{A}^{J_i})) s^{i'} \bar{t}^J$$

$$= \sum_{\substack{i',B,\\ J_{b_1},\ldots,J_{b_k}}} \sum_{n,p} \sum_{r} \dim \widetilde{H}^r(\Sigma K_B) \prod_{i=1}^{k} \sum_{r_1,\ldots,r_k,c_1,\ldots,c_k} \dim \widetilde{H}^{r_i+c_j}(\Sigma L_{b_i}|_{J_{b_i}} \wedge \widehat{A}^{J_{b_i}}) s^{i'} \bar{t}^{J_{b_1}} \ldots \bar{t}^{J_{b_k}}$$

Next, we will use a change of variables in order to rewrite this in a recognizable form, let

$$r_i + c_j = a_{i+j}$$

Then $i' = n + p = r + \sum r_i + \sum c_i = r + \sum a_{i+j} = r + \sum a_u$. Therefore,

$$\beta_{Z_{K(L_1,\ldots,L_m)}(\underline{CA},\underline{A})}(s,\bar{t})$$

$$= \sum_{i',B,J_{b_1},\ldots,J_{b_k}} \sum_{n,p} \sum_{r} \dim \widetilde{H}^r(\Sigma K_B) \prod_{i=1}^{k} \sum_{r_1,\ldots,r_k,c_1,\ldots,c_k} \dim \widetilde{H}^{r_i+c_j}(\Sigma L_{b_i}|_{J_{b_i}} \wedge \widehat{A}^{J_{b_i}}) s^{i'} \bar{t}^{J_{b_1}} \ldots \bar{t}^{J_{b_k}}$$

$$= \sum_{B,r} \dim \widetilde{H}^r(\Sigma K_B) s^r \prod_{i=1}^{k} \sum_{J_b,a_u} \dim \widetilde{H}^{a_u}(\Sigma L_{b_i}|_{J_{b_i}} \wedge \widehat{A}^{J_{b_i}}) s^{a_u} \bar{t}^{J_s}$$

$$= \beta_{Z_{K(D^1,S^0)}}(s, \widetilde{\beta}_{Z_{L_1}(CA,A)}(s,\bar{t}), \ldots, \widetilde{\beta}_{L_m(CA,A)}(s,\bar{t})) \qquad \square$$

Example 3.3.11. *Consider the composed simplicial complex from example 3.1.7. We need to compute the reduced beta-polynomials of each complex L_i. Since the full subcomplex of L_1 associated to $\{11\}$ is the empty set,*

$$\widetilde{\beta}_{Z_{L_1}(\underline{CA},\underline{A})} = \sum \dim \widetilde{H}^i(\Sigma\emptyset \wedge A_{11}) s^i t_{11}$$

The full subcomplexes of L_2 are all contractible, so its beta-polynomial is the zero polynomial. The only non-trivial full subcomplex of L_3 is associated to $\{31, 32\}$, and hence its reduced beta-polynomial is

$$\widetilde{\beta}_{Z_{L_3}(\underline{CA},\underline{A})} = \sum \dim \widetilde{H}^i(\Sigma\partial\Delta^1 \wedge A_{31} \wedge A_{32}) s^i t_{31} t_{32}$$

The non-contractible full subcomplexes of K *are* $\{1,2\},\{1,3\},\{1,2,3\}$. *Since the beta-polynomial of* L_2 *is zero, any subsets of* $[3]$ *that contain* 2 *do not contribute any non-trivial terms. Apply Proposition 3.3.10.*

$$\beta_{Z_{K(L_1,\dots L_m)}(\underline{CA},\underline{A})}(s,\bar{t})$$

$$\begin{aligned}
&= \beta_{Z_{K(D^1,S^0)}}(s,\widetilde{\beta}_{Z_{L_1}(CA,A)}(s,\bar{t}),\dots,\widetilde{\beta}_{Z_{L_m}(CA,A)}(s,\bar{t})) \\
&= \sum_{i\in\mathbb{Z},J\subset[m]} \dim \widetilde{H}^i(\Sigma\mathcal{K}_J)s^i(\widetilde{\beta}_{Z_{\underline{L}}(\underline{CA},\underline{A})})^J \\
&= 1+\sum \dim H^i(\Sigma\partial\Delta^1)s^i(\widetilde{\beta}_{Z_{L_1}(\underline{CA},\underline{A})})(\widetilde{\beta}_{Z_{L_3}(\underline{CA},\underline{A})}) \\
&= 1+s(\sum \dim \widetilde{H}^i(\Sigma\emptyset\wedge A_{11})s^it_{11})(\sum \dim \widetilde{H}^i(\Sigma\partial\Delta^1\wedge A_{31}\wedge A_{32})s^it_{31}t_{32})
\end{aligned} \tag{3.3.5}$$

Since all full subcomplexes of $K(L_1,L_2,L_3)$ *are contractible except those associated to the empty set and and the set* $\{11,31,32\}$, *its beta-polynomial is*

$$\begin{aligned}
\beta_{Z_{K(L_1,L_2,L_3)}(\underline{CA},\underline{A})}(s,\bar{t}) &= \sum_{i\in\mathbb{Z},J} \dim \widetilde{H}^i(\Sigma\mathcal{K}_J\wedge\widehat{A}^J)s^i\bar{t}^J \\
&= 1+\sum \dim \widetilde{H}^i(\Sigma\partial\Delta^2\wedge A_{11}\wedge A_{31}\wedge A_{32})s^it_{11}t_{31}t_{32}
\end{aligned} \tag{3.3.6}$$

If for example $A_i = S^2$ *for all* i, *then both expressions 3.3.6 and 3.3.5 simplify to*

$$1+s^8t_{11}t_{31}t_{32}$$

The definition of multigraded Betti numbers of a simplicial complex is given in terms of a Tor algebra and the Stanley-Reisner ring, which are in terms of indeterminates in degree 2. When Ayzenberg considers the $(-i,2j)$-th Betti number and applies Hochster's theorem, the $(-i,2j)$-th Betti number should be the dimension of the cohomology of K_J in degree

$(-i+2j)-j-1=-i+j-1$ where $|J|=j$. This is equivalent to Definition 3.3.9 since $\widetilde{H}^i(\Sigma\mathcal{K}_J\wedge\widehat{A}^J) = \widetilde{H}^{i-j-1}(K_J)$, with a change of variables for the shift in cohomological degree.

3.4 Cohomology Ring

In [5], Bahri, Bendersky, Cohen and Gitler also used their methods to describe the ring structure of a polyhedral product $Z_K(\underline{CA},\underline{A})$ in terms of the cohomology ring of the decomposition given in 3.1.15. The main idea is that given generators

$$\alpha = n_\alpha\otimes a_1\otimes\ldots\otimes a_m$$

$$\gamma = n_\beta\otimes g_1\otimes\ldots\otimes g_m$$

where $n_\alpha \in \widetilde{H}^*(\Sigma|lk(\sigma_1)|_{I_1^c})$, $n_\gamma \in \widetilde{H}^*(\Sigma|lk(\sigma_2)|_{I_2^c})$ and the remaining factors are in the appropriate E_i, B_i or C_i, the cup product $\alpha\cup\gamma$ is described in terms of coordinate-wise multiplication and a pairing

$$\widetilde{H}^*(\Sigma|lk(\sigma_1)|_{I_1^c})\otimes\widetilde{H}^*(\Sigma|lk(\sigma_2)|_{I_2^c})\to\widetilde{H}^*(\Sigma|lk(\sigma_3)|_{I_3^c})$$

where I_3 and σ_3 are described in terms of $I_1, I_2, \sigma_1, \sigma_2$. A complication that arises is that in the product the indexing set of the C_i's could be larger. In the decomposition (from Theorem 3.1.15), the C_i's in every term are indexed by a simplex in K. Therefore, if the larger indexing set of the C_i's does not correspond to a simplex in K, the cup product must be zero. We may think of $H^*(Z_K(\underline{X},\underline{A}))$ as living in the larger tensor algebra modulo the generalized Stanley-Reisner ideal of K.

Recall the generalized Stanley-Reisner ideal $I(K)$ in $\widetilde{H}^*(X_1) \otimes \ldots \otimes \widetilde{H}^*(X_m)$.

$$I(K) = \langle x_{i_1} \otimes \ldots \otimes x_{i_k} \mid \{i_1, \ldots, i_k\} \notin K \rangle$$

Since the generalized Stanley-Reisner ideal is crucial to understanding the ring structure of $H^*(Z_K(\underline{X}, \underline{A}))$, we will describe the generalized Stanley-Reisner ideal in the case that the underlying simplicial complex is a composition $K(L_1, \ldots, L_m)$ in terms of the generalized Stanley-Reisner ideal of $K, L_1, \ldots, L_m$.

Proposition 3.4.1. *The generalized Stanley-Reisner ideal of $K(L_1, \ldots, L_m)$ is the generalized Stanley-Reisner ideal of K in terms of the generalized Stanley-Reisner ideals of $L_1, \ldots, L_m$*

$$I(K(L_1, \ldots, L_m)) = \langle c_{i_1} \otimes \ldots \otimes c_{i_k} \mid \{i_1, \ldots, i_k\} \notin K \rangle$$

where $c_i \in I(L_i)$ for $i_1 \leq i \leq i_k$.

Proof. Recall from the proof of Theorem 3.3.1 that $C_i = I(L_i)$.

Suppose $c^\sigma \in I(K(L_1, \ldots, L_m))$ where $\sigma \subset [\sum l_i]$ such that $\sigma \notin K(L_1, \ldots, L_m)$. Recall from the equivalent definition of $K(L_1, \ldots L_m)$ (after Definition 3.1.2) that σ is of the form

$$\sigma = \bigcup_{i \in [m]} \sigma_i$$

where $\sigma_i \subset [l_i]$ and $\{i \in [m] \mid \sigma_i \notin L_i\} \notin K$. Let $A = \{i \in [m] \mid \sigma_i \notin L_i\} = \{i_1, \ldots, i_k\}$. In other words $c^\sigma = c^A = c_{i_1} \otimes \ldots \otimes c_{i_k}$ where $A \notin K$ and $c_i \in I(L_i)$. It follows that $c^\sigma \in \langle c_{i_1} \otimes \ldots \otimes c_{i_k} \mid \{i_1, \ldots, i_k\} \notin K \rangle$.

Suppose $c = c_{i_1} \otimes \ldots \otimes c_{i_k} \in \langle c_{i_1} \otimes \ldots \otimes c_{i_k} \mid \{i_1, \ldots, i_k\} \notin K \rangle$ where $c_i \in I(L_i)$ for

$i_1 \leq i \leq i_k$. Since $c_i \in I(L_i)$, it is of the form $c_i = c^{\tau_i}$ for some $\tau_i \notin L_i$. Since $\{i_1, \ldots, i_k\} \notin K$, we have that $\tau = \cup \tau_i \notin K(L_1, \ldots, L_m)$. Therefore, $c = c^{\tau} \in I(K(L_1, \ldots, L_m))$. □

Example 3.4.2. *The generalized Stanley-Reisner ideal is generated by the minimal non-faces, a set that is not a simplex but every subset is. Suppose we have the following simplicial complexes.*

$$\begin{array}{lll} L_1 \subset [2] & L_1 = \{\{11\}, \{12\}\} & I(L_1) = \langle c_{11} \otimes c_{12} \rangle \\ L_2 \subset [2] & L_2 = \{\{21\}\} & I(L_2) = \langle c_{22} \rangle \\ K \subset [2] & K = \{\{2\}\} & I(K) = \langle c_1 \rangle = \langle c_{11} \otimes c_{12} \rangle \end{array}$$

The composition $K(L_1, L_2) = L_1 * \Delta^1 = \{\{11, 21, 22\}, \{12, 21, 22\}\}$ *and hence*

$$I(K(L_1, L_2)) = \langle c_{11} \otimes c_{12} \rangle$$

.

Further research is necessary in order to fully describe the ring structure of $H^*(Z_{K(L_1,\ldots,L_m)}(\underline{X}, \underline{A}))$.

3.5 The pair $(L_i, \emptyset)$

In this section we will find a formula for the cohomology groups of $Z_{Z^*_K(L_i,\emptyset)}(\underline{X}, \underline{A})$, the polyhedral product over a polyhedral join given by the pairs $(L_i, \emptyset)$. In this case, we get a similar formula to theorem 3.1.8

$$Z_{Z^*_K(L_i,\emptyset)}(\underline{X}, \underline{A}) = Z_K(Z_{L_i}(\underline{X}, \underline{A}), \prod_{j \in [l_i]} A_j) \tag{3.5.1}$$

As an application, we can write the polyhedral product $Z_K(S^n, \vee S^0)$ as the real moment-angle complex $Z_{Z^*_K(\partial\Delta^{n_i},\emptyset)}(D^1, S^0)$.

It follows from the discussion in Section 3.2 that the kernel, cokernel and image can be computed if the links of simplices in L_i can be described in general. Note that L_i and its subsimplicial complex, $\emptyset$, do not have any (nontrivial) simplices in common, so the links do not present any issues. Equation 3.5.1 and theorem 3.1.15 imply the following formula.

Theorem 3.5.1. *Given simplicial complexes L_i on the vertex sets $[l_i]$ with no ghost vertices and pairs (X_{ij}, A_{ij}), where i varies in $[m]$ and j varies in $[l_i]$, that satisfy the freeness condition of Definition 3.1.13 with decompositions*

$$H^*(X_{ij}) = B_{ij} \oplus C_{ij}$$

$$H^*(A_{ij}) = B_{ij} \oplus E_{ij}$$

Then we have

$$H^*(Z_{Z^*_K(L_i,\emptyset)}(\underline{X}, \underline{A})) =$$

$$\bigoplus_{J,\tau,I,\sigma} E^T \otimes B^{(T\cup S)^c} \otimes C^S \otimes (\bigotimes_{v\in\tau} \widetilde{H}^*(\Sigma|lk(\sigma)|_I)) \otimes \widetilde{H}^*(\Sigma|lk(\tau)|_{J^c})$$

where

1. *$J \subset [m]$ with a simplex τ of K such that $\tau \subset J$*

2. *For $v \in \tau$, take subsets $I_v \subset [l_v]$ and a simplex $\sigma \in L_v$ such that $\sigma \subset I^c$. For $k \in [m]\backslash J$,*

consider subsets $I_k \subset [l_k]$. Then T and S are defined by

$$T = (\bigcup_{[m]\setminus J} I_k) \cup (\bigcup_{v\in\tau} I_v)$$

$$S = \bigcup_{v\in\tau} \sigma_v$$

Proof. From Definition 3.1.13, we need to find the kernel, E_i, image, B_i, and cokernel, C_i, of the map

$$H^*(Z_{L_i}(\underline{X},\underline{A})) \to H^*(\prod_{j\in[l_i]} A_j)$$

where $H^*(\prod_{j\in[l_i]} A_j) = E_i \oplus B_i$ and $H^*(Z_{L_i}(\underline{X},\underline{A})) = C_i \oplus B_i$. Since the cohomology of each space in the pair is given by

$$\begin{aligned}
H^*(\textstyle\prod_{j\in[l_i]} A_j) &= \bigoplus_{I\subset[l_i]} B^I \otimes E^{I^c} \\
H^*(Z_{L_i}(\underline{X},\underline{A})) &= \bigoplus_{\substack{\sigma\subset I\subset[m]\\ \sigma\in L_i}} E^{I^c} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)|_{I^c}) \otimes Y^{I,\sigma} \\
&= \big(\bigoplus_{I\subset[m]} E^{I^c} \otimes B^I \otimes \widetilde{H}^*(\Sigma\mathcal{L}_i|_{I^c})\big) \\
&\qquad\qquad \oplus \\
&\quad \big(\bigoplus_{\substack{\sigma\subset I\subset[m]\\ \emptyset\neq\sigma\in L_i}} E^{I^c} \otimes \widetilde{H}^*(\Sigma|lk(\sigma)|_{I^c}) \otimes Y^{I,\sigma}\big)
\end{aligned}$$

and the full subcomplex $\mathcal{L}_i|_{I^c}$ is only empty when $I^c = \emptyset$ (because L_i has no ghost vertices), we have that

$$B_i = B^{[l_i]}$$

$$E_i = \bigoplus_{I \subsetneq [l_i]} B^I \otimes E^{I^c}$$

$$C_i = \bigoplus_{\substack{\sigma \subset I \subsetneq [m] \\ \sigma \in L_i \\ \emptyset \neq \sigma \in L_i}} E^{I^c} \otimes \widetilde{H}^*(\Sigma |lk(\sigma)|_{I^c}) \otimes Y^{(I,\sigma)}$$

where "$\sigma, I \neq \emptyset$" means that σ and I are not both the empty set. Then substituting,

$$\begin{aligned}
& H^*(Z_{Z^*_K(L_i,\emptyset)}(\underline{X}, \underline{A})) \\
&= \bigoplus_{J,\tau} \widetilde{H}^*(\Sigma |lk(\tau)|_{J^c}) \otimes E^{[m]\setminus J} \otimes C^\tau \otimes B^{J\setminus\tau} \\
&= \bigoplus_{J,\tau} \Big(\widetilde{H}^*(\Sigma |lk(\tau)|_{J^c}) \otimes \bigotimes_{k\in[m]\setminus J} \big(\bigoplus_{L\subset [l_k]} E^L \otimes B^{L^c} \big) \\
&\qquad \otimes \bigotimes_{v\in\tau} \Big(\bigoplus_{\substack{I\subset[l_v], \\ \sigma\in L_v, \sigma, I\neq\emptyset}} E^I \otimes Y^{I^c,\sigma} \otimes \widetilde{H}^*(\Sigma |lk(\sigma)|_I) \Big) \\
&\qquad \otimes \bigotimes_{u\in J\setminus\tau} B^{[l_u]} \Big) \\
&= \bigoplus_{\substack{J,\tau, \\ I,L,\sigma}} \Big(\widetilde{H}^*(\Sigma |lk(\tau)|_{J^c}) \bigotimes_{k\in[m]\setminus J} (E^L \otimes B^{L^c}) \\
&\qquad \otimes \bigotimes_{v\in\tau} \big(E^I \otimes Y^{I^c,\sigma} \otimes \widetilde{H}^*(\Sigma |lk(\sigma)|_I) \big) \\
&\qquad \otimes \bigotimes_{u\in J\setminus\tau} B^{[l_u]} \Big)
\end{aligned}$$

□

Corollary 3.5.2. *Suppose $(X_{ij}, A_{ij}) = (CA_{ij}, A_{ij})$. Then*

$$H^*(Z_{Z^*_K(L_i,\emptyset)}(\underline{CA}, \underline{A})) = \bigoplus_{J,\tau,I_k,I_v\neq\emptyset} \big(\widetilde{H}^*(\widehat{A}^T) \otimes (\bigotimes_{v\in\tau} \widetilde{H}^*(\Sigma L_v|_{I_v})) \otimes \widetilde{H}^*(\Sigma |lk(\tau)|_{J^c})) \big)$$

where

1. *$J \subset [m]$ with a simplex τ of K such that $\tau \subset J$*

2. *For $k \in [m] \backslash J$ and $v \in \tau$, take subsets $I_v \subset [l_v]$ and $I_k \subset [l_k]$, and define T:*

$$T := \Big(\bigcup_{k \in [m]\backslash J} I_k\Big) \cup \Big(\bigcup_{v \in \tau} I_v\Big)$$

Proof. With the given pairs, we know that $C_{ij} = 0$, $B_{ij} = 1$ and $E_{ij} = \widetilde{H}^*(A_{ij})$ for all ij. Since $C_{ij} = 0$, σ in Proposition 3.5.1 must be the empty set. Therefore I cannot be the empty set and

$$\begin{aligned}
&H^*(Z_{Z^*_K(L_i,\emptyset)}(\underline{CA}, \underline{A})) \\
&\quad = \bigoplus_{\substack{J,\tau,I_k,\\ I_v \neq \emptyset}} \Big(\big(\bigotimes_{k \in [m]\backslash J} E^{I_k} \otimes B^{I_k^c}\big) \otimes \big(\bigotimes_{v \in \tau} E^{I_\alpha} \otimes B^{I_v^c} \otimes \widetilde{H}^*(\Sigma L_v|_{I_v})\big) \otimes \big(\bigotimes_{j \in J\backslash\tau} B^{[l_\beta]}\big) \otimes \widetilde{H}^*(\Sigma |lk(\tau)|_{J^c})\Big) \\
&\quad = \bigoplus_{J,\tau,I_k,I_\alpha \neq \emptyset} \Big(\big(\bigotimes_{k \in [m]\backslash J} E^{I_k}\big) \otimes \big(\bigotimes_{v \in \tau} E^{I_v} \otimes \widetilde{H}^*(\Sigma L_v|_{I_v})\big) \otimes \widetilde{H}^*(\Sigma |lk(\tau)|_{J^c})\Big)
\end{aligned}$$

□

3.6 Further Research

In [2], Bahri, Bendersky, Cohen and Gitler show that there is a stable splitting of the polyhedral product space 2.0.14. Though the decomposition cannot be desuspended in general, it is an open problem to characterize the simplicial complexes for which there is an unstable splitting. A simplicial complex is called *shifted* if there is an ordering of the vertices such that if v is a vertex of a simplex σ and v is greater than a vertex w, then $(\sigma - v) \cup w$ is also a simplex. In 2011, a significant advancement was showing that there is an unstable splitting for polyhedral products of $(\underline{CX}, \underline{X})$ over shifted simplicial complexes [19]. Grbić and Theriault also show that if K is shifted, then the splitting desuspends over $K(v)$ (the simplicial wedge). Since $K(v)$ may no longer be shifted, this shows that the shifted condition is not necessary for an unstable splitting, and that the result cannot be iterated.

They do, however, conjecture that the result holds for the J-construction. Since the polyhedral join is more flexible, I plan to investigate whether their results can be extended using polyhedral joins.

Chapter 4

Real Moment Angle Complexes

We turn to a study of the cohomology ring of real moment-angle complexes. This has applications in combinatorics and in algebraic geometry. In combinatorics, an important task is to find the homotopy type of the complement of a coordinate subspace arrangement associated to a simplicial complex, which was shown in [9] to be equivalent to real moment-angle complexes. Real moment-angle complexes are also equivalent to intersections of real quadrics, which are studied by algebraic geometers.

4.1 Background

For any space Y, it is easy to see that $Y \wedge S^0 = Y$. Thus the wedge lemma (2.0.13) implies that $\widehat{Z}_K(D^1, S^0)$ is the suspension of the geometric realization of K. It follows from this and the splitting theorem (2.0.14) that the stable homotopy type of the real moment-angle complex is a wedge sum of spaces $\Sigma\mathcal{K}_I$. Thus, computing the cohomology groups of real moment-angle complexes is a combinatorial process that involves examining only the simplicial complex. The generators of the cohomology ring are given by the subsets of $[m]$ that yield a noncontractible full subcomplex of K after suspension. This follows from

Theorem 4.1.1 (BBCG 2010)**.**

$$H^*(Z_K(D^1, S^0)) = \bigoplus_{I \subset [m]} H^*(\Sigma \mathcal{K}_{\mathcal{I}})$$

Moreover, this decomposition of real moment-angle complexes relates to the product structure for the more general space $H^*(Z_K(\underline{X}, \underline{A}))$. In [5], a description of the ring structure in $H^*(Z_K(\underline{X}, \underline{A}))$ is given using the decomposition from Theorem 3.1.15. It is induced by the ring structure in $H^*(X_i)$ and $H^*(A)$ and a multiplication given on the links.

For the pair (CA_i, A_i), $B_i = 1$, $C_i = 0$ and $E_i = \widetilde{H}^*(A)$. In particular, the links are all of the form K_I for $I \subset [m]$. It therefore follows from Theorem 3.1.15 that the product structure in $H^*(Z_K(\underline{CA}, \underline{A}))$ can be described in terms of the product structure in $H^*(A)$ and $\widetilde{H}^*(\Sigma \mathcal{K}_I)$. Since $H^*(Z_K(D^1, S^0)) = \oplus H^*(\Sigma \mathcal{K}_I)$, the ring structure in $H^*(Z_K(\underline{CA}, \underline{A}))$ can be described in terms of the ring structure in $H^*(A)$ and $H^*(Z_K(D^1, S^0))$. Restricting to the n-gon illustrates the structure that appears in the pairing of the cohomology of the links.

To compute the cohomology of a real moment-angle complex over an n-gon, we will use a filtered chain complex induced by the long exact sequence of the pair (D^1, S^0), denoted $C(K_I)$, constructed in [5] (however, this chain complex can be used to compute the cohomology of moment-angle complexes over arbitrary K). For $(\underline{X}, \underline{A}) = (D^1, S^0)$, we have

$\widetilde{H}^*(A_k) = \widetilde{H}^*(S^0)$ is generated by t_k and $\widetilde{H}^*(X_k/A_k) = \widetilde{H}^*(S^1)$ is generated by s_k.

Definition 4.1.2. *The complex $C(K_I)$ is generated by $y_\sigma := \otimes y_i$ where $\sigma \in K_I$ and*

$$y_i = \begin{cases} s_i & i \in \sigma \\ t_i & i \in I - \sigma \\ 1 & k \notin I \end{cases}$$

The differential is defined by

$$d_I(y_\sigma) = \sum_\tau (-1)^{n(\tau)} y_\tau$$

where $\sigma \subset \tau \in K_I$ and $\tau = \sigma \cup v$ for some vertex $v \in I$. The integer $n(\tau)$ is defined by the usual sign convention of a graded derivation. In particular, the coboundary δ acts on each factor of y_σ by $\delta(s) = 0$ and $\delta(t) = s$, and every time it passes an s a factor of (-1) is introduced.

Then

$$C_K = \bigoplus_{I \subset [n]} C(K_I)$$

and $H^*(Z_K(D^1, S^0)) = H^*(C_K)$.

It follows from [11] that the chain level cup product of two generators is induced by the following.

$$s_i \smile s_i = 0, \qquad t_i \smile t_i = t_i, \qquad s_i \smile t_i = s_i, \qquad t_i \smile s_i = 0$$

We will consider the case of K the boundary of a polygon. By Theorem 4.1.1, we need to find all subsets of $[n]$ to find the cohomology groups. By convention, when I is the empty set, $H^*(\Sigma\mathcal{K}_I) = 1$. The genus of the associated polyhedral product grows exponentially as n grows. The associated real moment-angle complex is a closed orientable surface of genus $g = 1 + (n-4)2^{n-3}$. Therefore, calculating $H^1(Z_K(D^1, S^0))$ means finding the $2g$ subsets of $[n]$ whose associated full subcomplexes are homotopy equivalent to a wedge of 0-spheres. Lastly, the suspension of the whole complex K is a degree two generator. The next step is to understand the product structure of the cohomology ring in terms of the full subcomplexes of the simplicial complex K by studying the cup product at the chain level.

The genus of the associated polyhedral product grows exponentially as n grows. It follows from [15] that the resulting real moment-angle complex is a closed orientable surface of genus $g = 1 + (n-4)2^{n-3}$. Therefore, calculating $H^1(Z_K(D^1, S^0))$ means finding the $2g$ subsets of $[n]$ whose associated full subcomplexes are homotopy equivalent to a wedge of 0-spheres.

4.2 Pentagon

To illustrate, we will consider the case when K is the boundary of the pentagon, in which case the associated real moment-angle complex has genus five. As an exercise it is easy to see how the real moment-angle complex over the square is a torus. The case of the pentagon takes a little more imagination. Below is a depiction of the real moment-angle complex over the boundary of the pentagon. An edge $\{i, j\}$ of the pentagon contributes 2^3 copies of $D^1 \times D^1$. The copies of $D^1 \times D^1$ are labeled in the picture by which edge of the pentagon they come from.

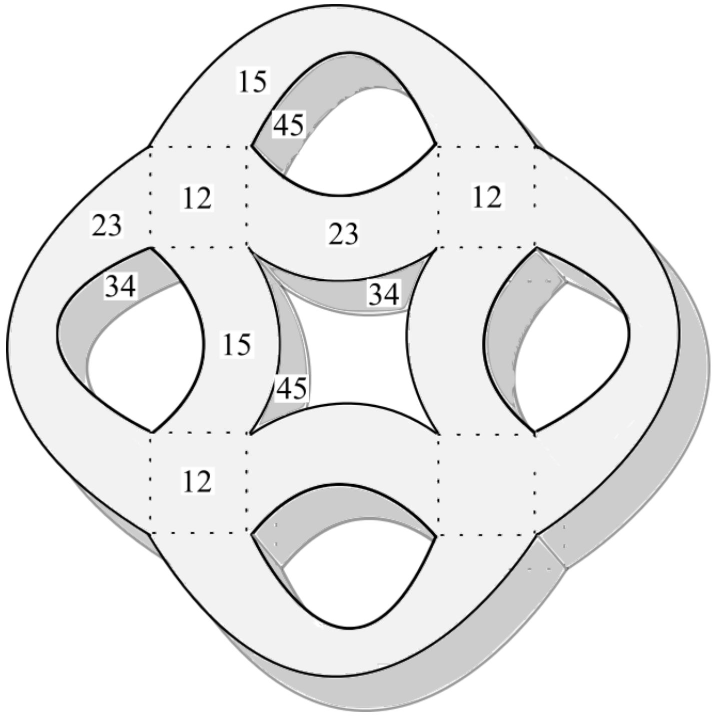

Figure 4.1: Real moment-angle complex over the boundary of a pentagon

For the combinatorial generators, we need to find ten subsets of [5] that yield a full subcomplex equivalent to a wedge of 0-spheres. Let K be the boundary of a pentagon with edges

$$\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}$$

Example 4.2.1. *Claim: The cohomology of $Z_K(D^1, S^0)$ has an identity, ten degree one generators $x_1, \ldots, x_5, w_1, \ldots, w_5$, and a degree two generator z, subject to a graded commutative product. The identity corresponds to the empty set. The generators x_i correspond to the subsets that yield a full subcomplex of K of an edge and the opposite vertex. The generators w_i correspond to the subsets that produce a full subcomplex of two disjoint vertices. Lastly,*

z corresponds to all of K.

$$H^*(Z_K(D^1,S^0)) =< 1, w_1, \ldots, w_5, x_1, \ldots, x_5, z \mid x_i x_j = z\delta_{j,i+1}, x_i w_j = z\delta_{i,j} >$$

where δ is the dirac delta function and the subscripts i, j are integers mod 5.

Take the subset $I = [5]$ of $[5]$. For this subset, the geometric realization of full subcomplex $\mathcal{K}_I$ is the boundary of the pentagon. Its suspension has the homotopy type of S^2. Now, computing the differentials, we can describe $H^2(Z_K(D^1,S^0))$.

$$\begin{aligned} d_I(s_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes t_5) = &\ \delta(s_1) \otimes t_2 \otimes t_3 \otimes t_4 \otimes t_5 \\ &+(-1)^{|s_1|} s_1 \otimes \delta(t_2) \otimes t_3 \otimes t_4 \otimes t_5 \\ &+ \ldots + (-1)^{|s_1|+|t_2|+|t_3|+|t_4|} s_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes \delta(t_5) \\ = &\ -s_1 \otimes s_2 \otimes t_3 \otimes t_4 \otimes t_5 - s_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes s_5 \end{aligned}$$

$$d_I(t_1 \otimes s_2 \otimes t_3 \otimes t_4 \otimes t_5) = s_1 \otimes s_2 \otimes t_3 \otimes t_4 \otimes t_5 - t_1 \otimes s_2 \otimes s_3 \otimes t_4 \otimes t_5$$

$$d_I(t_1 \otimes t_2 \otimes s_3 \otimes t_4 \otimes t_5) = t_1 \otimes s_2 \otimes s_3 \otimes t_4 \otimes t_5 - t_1 \otimes t_2 \otimes s_3 \otimes s_4 \otimes t_5$$

$$d_I(t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes t_5) = t_1 \otimes t_2 \otimes s_3 \otimes s_4 \otimes t_5 - t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes t_5$$

$$d_I(t_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes s_5) = s_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes s_5 + t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes s_5$$

Therefore all of the generators given by the edges are cohomologous, with the exception that the generator associated to the edge $\{1, 5\}$ is equivalent to the negative of the other generators. Denote the second degree generator by z.

For the subsets that yield degree one generators, the geometric realization of the full subcomplex is homotopy equivalent to S^0, and consequently the suspension is equivalent to

S^1. *The* x_i *are given by subsets*

$$I_1 = \{1,2,4\}, I_2 = \{2,3,5\}, I_3 = \{1,3,4\}, I_4 = \{2,4,5\}, I_5 = \{1,3,5\}$$

and the w_i *are from*

$$J_1 = \{3,5\}, J_2 = \{1,4\}, J_3 = \{2,5\}, J_4 = \{1,3\}, J_5 = \{2,4\},$$

Consider the subset $I_1 = \{1,2,4\}$. *We will omit the* $1s$ *as they do not affect the computations, and compute the differentials.*

$d_{I_1}(t_1 \otimes t_2 \otimes t_2) = s_1 \otimes t_2 \otimes t_4 + t_1 \otimes s_2 \otimes t_4 + t_1 \otimes t_2 \otimes s_4$

$d_{I_1}(s_1 \otimes t_2 \otimes t_4) = -s_1 \otimes s_2 \otimes t_4$

$d_{I_1}(t_1 \otimes s_2 \otimes t_4) = s_1 \otimes s_2 \otimes t_4$

$d_{I_1}(t_1 \otimes t_2 \otimes s_4) = 0$

Therefore, the first cohomology group of $\mathcal{K}_{I_1}$ *is*

$$\langle t_1 \otimes s_2 \otimes t_4 + t_1 \otimes s_2 \otimes t_4, t_1 \otimes t_2 \otimes s_4\rangle / \langle s_1 \otimes t_2 \otimes t_4 + t_1 \otimes s_2 \otimes t_4 + t_1 \otimes t_2 \otimes s_4\rangle$$

which means that from the set I_1, *we get a generator* x_1, *represented by* $t_1 \otimes s_2 \otimes t_4 + t_1 \otimes s_2 \otimes t_4$ *or* $t_1 \otimes t_2 \otimes s_4$. *The same happens with the remaining four subsets, where* $H^1(\mathcal{K}_{I_i})$ *is generated by an* x_i. *Similarly,* $H^1(\mathcal{K}_{J_i})$ *is generated by* w_i. *Now we can compute the cup products.*

$$\begin{aligned}
x_1 \smile x_2 &= t_1 \otimes t_2 \otimes s_4 \smile t_2 \otimes t_3 \otimes s_5 \\
&= t_1 \otimes (t_2 \smile t_2) \otimes t_3 \otimes s_4 \otimes s_5 \\
&= t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes s_5 \\
&= z
\end{aligned}$$

and

$$\begin{aligned}
x_2 \smile x_1 &= t_2 \otimes t_3 \otimes s_5 \smile t_1 \otimes t_2 \otimes s_4 \\
&= t_1 \otimes (t_2 \smile t_2) \otimes t_3 \otimes s_5 \otimes s_4 \\
&= t_1 \otimes t_2 \otimes t_3 \otimes (-1)^{1\cdot 1} s_4 \otimes s_5 \\
&= -t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes s_5 \\
&= -z
\end{aligned}$$

Similarly, we get that $x_i \smile x_{i+1} = z$ *and* $x_{i+1} \smile x_i = -z$. *If* $|i-j| \neq 1$, *then* $x_i \smile x_j = 0$ *since their respective sets do not have* $[5]$ *as their union. When the union of the associated sets is not* $[5]$, *the cup product is not even a* 2*-chain because the* 2*-chains are generated by* $s_1 \otimes s_2 \otimes t_3 \otimes t_4 \otimes t_5, \ldots, s_1 \otimes t_2 \otimes t_3 \otimes t_4 \otimes s_5$. *For the same reason we have that* $w_i \smile w_j = 0$ *for any* $i, j \in [5]$.

Lastly, it is clear that

$$\begin{aligned}
x_1 \smile w_1 = {} & t_1 \otimes t_2 \otimes s_4 \smile t_3 \otimes s_5 \\
= {} & t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes s_5 \\
= {} & z
\end{aligned}$$

, $x_i \smile w_i = z$, $w_i \smile x_i = -z$ *and for* $i \neq j$ *that* $x_i \smile w_j = 0$.

This basis for the cohomology ring is described in terms of the simplicial complex K. On the other hand, since $Z_K(D^1, S^0)$ is an orientable surface of genus five, we already have the symplectic basis for the cohomology ring. Recall that the symplectic basis for an orientable surface of genus five is given by an identity, ten degree one generators $\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5$, and a degree two generator γ subject to a graded commutative product:

$$< 1, \alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5, \gamma \mid \alpha_i\beta_j = \gamma\delta_{i,j}, \alpha_i\alpha_j = \beta_i\beta_j = 0 >$$

Next, we will relate the two bases.

Proposition 4.2.2. *Given the presentations discussed previously, we have that*

$$\alpha_i = x_i - w_{i-1} \text{ and } \beta_i = w_i$$

where i is reduced mod 5.

Proof. It is sufficient to check $\alpha_i\beta_j = \gamma\delta_{i,j}, \alpha_i\alpha_j = \beta_i\beta_j = 0$.

First, assume $i \neq j$

$$\begin{aligned} \alpha_i\beta_i &= (x_i - w_{i-1})w_i \\ &= x_iw_i - w_{i-1}w_i \\ &= \gamma - 0 = \gamma \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha_i\beta_j &= (x_i - w_{i-1})w_j \\ &= x_iw_j - w_{i-1}w_j \\ &= 0 - 0 = 0 \end{aligned}$$

Second, note that $\beta_i\beta_j$ is clearly always zero and that

$$\begin{aligned}
\alpha_i\alpha_j &= (x_i - w_{i-1})(x_j - w_{j-1}) \\
&= x_ix_j - w_{i-1}x_j - x_iw_{j-1} + w_{i-1}w_{j-1} \\
&= x_ix_j + x_jw_{i-1} - x_iw_{j-1} - 0 \\
&= \begin{cases}
0 + 0 - 0 = 0 & \text{if } i = j \\
x_ix_{i+1} + 0 - x_iw_i = \gamma - \gamma = 0 & \text{if } j = i+1 \\
x_ix_{i-1} + x_{i-1}w_{i-1} - x_iy_{i-2} = -\gamma + \gamma - 0 = 0 & \text{if } j = i-1 \\
0 + 0 - 0 = 0 & \text{otherwise}
\end{cases}
\end{aligned}$$

□

4.3 N-gon

Let K be an n-gon. Pick an ordering of the vertices of K, in ascending order. Following the notation in Definition 4.1.2, we give a description of the cohomology of a full subcomplex of K.

Lemma 4.3.1. *Suppose K is the boundary of an n-gon. Let $I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_p$ be a subset of $[n]$ such that K_I has exactly p maximal connected components, i.e. $\mathcal{K}_I \simeq \bigvee_1^{p-1} S^0$*

$$H^1(\Sigma\mathcal{K}_I) = \langle \sum_{i\in I_1} y_{\{i\}}, \sum_{i\in I_2} y_{\{i\}}, \ldots, \sum_{i\in I_p} y_{\{i\}} \rangle \Big/ \langle \sum_{i\in I} y_{\{i\}} \rangle$$

Proof. Let $I_1 = \{i_1, \ldots, i_c\}$. If $i_1 = i_c$, then $d(y_{\{i_1\}}) = 0$ and $y_{\{i_1\}}$ is clearly a cocycle. If $i_1 \neq i_c$, then the differential will not be trivial. If $+y_{\{e\}}$ for some edge $e \in K_{I_1}$ appears as a summand in the image of $d(y_{\{v\}})$ for some vertex $v \in I_1$, then $e = \{v-1, v\}$ or $v = n$ and $e = \{1, n\}$ (since $y_{\{e\}}$ was positive, we could not have passed an s). Additionally, since $y_{\{e\}}$ was in the image of $y_{\{v\}}$, it must be the case that $v - 1 \in I_1$, so $-y_{\{e\}}$ is a term in the image of $y_{\{v-1\}}$ under d. If it had been the case that $e = \{1, n\}$, then $\{1\} \in K_I$ and $-y_{\{e\}}$ would be in the image of $y_{\{1\}}$. Since it is only possible for $y_{\{e\}}$ to be in the image of $y_{\{v-1\}}$ or $y_{\{v\}}$, the terms $y_{\{e\}}$ cancel. This means that $d(y_{\{i_1\}} + \ldots + y_{\{i_c\}}) = 0$ since $i_1, \ldots, i_c$ are all the vertices in a connected component $\mathcal{K}_I$. Lastly, $d(\emptyset)$ is the sum of $y_{\{i\}}$ for $i \in I$. □

Note that this means $H^1(\Sigma|\mathcal{K}_I|)$ can be generated by any $p-1$ of the p generators.

Next, the following lemma will show how generators coming from different subsets of $[n]$ multiply. Consider subsets $I, J \subset [n]$ such that $I \cup J = [n]$. Consequently, we will employ a slight change in notation: replacing y's associated to I with a's and y's associated to J with b's to differentiate between generators in $C(K_I)$ and generators in $C(K_J)$. Recall that $a_{\{i\}}$ is the generator associated to the vertex i, whereas a_i is the ith factor of a generator.

Lemma 4.3.2. *Suppose $a_{\{i\}} \in C(K_I)$ and $b_{\{j\}} \in C(K_J)$. Then $a_{\{i\}} = a_1 \otimes a_2 \otimes \ldots \otimes\ a_n$ where*

$$a_k = \begin{cases} s_i & k = i \\ t_k & k \in I \backslash \{i\} \\ 1 & k \notin I \end{cases}$$

Define $b_{\{j\}}$ similarly. Then

$$a_{\{i\}} \smile b_{\{j\}} = \begin{cases} 0 & j \in I \text{ or } |i-j| \neq 1 \\ y_{\{i,j\}} & j = i+1 \\ -y_{\{j,i\}} & j = i-1 \end{cases}$$

Proof. If $|i - j| \neq 1$, then $\{i, j\}$ is not a simplex in K and $a_{\{i\}} \smile b_{\{j\}} = 0$. Therefore, we will now consider cases where $|i-j|=1$.

Recall that $s \smile t = s \smile 1 = s$ and $t \smile s = 0$.

Suppose $j \in I$. Since $j \in I$, $a_j = t_j$. In particular, in the jth spot of $a_{\{i\}} \smile b_{\{j\}}$, we will have $(a_j \smile b_j) = t_j \smile s_j = 0$ so $a_{\{i\}} \smile b_{\{j\}} = 0$

Next suppose $j \notin I$. Then $a_j = 1$ and $a_j \smile b_j = s_j$. If $j = i+1$, then

$$\begin{array}{cccccccccc}
\multicolumn{10}{l}{a_{\{i\}} \smile b_{\{j\}}} \\
= & (a_1 \smile b_1) & \otimes\ldots\otimes & (a_i \smile b_i) & \otimes & (a_j \smile b_j) & \otimes\ldots\otimes & (a_n \smile b_n) \\
= & t_1 & \otimes\ldots\otimes & s_i \smile b_i & \otimes & 1 \smile s_j & \otimes\ldots\otimes & t_n \\
= & t_1 & \otimes\ldots\otimes & s_i & \otimes & s_j & \otimes\ldots\otimes & t_n \\
= & y_{\{i,j\}}
\end{array}$$

If $j = i-1$, and since $a_i \smile b_{i-1} = (-1)^{|b_j||a_i|} b_{i-1} \smile a_i$, we have

$$\begin{array}{cccccccc}
\multicolumn{8}{l}{a_{\{i\}} \smile b_{\{j\}}} \\
= & (a_1 \smile b_1) & \otimes\ldots\otimes & (-1)^{|b_j||a_i|}(a_j \smile b_j) & \otimes & (a_i \smile b_i) & \otimes\ldots\otimes & (a_n \smile b_n) \\
= & t_1 & \otimes\ldots\otimes & (-1)1 \smile s_j & \otimes & s_i \smile b_i & \otimes\ldots\otimes & t_n \\
= & t_1 & \otimes\ldots\otimes & (-1)s_j & \otimes & s_i & \otimes\ldots\otimes & t_n \\
= & -y_{\{j,i\}}
\end{array}$$

$\square$

Theorem 4.3.3. *Let $I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_p$ and $J = J_1 \sqcup J_2 \sqcup \ldots \sqcup J_q$ be subsets of $[n]$ such that $I \cup J = [n]$ and $K_I \simeq \bigvee_1^{p-1} S^0$ and $K_J \simeq \bigvee_1^{q-1} S^0$. Given generators α and β of $H^*(Z_K(D^1, S^0))$ such that one is associated to some I_g for $1 \leq g \leq p$ and the other is associated to some J_h for some $1 \leq h \leq q$. If γ is the second degree generator of $H^*(Z_K(D^1, S^0))$, then $\alpha \smile \beta = \pm\gamma$ if and only if the following conditions are met*

- $I_g \not\subseteq J_h$
- $J_h \not\subseteq I_g$
- $K_{I_g \cup J_h}$ *is contractible*

Proof. We will compute $H^*(\widehat{Z}_K(D^1, S^0))$ using the chain complex described previously.

$$\begin{array}{rcl} d(y_\emptyset) & = & y_{\{1\}} + \ldots + y_{\{n\}} \\ d(y_{\{1\}}) & = & -y_{\{1,2\}} - y_{\{1,n\}} \\ d(y_{\{2\}}) & = & y_{\{1,2\}} - y_{\{2,3\}} \\ d(y_{\{3\}}) & = & y_{\{2,3\}} - y_{\{3,4\}} \\ \ldots & & \\ d(y_{\{n-1\}}) & = & y_{\{n-2,n-1\}} - y_{\{n-1,n\}} \\ d(y_{\{n\}}) & = & y_{\{n-1,n\}} + y_{\{1,n\}} \end{array}$$

Therefore, all the classes in $H^*(\widehat{Z}(K; (D^1, S^0)))$ represented by an edge are cohomologous, except $y_{\{1,n\}}$, which is the negative. Note that if 1 and n are in I, then 1 and n are in I_i for some $1 \leq i \leq p$ (since $\mathcal{K}_I \simeq \bigvee_1^{p-1} S^0$, it cannot be that 1 and n are in different subsets of I). This and Lemma 4.3.1 imply that we may only consider when $1, n \notin I_1 \cup J_1$ so that when we are computing $\alpha \smile \beta$ we do not encounter $y_{\{1,n\}}$.

Let $I_g = \{i_1, \ldots, i_c\}$ and $J_h = \{j_1, \ldots, j_d\}$. By Lemma 4.3.1, we have that

$$\alpha = \sum_{i \in I_g} a_{\{i\}}$$

Since $I \cup J = [n]$, we have that $i_1 \neq j_1$ and $i_c \neq j_d$.

First, suppose $I_g \cap J_h = \emptyset$. In the case that $i_c < j_1$, we must have $i_c = j_1 - 1$ so that there is at least one edge after expanding the product. Then there is only one nonzero term

$$\alpha \smile \beta = \sum_{i \in I_g, j \in J_h} a_{\{i\}} \smile b_{\{j\}} = y_{\{i_c, j_1\}}$$

since for any other i, j, $|i - j| \neq 1$. Similarly, if $j_d = i_1 - 1$, then

$$\alpha \smile \beta = \sum_{i \in I_g, j \in J_h} a_{\{i\}} \smile b_{\{j\}} = a_{\{i_1\}} \smile b_{\{j_d\}} = -y_{\{i-1, i_1\}}$$

Secondly, suppose $I_g \cap J_h \neq \emptyset$ and that neither set is contained in the other. If $j_1 \leq i_c$, then there exists $j \in J_h$ such that $j = i_c$. Note that $j + 1 \in J_h$ (because $i_c \neq j_d$ and so $j \neq j_d$). Since $\{i_c, j+1\}$ is an edge and $j + 1 = i_c + 1 \notin I$, by lemma 4.3.1 we have only one nonzero term

$$\alpha \smile \beta = a_{\{i_c\}} \smile b_{\{j+1\}} = y_{\{i_c, i_c+1\}}$$

Similarly, if $i_1 \leq j_d$ and $j = i_1$ for some $j \in J_1$, then $j - 1 \notin I$ and $i_1 - (j - 1) = 1$.

$$\alpha \smile \beta = a_{\{i_1\}} \smile b_{\{j-1\}} = -y_{\{i_1-1, i_1\}}$$

If $J_h \subset I_g$, then $\alpha \smile \beta = 0$ by Lemma 4.3.1. If $I_g \subset J_h$, then there exists $j \in J_1$ such that $j = i_1$.

$$\begin{aligned} \alpha \smile \beta &= a_{\{i_1\}} b_{\{j-1\}} + a_{\{i_c\}} b_{\{j+c\}} \\ &= -y_{\{j-1, i_1\}} + y_{\{i_c, i_c+1\}} \\ &= -\gamma + \gamma \\ &= 0 \end{aligned}$$

□

4.3.1 Hexagon

With the ring structure for the real moment-angle complex over the boundary of the hexagon, we illustrate the calculation in the previous section. Since the formula for the genus of $Z_K(D^1, S^0)$ is $1+(n-4)2^{n-3}$, the genus is growing exponentially, and thus the situation gets more complicated pretty quickly. If K is the boundary of a hexagon, then $Z_K(D^1, S^0)$ is a surface of genus seventeen. This means there are 34 generators of degree one. Therefore, we will not explicitly find all generators, and will instead do some examples.

First, consider $I = \{1, 2, 3, 5\}$ and $J = \{2, 3, 4, 6\}$, and compute their corresponding cohomology classes.

$d(a_{\{1\}}) = d(s_1 \otimes t_2 \otimes t_3 \otimes t_5) = -s_1 \otimes s_2 \otimes t_3 \otimes t_5$

$d(a_{\{2\}}) = d(t_1 \otimes s_2 \otimes t_3 \otimes t_5) = s_1 \otimes s_2 \otimes t_3 \otimes t_5 - t_1 \otimes s_2 \otimes s_3 \otimes t_5$

$d(a_{\{3\}}) = d(t_1 \otimes t_2 \otimes t_3 \otimes s_5) = t_1 \otimes s_2 \otimes s_3 \otimes t_5$

$d(a_{\{5\}}) = 0$

It follows that $H^1(\Sigma\mathcal{K}_I) = \langle a_{\{1\}} + a_{\{2\}} + a_{\{3\}}, a_{\{5\}}\rangle / \langle a_{\{1\}} + a_{\{2\}} + a_{\{3\}} + a_{\{5\}}\rangle$

$d(b_{\{2\}}) = d(s_2 \otimes t_3 \otimes t_4 \otimes t_6) = -s_2 \otimes s_3 \otimes t_4 \otimes t_6$

$d(b_{\{3\}}) = d(t_2 \otimes s_3 \otimes t_4 \otimes t_6) = s_2 \otimes s_3 \otimes t_4 \otimes t_6 - t_2 \otimes s_3 \otimes s_4 \otimes t_6$

$d(b_{\{4\}}) = d(t_2 \otimes t_3 \otimes s_4 \otimes t_6) = t_2 \otimes s_3 \otimes s_4 \otimes t_6$

$d(b_{\{6\}}) = 0$

Then we have that $H^1(\Sigma\mathcal{K}_J) = \langle b_{\{2\}} + b_{\{3\}} + b_{\{4\}}, b_{\{6\}}\rangle / \langle b_{\{2\}} + b_{\{3\}} + b_{\{4\}} + b_{\{6\}}\rangle$. Let α be $a_{\{5\}}$ and β be $b_{\{2\}} + b_{\{3\}} + b_{\{4\}}$. According to theorem 4.3.3, $\alpha \smile \beta = \pm\gamma$ since $\{2, 3, 4\} \nsubseteq \{5\}$, $\{5\} \nsubseteq \{2, 3, 4\}$ and $\mathcal{K}_{\{2,3,4,5\}}$ is contractible. At the chain level,

$$\begin{aligned}
\alpha \smile \beta &= a_{\{5\}} \otimes b_{\{2\}} + a_{\{5\}} \otimes b_{\{3\}} + a_{\{5\}} \otimes b_{\{4\}} \\
&= -t_1 \otimes s_2 \otimes t_3 \otimes t_4 \otimes s_5 \otimes t_6 + \\
&\quad -t_1 \otimes t_2 \otimes s_3 \otimes t_4 \otimes s_5 \otimes t_6 + \\
&\quad -t_1 \otimes t_2 \otimes t_3 \otimes s_4 \otimes s_5 \otimes t_6 \\
&= -y_{\{4,5\}}
\end{aligned}$$

Of course, the product will be cohomologous if we let $\alpha = a_{\{1\}}+a_{\{2\}}+a_{\{3\}}$. Nevertheless, it is still helpful to see what happens when the underlying sets intersect in this way. According to theorem 4.3.3, we will get the same answer because $\{2,3,4\} \nsubseteq \{1,2,3\}$, $\{1,2,3\} \nsubseteq \{2,3,4\}$ and $\mathcal{K}_{\{1,2,3,4\}}$ is contractible.

$$\begin{aligned}
\alpha \smile \beta &= a_{\{1\}} \otimes b_{\{2\}} + a_{\{1\}} \otimes b_{\{3\}} + a_{\{1\}} \otimes b_{\{4\}} + \\
&\quad a_{\{2\}} \otimes b_{\{2\}} + a_{\{2\}} \otimes b_{\{3\}} + a_{\{2\}} \otimes b_{\{4\}} + \\
&\quad a_{\{3\}} \otimes b_{\{2\}} + a_{\{3\}} \otimes b_{\{3\}} + a_{\{3\}} \otimes b_{\{4\}} \\
&= a_{\{1\}} \otimes b_{\{2\}} + a_{\{2\}} \otimes b_{\{3\}} + a_{\{3\}} \otimes b_{\{2\}} + a_{\{3\}} \otimes b_{\{4\}} \\
&= (s_1 \smile 1) \otimes (t_2 \smile s_2) \otimes t_3 \otimes t_4 \otimes t_5 \otimes t_6 + \\
&\quad t_1 \otimes (s_2 \smile t_2) \otimes (t_3 \smile s_3) \otimes t_4 \otimes t_5 \otimes t_6 + \\
&\quad (-1) t_1 \otimes (t_2 \smile s_2) \otimes (s_3 \smile t_3) \otimes t_4 \otimes t_5 \otimes t_6 + \\
&\quad t_1 \otimes t_2 \otimes (s_3 \smile t_3) \otimes (1 \smile s_4) \otimes t_5 \otimes t_6 \\
&= 0 + 0 + 0 + t_1 \otimes t_2 \otimes (s_3 \smile t_3) \otimes (1 \smile s_4) \otimes t_5 \otimes t_6 \\
&= y_{\{3,4\}}
\end{aligned}$$

Now let I and α be the same, $J = \{2,4,6\}$ and $\beta = b_{\{2\}}$. Since $\{2\} \subset \{2,4,6\}$, theorem

4.3.3 tells us that the product will be trivial. In particular,

$$\begin{aligned}
\alpha \smile \beta &= a_{\{1\}} \otimes b_{\{2\}} + a_{\{2\}} \otimes b_{\{2\}} + a_{\{3\}} \otimes b_{\{2\}} \\
&= s_1 \otimes (t_2 \smile s_2) \otimes t_3 \otimes t_4 \otimes t_5 \otimes t_6 + \\
&\quad t_1 \otimes (s_2 \smile s_2) \otimes t_3 \otimes t_4 \otimes t_5 \otimes t_6 + \\
&\quad -t_1 \otimes (t_2 \smile s_2) \otimes s_3 \otimes t_4 \otimes t_5 \otimes t_6 \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

Recall that the real moment-angle complex plays a large roll in describing the ring structure of the cohomology of a polyhedral product. In particular, with Theorem 4.3.3, the ring structure of the cohomology of a polyhedral product of pairs $(\underline{CA}, \underline{A})$ over a n-gon can be completely described.

4.4 Future Research

It would be interesting to compute the ring structure for other families of simplicial complexes. However, the preliminary examples do not provide interesting multiplicative structure. Those examples include the 1-skeleton and boundary of the 3-simplex, as well as the boundary of a triangulated 3-dimensional polytope. The case where K is a join of two other simplicial complexes is not an interesting example either. It remains to find a family of simplicial complexes whose real moment-angle complex has cohomology with a more complex multiplicative structure

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