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New Classical Solutions in Supergravity

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by

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Abstract

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In this Ph.D. thesis we construct three classes of new solutions to supergravity theories in various dimensions and study their properties. The first class is reduction ansatz of 10D and 11D supergravity on Ricci-flat and noncompact manifolds. These reductions are from a scaling limit of the famous spherical reductions, and can be solely supported by warp factors. The second class contains a large number of String/M theory solutions that have Lifshitz or Schrödinger scaling symmetry, obtained from marginally deforming the geometry of internal dimensions of previous solutions. We propose that these new solutions are dual to marginal deformations of certain holographic non-relativistic field theories. The last class are singly spinning non-spherical black holes named black rings in 5D $U(1)^3$ supergravity with three dipole charges and three electric charges, which lie in the classification of

5D nonsupersymmetric black holes. We analyze their thermodynamic and global properties. As a byproduct, we embed the three dipole black ring into spacetime that contains background magnetic fields.

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The bulk of this thesis contains three of my papers [17, 70, 111], which were previously published on Physical Review D and Journal of High Energy Physics.

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Chapter 1

Introduction

1.1 General Relativity

It has been an entire century since Einstein published the theory of General Relativity (GR) [1]. As a successor theory to Special Relativity (SR), which only concerns inertial reference frames, GR extends the scope of SR and considers covariant physics in generic reference frames, especially non-inertial reference frames. This is achieved by incorporating the *Equivalence Principle* which views acceleration in non-inertial frames equivalent as gravitational forces. Therefore GR is essentially a covariant theory of gravity. It further revolutionizes our view of gravity by relating the gravitational field to the metric of spacetime which is a pseudo-Riemannian manifold. In 4-dimensions, if only the gravity field is considered then the Lagrangian density is the simplest covariant function of the metric: the scalar curvature, and the action is

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} R, \tag{1.1}$$

where G_4 is the 4-dimensional Newton constant. Many other terms can be added covariantly for different scenarios such as a cosmological constant, electromagnetic field, scalar fields and so on.

Since its birth, GR has passed many nontrivial experimental tests including the newly announced direct observation of gravitational waves [2] and is well accepted as the modern theory of gravity. However, fundamental problems also manifest themselves along with the successes. A lot of solutions found in GR suffer from severe pathologies, such as singularities. Among them the most crucial ones are those at which some curvature invariant becomes infinite, and thus can not be resolved by any classical techniques. For instance, this happens inside of the horizon of a black hole. And it was proved that at general circumstances these singularities are unavoidable [3]. The existence and definiteness of these singularities demonstrates that GR as a theory of gravity is incomplete. Nevertheless this is compatible with the fact that GR is a classical theory. Modern physics has shown that once we focus on physics of small regions the quantum effects will not be negligible and a full quantum theory has to be used. In the case of singularities, they happen precisely when the gravity field and matter are compressed into a small region and involves an extremely small length scale. In this sense GR functions as well as Maxwell theory in which Coulomb's law is a solution (in Coulomb's gauge); it has a singularity at $r = 0$ and this is resolved by the quantum theory i.e. Quantum Electrodynamics (QED).

It is reasonable to expect that a quantum theory of gravity would resolve these singularities in GR, since they appear as extremely short scale (the Planck scale) phenomena. However, direct attempts to quantize GR as a field theory ran into troubles since by power counting GR is nonrenormalizable[1]. Being non-renormalizable does not mean that a theory is not quantizable. It means that the quantum theory gives the right description only below a momentum cutoff. Once we want to go beyond this cutoff, a new theory will replace the old one in the Wilsonian sense, e.g. such as the $SU(2) \times U(1)$ gauge theory replaces Fermi theory at Λ_{EW} (10^2 GeV). Therefore we need a theory that is UV complete to replace GR at Λ_{Planck} (10^{19} GeV).

[1]Of course one has to go beyond power counting to demonstrate that a theory is non-renormalizable, because miraculous cancellation might happen. In fact GR is renormalizable at and only at 1-loop [4]. The maximally supersymmetric cousin of GR in 4d namely the $\mathcal{N} = 8$ supergravity has been shown renormalizable up to 7-loop and conjectured to be perturbatively finite (see i.e. [5]), though it is also believed that the finiteness is broken by nonperturbative properties ([6]).

1.2 String Theory and Supergravity

Our current best candidate of a quantum theory of gravity is String Theory (see for example [7]). Initiated in the 70's, String Theory was first considered as a theory of the strong nuclear force since it is intuitive for phenomena such as Reggie trajectories. It was soon realized that String Theory contains massless spin-2 modes that describes gravitons. For this reason, a quantized string theory is necessarily a quantum gravity theory and therefore should be a theory of Planck scale (10^{19} GeV) rather than QCD scale (10^2 MeV) (although soon we will see briefly that String Theory does have abundant applications in QCD physics via certain dualities). A peculiar yet stunning property of string theory is that quantization requires a special dimensionality for a consistent quantized theory. To extend the world-sheet Weyl invariance to the quantum theory, string theory has to be either 26 dimensional (in the bosonic case) or 10 dimensional (in the supersymmetric case). Combining the tachyon free condition, 10 dimensional Superstring Theory is the most favorable model. In $d = 10$ there are 5 different string theories classified by orientability, spacetime/world-sheet chirality and gauge groups: type I, type IIA, type IIB, heterotic string theories $SO(32)$ and $E_8 \times E_8$. In the low energy or small string coupling limit these theories all reduce to certain Supergravity (SUGRA) theories [9].

SUGRA is a fascinating theoretical discovery from combining supersymmetry and general relativity. In GR one can couple both fermionic and bosonic field contents to gravity. In 1978 it was shown that it is possible to construct theories in which there are equal degrees of freedom for both bosonic and fermionic fields and they represent a local supersymmetry algebra. Taking the low energy limit, the spacetime action of each of the string theories would reduce to that of 10D SUGRA. For example, if one turns off all RR flux in IIA and IIB theories, both of their low energy bosonic Lagrangian are given by[2]:

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{3} H^2 \right), \tag{1.2}$$

which is 10D GR couples to a scalar and a 3-form B-field in the so-called string frame, with G_{10} being the 10d Newton constant. Note the fluxes and all fermionic fields are truncated here. When higher string coupling corrections are considered, there will be higher curvature terms such as R^2, $R_{\mu\nu}R^{\mu\nu}$ added to (1.2), along with other terms. That is when the low energy effective SUGRA is not valid any more.

[2]It is worth pointing out that in SUGRA when one only works on the classical solutions one only considers the bosonic part, which is no longer supersymmetric.

1.3 Classical solutions in GR and SUGRA

A great deal of effort has been made towards a better understanding of gravity by studying possible spacetime geometry as the classical solution that satisfies the equations of motion of the theory. There are several types of solutions that are of particular significance.

First, because GR and SUGRA are highly nonlinear theories, constructing soliton type solutions sheds light on the possible "state" of the theory. The most important ones among them are black holes. Some famous examples of this category are Schwarzschild and Kerr black holes, which describe an extreme condensation of matters and are one possible final stage of massive stars after supernovae, or intermediate stage of high energy particle collisions. A four dimensional Schwarzschild black hole in Schwarzschild coordinate is given by

$$ds^2 = (1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2, \tag{1.3}$$

where M is the parameter that is associated with the mass of this solution to an asymptotic observer. At $r = 0$ the solution has a curvature singularity, as we discussed previously. Many black hole type solutions in GR have been generalized to certain SUGRA theories. When classical solutions are concerned, one only considers the bosonic part of SUGRA, which besides the graviton might also contain scalars and gauge fields. Therefore one often says one is adding "charges" to black holes when it is generalized from GR to SUGRA. Beside adding charges, another generalization is to construct higher-dimensional black holes. Indeed black holes are generic in various gravity theories in many dimensions, and in higher dimensions there are also new classes of solutions, especially in string theory related SUGRA. One of them is the extended black objects, black p-branes, which are black hole solutions with non-compact event horzion [8]. They are especially important in connection with D-branes and the discovery of AdS/CFT which will be reviewed below. Another important class of higher-dimensional black hole is 5D black rings, which are black holes that have $S^1 \times S^2$ event horizons. Their existence demonstrates that several critical classification theorems for 4D black holes are not valid in 5D, e.g. the no-hair theorem. Later on black rings in string compacted SUGRA were identified as low energy excitations of the so called supertubes, which are extended objects in string theory. Therefore the study of black hole solutions in various dimensionality is of significance for understanding gravity in general.

In addition to solitonic soltuions, GR also contains time-dependent solutions that are employed in cosmology. One example is the renowned Friedmann-Lematre-Robertson-Walker (FLRW) metric, that describes a homogeneous, isotropically expanding/contracting universe. The third scope is the instanton type solutions which are influential in quantum gravity attempts and differential geometry. Many solutions of this kind including the famous Taub-NUT/bolt, Eguchi-Hanson and Euclidean black holes have been investigated by both physicists and mathematicians [10].

1.4 AdS/CFT correspondence

Starting from its formulation, string theory has manifested an amazing set of deep connections to quantum field theories. Some sort of "duality" between string theory and field theory had been speculated in the 80's and a major breakthrough happened in the 90's from the study of D-branes. D-branes are solitonic solutions of string theory. On the one hand open strings on D-branes are described by a certain gauge theory; on the other hand it accounts for classical black hole type solutions of SUGRA called black p-branes.

Eventually D-branes led to the discovery of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. AdS/CFT is a conjecture that certain conformal field theories are dual to some superstring theory compactified to Anti de Sitter space [11, 12, 13]. Though lacking a rigorous proof, there has been numerous works supporting this conjecture. One of the most interesting and favorable aspects of the correspondence is that when the coupling constant of the field theory is strong, the string coupling is weak and the string theory can be well approximated by its corresponding supergravity theory. Under this circumstance the gravity theory is a classical theory and by studying fields propagating in the background one can extract many analytic results of the strongly coupled field theory. One of the strongest evidences of AdS/CFT is the equivalence between the isometries of the geometry and the symmetries of the field theory. The isometry of the noncompact spacetime corresponds to the spacetime symmetry of the field theory. In most of the cases, the noncompact spacetime is a $d+1$-dimensional AdS space [3] whose isometry group is $SO(2,d)$ and is equivalent to the d-dimensional conformal group. On the other hand, the isometry of the "compact" manifold [4] gives rise to the global symmetry of the field theory.

The first example and the best studied case is the duality between IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ super Yang-Mills theory in $d=4$ with the gauge group $U(N_c)$. The (bosonic) symmetries of the field theory are encoded in the classical background as the following

$$\begin{aligned} AdS_5 \leftarrow SO(2,4) \rightarrow CFT_4, \\ S^5 \leftarrow SO(6) \cong SU(4) \rightarrow \mathrm{R}\,\text{symmetry}. \end{aligned} \tag{1.4}$$

[3]Alternatively one can have the AdS deformed or replaced.

[4]There has been various examples in which the subspace orthogonal to the AdS space is not compact.

The correspondence goes beyond the matching of the symmetries. It includes using the classical supergravity action to evaluate the partition function of the field theory, deriving correlation functions from AdS propagators, relating the expectation value of Wilson loops to areas of certain minimal surfaces in AdS and etc (for a review see [16]). Another line of generalization of the example discussed above is to consider other geometries and dimensions, and many solutions have been found as gravity duals of interesting field theories. See table 1.1 for some other examples.

String theory compactification	Field theory
Type IIB on $AdS_5 \times S^5$	$U(N)$ $\mathcal{N} = 4$ SYM
Type IIB on $AdS_5 \times S^5/Z_k$	$U(N)^k$ $\mathcal{N} = 2$ SYM + bifundamental matter
Type IIB on $AdS_5 \times L^{a,b,c}$	$U(N)^n$ $\mathcal{N} = 1$ quiver gauge theory
M theory on $AdS_4 \times S^7/Z_k$	ABJM theory

Table 1.1: Some prototypical examples of the AdS/CFT correspondence [11, 14, 15]. Left column: certain string theoriy compactifications. Right column: the dual field theories.

1.5 Summary

Inspired by the fascinating developments in modern gravity theories, we construct new classical solutions of GR and SUGRA[5]. By using various solution generating techniques we will construct three new classes of solutions of GR

[5]In this thesis we will consider SUGRA as low energy limits of string theory unless noticed otherwise, and therefore will often use the term "string theory" and "SUGRA" interchangeably.

and SUGRA. In the first part, we consider a reduction ansatz of 10D and 11D pure gravity on noncompact Ricci flat manifolds. No fluxes are turned on and the reduction is solely supported by warp factors. We found there are three classes of solutions and they are in one-to-one correspondence to the famous sphere reductions of 10D and 11D SUGRA. This is in a sense that they come as a scaling limit in which the spheres are blown up. Using these reduction ansatz we obtain several lower dimensional dS and AdS solutions, with the dS_3 solution being a new and regular solution.

In the second part, we use the T-duality-shift-T-duality (TsT) method to construct a large number of solutions that have either Lifshitz or Schrödinger scaling symmetry. They can be viewed as non-relativistic generalization of Lunin-Maldacena solutions [75]. Incorporating with the AdS/Condensed Matter Theory proposal, we conjecture that these solutions are marginal deformations of some field theories. In the last part we build 5-dimensional black rings in $U(1)^3$ SUGRA theory that carry 3-dipole charges, 3-electric charges and 1 angular momentum. We analyze their global and thermodynamical properties. As a byproduct, we also embed the 3-dipole solution into generalized Melvin's universe.

Chapter 2

Ricci-flat reductions and Holography

2.1 Introduction

Studying gravity in various dimensions has had many different motivations, at the classical and quantum levels, dating back to the 1920's and the seminal works of Kaluza and Klein[1]. String theory, of course, prefers ten or eleven-dimensional supergravity theories and related compactifications or reductions to lower dimensions. Except for rather limited though important cases in lower dimensions, such as supersymmetric solutions to ungauged supergravity [18, 19], we are not even close to classifying all solutions to a given (super)gravity theory. In the absence of supersymmetry, a specific class of simpler solutions that is often studied in gravity theories involves "vacuum" solutions, i. e. solutions to Einstein equations with vanishing

[1]This section is based on [17].

energy-momentum tensor, or alternatively, Ricci-flat geometries.

AdS/CFT motivations have directed a lot of activity in gravity solution construction towards finding and classifying solutions that involve an AdS factor and an internal compact space. Such solutions are almost always not vacuum solutions and involve various form-field fluxes present in supergravity theories. Moreover, for AdS/CFT purposes and also for stability requirements, as well as having (quantum) corrections under control, it is often demanded that such solutions preserve a fraction of the global supersymmetry of the theory. This tendency has led to Ricci-flat solutions with AdS factors being largely overlooked.

Separately, solutions with a de Sitter factor have also been of particular interest within higher-dimensional supergravity and string theory setups, as the observed universe we are living in seems to be an asymptotically de Sitter space. Nonetheless, it has proven to be notoriously difficult to construct four-dimensional de Sitter solutions in a string theory setting which are classically and quantum mechanically stable and do not have a moduli problem [20, 21]. A leading framework [22] and its uplifting procedure to produce dS vacua has recently been called into question [23], leading to a renewed interest in alternatives [24]. In this chapter, we consider simple higher-dimensional gravity with no local sources or non-perturbative contributions and simply solve the equations of motion. As a result, the de Sitter vacua we find are some of the simplest in the literature and, while we forfeit supersymmetry from the onset, we still retain some control through a scaling limit.

Although our AdS solutions are singular - albeit in a “good” sense [25] - it is a striking feature of our de Sitter constructions that they are completely smooth. This should be contrasted with recent studies of persistent singularities in non-compact geometries where anti-branes are used to uplift AdS vacua [26] (see also [27]). Here, since we are considering vacuum solutions, we have no branes, and thus, no singularities. Any branes that do exist only appear when we turn on fluxes in the lower-dimensional theories to construct dS_3 vacua. Regardless of whether fluxes are turned on or not, our construction evades the well-known “no-go” theorem [81] on the basis that the internal space is non-compact.

The scaling limit we employ may be traced to “near-horizon” limits of Extremal Vanishing Horizon (EVH) black holes [29]. Within an AdS/CFT context, such warped-product solutions have been studied previously in [31, 30] (more recently [32, 33]), where consistent KK reductions to lower-dimensional theories were constructed. In contrast to usual AdS/CFT setups, the vacua of the lower-dimensional theories are supported exclusively through the warp factor and lift to vacuum solutions in ten and eleven dimensions. We show that if one neglects the possibility of a large number of internal dimensions, there are just three example in this class. Furthermore, we demonstrate that the solutions can be easily generalized to de Sitter and that consistent KK reductions exist as scaling limits of well-known sphere reductions, for example [34].

We begin in the next section by considering a general D-dimensional warped-product spacetime Ansatz on the assumption that the internal space is Ricci-flat. While it is most natural to consider $\mathbb{R}^q$, the same analysis holds for Calabi-Yau cones and we expect a variant to hold for more generic Calabi-Yau. Locally, this construction encompasses cases for which the internal space is an Einstein space such as a sphere, which is conformally flat and the conformal factor is automatically included by way of the warp factor. We identify a class of vacuum solutions where the internal Ricci scalar can be made to vanish by tuning the dimensions. Somewhat surprisingly, this leads to three isolated examples, which only reside in ten or eleven dimensions, supporting (A)dS$_p$ vacua with $p = 5, 6, 8$. We remark that supersymmetry is broken.

In the following section, we focus on the warped (A)dS$_5$ solution to eleven-dimensional supergravity, where the internal space is $\mathbb{R}^6$. We explicitly construct a KK reduction Ansatz and note that the lower-dimensional theory one gets is five-dimensional $\mathrm{U}(1)^3$ gauged supergravity [34] on the nose. Importantly, this observation guarantees stability within our truncation, though of course instabilities can arise from modes that are truncated out [35]. In the absence of warping, the (A)dS$_5$ vacuum becomes a Minkowski vacuum and one can compactify the internal space to get the usual KK reduction to *ungauged* supergravity on a Calabi-Yau manifold. We show that the KK reduction naturally arises as a scaling limit of the KK reduction of eleven-dimensional supergravity on S^7, further truncated to the Cartan $\mathrm{U}(1)^4$ [34]. Using a similar scaling argument, we also exhibit KK reductions from ten and eleven-dimensional supergravity on $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively.

Since none of our solutions are manifestly supersymmetric, the final part of this chapter concerns (classical) stability. In constructing the KK reductions, we have assumed a product structure for the internal space and it is a well-known fact that reductions based on product spaces are prone to instabilities where the volume of one subspace increases whilst another decreases [36, 37, 38]. Neglecting all solutions to the five-dimensional theory, which are guaranteed to be stable within the truncation, we find that the AdS$_6$ and AdS$_8$ vacua are unstable. By constructing lower-dimensional Freund-Rubin type solutions within our AdS$_p$, $p = 5, 6, 8$ truncations, we show that AdS$_3$ solutions are stable. On physical grounds, since these arise as the near-horizon of EVH black holes [31, 30], they are expected to be classically stable.

In the final part of this chapter, we construct an example of a dS_3 vacuum and study its stability. We show that the vacuum energy of the de Sitter solution can be tuned so as to stabilise the vacuum against tunneling. This guarantees that the vacuum would be suitably long lived and, along with other dS_3 solutions to string theory [39]. We close the chapter with some discussion of related open directions.

2.2 Ricci-flat solutions

In this section we identify a class of Ricci-flat solutions in general dimension $D = p + q$. From the offset, we assume that the overall spacetime takes the form of a warped product,

$$\mathrm{d}s^2_{p+q} = \Delta^m \ \mathrm{d}s^2(M_p) + \Delta^n \ \mathrm{d}s^2(\Sigma_q), \tag{2.1}$$

decomposed into a p-dimensional external spacetime M_p and a q-dimensional Ricci-flat internal space Σ_q. m, n denote constant exponents and the warp factor Δ only depends on the coordinates of the internal space.

Denoting external coordinates, $a, b = 0, \ldots, p-1$ and internal coordinates $m, n = 1, \ldots q$, the vanishing of the internal Ricci scalar, i. e. $g^{mn} R_{mn}$, yields the equation:

$$\begin{aligned}[\frac{mp}{2} &+ n(q-1)]\nabla^2\Delta = \Big[n(q-1) - \frac{mp}{2}(m-1) \\ &- (q-2)[\frac{n^2}{4}(q-1) + \frac{mnp}{4}]\Big]\Delta^{-1}(\partial\Delta)^2. \end{aligned} \tag{2.2}$$

If the internal space is $\mathbb{R}^q$ and r denotes its radial coordinate, then simple solutions to (2.2) are given by

$$\Delta = \begin{cases} (c_1 + c_2 r^{2-q})^{\frac{1}{(1+\kappa)}}, & q \neq 2, \\ (c_1 + c_2 \log r)^{\frac{1}{(1+\kappa)}}, & q = 2, \end{cases} \tag{2.3}$$

where κ is a constant that depends on m, n, p, q and c_i denote integration constants. Geometries with these Δ are singular at the origin $r = 0$.

One may hope to find non-singular solutions to (2.2), by forcing the bracketed terms to vanish:

$$m = 2 - \frac{4}{q}, \quad n = -\frac{4}{q}, \quad p = \frac{4(q-1)}{q-2}. \tag{2.4}$$

This condition defines our class of Ricci-flat solutions. Note that $q = 2$ is not a legitimate choice and if one demands integer dimensions, we are hence led to only the following choices for (p, q): $(5, 6)$, $(6, 4)$ and $(8, 3)$. Besides these choices, one can also formally consider large D, $q \to \infty$ limit where one encounters a four-dimensional vacuum ($p = 4$).

It is a curious property of this class of solutions that they only exist in ten and eleven (and also infinite) dimensions, settings where we have low-energy effective descriptions for string theory. From the onset, there is nothing outwardly special about our Ansatz and one would assume that examples could be found for general D, yet we find that this is not the case.

To specify the overall spacetime, we simply now have to record the warp factor. Again, evoking the existence of a radial direction, we can write Δ as

$$\Delta = \sqrt{1 + \lambda r^2}, \tag{2.5}$$

where we have normalized the integration constants, one of which, $\lambda = -1$ or $+1$, dictates whether the vacuum is anti-de Sitter or de Sitter spacetime, respectively. For $\lambda = 0$, the warp factor becomes trivial and the solution reduces to D-dimensional Minkowski spacetime. The radius, ℓ, of M_p is expressed as

$$\ell^2 = \frac{1}{|\lambda|} \frac{(p-1)}{(q-2)}. \tag{2.6}$$

The other D-dimensional vacuum Einstein equations yield the following equation for Δ:

$$\Delta \nabla^2 \Delta + (\partial \Delta)^2 = q\lambda, \tag{2.7}$$

and, as a result, it is easy to infer through

$$\int_{\Sigma_q} \left(\Delta \nabla^2 \Delta + (\partial \Delta)^2 - q\lambda \right) = -q\lambda \mathrm{vol}(X) = 0, \tag{2.8}$$

that a compact internal space Σ_q requires a Minkowski vacuum, $\lambda = 0$. Therefore, all our (A)dS spacetimes, will have non-compact internal spaces.

Despite the existence of a covariantly constant spinor (which is a result of Ricci-flatness), it is easy to see that none of these geometries are supersymmetric, except when $\lambda = 0$. As a cross-check, we note that where complete classifications of supersymmetric solutions exist, for example [40], one can confirm that our solutions are not among them.

As an extension to the $\Sigma_q = \mathbb{R}^q$ internal space, for $q = 4$ and 6, Σ_q can easily be chosen to be a Calabi-Yau cone over a Sasaki-Einstein space, or more generally by a cone over an Einstein space. However, such cones have a conical singularity at their apex.

We also remark that for $\lambda = -1$, we encounter a curvature singularity at $r = 1$, where the warp factor vanishes. This does not affect the Ricci scalar, since the above solutions are Ricci-flat, but it does show up in contractions of the Riemann tensor $R_{MNPQ}R^{MNPQ}$. This singularity can be seen to be of "good" type [25], a point that was made recently in [33] and we will return to in due course. In contrast, the Minkowski and de Sitter solutions are smooth.

2.3 KK Reduction on the solutions

Taking each of the warped-product solutions identified in the previous section in turn, one can construct simple consistent Kaluza-Klein reductions to the lower-dimensional theory. By "simple", in contrast to traditional reductions, we mean that there is a clear division between scalars in the metric and gauge fields in the fluxes of the higher-dimensional supergravity. This means that even with scalars the overall spacetime is Ricci-flat[2], whereas the inclusion of gauge fields leads to a back-reaction externally, with the internal space remaining Ricci-flat. In this section we discuss the KK reduction of each of the three solutions independently.

2.3.1 KK Reduction on the $\mathbb{R}^6$

We start with a reduction from eleven-dimensional supergravity to five dimensions on $\mathbb{R}^6$. The main idea for such a reduction was presented in [30]. Our warped Ansatz naturally generalises the known reduction to a Minkowski vacuum on Calabi-Yau, for example [120]. Since the lower-dimensional theory in that case is *ungauged* supergravity, here we present a simple Ansatz that recovers the bosonic sector of $D = 5$ $\mathrm{U}(1)^3$ gauged supergravity [34], and via a flip in a sign (appropriate double Wick rotation and analytic continuation), the AdS_5 vacuum becomes dS_5. We recall the five-dimensional

[2]Superficially, this bears some similarity to the AdS/Ricci-flat correspondence [41] in that there is a connection between a Ricci-flat space and an AdS spacetime. However, our connection, which also works at the level of the equations of motion, does not involve an analytic continuation of dimensionality.

action of $D = 5$ $\mathrm{U}(1)^3$ gauged supergravity:

$$\begin{aligned}\mathcal{L}_5 &= R * \mathbf{1} - \frac{1}{2}\sum_i^2 \mathrm{d}\varphi_i \wedge *\mathrm{d}\varphi_i - \frac{1}{2}\sum_i^3 X_i^{-2} F^i \wedge *F^i \\ &- 4\lambda \sum_i^3 X_i^{-1} \mathrm{vol}_5 + F^1 \wedge F^2 \wedge A^3. \end{aligned} \tag{2.9}$$

In the above action, $F^i = \mathrm{d}A^i$ and the scalars X_i are subject to the constraint $\prod_{i=1}^3 X_i = 1$. In terms of the unconstrained scalars φ_i, they may be expressed as

$$X_1 = e^{-\frac{1}{2}\left(\frac{2}{\sqrt{6}}\varphi_1 + \sqrt{2}\varphi_2\right)}, \quad X_2 = e^{-\frac{1}{2}\left(\frac{2}{\sqrt{6}}\varphi_1 - \sqrt{2}\varphi_2\right)}, \tag{2.10}$$

where $X_3 = (X_1 X_2)^{-1}$. It is a well-known fact that the theory (2.9), with coupling $g^2 = -\lambda$, arises from a KK reduction of type IIB supergravity on S^5 [34]. Here, we provide an alternative higher-dimensional guise.

The $\mathrm{U}(1)^3$ theory (2.9) arises from the following KK reduction Ansatz:

$$\begin{aligned}\mathrm{d}s_{11}^2 &= \Delta^{\frac{4}{3}} \mathrm{d}s^2(M_5) + \Delta^{-\frac{2}{3}} \sum_{i=1}^3 X_i \left(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\psi_i^2\right), \\ G_4 &= -\sum_i \mu_i \mathrm{d}\mu_i \wedge \mathrm{d}\psi_i \wedge \mathrm{d}A^i, \end{aligned} \tag{2.11}$$

where the internal space is a product of three copies of $\mathbb{R}^2$ and the warp factor is given in (2.5) in terms of the overall radius r, where $r^2 = (\mu_1^2 + \mu_2^2 + \mu_3^2)$. The four-form is largely self-selecting, since it is only this combination of external and internal forms that will scale in the same way with the Ricci tensor, $R_{ab} \sim \Delta^{-\frac{4}{3}} \bar{R}_{ab}$. In general, constructing KK reductions for warped product spacetimes is tricky and a useful rule of thumb is that the fluxes should scale in the same way as the Ricci tensor, so that the warp factor drops out of the Einstein equation. Further discussion can be found in [43].

The internal part of metric in the reduction Ansatz (2.11) is rather unusual in the sense that we have isolated U(1)'s in the internal metric but have not gauged the isometries. In fact, it can be shown that the inclusion of traditional KK vectors along the U(1)'s in the metric would result in terms that scale differently with the warp factor. Therefore, on their own, they are inconsistent but it may be possible to restore consistency, essentially by mixing the metric with the fluxes so that the overall factor that appears with the gauge fields scales correctly. This would involve a more complicated Ansatz– potentially one where Ricci-flatness is sacrificed– and we leave this to future work.

This reduction is performed at the level of the equations of motion and is, by definition, consistent. The flux equation of motion in eleven-dimensional supergravity, $\mathrm{d} *_{11} G_4 + \frac{1}{2} G_4 \wedge G_4 = 0$, leads to the lower-dimensional flux equations of motion, while Ricci-flatness along each copy of $\mathbb{R}^2$ leads to an equation of motion for X_i. The constraint on the X_i comes from cross-terms in the Ricci tensor of the form

$$R_{a\mu_i} = -\mu_i \Delta^{-\frac{7}{3}} X_i^{-\frac{1}{2}} \partial_a \log \prod_i X_i = 0. \tag{2.12}$$

The reduction proves to be inconsistent at the level of the action. This is probably due to the non-compactness of the internal space [3] .

Some further comments are now in order. As stated, one may readily show that the $D = 11$ uplift of the AdS$_5$ vacuum is not supersymmetric. *A priori*, there is nothing to rule out the possibility that the solutions to U$(1)^3$ gauged supergravity, which are supported by the scalars X_i and the gauge fields A_i, are supersymmetric. However, we believe that this is unlikely. To back this up, we have confirmed that a three-parameter family of wrapped brane solutions considered in [89] (see [82] for earlier works) is not supersymmetric in the current context. The same solutions are supersymmetric when uplifted on S^5.

As for compactness, for $\lambda = -1$ we have a natural cut-off on the internal spaces, namely $\sum_{i=1}^{3} \mu_i^2 \leq 1$, thus leading to a finite-volume internal space, where each $\mathbb{R}^2$ subspace may be regarded as a disk. Curvature singularities of this type have been identified as the "good" type in the literature [25] and their CFT interpretation has been explored in [30]. For $\lambda = +1$ this is a

[3] It was also observed in [44] that a non-compact reduction was inconsistent when performed at the level of the action.

smooth embedding of $D = 5$ $\mathrm{U}(1)^3$ de Sitter gravity in eleven-dimensional supergravity, albeit with a non-compact internal space.

2.3.2 Origin of the KK reduction

It is striking that we have arrived at a class of vacua that only reside in ten and eleven-dimensional supergravity. In this section, we offer an explanation as to why that may be the case. Our observation hinges on a known "far from BPS" near-horizon limit of certain extremal black holes in $D = 4$ $\mathrm{U}(1)^4$ and $D = 5$ $\mathrm{U}(1)^3$ gauged supergravity [31, 30] (see also [33]), where an AdS_3 near-horizon is formed by incorporating an internal circular direction with the scalars, in this case X_i, all scaled appropriately.

Eschewing explicit solutions, we are free to apply the same scaling for the scalars X_i directly to the KK reduction Ansatz presented in [34]. For concreteness, we consider the SO(8) KK reduction on S^7, further truncated to the $\mathrm{U}(1)^4$ Cartan subgroup. The reduction Ansatz may be written as [34]

$$\begin{aligned} \mathrm{d}s^2_{11} &= \Delta^{\frac{2}{3}}\mathrm{d}s^2_4 + \Delta^{-\frac{1}{3}}\sum_{i=1}^{4} X_i^{-1}\left[\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{D}\phi_i^2\right], \qquad (2.13) \\ G_4 &= 2\sum_i \Big[\left(X_i^2\mu_i^2 - \Delta X_i\right)\mathrm{vol}_4 + \frac{1}{2} *_4 \mathrm{d}\log X_i \wedge \mathrm{d}(\mu_i^2) \\ &- \frac{1}{2} X_i^{-2}\mathrm{d}(\mu_i^2)\wedge \mathrm{D}\phi_i \wedge *_4 F^i\Big], \qquad (2.14) \end{aligned}$$

where we have defined $\mathrm{D} = \mathrm{d}\phi_i + A^i$, $\Delta = \sum_{i=1}^{4} X_i\mu_i^2$, $F^i = \mathrm{d}A^i$ and μ_i are constrained by $\sum_{i=1}^4 \mu_i^2 = 1$. The X_i are subject to the constraint $\prod_{i=1}^4 X_i = 1$.

We now isolate X_1 and blow it up by taking the limit[4]

$$\begin{aligned} X_1 &= \epsilon^{-\frac{3}{2}}\tilde{X}_1, \qquad X_i = \epsilon^{\frac{1}{2}}\tilde{X}_i, \quad i = 2,3,4, \\ \phi_1 &= \epsilon^{-1}\varphi_1, \qquad g_4 = \epsilon\tilde{g}_4, \end{aligned} \tag{2.15}$$

with $\epsilon \to 0$. In the process, the internal ϕ_1 direction migrates and combines with the original four-dimensional metric to form a five-dimensional subspace. Performing this scaling at the level of the Ansatz yields

$$\begin{aligned} ds_{11}^2 &= \mu_1^{\frac{2}{3}}\left[\tilde{X}_1^{\frac{2}{3}} d\tilde{s}_4^2 + \tilde{X}_1^{-\frac{4}{3}} d\varphi_1^2\right] \\ &+ \mu_1^{-\frac{2}{3}} \sum_{i=2}^4 \tilde{X}_1^{-\frac{1}{3}} \tilde{X}_i^{-1}\left[d\mu_i^2 + \mu_i^2 d\phi_i^2\right], \end{aligned} \tag{2.16}$$

$$G_4 = -\frac{1}{2}\sum_{i=2}^4 \tilde{X}_i^{-2} d(\mu_i^2) \wedge d\phi_i \wedge *_4 F^i, \tag{2.17}$$

where μ_1 is constrained, so the warp factor is

$$\mu_1 = \sqrt{1 - (\mu_2^2 + \mu_3^2 + \mu_4^2)}. \tag{2.18}$$

Up to redefinitions, and the introduction of λ, this Ansatz is the same as the consistent KK reduction Ansatz identified in the previous section. Note that the $*_4 F^i$ in the original notation refers to a two-form, with the Hodge dual

[4] In terms of the unconstrained scalars, $\varphi_i, i = 1,2,3$, this simply corresponds to the limit $\varphi_i \to -\infty$, so it is symmetric.

of the two-form leading to a two-form in the new five-dimensional spacetime, which appears wedged with the volume of the internal disks. It is interesting that the AdS vacuum is now sourced by the warp factor and not by the original vol_4 term in the four-form flux, G_4, which is suppressed in the limiting procedure.

Note that this limiting procedure naturally leads to a singularity, which conforms to the "good" type under the criterion of [25], since the scalar potential is bounded above in the lower-dimensional potential. In other words, from the perspective of the original $D = 4$ $\mathrm{U}(1)^4$ gauged supergravity, the limiting procedure results in a steadily more negative potential.

One could first take a limit where the S^7 degenerates to $\mathrm{S}^5 \times \mathbb{R}^2$ or $\mathrm{S}^3 \times \mathbb{R}^4$ [47] and then apply the scaling limit discussed here. The result of this two-step process is that the warp factor does not depend on the $\mathbb{R}^2$ or $\mathbb{R}^4$ portion of the internal space that resulted from taking the first limit. One could then perform dimensional reduction and T-duality along these flat directions so that our KK reductions are reinterpreted as arising from scaling limits of S^5 or S^3 reductions of type IIB theory, along with a toroidal reduction along the remaining flat directions. The resulting lower-dimensional theory admits a domain wall, rather than (A)dS, as a vacuum solution.

2.3.3 KK reductions on $\mathbb{R}^4$

In this subsection, we briefly record the other consistent KK reductions with warp factors. We begin by considering type IIB supergravity on the product $\mathbb{R}^4 \equiv \mathbb{R}^2 \times \mathbb{R}^2$. It was previously shown in [31] that there is a consistent KK reduction to a six-dimensional theory admitting an AdS$_6$ vacuum with a single scalar and three-form flux. We recall the ten-dimensional Ansatz [5] but allow for the opposite sign in the warp factor,

$$\begin{aligned} \mathrm{d}s_{10}^2 &= \Delta \mathrm{d}s^2(M_6) + \Delta^{-1} \sum_{i=1}^{2} L^2 e^{Y_i} (\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\psi_i^2), \\ F_5 &= (1 + *_{10}) H_3 \wedge \mu_1 \mathrm{d}\mu_1 \wedge \mathrm{d}\psi_1, \end{aligned} \tag{2.19}$$

where $\Delta = \sqrt{1 + \lambda(\mu_1^2 + \mu_2^2)}$, we have relabeled $X_i = e^{Y_i}$ to avoid logarithms and L is a length scale. The Bianchi identity for F_5 implies that we can define a two-form potential so that $H_3 = \mathrm{d}B_2$ and the self-duality requirement dictates that both H_3 and its Hodge dual appear in the Ansatz for the five-form flux. In [31], it was found in the absence of a dilaton and axion that consistency of the reduction (considering cross-terms in the Ricci tensor and flatness condition) requires $Y_1 = -Y_2$, leaving a single scalar in six dimensions. We recall that for the SO(6) reduction of type IIB supergravity on S^5 the dilaton and axion do not feature in the scalar potential (see for

[5] Our H_3 is related to F_3 in [31] by a factor of two, $F_3 = 2H_3$.

example [48]), so it is expected that one can reinstate them here.

After a conformal transformation to get to the Einstein frame, the six-dimensional action takes the form

$$\begin{aligned}\mathcal{L}_6 &= \sqrt{-\hat{g}_6}\bigg(\hat{R}_6 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}e^{2\Phi}(\partial\chi)^2 - \frac{1}{4}(\partial Z)^2 \\ &- \frac{8}{L^2}\lambda\cosh\frac{Z}{2} - \frac{1}{12}e^{-Z}H_3^2\bigg),\end{aligned} \tag{2.20}$$

where $\Phi = Y_1 + Y_2$ and $Z = Y_1 - Y_2$.

It is worth noting that the three-form entering in the lower-dimensional action is not self-dual from the six-dimensional perspective but it is self-dual in the five-form flux of type IIB supergravity. Truncating out either Y_1 or Y_2, we arrive at the action of [31], up to a symmetry of that action $Z \leftrightarrow Z^{-1}$.

Since the internal space is flat, the (A)dS$_6$ vacuum of the six-dimensional reduced theory could be embedded in either type IIB or type IIA supergravity, yet we have insisted on type IIB theory. This preference can be attributed to the fluxes, which will only scale correctly in type IIB supergravity, suggesting that the KK reduction carries some memory that it was originally a reduction of type IIB supergravity on S^5.

One could first take a limit where the S^5 degenerates to S$^3 \times \mathbb{R}^2$ [47] and then apply our scaling limit. This enables our KK reduction to be reinterpreted as arising from a scaling limit of an S^3 reduction of type IIA theory [49] followed by a toroidal reduction, though the lower-dimensional theory would have a domain wall as a vacuum solution rather than (A)dS.

We have omitted natural three-form fluxes, both NS-NS and RR, in our Ansatz. Though these scale correctly with the warp factor, we find that it is difficult to include them in an Ansatz, since they would require a decomposition into an external two-form and internal one-form. For the internal space $\mathbb{R}^4$, this is inconsistent with internal Ricci-flatness, and for Calabi-Yau reductions one generically does not have natural internal one-forms. It is expected that when $\lambda = 0$, modulo mirror symmetry (T-duality), we recover the Ansatz of the reduction of type IIB supergravity on CY_2, e. g. [50].

2.3.4 KK reductions on $\mathbb{R}^3$

Switching our attention to the final warped-product solution, which lives in eleven dimensions, we can construct a KK reduction Ansatz with the internal odd-dimensional space further split as $\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}$.

The Ansatz is motivated by a scaling limit of $D = 7$ gauged supergravity and is given as

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \Delta^{\frac{2}{3}} \mathrm{d}s^2(M_8) + \Delta^{-\frac{4}{3}} L^2 \Big[e^{-2Y} \mathrm{d}\mu_0^2 \\ &\qquad + e^{Y} (\mathrm{d}\mu_2^2 + \mu_2^2 \mathrm{d}\psi_2^2) \Big], \\ G_4 &= H \wedge \mathrm{d}\mu_0, \end{aligned} \tag{2.21}$$

where $\Delta = \sqrt{1+\lambda(\mu_0^2+\mu_2^2)}$ and $H = \mathrm{d}B_2$. The Ansatz for the four-form flux is largely self-selecting in that it preserves the symmetry of the $\mathbb{R}^2$ factor and scales correctly with the warp factor Δ in the Einstein equation. We remark that this choice appears to fall outside the Ansatz of Cvetic et al [34] but can be accommodated in reductions on S^4, where three-forms are retained [51, 52].

After the KK reduction, the eight-dimensional action is

$$\mathcal{L}_8 = \sqrt{-g_8}\Big[R_8 - \frac{3}{2}(\partial Y)^2 - \frac{1}{12}e^{2Y}H_{abc}H^{abc} - \frac{2}{L^2}\lambda(e^{2Y}+2e^{-Y})\Big]. \tag{2.22}$$

Note that one can also find a KK reduction Ansatz in type IIA theory by first taking a limit where the S^4 degenerates to $\mathrm{S}^3\times\mathbb{R}$ [47, 49], reducing along the $\mathbb{R}$ direction and then performing our scaling limit.

2.4 Stability of AdS vacua

It is a well-appreciated fact that in the absence of supersymmetry classical stability is a concern. In this section we focus on the stability of the (A)dS vacua. We begin with AdS stability, where violations of the Breitenlohner-Freedman (BF) bound [53] provide us with a simple litmus test for instability. We will turn our attention to dS vacua later.

An important caveat from the outset is that we will confine our attention to instabilities that arise *within* our truncation Ansatz. However, this does not preclude the possibility that instabilities arise from modes we have truncated out, for example see [35]. Within this restricted scope, we will explicitly show that lower-dimensional Freund-Rubin type solutions with AdS_3 factors enjoy greater stability than higher-dimensional vacua in the truncated theory. This may be attributed to the fact that these solutions correspond to the near-horizon of known black holes and are expected to be classically stable.

That some of our geometries are unstable may come as no surprise. It is known that product spaces can be prone to instabilities where one space becomes uniformly larger while another shrinks so that the overall volume is kept fixed. Earlier examples of this instability include the spacetimes $\mathrm{AdS}_4\times \mathrm{M}_n\times \mathrm{M}_{7-n}$ [36] and $\mathrm{AdS}_7\times \mathrm{S}^2\times \mathrm{S}^2$ [37]. Indeed, in a fairly comprehensive study of the classical stability of Freund-Rubin spacetimes of the form $\mathrm{AdS}_p\times \mathrm{S}^q$ [38], this is the primary instability observed.

In terms of our Ansatz, the scalars in the lower-dimensional theory control the volume of the internal spaces, which are further subject to a constraint. We will now investigate the stability of geometries with respect to these scalar modes, while at the same time taking into account breathing modes that arise from further reductions. Our analysis is not intended to be comprehensive but, since instabilities usually arise from products [38], experience suggests these are the most dangerous modes. We recall that the BF bound for AdS_p with radius R_{AdS} is [53]

$$m^2 R_{\mathrm{AdS}}^2 \geq -\frac{(p-1)^2}{4}. \tag{2.23}$$

Before proceeding to specialize to particular cases, here we record some preliminaries. We will in general consider spacetimes in dimension D and further reductions on constant curvature spaces of dimension $(D-p)$ to a p-dimensional spacetime:

$$\mathrm{d}s_D^2 = e^{2A\frac{(D-p)}{(2-p)}}\mathrm{d}s_p^2 + e^{2A}\mathrm{d}s^2(\Sigma_{D-p}), \tag{2.24}$$

which leads to scalars A, i.e. breathing modes, in the lower-dimensional theory.

The above reduction Ansatz is designed to bring us to the Einstein frame in p dimensions. The Einstein-Hilbert term in the higher-dimensional action reduces to

$$\begin{aligned}\mathcal{L}_p &= \sqrt{-g_p}\Big[R - \frac{(D-2)(D-p)}{(p-2)}(\partial A)^2 \\ &\qquad + e^{-\frac{2(D-2)}{(p-2)}A}\kappa(D-p-1)(D-p)\Big],\end{aligned} \tag{2.25}$$

where κ is the curvature of the internal space. When $\kappa > 0$, we will only consider the constant spherical harmonic, which appears with the lowest mass, $\nabla^2_{S^q} A = 0$. Higher spherical harmonics typically do not lead to instabilities, since they correspond to modes with a more positive mass squared. We can now specialize to the various potentials we have found.

2.4.1 $D = 11, p = 5,\ \Sigma_{D-p} = \mathbb{R}^6$

Since our lower-dimensional theory corresponds to the bosonic sector of a known supergravity, it is expected that solutions are stable. It is easy enough to check that the scalars precisely saturate the (unit radius) BF bound in five dimensions:

$$\nabla^2_{AdS_5} \delta\varphi_i + 4\delta\varphi_i = 0. \tag{2.26}$$

At this point, it is also instructive to make a comment regarding reductions in the *absence* of fluxes. The first non-trivial reduction with an AdS$_3$ vacuum involves a reduction on H^2. We omit the details since a related reduction appeared recently in [54] where, in addition, the underlying three-dimensional gauged supergravity was identified. The mass-squared matrix for the fluctuations takes the following form:

$$\nabla^2_{AdS_3} \begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \\ \delta A \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \\ \delta A \end{pmatrix}. \tag{2.27}$$

We observe that the scalars φ_i have mass $m^2 = -2$ which violates the BF bound for AdS_3. In contrast, the fluctuation of the scalar A does not affect the stability. Indeed, it is worth observing that we can also truncate out $\delta\varphi_i$, in which case the instability would not be observed. This is a pretty trivial example of the hidden instabilities noted in a higher-dimensional context in [35].

2.4.2 $D = 10, p = 6,\ \Sigma_{D-p} = \mathbb{R}^4$

In contrast to the AdS_5 vacuum, the AdS_6 vacuum is unstable to fluctuations of the scalar Z with a mass squared $m^2 = -10$ for unit-radius AdS_6. We will now consider whether a lower-dimensional $AdS_3 \times \Sigma_3$ vacuum is stable to this mode. Projecting out the massless axion and the dilaton, which are less likely to source instabilities, the theory may be dimensionally reduced to give

$$\begin{aligned}\mathcal{L}_3 &= \sqrt{-g_3}\bigg(R_3 - \frac{1}{4}(\partial Z)^2 - 12(\partial A)^2 + 6\kappa e^{-8A} \\ &+ \frac{8}{L^2}e^{-6A}\cosh\frac{Z}{2} - \frac{1}{2}e^{-12A}\left[p^2 e^Z + q^2 e^{-Z}\right]\bigg),\end{aligned}$$

where we have assumed

$$H_3 = p\, e^{Z-12A}\mathrm{vol}(M_3) + q\mathrm{vol}(\Sigma_3). \tag{2.28}$$

In the electric term in H_3, we have imposed the lower-dimensional equation of motion. Extremizing the potential, we get

$$\begin{aligned} p^2 &= 4e^{4A-Z}\left(\kappa + L^{-2}\, e^{\frac{1}{2}Z+2A}\right), \\ q^2 &= 4e^{4A+Z}\left(\kappa + L^{-2}\, e^{-\frac{1}{2}Z+2A}\right). \end{aligned} \tag{2.29}$$

Plugging these back into the action, we note that AdS_3 will have unit radius provided that

$$\frac{4}{L^2}\cosh\frac{Z}{2} = 2e^{6A} - 2\kappa e^{-2A}. \tag{2.30}$$

The $\mathrm{AdS}_3{\times}\mathrm{S}^3$ solutions appeared previously in [31].

If one considers variations of the breathing mode A and scalar Z, then the mass-squared matrix has eigenvalues

$$\begin{aligned} m^2 &= \frac{4e^{-16A}}{L^4}\Bigg(2e^{8A}L^4\kappa + 2e^{10A}L^2\cosh\frac{Z}{2} \\ &\pm \sqrt{e^{20A}L^4(2\cosh Z - 1)}\Bigg). \end{aligned} \tag{2.31}$$

To simplify expressions, we can solve (2.30) for the length scale L in terms of κ, A and Z:

$$L = \sqrt{2}e^{-A}\sqrt{\frac{\cosh\frac{Z}{2}}{e^{4A} - \kappa e^{-4A}}}. \tag{2.32}$$

Since (2.31) is symmetric under $Z \leftrightarrow -Z$, without loss of generality we can take $Z \geq 0$. For both $\kappa = 0$ (T^3) and $\kappa = 1$ (S^3), we find that the masses are always strictly positive. Indeed, for $\kappa = 0$, all dependence on the critical value of A drops out and $m^2 \to 0^+$ as $Z \to \infty$ for the lowest eigenvalue. When $\kappa = 1$, the dependence on A remains, but m^2 is again strictly positive.

When $\kappa = -1$, we have a constraint on the range of A and Z, namely $A \geq \frac{z}{8}$, such that the solution is real, i.e. $p^2, q^2 \geq 0$. In this range, m^2 is always positive. Thus, we conclude that a Freund-Rubin type AdS_3 solution is stable to fluctuations that destabilize the AdS_6 vacuum of the six-dimensional theory.

2.4.3 $D = 11, p = 8,\ \Sigma_{D-p} = \mathbb{R}^3$

The stability analysis for the eight-dimensional theory (2.22) parallels the six-dimensional theory (2.20), which we analyzed previously. Even in the absence of supersymmetry, where there is no dual (super)conformal theory in seven dimensions [55], the AdS_8 vacuum is puzzling, but it resolves itself by being unstable. In order to stabilize the vacuum, one can turn on the three-form. One can support a "magnetovac" solution, $\mathrm{AdS}_5 \times \Sigma_3$, however the fluctuations of the scalars A (breathing) and Y have the following mass eigenvalues (at unit radius):

$$m^2 = 4(3 - 4e^{-3Y} \pm \sqrt{25 - 32e^{-3Y} + 16e^{-6Y}}\ ). \tag{2.33}$$

and are thus unstable for all Σ_3.

We can also consider an "electrovac" with an AdS_3 factor but this solution can be incorporated in a more general case:

$$\begin{aligned} \mathrm{d}s^2 &= e^{-2A}\left[e^{-4B}\mathrm{d}s^2(M_3)+e^{2B}\mathrm{d}s^2(\Sigma_2)\right]+e^{2A}\mathrm{d}s^2(\Sigma_3), \\ H &= pe^{-2Y-4A-8B}\mathrm{vol}(M_3)+q\mathrm{vol}(\Sigma_3), \end{aligned} \tag{2.34}$$

where we now have two breathing modes A and B, two Freund-Rubin-type flux terms p, q and a transverse space Σ_2 of constant curvature κ_2. We can view the Ansatz as two successive reductions, one on Σ_3, followed by a second on Σ_2, where in each case one arrives in the Einstein frame.

The effective three-dimensional action may be written as

$$\begin{aligned} \mathcal{L}_3 &= \sqrt{-g_3}\bigg(R_3-\frac{3}{2}(\partial Y)^2-6(\partial A)^2-6(\partial B)^2 \\ &+ \frac{2}{L^2}(e^{2Y}+2e^{-Y})e^{-2A-4B}-\frac{1}{2}p^2e^{-2Y-4A-8B} \\ &+ 6\kappa_1 e^{-4A-4B}+2\kappa_2 e^{-6B}-\frac{1}{2}q^2e^{2Y-8A-4B}\bigg) \end{aligned} \tag{2.35}$$

where κ_1 denotes the constant curvature of Σ_3. Extremizing the potential, we arrive at the critical point:

$$\begin{aligned} e^{2Y} &= -L^2\kappa_2 e^{2A-2B}, \\ q^2 &= 4\kappa_1 e^{-2Y+4A}+\frac{2}{L^2}e^{6A}, \\ p^2 &= 4\kappa_1 e^{2Y+4B}+\frac{4}{L^2}(e^{-Y}-\frac{1}{2}e^{2Y})e^{2Y+2A+4B}. \end{aligned} \tag{2.36}$$

We observe that the Riemann surface Σ_2 should be negatively curved, thus making it H^2. We can set $q=0$, $\frac{1}{2}\kappa_1=\frac{1}{4}\kappa_2=\kappa$ and $B=2A$ to recover the electrovac solution, the details of which we have omitted. We can set AdS_3 to unit radius by choosing

$$L^2 = \frac{e^{-Y+2A}}{(e^{4A+4B} - \kappa_1)}. \tag{2.37}$$

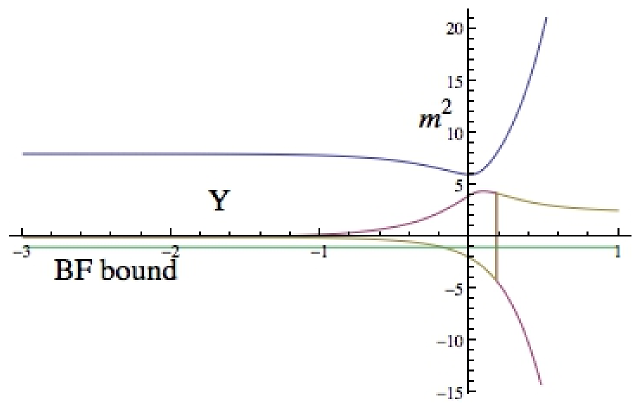

Figure 2.1: The mass-squared eigenvalues for scalar fluctuations around the geometry $AdS_3\times$ $T^3\times$ H^2 as a function of the critical value of Y.

With an additional breathing mode, the stability analysis is more complicated. In fact, even for the simpler case when $\kappa_1 = 0$, where expressions do not depend on the breathing modes A and B, we cannot find analytic expressions for the mass-squared matrix eigenvalues. For $\kappa_1 = 0$, we note that in order the solution to remain real, Y is restricted to the range $Y \leq \frac{1}{3}\log 2$. The mass as a function of the critical value of Y is plotted in FIG. 1. We see that for suitably negative values of Y, a range of stability exists. As can be seen from (2.36), this corresponds to values where the fluxes are larger.

When $\kappa_1 = \pm 1$, the mass-squared matrix only depends on Y and the combination $A+B$. It is easier to consider $\kappa_1 = -1$, since we have a constraint. From $p^2, q^2 \geq 0$, we find a constraint on Y in terms of $A+B$:

$$\frac{1}{2}(e^{4A+4B}+1) \quad \geq \quad e^{-3Y} \geq \frac{1}{2}\frac{(e^{4A+4B}+1)}{e^{4A+4B}}. \tag{2.38}$$

Taking $A+B$ to be a fixed value, for either value of κ_1, we see that the eigenvalues of the mass matrix vary with the critical value of Y in essentially the same way as they do for the AdS$_3\times$ T$^3\times$ H^2 geometry shown in FIG. 1. Thus, the eigenvalues of the mass-squared matrix are largely insensitive to curvature, given our choice of normalization. This means that the same range of stability will exist for $\kappa_1 = \pm 1$. The only caveat here is that for $\kappa_1 = -1$, there is the added constraint above (2.38), so $A+B$ has to be chosen to be large enough such that one has some overlap with the stable region.

2.5 de Sitter vacua

In this section we discuss a particular dS$_3$ solution in the five-dimensional theory (2.9), which via our consistent KK reduction may be regarded as a solution to eleven-dimensional supergravity. Neglecting time-dependent solutions, such as [56], static embeddings in the literature have either involved reductions on non-compact hyperbolic spaces, for example [59, 57, 58], or analytic continuations of known maximally supersymmetric solutions, such as $\mathrm{AdS}_5\times\ \mathrm{S}^5$, leading to solutions of so-called type II* theories and their dimensional reductions [60, 61]. Here we point out that the internal non-compact spaces need not be curved and can in fact be Ricci-flat. This evades the well-known "no-go" theorem [81] on non-compactness grounds.

Our effective three-dimensional theory supporting the dS_3, comes from a reduction of the $\mathrm{U}(1)^3$ theory (2.9) on a Riemann surface of constant curvature κ [6]. To do this, we employ the usual Ansatz,

$$\begin{aligned} \mathrm{d}s_5^2 &= e^{-4A}\mathrm{d}s_3^2 + e^{2A}\mathrm{d}s^2(\Sigma_2), \\ F^i &= -a_i \mathrm{vol}(\Sigma_2), \end{aligned} \tag{2.39}$$

where a_i denote constants. The three-dimensional action may be recast as

$$\mathcal{L}_3 = \sqrt{-g_3}\left[R_3 - \frac{1}{2}\sum_{i=1}^{3}(\partial W_i)^2 - V(W_i)\right] \tag{2.40}$$

where the potential V takes the form

$$V = -2\kappa e^K + \frac{4}{L^2}e^K\sum_{i=1}^{3}e^{W_i} + \frac{1}{2}e^{2K}\sum_{i=1}^{3}a_i^2 e^{2W_i}. \tag{2.41}$$

In expressing terms this way, we have made use of the Kähler potential of the three-dimensional gauged supergravity, $K = -(W_1 + W_2 + W_3)$, introduced

[6]It should be noted that the original five-dimensional vacuum corresponds to a local maximum and is unstable.

a length scale L for the internal space, and imported the notation of [54], $e^{W_i} = e^{2A} X_i^{-1}$, where A denotes the warp factor.

Up to the minus sign in front of the second term, this is the potential corresponding to magnetized wrapped brane solutions [54]. This potential has an underlying real superpotential provided $\kappa = -(a_1 + a_2 + a_3)$. One advantage of working with the type II* embeddings is that flux terms appear with the "wrong" sign and the theories may be regarded as "supersymmetric". Solutions then follow from extremizing the fake superpotential. This is not the case here, since the flux terms do not have the wrong sign. We have checked that a fake superpotential can be found, but only when all the constants are equal, $a_i = a$, and $\kappa = 5\frac{a}{L}$. One of the extrema of the potential in this case is AdS$_3$, so we will ignore this possibility.

Extremizing (2.41), we arrive at conditions on the fluxes for a critical point to exist:

$$a_i^2 = e^{\sum_{j\neq i} W_j} \left(\kappa e^{-W_i} - \frac{4}{L^2} \right). \tag{2.42}$$

For real solutions we recognize the immediate need for a reduction on a sphere ($\kappa > 0$). Inverting the above expression to get W_i in terms of a_i is, in general, problematic, so we consider the simplification where $a_i = a$, $W_i = W$. In this case, it is easy to locate the critical points of V,

$$e^{W_\pm} = \frac{L^2}{8}\left(1 \pm \sqrt{1 - 16a^2 L^{-2}}\right). \tag{2.43}$$

We note that we require $16a^2 < L^2$ for two real extrema. Examining the second derivative of the potential, we identify the upper sign as a local maximum and the lower sign as a local minimum corresponding to our de Sitter vacuum. By tuning the parameter a relative to L, as we show in FIG. 2, it is possible to find a de Sitter vacuum, where the cosmological constant is arbitrarily small and positive.

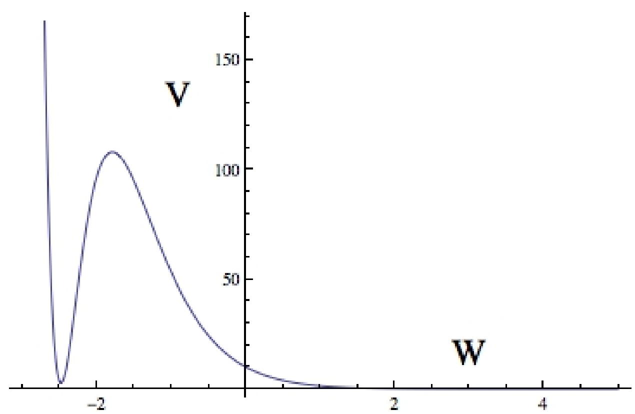

Figure 2.2: Plot of the potential for $L = 1$ and $a = 0.236$. By tuning a relative to L, we can increase the barrier to decay and stabilise the vacuum.

To address stability, we follow the treatment presented in [22], which is based in part on [62]. Since we are working in $D = 3$, it is natural to consider an O(3)-invariant Euclidean spacetime with the metric,

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + a(\tau)^2 \mathrm{d}\Omega_2^2, \tag{2.44}$$

where a is the Euclidean scale factor. The scalars obey the following equations of motion,

$$3W'' + 6\frac{a'}{a}W' = V_{,W}\,, \quad a'' = -\frac{a}{2}\left(\frac{3}{2}W'^2 + V\right), \tag{2.45}$$

where primes denote derivatives with respect to τ. These equations admit a simple instantonic three-sphere solution, where the scalar sits at one of the extrema of the potential, $W = W_\pm$, and

$$a(\tau) = \ell^{-1}\sin(\ell\tau). \tag{2.46}$$

Here ℓ is the inverse radius of the sphere, which in turn is related to the potential, $\ell^2 = \frac{V}{2}$. Given the two extrema, we have two trivial solutions of this type describing a time-independent field.

We now wish to consider Coleman-De Luccia instantons, which describe tunneling trajectories between the de Sitter vacuum and asymptotic Minkowski space ($W > W_+$). According to [62], the probability, P, for tunneling from from a false vacuum at $W = W_-$, with vacuum energy $V_0 \ll 1$ (in Planck units), to Minkowski space is to first approximation given by

$$P \approx \exp(S_0), \tag{2.47}$$

where $S_0 \equiv S(W_0)$ is the Euclidean action evaluated in the vicinity of the de

Sitter vacuum. S_0, in turn, is determined from the tunneling action,

$$S(W) = \int d^3x\sqrt{g}\left(-R_3 + \frac{3}{2}(\partial W)^2 + V(W)\right), \tag{2.48}$$

which describes trajectories beginning in the vicinity of the false vacuum, $W = W_-$, at $\tau = 0$ and reaching $W = 0$ (Minkowski) at $\tau = \tau_f$, where $a(\tau_f) = 0$. Using the trace of the Einstein equation, we can rewrite (2.48)

$$S(W) = -2\int d^3x\sqrt{g}V(W) = -8\pi\int_0^{\tau_f} d\tau a^2(\tau)V(W(\tau)). \tag{2.49}$$

The Euclidean action calculated for the false vacuum de Sitter solution at $W = W_-$ is

$$S_0 = -8\pi^2\sqrt{\frac{2}{V_0}}. \tag{2.50}$$

By tuning a and L appropriately, so that V_0 is small, we can find an arbitrarily long-lived dS_3 vacuum. We conclude that the dS_3 vacuum can be regarded as stable. Though we have only presented one example, we expect similar comments to hold for dS_3 vacua supported through the consistent reductions we have identified.

As it stands, our set-up needs some tweaking in order to incorporate dS_4 vacua. We have seen that a vacuum solution exists when the number of internal dimensions is large. In the absence of two-form flux in the six-dimensional theory (2.20), one could contemplate reducing on a 2d Riemann surface. For the $dS_4 \times S^2$ solution without flux threading the S^2, it is not surprising that one finds that the vacuum is unstable.

Before leaving the subject of de Sitter solutions, we make one final comment. The five-dimensional theory (2.9) also has solutions that smoothly interpolate between $dS_2 \times S^3$ in the infinite past and a dS_5-type spacetime in the infinite future [63]. These solutions can be obtained from the AdS black hole solutions in $D = 5$ $U(1)^3$ gauged supergravity simply by changing the sign of the scalar potential, and can all be embedded in eleven dimensions using the KK reduction Ansatz (2.11). Like the previously-mentioned dS_3, these solutions have a fake superpotential in the equal-charge case.

2.6 Conclusion

We have studied a class of non-supersymmetric Ricci-flat solutions which are warped products of a flat internal space and an anti-de Sitter or de Sitter spacetime. We have found that these Ricci-flat solutions are limited to three cases: warped products of (A)dS_5 and $\mathbb{R}^6$ in eleven dimensions, (A)dS_8 and $\mathbb{R}^3$ in eleven dimensions, and (A)dS_6 and $\mathbb{R}^4$ in ten dimensions. There is also a fourth potentially interesting case of (A)dS_4 in a spacetime with large dimension D. Given that these geometries are rather simple and do not involve any matter content, it is intriguing that so few examples exist and that they are mainly limited to ten and eleven dimensions. While singular in the anti-de Sitter cases, these geometries are completely smooth for de Sitter and are similar in structure to the "bubble of nothing" [64]. Unlike direct products of AdS and a sphere or warped products of de Sitter and a hyperbolic space, both of which are supported by flux, our solutions are supported entirely by the warp factor, are hence not bound by the no-go theorem [81], and do not appear to arise from a limit of the former solutions.

We construct consistent KK truncations for which the above solutions arise as vacuum solutions. These KK truncations are shown to arise as limits of the celebrated dimensional reductions on spheres (like those discussed in [34]) in which the lower-dimensional spacetime gets augmented by one of the spherical coordinates while the remaining directions along the sphere get flattened out. Unlike KK truncations on hyperbolic spaces which are associated with non-compact gauge groups, the truncations in this chapter lead to the bosonic sector of gauged supergravities with compact gauge groups. This is because the gauge fields are associated only with the flux in the higher-dimensional theory and, rather surprisingly, not the geometry. Therefore, within this truncation the isometries of the internal space do not play an explicit role in the lower-dimensional (bosonic sector of gauged) supergravity. This KK reduction enables one to embed five-dimensional $U(1)^3$ de Sitter gravity in eleven-dimensional supergravity. It is an interesting open direction to consider generalizations where the gauge groups are non-Abelian. We expect that one can achieve this by considering a similar limit of the maximally supersymmetric SO(4) [52], SO(6) [48] and SO(8) reductions [65].

Given that our solutions do not preserve supersymmetry, it is important to study their stability. We have focused on possible classical instabilities associated with breathing modes, though there could be other instabilities associated with massive modes that have been truncated out. Within this limited setting, one finds that the AdS_5 solution, although corresponding $D = 11$ solution is singular, is stable. A dual four-dimensional non-supersymmetric CFT is a rather intriguing notion, given that this is dual to pure $D = 11$ gravity (reduced on our Ricci-flat solutions), that D-branes are completely absent and that the eleven-dimensional solution is singular. Other stable solutions include $\mathrm{AdS}_3 \times \Sigma_3$, where Σ_3 is S^3, T^3 or H^3, as well as $\mathrm{AdS}_3 \times \Sigma_3 \times \mathrm{H}^2$ for a certain range of its parameters. On the other hand, the AdS_8, AdS_6 and $\mathrm{AdS}_5 \times \Sigma_3$ solutions are not stable. It would have been rather surprising if there had been a stable AdS_8 solution, since this would imply the existence of a corresponding seven-dimensional CFT, though it would be non-supersymmetric.

As for the de Sitter solutions, we find $\mathrm{dS}_3\times$ S^2 solutions that, in terms of the breathing modes, are stable. We expect similar solutions to of the form $\mathrm{dS}_3\times\ \mathrm{S}^3$ and $\mathrm{dS}_3\times\ \mathrm{S}^3\times\ \mathrm{H}^2$ to exist in the six-dimensional (2.20) and eight-dimensional theory (2.22), respectively. Although all the dS_4 vacua we have found are either i) unstable or ii) they require an infinite number of internal dimensions, and are thus unsatisfactory, we hope that this line of inquiry will lead to simple stable dS_4 in the future. The most positive angle is that a flat direction in the reduction on $\mathbb{R}^6$ to $D = 5$ can be found and a dS_4 vacuum can be engineered from the dS_5 vacuum using the approach of [66].

As with warped-product solutions of de Sitter and a hyperbolic space [57], our solutions appear to be intrinsically higher-dimensional, in that they are not amenable to either compactification or a braneworld scenario [67]. In particular, massless gravitational modes are not associated with normalizable wavefunctions and therefore cannot be localized on a brane. It is not clear as to whether one can use the proposed dS/CFT correspondence [68] to extract meaningful information directly from the eleven-dimensional embeddings of these de Sitter solutions.

While our construction remains intact if the internal space $\mathbb{R}^n$ is replaced by a cone over any Einstein space with positive curvature, there is a conical singularity at the apex of the cone. Replacing the internal space with a smooth cone, such as a resolved or deformed conifold [69] for the case of a six-dimensional space, would be interesting but would necessitate a slightly different Ansatz than was considered above.

Chapter 3

Lifshitz and Schrödinger type solutions

3.1 Introduction

The AdS/CFT correspondence provides a technique for studying certain strongly-coupled conformal field theories in terms of string theory on a weakly curved spacetime (see [16] for a review)[1]. In recent years, holographic techniques have been used to study strongly-coupled condensed matter systems, such as atomic gases at ultra-low temperature. Gravitational backgrounds that provide descriptions of these Lifshitz and Schrödinger-like fixed points have been proposed in [71] and [72, 73], respectively. In these systems, rather than obeying conformal scale invariance, the temporal and spatial coordinates scale anisotropically. Another example of a condensed matter system for which a dual gravitational description has been proposed is the effec-

[1]This section is based on [70].

tive field theory of the supersymmetric lowest Landau level, which is a $(0, 2)$ Landau-Ginsburg model [74].

All of these systems can undergo marginal deformations, in the sense that the deformed theories preserve the non-relativistic symmetries of the undeformed theory. We will employ a solution-generating technique that is based on U-duality in order to find gravitational backgrounds in type IIB theory that correspond to marginal deformations of various non-relativistic theories. The procedure can be outlined as follows. Begin with a theory that has a global symmetry group that includes $U(1) \times U(1)$, which corresponds to a two-torus subspace of the type IIB background. T-dualize along one of the $U(1)$ directions to type IIA theory. Lifting the solution to eleven dimensions provides a third direction which is associated with a $U(1)$ symmetry. One can now apply an $SL(3, R)$ transformation along these $U(1)^3$ directions. Dimensionally reducing to type IIA theory and T-dualizing back to type IIB theory along the shifted directions yields a new type IIB solution.

This method for finding new solutions has been widely used, one instance of which was to generate the type IIB supergravity background that corresponds to marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory [75]. In addition to the $U(1) \times U(1)$ symmetry, the theory has a $U(1)$ R-symmetry. Since the corresponding direction in the gravitational background was not involved in the solution-generating procedure, the deformed theory has $\mathcal{N} = 1$ supersymmetry. The deformation of the supergravity background was matched to an exactly marginal operator in the field theory, thereby providing a holographic test of the methods of Leigh and Strassler [76].

For the case of $\mathcal{N} = 4$ super Yang-Mills theory, one can perform a chain of T-duality-shift-T-duality transformations which involve the direction that corresponds to the $U(1)$ R-symmetry, which results in additional marginal deformations that do not preserve supersymmetry [77]. In contrast, some of the Lifshitz and Schrödinger-like fixed points that we consider have a global symmetry group that contains $U(1)^3$, which enables us to explore various marginal deformations that preserve a minimal amount of supersymmetry.

This chapter is organized as follows. In section 2, we construct the type IIB supergravity dual background that describes the marginal deformations of the effective theory of the supersymmetric lowest Landau level, which is a $(0, 2)$ Landau-Ginsburg theory. In section 3, we construct the supergravity duals of deformations of field theories which preserve Schrödinger symmetry. The Sasaki-Einstein spaces $L^{p,q,r}$ provide countably-infinite examples of such theories that can be deformed in this manner. In section 4, we consider the marginal deformations of theories that exhibit Lifshitz scaling. The Sasaki-Einstein spaces $Y^{p,q}$ are used to construct a countably-infinite familiy of such marginally deformed theories that all have dynamical exponent $z = 2$. We also consider a massive type IIA background that describes a marginally deformed theory with general dynamical exponent $z \geq 1$. Conclusions are presented in section 5.

3.2 Marginal deformations of $(0, 2)$ Landau-Ginsburg theory

The bosonic sector of $U(1)^3$ truncation of $D = 5$ $SO(6)$ gauged supergravity [78] that keeps two neutral scalar fields ϕ_a has the Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L}_5 &= R - \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}(\partial\varphi_2)^2 + 4\sum_{i=1}^{3} X_i^{-1} - \frac{1}{4}\sum_{i=1}^{3} X_i^{-2}(F^i_{(2)})^2 \\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda} F^1_{\mu\nu} F^2_{\rho\sigma} A^3_\lambda\,,
\end{aligned}
\tag{3.1}
$$

where the two scalars are expressed in terms of three constrained scalars X_i via

$$
X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}, \tag{3.2}
$$

with $X_1 X_2 X_3 = 1$. Also, we are taking $g = 1$. A family of magnetic $\mathrm{AdS}_3 \times \mathbb{R}^2$ solutions found in [79] and studied further in [74, 80] are given by[2]

$$\begin{aligned} ds_5^2 &= L^2 \, ds^2_{\mathrm{AdS}_3} + dy_1^2 + dy_2^2 \,, \\ F^i_{(2)} &= 2q_i \, dy_1 \wedge dy_2 \,, \\ \varphi_1 &= f_1 \,, \qquad \varphi_2 = f_2 \,, \end{aligned} \tag{3.3}$$

where f_1 and f_2 are constants and

$$L^{-2} = \sum_{i=1}^{3} X_i^{-1} , \qquad q_i^2 = X_i \,. \tag{3.4}$$

For $f_2 = 0$, these solutions reduce to a "magnetovac" solution of Romans' $D = 5$ gauged supergravity [84], which can be uplifted to ten [85] and eleven dimensions [86]. The case in which both f_1 and f_2 vanish is a non-supersymmetric solution of minimal gauged supergravity, which arises in the near-horizon limit of magnetic black brane solutions at zero temperature [87]. There is a subset of solutions which preserve supersymmetry provided that they satisfy the constraint

$$\sum_{i=1}^{3} q_i = 0 \,, \tag{3.5}$$

as well as

$$L^{-1} = \frac{1}{2} \sum_{i=1}^{3} X_i \,, \qquad 2 \sum_{i=1}^{3} X_i^2 = \left(\sum_{i=1}^{3} X_i \right)^2 . \tag{3.6}$$

[2]There are also families of supersymmetric magnetic $\mathrm{AdS}_3 \times H^2$ [81] and $\mathrm{AdS}_3 \times S^2$ [82] solutions whose marginal deformations were obtained in [83].

Since the solutions degenerate if any of the q_i vanish, we require all of them to be nonzero.

It has been proposed that the above supersymmetric subset of solutions provide a supergravity dual description of the effective field theory of the supersymmetric lowest Landau level [74]. This is a $(0,2)$ Landau-Ginsburg theory which has $\nu_1 = |q_1| N_\Phi N^2$ chiral multiplets Θ_{1i}, $\nu_2 = |q_2| N_\Phi N^2$ chiral multiplets Θ_{2j} and $\nu_3 = \nu_1 + \nu_2$ fermionic multiplets Ψ_{3k}, where $N_\Phi = BV_2/2\pi$, B is the magnitude of the magnetic field, V_2 is the volume of the magnetic plane and q_1 and q_2 are the charges of the chiral multiplets. The $(0,2)$ superpotential is given by

$$c_{ijk} \Theta_{1i} \Theta_{2j} \Psi_{3k} \,, \tag{3.7}$$

where c_{ijk} is from the overlap of the lowest Landau level wavefunctions. The $(0,2)$ Landau-Ginsburg theory can arise in the low energy limit of a flow from four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, which is described by a supergravity solution that smoothly interpolates from AdS_5 to $\mathrm{AdS}_3 \times \mathbb{R}^2$.

These solutions can be lifted to ten-dimensional type IIB supergravity on a 5-sphere using the Kaluza-Klein reduction ansatz in [78] to give

$$\begin{aligned} ds_{10}^2 &= \sqrt{\Delta}\, ds_5^2 + \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{3} X_i^{-1} \left(d\mu_i^2 + \mu_i^2 D\phi_i^2\right), \\ F_{(5)} &= L^3\, \epsilon_{(3)} \wedge \sum_{i=1}^{3} \left[2(X_i^2\mu_i^2 - \Delta X_i)\; dy_1 \wedge dy_2 + q^i X_i^{-1} d(\mu_i^2) \wedge D\phi_i\right] \\ &\quad +\text{dual}\,, \end{aligned} \tag{3.8}$$

where $\epsilon_{(3)}$ is the volume-form for AdS_3,

$$D\phi_i = d\phi_i + A^i, \qquad \Delta = \sum_{i=1}^{3} X_i \mu_i^2\,, \qquad dA^i = F^i_{(2)}\,, \tag{3.9}$$

and the μ_i obey the constraint $\sum_{i=1}^{3} \mu_i^2 = 1$.

We will consider deformations of these backgrounds which leave the $AdS_3 \times \mathbb{R}^2$ subspace intact. Such deformations can be considered to be marginal in the sense that they maintain the two-dimensional conformal symmetry of the undeformed field theory. In order to study deformations which preserve a $U(1) \times U(1)$ global symmetry as well as the $(0,2)$ supersymmetry, it is convenient to define the coordinates

$$\phi_1 = \alpha_1 \psi - \tilde{\phi}_2\,, \qquad \phi_2 = \alpha_2 \psi + \tilde{\phi}_1 + \tilde{\phi}_2\,, \qquad \phi_3 = \alpha_3 \psi - \tilde{\phi}_1\,, \tag{3.10}$$

where

$$\alpha_i = \frac{q_i^2}{q_1^2 + q_2^2 + q_3^2}\,. \tag{3.11}$$

Then the Killing vector ∂_ψ matches with the $U(1)$ superconformal R-symmetry computed through c-extremization [88, 89][3]. We T-dualize to type IIA theory along the $\tilde{\phi}_1$ direction and lift the resulting solution to eleven dimensions. Next, we perform the coordinate transformation

$$\tilde{\phi}_2 \to \tilde{\phi}_2 + \gamma\tilde{\phi}_1 + \sigma x_{11}\,, \tag{3.12}$$

where x_{11} is the eleventh direction. Reducing along the transformed x_{11} direction and T-dualizing back along the transformed $\tilde{\phi}_1$ direction yields the deformed type IIB solution

$$\begin{aligned}
ds^2_{10} &= \frac{1}{G^{\frac{1}{4}}\sqrt{\Delta}}\Big[\Delta\ ds_5^2 + \sum_{i=1}^{3} X_i^{-1}\left(d\mu_i^2 + G\mu_i^2 D\phi_i^2\right) \\
&\qquad + (\gamma^2+\sigma^2)\frac{G}{\Delta}\prod_{i=1}^{3}\frac{\mu_i^2}{X_i}\left(\sum_{i=1}^{3} D\phi_i\right)^2\Big]\,, \\
F_{(5)} &= L^3\ \epsilon_{(3)} \wedge \sum_{i=1}^{3}\left[2(X_i^2\mu_i^2 - \Delta X_i)\ dy_1 \wedge dy_2 + q^i X_i^{-1} d(\mu_i^2) \wedge D\phi_i\right] \\
&\quad + \mathrm{dual}\,, \\
F^{RR}_{(3)} &= \sigma\ dB_{(2)} - \gamma\ GHC_{(3)}\,, \qquad F^{NS}_{(3)} = \gamma\ d[GHB_{(2)}] + \sigma\ C_{(3)}\,, \\
e^{2\phi} &= GH^2\,, \qquad \chi = -\gamma\sigma g H^{-1}\,,
\end{aligned} \tag{3.13}$$

[3]See [90, 91] for additional examples of the supergravity dual of c-extremization applied to various AdS_3 backgrounds, including null-warped AdS_3 solutions obtained via TsT transformations.

where

$$
\begin{aligned}
B_{(2)} &= \frac{1}{\Delta H}\sum_{j<k}(-1)^{j+k}\frac{\mu_j^2\mu_k^2}{X_jX_k}\ D\phi_j\wedge D\phi_k\,, \\
*C_{(3)} &= L^3\ \epsilon_{(3)}\wedge\sum_{i=1}^{3}\big[2(X_i^2\mu_i^2-\Delta X_i)\ dy_1\wedge dy_2 \\
&\qquad +q^iX_i^{-1}d(\mu_i^2)\wedge D\phi_i\big]\wedge B_{(2)}\,, \qquad (3.14)
\end{aligned}
$$

and

$$
H = 1+\sigma^2 g\,,\quad G^{-1}=1+(\gamma^2+\sigma^2)g\,,\quad g=\frac{1}{\Delta}\sum_{j<k}\frac{\mu_j^2\mu_k^2}{X_jX_k}\,. \qquad (3.15)
$$

With the non-compact part of the metric intact, it's plausible to propose that the new solution (3.13) is dual to a marginally deformed $(0,2)$ Landau-Ginsburg theory. The dual field theory can arise in the low-energy limit of a flow from a marginal deformation of $\mathcal{N}=4$ super Yang-Mills theory, whose supergravity dual is described by the Lunin-Maldacena background [75]. One obtains the superpotential of the deformed $(0,2)$ theory directly from dimension reduction of the deformed superpotential in [75].

3.3 Marginal deformations of theories with Schrödinger symmetry

3.3.1 An example with a five-sphere

We will now consider gravity duals of theories which exhibit Schrödinger symmetry, which can be used to study non-relativistic systems such as atomic gases at ultra-low temperature [72, 73]. An example of such a solution in type IIB theory is given by [92, 93]

$$\begin{aligned} ds_{10}^2 &= r^2\left(-2dtdy - r^2dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + ds_{S^5}^2\,, \\ F_{(5)} &= 4\left(\Omega_{(5)} + *\Omega_{(5)}\right), \\ B_{(2)}^{NS} &= -r^2 dt \wedge \left(d\psi + A_{(1)}\right), \end{aligned} \tag{3.16}$$

where $F_{(3)}^{NS} = dB_{(2)}^{NS}$ and $\Omega_{(5)}$ is the volume-form on a unit S^5. The metric on S^5 has been expressed as a $U(1)$ bundle over $\mathbb{CP}^2$:

$$ds_{S^5}^2 = \left(d\psi + A_{(1)}\right)^2 + ds_{\mathbb{CP}^2}^2\,, \tag{3.17}$$

with the metric for the $\mathbb{CP}^2$ base space

$$ds_{\mathbb{CP}^2} = d\sigma^2 + \tfrac{1}{4}s_\sigma^2(d\theta^2 + s_\theta^2 d\phi^2) + \tfrac{1}{4}s_\sigma^2 c_\sigma^2(d\beta + c_\theta d\phi)^2\,, \tag{3.18}$$

and the Kähler potential

$$A_{(1)} = \tfrac{1}{2}s_\sigma^2(d\beta + c_\theta d\phi)\,. \tag{3.19}$$

The solution (3.16) can be obtained by applying a null Melvin twist to the $\mathrm{AdS}_5 \times S^5$ background, which enables the dual field theory interpretation of the result to be $\mathcal{N} = 4$ super Yang-Mills twisted by an R-charge [92, 93]. Rather than obeying conformal scale invariance, the temporal and spatial coordinates in the theory scale anisotropically. This corresponds to the metric in (3.16) obeying the scaling relation

$$t \to \lambda^z t\,, \qquad \vec{x} \to \lambda \vec{x}\,, \qquad r \to \lambda^{-1} r\,, \qquad y \to \lambda^{2-z} y\,, \tag{3.20}$$

where in this case the dynamical exponent $z = 2$. If the momentum along the y direction is interpreted as rest mass, then this geometry describes a system which exhibits time and space translation invariance, spatial rotational symmetry and invariance under the combined operations of time reversal and charge conjugation.

One can apply U-duality to generate deformations of the $\mathrm{AdS}_5 \times S^5$ solution using the β and ϕ directions, which correspond to marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theory that preserve conformal symmetry and $\mathcal{N} = 1$ supersymmetry [75]. A null Melvin twist of this deformed solution has been presented in [94]. Since this null Melvin twist does not involve the β and ϕ directions, one can obtain the same result by generating the marginal deformations directly on the solution (3.16). The final solution can be interpreted as describing marginally deformed super Yang-Mills theory twisted by an R-charge. Although the conformal symmetry has been broken by the null Melvin twist, the deformations are marginal with respect to the Schrödinger symmetry.

Alternatively, one can generate marginal deformations using the $U(1)$ symmetry associated with the y direction. Working directly from the solution given by (3.16), we perform T-duality along the β direction and lift to eleven dimensions. Then we perform the coordinate transformation $y \to y + \gamma_1 \beta + \gamma_2 x_{11}$, where x_{11} is the eleventh direction. Upon reducing to type IIA theory along the transformed x_{11} direction and T-dualizing along the transformed β direction, we obtain the deformed type IIB solution

$$\begin{aligned} ds_{10}^2 &= r^2\left(-2dtdy - r^2 H\ dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + ds^2_{S^5}\,, \\ F_{(5)} &= 4\left(\Omega_{(5)} + *\Omega_{(5)}\right), \\ C_{(2)}^{RR} &= \gamma_2\ r^2\ dt \wedge \left(s_\sigma^2\ d\psi + A_{(1)}\right), \\ B_{(2)}^{NS} &= -r^2\ dt \wedge \left[(1+\gamma_1^2 s_\sigma^2)d\psi + (1+\gamma_1)A_{(1)}\right], \\ \phi &= \chi = 0\,, \end{aligned} \tag{3.21}$$

where

$$H = 1 + (\gamma_1 + \gamma_1^2 + \gamma_2^2)s_\sigma^2\,, \tag{3.22}$$

and $F_{(3)}^{RR} = dC_{(2)}^{RR}$. Since the Schrödinger portion of the geometry remains intact for constant σ slicings, the dual field theory retains the Schrödinger symmetry.

In contrast to the previously-mentioned case, the procedure for obtaining these marginal deformations does not commute with the null Melvin twist, since the latter operation entails taking a lightlike boost in the y direction. In fact, the order of operations has a qualitative effect on the field theory interpretation of the deformations. Namely, if the above deformations had been generated on the $\mathrm{AdS}_5 \times S^5$ background, the result would have been a nonlocal field theory [95].

3.3.2 Countably-infinite examples with the $L^{p,q,r}$ spaces

The above construction can be generalized to gravity duals that involve the countably-infinite five-dimensional cohomogeneity two Sasaki-Einstein spaces $L^{p,q,r}$ [96]. The metric for the $L^{p,q,r}$ spaces can be written in the canonical form

$$ds^2_{L^{p,q,r}} = (d\tau + A_{(1)})^2 + ds_4^2\,, \tag{3.23}$$

where the metric of the four-dimensional Einstein-Kähler base space is

$$\begin{aligned} ds_4^2 &= \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2}\left(\frac{s_\theta^2}{\alpha}\,d\phi + \frac{c_\theta^2}{\beta}\,d\psi\right)^2 \\ &\quad + \frac{\Delta_\theta s_\theta^2 c_\theta^2}{\rho^2}\left(\frac{\alpha - x}{\alpha}\,d\phi - \frac{\beta - x}{\beta}\,d\psi\right)^2, \end{aligned} \tag{3.24}$$

the Kähler potential is

$$A_{(1)} = \frac{\alpha - x}{\alpha} s_\theta^2\,d\phi + \frac{\beta - x}{\beta} c_\theta^2\,d\psi\,, \tag{3.25}$$

and the various functions are given by

$$\Delta_x = x(\alpha - x)(\beta - x) - \mu\,, \qquad \Delta_\theta = \alpha c_\theta^2 + \beta s_\theta^2\,, \qquad \rho^2 = \Delta_\theta - x\,. \tag{3.26}$$

Details regarding the conditions that ensure that the $L^{p,q,r}$ metric extends smoothly onto a complete and non-singular manifold are given in [96]. The result is that the parameters α and β as well as the roots of Δ_x can be expressed in terms of coprime integer triples p, q and r which satisfy $0 < p \leq q$ and $0 < r < p+q$, with p and q each coprime to r and to $s = p+q-r$. The $L^{p,q,r}$ spaces have $U(1)^3$ isometry in general, which is enlarged to $SU(2) \times U(1)^2$ for $p+q = 2r$, which corresponds to the subset of $Y^{p,q} = L^{p-q,p+q,p}$ spaces found in [97, 98]. Note that the $L^{p,q,r}$ metric reduces to that of S^5 for $\mu = 0$, and μ can otherwise be rescaled to $\mu = 1$.

Consider the type IIB solution

$$\begin{aligned} ds^2_{10} &= r^2\left(-2dtdy - r^2dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + ds^2_{L^{p,q,r}}\,, \\ F_{(5)} &= 4\left(\epsilon_{(5)} + *\epsilon_{(5)}\right), \\ B^{NS}_{(2)} &= -r^2\, dt \wedge \left(d\tau + A_{(1)}\right), \end{aligned} \tag{3.27}$$

which can be obtained by applying a null Melvin twist to the $\mathrm{AdS}_5 \times L^{p,q,r}$ background. $\epsilon_{(5)}$ denotes the volume-form on the $L^{p,q,r}$ space. We will first consider marginal deformations which involve the $U(1)$ symmetries associated with the ϕ and ψ directions. We T-dualize the solution (3.27) along the ϕ direction, lift to eleven dimensions and perform the coordinate transformation $\psi \to \psi + \gamma\phi + \sigma x_{11}$, where x_{11} is the eleventh direction. Upon reducing and T-dualizing back to type IIB theory along the transformed x_{11} and ϕ directions, respectively, we obtain the deformed solution

$$\begin{aligned}
ds_{10}^2 &= G^{-1/4}\Bigg[r^2\left(-2dtdy - r^2dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + \frac{\rho^2}{4\Delta_x}\,dx^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 \\
&\quad + f\,d\tau^2 + \frac{Gc_\theta^2}{\beta^2\rho^2 b}\,D\psi^2 + \frac{Gs_\theta^2}{\alpha^2\rho^2 a}\,(d\phi + B_{(1)})^2\Bigg], \\
F_{(5)} &= \frac{2\rho^2 s_\theta c_\theta}{\alpha\beta}\,d\tau\wedge dx\wedge d\theta\wedge\Big(G\,d\psi + \frac{\gamma}{\alpha}(\alpha - x)r^2 s_\theta^2\,dt\Big)\wedge(d\phi + B_{(1)}) \\
&\quad + \text{dual}\,, \\
F_{(3)}^{RR} &= \frac{2\gamma}{\alpha\beta H}\rho^2 s_\theta c_\theta\,d\tau\wedge dx\wedge d\theta \\
&\quad + \frac{\sigma}{H}\Big(\frac{\gamma g}{\alpha}\,d[(\alpha - x)r^2 s_\theta^2 dt] - G^{-1}d[gG\,D\psi]\Big)\wedge(d\phi + B_{(1)})\,, \\
F_{(3)}^{NS} &= -d[r^2\,dt\wedge(d\tau + A_{(1)})] + \gamma\,d[gG\,D\psi\wedge(d\phi + B_{(1)})] \\
&\quad + \frac{2\sigma}{\alpha\beta}\rho^2 s_\theta c_\theta\,d\tau\wedge dx\wedge d\theta\,, \\
e^{2\phi} &= GH^2\,,\qquad \chi = -\gamma\sigma g H^{-1}\,,
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
B_{(1)} &= \alpha\rho^2(\alpha - x)a\,d\tau - \frac{\alpha}{\beta}ac_\theta^2\,d\psi - \frac{\gamma}{\beta}er^2c_\theta^2\,dt\,, \\
D\psi &= d\psi + \beta\rho^2 be\,d\tau + \gamma r^2 s_\theta^2\left(\frac{\alpha - x}{\alpha}\right)dt\,,
\end{aligned} \tag{3.29}$$

$$H = 1 + \sigma^2 g\,,\qquad G^{-1} = 1 + (\gamma^2 + \sigma^2)g\,,\qquad g = \frac{s_\theta^2 c_\theta^2}{\alpha^2\beta^2\rho^4 ab}\,, \tag{3.30}$$

and we have defined the functions

$$
\begin{aligned}
a^{-1} &= \alpha(\alpha - x)\rho^2 - \mu s_\theta^2 \,, \\
b^{-1} &= \beta(\beta - x)\rho^2 - \mu c_\theta^2 - a s_\theta^2 c_\theta^2 \,, \\
e &= \beta - x + (\alpha - x) a s_\theta^2 \,, \\
f &= 1 - (\alpha - x)^2 \rho^2 a s_\theta^2 - \rho^2 b e^2 c_\theta^2 \,.
\end{aligned} \tag{3.31}
$$

Since the above procedure for generating marginal deformations commutes with the null Melvin twist, the same result can be obtained by applying the null Melvin twist to the marginal deformations of $\mathrm{AdS}_5 \times L^{p,q,r}$ that are described in [99]. This enables us to interpret the final solution as describing marginal deformations of the corresponding quiver gauge theories twisted by an R-charge, for which conformal symmetry is broken but the Schrödinger symmetry is preserved.

The field theories have $p + 3q$ chiral fields which come in six different types: $q\ Y$, $(p+q-r)\ U_1$, $r\ U_2$, $p\ Z$, $(r-p)\ V_1$ and $(q-r)\ V_2$ fields. For the $Y^{p,q}$ subset, the U_i fields become a doublet under the $SU(2)$ flavor symmetry. For the undeformed $L^{p,q,r}$ theories, a superpotential can be built out of these fields which has the following schematic form [100]:

$$
W = 2p\ \mathrm{Tr}(YU_1ZU_2) + 2(q-r)\ \mathrm{Tr}(YU_1V_1) + 2(r-p)\ \mathrm{Tr}(YU_2V_2) \,. \tag{3.32}
$$

For the above marginal deformations, the quartic portion of the superpotential is altered as follows:

$$\mathrm{Tr}(YU_1ZU_2 - YU_2ZU_1) \to \mathrm{Tr}\left(e^{i\pi\tilde{\beta}}YU_1ZU_2 - e^{-i\pi\tilde{\beta}}YU_2ZU_1\right), \qquad (3.33)$$

where the complex deformation parameter[4] $\tilde{\beta} = \gamma - i\sigma$. This interpretation of the marginal deformations survives taking the null Melvin twist.

Now we consider marginal deformations which involve the $U(1)$ symmetries associated with the y and ϕ directions. We T-dualize along the ϕ direction, lift to eleven dimensions and perform the coordinate transformation $y \to y+\gamma_1\phi+\gamma_2 x_{11}$. Upon reducing to type IIA theory along the transformed x_{11} direction and T-dualizing along the transformed ϕ direction, we obtain the deformed type IIB solution

$$\begin{aligned}
ds^2_{10} &= r^2\left(-2dtdy - r^2H\ dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + ds^2_{L^{p,q,r}}\,,\\
F_{(5)} &= 4\left(\epsilon_{(5)} + *\epsilon_{(5)}\right),\\
C^{RR}_{(2)} &= \gamma_2\ \frac{r^2 s^2_\theta}{\alpha^2\rho^2 b}\ dt\wedge\left(d\phi + B_{(1)}\right),\\
B^{NS}_{(2)} &= -r^2\ dt\wedge\left(\left(d\tau + A_{(1)}\right) + \frac{\gamma_1 s^2_\theta}{\alpha^2\rho^2 b}\left(d\phi + B_{(1)}\right)\right),\\
\phi &= \chi = 0\,,
\end{aligned} \qquad (3.34)$$

where

$$B_{(1)} = \frac{\alpha\rho H}{\beta}\left(\beta(\alpha - x)\rho\ d\tau - c^2_\theta\ d\psi\right), \qquad (3.35)$$

[4]We put a tilde in order to distinguish this from the parameter β in the $L^{p,q,r}$ metric.

and

$$H = 1 + s_\theta^2 \left[\frac{\gamma_1^2 + \gamma_2^2}{\alpha^2 \rho^2 b} + 2\gamma_1 \left(\frac{x - \alpha}{\alpha} \right) \right] . \tag{3.36}$$

For constant slicings of θ and x, the Schrödinger portion of the geometry remains intact, and so the dual field theory still has Schrödinger symmetry.

3.4 Marginal deformations of Lifshitz vacua

3.4.1 Lifshitz-Chern-Simons gauge theories

We will now consider some examples of gravity dual descriptions of marginal deformations of field theories with Lifshitz scaling [71]. A candidate gravity dual for a class of $2+1$ dimensional Lifshitz-Chern-Simons field theories with dynamical exponent $z = 2$ was constructed in [101]. The type IIB solution is given by

$$\begin{aligned} ds_{10}^2 &= r^2(2dtdx_3 + d\vec{x}^2) + f\ dx_3^2 + \frac{dr^2}{r^2} + ds_{S^5}^2 \,, \\ F_{(5)} &= 2r^3\ d^4x \wedge dr + \text{dual} \,, \\ \chi &= \frac{Qx_3}{L_3} \,, \qquad \phi = 0 \,, \end{aligned} \tag{3.37}$$

where

$$f = \frac{Q^2}{4L_3^2} - \frac{r_0^4}{r^2} \,, \tag{3.38}$$

and $d\vec{x}^2 = dx_1^2 + dx_2^2$. It has been proposed that this background describes non-Abelian Lifshitz Chern-Simons gauge theories which can be realized as deformations of DLCQ $\mathcal{N} = 4$ super Yang-Mills theory. A Chern-Simons term explicitly breaks parity and time-reversal symmetries. The background (3.37) asymptotically exhibits the following scaling symmetry:

$$t \to \lambda^2 t\,, \qquad \vec{x} \to \lambda \vec{x}\,, \qquad r \to \lambda^{-1} r\,, \qquad x_3 \to x_3\,. \tag{3.39}$$

x_3 is a compact direction and therefore cannot scale, since scaling would change the compactification radius. The x_3 circle shrinks to zero size at

$$r = r_\star = \frac{2L_3 r_0^2}{Q}\,. \tag{3.40}$$

The metric is geodesically incomplete in the region $r \geq r_\star$, and a straightforward way of extending geodesics past $r_\star$ leads to closed timelike curves [102]. The implications of this hidden singularity on the dual field theory as well as its resolutions are discussed in [101].

The metric on S^5 can be expressed as

$$ds^2_{S^5} = \sum_{i=1}^{3} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right)\,, \tag{3.41}$$

where $\sum_{i=1}^{3} \mu_i^2 = 1$. Before applying the solution-generating technique, we perform the coordinate transformation (3.10). We T-dualize along the x_3 direction to obtain a solution in massive type IIA theory and perform the transformation

$$\tilde{\phi}_1 \to \tilde{\phi}_1 + \gamma_1 x_3\,. \tag{3.42}$$

Then we T-dualize back to type IIB theory along the transformed x_3 direction to obtain the deformed type IIB solution

$$
\begin{aligned}
ds^2_{10} &= \frac{1}{G^{1/4}}\Bigg[-\frac{r^4}{f}\,dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2} \\
&\qquad + Gf\left(dx_3 + \frac{r^2}{f}\,dt\right)^2 + ds^2_{S^5} - \gamma_1^2 fG\,d\phi_{23}^2\Bigg], \\
F_{(5)} &= 2r^3\,d^4x \wedge dr + \text{dual}\,, \\
F_{(3)}^{RR} &= \gamma_1 \frac{QGr^2}{L_3}\,dt \wedge dx_3 \wedge d\phi_{23}\,, \\
F_{(3)}^{NS} &= -\gamma_1\,d\left[G\left(f\,dx_3 + r^2 dt\right) \wedge d\phi_{23}\right], \\
e^{2\phi} &= G\,, \qquad \chi = \frac{Qx_3}{L_3}\,,
\end{aligned}
\tag{3.43}
$$

where

$$
d\phi_{23} \equiv \mu_2^2\,d\phi_2 - \mu_3^2\,d\phi_3\,, \qquad G^{-1} = 1 + \gamma_1^2 f(\mu_2^2 + \mu_3^2)\,. \tag{3.44}
$$

While the solution (3.37) has a null Killing vector, the deformed solution (3.43) has a timelike Killing vector ∂_t generating the time translations of the dual field theory for $r > 0$ and $\mu_2^2 + \mu_3^2 > 0$.

A different deformation can be obtained by T-dualizing along the $\tilde{\phi}_1$ direction to type IIA theory, performing the transformation

$$
\tilde{\phi}_2 \to \tilde{\phi}_2 + \gamma_2 \tilde{\phi}_1\,, \tag{3.45}
$$

and T-dualizing back to type IIB theory along the transformed $\tilde{\phi}_1$ direction.

The resulting deformed type IIB solution is given by

$$\begin{aligned}
ds^2_{10} &= \frac{1}{G^{1/4}}\Bigg[r^2(2dtdx_3 + d\vec{x}^2) + f\ dx_3^2 + \frac{dr^2}{r^2} \\
&\quad + \sum_{i=1}^{3}\left(d\mu_i^2 + G\mu_i^2 d\phi_i^2\right) + \gamma_2^2 G \prod_{j=1}^{3}\mu_j^2\Big(\sum_{k=1}^{3} d\phi_k\Big)^2\Bigg], \\
F_{(5)} &= 2r^3\ d^4x \wedge dr + \text{dual}\,, \\
F_{(3)}^{RR} &= \gamma_2\,\frac{QG}{L_3}\ dx_3 \wedge \sum_{j<k}(-1)^{j+k}\mu_j^2\mu_k^2\ d\phi_j \wedge d\phi_k \\
&\qquad + \frac{3}{2}\gamma_2\ d(\mu_2^2)\wedge d(\mu_3^2)\wedge d\psi\,, \\
F_{(3)}^{NS} &= \gamma_2\ d\left[G\sum_{j<k}(-1)^{j+k}\mu_j^2\mu_k^2\ d\phi_j \wedge d\phi_k\right], \\
e^{2\phi} &= G\,, \qquad \chi = \frac{Qx_3}{L_3}\,,
\end{aligned} \tag{3.46}$$

where

$$G^{-1} = 1 + \gamma_2^2\sum_{j<k}\mu_j^2\mu_k^2\,. \tag{3.47}$$

Note that the Killing vector ∂_t remains null for this deformation. If the γ_1 and γ_2 deformations are turned on at the same time, then the fiber structure of the resulting solution implies that the x_3 direction is periodic. However, we will restrict ourselves to the scenario in which the x_3 direction is extended.

3.4.2 Countably-infinite Lifshitz vacua with dynamical exponent $z = 2$

Infinite families of Lifshitz solutions of ten and eleven-dimensional supergravity with dynamical exponent $z = 2$ were constructed in [103]. As the starting point for applying the solution-generating technique, we will consider solutions in eleven-dimensional supergravity. Before turning to a countably infinite family of solutions involving the $Y^{p,q}$ spaces, we first consider the eleven-dimensional solution involving the space $T^{1,1}$, which is given by [103]

$$\begin{aligned} ds_{11}^2 &= ds_4^2 + \left(d\sigma + \frac{1}{\sqrt{18}}(c_1 d\phi_1 - c_2 d\phi_2)\right)^2 + dx_{11}^2 \\ &+ \frac{1}{9}(d\psi - c_1 d\phi_1 - c_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + s_1^2 d\phi_1^2 + d\theta_2^2 + s_2^2 d\phi_2^2)\,, \\ G_{(4)} &= dt \wedge d\left[r^4\ dx_1 \wedge dx_2 + r^2\ dx_{11} \wedge \left(d\sigma + \frac{1}{\sqrt{18}}(c_1 d\phi_1 - c_2 d\phi_2)\right)\right]\,, \end{aligned} \tag{3.48}$$

where

$$ds_4^2 = -r^4\ dt^2 + r^2(dx_1^2 + dx_2^2) + \frac{dr^2}{r^2}\,, \tag{3.49}$$

and we denote $c_1 \equiv \cos\theta_1$, $s_2 \equiv \sin\theta_2$, etc. We perform the coordinate transformation

$$\phi_2 \to \phi_2 + \gamma x_{11}\,, \tag{3.50}$$

reduce to type IIA theory along the transformed x_{11} direction and T-dualize along the transformed σ direction to obtain the type IIB solution

$$
\begin{aligned}
ds_{10}^2 &= G^{-1/4}\Bigg[ds_4^2 + \frac{1}{6}\left(d\theta_1^2 + s_1^2 d\phi_1^2 + d\theta_2^2 + GK s_2^2 d\phi_2^2\right) \\
&\quad + \frac{G}{9K}\left(d\psi - c_1 d\phi_1 - K c_2 d\phi_2\right)^2 + G(d\sigma + r^2 dt)^2\Bigg], \\
F_{(5)} &= 4r^3\, d\sigma \wedge dt \wedge dr \wedge dx_1 \wedge dx_2 \\
&\quad -\gamma H r^2 \left(1 + \tfrac{1}{18}\gamma G c_2^2\right) d\sigma \wedge dt \wedge C_{(2)} \wedge B_{(1)} + \text{dual}\,, \\
F_{(3)}^{RR} &= -\gamma H\ d\left[B_{(1)} \wedge (d\sigma + r^2 dt)\right] \\
&\quad - \tfrac{1}{\sqrt{18}}\gamma H c_2\ C_{(2)} \wedge d\sigma + \tfrac{1}{2} H G\ dG^{-1} \wedge B_{(1)} \wedge (d\sigma + r^2 dt)\,, \\
F_{(3)}^{NS} &= C_{(2)} \wedge d\sigma - \tfrac{1}{\sqrt{18}}\gamma\ d\left[G c_2\ B_{(1)} \wedge (d\sigma + r^2 dt)\right], \\
e^{2\phi} &= G H^{-2}\,, \qquad \chi = \tfrac{1}{\sqrt{18}}\gamma H c_2\,,
\end{aligned} \tag{3.51}
$$

where

$$
\begin{aligned}
B_{(1)} &= \tfrac{1}{9}c_2(d\psi - c_1 d\phi_1 - c_2 d\phi_2) - \tfrac{1}{6}s_2 d\phi_2\,, \\
C_{(2)} &= \tfrac{1}{\sqrt{18}}(s_2 d\theta_2 \wedge d\phi_2 - s_1 d\theta_1 \wedge d\phi_1)\,,
\end{aligned} \tag{3.52}
$$

and

$$
G^{-1} = 1 + \tfrac{\gamma^2}{9} + \tfrac{\gamma^2}{18}s_2^2\,, \qquad H^{-1} = 1 + \tfrac{\gamma^2}{6}\,, \qquad K^{-1} = 1 + \tfrac{\gamma^2}{6}s_2^2\,. \tag{3.53}
$$

Note that while the Killing vector ∂_t is null for $\gamma = 0$, it is timelike for $\gamma \neq 0$ and $r > 0$.

The above construction can be generalized to gravity duals that involve the Sasaki-Einstein spaces $Y^{p,q}$, which are characterised by two coprime positive integers p and q with $q < p$ [97, 98]. We begin with the eleven-dimensional solution [103]

$$\begin{aligned} ds_{11}^2 &= f^{1/3} ds_4^2 + f^{-2/3}\left[\left(d\sigma - \frac{D\beta}{\sqrt{72}(1-y)}\right)^2 + dx_{11}^2\right] \\ &+ f^{1/3}\left[\frac{1}{9}\left(d\psi + y\, D\beta - c_\theta d\phi\right)^2 + \frac{1-y}{6}(d\theta^2 + s_\theta^2 d\phi^2) \right. \\ &\left. + \frac{dy^2}{g} + \frac{g}{36}\, D\beta^2\right], \\ G_{(4)} &= dt \wedge d\left[r^4\, dx_1 \wedge dx_2 + \frac{r^2}{f}\, dx_{11} \wedge \left(d\sigma - \frac{D\beta}{\sqrt{72}(1-y)}\right)\right] \end{aligned} \tag{3.54}$$

where

$$ds_4^2 = -\frac{r^4}{f}\, dt^2 + r^2(dx_1^2 + dx_2^2) + \frac{dr^2}{r^2}\,,$$
$$D\beta = d\beta + c_\theta d\phi\,, \qquad g = \frac{2(a - 3y^2 + 2y^3)}{1-y}\,, \tag{3.55}$$

and $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$. We have expressed the metric of the $Y^{p,q}$ subspace in canonical form as a $U(1)$ bundle over an Einstein-Kähler metric. The function f satisfies

$$-4f + \frac{2}{1-y}\,\partial_y\left[(a - 3y^2 + 2y^3)\,\partial_y f\right] + \frac{1}{(1-y)^4} = 0\,. \tag{3.56}$$

Upon performing the coordinate transformation

$$\beta \to \beta + \gamma x_{11}\,, \tag{3.57}$$

reducing to type IIA theory along the transformed x_{11} direction and T-dualizing along the transformed σ direction, we obtain the type IIB solution

$$
\begin{aligned}
ds^2_{10} &= G^{-1/4}\Bigg[ds^2_4 + \frac{1-y}{6}\,(d\theta^2 + s^2_\theta d\phi^2) + \frac{dy^2}{g} + \frac{Kg}{36}\,D\beta^2 \\
&\quad + \frac{G}{9K}\left(D\psi - \tfrac{\gamma^2}{36}Kfgy\,D\beta\right)^2 + fG\left(d\sigma + \frac{r^2}{f}dt\right)^2\Bigg], \\
F_{(5)} &= 4r^3\,d\sigma\wedge dt\wedge dr\wedge dx_1\wedge dx_2 \\
&\quad +\gamma\,\frac{Gr^2}{f}\,d\sigma\wedge dt\wedge dB_{(1)}\wedge C_{(1)} + \text{dual}\,, \\
F^{RR}_{(3)} &= \gamma\,d\left[\frac{H}{\sqrt{72}(1-y)}\right]\wedge A_{(1)}\wedge\left(d\sigma + \frac{r^2}{f}\,dt\right) \\
&\quad +\gamma\,d\left[H\,C_{(1)}\wedge\left(d\sigma + \frac{r^2}{f}\,dt\right)\right] - \gamma\,\frac{H}{\sqrt{72}(1-y)}\,dB_{(1)}\wedge d\sigma\,, \\
F^{NS}_{(3)} &= d\left[B_{(1)}\wedge d\sigma + A_{(1)}\wedge\left(d\sigma + \frac{r^2}{f}\,dt\right)\right], \\
e^{2\phi} &= GH^{-2}\,, \qquad \chi = \frac{\gamma H}{\sqrt{72}(1-y)}\,,
\end{aligned}
\tag{3.58}
$$

where

$$
A_{(1)} = \frac{\gamma^2 G}{\sqrt{72}(1-y)}\,C_{(1)}\,, \quad B_{(1)} = -\frac{D\beta}{\sqrt{72}(1-y)}\,, \quad C_{(1)} = \frac{fy}{9}\,D\psi + \frac{fg}{36}\,D\beta\,,
$$

$$
D\sigma = d\sigma + B_{(1)}\,, \qquad D\psi = d\psi + y\,D\beta - c_\theta d\phi\,, \tag{3.59}
$$

and

$$
K^{-1} = 1 + \tfrac{\gamma^2}{36}fg\,, \quad G^{-1} = K^{-1} + \tfrac{\gamma^2}{9}fy^2\,, \quad H^{-1} = G^{-1} + \frac{\gamma^2}{72(1-y)^2}\,. \tag{3.60}
$$

The Killing vector ∂_t is null for $\gamma = 0$ and timelike for $\gamma \neq 0$ and $r > 0$.

Alternatively, one can perform the coordinate transformation

$$\phi \to \phi + \gamma x_{11}\,. \tag{3.61}$$

Then reducing along the transformed x_{11} direction and T-dualizing along the transformed σ direction yields the type IIB solution

$$\begin{aligned}
ds_{10}^2 &= G^{-1/4}\Bigg[ds_4^2 + \frac{1-y}{6}(d\theta^2 + L s_\theta^2 d\phi^2) + \frac{dy^2}{g} \\
&\quad + \frac{G}{9K}\left[D\psi + \tfrac{\gamma^2}{36}Kf(1-y)\left(gc_\theta D\beta + 6(1-y)s_\theta^2 d\phi\right)\right]^2 \\
&\quad + \frac{Kg}{36L}(D\beta - \tfrac{\gamma^2}{6}Lf(1-y)s_\theta^2 c_\theta d\phi)^2 + Gf(d\sigma + \frac{r^2}{f}\ dt)^2\Bigg], \\
F_{(5)} &= 4r^3\ d\sigma\wedge dt\wedge dr\wedge dx_1\wedge dx_2 \\
&\quad -\frac{\gamma}{9}Gr^2(1-y)c_\theta\ d\sigma\wedge dt\wedge dB_{(1)}\wedge \mathcal{D}\psi + \text{dual}\,, \\
F_{(3)}^{RR} &= \gamma\ d\left[\frac{Hc_\theta}{\sqrt{72}(1-y)}\right]\wedge A_{(1)}\wedge\left(d\sigma + \frac{r^2}{f}\ dt\right) \\
&\quad -\frac{\gamma}{9}\ d\left[Hf(1-y)c_\theta\mathcal{D}\psi\wedge(d\sigma + \frac{r^2}{f}\ dt)\right] - \frac{\gamma Hc_\theta}{\sqrt{72}(1-y)}\ dB_{(1)}\wedge d\sigma\,, \\
F_{(3)}^{NS} &= d\left[B_{(1)}\wedge d\sigma + A_{(1)}\wedge\left(d\sigma + \frac{r^2}{f}\ dt\right)\right], \\
e^{2\phi} &= GH^{-2}\,, \qquad \chi = \frac{\gamma Hc_\theta}{\sqrt{72}(1-y)}\,,
\end{aligned} \tag{3.62}$$

where now we take

$$A_{(1)} = -\frac{\gamma^2 Gfc_\theta^2}{9\sqrt{72}}\ \mathcal{D}\psi\,, \qquad B_{(1)} = -\frac{D\beta}{\sqrt{72}(1-y)}\,,$$

$$\mathcal{D}\psi = D\psi - \frac{g\ D\beta}{4(1-y)} - \frac{3s_\theta^2}{2c_\theta}\ d\phi\,, \tag{3.63}$$

and we are now defining

$$\begin{aligned} L^{-1} &= 1+\tfrac{\gamma^2}{6}f(1-y)s_\theta^2\,, \qquad K^{-1}=L^{-1}+\tfrac{\gamma^2}{36}fgc_\theta^2\,, \\ G^{-1} &= K^{-1}+\tfrac{\gamma^2}{9}f(1-y)^2c_\theta^2\,, \qquad H^{-1}=G^{-1}+\frac{\gamma^2c_\theta^2}{72(1-y)^2}\,. \end{aligned} \tag{3.64}$$

As with the previous deformation, the Killing vector ∂_t is null for $\gamma = 0$ and timelike for $\gamma \neq 0$ and $r > 0$.

3.4.3 An example with general dynamical exponent

Lifshitz solutions of Romans' six-dimensional gauged, massive, $\mathcal{N} = 4$ supergravity [104] were found in [105]. These solutions have with general dynamical exponent $z \geq 1$ and break supersymmetry. The geometry is a direct product of a four-dimensional Lifshitz geometry and a two-dimensional hyperboloid. The metric is given by

$$ds_6^2 = L^2\left(-r^{2z}dt^2 + r^2(dx_1^2+dx_2^2) + \frac{dr^2}{r^2}\right) + a^2\; dH_2^2\,, \tag{3.65}$$

where the metric for a hyperboloid dH_2^2 can be made compact by modding out a non-compact discrete subgroup of the isometry group. There is a topological restriction on z in terms of the gauge coupling g and the mass parameter m of the six-dimensional theory, due to flux quantization on the compact hyperbolic space.

These solutions can be lifted to massive type IIA theory by using the consistent reduction ansatz given in [107][5]. The resulting ten-dimensional solution is given by[6]

$$
\begin{aligned}
ds_{10}^2 &= S^{1/12} k_0^{1/8} \Delta^{3/8} \big[ds_6^2 + k_1 \; d\rho^2 \\
&\quad + k_2 \; \Delta^{-1} C^2 \; [d\theta^2 + s_\theta^2 d\phi^2 + (d\psi + c_\theta d\phi - g A_{(1)})^2] \big] \,, \\
F_{(4)} &= k_3 \; S^{1/3} C^3 \Delta^{-2} U s_\theta \; d\rho \wedge d\theta \wedge d\phi \wedge (d\psi - g A_{(1)}) \\
&\quad + k_4 \, \beta a^2 L r^z S^{1/3} C \; dt \wedge H_{(2)} \wedge d\rho \\
&\quad + k_5 \; S^{1/3} C \; G_{(2)} \wedge (d\psi + c_\theta d\phi - g A_{(1)}) \wedge d\rho \\
&\quad + k_6 \; S^{4/3} C^2 \Delta^{-1} s_\theta \; G_{(2)} \wedge d\theta \wedge d\phi - \frac{k_7 M}{2^{3/4} k_0^2} S^{4/3} *_6 B_{(2)} \,, \\
F_{(3)} &= k_7 \; \beta L^3 r S^{2/3} \; dx_1 \wedge dx_2 \wedge dr + \frac{2}{3} k_7 \, S^{-1/3} C B_{(2)} \wedge d\rho \,, \\
F_{(2)} &= 2^{-3/4} k_7 \, S^{2/3} M B_{(2)} \,, \\
e^{2\phi} &= k_0^{-5/2} S^{-5/3} \Delta^{1/2} \,,
\end{aligned}
\tag{3.66}
$$

where

$$
\begin{aligned}
dA_{(1)} &= G_{(2)} = \alpha L^2 r^{z-1} \; dt \wedge dr + \gamma a^2 \; H_{(2)} \,, \\
B_{(2)} &= \frac{\beta}{2} L^3 r^2 dx_1 \wedge dx_2 \,,
\end{aligned}
\tag{3.67}
$$

$H_{(2)}$ is the volume-form of a unit hyperboloid, and the type IIA mass param-

[5] These solutions have also been lifted to type IIB theory using a reduction ansatz generated via non-Abelian T-duality [108].

[6] We have included terms in the form fields associated with a 2-form $B_{(2)}$ in the six-dimensional solution that are missing in the uplifted solution presented in [105].

eter is given by

$$M = \left(\frac{2mg^3}{27}\right)^{1/4} . \tag{3.68}$$

The various functions are given by

$$\begin{aligned}
\Delta &= k_0\, C^2 + k_0^{-3} S^2 , \qquad U = k_0^{-6} S^2 - 3k_0^2\, C^2 + 4k_0^{-2} C^2 - 6k_0^{-2} , \\
C &= \cos\rho , \qquad S = \sin\rho , \qquad c_\theta = \cos\theta , \qquad s_\theta = \sin\theta ,
\end{aligned} \tag{3.69}$$

and the various constants are

$$\begin{aligned}
k_0 &= e^{\phi_0/\sqrt{2}} \left(\frac{g}{3m}\right)^{1/4} , \quad k_1 = \frac{8}{3mg}\, e^{\sqrt{2}\phi_0} , \quad k_2 = \frac{2}{g^2} \left(\frac{g}{3m}\right)^{1/4} e^{-\phi_0/\sqrt{2}} , \\
k_3 &= -\frac{4\sqrt{2}}{3g^3} \left(\frac{g}{3m}\right)^{3/4} , \quad k_4 = 3g^2\, e^{2\sqrt{2}\phi_0}\, k_3 , \quad k_5 = 3g\, k_3 , \\
k_6 &= -\frac{\sqrt{8}}{g^2}\, e^{-3\phi_0/\sqrt{2}} , \quad k_7 = \sqrt{\frac{12m}{g}} ,
\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}
L^2\beta^2\, e^{\sqrt{8}\phi_0} &= z - 1 , \\
\alpha^2 &= \gamma^2 (z-1) , \\
L^2\gamma^2\, e^{-\sqrt{2}\phi_0} &= \frac{(2+z)(z-3) \pm 2\sqrt{2(z+4)}}{2z} , \\
L^2 g^2\, e^{\sqrt{2}\phi_0} &= 2z(4+z) , \\
\tfrac{1}{2} L^2 m^2\, e^{-3\sqrt{2}\phi_0} &= \frac{6+z \mp 2\sqrt{2(z+4)}}{z} , \\
\frac{L^2}{a^2} &= 6 + 3z \mp 2\sqrt{2(z+4)} .
\end{aligned} \tag{3.71}$$

Since the metric in (3.66) is singular at $\rho = 0$ and π and the string coupling diverges there, we will consider this solution away from these regions.

T-dualizing to type IIB theory along the ψ direction using the extended T-duality transformation rules in [109], performing the transformation

$$\phi \to \phi + \sigma\psi\,, \tag{3.72}$$

and T-dualizing back to massive type IIA theory along the transformed ψ

direction yields the deformed solution

$$
\begin{aligned}
ds_{10}^2 &= G^{-1/4} S^{1/12} k_0^{1/8} \Delta^{3/8} \Big[ds_6^2 + k_1 \, d\rho^2 \\
&\quad + k_2 \, \Delta^{-1} C^2 [d\theta^2 + s_\theta^2 d\phi^2 + G(d\psi + c_\theta d\phi - gA_{(1)})^2] \Big], \\
F_{(4)} &= k_3 \, G \, S^{1/3} C^3 \Delta^{-2} U s_\theta \, d\rho \wedge d\theta \wedge d\phi \wedge (d\psi - gA_{(1)}) \\
&\quad + k_4 \, \beta a^2 L r^z S^{1/3} C \, dt \wedge H_{(2)} \wedge d\rho \\
&\quad + k_5 \, S^{1/3} C \, G_{(2)} \wedge (d\psi + c_\theta d\phi - gA_{(1)}) \wedge d\rho \\
&\quad + k_6 \, S^{4/3} C^2 \Delta^{-1} s_\theta \, G_{(2)} \wedge d\theta \wedge d\phi \\
&- \; k_7 \, \frac{M}{2^{3/4} k_0^2} \, S^{4/3} *_6 B_{(2)} + \frac{\sigma k_2^{5/4} k_4 \, \beta L^3 r s_\theta}{k_0 \Delta^{3/4} S^{1/3}} \sqrt{\frac{GC^7}{k_1}} \, dx_1 \wedge dx_2 \wedge d\theta \wedge dr \\
&- \; \frac{\sigma G k_7 k_2^2 M C^4 s_\theta^2}{2^{3/4} k_0 \Delta} \, B_{(2)} \wedge d\phi \wedge (d\psi - gA_{(1)}) \, , \\
F_{(3)} &= k_7 \, \beta L^3 r S^{2/3} \, dx_1 \wedge dx_2 \wedge dr + \tfrac{2}{3} k_7 \, S^{-1/3} C B_{(2)} \wedge d\rho \\
&\quad + d\Big[\frac{\sigma G k_2^2 C^4 s_\theta^2}{\Delta S^{2/3} k_0} d\phi \wedge (d\psi - gA_{(1)}) \Big], \\
F_{(2)} &= 2^{-3/4} k_7 \, S^{2/3} M B_{(2)} + \frac{\sigma G k_2^2 M C^4 s_\theta^2}{k_0 \Delta S^{2/3}} \, d\phi \wedge (d\psi - gA_{(1)}) \\
&\quad - \sigma k_3 S^{1/3} C^3 \Delta^{-2} U s_\theta \, d\theta \wedge d\rho \, , \\
e^{2\phi} &= G \, k_0^{-5/2} S^{-5/3} \Delta^{1/2} \, , \qquad (3.73)
\end{aligned}
$$

where

$$
G^{-1} = 1 + \sigma^2 \, \frac{k_2^2 C^4 s_\theta^2}{k_0 S^{2/3} \Delta} \, . \qquad (3.74)
$$

3.5 Conclusions

A solution-generating technique based on U-duality has been used to construct supergravity backgrounds that holographically describe the marginal deformations of various non-relativistic field theories which preserve a $U(1)\times U(1)$ global symmetry. For a $(0,2)$ Landau-Ginsburg theory describing the supersymmetric lowest Landau level, we have proposed that the marginal deformations are associated with the introduction of a phase in the $(0,2)$ superpotential. This can arise in the low-energy limit of a flow from marginal deformations of four-dimensional $\mathcal{N}=4$ super Yang-Mills theory described by the Lunin-Maldacena background [75].

We have generated the supergravity duals of marginally deformed field theories with Schrödinger symmetry or Lifshitz scaling. This includes a class of Lifshitz-Chern-Simons gauge theories, as well as countably-infinite families of field theories whose dual gravity description involves the Sasaki-Einstein spaces $Y^{p,q}$ and $L^{p,q,r}$. These theories all have dynamical exponent $z=2$. We have also considered massive type IIA backgrounds which are dual to marginal deformations of Lifshitz theories with general dynamical exponent $z\geq 1$.

With the exception of the last example, we have focused on marginal deformations that preserve supersymmetry, these constructions can be straightforwardly generalized to scenarios in which supersymmetry is not preserved. For instance, one could begin with a supersymmetric gravity background and perform a chain of T-duality-shift-T-duality transformations to generate multiple deformations which do not preserve supersymmetry. If this is done for the case of the $(0, 2)$ Landau-Ginsburg theory, then one obtains the description of a non-supersymmetric system that arises in the low-energy limit of a flow from the $6 + 2$-parameter deformation of super Yang-Mills theory found in [77]. Another possibility is to consider cases for which the undeformed field theory itself does not preserve any supersymmetry. For instance, one could study gravity duals of field theories with Schrödinger symmetry or Lifshitz scaling that involve five-dimensional Einstein spaces which are not Sasakian. Examples of such spaces include the $T^{p,q}$ spaces, as well as the $\Lambda^{p,q,r}$ spaces which encompass the Sasaki-Einstein spaces $L^{p,q,r}$ [110].

While examples without supersymmetry are more realistic, one then has less control over their behavior and must worry about instabilities. However, there are cases of non-supersymmetric string states that may be completely stable. For example, the gravity dual description for the effective field theory of the lowest Landau level presented in [74] has a regime of parameter space in which all known instabilities are apparently absent.

Chapter 4

Tri-dipole black rings in supergravity

4.1 Introduction

Black holes in five dimensions can have event horizons with $S^1 \times S^2$ topology, and are known as black rings in order to distinguish them from topologically spherical black holes (see [112] for a review)[1]. The existence of black ring solutions demonstrated that black hole uniqueness is violated in five dimensions. The neutral black ring in pure gravity was presented in [113], and charged black rings in supergravity were found in [114, 115, 116, 117, 118, 119, 120, 121, 122].

Various generalizations have been constructed, a spherical black hole surrounded by a black ring or multiple concentric rings [118, 123], a doubly-spinning black ring [124], and two concentric orthogonal rotating black rings

[1] This section is based on [111].

[125], as well as approximate solutions describing thin, neutral black rings in any dimension $D \geq 5$ in asymptotically flat spacetime [126] and asymptotically (anti)-de Sitter spacetime [127]. However, while the most general supersymmetric solution has been known for quite some time [117, 119, 120, 121], the general nonextremal solution in five-dimensional $U(1)^3$ supergravity has remained elusive until recently [128]. This nonextremal black ring has three independent dipole and electric charges, along with five additional parameters. If one imposes the condition that there are no conical singularities present, then the remaining parameters correspond to three electric charges, three dipole charges, two angular momenta and the mass.

One solution-generating technique that has been used to construct a number of the above black ring solutions is the inverse scattering method, which was first adapted to Einstein's equations in [129, 130]. In this chapter, we shall make use instead of a solution-generating technique that involves performing a dimensional reduction, and which hinges on the relation between black rings and the C-metric solution. In particular, the neutral black ring can be obtained by lifting the Euclideanized Kaluza-Klein C-metric of [131] to five dimensions on a timelike direction [132, 113].

A three-charge generalization of the C-metric has recently been constructed in STU gauged supergravity [133]. In the ungauged limit, the

Euclidean-signature version of this solution can be lifted on a timelike direction to give a black ring solution with three independent dipole charges and one non-vanishing rotation parameter. Furthermore, three independent electric charges can be generated by repeatedly lifting this solution to six dimensions and applying boosts. In a similar manner, one can repeatedly lift the C-metric solution to five dimensions and apply $SL(2, R)$ transformations to get a black ring solution with background magnetic fields, at the expense of altering the asymptotic geometry.

In order to avoid closed timelike curves for these solutions with a single angular momentum parameter, in turns out that one must turn off the electric charges. Moreover, one must impose a constraint in order to avoid conical singularities. Then one is left with solutions which have five nontrivial parameters corresponding to the mass, one nonvanishing angular momentum and three dipole charges. In addition to this, one can turn on three independent background magnetic fields, in which case the solution is no longer asymptotically flat but rather asymptotically approaches a five-dimensional Melvin fluxbrane [134].

The solutions discussed in this chapter all have a single non-vanishing rotation parameter, as opposed to the black ring presented in [128] which has two independent angular momenta. In particular, we expect that our black ring with three dipole charges and three electric charges arises as a specialization of the one in [128]. The relatively compact form of our solution enables one to verify its validity analytically, rather than numerically as done in [128]. Moreover, the various properties of the solution, its physical and thermodynamic charges and the relations between them can be studied explicitly.

This chapter is organized as follows. In the next section, we demonstrate how black rings with three dipole charges can be obtained from the C-metric solutions. We then use solution-generating techniques involving dimensional reductions to add three electric charges, as well as three background magnetic fields. In section 3, we study the global properties of the five and six-dimensional solutions in the Ricci-flat limit. In section 4, we perform the global analysis and study the thermodynamics of the general black ring solutions. Conclusions are presented in section 5. Lastly, in an appendix, we compile the dimensional reductions that relate the six, five and four-dimensional supergravity theories used for the purposes of solution generating.

4.2 Black rings in five-dimensional $U(1)^3$ supergravity

4.2.1 From C-metrics to dipole black rings

Our starting point is the four-dimensional ungauged supergravity theory that is known as the STU model, which can be obtained via a dimensional reduction from six-dimensional string theory. It has an $SL(2,R) \times SL(2,R) \times SL(2,R)$ global symmetry, corresponding to– when discretized– the (S,T,U) duality symmetries of the string theory. The theory contains four $U(1)$ vector fields $F_I = dA_I$ and three complex scalars $\tau_i = \chi_i + \mathrm{i}e^{\phi_i}$. The axions χ_i can be consistently truncated out, provided that one imposes the conditions

$$F_I \wedge F_J = 0\,, \qquad I \neq J\,. \tag{4.1}$$

The truncated bosonic Lagrangian is given by

$$e_4^{-1}\mathcal{L}_4 = R_4 - \tfrac{1}{2}\sum_{i=1}^{3}(\partial\phi_i)^2 - \tfrac{1}{4}\sum_{I=1}^{4} e^{\vec{a}_I\cdot\vec{\phi}}F_I^2\,, \tag{4.2}$$

where the dilaton vectors $\vec{a}_I$ satisfy

$$\vec{a}_I\cdot\vec{a}_J = 4\delta_{IJ} - 1\,, \qquad \sum_{I=1}^{4}\vec{a}_I = 0\,. \tag{4.3}$$

Our starting point will be the charged C-metric solutions obtained in [133], in the ungauged limit $g = 0$. In this case the solutions can be written as

$$
\begin{aligned}
ds_4^2 &= \frac{1}{\alpha^2(y-x)^2}\bigg[\frac{1}{\sqrt{\tilde{U}}}(G(y)d\tau^2 - \frac{\mathcal{H}(y)}{G(y)}dy^2) \\
&\quad +\sqrt{\tilde{U}}(\frac{\mathcal{H}(x)}{G(x)}dx^2 + G(x)d\varphi^2)\bigg]\,, \\
e^{\vec{a}_I\cdot\vec{\phi}} &= \frac{U_I^2}{\sqrt{\tilde{U}}}\,,\ F_I = \frac{4Q_I}{h_I(y)^2}\ dy\wedge d\tau\,,\ U_I = \frac{h_I(y)}{h_I(x)}\,,\ \tilde{U} = U_1U_2U_3U_4,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
h_I(\xi) &= 1+\alpha q_I\xi\,, \qquad \mathcal{H}(\xi) = \prod_{I=1}^{4} h_I(\xi)\,, \\
G(\xi) &= \mathcal{H}(\xi)\Big(b_0 + \sum_{i=1}^{4}\frac{16Q_i^2}{\alpha^2 q_i\prod_{j\neq i}(q_j-q_i)}\frac{1}{h_i(\xi)}\Big)\,.
\end{aligned}
\tag{4.5}
$$

The general solution contains 10 parameters: (α, b_0, q_I, Q_I) where $I = 1,\ldots,4$. The parameter b_0 can always be scaled to have one or another of the discrete values $(-1, 0, 1)$. The form of the solution is invariant under

$$
x = \frac{\tilde{x}}{1+b\tilde{x}}\,, \qquad y = \frac{\tilde{y}}{1+b\tilde{y}}\,,
\tag{4.6}
$$

which enables one to set one of the four scalar charges q_I to any value, including zero (the STU model has three independent scalars after all). In our construction, the parameters Q_I are independent of the scalar charges q_I. Therefore, we could set $Q_I = 0$ and obtain C-metric solutions supported solely by the scalar charges. On the other hand, for generic Q_I we cannot set the scalar charges to zero, as one would have expected from the theory. When all of the electric charge parameters are equal, namely $Q_I \equiv Q$, we can set all $q_I \equiv q$, the three scalar fields all vanish and we recover the charged C-metric solution of the Einstein-Maxwell theory. While conical singularities can be avoided only for a single point in the parameter space of the dilaton C-metric in [131], conical singularities are absent for a range of parameters for the present C-metric solutions. We will discuss global properties after we lift the solutions to five dimensions.

We shall be lifting the C-metric solution to a dipole black ring solution in five dimensions, but before that a number of preliminary steps are needed. First, we use electromagnetic duality to map the C-metric solution to one that has magnetic charges, for which

$$e^{\vec{a}_I\cdot\vec{\phi}} = \frac{\sqrt{\tilde{U}}}{U_I^2}\,, \qquad F_I = \frac{4Q_I}{h_I(x)^2}\;dx\wedge d\varphi\,. \tag{4.7}$$

Next, we Wick rotate $\tau \to \mathrm{i}\psi$ so that the C-metric has Euclidean signature. Then we dualize one of the gauge fields so that it can play the role of a Kaluza-Klein vector along a timelike direction:

$$\mathcal{F} = e^{\vec{a}_4\cdot\vec{\phi}} {*F_4} = \frac{4Q_4}{h_4^2}\;dy\wedge d\psi \qquad \Longrightarrow \qquad \mathcal{A} = \frac{4Q_4 y}{h_4(y)}d\psi\,. \tag{4.8}$$

Note that the kinetic term for the corresponding field strength now has the "wrong" sign in the Lagrangian, which is given by (102). However, this is not an issue once we lift to five dimensions. Using the reduction ansatz (101), we lift the solution along a timelike direction. The resulting five-dimensional solution is given by

$$\begin{aligned} ds_5^2 &= -\frac{U_4}{(U_1U_2U_3)^{\frac{1}{3}}}\Big(dt+\frac{4Q_4y}{h_4(y)}d\psi\Big)^2 \\ &\quad +\frac{1}{\alpha^2(x-y)^2}\Big[\frac{1}{U_4(U_1U_2U_3)^{\frac{1}{3}}}\Big(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\Big) \\ &\qquad +(U_1U_2U_3)^{\frac{2}{3}}\Big(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\varphi^2\Big)\Big]\,, \\ F_i &= \frac{4Q_i}{h_i^2(x)}dx\wedge d\varphi\,,\qquad e^{\vec{b}_i\cdot\vec{\phi}}=\frac{(U_1U_2U_3)^{\frac{2}{3}}}{U_i^2}\,,\qquad U_i=\frac{h_i(y)}{h_i(x)}\,. \end{aligned} \tag{4.9}$$

This is a black ring solution that has three independent dipole charges, which was first found in [116]. The (x,φ) subspace corresponds to the S^2 part of its $S^2\times S^1$ topology, and the S^1 is associated with ψ.

In the special case that Q_i and q_i are related as follows:

$$q_i=-\mu s_i^2\,,\qquad Q_I=\tfrac{1}{4}\mu s_i c_i\,,\qquad s_i=\sinh\delta_i\,,\qquad c_i=\cosh\delta_i\,, \tag{4.10}$$

we find that

$$G(\xi)=b_0\mathcal{H}(\xi)-\xi^2(1+\alpha\mu\xi)\,. \tag{4.11}$$

This specialisation enables us to take a Ricci-flat limit of the black ring solution, by setting $\delta_1=\delta_2=\delta_3=0$, and we shall return to it in section 4.3.

4.2.2 Adding electric charges

Using the reduction ansatz (93), we lift the black ring solution (4.9) to six dimensions on the z direction. Then we perform a Lorentz boost $t \to c_1 t + s_1 z,\ z \to c_1 z + s_1 t$, where $c_i = \cosh\delta_i,\ s_i = \sinh\delta_i$. The reduction back to five dimensions along the boosted z direction generates the first electric charge, which is associated with the KK vector A_3. Once we apply the discrete symmetry to interchange the gauge fields A_1 and A_3 (and rotate the dilatons accordingly), the electric charge is now associated with A_1. Next, we repeat the process of lifting to six dimensions, performing a boost with parameter δ_2 and reducing back to five dimensions in order to generate a second electric charge. Then we dualise the 2-form potential B in (93) to a 1-form potential A_2 and use the discrete symmetry again to interchange A_3 and A_2, so that the second electric charge is now associated with A_2. Dualising the resulting A_2 to a 2-form potential B again, lifting to six dimensions and boosting with parameter δ_3, we again reduce back to five dimensions. Finally, we dualise the 2-form B to a new 1-form A_2, thereby arriving at the black ring solution with three independent electric charges. After making a convenient transformation of the time coordinate, and a relabelling of the various functions in the solution, the 3-charge black ring takes the following form:

$$ds_5^2 = -\frac{U_4}{U^{1/3}(H_1H_2H_3)^{2/3}}\,(dt+\omega)^2 + (H_1H_2H_3)^{1/3}\,ds_4^2\,, \tag{4.12}$$

where

$$\begin{aligned}
\omega &= -4x\Big(\frac{s_1c_2c_3Q_1}{h_1(x)}+\frac{c_1s_2c_3Q_2}{h_2(x)}+\frac{c_1c_2s_3Q_3}{h_3(x)}+\frac{s_1s_2s_3Q_4}{h_4(x)}\Big)d\varphi \\
&\quad +4y\Big(\frac{c_1s_2s_3Q_1}{h_1(y)}+\frac{s_1c_2s_3Q_2}{h_2(y)}+\frac{s_1s_2c_3Q_3}{h_3(y)}+\frac{c_1c_2c_3Q_4}{h_4(y)}\Big)d\psi\,, \\
ds_4^2 &= \frac{1}{\alpha^2(x-y)^2}\left[\frac{1}{U_4U^{\frac{1}{3}}}\left(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\right)\right. \\
&\quad \left. +U^{\frac{2}{3}}\left(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\varphi^2\right)\right]\,,
\end{aligned} \tag{4.13}$$

and

$$H_i = 1+\left(1-\frac{U_i^2\,U_4}{U}\right)s_i^2\,, \qquad U = U_1U_2U_3\,. \tag{4.14}$$

The scalar fields are given by

$$e^{\vec{b}_i\cdot\vec{\phi}} = \frac{H_i^2}{U_i^2}\left(\frac{U}{H_1H_2H_3}\right)^{2/3}\,, \tag{4.15}$$

and the gauge potentials are

$$\begin{aligned}
A_1 &= \frac{s_1c_1}{H_1}\Big(1-\frac{U_1U_4}{U_2U_3}\Big)\,dt\,+ \\
&+ \frac{4x}{H_1}\Big(\frac{c_1c_2c_3Q_1}{h_1(x)}+\frac{s_1s_2c_3Q_2U_1U_4}{U_3h_2(y)}+\frac{s_1c_2s_3Q_3U_1U_4}{U_2h_3(y)}+\frac{c_1s_2s_3Q_4}{h_4(x)}\Big)\,d\varphi \\
&- \frac{4y}{H_1}\Big(\frac{s_1s_2s_3Q_1U_4}{U_2U_3h_1(x)}+\frac{c_1c_2s_3Q_2}{h_2(y)}+\frac{c_1s_2c_3Q_3}{h_3(y)}+\frac{s_1c_2c_3Q_4U_1}{U_2U_3h_4(x)}\Big)\,d\psi,
\end{aligned} \tag{4.16}$$

with A_2 and A_3 being obtained from A_1 by cycling the indices 1, 2 and 3 on all quantities appearing in (4.16).

4.2.3 Adding background magnetic fields

We can use a similar solution-generating procedure as in the previous section to obtain an Ernst-like generalization of the C-metric solution. Using the reduction ansatz (98), we lift the C-metric solution (4.4) to five dimensions on the spacelike direction z. Next, we perform the coordinate transformation $\varphi \to \varphi + Bz$ and reduce back to four dimensions, where we find that B parameterizes the strength of a background magnetic field. Applying the discrete symmetry to interchange gauge fields, we keep repeating this procedure until we have generated a solution with four background magnetic fields, given by

$$\begin{aligned}
ds_4^2 &= \frac{1}{\alpha^2(y-x)^2}\prod_{I=1}^{4}\sqrt{\frac{\Lambda_I}{U_I}}\Bigg[\left(G(y)d\tau^2 - \frac{\mathcal{H}(y)}{G(y)}dy^2\right) \\
&\quad + \prod_{I=1}^{4}U_I\left(\frac{\mathcal{H}(x)}{G(x)}dx^2 + \frac{G(x)}{\prod_{I=1}^{4}\Lambda_I}d\varphi^2\right)\Bigg], \\
A_I &= -\frac{1}{B_I\Lambda_I}\left(1+\frac{4B_IQ_Ix}{h_I(x)}\right)d\varphi\,, \qquad e^{\vec{a}_I\cdot\vec{\phi}} = \frac{\prod_{J=1}^{4}\sqrt{\Lambda_JU_J}}{\Lambda_I^2U_I^2}\,, \\
\Lambda_I &= \left(1+\frac{4B_IQ_Ix}{h_I(x)}\right)^2 + \frac{B_I^2G(x)}{\alpha^2(x-y)^2U_I^2}\,.
\end{aligned} \tag{4.17}$$

The various functions appearing here are defined in (4.5). The C-metric solution (4.4) is recovered for vanishing magnetic field parameters B_I. Note that one can tune the values of the B_I so as to avoid a conical singularity, even if one is present in the corresponding C-metric solution (meaning that all B_I

are taken to zero with all other parameters held fixed). Thus, the background magnetic fields associated with B_I play a role analogous to the cosmic string in the C-metric itself, by providing the force necessary to accelerate the black hole. (See [135] for a discussion of externally magnetised charged black hole solutions in STU supergravity.)

We can Wick rotate $\tau \to \mathrm{i}\psi$ in the solution (4.17) and then lift it to five dimensions on a timelike direction using the metric ansatz (101). This yields a dipole black ring with background magnetic fields, given by

$$\begin{aligned} ds_5^2 &= \frac{\Lambda^{\frac{2}{3}}}{\alpha^2(x-y)^2}\left[\frac{1}{U_4 U^{\frac{1}{3}}}\left(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\right)\right. \\ &\quad \left.+U^{\frac{2}{3}}\left(\frac{\mathcal{H}(x)}{G(x)}dx^2+\frac{G(x)}{\Lambda}d\varphi^2\right)\right]-\frac{U_4}{(\Lambda U)^{\frac{1}{3}}}\left(dt+\frac{4Q_4 y}{h_4(y)}d\psi\right)^2 , \\ A_I &= -\frac{1}{B_I\Lambda_I}\left(1+\frac{4B_I Q_I x}{h_I(x)}\right)d\varphi\,, \qquad e^{\vec{b}_i\cdot\vec{\phi}} = \frac{(\Lambda U)^{\frac{2}{3}}}{(\Lambda_i U_i)^2}\,, \\ U &\equiv U_1 U_2 U_3\,, \qquad \Lambda \equiv \Lambda_1\Lambda_2\Lambda_3\,, \end{aligned} \tag{4.18}$$

and $i = 1, 2, 3$. Note that this solution has three independent magnetic field parameters, since in five dimensions one can get rid of B_4 by performing the reverse of the coordinate transformation that was used to generate it in the first place.

4.3 Global analysis of Ricci-flat solutions

4.3.1 $D = 5$ Ricci-flat metric

Before turning to the general black ring solution, we shall first study the global structure of its Ricci-flat limit, as a warm-up exercise. If we set $Q_1 = Q_2 = Q_3 = 0 = q_1 = q_2 = q_3$, then the solution (4.9) becomes Ricci-flat and takes the form

$$\begin{aligned} ds_5^2 &= -\frac{h_4(y)}{h_4(x)}\Big(dt+\frac{4Q_4 y}{h_4(y)}d\psi\Big)^2+\frac{1}{\alpha^2(x-y)^2}\Big[\frac{h_4(x)}{h_4(y)}\Big(-G(y)d\psi^2-\frac{h_4(y)}{G(y)}dy^2\Big) \\ &\quad +\Big(\frac{h_4(x)}{G(x)}dx^2+G(x)d\varphi^2\Big)\Big]\,, \end{aligned} \tag{4.19}$$

where $h_4(\xi) = 1+\alpha q_4 \xi$. Setting $q_i = 0$ can be subtle, due to diverging terms in the expression for $G(\xi)$. Instead, we substitute the above form of the solution directly into the equations of motion in order to obtain the most general solution for $G(\xi)$. This is given by

$$G(\xi) = (c_0+c_1\xi+c_2\xi^2)h_4(\xi)-\frac{16Q_4^2\,\xi^2}{q_4^2}\,. \tag{4.20}$$

We can then shift c_0, c_1, c_2 to get rid of the ξ^2 factor in the Q_4^2 term, and shift ξ such that $h_4(\xi) \sim \xi$. This results in the expression

$$G(\xi) = (c_0+c_1\xi+c_2\xi^2)\xi-\frac{16Q_4^2}{\alpha^2 q_4^4}\,, \tag{4.21}$$

in which all the parameters of the solution are shown explicitly. Note that $G(\xi)$ cannot be written in this form for the general black ring solution. The curvature singularities are located at $x=\infty$, $y=\infty$ and $h_4(x)=0$, and the asymptotic region is at $x=y$.

Consider the coordinate transformations

$$x \to \frac{\tilde{x}-1}{2q_4^2}\,, \qquad y \to \frac{\tilde{y}-1}{2q_4^2}\,, \qquad t \to -\sqrt{a_0}(q_4^2\tilde{t}+\psi)\,, \tag{4.22}$$

where a_0 is related to Q_4 by $Q_4=\frac{1}{2}\sqrt{a_0}q_4^2$. Upon setting $\alpha=2q_4$, we find that the metric can be written as

$$\begin{aligned} ds_5^2 &= \frac{1}{(x-y)^2}\Big[\frac{x dx^2}{4G(x)}+G(x)d\phi^2-\frac{x dy^2}{4G(y)} \\ &\quad -\frac{xG(y)d\psi^2}{y}\Big]-\frac{a_0 y}{x}(dt+y^{-1}d\psi)^2\,, \\ G(\xi) &= -a_0+a_1\xi+a_2\xi^2+a_3\xi^3\,, \end{aligned} \tag{4.23}$$

where the parameters (a_1,a_2,a_3) are related to (c_0,c_1,c_2) in (4.21) by

$$c_0=\frac{a_1+a_2+a_3}{q_4^2}\,, \qquad c_1=2(a_2+2a_3)\,, \qquad c_2=4a_3q_4^2\,. \tag{4.24}$$

Then performing the triple Wick rotation

$$\psi \to \mathrm{i}\psi\,, \qquad \phi \to \mathrm{i}\phi\,, \qquad t \to \mathrm{i}t\,, \tag{4.25}$$

together with $a_0 \to -a_0$, we find that the metric (4.23) has the form given by (2.13)-(2.14) in [140]. Note that the above triple Wick rotation is equivalent to a double Wick rotation on (x,y), which corresponds to taking $a_3 \to -a_3$.

Since $a_3 = -\mu^2 < 0$ in the black ring analysis of [140], we can let $a_3 = \mu^2 > 0$ and hence

$$G(\xi) = \mu^2(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)\,, \qquad a_0 = \mu^2\xi_1\xi_2\xi_3 > 0\,. \tag{4.26}$$

We shall consider two choices for coordinate ranges.

Case 1

We can consider $x \in [\xi_1, \xi_2]$ and $y \in [\xi_2, \xi_3]$ with $\xi_3 > \xi_2 > \xi_1 > 0$, so that $G(x) \geq 0$ and $G(y) \leq 0$. In contrast to the point $y = 0$ for the solution discussed in [140], in the present case there is no ergo-region. In fact, it turns out that this case describes a smooth soliton which has a region with closed timelike curves (CTC's).

Case 2

Alternatively, we can consider

$$G(\xi) = -\mu^2(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)\,, \qquad a_0 = -\mu^2\xi_0\xi_1\xi_2 > 0\,, \tag{4.27}$$

with $x \in [\xi_1, \xi_2]$, $y \in [\xi_0, \xi_1]$ and $\xi_0 < 0 < \xi_1 < \xi_2$, so that $G(x) \geq 0$ and there is a horizon at $y = \xi_0 < 0$. Then $y = 0$ constitutes an ergo-region. This is rather similar to the case considered in [140], even though one needs to perform a triple Wick rotation in order to relate them.

Following the analysis in [140], we define

$$\xi_0 = -\eta_0^2\,, \qquad \xi_1 \equiv \eta_1^2 < \xi_2 \equiv \eta_2^2\,, \tag{4.28}$$

such that $a_0 = \mu^2\eta_0^2\eta_2^2\eta_3^2$ and all η_i are positive with $\eta_1 < \eta_2$. The ϕ direction collapses at $x = \eta_1^2$ and $x = \eta_2^2$. In order to avoid a conical singularity, we must have

$$\eta_0 = \sqrt{\eta_1\eta_2}\,, \tag{4.29}$$

as well as the periodicity condition $\Delta\phi_2 = 2\pi$, where

$$\phi_2 = \mu^2(\eta_2-\eta_1)(\eta_1+\eta_2)^2\phi\,. \tag{4.30}$$

In order to avoid CTC's at $y = \xi_1$, we take

$$t \to t - \frac{\psi}{\eta_1^2}\,. \tag{4.31}$$

Then in order to avoid a conical singularity at $y = \xi_1$ we must have $\Delta\phi_1 = 2\pi$, where

$$\phi_1 = \mu^2(\eta_2-\eta_1)(\eta_1+\eta_2)^2\psi\,. \tag{4.32}$$

In order to determine the horizon, we first need to find the asymptotic region where t is appropriately defined. This requires that

$$t \to \frac{t}{\mu(\eta_1\eta_2)^{\frac{3}{2}}}\,. \tag{4.33}$$

Making a coordinate transformation

$$\frac{\sqrt{x-\xi_1}}{x-y} = \frac{\mu(\eta_2+\eta_1)\sqrt{\eta_2-\eta_1}}{\sqrt{\eta_1}}\, r\cos\theta\,, \qquad \frac{\sqrt{\xi_1-y}}{x-y} = \frac{\mu(\eta_2+\eta_1)\sqrt{\eta_2-\eta_1}}{\sqrt{\eta_1}}\, r\sin\theta\,, \tag{4.34}$$

and then letting $r \to \infty$ yields

$$ds^2 = -dt^2+dr^2+r^2(d\theta^2+\cos^2\theta\, d\phi_2^2+\sin^2\theta\, d\phi_1^2)\,. \tag{4.35}$$

On the horizon $y = \xi_0 = -\eta_0^2$, the null Killing vector is given by

$$\ell = \frac{\partial}{\partial_t}+\Omega_\psi\frac{\partial}{\partial_\psi}\,, \qquad \Omega_\psi = \frac{\mu(\eta_2^2-\eta_1^2)\sqrt{\eta_1}}{\eta_2}\,, \tag{4.36}$$

and the surface gravity is

$$\kappa = \mu\eta_1(\eta_1+\eta_2)\,. \tag{4.37}$$

Therefore, this solution indeed describes a black ring. This solution, in slightly different coordinates, was first shown to describe a black ring in [113].

4.3.2 $D = 6$ Ricci-flat metric

Using the reduction ansatz (93) to lift the solution (4.9) to six dimensions yields

$$ds_6^2 \;=\; -\frac{U_4}{\sqrt{U_1U_2}}\Big(dt+\frac{4Q_4y}{h_4(y)}d\psi\Big)^2+\frac{\sqrt{U_1U_2}}{U_3}\Big(dz+\frac{4Q_3x}{h_3(x)}d\varphi\Big)^2$$

$$
\begin{aligned}
&+\frac{1}{\alpha^2(x-y)^2}\Big[\frac{1}{U_4\sqrt{U_1U_2}}\Big(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\Big) \\
&\qquad +U_3\sqrt{U_1U_2}\Big(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\varphi^2\Big)\Big]\,, \\
\hat{B} &= \frac{4Q_1x}{h_1(x)}d\varphi\wedge dz+\frac{4Q_2y}{h_2(y)}dt\wedge d\psi\,, \qquad e^{\sqrt{2}\,\phi_1}=\frac{U_2}{U_1}\,.
\end{aligned} \tag{4.38}
$$

As in the five-dimensional case, it is subtle to take the Ricci-flat limit of this solution by setting $Q_1=Q_2=0=q_1=q_2$, since information is lost upon setting $Q_1=Q_2$ and then $q_1=q_2$. However, this can be remedied by first taking the specialization

$$
q_i=-\mu s_i^2\,, \qquad Q_i=\mu s_i c_i\,, \qquad s_i=\sinh\delta_i\,, \qquad c_i=\cosh\delta_i\,, \qquad i=1,2. \tag{4.39}
$$

Then setting $s_i=0$ and renaming the integration constants yields

$$
G(\xi)=(b_0+b_1x)h_3(\xi)h_4(\xi)+\frac{16Q_4^2\,h_3(\xi)}{q_4^3(q_3-q_4)}+\frac{16Q_3^2\,h_4(\xi)}{q_3^3(q_4-q_3)}\,. \tag{4.40}
$$

This expression can also be obtained by inserting the form of the solution (4.38) directly into the equations of motion. Redefining (b_0,b_1) appropriately results in

$$
\begin{aligned}
ds_6^2 &= -U_4\Big(dt+\frac{4Q_4y}{h_4(y)}d\psi\Big)^2+\frac{1}{U_3}\Big(dz+\frac{4Q_3x}{h_3(x)}d\varphi\Big)^2 \\
&\quad +\frac{1}{\alpha^2(x-y)^2}\Bigg[\frac{1}{U_4}\Big(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\Big) \\
&\quad +U_3\Big(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\varphi^2\Big)\Bigg]\,,
\end{aligned}
$$

$$G(\xi) = (b_0+b_1x)h_3(\xi)h_4(\xi)+\frac{16Q_4^2\,h_3(\xi)\xi^2}{q_4(q_3-q_4)}+\frac{16Q_3^2\,h_4(\xi)\xi^2}{q_3(q_4-q_3)}. \quad (4.41)$$

Taking $q_i \to 1/q_i$ and then shifting (x,y) so that $U_4(x,y) = y/x$ enables one to redefine parameters such that the metric can be written as

$$\begin{aligned} ds^2 &= -\frac{y}{x}\Big(dt+Q_4y^{-1}d\psi\Big)^2+\frac{x+q_3}{y+q_3}\Big(dz+\frac{Q_3}{x+q_3}d\phi\Big) \\ &+\frac{1}{(x-y)^2}\Big[-\frac{x}{y}\Big(G(y)d\psi^2+\frac{\mathcal{H}(y)}{G(y)}dy^2\Big) \\ &+\frac{y+q_3}{x+q_3}\Big(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\phi^2\Big)\Big]\,, \end{aligned} \quad (4.42)$$

where

$$G(\xi) = (c_0+c_1\xi)\xi(\xi+q_3)+q_3^{-1}Q_3^2\xi-q_3^{-1}Q_4^2(\xi+q_3)\,, \quad \mathcal{H}(\xi) = \xi(\xi+q_3)\,. \quad (4.43)$$

Note also that the q_4 parameter is also trivial and can be absorbed. Curvature singularities occur at $x = \pm\infty$, $y = \pm\infty$, $x = 0$ and $y = -q_3$. We can express

$$G(\xi_i) = -\mu^2(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)\,, \quad (4.44)$$

where $\xi_0 = -\eta_0^2 < 0$ and $\xi_i = \eta_i^2$ $(i = 1,2)$ with $\eta_2 > \eta_1$. In this parametrization, we have

$$Q_4 = \mu\eta_0\eta_1\eta_2\,, \qquad Q_3 = \mu\sqrt{(\eta_0^2-q_3)(\eta_1^2+q_3)(\eta_2^2+q_3)}\,. \quad (4.45)$$

We can consider the coordinate ranges $x \in [\xi_1,\xi_2]$ and $y \in [\xi_0,\xi_1]$ so that $G(x) \geq 0$ and $G(y) \leq 0$. Next, we make coordinate transformations

$$t \;\to\; t+\frac{2\eta_0\eta_2\sqrt{\eta_1^2+q_3}}{(\eta_0^2+\eta_1^2)(\eta_2^2-\eta_1^2)\mu}\phi_1\,, \qquad \psi = \frac{2\eta_1\sqrt{\eta_1^2+q_3}}{(\eta_0^2+\eta_1^2)(\eta_2^2-\eta_1^2)\mu}\phi_1\,,$$

$$z \quad \to \quad z - \frac{2\eta_1\sqrt{(\eta_0^2 - q_3)(\eta_2^2 + q_3)}}{(\eta_0^2 + \eta_1^2)(\eta_2^2 - \eta_1^2)\mu}\phi_2\,,\; \phi = \frac{2\eta_1\sqrt{\eta_0^2 - q_3}}{(\eta_0^2 + \eta_1^2)(\eta_2^2 - \eta_1^2)\mu}\phi_2\,. \quad (4.46)$$

In the new coordinates, the absence of conical singularities at $x = \xi_1$ and $x = \xi_2$ requires that

$$\Delta\phi_2 = 2\pi = \Delta\phi_1\,. \qquad (4.47)$$

The absence of a conical singularity at $x = \xi_2$ implies that $\Delta z = 2\pi$. The horizon is located at $y = \xi_0$. The asymptotic region $x = \xi_1 = y$ has the geometry $(\mathrm{Mink})_5 \times S^1$, where S^1 corresponds to the z direction. The horizon topology is $S^3 \times S^1$, where the S^1 lies along the ϕ_1 direction and the S^3 corresponds to the (x, ϕ_2, z) directions.

In order for the asymptotic region to have the geometry $(\mathrm{Mink})_6$ instead of $(\mathrm{Mink})_5 \times S^1$, we need to decompactify the z direction, by taking $Q_3 = 0$, which corresponds to setting $q_3 = \eta_0^2$. One can then attempt to avoid a conical singularity at $x = \xi_{1,2}$ by taking the periodicity condition $\Delta\phi_2 = 2\pi$. However, it turns that this requires that $q_3 = 0$, and so a conical singularity cannot be avoided.

4.4 Global properties and thermodynamics of black rings

4.4.1 Black rings with triple dipole charges

We shall now study the global structure of black rings that carry three dipole charges and obtain the first law of thermodynamics. As it is written in (4.9), the local solution is over-parameterized. It is advantageous to make the reparametrization

$$q_i = \frac{1}{\tilde{q}_4 + \tilde{q}_i}\,, \quad Q_i = -\frac{\widetilde{Q}_i}{4(\tilde{q}_4 + \tilde{q}_i)^2\zeta}\,, \quad Q_4 = -\frac{\tilde{Q}_4}{4\tilde{q}_4^2\zeta}\,, \quad i = 1, 2, 3, \tag{4.48}$$

where $\zeta = \sqrt{\tilde{q}_4(\tilde{q}_4 + \tilde{q}_1)(\tilde{q}_4 + \tilde{q}_2)(\tilde{q}_4 + \tilde{q}_3)}$, followed by the coordinate transformation

$$x = \tilde{x} - \tilde{q}_4\,, \qquad y = \tilde{y} - \tilde{q}_4\,, \qquad \varphi = \zeta\tilde{\varphi}\,, \qquad \psi = \zeta\tilde{\psi}\,. \tag{4.49}$$

We then drop the tilde and redefine $h_i(\xi)$ as

$$h_i(\xi) \equiv \xi + q_i\,, \qquad i = 1, 2, 3\,, \tag{4.50}$$

and $h_4(\xi) = \xi$. It turns out that the U_i are given by the same expressions as before, namely

$$U_i = \frac{h_i(y)}{h_i(x)}\,, \qquad U = U_1 U_2 U_3\,. \tag{4.51}$$

The scalar fields and gauge potentials are now given by

$$e^{\vec{b}_i \cdot \vec{\phi}} = \frac{U^{\frac{2}{3}}}{U_i^2}\,, \qquad A_i = \frac{Q_i}{h_i(x)} d\varphi\,, \qquad i = 1, 2, 3. \tag{4.52}$$

The metric can now be written as

$$
\begin{aligned}
ds_5^2 &= U^{-\frac{1}{3}}\Big(-\frac{y}{x}(dt+Q_4 y^{-1}d\psi)^2+ds_4^2\Big)\,, &(4.53)\\
ds_4^2 &= \frac{1}{(x-y)^2}\Big[\frac{x}{y}\Big(-G(y)d\psi^2-\frac{\mathcal{H}(y)}{G(y)}dy^2\Big)\\
&\quad +U\Big(\frac{\mathcal{H}(x)}{G(x)}dx^2+G(x)d\phi^2\Big)\Big]\,, &(4.54)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}(\xi) &= \xi\, h_1(\xi)h_2(\xi)h_3(\xi)\,,\\
G(\xi) &= \mathcal{H}(\xi)\Big(b_0+\frac{Q_1^2}{q_1(q_1-q_2)(q_1-q_3)h_1(\xi)}+\frac{Q_2^2}{q_2(q_2-q_1)(q_2-q_3)h_2(\xi)}\\
&\quad +\frac{Q_3^2}{q_3(q_3-q_1)(q_3-q_2)h_3(\xi)}-\frac{Q_4^2}{q_1q_2q_3\xi}\Big)\,. &(4.55)
\end{aligned}
$$

An advantage of this parameterization is that q_4 drops out from the solution completely. We have also set $\alpha=1$ without loss of generality.

We are now in the position to study the global structure of the solution. The asymptotic region is located at $x=y$. Curvature singularities arise when either $h_i(x)$ or $h_i(y)$ vanishes or when either x or y diverges. Thus, we should arrange that the ranges of the coordinate x and y are confined to intervals that are specified by adjacent roots of $G(\xi)$, within which the $h_i(\xi)$ are non-vanishing. It is therefore more convenient to express $G(\xi)$ in terms of four roots, namely

$$
G(\xi)=-\mu^2(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)\,. \tag{4.56}
$$

Here we let $b_0 = -\mu^2 < 0$. The charge parameters Q_i and Q_4 can be expressed in terms of μ and the four roots as

$$\begin{aligned} Q_i^2 &= \mu^2(q_i+\xi_1)(q_i+\xi_2)(q_i+\xi_3)(q_i+\xi_4) \geq 0\,, \\ Q_4^2 &= \mu^2\xi_1\xi_2\xi_3\xi_4 \geq 0\,. \end{aligned} \tag{4.57}$$

As with the Ricci-flat metrics discussed in the previous section, the signs of $G(x) \geq 0$ and $G(y) \leq 0$ should be such that the metric has the proper signature. Without loss of generality, we consider $x \in [\xi_3, \xi_4]$ and $y \in [\xi_2, \xi_3]$.

The reality of Q_4 leads to two possibilities. The first one is where $0 < \xi_1 < \xi_2 < \xi_3 < \xi_4$. After a careful analysis, we find that the absence of naked curvature power-law singularities, together with the reality condition on Q_i, implies that the solution has an unavoidable naked conical singularity. This conclusion may not be too surprising, given that we would otherwise have a rotating black hole without an ergo-region. The natural location of the ergo-region is $y = 0$, which leads to the second case, for which

$$\xi_1 < \xi_2 < 0 < \xi_3 < \xi_4\,. \tag{4.58}$$

This ensures that the ergo-region at $y = 0$ lies in the range $y \in [\xi_2, \xi_3]$. The metric has no naked power-law curvature singularities provided that

$$h_i(x) > 0 \qquad \text{for} \qquad x \in [\xi_3, \xi_4]\,,$$

$$h_i(y) > 0 \qquad \text{for} \qquad y \in [\xi_2, \xi_3]\,. \tag{4.59}$$

Now we ensure that the solution does not have conical singularities. The null Killing vectors with unit Euclidean surface gravity at $x = \xi_3$ and $x = \xi_4$ are given by

$$\ell_{x=\xi_3} = \alpha_3 \partial_\phi\,, \qquad \ell_{x=\xi_4} = \alpha_4 \partial_\phi\,, \tag{4.60}$$

where

$$\alpha_3 = \frac{2\sqrt{\xi_3(\xi_3+q_1)(\xi_3+q_2)(\xi_3+q_3)}}{\mu^2(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_4-\xi_3)}\,, \quad \alpha_4 = \frac{2\sqrt{\xi_4(\xi_4+q_1)(\xi_4+q_2)(\xi_4+q_3)}}{\mu^2(\xi_4-\xi_1)(\xi_4-\xi_2)(\xi_4-\xi_3)}\,. \tag{4.61}$$

The absence of conical singularities then requires that $\alpha_3 = \alpha_4$. This condition and (4.59) can be shown to be simultaneously satisfied for the appropriate choice of parameters. As an example, consider $\xi_1 = -2$, $\xi_2 = -1$, $\xi_3 = 1$ and $\xi_4 = 2$, for which the absence of conical singularities requires

$$\frac{(q_1+2)(q_2+2)(q_3+2)}{2(q_1+1)(q_2+1)(q_3+1)} = 1\,. \tag{4.62}$$

This can be solved, for example, with $q_1 = 2$, $q_2 = 3$ and $q_3 = 4$. All the $q_i \geq -2$, so that $Q_i \geq 0$, $h_i(x) > 0$ and $h_i(y) > 0$. Thus, there are no singularities in the region of interest.

Once we have established that $\alpha_3 = \alpha_4$ and that the charge parameters Q_i are real, there are no further conditions on the parameters. The absence of a conical singularity at $y = \xi_3$ tells us the appropriate period for the coordinate ψ, and the horizon condition gives the temperature and entropy. However, the algebraic expression for $\alpha_3 = \alpha_4$ is rather complicated to solve, which means that quantities such as mass, charges and temperature can be quite complicated.

To proceed, it is convenient to rewrite the roots as

$$\xi_1 = -\eta_1^2\,, \qquad \xi_2 = -\eta_2^2\,, \qquad \xi_3 = \eta_3^2\,, \qquad \xi_4 = \eta_4^2\,, \tag{4.63}$$

with $\eta_1 > \eta_2 > 0$ and $\eta_4 > \eta_3 > 0$. The avoidance of naked conical singularities requires that

$$\frac{\eta_4(\eta_1^2+\eta_3^2)(\eta_2^2+\eta_3^2)\sqrt{(q_1+\eta_4^2)(q_2+\eta_4^2)(q_3+\eta_4^2)}}{\eta_3(\eta_1^2+\eta_4^2)(\eta_2^2+\eta_4^2)\sqrt{(q_1+\eta_3^2)(q_2+\eta_3^2)(q_3+\eta_3^2)}} = 1\,. \tag{4.64}$$

We define a new set of coordinates, given by

$$\begin{aligned} &\phi = a\,\phi_2\,, \qquad \psi = a\,\phi_1\,, \qquad t \to t - \frac{Q_4 a}{\eta_3^2}\phi_1\,, \\ &a = \frac{2\eta_3\sqrt{(q_1+\eta_3^2)(q_2+\eta_3^2)(q_3+\eta_3^2)}}{\mu^2(\eta_1^2+\eta_3^2)(\eta_2^2+\eta_3^2)(\eta_4^2-\eta_3^2)}\,. \end{aligned} \tag{4.65}$$

Then the periods of ϕ_1 and ϕ_2 are both 2π. Note that the shift in t ensures that only the spatial coordinate ϕ_1 collapses to zero size at $y = \xi_3$, and hence we avoid naked closed timelike curves (CTC's).

On the horizon at $y = \xi_2$, the null Killing vector is

$$\ell = \partial_t + \Omega_+ \partial_{\phi_1}\,, \qquad \Omega_+ = \frac{\eta_2^2\eta_3^2}{a(\eta_2^2+\eta_3^2)Q_4}\,. \tag{4.66}$$

The (Euclidean) surface gravity is given by

$$\kappa^2 = -\frac{\mu^4\eta_2^2\eta_3^2(\eta_1^2-\eta_2^2)^2(\eta_2^2+\eta_4^2)^2}{4(q_1-\eta_1^2)(q_2-\eta_2^2)(q_3-\eta_3^2)Q_4^2}\,. \tag{4.67}$$

For the solution to describe a black object with a horizon, we must have $\kappa^2 < 0$. It is worth checking that this condition can indeed be satisfied. As an example, we take

$$\xi_1 = -2\,, \qquad \xi_2 = -1\,, \qquad \xi_3 = 1\,, \qquad \xi_4 = 4\,. \tag{4.68}$$

An acceptable set of q_i with $i = 1,2,3$ must satisfy (4.64) and all have $q_i > -2$. Such solutions do in fact exist; for instance,

$$q_1 = \tfrac{7}{3}\,, \qquad q_2 = \tfrac{9}{4}\,, \qquad q_3 = \tfrac{68971}{27389}\,. \tag{4.69}$$

For this choice of parameters the Q_i are real, $h_i(x)$ and $h_i(y)$ are positive definite in the regions of concern, $\kappa^2 < 0$ and χ is real. Thus, a well-behaved black ring exists. The temperature and entropy are given by

$$T = \frac{\kappa}{2\pi}\,, \qquad S = \frac{\pi^2 a^2 Q_4(\eta_4^2-\eta_3^2)\sqrt{(q_1-\eta_2^2)(q_2-\eta_2^2)(q_3-\eta_2^2)}}{\eta_2\eta_3^3(\eta_2^2+\eta_4^2)}\,. \tag{4.70}$$

When $\eta_1 = \eta_2$, the temperature vanishes, corresponding to the extremal limit.

The asymptotic region is located at $x = \xi_3 = y$. To see this, we make the coordinate transformation

$$\frac{\sqrt{x-\xi_3}}{x-y} = b\,r\cos\theta\,, \qquad \frac{\sqrt{\xi_3-y}}{x-y} = b\,r\sin\theta\,,$$

$$b^2 = \frac{\mu^2(\eta_1^2+\eta_3^2)(\eta_2^2+\eta_3^2)(\eta_4^2-\eta_3^2)}{4\eta_3^2(\eta_3^2+q_1)(\eta_3^2+q_2)(\eta_3^2+q_3)}\,, \tag{4.71}$$

and then we take $r \to \infty$. The asymptotic geometry is five-dimensional Minkowski spacetime, with the metric written as

$$ds_5^2 = -dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\phi_1^2+\cos^2\theta d\phi_2^2)\,. \tag{4.72}$$

From the asymptotic falloffs of the metric, we can read off the ADM mass and the angular momenta as

$$M = \frac{\pi}{8b^2}\Big(\frac{3}{\eta_3^2}-\frac{1}{\eta_3^2+q_1}-\frac{1}{\eta_3^2+q_2}-\frac{1}{\eta_3^2+q_3}\Big)\,, \tag{4.73}$$

and

$$J_{\phi_1} = \frac{\pi a Q_4}{4b^2\eta_3^4}\,, \qquad J_{\phi_2} = 0\,, \tag{4.74}$$

respectively. The dipole charges are given by

$$D_i = \tfrac{1}{8}\int F_i = \tfrac{1}{4}\pi a Q_i\Big(\frac{1}{q_i+\eta_3^2}-\frac{1}{q_i+\eta_4^2}\Big)\,. \tag{4.75}$$

The only quantities left to determine are the dipole potentials Φ_{D_i}, which requires the dualization of A^i_μ to $B^i_{\mu\nu}$. Since $F_i\wedge F_j = 0$, we do not need to worry about the FFA term in the Lagrangian when performing the dualization. We find

$$e^{\vec{b}_i\cdot\vec{\phi}}{*F_i} = \frac{aQ_i}{(y+q_i)^2}dt\wedge d\phi_1\wedge dy\,, \qquad B_i = -\frac{aQ_i}{y+q_i}dt\wedge d\phi_1\,. \tag{4.76}$$

The 2-form potential difference between the horizon and the asymptotic region is then given by

$$\Phi_{D_i} = aQ_i\Big(\frac{1}{q_i-\eta_2^2} - \frac{1}{q_i+\eta_3^2}\Big)\,. \tag{4.77}$$

Having obtained all of the thermodynamic quantities, it is straightforward to verify that the first law of the thermodynamics,

$$dM = TdS + \Omega_+ dJ_{\phi_1} + \sum_{i=1}^{3} \Phi_{D_i} dD_i\,, \tag{4.78}$$

is obeyed. The Smarr formula is given by

$$M = \tfrac{3}{2}TS + \tfrac{3}{2}\Omega_{\phi_1} J_{\phi_1} + \sum_{i=1}^{3} \tfrac{1}{2}\Phi_{D_i} D_i\,. \tag{4.79}$$

Interestingly enough, the Smarr formula is actually valid even without imposing the condition (4.64) that ensures the absence of naked conical singularities.

After performing the various changes of variables discussed in this section, the solution can be written in terms of the eight parameters: μ, q_i and η_I, where $i = 1, 2, 3$ and $I = 1, \ldots, 4$. However, the solution is over-parameterized by two trivial parameters. While the μ parameter can be absorbed by a "trombone" scaling of the metric and other fields, the second extra parameter is more subtle. Although the five-dimensional theory has only two scalars, the solution has three scalar charges q_i, one of which is therefore trivial. Thus, removing μ and q_3, we are left with six nontrivial parameters: q_1, q_2 and η_I. After imposing the condition (4.64) to avoid naked conical singularities, we are left with five parameters associated with the mass, one non-vanishing angular momentum, and three dipole charges. These dipole black rings were first found in [116].

The horizon geometry has the metric

$$
\begin{aligned}
ds_3^2 \;=\; & \Big(\frac{(x+q_1)(x+q_2)(x+q_3)}{(q_1-\eta_2^2)(q_2-\eta_2^2)(q_3-\eta_2^2)^2}\Big)^{\frac{1}{3}}\Big[\frac{(q_1-\eta_2^2)(q_2-\eta_2^2)(q_3-\eta_3^2)\,x}{\mu^2(x+\eta_1^2)(x+\eta_2^2)^3(\eta_4^2-x)(x-\eta_3^2)}dx^2 \\
& +\frac{a^2\mu^2(q_1-\eta_2^2)(q_2-\eta_2^2)(q_3-\eta_3^2)(x+\eta_1^2)(\eta_4^2-x)(x-\eta_3^2)}{(x+q_1)(x+q_2)(x+q_3)(x+\eta_2^2)}d\phi_2^2 \\
& +\frac{a^2Q_4^2(\eta_2^2+\eta_3^2)^2}{\eta_2^2\eta_3^4\,x}d\phi_1^2\Big]\,. \qquad (4.80)
\end{aligned}
$$

Since $x \in [\eta_3^2, \eta_4^2]$, it is clear that the horizon topology is $S^2 \times S^1$. With non-vanishing dipole charges, the black ring solution does not have a limit with a spherical horizon.

The black ring solution has an Extremal Vanishing Horizon (EVH) case for which the near-horizon geometry has an AdS_3 factor and is contained within the large class of near-horizon geometries that have been studied and classified in [136, 137, 138]. According to the proposed EVH/CFT correspondence, there is a two-dimensional CFT description of the low-energy excitations of the black ring in this case [139].

4.4.2 Electrically-charged black rings and naked CTC's

We shall now study the global structure of the black ring solutions with three dipole charges and three electric charges, which was obtained in section 2.2. Note that these solutions generalize those found in [122], for which only two out of the three dipole charges were independent parameters.

As before, it is convenient to make the reparametrizations (4.48) and the coordinate transformations (4.49). Furthermore, we make a redefinition of the time coordinate,

$$\begin{aligned} t &= \tilde{t} - \Big(\frac{s_1 c_2 c_3 \widetilde{Q}_1}{\tilde{q}_4+\tilde{q}_1} + \frac{c_1 s_2 c_3 \widetilde{Q}_2}{\tilde{q}_4+\tilde{q}_2} + \frac{c_1 c_2 s_3 \widetilde{Q}_3}{\tilde{q}_4+\tilde{q}_3} + \frac{s_1 s_2 s_3 \widetilde{Q}_4}{\tilde{q}_4}\Big)\tilde{\varphi} \\ &\quad + \Big(\frac{c_1 s_2 s_3 \widetilde{Q}_1}{\tilde{q}_4+\tilde{q}_1} + \frac{s_1 c_2 s_3 \widetilde{Q}_2}{\tilde{q}_4+\tilde{q}_2} + \frac{s_1 s_2 c_3 \widetilde{Q}_3}{\tilde{q}_4+\tilde{q}_3} + \frac{c_1 c_2 c_3 \widetilde{Q}_4}{\tilde{q}_4}\Big)\tilde{\psi}\,. \end{aligned} \tag{4.81}$$

We may now drop the tilde, and redefine $h_i(\xi)$ as in (4.50). The functions H_i take the same form, namely

$$H_i = 1 + \Big(1 - \frac{y U_i^2}{xU}\Big) s_i^2\,, \tag{4.82}$$

where U_i and U are given by (4.51). The scalar fields retain the same form given by (4.15). The metric is now given by

$$ds_5^2 = \frac{1}{U^{1/3}(H_1H_2H_3)^{2/3}}\Big(-\frac{y}{x}\,(dt+\omega)^2 + H_1H_2H_3\, ds_4^2\Big)\,, \tag{4.83}$$

where ds_4^2 is precisely the same as (4.54) and ω is

$$\omega = -\Big(\frac{s_1 c_2 c_3 Q_1}{h_1(x)} + \frac{c_1 s_2 c_3 Q_2}{h_2(x)} + \frac{c_1 c_2 s_3 Q_3}{h_3(x)} + \frac{s_1 s_2 s_3 Q_4}{x}\Big) d\varphi$$

$$+\Big(\frac{c_1s_2s_3Q_1}{h_1(y)}+\frac{s_1c_2s_3Q_2}{h_2(y)}+\frac{s_1s_2c_3Q_3}{h_3(y)}+\frac{c_1c_2c_3Q_4}{y}\Big)d\psi\,. \qquad (4.84)$$

The gauge potentials are

$$\begin{aligned} A_1 &= \frac{s_1c_1}{H_1}\Big(1-\frac{U_1U_4}{U_2U_3}\Big)\,dt+ \\ &+\frac{1}{H_1}\,\Big(\frac{c_1c_2c_3Q_1}{h_1(x)}+\frac{s_1s_2c_3Q_2U_1U_4}{U_3h_2(y)}+\frac{s_1c_2s_3Q_3U_1U_4}{U_2h_3(y)}+\frac{c_1s_2s_3Q_4}{x}\Big)\,d\varphi \\ &-\frac{1}{H_1}\,\Big(\frac{s_1s_2s_3Q_1U_4}{U_2U_3h_1(x)}+\frac{c_1c_2s_3Q_2}{h_2(y)}+\frac{c_1s_2c_3Q_3}{h_3(y)}+\frac{s_1c_2c_3Q_4U_1}{U_2U_3\,x}\Big)\,d\psi, \qquad (4.85)\end{aligned}$$

with A_2 and A_3 being obtained from A_1 by cycling the indices 1, 2 and 3 on all quantities appearing in (4.85). Note that, aside from the coordinate transformations and reparametrizations, the A_i obtained above are related to those in (4.16) by gauge transformations.

It is of interest to note that the dipole charges are magnetic and are associated with the (x,φ) directions. Adding the electric charges has the effect of producing angular momentum in the φ direction as well. However, this also has the undesirable side effect of producing naked CTC's. To see this explicitly, it is useful to note that ds_4^2 is identical to that in the previous subsection and $G(\xi)$ can be expressed by (4.56) with (4.58). The null Killing vector associated with the collapsing circles at $x=\xi_3$ and $x-\xi_4$ are given by

$$\ell_{x=\xi_3}=\alpha_3\partial_\phi+\beta_3\partial_t\,,\qquad \ell_{x=\xi_4}=\alpha_4\partial_\phi+\beta_4\partial_t\,. \qquad (4.86)$$

The absence of CTC's requires that

$$\frac{\beta_3}{\alpha_3} = \frac{\beta_4}{\alpha_4}\,, \tag{4.87}$$

in which case we can shift $t \to t+\gamma\phi$, for an appropriate constant γ, such that the null Killing vectors do not involve the newly-defined time. For black rings with dipole charges but no electric charges, $\beta_3 = 0 = \beta_4$ and hence there are no naked CTC's. By contrast, we find that for black rings with both dipole and electric charges, the condition (4.87) cannot be satisfied and the solutions have naked CTC's. This parallels the situation for the five-dimensional black hole solutions studied in [140], for which naked CTC's appeared when the electric charges were nonvanishing. Turning on two independent angular momenta might result in black ring solutions with dipole and electric charges but without naked CTC's.

4.4.3 In background magnetic fields

We shall now consider some of the global properties of the solution with external magnetic fields, with the metric (4.18). This solution asymptotes to a five-dimensional Melvin fluxbrane, which is a higher-dimensional analog of Melvin's four-dimensional flux tube solution and describes the self-gravity of the background magnetic fields [134].

We use the same parameterizations as in section 4.1, with $G(\xi)$ given by

(4.56) and with the coordinate ranges $x \in [\xi_3, \xi_4]$ and $y \in [\xi_2, \xi_3]$. In order to avoid a conical singularity in the (x, ϕ) subspace, we must have

$$\prod_{i=1}^{3} \left(\frac{\xi_4 + q_i + B_i Q_i (\xi_4 - q_4)}{\xi_3 + q_i + B_i Q_i (\xi_3 - q_4)} \sqrt{\frac{\xi_3 + q_i}{\xi_4 + q_i}} \right) = \frac{(\xi_4 - \xi_1)(\xi_4 - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \sqrt{\frac{\xi_3}{\xi_4}} \,, \tag{4.88}$$

along with the appropriate periodicity for ψ.

First consider the case where $0 < \xi_1 < \xi_2 < \xi_3 < \xi_4$. While it is not possible to satisfy the condition (4.88) and the reality condition on the charge parameters Q_i at the same time for vanishing B_i, both conditions can be satisfied simultaneously when the B_i are turned on. A sample solution is given by

$$\begin{aligned} B_1 &= -1\,, \quad B_2 = -2\,, \quad B_3 = -3\,, \quad \xi_1 = 1\,, \quad \xi_2 = 2\,, \quad \xi_3 = 3\,, \quad \xi_4 = 4\,, \\ \mu &= 0.2\,, \quad q_2 = 1\,, \quad q_3 = 1.5\,, \quad q_4 = 2.5\,, \quad q_1 = 2.09789\,. \end{aligned} \tag{4.89}$$

Thus, at the expense of altering the asymptotic geometry, turning on background magnetic fields can have the effect of removing conical singularities for black rings, in much the same way as it does for Ernst solutions [141, 131].

Background magnetic fields also enable us to have multiple branches of solutions. For instance, in the case where $\xi_1 < \xi_2 < 0 < \xi_3 < \xi_4$ with vanishing B_i, a sample solution is given by

$$\xi_1 = -2\,, \; \xi_2 = -1\,, \; \xi_3 = 1\,, \; \xi_4 = 2\,, \; q_1 = 2\,, \; q_2 = 3\,, \; q_3 = 4\,. \tag{4.90}$$

Note that if only q_1 were to be left unspecified then the condition (4.88) would uniquely determine its value in terms of the other parameters. On the other hand, for nonvanishing B_i, sample solutions are given by

$$\begin{aligned} B_1 &= 1\,, \quad B_2 = 2\,, \quad B_3 = 3\,, \quad \xi_1 = -2\,, \quad \xi_2 = -1\,, \quad \xi_3 = 1\,, \quad \xi_4 = 2\,, \\ \mu &= 1\,, \quad q_2 = 2\,, \quad q_3 = 3\,, \quad q_4 = 4\,, \quad q_1 = 2.03367 \text{ or } 0.975202\,. \end{aligned} \tag{4.91}$$

Due to the presence of the B_i, using the condition (4.88) to solve for q_1 in terms of the other specified parameters yields two different solutions, both of which satisfy the reality condition on the charge parameters Q_i.

4.5 Conclusions

We have constructed black ring solutions in five-dimensional $U(1)^3$ supergravity, carrying three independent dipole charges, three electric charges and one non-vanishing angular momentum. We have also presented black ring solutions with three background magnetic fields. These various solutions have been obtained by lifting the Euclidean C-metric solution of four-dimensional ungauged STU supergravity [133] to five dimensions on a timelike direction, and then using solution-generating techniques involving dimensional reductions to add electric charges or background magnetic fields. We find that adding the electric charges gives rise to black rings with naked CTC's.

We expect that the solutions without the background magnetic fields should arise as special cases of the black ring solutions obtained in [128] if one of the angular momenta is set to zero. We have expressed this specialization in a form that is sufficiently compact that its various physical properties can be investigated explicitly. In particular, its global structure has been analyzed and the conditions determined in order for conical singularities and Dirac string singularities to be absent. Expressions for its mass, dipole charges, electric charges and angular momentum have been obtained, as well as the temperature and entropy. Moreover, we have analyzed the thermodynamics, finding that the Smarr formula is obeyed regardless of whether or not conical singularities are present. By contrast, the first law of thermodynamics is obeyed only in those cases where conical singularities are absent.

The four-dimensional Ernst-like generalization of the C-metric solution obtained in this chapter can be Wick rotated to a Euclidean instanton that describes the pair creation of black holes in magnetic fields. This generalizes the one-parameter family of instantons in [142, 143, 131] to multiple parameters. This substantially enhances the families of explicit examples for the creation of maximally entangled black holes, which have recently been proposed to be connected by some kind of Einstein-Rosen bridge [144].

The exact time-dependent C-metric solution constructed in [145] can be embedded in STU supergravity which, in the ungauged limit, can be lifted to five dimensions. It would be interesting to analyze the global structures of these five-dimensional solutions, especially with the prospect of finding time-dependent black rings.

Appendices

A. Dimensional Reduction Ansatz in D= 4, 5, 6

We present the Kaluza-Klein dimensional reductions that have been used to relate the four-dimensional C-metrics with the five-dimensional black ring solutions, as well as for the purposes of generating electric charges and background magnetic fields. We start with the $D=6$ theory whose Lagrangian is given by

$$e_6^{-1}\mathcal{L}_6 = R_6 - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{12}e^{\sqrt{2}\phi_1}\hat{H}^2\,, \tag{92}$$

where $\hat{H}=d\hat{B}$. Consider the reduction ansatz

$$\begin{aligned} ds_6^2 &= e^{-\frac{1}{\sqrt{6}}\phi_2}ds_5^2 + e^{\frac{3}{\sqrt{6}}\phi_2}(dz+A_3)^2\,,\\ \hat{B} &= B + A_1\wedge dz\,, \end{aligned} \tag{93}$$

where

$$F_2 = e^{-b_2\cdot\vec{\phi}}{*H}\,, \qquad H = dB - A_1\wedge A_3\,. \tag{94}$$

This yields the Lagrangian for the bosonic sector of five-dimensional $U(1)^3$ supergravity, given by

$$e_5^{-1}\mathcal{L}_5 = R_5 - \tfrac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{i=1}^{3}e^{\vec{b}_i\cdot\vec{\phi}}\hat{F}_i^2 + \mathcal{L}_{FFA}\,, \tag{95}$$

where $\vec{\phi}=(\phi_1,\phi_2)$ and the dilaton vectors $\vec{b}_i$ are given by

$$\vec{b}_1 = (\sqrt{2}, -\tfrac{2}{\sqrt{6}})\,, \qquad \vec{b}_2 = (-\sqrt{2}, -\tfrac{2}{\sqrt{6}})\,, \qquad \vec{b}_3 = (0, \tfrac{4}{\sqrt{6}})\,, \tag{96}$$

which obey

$$\vec{b}_i\cdot\vec{b}_j = 4\delta_{ij} - \tfrac{4}{3}\,, \qquad \sum_{i=1}^{3}\vec{b}_i = 0\,. \tag{97}$$

Next, we perform a dimensional reduction to the four-dimensional $U(1)^4$ theory.[2] Reducing on a spacelike direction with the metric ansatz

$$ds_5^2 = e^{-\frac{1}{\sqrt{3}}\phi_3} ds_4^2 + e^{\frac{2}{\sqrt{3}}\phi_3}(dz + A_4)^2\,, \tag{98}$$

yields

$$e_4^{-1}\mathcal{L}_4 = R_4 - \tfrac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{i=1}^{4} e^{\vec{a}_i\cdot\vec{\phi}}\hat{F}_i^2\,, \tag{99}$$

where we are now taking $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$. The dilaton vectors $\vec{a}_i$ are given by

$$\vec{a}_i = (\vec{b}_i, \tfrac{1}{\sqrt{3}})\,, \qquad \vec{a}_4 = (0, 0, -\sqrt{3})\,, \tag{100}$$

and they satisfy the conditions in (4.3).

Alternatively, we can reduce on the timelike direction with the metric ansatz

$$ds_5^2 = e^{-\frac{1}{\sqrt{3}}\phi_3} ds_4^2 - e^{\frac{2}{\sqrt{3}}\phi_3}(dt - \mathcal{A})^2\,, \tag{101}$$

which gives

$$e_4^{-1}\mathcal{L}_4 = R_4 - \tfrac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{4}\sum_{i=1}^{3} e^{\vec{a}_i\cdot\vec{\phi}}\hat{F}_i^2 + \frac{1}{4}e^{-\vec{a}_4\cdot\vec{\phi}}\mathcal{F}^2\,. \tag{102}$$

[2] Note that we are truncating out the three axions that would arise in the reductions of the three five-dimensional gauge fields. This truncation is consistent with the equations of motion provided that we restrict our attention to solutions for which $F^i\wedge F^j = 0$.

If we perform a Hodge dualization on the Kaluza-Klein vector, namely

$$e^{-\vec{a}_4\cdot\vec{\phi}} * \mathcal{F} = F_4\,, \tag{103}$$

then the kinetic term changes sign and the four-dimensional Lagrangian can be expressed in the more symmetric form given by (99).

B. Solution generating techniques

There are several classic solution generating techniques in GR and string theory. There are two of them that are frequently used in this thesis: Kaluza-Klein (KK) reduction and string dualities. We review them below.

B1. Kaluza-Klein reduction

Originally KK reduction was proposed as a way of unifying GR and electromagnetic force into a 5-dimensional gravity theory. Though the requirement of a nontrivial scalar renders this proposal implausible to direct phenomenology, it has become an important tool in string theory and higher-dimensional gravity theories. Here we review the simplest case of KK reduction by compactifying (D+1)-dimensional GR on an S^1 cycle. The 5d theory contains only gravitational field and the equation of motion is

$$\bar{R}_{\mu\nu} = 0, \tag{104}$$

where here and hereafter we refer to D+1 dimensional fields with bars and D-dimensional fields with no bars. Impose the condition that the solution is an S^1-bundle, and the solution ansatz can be written as

$$d\bar{s}^2_{D+1} = e^{2a\phi} ds^2_D + e^{2b\phi}(dz + A_\mu dx^\mu)^2. \tag{105}$$

z is the S^1 coordinate and x^μ ($\mu \in \{0, 1, 2, D-1\}$) is the D-manifold coordinate. Also note that the D-dimensional fields (the metric, gauge field and scalar) only depend on x^μ. The higher-dimensional EOM can be rewritten as

$$\begin{aligned}
\bar{R}_{\mu\nu} &= e^{-2a\phi}\left(R_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - a\eta_{\mu\nu}\Box\phi\right) - \frac{1}{2}e^{-2Da\phi}F^2 \\
\bar{R}_{\mu z} &= \frac{1}{2}e^{(D-3)a\phi}\nabla^\nu\left(e^{-2(D-1)a\phi}F_{\nu\mu}\right) \\
\bar{R}_{zz} &= (D-2)ae^{-2a\phi}\Box\phi + \frac{1}{4}e^{-2Da\phi}F^2,
\end{aligned} \tag{106}$$

with

$$a = \frac{1}{\sqrt{2(D-1)(D-2)}}, \quad b = -(D-2)a\,. \tag{107}$$

Therefore the higher-dimensional EOM become that of the lower dimensional theory with additional field content. An important fact is that, in order to make the reduction consistent, (107) has to be satisfied. In application, when a theory can be viewed as KK reduction from a higher-dimensional theory it is called a KK theory. Also note that the example can be generalized to

i) adding extra fields to the original theory and ii) replacing the S^1 with other manifolds. The latter is usually quite complicated when the compact manifold has nontrivial higher homology groups (i.e. S^n).

B2. String dualities

String theory has some unique and intrinsic dualities which relate two different string theories. Among them, T-duality is a very powerful tool to generate solutions. If we assume the spacetime background has a compact S^1 dimension, then by exchanging the Kaluza-Klein momentum with the winding mode and reversing the S^1 radius one can leave the string state spectrum intact. It therefore exchanges one string theory with a bigger cycle to another one with smaller cycle. From the spacetime point of view, this transformation changes the metric, gauge fields and scalars if they depend on the S^1 direction. In this thesis we will mostly consider T-duality in type II theories which exchanges IIA and IIB and swaps various field contents.

As low energy effective theory of string theory, the corresponding SUGRA's also inherit T-duality. One then can apply T-duality to a solution that has a S^1 direction and obtain another solution in dual theory. When there are more than one S^1 cycles in the original solution, one can extend this program further by adding an $SL(n, R)$ transformation, where n is the total number of

S^1's. For example, if one starts with a solution of IIA SUGRA that has an T^2 submanifold, one can T-dualize to a solution in IIB, $SL(2, R)$ transform and then T-dualize back along a new S^1 direction to IIA. As a consequence, the resultant solution automatically satisfies the EOM of IIA and are different from the original one.

Bibliography

[1] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1915**, 778 (1915) Addendum: [Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1915**, 799 (1915)].

[2] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. **116**, no. 6, 061102 (2016) doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

[3] S. W. Hawking and R. Penrose, Proc. Roy. Soc. Lond. A **314**, 529 (1970). doi:10.1098/rspa.1970.0021

[4] G. 't Hooft and M. J. G. Veltman, Annales Poincare Phys. Theor. A **20**, 69 (1974).

[5] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Fortsch. Phys. **59**, 561 (2011) doi:10.1002/prop.201100037 [arXiv:1103.1848 [hep-th]].

[6] T. Banks, arXiv:1205.5768 [hep-th].

[7] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond,"

[8] G. T. Horowitz and A. Strominger, Nucl. Phys. B **360**, 197 (1991). doi:10.1016/0550-3213(91)90440-9

[9] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D **13**, 3214 (1976). doi:10.1103/PhysRevD.13.3214

[10] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66**, 291 (1979). doi:10.1007/BF01197189

[11] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] doi:10.1023/A:1026654312961 [hep-th/9711200].

[12] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[13] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[14] S. Kachru and E. Silverstein, Phys. Rev. Lett. **80**, 4855 (1998) doi:10.1103/PhysRevLett.80.4855 [hep-th/9802183].

[15] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP **0810**, 091 (2008) doi:10.1088/1126-6708/2008/10/091 [arXiv:0806.1218 [hep-th]].

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[17] E. Ó Colgáin, M. M. Sheikh-Jabbari, J. F. Vzquez-Poritz, H. Yavartanoo and Z. Zhang, Phys. Rev. D **90**, no. 4, 045013 (2014) doi:10.1103/PhysRevD.90.045013 [arXiv:1406.6354 [hep-th]].

[18] G. W. Gibbons and C. M. Hull, Phys. Lett. B **109**, 190 (1982).

K. p. Tod, Phys. Lett. B **121**, 241 (1983).

[19] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, Class. Quant. Grav. **20**, 4587 (2003) [hep-th/0209114].

J. B. Gutowski, D. Martelli and H. S. Reall, Class. Quant. Grav. **20** (2003) 5049 [hep-th/0306235].

[20] C. P. Burgess, R. Kallosh and F. Quevedo, JHEP **0310**, 056 (2003) [hep-th/0309187].

A. Saltman and E. Silverstein, JHEP **0411**, 066 (2004) [hep-th/0402135].

V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, JHEP **0503**, 007 (2005) [hep-th/0502058].

[21] C. Caviezel, P. Koerber, S. Kors, D. Lust, T. Wrase and M. Zagermann, JHEP **0904**, 010 (2009) [arXiv:0812.3551 [hep-th]].

S. S. Haque, G. Shiu, B. Underwood and T. Van Riet, Phys. Rev. D **79**, 086005 (2009) [arXiv:0810.5328 [hep-th]].

L. Covi, M. Gomez-Reino, C. Gross, J. Louis, G. A. Palma and C. A. Scrucca, JHEP **0806**, 057 (2008) [arXiv:0804.1073 [hep-th]].

[22] S. Kachru, R. Kallosh, A. D. Linde and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003) [hep-th/0301240].

[23] I. Bena, M. Grana and N. Halmagyi, JHEP **1009**, 087 (2010) [arXiv:0912.3519 [hep-th]].

[24] U. Danielsson and G. Dibitetto, JHEP **1405**, 013 (2014) [arXiv:1312.5331 [hep-th]].

J. Blaback, D. Roest and I. Zavala, arXiv:1312.5328 [hep-th].

F. Hassler, D. Lust and S. Massai, arXiv:1405.2325 [hep-th].

R. Kallosh, A. Linde, B. Vercnocke and T. Wrase, arXiv:1406.4866 [hep-th].

[25] S. S. Gubser, Adv. Theor. Math. Phys. **4**, 679 (2000) [hep-th/0002160].

[26] I. Bena, A. Buchel and O. J. C. Dias, Phys. Rev. D **87**, no. 6, 063012 (2013) [arXiv:1212.5162 [hep-th]];

J. Blaback, U. H. Danielsson and T. Van Riet, JHEP **1302**, 061 (2013) [arXiv:1202.1132 [hep-th]];

I. Bena, A. Buchel and O. J. C. Dias, Phys. Rev. D **87**, no. 6, 063012 (2013) [arXiv:1212.5162 [hep-th]].

[27] D. Junghans, D. Schmidt and M. Zagermann, arXiv:1402.6040 [hep-th].

[28] J. M. Maldacena and C. Núñez, Int. J. Mod. Phys. A **16**, 822 (2001) [hep-th/0007018].

[29] M. M. Sheikh-Jabbari and H. Yavartanoo, JHEP **1110**, 013 (2011) [arXiv:1107.5705 [hep-th]].

[30] R. Fareghbal, C. N. Gowdigere, A. E. Mosaffa and M. M. Sheikh-Jabbari, Phys. Rev. D **81**, 046005 (2010) [arXiv:0805.0203 [hep-th]].

[31] R. Fareghbal, C. N. Gowdigere, A. E. Mosaffa and M. M. Sheikh-Jabbari, JHEP **0808**, 070 (2008) [arXiv:0801.4457 [hep-th]].

[32] J. de Boer, M. Johnstone, M. M. Sheikh-Jabbari and J. Simon, Phys. Rev. D **85** (2012) 084039 [arXiv:1112.4664 [hep-th]].

M. Johnstone, M. M. Sheikh-Jabbari, J. Simon and H. Yavartanoo, JHEP **1304** (2013) 045 [arXiv:1301.3387].

[33] O. DeWolfe, S. S. Gubser and C. Rosen, arXiv:1312.7347 [hep-th].

[34] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lü, J. X. Lu, R. Martinez-Acosta, C. N. Pope H. Sati and T. A. Tran, Nucl. Phys. B **558**, 96 (1999) [hep-th/9903214].

[35] N. Bobev, N. Halmagyi, K. Pilch and N. P. Warner, Class. Quant. Grav. **27**, 235013 (2010) [arXiv:1006.2546 [hep-th]].

[36] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Phys. Lett. B **139**, 154 (1984).

[37] M. Berkooz and S. -J. Rey, JHEP **9901**, 014 (1999) [Phys. Lett. B **449**, 68 (1999)] [hep-th/9807200].

[38] O. DeWolfe, D. Z. Freedman, S. S. Gubser, G. T. Horowitz and I. Mitra, Phys. Rev. D **65**, 064033 (2002) [hep-th/0105047].

[39] X. Dong, B. Horn, E. Silverstein and G. Torroba, Class. Quant. Grav. **27**, 245020 (2010) [arXiv:1005.5403 [hep-th]].

[40] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Class. Quant. Grav. **21**, 4335 (2004) [hep-th/0402153].

[41] M. M. Caldarelli, J. Camps, B. Goutraux and K. Skenderis, Phys. Rev. D **87**, no. 6, 061502 (2013) [arXiv:1211.2815 [hep-th]].

M. M. Caldarelli, J. Camps, B. Goutraux and K. Skenderis, JHEP **1404**, 071 (2014) [arXiv:1312.7874 [hep-th]].

[42] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, Phys. Rev. D **71**, 024033 (2005) [hep-th/0408120].

[43] E. Ó Colgáin and O. Varela, Phys. Lett. B **703**, 180 (2011) [arXiv:1106.4781 [hep-th]].

[44] G. Itsios, Y. Lozano, E. .O Colgain and K. Sfetsos, JHEP **1208**, 132 (2012) [arXiv:1205.2274 [hep-th]].

[45] F. Benini and N. Bobev, JHEP **1306**, 005 (2013) [arXiv:1302.4451 [hep-th]].

[46] S. Cucu, H. Lü and J. F. Vázquez-Poritz, Phys. Lett. B **568**, 261 (2003) [hep-th/0303211].

S. Cucu, H. Lü and J. F. Vázquez-Poritz, Nucl. Phys. B **677**, 181 (2004) [hep-th/0304022].

A. Almuhairi and J. Polchinski, arXiv:1108.1213 [hep-th].

[47] M. Cvetič, J. T. Liu, H. Lü and C. N. Pope, Nucl. Phys. B **560**, 230 (1999) [hep-th/9905096].

[48] M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh and T. A. Tran, Nucl. Phys. B **586** (2000) 275 [arXiv:hep-th/0003103]; A. Khavaev, K. Pilch and N. P. Warner, Phys. Lett. B **487** (2000) 14 [arXiv:hep-th/9812035].

[49] M. Cvetič, H. Lü, C. N. Pope, A. Sadrzadeh and T. A. Tran, Nucl. Phys. B **590**, 233 (2000) [hep-th/0005137].

[50] M. J. Duff, H. Lü and C. N. Pope, Nucl. Phys. B **544**, 145 (1999) [hep-th/9807173].

[51] J. T. Liu and R. Minasian, Phys. Lett. B **457**, 39 (1999) [hep-th/9903269].

[52] H. Nastase, D. Vaman and P. van Nieuwenhuizen, Phys. Lett. B **469**, 96 (1999) [hep-th/9905075].

[53] P. Breitenlohner and D. Z. Freedman, Annals Phys. **144**, 249 (1982).

[54] P. Karndumri and E. Ó Colgáin, Phys. Rev. D **87**, no. 10, 101902 (2013) [arXiv:1302.6532 [hep-th]].

P. Karndumri and E. Ó Colgáin, JHEP **1310**, 094 (2013) [arXiv:1307.2086].

[55] W. Nahm, Nucl. Phys. B **135**, 149 (1978).

[56] P. K. Townsend and M. N. R. Wohlfarth, Phys. Rev. Lett. **91**, 061302 (2003) [hep-th/0303097].

[57] G. W. Gibbons and C. M. Hull, hep-th/0111072.

[58] H. Lü and J. F. Vázquez-Poritz, Phys. Lett. B **597**, 394 (2004) [hep-th/0308104].

[59] C. M. Hull and N. P. Warner, Class. Quant. Grav. **5**, 1517 (1988).

[60] C. M. Hull, JHEP **9807**, 021 (1998) [hep-th/9806146].

[61] J. T. Liu, W. A. Sabra and W. Y. Wen, JHEP **0401**, 007 (2004) [hep-th/0304253].

H. Lü and J. F. Vázquez-Poritz, JCAP **0402**, 004 (2004) [hep-th/0305250].

[62] S. R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).

[63] H. Lü, C. N. Pope and J. F. Vázquez-Poritz, Nucl. Phys. B **709**, 47 (2005) [hep-th/0307001].

[64] I-S. Yang, Phys. Rev. D **81**, 125020 (2010) [arXiv:0910.1397 [hep-th]].

J. J. Blanco-Pillado and B. Shlaer, Phys. Rev. D **82**, 086015 (2010) [arXiv:1002.4408 [hep-th]].

[65] B. de Wit and H. Nicolai, Nucl. Phys. B **281** (1987) 211.

[66] I. R. Klebanov and J. M. Maldacena, Int. J. Mod. Phys. A **19**, 5003 (2004) [hep-th/0409133].

[67] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].

[68] A. Strominger, JHEP **0110**, 034 (2001) [hep-th/0106113].

D. Anninos, Int. J. Mod. Phys. A **27**, 1230013 (2012) [arXiv:1205.3855 [hep-th]].

[69] P. Candelas and X.C. de la Ossa, Nucl. Phys. **B342**, 246 (1990).

[70] D. Mallayev, J. F. Vazquez-Poritz and Z. Zhang, Phys. Rev. D **90**, no. 10, 106007 (2014) doi:10.1103/PhysRevD.90.106007 [arXiv:1309.3257 [hep-th]].

[71] S. Kachru, X. Liu and M. Mulligan, "Gravity Duals of Lifshitz-like Fixed Points," Phys. Rev. D **78**, 106005 (2008) [arXiv:0808.1725 [hep-th]].

[72] D. T. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry," Phys. Rev. D **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]].

[73] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," Phys. Rev. Lett. **101**, 061601 (2008) [arXiv:0804.4053 [hep-th]].

[74] A. Almuhairi and J. Polchinski, "Magnetic $AdS\times\mathbb{R}^2$: Supersymmetry and stability," arXiv:1108.1213 [hep-th].

[75] O. Lunin and J. M. Maldacena, "Deforming field theories with U(1) x U(1) global symmetry and their gravity duals," JHEP **0505**, 033 (2005) [hep-th/0502086].

[76] R. G. Leigh and M. J. Strassler, "Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory," Nucl. Phys. B **447**, 95 (1995) [hep-th/9503121].

[77] S. Frolov, "Lax pair for strings in Lunin-Maldacena background," JHEP **0505**, 069 (2005) [hep-th/0503201].

[78] M. Cvetič, M. J. Duff, P. Hoxha, J. T. Liu, H. Lü, J. X. Lu, R. Martinez-Acosta, C. N. Pope, H. Sati and T. A. Tran, "Embedding AdS black holes in ten-dimensions and eleven-dimensions," Nucl. Phys. B **558**, 96 (1999) [hep-th/9903214].

[79] A. Almuhairi, "AdS_3 and AdS_2 Magnetic Brane Solutions," arXiv:1011.1266 [hep-th].

[80] A. Donos, J. P. Gauntlett and C. Pantelidou, "Magnetic and Electric AdS Solutions in String- and M-Theory," Class. Quant. Grav. **29**, 194006 (2012) [arXiv:1112.4195 [hep-th]].

[81] J. M. Maldacena and C. Nuñez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. A **16**, 822 (2001) [hep-th/0007018].

[82] S. Cucu, H. Lü and J. F. Vázquez-Poritz, "A Supersymmetric and smooth compactification of M theory to AdS_5," Phys. Lett. B **568**, 261 (2003) [hep-th/0303211].

[83] C. h. Ahn and J. F. Vázquez-Poritz, "Deformations of flows from type IIB supergravity," Class. Quant. Grav. **23**, 3619 (2006) [hep-th/0508075].

[84] L. J. Romans, "Gauged N=4 Supergravities In Five-dimensions And Their Magnetovac Backgrounds," Nucl. Phys. B **267**, 433 (1986).

[85] H. Lü, C. N. Pope and T. A. Tran, "Five-dimensional N=4, SU(2) x U(1) gauged supergravity from type IIB," Phys. Lett. B **475**, 261 (2000) [hep-th/9909203].

[86] J. P. Gauntlett and O. Varela, "D=5 SU(2) x U(1) Gauged Supergravity from D=11 Supergravity," JHEP **0802**, 083 (2008) [arXiv:0712.3560 [hep-th]].

[87] E. D'Hoker and P. Kraus, "Magnetic Brane Solutions in AdS," JHEP **0910**, 088 (2009) [arXiv:0908.3875 [hep-th]].

[88] F. Benini and N. Bobev, "Exact two-dimensional superconformal R-symmetry and c-extremization," Phys. Rev. Lett. **110**, no. 6, 061601 (2013) [arXiv:1211.4030 [hep-th]].

[89] F. Benini and N. Bobev, "Two-dimensional SCFTs from wrapped branes and c-extremization," JHEP **1306**, 005 (2013) [arXiv:1302.4451 [hep-th]].

[90] P. Karndumri and E. Ó Colgáin, "3D Supergravity from wrapped D3-branes," JHEP **1310**, 094 (2013) [arXiv:1307.2086].

[91] J. Jeong, E. Ó Colgáin and K. Yoshida, "SUSY properties of warped AdS_3," JHEP **1406**, 036 (2014) [arXiv:1402.3807 [hep-th]].

[92] C. P. Herzog, M. Rangamani and S. F. Ross, "Heating up Galilean holography," JHEP **0811**, 080 (2008) [arXiv:0807.1099 [hep-th]].

[93] J. Maldacena, D. Martelli and Y. Tachikawa, "Comments on string theory backgrounds with non-relativistic conformal symmetry," JHEP **0810**, 072 (2008) [arXiv:0807.1100 [hep-th]].

[94] N. Bobev and A. Kundu, "Deformations of Holographic Duals to Non-Relativistic CFTs," JHEP **0907**, 098 (2009) [arXiv:0904.2873 [hep-th]].

[95] A. Bergman, K. Dasgupta, O. J. Ganor, J. L. Karczmarek and G. Rajesh, "Nonlocal field theories and their gravity duals," Phys. Rev. D **65**, 066005 (2002) [hep-th/0103090].

[96] H. Lü, D. N. Page and C. N. Pope, "New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kahler manifolds," Phys. Lett. B **593**, 218 (2004) [hep-th/0403079].

[97] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS_5 solutions of M theory," Class. Quant. Grav. **21**, 4335 (2004) [hep-th/0402153].

[98] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on $S^2 \times S^3$," Adv. Theor. Math. Phys. **8**, 711 (2004) [hep-th/0403002].

[99] C. -h. Ahn and J. F. Vázquez-Poritz, "Marginal deformations with $U(1)^3$ global symmetry," JHEP **0507**, 032 (2005) [hep-th/0505168].

[100] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, "Gauge theories from toric geometry and brane tilings," JHEP **0601**, 128 (2006) [hep-th/0505211].

[101] K. Balasubramanian and J. McGreevy, "String theory duals of Lifshitz-Chern-Simons gauge theories," Class. Quant. Grav. **29**, 194007 (2012) [arXiv:1111.0634 [hep-th]].

[102] K. Copsey and R. B. Mann, "Hidden Singularities and Closed Timelike Curves in A Proposed Dual for Lifshitz-Chern-Simons Gauge Theories," Phys. Rev. D **85**, 121902 (2012) [arXiv:1112.0578 [hep-th]].

[103] A. Donos and J. P. Gauntlett, "Lifshitz Solutions of D=10 and D=11 supergravity," JHEP **1012**, 002 (2010) [arXiv:1008.2062 [hep-th]].

[104] L. J. Romans, "The F(4) Gauged Supergravity in Six-dimensions," Nucl. Phys. B **269**, 691 (1986).

[105] R. Gregory, S. L. Parameswaran, G. Tasinato and I. Zavala, "Lifshitz solutions in supergravity and string theory," JHEP **1012**, 047 (2010) [arXiv:1009.3445 [hep-th]].

[106] L. J. Romans, "Massive N=2a Supergravity in Ten-Dimensions," Phys. Lett. B **169**, 374 (1986).

[107] M. Cvetič, H. Lü and C. N. Pope, "Gauged six-dimensional supergravity from massive type IIA," Phys. Rev. Lett. **83**, 5226 (1999) [hep-th/9906221].

[108] J. Jeong, Ö. Kelekci and E. Ó Colgáin, "An alternative IIB embedding of F(4) gauged supergravity," JHEP **1305**, 079 (2013) [arXiv:1302.2105 [hep-th]].

[109] M. Cvetič, H. Lü, C. N. Pope and K. S. Stelle, "T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map," Nucl. Phys. B **573**, 149 (2000) [hep-th/9907202].

[110] M. Cvetič, H. Lü, D. N. Page and C. N. Pope, "New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter," JHEP **0907**, 082 (2009) [hep-th/0505223].

[111] H. L, C. N. Pope, J. F. Vazquez-Poritz and Z. Zhang, JHEP **1506**, 112 (2015) doi:10.1007/JHEP06(2015)112 [arXiv:1411.1413 [hep-th]].

[112] R. Emparan and H. S. Reall, "Black rings," Class. Quant. Grav. **23**, R169 (2006) [hep-th/0608012].

[113] R. Emparan and H. S. Reall, "A Rotating black ring solution in five-dimensions," Phys. Rev. Lett. **88**, 101101 (2002) [hep-th/0110260].

[114] H. Elvang, "A Charged rotating black ring," Phys. Rev. D **68**, 124016 (2003) [hep-th/0305247].

[115] H. Elvang, R. Emparan and , "Black rings, supertubes, and a stringy resolution of black hole nonuniqueness," JHEP **0311**, 035 (2003) [hep-th/0310008].

[116] R. Emparan, "Rotating circular strings, and infinite nonuniqueness of black rings," JHEP **0403**, 064 (2004) [hep-th/0402149].

[117] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "A supersymmetric black ring," Phys. Rev. Lett. **93**, 211302 (2004) [hep-th/0407065].

[118] J. P. Gauntlett and J. B. Gutowski, "Concentric black rings," Phys. Rev. D **71**, 025013 (2005) [hep-th/0408010].

[119] I. Bena and N. P. Warner, "One ring to rule them all ... and in the darkness bind them?," Adv. Theor. Math. Phys. **9**, 667 (2005) [hep-th/0408106].

[120] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "Supersymmetric black rings and three-charge supertubes," Phys. Rev. D **71**, 024033 (2005) [hep-th/0408120].

[121] J. P. Gauntlett and J. B. Gutowski, "General concentric black rings," Phys. Rev. D **71**, 045002 (2005) [hep-th/0408122].

[122] H. Elvang, R. Emparan and P. Figueras, "Non-supersymmetric black rings as thermally excited supertubes," JHEP **0502**, 031 (2005) [hep-th/0412130].

[123] H. Elvang and P. Figueras, "Black Saturn," JHEP **0705**, 050 (2007) [hep-th/0701035].

[124] A. A. Pomeransky and R. A. Sen'kov, "Black ring with two angular momenta," hep-th/0612005.

[125] H. Elvang and M. J. Rodriguez, "Bicycling black rings," JHEP **0804**, 045 (2008) [arXiv:0712.2425 [hep-th]].

[126] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers and M. J. Rodriguez, "The phase structure of higher-dimensional black rings and black holes," JHEP **0710**, 110 (2007) [arXiv:0708.2181 [hep-th]].

[127] M. M. Caldarelli, R. Emparan and M. J. Rodriguez, "Black rings in (Anti)-de Sitter space," JHEP **0811**, 011 (2008) [arXiv:0806.1954 [hep-th]].

[128] A. Feldman and A. A. Pomeransky, "General triple charged black ring solution in supergravity," JHEP **1408**, 059 (2014) [arXiv:1406.4617 [hep-th]].

[129] V. A. Belinsky and V. E. Zakharov, "Integration of the Einstein equations by the inverse scattering problem technique and the calculation of the exact soliton solutions," Sov. Phys. JETP **48**, 985 (1978) [Zh. Eksp. Teor. Fiz. **75**, 1953 (1978)].

[130] V. A. Belinsky and V. E. Zakharov, "Stationary gravitational solitons with axial symmetry," Sov. Phys. JETP **50**, 1 (1979).

[131] F. Dowker, J. P. Gauntlett, D. A. Kastor and J. H. Traschen, "Pair creation of dilaton black holes," Phys. Rev. D **49**, 2909 (1994) [hep-th/9309075].

[132] A. Chamblin and R. Emparan, "Bubbles in Kaluza-Klein theories with space-like or time-like internal dimensions," Phys. Rev. D **55**, 754 (1997) [hep-th/9607236].

[133] H. Lü and J. F. Vázquez-Poritz, "C-metrics in gauged STU supergravity and beyond," JHEP **1412**, 057 (2014) [arXiv:1408.6531 [hep-th]].

[134] G. W. Gibbons and D. L. Wiltshire, "Space-Time as a Membrane in Higher Dimensions," Nucl. Phys. B **287**, 717 (1987) [hep-th/0109093].

[135] M. Cvetič, G. W. Gibbons, C. N. Pope and Z. H. Saleem, "Electrodynamics of black holes in STU supergravity," JHEP **1409**, 001 (2014) [arXiv:1310.5717 [hep-th], arXiv:1310.5717].

[136] H. Golchin, M. M. Sheikh-Jabbari and A. Ghodsi, "Dual 2d CFT Identification of Extremal Black Rings from Holes," JHEP **1310**, 194 (2013) [arXiv:1308.1478 [hep-th]].

[137] A. Ghodsi, H. Golchin and M. M. Sheikh-Jabbari, "More on Five Dimensional EVH Black Rings," JHEP **1409**, 036 (2014) [arXiv:1407.7484 [hep-th]].

[138] S. Sadeghian, M. M. Sheikh-Jabbari and H. Yavartanoo, "On Classification of Geometries with SO(2,2) Symmetry," JHEP **1410**, 81 (2014) [arXiv:1409.1635 [hep-th]].

[139] M. M. Sheikh-Jabbari and H. Yavartanoo, "EVH Black Holes, AdS3 Throats and EVH/CFT Proposal," JHEP **1110**, 013 (2011) [arXiv:1107.5705 [hep-th]].

[140] H. Lü, J. Mei and C. N. Pope, "New black holes in five dimensions," Nucl. Phys. B **806**, 436 (2009) [arXiv:0804.1152 [hep-th]].

[141] F.J. Ernst, "Removal of the nodal singularity of the C-metric," J. Math. Phys. **17**, 515 (1976).

[142] D. Garfinkle and A. Strominger, "Semiclassical Wheeler wormhole production," Phys. Lett. B **256**, 146 (1991).

[143] D. Garfinkle, S. B. Giddings and A. Strominger, "Entropy in black hole pair production," Phys. Rev. D **49**, 958 (1994) [gr-qc/9306023].

[144] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," Fortsch. Phys. **61**, 781 (2013) [arXiv:1306.0533 [hep-th]].

[145] H. Lü and J. F. Vázquez-Poritz, "Dynamic C-metrics in (gauged) supergravities," Phys. Rev. D **91**, no. 6, 064004 (2015) [arXiv:1408.3124 [hep-th]].