Quaternion Algebras and Hyperbolic 3-Manifolds

Joseph Quinn
Graduate Center, City University of New York

6-2016

How does access to this work benefit you? Let us know!
Follow this and additional works at: https://academicworks.cuny.edu/gc_etds
Part of the Geometry and Topology Commons

Recommended Citation
Quinn, Joseph, "Quaternion Algebras and Hyperbolic 3-Manifolds" (2016). CUNY Academic Works.
https://academicworks.cuny.edu/gc_etds/1354

This Dissertation is brought to you by CUNY Academic Works. It has been accepted for inclusion in All Dissertations, Theses, and Capstone Projects by an authorized administrator of CUNY Academic Works. For more information, please contact deposit@gc.cuny.edu.
Quaternions Algebras and Hyperbolic
3-Manifolds

by

Joseph A. Quinn

A dissertation submitted to the Graduate Faculty in Mathematics in partial
fulfillment of the requirements for the degree of Doctor of Philosophy, The
City University of New York.

2016
This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

(required signature)

Date

Chair of Examining Committee

(required signature)

Date

Executive Officer

Ara Basmajian

Abhijit Champanerkar

Ilya Kofman

John Voight

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK
Abstract

Quaternion Algebras and Hyperbolic 3-Manifolds

by

Joseph A. Quinn

Advisor: Abhijit Champanerkar

I use a classical idea of Macfarlane to obtain a complex quaternion model for hyperbolic 3-space and its group of orientation-preserving isometries, analogous to Hamilton’s famous result on Euclidean rotations. I generalize this to quaternion models over number fields for the action of Kleinian groups on hyperbolic 3-space, using arithmetic invariants of the corresponding hyperbolic 3-manifolds. The class of manifolds to which this technique applies includes all cusped arithmetic manifolds and infinitely many commensurability classes of cusped non-arithmetic, compact arithmetic, and compact non-arithmetic manifolds. I obtain analogous results for actions of Fuchsian groups on the hyperbolic plane. I develop new tools to study such manifolds, and then focus on a new algorithm for computing their Dirichlet domains.
Preface

In 1843, William Hamilton [26] invented the quaternions, fulfilling his vision of an algebraic structure on Euclidean space analogous to what the complex numbers provide for the Euclidean plane. The geometric applications of this four-dimensional, non-commutative, real algebra are well understood, having contributed for instance to group theory, number theory, crystallography, and applied sciences such as spatial navigation and three-dimensional computer graphics [39]. Leonard Dickson [16, 17, 18] generalized Hamilton’s quaternions in the early 1900’s to study quadratic Diophantine equations. This lead to a rich algebraic theory of (generalized) quaternion algebras, providing new methods for solving important problems in number theory [50]. More recently, in the study of hyperbolic surfaces and 3-manifolds, quaternion algebras over number fields appear as arithmetic invariants of manifolds, due to the work of Takeuchi [47], Maclachlan and Reid [36], and Neumann and Reid [40]. In this context, arithmetic properties of the algebras are used
to study commensurability of hyperbolic 3-manifolds, and give rise to the subclass of arithmetic manifolds in a natural way.

As I studied these things I formed the following question, which has become the underlying motivation for this thesis.

Is there a geometric interpretation for the quaternion algebra of a hyperbolic 3-manifold that puts an algebraic structure on motions in the manifold, generalizing what Hamilton’s quaternion do for Euclidean space?

In seeking an answer, I found Alexander Macfarlane’s paper *Hyperbolic quaternions* [34] from 1900, in which he developed a quaternionic precursor to Minkowksi space and used it to define hyperbolic trigonometric functions by modifying Hamilton’s multiplication rules. However, the algebraic properties of his construction were ambiguous, lacking the benefit of Dickson’s later formalism. The hyperbolic quaternions remained under-developed as the scientific community moved away from a quaternionic system for space analysis [14].

I used Dickson’s algebraic formalism to reinterpret Macfarlane’s idea, leading to my first main result (Theorem 3.2.18): a representation of the group of orientation-preserving isometries of hyperbolic 3-space via quater-
nion multiplication, where the points lie in Macfarlane’s quaternion embedding of Minkowski space. This gave a fresh perspective on a variety of classical ideas but at first I debated whether it would lead to an answer to my main question. I then received a motivational boost from Macfarlane himself when I came across the following passage.

“I have always felt that *Quaternions* is on the right track, and that Hamilton and Tait deserve and will receive more and more as time goes by thanks of the highest order. But at the same time I am convinced that the notation can be improved; that the principles require to be corrected and extended; that there is a more complete algebra which unifies Quaternions, Grassmann’s method and Determinants, and applies to physical quantities in space. The guiding idea in this paper is generalization. What is sought for is an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized will become ordinary algebra.”

Alexander Macfarlane, 1891 [33]

Inspired by the prospect of fulfilling Macfarlane’s vision, I expanded my first result to include a class of quaternion algebras over number fields
which arise as arithmetic invariants of hyperbolic 3-manifolds that I call
*Macfarlane manifolds*. My second main result (Theorem 4.3.3) was to es-
tablish that the action of the fundamental group of a Macfarlane manifold
by orientation-preserving isometries of hyperbolic 3-space admits a repre-
sentation via quaternion multiplication, where hyperbolic space is modeled
by a non-standard hyperboloid embedded in the same algebra. This gave
the simultaneous geometric and arithmetic interpretation that I desired, but
begged the question of how diverse this class of Macfarlane manifolds may
or may not be.

To address this, I found an algebraic characterization of Macfarlane man-
ifolds in terms of their quaternion algebras and identified an interesting
subclass geometrically, based on the existence of immersed totally geodesic
subsurfaces. Using this, I showed that the class of Macfarlane manifolds
includes all arithmetic cusped hyperbolic 3-manifolds and infinitely many
commensurability classes of non-arithmetic cusped, arithmetic compact, and
non-arithmetic compact examples. There are also analogous results for 2-
manifolds, which apply to all cusped hyperbolic surfaces and a broad class
of closed ones, including the aforementioned subsurfaces of the 3-manifolds.

Next I moved on to develop techniques to study these manifolds using
properties of the quaternion algebras that define them. I created a new set
of tools for this purpose, some of which are utilized here, others of which have yet to be explored. As an application, I produced an algorithm for computing Dirichlet domains for Macfarlane surfaces and 3-manifolds in my quaternion model (Theorem 5.1.10). This helps us study the geometry of Macfarlane manifolds and at the same time introduces a new technique for constructing Dirichlet domains. Notably, it applies to both arithmetic and non-arithmetic examples.

In my opinion, the current development is only the beginning of what is possible with this approach. While the examples included are interesting, I prefer to view them as illustrations of how to apply this theory, to be expanded upon in future work. Some possibilities for this, which I intend to pursue in the near future, are included in §6. In the spirit of Macfarlane, “The guiding idea in this paper is generalization.”
Acknowledgements

I thank my advisor, Professor Abhijit Champanerkar, for introducing me to the fascinating world of hyperbolic 3-manifolds, for enthusiastically encouraging me to find my own passion within it, and for persistently refining my ability to pursue, realize and communicate my ideas. I am very grateful for having received, due to his diligence and benevolence, such a warm and sincere welcome into the community of people working in this field. I thank Professor John Loustau from Hunter College for challenging me as an undergraduate student and convincing me of my ability to go further. It is due to his persuasion that I decided to pursue the PhD. I thank Professor John Voight from Dartmouth College for so generously sharing with me his wealth of knowledge on quaternion algebras, and for our long correspondence wherein he answered many of my questions, making it possible for me to take my results much further. I thank Professor Dennis Sullivan from CUNY for consistently placing my ideas in a much broader perspective, for building my
confidence in thinking about difficult mathematics, and for connecting me to other mathematicians in similar veins of thought. Thanks to the following additional people for helpful comments and discussions about the content: Ian Agol, Marcel Alexander, Ara Basmajian, John Biasis, Ben Bloom-Smith, Daniel Boyd, Erin Carmody, Michelle Chu, Jeff Danciger, Adam Hughes, Ilya Kofman, Ivan Levcovitz, Seungwon Kim, Stefan Kohl, Greg Laun, Ben Linowitz, Victoria Manning, Lee Mosher, Aurel Page, Alan Reid, Dennis Ryan, Dylan Sparrow, Roland van der Veen, Alberto Verjovsky, Jeff Weeks, Matthias Wendt and Alex Zorn.

I thank my parents Ann and Greg Quinn for making it possible for me to pursue higher mathematics, through their encouragement, love, and money! They have been a constant source of comfort and inspiration just by letting me know that whenever something good happened to me, it happened to them too. I thank my friends and my brother Bill Quinn, for their emotional support regarding the diverse and copious non-math-related problems that came up in my life along the way.

Especially I thank my wife Kat Byrne for marrying me, for staying married to me even though I spent years making no money and talking about hyperbolic manifolds, and for her endless love and support in this and every other aspect of my life.
Summary

Background material is covered in §1 through §3.1, original results are covered in §3.2 through §5, and ideas for future work are given in §6. The main framework for the theory developed here is found in §4, and the main results are Theorem 3.2.18, Theorem 4.3.3, Theorem 4.4.16, Theorem 4.1.10 and Theorem 5.1.10.

In §1 we discuss quaternion algebras, from their geometric origins to their arithmetic generalization. Historical content is included to motivate the current development, most importantly Macfarlane’s hyperbolic quaternions in §1.1.3. In §1.2 is a more comprehensive overview of the properties of quaternion algebras and the material in §1.2.1 is ubiquitous. Theorem 1.2.11 is also used often. The content of §1.2.3 and §1.2.4 is important in the development of Macfarlane spaces in §4.1.

In §2 we review the contemporary usage of quaternion algebras in the study of hyperbolic 3-manifolds and surfaces. In §2.1 we review some fun-
damental hyperbolic geometry. The upper half-space model is treated there in the usual way, but is defined differently and in more detail in §3.2.3. The unconventional Definition 2.1.10 of a hyperboloid model for hyperbolic space over an arbitrary real field is central to the theory of Macfarlane manifolds, as is Definition 2.2.1 of the trace field and quaternion algebra of a manifold. The relationship between quaternion algebras and arithmetic manifolds is expanded upon in Proposition 2.3.5, to clarify this topic.

In §3, we use classical results of Macfarlane [34] and Wigner [51] to develop standard Macfarlane spaces. Wigner’s Theorem 3.1.8 is a result from relativity theory [7], so an original proof is included that uses algebraic language more in line with the current methodology. Theorem 3.2.18 gives the quaternion representation of the group of orientation-preserving isometries of hyperbolic 3-space and is the first major original result. Theorem 3.2.26 gives a convenient way of transferring data from the quaternion model to the upper half-space model. In §3.3 we give analogous results that apply to the hyperbolic plane.

In §4, we develop the general theory of Macfarlane spaces (Definition 4.1.1), which are characterized algebraically by Theorem 4.1.10. Definition 4.3.1 introduces Macfarlane manifolds. Theorem 4.3.3 is imperative to the study of these manifolds but much of the groundwork for proving it is done
in the proof of Theorem 4.2.3. In §4.2.2, a variety of computational tools unique to the current approach are developed with an eye toward future work. Of these, Theorem 4.2.10 gives a geometric description of a point on a quaternionic hyperboloid acting on itself, and is applied to finding Dirichlet domains in §5. In §4.3 we establish the diversity of Macfarlane manifolds, giving classes of examples using arithmetic and geometric conditions. In §4.4, restricted Macfarlane spaces are defined (Definition 4.4.5) and characterized (Theorem 4.4.9) and a series of analogous results are proven for hyperbolic surfaces.

In §5, we give an algorithm for constructing Dirichlet domains for Macfarlane manifolds. We include a brief general discussion of Dirichlet domains, then the algorithm is detailed in §5.1. Theorem 5.1.10 shows its effectiveness. We discuss advantages of this technique over other methods, following from our use of quaternions. In §5.2 we illustrate the algorithm by working out arithmetic non-compact examples in dimensions 2 and 3, and we notice some properties made evident from the quaternion perspective.

In §6 we list some ideas for future developments. These include further study of hyperbolic 3-manifolds and surfaces as well as other geometries and higher dimensions.


## Contents

Abstract ............................................................ vi
Preface ............................................................... viii
Acknowledgements ................................................... xiii
Summary .............................................................. xvi
List of Tables ......................................................... xxiv
List of Figures ......................................................... xxv

1 Quaternions ......................................................... 1
   1.1 Geometric Origins ............................................. 2
      1.1.1 Hamilton’s Quaternions ............................... 2
      1.1.2 Biquaternions and Hyperbolic Space ............... 4
      1.1.3 Macfarlane’s Hyperbolic Quaternions .............. 4
   1.2 Number Theoretic Generalization .......................... 8
      1.2.1 Quaternion Algebras ................................ 8
      1.2.2 Ramification ......................................... 12

xxi
CONTENTS

1.2.3 Quadratic Forms ........................................ 19
1.2.4 Algebras with Involution .............................. 23

2 Arithmetic of Hyperbolic Manifolds .......................... 27
  2.1 Hyperbolic 3-Manifolds and Surfaces .................. 27
      2.1.1 The Upper Half-Space Model ...................... 28
      2.1.2 Hyperboloid Models ............................. 31
      2.1.3 Isometries ..................................... 34
      2.1.4 Some Applications of Number Theory ............ 37
  2.2 Quaternion Algebras as Invariants ..................... 39
      2.2.1 Manifold Invariants ............................ 40
      2.2.2 Commensurability Invariants .................... 43
  2.3 Arithmetic Kleinian and Fuchsian Groups ............. 46

3 Quaternion Models for Hyperbolic Space ..................... 55
  3.1 The Spinor Representation of SO(1,3) .................. 56
  3.2 A Quaternion Model for Hyperbolic 3-Space .......... 63
      3.2.1 The Standard Macfarlane Space ................ 64
      3.2.2 A Quaternion Representation of Isom\(^+(\mathbb{H}^3)\) .... 67
      3.2.3 Comparison to the Möbius Action ............... 73
  3.3 Quaternion Models for the Hyperbolic Plane ........ 84
3.3.1 The Standard Restricted Macfarlane Space . . . . . . . 85
3.3.2 Pure Quaternions and Hyperbolic Planes . . . . . . . 88

4 Macfarlane Spaces 91
4.1 Macfarlane Spaces . . . . . . . . . . . . . . . . . . . . . . . 92
4.2 Isometries in Macfarlane Quaternion Algebras . . . . . . . 104
4.2.1 The Action by Isometries . . . . . . . . . . . . . . . . . 104
4.2.2 Isometries as Points . . . . . . . . . . . . . . . . . . . . 108
4.3 Examples of Macfarlane 3-Manifolds . . . . . . . . . . . . . 122
4.3.1 Non-compact Macfarlane Manifolds . . . . . . . . . . . 124
4.3.2 Compact Macfarlane Manifolds . . . . . . . . . . . . . 125
4.4 Restricted Macfarlane Spaces and Hyperbolic Surfaces . . . 130
4.4.1 Real Quaternion Algebras and $\mathbb{H}^2$ . . . . . . . 131
4.4.2 Restricted Macfarlane Spaces . . . . . . . . . . . . . 135
4.4.3 Isometries in Restricted Macfarlane Spaces . . . . . . . 140

5 Dirichlet Domains for Macfarlane Manifolds 145
5.1 Quaternion Dirichlet Domains . . . . . . . . . . . . . . . . . 146
5.1.1 Orbit Points . . . . . . . . . . . . . . . . . . . . . . . . 149
5.1.2 The Algorithm . . . . . . . . . . . . . . . . . . . . . . . 153
5.2 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . 160
5.2.1 A Hyperbolic Punctured Torus .......................... 161
5.2.2 Torsion-Free Subgroups of Bianchi groups ........... 165

6 Future Research ................................................. 173

Index of Notation ................................................. 177

Bibliography ...................................................... 183
## List of Tables

3.1 Normalized isometries as quaternions. . . . . . . . . . . . . . . 76

4.1 Subspaces with signature $(1, 2)$, in real quaternion algebras. . 132

5.1 Points from a hyperbolic punctured torus group that lie on its
   quaternion hyperboloid model. . . . . . . . . . . . . . . . . . . . 164

5.2 First three heights of points in $\Gamma_1 \cap I^{\Gamma_1}$. . . . . . . . 168

5.3 First three heights of points in $\Gamma_3 \cap I^{\Gamma_3}$. . . . . . . . . 169

5.4 Lowest occurrences of points in $(\Gamma_d \cap I^{\Gamma_d}) \setminus \text{PSL}_2(\mathbb{Z})$ for first
   several $d$, and the directions of the corresponding geodesics. . . 172
# List of Figures

1.1 Macfarlane’s hyperboloids. ........................................ 7

3.1 Relationship between the quaternion approach and the approach from relativity theory. ........................................ 74

3.2 \( T^n \) and \( P^n \) lying in \( \mathbb{R}^{n+1} \), from [3]. .................. 78

3.3 \( P^n \) and \( H^n \) lying in \( \mathbb{R}^n \), from [3]. .................. 78

4.1 Pure quaternions in \( \mathcal{M} \). ................................. 111

4.2 An arithmetic link, from [37]. ................................. 125

5.1 A Dirichlet domain for the action of \( \mathbb{Z}^2 \) on \( \mathbb{R}^2 \). .................. 146

5.2 Orbit points by trace. ........................................ 150

5.3 Geodesics used to find sides of \( \mathcal{D}_T(1) \). .................. 154

5.4 Finding \( m \) in the compact (left) and cusped (right) cases. .. 155

5.5 Dirichlet domain for a hyperbolic punctured torus. ........ 166

5.6 Dirichlet domain for the Whitehead link complement. ........ 170
Chapter 1

Quaternions

In this chapter we give a historical overview of quaternion algebras, setting up conventions and basic properties as well as motivating later developments. In §1.1, we view quaternions from the classical perspective as a geometric construction. We summarize some ideas for generalizing Hamilton’s model of Euclidean rotations to hyperbolic space, then focus on a variation introduced by Macfarlane which will remain a central motivation throughout. In §1.2, we explain the modern generalization of quaternion algebras in the context of algebraic number theory. We study the classification of quaternion algebras using their ramification sets. We explore the relationship between the theory of quaternion algebras and the theories of quadratic forms and algebras with involution.
1.1 Geometric Origins

1.1.1 Hamilton’s Quaternions

William Hamilton was interested in finding a multiplicative structure on Euclidean 3-space analogous to what $\mathbb{C}$ does for vector multiplication in $\mathbb{R}^2$ [14]. He famously succeeded in 1843 by introducing the following 4-dimensional object.

Definition 1.1.1. Hamilton’s quaternions, denoted by $\mathbb{H}$, is the $\mathbb{R}$-algebra generated by $1, i, j, k$ with the following multiplication rules.

\[
\begin{align*}
  i^2 &= j^2 = k^2 = -1 \\
  ij &= k & jk &= i & ki &= j \\
  ji &= -k & kj &= -i & ik &= -j
\end{align*}
\]

Hamilton [26] used trigonometric functions to show that multiplication in this space corresponds to Euclidean rotations. In particular he gave a way of identifying $\mathbb{R}^3$ as well as the group of rotations in $\mathbb{R}^3$ centered at the origin, both as subsets of $\mathbb{H}$ where the group action can be defined multiplicatively as follows.

Definition 1.1.2. For a quaternion $q = w + xi + yj + zk \in \mathbb{H}$, its standard
1.1. GEOMETRIC ORIGINS

**involution** (or *quaternion conjugate*), **trace** and **norm**, respectively, are

\[
q^* := w - xi - yj - zk,
\]

\[
\text{tr}(q) := q + q^* = 2w,
\]

\[
n(q) := qq^* = w^2 + x^2 + y^2 + z^2.
\]

The **pure quaternions** in \(\mathbb{H}\) are defined as \(\mathbb{H}_0 := \{q \in \mathbb{H} \mid \text{tr}(q) = 0\}\).

As a metric space, \((\mathbb{H}_0, \sqrt{n})\) is isometric to \((\mathbb{R}^3, n_{\text{Euc}})\) where \(n_{\text{Euc}}\) is the Euclidean norm. Letting \(\mathbb{H}^1 := \{q \in \mathbb{H} \mid n(q) = 1\}\) and \(\mathbb{P}\mathbb{H}^1 := \mathbb{H}^1/\{\pm 1\}\), Hamilton’s theorem can be expressed in modern terminology.

**Theorem 1.1.3** (Hamilton, 1843). *An isomorphism from \(\mathbb{P}\mathbb{H}^1\) to the special orthogonal group \(\text{SO}(3)\) is defined by the group action*

\[
\mathbb{P}\mathbb{H}^1 \times \mathbb{H}_0 \to \mathbb{H}_0 : (u, p) \mapsto upu^*.
\]

Hamilton used quaternions to simultaneously represent isometries as well as points in the space they act upon. This allows the group action to be written multiplicatively as \(upu^*\) using the ambient algebraic structure of \(\mathbb{H}\), yielding various computational and theoretical advantages [14].
1.1.2 Biquaternions and Hyperbolic Space

Historically, Hamilton’s idea was intertwined with the development of modern vector analysis. For instance the use of $i, j, k$ to represent the standard basis of $\mathbb{R}^3$ in contemporary calculus textbooks is a result of Hamilton’s notation [14]. Hamilton’s theorem also motivated attempts to find variations for further applications to geometry and physics, the most relevant for our purposes being the applications to hyperbolic space.

The first generalization of quaternions is also due to Hamilton [27], who defined the biquaternions as the algebra generated over $\mathbb{C}$ by the same $1, i, j, k$. These have found some hyperbolic geometric interpretation in the context of special relativity [14] first due to A. Conway in 1911 [10], who later developed a use of biquaternions to compute 4-dimensional hyperbolic rotations [11]. Lanczos in 1949 [30] used biquaternions to compute Lorentz transformations for applications to relativistic mechanics. Quaternions in this context however were soon replaced with spin theory [14], where the duality between points and isometries is lost.

1.1.3 Macfarlane’s Hyperbolic Quaternions

Alexander Macfarlane was interested in generalizing $\mathbb{H}$ to real hyperbolic 3-space while maintaining 4-dimensionality over $\mathbb{R}$ (unlike the biquaternions).
In 1891, Macfarlane [33] introduced a preliminary version of “hyperbolic quaternions,” by altering Hamilton’s multiplication rules to the following.

\[ i^2 = j^2 = k^2 = 1 \]
\[ ij = k = -ji \quad jk = i = -kj \quad ki = j = -ik \]

The second line agrees with multiplication in \( \mathbb{H} \) while the first does not. Using this, Macfarlane worked out quaternionic definitions for hyperbolic trigonometric functions in analogy to what Hamilton had done, and proposed the result as an improved method of special analysis [35].

This was Macfarlane’s first attempt at generalizing \( \mathbb{H} \) to the hyperbolic setting. The idea was considered problematic [14] because the multiplication rules are non-associative. Given the other conditions inherited from Hamilton’s construction, for the algebra to be associative it is necessary that

\[ i^2 = j^2 = k^2 = -1. \]

**Proposition 1.1.4.** Let \( K \) be a field with \( \text{char}(K) \neq 2 \), and suppose

\[ K \oplus Ki \oplus Kj \oplus Kk \]
is an associative algebra where

\[ ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \]
\[ i^2 = a, \quad j^2 = b, \quad k^2 = c \]

for some \( a, b, c \in K^\times \). Then \( a = b = c = -1 \).

**Proof.** \( aj = i^2j = i(ij) = ik = -j \). Since \( j \) has an inverse, namely \( a^{-2}j \), this implies \( a^2 = -1 \). By symmetry, the same is true for \( b \) and \( c \).

In 1900, Macfarlane [34] made a second attempt and introduced a revised version of his “hyperbolic quaternions” where he recovered associativity by changing the multiplication rules to the following.

\[ i^2 = j^2 = k^2 = 1, \]
\[ ij = \sqrt{-1}k = -ji, \quad jk = \sqrt{-1}i = -kj, \quad ki = \sqrt{-1}j = -ik. \]

(1.1.5)

Although this gives an associative multiplication, it is unclear what base field is being used. With these rules, if one takes the additive and multiplicative closure of \( \{1, i, j, k\} \) over \( \mathbb{R} \), one gets an 8-dimensional \( \mathbb{R} \)-algebra. On the other hand, this can be seen as a 4-dimensional \( \mathbb{C} \)-algebra that is quaternion in the modern sense. In fact, we will see in §1.2 that the \( \mathbb{C} \)-algebra generated by these elements is isomorphic to Hamilton’s biquaternions.

Macfarlane’s decision to change the squares of \( i, j, k \) from \(-1\) to \(+1\) has a
1.1. GEOMETRIC ORIGINS

significant geometric consequence. Let $\mathcal{M}$ be the span of $\{1, i, j, k\}$ over $\mathbb{R}$. $\mathcal{M}$ is a 4-dimensional real vector space where the quaternion norm $n(q) = qq^*$ (as in Definition 1.1.2) restricts to a real-valued quadratic form of signature $(1, 3)$. To phrase this differently, $\mathcal{M}$ is a quaternionic version of Minkowski space. Yet Macfarlane’s construction predates Minkowski’s by 17 years [14]. Figure 1.1 is a drawing from Macfarlane’s paper [34] depicting an equivalent of what would later be termed the light cone, with light-like and time-like vectors.

Macfarlane studied the level sets $\{m \in \mathcal{M} \mid n(m) = c\}$ at $c \neq 0$, which are 3-dimensional hyperboloids over $\mathbb{R}$ (of one sheet when $c < 0$ and of two sheets when $c > 0$), and used these to carry out trigonometric analysis of hyperbolic functions. From this point of view, there is potential for a duality.
between points and isometries analogous to the identification of $\mathbb{R}^3$ with $\mathbb{H}_0$, but this analogy is not developed in Macfarlane’s papers.

Macfarlane’s idea did not gain popularity, having been proposed at a time when the math and physics community was moving away from a quaternionic system in setting the conventions for vector analysis [14]. It is unclear precisely what type of algebraic structure Macfarlane intended for his hyperbolic quaternions, as they were written before the formal development of generalized quaternion algebras. The main goal of this thesis is to complete Macfarlane’s idea using more modern techniques, then generalize it for applications to hyperbolic 3-manifolds and surfaces.

1.2 Number Theoretic Generalization

In a series of papers starting in 1912 [16, 17, 18], Dickson developed the generalization of $\mathbb{H}$ that we now call quaternion algebras, for the purpose of studying quadratic Diophantine equations. These algebras also provided the first examples of non-commutative division rings besides $\mathbb{H}$.

1.2.1 Quaternion Algebras

A quaternion algebra is a 4-dimensional central simple algebra: an algebra that is 4-dimensional over its center and has no nontrivial two-sided ideals
We will only be interested in quaternion algebras over \( \mathbb{R}, \mathbb{C} \), number fields (finite degree extensions of \( \mathbb{Q} \)), or \( p \)-adic fields, so we will use a definition more specific to these cases.

Let \( \mathbb{K} \) be a field with \( \text{char} \mathbb{K} \neq 2 \).

**Definition 1.2.1.** Let \( a, b \in \mathbb{K}^\times \), called the *structure constants* of the algebra. The *quaternion algebra* \( \left( \frac{a, b}{\mathbb{K}} \right) \) is the associative \( \mathbb{K} \)-algebra (with unity)

\[
\mathbb{K} \oplus Ki \oplus Kj \oplus Kij
\]

with multiplication rules \( i^2 = a, j^2 = b \) and \( ij = -ji \).

**Example 1.2.2.** \( \mathbb{H} = \left( \frac{-1\cdot -1}{\mathbb{R}} \right) \). With the \( k \) dropped, this captures Hamilton’s multiplication rules more concisely.

By writing \( ij \) instead of \( k \) we may now choose \( a, b \) to be arbitrary elements of \( \mathbb{K}^\times \) without losing associativity. In particular Proposition 1.1.4 no longer poses an obstruction to this.

Let \( \mathcal{B} = \left( \frac{a, b}{\mathbb{K}} \right) \).

**Definition 1.2.3.** Let \( q = w + xi + yj + zij \in \mathcal{B} \) with \( w, x, y, z \in \mathbb{K} \).

1. The *quaternion conjugate* of \( q \) is \( q^* := w - xi - yj - zk \).

2. The *norm* of \( q \) is \( n(q) := qq^* = w^2 - ax^2 - by^2 + abz^2 \).
(3) The trace of \( q \) is \( \text{tr}(q) := q + q^* = 2w \).

**Remark 1.2.4.**

(1) If \( K \subseteq \mathbb{C} \), then \( n \) and \( \text{tr} \) agree with the determinant and matrix trace under any faithful matrix representation of \( B \) into \( M_2(\mathbb{C}) \).

(2) In the literature [50] these are sometimes called the reduced norm and reduced trace, denoted respectively by \( \text{nr}(q) \) and \( \text{tr}(q) \). This is to distinguish them from the norm and trace of \( q \) under the left-regular representation

\[
B \to \text{End}_K(B) : p \mapsto (q \mapsto pq).
\]

The norm and trace of \( q \) in this sense are \( n(q)^2 \) and \( 2\text{tr}(q) \) respectively.

(3) \( \forall p, q \in B : \text{tr}(p + q) = \text{tr}(p) + \text{tr}(q) \).

**Proposition 1.2.5** ([50, 20]). For \( p, q \in B \), the following statements hold.

(1) \( q \) is invertible if and only if \( n(q) \neq 0 \), and in this case \( q^{-1} = \frac{q^*}{n(q)} \).

(2) \( (q^*)^* = q \).

(3) \( (pq)^* = q^*p^* \).

**Corollary 1.2.6.** For all \( p, q \in B \), \( n(pq) = n(p)n(q) \).
Definition 1.2.7. Let $q = w + xi + yj + zij \in B$ with $w, x, y, z \in K$ and let $S \subseteq B$ be a subset.

1. $q$ is a pure quaternion when $\text{tr}(q) = 0$.

2. The pure quaternion part of $q$ is $q_0 := xi + yj + zij$.

3. The set of pure quaternions in $S$ is $S_0 := \{q \in S \mid \text{tr}(q) = 0\}$. We will similarly indicate other conditions on the trace with a subscript, e.g. $S_+ := \{q \in S \mid \text{tr}(q) > 0\}$.

4. $S^1 := \{q \in S \mid n(q) = 1\}$. We will similarly indicate other conditions on the norm with a superscript, e.g. $S^0 := \{q \in S \mid n(q) = 0\}$ and the units in $S$ is the set $S^\times := \{q \in S \mid \exists q^{-1}\} = \{q \in S \mid n(q) \neq 0\}$.

Recall Hamilton’s Theorem 1.1.3 where the rotation action is accomplished by $p \mapsto upu^*$, with $u \in P\mathbb{H}_1$ and $p \in \mathbb{H}_0$. By Proposition 1.2.5, since $n(u) = 1$ this could just as well be written as a conjugation $p \mapsto upu^{-1}$.

As implied by the following Theorem, conjugation maps of this form play a larger role in the theory of quaternion algebras.

Theorem 1.2.8 (Skolem-Noether). [20, 50] Let $A$ be a central-simple $K$-algebra, and let $A'$ be a simple $K$-algebra. Let $\varphi_\ell : A \rightarrow A'$ be $K$-algebra homomorphisms, for $\ell = 1, 2$. Then $\exists \ c \in A'$ such that $\varphi_2(x) = c\varphi_1(x)c^{-1}$. 
Since quaternion algebras are central-simple, taking $A = A'$ to be quaternion and letting $\varphi_1 = \text{Id}$ in the notation of the theorem, we obtain the following.

**Corollary 1.2.9.** Let $\mathcal{B}$ be a quaternion algebra over $K$. If $\varphi \in \text{Aut}_K(\mathcal{B})$, then $\exists \ c \in \mathcal{B}$ such that $\varphi : p \mapsto cpc^{-1}$.

An equivalent way of stating this is every $K$-linear automorphism of $\mathcal{B}$ is inner, i.e. $\text{Aut}_K(\mathcal{B}) = \text{Inn}_K(\mathcal{B})$.

### 1.2.2 Ramification

The algebra $\left( \frac{a,b}{K} \right)$ is not unique up to isomorphism. It will be important in upcoming applications to know whether two given quaternion algebras are isomorphic. We do this by examining their ramification.

**Definition 1.2.10.** A quaternion algebra $\left( \frac{a,b}{K} \right)$ is called *split* if $\left( \frac{a,b}{K} \right) \cong M_2(K)$, and it is called *ramified* otherwise.

One can think of the term “split” here in relation to the splitting field of a polynomial, for a quaternion algebra is split when it contains the necessary square roots as described in part (3) below.
1.2. NUMBER THEORETIC GENERALIZATION

Theorem 1.2.11 ([20, 37, 50]).

(1) Up to $K$-algebra isomorphism, the only quaternion algebra over $K$ which is not a division algebra is $\left(\frac{1,1}{K}\right) \cong M_2(K)$.

(2) $\left(\frac{a,b}{K}\right) \cong \left(\frac{1,1}{K}\right) \iff \exists (x,y) \in K^2 : ax^2 + by^2 = 1$.

(3) $\forall x,y \in K^\times : \left(\frac{a,b}{K}\right) \cong \left(\frac{b,a}{K}\right) \cong \left(\frac{ax^2,by^2}{K}\right)$.

For example, since $\mathbb{C}$ is algebraically closed and $\mathbb{R}$ contains all its positive square roots, we have the following fact.

Corollary 1.2.12.

(1) Up to isomorphism, the only quaternion algebras over $\mathbb{R}$ are $\mathbb{H}$ (a division algebra) and $M_2(\mathbb{R})$, and

(2) Up to isomorphism, the only quaternion algebra over $\mathbb{C}$ is $M_2(\mathbb{C})$.

Places

Now let $K$ denote an arbitrary number field. An algebraic integer is a complex root of a monic polynomial in $\mathbb{Z}[x]$ (as opposed to an algebraic number, where we no longer require the polynomial to be monic). The ring of integers of $K$ is the ring formed by algebraic integers in $K$ and is denoted by $R_K$. An ideal of $R_K$ is denoted by $\mathfrak{J} \lhd R_K$. Let $p$ denote a prime ideal of $R_K$. 
A place of $K$ is an equivalence class of valuations on $K$ but for our purposes it is useful to define it as a type of field embedding, with the following justification. For each valuation $v$ there is a completion of $K$ at $v$, denoted by $K_v$. Since $K$ is a number field, $K_v$ will always be either $\mathbb{C}$, $\mathbb{R}$, or a $p$-adic field.

Consider first the cases where $K_v$ is either $\mathbb{C}$ or $\mathbb{R}$. Then (up to equivalence) $v$ is necessarily of the form $v : K \to \mathbb{R}^+, x \mapsto |\sigma(x)|$, where $|\cdot|$ is the complex modulus and $\sigma : K \hookrightarrow K_v$ is an embedding corresponding to an element of $\mathfrak{G}(K : \mathbb{Q})$, the Galois group of $K$ over $\mathbb{Q}$. When $K_v = \mathbb{R}$, there is a one-to-one correspondence between these choices for $\sigma$ and equivalence classes of $v$. When $K_v = \mathbb{C}$, there is a one-to-one correspondence between complex conjugate pairs $(\sigma, \overline{\sigma})$ and equivalence classes of $v$. Denote a real embedding of $K$ by $\theta$ and a complex embedding of $K$ by $\tau$.

In the case where $K_v$ is a $p$-adic field, there exists a non-zero prime ideal $p \in R_K$ so that $v$ is (up to equivalence) the $p$-adic valuation

$$v : K \to \mathbb{R}^+ \cup \{\infty\}, \quad x \mapsto \begin{cases} e^{-\text{ord}_p(x)} & ; \quad p \mid xR_K, \\ \infty & ; \quad p \nmid xR_K. \end{cases}$$

Here $\text{ord}_p(x)$ is the integer occurring as the exponent of $p$ in the unique prime factorization of the fractional ideal $xR_K$ into prime ideals $p, p_1, \ldots, p_m \in R_K$.

That is, $xR_K = p^{\text{ord}_p(x)} \prod_{\ell=1}^m p_\ell^{n_\ell}$ where $\forall \ell : p_\ell \neq p$. The point is that when
1.2. NUMBER THEORETIC GENERALIZATION

$K_v$ is $p$-adic it is a completion with respect to a prime ideal $p \ll R_K$, so we write $K_v = K_p$, and there is a natural embedding $\epsilon_p : K \hookrightarrow K_p$.

This justifies the following definition.

**Definition 1.2.13.**

1. A real place of $K$ is an embedding $\theta : K \hookrightarrow \mathbb{R}$.

2. A complex place of $K$ is an embedding $\tau : K \hookrightarrow \mathbb{C}$ where the image of $\tau$ is not totally real, and we identify $\tau$ with its complex conjugate $\overline{\tau}$.

3. An infinite place of $K$ is a real or complex place of $K$.

4. A finite place of $K$ is an embedding $\epsilon_p : K \hookrightarrow K_p$ where $p \ll R_K$. Since $p$ uniquely determines $\epsilon_p$, we often use $p$ to denote both the place and the ideal.

5. A place of $K$ is a finite or infinite place.

We gain a full description of isomorphism classes of quaternion algebras by examining the ramification at its field’s places. Let $B = \left( \frac{a,b}{K} \right)$.

**Definition 1.2.14.**

1. The completion of $B$ at the place $\sigma : K \hookrightarrow K_v$ is

$$B_\sigma := B \otimes_{K_v} \mathbb{K}_v = \left( \frac{\sigma(a), \sigma(b)}{K_v} \right).$$
(2) \( B \) is \textit{split} (or \textit{unramified}) at \( \sigma \) if \( B_\sigma \cong M_2(K_\nu) \), otherwise \( B \) is \textit{ramified} at \( \sigma \).

If \( \sigma \) is real, then \( B_\sigma = \left( \frac{\sigma(a), \sigma(b)}{K} \right) \). By Corollary 1.2.12, \( B_\sigma \) must be either \( \mathbb{H} \) or \( M_2(\mathbb{R}) \), thus the completion of \( B \) at a real place \( \theta \) is ramified if and only if \( B_\theta \cong \mathbb{H} \). By the same Corollary, if \( \tau \) is complex then \( B_\tau \cong M_2(\mathbb{C}) \), i.e. \( B_\tau \) is never ramified. This motivates the following.

\textbf{Definition 1.2.15.} The \textit{infinite ramification} of \( B \), denoted by \( \text{Ram}_\infty(B) \), is the set of real places \( \theta \) such that \( B_\theta \cong \mathbb{H} \).

Each finite place corresponds to some \( p \prec R_K \), and the completion takes the form \( B_p = \left( \frac{a, b}{K_p} \right) \). Quaternion algebras over a \( p \)-adic field, like those over \( \mathbb{R} \), fall into only two isomorphism classes.

\textbf{Proposition 1.2.16 ([37])}. If \( B \) is a quaternion algebra over the \( p \)-adic field \( K_p \) then either \( B \cong M_2(K_p) \), or \( B \) is a division algebra. Moreover, up to isomorphism there is only one quaternion division algebra over \( K_p \).

\textbf{Definition 1.2.17.} The \textit{finite ramification} of \( B \), denoted by \( \text{Ram}_p(B) \) is the set of prime ideals \( p \prec R_K \) such that \( B_p \) is ramified.

\textbf{Definition 1.2.18.} The \textit{ramification} of \( B \) is the set

\[ \text{Ram}(B) := \text{Ram}_\infty(B) \cup \text{Ram}_p(B). \]
We summarize some important properties of ramification.

**Theorem 1.2.19** (§2.7 of [37], [50]). Let $\mathcal{B}$ be a quaternion algebra.

1. $\mathcal{B}$ is split if and only if $\text{Ram}(\mathcal{B}) \neq \emptyset$.

2. $\text{Ram}(\mathcal{B})$ is a finite set of even cardinality.

3. For every number field $K$ and every set $S$ of complex places of $K$ having even cardinality, there exists a quaternion algebra $\mathcal{B}$ such that $\text{Ram}(\mathcal{B}) = S$.

4. If $\mathcal{B}'$ is a quaternion algebra over the same number field as $\mathcal{B}$, then $\mathcal{B} \cong \mathcal{B}'$ as $K$-algebras if and only if $\text{Ram}(\mathcal{B}) = \text{Ram}(\mathcal{B}')$.

**Computation**

When computing ramification, the infinite case is more straightforward because of the analogy with $\mathfrak{O}(K : \mathbb{Q})$. Let $d = [K : \mathbb{Q}]$, the degree of the field extension $K \supseteq \mathbb{Q}$. Then the minimal polynomial of $K$ over $\mathbb{Q}$ has $d$ roots with complex roots occurring in conjugate pairs. Let $r$ be the number of real roots and let $2c$ be the number of complex roots. Then $d = r + 2c$, and $K$ has $r$ real places and $c$ complex places. If $t \in \mathbb{C}$ is the primitive element of $K$, then $K = \mathbb{Q}(t)$ and the infinite places of $K$ are the results of replacing $t$ with these roots.
Finite ramification is more subtle but there is often a straightforward method for computing it.

**Definition 1.2.20.** The *ideal norm* for $K$ is the function

$$N : \{ \mathfrak{I} \mid \mathfrak{I} \subseteq R_K \}, \mathfrak{I} \mapsto |R_K/\mathfrak{I}|.$$

**Example 1.2.21.**

(1) When $K = \mathbb{Q}$, ideals are of the form $(n)$ with $n \in \mathbb{N}$, and we have $N((n)) = n$.

(2) When $K = \mathbb{Q}(\sqrt{10})$, we have $N((2)) = 4$ and $N((2, \sqrt{10})) = 2$.

**Definition 1.2.22.** A prime ideal $\mathfrak{p}$ is *dyadic* when $N(\mathfrak{p}) = 2^n$ for some $n \in \mathbb{N}$.

Computing ramification at dyadic primes is more complicated, and usually handled case-by-case, but for the other primes we have the theorem below. For $S \subseteq \mathbb{Q}$, let $S^{(2)} := \{ s^2 \mid s \in S \}$.

**Theorem 1.2.23** (§2.6 of [37]). Let $\mathcal{B} = \left( \frac{a,b}{K} \right)$, and let $\mathfrak{p} \ll R_K$ be non-dyadic.

(1) If $a, b \notin \mathfrak{p}$, then $\mathcal{B}_p$ is split (unramified).
1.2. NUMBER THEORETIC GENERALIZATION

(2) When $a \notin \mathfrak{p}$ and $b \in \mathfrak{p} \setminus \mathfrak{p}^{(2)}$:

\[ \mathcal{B}_p \text{ is split } \iff a \text{ is congruent to a square modulo } \mathfrak{p}. \]

(3) When $a, b \in \mathfrak{p} \setminus \mathfrak{p}^{(2)}$:

\[ \mathcal{B}_p \text{ is split } \iff -a^{-1}b \text{ is congruent modulo } \mathfrak{p} \text{ to a square in } R_K. \]

1.2.3 Quadratic Forms

The theory of quaternion algebras is deeply related to the theory of quadratic forms. Let $K$ be a field with $\text{char}(K) \neq 0$. Recall that the norm on $\mathcal{B} = \left( \frac{a,b}{K} \right)$ is the map

\[ n : \mathcal{B} \to K \quad w + xi + yj + zi j \mapsto w^2 - ax^2 - by^2 + abz^2. \]

The norm is a quadratic form over $K$, and in this sense $\mathcal{B}$ is a quadratic space. However for any $\mathcal{B}$, the coefficient of $w^2$ will always be 1, thus there is no loss of information in considering only how $n$ behaves on the pure quaternions $\mathcal{B}_0$.

Also, the coordinate $ab$ of $z^2$ depends only on the previous two coordinates, so in a sense even the restriction $n|_{\mathcal{B}_0}$ contains some redundancy. This can be made precise as follows.

Let $V$ and $V'$ be quadratic spaces over $K$, with respective quadratic forms $\phi$ and $\phi'$.

**Definition 1.2.24.** We say that $V$ and $V'$ (or $\phi$ and $\phi'$) are *similar* if $\exists a$
K-linear isomorphism $\psi : V \to V'$ and $\exists u \in K^\times$ such that

$$\forall v \in V : \phi(v) = u(\phi' \circ \psi)(v).$$

**Theorem 1.2.25 ([50]).** The map $B \mapsto n|_{B_0}$ yields a bijection from the set of quaternion algebras over $K$ up to isomorphism, and the set of non-degenerate quadratic forms on $K^3$ up to similarity.

The following tools will be useful in discussing the behavior of $n$.

**Definition 1.2.26.** Let $V$ be a finite-dimensional vector space over $K$ and let $\phi$ be a quadratic form on $V$.

1. The **associated symmetric bilinear form** of $\phi$ is the map

$$\beta : V \times V \to K \quad (v, w) \mapsto \phi(v + w) - \phi(v) - \phi(w).$$

2. Vectors $v$ and $w$ are **orthogonal (with respect to $\phi$)**, if $\beta(v, w) = 0$, and this is denoted by $v \perp w$.

3. For a subspace $S \subset V$, the **orthogonal complement of $S$ (with respect to $\phi$)** is $S^\perp := \{v \in V \mid \forall s \in S : v \perp s\}$.

4. Given any basis $B = \{b_1, \ldots, b_\ell\}$ for $V$, the **Gram matrix** of $\phi$ with respect to $B$, denoted by $G^B_\phi$, is the $\ell \times \ell$ matrix where entry $(m, n)$ is $\beta(b_m, b_n)$. 
1.2. NUMBER THEORETIC GENERALIZATION

(5) A basis $B$ for $V$ diagonalizes $\phi$ if $G^B_\phi$ is a diagonal matrix, and in this case $G^B_\phi$ is a diagonalization of $\phi$.

(6) Let $\text{diag}(d_1, \ldots, d_\ell)$ denote the diagonal matrix with the listed entries as its diagonal.

Remark 1.2.27. Letting $\vec{v} := (v_1, \ldots, v_n) \in V$ where the coordinates are with respect to some basis $B$, 

$$\phi(\vec{v}) = \sum_{k,\ell=1}^{n} a_{k,\ell} v_k v_\ell = \phi(\vec{v}) = \vec{v}(a_{k,\ell}) \vec{v}^T,$$

where $\forall k, \ell : a_{k,\ell} = a_{\ell,k}$. Therefore $G^B_\phi = (a_{k,\ell})$, a symmetric matrix.

Definition 1.2.28. Let $V$ be an $n$-dimensional vector space over $K$, and let $\phi$ and $\phi'$ be quadratic forms on $V$. We say $\phi$ is isometric over $K$ to $\phi'$, denoted by $\phi \simeq \phi'$, if there exists some $m \in \text{GL}_n(K)$ such that $mG^B_\phi m^\top = G^B_{\phi'}$.

Theorem 1.2.29 (The Inertia Theorem [32]). There exists a basis $B$ that diagonalizes $\phi$. Moreover, this diagonal basis takes the following forms.

(1) If $K = \mathbb{C}$, then $B$ can be chosen so that $G^B_\phi = \text{diag}(1, \ldots, 1, 0, \ldots, 0)$, and the number of occurrences of 1 and of 0 is independent of the choice of $B$. 

(2) If $K = \mathbb{R}$, then $B$ can be chosen so that
\[ G_B^\phi = \text{diag}(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1) \]
and the number of occurrences of each of $1, 0, -1$ is independent of the choice of $B$.

(3) If $K \subset \mathbb{R}$, then $B$ can be chosen so that $G_B^\phi$ is diagonal, but the number of occurrences of each of positive, negative, and zero entries in $G_B^\phi$ is independent of the choice of $B$.

By this theorem the following terms are well-defined.

**Definition 1.2.30.** A quadratic form $\phi$ with a diagonalization $\text{diag}(d_1, \ldots, d_n)$ is nondegenerate if $d_1, \ldots, d_n \neq 0$. If $\phi$ is nondegenerate and $K \subset \mathbb{R}$, the signature of $\phi$, denoted by $\text{sig}(\phi)$, is the pair of integers $(p, m)$ where $p$ and $m$ are the numbers of positive and negative entries in $\text{diag}(d_1, \ldots, d_n)$, respectively.

**Proposition 1.2.31.** [50] Let $\mathcal{B}$ be a quaternion algebra over $K$. Consider $\mathcal{B}$ as a quadratic space over $K$ with quadratic form $n$.

1. The symmetric bilinear form associated to $n$ is $\beta(v, w) = \text{tr}(v^*w)$.

2. $K^\perp = \mathcal{B}_0$.

3. $\{1, i, j, ij\}$ is an orthogonal basis and $G_n^\mathcal{B} = \text{diag}(1, -a, -b, ab)$.
1.2. NUMBER THEORETIC GENERALIZATION

1.2.4 Algebras with Involution

Quaternion division algebras admit an alternative characterization in the language of algebras with involution [29, 50]. We give some basic definitions and results which will be useful later. Let $K$ be a field where $\text{char}(K) \neq 2$, and let $A$ be a central simple $K$-algebra.

Definition 1.2.32. An involution on $A$ is a map $\star : A \to A : x \mapsto x^\star$ such that $\forall x, y \in A$:

1. $(x + y)^\star = x^\star + y^\star$

2. $(xy)^\star = y^\star x^\star$

3. $(x^\star)^\star = x$

Let $\star$ be an involution on $A$.

Definition 1.2.33. For $S \subset A$, the set of symmetric elements of $\star$ in $S$ is

$\star S := \{ x \in S \mid x^\star = x \}$.

Remark 1.2.34. In [29] this is denoted by $\text{Sym}(S, \star)$. In other contexts one might see it written as $S^{\star}$ or $S^{(\star)}$. 
Definition 1.2.35.

(1) $\star$ is of the first kind if $K = \, ^*K$.

(2) $\star$ is standard if it is of the first kind and $\forall \, x \in A : xx^* \in K$.

(3) $\star$ is of the second kind if $K \neq \, ^*K$.

Proposition 1.2.36.

(1) $1^* = 1$.

(2) If $\star$ is of the first kind then it is $K$-linear.

(3) If $\star$ is of the second kind then $[K : \, ^*K] = 2$.

Example 1.2.37.

(1) Complex conjugation on the $\mathbb{R}$-algebra $\mathbb{C}$ is a standard involution.

(2) Matrix transposition on the $K$-algebra $M_n(K)$ is an involution of the first kind, but not a standard involution.

(3) If $K$ is a number field of the form $K = F(\sqrt{-d})$ where $F \subset \mathbb{R}$ and $d \in F^+$, then the conjugate transpose is an involution of the second kind on the $K$-algebra $M_n(K)$. 
1.2. NUMBER THEORETIC GENERALIZATION

(4) An algebra over the field $\mathbb{Q}(\sqrt{-2})$ does not admit an involution of the second kind because it has no subfield of index 2.

The elements of an algebra with involution admit a convenient description in terms of the symmetric elements and the complementary set defined below.

**Definition 1.2.38.** Let $A$ be a $K$-algebra with involution $\ast$. The set of skew-symmetric elements of $A$ (with respect to $\ast$) is defined as

$$\text{Skew}(A, \ast) := \{ q \in \mathbb{Q} : q^* = -q \}. $$

**Proposition 1.2.39.** $A = A \oplus \text{Skew}(A, \ast)$

**Proof.** For an arbitrary element $x \in A$, let $m = \frac{x + x^*}{2}$ and $s = \frac{x - x^*}{2}$. Then $x = m + s$. Observe that the elements $m$ and $s$ are in the desired sets.

$$m^* = \left( \frac{x + x^*}{2} \right)^* = \frac{x^* + (x^*)^*}{2} = \frac{x^* + x}{2} = m$$

$$s^* = \left( \frac{x - x^*}{2} \right)^* = \frac{x^* - (x^*)^*}{2} = \frac{x^* - x}{2} = -s$$

It remains to show the decomposition is unique. Suppose $x = m' + s'$ where $m' \in A$ and $s' \in \text{Skew}(A, \ast)$. Then $x^* = m'^* + s'^* = m' - s'$, giving $x + x^* = 2m'$ and $x - x^* = 2s'$, giving the result. \hfill $\square$

Let us now look at some less obvious properties of involutions. Standard involutions can be used to characterize quaternion division algebras in the
following sense.

**Theorem 1.2.40.** [50] A non-commutative division algebra is quaternion if and only if it admits a standard involution. Moreover this standard involution is the quaternion conjugate.

Not every quaternion algebra over \( K \) admits involutions of the second kind. When one does, the following theorem due to Albert [29] gives a correspondence between such involutions and quaternion subalgebras over \( ^*K \).

**Theorem 1.2.41.** [29] Let \( F \) be a field and \( K \supseteq F \) be a separable quadratic field extension. Let \( \tau \) generate \( \mathfrak{S}(K: F) \). Let \( \mathcal{B} \) be a quaternion algebra over \( K \) and \( * \) be an involution of the second kind on \( \mathcal{B} \) such that \( *|_K = \tau \).

Then there exists a unique quaternion \( F \)-subalgebra \( \mathcal{A} \subset \mathcal{B} \) such that \( \mathcal{B} = \mathcal{A} \otimes_F K \) and \( (q \otimes c)^* = q^* \otimes \tau(c) \). In particular, \( \mathcal{A} \) is the algebra generated over \( F \) by \( \text{Skew}(\mathcal{B}, *)_0 = \{ p \in \mathcal{B}_0 \mid p^* = -p \} \).
Chapter 2

Arithmetic of Hyperbolic Manifolds

In this chapter we give a brief introduction to hyperbolic 3-manifolds and surfaces and their arithmetic invariants. In §2.1 we introduce hyperbolic $n$-space, the upper half-space model, hyperboloid models and other aspects of hyperbolic geometry (see [5, 42]). In §2.2 we summarize the use of number fields and quaternion algebras to distinguish manifolds and commensurability classes of manifolds (see [37]). In §2.3, we explain how hyperbolic 3-manifolds and surfaces whose fundamental groups are arithmetic in the sense of Borel arise naturally from quaternion algebras.

2.1 Hyperbolic 3-Manifolds and Surfaces

We will denote hyperbolic $n$-space by $\mathbb{H}^n$. All hyperbolic $n$-manifolds are assumed to be complete and orientable. We focus primarily on $n = 3$ with
28  \textit{CHAPTER 2. ARITHMETIC OF HYPERBOLIC MANIFOLDS}

analogous commentary for $n = 2$.

A \textit{geodesic} is a path that has the shortest possible length among paths connecting its endpoints, and a \textit{complete geodesic} in $\mathcal{H}^n$ is a geodesic whose endpoints lie on $\partial \mathcal{H}^n$. A lower-dimensional submanifold $S$ of a manifold $X$ is \textit{totally geodesic} in $X$ if every geodesic in $S$ is a geodesic in $X$. We will be interested later in hyperbolic surfaces that admit immersions in hyperbolic 3-manifolds as totally geodesic surfaces.

\section*{2.1.1 The Upper Half-Space Model}

The $n$-dimensional upper half-space model for $\mathcal{H}^n$ is

$$\mathcal{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

endowed with the metric $ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$. We call $\mathcal{H}^3$ the \textit{upper half-space} and $\mathcal{H}^2$ the \textit{upper half-plane}, and one can make the following identifications.

$$\mathcal{H}^3 = \mathbb{C} \times \mathbb{R}^+ \quad \partial \mathcal{H}^3 = \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

$$\mathcal{H}^2 = \mathbb{R} \times \mathbb{R}^+ \sqrt{-1} \quad \partial \mathcal{H}^2 = \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$$

The standard approach then is to explicitly define the Möbius action by $\text{PSL}_2(\mathbb{C})$ (respectively $\text{PSL}_2(\mathbb{R})$) on $\partial \mathcal{H}^3$ (respectively $\partial \mathcal{H}^2$), and define the
action by orientation-preserving isometries on the whole space abstractly as the unique isometric extension of the Möbius action on the boundary. This gives that the group \( \text{Isom}^+(\mathcal{H}^3) \) of orientation-preserving isometries of \( \mathcal{H}^3 \) is isomorphic to \( \text{PSL}_2(\mathbb{C}) \), and \( \text{Isom}^+(\mathcal{H}^2) \cong \text{PSL}_2(\mathbb{R}) \).

**Definition 2.1.1.** A *Kleinian group* is a discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \), and a *Fuchsian group* is a discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \).

**Remark 2.1.2.** In some sources, a Kleinian (Fuchsian) group is defined as a group admitting a discrete faithful representation into \( \text{PSL}_2(\mathbb{C}) \) (\( \text{PSL}_2(\mathbb{R}) \)), while here we define them concretely as having a fixed embedding.

Let \( \Gamma \) be a Kleinian (or Fuchsian) group. With \( n = 3 \) (or 2), we denote by \( \mathcal{H}^n/\Gamma \) the quotient of \( \mathcal{H}^n \) by the action of \( \Gamma \). That is, \( \mathcal{H}^n \) is the universal covering space of \( \mathcal{H}^n/\Gamma \) with covering group \( \Gamma \), giving \( \mathcal{H}^n/\Gamma \) a geometric structure modeled on \( \mathcal{H}^n \). It follows that \( \mathcal{H}^n \) is a hyperbolic \( n \)-orbifold. When \( \Gamma \) is torsion-free, \( \mathcal{H}^n/\Gamma \) is a hyperbolic \( n \)-manifold. We are interested in these manifolds up to homeomorphism and in this sense, all oriented hyperbolic \( n \)-manifolds arise in this way.

**Definition 2.1.3.** \( \Gamma \) is called *finite covolume* when \( \text{vol}(\mathcal{H}^3/\Gamma) < \infty \) and is called *cocompact* when \( \mathcal{H}^3/\Gamma \) is compact.
CHAPTER 2. ARITHMETIC OF HYPERBOLIC MANIFOLDS

In dimensions $n \leq 3$, there is a correspondence between the geometry and the topology of $\mathcal{H}^n$, allowing us to work with Kleinian groups up to isomorphism.

**Theorem 2.1.4.** [3] (The Mostow-Prasad Rigidity Theorem) If $X$ and $Y$ are finite-volume, complete, connected hyperbolic $n$-manifolds with $n \geq 3$, where $\pi_1(X) \cong \pi_2(Y)$, then $X \cong Y$.

**Example 2.1.5.** [37] The figure-8 knot group has the presentation

$$\langle x, y \mid xyx^{-1}yx = yxy^{-1}xy \rangle.$$ 

A matrix representation of this is $\Gamma = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{1 + \sqrt{-3}}{2} & 1 \end{pmatrix} \rangle$, thus $\mathcal{H}^3/\Gamma$ is the figure-8 knot complement equipped with hyperbolic structure.

Any isomorphism of Kleinian groups is realized via conjugation in $\text{PSL}_2(\mathbb{C})$, thus such conjugation induces a homeomorphism of the corresponding manifolds. While the Mostow-Prasad Rigidity Theorem does not hold in dimension 2, we do have something similar for subsurfaces of a hyperbolic 3-manifold.
2.1. HYPERBOLIC 3-MANIFOLDS AND SURFACES

**Theorem 2.1.6.** Let $X$ be a finite-volume hyperbolic 3-manifold and $\Gamma \cong \pi_1(X)$ be a Kleinian group. Then there exists a compact hyperbolic surface $S$ which admits a totally geodesic immersion in $X$ if and only if $\exists \Delta < \Gamma$ and $\exists c \in \text{PSL}_2(\mathbb{C})$ such that

1. $\Delta \cong \pi_1(S)$,
2. $c\Delta c^{-1}$ is Fuchsian, and
3. $S \approx \mathcal{H}^2/c\Delta c^{-1}$.

2.1.2 Hyperboloid Models

While the upper half-space model for $\mathbb{H}^n$ is more conventional, we will be more interested, in §3 onward, in working with hyperboloid models. A key difference between these two approaches is that in $\mathcal{H}^n$ points on the boundary are more accessible while in a hyperboloid model points in hyperbolic space are more accessible.

Many interesting results have been accomplished using hyperboloid models, such as Epstein and Penner’s decomposition algorithm for non-compact manifolds [19], Vinberg and Shvartsman’s classification of hyperbolic reflection groups [48], and Gromov and Piatetski-Shapiro’s realization of non-arithmetic lattices [25].
**Definition 2.1.7.** Let $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a quadratic form with $\text{sig}(\phi) = (1, n)$. A hyperboloid model for hyperbolic $n$-space is defined as

$$\mathcal{I}^\phi := \left\{(w, x, y, z) = p \in \mathbb{R}^{(1,n)} \mid \phi((1,n))(p) = 1, w > 0\right\},$$

equipped with the metric induced by the restriction of $\phi$ to the tangent space of $\mathcal{I}^\phi$.

**Remark 2.1.8.** In the literature, $\text{sig}(\phi) = (1, n)$ is instead $(n, 1)$, and the requirement $\phi(p) = 1$ is instead $\phi(p) = -1$. The version given above is equivalent and better suited to the constructions in the next chapter.

The standard choice for $\phi$ is the standard quadratic form over $\mathbb{R}$ of signature $(1, n)$, which is

$$\phi_{(1,n)} : \mathbb{R}^{n+1} \to \mathbb{R} : (x_1, \ldots, x_{n+1}) \mapsto x_1^2 - x_2^2 - \cdots - x_{n+1}^2. \quad (2.1.9)$$

When $\phi = \phi_{(1,n)}$, we call $\mathcal{I}^\phi$ the standard hyperboloid model for hyperbolic $n$-space, and denote this by $\mathcal{I}^n$.

More generally, we introduce hyperboloid models over arbitrary subfields of $\mathbb{R}$.

**Definition 2.1.10.** Let $F \subset \mathbb{R}$ be a field and let $\phi : F^{n+1} \to F$ be a quadratic form of signature $(1, n)$. The vector model defined by the form $\phi$ is
2.1. HYPERBOLIC 3-MANIFOLDS AND SURFACES

defined as

\[ \mathcal{I}^\phi := \{(x_1, \ldots, x_{n+1}) \in F^{n+1} \mid \phi(p) = 1, x_1 > 0\}, \]

equipped with the metric induced by the restriction of \( \phi \) to the tangent space of \( \mathcal{I}^n \). We will call this a hyperboloid model for hyperbolic \( n \)-space defined over \( F \), or more specifically the hyperboloid model induced by \( \phi \).

**Proposition 2.1.11.** Let \( \phi : F^{n+1} \to F \) be as in the above definition and let \( \epsilon : \mathcal{I}^\phi \hookrightarrow \mathbb{R}^{n+1} \) be the natural embedding. Then over \( \mathbb{R} \) we have \( \epsilon(\mathcal{I}^\phi) \cong \mathcal{I}^n \).

**Proof.** The statement holds if and only if \((\mathbb{R}^{n+1}, \phi)\) and \((\mathbb{R}^{n+1}, \phi_{(1,n)})\) are isometric as quadratic spaces, which is true if and only if \( \phi \) (under an extension of scalars from \( F \) to \( \mathbb{R} \)) and \( \phi_{(1,n)} \) are isometric as quadratic forms over \( \mathbb{R} \). This follows from Theorem 1.2.29. \( \square \)

The distance between two points \( p, q \in \mathcal{I}^\phi \) is given by

\[ d_{\mathcal{I}^\phi}(p, q) = \text{arcosh}(\beta(p, q)) \quad (2.1.12) \]

where \( \beta \) is the symmetric bilinear form associated to \( \phi \) as in Definition 1.2.26 [42]. Taking \( p = (x_1, \ldots, x_{n+1}) \), \( q = (x'_1, \ldots, x'_{n+1}) \) and \( \phi = \phi_{(1,n)} \) yields

\[ d_{\mathcal{I}^n}(p, q) = \text{arcosh}\left(\frac{\phi(p + q) - \phi(p) - \phi(q)}{2}\right) \]

\[ = \text{arcosh}(x_1x'_1 - x_2x'_2 - \cdots - x_{n+1}x'_{n+1}). \quad (2.1.13) \]
A convenient property of a hyperboloid model $T^\phi$ is that under its inclusion into $F^{(1,n)}$, linear maps on $F^{(1,n)}$ preserving $\phi$ induce isometries of $T^\phi$. For instance, geodesics on $T^\phi$ take the form of intersections of $T^\phi$ with 2-dimensional Euclidean planes passing through the origin of $F^{(1,n)}$.

2.1.3 Isometries

We characterize elements of $\text{Isom}^+(\mathcal{H}^3)$ and $\text{Isom}^+(\mathcal{H}^2)$ as follows.

**Definition 2.1.14.**

(1) An **elliptic isometry** of $\mathcal{H}^3$ is a rotation about some fixed complete geodesic, called the **axis** of the isometry, and has two fixed points in $\partial \mathcal{H}^3$, one at each end of its axis.

(2) A **parabolic isometry** of $\mathcal{H}^3$ is a translation having no fixed points in $\mathcal{H}^3$ and a single fixed point in $\partial \mathcal{H}^3$.

(3) A **hyperbolic isometry** of $\mathcal{H}^3$ is a translation along a complete geodesic, called the **axis** of the isometry, fixing no points of $\mathcal{H}^3$ and fixing two points of $\partial \mathcal{H}^3$, one on each end of its axis.

(4) A **purely loxodromic isometry** of $\mathcal{H}^3$ is a simultaneous translation along and rotation about a complete geodesic (its **axis**) in $\mathcal{H}^3$, fixing no points of $\mathcal{H}^3$, and fixing two points of $\partial \mathcal{H}^3$, one on each end of its axis.
(5) A loxodromic isometry of $\mathbb{H}^3$ is a hyperbolic or purely loxodromic isometry of $\mathbb{H}^3$.

**Definition 2.1.15.**

(1) An elliptic isometry of $\mathbb{H}^2$ is a rotation about a point in $\partial\mathbb{H}^2$.

(2) A parabolic isometry of $\mathbb{H}^2$ is a translation having no fixed points in $\mathbb{H}^2$ and a single fixed point in $\partial\mathbb{H}^2$.

(3) A hyperbolic isometry of $\mathbb{H}^2$ is a translation along a complete geodesic, called the axis of the isometry, fixing no points of $\mathbb{H}^2$ and fixing two points of $\partial\mathbb{H}^2$, one on each end of its axis.

Using the model $\mathcal{H}^n$, and the group $\text{PSL}_2(\mathbb{C})$ (or $\text{PSL}_2(\mathbb{R})$), there is an equivalent, algebraic characterization.

**Proposition 2.1.16.** Let $\gamma \in \text{PSL}_2(\mathbb{C})$, $\gamma \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $\gamma \in \text{PSL}_2(\mathbb{R})$, then only the first three definitions apply.

(1) $\gamma$ is elliptic $\iff$ $\text{tr}(\gamma) \in \mathbb{R}$ and $|\text{tr}(\gamma)| < 2$.

(2) $\gamma$ is parabolic $\iff$ $\text{tr}(\gamma) = \pm 2$.

(3) $\gamma$ is hyperbolic $\iff$ $\text{tr}(\gamma) \in \mathbb{R}$ and $|\text{tr}(\gamma)| > 2$.

(4) $\gamma$ is purely loxodromic $\iff$ $\text{tr}(\gamma) \in \mathbb{C} \setminus \mathbb{R}$. 
(5) \( \gamma \) is loxodromic \( \iff \) \( \text{tr}(\gamma) \notin [-2, 2] \).

**Definition 2.1.17.** If \( \gamma \in \text{PSL}_2(\mathbb{C}) \) is loxodromic isometry, the *translation length* of \( \gamma \), denoted by \( \ell(\gamma) \), is the hyperbolic distance by which \( \gamma \) moves points along its axis.

**Proposition 2.1.18.** If \( \gamma \in \text{PSL}_2(\mathbb{C}) \) is loxodromic (hyperbolic when \( \gamma \in \text{PSL}_2(\mathbb{R}) \)), then \( \ell(\gamma) = 2 \arccosh\left(\frac{\text{tr}(\gamma)}{2}\right) \).

**Proof.** Since \( \det(\gamma) = 1 \), using the characteristic polynomial of \( \gamma \), an eigenvalue \( \lambda \) of \( \gamma \) satisfies \( \lambda^2 - \text{tr}(\gamma)\lambda + 1 = 0 \), which implies

\[
\lambda = \frac{\text{tr}(\gamma) \pm \sqrt{\text{tr}^2(\gamma) - 4}}{2}.
\]

Since \( \gamma \) is loxodromic, \( \gamma \) has two fixed points. Up to conjugation these are 0 and \( \infty \) so without loss of generality \( \gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \). Then the Möbius action of \( \gamma \) on 1 is \( \gamma(1) = \frac{\lambda + 0}{0 + \lambda^{-1}} = \lambda^2 \), so in the hyperbolic metric, the translation length of \( \gamma \) is \( \log(\lambda^2) = 2 \log \left| \frac{\text{tr}(\gamma) + \sqrt{\text{tr}^2(\gamma) - 4}}{2} \right| \), where without loss of generality the sign choice reflects the eigenvalue satisfying \( |\lambda| > 1 \). Lastly,

\[
\log \left| \frac{\text{tr}(\gamma) + \sqrt{\text{tr}^2(\gamma) - 4}}{2} \right| = \log \left( \frac{\text{tr}(\gamma)}{2} + \sqrt{\left(\frac{\text{tr}(\gamma)}{2}\right)^2 - 1} \right)
= \arccosh\left(\frac{\text{tr}(\gamma)}{2}\right).
\]

The classification of isometries in \( \Gamma \) also provides topological information
about $\mathcal{H}^3/\Gamma$. When $\Gamma$ includes a rotation, this contributes a *cone point* to $\mathcal{H}^3/\Gamma$ so that it cannot be a manifold. When $\Gamma$ includes parabolic elements, these contribute *cusps* to $\mathcal{H}^3/\Gamma$. Each distinct cusp of $\mathcal{H}^n/\Gamma$ corresponds to a distinct conjugacy class of maximal parabolic subgroups in $\Gamma$. We summarize these observations below.

**Theorem 2.1.20** ([37]). Let $\Gamma$ be a Kleinian group (or a Fuchsian group when $n = 2$).

1. $\mathcal{H}^n/\Gamma$ is a manifold $\iff$ $\Gamma$ is torsion-free $\iff$ $\Gamma$ contains no elliptic elements.

2. $\mathcal{H}^n/\Gamma$ is cocompact $\iff$ $\Gamma$ contains no parabolic elements.

### 2.1.4 Some Applications of Number Theory

Recall that $R_K$ denotes the ring of integers of $K$. When $K$ is an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ (with $d \in \mathbb{N} \setminus \mathbb{N}^{(2)}$), we write $\mathcal{O}_d := R_{\mathbb{Q}(\sqrt{-d})}$.

**Example 2.1.21.** A *Bianchi group*, denoted by $\Gamma_d$ ($d \in \mathbb{N}$), is the group $\text{PSL}_2(\mathcal{O}_d)$. The corresponding *Bianchi orbifold* is $M_d := \mathcal{H}^3/\Gamma_d$. $M_d$ is not a manifold because $\Gamma_d$ contains torsion elements, e.g. $\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$. $M_d$ is non-compact because $\Gamma_d$ contains parabolic elements, e.g. $\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$. The
Bianchi groups provide an important set of representatives of commensurability classes which will be explored later.

**Remark 2.1.22.** If \( K \) not a complex quadratic field, then \( \text{PSL}_2(R_K) \) is not a Kleinian group because it is not discrete. This is because, since \( [K : \mathbb{Q}] > 2 \), \( \exists \ x \in R_K \) with complex modulus \( |x| < 1 \). Thus \( \text{PSL}_2(R_K) \) contains the infinite sequence \( \left\{ \begin{pmatrix} 1 & x^n \\ 0 & 1 \end{pmatrix} \right\}_{n \in \mathbb{N}} \) which approaches \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). So the Bianchi groups are the only Kleinian groups of the form \( \text{PSL}_2(R_K) \).

Two ideals \( I_1, I_2 \triangleleft \mathcal{O}_d \) are **equivalent** if there exist \( x \in \mathcal{O}_d \setminus \{0\} \) such that \( (x)I = I' \), where \( (x) \) is the principal ideal generated by \( x \). The **class number** of \( \mathbb{Q}(-d) \) is the number of distinct equivalence classes of ideals under this relation. A classical result relates the class number to cusps.

**Theorem 2.1.23** (Bianchi, 1892 [4]). The class number of \( \mathbb{Q}(-d) \) is equal to the number of cusps of \( M_d \).

**Example 2.1.24.** Let \( W \) be the Whitehead link group.

Using the diagram above for this link and computing the Wirtinger presentation...
2.2. QUATERNION ALGEBRAS AS INVARIANTS

Let $\Gamma$ be a torsion-free, finitely generated and finite covolume Kleinian group. The Mostow-Prasad Rigidity Theorem tells us that there is a one-to-one correspondence between such groups up to isomorphism and hyperbolic 3-manifolds up to homeomorphism.

The following method of associating quaternion algebras to such groups is due to Takeuchi (1975) [47] in the Fuchsian case, with the extension to the Kleinian case by Maclachlan and Reid (1987) [36] and Neumann and Reid (1992) [40]. As before both cases are treated simultaneously where convenient, letting $n = 3$ when $\Gamma$ is Kleinian and letting $n = 2$ and when $\Gamma$ is Fuchsian. Since $\text{PSL}_2(\mathbb{R}) \subset \text{PSL}_2(\mathbb{C})$ we will often refer to $\Gamma$ as sitting in $\text{PSL}_2(\mathbb{C})$ when addressing both cases.

By [31], one can obtain the matrix representation

$$W = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 - \sqrt{-1} & 1 \end{pmatrix} \right\rangle.$$

This is a finite-index, torsion-free subgroup of $\Gamma_1$. The Whitehead link complement $\mathcal{H}^3/W$ has two cusps corresponding to the two link components, even though the class number of $\mathbb{Q}(\sqrt{-1})$ is 1. This shows that subgroups of Bianchi groups do not inherit the property observed in the previous example.

2.2 Quaternion Algebras as Invariants

Let $\Gamma$ be a torsion-free, finitely generated and finite covolume Kleinian group. The Mostow-Prasad Rigidity Theorem tells us that there is a one-to-one correspondence between such groups up to isomorphism and hyperbolic 3-manifolds up to homeomorphism.

The following method of associating quaternion algebras to such groups is due to Takeuchi (1975) [47] in the Fuchsian case, with the extension to the Kleinian case by Maclachlan and Reid (1987) [36] and Neumann and Reid (1992) [40]. As before both cases are treated simultaneously where convenient, letting $n = 3$ when $\Gamma$ is Kleinian and letting $n = 2$ and when $\Gamma$ is Fuchsian. Since $\text{PSL}_2(\mathbb{R}) \subset \text{PSL}_2(\mathbb{C})$ we will often refer to $\Gamma$ as sitting in $\text{PSL}_2(\mathbb{C})$ when addressing both cases.
Elements of $\Gamma$ are technically equivalence classes of the form $\{\pm \gamma\}$ where $\gamma \in \text{SL}_2(\mathbb{C})$. Often this distinction can be ignored and $\Gamma$ can be treated as lying in $\text{SL}_2(\mathbb{C})$, but when more precision is needed we use the notation $P^{-1}(\Gamma) := \{ \pm \gamma \in \text{SL}_2(\mathbb{C}) \mid \{\pm \gamma\} \in \Gamma\}$.

2.2.1 Manifold Invariants

**Definition 2.2.1.** The *trace field* of $\Gamma$, or of $X = \mathcal{H}^n/\Gamma$, is defined as

$$KX = K\Gamma := \mathbb{Q}\langle \{\text{tr}(\gamma) \mid \gamma \in P^{-1}\Gamma\} \rangle,$$

and the *quaternion algebra* of $\Gamma$, or of $X = \mathcal{H}^n/\Gamma$, is defined as

$$BX = B\Gamma := \left\{ \sum_{\ell=1}^{n} t_\ell \bar{\gamma}_\ell \mid t_\ell \in K\Gamma, \gamma_\ell \in P^{-1}\Gamma \right\}.$$

**Remark 2.2.2.**

(1) In [37] these are denoted respectively by $\mathbb{Q}(\text{tr}\Gamma)$ and $A_0\Gamma$.

(2) $K\Gamma$ and $B\Gamma$ are also defined when $\Gamma$ has torsion, in which case $\mathcal{H}^n/\Gamma$ is not a manifold but an orbifold. In this case however one must be careful about using $KX$ and $BX$ as orbifold invariants up to homotopy equivalence.
Proposition 2.2.3.

(1) $B\Gamma$ is a quaternion algebra over $K\Gamma$.

(2) If $\Gamma$ is Kleinian, then $K\Gamma$ is a number field.

The first of these two facts follows from the Skolem-Noether Theorem and the characterization of quaternion algebras over $K\Gamma$ as the unique 4-dimensional central simple algebras over $K\Gamma$ [20]. The second fact is proven using the representation variety of $\Gamma$ and some standard results from algebraic geometry [37]. In the Fuchsian case since Mostow-Prasad rigidity does not hold there $K\Gamma$ need not be a number field, but for finite-covolume surfaces there are many interesting examples where $K\Gamma$ is a number field [47].

In dimension 3, Mostow-Prasad Rigidity implies the following.

**Theorem 2.2.4 ([37]).** Let $\Gamma, \Gamma'$ be Kleinian groups. If $\mathcal{H}^3/\Gamma$ and $\mathcal{H}^3/\Gamma'$ are homeomorphic, then $K\Gamma = K\Gamma'$, and $B\Gamma \cong B\Gamma'$ as $K\Gamma$-algebras.

Observe that the concrete field $K\Gamma$ is a manifold invariant, but that only the $K\Gamma$-algebra isomorphism class of $BX$ is a manifold invariant. Thus when working with specific quaternion algebras $\left(\frac{a \cdot b}{KX}\right) = BX$, if one wants to infer a topological property of $X$, one must show invariance of the $KX$-algebra isomorphism class of $BX$.

The converse to the Theorem does not hold.
Example 2.2.5. Let $\Gamma, \Gamma'$ be Kleinian representations of the Whitehead link group and Borromean rings group, respectively. Then $K\Gamma = K\Gamma' = \mathbb{Q}(\sqrt{-1})$ and $B\Gamma = B\Gamma' = \left( \frac{1.1}{\mathbb{Q}(\sqrt{-d})} \right)$. But vol($\mathcal{H}^3/\Gamma$) $\approx$ 3.663862377 and vol($\mathcal{H}^3/\Gamma'$) $\approx$ 7.327724753, hence the manifolds are not homeomorphic.

There are many useful geometric interpretations for the trace field. For instance when $X$ is a hyperbolic knot or link complement, $KX$ is equal to the field generated by the shape parameters of an ideal triangulation of the manifold [40].

Since $B\Gamma \cong B\Gamma'$ implies $K\Gamma \cong K\Gamma'$, a group’s quaternion algebra contains more information about $\mathcal{H}^n/\Gamma$ than $K\Gamma$, yet a geometric interpretation for $B\Gamma$ has been more elusive. In [23] Hamilton’s theorem is used to draw some geometric information from quaternion algebras with the same structure parameters as $\mathbb{H}$. In [9] a geometric interpretation of the trace is used to show that if $\mathcal{H}^3/\Gamma$ contains a closed, self-intersecting geodesic, then the invariant quaternion algebra of $\Gamma$ (defined in the next subsection) has at least one real structure parameter. More commonly though $B\Gamma$ is not interpreted geometrically but rather as an arithmetic tool for classifying hyperbolic 3-manifolds.
2.2.2 Commensurability Invariants

**Definition 2.2.6.** Kleinian (or Fuchsian) groups \( \Gamma \) and \( \Gamma' \) are *commensurable (in the wide sense)* if there exists \( \gamma \in \text{PSL}_2(\mathbb{C}) \) (or \( \text{PSL}_2(\mathbb{R}) \)) such that

\[
[\Gamma : \Gamma \cap (\gamma \Gamma' \gamma^{-1})] < \infty \quad \text{and} \quad [\Gamma' : \Gamma \cap (\gamma \Gamma' \gamma^{-1})] < \infty.
\]

The topological interpretation of this is that \( \Gamma \) and \( \Gamma' \) are commensurable if and only if \( \mathcal{H}^n/\Gamma \) and \( \mathcal{H}^n/\Gamma' \) share a finite-sheeted cover. In this case we will also say that \( \mathcal{H}^n/\Gamma \) and \( \mathcal{H}^n/\Gamma' \) are *commensurable*.

**Definition 2.2.7.** Let \( \Gamma^{(2)} := \langle \gamma^2 \mid \gamma \in \Gamma \rangle \).

1. The *invariant trace field* of \( \Gamma \), or of \( X = \mathcal{H}^n/\Gamma \), is \( kX = k\Gamma := K\Gamma^{(2)} \).
2. The *invariant quaternion algebra* of \( \Gamma \), or of \( X = \mathcal{H}^n/\Gamma \), is \( AX = A\Gamma := B\Gamma^{(2)} \).

**Remark 2.2.8.** In the literature the group generated by squares in \( \Gamma \) is denoted by \( \Gamma^{(2)} \), which we have adapted to distinguish it from \( S^{(2)} \), the set of squares in \( S \) from §1.2.

Since \( \Gamma^{(2)} \subset \Gamma \), we have \( A\Gamma \subset B\Gamma \) and \( k\Gamma \subset K\Gamma \). Moreover \( \Gamma^{(2)} \) is a finite index normal subgroup of \( \Gamma \) [37], and for all finite index subgroups \( \Gamma' \subset \Gamma \) we have \( k\Gamma' \subset K\Gamma \) [40]. For the same reasons as before, \( A\Gamma \) is a quaternion
algebra over $k\Gamma$ and when $\Gamma$ is Kleinian, $k\Gamma$ is a number field. It is immediate that in dimension 3 these are manifold invariants, but also we gain something stronger.

**Theorem 2.2.9** (Maclachlan, Reid [37]). *If $\Gamma$ and $\Gamma'$ are commensurable, then $k\Gamma = k\Gamma'$ and $A\Gamma \cong A\Gamma'$ as $k\Gamma$-algebras.*

When $\Gamma$ is non-cocompact, $A\Gamma$ and $B\Gamma$ are completely determined by $k\Gamma$ and $K\Gamma$ for the following reason.

**Theorem 2.2.10** ([37]). *Let $\Gamma$ be non-cocompact.*

1. $B\Gamma \cong \left( \frac{1}{K\Gamma} \right)$ and $A\Gamma \cong \left( \frac{1}{K\Gamma} \right)$.

2. There is a discrete faithful representation of $\Gamma$ into $\text{PSL}_2(K\Gamma)$.

**Proof.** Since $\Gamma$ is non-cocompact, it contains a parabolic element $\gamma$ which, up to conjugation, is of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in \mathbb{C}$. But $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B\Gamma$, therefore $I - \gamma = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \in B\Gamma$, and is non-invertible, therefore $B\Gamma$ is not a division algebra. $\gamma^2 \in \Gamma^{(2)}$ is parabolic as well, so the same argument applies to $A\Gamma$. By Theorem 1.2.11, this means $B\Gamma \cong \left( \frac{1}{K\Gamma} \right)$ and $A\Gamma \cong \left( \frac{1}{K\Gamma} \right)$, and then (2) follows from Theorem 1.2.11(1). 

**Example 2.2.11.** With $\Gamma_d$ and $W$ as they were in Examples 2.1.21 and 2.1.24, we have $A\Gamma_1 = AW = \left( \frac{1}{Q(\sqrt{-1})} \right)$. More generally, for any Bianchi
2.2. QUATERNION ALGEBRAS AS INVARIANTS

group $\Gamma_d = \text{PSL}_2(\mathcal{O}_d)$ and any non-cocompact finite index subgroup $\Gamma < \Gamma_d$, we have $A\Gamma = A\Gamma_d = \left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$.

In the non-compact case, $k\Gamma$ gains additional geometric interpretations.

**Theorem 2.2.12 ([40]).** Let $\Gamma$ be Kleinian and non-cocompact.

1. $k\Gamma$ is the field generated by the shape parameters of an ideal triangulation of $\mathcal{H}^3/\Gamma$.

2. If $\Gamma$ is chosen so that $0, 1, \infty \in \mathcal{C} = \partial \mathcal{H}^3$ are parabolic fixed points, then $k\Gamma$ is the field generated by all parabolic fixed points in $\mathcal{C}$.

In the case where $\Gamma$ is cocompact, $B\Gamma$ and $A\Gamma$ can be split or can be quaternion division algebras. Theorem 1.2.19 implies there are infinitely many possibilities of these for any given $K\Gamma$ or $k\Gamma$.

**Example 2.2.13.** The Weeks manifold $M$ is obtained by performing $(5,1)$ and $(5,2)$ Dehn filling on the cusps of the Whitehead link complement. $M$ has the smallest volume of all closed orientable hyperbolic 3-manifolds [22]. We have $kM = \mathbb{Q}(t)$ where $t$ is a complex root of $t^3 - t + 1$. This polynomial has a single pair of conjugate complex roots and a single real root, so this is enough to specify the concrete number field and also implies that $kM$ has one real place.
$AM$ is ramified at the real place, at the prime ideal generated by 5, and nowhere else. This specifies the isomorphism class of $AM$. [37]

Finding $A\Gamma$ for a given $\Gamma$ is well-studied (see Maclachlan and Reid (2003) [37] and the references therein), but we will be more interested (from §3.2 onward) in beginning with a quaternion algebra over a number field and deriving topological properties of manifolds $H^n/\Gamma$ having this algebra as $B\Gamma$.

### 2.3 Arithmetic Kleinian and Fuchsian Groups

Algebraic groups were defined by Borel and Harish-Chandra in 1962 [6] as groups of matrices in $\text{GL}_n(\mathbb{C})$ having entries that vanish on some set of polynomials over $\mathbb{C}$. For a field $K \subset \mathbb{C}$, such a group $G$ is defined over $K$ when the coefficients of the polynomials can be chosen in $K$. If $G$ is defined over $\mathbb{Q}$, an arithmetic subgroup of $G$ is a subgroup $H \leq G$ defined over $\mathbb{Z}$ where $\forall h \in H$, $\det(h) = \pm 1$. For arithmetic subgroups of $\text{PSL}_2(\mathbb{C})$ we give an equivalent definition which is useful for our context, but we briefly describe the idea before giving the technical definition.

For a Kleinian or Fuchsian group $\Gamma$ to be arithmetic, it must have a faithful representation in some $G < \text{GL}_n(\mathbb{C})$, where $G$ is defined over $\mathbb{Q}$, and $\Gamma$ is defined over $\mathbb{Z}$. This $n$ need not be 2, but we also know that $\Gamma$ has a faithful representation into $\text{PSL}_2(\mathbb{C})$. This forces the existence of a number
field $K$ so that $G$ embeds into $\text{GL}_2(K)$.

Since $M_2(K) \cong \left( \frac{1}{K} \right)$ this implies that $G$ is isomorphic to a subgroup of $\mathcal{B}^\times / \mathbb{Z}(\mathcal{B}^\times)$, for some quaternion algebra $\mathcal{B}$ over $K$ [50]. The arithmetic group $\Gamma$ then, up to commensurability, is a subgroup of $\text{PO}^1$ where $\mathcal{O}$ is an order in $\mathcal{B}$.

**Definition 2.3.1.** Let $K$ be a number field with ring of integers $R_K$, and let $\mathcal{B} = \left( \frac{a,b}{K} \right)$. A lattice $L$ in $\mathcal{B}$ is an $R_K$-submodule of $\mathcal{B}$, and $L$ is complete if $L \otimes K = \mathcal{B}$.

**Definition 2.3.2.** An order in $\mathcal{A}$ is a complete $R_K$-lattice that also is a ring with unity.

**Example 2.3.3.** $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ is an order in $\left( \frac{a,b}{\mathbb{Q}} \right)$ whenever $a, b \in \mathbb{Z}$.

We can now define an arithmetic Kleinian or Fuchsian group in terms of how it arrises from an order in a quaternion algebra. This also gives the definition of an arithmetic hyperbolic 3-manifold or surface.

**Definition 2.3.4.** Let $\mathcal{B}$ be a quaternion algebra over a number field $K$. If $\Gamma$ is Kleinian: let $K$ have exactly one complex place (which is necessarily split) and let $\mathcal{B}$ be ramified at every real place of $K$. If $\Gamma$ is Fuchsian: let $K$ have no complex places and let $\mathcal{B}$ be split at exactly one real place of $K$. In both cases, let $\sigma : \mathcal{B} \hookrightarrow M_2(\mathbb{C})$ be the split place.
CHAPTER 2. ARITHMETIC OF HYPERBOLIC MANIFOLDS

(1) \( \Gamma \) (or \( \mathcal{H}^n / \Gamma \)) is derived from \( B \) if there is an order \( \mathcal{O} \) in \( B \), such that
\( \Gamma \) is a finite index subgroup of \( P\sigma(\mathcal{O}^1) \), where \( P \) is the quotient by the center (as in PSL).

(2) \( \Gamma \) (or \( \mathcal{H}^n / \Gamma \)) is derived from a quaternion algebra if there is some \( B \) as above from which it is derived, up to isomorphism.

(3) \( \Gamma \) (or \( \mathcal{H}^n / \Gamma \)) is arithmetic if it is commensurable to one that is derived from a quaternion algebra.

For a group arising as above, the ramification conditions that we placed on \( B \) not only force the group’s arithmeticity (in the sense of Borel and Harish-Chandra [6]) but also require the group to be discrete, as explained in the Proposition below. So groups arising in this way are necessarily Kleinian or Fuchsian in addition to being arithmetic.

Proposition 2.3.5. Let \( B = \left( \frac{a,b}{K} \right) \) and let \( \sigma \) be place the of \( B \) corresponding to the identity in \( \mathfrak{B}(K : Q) \). If \( B \) contains an order \( \mathcal{O} \subset B \) such that \( \Gamma := P\sigma(\mathcal{O}^1) \) is discrete and non-elementary, then \( B \) is split at \( \sigma \) and ramified all of its other places.

Proof. If \( \sigma \) is a complex embedding then \( B_\sigma \) is split by Corollary 1.2.12. If \( \sigma \) is a real embedding and \( B_\sigma \) is not split, then \( B_\sigma \cong \mathbb{H} \), which would mean
\(\sigma(O_1) \subset \mathbb{H}^1 \cong S^3\), but since \(S^3\) is compact this would make \(\Gamma\) either finite (elementary) or not discrete, a contradiction.

Denote the remaining places of \(K\) by \(\tau_1, \ldots, \tau_c, \theta_1, \ldots, \theta_r\), where the \(\tau_n\) are complex places, and the \(\theta_n\) are real places. Then

\[
B \otimes_{\mathbb{Q}} \mathbb{R} = B_\sigma \oplus (B_{\tau_1} \oplus \cdots \oplus B_{\tau_c}) \oplus (B_{\theta_1} \oplus \cdots \oplus B_{\theta_r}).
\]

However, by Corollary 1.2.12, we know that \(\forall \ell : B_{\tau_\ell} \cong M_2(\mathbb{C})\), and that each \(B_{\theta_\ell}\) is either \(\mathbb{H}\) or \(M_2(\mathbb{R})\) depending on whether or not \(B\) is ramified at \(\theta_\ell\). So suppose \(B\) is split at \(\theta_1, \ldots, \theta_s\) and ramified at \(\theta_{s+1}, \ldots, \theta_r\). Then

\[
B \otimes_{\mathbb{Q}} \mathbb{R} = B_\sigma \oplus M_2(\mathbb{C})^c \oplus M_2(\mathbb{R})^s \oplus \mathbb{H}^{r-s}.
\]

Now, \(O_1 \subset B_1\), so taking \((B \otimes_{\mathbb{Q}} \mathbb{R})^1\) gives an embedding via direct sum of the places:

\[
O_1 \hookrightarrow B_1^1 \oplus SL_2(\mathbb{C})^c \oplus SL_2(\mathbb{R})^s \oplus (H^1)^{r-s}.
\]

Since \(O_1\) is discrete in \(B_1^1\), once again using the compactness of \(\mathbb{H}^1 \cong S^3\) we may project onto the first \(1 + c + s\) components of the direct sum while preserving discreteness of the embedding of \(O_1\). Any further projection onto \(B_1^1\) however would not be discrete (see Theorem 8.1.2 of [37]). Therefore it must be that \(c = s = 0\), i.e. the only places of \(B\) are \(\sigma\) (at which \(B\) is split) and the real places \(\theta_{s+1}, \ldots, \theta_r\), at which \(B\) is ramified. \(\square\)
Arithmetic Kleinian and Fuchsian groups, and those derived from quaternion algebras, can alternatively be characterized by the traces of the elements of the groups.

**Theorem 2.3.6.** [37] A finite covolume Kleinian group $\Gamma$ is arithmetic if and only if the following two conditions hold.

1. $k\Gamma$ has a unique complex place, and $A\Gamma$ is ramified at all real places.
2. $\forall \gamma \in \Gamma : \text{tr}(\gamma)$ is an algebraic integer.

If $\Gamma$ is derived from a quaternion algebra, if and only if additionally $\forall \gamma \in \Gamma : \text{tr}(\gamma) \in k\Gamma$.

**Theorem 2.3.7.** [37] A finite coarea Fuchsian group $\Gamma$ is arithmetic if and only if the following two conditions hold.

1. $k\Gamma$ has no complex places, and $A\Gamma$ is split at exactly one (real) place of $k\Gamma$.
2. $\forall \gamma \in \Gamma : \text{tr}(\gamma)$ is an algebraic integer.

We now discuss some properties of arithmetic groups. For these the invariant quaternion algebra is upgraded to a complete commensurability invariant, in the sense of (3) below.
Theorem 2.3.8 ([37]). Let \( \Gamma \) be an arithmetic Kleinian (Fuchsian) group, and let \( B \) be a quaternion algebra containing an order \( \mathcal{O} \) such that \( \Gamma \) is commensurable to \( \text{P} \rho(\mathcal{O}^1) \). Then the following hold.

1. \( \Gamma^{(2)} \) is derived from \( B \).

2. \( \Gamma \) is non-cocompact if and only if \( \Gamma \) is commensurable to a Bianchi group \( \Gamma_d \) (or the modular group \( \text{PSL}_2(\mathbb{Z}) \) when \( \Gamma \) is Fuchsian).

3. Let \( \Gamma' \) be another arithmetic Kleinian (Fuchsian) group. \( \Gamma' \) is commensurable to \( \Gamma \) if and only if \( k\Gamma = k\Gamma' \), and \( A\Gamma \cong A\Gamma' \) as \( k\Gamma \)-algebras.

Since there is only one modular group, and only one Bianchi group over each quadratic field, Theorem 2.3.8(2) implies the following.

Corollary 2.3.9. Let \( \Gamma \) and \( \Gamma' \) be arithmetic and noncocompact.

1. If \( \Gamma, \Gamma' \) are Fuchsian, then they are commensurable.

2. If \( \Gamma, \Gamma' \) are Kleinian, then they are commensurable if and only if \( k\Gamma = k\Gamma' \).

In some cases, the commensurability invariants coincide with the manifold invariants.
Theorem 2.3.10.

(1) If \( k\Gamma = K\Gamma \), then \( A\Gamma = B\Gamma \).

(2) If \( \Gamma \) is derived from a quaternion algebra, then \( k\Gamma = K\Gamma \) and \( A\Gamma \cong B\Gamma \).

(3) If \( \mathcal{H}^3 / \Gamma \) is a knot or link complement, then \( k\Gamma = K\Gamma \) and \( A\Gamma \cong B\Gamma \).

Proof. (1) Since \( A\Gamma \subset B\Gamma \) and both are 4-dimensional vector spaces, if they are over the same field then they must be equal.

(2) Let \( \Gamma \) be derived from the quaternion algebra \( \mathcal{B} \). Then \( B\Gamma \subset \mathcal{B} \) and by Theorem 2.3.8, \( \mathcal{B} = A\Gamma \), so then \( B\Gamma \subset A\Gamma \). But we have just seen that \( A\Gamma \subset B\Gamma \), so \( A\Gamma = B\Gamma \).

See [37] for a proof of (3). \( \square \)

Example 2.3.11. Reid showed in 1991 that the figure-8 knot is the only arithmetic knot complement [43]. To do this, he used the fact that an arithmetic cocompact group has a trace field of the form \( \mathbb{Q}(\sqrt{-d}) \), and since a knot complement has one cusp this field must have class number one (by Theorem 2.1.23). Swan [46] had determined the list of nine \( d \)-values that give this class number. Reid used a case by case analysis to eliminate all of these but \( d = 3 \), which lead to a presentation for the figure-8 knot group.
We conclude this section with another relationship between arithmeticity and commensurability due to Margulis.

**Definition 2.3.12.** The commensurator of a Kleinian group $\Gamma$, denoted by $\text{Comm}^{p}_q \Gamma$, is the set of elements $\gamma \in \text{PSL}_2(\mathbb{C})$ such that $[\Gamma : \Gamma \cap (\gamma \Gamma \gamma^{-1})] < \infty$ and $[\gamma \Gamma \gamma^{-1} : \Gamma \cap (\gamma \Gamma \gamma^{-1})] < \infty$ (i.e. $\Gamma$ and $\gamma \Gamma \gamma^{-1}$ are commensurable in the narrow sense). The commensurator of a Fuchsian group is defined analogously, with $\text{PSL}_2(\mathbb{C})$ replaced with $\text{PSL}_2(\mathbb{R})$.

**Theorem 2.3.13.** [3] A Kleinian or Fuchsian group $\Gamma$ is arithmetic if and only if $\text{Comm}(\Gamma)$ is dense in $\text{PSL}_2(\mathbb{C})$ or $\text{PSL}_2(\mathbb{R})$, respectively.
Chapter 3

Quaternion Models for Hyperbolic Space

In this chapter we develop the standard Macfarlane space, which is inspired by the 1900 version of Macfarlane’s hyperbolic quaternions [34]. The main theorem is given in §3.2.

In §3.1 we summarize an interpretation from relativity theory of $\operatorname{PSL}_2(\mathbb{C})$ as a representation of $\operatorname{Isom}^+(\mathbb{H}^3)$, which is different from the representation in §2.1 usually used in topology. In §3.2, we adapt this to a quaternionic setting and reinterpret Macfarlane’s concept using modern notation. This allows us to define a hyperboloid model for $\mathbb{H}^3$ as well as $\operatorname{Isom}^+(\mathbb{H}^3)$ within the quaternions over $\mathbb{C}$. We also establish a concise relationship between this new approach and the upper half-space model. In §3.3, we give an analogous construction for $\mathbb{H}^2$ and $\operatorname{Isom}^+(\mathbb{H}^2)$. 

55
The method of studying hyperbolic 3-space described here is standard in relativity theory [7], though not as common in topology. The key ideas behind Theorem 3.1.8 were first introduced by Wigner [51] in 1937 as a way of applying group theory to study of symmetries in the atom. In the literature, these ideas are developed in the language of differential geometry. Here they have been reformulated to a more algebraic context (especially the proof of Theorem 3.1.8) so as to transfer easily to the quaternion setting we will soon describe.

**Definition 3.1.1.** Minkowski spacetime, denoted by \( \mathbb{R}^{(1,3)} \) is the vector space \( \mathbb{R}^4 \) endowed with a quadratic form \( \phi \) with \( \text{sig}(\phi) = (1,3) \). The **indefinite special orthogonal group** \( \text{SO}(1,3) \) is the group of linear transformations on \( \mathbb{R}^{(1,3)} \) fixing \( \phi \) and having determinant 1.

We will speak of quadratic forms up to congruence (see Definition 1.2.28), and up to congruence the quadratic form used here is the standard one:

\[
\phi_{(1,3)} : \mathbb{R}^4 \to \mathbb{R}, (w, x, y, z) \mapsto w^2 - x^2 - y^2 - z^2.
\] (3.1.2)

The next step is to identify the points in \( \mathbb{R}^{(1,3)} \) with the Hermitian matrices. For a matrix \( m \in M_2(\mathbb{C}) \), let \( \overline{m} \) denote its complex conjugate, let \( m^\top \)}
3.1. THE SPINOR REPRESENTATION OF SO(1, 3)

denote its transpose, and let $m^\dagger := \overline{m}^\top$ denote its conjugate transpose. For a ring $R$ (often a field), the set of $2 \times 2$ Hermitian matrices over $R$ is defined as

$$\text{Herm}_2(R) := \{ m \in M_2(R) \mid m = m^\dagger \}. \quad (3.1.3)$$

Notice that $\text{Herm}_2(\mathbb{C})$ is not closed under matrix multiplication, so it cannot be thought of as an algebra but it can be given geometric meaning. Consider the quadratic space $(\text{Herm}_2(\mathbb{C}), \det)$.

**Lemma 3.1.4.** $\mathbb{R}^{(1,3)}$ and $(\text{Herm}_2(\mathbb{C}), \det)$ are isometric as quadratic spaces.

**Proof.** Define

$$\eta : \mathbb{R}^{(1,3)} \to (\text{Herm}_2(\mathbb{C}), \det), \quad (w, x, y, z) \mapsto \begin{pmatrix} w - x & y - \sqrt{-1}z \\ y + \sqrt{-1}z & w + x \end{pmatrix}. \quad (3.1.5)$$

$\eta$ is bijective because it has an inverse:

$$\eta^{-1} : (\text{Herm}_2(\mathbb{C}), \det) \to \mathbb{R}^{(1,3)}, \quad \begin{pmatrix} r & s \\ t & u \end{pmatrix} \mapsto \begin{pmatrix} u + r & u - r \\ t + s & t - s \end{pmatrix}. \quad (3.1.6)$$

$\eta$ is an isometry because $\forall \ p \in \mathbb{R}^{(1,3)} : \phi(p) = w^2 - x^2 - y^2 - z^2 = (\det \circ \eta)(p)$.

$\square$

Recall Definition 2.1.7 where the (standard) hyperboloid model for hyper-
bolic 3-space is defined as
\[ \mathcal{I}^3 := \{ p = (w, x, y, z) \in \mathbb{R}^{(1,3)} \mid \phi(p) = 1, w > 0 \} . \]

The spinor representation of \( \text{SO}(1, 3) \) into \( \text{SL}_2(\mathbb{C}) \) is defined as the following action on \( \mathbb{R}^{(1,3)} \), together with the restriction of this to \( \mathcal{I}^3 \).
\[ \tilde{\Lambda} : \text{SL}_2(\mathbb{C}) \times \mathbb{R}^{(1,3)} \to \mathbb{R}^{(1,3)}, \quad (m, p) \mapsto \eta^{-1}(m^\dagger \eta(p)m). \]
\[ \Lambda : \text{PSL}_2(\mathbb{C}) \times \mathcal{I}^3 \to \mathcal{I}^3, \quad (m, p) \mapsto P(\tilde{\Lambda}(P^{-1}(m), p)). \quad (3.1.7) \]

**Theorem 3.1.8** (Wigner, 1937 [7]). \( \Lambda \) is a left group action and gives a faithful representation of \( \text{Isom}^+(\mathcal{I}^3) \) onto \( \text{PSL}_2(\mathbb{C}) \).

**Lemma 3.1.9.** Let \( R \subseteq \mathbb{C} \) be a ring that is closed under complex conjugation. For all \( h \in \text{Herm}_n(R) \) and for all \( m \in \text{M}_n(R) \): \( mh m^\dagger \in \text{Herm}_n(R) \).

**Proof.** Since \( \overline{R} \subseteq R \), we have \( \forall m \in \text{M}_n(R) : m^\dagger \in \text{M}_n(R) \), thus the entries of \( mh m^\dagger \) are all in \( R \). Using the fact that the conjugate transpose is an involution of \( \text{M}_n(R) \) with symmetric elements \( \text{Herm}_n(R) \), we have
\[ (mh m^\dagger)^\dagger = (m^\dagger)^\dagger h m^\dagger = mh m^\dagger \implies mh m^\dagger \in \text{Herm}_n(R). \]

Let the prefix \( S \) indicate the subgroup of elements of determinant one and let the prefix \( P \) indicate taking the quotient by the center (as in \( \text{SL}_2 \) and \( \text{PSL}_2 \)).
Lemma 3.1.10. \( \forall h \in \text{PSHerm}_2(R), \forall m \in \text{PSL}_2(R) : mhm^\dagger \in \text{PSHerm}_2(R) \).

Proof. First work with representatives in \( \text{SHerm}_2(R) \) and \( \text{SL}_2(R) \).

\[
\det(mhm^\dagger) = \det(m) \det(h) \det(m^\dagger) = 1 \cdot 1 \cdot \det(m^\top) = \det(m) = \overline{\det(m)} = \overline{1} = 1.
\]

So by Lemma 3.1.9, \( mhm^\dagger \in \text{SHerm}_2(R) \). Taking the quotient by the center for both sets of matrices is equivalent to identifying each matrix with its negative, thus the above remains valid under P.

Remark 3.1.11. The motivation for choosing the nonstandard signature and sign convention for \( \mathcal{T}_\phi \), as mentioned in Remark 2.1.8, is as follows. Since \( \mathcal{T}_\pm^3 \) is the set in \( \mathbb{R}^{(1,3)} \) where \( \phi \) takes the value 1, the image of \( \mathcal{T}^3 \) under \( \phi \) will be the image of \( \text{PSHerm}_2(\mathbb{C}) \) under the determinant.

Lemma 3.1.12. \( \eta \) maps \( \mathcal{T}_\pm^3 \) bijectively to \( \text{SHerm}_2(\mathbb{C}) \), and under the projection it maps \( \mathcal{T}^3 \) bijectively to \( \text{PSHerm}_2(\mathbb{C}) \).

Proof. Since \( p \in \mathcal{T}_\pm^3 \iff \phi(p) = 1 \), Lemma 3.1.4 gives that the restricted
map $\eta|_{I^3_\pm} : I^3_\pm \rightarrow \text{SHer}_2(\mathbb{C})$ is a bijection. The cokernel of $(\det \circ \eta)|_{I^3_\pm}$ is $\{ \pm I \}$, so taking the quotient by this gives the second result.

Now in proving the Theorem we write, for $m \in \text{SL}_2(\mathbb{C})$:

$$\Lambda_m : I^3 \rightarrow I^3, \quad m \mapsto \Lambda(m, p),$$

$$\tilde{\Lambda}_m : I^3_\pm \rightarrow I^3_\pm, \quad m \mapsto \tilde{\Lambda}(m, p),$$

$$\Xi : \text{SL}_2(\mathbb{C}) \rightarrow \text{Isom}(I^3_\pm), \quad m \mapsto \tilde{\Lambda}_m.$$

**Proof of Theorem 3.1.8.** We know from §2 that $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C})$ and that $I^3$ is a model for hyperbolic 3-space. We know from Lemmas 3.1.10 and 3.1.12 that $I^3$ is invariant under $\Lambda$, so the map $\Lambda_m$ is well-defined. It remains to show that $\Lambda$ preserves distance, and acts faithfully and transitively on the left. To do this we will look at how the extended map $\tilde{\Lambda}$ acts on $\text{SL}_2(\mathbb{C}) \times I^3_\pm$. Also, we will avoid some notational awkwardness by identifying points in $I^3_\pm$ with their images under $\eta$ in $\text{SH}_2(\mathbb{C})$. 
3.1. THE SPINOR REPRESENTATION OF SO(1, 3)

\( \tilde{\Lambda} \) preserves distance because \( \forall m \in \text{SL}_2(\mathbb{C}) \) and \( \forall p, q \in \mathbb{R}^{(1,3)} \):

\[
2 \cosh \left( d_{\mathbb{R}^{(1,3)}}(\tilde{\Lambda}(m, p), \tilde{\Lambda}(m, q)) \right) = 2 \cosh \left( d_{\mathbb{R}^{(1,3)}}(mpm^\dagger, mqm^\dagger) \right)
\]

\[
= \phi(mpm^\dagger + mqm^\dagger) - \phi(mpm^\dagger) - \phi(mqm^\dagger)
\]

\[
= \phi(m(p + q)m^\dagger) - \phi(mpm^\dagger) - \phi(mqm^\dagger)
\]

\[
= \det(m) \left( \phi(p + q) - \phi(p) - \phi(q) \right) \det(m^\dagger)
\]

\[
= \phi(p + q) - \phi(p) - \phi(q)
\]

\[
= 2 \cosh \left( d_{\mathbb{R}^{(1,3)}}(p, q) \right).
\]

Since \( \mathcal{I}^3 \subset \mathbb{R}^{(1,3)} \), \( \Lambda \) preserves distance as well.

To show injectivity, we show that the kernel of \( \Xi \) is \( \{ \pm I \} \) where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Suppose \( \tilde{\Lambda}_m \in \ker(\Xi) \). Then \( \forall p \in \mathcal{I}^3_\pm : \tilde{\Lambda}_m(p) = p \). Taking \( p = 1 \) gives

\[
1 = \tilde{\Lambda}_m(1) = m1m^\dagger = mm^\dagger \implies m^\dagger = m^{-1}.
\]

So then for any \( p \in \mathcal{I}^3_\pm : \)

\[
p = \tilde{\Lambda}_m(p) = mpm^\dagger = mp^{-1} \implies pm = mp.
\]

Since \( \text{Herm}_2 \otimes \mathbb{C} = \text{M}_2(\mathbb{C}) \) and \( \tilde{\Lambda}_m \) acts linearly, \( \forall n \in \text{M}_2(\mathbb{C}) : nm = mn \). In other words \( m \) lies in the center of \( \text{M}_2(\mathbb{C}) \), so \( m = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \) for some \( c \in \mathbb{C} \). But \( m \in \text{SL}_2(\mathbb{C}) \), so \( \det(m) = c^2 = 1 \), thus \( c = \pm 1 \) and \( m = \pm I \). Conversely,
\[ \forall p \in \mathcal{T}_3^3 : \bar{\Lambda}_{\pm I}(p) = \pm Ip(\pm I)^\dagger = \pm Ip(\pm I) = (\pm I)^2p = p, \text{ so } \pm I \text{ both act trivially on } \mathcal{T}_3^3. \] Thus the kernel of \( \Xi \) is \( \{\pm 1\} \).

As a result, taking the quotient by \( \{\pm 1\} \) yields an injective action by \( \text{PSL}_2(\mathbb{C}) \) on \( \mathcal{T}_3^3 \), by Noether’s isomorphism theorem. However, \( \mathcal{T}_3^3 \) consists of two components where one is the negation of the other, and the action by \( \text{PSL}_2(\mathbb{C}) \) is continuous. In particular any \( m \in \text{PSL}_2(\mathbb{C}) \) preserves components of \( \mathcal{T}_3^3 \) and \( \Lambda(m, -p) = -\Lambda(m, p) \), so we may identify the two components and represent this by the upper sheet \( \mathcal{T}^3 \). Therefore \( \Lambda \) is well-defined and gives a faithful action by \( \text{PSL}_2(\mathbb{C}) \) on \( \mathcal{T}^3 \).

\( \Lambda \) is a left group action because for any \( m, n \in \text{PSL}_2(\mathbb{C}) \) and any \( p \in \mathcal{T}^3 \):

\[ \Lambda_{mn}(p) = (mn)p(mn)^\dagger = mnpn^\dagger m^\dagger = m(npn^\dagger)m^\dagger = \Lambda_m(npn^\dagger) = \Lambda_m(\Lambda_n(p)). \]

Since \( \Lambda \) is a left group action, we can show transitivity by showing \( \mathcal{T}^3 = \text{Orb}_{\text{PSL}_2(\mathbb{C})}(1) \), and we know that \( \text{Orb}_{\text{PSL}_2(\mathbb{C})}(1) = \{mm^\dagger \mid m \in \text{PSL}_2(\mathbb{C})\} \).

Claim: \( \forall h \in \text{PSHerm}_2(\mathbb{C}) \exists m \in \text{PSL}_2(\mathbb{C}) \) such that \( h = mm^\dagger \).

Proof of claim: Suppose \( h \in \text{PSHerm}_2(\mathbb{C}) \). Since \( h \) is Hermitian, it has real eigenvalues and can be diagonalized by a unitary matrix (by the finite-dimensional spectral theorem, see [32]). That is, letting \( d \) be the diagonal
matrix with the eigenvalues $\lambda, \lambda' \in \mathbb{R}$ of $h$ as its diagonal, there exists a matrix $u$ such that $u^\dagger = u^{-1}$ and $u^\dagger hu = d$. This implies $h = udu^\dagger$. Since $h \in \mathrm{PSHerm}_2(\mathbb{C})$ we can choose the representative of $d$ where $\lambda, \lambda' > 0$. Let $d'$ be the diagonal matrix with diagonal $\sqrt{\lambda}, \sqrt{\lambda'}$ and let $m = ud'u^\dagger$. The proof follows from the following two observations:

$$mm^\dagger = (ud'u^{-1})(ud'u^{-1}) = ud'^2u^\dagger = udu^\dagger = h,$$

and

$$\det(m) = \det(ud'u^{-1}) = \det(d') = \sqrt{\lambda}\sqrt{\lambda'} = \sqrt{\lambda\lambda'}$$

$$= \sqrt{\det(d)} = \sqrt{\det(udu^{-1})} = \sqrt{\det(h)} = 1. \quad \square$$

### 3.2 A Quaternion Model for Hyperbolic 3-Space

Quaternion algebras over $\mathbb{C}$ were applied to special relativity using hyperbolic geometry in the form of the biquaternions (as mentioned at the end of §1.1). In this case the structure parameters $i^2$ and $j^2$ were always taken to be $-1$. However Macfarlane insisted that $i^2 = j^2 = +1$ [14]. While this led him to interesting geometric observations, the relevance of Macfarlane’s work to arithmetic properties of algebras has been obscure since it predated the theory introduced by Dickson.
64 CHAPTER 3. QUATERNION MODELS FOR HYPERBOLIC SPACE

In this section, we unite Macfarlane’s geometric concept with Dickson’s algebraic formalism, obtaining a new interpretation of Wignor’s construction (§3.1) in the language of quaternions, thereby generalizing Hamilton’s Theorem 1.1.3 to hyperbolic geometry.

3.2.1 The Standard Macfarlane Space

We begin by writing the multiplication rules (1.1.5) from Macfarlane’s associative hyperbolic quaternions in the notation of Definition 1.2.1.

Recall from (1.1.5), Macfarlane’s equation

\[ ij = \sqrt{-1}k. \]

Solving for \( k \) in terms of \( i \) and \( j \) gives

\[ k = -\sqrt{-1}ij. \tag{3.2.1} \]

Since \( ij = -ji \) and \( i^2 = j^2 = 1 \), observe that replacing \( k \) with \(-\sqrt{-1}ij\) preserves the rest of Macfarlane’s multiplication rules as well.

\[ k^2 = (-\sqrt{-1}ij)^2 = -iji = ij^2i = i^2 = 1. \]

\[ jk = j(-\sqrt{-1}ij) = -\sqrt{-1}iji = \sqrt{-1}ij^2 = \sqrt{-1}i. \]

\[ ki = (-\sqrt{-1}ij)i = \sqrt{-1}i^2j = \sqrt{-1}j. \]
This multiplication also agrees with the choice $a = b = 1$ in Definition 1.2.1. So we eliminate the $k$ and study

$$\text{Span}_\mathbb{R}(1, i, j, k) = \text{Span}_\mathbb{R}(1, i, j, \sqrt{-1}ij) \subset \left(\frac{1,1}{C}\right),$$

motivating the following definition.

**Definition 3.2.2.** The (standard) Macfarlane space is the normed vector space

$$\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}\sqrt{-1}ij \subset \left(\frac{1,1}{C}\right)$$

over $\mathbb{R}$, where the norm is the restriction of the quaternion norm.

Let $\mathcal{M}$ be the standard Macfarlane space and $n$ be the norm from $\left(\frac{1,1}{C}\right)$.

**Proposition 3.2.3.** $(\mathcal{M}, n)$ is a real quadratic space of signature $(1, 3)$.

**Proof.** Let $m = w + xi + yj + \sqrt{-1}zij \in \mathcal{M}$ where $w, x, y, z \in \mathbb{R}$ and refer to Definition 1.2.3, taking $a = b = 1$.

$$n(m) = w^2 - x^2 - y^2 + (\sqrt{-1}z)^2 = w^2 - x^2 - y^2 - z^2.$$

A hyperboloid model occurs naturally in $\mathcal{M}$. Define the metric space

$$\mathcal{M}_+^1 := \{ m \in \mathcal{M} \mid n(m) = 1, \text{tr}(m) > 0 \} \quad (3.2.4)$$
CHAPTER 3. QUATERNION MODELS FOR HYPERBOLIC SPACE

where the metric is induced by the restriction of the norm to the tangent space.

Proposition 3.2.5. $M_+^1$ is a hyperboloid model for hyperbolic 3-space.

Proof. Let $\phi : \mathbb{R}^4 \to \mathbb{R}, (w, x, y, z) \mapsto w^2 - x^2 - y^2 - z^2$, as in the definition of Minkowski spacetime (3.1.2). Under the natural identification of $m = w + xi + yz + \sqrt{-1} \omega_{ij} \in \mathcal{M}$ with $p = (w, x, y, z) \in \mathbb{R}^{(1, 3)}$, we have that $n(m) = \phi(p)$. Therefore $(\mathcal{M}, n)$ and $\mathbb{R}^{(1, 3)}$ are isometric quadratic spaces, and then $\mathcal{M}^1 \simeq \mathcal{I}^1_\pm$. Since $w > 0$ if and only if $\text{tr}(m) = 2w > 0$, $\mathcal{M}_+^1 \simeq \mathcal{I}^3$ is a hyperboloid model for hyperbolic 3-space. \hfill \Box

Corollary 3.2.6. $\mathcal{M}_+^1$ identifies naturally with a subset of $P\left(\frac{1}{1}c\right)^1$.

Proof. $\mathcal{M}^1 = \left\{ q \in \left(\frac{1}{1}c\right) \mid n(q) = 1 \right\} \subset \left(\frac{1}{1}c\right)^1$ and contains no elements of zero trace. To see this geometrically, observe that $\mathcal{M}^1$ is a hyperboloid of two sheets with one sheet on either side of the pure quaternion part $\mathcal{M}_0$ of $\mathcal{M}$. To see this algebraically, observe that if $m \in \mathcal{M}_0$, then $\exists x, y, z \in \mathbb{R}$ such that $m = xi + yj + z\sqrt{-1}ij$, but then $n(m) = -x^2 - y^2 - z^2 = 1$, which is a contradiction. So identify each $p \in \mathcal{M}_+^1$ with the corresponding $\{\pm p\} \in P\left(\frac{1}{1}c\right)^1$. \hfill \Box
3.2. A Quaternion Model for Hyperbolic 3-Space

3.2.2 A Quaternion Representation of $\text{Isom}^+(S^3)$

Let $K$ be a field where $\text{char}(K) \neq 2$, and let $B = \left( \frac{a}{b} \right)$ with $a, b \in K^\times$. We focus here on the cases where $B = \mathbb{H}$ or $\left( \frac{1}{\mathbb{C}} \right)$, but since it will be useful for future reference and requires little additional effort, we define the following matrix representation of $B$ more generally:

$$
\rho_B : B \rightarrow \text{M}_2(K(\sqrt{a}, \sqrt{b})),
$$

$$
w + xi + yj + zij \mapsto \begin{pmatrix} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{pmatrix}.
$$

(3.2.7)

**Remark 3.2.8.** It is more typical in the literature [37, 50] to work with a representation into $\text{M}_2(K(\sqrt{a}))$. The reason for instead defining $\rho_B$ as above is to use the resulting correspondence with $\text{Herm}_2(K(\sqrt{a}, \sqrt{b}))$.

Taking $B = \mathbb{H}$ gives

$$
\rho_B : \mathbb{H} \rightarrow \text{M}_2(\mathbb{C}), w + xi + yj + zij \mapsto \begin{pmatrix} w - x\sqrt{-1} & y\sqrt{-1} - z \\ y\sqrt{-1} + z & w + x\sqrt{-1} \end{pmatrix}
$$

and taking $B = \left( \frac{1}{\mathbb{C}} \right)$ gives

$$
\rho_B : \left( \frac{1}{\mathbb{C}} \right) \rightarrow \text{M}_2(\mathbb{C}), w + xi + yj + zij \mapsto \begin{pmatrix} w - x & y - z \\ y + z & w + x \end{pmatrix}.
$$

**Proposition 3.2.9.** $\rho_B$ is an injective $K$-algebra homomorphism.

**Proof.** Since $\rho_B$ is linear over $K$, we can show it is a $K$-algebra homomorphism by showing it preserves multiplication. Since $B$ is generated over $K$
by \( i \) and \( j \), this can be done by showing that \( \rho_B \) preserves the multiplication laws on \( i \) and \( j \).

\[
\rho_B(i)^2 = \left( -\sqrt{a} \begin{array}{c} 0 \\ \sqrt{a} \end{array} \right)^2 = \left( \begin{array}{c} a \\ 0 \end{array} \right) = \rho_B(a) = \rho_B(i^2)
\]

\[
\rho_B(j)^2 = \left( \begin{array}{c} 0 \\ \sqrt{b} \end{array} \right)^2 = \left( \begin{array}{c} b \\ 0 \end{array} \right) = \rho_B(b) = \rho_B(j^2)
\]

\[
\rho_B(ij) = \left( \begin{array}{c} 0 \\ -\sqrt{ab} \end{array} \right) = \left( \begin{array}{cc} -\sqrt{a} & 0 \\ 0 & \sqrt{a} \end{array} \right) \left( \begin{array}{c} 0 \\ \sqrt{b} \end{array} \right) = \rho_B(i)\rho_B(j)
\]

\[
\rho_B(-ij) = \left( \begin{array}{c} 0 \\ \sqrt{ab} \end{array} \right) = \left( \begin{array}{cc} 0 & \sqrt{b} \\ -\sqrt{a} & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ \sqrt{b} \end{array} \right) = \rho_B(j)\rho_B(i)
\]

\( \rho_B \) is injective because if \( \rho_B(w + xi + yj + zij) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \), then we have the following equations.

\[
w - x\sqrt{a} = 0 \quad y\sqrt{b} - z\sqrt{ab} = 0
\]

\[
w + x\sqrt{a} = 0 \quad y\sqrt{b} + z\sqrt{ab} = 0
\]

Since \( a, b \neq 0 \), this gives \( w = x = y = z = 0 \).

**Corollary 3.2.10.** If \( B = \left( \begin{array}{c} 1 \\ 1 \\ \mathbb{C} \end{array} \right) \), then \( \rho_B \) is a \( \mathbb{C} \)-algebra isomorphism.

**Proof.** In this case \( \rho_B \) is surjective by Theorem 1.2.11(1).

**Remark 3.2.11.** Since \( \mathbb{H} \) is a division algebra and \( M_2(\mathbb{C}) \) is not, \( \rho_\mathbb{H} \) is not surjective.

If we restrict the codomain of \( \rho \) to its image, we can write its inverse as
follows (when $\rho_B$ is not surjective, the square roots in this map will cancel with the square roots from the image of $\rho_B$).

$$\rho_B^{-1}|_{\rho(B)} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \frac{u + r}{2} + \frac{u - r}{2\sqrt{a}} i + \frac{t + s}{2\sqrt{b}} j + \frac{t - s}{2\sqrt{ab}} \hat{ij}. \quad (3.2.12)$$

The next step is to introduce an involution in terms of $\rho_B$ which, when $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, plays an equivalent role to the matrix conjugate transpose $\dagger$ in §3.1, and when $B = H$, generalizes the role of the standard involution $\ast$ in Hamilton’s Theorem 1.1.3. This involution is the key step in generalizing Hamilton’s theorem.

**Definition 3.2.13.** For the quaternion algebra $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $H$, the adjoint of $q \in B$ is $q^\dagger := \rho_B^{-1}(\overline{\rho_B(q)}^\top)$, where by $\rho_B^{-1}$ we mean the map from (3.2.12) with domain $\rho_B(B)$.

**Remark 3.2.14.** This Definition is not a canonical quaternionic equivalent of the matrix conjugate transpose because it depends on the choice of matrix representation $\rho_B$. However since our use of $\rho_B$ will be consistent, we will use $\dagger$ to indicate both the adjoint as defined above and the conjugate transpose of a matrix.

**Proposition 3.2.15.** When $B = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ with $K \subset \mathbb{C}$ and $K \not\subset \mathbb{R}$, $\dagger$ is an involution of the second kind.
Proof. \( \forall x, y \in M_2(\mathbb{C}) \):

\[
\begin{align*}
(x + y)^\top &= x^\top + y^\top \\
(x^{}y^{})^\top &= (yx^{})^\top = y^\top x^\top \\
(m^\top)^\top &= m
\end{align*}
\]

and \( \{ w \in K \mid \bar{w}^\top = w \} = \mathbb{R} \cap K \neq K \). By Proposition 3.2.9, \( \rho_B \) transfers these properties to \( \dagger \).

Since the standard involution \( * \) on \( B \) is an involution of the first kind, in general \( \dagger \) and \( * \) will not be the same. This does occur however in \( \mathbb{H} \).

**Lemma 3.2.16.** \( \forall q \in \mathbb{H} : q^\dagger = q^* \).

**Proof.** Recall that \( \mathbb{H} = \left( -\frac{1}{\mathbb{R}}, -\frac{1}{\mathbb{R}} \right) \) and let \( q = w + xi + yj + zij \in \mathbb{H} \) with \( w, x, y, z \in \mathbb{R} \).

\[
q^\dagger = \rho_H^{-1}(\rho_B(w + xi + yj + zij)^\dagger)
\]

\[
= \rho_H^{-1}\left(\begin{pmatrix}
w - x\sqrt{-1} & y\sqrt{-1} - z \\
y\sqrt{-1} + z & w + x\sqrt{-1}
\end{pmatrix}\right)^\top
\]

\[
= \rho_H^{-1}\left(\begin{pmatrix}
w + x\sqrt{-1} & -y\sqrt{-1} + z \\
-y\sqrt{-1} - z & w - x\sqrt{-1}
\end{pmatrix}\right)
\]

\[
= w - xi - yj - zij
\]

\[
= q^*
\]
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

We now rephrase Hamilton’s Theorem 1.1.3.

**Theorem 3.2.17.** An isomorphism $P\mathbb{H}^1 \cong \text{Isom}^+(S^2)$ is defined by the group action

$$P\mathbb{H}^1 \times \mathbb{H}_0^1 \rightarrow \mathbb{H}_0^1, \quad (u, p) \mapsto upu^\dagger.$$

**Proof.** By the Lemma, this is the same action by $P\mathbb{H}^1 \cong \text{SO}(3)$ as in Hamilton’s original Theorem, but $\text{SO}(3) \cong \text{Isom}^+(S^2)$. Also $(\mathbb{H}_0^1, \sqrt{n}) \cong (\mathbb{R}^3, n_{\text{Euc}})$, thus we have $(\mathbb{H}_0^1, \sqrt{n(r)}) \cong (S^2, n_{\text{Euc}})$.

We next establish a generalization to $\mathcal{S}^3$. Let $\mathcal{B} = \left(\begin{smallmatrix} 1 & 1 \\ \ell & \ell \end{smallmatrix}\right)$.

**Theorem 3.2.18.** An isomorphism $PB^1 \cong \text{Isom}^+(\mathcal{S}^3)$ is defined by the group action

$$\mu : PB^1 \times M^1_+ \rightarrow M^1_+, \quad (u, m) \mapsto umu^\dagger.$$

**Lemma 3.2.19.** $M = \uparrow \mathcal{B}$.

**Proof.** Let $q = w + xi + yj + zij \in \left(\begin{smallmatrix} 1 & 1 \\ \ell & \ell \end{smallmatrix}\right)$. Then $\rho(q) = \left(\begin{smallmatrix} w - x & y - z \\ y + z & w + x \end{smallmatrix}\right)$, and then $\overline{\rho(p)^\dagger} = \left(\begin{smallmatrix} \overline{w - x} & \overline{y + z} \\ \overline{y - z} & \overline{w + x} \end{smallmatrix}\right)$, giving that $q^\dagger = \overline{w} + \overline{x}i + \overline{y}j - \overline{z}ij$. So, $q = q^\dagger$ if and only if $w = \overline{w}, x = \overline{x}, y = \overline{y},$ and $z = -\overline{z}$, which is equivalent to saying $w, x, y \in \mathbb{R}$ and $z \in \mathbb{R}\sqrt{-1}$, i.e. $q \in \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}\sqrt{-1}ij = M$. □
Proof of Theorem 3.2.18. The proof is essentially the same as the proof of Theorem 3.1.8, except there is no longer any need for pulling points back to matrices, since here points and isometries are both quaternions and reside in the same ambient structure. In particular, the map \( \eta \) can be done away with and there is no abuse of notation in writing \( q pq^\dagger \) where \( q \) represents an isometry and \( p \) represents a point.

It is sufficient to verify that the objects in the Theorem are mapped by \( \rho_B \) to the corresponding objects in the proof of Wigner’s Theorem 3.1.8. There we gave an action by \( \text{PSL}_2(\mathbb{C}) \) on \( \mathcal{I}^3 \) by identifying \( \mathcal{I}^3 \) with \( \text{PSHerm}_2(\mathbb{C}) \). Here we transfer this to an action by \( \text{P}(\mathbb{C})^1 \) on \( M^1_+ \).

\[
\rho_B \left( \frac{1,1}{c} \right) = M_2(\mathbb{C}), \text{ therefore } \rho_B \left( \left( \frac{1,1}{c} \right)^\dagger \right) = \text{SL}_2(\mathbb{C}), \text{ and thus } \text{P}\left( \frac{1,1}{c} \right)^1 \text{ identifies naturally under } \rho_B \text{ with } \text{PSL}_2(\mathbb{C}). \text{ Since } \text{PSL}_2(\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3), \text{ we have } \text{P}\left( \frac{1,1}{c} \right)^1 \cong \text{Isom}^+(\mathbb{H}^3).
\]

By Lemma 3.2.19, \( \rho(M) = \text{Herm}_2(\mathbb{C}) \), therefore \( \rho(M^1_+) = \text{PSHerm}_2(\mathbb{C}) \cong \mathcal{I}^3 \).

Lastly, using \( \dagger \) simultaneously as the adjoint involution on \( B \) and the
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

conjugate transpose on \( \text{PSL}_2(\mathbb{C}) \) as mentioned in Remark 3.2.14, we have

\[
\forall (q, p) \in P \left( \frac{1}{1} \right) \times \mathcal{M}_+^1 : \rho(\mu(q, p)) = \rho(q p q^\dagger)
\]
\[= \rho(q) \rho(p) \rho(q)^\dagger
\]
\[= \Lambda(\rho(q), \rho(p)). \quad \square
\]

We summarize the correspondence between the approach from relativity theory and our approach with quaternion algebras. Each object from the former is replaced by the latter as follows. Figure 3.1 gives more detail.

\[
\begin{align*}
\text{M}_2(\mathbb{C}) & \rightsquigarrow \left( \frac{1}{1} \right) & \text{Herm}_2(\mathbb{C}) & \rightsquigarrow \mathcal{M} & \phi & \rightsquigarrow n & \dagger & \rightsquigarrow \dagger \\
\text{SL}_2(\mathbb{C}) & \rightsquigarrow \left( \frac{1}{1} \right)^1 & \text{SH}_2(\mathbb{C}) & \rightsquigarrow \mathcal{M}^1 & \mathcal{I}_\pm^3 & \rightsquigarrow \mathcal{M}_+^1 & \Lambda & \rightsquigarrow \mu \\
\text{PSL}_2(\mathbb{C}) & \rightsquigarrow P \left( \frac{1}{1} \right)^1 & \text{PSHerm}_2(\mathbb{C}) & \rightsquigarrow \mathcal{M}_+^1 & \mathcal{I}^3 & \rightsquigarrow \mathcal{M}_+^1
\end{align*}
\]

3.2.3 Comparison to the Möbius Action

In this subsection, we compare our model with the upper half-space model and establish a concise formula for transferring data between the two.

Let \( B = \left( \frac{1}{1} \right) \) with \( i, j \in B \) as usual. To distinguish elements of \( \mathbb{H} \) from
Figure 3.1: Relationship between the quaternion approach and the approach from relativity theory.

\[
\begin{array}{c}
\mathbb{R}^{(1,3)} \xrightarrow{\eta \approx} \text{Herm}_2(\mathbb{C}) \xleftarrow{\rho \approx} \mathcal{M} \\
\mathcal{I}^3 \xrightarrow{\eta \approx} \text{PSHerm}_2(\mathbb{C}) \xleftarrow{\rho \approx} \mathcal{M}_+^1 \xrightarrow{\rho \approx} \text{PSL}_2(\mathbb{C}) \xrightarrow{\rho \approx} \text{M}_2(\mathbb{C}) \\
\text{Isom}^+(\mathcal{I}^3) \xleftarrow{\approx} \text{PSL}_2(\mathbb{C}) \xrightarrow{\rho \approx} \text{P} \left( \frac{1,1}{\mathbb{C}} \right)^1
\end{array}
\]

“\(\approx\)” denotes an isomorphism of groups on the bottom and of \(\mathbb{C}\)-algebras on the right. As usual “\(\approx\)” denotes a quadratic space isometry. In the direction of the embedding arrows the diagram commutes. The dashed hook arrow on the left is made concrete by the embedding of \(\mathcal{M}_+^1\) into \(\text{P} \left( \frac{1,1}{\mathbb{C}} \right)^1\) on the right.

elements of \(\mathcal{B}\), we write \(\mathbb{H}\) as follows.

\[
\mathbb{H} = \mathbb{R} \oplus \mathbb{R}I \oplus \mathbb{R}J \oplus \mathbb{R}IJ,
\]

\[
I^2 = J^2 = -1,
\]

\[
IJ = -JI.
\]

Then \(\mathbb{C} \subset \mathcal{B}\) as the basefield, and also \(\mathbb{C} \subset \mathbb{H}\) as \(\mathbb{C} = \mathbb{R} \oplus \mathbb{R}I\). This identification between subspaces of \(\mathcal{B}\) and \(\mathbb{H}\) may seem incidental at this point, but in fact it is intentional and significant. We will write \(\mathbb{C}\) as \(\mathbb{R} \oplus \mathbb{R}I\) in both instances.

The upper half-space model admits an alternative definition to the one
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

given in §2.1, as a subspace of Hamilton’s quaternions.

Definition 3.2.20.

(1) The \textit{upper half-space} model \( \mathcal{H}^3 \) is \( \mathbb{R} \oplus \mathbb{R}I \oplus \mathbb{R}^+ J \subset \mathbb{H} \) endowed with the metric induced by the map \( p = x_1 + x_2 I + x_3 J \mapsto \frac{\sqrt{n(p)}}{|x_3|} \).

(2) The \textit{upper half-plane} model \( \mathcal{H}^2 \) is \( \mathbb{R} \oplus \mathbb{R}^+ I \subset \mathbb{C} \subset \mathbb{H} \), endowed with the metric induced by the map \( p = x_1 + x_2 I \mapsto \frac{\sqrt{n(p)}}{x_2} \).

This provides a way of computing explicitly the isometric extension of the Möbius action from \( \partial \mathcal{H}^3 \) to \( \mathcal{H}^3 \), mentioned at the beginning of §2.1.1.

Definition 3.2.21. The \textit{(quaternionic) Möbius action} by \( \text{PSL}_2(\mathbb{C}) \) on \( \mathcal{H}^3 \) is

\[
\alpha : \text{PSL}_2(\mathbb{C}) \times \mathcal{H}^3 \to \mathcal{H}^3, \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, p \right) \mapsto (ap + b)(cp + d)^{-1},
\]

and the \textit{(quaternionic) Möbius action} by \( \text{PSL}_2(\mathbb{R}) \) on \( \mathcal{H}^2 \) is given by the restriction \( \alpha|_{\text{PSL}_2(\mathbb{R}) \times \mathcal{H}^2} \).

Remark 3.2.22. The expression \( (ap + b)(cp + d)^{-1} \) cannot, in general, be written as a fraction because \( \mathbb{H} \) is noncommutative, but by multiplying the inverse on the right this agrees with the usual interpretation.

Recall the classification of isometries from Proposition 2.1.16. For \( \gamma \in \text{PSL}_2(\mathbb{C}) \setminus \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \) we have the following (where only the first three apply if \( \gamma \in \text{PSL}_2(\mathbb{R}) \)).
Table 3.1: Normalized isometries as quaternions.

<table>
<thead>
<tr>
<th>Type</th>
<th>Quaternion Form</th>
<th>Normalized Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>parabolic</td>
<td>$c \in \mathbb{C} \setminus {0}$</td>
<td>$1 + cj - cij$</td>
</tr>
<tr>
<td>elliptic</td>
<td>$c \in (-1, 1)$</td>
<td></td>
</tr>
<tr>
<td>hyperbolic</td>
<td>$c \in \mathbb{R} \setminus (-1, -1)$</td>
<td>$\frac{c + c^{-1}}{2}j + \frac{c - c^{-1}}{2}ij$</td>
</tr>
<tr>
<td>purely loxodromic</td>
<td>$c \in \mathbb{C} \setminus \mathbb{R}$</td>
<td></td>
</tr>
</tbody>
</table>

(1) $\gamma$ is elliptic $\iff$ $\text{tr}(\gamma) \in \mathbb{R}$ and $|\text{tr}(\gamma)| < 2$.

(2) $\gamma$ is parabolic $\iff$ $\text{tr}(\gamma) = \pm 2$.

(3) $\gamma$ is hyperbolic $\iff$ $\text{tr}(\gamma) \in \mathbb{R}$ and $|\text{tr}(\gamma)| > 2$.

(4) $\gamma$ is purely loxodromic $\iff$ $\text{tr}(\gamma) \in \mathbb{C} \setminus \mathbb{R}$.

(5) $\gamma$ is loxodromic $\iff$ $\text{tr}(\gamma) \notin [-2, 2]$.

Since $\forall q \in B : \text{tr}(q) = (\text{tr} \circ \rho)(q)$ (where the second tr is the matrix trace), the same classification applies to isometries of $\mathcal{M}_1^1$ in $PB^1$. Each type of isometry, as an element of $\text{PSL}_2(\mathbb{C})$ can be conjugated to a normalized form that simplifies computations. Conjugation in $\text{PSL}_2(\mathbb{C})$ corresponds under $\rho_B$ to conjugation in $PB^1$.

**Proposition 3.2.23.** Up to conjugation in $B$, isometries in $PB^1$ admit the following normalized forms.
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

Proof. Let us look at this first in $\text{PSL}_2(\mathbb{C})$ then transfer it to $\mathbb{B}^1$. Let $\gamma \in \text{PSL}_2(\mathbb{C})$. If $\gamma$ is parabolic then $\gamma$ is conjugate to $\begin{pmatrix} 1 & 2c \\ 0 & 1 \end{pmatrix}$ for some $c \in \mathbb{C} \setminus \{0\}$; if $\gamma$ is elliptic, hyperbolic, or purely loxodromic, then $\gamma$ is conjugate to $\begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}$ with $c \in (-1, 1), \mathbb{R} \setminus (-1, -1)$ or $\mathbb{C} \setminus \mathbb{R}$, respectively [5]. By (3.2.12), Table 3.1 gives the corresponding normalized forms for $\mathbb{B}^1$. 

We will develop a convenient way of mapping points between $\mathcal{M}_+^1$ and $\mathcal{H}^3$ that behaves well with our existing method of mapping isometries via $\rho|_{\mathcal{B}^1} : \mathbb{B}^1 \xrightarrow{\cong} \text{PSL}_2(\mathbb{C})$. We will do this by composing standard isometries between different models for hyperbolic 3-space, written quaternionically. We will have need for the following model as an intermediary step.

**Definition 3.2.24.** The Poincaré ball model for hyperbolic $n$-space is

$$\mathcal{P}^n := \{ p \in \mathbb{R}^n \mid \text{n}_{\text{Euc}}(p) < 1 \},$$

with the metric induced by the stereographic projection from the point $(-1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$:

$$\mathcal{I}^n \rightarrow \mathcal{P}^n : \ (x_0, \ldots, x_n) \mapsto \frac{(x_1, \ldots, x_n)}{1 + x_0}.$$ 

Figure 3.2 shows the positioning of $\mathcal{I}^n$ and $\mathcal{P}^n$ within a common $\mathbb{R}^{n+1}$, making possible the stereographic projection from the Definition.
There is also a standard isometry from $\mathcal{P}^n \subset \mathbb{R}^n$ to the upper half-space model $\mathcal{H}^n \subset \mathbb{R}^n$ given by inversion through the sphere of radius $\sqrt{2}$ centered at the point $(-1, 0, \ldots, 0) \in \mathbb{R}^n$. Figure 3.3 shows $\mathcal{P}^n$ and $\mathcal{H}^n$ as subsets of a common $\mathbb{R}^n$, and the dotted circle is the one we invert through to map one model to the other.
In coordinates, this map is
\[ P^n \to \mathcal{H}^n, \quad (x_1, \ldots, x_n) \mapsto \frac{(2x_1, \ldots, 2x_{n-1}, (1 - x_1^2 - \cdots - x_n^2)x_n)}{x_1^2 + \cdots + x_{n-1}^2 + (x_n + 1)^2}. \]
(3.2.25)

We now obtain a concise relationship between the Möbius action \( \alpha \) on \( \mathcal{H}^3 \) from Definition 3.2.21, and our isometric action \( \mu \) on \( \mathcal{M}_+^1 \) from (Theorem 3.2.18).

**Theorem 3.2.26.** Let
\[ \iota : \mathcal{M}_+^1 \to \mathcal{H}^3, \quad w + xi + j(y + z I i) \mapsto \frac{y + zI + J}{w + x}. \]

(1) \( \iota \) is an orientation-preserving isometry.

(2) \( \forall (q, m) \in PB^1 \times \mathcal{M}_+^1 : \iota(\mu(q, m)) = \alpha(\rho(q), \iota(m)) \).

**Remark 3.2.27.** With \( k \) as in Macfarrlane’s original notation from (1.1.5),
\[ w + xi + j(y + z I i) = w + xi + yj + zk. \]

**Proof.** We prove (1) first. Taking \( n = 3 \) in Definition 3.2.24, \( P^3 \) can be identified within the same 4-dimensional Minkowski space \( \mathbb{R}^{(1,3)} \) where \( T^3 \) lies, with the given stereographic projection providing a conformal map between the models. Using instead \( \mathcal{M}_+^1 \) as our hyperboloid model, and the Macfarrlane space \( \mathcal{M} \) instead of Minkowski space, \( P^3 \) is identified with the set of
Converting the stereographic projection map to quaternion notation gives the isometry

$$\iota_{\text{proj}} : \mathcal{M}_1^1 \rightarrow \mathcal{M}_0^{<1}, \quad w + xi + yj + zIij \mapsto \frac{xi + yj + zIij}{1 + w}.$$ 

Next, we find our quaternion equivalent of the map (3.2.25) with $n = 3$. The domain will now be $\mathcal{M}_0^{<1} \subset \left( \frac{1}{\xi} \right)$ but in the range we must relabel our axes so that the image lies in $\mathcal{H}^3 \subset \mathbb{H}$. We do this by replacing the axes $i\mathbb{R}, j\mathbb{R}$ and $Iij\mathbb{R}$ with the axes $\mathbb{R}, I\mathbb{R}$ and $J\mathbb{R}$, respectively. Under this relabeling, our isometry is an inversion through the sphere of radius $\sqrt{2}$ centered at $-J \in \mathbb{H}$, and is given by

$$\iota_{\text{inv}} : \mathcal{M}_0^{<1} \rightarrow \mathcal{H}^3, \quad xi + yj + zIij \mapsto \frac{2x + 2yI + (1 - x^2 - y^2 - z^2)J}{x^2 + y^2 + (z + 1)^2}.$$ 

Composing $\iota_{\text{inv}}$ with $\iota_{\text{proj}}$ would yield an isometry from $\mathcal{M}_1^1$ to $\mathcal{H}^3$, but since $\iota_{\text{inv}}$ is an inversion, this would be orientation reversing. To remedy this and achieve the map $\iota$, we include the following sign change and (even) permutation of the coordinates, which is a second orientation-reversing isom-
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

entry.

\[ \iota_{\text{perm}} : \mathcal{M}_0^{<1} \to \mathcal{M}_0^{<1}, \quad xi + yj + z I i j \mapsto y i - z j + I x i j \]

Using the fact that \( \forall (w + xi + yj + z I i j) \in \mathcal{M}_1^1 : x^2 + y^2 + z^2 = w^2 - 1 \), a computation shows

\[ (\iota_{\text{inv}} \circ \iota_{\text{perm}} \circ \iota_{\text{proj}})(w + xi + yj + z I i j) = \frac{y - z I + J}{w + x}. \]

If we negate the \( z \)-coordinate in the input, the negative sign on the right is eliminated, and the input can be written as in the definition of \( \iota \).

We will prove (2) by first showing that the statement holds when \( m = 1 \in \mathcal{M}_1^1 \) and then showing how this generalizes. Let \( * \) be the standard involution on \( \mathbb{H} \), \( n \) the quaternion norm in \( \mathbb{H} \), and \( | \cdot | \) the complex modulus.

Here are some observations that are used in the computation to follow. Since we identify \( \mathbb{C} \) with \( \mathbb{R} \oplus \mathbb{R} I \subset \mathbb{H} \), we have that \( \forall \ c \in \mathbb{C} : c^* = \overline{c} \), \( Jc = \overline{c} J \), and \( n(c) = |c|^2 \) (as opposed to in \( \mathcal{B} \) where \( c \in \mathbb{C} \) would be a scalar with norm \( c^2 \)). So for \( c_1, c_2 \in \mathbb{C} \), we have \( (c_1 + c_2 J)^* = \overline{c_1} - c_2 J \), and \( n(c_1 + c_2 J) = |c_1|^2 + |c_2|^2 \). It will also be useful to write \( ij \in \mathcal{B} \) as \( I(-I i j) \) to better see the image of a point under \( \iota \).

Let \( q = w + xi + yj + z i j \in \mathcal{P} \mathcal{A}_1^1 \) where \( w, x, y, z \in \mathbb{C} \), then (as is used in the fourth equality below) by the norm in \( \mathcal{B} \), \( w^2 - x^2 - y^2 + z^2 = 1 \) and we
have the following.

\[ \iota(qq^1) = \iota((w + xi + yj + zij)(\bar{w} + \bar{x}i + \bar{y}j - \bar{z}ij)) = \iota(|w|^2 + |x|^2 + |y|^2 + |z|^2 + (w\bar{x} + x\bar{w} + y\bar{z} + z\bar{y})i) \]

\[ + (w\bar{y} - x\bar{z} + y\bar{w} - z\bar{x})j + (-w\bar{z} + x\bar{y} - y\bar{x} + z\bar{w})I(-Iij) \]

\[ = \frac{w\bar{y} - x\bar{z} + y\bar{w} - z\bar{x} + (-w\bar{z} + x\bar{y} - y\bar{x} + z\bar{w})I \cdot I + J}{|w|^2 + |x|^2 + |y|^2 + |z|^2 + w\bar{x} + x\bar{w} + y\bar{z} + z\bar{y}} \]

\[ = \frac{y\bar{w} + y\bar{x} - z\bar{w} - z\bar{x} + w\bar{y} + w\bar{z} - x\bar{y} - x\bar{z} + (w^2 - x^2 - y^2 + z^2)J}{(w + x)(w + x) + (y + z)(y + z)} \]

\[ = \frac{(y - z)(\bar{w} + \bar{x}) + (w - x)(\bar{y} + \bar{z}) + (w - x)(w + x) - (y - z)(y + z) \cdot J}{|w + x|^2 + |y + z|^2} \]

\[ = \frac{(y - z + (w - x)J)(\bar{w} + \bar{x} - (y + z)J)}{n(w + x + (y + z)J)} \]

\[ = (y - z + (w - x)J)(w + x + (y + z)J)^{-1} \]

\[ = \alpha \left( \begin{pmatrix} w - x & y - z \\ y + z & w + x \end{pmatrix}, J \right) \]

\[ = \alpha(\rho(q), \iota(1)). \]  

(3.2.28)

This establishes the equation in the case where \( m = 1 \).

Now suppose \( m \neq 1 \). Then \( \exists r \in \mathbb{P}^1 : \mu(r, m) = 1 \) and we use the facts that \( \mu \) and \( \alpha \) define left group actions and that \( \rho \) is a \( \mathbb{C} \)-algebra isomorphism
3.2. A QUATERNION MODEL FOR HYPERBOLIC 3-SPACE

(by Proposition 3.2.9). Let $q \in \mathcal{PB}^1$. Then

\[
\iota(\mu(q,m)) = \iota(\mu(q,\mu(r,1))) = \iota(\mu(qr,1)) \\
= \alpha(\rho(qr),\iota(1)) = \alpha(\rho(q)\rho(r),\iota(1)) \\
= \alpha(\rho(q),\alpha(\rho(r),1)) = \alpha(\rho(q),\iota(\mu(r,1))) \\
= \alpha(\rho(q),\iota(m)).
\]

Example 3.2.29. The isometry $ij \in \mathcal{PB}^1$ is a rotation fixing the point 1.

\[
\mu(ij,1) = ij(ij)^1 = -iji = ij^2i = i^2 = 1
\]

We can also see this in $\mathcal{H}^3$.

\[
\alpha(\rho(ij),\iota(1)) = \alpha\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right), J = \frac{0 \cdot J - 1}{J + 0} = -J^{-1} = J = \iota(1)
\]

Example 3.2.30. Suppose $q$ is a normalized parabolic isometry as listed in Table 3.1, acting on 1. The computation at (3.2.28) simplifies to:

\[
\iota(\mu(q,1)) = \iota((1 + cj - ci)j(1 + \bar{c}j + \bar{c}ij)) \\
= \iota(1 + 2|c|^2 - 2|c|^2i + (c + \bar{c})j + (\bar{c} - c)I(-I)ij) \\
= \frac{c + \bar{c} + (\bar{c} - c)I \cdot I + J}{1 + 2|c|^2 - 2|c|^2} = 2c + J = \frac{1 \cdot J + 2c}{0 \cdot J + 1} \\
= \alpha\left(\begin{array}{cc} 1 & 2c \\ 0 & 1 \end{array}\right), J = \alpha(\rho(q),\iota(1))
\]
Example 3.2.31. Suppose $q$ is a normalized non-parabolic isometry as listed in Table 3.1, acting on 1. The computation at (3.2.28) simplifies to:

\[
\iota(\mu(q, 1)) = \iota \left( \left( \frac{c + c^{-1}}{2} j + \frac{c - c^{-1}}{2} ij \right) \left( \frac{\bar{c} + \bar{c}^{-1}}{2} j - \frac{\bar{c} - \bar{c}^{-1}}{2} ij \right) \right)
\]

\[
= \iota \left( \frac{1}{4} \left( (c + c^{-1})^2 + (c - c^{-1})^2 + ((c + c^{-1})(c - c^{-1}) + (c - c^{-1})(c + c^{-1}))i \right) \right)
\]

\[
= \iota \left( \frac{1}{2} \left( (c^2 + c^{-2}) + (c^2 - c^{-2})i \right) \right)
\]

\[
= \frac{J}{\frac{1}{2}((c^2 + c^{-2}) + (c^2 - c^{-2}))} = J/c^2 = \frac{c^{-1}J + 0}{0 + c}
\]

\[
= \alpha \left( \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}, J \right) = \alpha(\rho(q), \iota(1))
\]

3.3 Quaternion Models for the Hyperbolic Plane

In the previous section, a method was given for identifying $\mathbb{H}^3$ and Isom$^+(\mathbb{H}^3)$ within $\left( \frac{1}{\sqrt{c}} \right)$. In this section we will introduce two methods of similarly identifying $\mathbb{H}^2$ and Isom$^+(\mathbb{H}^2)$ within $\left( \frac{1}{\sqrt{c}} \right)$. The first method is developed with a view toward studying immersed hyperbolic surfaces in hyperbolic 3-manifolds, and the second method will result in interesting analogies to Hamilton’s Theorem 1.1.3.
Recall that by Definition 1.2.1 we have

\[
\begin{align*}
\left( \frac{1,1}{\mathbb{R}} \right) &= \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij, \\
\i^2 &= j^2 = 1, \\
i_1 &= -ji.
\end{align*}
\]

3.3.1 The Standard Restricted Macfarlane Space

For this subsection let \( B = \left( \frac{1,1}{\mathbb{C}} \right) \) and let \( A = \left( \frac{1,1}{\mathbb{R}} \right) \subset B \). It will be of particular interest that the construction here occurs as a subspace of the standard Macfarlane space, and can be extended to that by appending the \( i_1 \sqrt{-1} \) coordinate.

**Definition 3.3.1.** The (standard) restricted Macfarlane space \( \mathcal{L} \) is the 3-dimensional \( \mathbb{R} \)-space

\[
\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \subset \left( \frac{1,1}{\mathbb{R}} \right)
\]
equipped with the restriction of the quaternion norm.

Several results analogous to those of the previous section follow easily.

**Proposition 3.3.2.** \( (\mathcal{L}, n) \simeq \mathbb{R}^{1,2} \) and \( \mathcal{L}^1_+ \simeq T^2 \).

**Proof.** For the first isometry it suffices (by Definition 3.1.1) to show that the restriction of the norm from \( A \) to \( \mathcal{L} \) is a real-valued quadratic form of
CHAPTER 3. QUATERNION MODELS FOR HYPERBOLIC SPACE

signature (1, 2). From Definition 1.2.3, for \( q = w + xi + yj + zij \in \mathcal{A} \) with \( w, x, y, z \in \mathbb{R} \), the norm on \( \mathcal{A} \) takes the form \( n(q) = w^2 - x^2 - y^2 + z^2 \). So for \( p = w + xi + yj \in \mathcal{L} \) we get \( n(p) = w^2 - x^2 - y^2 \in \mathbb{R} \) as desired.

To show the second isometry recall that \( \text{tr}(p) > 0 \iff w > 0 \) and compare \( \mathcal{L}^1_+ \) to Definition 2.1.7.

\[
\mathcal{L}^1_+ = \{ q \in \mathcal{L} \mid n(q) = 1, \text{tr}(q) > 0 \}
\]

\[
\simeq \{ (w, x, y) \in \mathbb{R}^3 \mid w^2 - x^2 - y^2 = 1, w > 0 \}
\]

\[
= \mathcal{I}^2.
\]

Let \( \rho_A \) be the matrix representation given by 3.2.7. That is, taking \( w, x, y, z \) as variables in \( \mathbb{R} \) gives

\[
\rho_A : \left( \frac{1}{\mathbb{R}} \right) \to \text{M}_2(\mathbb{R}), \quad w + xi + yj + zij \mapsto \begin{pmatrix} w - x & y - z \\ y + z & w + x \end{pmatrix}.
\]

**Proposition 3.3.3.** \( \rho_A \) is an \( \mathbb{R} \)-algebra isomorphism.

**Proof.** \( \rho_A \) is an injective \( \mathbb{R} \)-algebra homomorphism by Proposition 3.2.9. We know from Theorem 1.2.11 (1) that \( \mathcal{A} \cong \text{M}_2(\mathbb{R}) \), forcing \( \rho_A \) to be surjective as well. \( \square \)

The set of symmetric matrices is \( \{ s \in \text{M}_2(\mathbb{R}) \mid s^\top = s \} \). Since a symmetric matrix has no complex entries this is equivalent to \( \text{Herm}_2(\mathbb{R}) \) from (3.1.3).
Proposition 3.3.4. \( \rho_A|_{L} : (L, n) \to (\text{Herm}(\mathbb{R}), \det) \) is a quadratic space isometry.

Proof. Since \( \rho_A \) is bijective, \( \rho_A|_{L} \) is bijective onto its image. \( \forall p \in L \)
\( \exists w, x, y \in \mathbb{R} \) such that \( p = w + xi + yj \) and then
\[
\rho_A(p) = \begin{pmatrix}
w - x & y \\
y & w + x
\end{pmatrix}.
\]
As \( w, x, y \) range over \( \mathbb{R} \) we attain precisely the symmetric matrices \( \text{Herm}_2(\mathbb{R}) \),
so that \( \rho_A|_{L} \) is bijective onto \( \text{Herm}_2(\mathbb{R}) \).

Finally, \( \rho_A \) is an isometry because
\[
\det(\rho_A(p)) = \begin{vmatrix}
w - x & y \\
y & w + x
\end{vmatrix} = w^2 - x^2 - y^2 = n(p). \quad \Box
\]

Just as \( \mathbb{PB}^1 \cong \text{PSL}_2(\mathbb{C}) \), we now have \( \mathbb{PA}^1 \cong \text{PSL}_2(\mathbb{R}) \), and we gain
the 2-dimensional equivalent of Theorem 3.2.18 simply by restricting \( \mu \) of
Theorem 3.2.18 appropriately.

The adjoint map (Definition 3.2.13) on \( \mathcal{A} \) is the restriction over \( \mathbb{R} \) of the
adjoint map on \( \mathcal{B} \), and takes the form
\[
\uparrow : \mathcal{A} \to \mathcal{A}, \quad w + xi + yj + zij \mapsto w + xi + yj - zij.
\]
Finally, we get the desired modification for \( \mathcal{H}^2 \) as follows.

Theorem 3.3.5. An isomorphism \( \mathbb{PA}^1 \cong \text{Isom}^+(\mathcal{H}^2) \) is defined by the group
action

\[ \mu|_{\mathcal{P}\mathcal{A}^1 \times \mathcal{L}_+^1} : \mathcal{P}\mathcal{A}^1 \times \mathcal{L}_+^1 \rightarrow \mathcal{L}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger. \]  

(3.3.6)

**Proof.** Using the Propositions from this section thus far and Theorem 3.2.18 it suffices to observe that \( \nu \) fixes \( \mathcal{L}_+^1 \), which is immediate. \( \square \)

### 3.3.2 Pure Quaternions and Hyperbolic Planes

Let \( \mathcal{B} = \left( \begin{smallmatrix} a & b \\ -b & K \end{smallmatrix} \right) \) be an arbitrary quaternion algebra.

There is a more direct analogy to Hamilton’s Theorem 1.1.3. For completeness we include a brief discussion of this but note that for our purposes this approach has limitations. Although it provides another model for \( \mathcal{H}^2 \), this model fails to extend to \( \mathcal{H}^3 \) in the natural way we observed in the previous subsection.

Consider the pure quaternions \( \mathcal{B}_0 \subset \mathcal{B} \). The restriction of the quaternion norm to this space is the quadratic form with Graham matrix \( \text{diag}(a, b, ab) \).

Since \( n \) arises from the standard involution on \( \mathcal{B} \), i.e. \( n(q) = qq^* \), and with respect to this we have \( \mathcal{K} = \mathcal{B}^1 = \mathcal{B}_0^\perp \). We get an isometric action

\[ \mathcal{B}^1 \times \mathcal{B}_0 \rightarrow \mathcal{B}_0, \quad (\gamma, p) \mapsto \gamma p \gamma^* \]  

(3.3.7)

which defines an isomorphism from \( \mathcal{P}\mathcal{B}^1 \) to the orthogonal group associated to the quadratic form \( n|_{\mathcal{B}_0} \) [29].
Now consider the case where $K = \mathbb{R}$. Up to isomorphism, $\left( \frac{a, b}{\mathbb{R}} \right) = \mathbb{H}$ or $\left( \frac{1, 1}{\mathbb{R}} \right)$. Likewise, up to similarity the Graham matrix of the quadratic form $n|_{B_0}$ is either $\text{diag}(1, 1, 1)$ or $\text{diag}(-1, -1, 1)$. Thus the action given by (3.3.7) represents either $\text{SO}(3)$ or $\text{SO}(1, 2)$, and gives rise to either spherical or hyperbolic geometry in 2 dimensions, respectively. The former case is the familiar Theorem 1.1.3, so we now focus on the latter.

Let $\mathcal{A} = \left( \frac{1, 1}{\mathbb{R}} \right)$, and consider

$$\mathcal{A}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij \subset \left( \frac{1, 1}{\mathbb{R}} \right)$$

equipped with the restriction of the quaternion norm.

We obtain a 2-dimensional hyperboloid of 2 sheets by forming the space $\mathcal{A}_0^1 = \{ q \in \mathcal{A} \mid \text{tr}(q) = 0, n(q) = 1 \}$, but if we desire a hyperboloid model for $\mathcal{H}^2$, we must identify $ij \sim -ij \in \mathcal{A}_0^1$. Since $\mathcal{A} \subset \left( \frac{1, 1}{\mathbb{R}} \right)$, restricting the action given by (3.3.7) then gives an action by isometries on a hyperboloid model for $\mathcal{H}^2$.

$$\mathbb{P} \mathcal{A}^1 \times \mathcal{A}_0^1/ \sim \to \mathcal{A}_0^1/ \sim, \quad (\gamma, p) \mapsto \gamma p \gamma^*$$

Due to the difference between $\mathbb{P}$ and $\sim$, the interplay between points and isometries is not as clear as in the earlier constructions. Nonetheless, notice that if we identify $\mathcal{A}_0^1/ \sim$ with the sheet of $\mathcal{A}_0^1$ where the $ij$ coordinate is
positive, and take a point $p$ on this sheet, then $p$ acts as a rotation that fixes itself:

$$(p, p) \mapsto ppp^* = pn(p) = p.$$ 

This is in direct analogy to what occurs in Theorem 1.1.3.

As mentioned earlier, this model does not give rise to an embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$ as we achieved using the restricted standard Macfarlane space in the previous subsection. Extending the quadratic space $\mathcal{A}_0$ to $\mathcal{A}$, the signature changes from $(1, 2)$ to $(2, 2)$, not to the desired $(1, 3)$. Alternatively, extending the quadratic space $\mathcal{A}_0$ to $\left(\frac{1, 1}{0}\right)$, the signature (over $\mathbb{R}$) changes from $(1, 2)$ to $(3, 3)$. Thus from here on when working with $\mathbb{H}^2$ we will use the approach of the previous subsection.
Chapter 4

Macfarlane Spaces

In this chapter, we generalize Macfarlane spaces for applications to hyperbolic 3-manifolds and surfaces. Using a non-standard hyperboloid model defined over an algebraic number field, we define its group of isometries quaternionically, giving rise to new tools for studying the action of Kleinian and Fuchsian groups on $H^3$ and $H^2$. This sets the stage for new interactions between algebraic number theory and hyperbolic 3-manifolds and surfaces using the arithmetic and geometric properties of quaternion algebras.

In §4.1 we define Macfarlane spaces, give some of their basic properties, and characterize them from the perspectives of geometry and arithmetic. In §4.2 we establish the action by isometries on a 3-dimensional hyperboloid model lying in a Macfarlane space analogous to Theorem 3.2.18, and develop some new tools for understanding isometries in terms of their properties as quaternions. In §4.3 we focus on the role of Macfarlane spaces in the action
of Kleinian or Fuchsian groups, and define Macfarlane manifolds. We discuss some naturally arising objects and properties of these manifolds, and prove that they include diverse classes of examples. In §4.4 we establish analogous results for 2-dimensional hyperboloid models and restricted Macfarlane spaces. We also discuss quaternion algebras over real fields as subalgebras of Macfarlane quaternion algebras, and their relevance to restricted Macfarlane spaces.

4.1 Macfarlane Spaces

Let $K \subset \mathbb{C}$ be a field with $K \notin \mathbb{R}$ and let $\mathcal{B}$ be a quaternion algebra over $K$, with norm $n$. Recall (3.2.7) where we introduced the faithful matrix representation $\rho_{\mathcal{B}}: \mathcal{B} \to M_2(K(\sqrt{a}, \sqrt{b}))$.

$w + xi + yj + zij \mapsto \begin{pmatrix} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{pmatrix}.$

We used this to define the adjoint of $q \in \left(\frac{1}{2}\right)$ or $\mathbb{H}$ as $q^\dagger = \rho^{-1}(\overline{\rho(q)^T})$, pulling back the conjugate transpose. We will reuse the notation $\dagger$ in defining Macfarlane spaces and, as we will see in this section, this usage is a generalization of the prior one.

Recalling the terminology and notation from Definition 1.2.32, if $\star$ is an
involution of the second kind on $B$, then its set of symmetric elements is
\[ \ast B = \{ q \in B \mid q^* = q \}, \] and $\ast K = \{ w \in K \mid w^* = w \}$ is a subfield of $K$ of index 2.

**Definition 4.1.1.** $B$ is Macfarlane if it admits an involution of the second kind $\dagger$ such that $(\mathcal{B}, n|_B)$ is a quadratic space of signature $(1, 3)$ over $F := \dagger K \subset \mathbb{R}$. In this case $\mathcal{M} := \dagger B$ is a Macfarlane space.

**Example 4.1.2.**

(1) From the discussion in §3.2, $\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}\sqrt{-1}ij \subset \left( \frac{1,1}{\mathbb{R}} \right)$ is a Macfarlane space.

(2) $F \oplus Fi \oplus Fj \oplus F\sqrt{-dij} \subset \left( \frac{a,b}{F(\sqrt{-d})} \right)$, where $F \subset \mathbb{R}$ and $a, b, d \in F^+$, is a Macfarlane space.

(3) [Non-example] Let $B = \left( \frac{a,b}{K} \right)$, and $[K : \mathbb{Q}]$ be odd $\implies B$ is not Macfarlane because $K$ has no subfield of index two, and thus by Proposition 1.2.36, $B$ does not admit an involution of the second kind.

In this section we will show that up to isomorphism of $B$, all Macfarlane spaces take the form of Example (2) above. First let us see some motivation for the construction.
Define

\[ \mathcal{M}_+^1 := \{ p \in \mathcal{M} \mid n(p) = 1, \text{tr}(p) > 0 \}, \]  

(4.1.3)

and recall (Definition 2.1.10) that a hyperboloid model for \( \mathbb{H}^3 \) defined over \( F \) is a hyperboloid

\[ \{ (w, x, y, z) = p \in F^4 \mid \phi(p) = 1, w > 0 \} \]  

(4.1.4)

where \( \phi : F^4 \to F \) is a quadratic form of signature \((1, 3)\).

**Proposition 4.1.5.** \( \mathcal{M}_+^1 \) is a hyperboloid model for hyperbolic 3-space defined over \( F \).

**Proof.** Since \( \text{sig}(n|_{\mathcal{M}}) = (1, 3) \) it is immediate that \( \mathcal{M}^1 \) is a hyperboloid of 2 sheets defined over \( F \). We know that \( F \subset \mathcal{M} \), and there it forms a scalar axis along which \( n|_{\mathcal{M}} \) is positive definite. Since the scalar part of some \( p \in \mathcal{M} \) is \( \frac{\text{tr}(p)}{2} \), we can specify the upper sheet of \( \mathcal{M}^1 \) by requiring points to have positive trace. This works because \( n|_{\mathcal{M}_0} \) is negative definite and a point \( p \in \mathcal{M}^1 \) satisfies \( n(p) = 1 \), so no members of \( \mathcal{M}^1 \) have zero trace. Thus \( \mathcal{M}_+^1 \) has the desired form. \( \square \)

We will be now interested in working with a hyperboloid model defined over the algebraic number field \( F \), rather than working over \( \mathbb{R} \). So we allow
4.1. MACFARLANE SPACES

our hyperboloid model to be stretched and skewed as necessary up to isometry over $F$, while remembering (Proposition 2.1.11) that the standard model can be recovered if needed by extending scalars to $\mathbb{R}$.

**Corollary 4.1.6.** $(M \otimes_F \mathbb{R})^1_+ \textit{ is isometric to } I^3$.  

We will soon see how to normalize Macfarlane spaces to a particular convenient form, and how to recognize Macfarlane quaternion algebras. But first let us build some intuition by taking a more naive approach in the following examples.

**Example 4.1.7.** Let $B = \left( \frac{-1, \sqrt{-1}}{\mathbb{Q}(\sqrt{-1})} \right)$. For $p = w + xi + yj + zij \in B$ with $w, x, y, z \in \mathbb{Q}$, the norm on $B$ takes the form

$$n(p) = w^2 + x^2 - \sqrt{-1}y^2 - \sqrt{-1}z^2.$$ 

We cannot convert the coefficients of this to the desired form merely by introducing factors of $\sqrt{-1}$ on some choice of basis elements, as was done in §3.2. Nonetheless it is possible to produce a $\mathbb{Q}$-space of signature $(1, 3)$ by using the basis $S = \{1, \sqrt{-1}i, (1 - \sqrt{-1})j, (1 - \sqrt{-1})ij\}$ because

$$n(\sqrt{-1}i) = -(-1)^2(-1) = -1,$$
$$n((1 - \sqrt{-1})j) = -(1 - \sqrt{-1})^2\sqrt{-1} = -(2\sqrt{-1})\sqrt{-1} = -2,$$
$$n((1 - \sqrt{-1})ij) = (1 - \sqrt{-1})^2(-\sqrt{-1}) = -2.$$
That is, \( G_n^S = \text{diag}(1, -1, -2, -2) \), and so \( \mathcal{M} := \text{Span}_Q(S) \) has the appropriate signature. With some more work one can show that \( \mathcal{M} \) is the set of symmetric elements of the involution \( \dagger \) given by linearly extending the rules

\[
\sqrt{-1}^\dagger = -\sqrt{-1} \quad i^\dagger = -i \quad j^\dagger = -\sqrt{-1}j
\]

over \( Q \). Therefore \( \mathcal{M} \) is a Macfarlane space.

If we can establish that the property of being Macfarlane is isomorphism invariant, this example admits an easier solution by working up to isomorphism and using Theorem 1.2.11. In particular note that

\[-(\sqrt{-1})^2 + \sqrt{-1}(0)^2 = 1.
\]

From this we have \( \mathcal{B} \cong \left( \frac{1}{\sqrt{-1}} \right) \), which contains the more convenient Macfarlane space \( Q \oplus Q i \oplus Q j \oplus \sqrt{-1}Q ij \).

Since we will be applying this theory to arithmetic invariants of hyperbolic 3-manifolds, we are interested in the isomorphism classes of quaternion algebras over a fixed concrete number field. Once the desired isomorphism invariance is established, we can avoid complications as in the above examples by choosing an appropriate representative of the isomorphism class.

**Definition 4.1.8.** A Macfarlane quaternion algebra is *normalized* if it is of the form \( \left( \frac{a, b}{F(\sqrt{-d})} \right) \), where \( F \subset \mathbb{R} \) and \( a, b, d \in F^+ \), and a Macfarlane space
is normalized if it is of the form

\[ F \oplus Fi \oplus Fj \oplus F\sqrt{-dij} \subset \left( \frac{a, b}{F(\sqrt{-d})} \right). \]

**Proposition 4.1.9.** Let \( \mathcal{B} \) be a normalized Macfarlane quaternion algebra with normalized Macfarlane space \( \mathcal{M} \subset \mathcal{B} \), and for \( q \in \mathcal{B} \) define \( q^\dagger = \rho_{\mathcal{B}}^{-1}(\rho_{\mathcal{B}}(q)^\top) \). Then \( q^\dagger \) is an involution of the second kind and \( \mathcal{M} = \mathcal{B} \).

**Proof.** We proved (Proposition 3.2.15) that \( \rho_{\mathcal{B}} \) is an involution of the second kind for more general \( \mathcal{B} \). We also showed (Lemma 3.2.19) that \( \mathcal{M} = \mathcal{B} \) in the case where \( \mathcal{B} = \left( \frac{1, 1}{\mathbb{C}} \right) \). We now use a similar argument to generalize this.

Let \( \mathcal{B} = \left( \frac{a, b}{F(\sqrt{-d})} \right) \) and \( q = w + xi + yj + zij \in \mathcal{B} \) with \( w, x, y, z \in F(\sqrt{-d}) \). Then \( \rho_{\mathcal{B}}(q) = \left( \begin{array}{cc} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{array} \right) \). Since \( a, b > 0 \), \( \rho_{\mathcal{B}}(q)^\top = \left( \begin{array}{cc} \overline{w} - \overline{x}\sqrt{a} & \overline{y}\sqrt{b} + \overline{z}\sqrt{ab} \\ \overline{y}\sqrt{b} - \overline{z}\sqrt{ab} & \overline{w} + \overline{x}\sqrt{a} \end{array} \right) \), so applying \( \rho_{\mathcal{B}}^{-1} \) gives \( q^\dagger = \overline{w} + \overline{x}i + \overline{y}j - \overline{z}ij \). Then \( q = q^\dagger \iff w = \overline{w}, x = \overline{x}, y = \overline{y} \) and \( z = -\overline{z} \iff w, x, y \in F \) and \( z \in \sqrt{-d}F \).

Up to isomorphism all Macfarlane spaces occur in this way, giving the desired result that the property of being Macfarlane is isomorphism invariant.

**Theorem 4.1.10.** A quaternion algebra \( \mathcal{B} \) over a field \( K \) is Macfarlane if and only if both of the following conditions hold.

1. \( K = F(\sqrt{-d}) \) for some \( F \subset \mathbb{R} \) and \( d \in F^+ \).
(2) There is a $K$-algebra isomorphism from $\mathcal{B}$ to a normalized Macfarlane quaternion algebra (i.e. $\exists a, b \in F^+: \mathcal{B} \cong \left( \frac{a, b}{F(\sqrt{-d})} \right)$).

Moreover, the Macfarlane space in $\mathcal{B}$ is unique.

Before proving this Theorem, we first establish some lemmas.

**Lemma 4.1.11.** If $\mathcal{B}$ over $K$ is Macfarlane with corresponding involution $\dagger$, then $\exists c \in K \setminus F^{(2)}$ such that $K = F(c)$ and $c^\dagger = -c$.

**Proof.** By Proposition 1.2.36, $[K : F] = 2$, so there exists some $c \in K \setminus F$ such that $c^2 \in F$ and $K = F(c)$. Since $F = F^\dagger$, we have $(c^\dagger)^2 = (c^2)^\dagger = c^2$, thus $c^\dagger = \pm c$. If $c^\dagger = c$ then $F^\dagger = F(c) = F(c^\dagger) = F(c) = K$, which would mean $F = K$, but $[K : F] = 2$. Therefore $c^\dagger = -c$. \hfill $\Box$

**Lemma 4.1.12.** If $\mathcal{B}$ over $K$ is Macfarlane, then $\text{Span}_K(\mathcal{M}) = \mathcal{B}$.

**Proof.** By the previous lemma, we know that $K = F \oplus Fc$, and that $c^\dagger = -c$. This implies that $F \subset \mathcal{M}$ and that $Fc \cap \mathcal{M} = \{0\}$. Since $\text{Span}_K(F) = K$, it remains to show that $\text{Span}_K(\mathcal{M}_0) = \mathcal{B}_0$.

Let $S = \{s_1, s_2, s_3\}$ be a basis for $\mathcal{M}_0$ over $F$ and suppose by way of contradiction that $S$ is not linearly independent over $K$. Then $\exists k_\ell \in K$ such that $\sum_{\ell=1}^3 k_\ell s_\ell = 0$. Since $K = F \oplus Fc$, we have $\forall \ell: k_\ell = f_{\ell,1} + f_{\ell,2}c$ for some $f_{\ell,1}, f_{\ell,2} \in F$. Making this substitution in the sum and then rearranging
4.1. MACFARLANE SPACES

terms in the equation gives the following.

\[ f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = -c(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3) \]

But \( f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 \) and \( f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3 \) both lie in \( \mathcal{M}_0 \) so each is fixed by \( \dagger \), while \( c^\dagger = -c \). So applying \( \dagger \) to both sides of the previous equation gives

\[ f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = c(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3). \]

Adding the two equations then gives that \( f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3 = 0 \). Since \( f_{1,2}, f_{2,2}, f_{3,2} \in F \), this contradicts that \( S \) is a basis for \( \mathcal{M}_0 \) over \( F \).

We conclude that \( S \) is linearly independent over \( K \), giving that

\[ \dim_K \left( \text{Span}_K(S) \right) = \dim_K \left( \text{Span}_K(\mathcal{M}_0) \right) = 3, \]

which forces \( \text{Span}_K(\mathcal{M}_0) = B_0 \) as desired.

\[ \square \]

**Proof of Theorem 4.1.10.** We first show that conditions (1) and (2) imply that \( B \) is Macfarlane. Let \( C \) be a normalized Macfarlane quaternion \( K \)-algebra and let \( \psi : C \to B \) be a \( K \)-algebra isomorphism. Then \( \psi \) is also an isometry of quadratic spaces respecting the quaternion norms, so that if \( \mathcal{M} \) is the Macfarlane space in \( C \) then \( \psi(\mathcal{M}) \) is the Macfarlane space in \( B \).

For the forward implication, let \( B \) be a Macfarlane quaternion algebra
with Macfarlane space $\mathcal{M} \subset \mathcal{B}$. By Lemma 4.1.11, $\mathcal{B} = \left( \begin{array}{cc} a & b \\ \frac{a^2 + b^2}{d} & -\frac{a}{d} \end{array} \right)$ for some $F \subset \mathbb{R}$ and $d \in F^+$. The norm $n|_{\mathcal{M}}$ is a real-valued quadratic form of signature $(1, 3)$, so there exists an orthogonal basis $B$ for $\mathcal{M}$ so that $G^B_n = \text{diag}(\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4) \in \text{GL}_4(F)$ where $\forall \ell : \alpha_\ell \in F^+$.

Since $B$ is an orthogonal basis for $\mathcal{M}$ over $F$, Lemma 4.1.12 gives that $B$ is also an orthogonal basis for $\mathcal{B}$ over $K$. Let $S$ be the standard basis $\{1, i, j, ij\}$ for $\mathcal{B}$. Then $S$ is an orthogonal basis for $\mathcal{B}$ over $K$ as well, and in particular, $G^S_n = \text{diag}(1, -a, -b, ab)$. Now while $G^B_n$ and $G^S_n$ are not congruent over $F$, they are congruent over $K$, i.e. $\exists D \in \text{GL}_4(K)$ such that

$$D^T G^B_n D = G^S_n. \quad (4.1.13)$$

But since $G^B_n$ and $G^S_n$ are diagonal and nonzero on their diagonals, $D$ must also be diagonal and nonzero on its diagonal, i.e. $\exists x_\ell \in K \setminus \{0\}$ such that $D = \text{diag}(x_1, x_2, x_3, x_4)$. Plugging in to (4.1.13) and solving for the $\alpha_\ell$ gives

$$\alpha_1 = \frac{1}{x_1^2}, \quad \alpha_2 = \frac{a}{x_2^2}, \quad \alpha_3 = \frac{b}{x_3^2}, \quad \alpha_4 = \frac{-ab}{x_4^2},$$

and recall that $B$ was chosen so that $\forall \ell : \alpha_\ell \in F^+$.

Let $\mathcal{B}' := \left( \begin{array}{cc} a_2 & a_3 \\ \frac{a_2^2 + a_3^2}{d} & -\frac{a_2}{d} \end{array} \right)$ with $i'^2 := a_2, j'^2 := a_3$, and let $n'$ be the norm on
4.1. MACFARLANE SPACES

\( B' \). By Theorem 1.2.11 (3) \( B \cong B' \). Let \( c := \frac{x_2 x_3}{x_4} \) and define

\[ \mathcal{M}' := F \oplus F i' \oplus F j' \oplus F c i' j' \subset B'. \]

Then \( \text{sig}_F(n'|_{\mathcal{M}'}) = (1, 3) \) because \( \alpha_1, \ldots, \alpha_4 \in F^+ \) and

\[
n'(p) = w^2 - \alpha_2 x^2 - \alpha_3 y^2 + \alpha_2 \alpha_3 c^2 z^2 \\
= w^2 - \alpha_2 x^2 - \alpha_3 y^2 + \frac{ab}{x_2^2 x_3^2} \left( \frac{x_2^2 x_3^2}{x_4^2} \right) z^2 \\
= w^2 - \alpha_2 x^2 - \alpha_3 y^2 - \left( \frac{-ab}{x_4^2} \right) z^2 \\
= w^2 - \alpha_2 x^2 - \alpha_3 y^2 - \alpha_4 z^2.
\]

Also, \( \mathcal{M}' \) is unaffected by replacing \( c \) with \( \sqrt{-d} \) for the following reason. \( \alpha_2 \alpha_3 c^2 = n'(ci'j') \in F^- \), and \( \alpha_2, \alpha_3 \in F^+ \), therefore \( c^2 \in F^- \). But \( c \in K = F(\sqrt{-d}) \) so that \( c \) must be of the form \( f\sqrt{-d} \) with \( f \in F \). This proves that \( \mathcal{M}' \) is a normalized Macfarlane space.

Lastly, we now show uniqueness. Since \( K = F(\sqrt{-d}) \) as above, there exists subspaces \( S_\ell \subset B \) so that \( B = K \oplus S_1 \oplus S_2 \oplus S_3 \), where \( \forall \ell : \dim_F(S_\ell) = 2 \) and \( \text{sig}_F(n|_{S_\ell}) = (1, 1) \) over \( F \). Let \( \mathcal{M} \subset B \) be Macfarlane. Since \( F \subset \mathcal{M} \) and \( F(\sqrt{-d}) \cap \mathcal{M} = \emptyset \), the positive definite portion of \( \mathcal{M} \) is forced, so then its remaining 3-dimensional negative-definite portion is forced as well, by the following argument.

Let \( \dagger \) be the involution giving \( \mathcal{M} = \dagger B \). Then by Theorem 1.2.41, \( \dagger \)
is uniquely determined by the quaternion $F$-subalgebra of $\mathcal{B}$ generated by 
$\{p \in \mathcal{B}_0 \mid p^\dagger = -p\}$, and by Proposition 1.2.39, this set is in turn uniquely
determined by $\mathcal{M}$.

Since this Theorem characterizes Macfarlane quaternion algebras by a
property that holds up to $K$-algebra isomorphism, and noting Theorem 2.2.4,
we have the following important corollary.

**Corollary 4.1.14.** For finite-volume, complete hyperbolic 3-manifolds, the
property of having a Macfarlane quaternion algebra is a manifold invariant.

Often the quaternion algebra of a hyperbolic 3-manifold is given not by a
symbol $\left( \frac{a,b}{F} \right)$ but by its field and ramification set [13, 37], so it will be useful
to have a way of detecting if $\mathcal{B}$ is Macfarlane in terms of this data.

**Theorem 4.1.15.** Let $\mathcal{B}$ be a quaternion algebra over $F(\sqrt{-d})$ where $F \subset \mathbb{R}$
and $d \in F^+$, and let $\tau$ be the place of $F(\sqrt{-d})$ corresponding to complex
conjugation. $\mathcal{B}$ is Macfarlane if and only if $\tau(\text{Ram}(\mathcal{B})) = \text{Ram}(\mathcal{B})$.

**Proof.** If $\mathcal{B}$ is Macfarlane, then by Theorem 4.1.10, $\mathcal{B} \cong \left( \frac{a,b}{F(\sqrt{-d})} \right)$ for some
$a, b \in F^+$. Since $\tau(\text{Ram}(\mathcal{B})) = \text{Ram}(\mathcal{B}_\tau)$, we have

$$
\tau(\text{Ram}(\mathcal{B})) = \text{Ram}\left( \frac{\overline{a}, \overline{b}}{F(\sqrt{-d})} \right) = \text{Ram}\left( \frac{a, b}{F(\sqrt{-d})} \right) = \text{Ram}(\mathcal{B}).
$$
4.1. MACFARLANE SPACES

Suppose conversely that $\tau(\text{Ram}(B)) = \text{Ram}(B)$. We prove that $B$ is Macfarlane by showing that there is a quaternion algebra $A$ over $F$ such that $A \otimes_F F(\sqrt{-d}) \cong B$. This will suffice because the structure parameters of $A \otimes_F F(\sqrt{-d})$ must be real.

The field $F(\sqrt{-d})$ has no real places and $B$ is split at all complex places, so $\text{Ram}(B)$ is a set of prime ideals of $R_{F(\sqrt{-d})}$ of even cardinality. Suppose $p' \in \text{Ram}(B)$ lies over an ideal $p \prec R_F$ which splits in the extension $F(\sqrt{-d}) \supseteq F$. Then $pR_K = p\tau(p')$ and by hypothesis $\tau(p') \in \text{Ram}(B)$ so every pair of such split primes is in $\text{Ram}(B)$. If $p' \prec R_{F(\sqrt{-d})}$ does not arise from a splitting of some $p \prec R_F$, then the completion $B_{p'}$ is split, so in this case $p' \notin \text{Ram}(B)$.

Therefore $\text{Ram}(B)$ is a set of pairs of conjugate prime ideals, i.e. $\exists p_\ell \prec R_{F(\sqrt{-d})}$ such that $\text{Ram}(B) = \{p_1, \tau(p_1'), \ldots, p_n, \tau(p_n')\}$ and $\forall \ell \exists p_\ell \prec R_F$ such that $p_\ell R_{F(\sqrt{-d})} = p_\ell \tau(p_\ell')$.

Now let $p \prec R_F$ be some prime ideal that does not split in the extension $F(\sqrt{-d}) \supseteq F$ and let $A$ be a quaternion algebra over $F$ satisfying

$$\text{Ram}(A) = \begin{cases} \{p_1', \ldots, p_n'\} & \text{if } n \text{ is even} \\ \{p, p_1', \ldots, p_n'\} & \text{if } n \text{ is odd}. \end{cases}$$

Since $|\text{Ram}(A)|$ is even, Theorem 1.2.19(3) guarantees the existence of $A$, and by the previous paragraph all of the pairs of primes lying above the $p_\ell'$ are preserved in $\text{Ram}(A \otimes_F F(\sqrt{-d}))$, but $p$ vanishes. Therefore $\text{Ram}(B) =$
Ram(\(A \otimes_F F(\sqrt{-d})\)), which gives \(B \cong A \otimes_F F(\sqrt{-d})\) as desired. \(\square\)

### 4.2 Isometries in Macfarlane Quaternion Algebras

In this section, we provide an intrinsic quaternionic representation for the group of orientation-preserving isometries of a hyperboloid within a Macfarlane space, generalizing Theorem 3.2.18. With this we will see the full motivation for defining Macfarlane spaces as we did. In particular, non-Macfarlane quaternion algebras (for example \((\frac{1}{\sqrt{1,\sqrt{-2}}})\)) might similarly contain a hyperboloid but our conditions are necessary to define these isometric actions.

#### 4.2.1 The Action by Isometries

Recall the representation \(\rho_Q\) defined at (3.2.7). With \(B = \left(\frac{11}{\sqrt{1,\sqrt{-2}}}\right)\), we showed in §3.2.2 that \(\rho_B\) is a quadratic space isometry from \(B\) to \(\text{Herm}_2(\mathbb{C})\). We then drew upon Wigner’s Theorem 3.1.8 to show that this leads to a representation of Isom\(^+\)(\(\mathfrak{H}^3\)) in the form of \(PB^1\) acting on the hyperboloid model within the standard Macfarlane space. That argument can be generalized to give an analogous action by isometries on a normalized Macfarlane space.

We are interested in actions by isometries in arbitrary Macfarlane spaces, but it is instructive to see the details of the normalized case. Firstly, it is
4.2. ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS

used frequently in computations of examples. Secondly, it makes the con-
nection to Hermitian matrices more explicit, which clarifies the relationship
between the Macfarlane approach and the more conventional use of Möbius
transformations in the upper half-space model. Our strategy here is thus to
develop the more general construction by using the normalized version as a
lemma, with explicit details included that are not necessarily needed for the
general case.

Let \( \mathcal{B} \) be a Macfarlane quaternion algebra with Macfarlane space \( \mathcal{M} = \uparrow \mathcal{B} \),
where \( \uparrow \) is the corresponding involution. The goal is to establish the existence
and desired properties of the map

\[
\mu_{\mathcal{B}} : \mathbb{P}\mathcal{B}^1 \times \mathcal{M}_+^1 \rightarrow \mathcal{M}_+^1, \quad (q, p) \mapsto qp\uparrow,
\]

(4.2.1)

so that we have \( \mu_{\left( \frac{1}{1} \right)} = \mu \) as given by (Theorem 3.2.18). By Theorem 4.1.10,
the symbol \( \mu_{\mathcal{B}} \) describes this action entirely because \( \mathcal{M} \) is unique in \( \mathcal{B} \).

Define

\[
\Phi_{\mathcal{B}} : \mathbb{P}\mathcal{B}^1 \rightarrow \text{End}_K(\mathcal{M}_+^1), \quad \Phi_{\mathcal{B}}(q)(p) = \mu_{\mathcal{B}}(q, p).
\]

When the context is clear, we will abbreviate \( \Phi_{\mathcal{B}}(q)(p) \) by writing \( q(p) \).

Lemma 4.2.2. Let \( \mathcal{B} \) be a normalized Macfarlane quaternion algebra where
\( \mathcal{M} \subset \mathcal{B} \) is the Macfarlane space. Then \( \mathbb{P}\mathcal{B}^1 \cong \text{Isom}^+(\mathcal{M}_+^1) \), and this isomor-
phantom is realized by $\Phi_B$.

Proof. $\mathcal{M}$ is normalized, therefore $\exists F \subset \mathbb{R}$ and $\exists a, b, d \in F^+$ such that $B = \left( \begin{smallmatrix} a & b \\ \sqrt{-d} & c \end{smallmatrix} \right)$, where $K = F(\sqrt{-d})$. Then by Proposition 3.2.9 there is an injective $K$-algebra homomorphism

$$\rho_B : \mathcal{B} \to M_2(K(\sqrt{a}, \sqrt{b})),$$

$$w + xi + yj + zi \mapsto \begin{pmatrix} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{pmatrix}$$

where $w, x, y, z$ are variables in $K$.

Again since $\mathcal{M}$ is normalized, $\mathcal{M} = F \oplus Fi \oplus Fj \oplus F\sqrt{-dij}$ and $a, b, d > 0$, thus

$$\rho_B(\mathcal{M}) = \left\{ \begin{pmatrix} w + x\sqrt{a} & y\sqrt{b} + z\sqrt{-abcd} \\ y\sqrt{b} + z\sqrt{ab} & w - x\sqrt{a} \end{pmatrix} \middle| w, x, y, z \in F \right\}$$

$$= \text{Herm}_2(F(\sqrt{a}, \sqrt{b}, \sqrt{d})).$$

Now let $n$ be the norm on $\mathcal{B}$ as usual, let $w, x, y, z$ be variables in $F$, and observe that

$$\forall q = w + xi + yj + z\sqrt{-dij} \in \mathcal{M} :$$

$$n(q) = w^2 - ax^2 - by^2 - abdz^2 = \det (\rho_B(q)).$$

Thus $\rho_B$ is a quadratic space isometry, so $(\mathcal{M}, n) \simeq \left( \text{Herm}_2(K(\sqrt{a}, \sqrt{b})), \det \right)$, which implies that $(\mathcal{M}_1^+, n) \simeq \left( \text{PSHerm}_2(K(\sqrt{a}, \sqrt{b})), \det \right)$. 
4.2. ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS

Therefore \( \forall (p, q) \in PB^1 \times M^1_+ : \rho(p)\rho(q)^\dagger = \rho^{-1}(\rho(p)\rho(q)^\dagger) \), so by a similar argument to the proof of Theorem 3.1.8, the action given in the statement of the Lemma is as desired. \( \square \)

**Theorem 4.2.3.** Let \( \mathcal{B} \) be any Macfarlane quaternion algebra over \( K \), and let \( \dagger \) be the involution so that \( \mathcal{M} = \mathcal{B} \) is the Macfarlane space. Then \( PB^1 \cong \text{Isom}^+(M^1_+) \) and this isomorphism is achieved by \( \Phi_B \).

**Proof.** By Theorem 4.1.10, there exists a normalized Macfarlane quaternion algebra \( \mathcal{B}' \) with normalized Macfarlane space \( \mathcal{N} = \mathcal{B}' \) where \( \bullet \) is the corresponding involution on \( \mathcal{B}' \), and there exists a \( K \)-algebra isomorphism \( \psi \) which restricts to a quadratic space isometry as follows.

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\psi} & \mathcal{B}' \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{M} & \xrightarrow{\psi|_\mathcal{M}} & \mathcal{N}
\end{array}
\]

By Lemma 4.2.2, \( \mu_{\mathcal{B}'} \) defines a group action of \( PB^1 \) on \( \mathcal{N} \) which gives an isomorphism \( PB^1 \cong \text{Isom}^+(\mathcal{N}^1_+) \). By applying \( \psi \) to each element in that action we get \( PB^1 \cong \text{Isom}^+(M^1_+) \), given by the group action

\[
PB^1 \times M^1_+ \to M^1_+: \ (q, p) \mapsto qp\psi(\psi^{-1}(q)^\bullet).
\]

Now, \( q \mapsto \psi(\psi^{-1}(q)^\bullet) \) and \( q \mapsto q^\dagger \) are involutions of the second kind on \( \mathcal{B} \),
both of whose sets of symmetric elements are Macfarlane spaces. By Theorem 4.1.10 these two Macfarlane spaces are the same, therefore $\psi(\psi^{-1}(q)^*) = q^t$.

Remark 4.2.4. When $\mathcal{B}$ is split, i.e. $\mathcal{B} \cong M_2(K)$, the isometry group $P\mathcal{B}^1$ is isomorphic to $M_2(K)$. When $\mathcal{B}$ is ramified, $P\mathcal{B}^1$ is isomorphic to a proper (division) subalgebra of $M_2(K(\sqrt{a},\sqrt{b}))$.

4.2.2 Isometries as Points

Let $\mathcal{B} = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ be a normalized Macfarlane quaternion algebra, i.e. $K = F(\sqrt{-d})$ where $F \subset \mathbb{R}$ and $a, b, d \in F^+$. Let $\mathcal{M} = \mathcal{B}^\dagger$ be its Macfarlane space, where $\dagger$ is the corresponding involution. We develop techniques for understanding isometries in $P\mathcal{B}^1$ of $\mathcal{M}^1_+$, in terms of their properties as quaternions.

Fixed Points

Recall the discussion from §2.1.3. Every isometry of $\mathcal{H}^3$ has one or two fixed points on the boundary according to whether or not it is parabolic. A non-parabolic isometry has a unique axis of translation or rotation, which is determined by this pair of fixed points. This isometry translates, translates and rotates, or just rotates about this axis depending on whether it is hyperbolic, purely loxodromic or elliptic, respectively. [5]

For $q \in P\mathcal{B}$, its pure quaternion part $q_0$ gives the location of the fixed
4.2. ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS

point(s) of \( q \). The upper half-space model \( \mathcal{H}^3 \) is useful for working with points on the boundary, as \( \partial \mathcal{H}^3 = \mathring{\mathbb{C}} \), so we employ Theorem 3.2.26 to combine the quaternion notation with the conventional approach.

In the proof below we use the fact that every quaternion algebra over a number field can be thought of as embedded in \( \left( \frac{1}{1} \right) \). This enables us to avoid working with \( \sqrt{a}, \sqrt{b} \), as well as the step of converting \( (\mathcal{M} \otimes \mathbb{R})^1 \) to the standard model because using the embedding in \( \left( \frac{1}{1} \right) \) cancels out these two steps. Recall the quaternionic Möbius transformation

\[
\alpha : \text{PSL}_2(\mathbb{C}) \times \mathcal{H}^3 \rightarrow \mathcal{H}^3, \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, p \right) \mapsto (ap + b)(cp + d)^{-1}
\]

from Definition 3.2.21, where \( \mathcal{H}^3 = \mathbb{R} \oplus i \mathbb{R} \oplus R^+ j \subset \mathbb{H} \), and \( \mathbb{C} \subset \text{PSL}_2(\mathbb{C}) \) is identified with \( \mathbb{C} \subset \partial \mathcal{H}^3 \).

**Theorem 4.2.5.** Let \( \mathcal{B} = \left( \frac{1}{1} \right) \) and \( q = w + xi + yj + zij \in \mathcal{PB}^1 \), where \( w, x, y, z \in K \). The fixed point(s) of \( \rho_\mathcal{B}(q) \) in \( \mathring{\mathbb{C}} = \partial \mathcal{H}^3 \) are \( \frac{-x \pm \sqrt{-n(q_0)}}{y + z} \).

**Proof.** Let \( c \in \mathring{\mathbb{C}} = \partial \mathcal{H}^3 \) satisfy \( \alpha(\rho_\mathcal{B}(q), c) = c \)

\[
\Rightarrow \rho_\mathcal{B}(q) = \begin{pmatrix} w - x & y - z \\ y + z & w + x \end{pmatrix} \\
\Rightarrow \frac{(w - x)c + y - z}{(y + z)c + w + x} = c \\
\Rightarrow (y + z)c^2 + 2xc - y + z = 0.
\]
Applying the quadratic formula yields
\[ c = \frac{-x \pm \sqrt{x^2 + y^2 - z^2}}{y + z}. \]

Lastly, \( q_0 = xi + yz + zij \) thus \( n(q_0) = -x^2 - y^2 + z^2. \)

**Corollary 4.2.6.** Let \( q = w + xi + yj + zij \in P\left(\frac{1}{\ell^2}\right) \) be a parabolic isometry. Then the unique fixed point of \( \rho_B(q) \) is \( \frac{-x}{y + z} \in \hat{C} = \partial H^3. \)

**Proof.** Since \( q \) is parabolic, \( 2w = \text{tr}(q) = \pm 2 \) so \( w = \pm 1. \) Since \( q \in PA^1 \) we have \( 1 = n(q) = 1 - x^2 - y^2 + z^2, \) thus \( x^2 + y^2 - z^2 = 0. \)

**Pure Quaternions and Geodesics**

The pure quaternions in \( M \) are the elements of \( M_0 = Fi \oplus Fj \oplus \sqrt{-d} FiFj. \)

These comprise the 3-dimensional coordinate space lying beneath our hyperboloid model \( M^1_+ \), orthogonal to the scalar axis which passes through the center of the hyperboloid. There is a natural one-to-one correspondence

\[ M^1_+ \leftrightarrow M_0 : m \leftrightarrow m_0 \]

given by orthogonal projection to \( M_0. \)

The complete geodesic \( \tilde{g}(p, q) \) through two points \( p, q \in M^1_+ \) is formed by the intersection of the hyperboloid with the 2-dimensional plane containing \( p, q \) and 0. This gives a convenient way of parametrizing geodesics on \( M^1_+ \).
that pass through 1, as illustrated in Figure 4.1. Let $\mathbb{P}^2(F) := F^3/\sim$ where $(x, y, z) \sim (x', y', z') \iff \exists \lambda \in F^\times$ such that $(x, y, z) = \lambda (x', y', z')$, i.e. $\mathbb{P}^2(F)$ is the 2-dimensional projective space over $F$. Denote an equivalence class in this space by $[x : y : z]$.

**Proposition 4.2.7.** The set of geodesics on $\mathcal{M}_1^+$ passing through 1 is in one-to-one correspondence with the projective space $\mathbb{P}^2(F)$.

**Proof.** A geodesic on $\mathcal{M}_1^+$ passing through 1 can be parametrized by

$$\tilde{g}(1, p) = \{ q \in \mathcal{M}_1^+ \mid \exists \lambda \in F : p_0 = \lambda q_0 \}. \quad (4.2.8)$$

In this notation, $\tilde{g}(1, p) = \tilde{g}(1, q) \iff \exists \lambda \in F$ such that $p_0 = \lambda q_0$. Since $\mathcal{M}_0 = Fi \oplus Fj \oplus F\sqrt{-dij}$, there is a natural map to $F^3$ followed by a
projection, as follows.

\[ \mathcal{M}_0 \to F^3 \to \mathbb{P}^2(F) : \quad xi + yj + z\sqrt{-d}ij \mapsto (x, y, z) \mapsto [x : y : z]. \]

Under this composition we have that \( p_0 \sim q_0 \iff \tilde{g}(1, p) = \tilde{g}(1, q). \)

**Remark 4.2.9.** For \( p_0 = xi + yj + z\sqrt{-d}ij \), write \([p_0] := [x : y : z]\). This will be used in §5.

**The Boundary of a Quaternion Hyperboloid Model**

\( \partial \mathcal{M}_+^1 \) identifies in a natural way with projective rays through 0 on the light-cone \( \mathcal{M}^0 := \{ p \in \mathcal{M} \mid n(p) = 0 \} \). In the standard hyperboloid model, this concept was used in the Epstein-Penner decomposition [19]. Interestingly, in the case where \( \mathcal{B} \) is a division algebra, \( \mathcal{M}^0 = \emptyset \) which is suggestive of the fact that a manifold \( X \) with \( BX = \mathcal{B} \) cannot have cusps.

**Points on the Hyperboloid as Isometries**

Since \( \mathcal{M}_+^1 \subset \mathcal{B}^1 \), the hyperboloid can be thought of as a collection of isometries as well as points.

Recall the comparison between the Macfarlane model and Hamilton’s classical result as phrased in Theorem 3.2.17. Elaborating further on that, since \( \mathbb{H}_0^1 \subset \mathbb{H}^1 \), Hamilton’s action similarly defines an action of \( S^2 \) on itself. In Hamilton’s model, a point \( q \) on \( \mathbb{H}_0^1 \cong S^2 \) acts on \( \mathbb{H}_0^1 \) as a rotation by \( \frac{\pi}{2} \).
4.2. ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS

about the axis $q_0$ [26]. We can similarly investigate the behavior of a point on the 3-dimensional hyperboloid $\mathcal{M}_+^1$ as an isometry of $\mathcal{M}_+^1$.

Theorem 4.2.10. Let $p \in \mathcal{M}_+^1$.

(1) If $p = 1$ then $p$ is the identity isometry.

(2) If $p \neq 1$, then $p$ acts on $\mathcal{M}_+^1$ as a hyperbolic isometry.

(3) If $w$ is the scalar part of $p \neq 1$, then $\Phi(p)$ is a translation by $2\text{arcosh}(w)$ along the geodesic that passes through 1 and $p$, in the direction of $p_0$.

Proof. (1) is immediate, since $\forall p \in \mathcal{M}_+^1 : \mu(1, p) = 1p1^\dagger = 1$.

Since $p \in \mathcal{M}_+^1$, we have $p = w + xi + yj + z\sqrt{-1}dj$ for some $w, x, y, z \in F$. Since $n(p) = 1$, we have $w^2 = 1 + ax^2 + by^2 + abdz^2 \geq 1$. Since $a, b, d > 0$, the only way to get $w = \pm 1$ is if $x = y = z = 0$. Otherwise $|w| > 1$, so $\text{tr}(p) = 2w \in \mathbb{R} \setminus [-2, 2]$, proving (2).

Since $p$ is a hyperbolic isometry, there is a unique geodesic in $\mathcal{M}_+^1$ that is invariant under the action of $p$, which $p$ translates along. As given by (4.2.8), the geodesic passing through 1 and $p$ is the collection of points

$$g := \{ q \in \mathcal{M}_+^1 \mid \exists \lambda \in F : q_0 = \lambda p_0 \}.$$ 

Let $q \in g$, so then $\exists \lambda \in F$ such that $q = \frac{\text{tr}(q)}{2} + \lambda p_0$. Let $\dagger$ be the involution
CHAPTER 4. MACFARLANE SPACES


giving $\mathcal{M} = \mathcal{B}$. Then since $p \in \mathcal{M}_1^+$ we know that $p^\dagger = p$ and we have

$$\mu_B(p, q) = pqp^\dagger = pqp = \left( \frac{\text{tr}(p)}{2} + p_0 \right) \left( \frac{\text{tr}(p)}{2} + \lambda p_0 \right) \left( \frac{\text{tr}(p)}{2} + p_0 \right).$$

If we multiply this out, there will be scalars $r, s, t, u \in F$ so that the expression has the form

$$\mu_B(p, q) = r + sp_0 + tp_0^2 + up_0^3 = (r + tp_0^2) + (s + up_0^2)p_0.$$

Next, observe that since $p_0^* = -p_0$, we have $p_0^2 = -p_0p_0^* = -n(p_0)$. Since $p \in \mathcal{M}_1^+$, we have $p_0^2 = -n(p_0) \in F$, thus $r + tp_0^2 \in F$ and $s + up_0^2 \in F$. Then $\mu(p, q)_0 = (s + up_0^2)p_0$ is a real multiple of $p_0$ and so $\mu(p, q)$ is of a form that makes it a point on $g$. Therefore $g$ is invariant under the action of $p$, as desired.

Lastly, since $p$ is a loxodromic isometry, by Proposition 2.1.18 its translation length is $2\text{arcosh}\left(\frac{\text{tr}(p)}{2}\right) = 2\text{arcosh}\left(\frac{2w}{2}\right) = 2\text{arcosh}(w)$. □

Remark 4.2.11. There is an important distinction between what happens here and what happens in Hamilton’s model.

1. In Hamilton’s model, the action of $\mathbb{H}_0^1$ on itself is closed under multiplication, whereas the action of $\mathcal{M}_1^+$ on itself is not.

2. The action of $\mathbb{H}_0^1$ on itself includes a rotation about every possible axis.
4.2. **ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS**

\(\mathbb{H}_0^1\) might not include a rotation by every possible angle, but the isometry group \(\text{Isom}^+(\mathbb{H}_0^1) \cong \text{Isom}^+(S^2)\) consists only of rotations, so in a sense every type of isometry is represented. The action of \(\mathcal{M}_+^1\) on the other hand leaves out entire classes of isometries from \(\text{Isom}^+(\mathcal{M}_+^1)\): it does not include any elliptic, parabolic, or purely loxodromic elements. Focusing on the hyperbolic isometries, we do attain every possible translation length but we are limited to translations whose axes pass through the point 1 on the hyperboloid.

**The Anti-Hermitian Complement of \(\mathcal{M}\)**

To understand isometries of \(\mathcal{M}_+^1\) more generally in terms of their properties as quaternions, we look at points in \(\mathcal{B}^1\) not lying in \(\mathcal{M}\).

**Definition 4.2.12.** The **anti-Hermitian complement of \(\mathcal{M}\)**, denoted by \(\mathcal{W}\), is the 4-dimensional \(\mathbb{F}\)-space

\[
\sqrt{-d}\mathcal{M} = F\sqrt{-d} \oplus F\sqrt{-d} i \oplus F\sqrt{-d} j \oplus Fij \subset \mathcal{B}.
\]

A matrix \(s \in M_2(\mathbb{C})\) is **anti-Hermitian** (or **Skew-Hermitian**) if \(\overline{s}^T = -s\). Just as \(\mathcal{M}\) corresponds to the Hermitian matrices under \(\rho_\mathcal{B}\), its anti-Hermitian complement \(\mathcal{W}\) corresponds to the anti-Hermitian matrices under \(\rho_\mathcal{B}\), by an argument similar to that given in Lemma 3.2.19. Equivalently, \(\mathcal{W}\)
can be characterized as the set \( \{ q \in B \mid q^\dagger = -q \} \).

**Proposition 4.2.13.** \( B = M \oplus W \).

**Proof.** Just as in Proposition 1.2.39, for each \( q \in B \)

\[
q = \frac{q + q^\dagger}{2} + \frac{q - q^\dagger}{2}
\]

where \( \frac{q + q^\dagger}{2} \in M \) and \( \frac{q - q^\dagger}{2} \in W \).

To see this more explicitly, notice that any element \( q \in B \) has the form

\[
q = q_1 + q_2 i + q_3 j + q_4 i j
\]

where for each \( q_\ell \) there is a unique \( r_\ell, s_\ell \in F \) such that \( q_\ell = r_\ell + s_\ell \sqrt{-d} \). We then have

\[
q = (r_1 + r_2 i + r_3 j + s_4 \sqrt{-d}ij) + (\sqrt{-d}s_1 + \sqrt{-d}s_2 i + \sqrt{-d}s_3 j + r_4 ij).
\]

The first summand is in \( M \), the second is in \( W \), and the decomposition is unique. \( \square \)

The portion of \( W \) which, up to sign, contributes to the group of isometries \( P B^1 \) is \( W^1 = \{ q \in W \mid n(q) = 1 \} \). This set has an interesting geometric structure which can be used to describe the behavior of its elements as isometries of \( M^1 \).

**Proposition 4.2.14.**

1. \( W^1 \) is a hyperboloid of one sheet and \( W_0^1 \) is an ellipsoid.
4.2. ISOMETRIES IN MACFARLANE QUATERNION ALGEBRAS 117

(2) If \( p \in \mathcal{W}^1 \setminus \mathcal{W}^1_0 \), then \( \Phi_B(p) \) is a purely loxodromic isometry with axis 
\( \tilde{g}(1,-p^2) \), rotation angle \( \frac{\pi}{2} \), and translation length \( d_{\mathcal{M}^1}(1,-p^2) \).

Proof. Let \( p \in \mathcal{W}_1 \). Then \( p = w\sqrt{-d} + x\sqrt{-d}i + y\sqrt{-d}j + zj \) and then 
\[ 1 = n(p) = -dw^2 + ax^2 + bdy^2 + abz^2. \]
Since \( a, b, d \in F^+ \), the collection of such points forms a hyperboloid of one sheet. Eliminating the first coordinate gives the ellipsoid
\[ \{adx^2 + bdy^2 + abz^2 \mid x, y, z \in F \} \]
of pure quaternions.

If \( \text{tr}(p) \neq 0 \), then \( \text{tr}(p) = 2w\sqrt{-d} \) where \( w \in F \subset \mathbb{R} \), making \( p \) purely loxodromic. Taking \( \text{arcosh} \) of a purely imaginary number yields a number of the form \( r + \frac{\pi}{2} \sqrt{-1} \), which means that the rotation angle of \( p \) is \( \frac{\pi}{2} \). We find the translation length by using that \( p^\dagger = -p \) and computing \( \mu_B(p,1) = pp^\dagger = -p^2 \). We now show that the axis of \( p \) is \( \tilde{g}(1,-p^2) \). Let \( q \in \tilde{g}(1,-p^2) \). Then using (4.2.8), \( \exists \lambda \in \mathbb{R} \) such that 
\[ q = \frac{\text{tr}(q)}{2} + (-p^2)_0 = \frac{\text{tr}(q)}{2} - (p^2)_0. \]
(Note that \((p^2)_0 \neq (p_0)^2\).) Again using \(p^\dagger = p\), we compute

\[
\mu_B(p, q) = p \left( \frac{\text{tr}(q)}{2} - (p^2)_0 \right) p^\dagger \\
= -p \left( \frac{\text{tr}(q)}{2} - (p^2)_0 \right) p \\
= \frac{\text{tr}(q)}{2}(-p^2) + (p^2)_0 p^2.
\]

Thinking of these two summands as Euclidean vectors, the first is co-linear with the vector pointing to \(-p^2 \in \mathcal{M}_{\pi}^1\). If we can show that adding the second summand results in a point on \(\mathcal{g}(-p^2)\) we will be done. We do this by proving that the pure quaternion part of \((p^2)_0 p^2\) is parallel to the pure quaternion part of \(-p^2\). Since \(p^2 \in \mathcal{M}\), we have \(p^2 = r + si + yj + z\sqrt{-d}ij\) for some \(r, s, t, u \in F\). Then

\[
((p^2)_0 p^2)_0 = ((si + yj + z\sqrt{-d}ij)(r + si + yj + z\sqrt{-d}ij))_0 \\
= (rs - byz\sqrt{-d} + byz\sqrt{-d})i + (asz\sqrt{-d} + ry - asz\sqrt{-d})j \\
+ (sy - sy + rz\sqrt{-d})ij \\
= rsi + ryz + rz\sqrt{-d}ij \\
= r(si + yj + z\sqrt{-d}) \\
= r(p^2)_0.
\]

\[\square\]

**Remark 4.2.15.** Just as there are hyperbolic isometries not lying on \(\mathcal{M}_{\pi}^1\), there are...
there are purely loxodromic isometries not lying on \( W^1 \), as well as many
other isometries in \( PB^1 \).

For arbitrary isometries in \( PB^1 \) we can still compute information about
them using the decomposition of \( B \) into \( M \oplus W \) but, as suggested by the
following Lemma, this will entail working with elements of \( M \) and \( W \) that
are not isometries.

Define the following lifts of prior maps similarly to in §3.1 (where here
the \( \sim \) indicates a broader extension of \( \mu \) than it did of \( \Lambda \)).

\[
\tilde{\mu}_B : B \times M \to M, \quad (q, p) \mapsto qpq^1
\]

(4.2.16)

\[
\tilde{\Phi}_B : B \to \text{End}_K(B), \quad q \mapsto \tilde{\mu}_B(q, \cdot)
\]

\textbf{Lemma 4.2.17.} If \( q = m + w \in B^1 \), where \( m \in M \) and \( w \in W \), then exactly
one of the following is true:

\( (1) \) \( q = m \), or

\( (2) \) \( q = w \), or

\( (3) \) \( m, w \notin B^1 \).

\textit{Proof.} Let \( q = m + w \) as in the statement. Since \( M \cap W = 0 \notin A^1 \), the first
two possibilities are mutually exclusive. So suppose neither of those are true.
Suppose by way of contradiction that \( n(m) = 1 \). Since \( \rho_B(m) \) is Hermitian, up to conjugation by a unitary matrix, it has the form \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) for some \( \lambda \in \mathbb{C} \). Since \( \rho(w) \) is anti-Hermitian, under the same conjugation it has the form \( \begin{pmatrix} r\sqrt{-1} & c \\ -\bar{c} & s\sqrt{-1} \end{pmatrix} \) for some \( c \in \mathbb{C} \) and \( r, s \in \mathbb{R} \). Now write the determinate of this conjugation of \( \rho_B(q) \) (which equals 1 since \( n(q) = 1 \)) in terms of \( \lambda, c, r, s \).

\[
1 = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} + \begin{pmatrix} r\sqrt{-1} & c \\ -\bar{c} & s\sqrt{-1} \end{pmatrix} \right| \\
= \left| \lambda + r\sqrt{-1} & c \\ -\bar{c} & \lambda^{-1} + s\sqrt{-1} \right| \\
= (\lambda + r\sqrt{-1})(\lambda^{-1} + s\sqrt{-1}) + \bar{c}c \\
= 1 - rs + |c|^2 + (\lambda s + \lambda^{-1}r)\sqrt{-1}
\]

Taking the real and imaginary parts gives the following.

\[
|c|^2 - rs = 0 \quad \lambda s + \lambda^{-1}r = 0
\]

By the equation on the right, \( r = -\lambda^2 s \), then via the equation on the left, \( |c| + (\lambda s)^2 = 0 \). By the assumption that \( n(m) = 1 \), we know that \( \lambda \neq 0 \), thus \( c = s = 0 \), and then \( r = \lambda^2 \cdot 0 = 0 \), which means that \( \rho_B(w) \) is the zero matrix, since this property is invariant of conjugation. Therefore \( q = m + 0 = m \in \mathcal{M}^1 \), contradicting the assumption.

A similar argument gives \( n(w) \neq 1 \). \qed
Remark 4.2.18. Thanks to Alex Zorn for pointing out the property of Hermitian matrices to use for this proof.

For \( p, q \in \mathcal{B} \), their commutator is defined as \([p, q] := pq - qp\).

**Proposition 4.2.19.** If \( q = m + w \in \mathcal{PB}^1 \) where \( m \in \mathcal{M} \) and \( w \in \mathcal{W} \), then \([m, w] \in \mathcal{M}_0 \) and

\[
\mu_\mathcal{B}(q, 1) = \tilde{\mu}_\mathcal{B}(m, 1) + \tilde{\mu}_\mathcal{B}(w, 1) - [m, w].
\]

**Proof.** Since \( \text{tr}(mw) = \text{tr}(wm) \), we have \([m, w] \in \mathcal{B}_0 \) (this holds for any pair of quaternions). Since

\[
[m, w]^\dagger = (mw - wm)^\dagger = (mw)^\dagger - (wm)^\dagger = w^\dagger m^\dagger - m^\dagger w^\dagger
\]

\[
= -wm + mw = mw - wm = [m, w].
\]

we have \([m, w] \in \mathcal{M} \). Then \([m, w] \in \mathcal{B}_0 \cap \mathcal{M} = \mathcal{M}_0 \).

Also,

\[
\mu_\mathcal{B}(q, 1) = \tilde{\mu}_\mathcal{B}(m + w, 1)
\]

\[
= (m + w)(m + w)^\dagger = (m + w)(m^\dagger + w^\dagger)
\]

\[
= (m + w)(m - w) = m^2 - mw + wm - w^2
\]

\[
= mm^\dagger + ww^\dagger - (mw - wm)
\]

\[
= \tilde{\mu}_\mathcal{B}(m, 1) + \tilde{\mu}_\mathcal{B}(w, 1) - [m, w]. \quad \square
\]
Corollary 4.2.20. \( \text{tr}(\mu_B(q, 1)) = \text{tr}(\tilde{\mu}_B(m, 1) + \tilde{\mu}_B(w, 1)) \).

Proof. \( \text{tr}([m, w]) = 0. \) \( \square \)

4.3 Examples of Macfarlane 3-Manifolds

In this section we find infinite families of finite-volume hyperbolic 3-manifolds whose quaternion algebras are Macfarlane. These include both compact and non-compact families and within each of these, infinitely many commensurability classes of both arithmetic and non-arithmetic examples.

Let \( X \) denote a complete, finite-volume hyperbolic 3-manifold and let \( \Gamma \cong \pi_1(X) \) be a Kleinian group. Recall Definition 2.2.1 of \( B\Gamma \), the quaternion algebra of \( \Gamma \). Let \( K = K\Gamma \) and \( B = B\Gamma \). If \( B \) is Macfarlane then \( K = F(\sqrt{-d}) \) for some number field \( F \subset \mathbb{R} \) and \( d \in F^+ \). In this case let \( \mathcal{M} \subset B \) be the Macfarlane space, with corresponding involution \( \dagger \).

Definition 4.3.1. When \( B\Gamma \) is Macfarlane, we call \( X \) a Macfarlane manifold, we call \( \Gamma \) a Macfarlane group, and we define the (quaternionic) hyperboloid model for \( \Gamma \) as \( \mathcal{I}^\Gamma := \mathcal{M}^1_+ \).

Remark 4.3.2. While the property of being Macfarlane is a manifold invariant (by Corollary 4.1.14), \( X \) does not determine a specific \( B\Gamma \), since \( B\Gamma \) is only a manifold invariant up to \( K\Gamma \)-algebra isomorphism. But when our
4.3. EXAMPLES OF MACFARLANE 3-MANIFOLDS

choice of \(\Gamma\) is clear we will also write \(BX := B\Gamma\) and \(\mathcal{I}^X := \mathcal{I}^\Gamma\).

Recall from (4.2.1) the map
\[
\mu_B : P\mathcal{B}^1 \times \mathcal{M}_+^1 \rightarrow \mathcal{M}_+^1, \quad (q, p) \mapsto qp^\dagger.
\]

**Theorem 4.3.3.** If \(\Gamma\) is Macfarlane, then the action of \(\Gamma\) by isometries of \(\mathfrak{H}^3\) is faithfully represented by the restriction
\[
\mu_B|_{\Gamma \times \mathcal{H}^\Gamma} : \Gamma \times \mathcal{H}^\Gamma \rightarrow \mathcal{H}^\Gamma, \quad (\gamma, p) \mapsto \gamma p^\dagger.
\]

**Proof.** \(\mu_B\) gives an action of \(P(B\Gamma)^1\) on \(\mathcal{M}_+^1 = \mathcal{I}^\Gamma\). Recall from the beginning of §2.2 the notation \(P^{-1}(\Gamma) := \{ \pm \gamma \mid \{\pm\gamma\} \in \Gamma\}\). By definition of \(B\Gamma\), \(P^{-1}(\Gamma) \subset B\Gamma\). Then \(\Gamma \subset P(B\Gamma) \subset P(B\Gamma)^1\), so \(\mu_B\) defines an action of \(\Gamma\) which, by Theorem 4.2.3, is its action by isometries of \(\mathcal{I}^\Gamma\). By Corollary 4.1.6 this gives the action of \(\Gamma\) by isometries of \(\mathfrak{H}^3\). \(\square\)

**Corollary 4.3.4.** If \(X\) is a Macfarlane manifold, then there exists a unique normalized Macfarlane space \(\mathcal{M}\) over \(F\), such that \(X \approx (\mathcal{M} \otimes_F \mathbb{R})_+^1/\Gamma\).

**Proof.** This follows from Theorem 4.1.10 and the Mostow-Prasad Rigidity Theorem. \(\square\)
4.3.1 Non-compact Macfarlane Manifolds

Proposition 4.3.5. If $KX = F(\sqrt{-d})$ for some $F \subset \mathbb{R}$ and $d \in F^+$, and $X$ is non-compact, then $X$ is Macfarlane.

Proof. Let $KX$ be as in the Theorem. If $X$ is non-compact, then by Theorem 2.2.12, $BX \cong \left(\frac{1}{KX}\right)$, and then by Theorem 4.1.10, $X$ is Macfarlane. \hfill \Box

Corollary 4.3.6. If $X$ is arithmetic and non-compact, then $X$ is Macfarlane.

Proof. When $X$ is arithmetic and non-compact, $BX \cong \left(\frac{1}{\mathbb{Q}(\sqrt{-d})}\right)$ for some $d \in \mathbb{N}$, by Theorem 2.3.8(2). \hfill \Box

Example 4.3.7.

1. The figure-8 knot complement is Macfarlane, with trace field $\mathbb{Q}(\sqrt{-3})$.

2. Arithmetic link complements are Macfarlane, for instance

   (a) the Whitehead link has trace field $\mathbb{Q}(\sqrt{-1})$,
   
   (b) the arithmetic link shown in Figure 4.2 has trace field $\mathbb{Q}(\sqrt{-11})$,
   
   (c) and the six-component chain link has trace field $\mathbb{Q}(\sqrt{-15})$.

However, there are only finitely many $d \in \mathbb{N}$ such that some arithmetic link has trace field $\mathbb{Q}(\sqrt{-d})$. \cite{37}. 
There are many arithmetic non-compact examples besides knot and link complements, for instance certain punctured torus bundles, and certain face pairings on ideal tetrahedra [37].

One can also find many examples in the census of non-arithmetic non-compact examples meeting our conditions.

**Example 4.3.8.** In [8], an infinite class of non-commensurable link complements are generated all having trace field \( \mathbb{Q}(\sqrt{-1}, \sqrt{2}) \), by gluing along totally geodesic 4-punctured spheres. This field satisfies our conditions (take \( F = \mathbb{Q}(\sqrt{2}) \) and \( d = 1 \)), and since it is not of the form \( \mathbb{Q}(\sqrt{-d}) \) with \( d \in \mathbb{N} \), by Theorem 2.3.8 these link complements are non-arithmetic.

### 4.3.2 Compact Macfarlane Manifolds

In contrast to the non-compact case, not every compact arithmetic manifold is Macfarlane. For instance the Weeks manifold is compact and arithmetic but has a cubic trace field. Yet there is still a wide ranging class of compact
arithmetic Macfarlane manifolds.

**Proposition 4.3.9.** For every square-free $d \in \mathbb{N}$, there are infinitely non-commensurable compact arithmetic Macfarlane manifolds having the trace field $\mathbb{Q}(\sqrt{-d})$.

*Proof.* With $d$ as in the statement, let $\mathcal{B} = \left( \frac{a,b}{\mathbb{Q}(\sqrt{-d})} \right)$ with $a, b \in \mathbb{Q}^+$. First observe that for each choice of $d$, we can attain infinitely many non-isomorphic $\mathcal{B}$ by varying our choices of $a, b$ because there are infinitely many non-similar quadratic forms $-ax^2 - by^2 + abz^2 \in \mathbb{Q}(\sqrt{-d})[x, y, z]$ with $a, b \in \mathbb{Q}^+$ (see Theorem 1.2.25). Since only one of these isomorphism classes is split, there are infinitely many such $\mathcal{B}$ that are ramified. Assume from here on that $\mathcal{B}$ is ramified.

Since $\mathbb{Q}(\sqrt{-d})$ has a unique complex place and no real places, any group $\Gamma$ derived from $\mathcal{B}$ (in the sense of Definition 2.3.4) will be arithmetic. Since $\mathcal{B}$ is ramified, this $\Gamma$ will be cocompact, and since $a, b, d \in \mathbb{Q}^+$, it will be Macfarlane. Also, when $\Gamma$ is derived from $\mathcal{B}$, we have $A\Gamma = B\Gamma = \mathcal{B}$ (see Theorem 2.3.10) so that $\mathcal{B}$ determines the commensurability class of $\Gamma$. Therefore if we can construct an arithmetic Macfarlane manifold $X$ having $BX = \left( \frac{a,b}{\mathbb{Q}(\sqrt{-d})} \right)$ for each choice of $a, b \in \mathbb{Q}^+$, we will obtain an infinite family of non-commensurable Macfarlane manifolds all having the trace field $\mathbb{Q}(\sqrt{-d})$. 

4.3. EXAMPLES OF MACFARLANE 3-MANIFOLDS

Let \( d \in \mathbb{N} \) and \( a, b \in \mathbb{Q}^+ \) be arbitrary. Let \( \mathcal{O} = \mathcal{O}_d \oplus \mathcal{O}_d i \oplus \mathcal{O}_d j \oplus \mathcal{O}_d ij \).
Then \( \mathcal{O} \) is an order and its coefficients are algebraic integers. Since \( \mathcal{B} \) has a unique complex place and no real places, \( \mathcal{P}\mathcal{O}^1 \) is discrete and arithmetic by Theorem 2.3.6. By Selberg’s Lemma [42], \( \exists \) torsion-free \( \Gamma < \mathcal{P}\mathcal{O}^1 \) such that \([\mathcal{P}\mathcal{O}^1 : \Gamma] < \infty \). Then \( \mathcal{H}^3/\Gamma \) is a hyperbolic 3-manifold derived from \( \mathcal{B} \), thus \( B\Gamma = \mathcal{B} \).

Before discussing classes of compact non-arithmetic Macfarlane manifolds, we give a geometric condition which can apply to all manifolds regardless of arithmeticity.

**Theorem 4.3.10.** Let \( K\Gamma = F(\sqrt{-d}) \) for some \( F \subset \mathbb{R} \) and \( d \in F^+ \). If \( X \) contains an immersed, closed, totally geodesic surface, then \( X \) is Macfarlane.

**Proof.** Let \( S \subset X \) be a closed, totally geodesic immersion of a surface. Theorem 2.1.6 tells us that \( \pi_1(S) \cong \Delta \) for some \( \Delta < \Gamma \). Without loss of generality we can assume \( \Gamma \) is conjugated so that \( \Delta < \Gamma \). Then \( K\Delta \subset K\Gamma \), and since \( K\Delta \subset \mathbb{R} \) and \( K \cap \mathbb{R} = F \), then \( K\Delta \subset F \). Therefore \( B\Delta \) is a quaternion subalgebra of \( \mathcal{B} \) over a subfield of \( F \). Hence \( \exists a, b \in F \) so that \( B\Delta = \left( \frac{a}{b} \right) \). Then \( B\Delta \otimes_{K\Delta} K\Gamma = \left( \frac{a}{F(\sqrt{-d})} \right) \subset \mathcal{B} \), but since \( \left( \frac{a}{b} \right) \) and \( \mathcal{B} \) are both 4-dimensional vector spaces over \( K\Gamma \), this forces \( \left( \frac{a}{F(\sqrt{-d})} \right) = \mathcal{B} \).
CHAPTER 4. MACFARLANE SPACES

Let \( a' = a \) if \( a > 0 \) and let \( a' = -ad \) otherwise, and define \( b' \) similarly.

Then by Theorem 1.2.11, \( \left( \frac{a', b'}{F(\sqrt{-d})} \right) \cong \left( \frac{a, b}{F(\sqrt{-d})} \right) \) and contains the Macfarlane space \( \mathcal{M} := F \oplus Fi \oplus Fj \oplus F\sqrt{-d}ij \) where \( i^2 = a' \) and \( j^2 = b' \). Since \( B\Gamma \cong \left( \frac{a', b'}{F(\sqrt{-d})} \right) \), it contains a Macfarlane space as well. \( \square \)

**Remark 4.3.11.**

(1) The converse to this holds when \( X \) is arithmetic [36].

(2) A theorem of Chinburg and Reid (1993) [9] implies that if \( X \) is arithmetic and contains a non-simple closed geodesic, then \( \exists a \in KX \cap \mathbb{R} \) so that \( BX \cong \left( \frac{a, b}{KX} \right) \). In view of the theorem above, one can show that if this geodesic lies on a hyperbolic subsurface \( S \) as in the theorem, meaning \( S \) is immersed (but not embedded), then \( \left( \frac{a, b}{KX} \right) \) can be chosen so that \( b \) also lies in \( BX \cap \mathbb{R} \).

Examples of this form can alternatively be realized by starting with a Fuchsian group and introducing matrices with complex quadratic entries.

**Compact Non-arithmetic Examples**

We can also generate infinitely many non-commensurable examples of non-arithmetic, compact, Macfarlane manifolds by using Theorem 4.3.10 and applying the technique of interbreeding introduced by Gromov and Piatetski-
4.3. **EXAMPLES OF MACFARLANE 3-MANIFOLDS**

Shapiro [25]. This entails gluing together a pair of non-commensurable arithmetic manifolds along a pair of totally geodesic and isometric subsurfaces, resulting in a non-arithmetic manifold. We use a variation on this technique introduced by Agol [1] called inbreeding, whereby one glues together a pair of geodesic subsurfaces bounding non-commensurable submanifolds of the same arithmetic manifold.

**Proposition 4.3.12.** For every Macfarlane manifold $X$ containing an immersed, totally geodesic surface, there exist infinitely many commensurability classes of non-arithmetic Macfarlane manifolds having the same quaternion algebra.

*Proof.* Let $X$ be as in the Proposition and $\Gamma = \pi_1(X)$ a Kleinian group. By [37](§9.5), $X$ contains infinitely many non-commensurable, immersed, totally geodesic, arithmetic, hyperbolic subsurfaces. Let $S_1, S_2$ be two of these subsurfaces. Since they are arithmetic they each correspond to a lattice in a quaternion subalgebra over a real subfield of $F$, which we then deform so that $S_1$ and $S_2$ can be glued them together via the identification map $f : S_1 \to S_2$ as in [1]. Then $X_{1,2} := X \ast_f X$ is non-arithmetic, and the reflection involution through the identified subsurface lies in $\text{Comm}(\Gamma)$. So with $\Gamma_{1,2} = \pi_1(X_{1,2})$ we have $K\Gamma = K\Gamma_{1,2}$ and $B\Gamma \cong B\Gamma_{1,2}$.
By the results in [1] there exists an infinite sequence of choices for pairs of surfaces $S_\ell, S_m$ as above, so that the injectivity radius of $X_{\ell,m}$ gets arbitrarily small. But by Theorem 2.3.13 there is a lower bound to the injectivity radius of any class of commensurable non-arithmetic manifolds. Thus, in our sequence $\{X_{\ell,m}\}$, we enter a new commensurability class infinitely often as this radius approaches 0.

4.4 Restricted Macfarlane Spaces and Hyperbolic Surfaces

In this section we develop a theory of restricted Macfarlane spaces for hyperbolic surfaces and Fuchsian groups analogous to the theory of Macfarlane spaces for hyperbolic 3-manifolds and Kleinian groups. We look at quaternion algebras over real fields that contain 2-dimensional hyperboloid models, then we study embeddings of quaternion algebras over real fields into quaternion algebras over complex fields. We define restricted Macfarlane spaces, generalizing Definition 3.3.1, and hyperboloid models for Fuchsian groups. We prove that these are uniquely characterized by a natural algebraic property (Theorem 4.4.9). We also see that the isometric action on the resulting 2-dimensional hyperboloid is the restriction of the isometric action for 3 dimensions (Theorem 4.4.13). Finally we give some geometric results on
4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

isometries as points, following easily from the 3-dimensional case.

4.4.1 Real Quaternion Algebras and $\mathfrak{H}^2$

Let $F \subset \mathbb{R}$ be a field and $\mathcal{A} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$. The conditions for $\mathcal{A}$ to contain a 2-dimensional hyperboloid defined over $F$ are much less restrictive than in the 3-dimensional complex case.

**Proposition 4.4.1.** $\mathcal{A}$ contains a 2-dimensional hyperboloid over $F$ if and only at least one of $a, b$ is positive, i.e. $\mathcal{A} \otimes_F \mathbb{R} \cong M_2(\mathbb{R})$.

**Proof.** The norm $n$ on $\mathcal{A}$ is a real quadratic form, and $S = \{1, i, j, ij\}$ is an orthogonal basis, therefore the signature of $n$ is determined by the signs in the diagonal of $G^S_n$. Define

$$s := (+, \text{sign}(-a), \text{sign}(-b), \text{sign}(ab)),$$

the list of these signs. If $a, b < 0$, then $n$ is positive definite, and there is no congruence over $F$ which will change this. In this case, on any 3-dimensional subspace of $\mathcal{A}$, $n$ will remain positive definite, so that $\mathcal{A}^1$ will be a sphere and not a hyperboloid. This proves the forward direction.

Table 4.1 gives two spaces of signature $(1, 2)$ within $\mathcal{A}$ for each remaining sign combination, proving the converse.
Next we observe that if we embed $A$ into the quaternion algebra $\left(\frac{a,b}{F(\sqrt{-d})}\right)$, then up to isomorphism of the larger algebra, there is no obstruction to it containing a 2-dimensional hyperboloid over $F$. Moreover, every Macfarlane quaternion algebra can be seen as occurring from an extension like this.

**Theorem 4.4.2.** Let $F \subset \mathbb{R}$, let $A$ be a quaternion algebra over $F$, and let $K$ be an imaginary quadratic extension of $F$. Then the following statements hold.

(1) $A \otimes_F K$ is Macfarlane.

(2) $A \otimes_F K$ contains a 2-dimensional hyperboloid model defined over $F$.

(3) Every Macfarlane quaternion algebra arises in this way up to isomorphism.

**Proof.** With $F, K, A$ as in the Proposition, we have $A = \left(\frac{a,b}{F}\right)$ for some $a, b \in F \subset \mathbb{R}$, and $K = (\sqrt{-d})$ for some $d \in F^+$. Then $A \otimes_F K = \left(\frac{a,b}{K}\right)$. If
a < 0 then by Theorem 1.2.11 we may replace it with \(-ad\) without leaving the isomorphism class, and similarly for \(b\). Therefore \(\mathcal{A} \otimes_F K\) is Macfarlane, proving (1), and thus \(\mathcal{A} \otimes_F K\) contains a 3-dimensional hyperboloid model defined over \(F\). Omitting one of the positive-definite coordinates gives (2).

Let \(\mathcal{B}\) be a Macfarlane quaternion algebra. By Theorem 4.1.10, \(\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)\) for some \(F \subset \mathbb{R}\) and \(a,b,d \in F^+\). Lastly, \(\left(\frac{a,b}{F}\right)\) is a quaternion algebra over \(F\) and \(\left(\frac{a,b}{F}\right) \otimes_F K = \left(\frac{a,b}{F(\sqrt{-d})}\right)\), proving (3).

Not every embedding of a quaternion \(F\)-algebra into a quaternion \(F(\sqrt{-d})\)-algebra will be as clean as in the above Proposition, where we worked in a normalized setting. In particular, the subalgebra may have different structure parameters from the larger algebra. This can be seen by considering Albert’s Theorem 1.2.41 in the context of Macfarlane spaces.

Let \(\mathcal{B}\) be a Macfarlane quaternion algebra over the field \(K = F(\sqrt{-d})\) with \(F \subset \mathbb{R}\) and \(d \in F^+\). Then by Theorem 1.2.41, there is a unique quaternion subalgebra \(\mathcal{A} \subset \mathcal{B}\) over \(F\) so that \(\mathcal{A} \otimes_F K = \mathcal{B}\) and for \(q \in \mathcal{A}, w \in K\), we have \((q \otimes w)^\dagger = q^* \otimes \overline{w}\), where \(^*\) is the quaternion conjugate on \(\mathcal{A}\). In particular, this tells us that \(\mathcal{A}\) is the \(F\)-algebra generated by \(\text{Skew}(\mathcal{B}, \dagger)_{\{0\}}\).

We show that the pure quaternion part of this algebra does not intersect the pure quaternion part of the Macfarlane space. This fact will be used
later. We also notice that this subalgebra does not contain a 2-dimensional hyperboloid over $F$.

**Proposition 4.4.3.** Let $\mathcal{B}$ be a Macfarlane quaternion algebra over $F(\sqrt{-d})$, where $F \subset \mathbb{R}, d \in F^+$, and let $\dagger$ be the involution of the second kind providing $\mathcal{M} = \mathcal{B}$. Let $\mathcal{A}$ be the unique quaternion $F$-subalgebra such that $\mathcal{A} \otimes_F K = \mathcal{B}$ and $(q \otimes w)^\dagger = q^* \otimes \overline{w}$ for $q \in \mathcal{A}, w \in K$, where $*$ is the quaternion conjugate on $\mathcal{A}$. Then

(1) $\mathcal{A} \cap \mathcal{M} = F$, and

(2) $n|_\mathcal{A}$ is positive definite.

**Proof.** Let $\mathcal{B} = \left( \begin{array}{cc} a & b \\ \frac{c}{d} & \frac{e}{f} \end{array} \right)$. By Theorem 4.1.10, up to $K$-algebra isomorphism we may assume $a, b \in F^+$ and $\mathcal{M} = F \oplus Fi \oplus Fj \oplus F\sqrt{-dij}$. This gives that $\text{Skew}(Q, \dagger)_0 = F\sqrt{-di} \oplus F\sqrt{-dj} \oplus Fij$, which implies

$$\mathcal{A} = F \oplus F\sqrt{-di} \oplus F\sqrt{-dj} \oplus Fij.$$  

This proves (1).

To see $\mathcal{A}$ as a quaternion $F$-algebra, let $i' = \sqrt{-di}$ and $j' = \sqrt{-dj}$, i.e. $i'^2 = ad > 0, j'^2 = bd > 0$ and $i'j' = -j'i'$, and we have

$$\mathcal{A} = F \oplus F' \oplus F'j' \oplus Fi'j' = \left( \frac{-ad, -bd}{F} \right).$$  

(4.4.4)
4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

If \( n' \) is the norm on \( A \), and \( S = \{1, i', j', i'j'\} \) then \( G'_n^{S} = \text{diag}(1, ad, bd, abd^2) \).

\[ \square \]

4.4.2 Restricted Macfarlane Spaces

We define the lower-dimensional analogue of a Macfarlane space in terms of its relationship to a (complex) Macfarlane quaternion algebra. According to Theorem 4.4.2, this does not lose any generality. Let \( \mathcal{B} = \left( \frac{a,b}{F[\sqrt{-d}]} \right) \) be a Macfarlane quaternion algebra where \( \mathcal{M} \subset \Gamma \) is the Macfarlane space, and \( \dagger \) is the corresponding involution.

Definition 4.4.5. A **restricted Macfarlane space** in \( \mathcal{M} \) is a 3-dimensional \( F \)-space \( \mathcal{L} \subset \mathcal{M} \) such that \( \text{sig}_F(n|_\mathcal{L}) = (1,2) \) and the \( F \)-algebra generated by \( \mathcal{L} \), denoted by \( \langle \mathcal{L} \rangle_F \), is a quaternion algebra over \( F \).

Proposition 4.4.6. \( \mathcal{L}^1_+ \) is a hyperboloid model for hyperbolic 2-space defined over \( F \).

**Proof.** Since \( \text{sig}_F(n|_\mathcal{L}) = (1,2) \), \( \mathcal{L}^1 \) is a hyperboloid of two sheets over \( F \). Since the only positive definite part of \( \mathcal{M} \) is \( F \), and \( \mathcal{L} \subset \mathcal{M} \) contains a 1-dimensional positive definite subspace, this must also be \( F \). Therefore, the points of positive trace \( \mathcal{L}^1_+ \) comprise the upper sheet of the hyperboloid. \( \square \)

Corollary 4.4.7. \( (\mathcal{L} \otimes \mathbb{R})^1_+ \cong \mathcal{T}^2 \).
Let $L$ be the restricted Macfarlane space in $M$ and let $A = \langle L \rangle_F$, the quaternion $F$-algebra generated by $L$.

**Proposition 4.4.8.** Suppose $M$ is normalized, i.e. $i^2 = a, j^2 = b \in F^+$ and $M = F \oplus Fi \oplus Fj \oplus F\sqrt{-dij}$. Then the following statements hold.

1. $\langle L \rangle_F = \left( \frac{a,b}{F} \right)$.
2. $F \oplus Fi \oplus Fj$ is a restricted Macfarlane space.

**Proof.** Taking sums and products of elements of the form $w + xi + yj$ generates $\left( \frac{a,b}{F} \right)$, and since $a, b > 0$, $\text{sig}_F(n|_{F \oplus Fi \oplus Fj}) = (1,2)$. \qed

Beyond the normalized setting, the restriction of scalars to $F$ need not generate the $F$-algebra generated by the restricted Macfarlane space. Rather, one often has to use a different pair $i, j$ than the ones of the ambient quaternion $K$-algebra, as was suggested by Proposition 4.4.3. But as we will see, much of the important information can be derived from the normalized setting.

**Theorem 4.4.9.** The restricted Macfarlane space in $M$ is unique. Moreover, if $\psi : B \rightarrow B' = K \oplus Ki' \oplus Kj' \oplus Ki'j'$ is a $K$-algebra isomorphism where $B'$ is normalized, then $\psi(L) = F \oplus Fi' \oplus Fj'$. 


4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

Proof. We will show uniqueness first in the normalized case, then in general.
The “moreover” part will follow from the construction used.

Suppose $\mathcal{M} \subset \left(\frac{a \cdot b}{\overline{K}}\right)$ is a normalized Macfarlane space, i.e.

$$\mathcal{M} = F \oplus F_i \oplus F_j \oplus F\sqrt{-dij}.$$ 

From Proposition 4.4.8 we know this contains the restricted Macfarlane space $F \oplus F_i \oplus F_j$. So to show uniqueness, suppose $\mathcal{L} \subset \mathcal{M}$ is an arbitrary restricted Macfarlane space and we show that $\mathcal{L} = F \oplus F_i \oplus F_j$.

As in the proof of Proposition 4.4.6, $F \subset \mathcal{L}$, thus $\mathcal{L}_0$ is a 2-dimensional $F$-subspace of $\mathcal{M}_0$ where $n$ is negative definite. We aim to prove that $\mathcal{L}_0 = F_i \oplus F_j$, so suppose by way of contradiction that there exists some $x''i + y''j + z''\sqrt{-dij} \in \mathcal{L}_0$ with $x'', y'', z'' \in F$ and $z'' \neq 0$. Hence we can use $z''$ to eliminate the $\sqrt{-dij}$-coordinate of one of the basis elements, forming a basis $\{s, s'\}$ for $\mathcal{L}_0$ over $F$ of the form

$$s = xi + yj + \sqrt{-dij}$$
$$s' = x'i + y'j$$

where $x, x', y, y' \in F$.

Let $\mathcal{A}$ be the $F$-algebra generated by $\mathcal{L}$. Since $F \subset \mathcal{A}$, by taking products
of these basis elements and subtracting the portion in $F$, we get that

$$(ss')_0 = ax'\sqrt{-d}i + by'\sqrt{-d}j + (xy' - x'y)ij$$

$$(s's)_0 = -by'\sqrt{-d}i - ax'\sqrt{-d}j + (x'y - xy')ij$$

are in $\mathcal{A}_0$. Since $s, s' \in \mathcal{A}_0$ and $\mathcal{A}_0$ is 3-dimensional over $F$, there exist $f_\ell \in F$ such that $s = f_1 s' + f_2 (ss')_0 + f_3 (s's)_0$. Looking at the coefficients of $ij$, this implies that $\sqrt{-d} = (f_2 - f_3)(xy' - x'y)$, but the right hand side is in $F$ while $\sqrt{-d}$ is not. Thus by contradiction, $\mathcal{L}_0 = Fi \oplus Fj$. Then $\mathcal{L} = F \oplus Fi \oplus Fj$.

Now suppose that $\mathcal{M}' \subset \begin{pmatrix} a' & b' \\ K \end{pmatrix}$ is a Macfarlane space in a quaternion algebra $\mathcal{B}$ which is not normalized. By Theorem 4.1.10, there exists a normalized $\mathcal{M} \subset \begin{pmatrix} a & b \\ K \end{pmatrix}$ as above, and a $K$-algebra isomorphism $\psi : \begin{pmatrix} a & b \\ K \end{pmatrix} \rightarrow \begin{pmatrix} a' & b' \\ K \end{pmatrix}$. Let $\mathcal{L} \subset \mathcal{M}$ be as above. Since $\psi|_\mathcal{M}$ is an isometry from $\mathcal{M}$ to $\mathcal{M}'$, it follows that $\psi(\mathcal{L}) \subset \mathcal{M}'$ is a 3-dimensional $F$-space where the quaternion norm from $\begin{pmatrix} a' & b' \\ K \end{pmatrix}$ has signature $(1, 2)$. So to show that $\psi(\mathcal{L})$ is a restricted Macfarlane space, we must show that $\langle \psi(\mathcal{L}) \rangle_F$ is quaternion.

With $i, j \in \begin{pmatrix} a & b \\ K \end{pmatrix}$ as above, we have seen that $\langle \mathcal{L} \rangle_F = F \oplus Fi \oplus Fj \oplus Fi$. Since $\psi$ is $K$-linear and $F \subset K$, we have

$$\langle \psi(\mathcal{L}) \rangle_F = \psi(\langle \mathcal{L} \rangle_F) = F \oplus F\psi(i) \oplus F\psi(j) \oplus F\psi(i)\psi(j).$$
4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

This is a quaternion $F$-algebra (in fact, isomorphic to $\langle L \rangle_F$) because

\[
\psi(i)^2 = \psi(i^2) = \psi(a) = a \\
\psi(j)^2 = \psi(j^2) = \psi(b) = b \\
\psi(i)\psi(j) = \psi(ij) = \psi(-ji) = -\psi(j)\psi(i).
\]

Finally, we show uniqueness for the non-normalized case. Suppose $L' \subset M'$ is an arbitrary restricted Macfarlane space. By a similar argument to the above, $\psi^{-1}(L') \subset M$ must also be a restricted Macfarlane space. But since we have shown this is unique in the normalized setting, we get $\psi^{-1}(L') = L$. That is, $L' = \psi(L)$ as desired. \hfill \Box

Corollary 4.4.10. The restricted Macfarlane space is unique in $B$.

Proof. Recall that $M$ is unique in $B$ (Theorem 4.1.10). \hfill \Box

This allows much of the general theory to follow directly from the simpler setting of normalized quaternion algebras. We make use of this shortly in proving Theorem 4.4.13, the main result of this section.
4.4.3 Isometries in Restricted Macfarlane Spaces

Recall the map $\mu_B$ from (4.2.1). Let $L$ be the restricted Macfarlane space in $B$, and let $A = \langle L \rangle_F$. Then $A \otimes_F K = B$, so we may define

$$\mu_A := \mu_B|_{P_A \times L_1^+}. \quad (4.4.11)$$

Also define

$$\Phi_A : P_A \to \text{Isom}^+(L_1^+), \quad \Phi_A(q)(p) = \mu_A(p, q). \quad (4.4.12)$$

**Theorem 4.4.13.** $\mu_A$ is a faithful action by isometries on $L_1^+$ and $\Phi_A$ is an isomorphism.

**Proof.** Since we already established this for $\mu_B$ (Theorem 4.2.3), it will suffice to show that for all $q \in A$, the image of $\Phi_A(q)$ is $L_1^+$. By Theorem 4.4.9, we need only establish this under the assumption that $M$ is normalized, and in this case we know that $L = F \oplus Fi \oplus Fj$.

The involution $\dagger$ that gives $B = M$ is $(w + xi + yj + zij)^\dagger = \overline{w + xi + yj - zij}$ with $w, x, y, z \in K$. Restricting this to $A$ yields an involution of the first kind $(w + xi + yj + zij)^\dagger = w + xi + yj - zij$. Therefore

$$\forall (q, p) \in A \times L : (q^p q^\dagger)^\dagger = (q^\dagger)^p q^\dagger = q p q^\dagger,$$

i.e. $\mu_B(A \times L_1^+) \subset L_1^+$. Since $\mu_A$ is a restriction of $\mu_B$, it inherits the following
4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

properties: it preserves the norm, it preserves the sign of the trace, and \( \Phi(q) \) is injective for all \( q \in A \). Thus \( \mu_A(q, L_1^+) \subseteq L_1^+ \), and it remains to show that this is in fact an equality.

Let \( p = t + ui + vj \in L_1^+ \) and \( q = w + xi + yj + zij \in A \) where \( t, u, v, w, x, y, z \in F \). By properties of \( \mu_B \), we know there exists some \( p' = w' + x'i + y'j + z'\sqrt{-dij} \in N_1^+ \), where \( w', x', y', z' \in F \), so that \( \mu_B(q, p') = p \). That is, \( qp'q^\dagger = p \). Now since \( p \) has no term containing \( \sqrt{-dij} \), neither does \( qp'q^\dagger \). But if we expand this product, the coefficient of the \( \sqrt{-dij} \) term is

\[
0 = (w^2 - ax^2 - by^2 + abz^2)z' = n(q)z',
\]

where \( n \) is the norm from \( A \) (or equivalently the norm from \( B \) since we are in the normalized setting). Since \( n(q) = 1 \), this forces \( z' = 0 \), which means that \( p' \in L_1^+ \). Then we have \( \mu_A(q, L_1^+) = L_1^+ \) as desired.

**Remark 4.4.14.** Another way of doing this would be to use the analogy between \( N \) and the subset \( \rho_B(N) \) of the Hermitian matrices \( \text{Herm}_2(K(\sqrt{a}, \sqrt{b})) \), extending what was done in §3.2.2. From this viewpoint, \( L \) corresponds to the subset \( \rho_A(L) \) of the symmetric matrices \( \{ m \in M_2(F(\sqrt{a}, \sqrt{b})) \mid m^\top = m \} \), extending what was done in §3.3.1.

It now follows easily to introduce Fuchsian groups \( \Delta \) as subgroups of \( \text{Isom}^+(L_1^+) \) where \( L \) lies in \( B\Delta \), but there is one important difference. The
trace field of a Fuchsian group can be transcendental. So when discussing Fuchsian subgroups of Kleinian groups, one is only considering groups having number fields as trace fields. For our purposes, those are the groups we consider.

**Definition 4.4.15.** Let $X$ be a hyperbolic surface and let $\Delta \cong \pi_1(X)$ be a Fuchsian subgroup of a Kleinian group $\Gamma$. If $B\Gamma$ is Macfarlane, we say that $X$ is a *Macfarlane surface* and that $\Delta$ is a *Macfarlane surface group*. In this case, the *(quaternionic) hyperboloid model* for $\Delta$ is $I^\Delta := (\mathcal{L})^1_+$, where $\mathcal{L}$ is the restricted Macfarlane space in $B\Gamma$.

**Theorem 4.4.16.** The action of $\Delta$ by isometries of $S^2$ is faithfully represented by the restriction

$$\mu_A|_{\Delta \times \mathcal{T}^2} : \Delta \times \mathcal{T}^2 \to \mathcal{T}^2, \quad (\delta, p) \mapsto \delta p \delta^t.$$

**Proof.** This follows from a parallel argument to that given for Theorem 4.3.3.

Recall §4.2.2 where we computed the behavior of various isometries of the hyperboloid model for a Macfarlane manifold based on their properties as quaternions. A portion of that has a direct analogy for the 2-dimensional case and we include one such result by way of example.
4.4. RESTRICTED MACFARLANE SPACES AND HYPERBOLIC SURFACES

Proposition 4.4.17. Let \( q \in \mathcal{L}_+^1 \).

1. If \( q = 1 \) then \( q \) is the identity isometry.

2. If \( q \neq 1 \), then \( q \) acts on \( \mathcal{M}_+^1 \) as a hyperbolic isometry.

3. If \( w \) is the scalar part of \( q \), then \( \Phi(q) \) is a translation by \( 2\text{arcosh}(w) \) along the geodesic that passes through 1 and \( q \), in the direction of \( q_0 \).

Proof. Repeat the argument from Theorem 4.2.10 but without the \( z \)-coordinate:

\[ q = w + xi + yj, \text{ then } n(q) = 1, \text{ and then } w^2 = 1 + ax^2 + by^2 \geq 1, \text{ with } q = 1 \iff w = 1. \]

There is no useful analogy however to the anti-Hermitian complement of \( \mathcal{M} \) for the 2-dimensional setting, as Fuchsian groups contain no purely loxodromic isometries. Also, the anti-symmetric complement of \( \mathcal{L} \) is just \( Fij \), and \( (Fij)^1 = \{ \pm \frac{1}{\sqrt{ab}}ij \} \), which maps to a single elliptic element under \( P \).
Chapter 5

Dirichlet Domains for Macfarlane Manifolds

In this chapter we apply some of the tools from §4 to construct Dirichlet domains of Macfarlane manifolds using their quaternion hyperboloid models. In §5.1, we explain the algorithm for constructing these fundamental domains and we discuss properties of quaternions that facilitate the process. In §5.2, we illustrate this with some examples. We compute Dirichlet domains for the hyperbolic punctured torus and the Whitehead link, and we notice some properties of arithmetic non-compact examples made evident by this technique.
5.1 Quaternion Dirichlet Domains

**Definition 5.1.1.** The *Dirichlet domain* for a group $\Gamma$ acting on a metric space $\mathcal{X}$, centered at $c \in \mathcal{X}$, is defined as

$$D_\Gamma(c) := \left\{ p \in \mathcal{X} \mid \forall \gamma \in \Gamma \setminus \{1\} : d(c, p) \leq d(c, \gamma(p)) \right\}.$$  

When $\mathcal{X}$ is a geometry, this is the intersection of all half-spaces containing $c$, formed by perpendicular bisectors of geodesics from $c$ to the points in the orbit $\text{Orb}_\Gamma(c) := \{ \gamma(c) \mid \gamma \in \Gamma \}$.

Choosing $c$ so that $\text{Stab}_\Gamma(c) = \{1\}$, we get that $D_\Gamma(c)$ is a fundamental domain for $\mathcal{X}/\Gamma$. If $\{\gamma_\ell\}_\ell \subset \Gamma$ is the set such that the sides of $D_\Gamma(c)$ are portions of perpendicular bisectors of geodesics from $c$ to the points $\gamma_\ell(c)$, then the side-pairing identifications are given by applying $\gamma_\ell^{-1}$ to the side contributed by $\gamma_\ell$, for each $\gamma_\ell$. (For a discussion of Dirichlet domains and
their properties, see [5]).

Let $X$ be a finite-volume, complete, oriented hyperbolic 3-manifold, and $\Gamma \cong \pi_1(X)$ a Kleinian group. Only elliptic elements of $\text{Isom}^+(S^3)$ have fixed points inside $S^3$, so by Theorem 2.1.20 we have $\text{Stab}_\Gamma(c) = \{1\}$ for any $c$ in any model of $S^3$, thus $D_\Gamma(c)$ is always a fundamental domain for $\Gamma$. The idea now is to understand $X$ by studying $D_\Gamma(c)$ equipped with side-pairing maps on its boundary.

Since $\text{vol}(X) < \infty$, the group $\Gamma$ is finitely-generated and so by the Poincaré Polyhedral Theorem, any Dirichlet domain for $\Gamma$ has finitely many sides [5], which are geodesic surfaces in $S^3$. Provided one has a way of systematically checking whether each point in $\text{Orb}_\Gamma(c)$ contributes a side to $D_\Gamma(c)$, and a way of telling when all of the sides have been found, one has an algorithm to compute $D_\Gamma(c)$. In this sense there always exists such an algorithm to compute a Dirichlet domain, for any $\Gamma$. However, the algorithms which are efficient enough to be useful in practice are limited.

Hurwitz [28] computed fundamental domains in $H^2$ for the modular group $\text{PSL}_2(\mathbb{Z})$ and its congruence subgroups as early as 1881, and Bianchi [4] extended this to find fundamental domains in $H^3$ for the Bianchi groups $\text{PSL}_2(\mathcal{O}_d)$ ($d \in \mathbb{N} \setminus \mathbb{N}^{(2)}$) in 1892. Since all non-compact, arithmetic hyperbolic 3-manifolds are commensurable to finite covers of Bianchi groups, this
case has been better understood historically. An algorithm was given by Swan [46] to compute Dirichlet domains for manifolds in this class in 1971, and by Riley [45] in 1983 to compute Ford domains (another type of fundamental domain, see [5, 42]) using representations over algebraic numbers. In 2004 Corrales, Jespers, Leal and del Rio [12] gave an algorithm to compute Dirichlet domains for all arithmetic 3-manifolds. This was later improved using the relationship between arithmetic manifolds and orders in quaternion algebras (as in Definition 2.3.4). For arithmetic hyperbolic surfaces, Voight [49] provided a more efficient algorithm using reduction theory, which Page [41] extended to arithmetic hyperbolic 3-manifolds. For arbitrary closed manifolds, Manning’s algorithm [38] to detect hyperbolicity implies an algorithm for finding fundamental domains. For arbitrary cusped manifolds, a construction of Ford domains follows from the Epstein-Penner decomposition [19]. There is no effective general approach that includes examples from every combination of arithmetic and non-arithmetic, compact and non-compact.

The technique given here differs from prior algorithms in the following ways. The Macfarlane model identifies the points in hyperbolic space with quaternions, giving them additional algebraic properties. The points in the orbit have traces and pure quaternion parts which can be used to systematically order them and keep track of side-pairings. Also, there exist elements
5.1. QUATERNION DIRICHLET DOMAINS

of $\Gamma$ lying on the hyperboloid $\mathcal{I}^\Gamma$ as points, and these allow us to locate sides of the Dirichlet domain more quickly than if we were limited to searching through $\text{Orb}_\Gamma(c)$. Another advantage is that the location of points can often be translated into systems of Diophantine equations, to which the classical quaternion arithmetic gives efficient solutions.

5.1.1 Orbit Points

Let $X$ be a Macfarlane manifold and choose a Kleinian group $\Gamma \cong \pi_1(X)$ so that $\mathcal{B} := B\Gamma = \left(\frac{a,b}{F(\sqrt{-d})}\right)$ is a normalized Macfarlane quaternion algebra ($F \subset \mathbb{R}$ and $a, b, d \in F^+$). Let $\dagger$ be the involution so that $\mathcal{M} = \mathcal{B}^\dagger$ is the Macfarlane space. Then $\mathcal{I}^\Gamma = \mathcal{M}^\dagger_1$ is the hyperboloid model for $X$.

In conventional methods there is no natural choice of a center for a Dirichlet domain, but here $1 \in \mathcal{I}^\Gamma$ is the point lying at the bottom of the hyperboloid, so $\mathcal{D}_\Gamma(1)$ is a natural choice for the Dirichlet domain. Recalling (4.2.1), for $\gamma \in \Gamma$ we have $\mu_\mathcal{B}(\gamma, p) = \gamma p \gamma^\dagger$ and therefore

$$\text{Orb}_\Gamma(1) = \{\gamma \gamma^\dagger \mid \gamma \in \Gamma\}. \quad (5.1.2)$$

By Theorem 4.3.3, $\text{Orb}_\Gamma(1) \subset \mathcal{I}^\Gamma$. In particular $\forall \ p \in \text{Orb}_\Gamma(1) : \text{tr}(p) \in F$, and we use this to order the elements of $\text{Orb}_\Gamma(1)$.
For $t \in F$ define

$$V_t := \{ p \in \text{Orb}_\Gamma(1) \mid \text{tr}(p) = t \}. \quad (5.1.3)$$

Another way to describe $V_t$ is as follows. Take the cross-section of $\mathcal{I}^\Gamma$ by the hyper-plane in $\mathcal{M}$ which is parallel to $\mathcal{M}_0$ and at a constant height of $t/2$. (Notice that the trace of a pure quaternion is twice its scalar part, hence dividing by 2 here). This yields an ellipsoid, and $V_t$ is the set of points from $\text{Orb}_\Gamma(1)$ lying on it. That is,

$$V_t = \text{Orb}_\Gamma(1) \cap \{ (t/2) \oplus \mathcal{M}_0 \}.$$

Figure 5.2 shows a 2-dimensional version of this, where orbit points lie in the $V_t$ as subsets of ellipses.
5.1. QUATERNION DIRICHLET DOMAINS

Proposition 5.1.4.

\[ V_t = \left\{ \frac{t}{2} + xi + yj + z\sqrt{-dij} \in \text{Orb}_\Gamma(1) \mid \frac{x^2}{bd} + \frac{y^2}{ad} + z^2 = \frac{t^2 - 4}{4abd} \right\}. \]

Proof. Use the fact that the norm of an arbitrary \( p \in V_t \) is \( n(p) = 1 \), then put this equation into the standard form of an ellipse. \( \square \)

The distribution of the \( V_t \) and of the points within them makes these sets a convenient tool for searching through \( \text{Orb}_\Gamma(1) \).

Proposition 5.1.5.

\( (1) \forall h < 1 : V_t = \Ø. \)

\( (2) V_1 = \{1\}. \)

\( (3) \forall t \in F : |V_t| < \infty. \)

\( (4) \{t \in F \mid V_t \neq \Ø\} \text{ is a discrete subset of } F. \)

Proof. \( \mathcal{I}_F \) intersects the \( F \)-axis at 1, and stretches upward from there (with increasing trace) in every direction. Since \( V_t \subset \mathcal{I}_F \cap (\{t/2\} \oplus \mathcal{M}_0) \), this proves (1) and (2).

\( \text{Orb}_\Gamma(1) \) is discrete and \( V_t \) is contained in an ellipsoid on \( \mathcal{I}_F \), thus \( V_t \) is a discrete subset of a compact set, proving (3). To prove (4), suppose there is some sequence of nonempty \( V_{t_\ell} \) so that \( \{t_\ell\} \) accumulates to \( t \). Let
\( C = \{ p \in \mathcal{I} \Gamma \mid \text{tr}(p) \leq \frac{t}{2} + 1 \} \). Then \( C \) is compact and contains infinitely many points from \( \bigcup V_e \), thus these points converge to a limit point in \( C \) contradicting the discreteness of \( \text{Orb}_\Gamma(1) \).

Recall that for \( \gamma \in \Gamma \) we write \( \gamma(1) = \mu_B(\gamma, 1) \). When \( \gamma \) is given as a matrix, we can easily write the height of the point \( \gamma(1) \) along the \( F \)-axis in terms of the entries of \( \gamma \), or by multiplying by 2, the trace.

**Proposition 5.1.6.**

1. If \( \gamma = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \subseteq \text{PSL}_2(\mathbb{C}) \), then \( \rho_B^{-1}(\gamma)(1) \in V_t \) where

\[
    t = |r|^2 + |s|^2 + |u|^2 + |v|^2.
\]

2. If \( \gamma = w + xi + yj + zij \in \left( \frac{a, b}{F(\sqrt{-d})} \right) \), then \( \gamma(1) \in V_t \) where

\[
    t = 2(|w|^2 + a|x|^2 + b|y|^2 + ab|z|^2).
\]

**Proof.** With \( \gamma \) as in (1),

\[
    \rho_B(\rho_B^{-1}(\gamma)(1)) = \gamma \gamma^\dagger = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} \overline{r} & \overline{s} \\ \overline{u} & \overline{v} \end{pmatrix} = \begin{pmatrix} |r|^2 + |s|^2 & r\overline{u} + s\overline{v} \\ u\overline{r} + v\overline{s} & |u|^2 + |v|^2 \end{pmatrix}.
\]

Then \( \text{tr}(\rho_B^{-1}(\gamma)(1)) = |r|^2 + |s|^2 + |u|^2 + |v|^2 \).

With \( \gamma \) as in (2), recall that \( a, b > 0 \) and that \( \text{tr} \) is invariant under \( \rho_B \),
then apply (1) to
\[ \rho(\gamma) = \left( \begin{array}{cc} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{array} \right). \]

The computation simplifies to the given formula.

**Remark 5.1.7.** The Frobenius norm of a matrix \((a_{k,\ell}) \in M_{m,n}(\mathbb{C})\) is
\[ \| (a_{k,\ell}) \|_{\text{Frob}} := \sqrt{\sum_{k,\ell=1}^{m,n} |a_{k,\ell}|^2}. \]

Thus with \(\gamma\) and \(t\) as in (1) we have that \(t = \|\gamma\|_{\text{Frob}}^2\).

### 5.1.2 The Algorithm

**Definition 5.1.8.** Let \(\gamma \in \Gamma\) and \(p = \gamma(1)\).

(1) Let \(g(p) := g(1,p)\), the geodesic segment from 1 to \(p\), and let \(\widetilde{g}(p)\) be the complete geodesic containing \(g(p)\).

(2) Let \(\tilde{s}(p)\) be the complete geodesic hyperplane perpendicularly bisecting \(g(p)\).

(3) Let \(E(p) \subset T^\Gamma\) be the closed half-space satisfying \(\partial E(p) = \tilde{s}(p)\) and \(1 \in E(p)\).

(4) We say that \(\gamma\) contributes a side to \(D_\Gamma(1)\) if the intersection of \(\tilde{s}(p)\) with \(\partial D_\Gamma(1)\) is codimension-1, and in this case the side contributed to
Remark 5.1.9.

(1) It follows that \( s(p) \) is the complete geodesic hyperplane containing \( s(p) \), consistent with notation.

(2) Figure 5.3 shows how \( \tilde{g}(p) \), \( \tilde{s}(p) \) and \( E(p) \) are situated in the 2-dimensional analogy.

The strategy now is to search through the non-empty \( V_t \) as \( t \) increases from 2 and compute the sides contributed along the way. We show that this gives an effective way of finding all of the sides of \( \mathcal{D}_T(1) \), and that there is an efficient way to check when all the sides have been found.
5.1. QUATERNION DIRICHLET DOMAINS

**Theorem 5.1.10.** By checking the $V_t$ in order of increasing $t$ for points that contribute sides to $\mathcal{D}_\Gamma(1)$, the following things will occur.

1. All of the sides will be found in order of their distance from the center.

2. There is a computable value $m \in F$ such that $\forall \ t > m$ and $\forall \ p \in V_t$, $p$ does not contribute a side to $\mathcal{D}_\Gamma(1)$.

![Figure 5.4: Finding $m$ in the compact (left) and cusped (right) cases.](image)

**Proof.** $\mathcal{D}_\Gamma(1)$ has finitely many sides, the set $\{t \mid V_t \neq \emptyset\}$ is discrete, and for $p \in V_t$ we know $d_{\mathcal{Z}}(1, p) = \text{arcosh}(\text{tr}(p)/2) = \text{arcosh}(t/2)$, proving (1).

To prove (2) define

$$\mathcal{R}_u := \bigcap_{t \leq u} E_p.$$
Then \( R_u \) is the region containing 1 delineated by all sides contributed up to trace \( u \). When these sides include all sides of \( D_\Gamma(1) \) we have \( R_u = D_\Gamma(1) \).

Suppose first that \( \Gamma \) is cocompact. Then \( D_\Gamma(1) \) is compact, thus there will be a minimal \( m' \in K \) so that \( R_{m'} \) is a compact region. To detect when this occurs, compute the circles that the \( \tilde{s}(p) \) approach on \( \partial I^\Gamma \) (using the identification \( \partial I^\Gamma \leftrightarrow \mathcal{M}^0/\sim \) from §4.2.2, or by working in \( \partial H^3 \) using the map \( \iota \) from Theorem 3.2.26). As soon as every point of \( \partial I^\Gamma \) is contained in the interior of one of these circles, let \( m' \) be the most recent trace searched and then \( R_{m'} \) will have enclosed a compact region. \( R_{m'} \) is not necessarily \( D_\Gamma(1) \), but from here we can find the trace \( m \) sufficiently high to check up to, so that \( R_m = D_\Gamma(1) \). Let \( m = m'^2 \), then \( d_{I^\Gamma}(1, m) = 2d_{I^\Gamma}(1, m') \), so that for any point \( p \) with \( \text{tr}(p) > m \), the lowest point on \( \tilde{s}(p) \) will be above \( R_m \), as illustrated on the left side of Figure 5.4 (in the 2-dimensional analogy). As we check points up to \( m \), it is possible that new sides will be contributed, truncating \( R_{m'} \) but after trace \( m \) this becomes impossible.

Now suppose \( \Gamma \) is non-cocompact. Then \( D_\Gamma(1) \) will not be a region of bounded trace but it will take the form of a region in \( I^\Gamma \) together with some finite number of points on \( \partial I^\Gamma \), each having a neighborhood in \( I^\Gamma \) homeomorphic to \( T^2 \times [0, \infty) \). Thus there will be a first \( m' \in K \) where \( R_{m'} \) takes this same form. We can determine when this has happens by checking
the boundary circles approached by the $\mathcal{s}(p)$, as in the compact case, but
now looking for when the only points not included in the interiors of these
circles are some finitely many points $q$ where four bisectors $\mathcal{s}(p_1), \ldots , \mathcal{s}(p_4)$
intersect. When this happens, for each $q$ compute the pair of parabolic
isometries pairing oppose sides and check whether these parabolic isometries
lie in the group. If so, then this neighborhood corresponds to a cusp of the
manifold, and will not be truncated by any future $\mathcal{s}(\gamma)$. If this does not
happen, continue finding sides in the usual manner until it does. Once every
one of these cuspoidal neighborhood of points $q$ has been found, there is a
computable neighborhood $N(q)$ of $q$ so that any geodesic $\mathcal{g}(p)$ entering $N(q)$
would converge to $[q]$ in the quotient $\mathcal{I}/\Gamma [3]$. Therefore, all future $\mathcal{g}(p)$
must avoid $N(q)$. So when this happens, let $m'$ be the supremum of traces of
points on $\left( \bigcup_{\ell=1}^{4} \mathcal{s}(\gamma_\ell) \right) \setminus N(p)$. The right side of Figure 5.4 illustrates this in
the case where there is a single cusp (in the 2-dimensional analogy), and in
that picture $N(p)$ is the yellow cuspoidal region above the red circle. Let $m''$
be the supremum among all these $m'$, or the maximal height in the compact
portion if this is higher. Then let $m = m''^2$ and proceed as in the compact

case. \hfill $\square$

We now give the outline of the general algorithm, which lends itself to
computer implementation.

1. Obtain a set of matrix generators for $\Gamma \cong \pi_1(X)$ so that $B\Gamma$ is a normalized Macfarlane quaternion algebra. Let $t_0 = 2$.

2. Find $t := \min \{ \text{tr}(p) \neq t_0 \mid p \in \text{Orb}_1(1) \}$, then find the elements of $V_t$.

3. For each $p \in V_t$, compute $\tilde{s}(p)$. Then determine whether $R_t$ encloses a region (compact case) or approaches only finitely many boundary points. If not, replace $t_0$ with $t$ and return to step (2). If so, continue to step (4).

4. Find a value $m$ so that if $\text{tr}(\gamma(1)) > m$, then $\tilde{s}(\gamma)$ does not intersect $R_m$. If $t \leq m$, replace $t_0$ with $t$ and return to step (2). If $t > m$, then continue to step (5).

5. For each side $s(\gamma)$, compute $\gamma^{-1}(b(\gamma))$. The intersection of this with $\partial R$ will be some other side $s$, and gives a side-pairing $s(\gamma) \sim s$.

Remark 5.1.11. It is always possible to check, given a set of generators, whether some element of $B\Gamma$ is a member of $\Gamma$ [21]. The methods differ depending on the complexity of $\Gamma$ and on the number of generators.

For step (2), Proposition 5.1.5 implies there is always a minimal $t > t_0$ such that the $V_t$ is non-empty. When $\Gamma$ has integral traces, the $t$-values
to be searched lie in a ring of integers (often an $R_F$-order where $R_F$ is the ring of integers of the real part of the trace field $K\Gamma$). In this case, finding points in $V_t$ as in step (3) corresponds to finding solutions to Diophantine equations as in Proposition 5.1.4. More generally one can use constraints on the coefficients of the elements $\gamma \in \Gamma$ that might satisfy $\gamma(1) \in V_t$, such as in Proposition 5.1.6.

It is also useful to consider points lying on the hyperboloid model that are not necessarily orbit points.

**Theorem 5.1.12.** If $p \in \Gamma \cap \mathcal{T}_\Gamma$, but $p \notin \operatorname{Orb}_\Gamma(1)$, then

1. $p^2 \in \operatorname{Orb}_\Gamma(1),$
2. $\forall n \in \mathbb{Z} : \tilde{g}(p) = \tilde{g}(p^n),$
3. $\tilde{g}(p)$ intersects $\tilde{s}(p)$ at $p$.

**Proof.** If $p \in \Gamma \cap \mathcal{T}_\Gamma$, then $p^\dagger = p$, so then $p(1) = pp^\dagger = p^2 \in \operatorname{Orb}_\Gamma(1)$, proving (1). By Theorem 4.2.10, $p$ acts on $\mathcal{T}_\Gamma$ as a hyperbolic translation along $\tilde{g}(p)$. Thus $p(p(1)) = pp^2p^\dagger = p^3$ also lies on $\tilde{g}(p)$, and iterating this gives a sequence of points $p^n(1) = p^{2n}$ climbing up $\tilde{g}(p)$, for $n \in \mathbb{N}$. Similarly, $p(p) = ppp^\dagger = p^3$, so that iterating this gives a sequence of points $p^n(p) = p^{2n+1}$ climbing up $\tilde{g}(p)$, for $n \in \mathbb{N}$. Also, $p^{-1}$ acts as a hyperbolic
translation along the same geodesic in the opposite direction so applying the
same argument proves (2).

Since $\tilde{s}(p)$ intersects $\tilde{g}(p)$ by definition, to prove (3) we show that $p$ lies
halfway between 1 and $p^2$. Let $q \in (\mathcal{M} \otimes_{K\Gamma} \mathbb{R})^1_+$ be the hyperbolic translation
along $\tilde{g}(p)$ that takes 1 to $p$ ($q$ might not have coordinates in $K\Gamma$ but up
to extending scalars to $\mathbb{R}$ it exists). Then $q(1) = qq^1 = p$, and since $q$
and $p$ are both translations along the same geodesic, they commute. Then
$q(p) = qpq^1 = qq^1p = p^2$. Therefore $d(1,p)$ and $d(p,p^2)$ both equal the
translation length $2\text{arcosh} \left( \frac{\text{tr}(q)}{2} \right)$ of $q$. \hfill $\Box$

There are analogous results to all of the above for hyperbolic surfaces,
using restricted Macfarlane spaces. An example of this is included in the
following section.

5.2 Examples

We now carry out the algorithm for some basic examples and note some
properties which are readily observable from the data. We focus on the
non-compact arithmetic case. Here the desired points can be found using
Diophantine equations that arise from the modular group in the case of
surfaces, and from the Bianchi groups in the case of 3-manifolds. We also
benefit from the following additional fact.
Proposition 5.2.1. If $B \Gamma = B \Gamma'$, $\Gamma < \Gamma'$, and $\Gamma'$ is closed under complex conjugation, then $\text{Orb}_\Gamma(1) \subset \Gamma'$.

Proof. Since $B \Gamma = B \Gamma'$, elements of both $\Gamma$ and $\Gamma'$ identify with quaternions in $B \Gamma$, and $\mathcal{T}^\Gamma = \mathcal{T}^{\Gamma'}$. Let $p \in \text{Orb}_\Gamma(1)$. Then $\exists \gamma \in \Gamma$ such that $p = \gamma \gamma^\dagger$. Since $\gamma, \gamma^\dagger \in \Gamma'$ we have $\gamma \gamma^\dagger \in \Gamma'$. \hfill $\square$

5.2.1 A Hyperbolic Punctured Torus

Let $X$ be the hyperbolic punctured torus with $\pi_1(X) \cong \Gamma$, the torsion-free subgroup of the modular group $\text{PSL}_2(\mathbb{Z})$. Then $\Gamma = \langle \gamma, \delta \rangle$ where

$$\gamma := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \delta := \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

and $\mathcal{B} = B \Gamma = \left( \frac{1,1}{\mathbb{Q}} \right)$. Then $\mathcal{B} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ (with $i^2 = j^2 = 1$ and $ij = -ji$) contains the restricted Macfarlane space $\mathcal{L} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j$, and $\mathcal{T}^\Gamma := \mathcal{M}_+^1$ is the hyperboloid model for $\Gamma$.

We may move between matrices and quaternions via the $\mathbb{Q}$-algebra isomorphisms

$$\rho_\mathcal{B} : \mathcal{B} \to M_2(\mathbb{Q}), \quad \begin{bmatrix} w + xi + yj + zij \end{bmatrix} \mapsto \begin{pmatrix} w - x & y - z \\ w + z & y + z \end{pmatrix},$$

$$\rho_{\mathcal{B}}^{-1} : M_2(\mathbb{Q}) \to \mathcal{B}, \quad \begin{bmatrix} s & t \\ u & v \end{bmatrix} \mapsto \frac{v + s}{2} + \frac{v - s}{2}i + \frac{u + t}{2}j + \frac{u - t}{2}ij.$$

We may move from the hyperboloid model to the upper half-plane model via
the isometry

\[ \iota : \mathcal{I}^\Gamma \to \mathcal{H}^2, \quad w + xi + yj \mapsto \frac{y + J}{w + x}. \]

Here \( J \) is used so that \( \mathcal{H}^2 = \mathbb{R} \oplus \mathbb{R}^+ J \subset \mathbb{H} \) and \( J^2 = -1 \), as in \( \S 3.2.3 \). By Theorem 3.2.26, \( \rho_B \) and \( \iota \) preserve the isometric action in the two models, so when the context is clear we use \( \Gamma = \langle \gamma, \delta \rangle \) to mean both the matrix group and the corresponding quaternion group.

We are interested in finding points in \( \text{Orb}_\Gamma(1) \), but since \( \Gamma \) consists of all non-elliptic elements of \( \text{PSL}_2(\mathbb{Z}) \), and an elliptic element remains elliptic under transposition, \( \Gamma \) is closed under transposition. Thus by Proposition 5.2.1 \( \text{Orb}_\Gamma(1) \subset \Gamma \), which implies that \( \forall t \in \mathbb{N} : \)

\[ V_t = \{(\gamma \in \Gamma \cap \mathcal{I}^\Gamma \mid \text{tr}(\gamma) = t) \} \]

\[ = \rho_B^{-1}\left( \left\{ \gamma = \begin{pmatrix} x & y \\ y & t-x \end{pmatrix} \mid \text{det}(\gamma) = 1, x, y \in \mathbb{Z} \right\} \right) \]

\[ = \left\{ \frac{t}{2} + \frac{t - 2x}{2}i + y \mid x^2 + y^2 = tx - 1, x, y \in \mathbb{Z} \right\}. \]

Now, we know that any \( \gamma \) taking this form lies in \( \Gamma \), but must determine for each \( \gamma \) whether or not \( \gamma \in \text{Orb}_\Gamma(1) \). This can be done by checking whether it can be written in the form \( \gamma = \delta \delta^\dagger \) for some word \( \delta \) in the generators. If it can we get that \( \tilde{s}(\gamma) \) passes halfway between 1 and \( \gamma \), and if it cannot, then by Theorem 5.1.12 \( \tilde{s}(\gamma^2) \) passes through \( \gamma \).
5.2. EXAMPLES

Applying $\rho_{B}^{-1}$ to the given generators as matrices, we obtain the quaternions

$$\gamma = \frac{3}{2} + \frac{1}{2}i + j, \quad \delta = \frac{3}{2} + \frac{1}{2}i - j.$$ 

The strategy now is to, as $t$ increments through the sequence $(2, 3, 4, \ldots)$, find the solutions to the Diophantine equation

$$x^2 + y^2 = tx - 1,$$

then for each corresponding point $p = \frac{t}{2} + \frac{t-2x}{2}i + y$, determine whether the new potential side passes through $p$ or halfway between $p$ and 1.

Table 5.1 lists the data from implementing this process, giving the points in $I^\Gamma \cap \Gamma$ up to trace 18. Points in $\text{Orb}_\Gamma(1)$ that contribute sides to $D_\Gamma(1)$ are in bold. For each $p \in I^\Gamma \cap \Gamma$, the direction of the complete geodesic $\tilde{g}(p)$ is given as a “slope” by converting $[\tilde{g}(p)_0] \in \mathbb{P}^1(\mathbb{Z})$ (in the sense of Proposition 4.2.7) to a number in $\mathbb{Q}$ in the natural way. In the rightmost column the corresponding points in $H^2$ are given, as obtained by applying the map $\iota$ from Theorem 3.2.26. Notice that (as predicted by Theorem 5.1.12), if a point $p$ in the chart does not lie in $\text{Orb}_\Gamma(1)$, then later the point $p^2$ creates a bisector $\tilde{s}(p^2)$ passing through $p$. For example, the elements of $I^\Gamma \cap \Gamma$ at trace 3 lead to sides contributed at trace 6.
Table 5.1: Points from a hyperbolic punctured torus group that lie on its quaternion hyperboloid model.

<table>
<thead>
<tr>
<th>trace</th>
<th>$q \in T^1 \cap \Gamma$</th>
<th>slope of $q_0$</th>
<th>in $\text{PSL}_2(\mathbb{R})$</th>
<th>in $\mathcal{H}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1$</td>
<td>$-$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$J$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{2} + \frac{1}{2}i \pm j$</td>
<td>$\pm 2$</td>
<td>$\begin{pmatrix} 1 &amp; \pm 1 \ 2 &amp; \pm 1 \end{pmatrix}$</td>
<td>$\pm \frac{1}{2} + \frac{1}{2}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2}i \pm j$</td>
<td>$\mp 2$</td>
<td>$\begin{pmatrix} 2 &amp; \pm 1 \ \pm 1 &amp; 1 \end{pmatrix}$</td>
<td>$\pm 1 + J$</td>
</tr>
<tr>
<td>6</td>
<td>$3 + 2i \pm 2j$</td>
<td>$\pm 1$</td>
<td>$\begin{pmatrix} 1 &amp; \pm 2 \ 5 &amp; \pm 2 \end{pmatrix}$</td>
<td>$\pm \frac{2}{5} + \frac{1}{5}J$</td>
</tr>
<tr>
<td></td>
<td>$3 - 2i \pm 2j$</td>
<td>$\mp 1$</td>
<td>$\begin{pmatrix} 5 &amp; \pm 2 \ \pm 2 &amp; 1 \end{pmatrix}$</td>
<td>$\pm 2 + J$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{7}{2} + \frac{3}{2}i \pm 3j$</td>
<td>$\pm 2$</td>
<td>$\begin{pmatrix} 2 &amp; \pm 3 \ 5 &amp; \pm 3 \end{pmatrix}$</td>
<td>$\pm \frac{3}{5} + \frac{1}{5}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{7}{2} - \frac{3}{2}i \pm 3j$</td>
<td>$\mp 2$</td>
<td>$\begin{pmatrix} 5 &amp; \pm 3 \ \pm 3 &amp; 2 \end{pmatrix}$</td>
<td>$\pm \frac{3}{2} + \frac{1}{2}J$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{11}{2} + \frac{9}{2}i \pm 3j$</td>
<td>$\pm \frac{2}{3}$</td>
<td>$\begin{pmatrix} 1 &amp; \pm 3 \ 10 &amp; \pm 3 \end{pmatrix}$</td>
<td>$\pm \frac{3}{10} + \frac{1}{10}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{11}{2} - \frac{9}{2}i \pm 3j$</td>
<td>$\mp \frac{2}{3}$</td>
<td>$\begin{pmatrix} 10 &amp; \pm 3 \ \pm 3 &amp; 1 \end{pmatrix}$</td>
<td>$\pm 3 + J$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{15}{2} + \frac{11}{2}i \pm 5j$</td>
<td>$\pm \frac{10}{11}$</td>
<td>$\begin{pmatrix} 2 &amp; \pm 5 \ 13 &amp; \pm 5 \end{pmatrix}$</td>
<td>$\pm \frac{5}{13} + \frac{1}{13}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{15}{2} - \frac{11}{2}i \pm 5j$</td>
<td>$\mp \frac{10}{11}$</td>
<td>$\begin{pmatrix} 13 &amp; \pm 5 \ \pm 5 &amp; 2 \end{pmatrix}$</td>
<td>$\pm \frac{5}{2} + \frac{1}{2}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{15}{2} + \frac{5}{2}i \pm 7j$</td>
<td>$\pm \frac{14}{5}$</td>
<td>$\begin{pmatrix} 5 &amp; \pm 7 \ 10 &amp; \pm 7 \end{pmatrix}$</td>
<td>$\pm \frac{7}{10} + \frac{1}{10}J$</td>
</tr>
<tr>
<td></td>
<td>$\frac{15}{2} - \frac{5}{2}i \pm 7j$</td>
<td>$\mp \frac{14}{5}$</td>
<td>$\begin{pmatrix} 10 &amp; \pm 7 \ \pm 7 &amp; 5 \end{pmatrix}$</td>
<td>$\pm \frac{7}{5} + \frac{1}{5}J$</td>
</tr>
<tr>
<td>18</td>
<td>$9 + 8i \pm 4j$</td>
<td>$\pm \frac{1}{2}$</td>
<td>$\begin{pmatrix} 1 &amp; \pm 4 \ 17 &amp; \pm 4 \end{pmatrix}$</td>
<td>$\pm \frac{4}{17} + \frac{1}{17}J$</td>
</tr>
<tr>
<td></td>
<td>$9 - 8i \pm 4j$</td>
<td>$\mp \frac{1}{2}$</td>
<td>$\begin{pmatrix} 17 &amp; \pm 4 \ \pm 4 &amp; 1 \end{pmatrix}$</td>
<td>$\pm 4 + J$</td>
</tr>
<tr>
<td></td>
<td>$9 + 4i \pm 8j$</td>
<td>$\pm 2$</td>
<td>$\begin{pmatrix} 5 &amp; \pm 8 \ 13 &amp; \pm 8 \end{pmatrix}$</td>
<td>$\pm \frac{8}{13} + \frac{1}{13}J$</td>
</tr>
<tr>
<td></td>
<td>$9 - 4i \pm 8j$</td>
<td>$\mp 2$</td>
<td>$\begin{pmatrix} 13 &amp; \pm 8 \ \pm 8 &amp; 5 \end{pmatrix}$</td>
<td>$\pm \frac{8}{5} + \frac{1}{5}J$</td>
</tr>
</tbody>
</table>
5.2. EXAMPLES

Figure 5.5 shows in $H^2$ which sides are contributed at each trace until $D_{\Gamma}(1)$ is complete, and illustrates how the induced side-pairings create a punctured torus.

5.2.2 Torsion-Free Subgroups of Bianchi groups

Let $\Gamma$ be a torsion-free finite index subgroup of a Bianchi group $\text{PSL}_2(O_d)$. Since $\text{PSL}_2(O_d)$ is closed under complex conjugation, Proposition 5.2.1 gives that $\text{Orb}_{\Gamma}(1) \subset \text{PSL}_2(O_d)$, and the entries of matrices in $\Gamma$ lie in

$$\{m + n\sqrt{-d} \mid m, n \in \mathbb{Z}\} \quad \text{if} \quad d \not\equiv 1$$

or

$$\left\{m + \frac{1 + n\sqrt{-d}}{2} \mid m, n \in \mathbb{Z}\right\} \quad \text{if} \quad d \equiv 1.$$

We can find the points in $V_t = \{p = \gamma(1) \mid \gamma \in \Gamma, \text{tr}(p) = t\}$ by first finding the points in $\{\gamma \in \text{PSL}_2(O_d) \mid \text{tr}(\gamma) = t\}$ and then using the generators of $\Gamma$ to determine which of these lie in $V_t$.

Since the only real traces found in $\text{PSL}_2(O_d)$ lie in $\mathbb{Z}$, the $V_t$ can only be non-empty when $t \in \{2, 3, 4, \ldots\}$. For each of these $t$-values, elements of $\{\gamma \in \text{PSL}_2(O_d) \mid \text{tr}(\gamma) = t\}$ are given by the solutions to the Diophantine equation arising from $\det \begin{pmatrix} r & s \\ t & \end{pmatrix} = 1$ where $t \in \mathbb{N}$ is fixed, $r \in \mathbb{Z}$ and
Figure 5.5: Dirichlet domain for a hyperbolic punctured torus.

Trace 6

Trace 7

Trace 11: a region is enclosed.

Trace 15: the Dirichlet domain is complete.
s \in \mathcal{O}_d$. Tables 5.2 and 5.3 give these for the first few $t$-values in the cases $d = 1$ and $d = 3$.

**Example 5.2.2** (The Whitehead link complement). Recall from Example 2.1.24 that the Whitehead link complement is a finite-index subgroup of $\Gamma_1$. Suppressing some details, we use some computer implementation to apply the above method to this example. We then describe how our Dirichlet domain is formed by drawing the sides contributed at each trace, in $\mathbb{H}^3$. Figure 5.6 shows the result of this process.

Lastly, as an example of a phenomenon made evident by this technique, we observe some data about the heights of the points in $\text{PSL}_2(\mathcal{O}_d) \cap \mathcal{I}^{\text{PSL}_2(\mathcal{O}_d)}$ as $d$ varies. Notice that $\forall \ d : \text{PSL}_2(\mathbb{Z}) < \text{PSL}_2(\mathcal{O}_d)$. Therefore all orbit points of the group for the hyperbolic punctured torus from §5.2.1 will also occur as orbit points of the torsion-free subgroup of $\text{PSL}_2(\mathcal{O}_d)$. It is then of interest to see what is the lowest height where additional orbit points are introduced, for various values of $d$. Table 5.4 gives this information up to $d = 41$, as well as the directions of the geodesics $\hat{g}(p)$ passing through each point in projective coordinates, in the sense of Proposition 4.2.7.

Due to symmetry in these groups, points occur in pairs so that each geodesic passes through two points. Moreover, for each geodesic passing
Table 5.2: First three heights of points in $\Gamma_1 \cap \mathcal{I}^r$.

<table>
<thead>
<tr>
<th>trace</th>
<th>in $\mathcal{I}^r$</th>
<th>in $\text{PSL}_2(\mathbb{C})$</th>
<th>in $\mathbb{H}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{3}{2} + \frac{1}{2} i \pm j$</td>
<td>$\left(\begin{array}{cc} 1 &amp; \pm 1 \ \pm 1 &amp; 2 \end{array}\right)$</td>
<td>$(\pm \frac{1}{2}, 0, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2} i \pm j$</td>
<td>$\left(\begin{array}{cc} 2 &amp; \pm 1 \ \pm 1 &amp; 1 \end{array}\right)$</td>
<td>$(\pm 1, 0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2} i \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; \pm \sqrt{-1} \ \pm \sqrt{-1} &amp; 1 \end{array}\right)$</td>
<td>$(0, \pm 1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} + \frac{1}{2} i \pm ij$</td>
<td>$\left(\begin{array}{cc} 1 &amp; \pm \sqrt{-1} \ \pm \sqrt{-1} &amp; 2 \end{array}\right)$</td>
<td>$(0, \pm \frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td>4</td>
<td>$2 + i - j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 1 &amp; -1 \pm \sqrt{-1} \ -1 \mp \sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(-\frac{1}{3}, \pm \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$2 + i + j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 1 &amp; -1 \pm \sqrt{-1} \ -1 \mp \sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(\frac{1}{3}, \pm \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$2 - i - j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; -1 \pm \sqrt{-1} \ -1 \mp \sqrt{-1} &amp; 1 \end{array}\right)$</td>
<td>$(-1, \pm 1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$2 - i + j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; 1 \pm \sqrt{-1} \ 1 \mp \sqrt{-1} &amp; 1 \end{array}\right)$</td>
<td>$(1, \pm 1, 1)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5}{2} + \frac{1}{2} i - 2j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; -2 \pm \sqrt{-1} \ -2 \mp \sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(-\frac{2}{3}, \pm \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{1}{2} i - j \pm 2\sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; -1 \pm 2\sqrt{-1} \ -1 \mp 2\sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(-\frac{1}{3}, \pm \frac{2}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{1}{2} i + j \pm 2\sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; 1 \pm 2\sqrt{-1} \ 1 \mp 2\sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(\frac{1}{3}, \pm \frac{2}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{1}{2} i + 2j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; -1 \pm 2\sqrt{-1} \ -1 \mp 2\sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(-\frac{1}{3}, \pm \frac{2}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{1}{2} i + 2j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 2 &amp; 1 \pm 2\sqrt{-1} \ 1 \mp 2\sqrt{-1} &amp; 3 \end{array}\right)$</td>
<td>$(\frac{2}{3}, \pm \frac{1}{3}, \frac{1}{3})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{1}{2} i - 2j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; -2 \pm \sqrt{-1} \ -2 \mp \sqrt{-1} &amp; 2 \end{array}\right)$</td>
<td>$(1, \pm \frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{1}{2} i - j \pm 2\sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; -1 \pm 2\sqrt{-1} \ -1 \mp 2\sqrt{-1} &amp; 2 \end{array}\right)$</td>
<td>$(-\frac{1}{2}, \pm 1, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{1}{2} i + j \pm 2\sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; 1 \pm 2\sqrt{-1} \ 1 \mp 2\sqrt{-1} &amp; 2 \end{array}\right)$</td>
<td>$(\frac{1}{2}, \pm 1, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{1}{2} i + 2j \pm \sqrt{-1}ij$</td>
<td>$\left(\begin{array}{cc} 3 &amp; 2 \pm \sqrt{-1} \ 2 \mp \sqrt{-1} &amp; 2 \end{array}\right)$</td>
<td>$(1, \pm \frac{1}{2}, \frac{1}{2})$</td>
</tr>
</tbody>
</table>
5.2. EXAMPLES

Table 5.3: First three heights of points in $\Gamma_3 \cap \mathcal{T}_3$.

<table>
<thead>
<tr>
<th>Trace</th>
<th>in $\mathcal{T}_3$</th>
<th>in $\text{PSL}_2(\mathbb{C})$</th>
<th>in $\mathbb{H}^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{3}{2} + \frac{1}{2} j \pm i$</td>
<td>$\begin{pmatrix} 1 &amp; \pm 1 \ 1 &amp; 2 \end{pmatrix}$</td>
<td>$(\pm \frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2} j \pm i$</td>
<td>$\begin{pmatrix} 2 &amp; \pm 1 \ 1 &amp; 1 \end{pmatrix}$</td>
<td>$(\pm 1, 0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} + \frac{1}{2} i + \frac{\sqrt{-3}}{2} j$</td>
<td>$\begin{pmatrix} \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; 2 \end{pmatrix}$</td>
<td>$(\frac{1}{4}, \pm \frac{\sqrt{3}}{4}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} + \frac{1}{2} i - \frac{1}{2} j \pm \frac{\sqrt{-3}}{2} i$</td>
<td>$\begin{pmatrix} \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; 2 \end{pmatrix}$</td>
<td>$(-\frac{1}{4}, \pm \frac{\sqrt{3}}{4}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2} i \pm \frac{1}{2} j + \frac{\sqrt{-3}}{2} i$</td>
<td>$\begin{pmatrix} \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; 1 \end{pmatrix}$</td>
<td>$(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2} - \frac{1}{2} i \pm \frac{1}{2} j - \frac{1}{2} \sqrt{-3} i$</td>
<td>$\begin{pmatrix} \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{1}{2} \pm \frac{\sqrt{-3}}{2} &amp; 1 \end{pmatrix}$</td>
<td>$(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1)$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} 2 &amp; \pm \sqrt{-3} \ \pm \sqrt{-3} &amp; 2 \end{pmatrix}$</td>
<td>$(0, \pm \frac{\sqrt{3}}{4}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$2 - \frac{3}{2} j \pm \frac{\sqrt{-3}}{2} i j$</td>
<td>$\begin{pmatrix} 2 &amp; \frac{3}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{3}{2} \pm \frac{\sqrt{-3}}{2} &amp; 2 \end{pmatrix}$</td>
<td>$(-\frac{3}{4}, \pm \frac{\sqrt{3}}{4}, \frac{1}{2})$</td>
</tr>
<tr>
<td></td>
<td>$2 + \frac{3}{2} j \pm \frac{\sqrt{-3}}{2} i j$</td>
<td>$\begin{pmatrix} 2 &amp; \frac{3}{2} \pm \frac{\sqrt{-3}}{2} \ \frac{3}{2} \pm \frac{\sqrt{-3}}{2} &amp; 2 \end{pmatrix}$</td>
<td>$(\frac{3}{4}, \pm \frac{\sqrt{3}}{4}, \frac{1}{2})$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{5}{2} + \frac{3}{2} i \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} 1 &amp; \pm \sqrt{-3} \ \pm \sqrt{-3} &amp; 4 \end{pmatrix}$</td>
<td>$(0, \pm \frac{\sqrt{3}}{4}, \frac{1}{4})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{3}{2} i \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} 4 &amp; \pm \sqrt{-3} \ \pm \sqrt{-3} &amp; 1 \end{pmatrix}$</td>
<td>$(0, \pm \sqrt{3}, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{3}{2} i + \frac{3}{2} j \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; \frac{3}{2} \pm \sqrt{-3} \ \frac{3}{2} \pm \sqrt{-3} &amp; 4 \end{pmatrix}$</td>
<td>$(\frac{3}{8}, \pm \frac{\sqrt{3}}{8}, \frac{1}{4})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} + \frac{3}{2} i - \frac{3}{2} j \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; \frac{3}{2} \pm \sqrt{-3} \ \frac{3}{2} \pm \sqrt{-3} &amp; 4 \end{pmatrix}$</td>
<td>$(-\frac{3}{8}, \pm \frac{\sqrt{3}}{8}, \frac{1}{4})$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{3}{2} i + \frac{3}{2} j \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; \frac{3}{2} \pm \sqrt{-3} \ \frac{3}{2} \pm \sqrt{-3} &amp; 1 \end{pmatrix}$</td>
<td>$(\frac{3}{2}, \pm \sqrt{3}, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{5}{2} - \frac{3}{2} i - \frac{3}{2} j \pm \sqrt{-3} i j$</td>
<td>$\begin{pmatrix} \frac{1}{2} &amp; \frac{3}{2} \pm \sqrt{-3} \ \frac{3}{2} \pm \sqrt{-3} &amp; 1 \end{pmatrix}$</td>
<td>$(-\frac{3}{2}, \pm \sqrt{3}, 1)$</td>
</tr>
</tbody>
</table>
Figure 5.6: Dirichlet domain for the Whitehead link complement.

**Trace 3:** a pair of parallel half-planes.

**Trace 4:** a pair of hemispheres tangent at zero.

**Trace 5:** four overlapping hemispheres symmetric around the vertical axis.

**Trace 6:** two parallel half-planes completing a cusp at infinity, and two larger hemispheres.

**Trace 7:** two smaller hemispheres centered on the real axis further truncate the region.

The completed Dirichlet domain. At trace 10 a pair of hemispheres tangent at zero completes the cusp at infinity suggested at trace 3. Several other orbit points, starting at trace 6, are not included because they are too far away to contribute sides. After trace 10, this is true for all further orbit points.
through a pair of points at some trace value, if we alter the signs in its projective coordinates we get another geodesic passing through another pair of points at that trace value. So in the rightmost column of Table 5.4 only the geodesics with positive signs are listed. For example, the entry \([1 : 0 : 2]\) indicates the occurrence of the two geodesics \([1 : 0 : 2]\) and \([1 : 0 : -2]\), each of which passes through two points; and the entry \([1 : 1 : 1]\) indicates the occurrence of the four geodesics \([1 : 1 : 1]\), \([1 : 1 : -1]\), \([1 : -1 : 1]\) and \([1 : -1 : -1]\), each of which passes through two points.
Table 5.4: Lowest occurrences of points in $(\Gamma_d \cap T^d) \setminus \text{PSL}_2(\mathbb{Z})$ for first several $d$, and the directions of the corresponding geodesics.

<table>
<thead>
<tr>
<th>$d$</th>
<th>Lowest $t$-value with new points</th>
<th># new points (# in PSL$_2(\mathbb{Z})$)</th>
<th>directions of geodesics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4 (4)</td>
<td>[1 : 0 : 2]</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8 (0)</td>
<td>[1 : 0 : 1], [0 : 1 : 1]</td>
</tr>
<tr>
<td>3*</td>
<td>3</td>
<td>8 (4)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4 (0)</td>
<td>[1 : 0 : 2]</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>8 (4)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>7*</td>
<td>4</td>
<td>8 (0)</td>
<td>[2 : 1 : 1]</td>
</tr>
<tr>
<td>10</td>
<td>7</td>
<td>8 (4)</td>
<td>[1 : 2 : 2]</td>
</tr>
<tr>
<td>11*</td>
<td>4</td>
<td>4 (0)</td>
<td>[0 : 1 : 1]</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>8 (0)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>8 (0)</td>
<td>[1 : 0 : 1], [0 : 1 : 1]</td>
</tr>
<tr>
<td>15*</td>
<td>6</td>
<td>8 (4)</td>
<td>[4 : 1 : 1]</td>
</tr>
<tr>
<td>17</td>
<td>9</td>
<td>4 (0)</td>
<td>[3 : 0 : 2]</td>
</tr>
<tr>
<td>19*</td>
<td>5</td>
<td>8 (0)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>21</td>
<td>13</td>
<td>4 (0)</td>
<td>[9 : 0 : 2]</td>
</tr>
<tr>
<td>22</td>
<td>10</td>
<td>8 (0)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>23*</td>
<td>6</td>
<td>4 (4)</td>
<td>[0 : 3 : 1]</td>
</tr>
<tr>
<td>26</td>
<td>11</td>
<td>8 (4)</td>
<td>[3 : 2 : 2]</td>
</tr>
<tr>
<td>27*</td>
<td>6</td>
<td>8 (4)</td>
<td>[2 : 1 : 1]</td>
</tr>
<tr>
<td>29</td>
<td>11</td>
<td>4 (4)</td>
<td>[1 : 0 : 2]</td>
</tr>
<tr>
<td>30</td>
<td>12</td>
<td>16 (0)</td>
<td>[1 : 2 : 1], [2 : 1 : 1]</td>
</tr>
<tr>
<td>31*</td>
<td>6</td>
<td>4 (4)</td>
<td>[0 : 1 : 1]</td>
</tr>
<tr>
<td>33</td>
<td>12</td>
<td>8 (0)</td>
<td>[1 : 1 : 1]</td>
</tr>
<tr>
<td>34</td>
<td>12</td>
<td>8 (0)</td>
<td>[0 : 1 : 1], [1 : 0 : 1]</td>
</tr>
<tr>
<td>35*</td>
<td>7</td>
<td>16 (4)</td>
<td>[1 : 3 : 1], [3 : 1 : 1]</td>
</tr>
<tr>
<td>37</td>
<td>13</td>
<td>8 (0)</td>
<td>[1 : 4 : 2]</td>
</tr>
<tr>
<td>39*</td>
<td>12</td>
<td>8 (0)</td>
<td>[10 : 1 : 1]</td>
</tr>
<tr>
<td>41</td>
<td>13</td>
<td>4 (0)</td>
<td>[1 : 0 : 2]</td>
</tr>
</tbody>
</table>

The * indicates $d \equiv_4 3$. 


Chapter 6

Future Research

We have laid the foundations for the theory of Macfarlane spaces and their use in the study of hyperbolic 3-manifolds and surfaces. Many tools developed here would benefit from further development in this same context. More broadly, by generalizing some of the ideas we can build analogous constructions to study manifolds in higher dimensions and other geometries. We now conclude with a brief list of possibilities for future work.

1. **Anti-de Sitter Space.** Definition 3.2.2 of the standard Macfarlane space picks out a real slice of $\left( \frac{1}{2} \right)$ where the quaternion norm restricts to a quadratic form of signature (1, 3). One could similarly define other real slices of $\left( \frac{1}{2} \right)$ where the quaternion norm restricts to a quadratic form of signature (2, 2), by introducing other coefficients $\sqrt{-1}$ in the appropriate places. This approach could be used to study anti-de Sitter
CHAPTER 6. FUTURE RESEARCH

space, benefitting from recently established analogies between that and hyperbolic space [15, 24]. As with Macfarlane spaces, one would gain geometric and arithmetic relationships between points and isometries that may shed light on various open problems currently of interest [2].

(2) **Length Spectra**. Proposition 4.2.7 gives a way of identifying certain geodesics in a manifold with projective points over the manifold’s trace field, using pure quaternions. We saw the usefulness of this in finding sides and face-pairings when constructing Dirichlet domains in §5. This idea could be used more broadly as a new way of studying geodesics in hyperbolic 3-manifolds and surfaces, for instance in distinguishing commensurability classes using length spectrums as discussed in recent work by Reid [44].

(3) **Macfarlane Trigonometry.** Proposition 4.2.19 gives a way of computing the action of some $\gamma \in \text{Isom}^+(\mathbb{H}^3)$ using the decomposition $B = M + W$ of a normalized Macfarlane quaternion algebra $B$ into its Macfarlane space $M$ and that space’s anti-Hermitian complement $W$. This shows potential for more efficiently determining the axis and translation length of an isometry based on the location of its components on hyperboloid slices in $M$ and $W$. This concept parallels the
trigonometric analysis originally carried out by Macfarlane [34] but modernizes his notation with potential applications to the study of Macfarlane manifolds.

(4) **Volume Bounds.** The development of restricted Macfarlane spaces in §4.4 gives a way of studying hyperbolic surfaces admitting totally geodesic immersions in a Macfarlane 3-manifold. We see one application of this at the end of §5.2.2 where the immersion of the hyperbolic punctured torus in a Bianchi group leads to a lower bound on the height of orbit points for certain 3-manifolds, in the quaternion hyperboloid model. Another way of viewing this is that the subsurface leads to a lower bound on the injectivity radius at the center of the Dirichlet domain, which in turn puts a lower bound on the volume of the 3-manifold. It would be interesting to see how this could be generalized for more complicated classes of Macfarlane manifolds and their subsurfaces.

(5) **Non-arithmetic Examples.** The algorithm for computing Dirichlet domains given in §5.1 is carried out for some non-compact arithmetic examples in §5.2. The full power of this technique however is that unlike other uses of quaternion algebras in computing Dirichlet do-
mains for hyperbolic 3-manifolds and surfaces [41, 49], it works for classes of non-arithmetic manifolds. It would be valuable to see what new information can be gained in that setting, for instance in finding Dirichlet domains for the link complements studied by Chesebro and Debois [8]. It would be especially interesting to see the outcome of using non-standard hyperboloid models to find Dirichlet domains for non-arithmetic compact examples, such as those arising from Agol’s [1] inter-breeding technique.

(6) **Higher Dimensions.** The quaternion algebra containing a Macfarlane space $\mathcal{M}$ could equivalently be formulated as the even part of the Clifford algebra defined by the quadratic form on $\mathcal{M}$. One can similarly define a Clifford algebra from an arbitrary quadratic form over a space of any dimension and having any signature [29]. The Macfarlane space concept, in this way, suggests a generalization for the study of higher-dimensional geometries that can be characterized using quadratic spaces.

(7) **Solv 3-Manifolds.** There is a conjecture of Dennis Sullivan and Alberto Verjovsky that commensurability classes of 3-dimensional solv manifolds are in bijective correspondence with real quadratic fields
(Dennis Sullivan, personal communication, March 15, 2016). As a special case of the previous idea, one could gain insight into this by focusing on quadratic forms that arise as field norms of real quadratic fields, and checking if this leads to models for commensurability classes of solv 3-manifolds. Moreover, if the conjecture is true, then one could use this technique to learn more about solv 3-manifolds by developing explicit tools similar to those given for Macfarlane manifolds.
Index of Notation

* involution
* standard involution on $\mathcal{B}$ (quaternion conjugation)
† involution fixing a Macfarlane space
∼ equivalence relation
≈ homeomorphism, approximation
≅ isometry
≌ isomorphism
≡ congruence module $d$
α quaternionic Möbius action
$\text{Aut}_F(\mathcal{B})$ automorphism group of $\mathcal{B}$ over $F$
$\beta$ symmetric bilinear form
$\mathcal{A}$ quaternion algebra over a real field
$\mathcal{A}\Gamma$ invariant quaternion algebra of $\Gamma$
$\mathcal{B}$ quaternion algebra
$\ast S$ symmetric elements of $\ast$ in $S \subseteq \mathcal{B}$
$S_0$ pure quaternions in $S \subseteq \mathcal{B}$
$S_+$ quaternions of positive trace in $S \subseteq \mathcal{B}$
$S^1$ quaternions of norm 1 in $S \subseteq \mathcal{B}$
$S^0$ quaternions of norm 0 in $S \subseteq \mathcal{B}$
$\langle S \rangle_K$ $K$-algebra generated by $S \subset \mathcal{B}$
$B\Gamma$ quaternion algebra of $\Gamma$
$\mathcal{B}_\sigma$ completion of $\mathcal{B}$ at the place corresponding to $\sigma$
$|\cdot|$ complex modulus, set cardinality
$\mathbb{C}$ complex numbers
$D_\Gamma(c)$ Dirichlet domain for $\Gamma$ with center $c$
$\text{diag}(x_1, \ldots, x_\ell)$ diagonal matrix with diagonal $(x_1, \ldots, x_\ell)$
$\partial$ boundary
$\Delta$ Fuchsian group
index of notation

embedding
\(E(p)\) half-space containing 1 with boundary \(\bar{s}(p)\)
\(\text{End}_F(\mathcal{B})\) endomorphism ring of \(\mathcal{B}\) over \(F\)
\(\mathcal{G}(K : K')\) Galois group of \(K\) over \(K'\)
\(g(p, q)\) geodesic from \(p\) to \(q\)
\(\tilde{g}(p, q)\) complete geodesic passing through \(p\) and \(q\)
\(F\) real field
\(G_\phi^S\) Graham matrix of \(\phi\) with respect to the basis \(S\)
\(\Gamma\) Kleinian or Fuchsian group
\(\Gamma^{(2)}\) group generated by squares of elements in \(\Gamma\)
\(S^{(2)}\) set of squares in (group, ring, field, etc.) \(S\)
\(\Gamma_d\) Bianchi group \(\text{PSL}_2(\mathcal{O}_d)\)
\(\mathcal{H}^n\) upper half-space model for hyperbolic \(n\)-space
\(\mathcal{H}^n\) abstract hyperbolic \(n\)-space
\(\mathbb{H}\) Hamilton’s quaternions
\(\text{Herm}_n(S)\) \(n \times n\) Hermitian matrices with entries in \(S\)
\(\mathcal{I}^n\) standard hyperboloid model for hyperbolic \(n\)-space
\(\mathcal{I}^\phi\) hyperboloid model induced by \(\phi\)
\(\mathcal{I}^\Gamma\) quaternion hyperboloid model for \(\Gamma\)
\(\text{Inn}_F(\mathcal{B})\) inner automorphism group of \(\mathcal{B}\) over \(F\)
\(\text{Isom}^+(S)\) group of orientation-preserving isometries of the space \(S\)
\(i, j\) standard generators of a quaternion algebra
\(i\) preferred isometry from \(M_+^1\) to \(\mathcal{H}^3\)
\(\mathcal{J}\) ideal
\(K\) field
\(S^\times\) invertible elements of (field, group, algebra, order, etc) \(S\)
\([K : K']\) degree of the field extension \(K\) over \(K'\)
\(K_v\) completion of \(K\) at valuation \(v\)
\(k\Gamma\) invariant trace field of \(\Gamma\)
\(K\Gamma\) trace field of \(\Gamma\)
\(\ell(\gamma)\) translation length of \(\gamma\)
\(\mathcal{L}\) restricted Macfarlane space
\(\Lambda\) spinor representation of \(\text{PSL}_2(\mathbb{C})\)
\(\mathcal{M}\) Macfarlane space
\(M_d\) Bianchi orbifold \(\mathcal{H}^3/\mathcal{O}_d\)
\(\mu_B\) action by isometries of \(\mathbb{P}\mathcal{B}^1\) on \(M_+^1\)
\(\tilde{\mu}_B\) extension of \(\mu\) to \(\mathcal{B} \times \mathcal{M}\)
\(\mathbb{N}\) natural numbers
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>ideal norm of a field</td>
</tr>
<tr>
<td>n</td>
<td>quaternion (reduced) norm</td>
</tr>
<tr>
<td>N</td>
<td>normalized Macfarlane space</td>
</tr>
<tr>
<td>(\mathcal{O}_d)</td>
<td>ring of integers of the quadratic field (\mathbb{Q}(\sqrt{-d}))</td>
</tr>
<tr>
<td>Orb(_\Gamma)(c)</td>
<td>orbit of c under (\Gamma)</td>
</tr>
<tr>
<td>p</td>
<td>prime ideal</td>
</tr>
<tr>
<td>[p,q]</td>
<td>ring commutator of (p,q \in \mathcal{B})</td>
</tr>
<tr>
<td>(\Phi_B)</td>
<td>isomorphism from (\mathcal{B}) into \text{End}(\mathcal{B})) induced by (\widetilde{\mu}_B)</td>
</tr>
<tr>
<td>(\phi)</td>
<td>quadratic form</td>
</tr>
<tr>
<td>(\phi_{(m,n)})</td>
<td>standard quadratic form over (\mathbb{R}) of signature ((m,n))</td>
</tr>
<tr>
<td>(\rho_B)</td>
<td>preferred matrix representation of (\mathcal{B})</td>
</tr>
<tr>
<td>P</td>
<td>projection which takes the quotient by the center</td>
</tr>
<tr>
<td>(\mathbb{Q})</td>
<td>rational numbers</td>
</tr>
<tr>
<td>(\text{Ram}(\mathcal{B}))</td>
<td>ramification set of (\mathcal{B})</td>
</tr>
<tr>
<td>(\text{Ram}_p(\mathcal{B}))</td>
<td>finite ramification set of (\mathcal{B})</td>
</tr>
<tr>
<td>(\text{Ram}_\infty(\mathcal{B}))</td>
<td>infinite ramification set of (\mathcal{B})</td>
</tr>
<tr>
<td>(\mathbb{R})</td>
<td>real numbers</td>
</tr>
<tr>
<td>(\mathbb{R}_K)</td>
<td>ring of integers of (K)</td>
</tr>
<tr>
<td>(s(p))</td>
<td>side contributed to (\mathcal{D}<em>\Gamma(1)) by (p \in \mathcal{I}</em>\Gamma)</td>
</tr>
<tr>
<td>(\tilde{s}(p))</td>
<td>perpendicular bisector of (g(1,p))</td>
</tr>
<tr>
<td>(\text{sig}(\phi))</td>
<td>signature of (\phi) over its base field</td>
</tr>
<tr>
<td>(\text{sig}_F(\phi))</td>
<td>signature of (\phi) over (F)</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>field embedding</td>
</tr>
<tr>
<td>(\tau)</td>
<td>non-real field embedding into (\mathbb{C})</td>
</tr>
<tr>
<td>(\theta)</td>
<td>field embedding into (\mathbb{R})</td>
</tr>
<tr>
<td>tr</td>
<td>quaternion (reduced) trace, matrix trace</td>
</tr>
<tr>
<td>(V_t)</td>
<td>orbit points on (\mathcal{I}_\Gamma) at trace (t)</td>
</tr>
<tr>
<td>(\mathcal{W})</td>
<td>anti-Hermitian complement of (\mathcal{M})</td>
</tr>
<tr>
<td>(X)</td>
<td>manifold</td>
</tr>
<tr>
<td>(\mathcal{X})</td>
<td>geometry</td>
</tr>
</tbody>
</table>
Bibliography


