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Some 2-Categorical Aspects in Physics

Arthur Parzygnat
The Graduate Center, City University of New York

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Some 2-Categorical Aspects in Physics

by

Arthur J. Parzygnat

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

(required signature)

Date

Dist. Professor V. Parameswaran Nair
Chair of Examining Committee

Date

Professor Scott O. Wilson
Mathematics Advisor

(required signature)

Date

Professor Igor Kuskovsky
Executive Officer

Professor Alexios Polychronakos

Professor Thomas Tradler

Professor S. G. Rajeev

Supervisory Committee
Abstract

Some 2-Categorical Aspects in Physics

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Arthur J. Parzygnat

Advisors: V. Parameswaran Nair & Scott O. Wilson

2-categories provide a useful transition point between ordinary category theory and \( \infty \)-category theory where one can perform concrete computations for applications in physics and at the same time provide rigorous formalism for mathematical structures appearing in physics. We survey three such broad instances. First, we describe two-dimensional algebra as a means of constructing non-abelian parallel transport along surfaces which can be used to describe strings charged under non-abelian gauge groups in string theory. Second, we formalize the notion of convex and cone categories, provide a preliminary categorical definition of entropy, and exhibit several examples. Thirdly, we provide a universal description of the Gelfand-Naimark-Segal construction as a canonical procedure from states on \( C^* \)-algebras to representations of \( C^* \)-algebras equipped with pure state.
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Chapter 1

Introduction

1.1 Motivation

Category theory plays an important role by abstracting common structures among different mathematical disciplines, and hence physics. This allows one to (1) organize data more precisely, (2) relate seemingly different structures, and (3) prove statements applicable to many areas simultaneously and more easily. These three ramifications equally apply to nature, whose description is inherently mathematical. Indeed, the goal of physics is to (1) understand the basic structure (data) of nature organized into a (potentially unified) theory, (2) extrapolate this structure into one that is perceived by observers, and (3) and develop precise predictions that can be tested.\textsuperscript{1}

Practically every mathematical structure together with its symmetries

\textsuperscript{1}A more philosophical description of these concepts and why category theory is a preferred mathematical framework for physics is deferred to Section 1.3.
fits into the framework of category theory, though categories also allow non-invertible symmetries, i.e. non-invertible processes or relations. Categories consist of a collection of objects together with such relations and a way to compose relations. Consider, for example, the gauge theory of particles. The main category of study here is one of principal bundles with connection over a manifold $M$, the underlying manifold of spacetime, together with its gauge transformations. More precisely, an object consists of a principal group bundle over $M$ together with a connection. A morphism between two such objects is a smooth and equivariant connection-preserving map of principal bundles. One of the reasons to consider the full category as opposed to the set of connections over a fixed bundle is because the latter fixes a particular topological sector, sometimes called an instanton sector, whereas the former includes all such sectors [Sc16a].

A closely related example occurs in prequantum string theory [Sc16a]. Abelian gauge theory for strings was initiated by Kalb and Ramond in 1974 [KaRa74] and its geometrical description in terms of gerbes was provided by Gawedzki in 1987 [Ga88]. Gerbes are geometric structures on the target manifold analogous to bundles but used for higher-dimensional objects such as strings. Between 2009 and 2013, Waldorf showed that these geometric objects correspond to the usual notions of bundles with connection on the
Motivation

loop space [Wa12a], [Wa16], [Wa12b]. However, coupling gauge fields to matter on the loop space, for instance with a Dirac operator [Wit86], is still not fully understood even though a lot of recent progress has been made [StTe05], [St08a], [St08b], [KoMe13].

Because gerbes are formulated directly on the manifold, a different approach is to construct the analogues of the expected mathematical structures without referring to loop spaces. Non-abelian generalizations of bundles, known either as non-abelian gerbes or principal 2-bundles with 2-connection, were made possible with the use of 2-category theory and appeared later [BrMe05] eventually with a description of parallel transport for strings [BaSc04], [ScWa13]. Besides strings, higher-dimensional branes can also be charged under gauge groups that need not be abelian. It has been suggested that the proper mathematical framework for such higher-dimensional branes involves higher categorical analogues of bundles with connection known as principal $n$-bundles or $(n-1)$-gerbes with connection [Sc16b]. Although this has not been verified across all models currently studied in string theory, recent work indicates this may be the case [PaSa12], [FSS14], [JSW16]. This leads one to the notion of $n$-categories [ChLa04], which are known to play a role in physics ranging from loop quantum gravity to quantum mechanics [BaLa11], particularly in topological quantum field theory [Lu09]. $n$-categories consist
of objects, relations, relations between relations, and so on up until level $n$. Thus, $n$-categories get a bit more complicated as $n$ increases.

2-categories provide some intuition about $n$-categories yet are simple enough to work with. In particular, they can be conveniently used to define two-dimensional algebra, a notion of algebra that allows one to multiply in various directions along a surface instead of just along a single direction and hence might be used to describe gauge theory for strings. It is therefore important to explore the implications of this structure, which is done in Chapters 2 and 3 of this thesis. In Chapter 2, we construct, from elementary building blocks, a surface-ordered integral providing a construction of parallel transport for surfaces that includes non-abelian gauge fields. Although we do not explore this here, this may provide a step towards non-perturbative string theory. In Chapter 3, we provide several example computations and we show that magnetic flux for magnetic monopoles can be described in terms of such parallel transport.

Another context where category theory might play a vital role is in the concept of entropy. Although entropy has had an enormous impact on all of science ranging from physics to computer science to biology and more, it is still not a completely understood concept. Because entropy shows up in a variety of contexts such as classical mechanics [Bo77], information the-
MOTIVATION

ory [Sh48], quantum mechanics [EPR35], black hole thermodynamics [Be73], dynamical systems [Bo71], quantum field theory [CaCa04], etc., one might suspect that categorical language can be used to isolate key features of entropy common to all such scenarios. For instance, one of the main properties of entropy is that it is a convex function [Li75]. Recently, Baez, Fritz, and Leinster showed that entropy can be better thought of as a convex functor [BFL11], [BF14]. In the special cases considered, one can emphasize their result with the following rough slogan: “Although not every continuous convex function on probability spaces is proportional to the Shannon entropy function, every continuous convex functor is!” Actually, convex functors have not been adequately defined. In Chapter 4, we therefore develop the appropriate mathematical structure needed to make sense of these theorems and to provide a foundation to study many (if not all) forms of entropy from a categorical perspective.

Whether the above slogan is true in greater generality is one of the main questions of this project though we do not answer this in the present work. An answer in either direction would be useful. If all entropy functors are proportional to each other, we have a new characterization of entropy and hence a concrete approach to precisely defining entropy in contexts where it is less understood such as quantum field theory or black hole thermodynamics.
If not, then much like Rényi entropy generalizes Shannon entropy [Re61], this form of entropy will also provide another perspective that should be explored in more detail. This is especially important now due to the recent interest in entropy and entanglement and its connection between field theory and geometry [RyTa06].

Yet another instance where 2-categories play a crucial role is in the description of the Gelfand-Naimark-Segal (GNS) construction [Se47], which roughly says the following. Given an algebra of observables together with a state, which need not be pure, there exists a canonical representation of this algebra together with a (cyclic) vector on the associated Hilbert space whose restriction, when viewed as a state, to the algebra agrees with the original state. This statement alone can be described in terms of ordinary categories. However, if one wants to include the possibility of changing the algebra of observables, ordinary categories are not enough. For example, observers typically do not have access to the entire algebra of observables and hence can only observe the state with respect to a subalgebra. The GNS construction can be applied to the two resulting states and their associated representations are in general inequivalent. Nevertheless, they are related and one embeds into the other. These descriptions can be formulated using 2-categorical adjunctions.
In summary, this thesis explores three applications of category theory in
gauge theory, entropy, and the observables and states of quantum theory.
Two-dimensional algebra is explored from a computational perspective and
is used to describe certain surface observables in gauge theory. The notion of
convex category is introduced and lays the foundation for future work aimed
at understanding entropy. Finally, a higher-categorical phrasing of the GNS
construction brings in an entirely new set of tools which could be used to
study observables, states, and their relations in more detail.

1.2 Contents

In Chapter 2, specifically Sections 2.2.1 and 2.2.2, we begin with a visual
and computationally accessible introduction to categories and 2-categories
in terms of string diagrams. In the latter case, these string diagrams provide
a realization of what we call two-dimensional algebra. We then focus on the
special case where the two-dimensional operations are invertible in Section
2.2.3 providing the main class of examples given by 2-groups.

Before moving to parallel transport for strings, we review parallel trans-
port for particles, particularly the path-ordered integral in Section 2.3.1.
We then apply these techniques to non-abelian parallel transport in gauge
theory for strings in Section 2.3.2. As an explicit calculation and illustra-
ition of two-dimensional algebra, we break up an arbitrary worldsheet for an open string (defined as a smooth map of a square, equipped with appropriate orientation data, into a manifold) using a lattice approximation. We then organize the terms explicitly exhibiting the non-abelian nature of the natural surface-ordered product that emerges in terms of infinitesimal group elements associated to links and plaquettes. This seems to be the first such fully worked out calculation in the literature and provides a connection between the heuristic argument of Baez and Schreiber in [BaSc04] and the formula provided by Schreiber and Waldorf who proved that their formula satisfies the necessary functorial properties but did not construct the expression from scratch [ScWa11]. In particular, we prove not only convergence but also a simplification of the formula in such a way so that the surface-ordered integral can be described by first performing an ordinary integral in one direction followed by a path-ordered integral in the leftover direction.

We also derive formulas for the 3-form curvature of a connection for surface transport in Section 2.3.5 by calculating an infinitesimal Wilson cube. The result is an illustration of the idea behind two-dimensional algebra, which is depicted in Figure 1.1. We also obtain all types of gauge transformations in theories with 1-form and 2-form gauge fields, which are described in detail in Section 2.3.3. This is merely a review and most details can be found in
Girelli and Pfeiffer’s work [GiPf04] though we use this opportunity to exhibit more examples of how to use two-dimensional algebra. A global geometric description of such forms is given by principal 2-bundles with 2-connections, though in this chapter, we only focus on topologically trivial 2-bundles and therefore do not require this concept.

Much, though not all, of Chapter 2 contains results that date back to work of Attal, Baez, Breen, Girelli, Messing, Pfeiffer, and Schreiber on non-abelian gerbes with connection [At04], [BaSc04], [BrMe05], [GiPf04]. However, these results are scattered, categorical language is used throughout, notations differ from author to author, and many include technicalities which we feel are not necessary for a working knowledge of the main results. Furthermore, string
diagram notation seems to not have been implemented at all even though its usefulness is known in many areas of physics (string diagrams are similar to tensor networks and have their origin in work of Penrose [Pe71]).

In Chapter 3, we describe global non-abelian parallel transport (as opposed to the local description of the preceding chapter) and follow the work of Schreiber and Waldorf closely summarizing many of their results in their four papers [ScWa09], [ScWa11], [ScWa], and [ScWa13]. We review global transport functors for ordinary gauge theory of particles relying on Čech covers in Section 3.2 and generalize this to gauge theory for strings in Section 3.3. In particular, we recall the definition of global transport 2-functors, which provide a geometric analogue for globally non-trivial bundles with connections used to describe parallel transport along paths and surfaces. Using these definitions, in Section 3.3.8 we explore the meaning of gauge invariance and prove that parallel transport along spheres can be used to define gauge invariant quantities after taking a quotient analogous to conjugacy classes used in the parallel transport along closed loops in gauge theory. This is an improvement of results by Schreiber and Waldorf who quotient out to a group analogous to the abelianization in order to have a well-defined surface transport [ScWa13]. We provide examples that illustrate how one loses information this way in general. Furthermore, we give a functorial description
for what it means to locally trivialize a transport 2-functor. By “abstract nonsense,” any two such choices of local trivializations are gauge equivalent.

There are several subtle issues that are addressed. Our infinite-dimensional manifolds are modeled on diffeological spaces, which are motivated by foundational work of Chen [Ch86]. One of the subtle issues regarding parallel transport is the choice of a basepoint for paths and the choice of a based path/loop for surfaces. This is made precise by the introduction of markings. Furthermore, besides being invariant under reparametrization, parallel transport along surfaces is invariant under thin homotopy (homotopies that do not sweep out any volume). We construct a diffeological space modeling the thin homotopy classes of unmarked spheres and show that surface holonomy is still well-defined as a map to the quotient mentioned above in the previous paragraph.

In Section 3.5, we give new examples of non-abelian transport 2-functors that show up in gauge theories with magnetic monopoles. Parallel transport along paths provides elements in the gauge group while parallel transport along surfaces provides elements in a cover of the gauge group. Thus, for non-trivial covers, the parallel transport along surfaces contains more information.

---

2 “Abstract nonsense” refers to arguments made in category theory.

3 Actually, Chen’s first definition was written down 13 years before this article. The present one cited is one that focuses on just the notion of smooth space.
than just the holonomy. We show it also contains information about magnetic monopoles, namely the magnetic charge obtained as a magnetic flux. We finally calculate this in several explicit examples starting with the Dirac monopole, moving to $SO(3)$, $SU(N)/Z(N)$, and $U(n)$ monopoles. Results of the previous section prove that magnetic flux is described as a gauge invariant surface holonomy.

Chapter 4 of this thesis pertains to analyzing the mathematical structure relevant to defining entropy in categorical terms. In this chapter, we define convex categories and convex functors. We jump straight into the abstract definition of a convex category, motivated by Šwirszcz’s algebraic definition of a convex set,\footnote{Unfortunately, we do not have access to Šwirszcz’s work \cite{Sw74} and will rely on Flood’s (well-written) exposition in \cite{Fl80}.} which we also internalize in any cartesian monoidal category (cartesian monoidal categories are reviewed in an Appendix). To my knowledge, this is the first appearance of such a definition in the literature though Leinster has some informal notes and blogs on an operadic definition \cite{Le11} that we avoid since Šwirszcz’s definition can be categorified without too much work. In fact, our approach clarifies several difficulties with respect to “multiplication by 0” which was mentioned in \cite{Le11} and \cite{BF14}. Afterwards, we provide some examples of convex categories and show how
some are related to each other. We focus on finite probability spaces with measure-preserving maps, finite probability spaces with measure-preserving stochastic maps, probability density functions on measure spaces, and a few more.

Before a robust and accurate definition of entropy can be made, we introduce cone categories, which are categories with an action of positive real numbers on them together with additional structure. Every cone category has a canonical convex structure. Convex functors between convex categories (and hence cone categories) are also introduced including appropriate notions of structure preserving natural transformations. Cone categories are important because they are the target categories of entropy when viewed as a functor. In the process, we provide definitions and results towards a theory of convex analysis in categories. Following this, we provide a preliminary definition of entropy as a convex functor from a convex category into a cone category. We define the notion of proportionality between such functors and prove several statements about these structures. In this thesis, we have only been able to provide the motivating example of Baez, Fritz, and Leinster [BFL11]. In the near future, we expect to provide several more in the context of quantum probability theory.

Chapter 5 contains the final part of this thesis and begins with a self-
contained introduction to the concepts of $C^*$-algebras, representations, and the Gelfand-Naimark-Segal (GNS) construction. We provide physical insight and relate the abstract concepts to simple examples from quantum mechanics, field theory, and quantum field theory in curved spacetime. We describe states and representations as presheaves (technically prestacks), and show that the GNS construction can be viewed as a left-adjoint to the operation that takes a representation and a vector and produces a state via restriction:

$$
\begin{tikzcd}
\text{C}^*\text{-Alg}^\text{op} \arrow[r, swap, bend left=30, yshift=1em] \arrow[r, bend left=30, yshift=-1em] & \text{States} \arrow[r, swap, bend left=30, yshift=1em] \arrow[r, bend left=30, yshift=-1em] & \text{Cat}.
\end{tikzcd}
$$

For the reader unfamiliar with adjunctions, one can think of them as the next best thing to equivalences (see Section 1.3 for a philosophical description)—they are defined precisely in an Appendix contained in Chapter 5. In the present context, this adjunction says the following. Given a $C^*$-algebra, a representation on a Hilbert space, and a vector state in that Hilbert space, one can pull back that state to the algebra (the resulting state need not be pure anymore). Now, forget the representation and only remember the state on the algebra. Can one reconstruct the representation and original vector state back? The answer is no in general, but what one can do is construct the “optimal” representation back which sits inside the original one as a sub-
space on which the vector state is cyclic. This characterization extends to the
category of $C^*$-algebras and hence includes the notion of restricting to a sub-
 algebra of observables (or perhaps an operation that identifies observables).
As an example, such an operation occurs for observers outside a stationary
black hole: the initial algebra is restricted to the algebra of observables acces-
sible by an outside observer and the pure vacuum state becomes the thermal
state with temperature given by the Hawking temperature [Wa94]. Because
it is important to allow for such changes of $C^*$-algebras, depending, for in-
stance, on the observer, the adjunction necessarily becomes 2-categorical.

Finally, an overall Appendix contains background material on 2-categories,
particularly compositions of functors, natural transformations, and modifi-
cations. These concepts are used in Chapters 3, 4, and 5. Chapter 2 can
be read without this Appendix. All references are listed at the end of the
thesis.

1.3 Discussion on category theory

The following is not necessary to understand the body of work but is meant to
place the reader in a state of mind that justifies the presentation of material
and particularly for choosing category theory as the underlying framework.

Over the years, I have discovered a language that not only helps organize
facts, but provides unifying perspectives on concepts from various fields. This language is category theory. I will sketch below three reasons why I think it is an essential tool for the next generation of physics. The first is an organizational tool to distinguish data and structures from conditions and properties. The second is a common language that would allow for easier interdisciplinary interactions. The third is to unify concepts from various fields and explore consequences from essential structure to the key problem at hand, i.e. localizing at relevant information. These three reasons were briefly mentioned at the beginning of Section 1.1. The reader might agree that these three reasons are important, but perhaps might not see how category theory addresses them. I will attempt to explain this now.

1. The first thing we must understand is the difference between structure and property. When we say whether or not something satisfies a particular property, we are asking a “yes” or “no” question. When we say something has structure, we specify data from a potentially large collection of choices. The two have different information-theoretic measure. Namely, a property is boolean whereas a structure may be infinite.

A simple example will help to clarify. A semigroup is defined in terms of its structure as a set $M$ together with a binary operation $M \times M \rightarrow M$. 
In general, the choice of such an operation is far from unique.\(^5\) The associativity of \(\mu\), however, is a property: is or is not \(\mu\) associative? Sometimes there are instances where some confusion may arise. For instance, if an element \(e \in M\) exists that acts as an identity for \(\mu\), then it is unique. Hence, the choice of such an \(e\) (structure) and the mere existence (property) are closely related: namely, if \((M, \mu)\) has the property that such an \(e\) exists, then it is unique and there is no additional data in choosing it and hence no additional structure.\(^6\)

Category theory helps to organize structures and properties by describing its objects and morphisms in terms of data satisfying certain conditions.

More importantly, it is likely that information-theoretic concepts underlie the structure of nature. For example, to name a few:

- Special relativity is based on the assumption that there exists a maximum speed at which information can travel.

\(^5\)Consider for instance the semigroups \(\mathbb{Z}_4 := \{0, 1, 2, 3\}\) with addition modulo 4 and \(\mathbb{Z}_2 \times \mathbb{Z}_2 := \{0, 1\} \times \{0, 1\}\) with addition component wise modulo 2. These two sets are (essentially) the same (one could have represented them as the same 4-element set—we have chosen not to in order to make them appear more familiar) yet their binary operations are different and there is no isomorphism between them preserving this structure.

\(^6\)There have been modifications to this principle in recent years due to the advent of abstract homotopy theory and \(\infty\)-categories. Namely, if the space of choices for structure is contractible, then there is no additional data in choosing one (even if the number of options is infinite). The case I am talking about is when this space is a point.
• Unitarity in quantum mechanics is a realization of conservation of information.

• The Bekenstein bound indicates that there is a maximum storage capacity for information in space.

Standard mathematics has helped an enormous amount towards our understanding of nature, but to go beyond our current state of knowledge, it may be fruitful to use a language more suitable for information-theoretic purposes. This language might be category theory.

2. The number of specific subjects in physics is overwhelming. As a result, a specialist in one field may find it difficult to study a completely different field. However, we know that many big advancements in physics, mathematics, and all of science have been made thanks to connections between different fields. Such connections can be made more accessible by providing data as described in the previous point and using said information to create a new mathematical object in another category. Such constructions are described by functors between categories when relations are transferred in a reasonable manner. Invariants are common examples of functors since they are described in terms of data associated to objects that do not change (or change in a well-understood
and invertible way) under symmetries, special kinds of relations.

There are only a few general kinds of functors that can be constructed, and in general, functors can only forget things—you can never gain more information than you began with from a single functor alone. However, often in practice, functors remember some information. This happens for instance if they are full, faithful, essentially surjective, or some combination of these. A functor which is all of these is one that is an equivalence and remembers all the essential information. Adjunctions are the next best thing after equivalences in terms of remembering data.

This is part of my motivation for including Chapter 5 in this thesis. I knew nothing about $C^*$-algebras, their representation theory, nor the Gelfand-Naimark-Segal (GNS) construction up until a few months ago. However, as suggested above, constructions are almost always functors or some variant, and if you are lucky and your construction satisfies some universal property, then it is also likely this functor is part of an adjunction. This was indeed the case and it helped me understand what the GNS construction does. Now, for somebody who understands category theory, they can understand what the statement of the GNS
construction is after just learning the corresponding categories, which merely involves learning the definition of $C^*$-algebra, states, and representations. Due to the universality of adjunctions, there is no need to actually know the GNS construction to see what it is saying!

3. Finally, category theory provides a framework to relate different concepts between categories that one might not have been able to easily express otherwise. For instance, the Gelfand-Naimark theorem expresses an equivalence of categories between locally compact Hausdorff topological spaces and commutative $C^*$-algebras. Thus, the two descriptions of information—algebra on the one hand and topology on the other—are equivalent. As a result, techniques that have been developed in one field might be used to provide previously unknown results in the other. It also brings new perspectives to different regimes allowing further development. A great example of this, related to the equivalence mentioned above, is non-commutative geometry.
Chapter 2

Two-dimensional algebra and gauge theory

2.1 Introduction

In this chapter, we use string diagrams to express many concepts in gauge theory in the broader context of two-dimensional algebra. By two-dimensional algebra, we mean the manipulation of algebraic quantities along surfaces. Such manipulations are dictated by 2-category theory and we include a thorough and visual introduction to 2-categories based on string diagrams. We postulate simple rules for associating algebraic data to surfaces with boundary and use the rules of two-dimensional algebra to derive non-abelian surface transport from infinitesimal pieces arising from a triangulation/cubulation of the surface. One of the novelties in this work is an analytic proof for the convergence of surface transport together with a more direct derivation of the
iterated surface integral. To provide a more or less self-contained reference, we also include discussions on gauge transformations, orientation data on surfaces, and a two-dimensional calculation of a Wilson cube deriving the curvature 3-form. We also review ordinary transport for particles to make the transition from one-dimensional algebra to two-dimensional algebra less mysterious.

Ordinary algebra, matrix multiplication, group theory, etc. are special cases of one-dimensional algebra in the sense that they can all be described by ordinary category theory. For example, a group is a type of category that consists of only a single object. Thanks to the advent of higher category theory, beginning with the work of Bénabou on 2-categories [Bé67], it has been possible to conceive of a general framework for manipulating algebraic quantities in higher dimensions. In particular, monoidal categories and the string diagrams associated with them [JSV96] can be viewed as 2-categories with a single object. The special case of this where all algebraic quantities have inverses are known as 2-groups, with a simple review given in [BaHu11] and a more thorough investigation in [BaLa04]. We do not expect the reader to be knowledgeable of these definitions and we only assume the reader knows about Lie groups (even a heuristic knowledge will suffice since our formulas will be expressed for matrix groups).
While there already exist several articles [BaHu11], [Pf03], [GiPf04], [ScWa13], introducing the conceptual basic ideas of higher gauge theory and parallel transport for strings in terms of category theory and even a book by Schreiber describing the mathematical framework of higher-form gauge theories [Sc16b], few provide explicit and computationally effective methods for calculating such parallel transport. Although Girelli and Pfeiffer explain many ideas, most results useful for computations are infinitesimal and it is not clear how to build local quantities from the infinitesimal ones [GiPf04]. Baez and Schreiber focus on similar aspects as we do in this article, but our presentation is significantly simplified since we do not work on path spaces [BaSc04]. Our goal is to provide tools and visualizations to perform calculations.

2.1.1 Background

In 1973, Kalb and Ramond first introduced the idea of coupling classical abelian gauge fields to strings in [KaRa74]. Actions for interacting charged strings were written down together with equations of motions for both the fields and the strings themselves. Furthermore, a little bit of the quantization of the theory was discussed. The next big step took place in 1985 with the work of Teitelboim (aka Bunster) and Henneaux, who introduced higher
form abelian gauge fields which could couple to higher-dimensional manifolds [Te86], [HeTe86]. In [Te86], Teitelboim studied the generalization of parallel transport for higher dimensional surfaces and concluded that non-abelian $p$-form gauge fields for $p \geq 2$ cannot be coupled to $p$-dimensional manifolds in order to construct parallel transport. The only possibilities are abelian gauge fields.

As a result, it seemed that only a few tried to get around this in the early 1980’s. For example, the non-abelian Stoke’s theorem came from analyzing these issues in the context of Yang-Mills theories and confinement [Ar80] (see also for instance Section 5.3 of [Ma02]). Although such calculations led people to believe non-abelian surface parallel transport is possible, reparametrizations and gauge transformations caused some issues. Without a different perspective, interest in it seemed to fade.

The issue in the argument of Teitelboim is related to the fact that higher homotopy groups are abelian and is sometimes also known as the Eckmann-Hilton argument [BaHu11]. However, J. H. C. Whitehead in 1949 realized that higher relative homotopy groups can be described by non-abelian groups [Wh49]. In fact, it was Whitehead who introduced the concept of a crossed module to describe homotopy 2-types. This work was in the area of algebraic topology and the connection between crossed modules and
higher groups were not made until much later. A review of this is given in [BaHu11]. Eventually, non-abelian generalizations of parallel transport for surfaces were made using category theory and ideas from homotopy theory stressing that one should also associate differential form data to lower-dimensional submanifolds beginning with the work of Girelli and Pfeiffer [GiPf04]. Before this, most of the work on non-abelian forms associated to higher-dimensional objects did not discuss parallel transport but developed the combinatorial and cocycle data [At04], [Pf03] building on the foundational work of Breen and Messing [BrMe05]. The idea of decorating lower-dimensional manifolds is consistent with the explicit locality exhibited in the extended functorial field theory approach to axiomatizing quantum field theories [Se88], [At88], [BaDo95], [Lu09]. Most recently, in a series of four papers, Schreiber and Waldorf axiomatized parallel transport along curves and surfaces [ScWa09], [ScWa11], [ScWa], [ScWa13], building on earlier work of Caetano and Picken [CaPi94].

2.1.2 Motivation

We have already indicated one of the motivations of pursuing an understanding of parallel transport along surfaces, namely in the context of string theory. Strings can be charged under non-abelian groups and interact via non-abelian
differential forms. Just as parallel transport can be used to described non-perturbative effects in ordinary gauge theories for particles, parallel transport along higher-dimensional surfaces might be used to describe non-perturbative effects in string theory and M-theory.

Furthermore, higher form symmetries have been of recent interest in high energy physics and condensed matter in the exploration of surface operators and charges for higher-dimensional excitations [GKSW15]. However, the forms in the latter are strictly abelian and the proper mathematical framework for describing them is provided by abelian gerbes (aka higher bundles) [MaPi02], [TWZ12]. Higher non-abelian forms appear in many other contexts in physics, such as in a stack of D-branes in string theory [My99], in the ABJM model [PaSa12], and in the quantum field theory on the M5-brane [FSS14]. In fact, [PaSa12] show how higher gauge theories provide a unified framework for describing certain M-brane models and how the 3-algebras of [BaLa07] can be described in this framework.

Although a description of the non-abelian forms themselves is described by higher differential cohomology [Sc16b], parallel transport seems to require additional flatness conditions on these forms. For example, in the special case of surfaces, this condition is known as the vanishing of the fake curvature. Some argue that this condition should be dropped and issues of parallel
transport are not as important [Ch11]. However, our perspective is to take this condition seriously and work out some of its consequences. Indeed, since higher-dimensional objects can be charged in many physical models, parallel transport might be used to study non-perturbative aspects of theories, an important tool to understand quantization (see the illuminating discussion at the end of [Sc15]). Because it is not yet known how to avoid these flatness conditions, further investigation is necessary.

Therefore, because of the subject’s infancy, it is a good idea to devote some time into understanding how to calculate surface transport explicitly to better understand how branes of different dimensions can be charged under various gauge groups. Here, we focus on the case of two-dimensional surfaces such as strings, or D1-branes.

2.1.3 Outline

In Section 2.2, we describe how categorical ideas can be used to express algebraic concepts. Namely, in Section 2.2.1, we review in detail “string diagrams” for ordinary categories and how group theory arises as a special case of ordinary category theory. In Section 2.2.2, we define 2-categories and other relevant structures providing a two-dimensional visualization of the algebraic quantities in terms of string diagrams. In Section 2.2.3, we specialize to the
case where the algebraic data are invertible. We restrict our attention to strict 2-groups, which is sufficient for many interesting applications.

In Section 2.3, we describe how certain aspects of gauge theory for 0-dimensional objects (particles) and 1-dimensional objects (strings) can be expressed conveniently in the language of two-dimensional algebra. In detail, in Section 2.3.1, we review how classical gauge theory for particles is described categorically. We include a review of the formula for parallel transport describing it in terms of one-dimensional algebra as an iterated integral obtained from a discretization and a limiting procedure. In Section 2.3.2, we include several crucial calculations for gauge theory for 1-dimensional objects (strings) expressing everything in terms of two-dimensional algebra. In particular, we derive the local infinitesimal data of a gauge theory. To our knowledge, these ideas seem to have first been analyzed in [At04], [GiPf04], and [BaSc04], though our inspiration for this viewpoint came from [ChTs93]. Furthermore, we use the rules of two-dimensional algebra to construct an explicit formula for the discretized and continuous limit versions of the local parallel transport of non-abelian gauge fields along a surface. Although the resulting formula appears in the literature [BaSc04], [ScWa11], we provide a more direct and tractable expression useful for computations along with an analytic justification for the formula. In the process, we illustrate the correct
surface ordering needed to describe parallel transport along surfaces with non-abelian gauge fields. In Section 2.3.3, we study the gauge covariance of the earlier expressions and derive the infinitesimal counterparts in terms of differential forms. In Section 2.3.4, we discuss the subtle issue of orientations of surfaces and how our formalism incorporates them. In Section 2.3.5, we again use two-dimensional algebra to calculate a Wilson cube on a lattice and from it obtain the 3-form curvature. We then study how it changes under gauge transformations.

Finally, in Section 2.4 we discuss some indication as to how these ideas might be used in physical situations and indicate lines of future work and open questions.

2.1.4 Acknowledgements

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2.2 Categorical algebra

2.2.1 Categories as one-dimensional algebra

We do not assume the reader is familiar with categories in this paper. We will present categories in terms of what are known as “string diagrams” since we find that they are simpler to manipulate and compute with when working with 2-categories. Therefore, we will define categories, functors, and natural transformations in terms of string diagrams. Afterwards, we will make a simplification and discuss special examples of categories known as groups.

Definition 2.2.1. A category, denoted by \( \mathcal{C} \), consists of

i) a collection of 1-d domains (aka objects)

\[
\begin{array}{ccc}
\text{R} & \text{V} & \text{A} \\
\end{array}
\]

(labelled for now by some color),

ii) between any two 1-d domains, a collection (which could be empty) of 0-d defects (aka morphisms)\(^1\)

---

\(^1\)Technically, 0-d defects have a direction/orientation. See Remark 2.2.2 for further details. In this paper, the convention is that we read the expressions from right to left (this will be more consistent with what we’re used to). Hence, \( g \) is thought of as “beginning” at \( A \) and “ending” at \( R \) or transitioning from \( A \) to \( R \). In many cases, as in the theory of...
iii) an “in series” composition rule

$$g_2 g_1$$

whenever 1-d domains match,

iv) and between every 1-d domain and itself, a specified 0-d defect

$$e$$

called the identity.

In groups, we will always be able to go back by an inverse operation. However, in general, $g$ will merely be a transformation from $A$ to $R$. If at any point confusion may arise as to the direction, we will signify with an arrow close to the 0-d defect.
These data must satisfy the following conditions

(a) The composition rule is associative.

(b) The identity 0-d defect is an identity for the composition rule.

**Remark 2.2.2.** For the reader familiar with categories, we are defining them in terms of their Poincaré duals. The relationship can be visualized by the following diagram.

In this article, we may occasionally use the notation

\[ R \leftarrow \begin{array}{c} g \\ \end{array} \rightarrow A \]  

instead and denote the 1-d domains as “objects” and the 0-d defects as “morphisms.” The motivation for using the terminology of domains and defects comes from physics.

**Example 2.2.4.** Let \( G \) be a group. From \( G \), one can construct a category, denoted by \( \mathbb{B}G \), consisting of only a single domain (say, red) and the 0-d defects from that domain to itself consist of all the elements of \( G \). The composition is group multiplication. The identity at the single domain is the identity of the group.
The previous example of a category is one in which all 0-d defects are invertible.

**Definition 2.2.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an assignment sending 1-d domains in $\mathcal{C}$ to 1-d domains in $\mathcal{D}$ and 0-d defects in $\mathcal{C}$ to 0-d defects in $\mathcal{D}$ satisfying

(a) the source-target matching condition

(b) preservation of the identity

(c) and preservation of the composition in series

This last condition can be expressed by saying that the following triangle of defects commutes
meaning that going left along the top two parts of the triangle and composing in series is the same as going left along the bottom. There are several ways to think about what functors do. On the one hand, they can be viewed as a construction in the sense that one begins with data and from them constructs new data in a consistent way. Another perspective is that functors are invariants and give a way of associating information that only depends on the equivalence class of 1-d defects. Another perspective that we will find useful in this article is to think of a functor as attaching algebraic data to geometric data. We will explore this last idea in Section 2.3.1. Yet another perspective is to view categories more algebraically and think of a functor as a generalization of a group homomorphism since the third condition in Definition 2.2.5 is precisely a generalization of this concept. We will explore this last perspective in in the following example.

**Example 2.2.6.** Let $G$ and $H$ be two groups and let $\mathbb{B}G$ and $\mathbb{B}H$ be their associated one-object categories as discussed in Example 2.2.4. Functors
$F : \mathbb{B}G \rightarrow \mathbb{B}H$ are in one-to-one correspondence with group homomorphisms $f : G \rightarrow H$.

**Definition 2.2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A *natural transformation* $\sigma : F \Rightarrow G$ is an assignment sending 1-d domains of $\mathcal{C}$ to 0-d defects of $\mathcal{D}$ in such a way so that

![Diagram](image)

and to every 0-d defect

![Diagram](image)

the condition

![Diagram](image)

must hold.
The last condition in the definition of a natural transformation can be thought of as saying both ways of composing in the following “square”

\[
\begin{array}{ccc}
G(R) & \sigma(R) & F(R) \\
\downarrow & \uparrow & \downarrow \\
G(A) & \sigma(A) & F(A)
\end{array}
\]

are equal (the arrows have been drawn to be clear about the order in which one should multiply), i.e. as an algebraic equation without pictures

\[\sigma(R) F(g) = G(g) \sigma(A).\] (2.2.8)

Natural transformations can be composed though we don’t need this now and will instead discuss this in greater generality for 2-categories later.

**Example 2.2.9.** Let \(G\) be a group and \(\mathbb{B}G\) its associated category. Let \(\text{Vect}_\mathbb{K}\) be the category of vector spaces over a field \(\mathbb{K}\). Namely, the 1-d domains are vector spaces and the 0-d defects are \(\mathbb{K}\)-linear operators between vector spaces. Let us analyze what a functor \(\rho : \mathbb{B}G \to \text{Vect}_\mathbb{K}\) is. To the single 1-d domain of \(\mathbb{B}G\), \(\rho\) assigns to it some vector space, denoted by \(V\). To every
group element $g \in G$, i.e. to every 0-d defect, $\rho$ assigns an invertible operator $\rho(g) : V \rightarrow V$. This assignment satisfies $\rho(e) = \text{id}_V$ and $\rho(gh) = \rho(g)\rho(h)$.

Thus, the functor $\rho$ encodes the data of a representation of $G$. Now, let $\rho$ and $\rho'$ be two representations, where the vector space associated to $\rho'$ is denoted by $V'$. A natural transformation $\sigma : \rho \Rightarrow \rho'$ consists of a single linear operator $\sigma : V \rightarrow V'$ satisfying the condition that

$$\sigma \rho(g) = \rho'(g)\sigma$$

(2.2.10)

for all $g \in G$. In other words, a natural transformation encodes the data of an intertwiner of representations of $G$.\(^2\)

### 2.2.2 2-categories as two-dimensional algebra

2-categories provide one realization of 2-d algebra.

**Definition 2.2.11.** A 2-category, also denoted by $\mathcal{C}$, consists of

1) a collection of 2-d domains (aka objects)

\[ R \quad V \quad A \]

\(^2\)For the reader not familiar with intertwiners, these are used to relate two different representations. For instance, the Fourier transform is a unitary intertwiner between the position and momentum representations of the Heisenberg algebra in quantum mechanics. As another example, all tensor operators are intertwiners.
(labelled for now by some color),

ii) between any two 2-d domains, a collection (which could be empty) of 1-d defects (aka 1-morphisms)

\[
\begin{array}{|c|c|c|}
\hline
R & g & A \\
\hline
\end{array}
\]

(often labelled by lower-case Roman letters),

iii) between any two 1-d defects that are themselves between the same two 2-d domains, a collection (which could be empty) of 0-d defects (aka 2-morphisms)\(^3\)

\[
\begin{array}{|c|c|c|}
\hline
R & g & V \\
\hline
\end{array}
\]

(often labelled by lower case Greek letters),

\(^3\)Technically, both 1-d defects and 0-d defects have direction as depicted in Remark 2.2.12. Our convention here is that 1-d defects go from right to left and 0-d defects go from top to bottom on the page. Occasionally, it will be convenient to move diagrams around and draw them sideways or in other directions for visual purposes. In these cases, we will label the directionality when it might be unclear.
iv) an “in parallel” composition rule for 1-d defects

\[ \text{Diagram} \]

v) an “in series” composition rule for 0-d defects

\[ \text{Diagram} \]

vi) an “in parallel” composition rule for 0-d defects

\[ \text{Diagram} \]
vii) Every 2-d domain $R$ has both an identity 1-d defect and an identity 0-d defect

\[ R \quad \text{id}_R \quad R \]

\[ R \quad \text{id}_{id_R} \quad R \]

respectively, and every 1-d defect has an identity 2-d defect

\[ R \quad \text{id}_g \quad V \]

These data must satisfy the following conditions.

(a) All composition rules are associative.\(^4\)

(b) The identities obey rules exhibiting them as identities for the two compositions.

\(^4\)This will be implicit in drawing the diagrams as we have.
The composition in series and in parallel must satisfy the “interchange law”

\[
\begin{array}{cc}
& h & k \\
g & j \\
\lambda & \sigma \\
f & i \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cc}
& h & k \\
\mu & \tau \\
g & j \\
\lambda & \sigma \\
\mu & \tau \\
\lambda & \sigma \\
fi & i \\
\end{array}
\]

Remark 2.2.12. The above depiction of 2-categories is related to the usual presentation of 2-categories via

and are called “string diagrams.”
Using this definition, we can actually make sense of combinations of defects such as

interpreting it as the composition in parallel of the top two 1-d defects along the common 2-d domain (drawn in green)

In fact, a 0-d defect can have any valence with respect to 1-d defects

but it is important to keep in mind which 1-d defects are incoming and outgoing. Our convention is that all incoming 1-d defects come from above
and all outgoing 1-d defects go towards the bottom of the page. Occasionally, we will go against this convention, and we will rely on the context to be clear, or to be cautious, we may even include arrows to indicate the direction. For example, this last 4-valence diagram might be drawn as

Furthermore, we can define composition in parallel between a 1-d defect and a 0-d defect as in

by viewing the 1-d defect as an identity 0-d defect for \( g \) and then use the already defined composition of 0-d defects in parallel
A similar idea can be used if the right side was just a 1-d defect. Using these rules, we can make sense of diagrams such as

by extending the left “dangling” 1-d defect to the bottom and the right “dangling” 1-d defect to the top as follows
Then we can compose in parallel to obtain

\[ h_{i} \sigma \text{id}_{i} \text{id} g_{k} \tau g_{j} \]

and finally compose in series

\[ h_{i} \sigma \text{id}_{i} \text{id} g_{k} \tau g_{j} \]
One must be cautious in such an expression. It does not make sense to compose $\sigma$ with $\text{id}_g$ alone in series because $k$ is an outgoing 1-d defect from $\sigma$. Therefore, the expression $\sigma^{\text{id}_i}_{\text{id}_g^\tau}$ must be calculated by first composing in parallel and then one can compose the results in series as we have done.

Examples of 2-categories related to groups will be given in Section 2.2.3.

**Example 2.2.13.** Let $\text{Hilb}$ be the category of Hilbert spaces, i.e. 1-d domains are Hilbert spaces and 0-d defects are linear operators. Let $\text{Hilb}_{\text{Isom}}$ be the subcategory whose 1-d domains are Hilbert spaces and 0-d defects are isometries. Finally, let $\text{Hilb}_{\text{Isom}}^{\text{proj}}$ be the two-category whose 2-d domains are Hilbert spaces, 1-d defects are isometries, and 0-d defects are elements of $U(1)$. More precisely, given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ and two isometries $L, K : \mathcal{H}' \rightarrow \mathcal{H}$ a 0-d defect from $L$ to $K$ is an element $\lambda \in U(1)$ such that $K = \lambda L$. The composition in series is given by the product of elements in $U(1)$.
and the in parallel composition is also defined by the product of elements in $U(1)$

The products $LL'$ and $KK'$ are given by the composition of linear operators. The reader should check that this is indeed a 2-category. Similarly, let $\text{Hilb}^{\text{proj}}$ be the 2-category whose 2-d defects are Hilbert spaces, 1-d defects are linear maps, and 0-d defects are non-zero complex numbers. All compositions are analogous to those of $\text{Hilb}^{\text{proj}}_{\text{Isom}}$.

**Definition 2.2.14.** Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A *(normalized) weak functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is an assignment sending $d$-dimensional domains/defects of $\mathcal{C}$ to $d$-dimensional domains/defects of $\mathcal{D}$ together with an assignment $c^F$ that associates to every pair of “in parallel” composable 1-d defects $f$ and $g$ in $\mathcal{C}$ an invertible 0-d defect in $\mathcal{D}$ interpolating from $F(f)F(g)$ to $F(fg)$ as in
These assignments must satisfy the following conditions.

(a) The assignment $F$ is such that

(b) All identities are preserved (this is the “normalized” condition).

(c) For any 1-d defect $f$

the equalities
i.e.

\[ c^F_{f, \text{id}_V} = \text{id}_{F(f)} = c^F_{\text{id}_R, f} \]  \hspace{1cm} (2.2.15)

must hold.

(d) To every triple of parallel composable 1-d defects

the equality
must hold.

If \( c_{f,g}^F \) is the identity for all \( f \) and \( g \) in \( C \), then \( F \) is said to be a strict functor.

**Remark 2.2.17.** For each pair of composable 1-d defects \( f \) and \( g \), the 0-d defect \( c_{f,g}^F \) can be thought of as filling in the triangle from the comments after Definition 2.2.5. Condition (d) resembles associativity. In fact, it is an example of a cocycle condition and will be discussed more in the following example (in particular, this definition allows one to define higher cocycles for non-abelian groups). Condition (d) can also be re-written as

\[
\frac{c_{f,g}^F \circ \text{id}_{F(h)}}{c_{f,g,h}^F} = \frac{\text{id}_F(c_{g,h}^F)}{c_{f,gh}^F}
\]  

(2.2.16)
which illustrates more of a connection to Pachner moves for triangulations of surfaces. However, this latter presentation requires arrows to keep track of incoming versus outgoing directions.

Examples of weak functors abound. In fact, projective representations are examples of weak functors that are not strict functors as will be explained in the following example. Furthermore, strict functors will be used as a means of defining parallel transport along surfaces in gauge theory in Section 2.3.2. Natural transformations will be used to define gauge transformations of such functors and their infinitesimal counterparts will be derived from these definitions.

**Example 2.2.18.** Let \( G \) be a group and \( \mathbb{B}G \) its associated category. Every category can be thought of as a 2-category by only allowing identity 0-d de-
fects. Namely, there is only a single 2-d domain, the 1-d defects are elements of $G$, and the 0-d defects are all identities. This 2-category will also be denoted by $\mathbb{B}G$. Let $\text{Hilb}_{\text{proj}}^\text{isol}$ be the 2-category introduced in Example 2.2.13.

A weak normalized functor $\rho : \mathbb{B}G \rightarrow \text{Hilb}_{\text{proj}}^\text{isol}$ encodes the data of a Hilbert space $\mathcal{H}$, an assignment $\rho : G \rightarrow U(\mathcal{H})$, and a function $c^\rho : G \times G \rightarrow U(1)$ in such a way so that to every pair of elements $g, h \in G$

\[
\rho(gh) = c^\rho_{g,h} \rho(g) \rho(h) \tag{2.2.19}
\]

and also

\[
\rho(e) = \text{id}_\mathcal{H}. \tag{2.2.20}
\]

Furthermore, $c$ satisfies the condition that to every triple $g, h, k \in G$,

\[
c^\rho_{gh,k} c^\rho_{g,h} = c^\rho_{g,hk} c^\rho_{h,k}. \tag{2.2.21}
\]

This is precisely the definition of a (normalized) projective unitary representation of $G$ on a Hilbert space $\mathcal{H}$. If we had not used $\text{Hilb}_{\text{isol}}^\text{proj}$ but in-
stead $\text{Hilb}^{\text{proj}}$, then $\rho : B G \longrightarrow \text{Hilb}^{\text{proj}}$ provides a Hilbert space $\mathcal{H}$, a map $\rho : G \longrightarrow \text{GL}(\mathcal{H})$, and a cocycle $c^\rho : G \times G \longrightarrow \mathbb{C}^\times$. Here $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$.

**Definition 2.2.22.** Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two weak functors between two 2-categories. A *natural transformation* $\sigma : F \Rightarrow G$ is an assignment sending $k$-d domains/defects of $\mathcal{C}$ to $(k - 1)$-d defects of $\mathcal{D}$ for $k = 1, 2$ satisfying the following conditions.

(a) The assignment is such that

\[
\begin{array}{ccc}
R & \longrightarrow & \sigma \\
\uparrow & & \uparrow \\
G(R) & \longrightarrow & \sigma(R) \\
& & \uparrow \\
& & F(R)
\end{array}
\]

and

\[
\begin{array}{ccc}
R & \Downarrow \sigma \\
g & \rightarrow & A \\
\sigma(R) & \longrightarrow & F(g) \\
G(g) & \rightarrow & \sigma(A)
\end{array}
\]

(b) To every pair of parallel composable 1-d defects

---

5The diagram on the right can be thought of as filling in the square from the comments after Definition 2.2.7 (rotate the square by 45° counterclockwise).
the equality

\[ \sigma_p R q F_p g q \sigma_p V q G_p g q c G_f g \sigma_p A q G_p fg q \sigma_p R q c F_f g \sigma_p fg q \]

must hold.

(i.e.)

\[
\frac{\sigma(f) \text{id}_{F(f)}}{\text{id}_{G(f)} \sigma(g)} = \frac{\sigma(R)c_{f,g}^F}{c_{f,g}^G \sigma(A)} = \frac{\sigma(R)c_{f,g}^F}{\sigma(fg)} \quad (2.2.23)
\]

must hold.

(c) To every identity 1-d defect \( \text{id}_R \) the equality

\[ \sigma(\text{id}_R) = \text{id}_{\sigma(R)} \quad (2.2.24) \]

must hold.
(d) To every 0-d defect

\[
\sigma_p f q \sigma_p R q F p f q \sigma_p A q G p f q G p \lambda q G p g q \sigma_p A q G p f q G p \lambda q \]

i.e.

\[
\sigma(f) G(\lambda) \text{id}_{\sigma(A)} = \text{id}_{\sigma(R)} F(\lambda) \sigma(g) \]

must hold.

Such string diagram pictures facilitate certain kinds of computations

[PoSh13] (for instance, compare the definition of natural transformation in
Natural transformations between functors can be thought of as symmetries. For example, just as natural transformations of functors between ordinary categories describe intertwiners for ordinary representations, natural transformations of functors between 2-categories describe intertwiners of projective representations.

**Example 2.2.26.** Using the notation of Example 2.2.18, let $\rho, \pi : B G \to \text{Hilb}_{\text{proj}}$ be two projective unitary representations on $\mathcal{H}$ and $\mathcal{K}$ with cocycles $c^\rho$ and $c^\pi$, respectively. A natural transformation $\sigma : \rho \Rightarrow \pi$ provides an isometry $\sigma^\mathcal{H}_\mathcal{K} : \mathcal{H} \to \mathcal{K}$ and a function $\sigma : G \to U(1)$, whose value on $g$ is denoted by $\sigma_g$ and fits into

\[
\begin{array}{c}
\mathcal{H} \\
\sigma^\mathcal{H}_{\mathcal{K}} \\
\mathcal{K} \\
\sigma_g \\
\mathcal{H} \\
\rho(g) \\
\mathcal{K} \\
\pi(g) \\
\end{array}
\]

which in particular says

\[
\pi(g)\sigma^\mathcal{H}_{\mathcal{K}} = \sigma_g\sigma^\mathcal{H}_{\mathcal{K}}\rho(g),
\]

(2.2.27)

satisfying the condition

\[
\sigma_{gh}c^\rho_{g,h} = c^\pi_{g,h}\sigma_g\sigma_h
\]

(2.2.28)
for all \( g, h \in G \). This provides the data of an intertwiner of projective unitary representations. If \( \rho, \pi : B \rightarrow \text{Hilb}^{\text{proj}} \) and \( \sigma : \rho \Rightarrow \pi \), then this gives a linear map \( \sigma^H_K : \mathcal{H} \rightarrow \mathcal{K} \) and \( \sigma : G \rightarrow \mathbb{C}^\times \) satisfying the above conditions.

It will be important to compose natural transformations. This will correspond to iterating gauge transformations successively.

**Definition 2.2.29.** Let \( E, F, G : C \rightarrow D \) be two weak functors between two 2-categories and let \( \sigma : F \Rightarrow G \) and \( \lambda : E \Rightarrow F \) be two natural transformations. The vertical composition of \( \sigma \) with \( \lambda \), written as (read from top to bottom)

\[
\lambda_{\sigma},
\]

is a natural transformation \( E \Rightarrow G \) defined by the assignment

\[
\begin{array}{cccc}
R & \rightarrow & G(R) & \sigma(R) & F(R) & \lambda(R) & E(R) \\
\end{array}
\]

on 2-d domains and

\[
\begin{array}{cccc}
R & g & A & \rightarrow & G(g) & \sigma(g) & F(g) & \lambda(A) \\
\end{array}
\]
on 1-d domains.

Technically, one should check this indeed defines a natural transformation. This is a simple exercise in two-dimensional algebra. There are actually similar symmetries between natural transformations, called *modifications*, which we define for completeness.

**Definition 2.2.31.** Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two weak functors between two 2-categories and $\sigma, \rho : F \Rightarrow G$ two natural transformations. A *modification* $m : \sigma \Rightarrow \rho$ assigns to every 2-d domain of $\mathcal{C}$ a 0-d defect in $\mathcal{D}$ such that the following conditions hold.

(a) The assignment is such that

(b) To every 1-d defect
the equality

\[ m(R) \rho(g) = \sigma(g) \rho(A) \]

i.e.

\[ m(R) \text{id}_{F(g)} \rho(g) = \sigma(g) \text{id}_{G(g)} m(A) \quad (2.2.32) \]

must hold.

### 2.2.3 Two-dimensional group theory

A convenient class of 2-categories are those for which there is only a single 2-d domain and all defects are invertible under all compositions. Such a 2-category is called a 2-group. 2-groups therefore only have labels on 1-d and 0-d defects. They can be described more concretely in terms of more familiar objects, namely ordinary groups.
**Definition 2.2.33.** A *crossed module* is a quadruple $\mathcal{G} := (H, G, \tau, \alpha)$ of two groups, $G$ and $H$, group homomorphisms $\tau : H \to G$ and $\alpha : G \to \text{Aut}(H)$, satisfying the two conditions

$$
\alpha_{\tau(h)}(h') = h h' h^{-1}
$$

(2.2.34)

and

$$
\tau(\alpha_g(h)) = g \tau(h) g^{-1}.
$$

(2.2.35)

Here $\text{Aut}(H)$ is the automorphism group of $H$, i.e. invertible group homomorphisms from $H$ to itself. If the groups $G$ and $H$ are Lie groups and the maps $\tau$ and $\alpha$ are smooth, then $(H, G, \tau, \alpha)$ is called a *Lie crossed module*.

Examples of crossed modules abound.

**Example 2.2.36.** Let $G$ be any group, $H := G$, $\tau := \text{id}_G$, and let $\alpha$ be conjugation.

**Example 2.2.37.** Let $H$ be any group, $G := \text{Aut}(H)$, let $\tau(h)$ be the automorphism defined by $\tau(h)(h') := hh'h^{-1}$, and $\alpha := \text{id}_{\text{Aut}(H)}$.

**Example 2.2.38.** Let $N$ be a normal subgroup of $G$. Set $H := N$, $\tau$ the inclusion, and $\alpha$ conjugation.

**Example 2.2.39.** Let $G$ be a Lie group, $\tau : H \to G$ a covering space, and $\alpha$ conjugation by a lift. For instance, $\exp\{2\pi i \cdot \} : \mathbb{R} \to S^1$ and the quotient
map $SU(n) \twoheadrightarrow SU(n)/Z(n)$ give examples. Here $SU(n)$ is the set of $n \times n$ special unitary matrices and $Z(n)$ is its center, i.e. elements of the form $e^{2\pi i/k} \text{id}_n$ with $k \in \mathbb{Z}$.

**Example 2.2.40.** Let $G := \{\ast\}$, the trivial group, $H$ any abelian group, $\tau$ the trivial map, and $\alpha$ the trivial map.

**Remark 2.2.41.** It is *not* possible for $H$ to be a non-abelian group if $G$ is trivial! In fact, for an arbitrary crossed module $(H, G, \tau, \alpha)$, $\ker(\tau)$ is always a central subgroup of $H$.

We now use crossed modules to construct examples of 2-categories, specifically 2-groups.

**Example 2.2.42.** Let $\mathcal{G} := (H, G, \tau, \alpha)$ be a crossed module. From $\mathcal{G}$, one can construct a 2-category, denoted by $\mathcal{B} \mathcal{G}$, consisting only of a single 2-d domain, the 1-d defects are labelled by elements of $G$ and the 0-d defects are labelled by elements of $H$. However, such labels must be of the form
Composition of 1-d defects in parallel is the group multiplication in $G$ just as in $\mathbb{B}G$. Composition of 0-d defects in series is defined by

\[
\tau(h)g_1 \tau(h') \rightarrow \tau(h'h)g_1
\]

Composition of 0-d defects in parallel is defined by

\[
\tau(h_2)g_2 \tau(h_1) \rightarrow \tau(h_2\alpha_{g_2}(h_1))g_2g_1 \tau(h_2)g_2\tau(h_1)g_1
\]

Notice that the outgoing edge is consistent with our definitions

\[
\tau(h_2\alpha_{g_2}(h_1))g_2g_1 = \tau(h_2)g_2\tau(h_1)g_2^{-1}g_2g_1 = \tau(h_2)g_2\tau(h_1)g_1
\]

(2.2.43)
due to (2.2.35).

The identities are given as follows. The 1-d defect identity associated to the single 2-d domain is the 1-d defect labelled by $e$, the identity of $G$. The
identity 0-d defect associated to a 1-d defect labelled by $g$ is labelled by slight abuse of notation $e$, the identity of $H$. It follows from these two definitions that the identity 0-d defect associated to the single 2-d domain is labelled by the identity on both the 1-d and 0-d defects. These three identities are depicted visually as

![Diagram](image1)

respectively.

The inverse of the 1-d defect labelled by $g$ for the parallel composition of 1-d defects is just the 1-d defect labelled by $g^{-1}$. Inverses for 0-d defects are depicted for “in series” composition by

![Diagram](image2)
and “in parallel” composition by

and similarly on the left. Notice that 0-d defects have two inverses for the two compositions.

This last class of examples of 2-groups from crossed modules will be used throughout this paper. In fact, all 2-groups arise in this way.

Theorem 2.2.44. For every 2-group, let $G$ be the set of 1-d defects and let $H$ be the set of 0-d defects of the form
(i.e. 0-d defects whose source 1-d defect is \(e\)). Define \(\tau : H \rightarrow G\) by \(\tau(h) := g\) from 0-d defects of the above form. Set \(\alpha_g(h)\) to be the resulting 0-d defect obtained from the composition

\[
\begin{array}{c}
g & \xrightarrow{h} & g^{-1} \\
\tau(h) & & \\
\end{array}
\]

The product in \(G\) is obtained from the composition of 1-d defects in parallel and the product in \(H\) is obtained from the composition of 0-d defects in series. With this structure, \((G, H, \tau, \alpha)\) is a crossed module. Furthermore, this correspondence between crossed modules and 2-groups extends to an equivalence of 2-categories [BaHu11].

We now provide some examples of 2-groups along with weak functors between them to illustrate their meaning.
Example 2.2.45. Let $G$ be a group and $\mathcal{H}$ a Hilbert space. Let $U(\mathcal{H})$ denote the unitary operators of $\mathcal{H}$. Let $\mathcal{G}$ be the crossed module $\langle \{1\}, G, !, ! \rangle$, where the $!$ stand for the trivial map and trivial action, respectively. Let $U(\mathcal{H})$ be the crossed module $(U(1), U(\mathcal{H}), \tau, \alpha)$ with $\tau(e^{i\theta}) := e^{i\theta}1_\mathcal{H}$ and $\alpha$ the trivial action. By definition, a weak functor $\rho : \mathcal{G} \longrightarrow U(\mathcal{H})$ consists of a function $\rho : G \longrightarrow U(\mathcal{H})$ and a function $c^\rho : G \times G \longrightarrow U(1)$ of the form sending $(g, h)$ to

\[
\rho(g) \rho(h) = \rho(gh),
\]

which in particular says

\[
c^\rho_{g,h} \rho(g) \rho(h) = \rho(gh), \quad (2.2.46)
\]

satisfying

\[
c^\rho_{g,e} = 1 = c^\rho_{e,g} \quad (2.2.47)
\]

for all $g \in G$ and

\[
c^\rho_{gh,k} c^\rho_{g,h} = c^\rho_{g,hk} c^\rho_{h,k} \quad (2.2.48)
\]
for all $g, h, k \in G$. This is precisely the definition of a (normalized) projective representation of $G$ on $\mathcal{H}$ and is really a special case of Example 2.2.18, where the Hilbert space is fixed from the start. The crossed module $\mathcal{U}(\mathcal{H})$ introduced here is actually the automorphism crossed module (in analogy to the automorphism group) of the Hilbert space $\mathcal{H}$ viewed as a 2-d domain in the 2-category $\text{Hilb}^{\text{proj}}_{\text{Isom}}$.

The following fact will be used in distinguishing two types of gauge transformations. It allows one to decompose an arbitrary gauge transformation into a composition of these two types.

**Proposition 2.2.49.** Let $\mathcal{C}$ be a category, $\mathcal{G} := (H, G, \tau, \alpha)$ a crossed module with associated 2-group $\mathbb{B}G$, and $F, F' : \mathcal{C} \rightarrow \mathbb{B}G$ two strict functors (so that $c^F$ and $c^{F'}$ are identities). A natural transformation $\sigma : F \Rightarrow F'$ consists of a function from 2-d defects of $\mathcal{C}$ to $G$, denoted by $g$

\[
\begin{array}{c}
\text{z} \\
\end{array}
\xrightarrow{\sigma}

\begin{array}{c}
g(z) \\
\end{array}
\]

and a function from 1-d defects of $\mathcal{C}$ to $H$, denoted by $h$. 

which says that
\[ \tau(h(\gamma))g(z)F(\gamma) = F'(\gamma)g(y), \] (2.2.50)
satisfying the axioms in the definition of a natural transformation. Thus, \( \sigma \) can be written as the pair \((g, h)\). Furthermore, there exists a strict functor \( F'' : C \rightarrow B \mathbb{G} \) such that the natural transformation \( \sigma \) is the vertical composition (recall Definition 2.2.29)
\[ \sigma = (g, e) \quad (e, h) \] (2.2.51)
namely, for any 1-d defect \( z \xleftarrow{\gamma} y \),

**Proof.** Define \( F'' : C \rightarrow B \mathbb{G} \) by sending a 1-d defect \( z \xleftarrow{\gamma} y \) of \( C \) to
and sending a 0-d defect

Using these definitions, one should check $F''$ is indeed a strict functor, both $(g, e) : F \Rightarrow F''$ and $(e, h) : F'' \Rightarrow F'$ are natural transformations, and $\sigma$ is the composition of $(g, e)$ with $(e, h)$. ■
2.3 Local prequantum gauge theory

Before proceeding, we comment on the terminology of “prequantum” and why we use it as opposed to “classical.” In classical electromagnetism, or gauge theory in general, only the equations of motion are relevant. In particular, the field strength, and not the gauge potential, appear in the equations of motion. The vector potential becomes relevant when formulating the equations of motion as a variational principle which is itself a reference point towards quantization [Sc16a], [Sc16b]. The exponentiated Action and parallel transports of gauge theory are realized precisely in this intermediate stage of local prequantum field theory which lies between classical field theory and quantum field theory. We will focus on special 1-d and 2-d field theories, i.e. particle mechanics and string theory. More justifications for our presentation can be found in Sections 3.2.6 and 3.3.6 of this thesis.

2.3.1 One-dimensional algebra and parallel transport

The solution to the initial value problem

$$\frac{d\psi(t)}{dt} = -A(t)\psi(t), \quad \psi(0) = \psi_0 \in \mathbb{R}^n$$ (2.3.1)

with $A(t)$ a time-dependent $n \times n$ matrix is

$$\psi(t) = \psi_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_0^t dt_k \cdots \int_0^t dt_1 \, \mathcal{T} [A(t_k) \cdots A(t_1)] \psi_0$$ (2.3.2)
where $T$ stands for time-ordering with earlier times appearing to the right. The choice of sign convention (2.3.1) is to be consistent with references [Pa15], [BaMu94], and [ScWa09].\footnote{Be warned, however, as this sign will lead to different conventions for other related forms such as the curvature 2-form, the connection 2-form, and gauge transformation relations. Certain authors use this other convention [Hu94], [MiSt74].} This shows up in several contexts such as (a) solving Schrödinger’s equation with $A(t) = iH(t)$ for a time-dependent Hamiltonian and $\psi$ a vector in the space on which $H$ acts and (b) calculating the parallel transport along a curve in gauge theory, where $A$ is the local vector potential. This integral goes under many names: Dyson series, Picard iteration, path/time-ordered exponential, Berry phase, etc.

As an approximation, the solution to this differential equation can be obtained by breaking up a curve into infinitesimal paths

\[
\exp \left\{ -A_{\mu_i}(x(t_i)) \frac{dx^\mu_i}{dt} \bigg|_{t_i} \Delta t_i \right\}
\]

(2.3.3)

and associating the group elements to these infinitesimal paths and multiplying those group elements in the order
dictated by the path. Here \( \Delta t_i \) should be thought of as the length of the infinitesimal interval from \( t_i \) to \( t_{i+1} \) hence \( \Delta t_i = t_{i+1} - t_i \) and will be used later as an approximation for calculating integrals. For simplicity, we may take it to be \( \Delta t_i = \frac{1}{n} \) if our parametrization is defined on [0, 1] and if there are \( n \) subintervals. Furthermore, by locality, the group elements should be of this form to lowest order in approximation. Preserving the order dictated by the path, the result of multiplying all these elements is

\[
\exp \left\{ -A_{\mu_n} (x(t_n)) \frac{dx^{\mu_n}}{dt} \bigg|_{t_n} \Delta t_n \right\} \cdots \exp \left\{ -A_{\mu_1} (x(t_1)) \frac{dx^{\mu_1}}{dt} \bigg|_{t_1} \Delta t_1 \right\} \ . \tag{2.3.4}
\]

Expanding out to lowest order (since the paths are infinitesimal) gives

\[
\left( 1 - A_{\mu_n} (x(t_n)) \frac{dx^{\mu_n}}{dt} \bigg|_{t_n} \Delta t_n \right) \cdots \left( 1 - A_{\mu_1} (x(t_1)) \frac{dx^{\mu_1}}{dt} \bigg|_{t_1} \Delta t_1 \right) \tag{2.3.5}
\]

and reorganizing terms results in

\[
1 - \sum_{i=1}^{n} A_{\mu_i} (x(t_i)) \frac{dx^{\mu_i}}{dt} \bigg|_{t_i} \Delta t_i + \sum_{i>j \geq 1} A_{\mu_i}(x(t_i)) A_{\mu_j}(x(t_j)) \frac{dx^{\mu_i}}{dt} \bigg|_{t_i} \frac{dx^{\mu_j}}{dt} \bigg|_{t_j} \Delta t_i \Delta t_j \pm \cdots \ , \tag{2.3.6}
\]

which is exactly the path-ordered integral in (2.3.1) after taking the \( n \rightarrow \infty \) limit in which \( \Delta t_j \) are replaced by \( dt_j \). Actually, to see this, one should note that the above sum becomes an integral over simplices and there is an equality

\[
\int_0^1 dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \mathcal{T} \left[ A(t_k) \cdots A(t_1) \right] = \frac{1}{k!} \int_0^t dt_k \cdots \int_0^t dt_1 \mathcal{T} \left[ A(t_k) \cdots A(t_1) \right] \ , \tag{2.3.7}
\]
giving an additional $\frac{1}{k!}$ from the volume of the $k$-simplex. We picture the

group element (2.3.6) as all the number of ways in which $A$ interacts with
the particle preserving the order of the path

\[
-\int A_{\mu_1}(t_1) \frac{dx_\mu_1}{dt} \bigg|_{t_1} \, dt_1 + \int \int A_{\mu_2}(t_2) \frac{dx_\mu_2}{dt} \bigg|_{t_2} A_{\mu_1}(t_1) \frac{dx_\mu_1}{dt} \bigg|_{t_1} \, dt_1 dt_2 + \cdots
\]

(2.3.8)

Thus, given a path $\gamma : [0, 1] \rightarrow M$, we denote the parallel transport group

element in (2.3.6) by $\text{triv}(\gamma)$.\textsuperscript{7} Three key properties of the parallel transport
are that (a) it is reparametrization invariant and (b) if one had two paths
connected at their endpoints as in

\[
\gamma \quad \delta
\]

then\textsuperscript{8}

\[
\text{triv}(\gamma\delta) = \text{triv}(\gamma)\text{triv}(\delta)
\]

\textsuperscript{7}The reason for the notation $\text{triv}(\gamma)$ is because we will always work in a local trivialization of a bundle with connection. This choice is also made to be consistent with earlier work [Pa15], Chapter 3 of this thesis, as well as the reference [ScWa09].

\textsuperscript{8}One proof of this uses the useful identity

\[
\sum_{k=0}^{\infty} \sum_{n=0}^{k} a_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k,n-k}
\]
and finally (c) it is a smooth function from paths in $M$ to the group $G$. This resembles the definition of a functor. To state the relationship between parallel transport and functors more precisely, we note that $\text{triv}(\gamma)$ is invariant under more than just reparametrizations of $\gamma$. It is also invariant under thin homotopy. The appropriate domain on which $\text{triv}$ is therefore defined is a (smooth) category $\mathcal{P}^1(M)$ known as the thin path groupoid of $M$.

**Definition 2.3.10.** A groupoid is a category all of whose 0-d defects are invertible.

**Definition 2.3.11.** The thin path groupoid of $M$, denoted by $\mathcal{P}^1(M)$, has objects consisting of points of $M$ has morphisms certain equivalence classes of paths of $M$.

More details on the thin path groupoid can be found in [Pa15], Chapter 3 in this thesis, and [ScWa09]. In terms of 1-d domains and 0-d defects, we use the Poincaré dual so that points in $M$ correspond to 1-d domains (which are now better thought of as objects) and paths in $M$ correspond to 0-d defects (which are now better thought of as morphisms). Fortunately, we will not need the technical details of thin homotopy equivalence classes with

$$a_{k,n} := \frac{(-1)^{n+k}}{n!k!} T \left[ \left( \int_1^2 A(s)ds \right)^n \left( \int_0^1 A(s)ds \right)^k \right].$$
for our calculations. All we should keep in mind is that \( \text{triv} : \mathcal{P}^1(M) \to \mathbb{B}G \) associates group elements to paths

\[
\gamma \to \text{triv}(\gamma)
\]

smoothly and the path ordered integral arises from smoothness, breaking up the path into infinitesimal pieces, and using the generalized group homomorphism property. Namely, associated to such a path \( \gamma \) and a decomposition

\[
\gamma = \gamma_n \cdots \gamma_1
\]

let

\[
a_i := \text{triv}(\gamma_i) \cong \exp \left\{ -A_{\mu_i}(x(t_i)) \frac{dx^{\mu_i}}{dt} \bigg|_{t_i} \Delta t_i \right\}.
\]

Then the parallel transport is the product

\[
a_n a_{n-1} \cdots a_2 a_1
\]

given in (2.3.4). This is essentially what we mean by one-dimensional algebra: one-dimensional algebra is the theory of categories and functors.

The symmetries associated with the parallel transport are given by functions \( M \to G \).
Definition 2.3.14. Let \( \text{triv}, \text{triv}' : \mathcal{P}^1 M \to \mathcal{B} G \) be two parallel transport functors defined by vector potentials \( A \) and \( A' \), respectively. A \textit{finite gauge transformation} from \( A \) to \( A' \) is a function \( g : M \to G \) satisfying\(^9\)

\[
A' = g A g^{-1} - d g g^{-1}.
\] (2.3.15)

This definition of gauge transformation is equivalent [ScWa09] to the condition that for any path \( \gamma \) from \( y \) to \( z \),

\[
\text{triv}'(\gamma)g(y) = g(z)\text{triv}(\gamma)
\] (2.3.16)

which in turn is equivalent to the statement that \( g : M \to G \) defines a natural transformation from \( \text{triv} \) to \( \text{triv}' \) (see Definition 2.2.7). A sketch of this equivalence can be seen by discretizing a path \( t \mapsto x(t) \) into \( n \) pieces and using the expression (2.3.5) for the approximation of the parallel transport.

Applying a gauge transformation to each piece gives

\[
\text{triv}'(\gamma) \cong \prod_{i=1}^{n} g(x(t_{i+1})) \left( 1 - A_{\mu_i}(x(t_i)) \frac{dx^\mu_i}{dt} \bigg|_{t_i} \right) \Delta t_i g(x(t_i))^{-1}
\] (2.3.17)

where the product is in the specified order as in (2.3.5). Taylor expanding out the latter group element gives

\[
g(x(t_{i+1})) \cong g(x(t_i)) + \frac{\partial g}{\partial x^\mu_i} \bigg|_{t_i} \Delta t_i
\] (2.3.18)

\(^9\)As usual, we are thinking of \( G \) as a matrix group, though we do not need to be for any statements made. It is only meant to facilitate computations.
to first order. Using this gives

$$\text{triv}'(\gamma) \approx \prod_{i=1}^{n} \left( g(x(t_i)) + \frac{\partial g}{\partial x^{\mu_i}} \left|_{t_i} \Delta t_i \right. \right)$$

$$\times \left( 1 - A_{\mu_i}(x(t_i)) \frac{dx^{\mu_i}}{dt} \left|_{t_i} \Delta t_i \right. \right) g(x(t_i))^{-1}$$

$$\approx \prod_{i=1}^{n} \left( 1 - g(x(t_i)) A_{\mu_i}(x(t_i)) \frac{dx^{\mu_i}}{dt} \left|_{t_i} \Delta t_i \right. \right)$$

$$+ \frac{\partial g}{\partial x^{\mu_i}} \left|_{t_i} \right. g(x(t_i))^{-1} \Delta t_i,$$

where we have dropped the term

$$\left( \frac{\partial g}{\partial x^{\mu_i}} \left|_{t_i} \Delta t_i \right. \right) \left( -A_{\mu_i}(x(t_i)) \frac{dx^{\mu_i}}{dt} \left|_{t_i} \Delta t_i \right. \right)$$

(2.3.20)

since it is second order in $\Delta t_i$. Finally, since

$$\text{triv}'(\gamma) \approx \prod_{i=1}^{n} \left( 1 - A'_{\mu_i}(x(t_i)) \frac{dx^{\mu_i}}{dt} \left|_{t_i} \Delta t_i \right. \right)$$

(2.3.21)

it is reasonable to identify corresponding terms

$$A'_{\mu} = g A_{\mu} g^{-1} - \frac{\partial g}{\partial x^{\mu}} g^{-1},$$

(2.3.22)

which reproduces (2.3.15). This latter perspective of functors and natural transformations will be used in the sequel to define parallel transport along two-dimensional surfaces (with some data on orientations). This was first made precise in [ScWa09] though the formulation in terms of functors had been expressed earlier [BaSc04].
Remark 2.3.23. Most of the calculations in this chapter will follow this sort of logic rather than dry style. Although similar techniques were used in [GiPf04] and [BaSc04], we were largely motivated by the kinds of calculations in [ChTs93] and hope that our treatment will be more accessible to a wider audience.

2.3.2 Two-dimensional algebra and surface transport

Understanding higher form non-abelian gauge fields has been a long-standing problem in physics, particularly in string theory and M-theory (see for instance the end of [Wit02]). Although we do not aim to solve all of these problems, we hope to indicate the important role played by category theory in understanding certain non-local aspects in these theories. What we will do, however, is show how 2-categories and the laws set up in the previous sections naturally lead to the notion of parallel transport along surfaces. Parallel transport will obey an important gluing condition analogous to the gluing condition for paths. Gauge transformations will be studied in the next section. Furthermore, we will produce an explicit formula analogous to the Dyson series expansion for paths. Although an integral formula is known in the literature [ScWa11], a complete derivation of this integral from infinitesimal components seems to be lacking. A sketch is included in [BaSc04] in
Section 2.3.2 but further analysis was done in path space, which we feel is more difficult—indeed, the goal of that work was to relate gerbes with connection on path spaces to connections on path spaces. Furthermore, although experts are aware of how bigons are related to more general surfaces, we explicitly perform our calculations on “reasonable” surfaces, namely squares, for clearer visualization.

We feel it is important to express surface transport in a more computationally explicit manner using a lattice and prove from the ground up a visualization of the surface-ordered integral sketched in Figure 3.15 in Chapter 3. Just as the group $G$-valued parallel transport along paths in a manifold $M$ is described by a functor $\text{triv} : \mathcal{P}_1(M) \to \mathbb{B}G$, crossed-module $\mathcal{G}$-valued parallel transport along surfaces should be described by a functor from some 2-category associated with paths and surfaces in $M$ to the 2-group $\mathbb{B}\mathcal{G}$. Ideally, such a 2-category should be a version of the (extended) 2-dimensional cobordism 2-category over the manifold $M$ to mimic the ideas of functorial field theories. However, this has not yet been achieved in this form for non-abelian 2-groups. In fact, it has only recently been achieved for the 1-dimensional case by Berwick-Evans and Pavlov [BEPa15]. Earlier work on abelian gerbes indicates this should be the case in general [Pi04] though this has not been fully worked out. Part of the reason is due to the fact that the
representation theory for higher groups is a rather young subject [BBFW12].

Fortunately, a related solution exists if one works with a 2-category of paths and homotopies. This 2-category is denoted by \( \mathcal{P}^2(M) \). It is more natural to describe this category in terms of the Poincaré dual of string diagrams. Namely, objects of \( \mathcal{P}^2(M) \) are points of \( M \), 1-morphisms of \( \mathcal{P}^2(M) \) are thin homotopy classes of paths in \( M \), and 2-morphisms are thin-homotopy classes of bigons in \( M \). A bigon is essentially a homotopy \( \Sigma \) between two paths whose endpoints agree.

**Definition 2.3.24.** Let \( \gamma \) and \( \delta \) be two paths from \( x \) to \( y \) parametrized by \( t \in [0, 1] \) such that there exists an \( \epsilon > 0 \) with \( \gamma(t) = \delta(t) \) for all \( t \in [0, \epsilon] \cup [1 - \epsilon, 1] \). A **bigon** from \( \gamma \) to \( \delta \) is a map \( \Sigma : [0, 1] \times [0, 1] \to M \) such that there exists an \( \epsilon > 0 \) with

\[
\Sigma(t, s) = \begin{cases} 
  x & \text{for all } (t, s) \in [0, \epsilon] \times [0, 1] \\
  y & \text{for all } (t, s) \in [1 - \epsilon, 1] \times [0, 1] \\
  \gamma(t) & \text{for all } (t, s) \in [0, 1] \times [0, \epsilon] \\
  \delta(t) & \text{for all } (t, s) \in [0, 1] \times [1 - \epsilon, 1]
\end{cases}
\]

(2.3.25)

It is helpful to visualize a bigon as
**Definition 2.3.26.** Two bigons $\Sigma$ and $\Gamma$ from paths $\gamma$ to $\delta$ are *thinly homotopic* if there exists a smooth map of a 3-dimensional cube into $M$ whose top face is $\Sigma$, whose bottom face is $\Gamma$, and similarly for the other face for the paths $\gamma$ and $\delta$ along with their endpoints (all of these assume some constancy in a small neighborhood of each face). Furthermore, and most importantly, this map cannot sweep out any volume in $M$, i.e. its rank is strictly less than 3.

More details can be found in [Pa15] and [ScWa11] though again such technicalities will be avoided here. Thus, a strict functor $\text{triv} : P^2(M) \rightarrow BG$ is an assignment sending

\[
\begin{array}{ccc}
\gamma & \downarrow^\Sigma & \delta \\
y & \rightarrow & x \\
\end{array}
\]

which in particular says

\[
\tau(\text{triv}(\Sigma)) \text{triv}(\gamma) = \text{triv}(\delta),
\]

and satisfying a homomorphism property in the following sense. Bigons can be glued together in series and in parallel by a choice of parametrization. By the thin homotopy assumption, the value of the bigons is independent of such
parametrizations. It might seem undesirable to restrict ourselves to surfaces of this form. However, this is no serious matter because every compact surface can be expressed in this manner under suitable identifications living on sets of measure zero. For example, a surface of genus two with three boundary components with orientations shown (the orientation of the surface itself is clockwise)

is depicted on the right as a bigon beginning at the path $\gamma$ (in blue) and ending at the path $\delta$ (in yellow) both of which are loops beginning at the same basepoint which is the top left corner of the octagon on the left. The identifications on the outer boundary of the octagon are standard ways of representing a genus two surface. Furthermore, one can always triangulate or cubulate such a surface. If one chooses triangulations, then one merely needs to know the parallel transport on triangles
and if one cubulates a surface, then one needs to know it for squares

Thus, in order to find an explicit formula for the parallel transport along surfaces with non-trivial topology, it suffices to calculate the parallel transport along a square, say. Squares are also more convenient to use for continuum limiting procedures as opposed to triangles [Su10]. Functoriality for gluing squares together implies
and using the rules of two-dimensional algebra, this composition is

\[ \text{triv}(\Omega) \alpha_{\text{triv}(\beta)}(\text{triv}(\Sigma)). \]  

(2.3.28)

Similarly, for gluing along a different edge

the composition of the 0-d defects is

\[ \alpha_{\text{triv}(\zeta)}(\text{triv}(\Pi))\text{triv}(\Sigma). \]  

(2.3.29)

If one also wishes to attach a square in a somewhat arbitrary way such as
then this attachment must be oriented in such a way that (a) the boundary orientation agrees with the orientation of the first surface and (b) the two surface orientations combine to form a consistent orientation when glued together. So, for example,

is an allowed glueing orientation (more on orientations are discussed in Section 2.3.4). In this case, if we label all the vertices, edges, and squares, then the parallel transport along the glued surface is
which reads

$$\alpha_{\text{triv}(\zeta_3)}(\text{triv}(\Pi))\text{triv}(\Sigma)$$

(2.3.30)

on the resulting 0-d defect.

Using all of these results and in complete analogy with Section 2.3.1, we can take an arbitrary worldsheet (with orientations giving it the structure of a bigon), break it up into infinitesimal squares

and approximate the parallel transport along an infinitesimal square
where

\[ a_{ij}^s := \exp \left\{ -A_{\mu i} (x(s_i, t_j)) \frac{\partial x^{\mu i}}{\partial s} \bigg|_{(s_i, t_j)} \Delta s_i \right\} \]  \hspace{1cm} (2.3.31)

and

\[ a_{ij}^t := \exp \left\{ -A_{\nu j} (x(s_i, t_j)) \frac{\partial x^{\nu j}}{\partial t} \bigg|_{(s_i, t_j)} \Delta t_j \right\} \]  \hspace{1cm} (2.3.32)

denote the parallel transport along infinitesimal paths and 

\[ b_{ij} := \exp \left\{ B_{\mu i \nu j} (x(s_i, t_j)) \frac{\partial x^{\mu i}}{\partial s} \frac{\partial x^{\nu j}}{\partial t} \bigg|_{(s_i, t_j)} \Delta s_i \Delta t_j \right\} \]  \hspace{1cm} (2.3.33)

denotes the parallel transport along infinitesimal squares. Here

\[ \Delta s_i = s_{i+1} - s_i \quad \& \quad \Delta t_j = t_{j+1} - t_j \]  \hspace{1cm} (2.3.34)

and for an \( n \times n \) square grid these are both \( \Delta s_i = \frac{1}{n} = \Delta t_j \). Note that in order for this association to be consistent with our description of 2-groups,

---

Our convention is to ignore combinatorial factors from our Einstein summation convention since these are cumbersome to carry. Normally, such an expression in the exponential (2.3.33) would have a \( \frac{1}{2} \).
it must be true that
\[ a_{i,j+1}^s a_{ij}^t = \tau(b_{ij}) a_{i+1,j}^t a_{ij}^s, \]  
(2.3.35)
or equivalently
\[ \tau(b_{ij}) = a_{i,j+1}^s a_{ij}^t \left( a_{ij}^s \right)^{-1} \left( a_{i+1,j}^t \right)^{-1}, \]  
(2.3.36)
at least to lowest non-trivial order. The term on the right-hand-side is precisely the parallel transport along an infinitesimal square\(^{11}\)
\[ a_{i,j+1}^s a_{ij}^t \left( a_{ij}^s \right)^{-1} \left( a_{i+1,j}^t \right)^{-1} \]
\[ \cong \left( 1 - A_{\mu_i} \left( x(s_i, t_{j+1}) \right) \frac{\partial x_{\mu_i}'}{\partial s} \bigg|_{(s_i, t_{j+1})} \right) \left( 1 - A_{\nu_j} \left( x(s_i, t_j) \right) \frac{\partial x_{\nu_j}'}{\partial t} \bigg|_{(s_i, t_j)} \right) \]
\[ \times \left( 1 + A_{\mu_i} \left( x(s_i, t_j) \right) \frac{\partial x_{\mu_i}}{\partial s} \bigg|_{(s_i, t_j)} \right) \left( 1 + A_{\nu_j} \left( x(s_{i+1}, t_j) \right) \frac{\partial x_{\nu_j}'}{\partial t} \bigg|_{(s_i, t_j)} \right) \]
\[ \cong \left( 1 - A_{\mu_i} \frac{\partial x_{\mu_i}}{\partial s} - A_{\nu_j} \frac{\partial x_{\nu_j}}{\partial t} \right) \left( 1 - A_{\nu_j} \frac{\partial x_{\nu_j}}{\partial t} \right) \left( 1 + A_{\mu_i} \frac{\partial x_{\mu_i}}{\partial s} \right) \left( 1 + A_{\nu_j} \frac{\partial x_{\nu_j}}{\partial t} \right) \]
\[ \cong \left( 1 - A_{\mu_i} \frac{\partial x_{\mu_i}}{\partial s} - A_{\nu_j} \frac{\partial x_{\nu_j}}{\partial t} \right) \left( 1 - A_{\nu_j} \frac{\partial x_{\nu_j}}{\partial t} \right) \]
\[ = 1 + \left( \frac{\partial A_{\nu_j}}{\partial x_{\mu_i}} - A_{\mu_i} A_{\nu_j} - A_{\nu_j} A_{\mu_i} \right) \frac{\partial x_{\mu_i}'}{\partial s} \frac{\partial x_{\nu_j}'}{\partial t} \bigg|_{(s_i, t_j)} \]
\[ = 1 + F_{\mu_i \nu_j} \frac{\partial x_{\mu_i}'}{\partial s} \frac{\partial x_{\nu_j}'}{\partial t} \bigg|_{(s_i, t_j)} \],

\(^{11}\)For the purpose of this calculation, we have dropped the \(\Delta s_i\) and \(\Delta t_j\) from the notation to avoid clutter. This should cause no confusion because these quantities are always coupled with their corresponding derivatives \(\frac{\partial}{\partial s}\) and \(\frac{\partial}{\partial t}\), respectively.
to lowest order, which is a standard result. Here

\[ F := dA + A \wedge A \]  

(2.3.38)

is the curvature of \( A \). Meanwhile, the left-hand-side is

\[ \tau(b_{ij}) \approx 1 + \tau \left( B_{\mu \nu j}(x(s_i, t_j)) \frac{\partial x^{\mu i}}{\partial s} \frac{\partial x^{\nu j}}{\partial t} \bigg|_{(s_i, t_j)} \right) \]  

(2.3.39)

to lowest order. Here \( \tau : \mathfrak{h} \rightarrow \mathfrak{g} \) is the derivative of the map \( \tau : H \rightarrow G \) at the identity, i.e. on the Lie algebras (at this point, it might be a good idea for the reader to review the Appendix on the infinitesimal version of \((H, G, \tau, \alpha)\)). This therefore forces the condition

\[ \tau(B) - F = 0 \]  

(2.3.40)

and is known in the literature as the \textit{vanishing of the fake curvature}. Finally, we can expand out these exponentials of differential forms and multiply all terms together analogously to what was done for a path. An arbitrary worldsheet, which is a map (via some reparametrization if necessary) from \([0, 1] \times [0, 1]\) to some target manifold, is naturally a bigon with the orientation induced by having \((s, t)\) a right-handed coordinate system. Breaking up such a bigon into infinitesimal squares allows one to associate the above exponentials on the Poincaré dual of the cubulation of the worldsheet.
where a cubulation of the domain $[0, 1] \times [0, 1]$ is shown on the right together with its Poincaré dual. This rotated $(s, t)$ coordinate system was chosen to agree with our earlier convention on two-dimensional algebra. We will consider a $5 \times 5$ grid for concreteness. Using the rules set up earlier on how to read such diagrams, we extend the 1-d defects of the Poincaré dual to the top and bottom of the page using identity 0-d defects drawn on the $(s, t)$ domain of the worldsheet (the identities are drawn in yellow to illustrate where they are and not because the 2-d domain is different—there is only a single 2-d domain in a 2-group—see Section 2.2.3).\footnote{One could have also added identities in many other consistent ways and performed similar calculations. The end results would all be the same (to lowest order) due to the interchange law.}
To more easily relate this picture to earlier ones, it helps to draw horizontal lines to distinguish the order of multiplication and then to tilt the angles of the identities (only the top half is drawn)
which now makes it easy to see we can first compose each row in parallel and then compose the results in the remaining column in series. We explicitly label (some of) the 1-d and 0-d defects

and then multiply each row in parallel. The first row looks like
The result on the 1-d defects is just the usual group multiplication product while the result on the 0-d defects is

\[ \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{t}} (b_{51}). \]  

(2.3.41)

The 0-d defects of the next several rows are all given by the following

\[ = \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t}} (b_{52}) \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{s} a_{52}^{t}} (b_{41}) \]  

(2.3.42)

\[ = \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{s} a_{52}^{t}} (b_{53}) \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{s} a_{52}^{t} a_{42}^{t}} (b_{42}) \]  

(2.3.43)

\[ = \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{s} a_{52}^{t} a_{42}^{t}} (b_{43}) \alpha_{a_{65}^{t} a_{64}^{t} a_{63}^{t} a_{62}^{s} a_{52}^{t} a_{42}^{t} a_{32}^{t}} (b_{31}) \]  

(2.3.44)
\[2\text{-D ALGEBRA AND GAUGE THEORY}\]

\[= b_{55} \alpha_{a_5 a_5 a_5} (b_{44}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{33}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{22}) \times \alpha_{a_6 a_5 a_5 a_4 a_4 a_4 a_3 a_2 a_2} (b_{11}) \]  
\[= \alpha_{\alpha_{a_5 a_5} a_5 a_5} (b_{44}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{33}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{22}) \times \alpha_{a_6 a_5 a_5 a_4 a_4 a_4 a_3 a_2 a_2} (b_{11}) \]  
\[= \alpha_{\alpha_{a_5 a_5} a_5 a_5} (b_{44}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{33}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{22}) \times \alpha_{a_6 a_5 a_5 a_4 a_4 a_4 a_3 a_2 a_2} (b_{11}) \]  
\[= \alpha_{\alpha_{a_5 a_5} a_5 a_5} (b_{44}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{33}) \alpha_{a_6 a_5 a_5 a_4 a_4} (b_{22}) \times \alpha_{a_6 a_5 a_5 a_4 a_4 a_4 a_3 a_2 a_2} (b_{11}) \]  

and finally
The result of composing all of these in parallel gives the following sequence of in series composable 0-d defects

\[
\begin{align*}
\alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{51}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{52}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{41}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{53}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{42}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{34}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{54}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{43}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{35}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{44}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{55}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{45}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{56}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{46}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{57}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{47}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{58}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{48}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{59}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{49}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{60}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{50}) \\
\alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{61}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{51})
\end{align*}
\]

\[(2.3.50)\]

which, after performing the in series composition in $H$ gives

\[
\begin{align*}
\alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{15}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{25}) \alpha_{a_{56}^t a_{46}^t a_{36}^t a_{26}^t} (b_{14}) \cdots \\
\cdots \times \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{41}) \alpha_{a_{65}^t a_{64}^t a_{63}^t a_{52}^t} (b_{51})
\end{align*}
\]

\[(2.3.51)\]

We can visualize this mess more easily by expanding out each $b_{ij}$ to lowest order (since we already know that the $a$'s give the one-dimensional parallel transport, we do not have to expand them out) and examining the terms with zero $b_{ij}$'s, terms with one $b_{ij}$, terms which include a product of some $b_{ij}$ with another $b_{kl}$, and so on. The zeroth order term is just the identity. There are 25 terms with a single $B$ (some of these terms are written underneath the pictures to more clearly illustrate our convention)
These pictures express the fact that we calculate the ordinary parallel transport along a specified path between the point \((s, t) = (s_6, t_6)\) and another point \((s_{i+1}, t_{j+1})\) (represented by a blue line) and conjugate each \(B\) field at \((s_i, t_j)\) (represented by a blue square) by that parallel transport using \(\alpha\). Then we sum over all points at which \(B\) has been specified. There are \(\sum_{k=1}^{24} k = \frac{24(25)}{2} = 600\), i.e. \(\binom{25}{2}\), terms with two \(B\)'s:
In this long expression, there are 24 terms in the first 3 rows of pictures, 23 in the two rows after that, up until we get 2 + 1 shown in the last row. This is consistent with the counting \( \binom{25}{2} \). Now, we should do this sum for all products of \( B \)'s ranging from terms with 0 \( B \)'s to terms with 25 \( B \)'s. Just to be clear, for example, a term with 4 \( B \)'s might look like

but a term such as
does not appear due to the automatic ordering. The total number of all terms in such an expansion is enormous and is given by

$$\sum_{k=0}^{25} \binom{25}{k} = 2^{25}, \quad (2.3.52)$$

which is ridiculously huge (near the order of Avogadro’s number) or more generally

$$\sum_{k=0}^{n^2} \binom{n^2}{k} = 2^{n^2} \quad (2.3.53)$$

if we have an \( n \times n \) grid.

**Definition 2.3.54.** Let \( \text{triv}_{5}^{\text{full}} \) be the expansion of (2.3.51) consisting of the \( 2^{25} \) terms described above. Let \( \text{triv}_{n}^{\text{full}} \) be the generalization of this expression for an \( n \times n \) grid (this consists of \( 2^{n^2} \) terms).

Firstly, we should be sure that the sum of all such terms coming from (2.3.51) sum converges as the spacing goes to zero, i.e. as \( n \to \infty \). The terms with \( k \) \( B \)'s have an additional factor of \( \frac{1}{n^k} \) associated with the area element on which they are approximated. The ratio of the number of all such terms for \( k \leq \left\lfloor \frac{n^2}{2} \right\rfloor \) to this factor is

$$\frac{n^2}{n^2} = \frac{n^2!}{k!(n^2-k)!n^{2k}} = \frac{1}{k!} \prod_{i=1}^{k} \left( 1 - \frac{i-1}{n^2} \right), \quad (2.3.55)$$

where \( \lfloor \cdot \rfloor \) denotes the floor function. Note that the product term satisfies

$$0 \leq \prod_{i=1}^{k} \left( 1 - \frac{i-1}{n^2} \right) \leq 1 \quad (2.3.56)$$
because it is a product of numbers strictly less than or equal to 1 for all $i$.

Hence,

$$\frac{n^2}{n^{2k}} \leq \frac{1}{k!}. \quad (2.3.57)$$

For $k \geq \left\lfloor \frac{n^2}{2} \right\rfloor$, this decays even more strongly because $\binom{n^2}{k}$ is symmetric at $\left\lfloor \frac{n^2}{2} \right\rfloor$ and hence $\binom{n^2}{k}$ begins to decrease for larger values of $k$ while the $\frac{1}{n^{2k}}$ factor remains and increases as $k$ gets larger.

**Proposition 2.3.58.** The sequence $\{\text{triv}_{n}^{\text{full}}\}_n$ converges as $n \to \infty$.

**Proof.** Let $M$ be the absolute value of the maximum value of the expressions of the form $\alpha_{a_{36}^e a_{40}^e a_{36}^e a_{35}^e a_{25}^e} (B_{14})$. By smoothness of the forms $A$ and $B$ and compactness of $[0,1] \times [0,1]$, such a maximum $M$ always exists. The $k$-th order term in the expression for $\text{triv}_{n}^{\text{full}}$ has norm bounded by

$$\left( \frac{M^k}{n^2} \right)^k \binom{n^2}{k} = M^k \left( \frac{n^2}{n^{2k}} \right) \leq \frac{M^k}{k!} \quad (2.3.59)$$

for all $k \in \{0,1,\ldots,n^2\}$. This statement is true for all $n \in \mathbb{Z}$. This shows that the sequence is bounded by the sequence for $\exp\{M\}$ and so converges absolutely. Hence, $\{\text{triv}_{n}^{\text{full}}\}_n$ converges.

Fortunately, we can simplify the expression $\text{triv}_{n}^{\text{full}}$ by rearranging and reorganizing all of these terms. For example, consider terms with two $B$’s. There are terms with two $B$’s at different “heights” such as
As explained above, note that there do not exist terms with the order flipped in the above two images. This is due to the automatic ordering. Therefore, the number of terms with two $B$'s at the same height is ($n = 5$ in our picture)

\[
\sum_{m=2}^{n-1} \binom{m}{2} + \binom{n}{2} + \sum_{m=2}^{n-1} \binom{m}{2} = 2 \sum_{m=2}^{n-1} \binom{m}{2} + \binom{n}{2} \\
= 2\binom{n}{3} + \binom{n}{2} \\
= \frac{2n!}{3!(n-3)!} + \frac{n!}{2!(n-2)!} \\
= \frac{n(n-1)(2n-1)}{3!},
\]

where the second second equality comes from a cute fact about Pascal’s
triangle

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 3 & 2 & & \\
1 & 3 & 6 & 7 & 6 & 1 \\
1 & 4 & 10 & 10 & 6 & 1 \\
1 & 5 & 15 & 20 & 15 & 6 & 1
\end{array}
\] (2.3.61)

The ratio of terms with two \( B \)'s at the same height to the total number of terms with two \( B \)'s is

\[
\frac{n(n-1)(2n-1)/3!}{\binom{n^2}{2}} = \frac{2n-1}{3(n+1)n}.
\] (2.3.62)

Note that the limit of this quantity as \( n \to \infty \) is

\[
\lim_{n \to \infty} \frac{2n-1}{3(n+1)n} = 0.
\] (2.3.63)

Hence, terms that involve a product of two \( B \)'s that appear at the same height become negligible in the \( n \to \infty \) limit. One might wonder if this is true for any product of \( B \)'s. Clearly, this is false when we have a product of \( k \) \( B \)'s and \( k > 2n - 1 \) since every configuration has at least one row in which \( B \) occurs at least twice. However, it is true for \( k \) sufficiently smaller than \( n \). This leads us to an interesting combinatorial problem in its own right.

The number of configurations of \( k \) blocks in an \( n \times n \) grid tilted 45° such
that no two blocks appear at the same height is

\[ S_{n,k} := \sum_{2n-1 \geq i_k > i_{k-1} > \cdots > i_1 \geq 1} l_n(i_1) \cdots l_n(i_k), \quad (2.3.64) \]

where

\[ l_n(i) := \begin{cases} 
    i & \text{if } 1 \leq i \leq n \\
    2n - i & \text{if } n < i \leq 2n - 1
\end{cases} \quad (2.3.65) \]

denotes the number of blocks of a given height \( i \). The ratio of this number to the total number of configurations of \( k \) blocks is

\[ R_{n,k} := \frac{S_{n,k}}{n^2} = \frac{(n^2 - k)!k!S_{n,k}}{n^2!}. \quad (2.3.66) \]

**Lemma 2.3.67.** For any \( \epsilon > 0 \) and \( K \in \mathbb{N} \), there exists an integer \( N \gg K \) such that

\[ 1 - R_{n,k} \leq \epsilon \quad (2.3.68) \]

for all \( n \geq N \) and \( k \leq K \), i.e.

\[ \lim_{n \to \infty} R_{n,k} = 1 \quad (2.3.69) \]

for all \( k \in \mathbb{N} \).

The graph in Figure 2.1 should be convincing\(^{14}\) though of course it is not a substitute for a proof.

\(^{13}\) We’d like to thank Zhibai Zhang and Scott O. Wilson who both independently suggested the currently used approach for this problem and for discussions leading to this formula.

\(^{14}\) Special thanks goes to Steven Vayl for teaching me some basics of C++ that gave me the necessary tools to make this plot.
The proof of Lemma 2.3.67 is quite involved and is given in the second Appendix. Instead, we offer a rough estimate analysis via averaging. The average value of $l_n$ is

$$\text{avg}(l_n) := \frac{\sum_{i=1}^{2n-1} l_n(i)}{2n-1} = \frac{n^2}{2n-1}. \quad (2.3.70)$$

Hence, to a good approximation for large $n$ and small $k$,

$$S_{n,k} \approx \sum_{2n-1 \geq i_k > i_{k-1} > \cdots > i_1 \geq 1} \left[ \text{avg}(l_n) \right]^k$$

$$= \binom{n^2}{2n-1}^k \binom{2n-1}{k}$$

$$= \frac{n^{2k}(2n-1)(2n-2) \cdots (2n-k)}{k!(2n-1)^k}, \quad (2.3.71)$$

where the second line comes from the fact that there are $\binom{2n-1}{k}$ terms in the
summation. Hence, to a good approximation

$$R_{n,k} \approx \frac{n^{2k}(2n - 1)(2n - 2) \cdots (2n - k)}{(2n - 1)^kn^2(n^2 - 1) \cdots (n^2 - (k - 1))}$$

$$= \frac{(1 - \frac{1}{2n}) \cdots (1 - \frac{k}{2n})}{(1 - \frac{1}{2n})^k (1 - \frac{1}{n^2}) \cdots (1 - \frac{k-1}{n^2})}$$

(2.3.72)

Since $k$ is fixed, the right-hand-side tends to one as $n \to \infty$. Again, the precise proof is given in the second Appendix of this Chapter.

**Definition 2.3.73.** Let $\text{triv}_{n}^{\text{red}}$ be the same expression as $\text{triv}_{n}^{\text{full}}$ but with all terms in which $B$ occurs at least twice at the same height for some height removed.

**Theorem 2.3.74.** For any $\epsilon > 0$, there exists an $N$ such that

$$\| \text{triv}_{n}^{\text{full}} - \text{triv}_{n}^{\text{red}} \| \leq \epsilon$$

(2.3.75)

for all $n \geq N$.

**Proof.** Let $M$ be the maximum value of the norms of all quantities of the form $\alpha_{a_5^*a_6^*a_6^*a_6^*a_{35}^*a_{25}^*}(B_{14})$. The difference $\text{triv}_{n}^{\text{full}} - \text{triv}_{n}^{\text{red}}$ only consists of contributions from terms in which there exist at least two $B$’s that occur at the same height. Fix $\epsilon > 0$. To begin, let $K$ be large enough so that

$$\sum_{k=K+1}^{\infty} \frac{M^k}{k!} \leq \frac{\epsilon}{2}.$$
which is possible since the series for the exponential converges. Furthermore, by Lemma 2.3.67, for any \( \epsilon > 0 \), there exists an \( N \) large enough so that

\[
1 - R_{n,k} \leq \frac{\epsilon}{2eM} \quad \forall \ k \leq K, n \geq N.
\]  

(2.3.77)

Using these two results, the value of the norm of the difference \( \text{triv}_{n}^{\text{full}} - \text{triv}_{n}^{\text{red}} \) is bounded by

\[
\| \text{triv}_{n}^{\text{full}} - \text{triv}_{n}^{\text{red}} \| \leq \sum_{k=1}^{n^2} \left( \frac{M}{n^2} \right)^k \left( \binom{n^2}{k} - S_{n,k} \right)
\]

\[
= \sum_{k=1}^{n^2} \left( \frac{M}{n^2} \right)^k \binom{n^2}{k} [1 - R_{n,k}]
\]

\[
\leq \sum_{k=1}^{n^2} \frac{M^k}{k!} [1 - R_{n,k}]
\]

\[
= \sum_{k=1}^{K} \frac{M^k}{k!} [1 - R_{n,k}] + \sum_{k=K+1}^{n^2} \frac{M^k}{k!} [1 - R_{n,k}]
\]  

(2.3.78)

\[
\leq \sum_{k=1}^{K} \frac{M^k}{k!} \left( \frac{\epsilon}{2eM} \right) + \sum_{k=K+1}^{n^2} \frac{M^k}{k!}
\]

\[
\leq \left( \frac{\epsilon}{2eM} \right) \sum_{k=1}^{\infty} \frac{M^k}{k!} + \sum_{k=K+1}^{\infty} \frac{M^k}{k!}
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]

\[
= \epsilon.
\]

Thus, heuristically, as \( n \to \infty \), the number of terms for which at least two \( B \)'s are at the same height is a set of measure zero with respect to all
possibilities and hence we can ignore them in the calculation of the surface parallel transport after taking the $n \to \infty$ limit. This gives the following picture for the surface-iterated integral.

**Theorem 2.3.79.** Let $\gamma_{s,t}$ be the path

The limit of the expression (2.3.51) as $n \to \infty$ is given by an iterated integral

with path-ordering only in the vertical direction.

In more detail, the surface-ordered integral is depicted schematically as an infinite sum of terms expressed by placing $B$ at the endpoints of the drawn
paths and conjugating it by parallel transport along the path connecting to it using $A$ and $\alpha$. Then we use an ordinary integral over the horizontal direction to get a 1-form (similar to what is done in [BaSc04] and [ScWa11]). Finally we use the usual path-ordered integral in the vertical direction. More explicitly, by changing coordinates to

$$u := \frac{s + t}{\sqrt{2}} \quad \& \quad v := \frac{s - t}{\sqrt{2}},$$

one can express $\gamma_{s,t}$ in terms of $u$ and $v$. We write this path as $\gamma_{u,v}$. Using this, the surface parallel transport is given by

$$1 + \int \left( \int_{u_2 > u_1}^{\alpha_{\text{triv}}(\gamma_{u_2,v_2})} (B(u, v)) \, dv \right) \, du + \int \left( \int_{u_2 > u_1}^{\alpha_{\text{triv}}(\gamma_{u_2,v_2})} (B(u_1, v_1)) \, dv_2 \, dv_1 \right) \, du_2 \, du_1 + \cdots$$

$$+ \int \left( \int_{u_n > \cdots > u_1}^{\alpha_{\text{triv}}(\gamma_{u_n,v_n})} \cdots (B(u_1, v_1)) \, dv_n \cdots dv_1 \right) \, du_n \cdots du_1$$

$$+ \cdots,$$

where $B(u, v)$ stands for

$$B(u, v) := B \left( \frac{\partial \Sigma}{\partial s}, \frac{\partial \Sigma}{\partial t} \right) \bigg|_{\left( s = \frac{u + v}{\sqrt{2}}, \ t = \frac{u - v}{\sqrt{2}} \right)}.$$

**Remark 2.3.83.** It is not clear to us whether the surface iterated integral (2.3.81) agrees with the result of Schreiber and Waldorf in [ScWa11], which is the special case when our $s$ directions are pinched at $t = 0$ and $t = 1$. 
Although the result is very similar, [ScWa11] have the parallel transport along the path beginning at $t = 0$ and ending at the points we have drawn act on $B$ (see their equations (2.26) and (2.27)). Let us try to heuristically explain this. First off, [ScWa11] use the opposite frame with respect to ours (hence our $s$ is their $t$ and vice versa). Secondly, and much more importantly, they use bigons so that their $s$ coordinate is degenerate at $t = 0$ and $t = 1$. By a reparametrization and taking these differences into account, our picture and [ScWa11]'s picture for the surface transport look like

![Diagram](image)

respectively. In other words, [ScWa11] use the $\alpha$ action to conjugate the $B$ field along a path that is homotopic to ours but not thinly homotopic to ours. These issues will be clarified in future work.

### 2.3.3 Gauge transformations for surface transport

In Section 2.3.1, we described gauge transformations as natural transformations of parallel transport functors for paths. In this section, we will use this as the definition of a gauge transformation and derive the corresponding
formulas for differential forms. As before, let $\mathcal{G} := (H, G, \tau, \alpha)$ be a crossed module, $\mathbb{B}\mathcal{G}$ its associated 2-group, and $M$ a smooth manifold.

**Definition 2.3.84.** A *(first order) gauge transformation* from a parallel transport functor $\text{triv} : \mathcal{P}^2(M) \rightarrow \mathbb{B}\mathcal{G}$ to another $\text{triv}^\prime : \mathcal{P}^2(M) \rightarrow \mathbb{B}\mathcal{G}$ is a (smooth) natural transformation $\text{triv} \Rightarrow \text{triv}^\prime$.

By Definition 2.2.22 and Proposition 2.2.49, such a natural transformation consists of a pair of smooth functions $g : M \rightarrow G$ and $h : \mathcal{P}^1 M \rightarrow H$ satisfying the conditions described in that Proposition. Namely, to every thin path $z \xrightarrow{\gamma} y$,

$$\tau(h(\gamma))g(z)\text{triv}(\gamma) = \text{triv}^\prime(\gamma)g(y), \quad (2.3.85)$$

to every pair of composable thin paths $z \xrightarrow{\gamma} y \xleftarrow{\delta} x$,

$$h(\gamma\delta) = \alpha_{\text{triv}^\prime(\gamma)}(h(\delta))h(\gamma), \quad (2.3.86)$$

to every point $x \in M$,

$$h(\text{id}_x) = e, \quad (2.3.87)$$

and finally to any worldsheet
viewed as a bigon from $\gamma \delta$ to $\zeta \xi$, the equality\footnote{This follows from condition (d) in Definition 2.2.22.} holds. Reading this diagram is a bit tricky without the arrows (recall Remark 2.2.12). More explicitly, this equality says

\begin{equation}
\text{triv}
\end{equation}
i.e.

\[ \alpha_{\text{triv}'(\zeta)}(h(\xi)) h(\zeta) \alpha_{g(z)}(\text{triv}(\Sigma)) = \text{triv}'(\Sigma) \alpha_{\text{triv}'(\gamma)}(h(\delta)) h(\gamma) \] (2.3.88)

or equivalently by our earlier condition (2.3.86)

\[ h(\zeta \xi) \alpha_{g(z)}(\text{triv}(\Sigma)) = \text{triv}'(\Sigma) h(\gamma \delta). \] (2.3.89)

By Proposition 2.2.49, such a natural transformation can be decomposed into

\[ (g, h) = \begin{pmatrix} g, e \\ e, h \end{pmatrix}. \] (2.3.90)

**Definition 2.3.91.** A gauge transformation of the type \((g, e)\) is typically called a (first order) *thin gauge transformation* and one of the type \((e, h)\) is called a (first order) *fat gauge transformation* [MaMi10].
Thus, Proposition 2.2.49 implies that an arbitrary gauge transformation of the first kind can be decomposed into a thin and fat gauge transformation. Using this, we can calculate infinitesimal versions of the functions $g : M \rightarrow G$ and $h : P^1M \rightarrow H$ for small paths, i.e. for a point $x \in M$ and a tangent vector at $x$. This was already done for $g : M \rightarrow G$ at the end of Section 2.3.1 with result (2.3.15). For $h : P^1M \rightarrow H$, let $t \mapsto x(t)$ parametrize an infinitesimal path $\gamma$, then to lowest order

$$ h(\gamma) = \exp \left\{ \varphi_\mu(x(t)) \frac{dx_\mu}{dt} \bigg|_t \Delta t \right\} $$

(2.3.92)

for some 1-form $\varphi \in \Omega^1(M; h)$ by smoothness of $h$.\footnote{Smoothness of functions on infinite-dimensional spaces such as $h$ is discussed in more detail in [ScWa11] and in Chapter 3 of this thesis. In this case, one can take (2.3.92) as the definition of smoothness.} Thus, following (2.3.85), a fat gauge transformation from $(A, B)$ to $(A', B')$ infinitesimally gives

$$ \tau \left( \exp \left\{ \varphi_\mu(x(t)) \frac{dx_\mu}{dt} \bigg|_t \Delta t \right\} \right) \exp \left\{ -A_\nu(x(t)) \frac{dx_\nu}{dt} \bigg|_t \Delta t \right\} = \exp \left\{ -A'_\nu(x(t)) \frac{dx'_\nu}{dt} \bigg|_t \Delta t \right\} $$

(2.3.93)

expanding out to lowest order gives

$$ 1 + \tau \left( \varphi_\mu(x(t)) \frac{dx_\mu}{dt} \bigg|_t \right) \Delta t - A_\nu(x(t)) \frac{dx_\nu}{dt} \bigg|_t \Delta t = 1 - A'_\nu(x(t)) \frac{dx'_\nu}{dt} \bigg|_t \Delta t $$

(2.3.94)

giving the relationship

$$ A' = A - \tau(\varphi) $$

(2.3.95)
for a fat gauge transformation. We already calculated what happens for a thin gauge transformation in Section 2.3.1. Using Proposition 2.2.49, combining (2.3.15) with this gives

$$A' = gAg^{-1} - dgg^{-1} - \tau(\varphi)$$ (2.3.96)

for an arbitrary gauge transformation. The $B$ field under an arbitrary gauge transformation changes according to (2.3.88). By substituting the necessary forms, this expression on the left-hand-side of (2.3.88) becomes (to avoid clutter, we have not explicitly written $\Delta s$ and $\Delta t$)

$$\alpha \exp\left\{-A'_e \left(x(s,t+e)\right)|_{(s,t+e)}\right\} \left(\exp\left\{\varphi_\mu(x(s,t))\frac{\partial x^\mu}{\partial t}\right\}\right)$$

$$\times \exp\left\{\varphi_\rho(x(s,t+\epsilon))\frac{\partial x^\rho}{\partial s}\right\}$$

$$\times \alpha_{g(x(s+\epsilon,t+\epsilon))} \left(\exp\left\{B_{\sigma\tau}(x(s,t))\frac{\partial x^\sigma}{\partial s}\frac{\partial x^\tau}{\partial t}\right\}\right)$$

$$= \alpha_{1-A'_{e\frac{\partial x^\nu}{\partial x^\sigma}} - A'_{e\frac{\partial x^\lambda}{\partial x^\sigma}} - A'_{e\frac{\partial x^\nu}{\partial x^{\sigma\tau}}}} \left(1 + \varphi_\mu \frac{\partial x^\mu}{\partial t}\right)$$

$$\times \left(1 + \varphi_\rho \frac{\partial x^\rho}{\partial s} + \frac{\partial \varphi_\rho}{\partial x^\beta} \frac{\partial x^\beta}{\partial t} \frac{\partial x^\rho}{\partial s} + \varphi_\rho \frac{\partial x^\rho}{\partial s}\right)$$

$$\times \alpha_{g^{e\frac{\partial x^\sigma}{\partial x^\tau}} + g^{e\frac{\partial x^\lambda}{\partial x^\tau}} + g^{e\frac{\partial x^\nu}{\partial x^{\sigma\tau}}}} \left(1 + B_{\sigma\tau}\frac{\partial x^\sigma}{\partial s}\frac{\partial x^\tau}{\partial t}\right)$$

$$= 1 + \varphi_\mu \varphi_\rho \frac{\partial x^\mu}{\partial t} \frac{\partial x^\rho}{\partial s} + \frac{\partial \varphi_\rho}{\partial x^\beta} \frac{\partial x^\beta}{\partial t} \frac{\partial x^\rho}{\partial s} + \varphi_\rho \frac{\partial x^\rho}{\partial s}\frac{\partial x^\rho}{\partial t}$$

$$+ \alpha_{g}\left(B_{\sigma\tau}\frac{\partial x^\sigma}{\partial s}\frac{\partial x^\tau}{\partial t} - \alpha_{A'_e}(\varphi_\mu)\frac{\partial x^\mu}{\partial s}\frac{\partial x^\mu}{\partial t}\right)$$

$$= 1 + \varphi_\lambda \frac{\partial^2 x^\lambda}{\partial s \partial t} + \left(\varphi_\nu \varphi_\mu + \frac{\partial \varphi_\mu}{\partial x^\nu} + \alpha_{g}(B_{\mu\nu}) - \alpha_{A'_\mu}(\varphi_\nu)\right)\frac{\partial x^\mu}{\partial s} \frac{\partial x^\nu}{\partial t}$$
where it is understood that all terms now are evaluated at $(s, t)$. Meanwhile, the right-hand-side of (2.3.88) is

\[
\exp \left\{ B'_{\sigma \tau}(x(s, t)) \frac{\partial x^\sigma}{\partial s} \frac{\partial x^\tau}{\partial t} \right\}_{(s, t)} \times \alpha \exp \left\{ -\Lambda'_\nu \left( x(s + \epsilon, t) \right) \frac{\partial x^\nu}{\partial \epsilon} \right\}_{(s + \epsilon, t)} \left( \exp \left\{ \varphi_{\mu}(x(s, t)) \frac{\partial x^\mu}{\partial s} \right\}_{(s, t)} \right) \times \exp \left\{ \varphi_\lambda(x + \epsilon, t) \frac{\partial x^\lambda}{\partial \epsilon} \right\}_{(s + \epsilon, t)} \right.
\]

\[
= \left(1 + B'_{\sigma \tau} \frac{\partial x^\sigma}{\partial s} \frac{\partial x^\tau}{\partial t} \right) \alpha_1 - \Lambda'_\nu \left( x(s + \epsilon, t) \right) \frac{\partial x^\nu}{\partial \epsilon} + \Lambda'_\nu \left( x(s, t) \right) \frac{\partial x^\nu}{\partial \epsilon} \left(1 + \varphi_{\mu}(x(s, t)) \frac{\partial x^\mu}{\partial s} \right) \right.
\]

\[
= 1 + B'_{\sigma \tau} \frac{\partial x^\sigma}{\partial s} \frac{\partial x^\tau}{\partial t} + \varphi_{\mu} \Lambda'_\nu \left( x(s, t) \right) \frac{\partial x^\mu}{\partial s} - \Lambda'_\nu \left( x(s, t) \right) \frac{\partial x^\mu}{\partial s} \frac{\partial x^\mu}{\partial t} - \alpha_{A'_\nu} \left( x(s, t) \right) \frac{\partial x^\nu}{\partial t} \frac{\partial x^\mu}{\partial s} \right.
\]

\[
= 1 + \varphi_\lambda \frac{\partial^2 x^\lambda}{\partial t \partial s} + \left( \varphi_{\mu} \varphi_{\nu} + B'_{\mu \nu} - \alpha_{A'_\nu} \left( x(s, t) \right) \frac{\partial x^\nu}{\partial t} \right) \frac{\partial x^\mu}{\partial s} \frac{\partial x^\nu}{\partial t}
\]

Equating these two expressions gives

\[
B'_{\mu \nu} = \alpha_{\bar{g}}(B_{\mu \nu}) - \varphi_{\mu} \varphi_{\nu} - \varphi_{\nu} \varphi_{\mu} - \partial_{\mu} \varphi_{\nu} - \partial_{\nu} \varphi_{\mu} - \alpha_{A'_\nu} \left( x(s, t) \right) \frac{\partial x^\nu}{\partial t} - \alpha_{A'_\mu} \left( x(s, t) \right) \frac{\partial x^\mu}{\partial t} \right.
\]

(2.3.99)

in components or

\[
B' = \alpha_{\bar{g}}(B) - \varphi \wedge \varphi - d\varphi - \alpha_{A'}(\varphi)
\]

(2.3.100)

as an equation in terms of differential forms. We write such a gauge transformation as

\[
(A, B) \xrightarrow{(g, \varphi)} (A', B').
\]

(2.3.101)
This and (2.3.96) agrees with Proposition 2.10 of [ScWa11]. We can express this purely in terms of \( A, B, g, \) and \( \varphi \) as

\[
B' = \alpha_g(B) - \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1}-dgg^{-1}}(\varphi) \wedge (\varphi)
\]

\[
= \alpha_g(B) - \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1}-dgg^{-1}}(\varphi) + [\varphi, \varphi] \quad (2.3.102)
\]

\[
= \alpha_g(B) + \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1}-dgg^{-1}}(\varphi).
\]

This will be useful later.

**Definition 2.3.103.** Let \((g, h), (g', h') : \text{triv} \Rightarrow \text{triv}'\) be two first order gauge transformations. A **second order gauge transformation** \(a : (g, h) \Rightarrow (g', h')\) is a (smooth) modification from \((g, h)\) to \((g', h')\).

By Definition 2.2.31, this consists of a smooth function \(a : M \rightarrow H\) fitting into

\[
\begin{array}{c}
g(x) \\
\uparrow \\
a(x) \\
\uparrow \\
g'(x)
\end{array}
\]

which in particular says

\[
\tau(a)g = g', \quad (2.3.104)
\]

satisfying the condition that to any path \(y \xleftarrow{\gamma} x\),

\[
h'(\gamma)a(y) = \alpha_{\text{triv}'(\gamma)}(a(x))h(\gamma). \quad (2.3.105)
\]
Expanding out this expression on infinitesimal paths gives

\[ \left(1 + \varphi'_\mu \frac{dx^\mu}{dt}\right) \left( a + \frac{\partial a}{\partial t} \right) = \alpha_{1-A'_\mu \frac{dx^\mu}{dt}}(a) \left(1 + \varphi'_\nu \frac{dx^\nu}{dt}\right), \quad (2.3.106) \]

which is

\[ 1 + \varphi'_\mu \frac{dx^\mu}{dt} + \frac{\partial a}{\partial x^\nu} \frac{dx^\nu}{dt} = 1 + a \varphi'_\nu \frac{dx^\nu}{dt} - \alpha_{A'_\nu}(a) \frac{dx^\mu}{dt}, \quad (2.3.107) \]

which gives the condition (after multiplying by \( a^{-1} \) on the right)

\[ \varphi'_\mu = a \varphi'_\mu a^{-1} - \alpha_{A'_\mu}(a)a^{-1} - (\partial_\mu a)a^{-1} \quad (2.3.108) \]

on components and

\[ \varphi'_\nu = a \varphi'_\nu a^{-1} - d a a^{-1} - \alpha_{A'_\nu}(a)a^{-1} \quad (2.3.109) \]

as \( \mathfrak{g} \)-valued differential forms. This and (2.3.104) exactly agree with Proposition 2.11 of [ScWa11].

### 2.3.4 Orientations and inverses

It is well-known that given a path \( y \xrightarrow{\gamma} x \), the parallel transport along the reversed oriented path \( \gamma^{-1} \) is the inverse

\[ \text{triv}(\gamma^{-1}) = \text{triv}(\gamma)^{-1}, \quad (2.3.110) \]

where \( \text{triv} : \mathcal{P}^1(M) \longrightarrow \mathbb{B}G \) is the (local) parallel transport functor. This can be viewed as a consequence of thin homotopy invariance and functoriality of
parallel transport. Namely, although the paths $\gamma^{-1}\gamma$ and $\gamma^{-1}\gamma$ are not the constant paths (the notation $\gamma^{-1}$ is therefore a bit abusive), they are thinly homotopic to constant paths and hence give the same value on triv. Thus,

$$\text{triv}(\gamma^{-1}\gamma) \quad \quad \quad \quad \quad \quad \text{triv}(\text{id}_x) \quad \quad \quad \quad \quad \text{triv}(\gamma^{-1})\text{triv}(\gamma)$$

verifying (2.3.110). In this section, we will explore analogous results for reversing different kinds of orientations on bigons. We therefore include arrows for clarity.

Note that the different orientations on a bigon can be expressed as an orientation of edges on the boundary and an orientation of the surface. The above bigons correspond to the following surfaces with associated orientations.
A necessary and sufficient condition for such orientations on surfaces and edges to give rise to a bigon is the following. Given a map of a polygon \( \Sigma \) into \( M \), the boundary consists of the edges of the polygon. The union of the oriented edges consistent with the orientation of the polygon must be connected. Similarly, the union of the orientated edges with negative orientation with respect to the induced one from the polygon must also be connected. Then, the source of the bigon is the union of the consistent edges and the target is the union of the oppositely oriented edges. An example together with a non-example are

\[
\text{\begin{tikzpicture}
\end{tikzpicture}} \quad \& \quad \text{\begin{tikzpicture}
\end{tikzpicture}},
\]

respectively (blue corresponds to an orientation agreeing with the induced one from the surface while yellow disagrees with that orientation).

Going back to the three bigons and their orientations at the beginning of this section, we notice that several of these bigons can be composed with one another. For instance,
and

after applying a thin homotopy. Therefore, these bigons provide inverses in series and in parallel, respectively, of $\Sigma$. This implies, together with functoriality of triv and the inverses discussed in Example 2.2.42,

$$\text{triv}(\Sigma) = \text{triv}(\Sigma)^{-1} \quad \& \quad \text{triv}(\Sigma^{-1}) = \alpha_{\text{triv}(\gamma)^{-1}}(\text{triv}(\Sigma)^{-1}) \quad (2.3.112)$$

and therefore describes how parallel transport along surfaces changes under reversals in surface orientations and boundary orientations, respectively.

### 2.3.5 The 3-curvature

In the following, we make some further calculations. Just as the curvature $F$ of a 1-form connection $A$ can be obtained by calculating the parallel trans-
port along an infinitesimal loop, the 2-curvature of a 2-form connection \((A, B)\) can be obtained by calculating the surface transport along an infinitesimal sphere, which on a lattice corresponds to a cube. We will perform this calculation explicitly and study some properties of the resulting 3-form curvature. Similar analysis was done on a tetrahedron in [GiPf04].

Let \((r, s, t) \mapsto x(r, s, t)\) be an infinitesimal cube and consider the following domain for that cube along with the infinitesimal path that goes first along the \(r\) direction, then in the \(s\) direction, and finally in the \(t\) direction. Our convention is that \((r, s, t)\) is a right-handed coordinate frame, i.e. \(dr \wedge ds \wedge dt\) is the volume form.

![Diagram](image.png)

Such a cube can be expressed as a bigon by the following sequence of plaquette bigons that begin and end at the same path starting at the top left and moving clockwise.
The corresponding 2-group elements are given as follows. We begin with the first surface introducing some shorthand notation

\begin{equation}
\partial_r x := \frac{\partial x}{\partial r}, \quad \partial_s x := \frac{\partial x}{\partial s}, \quad \partial_t x := \frac{\partial x}{\partial t} \tag{2.3.113}
\end{equation}
as well as

\[ e^{-A_\mu \hat{\partial}_t x^\mu \mid_{(\epsilon, \epsilon, 0)}} := \exp \left\{ -A_\mu \left( x(\epsilon, \epsilon, 0) \right) \frac{\hat{\partial}_t x^\mu}{\partial t} \mid_{(\epsilon, \epsilon, 0)} \right\} \quad (2.3.114) \]

and similarly for the other terms. We also write \( \epsilon \) instead of \( \Delta r, \Delta s, \) or \( \Delta t \) and use the derivatives to remind ourselves of the direction. We have also assumed for simplicity that our coordinates are centered at the origin and the lattice spacing is \( \epsilon \) in each direction. Working out this diagram infinitesimally on the 0-d defect gives

\[ \alpha \frac{e^{-A_\mu \hat{\partial}_t x^\mu \mid_{(\epsilon, \epsilon, 0)}} \left( e^{B_{\lambda \nu} \hat{\partial}_r x^\lambda \hat{\partial}_s x^\nu \mid_{(0,0,0)}} \right)}{e^{B_{\lambda \nu} \hat{\partial}_r x^\lambda \hat{\partial}_s x^\nu \mid_{(0,0,0)}}} - B_{\lambda \nu} \hat{\partial}_r x^\lambda \hat{\partial}_s x^\nu \]

\[ - \frac{\alpha A_\mu (B_{\lambda \nu}) \hat{\partial}_r x^\mu \hat{\partial}_s x^\nu}{2} \quad (2.3.115) \]

to lowest order. As usual, rather than writing out the \( \Delta r, \Delta s, \Delta t \), we use the number and type of derivatives appearing to keep track of the order. The other terms are given by the following
\[ e^{|_{(0,\epsilon,0)}|_{0,0,0}} = 1 + B^B_{\rho\sigma} \partial_\rho^x \partial_\sigma^x + \partial_\rho^x B^B_{\rho\pi} \partial_\pi^x \partial_\tau^x \]

\[ + B^B_{\rho\pi} \partial_\pi^x \partial_\tau^x \partial_\rho^x + B^B_{\rho\pi} \partial_\rho^x \partial_\pi^x \partial_\tau^x \]

\[ (2.3.116) \]

\[ \alpha e^{-A_\alpha \partial_\alpha x^\alpha |_{(0,0,0)}} \left( e^{B^B_{\sigma\pi} \partial_\sigma x^\sigma \partial_\tau^x |_{(0,0,0)}} \right) = 1 + B^B_{\sigma\tau} \partial_\sigma x^\sigma \partial_\tau^x \]

\[ - \alpha A_\alpha (B^B_{\sigma\tau}) \partial_\alpha x^\alpha \partial_\beta x^\beta \partial_\gamma \]

\[ (2.3.117) \]

\[ e^{B^B_{\beta\kappa} \partial_\beta x^\beta \partial_\kappa x^\kappa |_{(0,0,0)}} = 1 + B^B_{\beta\kappa} \partial_\beta x^\beta \partial_\kappa x^\kappa + \partial_\beta^B B^B_{\beta\kappa} \partial_\tau^x \partial_\beta x^\kappa \]

\[ + B^B_{\beta\kappa} \partial_\beta^B \partial_\kappa x^\kappa + B^B_{\beta\kappa} \partial_\beta x^\beta \partial_\kappa x^\kappa \]

\[ (2.3.118) \]
The composition of all of these elements is given by the following diagram (with the light shaded blue squares depicting the faces of the cube).

\[
\alpha e^{-A_\theta \partial_\tau x^\theta|_{(\epsilon,0,0)}} (e^{B_{\gamma\eta} \partial_t x^\gamma \partial_r x^\eta|_{(0,0,0)}}) = 1 + B_{\gamma\eta} \partial_t x^\gamma \partial_r x^\eta - \alpha A_B (B_{\gamma\eta}) \partial_s x^\theta \partial_t x^\gamma \partial_r x^\eta \tag{2.3.119}
\]

\[
e^{B_{\omega\psi} \partial_t x^\omega \partial_s x^\psi|_{(\epsilon,0,0)}} = 1 + B_{\omega\psi} \partial_t x^\omega \partial_s x^\psi + \partial_\alpha B_{\omega\psi} \partial_r x^\alpha \partial_t x^\omega \partial_s x^\psi + B_{\omega\psi} \partial_r x^\omega \partial_s x^\psi + B_{\omega\psi} \partial_t x^\omega \partial_r x^\psi \tag{2.3.120}
\]

The composition of all of these elements is given by the following diagram (with the light shaded blue squares depicting the faces of the cube).
And the result of multiplying these out gives

\[
e^{B_{\omega\phi} \partial_x^\omega \partial_\phi x^\phi |(\epsilon,0,0)} \alpha_{-A_{\phi} \partial_x^\phi |(\epsilon,0,0)} \left( e^{B_{\gamma\eta} \partial_x^\gamma \partial_\eta x^\eta |(0,0,0)} e^{B_{\beta\alpha} \partial_\beta x^\beta \partial_\alpha x^\alpha |(0,0,\epsilon)} \right) \\
\times \alpha_{-A_{\alpha} \partial_x^\alpha |(0,0,\epsilon)} \left( e^{B_{\sigma\tau} \partial_x^\sigma \partial_\tau x^\tau |(0,0,0)} e^{B_{\rho\pi} \partial_x^\rho \partial_\pi x^\pi |(0,\epsilon,0)} \right) \\
\times \alpha_{-A_{\mu} \partial_x^\mu |(\epsilon,0,0)} \left( e^{B_{\lambda\nu} \partial_x^\lambda \partial_\nu x^\nu |(0,0,0)} \right). \tag{2.3.121}
\]
This is yet another manifestation of two-dimensional algebra. The result of multiplying all these terms is given as follows, order by order. The zeroth order term is $1$.

There are no first order terms. The second order terms are given by

$$B_\omega \xi \xi \hat{\xi} \hat{\xi} + B_\gamma \eta \xi \hat{\xi} \hat{\xi} + B_\beta \kappa \xi \hat{\xi} \hat{\xi}$$

$$+ B_\sigma \tau \xi \hat{\xi} \hat{\xi} + B_\rho \pi \xi \hat{\xi} \hat{\xi} + B_\lambda \nu \xi \hat{\xi} \hat{\xi}$$

$$= (B_\sigma \tau + B_\tau \sigma) \xi \hat{\xi} \hat{\xi} + (B_\lambda \nu + B_\nu \lambda) \xi \hat{\xi} \hat{\xi} \quad (2.3.122)$$

$$+ (B_\rho \pi + B_\pi \rho) \xi \hat{\xi} \hat{\xi} \hat{\xi}$$

$$= 0$$

by anti-symmetry of $B_{\mu \nu}$ in the $\mu$ and $\nu$ indices. Thus, the only non-zero terms are the zeroth and third order terms (up to third order). One type of the third order terms are given by

$$\left( B_\omega \xi \xi \hat{\xi} \hat{\xi} + B_\omega \xi \xi \hat{\xi} \hat{\xi} + B_\beta \kappa \xi \hat{\xi} \hat{\xi} \hat{\xi} \right)$$

$$+ \left( B_\sigma \tau \xi \hat{\xi} \hat{\xi} \hat{\xi} + B_\rho \pi \xi \hat{\xi} \hat{\xi} \hat{\xi} \right)$$

$$= (B_\beta \kappa + B_\kappa \beta) \xi \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi} + (B_\omega \psi + B_\psi \omega) \xi \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi}$$

$$+ (B_\rho \pi + B_\pi \rho) \xi \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi}$$

and vanish again by anti-symmetry of $B_{\mu \nu}$ and commutativity of partial
derivatives. The final result is of (2.3.121) to lowest nontrivial order is
\[ 1 + \partial_a B_{\omega \psi} \partial_t x^a \partial_t x^\omega \partial_r x^\psi - \alpha_{A_\theta} (B_{\gamma \eta}) \partial_s x^\theta \partial_t x^\gamma \partial_r x^\eta + \partial_t B_{\beta \mu} \partial_t x^\beta \partial_s x^\mu \partial_r x^\gamma \]
\[ - \alpha_{A_\alpha} (B_{\sigma \tau}) \partial_r x^\alpha \partial_s x^\beta \partial_t x^\gamma + \partial_{\lambda} B_{\rho \nu} \partial_s x^\rho \partial_r x^\nu \partial_t x^\gamma - \alpha_{A_\mu} (B_{\lambda \nu}) \partial_t x^\mu \partial_r x^\lambda \partial_s x^\nu \]
\[ = 1 - \left( \partial_{\mu} B_{\nu \lambda} + \partial_{\lambda} B_{\mu \nu} + \partial_{\nu} B_{\lambda \mu} + \alpha_{A_\mu} (B_{\nu \lambda}) + \alpha_{A_\nu} (B_{\lambda \mu}) + \alpha_{A_\lambda} (B_{\mu \nu}) \right) \partial_r x^\mu \partial_s x^\nu \partial_t x^\lambda \]
\[ = 1 - H_{\mu \nu \lambda} \partial_r x^\mu \partial_s x^\nu \partial_t x^\lambda. \quad (2.3.124) \]

In analogy to the curvature 2-form associated to a 1-form potential \( A \) obtained by calculating the holonomy along an infinitesimal square, we define this third order term to be the 3-form curvature associated to the pair \( (A, B) \) and denote it by \( H \). In terms of components, it is given by
\[ H_{\mu \nu \lambda} := \partial_{\mu} B_{\nu \lambda} + \partial_{\lambda} B_{\mu \nu} + \partial_{\nu} B_{\lambda \mu} + \alpha_{A_\mu} (B_{\nu \lambda}) + \alpha_{A_\nu} (B_{\lambda \mu}) + \alpha_{A_\lambda} (B_{\mu \nu}) \quad (2.3.125) \]
and using differential form notation
\[ H := dB + \alpha_A (B). \quad (2.3.126) \]

This definition and result agrees with (3.28) of [GiPf04] and Lemma A.11 in [ScWa11]. As was also pointed out in [GiPf04],
\[ \tau (H) = \tau (dB) + \tau (\alpha_A (B)) = d \tau (B) + [A, \tau (B)] = dF + [A, F] = 0 \quad (2.3.127) \]
by the Bianchi identity. Since \( \ker \tau \) is a central Lie subalgebra of \( h \), this means \( H \) is a 3-form with values in an abelian Lie algebra (see Remark 2.2.41).
At this point, the reader should consult the Appendix on differential Lie crossed modules if the following calculations are mysterious.

**Proposition 2.3.128.** Under a first order gauge transformation \((A, B) \xrightarrow{(g, \varphi)} (A', B')\) as in (2.3.101) and using (2.3.102), the 3-form curvature changes to

\[
H' = \alpha_g(H) - [\varphi, \alpha_g(B)] - \alpha_{gFg^{-1}}(\varphi). \tag{2.3.129}
\]

**Proof.** To see this, first note that

\[
H' = dB' + \alpha_{A'}(B')
\]

\[
= d\left(\alpha_g(B) + \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1} - dgg^{-1}}(\varphi)\right)
\]

\[
+ \alpha_{gAg^{-1} - dgg^{-1} - \tau(\varphi)}\left(\alpha_g(B) + \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1} - dgg^{-1}}(\varphi)\right)
\]

\[
= d\left(\alpha_g(B)\right) + d\varphi \wedge \varphi - d\varphi - d\left(\alpha_{gAg^{-1} - dgg^{-1}}(\varphi)\right)
\]

\[
+ \alpha_{gAg^{-1} - dgg^{-1}}\left(\alpha_g(B) + \varphi \wedge \varphi - d\varphi - \alpha_{gAg^{-1} - dgg^{-1}}(\varphi)\right)
\]

\[
- [\varphi, \alpha_g(B)] - \underbrace{[\varphi, \varphi \wedge \varphi] + [\varphi, d\varphi] + [\varphi, \alpha_{gAg^{-1} - dgg^{-1}}(\varphi)]}_{0},
\]

where the underlined terms cancel. It will take a bit of work to simplify all of this. At this point, it is useful to figure these out by applying \(\tau\) and
calculating the results in terms of commutators and such. For example,
\[ \tau \left( d_{\alpha_g} (B) \right) = d \left( \tau(\alpha_g(B)) \right) \]
\[ = d(g \tau(B) g^{-1}) \]
\[ = dg \tau(B) g^{-1} + g \tau(dB) g^{-1} + g \tau(B) dg^{-1} \]
\[ = dgg^{-1} g \tau(B) g^{-1} + \tau(\alpha_g(dB)) - g \tau(B) g^{-1} dgg^{-1} \]
\[ = \tau(\alpha_g(dB) + \alpha_{dgg^{-1}}(\alpha_g(B))) . \]

This is a helpful trick and we will use it to calculate all other terms (one can be more rigorous without using this trick, but our results are not changed).

For instance,
\[ \alpha_{Ag^{-1}}(\alpha_g(B)) = \alpha_g(\alpha_A(B)). \]

Since \( \alpha \) is a derivation,
\[ \alpha_{Ag^{-1} - dgg^{-1}}(\varphi \wedge \varphi) = \alpha_{Ag^{-1} - dgg^{-1}}(\varphi) \wedge \varphi - \varphi \wedge \alpha_{Ag^{-1} - dgg^{-1}}(\varphi), \]

which cancels with the term \( [\varphi, \alpha_{Ag^{-1} - dgg^{-1}}(\varphi)] \). Furthermore, note that
\[ \tau \left( \alpha_X (\alpha_X(Y)) \right) = \left[ X, \tau(\alpha_X(Y)) \right] \]
\[ = \left[ X, [X, \tau(Y)] \right] \]
\[ = \left[ X, X \wedge \tau(Y) + \tau(Y) \wedge X \right] \]
\[ = X \wedge X \wedge \tau(Y) - X \wedge \tau(Y) \wedge X \]
\[ + X \tau(Y) \wedge X - \tau(Y) \wedge X \wedge X \]
\[ = \tau(\alpha_{X \wedge X}(Y)) \]
for any \( g \)-valued 1-form \( X \) and for any \( h \)-valued 1-form \( Y \). This implies
\[
\Omega_{gAg^{-1}-dgg^{-1}}(\Omega_{gAg^{-1}-dgg^{-1}}(\varphi)) = \Omega_{gAg^{-1}}(\varphi) - \Omega_{dgAg^{-1}}(\varphi) - \Omega_{Ag^{-1}dgg^{-1}}(\varphi) + \Omega_{dgg^{-1}Ag^{-1}}(\varphi).
\] (2.3.135)

One of the more cumbersome set of terms is
\[
\tau\left(d(\Omega_{gAg^{-1}-dgg^{-1}}(\varphi)) + \Omega_{gAg^{-1}-dgg^{-1}}(d\varphi)\right) = d[gAg^{-1}, \tau(\varphi)] - d[dgg^{-1}, \tau(\varphi)] + [gAg^{-1}, \tau(d\varphi)] - [dgg^{-1}, \tau(d\varphi)]
\]
\[
= dgAg^{-1}\tau(\varphi) + gdAg^{-1}\tau(\varphi) - gAg^{-1}\tau(d\varphi) - dgg^{-1}\tau(d\varphi)
\]
\[
+ \tau(d\varphi)gAg^{-1} - \tau(\varphi)dgAg^{-1} - \tau(\varphi)gdAg^{-1} + \tau(\varphi)gAdg^{-1}
\]
\[
+ dgAg^{-1}\tau(\varphi) - dgg^{-1}\tau(d\varphi) - dgg^{-1}\tau(\varphi) + dgg^{-1}\tau(d\varphi)
\]
\[
= \tau\left(\Omega_{gAg^{-1}}(\varphi) + \Omega_{gAg^{-1}}(\varphi) - \Omega_{AgAg^{-1}}(\varphi) + \Omega_{dgg^{-1}Ag^{-1}}(\varphi)\right).
\] (2.3.136)

Combining this with the result preceding it gives just a single term \( \Omega_{gFF^{-1}}(\varphi) \).

Putting all of this together into (2.3.130), we obtain
\[
H' = \omega_g(dB + \omega_A(B)) - [\varphi, \omega_g(B)] - \omega_{gFG^{-1}}(\varphi)
\]
\[
= \omega_g(H) - [\varphi, \omega_g(B)] - \omega_{gFG^{-1}}(\varphi).
\] (2.3.137)

\[\blacksquare\]

In the special case of a thin gauge transformation, where \( \varphi = 0 \), this becomes
\[
H' = \omega_g(H)
\] (2.3.138)
and in the special case of a fat gauge transformation, where $g = e$, this becomes

$$H' = H - [\varphi, B] - \alpha_F(\varphi). \quad (2.3.139)$$

### 2.4 Discussion, conclusion, and future work

We have illustrated that 2-category theory can be implemented and used in such a way as to calculate parallel transport along two-dimensional surfaces, such as worldsheets of strings, explicitly for gauge groups that are not necessarily abelian via an approximation technique that can be implemented numerically. We have done this using string diagram techniques to facilitate 2-categorical techniques and bring category theory to a wider audience. Although Girelli and Pfeiffer have calculated infinitesimal gauge transformations and curvature forms via similar techniques [GiPf04] and Schreiber and Waldorf provided formula for the parallel transport along a surface [ScWa11], our infinitesimal methods give a much more explicit and direct construction of the iterated surface integral from elementary building blocks filling in some of the arguments sketched by Baez and Schreiber in [BaSc04], particularly in Section 2.3.2 (Section 5.1 of a draft of this paper even contains a nice picture that unfortunately did not make it to the final version of their paper). Schreiber and Waldorf’s integral in [ScWa11] was obtained from
consistency conditions and then they proved that it satisfies the necessary functorial properties expected of surface holonomy. The novelty of our result is that we derived the formula from scratch using discretizations of our surface. To our knowledge, this is the first appearance of such an explicit construction together with analytical results on convergence and a simplification providing a manageable surface-ordered integral. In relation to other work, such surface-ordered integrals have been used recently in constructing a Hochschild complex for surface transport [Mi15]. Our approach offers a more detailed analysis including verification of convergence and arguments supporting the idea that the path ordering can be done in a single direction as opposed to two. By implementing string diagrams, we have also provided a more friendly visualization. Furthermore, we have avoided using path spaces and have dramatically simplified the arguments.

We hope that we have opened a new realm to two-dimensional algebra illustrating how it can be used for explicit calculations. If taken further, these ideas may be used to explain two-dimensional physical phenomena more naturally. Although slightly speculative, consider elements and molecules for instance. These are the building blocks of chemical compounds in nature and can be used to build up more complicated structures such as amino acids and proteins. These are objects that use three dimensions to configure themselves
and therefore a natural and faithful representation of them would involve a sort of 3-dimensional algebra. As another even more speculative example, it is known that the entropy of a black hole is proportional to the surface area of the horizon. This may lead one to believe that the microstates of the theory can be expressed as living on a lower-dimensional world. This in turn then suggests the possibility that a lower-dimensional algebra may be useful in describing some of the properties of the theory that describes these microstates.

We hope to address additional issues in future work. These include a further analysis of lattice gauge theory including matter fields [Wi74]. Work on the pure gauge field side was initiated in the work of Pfeiffer [Pf03] using a 2-categorical approach. To proceed, it seems that a better suited representation theory for 2-categories will be useful [BBFW12]. Furthermore, characters for 2-groups [GaKa08], [GaUs14] and traces [PoSh13], [HPT15] need to be studied further. Other lattice gauge theory approaches existed earlier [Or80], [Or83], [Or84] with a renewed interest in [LiRE14] but it is not clear to us how these approaches to higher lattice gauge theory are related to the rest of the literature.

In the realm of string theory and M-theory, a more precise construction of the non-abelian gauge theories on a stack of \( D \)-branes [Zw09] and its low
energy effective Action, beginning with early work of Witten, Myers, and others [Wit96], [My99], is still lacking. These effective Actions are swarmed with higher form non-abelian gauge fields, but the precise mathematical formulation is still lacking though it is likely that non-abelian differential cohomology [Sc16b] is relevant suggested by recent work on M5-branes in which it plays an essential role [FSS14]. Most arguments used to describe such effective Actions are not always entirely correct and involve consistency conditions (such as T-duality [My99] and scattering amplitude calculations [DST00]) rather than derivations. It is therefore possible that a more thorough investigation may involve understanding the nonperturbative effects, one of which is dictated by transport. On the other hand, due to the non-commutative nature of the normal coordinates to these branes [Mo05], this may involve a modification of such transport to the setting of non-commutative geometry.

**Appendix: Differential Lie crossed modules**

Here we briefly review the infinitesimal version of a Lie crossed module $(H, G, \tau, \alpha)$, which we write as $(\mathfrak{h}, \mathfrak{g}, \tau, \alpha)$. There are many relations that these maps satisfy that are used throughout, which we review here. We also make some comments on how this is used for differential forms with values in $\mathfrak{g}$ and $\mathfrak{h}$. This information can also be found in many articles on the subject
of higher gauge theory such as [BaSc04], [GiPf04]. Martins and Miković also
have an exceptionally clear and thorough exposition in Section 2.1 of their
document [MaMi10].

$\tau : \mathfrak{h} \rightarrow \mathfrak{g}$ is the derivative of $\tau : H \rightarrow G$ at the identity and is a Lie
algebra homomorphism since $\tau$ is a Lie group homomorphism. Notice that
$\alpha$ can be equivalently described as a function $\alpha : G \times H \rightarrow H$ that is a
group homomorphism in each component separately. As a result, for any
fixed $g \in G$, $\alpha_g : H \rightarrow H$ is a Lie group homomorphism and hence has a
derivative at the identity. We denote this by $\alpha_g : \mathfrak{h} \rightarrow \mathfrak{h}$. This map, besides
being a Lie algebra homomorphism, satisfies the additional property that

$$\tau(\alpha_g(Y)) = g\tau(Y)g^{-1}$$  \hspace{1cm} (2.4.1)

for all $Y \in \mathfrak{h}$ and $g \in G$. Similarly, although $\alpha : G \times H \rightarrow H$ is not a group
homomorphism, it is smooth and its derivative $\alpha : \mathfrak{g} \times \mathfrak{h} \rightarrow \mathfrak{h}$ makes sense.

It is a derivation once the $\mathfrak{g}$ coordinate is fixed, i.e.

$$\alpha_X([Y, Z]) = [\alpha_X(Y), Z] + [Y, \alpha_X(Z)]$$  \hspace{1cm} (2.4.2)

for all $X \in \mathfrak{g}$ and $Y, Z \in \mathfrak{h}$. $\alpha$ also satisfies

$$\alpha_{[X, Y]}(Y) = \alpha_X(\alpha_Y(Y)) - \alpha_Y(\alpha_X(Y))$$  \hspace{1cm} (2.4.3)
for all $X, X' \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Finally,

$$\tau(\alpha_X(Y)) = [X, \tau(Y)] \quad (2.4.4)$$

and

$$\alpha_{\tau(Y)}(Z) = [Y, Z] \quad (2.4.5)$$

for all $X \in \mathfrak{g}$ and $Y, Z \in \mathfrak{h}$.

Once combined with differential forms, the maps $\alpha$ and $\tau$ are extended in the appropriate way (see Part II Chapter 3 in the section on the Bianchi Identity in [BaMu94] for details on differential forms with values in Lie algebras).

For instance, $\alpha$ is a graded derivation in its second coordinate. To clarify the notation used throughout, consider differential forms $A \in \Omega^1(M; \mathfrak{g})$, $F \in \Omega^2(M; \mathfrak{g})$, $\varphi \in \Omega^1(M; \mathfrak{h})$, $B \in \Omega^2(M; \mathfrak{h})$. When we write expressions such as $\alpha_A(\varphi)$ or $\alpha_A(B)$ we mean the following. First, let $\{t^a\}_a$ be a basis for $\mathfrak{g}$ and $\{s^b\}_b$ be a basis for $\mathfrak{h}$. Then

$$A = A_a t^a, \quad F = F_a t^a, \quad \varphi = \varphi_b s^b, \quad \& \quad B = B_b s^b \quad (2.4.6)$$

with a summation over indices implied and where $A_a, \varphi_b \in \Omega^1(M)$ and $F_a, B_b \in \Omega^2(M)$ for all indices. Then by definition,

$$\alpha_A(\varphi) \equiv \alpha_{A_a t^a}(\varphi_b s^b) := (A_a \wedge \varphi_b)\alpha_{t^a}(s^b) \quad (2.4.7)$$
and similarly for any other forms. Because we use Lie algebra valued forms, the bracket is graded, so for instance

\[ [\varphi, \varphi] := (\varphi_b \wedge \varphi_b)[s^b, s^{b'}] = \varphi \wedge \varphi + \varphi \wedge \varphi \quad (2.4.8) \]

but

\[ [\varphi, B] := (\varphi_b \wedge B_b)[s^b, s^{b'}] = \varphi \wedge B - B \wedge \varphi \quad (2.4.9) \]

The second equalities follow if we think of our Lie algebras as coming from matrix Lie algebras, which we often do. The general formula is

\[ [\omega, \eta] = \omega \wedge \eta - (-1)^{||\omega||\eta} \eta \wedge \omega, \quad (2.4.10) \]

where \(|\omega|\) and \(|\eta|\) are the degrees of the forms \(\omega \in \Omega^{|\omega|}(M; \mathfrak{h})\) and \(\eta \in \Omega^{|\eta|}(M; \mathfrak{h})\). Other properties are derived as needed in calculations in the body of the article.

**Appendix: Proof of configurations Lemma**

This appendix serves to give a rigorous proof of Lemma 2.3.67. Recall, \(S_{n,k}\) is the number of configurations of \(k\) blocks on an \(n \times n\) grid tilted 45° such that no two blocks on the same horizontal row are occupied. \(R_{n,k}\) is the ratio of this to the total number of configurations. Lemma 2.3.67 states that \(\lim_{n \to \infty} R_{n,k} = 0\) for all \(k\). For the proof of this Lemma, it is useful to rewrite
\[ S_{n,k} = \frac{1}{k!} \sum_{2n-1 \geq i_k \neq i_{k-1} \neq \ldots \neq i_1 \geq 1} l_n(i_1) \cdots l_n(i_k) \]

\[ = \frac{1}{k!} \sum_{i_k = 1}^{2n-1} l_n(i_k) \sum_{i_{k-1} = 1}^{2n-1} l_n(i_{k-1}) \cdots \sum_{i_2 = 1}^{2n-1} l_n(i_2) \sum_{i_1 = 1}^{2n-1} l_n(i_1), \quad (2.4.11) \]

where it is understood that any sum operation on the left acts on everything to the right. Before working out this summation to obtain a more explicit formula, for each \( n \in \mathbb{Z}^+ \), define the function \( \phi_n : \{1, 2, \ldots, 2n-1\} \rightarrow \mathbb{Z} \) by

\[ \phi_n(p) := \sum_{i=1}^{2n-1} l_n(i)^p \quad (2.4.12) \]

where \( z \in \mathbb{Z}^+ \). Explicitly, this can be calculated as follows \cite{We02a}.

\[ \phi_n(p) = 2 \sum_{q=1}^{n} q^p - n^p = \frac{2}{p+1} \sum_{q=1}^{p+1} (-1)^{\delta_{qp}} \binom{p+1}{q} B_{p+1-q} n^q - n^p, \quad (2.4.13) \]

where \( \delta_{qp} \) is the Kronecker delta function and \( B_r \) is the Bernoulli number defined, for instance, by the power series expansion (thought of as a formal power series in the variable \( x \)) \cite{We02b}

\[ \frac{x}{e^x - 1} = \sum_{r=0}^{\infty} \frac{B_r x^r}{r!}. \quad (2.4.14) \]
The first few of these Bernoulli numbers are
\[ B_0 = 1 \]
\[ B_1 = -\frac{1}{2} \]
\[ B_2 = \frac{1}{6} \]
\[ B_3 = 0 \]
\[ B_4 = -\frac{1}{30} \] (2.4.15)

while the first few \( \phi_n \) are
\[ \phi_n(1) = n^2 \]
\[ \phi_n(2) = \frac{n(n^2 + 1)}{3} \]
\[ \phi_n(3) = \frac{n^2(n^2 + 1)}{2} \]
\[ \phi_n(4) = \frac{n(6n^4 + 10n^2 - 1)}{15} \] (2.4.16)

Examining \( \phi_n(p) \) a little more, one sees immediately a crucial result for the proof of this lemma
\[ \lim_{n \to \infty} \frac{\phi_n(p)}{n^{2p}} = 0 \quad \text{for } p \geq 2. \] (2.4.17)
Now, $S_{n,k}$ can be written as a polynomial in the $\phi_n$'s

\[
S_{n,k} = \frac{1}{k!} \sum_{i_k=1}^{2n-1} l_n(i_k) \sum_{i_{k-1}=1}^{2n-1} l_n(i_{k-1}) \cdots \sum_{i_2=1}^{2n-1} l_n(i_2) \left[ \phi_n(1) \right] + \sum_{j_1=2}^{k} l_n(i_{j_1})
\]

\[
= \frac{1}{k!} \sum_{i_k=1}^{2n-1} l_n(i_k) \cdots \sum_{i_4=1}^{2n-1} l_n(i_4) \left[ \phi_n(1)^2 - \phi_n(2) - 2\phi_n(1) \sum_{j_1=3}^{k} l_n(i_{j_1}) \right] + \sum_{j_2=3}^{k} l_n(i_{j_2})^2 + \sum_{j_2=3}^{k} \sum_{j_1=3}^{k} l_n(i_{j_2})l_n(i_{j_1})
\]

\[
= \frac{1}{k!} \sum_{i_k=1}^{2n-1} l_n(i_k) \cdots \sum_{i_4=1}^{2n-1} l_n(i_4) \left[ *_{n,4} \right],
\]

where

\[
*_{n,4} = \phi_n(1)^3 - 3\phi_n(1)\phi_n(2) + 2\phi_n(3) + 3(\phi_n(2) - \phi_n(1)^2) \sum_{j_1=4}^{k} l_n(i_{j_1})
\]

\[
+ 3\phi_n(1) \left( \sum_{j_1=4}^{k} l_n(i_{j_1})^2 + \sum_{j_2=4}^{k} \sum_{j_1=4}^{k} l_n(i_{j_1})l_n(i_{j_2}) \right)
\]

\[
- 2 \sum_{j_1=4}^{k} l_n(i_{j_1})^3 - 3 \sum_{j_2=4}^{k} \sum_{j_1=4}^{k} l_n(i_{j_1})^2l_n(i_{j_2}) - \sum_{j_3=4}^{k} \sum_{j_2=4}^{k} \sum_{j_1=4}^{k} l_n(i_{j_1})l_n(i_{j_2})l_n(i_{j_3}),
\]

(2.4.18)
and so on (a more explicit formula will be given momentarily). For example, one obtains the following expressions for small values of $k$:

\[
S_{n,1} = \phi_n(1) \\
S_{n,2} = \frac{1}{2!}(\phi_n(1)^2 - \phi_n(2)) \\
S_{n,3} = \frac{1}{3!}(\phi_n(1)^3 - 3\phi_n(1)\phi_n(2) + 2\phi_n(3)) \\
S_{n,4} = \frac{1}{4!}(\phi_n(1)^4 - 6\phi_n(1)^2\phi_n(2) + 8\phi_n(1)\phi_n(3) + 3\phi_n(2)^2 - 6\phi_n(4)) \\
S_{n,5} = \frac{1}{5!}(\phi_n(1)^5 + 10\phi_n(1)^3\phi_n(2) + 20\phi_n(1)^2\phi_n(3) + 15\phi_n(1)\phi_n(2)^2 \\
- 30\phi_n(1)\phi_n(4) - 20\phi_n(2)\phi_n(3) + 24\phi_n(5))
\]

(2.4.20)

Looking back at the expressions for $S_{n,k}$, one sees that there is a recursion relation for $S_{n,k}$. Setting $S_{n,0} := 1$, this recursion relation reads

\[
S_{n,k} = \frac{1}{k} \sum_{j=1}^{k} (-1)^{j+1} S_{n,k-j} \phi_n(j)
\]

(2.4.21)

and with some algebra, one can check that this recursion relation works. This recursion relation can be used to express $S_{n,k}$ purely in terms of the $\phi_n$’s and is given by

\[
S_{n,k} = \sum_{j_1=1}^{k} \sum_{j_2=1}^{k-j_1} \sum_{j_3=1}^{k-j_1-j_2} \cdots \sum_{j_k=1}^{k-j_{k-1}} \frac{(-1)^{k+j_1+j_2+\cdots+j_k} \phi_n(j_1)\phi_n(j_2)\cdots\phi_n(j_k)}{k(k-j_1)\cdots(k-j_1-j_2-\cdots-j_{k-1})}
\]

(2.4.22)
where it is understood that the sum terminates earlier if any of the $j$’s are larger than 1. For example, if there are $s$ of them, then

$$\sum_{r=1}^{s} j_r = k.$$  \hspace{1cm} (2.4.23)

Therefore, fix $k$ and consider the product

$$\prod_{r=1}^{s} \phi_n(j_r)$$  \hspace{1cm} (2.4.24)

The first claim is that

$$\lim_{n \to \infty} \frac{\prod_{r=1}^{s} \phi_n(j_r)}{\phi_n(1)^k} = \begin{cases} 0 & \text{if } s < k \\ 1 & \text{if } s = k \end{cases}$$  \hspace{1cm} (2.4.25)

The result when $s = k$ is obvious so suppose $s < k$. By the formula for $\phi_n(p)$ in (2.4.13) and the asymptotics of this given in (2.4.17)

$$\lim_{n \to \infty} \frac{\prod_{r=1}^{s} \phi_n(j_r)}{\phi_n(1)^k} = \lim_{n \to \infty} \frac{\prod_{r=1}^{s} \phi_n(j_r)}{n^{2k}}$$  \hspace{1cm} (2.4.26)

$$= \lim_{n \to \infty} \frac{\prod_{r=1}^{s} \phi_n(j_r)}{n^{2(j_1 + \ldots + j_s)}}$$

$$= \lim_{n \to \infty} \prod_{r=1}^{s} \frac{\phi_n(j_r)}{n^{2j_r}}$$

$$= 0.$$

Hence,

$$\lim_{n \to \infty} \frac{k! S_{n,k}}{n^{2k}} = 1.$$  \hspace{1cm} (2.4.27)
Finally going back to $R_{n,k}$ and using this fact gives

$$
\lim_{n \to \infty} R_{n,k} = \lim_{n \to \infty} \frac{k! S_{n,k}}{n^2 (n^2 - 1) \cdots (n^2 - k + 1)}
= \lim_{n \to \infty} \frac{k! S_{n,k}}{n^{2k} \left(1 - \frac{1}{n^2}\right) \cdots \left(1 - \frac{k-1}{n^2}\right)}
= 1.
$$

(2.4.28)

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Chapter 3

Gauge invariant surface holonomy and monopoles

The present chapter of this thesis comprises the contents of the paper [Pa15].

3.1 Introduction

3.1.1 Background, motivation, and overview

Ordinary holonomy along paths for principal group bundles has been studied for over 40 years in the context of gauge theories in physics and in the context of fiber bundles in mathematics. Recently, with ideas from higher category theory, it has been possible to extend these ideas to holonomy along surfaces. Although higher holonomy, and more generally higher gauge theory, has been studied in the context of abelian gauge theory for higher-dimensional manifolds, it was thought for some time that non-abelian generalizations were not possible [Te86]. Today, we understand this as being due to the fact that a
group object in the category of groups is an abelian group. By “categorifying” well-known concepts, and considering group objects in the category of categories, one can avoid this restriction. The language of higher categories allows us to give a resolution to this problem.

The data needed for defining surface holonomy for abelian structure groups has been known for quite some time under the name *abelian gerbes with connection* with a formal presentation offered by Gawedzki [Ga88] in 1988 in the context of the WZW model, with further work in 2002 with Reis [GaRe02]. Further development under the name of *non-abelian gerbes, higher bundles*, and so on were carried out in the following years starting with the foundational work of Breen and Messing [BrMe05] in 2001, where the data for connections on non-abelian gerbes first appeared. In [BaSc04], Baez and Schreiber gave a definition of non-abelian gerbes with connection in terms of parallel transport using the notion of a 2-group. The most up-to-date theoretical framework in terms of category theory, which provides a language easily adaptable for non-abelian generalizations, was established by Schreiber and Waldorf in [ScWa13]. In this categorical setting, higher principal bundles with connections are described by transport functors.

The motivation for transport functors comes from observations originally made by Barrett in [Ba91] and expanded on by Caetano and Picken
in [CaPi94] by describing a bundle with connection in terms of its holonomies. In [ScWa09], Schreiber and Waldorf use a categorical perspective to prove that a principal group bundle with connection over a smooth manifold determines, and is determined by, a transport functor defined on the thin path groupoid of that manifold with values in a fattened version of the structure group viewed as a one-object category. The upshot of this equivalence is that it is conceptually simple to go from categories and functors to 2-categories and 2-functors. In [ScWa11], [ScWa], and [ScWa13], Schreiber and Waldorf take advantage of this equivalence and abstract the definition so that it can be used to define principal 2-group 2-bundles with connection allowing a conceptually simple formulation of surface holonomy.

In the present article, we review the theory of transport functors formalized by Schreiber and Waldorf in [ScWa09], [ScWa11], [ScWa], and [ScWa13] with an emphasis on examples and explicit computations. Besides this, we accomplish several new results. First, we provide a definition of holonomy along spheres modulo thin homotopy without representing a sphere as a bigon (Definition 3.3.161). The target of this holonomy is an analogue of conjugacy classes, which is used for ordinary holonomy along loops, called \(\alpha\)-conjugacy classes. To prove this, we introduce a procedure that turns an arbitrary transport functor into a group-valued transport functor. In [ScWa13], the
authors forced their surface holonomy to land in a rather restrictive quotient of the structure 2-group to prove gauge invariance of holonomy. Our perspective is to take the smallest quotient possible, and we show our quotient surjects onto the one of \[ScWa13\].

We then focus on transport functors with a particular class of 2-groups, termed covering 2-groups, given by a Lie group $G$ and a covering space of $G$. We provide a simple formula, motivated by constructions in \[ChTs93\], for holonomy along surfaces in a local trivialization and show that this formula agrees with the surface-ordered integral in \[ScWa11\]. This gives an interesting relationship between (i) well-known formulas in the physics literature for computing the magnetic flux in terms of a loop of holonomies and (ii) non-abelian surface-ordered integrals in terms of 1- and 2-forms of \[ScWa11\]. Physically, we argue that the latter is the correct analogue to computing the magnetic flux as a surface integral and our formula tells us that this agrees with the usual definition given in the physics literature. This is all done without the introduction of a Higgs field, completing the ideas in \[GoNuOl77\].

Then we consider an entire collection of examples of transport 2-functors constructed from an ordinary principal $G$-bundle with connection along with a choice of a subgroup $N$ of $\pi_1(G)$, the fundamental group of $G$ (such a choice of subgroup determines a covering 2-group). We show that when the
subgroup $N$ is chosen to be $\pi_1(G)$ itself, our example reduces to the curvature 2-functor defined by Schreiber and Waldorf in [ScWa13]. We instead focus on the other extreme, namely when the subgroup $N$ is chosen to be the trivial group $\{1\}$, to calculate four examples of surface holonomies associated to both abelian and non-abelian magnetic monopoles. But just as ordinary holonomy is not exactly group-valued on the space of all loops (due to conjugation issues), surface holonomy isn’t in general either. Using our results on gauge invariance of sphere holonomy for arbitrary 2-groups, we prove that the surface holonomies for magnetic monopoles are not only gauge invariant but also form an abelian group.

3.1.2 Outline of chapter along with main results

In Section 3.2, we review the main definitions of transport functors along with an equivalence between local descent data and global transport functors. We follow the recent work of Schreiber and Waldorf [ScWa09] who describe it precisely and categorically in a framework that is suitable for generalizations to surfaces. We briefly discuss the relationship to principal $G$-bundles with connection, where $G$ is a Lie group, in their usual formulation by introducing the category of $G$-torsors (manifolds with free and transitive right $G$-actions). The equivalence between the two descriptions was proved
in [ScWa09]. We also review the relationship between local descent data and differential cocycle data for principal group bundles, recalling the well-known formula for parallel transport in terms of a path-ordered integral. To obtain group-valued holonomies, we introduce a procedure (3.2.86) described as a functor that takes an arbitrary transport functor and produces a group-valued transport functor in Section 3.2.8. The presentation differs a bit from that of [ScWa09] so we describe it in some detail.

In Section 3.3, we review how to ‘categorify’ the definitions and statements of Section 3.2 in order to define transport 2-functors. The main references for this section include [ScWa11], [ScWa], and [ScWa13]. We only briefly review the technical points but spend more time on a computational understanding of surface holonomy and also supply an iterated integral expression for surface holonomy including a picture (Figure 3.15) that we think will be useful for lattice gauge theory. This picture was explored in more detail in Chapter 2 of this thesis. As in the case of holonomy along loops, we introduce a procedure (3.3.121) to obtain group-valued surface holonomy. This lets us discuss gauge covariance and gauge invariance simply and in full detail without referring to the reduced group of [ScWa13]. However, we restrict ourselves to holonomy along spheres as opposed to surfaces of arbitrary genus. We show, in Theorem 3.3.159, that our holonomy along spheres lands
in a set that surjects onto the reduced group and give a simple example, in Lemma 3.3.167, where this surjection has nontrivial kernel.

In Section 3.4, we consider transport 2-functors with structure 2-group given by a covering 2-group. We give a new and simple formula valid for all such transport 2-functors in Corollary 3.4.75 for surface holonomy in a local trivialization in terms of homotopy classes of paths of holonomies along loops. This construction was inspired by work of physicists for computing magnetic charge as a topological number [ChTs93]. In Definition 3.4.57, we give our main construction of a transport 2-functor, called the path-curvature 2-functor, associated to every principal $G$-bundle with connection and to any subgroup of $\pi_1(G)$. We prove that this assignment is functorial. Furthermore, the path-curvature 2-functor is shown to reduce to the example of Schreiber and Waldorf known as the curvature 2-functor in [ScWa13] when the subgroup of $\pi_1(G)$ is chosen to be $\pi_1(G)$ itself. We describe this construction on four levels: (i) global transport functors (ii) functors with smooth trivialization data chosen (iii) descent data (iv) differential cocycle data. This allows one to work with either construction at whatever level he or she pleases. We then summarize our result as a list of commutative diagrams of functors in (3.4.66), (3.4.73), and (3.4.77).

In Section 3.5, we consider special cases of covering 2-groups and give
several examples all of which are known as magnetic monopoles \cite{ChTs93}. The first example is obtained from any principal $U(1)$-bundle with connection over the two-sphere $S^2$. It is shown that the surface holonomy along this sphere coming from the path-curvature 2-functor defined in Section 3.4 is precisely the integral of the curvature form of the principal $U(1)$-bundle along this sphere, which in this case is the integral of the first Chern class over the sphere. This example is precisely the Dirac monopole \cite{Di31} and the surface holonomy gives the magnetic charge as the integral of a magnetic flux. We then discuss non-abelian examples starting with a principal $SO(3)$-bundle with connection over the sphere and compute the surface holonomy explicitly using both our simple formula and the formula in terms of path-ordered integrals using differential forms. In the case of a non-trivial bundle, the surface holonomy along the sphere is given by the element \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) in $SU(2)$, the universal cover of $SO(3)$, which is the nontrivial element in the kernel of the covering map $\tau : SU(2) \longrightarrow SO(3)$. We do this same computation in other examples including $SU(n) \longrightarrow SU(n)/Z(n)$, where $Z(n)$ is the center of $SU(n)$, and also for the case $SU(n) \times \mathbb{R} \longrightarrow U(n)$. This gives a rigorous meaning to the notion of non-abelian magnetic flux as a surface holonomy along a sphere (see Definition 3.5.65). Furthermore, it is shown that magnetic flux is a gauge-invariant quantity in Corollary 3.5.66.
Finally, the Appendix includes an overview of diffeological spaces which are used to describe several of the constructions involving infinite-dimensional manifolds and smooth maps between them.

In short, this chapter contains the following results.

- Theorem 3.2.117 allows one to define gauge-invariant holonomy along loops in the language of transport functors via Definition 3.2.119. The image lands in conjugacy classes instead of the abelianization.

- Theorem 3.3.159 accomplishes the analogous result for surface holonomy along spheres in Definition 3.3.161. The image lands in $\alpha$-conjugacy classes (Definition 3.3.157) instead of the reduced group of [ScWa13]. The set of $\alpha$-conjugacy classes surjects to the reduced group but is not in general injective as shown in Lemma 3.3.167. We also prove that the fixed points of this $\alpha$ action form a central subgroup of the group of surface holonomies in Lemma 3.3.170.

- The rest of this chapter focuses on transport 2-functors whose structure 2-groups are covering 2-groups (Definition 3.4.20). They are called path-curvature 2-functors (Definition 3.4.57). These transport 2-functors are defined without using surface integrals, and we show, in Theorem 3.4.74 and Corollary 3.4.75, that locally, any transport 2-functor (de-
fined as in [ScWa11] using surface integrals) with structure 2-group a covering 2-group, coincides with ours, thus enabling a simple formula for calculating surface holonomy. This formula is to be contrasted with the surface ordered integral described in Chapter 2.

- Section 3.5 includes several examples and explicit computations of surface holonomy. Due to the previously mentioned theorem, these examples can rightfully be called magnetic fluxes of magnetic monopoles from physics. We include several examples of non-abelian surface holonomy. We conclude with Corollary 3.5.66 that shows that the magnetic flux is a fixed point under the $\alpha$ action and therefore lands in the central subgroup mentioned earlier. In particular, this implies that the magnetic charge is an abelian group-valued quantity known as a topological number.

### 3.1.3 Acknowledgments

Firstly, we thank Scott O. Wilson who helped greatly during the entire process of this work, providing ideas and proofreading drafts. Secondly, we thank V. Parameswaran Nair who made suggestions related to this work and informed us of references including [GoNuOl77]. We have benefited from conversations with Gregory Ginot, Jouko Mickelsson, Urs Schreiber, Stefan
We are also grateful to the referee of TAC for making several useful suggestions and corrections to our first draft. We thank Aaron Lauda for his tutorial on xypic, which we relied on to make many diagrams in this paper. All other figures were done in Gimp. This material is based on work supported by the National Science Foundation Graduate Research Fellowship under Grant No. 40017-06 05 and 40017-06 06.

3.1.4 Notations and conventions

We assume the reader is familiar with some basic concepts of 2-categories (the Appendix of [ScWa] explains most details needed for this paper though Appendix A contains more) but our notation differs from the norm so we set it now.

Compositions of 1-morphisms is usually written from right to left as in

\[ z \leftarrow^\alpha y \leftarrow^\beta x \mapsto z \leftarrow^{\alpha \circ \beta} x. \]  \hspace{1cm} (3.1.1)

Vertical composition is written from top to bottom as

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\uparrow \Sigma \\
\downarrow \Omega \\
\downarrow \zeta
\end{array}
\end{array}
\end{align*}
\[ y \leftarrow^\delta x \mapsto y \leftarrow^{\delta \circ \beta} x. \]   \hspace{1cm} (3.1.2)
Horizontal composition is written as

\[
\begin{array}{c}
\alpha \downarrow \\
\gamma \\
\downarrow \\
\Sigma \\
\downarrow \\
\delta \\
\downarrow \\
\Omega \\
\downarrow \\
x \\
\downarrow \\
\downarrow \\
\downarrow \\
\Sigma \Omega \\
\downarrow \\
z \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\gamma \cdot \delta
\end{array}
\] \quad \iff \quad
\begin{array}{c}
\alpha \circ \beta \\
\downarrow \\
\Sigma \Omega \\
\downarrow \\
x \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\gamma \cdot \delta
\end{array}
\] \quad (3.1.3)

Sources, targets, and identity-assigning functions are denoted by \( s, t, \) and \( i \), respectively. We will always write the identity \( i(x) \) at an object \( x \) as \( \text{id}_x \), \( \text{id}_\alpha \) for the vertical identity at a 1-morphisms \( \alpha \), and \( \text{id}_{\text{id}_x} \) for the horizontal identity at an object \( x \). Given a 2-category \( C \), the set of objects is typically denoted by \( C_0 \), 1-morphisms by \( C_1 \) and 2-morphisms by \( C_2 \). In general, an overline such as \( \overline{f} \) will denote weak inverses, vertical inverses, and reversing paths/bigons. It will be clear from context which is which. The first form of 2-categories appeared under the name \textit{bi-categories} and were introduced by Bénabou [Bé67].

### 3.2 Principal bundles with connection are transport functors

In this section, we review the notion of transport functors mainly following [ScWa09]. We split up the discussion into several parts. We first discuss a Čech description of principal \( G \)-bundles (without connection), where \( G \) is a Lie group, in terms of smooth functors. Then we attempt a guess for describing principal \( G \)-bundles with connections in terms of smooth functors. This
attempt fails as it only gives topologically trivialized bundles, motivating the need to use transport functors. We then proceed to describing local trivialization data, descent data, and finally transport functors. The key feature of descent data is that it enables us to encode smoothness while still allowing the ‘bundle’ to have nontrivial topology. We then discuss a reconstruction functor that takes us from the category of descent data to the category of transport functors with chosen trivializations. It is here that we discuss a version of the Čech groupoid incorporating paths and ‘jumps’ that are necessary for transition functions. Then we move in the other direction and go from smooth descent data to locally defined differential forms, or more generally differential cocycle data. We also describe how to go from differential cocycle data back to smooth descent data. We then summarize the four different levels describing transport functors and their relationship to one another. Finally, we use these results to formulate a procedure that sends an arbitrary transport functor to a transport functor with group-valued parallel transport and discuss its gauge covariance and invariance stressing the use of conjugacy classes.
3.2.1 A Čech description of principal $G$-bundles

Let $G$ be a Lie group. Principal $G$-bundles over a smooth manifold $M$ can be described simply in terms of functors. Furthermore, an isomorphism of such bundles corresponds to a natural transformation of the corresponding functors. This is done as follows (this is an expansion of Remark II.13. in [Wo11]).

**Definition 3.2.1.** Given an open cover $\{U_i\}_{i \in I}$ of $M$, the Čech groupoid $\mathcal{U}$ is the category whose set of objects is given by

$$\mathcal{U}_0 := \coprod_{i \in I} U_i$$

and whose morphisms, called ‘jumps,’ are given by

$$\mathcal{U}_1 := \coprod_{i,j \in I} U_{ij},$$

where $U_{ij} := U_i \cap U_j$ and the order of the index is kept track of in the disjoint union. Explicitly, elements of $\mathcal{U}_0$ are written as $(x, i)$ and elements of $\mathcal{U}_1$ are written as $(x, i, j)$. The source and target maps are given by $s((x, i, j)) := (x, i)$ and $t((x, i, j)) := (x, j)$ for $(x, i, j) \in \mathcal{U}_1$. The identity-assigning map is given by\(^1\) $i((x, i)) := (x, i, i)$. Let $(x, i, j)$ and $(x', i', j')$ be two morphisms

---

\(^1\)Our apologies for this double usage of the letter $i$ to mean both the identity-inclusion map and the index letter. We hope that it is not too confusing. Later, we will also use the letter $i$ for several other purposes.
with \( t((x, i, j)) = s((x', i', j')) \), i.e. \((x, j) = (x', i')\). Renaming the index \( j' \) to \( k \), the composition is defined to be

\[
(x, j, k) \circ (x, i, j) := (x, i, k).
\] (3.2.4)

**Definition 3.2.5.** For every Lie group \( G \), there is a one-object groupoid \( BG \) defined as follows. Denote the one object by \( \bullet \). Let the set of morphisms from \( \bullet \) to itself be given by the set \( G \). Composition is given by group multiplication.

The previous two groupoids have a smooth structure, formalized in the following definition.

**Definition 3.2.6.** A **Lie groupoid** is a (small) category, typically denoted by \( Gr \), whose objects, morphisms, and sets of composable morphisms all form smooth manifolds. Furthermore, the source, target, identity-assigning, and composition maps are all smooth. In addition, every morphism has an inverse and the map that sends a morphism to its inverse is smooth.

**Example 3.2.7.** The \( \check{\text{C}} \text{ech} \) groupoid of Definition 3.2.1 and \( BG \) of Definition 3.2.5 are Lie groupoids with the appropriate (obvious) smooth structures.

**Definition 3.2.8.** A **smooth functor** from one Lie groupoid to another is an ordinary functor that is smooth on objects and morphisms. Likewise, a **smooth natural transformation** is a natural transformation whose function from objects to morphisms is smooth.
Any smooth functor $\mathcal{U} \to BG$ gives the Čech cocycle data of a principal $G$-bundle over $M$ subordinate to the cover $\{U_i\}_{i \in I}$. To see this, simply recall what a functor does. To each object $(x, i)$ in $\mathcal{U}$, it assigns the single object $\bullet$ in $BG$. To each jump $(x, i, j)$, it assigns an element denoted by $g_{ij}(x) \in G$ in such a way that we get a smooth 1-cochain $g_{ij} : U_{ij} \to G$

\begin{equation}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
g_{ij}
\end{array}
\end{equation}

(3.2.9)

This picture should be interpreted as follows. To each $x \in U_{ij}$, we draw the jump $(x, i, j)$ as the figure on the left. Its image under $\mathcal{U} \to BG$ is $g_{ij}(x)$ drawn on the right (without explicitly writing $x$). To each triple intersection $U_{ijk}$, which corresponds to the composition of $(x, i, j)$ in $U_{ij}$ with $(x, j, k)$ in $U_{jk}$ as in (3.2.4), functoriality gives a cocycle condition

\begin{equation}
\begin{array}{c}
g_{jk}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
g_{ij}
\end{array}
\end{equation}

(3.2.10)

which says

\begin{equation}
g_{jk}g_{ij} = g_{ik}.
\end{equation}

(3.2.11)
This convention was chosen to match that of [ScWa09] and [ScWa13] so that the reader who is interested in further details can consult without too much trouble.

We now discuss refinements and morphisms between two such functors. Let \( \{U'_i\}_{i \in I'} \) be another cover of \( M \) with associated Čech groupoid \( \mathfrak{U}' \). Let \( P : \mathfrak{U} \rightarrow BG \) and \( P' : \mathfrak{U}' \rightarrow BG \) be two smooth functors. A morphism from \( P \) to \( P' \) consists of a common refinement \( \{V_a\}_{a \in A} \) with associated Čech groupoid \( \mathfrak{V} \), of both \( \{U_i\}_{i \in I} \) and \( \{U'_i\}_{i' \in I'} \) along with a smooth natural transformation

\[
\begin{tikzcd}
\mathfrak{V} & \mathfrak{U} \\
\mathfrak{V}' & BG \\
\alpha & P \\
\alpha' & P'
\end{tikzcd}
\]

(3.2.12)

The refinement condition means that there are associated functions \( \alpha : A \rightarrow I \) and \( \alpha' : A \rightarrow I' \) so that \( V_a \subset U_{\alpha(a)} \) and \( V_a \subset U'_{\alpha(a)} \) for all \( a \in A \). These functions determine the functors \( \alpha : \mathfrak{V} \rightarrow \mathfrak{U} \) and \( \alpha' : \mathfrak{V} \rightarrow \mathfrak{U}' \) drawn above. We denote the restrictions of \( g_{\alpha(a)(b)} \) and \( g'_{\alpha'(a')(b')} \) to \( V_{ab} \) by \( g_{ab} \) and \( g'_{ab} \), respectively. Any such smooth natural transformation gives an equivalence of Čech cocycle data of principle G-bundles. To see this, simply recall what a natural transformation does. To each object \( (x, a) \) in \( \mathfrak{V} \) it assigns a group element \( h_a(x) \in G \) in a smooth way. In other words, it gives a smooth
function \( h_a : V_\alpha \longrightarrow G \). To each jump \((x, a, b)\) in \( \mathcal{W} \), the naturality condition

\[
\begin{array}{c}
\bullet & h_b & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bullet & h_a & \bullet \\
\end{array}
\]

(3.2.13)

says that

\[
h_b g_{ab} = g'_{ab} h_a
\]

(3.2.14)

on \( V_{ab} \). This is precisely the condition that says the principal \( G \)-bundles \( P \) and \( P' \) are isomorphic [St99].

### 3.2.2 A naive guess for transport functors

Our goal in this section is to guess what a connection on a principal \( G \)-bundle over \( M \) should be in terms of functors. We will fail at this attempt, but will learn an important lesson which will motivate the modern definition in terms of transport functors. First, recall that in a principal \( G \)-bundle \( P \longrightarrow M \), every fiber is a right \( G \)-torsor.

**Definition 3.2.15.** Let \( G \) be a Lie group. Let \( G\)-Tor be the category whose objects are right \( G \)-torsors, i.e. smooth manifolds equipped with a free and transitive right \( G \)-action, and whose morphisms are right \( G \)-equivariant maps.
Furthermore, a connection on a principal $G$-bundle over $M$ gives an assignment from paths in $M$ to isomorphisms of fibers between the endpoints. This assignment is independent of the parametrization of the path, but it is even independent of the thin homotopy class of a path as discussed in [CaPi94]. To define this, we use the theory of smooth spaces, reviewed in the Appendix on smooth spaces, which give natural definitions for smooth structures on subsets, mapping spaces, and quotient spaces.

**Definition 3.2.16.** Let $X$ be a smooth manifold. A *path with sitting instants* is a smooth map $\gamma : [0, 1] \to X$ such that there exists an $\epsilon$ with $\frac{1}{2} > \epsilon > 0$ and $\gamma(t)$ is constant for all $t \in [0, \epsilon] \cup [1 - \epsilon, 1]$. Such a path $\gamma$ with $\gamma(0) = x$ and $\gamma(1) = y$, will be written as

$$y \overset{\gamma}{\longrightarrow} x.$$  \hspace{1cm} (3.2.17)

The set of paths with sitting instants in $X$ will be denoted by $PX$.

Paths with sitting instants were first described in [CaPi94]. We reserve the notation $X^{[0,1]}$ for the set of (ordinary) smooth paths in $X$. Thus, $PX \subset X^{[0,1]}$.

**Definition 3.2.18.** Two paths in $X$ with sitting instants $\gamma$ and $\gamma'$ with the same endpoints, i.e. $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$, are said to be
**thinly homotopic** if there exists a smooth map $\Gamma : [0, 1] \times [0, 1] \longrightarrow X$ with the following two properties.

(a) First, there exists an $\epsilon$ with $\frac{1}{2} > \epsilon > 0$ such that

\[
\Gamma(t, s) = \begin{cases} 
  x & \text{for all } (t, s) \in [0, \epsilon] \times [0, 1] \\
  y & \text{for all } (t, s) \in [1 - \epsilon, 1] \times [0, 1] \\
  \gamma(t) & \text{for all } (t, s) \in [0, 1] \times [0, \epsilon] \\
  \gamma'(t) & \text{for all } (t, s) \in [0, 1] \times [1 - \epsilon, 1]
\end{cases}
\]  

(3.2.19)

A map $\Gamma : [0, 1] \times [0, 1] \longrightarrow X$ satisfying just (3.2.19) is called a **bigon** in $X$ and is typically denoted by

\[
y \xRightarrow{\gamma} \Gamma \xRightarrow{x} y \\
\downarrow \downarrow \downarrow \\
\gamma' \xRightarrow{\Gamma} \\
x
\]  

(3.2.20)

The set of bigons in $X$ is denoted by $BX$.

(b) Second, the rank of $\Gamma$ is strictly less than 2, i.e. the differential $D_{(t,s)}\Gamma : T_{(t,s)}([0, 1] \times [0, 1]) \longrightarrow T_{\Gamma(t,s)}X$, where $T_y Y$ denotes the tangent space to $Y$ at the point $y \in Y$, has kernel of dimension at least one for all $(t, s) \in [0, 1] \times [0, 1]$.

Thin homotopy is an equivalence relation and the equivalence classes are called **thin paths**. Denote the set of thin paths in $X$ by $P^1 X$.

$P^1 X$ is naturally a smooth space since it is a quotient of $PX$, which is itself a subset of $X^{[0,1]}$, which has a natural smooth space structure as a mapping
space. With these preliminaries, the definition of the thin path-groupoid of a smooth manifold $X$ can be given (we refer the reader to [CaPi94] and [ScWa09] for more details).

**Definition 3.2.21.** Let $X$ be a smooth manifold. Let $\mathcal{P}_1(X)$ be the category whose objects are the points of the smooth manifold $X$ and whose morphisms are the thin paths of $X$. The source and target of a thin path are defined by choosing a representative and taking the source and target, respectively. The identity at each point $x \in X$ is the thin path associated to the constant path at $x$. The composition of two thin paths is defined by choosing representatives and concatenating with double-speed parametrization. Namely, given two thin paths

$$z \leftarrow \gamma' \rightarrow y \leftarrow \gamma \rightarrow x,$$  \hspace{1cm} (3.2.22)

the composition is given by the thin homotopy class associated to

$$(\gamma' \circ \gamma)(t) := \begin{cases} \gamma(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \gamma'(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \hspace{1cm} (3.2.23)$$

Under the sitting instants assumption and the thin homotopy equivalence relation, the composition is well-defined, smooth, associative, has left and right units given by constant paths, and right and left inverses by reversing paths. By replacing the word “smooth manifold” with “smooth space” in Definition 3.2.6, $\mathcal{P}_1(X)$ is therefore a Lie groupoid.
With this definition of the thin path-groupoid of $M$, one might guess that a transport functor should be a smooth functor $\mathcal{P}_1(M) \to G$-Tor. However, since $G$-Tor is not a Lie groupoid, there is no obvious way of demanding such a functor to be smooth. One might therefore try to use $BG$ instead of $G$-Tor. Indeed, notice that there is a natural functor $i : BG \to G$-Tor defined by

\[ \begin{align*}
\bullet & \mapsto G \\
g & \mapsto L_g,
\end{align*} \tag{3.2.24} \]

where $G$ is viewed as a right $G$-torsor and $L_g$ is left multiplication on $G$ by $g$. One can think of $G$-Tor as a ‘thickening’ of $BG$ because $i$ is an equivalence of categories. We can then try to use $BG$ for our target instead of $G$-Tor so that we can ask for smoothness. Then one might guess that a transport functor should be a smooth functor $\mathcal{P}_1(M) \to BG$. Unfortunately, now that we have smoothness, we’ve lost non-triviality because such smooth functors describe parallel transport on trivialized principal $G$-bundles (this fact follows from Section 3.2.6 particularly around equation (3.2.70)).

In order to encode local instead of global triviality, we have to combine these ideas with those of the previous section in terms of the Čech groupoid (we will also return to a more suitable combination of the path groupoid and the Čech groupoid in Section 3.2.5). To avoid a huge collection of indices again, we denote our open cover $\{U_i\}_{i \in I}$ of $M$ simply by $Y := \bigsqcup_{i \in I} U_i$, and we
let $\pi : Y \hookrightarrow M$ be the inclusion of these open sets into $M$. Note that $\pi$ is a surjective submersion. Then, the next guess might be that we need to have a smooth functor $\mathcal{P}_1(Y) \rightarrow BG$, but we still need an assignment of fibers $\mathcal{P}_1(M) \rightarrow G\text{-Tor}$. These assignments should be compatible with respect to the functor $i : BG \rightarrow G\text{-Tor}$ and the submersion $\pi$. This is exactly what is done in [ScWa09] and we therefore now proceed to discussing local triviality of functors.

### 3.2.3 Local triviality of functors

Our first goal is to discuss local triviality of functors without making any assumptions on smoothness, which is left to the next section. The fibers of principal $G$-bundles were right $G$-torsors, which led us to consider the category $G\text{-Tor}$ of $G$-torsors. One of the great features of Schreiber’s and Waldorf’s work [ScWa09] is their generality on the different flavors of bundles. If one wants to work with vector bundles one simply replaces $G\text{-Tor}$ with Vect, the category of vector spaces (over some appropriate field such as $\mathbb{R}$ or $\mathbb{C}$), and if this vector bundle is an associated bundle for some representation of $G$, then this representation is precisely encoded by a functor $i : BG \rightarrow $ Vect. Fiber bundles can be defined similarly. Therefore, we have made two important observations. The first is that fibers of a bundle are objects of some category...
The second is that the structure group of the bundle is encoded by a functor $i : BG \to T$. Schreiber and Waldorf generalize this even further by considering any Lie groupoid $Gr$ instead of the special one $BG$. They define a $\pi$-local trivialization as follows (Definition 2.5. of [ScWa09]).

**Definition 3.2.25.** Let $Gr$ be a Lie groupoid, $T$ a category, $i : Gr \to T$ a functor, and $M$ a smooth manifold. Fix a surjective submersion $\pi : Y \to M$. A $\pi$-local $i$-trivialization of a functor $F : P_1(M) \to T$ is a pair $(\text{triv}, t)$ of a functor $\text{triv} : P_1(Y) \to Gr$ and a natural isomorphism $t : \pi^* F \Rightarrow \text{triv}_i$ as in the diagram

$$
\begin{array}{ccc}
P_1(M) & \xrightarrow{\pi^*} & P_1(Y) \\
F \downarrow & & \downarrow \text{triv} \\
T & \leftarrow & i \text{Gr}
\end{array}
$$

(3.2.26)

The groupoid $Gr$ is called the *structure groupoid* for $F$.

In this definition $\pi_*$ is the pushforward, which sends a point $y \in Y$ to $\pi(y)$ and sends a thin path $\gamma \in P^1 Y$ to the thin homotopy class of $\pi \circ \gamma$ in $M$ (after choosing a representative). $\pi^* F := F \circ \pi_*$ is the pullback of $F$ along $\pi$ and $\text{triv}_i := i \circ \text{triv}$. Functors $F : P_1(M) \to T$ equipped with $\pi$-local $i$-trivializations $(\text{triv}, t)$ form the objects, written as triples $(F, \text{triv}, t)$, of a category denoted by $\text{Triv}_{\pi}^1(i)$. 
**Definition 3.2.27.** A morphism $\alpha : (F, \text{triv}, t) \to (F', \text{triv}', t')$ in $\text{Triv}_{i}^{\pi}(i)$ of $\pi$-local $i$-trivializations is a natural transformation $\alpha : F \Rightarrow F'$. Composition is given by vertical composition of natural transformations.

**Remark 3.2.28.** One might expect a morphism $(F, \text{triv}, t) \to (F', \text{triv}', t')$ to consist of $\alpha : F \Rightarrow F'$ as well as a natural transformation $h : \text{triv} \Rightarrow \text{triv}'$ satisfying some compatibility condition with $\alpha$, $t$, and $t'$. This natural compatibility condition completely determines $h$ which is why it is excluded from the definition.

In this description, it is not immediately obvious what transition functions are. This is part of the motivation for introducing descent objects (Definition 2.8. of [ScWa09]). We use the notation $Y^{[n]}$ associated to a surjective submersion $\pi : Y \to M$ to mean the $n$-fold fiber product defined by

$$Y^{[n]} := \{(y_1, \ldots, y_n) \in Y \times \cdots \times Y \mid \pi(y_1) = \cdots = \pi(y_n)\}. \quad (3.2.29)$$

There are several projection maps $\pi_{i_1\cdots i_k} : Y^{[n]} \to Y^{[n-k]}$ for all $n \geq 2$ and $0 < k < n$ with $1 < i_1 < \cdots < i_k < n$ that are defined by

$$Y^{[n]} \ni (y_1, \ldots, y_n) \mapsto (y_{i_1}, \ldots, y_{i_k}). \quad (3.2.30)$$

$Y^{[n]}$ is a smooth manifold for all $n$ and all $\pi_{i_1\cdots i_k}$ are smooth since $\pi$ is a surjective submersion.
Definition 3.2.31. Let Gr be a Lie groupoid, $T$ a category, and $i : Gr \to T$ a functor. Fix a surjective submersion $\pi : Y \to M$. A descent object is a pair $(\text{triv}, g)$ consisting of a functor $\text{triv} : P_1(Y) \to \text{Gr}$, a natural isomorphism

$$P_1(Y) \xrightarrow{\pi_{1*}} P_1(Y^{[2]})$$

$$\text{triv} \downarrow \quad g \quad \downarrow \pi_{2*}.$$

(3.2.32)

The pair $(\text{triv}, g)$ must satisfy

$$\pi^{12}_{13}g = \pi^{*}_{13}g,$$

where the left-hand-side is vertical composition of natural transformations (read from top to bottom), and

$$\text{id}_{\text{triv}_i} = \Delta^{*}g,$$

where $\Delta$ is the diagonal $\Delta : Y \to Y^{[2]}$ sending $y$ to $(y, y)$.

Descent objects form the objects of a category denoted by $\text{Des}^1_{\pi}(i)$.

Definition 3.2.35. A descent morphism $h : (\text{triv}, g) \to (\text{triv}', g')$ is a natural transformation $h : \text{triv}_i \Rightarrow \text{triv}'_i$ satisfying

$$\pi^{*}_{1h} \circ g' = g = \pi^{*}_{2h}.$$
There is a functor $\text{Ex}^1_\pi : \text{Triv}^1_\pi (i) \to \text{Des}^1_\pi (i)$ that extracts descent data from trivialization data. At the level of objects, this functor is defined as follows. Let $(F, \text{triv}, t)$ be an object in $\text{Triv}^1_\pi (i)$. For the pair $(\text{triv}, g)$, take triv to be exactly the same. For $g$ take the composition $g := \pi^1_1 t$ coming from the composition in the diagram

$\begin{array}{ccc}
\mathcal{P}_1(Y) & \xleftarrow{\pi^1_1} & \mathcal{P}_1(Y) \\
\downarrow \pi_* & \downarrow \pi_* & \downarrow \pi_* \\
\mathcal{P}_1(M) & \xleftarrow{\pi_*} & \mathcal{P}_1(Y), \\
\downarrow t & \downarrow t & \downarrow t \\
T & \xleftarrow{\alpha} & \text{Gr}
\end{array}$

(3.2.37)

where $\bar{t}$ is the (vertical) inverse of $t$. This defines a descent object (Section 2.2 of [ScWa09]). On a morphism $\alpha : (F, \text{triv}, t) \to (F', \text{triv}', t')$, the functor $\text{Ex}^1_\pi$ is defined by setting

$h := \pi^1_1 \alpha \bar{t}$(3.2.38)

coming from the composition in the diagram

$\begin{array}{ccc}
T & \xleftarrow{\alpha} & \mathcal{P}_1(M) \\
\downarrow F & \downarrow F' & \downarrow F' \\
\mathcal{P}_1(M) & \xleftarrow{\pi_*} & \mathcal{P}_1(Y) \\
\downarrow t & \downarrow t' & \downarrow t' \\
\text{triv}_i & \xleftarrow{\text{triv}_{i'}} & \text{Gr}
\end{array}$

(3.2.39)

The functor $\text{Ex}^1_\pi$ is part of an equivalence of categories between $\text{Triv}^1_\pi (i)$ and $\text{Des}^1_\pi (i)$. This is done by constructing a weak inverse functor $\text{Rec}^1_\pi : \text{Des}^1_\pi (i) \to \text{Triv}^1_\pi (i)$, which we will describe in Section 3.2.5.
Definition 3.2.40. Let $(F, \text{triv}, t)$ be a $\pi$-local $i$-trivialization of a functor $F : \mathcal{P}_1(M) \rightarrow T$, i.e. an object of Triv$_p^1(i)$. The descent object associated to the $\pi$-local $i$-trivialization of $F$ is $\text{Ex}^1_\pi(F, \text{triv}, t)$. Let $\alpha : (F, \text{triv}, t) \rightarrow (F', \text{triv}', t')$ be a morphism in Triv$_p^1(i)$. The descent morphism associated to the $\pi$-local $i$-trivialization of $\alpha$ is $\text{Ex}^1_\pi(\alpha)$.

3.2.4 Transport functors

We now discuss smoothness of descent data and finally give a definition of transport functors.

Definition 3.2.41. A descent object $(\text{triv}, g)$ as above is said to be smooth if $\text{triv} : \mathcal{P}_1(Y) \rightarrow \text{Gr}$ is a smooth functor and if there exists a smooth natural isomorphism $\tilde{g} : \pi_1^*\text{triv} \Rightarrow \pi_2^*\text{triv}$ with $g = \text{id}_i \circ \tilde{g}$, the horizontal composition of natural transformations $\text{id}_i$ and $\tilde{g}$. A descent morphism $h : (\text{triv}, g) \rightarrow (\text{triv}', g')$ as above is said to be smooth if there exists a smooth natural isomorphism $\tilde{h} : \text{triv} \Rightarrow \text{triv}'$ with $h = \text{id}_i \circ \tilde{h}$.

Smooth descent objects and morphisms form the objects and morphisms of a category denoted by $\mathcal{O} \text{es}^1_\pi(i)^\text{op}$ and form a sub-category of $\mathcal{O} \text{es}^1_\pi(i)$.

Definition 3.2.42. A $\pi$-local $i$-trivialization $(F, \text{triv}, t)$ is said to be smooth if the associated descent object $\text{Ex}^1_\pi(F, \text{triv}, t)$ is smooth. A morphism $\alpha :
(F, triv, t) \rightarrow (F', triv', t') is said to be **smooth** if the associated descent morphism $\text{Ex}_n^1(\alpha)$ is smooth.

Smooth local trivializations and their morphisms form the objects and morphisms of a category denoted by $\text{Triv}_n^1(i)\otimes$ and form a sub-category of $\text{Triv}_n^1(i)$. $\text{Ex}_n^1$ restricts to an equivalence of categories $\text{Triv}_n^1(i)\otimes \cong \text{Des}_n^1(i)\otimes$ of smooth data. Again, we will discuss an inverse functor in Section 3.2.5 since it will be necessary in discussing gauge invariant holonomy in Section 3.2.8. We now come to the definition of a transport functor (Definition 3.6 of [ScWa09]).

**Definition 3.2.43.** Let $\text{Gr}$ be a Lie groupoid, $T$ a category, $i : \text{Gr} \rightarrow T$ a functor, and $M$ a smooth manifold. A transport functor on $M$ with values in a category $T$ and with $\text{Gr}$-structure is a functor $\text{tra} : \mathcal{P}_1(M) \rightarrow T$ such that there exists a surjective submersion $\pi : Y \rightarrow M$ and a smooth $\pi$-local $i$-trivialization $(\text{triv}, t)$ of $\text{tra}$.

Transport functors with values in $T$ with $\text{Gr}$-structure form the objects of a category $\text{Trans}_n^1(\text{Gr}, (M, T))$. We also define the morphisms of transport functors.

**Definition 3.2.44.** A morphism $\eta$ of transport functors on $M$ from $\text{tra}$ to $\text{tra'}$ is a natural transformation $\eta : \text{tra} \Rightarrow \text{tra'}$ such that there exists a sur-
jective submersion \( \pi : Y \rightarrow M \) and smooth \( \pi \)-local \( i \)-trivializations \( (\text{triv}, t) \), \( (\text{triv}', t') \), and \( h : (\text{triv}, t) \rightarrow (\text{triv}', t') \) of \( \text{tra} \), \( \text{tra}' \), and \( \eta \) respectively.

By using pullbacks, one can define the composition of such morphisms. We will not explicitly describe this now since we will come back to it later when discussing limit categories over surjective submersions in Section 3.2.7.

### 3.2.5 The reconstruction functor: local to global

In many situations, one works locally and pieces together data to construct globally defined quantities. In the case of parallel transport, one obtains group elements. An explicit construction of a (weak) inverse

\[
\text{Rec}_\pi^1 : \text{Des}_\pi^1(i) \longrightarrow \text{Triv}_\pi^1(i)
\]  

(3.2.45)

will assist in this direction. Following Section 2.3 of [ScWa09], we introduce a category that combines the Čech groupoid with the path groupoid utilizing the surjective submersion \( \pi : Y \rightarrow M \).

**Definition 3.2.46.** Let \( \mathcal{P}_1^\pi(M) \) be the category, called the Čech path groupoid, whose set of objects are the elements of \( Y \). The set of morphisms are freely generated by two types of morphisms (the generators) which are given as follows

i) thin paths (see Definition 3.2.18) \( \gamma \) in \( Y \) and
ii) points $\alpha$ in $Y^{[2]}$ (thought of as morphisms $\pi_1(\alpha) \xrightarrow{\alpha} \pi_2(\alpha)$ and called \textit{jumps}).

There are several relations imposed on the set of morphisms.

(a) For any thin path $\Theta : \alpha \xrightarrow{} \beta$ in $Y^{[2]}$ the diagram

\[
\begin{array}{ccc}
\pi_1(\beta) & \xrightarrow{\pi_1(\Theta)} & \pi_1(\alpha) \\
\downarrow & & \downarrow \\
\beta & \xrightarrow{} & \alpha \\
\pi_2(\beta) & \xleftarrow{\pi_2(\Theta)} & \pi_2(\alpha)
\end{array}
\]  

(3.2.47)

commutes (see Figure 3.1 for a visualization of this).

![Diagram](image)

Figure 3.1: Thinking in terms of an open cover as a submersion, condition i) above says that if a path $\Theta : \alpha \rightarrow \beta$ is in a double intersection, it doesn’t matter whether or not the jump is performed first and then the thin path is traversed or vice versa. One should compare these jumps to those in (3.2.9) and (3.2.10).

(b) For any point $\Xi \in Y^{[3]}$ the diagram

\[
\begin{array}{ccc}
\pi_3(\Xi) & \xrightarrow{\pi_{13}(\Xi)} & \pi_1(\Xi) \\
\pi_{23}(\Xi) & \xrightarrow{} & \pi_{12}(\Xi) \\
\pi_2(\Xi) & \xleftarrow{\pi_2(\Theta)} & \pi_2(\Xi)
\end{array}
\]  

(3.2.48)
commutes.

(c) The free composition of two thin free paths is the usual composition of thin paths and for every point $y \in Y$, the thin homotopy class representing the constant path at $y$ is equal to $\Delta(y) \in Y^{[2]}$ which is the formal identity for the composition.

The notation for the free composition will be $\ast$.

Item (b) together with item (c) demands that the jumps $\alpha \in Y^{[2]}$ are isomorphisms. A typical morphism in $\mathcal{P}_1^\pi(M)$ is depicted in Figure 3.2.

Figure 3.2: A generic representative of a morphism in $\mathcal{P}_1^\pi(M)$ is shown above for $Y = \coprod_{i \in I} U_i$, the disjoint union over an open cover. The larger ellipses indicate open sets and the smaller ones in the middle indicate intersections. The curves in the open sets indicate the paths and the dotted vertical lines indicate the jumps.

Associated to every descent object $(\text{triv}, g)$ in $\mathcal{D}_{\pi}^1(i)$ is a functor $R_{(\text{triv}, g)}$ :
\[
\mathcal{P}_1^\pi(M) \longrightarrow T \text{ defined (on objects and generators) by }
\]
\[
Y \ni y \mapsto \text{triv}_i(y),
\]
\[
P^1Y \ni \gamma \mapsto \text{triv}_i(\gamma), \quad \text{and}
\]
\[
Y^{[2]} \ni \alpha \mapsto \left(g(\alpha) : \text{triv}_i(\pi_1(\alpha)) \longrightarrow \text{triv}_i(\pi_2(\alpha))\right).
\]

This assignment extends to a functor
\[
R : \mathfrak{Des}_\pi(i) \longrightarrow \text{Funct}(\mathcal{P}_1^\pi(M), T)
\]

(Lemma 2.14. of [ScWa09]). To a descent morphism \( h : (\text{triv}, g) \longrightarrow (\text{triv}', g') \) it gives a natural transformation \( R_h : R_{(\text{triv}, g)} \Rightarrow R_{(\text{triv}', g')} \) defined by sending \( y \in Y \) to \( h(y) \) for all \( y \in Y \).

The functor \( \text{Rec}_\pi : \mathfrak{Des}_\pi(i) \longrightarrow \text{Triv}_\pi(i) \) will be defined so that it factors through \( R \). What will then remain is to define a functor

\[
\text{Funct}(\mathcal{P}_1^\pi(M), T) \longrightarrow \text{Funct}(\mathcal{P}_1(M), T).
\]

In order to do this, we need to “lift” paths. First, notice that there is a canonical projection functor \( p^\pi : \mathcal{P}_1^\pi(M) \longrightarrow \mathcal{P}_1(M) \) which sends objects \( y \in Y \) to \( \pi(y) \), thin paths \( \gamma \) to \( \pi(\gamma) \), and points \( \alpha \in Y^{[2]} \) to the identity at \( \pi_1(\alpha) = \pi_2(\alpha) \). We will now construct a weak inverse \( s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M) \).

Since \( \pi : Y \longrightarrow M \) is surjective, for every \( x \in M \), there exists a \( y \in Y \) such that \( \pi(y) = x \). Therefore, define \( s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M) \) on objects to be this
assignment. Because \( \pi : Y \to M \) is a surjective submersion, there exists an open cover \( \{ U_i \}_{i \in I} \) of \( M \) with local sections \( s_i : U_i \to Y \) of \( \pi \). Using these local sections, we can define \( s^{\pi} : \mathcal{P}_1(M) \to \mathcal{P}^{\pi}_1(M) \) on morphisms as follows.

For every thin path \( \gamma : x \to x' \) in \( M \) there exists a collection of thin paths \( \gamma_1, \ldots, \gamma_n \) with (representatives of) \( \gamma_j \) inside \( U_{ij} \) for all \( j = 1, \ldots, n \) and

\[
x' \xleftarrow{\gamma_n} x = x' \xleftarrow{\gamma_{n-1}} x_{n-1} \xleftarrow{\gamma_{n-2}} \ldots \xleftarrow{\gamma_2} x_1 \xleftarrow{\gamma_1} x.
\]

For such a choice define (we write \( s_j \) instead of \( s_{ij} \) to avoid too many indices)

\[
s^{\pi}(\gamma) := \alpha_{x'} \ast s_n(\gamma_n) \ast \alpha_{n-1} \ast s_{n-1}(\gamma_{n-1}) \ast \cdots \ast s_2(\gamma_2) \ast \alpha_1 \ast s_1(\gamma_1) \ast \alpha_x.
\]

where \( \alpha_x \) is the unique isomorphism from \( s^{\pi}(x) \) to \( s_1(x) \), \( \alpha_j \) is the unique isomorphism from \( s_{j-1}(x_j) \) to \( s_j(x_j) \), and \( \alpha_{x'} \) is the unique isomorphism from \( s_n(x) \) to \( s_\pi(x') \). This definition comes from Figure 3.3.

The functor \( s^{\pi} \) is a weak inverse to \( p^{\pi} \) (Lemma 2.15. of [ScWa09]). For reference, by definition this means there exists a natural isomorphism

\[
\zeta : s^{\pi} \circ p^{\pi} \Rightarrow \text{id}_{\mathcal{P}_1(M)}.
\]

that is part of an adjoint equivalence given by the quadruple \( (s^{\pi}, p^{\pi}, \zeta, \text{id}_{p^{\pi} \circ s^{\pi}}) \) since \( p^{\pi} \circ s^{\pi} = \text{id}_{\mathcal{P}_1(M)} \). This natural isomorphism \( \zeta \) is the one that sends \( y \in Y \) to the unique jump, an isomorphism, from \( y \) to \( s^{\pi}(\pi(y)) \). It is natural by relation i) in Definition 3.2.46.
Figure 3.3: By choosing a decomposition of every path to land in open sets one can lift using the locally defined sections. At the beginning and end of the path, one must apply a jump since the map $s$ defined on objects might not coincide with the lift of the endpoint of the path.

**Remark 3.2.55.** Note that we have not put a smooth structure on $\mathcal{P}^\pi_1(M)$ nor will we (although it is done in [ScWa09]). Indeed, the choice of lifts for the points could be sporadic. All the smoothness for transport functors is contained in the descent data.

The functor $s^\pi : \mathcal{P}_1(M) \rightarrow \mathcal{P}^\pi_1(M)$ induces a pullback functor

$$s^\pi_* : \text{Funct}(\mathcal{P}^\pi_1(M), T) \rightarrow \text{Funct}(\mathcal{P}_1(M), T)$$

(3.2.56)
defined by $s^\pi_*(F) := F \circ s^\pi$ on functors $F : \mathcal{P}^\pi_1(M) \rightarrow T$ and by $s^\pi_*(\rho) := \rho \circ \text{id}_{s^\pi}$ on natural transformations $\rho : F \Rightarrow G$. Finally, $\text{Rec}_\pi^1$ is defined as the composition in the diagram

$$\text{Funct}(\mathcal{P}_1(M), T) \xleftarrow{\text{Rec}_\pi^1} \text{Funct}_D^1(i) \xrightarrow{R} \text{Funct}(\mathcal{P}^\pi_1(M), T).$$

(3.2.57)
The image of $\mathfrak{Des}_\pi^1(i)$ under $\text{Rec}_\pi^1$ is actually in $\text{Triv}_\pi^1(i)$. This means at the level of objects that associated to $R_{(\text{triv,}g)} \circ s^\pi$ there exists a $\pi$-local $i$-trivialization. We take $\text{triv}$ itself for the first part of this datum. To define $t : \pi^* \left( s^{\pi*}(R_{(\text{triv,}g)}) \right) \Rightarrow \text{triv}_i$ we take the composition defined by the diagram

\[
\begin{array}{ccc}
\mathcal{P}_1(M) & \xrightarrow{\pi_*} & \mathcal{P}_1(Y) \\
\downarrow s^* & & \downarrow \text{id} \\
\mathcal{P}_1^\pi(M) & \xrightarrow{i} & \text{triv,} \\
\downarrow R_{(\text{triv,}g)} & & \downarrow \text{id} \\
T & \xrightarrow{i} & \text{Gr}
\end{array}
\]

where the functor $\mathcal{P}_1(Y) \hookrightarrow \mathcal{P}_1^\pi(M)$ is the inclusion. The rest of the proof, namely the fact that the image of a morphism lands in $\text{Triv}_\pi^1(i)$ under $\text{Rec}_\pi^1$, is explained in Appendix B.1. of [ScWa09].

### 3.2.6 Differential cocycle data

We now switch gears a bit and go in the other (infinitesimal) direction. We describe this in several parts. We focus on a local description first in terms of ‘trivialized’ transport functors. We extract the differential cocycle data from functors and then we construct functors from differential cocycle data. This is a brief and simplified account of the material covered in Section 4 of [ScWa09] since we do not prove any results. Chapter 2 of this thesis
provides more physics-style derivations along the lines of [ChTs93].

**From functors to 1-forms**

Throughout this chapter, let $G$ denote the Lie algebra of $G$. Given a smooth functor $F : P_1(X) \to BG$, we will define a $G$-valued 1-form $A$ pointwise for every $x \in X$ and $v \in T_xX$ as follows. Let $\gamma : \mathbb{R} \to X$ be a curve in $X$ with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. $\gamma : \mathbb{R} \to X$ induces a smooth pushforward functor $\gamma_* : P_1(\mathbb{R}) \to P_1(X)$. At the level of morphisms, the space $P^1\mathbb{R}$ of thin homotopy classes of paths in $\mathbb{R}$ is actually smoothly equivalent to $\mathbb{R} \times \mathbb{R}$.

The diffeomorphism $\gamma_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to P^1\mathbb{R}$ is defined by sending $(s,t)$ to the thin homotopy class of a path in $\mathbb{R}$ determined by its source point $s$ and target $t$ as shown schematically in Figure 3.4.

![Figure 3.4](image)

Figure 3.4: A point $(s,t)$ in $\mathbb{R}^2$ is drawn as two points on $\mathbb{R}$ and under the map $\gamma_{\mathbb{R}}$ gets sent to the thin path in $\mathbb{R}$ from the point $s$ to the point $t$ with a representative shown on the right.

Therefore, we obtain a function $F_1 \circ \gamma_* \circ \gamma_{\mathbb{R}}$ from the composition

$$G \xrightarrow{F_1} P^1 X \xleftarrow{\gamma_*} P^1 \mathbb{R} \xleftarrow{\gamma_{\mathbb{R}}} \mathbb{R} \times \mathbb{R}.$$  (3.2.59)
Here $F_1$ is $F$ restricted to the set of morphisms $P^1 X$. Using this, we define

$$A_x(v) := -\frac{d}{dt} \bigg|_{t=0} F_1 \left( \gamma_*(\gamma_{\mathbb{R}}(0, t)) \right).$$

(3.2.60)

$A_x(v)$ is independent of $\gamma$ and only depends on $x$ and $v$. Furthermore, it defines a 1-form $A \in \Omega^1(X; G)$. This result is what allowed us to assume that infinitesimally $F$ is of the form (2.3.3) in Section 2.3.1.

**From 1-forms to functors**

Starting with a $G$-valued 1-form $A \in \Omega^1(X; G)$ on $X$ we want to define a smooth functor $P_1(X) \to BG$. To do this, we first define a function, referred to as the *path transport*, $k_A : PX \to G$ on paths in $X$ with sitting instants (we do not mod out by thin homotopy). Given $\gamma \in PX$, we can pull back the 1-form $A$ to $\mathbb{R}$, namely $\gamma^*(A) \in \Omega^1([0, 1]; G)$. We then define $k_A(\gamma)$ using the path-ordered-exponential

$$k_A(\gamma) := \mathcal{P} \exp \left\{ \int_0^1 A_t \left( \frac{\partial}{\partial t} \right) dt \right\}. \quad (3.2.61)$$

Recall that this path-ordered exponential is defined by\(^2\)

\[
\mathcal{P} \exp \left\{ \int_0^1 A_t \left( \frac{\partial}{\partial t} \right) dt \right\} 
= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \int_0^1 dt_n \cdots \int_0^1 dt_1 \left[ A_{t_n} \left( \frac{\partial}{\partial t} \right) \cdots A_{t_1} \left( \frac{\partial}{\partial t} \right) \right] \right]. \quad (3.2.62)
\]

\(^2\)In this expression, we are assuming that $G$ is a matrix Lie group.
where the time-ordering operator $\mathcal{T}$ is defined by

$$
\mathcal{T}[A_tA_s] := \begin{cases} 
A_tA_s & \text{if } t \geq s \\
A_sA_t & \text{if } s \geq t 
\end{cases}
$$

(3.2.63)

The $n = 0$ term on the right-hand side of equation (3.2.62) is the identity.

We can picture the path-ordered exponential schematically as a power series of graphs with marked points as in Figure 3.5. This result has been justified in Section 2.3.1 and derived more systematically.

$k_A$ only depends on the thin homotopy class of $\gamma$ and therefore factors through a smooth map $F_A : P^1X \longrightarrow G$ on thin paths (see Definition 3.2.18). This map defines a smooth functor $F_A : \mathcal{P}_1(X) \longrightarrow \mathcal{B}G$ (see Proposition 4.3. and Lemma 4.5. of [ScWa09]).
Local differential cocycles for transport functors

The above constructions can be extended to smooth natural transformations between smooth functors. Given a smooth natural transformation \( h : F \Rightarrow F' \) of smooth functors \( F, F' : \mathcal{P}_1(X) \to BG \) we obtain a function, written somewhat abusively also as \( h : X \to G \) satisfying

\[
h(y)F(\gamma) = F'(\gamma)h(x)
\]  
(3.2.64)

for all thin paths \( \gamma : x \to y \) in \( X \). If we differentiate this condition, we obtain

\[
A' = \text{Ad}_h(A) - h^*\overline{\theta},
\]  
(3.2.65)

where \( \overline{\theta} \) is right Maurer-Cartan form, sometimes written as \( dgg^{-1} \) for matrix groups, \( A \) is the 1-form corresponding to \( F \), \( A' \) is the 1-form corresponding to \( F' \), and \( \text{Ad} \) is the adjoint action on the Lie algebra \( G \) defined by

\[
\text{Ad}_h(T) := \frac{d}{dt} \bigg|_{t=0} \left( h \exp\{tT\}h^{-1}\right)
\]  
(3.2.66)

for all \( T \in G \). This was also derived in Section 2.3.1. This motivates the following definition.

**Definition 3.2.67.** Let \( Z^1_X(G)^\infty \) be the category whose objects are 1-forms \( A \in \Omega^1(X; G) \) and a morphism from \( A \) to \( A' \) is a function \( h : X \to G \) satisfying

\[
A' = \text{Ad}_h(A) - h^*\overline{\theta}.
\]  
(3.2.68)
The composition is defined by
\[
\left( A'' \overset{h'}{\leftarrow} A' \overset{h}{\leftarrow} A \right) \mapsto \left( A'' \overset{h'h}{\leftarrow} A \right),
\]
where \( h'h \) is (pointwise) multiplication of \( G \)-valued functions.

This (and the previous section) defines two functors
\[
\begin{array}{c}
\mathcal{Z}_1^X (G) \overset{\mathcal{P}_X}{\rightarrow} \text{Funct}^\times (\mathcal{P}_1 (X), \mathcal{B}G), \\
\downarrow \mathcal{D}_X
\end{array}
\]
where \( \text{Funct}^\times (\mathcal{P}_1 (X), \mathcal{B}G) \) is the category of smooth functors and smooth natural transformations from the thin path groupoid of \( X \) to \( \mathcal{B}G \). These functors are defined on objects by \( \mathcal{D}_X (F) := A \) from (3.2.60) and \( \mathcal{P}_X (A) := F_A \) from (3.2.61). These two functors are inverses of each other, and not just an equivalence pair (Proposition 4.7. of [ScWa09]).

All of this was for globally defined smooth functors, which was explained in much more detail in Section 2.3.1. Topologically non-trivial global transport functors, however, were not discussed there. Transport functors on \( M \) are not necessarily smooth globally. However, there must exist a surjective submersion \( \pi : Y \rightarrow M \) with a smooth \( \pi \)-local \( i \)-trivialization. The smooth functor \( \mathcal{P}_1 (Y) \rightarrow \mathcal{B}G \) corresponds to a 1-form \( A \in \Omega^1 (Y; G) \), which is an object in \( \mathcal{Z}_1^Y (G) \). The natural transformation \( g : \pi_1^* \text{triv}_i \Rightarrow \pi_2^* \text{triv}_i \) factors through a smooth natural transformation \( \tilde{g} : \pi_1^* \text{triv} \Rightarrow \pi_2^* \text{triv} \), which
is a morphism in the category $Z^1_{Y;r}(G)\times$ from $\pi_1^*A$ to $\pi_2^*A$. This means

$$\pi_2^*A = \text{Ad}_{\tilde{g}}(\pi_1^*A) - \tilde{g}^*\tilde{g}. \quad (3.2.71)$$

The condition

$$\pi_{23}^*g = \pi_{13}^*g \quad (3.2.72)$$

translates to

$$\pi_{23}^*\tilde{g} \pi_{12}^*\tilde{g} = \pi_{13}^*\tilde{g}, \quad (3.2.73)$$

where the concatenation indicates multiplication of $G$-valued functions. A morphism of transport functors subordinate to the same surjective submersion is a natural transformation $h : \text{triv}_i \Rightarrow \text{triv}'_i$ that factors through a smooth natural transformation $\tilde{h} : \text{triv} \Rightarrow \text{triv}'$ and therefore defines a morphism from $A$ to $A'$ in $Z^1_Y(G)\times$. This motivates the following definition of local differential cocycles.

**Definition 3.2.74.** Let $\pi : Y \longrightarrow M$ be a surjective submersion. Define the category $Z^1_{\pi}(G)\times$ of **differential cocycles subordinate to** $\pi$ as follows. An object of $Z^1_{\pi}(G)\times$ is a pair $(A, g)$, where $A$ is an object in $Z^1_Y(G)\times$, $g$ is a morphism from $\pi_1^*A$ to $\pi_2^*A$ in $Z^1_{Y;r}(G)\times$. A morphism from $(A, g)$ to $(A', g')$ is a morphism $h$ from $A$ to $A'$ in $Z^1_Y(G)\times$. The composition of morphisms in $Z^1_{\pi}(G)\times$ is defined by

$$\left( (A'', g'') \xleftarrow{h'} (A', g') \xleftarrow{h} (A, g) \right) \mapsto \left( (A'', g'') \xleftarrow{h'h} (A, g) \right). \quad (3.2.75)$$
The above generalizations produce functors
\[ Z^1_\pi(G)_\infty \xrightarrow{p_\pi} \text{Des}^1_\pi(i)_\infty \]
(3.2.76)
emanating an equivalence of categories whenever \( i : BG \rightarrow T \) is an equivalence (Corollary 4.9. in [ScWa09]).

3.2.7 Limit over surjective submersions

Here we give a brief summary of the four levels of construction introduced and the notation of the functors relating these categories. To do this, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our categories depended on the choice of a surjective submersion. These categories were \( \text{Triv}^1_\pi(i)_\infty, \text{Des}^1_\pi(i)_\infty, \) and \( Z^1_\pi(G)_\infty. \) On the contrast, the category of transport functors \( \text{Trans}^1_{BG}(M,T) \) does not depend on \( \pi. \) To relate these categories better, we will take limits over \( \pi. \) Changing the surjective submersion gives a collection of categories depending on such surjective submersions. One can take a limit over the collection of surjective submersions in this case.

The general construction is done as follows. Let \( S_\pi \) be a family of categories parametrized by surjective submersions \( \pi : Y \rightarrow M \) and let \( F(\zeta) : S_\pi \rightarrow S_{\pi \circ \zeta} \) be a family of functors for every refinement \( \zeta : Y' \rightarrow Y \) of \( \pi \) satisfying the condition that for any iterated refinement \( \zeta' : Y'' \rightarrow Y' \)
and $\zeta : Y' \to Y$ of $\pi : Y \to M$ then $F(\zeta' \circ \zeta) = F(\zeta') \circ F(\zeta)$. In this case, an object of $\varprojlim \pi S_\pi$ is given by a pair $(\pi, X)$ of a surjective submersion $\pi : Y \to M$ and an object $X$ of $S_\pi$. A morphism from $(\pi_1, X_1)$ to $(\pi_2, X_2)$ consists of an equivalence class of a common refinement

and

\[
\begin{array}{c}
Z \\
| \\
| \\
| \\
Y_1 \xleftarrow{\pi_1} \downarrow \zeta \downarrow \pi_2 \\
| \\
M
\end{array}
\]

\[
(3.2.77)
\]

together with a morphism $f : (F(y_1))(X_1) \to (F(y_2))(X_2)$ in $S_\zeta$. It is written as a pair $(\zeta, f)$. Two such $(\zeta, f)$ and $(\zeta', f')$ are equivalent if they agree (on the nose) on their common pullback. The composition

\[
(\pi_3, X_3) \xleftarrow{(\zeta_{23}, g)} (\pi_2, X_2) \xleftarrow{(\zeta_{12}, f)} (\pi_1, X_1)
\]

\[
(3.2.78)
\]

is defined by choosing representatives and taking the pullback refinement

and

\[
\begin{array}{c}
Z_{13} \\
| \\
| \\
| \\
Z_{12} \xleftarrow{pr_{12}} \downarrow \zeta_{12} \downarrow \pi_1 \\
Z_{23} \xleftarrow{pr_{23}} \\
Y_1 \xleftarrow{\pi_1} \downarrow \pi_2 \\
| \\
| \\
M
\end{array}
\]

\[
(3.2.79)
\]

along with the composition $(F(pr_{23}))(g) \circ (F(pr_{12}))(f)$. One can check this definition does not depend on the equivalence class chosen.
After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We set the following notation, slightly differing from that of [ScWa13]:

\[ \lim_{\pi} \text{Triv}^1_{M}(i)^{\infty} =: \text{T} \]
\[ \lim_{\pi} \text{Des}^1_{M}(i)^{\infty} =: \text{D} \]
\[ \lim_{\pi} Z^1(M;G)^{\infty} =: \text{Z} \]

Because a limit of such equivalences is still an equivalence, the following facts, summarizing the several previous sections, hold. The categories \( Z^1(M;G)^{\infty} \) and \( \text{Des}^1_{M}(i)^{\infty} \) are equivalent under the condition that \( i : BG \to T \) is an equivalence of categories. \( \text{Des}^1_{M}(i)^{\infty} \) and \( \text{Triv}^1_{M}(i)^{\infty} \) are equivalent for any \( i \). Let \( v : \text{Triv}^1_{M}(i)^{\infty} \to \text{Trans}^1_{BG}(M,T) \) be the forgetful functor which forgets the specific local trivialization. If \( i \) is full and faithful, then \( v : \text{Triv}^1_{M}(i)^{\infty} \to \text{Trans}^1_{BG}(M,T) \) is part of an equivalence of categories. All these statements are proved in [ScWa09] (except the last one, but it follows from Lemma 3.3 in [ScWa09]).

For the reader’s convenience, we collect the categories and equivalences (assuming \( i \) is an equivalence) introduced in the past few sections

\[ Z^1(M;G)^{\infty} \xrightarrow{\mathcal{P}} \text{Des}^1_{M}(i)^{\infty} \xrightarrow{\mathcal{R}} \text{Triv}^1_{M}(i)^{\infty} \xrightarrow{v} \text{Trans}^1_{BG}(M,T), \quad (3.2.83) \]
where we’ve introduced the notation $\mathcal{P} := \lim_{\pi} \mathcal{P}_\pi$ and similarly for the other functors. $c$ is a weak inverse to $v$ and chooses a $\pi$-local $i$-trivialization for transport functors.

### 3.2.8 Parallel transport, holonomy, and gauge invariance

Holonomy for principal $G$-bundles with connection is defined in several different ways. In all cases, it is a special case of parallel transport where one restricts attention to paths whose target match their source, i.e. marked loops. Holonomy along a marked loop is an isomorphism of the fiber over the endpoint. However, for computational purposes, it is convenient to express such isomorphisms as group elements. One common way of doing this is to choose an open cover over which the bundle trivializes, choose a trivialization, and for each path, choose a decomposition of that path subordinate to the cover and parallel transport along each piece while patching the terms together using the transition functions. This is the procedure we discussed in Section 3.2.5. The problem with this procedure is that it depends on several choices. One purpose of this section is to analyze the dependence on these choices. The second purpose is to discuss (and make precise) the dependence

---

3. The terminology “marked” is chosen over “based” to avoid confusion with the based loop space, which is the space of loops with a single base point. We allow our basepoints to vary.
of such group elements on the marking chosen for loops. The punchline is that to obtain a well-defined holonomy independent of such choices, one needs to pass to conjugacy classes in $G$.

The first goal is accomplished by starting with a transport functor $F : \mathcal{P}_1(M) \rightarrow T$, choosing a local trivialization, extracting the descent data, and using the descent data to reconstruct a transport functor. This procedure can be described as a functor, which we denote by $\mathcal{C}$, from $\text{Trans}^1_{BG}(M, T)$ to itself (see Definition 3.2.85). Although all the ingredients for the functor $\mathcal{C}$ were described in [ScWa09], this procedure was not discussed. Here, we formulate this procedure and use it to analyze holonomy along loops. Thus, starting with a transport functor $F$ we obtain a new transport functor $\mathcal{C}(F)$ that produces group-valued holonomies along loops under suitable assumptions. The first choice we made in this procedure is the transport functor $F$ itself. One could have chosen a different, but naturally isomorphic, transport functor $F'$ to obtain $\mathcal{C}(F')$. The other choices made were those defining $\mathcal{C}$. Abstract nonsense tells us there is a morphism $F \rightarrow \mathcal{C}(F)$ of transport functors. Different choices of local trivializations and reconstructions are thus described in terms of natural isomorphisms. Formulated this way, it becomes a tautology that holonomy along loops is independent of these choices once one passes to conjugacy classes in $G$. 
Remark 3.2.84. One might argue that such a complicated formalism to obtain the well-known fact that holonomy is defined only with respect to conjugacy classes of $G$ is overkill. While this is true for holonomy along loops, this formalism extends naturally to holonomy along surfaces, which is our main objective, and the proofs are similar since they are expressed in terms of category theory. In the case of surfaces, we will use these ideas to generalize the results of Section 5.2 of [ScWa13]. It is therefore important to study the simpler case of holonomy along loops first.

The second goal, namely the dependence on markings, is accomplished by showing that for any two loops that are thinly homotopic, but not necessarily thinly homotopic preserving their marking, the group-valued holonomy using $\mathcal{F}(F)$ is well-defined up to conjugation. Using all these observations, we define, for every isomorphism class of transport functors, a holonomy map $L^1\mathcal{M} \to \text{G/Inn}(G)$ from the space of thin homotopy classes of free loops (see Definition 3.2.94) to the conjugacy classes of $G$.

We now define precisely what we mean by (functorially) extracting group-valued parallel transport from arbitrary transport functors. In order to accomplish this, we restrict our discussion to transport functors with $BG$-structure and with values in $T$ and assume once and for all that $i : BG \to T$ is full and faithful.
Definition 3.2.85. A group-valued transport extraction is a composition of functors (starting at the left and moving clockwise)

\[
\begin{array}{c}
\text{Trans}_{SG}^1(M, T) \xrightarrow{\text{Triv}_1^1(i) \circ \text{Ex}_1^1} \\
\text{Des}_1^1(i) \circ \text{Rec}_1^1 \xleftarrow{\text{Triv}_1^1(i) \circ \text{Rec}_1^1} \\
\end{array}
\]  \quad (3.2.86)

and consists of a choice of a weak inverse \( c \) of the forgetful functor \( v \) and a reconstruction functor \( \text{Rec}_1^1 \) (which itself depends on the choice of a lifting of paths as in (3.2.53)). Such a functor is written as \( \ell := v \circ \text{Rec}_1^1 \circ \text{Ex}_1^1 \circ c \).

The notation \( \ell \) stands for (local) trivialization.

Remark 3.2.87. Although the functor \( \ell \) depends on both \( c \) and \( s^\pi \) (which defines \( \text{Rec}_1^1 \)) we suppress the notation. The reason is because if we change \( c \) and/or \( s^\pi \), the functor \( \ell \) will change to a naturally isomorphic one and only this fact will matter in any computation.

The purpose of \( \ell \) is that it assigns group elements to thin paths for every transport functor \( F \) and also assigns group-valued gauge transformations for every morphism \( \eta : F \rightarrow F' \) of transport functors (this will be reviewed in the following paragraphs). Furthermore, we know that a natural isomorphism \( \varepsilon : \text{id} \Rightarrow \ell \) exists because all the functors in (3.2.86) are (part of) equivalences of categories. Choosing such a natural isomorphism specifies isomorphisms...
from the original fibers to the fiber $G$ viewed as a $G$-torsor and relates our original parallel transports to the group elements defined from $\mathcal{C}$.

To see this, first recall what $\mathcal{C}$ does. For a transport functor $F$, $c$ chooses a local trivialization $c(F) := (\pi, F, \text{triv}, t)$. Then we extract the smooth local descent object $\text{Ex}^1(\pi, F, \text{triv}, t) := (\pi, \text{triv}, g)$. Then, we reconstruct a $\pi$-local $i$-trivialization $\text{Rec}^1(\pi, \text{triv}, g)$ and then forget the trivialization data keeping just the transport functor $v(\text{Rec}^1(\pi, \text{triv}, g))$. The resulting transport functor, written as $\mathcal{C}$ (as opposed to $\mathcal{C}(F)$), is defined by (see the paragraph after Definition 3.2.46)

$$
\mathcal{P}_1(M) \xrightarrow{\mathcal{C}} T
$$

$$
P^1M \ni \gamma \mapsto R_{\text{Ex}^1(c(F))}(s^\pi(\gamma)).
$$

Here $\text{triv} : \mathcal{P}_1(Y) \longrightarrow BG$ is the "local" transport, $s^\pi : \mathcal{P}_1(M) \longrightarrow \mathcal{P}_1^\pi(M)$ is a choice of lifting points and paths, and $R_{\text{Ex}^1(c(F))}(s^\pi(\gamma)) : i(\bullet) \longrightarrow i(\bullet)$ is an element of $G$ because $i$ is full and faithful. This element of $G$ is defined by choosing a lift of the path $\gamma$ (see Figure 3.3) and applying trivialized transport on the pieces and transition functions on the jumps (see Section 3.2.5). Note that in the special case that $T = G$-Tor, $i(\bullet)$ can be taken to be $G$ itself and then $\mathcal{C}(\gamma)$ for a thin path $\gamma$ is left multiplication by some uniquely specified group element.
To a morphism $\eta : F \to F'$ of transport functors, the resulting morphism of transport functors, written as $\xi_\eta$, is defined as follows. First, $c$ chooses surjective submersions $\pi : Y \to M$ and $\pi' : Y' \to M$ for $F$ and $F'$, respectively, along with local trivializations $(\text{triv}, t)$ and $(\text{triv}', t')$. This means that under $c$ the morphism $c(\eta)$ is defined on a common refinement $\zeta : Z \to M$ of both $\pi$ and $\pi'$. The same thing applies to the extracted descent morphism $\text{Ex}^1(c(\eta)) = (\zeta, h)$. Since our domain is changed under the refinement, $h$ is defined by the composition

\[
\begin{align*}
\begin{array}{c}
\text{triv}_1 \\
\downarrow y \\
T \\
\downarrow \eta \\
P_1(M) \\
\downarrow t' \\
P_1(Y') \\
\text{triv}'_1
\end{array}
\quad
\begin{array}{c}
\downarrow \pi_* \\
\downarrow \text{id} \\
\downarrow \pi'_* \\
\downarrow \text{id}
\end{array}
\begin{array}{c}
P_1(Y) \\
P_1(Z)
\end{array}
\end{align*}
\]

This composition satisfies the condition

\[
y^{[2]*}g = \zeta_1^*h = y^{[2]*}y'g'.
\]

The notation means the following. A map $y : Z \to Y$ (and similarly for $y' : Z \to Y'$) determines a unique map $y^{[2]} : Z^{[2]} \to Y^{[2]}$ defined by $y^{[2]}(z, z') := (y(z), y(z'))$. The maps $\zeta_1, \zeta_2 : Z^{[2]} \to Z$ are the two projections.

The reconstruction functor $\text{Rec}^1 : \mathcal{O} \text{Ces}^1_M(i) \to \text{Triv}^1_M(i)$ sends the morphism $h$ in (3.2.89) to $\text{Rec}^1(\zeta, h) := s^*R(\zeta, h)$ which is a morphism of trans-
port functors from $\text{Rec}^1(y^*(\pi, \text{triv}, g))$ to $\text{Rec}^1(y'^*(\pi', \text{triv}', g'))$ with respect to this common refinement and where $s^\zeta : P_1(M) \to \mathcal{P}_1^\zeta(M)$. $\text{Rec}^1(\zeta, h)$ is defined by sending $x \in M$ to $h(s^\zeta(x))$ which is a morphism from $\text{triv}_i(y(s^\zeta(x)))$ to $\text{triv}_i(y'(s^\zeta(x)))$.

Now, the natural isomorphism $\varepsilon : \text{id} \Rightarrow \mathcal{F}$ assigns to every transport functor $F$ a morphism of transport functors $\varepsilon_F : F \to \mathcal{F}$. This means (see Definition 3.2.44) that associated to every $x \in M$ is an isomorphism $\varepsilon_F(x) : F(x) \to i(\bullet)$ satisfying naturality, which means that to every thin path $\gamma \in P^1M$ from $x$ to $y$, the diagram

\[
\begin{array}{ccc}
i(\bullet) & \xrightarrow{\varepsilon_F(x)} & F(x) \\
\downarrow & & \downarrow \varepsilon_F(\gamma) & \varepsilon_F(\gamma) \\
i(\bullet) & \xrightarrow{\varepsilon_F(y)} & F(y)
\end{array}
\]

(3.2.91)

commutes.

**Remark 3.2.92.** In Section 3.2, [ScWa09] define the Wilson line, what we’re calling $\mathcal{A}_F(\gamma)$, in terms of (3.2.91) as the composition $\varepsilon_F(y) \circ F(\gamma) \circ \varepsilon_F(x)^{-1} : i(\bullet) \to i(\bullet)$ using that $i$ is full and faithful so that this composition defines a unique group element. Our viewpoint is to define the Wilson line functorially and globally by using the group-valued transport extraction procedure $\mathcal{F}$.

Since $\varepsilon$ itself is a natural transformation, to every morphism $\eta : F \to F'$
of transport functors, the diagram

\[
\begin{array}{ccc}
F & \xleftarrow{r_F} & F' \\
\downarrow{\eta} & & \downarrow{\eta} \\
F'' & \xleftarrow{r_{F''}} & F'''
\end{array}
\]

commutes.

To analyze holonomy, we need to restrict parallel transport to thin paths whose source and target are the same, i.e. thin marked loops, and eventually thin free loops.

**Definition 3.2.94.** The **marked loop space of** \( M \) **is the set**

\[
\mathfrak{L} M := \{ \gamma \in P M \mid s(\gamma) = t(\gamma) \}
\]

**Definition 3.2.97.** The **\( t \)-holonomy of** \( F \), written as \( \text{hol}_F^t \), **is defined as the restriction of parallel transport of a transport functor** \( F \) **to the thin marked loop space** \( \mathfrak{L}^1 M \) **of** \( M \):

\[
\text{hol}_F^t := \left. \mathcal{L}_F \right|_{\mathfrak{L}^1 M} : \mathfrak{L}^1 M \longrightarrow G.
\]
We now pose three questions that will motivate the rest of our discussion on holonomy along thin marked loops.

i) How does $\text{hol}^F_t$ depend on the choice of basepoint? Namely, suppose that two thin marked loops $\gamma : x \to x$ and $\gamma' : x' \to x'$ are thinly homotopic without preserving the marking\(^4\) (see Definition 3.2.99). Then, how is $\text{hol}^F_t(\gamma)$ related to $\text{hol}^F_t(\gamma')$?

ii) How does $\text{hol}^F_t$ depend on $F$? Namely, suppose that $\eta : F \to F'$ is a morphism of transport functors. How is $\text{hol}^F_t$ related to $\text{hol}^{F'}_t$ in terms of $\eta$?

iii) How does $\text{hol}^F_t$ depend on $\mathcal{L}$, i.e., the choices of $c$ and $s^\pi$? Namely, suppose that $\mathcal{L}'$ is another trivialization. Then how is $\text{hol}^F_t$ related to $\text{hol}^{F'}_t$?

We first define what we mean by the thin free loop space and then we proceed to answer the above questions. Denote the smooth space of loops in $M$ by $LM := \{\gamma : S^1 \to M \mid \gamma \text{ smooth}\}$.

**Definition 3.2.99.** Two smooth loops $\gamma, \gamma' \in LM$ are thinly homotopic if there exists a smooth map $h : S^1 \times [0, 1] \to M$ such that

\(^4\)The notion of thin homotopy introduced in Definition 3.2.18 does not make sense when $x \neq x'$. 
i) there exists an $\epsilon > 0$ with $h(t, s) = \gamma(t)$ for $s \leq \epsilon$ and $h(t, s) = \gamma'(t)$ for $s \geq 1 - \epsilon$ and for all $t \in S^1$ and

ii) the smooth map $h$ has rank $\leq 1$.

Such a smooth map $h$ is called an unmarked thin homotopy. The smooth space of such thin homotopy classes of loops is denoted by $L^1M$ and is called the thin free loop space of $M$. Elements of $L^1M$ are called thin free loops or just thin loops.

The first condition guarantees that unmarked thin homotopy defines an equivalence relation and $L^1M$ is well-defined. The second condition is where the thin structure is buried. We need to discuss a few definitions and facts before relating thin loops to thin marked loops. For the purposes of being absolutely clear, from Lemma 3.2.101 through Lemma 3.2.107 we will distinguish between representatives of loops and thin homotopy equivalence classes by using brackets [ ]. However, afterwards, we will abuse notation and will rarely make the distinction.

**Definition 3.2.100.** The function $f : \mathcal{LM} \to LM$ defined by sending a marked loop $\gamma : [0, 1] \to M$ to the associated map $f(\gamma) : S^1 \to M$ obtained from identifying the endpoints of $[0, 1]$ is called the forgetful map.
Lemma 3.2.101. There exists a unique map \( f^1 : \mathcal{L}^1 M \rightarrow L^1 M \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{L} M & \rightarrow & \mathcal{L}^1 M \\
\downarrow f & & \downarrow f^1 \\
L M & \rightarrow & L^1 M
\end{array}
\]  

(3.2.102)

commutes (the horizontal arrows are the projections onto thin homotopy classes).

Proof. The map is constructed by choosing a representative, applying \( f \), and then projecting to \( L^1 M \). Let \([\gamma] : x \rightarrow x\) be an element of \( \mathcal{L}^1 M \) and let \( \gamma : x \rightarrow x \) and \( \gamma' : x \rightarrow x \) be two representatives in \( \mathcal{L} M \). Then there exists a thin homotopy \( h : [0, 1] \times [0, 1] \rightarrow M \) from \( \gamma \) to \( \gamma' \). Because \( h(t, s) = x \) for all \( s \in [0, 1] \) and all \( t \in [0, \epsilon] \cup [1 - \epsilon, 0] \) for some \( \epsilon > 0 \), the two ends of the first \([0, 1]\) factor can be identified resulting in a smooth map \( \tilde{h} : S^1 \times [0, 1] \rightarrow M \).

This gives the desired homotopy from \( f(\gamma) \) to \( f(\gamma') \).

\[\blacksquare\]

Note that there is also a function \( \text{ev}_0 : \mathcal{L}^1 M \rightarrow M \) given by evaluating a thin loop at its endpoint. This function forgets the loop and remembers only the basepoint.

Definition 3.2.103. A **marking of thin loops** is a section (not necessarily smooth) \( \mathbf{m} : L^1 M \rightarrow \mathcal{L}^1 M \) of \( f^1 : \mathcal{L}^1 M \rightarrow L^1 M \), i.e. \( f^1 \circ \mathbf{m} = \text{id} \).
Remark 3.2.104. A marking of ordinary loops cannot be defined in this way as a section of $f : \mathcal{L}M \rightarrow LM$ because an arbitrary smooth map $S^1 \rightarrow M$ need not have a sitting instant at any point.

Proposition 3.2.105. A marking of thin loops exists.

Actually, much more is true. Because the fact is somewhat surprising and interesting (and only holds due to the thin homotopy equivalence relation), we include it here. Let $\pi_0 M$ denote the set of components of $M$ and $p : M \rightarrow \pi_0 M$ the canonical function sending a point to its component. Let $c_0 : L^1 M \rightarrow \pi_0 M$ denote the canonical function sending a thin loop to the component in which it (every representative) lies. A marking of thin loops $m$ determines a function $\beta : L^1 M \rightarrow M$ given by $\beta := \text{ev}_0 \circ m$ that satisfies the condition that

\[
\begin{array}{ccc}
L^1 M & \xrightarrow{c_0} & \pi_0 M \\
\downarrow & & \downarrow \\
M & \xrightarrow{p} & \pi_0 M \\
\end{array}
\]

(commutes).

Lemma 3.2.107. Let $\beta : L^1 M \rightarrow M$ be any function such that the diagram in (3.2.106) commutes. Then there exists a marking of thin loops
m : L^1 M \rightarrow \mathcal{L}^1 M such that the diagram

\[
\begin{array}{ccc}
\mathcal{L}^1 M & \overset{\text{ev}_0}{\longrightarrow} & M \\
\downarrow{m} & & \downarrow{\beta} \\
L^1 M & \end{array}
\] (3.2.108)

commutes.

Proof. A function m can be defined as follows. For any thin loop [\gamma] \in L^1 M, let \gamma : S^1 \rightarrow M be a representative. Then there exists an unmarked thin homotopy h from \gamma to a loop \gamma_\beta with sitting instants at \beta([\gamma]) because (3.2.106) commutes. To see this, one can simply pick a point on the loop and extend the loop out to the basepoint and come back without sweeping out any area (see Figure 3.6). Then project \gamma_\beta to \mathcal{L}^1 M. Thus, set m([\gamma]) := [\gamma_\beta].

Figure 3.6: Let [\gamma] be a thin free loop, x := \beta([\gamma]) a point in the same connected component as [\gamma], and \gamma a representative loop (in red). Then there exists a path \gamma' : x \rightarrow x with sitting instants (in blue) and an unmarked thin homotopy h : \gamma \Rightarrow \gamma'. The cylinder depicts such a homotopy with the middle loop (in purple) indicating an intermediate loop. The dashed line on the cylinder indicates that the loops begin to extend outwardly towards the marking without sweeping any area. The “mouse-hole” on the cylinder indicates that the loops from the homotopy eventually sit at x.
To see that this is well-defined, let $\gamma'$ be another representative of $[\gamma]$ and let $\tilde{h}$ be an unmarked thin homotopy from $\gamma'$ to $\gamma$. Then composing the two unmarked thin homotopies $h \circ \tilde{h}$ gives an unmarked thin homotopy from $\gamma'$ to $\gamma_\beta$. Of course, there are many possible choices for $\gamma_\beta$ for a given $\beta$ that will give different markings $m$. 

\[ \square \]

**Remark 3.2.109.** If $\beta$ is chosen so that the diagram in (3.2.106) does not commute, a marking $m$ satisfying (3.2.108) does not exist.

We now proceed to answering the above questions in order.

i) Let $m, m' : L^1 M \to L^1 M$ be two markings of thin loops in $M$. Let $[\gamma] \in L^1 M$ and denote $x := ev_0(m([\gamma]))$ and $x' := ev_0(m'([\gamma]))$. A choice of representatives $\gamma : x \to x$ and $\gamma' : x' \to x'$ as paths with sitting instants of $m([\gamma])$ and $m'([\gamma])$, respectively, need not have the same image. In particular, $x$ and $x'$ might not lie on each others images. Figure 3.7 gives an example. This makes it impossible to compare their holonomies using thin bigons in the usual way (because no such bigon exists).

However, there is an unmarked thin homotopy $h : S^1 \times [0, 1] \to M$ with $h(t, s) = \gamma(t)$ for $s \leq \epsilon$ and $h(t, s) = \gamma'(t)$ for $s \geq 1 - \epsilon$ for some $\epsilon > 0$. Therefore, one can choose a loop $\tilde{\gamma}$ and two paths with sitting instants $\gamma_{x'x} : x \to x'$ and $\gamma_{xx'} : x' \to x$ with the following three properties.
First, as a loop, \( \tilde{\gamma} \) can be written as the composition \( \gamma_{x'x} \) and \( \gamma_{xx} \) in some order, i.e. using the map \( f \) of Definition 3.2.100, \( \tilde{\gamma} = f(\gamma_{x'x} \circ \gamma_{xx}) \) or \( f(\gamma_{xx} \circ \gamma_{x'x}) \). Second, the composition \( \gamma_{xx} \circ \gamma_{x'x} \) is thinly homotopic to \( \gamma \) preserving the basepoint \( x \). Third, the composition \( \gamma_{x'x} \circ \gamma_{xx} \) is thinly homotopic to \( \gamma' \) preserving the basepoint \( x' \). This is depicted in Figure 3.8.

This says that given two marked loops, with possibly different markings, that are thinly homotopic \textit{without} preserving the marking, one can always choose a representative of such a thin loop in \( M \) with \textit{two} marked points so that the associated two \textit{marked} loops (coming from starting at either marking) are thinly homotopic to the original two with a thin homotopy that preserves the marking. More precisely, we proved the following fact.
FIGURE 3.8: The domain of the unmarked thin homotopy $h : S^1 \times [0, 1] \to M$ is drawn as an annulus depicting the domain of $\gamma$ as the inner circle and that of $\gamma'$ as the outer circle. The homotopy allows us to choose a loop $\tilde{\gamma}$, drawn somewhat in the middle (in orange), that contains both $x$ and $x'$ and is thinly homotopic to both $\gamma$ and $\gamma'$. This loop $\tilde{\gamma}$ is decomposed into two paths $\gamma_{x'x} : x \to x'$ and $\gamma_{xx'} : x' \to x$. The dashed lines indicate the regions of sitting instants. All paths are oriented counter-clockwise. Note that, by a reparametrization if necessary, the homotopy $h$ may be chosen to separate the two basepoints into the northern and southern hemispheres as drawn.

**Lemma 3.2.110.** Let $m, m' : L^1 M \to \mathcal{L}^1 M$ be two markings. Let $[\gamma] \in L^1 M$ be a thin loop in $M$ and write $x := ev_0(m([\gamma]))$ and $x' := ev_0(m'([\gamma]))$. Then, there exist two paths $\gamma_{x'x} : x \longrightarrow x'$ and $\gamma_{xx'} : x' \longrightarrow x$ with sitting instants such that the following three properties hold (see Figure 3.9).

i) The composition of $\gamma_{xx'}$ and $\gamma_{x'x}$ (in either order) and forgetting the marking is a representative of $[\gamma]$.

ii) $\gamma_{xx'} \circ \gamma_{x'x}$ is a representative of $m([\gamma])$ as a path with sitting instants.
iii) \( \gamma_{x'} \circ \gamma_{xx'} \) is a representative of \( \mathfrak{m}'([\gamma]) \) as a path with sitting instants.

Figure 3.9: For two markings with associated basepoints \( x \) and \( x' \) of a thin loop \( [\gamma] \), there exist representatives paths with sitting instants (shown on the right) \( \gamma_{x':x} : x \to x' \) (in red) and \( \gamma_{xx'} : x' \to x \) (in blue) such that \( \gamma := \gamma_{xx'} \circ \gamma_{x':x} \) (shown on the left) represents one marking and \( \gamma' := \gamma_{x':x} \circ \gamma_{x} \) (shown in the middle) represents the other. Note that \( \gamma' \) and \( \gamma_{xx'} \circ \gamma \circ \gamma_{xx'} \) are thinly homotopic.

Therefore, without loss of generality, we can choose a single representative \( \tilde{\gamma} \) of a thin free loop \( [\gamma] \) with a decomposition as in the Lemma. We denote \( \gamma' := \gamma_{x'} \circ \gamma_{xx'} \) and \( \gamma := \gamma_{xx'} \circ \gamma_{x} \). Thus \( \tilde{\gamma} \) is one of \( f(\gamma) \) or \( f(\gamma') \). Note that \( \gamma' \) and \( \gamma_{xx'} \circ \gamma \circ \gamma_{xx'} \) are thinly homotopic. For convenience, from now on we abuse notation often and do not distinguish between the actual paths versus the thin homotopy classes as elements of \( P^1 M \).

By functoriality of the transport functor \( \mathcal{F} \), we have

\[
\text{hol}^F(\gamma') = \mathcal{F}(\gamma')
\]

\[
= \mathcal{F}(\gamma_{xx'} \circ \gamma \circ \gamma_{xx'})
\]

\[
= \mathcal{F}(\gamma_{xx'}) \mathcal{F}(\gamma) \mathcal{F}(\gamma_{xx'})
\]

\[
= (\mathcal{F}(\gamma_{xx'}))^{-1} \text{hol}^F(\gamma) \mathcal{F}(\gamma_{xx'})
\]

(3.2.111)
so that $\text{hol}^F$ changes by conjugation in $G$ when the marking is changed.

ii) Suppose that $\eta : F \rightarrow F'$ is a morphism of transport functors. Then, for every thin path $\gamma : x \rightarrow y$ we have a commutative diagram

$$
\begin{array}{c}
\mathcal{F}'(x) \\ \downarrow \mathcal{F}'(\gamma) \end{array} \quad \begin{array}{c}
\mathcal{F}(x) \\ \downarrow \mathcal{F}(\gamma) \end{array}
\begin{array}{c}
\mathcal{F}'(y) \\ \downarrow \mathcal{F}'(\gamma) \end{array} \quad \begin{array}{c}
\mathcal{F}(y) \\ \downarrow \mathcal{F}(\gamma) \end{array}
$$

which says

$$
\mathcal{F}'(\gamma) \mathcal{F}(\gamma) = \mathcal{F}(\gamma) \mathcal{F}'(\gamma). 
$$

(3.2.113)

If we restrict this to a thin marked loop $\gamma$ with $y = x$, then

$$
\text{hol}^F(\gamma) = (\mathcal{F}'(\gamma) \mathcal{F}'(\gamma) \mathcal{F}(\gamma))^{-1} \text{hol}^F(\gamma) \mathcal{F}(\gamma) 
$$

(3.2.114)

so that again, $\text{hol}^F$ changes under conjugation when the functor $F$ is changed to an isomorphic one.

iii) Suppose that another trivialization $\mathcal{T}'$ was chosen. Then following the comments after Remark 3.2.87, we can choose natural isomorphisms $\mathcal{T} : \text{id} \Rightarrow \mathcal{T}$ and $\mathcal{T}' : \text{id} \Rightarrow \mathcal{T}'$ resulting in a natural isomorphism $\mathcal{S} := \mathcal{T}' \circ \mathcal{T} : \mathcal{T}' \Rightarrow \mathcal{T}$.

This means every transport functor $F$ gets assigned a morphism of transport functors $\mathcal{S}_F : \mathcal{F}' \rightarrow \mathcal{F}$ satisfying naturality.

This means to every $x \in M$ we have a morphism $\mathcal{S}_F(x) : \mathcal{F}'(x) \rightarrow \mathcal{F}(x)$
satisfying naturality, i.e. to every path $\gamma : x \rightarrow y$ the diagram

\[
\begin{array}{ccc}
\mathcal{G}(x) & \xrightarrow{\mathcal{B}(x)} & \mathcal{G}'(x) \\
\downarrow \mathcal{G}(\gamma) & & \downarrow \mathcal{G}'(\gamma) \\
\mathcal{G}(y) & \xleftarrow{\mathcal{B}(y)} & \mathcal{G}'(y)
\end{array}
\]  

(3.2.115)

commutes. In case $\gamma$ is a thin loop at $x$, this gives

\[
\text{hol}_{\mathcal{G}}^F(\gamma) = (\mathcal{B}(x))^{-1}\text{hol}_{\mathcal{G}}^F(\gamma)\mathcal{B}(x). 
\]  

(3.2.116)

In conclusion, the answer to every one of the three questions is conjugation. This is what is called \textit{gauge covariance}. To get something gauge invariant, we first denote the quotient map from $G$ to its conjugacy classes by $q : G \rightarrow G/\text{Inn}(G)$, where $\text{Inn}(G)$ stands for the inner automorphisms of $G$ and the quotient $G/\text{Inn}(G)$ is given by the conjugation action of $G$ on itself. All of the above considerations show that the following theorem holds.

\textbf{Theorem 3.2.117.} Let $M$ be a smooth manifold, $G$ be a Lie group, $T$ a category, and suppose that $i : \mathcal{B}G \rightarrow T$ is full and faithful. Let $F \in \text{Trans}^1_{\mathcal{B}G}(M,T)$ be a transport functor and $\mathcal{I}$ a group-valued transport extraction. Let $L^1M, \mathcal{L}^1M, m, \text{hol}_{\mathcal{G}}^F$ and $q$ be defined as above. Then the composition

\[
G/\text{Inn}(G) \xrightarrow{q} G \xleftarrow{\text{hol}_{\mathcal{G}}^F} \mathcal{L}^1M \xrightarrow{m} L^1M
\]  

(3.2.118)
\textbf{i}) independent of } m, \\
\textbf{ii}) independent of the isomorphism class of } F, \\
\textbf{iii}) and independent of the isomorphism class of } \mathcal{L}.

Notice that this theorem lets us make the following definition.

\textbf{Definition 3.2.119.} Let \([F]\) be an isomorphism class of transport functors. The \textit{gauge invariant holonomy} of \([F]\) is defined to be the map in the previous theorem, namely

\[
\text{hol}^{[F]} : q \circ \text{hol}^F \circ m : L^1 M \to \frac{G}{\text{Inn}(G)}
\]

where \(F\) is a representative of \([F]\), \(\mathcal{L}\) is a group-valued transport extraction, and \(m : L^1 M \to L^1 M\) is a marking of thin loops in \(M\). Let \(\gamma \in L^1 M\). If \(\text{hol}^{[F]}(\gamma)\) is such that \(q^{-1}(\text{hol}^{[F]}(\gamma))\) is a single element, we will say that \(\text{hol}^{[F]}(\gamma)\) is \textit{gauge invariant} and abusively write \(\text{hol}^{[F]}(\gamma)\) instead of \(q^{-1}(\text{hol}^{[F]}(\gamma))\).

\section{3.3 Transport 2-functors and gauge invariant surface holonomy}

In the present section, we review the basics of transport 2-functors and also provide some new results. As a preliminary, we briefly set our notation and
review some facts about (strict) 2-groups and crossed modules. Then we split
up the discussion into several parts and follow a similar pattern to the trans-
port functor case. However, since we are now aware of what local triviality
should mean, we skip the guess-work and head straight to the correct the-
ory. We start with a Čech description of ordinary principal (strict) 2-group
2-bundles (without connection) in terms of smooth 2-functors. We then dis-
cuss how to add connection data by introducing transport functors, local
triviality, and descent data. The discussion of the reconstruction functor is
more involved, and because it is important for the calculation, we spend some
time on it. Nevertheless, we skip some technical details (such as compositors
and unifiers). Then we consider the differential cocycle data and discuss a
formula for higher holonomy in terms of an iterated surface integral. We
summarize the results as before. Sections 3.3.1 through 3.3.7 are a summary
of [ScWa11], [ScWa], and [ScWa13].

Finally, in Section 3.3.8, we discuss some results on surface holonomy
and its gauge covariance. We introduce a notion of $\alpha$-conjugacy classes for
a 2-group in Definition 3.3.157 and prove in Theorem 3.3.159 that surface
holonomy along spheres is well-defined in $\alpha$-conjugacy classes generalizing the
reduced group of [ScWa13] (it is not yet known whether this generalization
will work for more general surfaces). In the process, the procedure of group-
valued transport extraction is categorified for the purposes of (i) proving this theorem and (ii) providing a functorial description for computing transport locally, which we utilize in Section 3.5.

We assume the reader is familiar with the basics of 2-categories. A review sufficient for most of our purposes can be found in Appendix A of [ScWa] or Appendix A of this thesis.

### 3.3.1 2-group conventions

The theory of 2-groups is discussed in great detail in the article [BaLa04]. However, to simplify the discussion, we will define a (strict) 2-group as a strict one-object 2-groupoid, i.e. a strict 2-category with inverses for all 1- and 2-morphisms. Normally, one defines a 2-group as a groupal groupoid as in [BaLa04], but we find this unnecessary. However, to be consistent with notation in the literature, we will write our 2-groups as $\mathcal{B}\mathfrak{G}$ and use the notation $\mathfrak{G}$ where appropriate.

There is a 2-category of strict 2-groups denoted by 2-Grp whose 1-morphisms and 2-morphisms are functors and natural transformations, respectively. It is useful to relate this higher-categorical definition to one involving ordinary groups. Although this is standard, we set the notation, which may differ from some authors (we have also described some aspects of this in Section
Definition 3.3.1. A crossed module is a quadruple \((H, G, \tau, \alpha)\) of two groups, \(G\) and \(H\), and group homomorphisms \(\tau : H \rightarrow G\) and \(\alpha : G \rightarrow \text{Aut}(H)\) satisfying the two conditions

\[
\alpha_{\tau(h)}(h') = hh'h^{-1}
\]  \hspace{1cm} (3.3.2)

and

\[
\tau(\alpha_g(h)) = g\tau(h)g^{-1}.
\]  \hspace{1cm} (3.3.3)

In this definition, \(\text{Aut}(H)\) is the automorphism group of \(H\). The collection of crossed modules form the objects of a 2-category \(\text{CrsMod}\).

Theorem 3.3.4. The 2-categories \(\text{CrsMod}\) and \(\text{2-Grp}\) are equivalent.

This theorem has been known for quite some time in several different forms. A simple place to start for this is in the article \([\text{BaHu11}]\) with more information in \([\text{BaLa04}]\).

Proof. We only prove the equivalence at the level of objects and in only one direction. This will set up our conventions throughout the paper. Given a crossed module \((H, G, \tau, \alpha)\) the associated 2-group \(\mathcal{B}\mathcal{G}\) is defined to have a single object \(\bullet\), \(G\) as its set of 1-morphisms, and \(H \times G\) as its set of 2-morphisms. Composition of 1-morphisms is given by multiplication in \(G\).
The source and target maps of 2-morphisms are defined pictorially by
\[
\begin{array}{c}
\bullet \\
\bigcirc \hspace{1cm} (h,g) \\
\downarrow \\
\tau(h)g
\end{array}
\]
(3.3.5)

Vertical and horizontal compositions are defined by
\[
\begin{array}{c}
\bullet \\
\bigcirc \hspace{1cm} \tau(h)g \\
\downarrow \\
\tau(h)\tau(h)g
\end{array} \quad \rightarrow \quad 
\begin{array}{c}
\bullet \\
\bigcirc \hspace{1cm} (h',g') \\
\downarrow \\
\tau(h')g'
\end{array}
\]
and
\[
\begin{array}{c}
\bullet \\
\bigcirc \hspace{1cm} (h,g) \\
\downarrow \\
\tau(h)g
\end{array} \quad \rightarrow \quad 
\begin{array}{c}
\bullet \\
\bigcirc \hspace{1cm} (h',g') \\
\downarrow \\
\tau(h')g'
\end{array}
\]
respectively. When writing 2-group multiplication, we will always drop the composition symbol \( \circ \), which is a common practice for ordinary group multiplication.

The above proof sets up our convention for 2-group multiplication. Equation (3.3.5) shows that what is needed to specify a 2-morphism is an element of \( G \), the source of the 2-morphism, and an element of \( H \). Thus, if the source is already known, the element in \( H \) specifies the 2-morphism. Equation (3.3.6) defines vertical composition and equation (3.3.7) defines horizontal composi-
tion. Please be aware that different authors have different conventions (since the 2-categories CrsMod and 2-Grp are equivalent in many ways).

The following is a simple but important fact (which we use in studying gauge invariance, mainly Corollary 3.5.66).

**Lemma 3.3.8.** Let \((H, G, \tau, \alpha)\) be a crossed module. Then \(\ker \tau := \{h \in H \mid \tau(h) = e\}\) is a central subgroup of \(H\).

**Proof.** Let \(k \in \ker \tau\) and \(h \in H\). Then

\[kh = khk^{-1} = \alpha_{\tau(k)}(h)k = \alpha_e(h)k = hk.\]  

(3.3.9)

\[\square\]

**Definition 3.3.10.** A **Lie crossed module** is a crossed module \((H, G, \tau, \alpha)\) with \(G\) and \(H\) Lie groups and where \(\tau\) and \(\alpha\) are smooth maps, where \(\alpha\) being smooth technically means that the adjoint map \(G \overset{\hat{}}{\to} H\) is smooth.

**Definition 3.3.11.** A **Lie 2-groupoid** is a strict 2-category Gr whose objects, 1-morphisms, and 2-morphisms are all smooth spaces and all structure maps are smooth. Furthermore, all 1- and 2-morphisms are invertible and the inversion maps are all smooth.

**Definition 3.3.12.** A **Lie 2-group** is a Lie 2-groupoid with a single object.
Remark 3.3.13. Lie crossed modules form the objects of a 2-category and Lie 2-groups form the objects of a 2-category. A similar proof shows that these 2-categories are also equivalent.

3.3.2 A Čech description of principal $\mathfrak{G}$-2-bundles

Let $\mathcal{B}\mathfrak{G}$ be a Lie 2-group and denote the associated crossed module by $(H, G, \tau, \alpha)$. Principal $\mathfrak{G}$-2-bundles over a manifold $M$ can be described in terms of 2-functors using the Čech groupoid as well (this also comes from Remark II.13. of [Wo11]). However, since we are dealing with 2-categories we need to slightly modify the Čech groupoid of Definition 3.2.1. The way we do this is just by throwing on identity 2-morphisms. In other words, given an open cover $\{U_i\}_{i \in I}$ of $M$, a 2-morphism from $(x, i, j)$ to $(x', i', j')$ exists only if $x' = x$, $i' = i$, and $j' = j$ and in this case there is only the identity 2-morphism. Composition is uniquely defined by this. This defines the Čech 2-groupoid, also written as $\mathfrak{U}$. This is a Lie 2-groupoid.

Definition 3.3.14. 2-functors between Lie 2-groupoids are smooth if they assign data smoothly. Similarly, pseudonatural transformations and modifications are smooth when the assignments defining them are smooth.

Any smooth 2-functor $\mathfrak{U} \to \mathcal{B}\mathfrak{G}$ gives the Čech cocycle data of a principal $\mathfrak{G}$-2-bundle over $M$ subordinate to the cover $\{U_i\}_{i \in I}$. To see this, simply recall
what a 2-functor does (see Definition A.5. of [ScWa] or Definition A.44 in Appendix A in this thesis). To each object \((x, i)\) in \(\mathfrak{U}\) it assigns the single object \(\bullet\) in \(\mathcal{B\Phi}\). To each jump \((x, i, j)\), it assigns an element denoted by \(g_{ij}(x) \in G\) in such a way that we get a smooth 1-cochain \(g_{ij} : U_{ij} \longrightarrow G\) as in Section 3.2.1. However, to each triple intersection \(U_{ijk}\), which corresponds to the composition of \(U_{ij}\) with \(U_{jk}\), it assigns an element \(f_{ijk}(x) \in H\) in such a way that we get a smooth 2-cochain \(f_{ijk} : U_{ijk} \longrightarrow H\)

\[
\begin{array}{c}
\text{which says}\\
\tau(f_{ijk})g_{jk}g_{ij} = g_{ik}.
\end{array}
\] (3.3.16)
condition on quadruple intersections giving a “cocycle condition”

\[ (f_{jkl}, g_{kli}g_{jkl}) (e, g_{ij}) = (e, g_{kli})(f_{jkl}, g_{kli}g_{ij}), \]

which after multiplying out (using the rules of Section 3.3.1) and projecting both sides to $H$ gives

\[ f_{ijl}f_{jkl} = f_{ikl}g_{kli}(f_{ijl}), \]

The 2-functor also assigns 0-cochains $\psi_i : U_i \rightarrow H$

which says

\[ \tau(\psi_i) = g_{ii}. \]
These satisfy two "degenerate" cocycle conditions on each double intersection $U_{ij}$ of $M$ for the two ways one edge can be collapsed on the triangle. One is

$$f_{iij}g_{ij}(\psi_i) = e. \quad (3.3.23)$$

The other cocycle condition is

$$f_{ijj}(\psi_j) = e. \quad (3.3.25)$$

Refinements and 1-morphisms between two such 2-functors is similar to the ordinary functor case from Section 3.2.1 but a bit more subtle due to
modifications (which we won’t discuss now anyway). Let \( \{ U_i \}_{i \in I} \) be another cover of \( M \) with associated Čech 2-groupoid \( \mathfrak{U} \). Let \( P : \mathfrak{U} \rightarrow \mathcal{B} \mathfrak{G} \) and \( P' : \mathfrak{U}' \rightarrow \mathcal{B} \mathfrak{G} \) be two smooth 2-functors. A 1-morphism from \( P \) to \( P' \) consists of a common refinement \( \{ V_a \}_{a \in A} \) of both \( \{ U_i \}_{i \in I} \) and \( \{ U_i' \}_{i' \in I'} \) along with a smooth pseudo-natural transformation

\[
\begin{array}{ccc}
\mathfrak{U} & \xrightarrow{\alpha} & \mathfrak{U}' \\
\downarrow{h} & & \downarrow{h'} \\
\mathcal{B} \mathfrak{G} & \xrightarrow{P} & \mathcal{B} \mathfrak{G} \\
\end{array}
\tag{3.3.26}
\]

By definition (see Definition A.6. in [ScWa] or Definition A.66 in Appendix A of this thesis), to each object \((x, a)\) in \( \mathfrak{U} \) such a pseudo-natural transformation gives a smooth function \( h_a : V_a \rightarrow G \) as before, but also to each jump \((x, a, b)\) in \( \mathfrak{U} \), it gives another smooth function \( \epsilon_{ab} : V_{ab} \rightarrow H \) fitting in the diagram

\[
\begin{array}{ccc}
\bullet & \xrightarrow{g_{ab}} & \bullet \\
\downarrow{h_b} & & \downarrow{h_a} \\
\bullet & \xleftarrow{(\epsilon_{ab}, h_bg_{ab})} & \bullet \\
\end{array}
\tag{3.3.27}
\]

which says that

\[
\tau(\epsilon_{ab})h_bg_{ab} = g'_{ab}h_a.
\tag{3.3.28}
\]

The higher naturality conditions of a pseudo-natural transformation are given as follows. In general, to every 2-morphism, there is an associated naturality
condition, but because the 2-morphisms in $\mathcal{U}$ are all identities, this condition is vacuously true. To every pair of composable 1-morphisms $(x, i, j)$ and $(x, j, k)$ we get

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g_{bc} \\
\downarrow g_{bc} \\
\downarrow g_{ab}
\end{array}
\begin{array}{c}
\begin{array}{c}
f_{abc} \\
\downarrow f_{abc} \\
\downarrow f_{abc}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
g_{ac} \\
\downarrow g_{ac} \\
\downarrow g_{ac}
\end{array}
\begin{array}{c}
\begin{array}{c}
h_c \\
\downarrow h_c \\
\downarrow h_a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
g'_{bc} \\
\downarrow g'_{bc} \\
\downarrow g'_{ab}
\end{array}
\begin{array}{c}
\begin{array}{c}
f'_{abc} \\
\downarrow f'_{abc} \\
\downarrow f'_{abc}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
g'_{ac} \\
\downarrow g'_{ac} \\
\downarrow g'_{ac}
\end{array}
\begin{array}{c}
\begin{array}{c}
h_a \\
\downarrow h_a \\
\downarrow h_a
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Commutativity of this diagram says

\[
(e, h_c)(f_{abc}, g_{bc}g_{ab}) = (e, h_c h_{bc})(e, g_{ab}),
\]

\[
(e, g_{ac})(e, h_{bc} h_{g_{bc}}) = (e, g_{ac})(e, h_{bc} g_{ab}),
\]

which after equating both projections to $H$ gives

\[
\epsilon_{ac} \alpha_{hc}(f_{abc}) = f'_{abc} \alpha_{g_{bc}}(\epsilon_{ab}) \epsilon_{bc}
\]

for all $a, b, c \in A$. 

(3.3.29)

(3.3.30)

(3.3.31)
Finally, to every object \((x, i)\) we get on each open set \(U_i\)

\[
\begin{aligned}
\psi_a, e \\
\downarrow \quad \downarrow \quad \downarrow \\
\epsilon_{aa}, h_a g_{aa} \\
\downarrow \quad \downarrow \\
\psi_a, e \\
\downarrow \\
\epsilon_{aa}, h_a \\
\end{aligned}
\]

where the back face of the cylinder is the identity 2-morphism \((e, h_a)\). This reads

\[
(e, h_a)(\psi_a, e) = (e, h_a) \quad (\epsilon_{aa}, h_a g_{aa}) = (\psi_a', e)(e, h_a) ,
\]

which after projecting to \(H\) says

\[
\epsilon_{aa} \alpha h_a (\psi_a) = \psi_a'.
\]

Therefore, a 1-morphism of such principal 2-bundles as described above defines an equivalence of principal 2-bundles as described in [Wo11].

We won’t discuss 2-morphisms now because we will see that the above construction is a special case of the concept of limits of 2-categories in Section 3.3.7.
3.3.3 Local triviality of 2-functors

Just as transport functors describe parallel transport along paths, transport 2-functors describe parallel transport along paths and surfaces. They exhibit a formulation of a generalization of bundles with connection that describe such transport. We start by generalizing the thin path groupoid $\mathcal{P}_1(X)$ to the thin path 2-groupoid $\mathcal{P}_2(X)$. At this point, one should recall Definition 3.2.99 where bigons are introduced.

**Definition 3.3.35.** Let $X$ be a smooth manifold. Two bigons $\Gamma$ and $\Gamma'$ are said to be *thinly homotopic* if there exists a smooth map $A : [0, 1] \times [0, 1] \times [0, 1] \rightarrow X$ with the following two properties.

i) First, there exists an $\epsilon$ with $\frac{1}{2} > \epsilon > 0$ such that

$$A(t, s, r) = \begin{cases} x & \text{for all } (t, s, r) \in [0, \epsilon] \times [0, 1] \times [0, 1] \\ y & \text{for all } (t, s, r) \in [1 - \epsilon, 1] \times [0, 1] \times [0, 1] \\ \gamma(t) & \text{for all } (t, s, r) \in [0, 1] \times [0, \epsilon] \times [0, 1] \\ \gamma'(t) & \text{for all } (t, s, r) \in [0, 1] \times [1 - \epsilon, 1] \times [0, 1] \\ \Gamma(t, s) & \text{for all } (t, s, r) \in [0, 1] \times [0, 1] \times [0, \epsilon] \\ \Gamma'(t, s) & \text{for all } (t, s, r) \in [0, 1] \times [0, 1] \times [1 - \epsilon, 1] \end{cases}$$

(3.3.36)

ii) Second, the rank of $A$ is strictly less than 3 for all $(t, s, r) \in [0, 1] \times [0, 1] \times [0, 1]$ and is strictly less than 2 for all $(t, s, r) \in [0, 1] \times ([0, \epsilon] \cup [1 - \epsilon, 1]) \times [0, 1]$. 
In this case, \( A \) is said to be a \textit{thin homotopy} from \( \Gamma \) to \( \Gamma' \). The set of equivalence classes of bigons under thin homotopy is denoted by \( P^2X \). Elements of \( P^2X \) are called \textit{thin bigons}.

**Definition 3.3.37.** Let \( X \) be a smooth manifold. The \textit{thin path-2-groupoid} is a 2-category \( \mathcal{P}_2(X) \) defined as follows. The set of objects and 1-morphisms of \( \mathcal{P}_2(X) \) coincide with those of \( \mathcal{P}_1(X) \). The set of 2-morphisms of \( \mathcal{P}_2(X) \) is \( P^2X \). Let \( [\Gamma] \) be a thin bigon. The source and targets are defined by choosing a representative bigon \( \Gamma \) and taking the thin homotopy equivalence classes of the paths \( t \mapsto \Gamma(t, 0) \) and \( t \mapsto \Gamma(t, 1) \), respectively. For a thin path \( [\gamma] \), the identity at \( [\gamma] \) is the thin homotopy class of the bigon \( (t, s) \mapsto \gamma(t) \).

The various compositions in \( \mathcal{P}_2(X) \) are the usual ones of composing paths and homotopies by either stacking squares vertically or horizontally and parametrizing via double speed vertically or horizontally, respectively. More concretely, given two vertically composable thin bigons\(^5\)

\[
\begin{array}{c}
\downarrow [\delta'] \\
\downarrow [\Delta] \\
\downarrow [\epsilon]
\end{array}
\begin{array}{c}
\downarrow [\Gamma] \\
\downarrow [\gamma] \\
y
\end{array}
\begin{array}{c}
x
\end{array}
\]

(3.3.38)

the vertical composition is given by first choosing representatives \( \delta \) for the target of \( \Gamma \) and \( \delta' \) for the source of \( \Delta \). Then, there exists a thin (rank strictly

\(^5\)To be absolutely clear, we write square brackets to denote the thin homotopy equivalence classes. After this definition, we will generally \textit{not} do this, unless otherwise specified.
less than 2) homotopy Σ : δ ⇒ δ′. Using this thin homotopy, the vertical composition is the thin homotopy class associated to the bigon

\[ \Gamma(\Delta)(t, s) := \begin{cases} 
\Gamma(t, 3s) & \text{for } 0 \leq s \leq \frac{1}{3} \\
\Sigma(t, 3s - 1) & \text{for } \frac{1}{3} \leq s \leq \frac{2}{3}, \ t \in [0, 1]. \\
\Delta(t, 3s - 2) & \text{for } \frac{2}{3} \leq s \leq 1
\end{cases} \] (3.3.39)

Given two horizontally composable thin bigons

![Diagram](3.3.40)

the horizontal composition is given by the thin homotopy class associated to

\[ (\Gamma' \circ \Gamma)(t, s) := \begin{cases} 
\Gamma(2t, s) & \text{for } 0 \leq t \leq \frac{1}{2} \\
\Gamma'(2t - 1, s) & \text{for } \frac{1}{2} \leq t \leq 1, \ s \in [0, 1].
\end{cases} \] (3.3.41)

All such compositions are well-defined, smooth, associative, have left and right units given by constant bigons for horizontal composition and paths viewed as bigons for vertical composition respectively, and satisfy the interchange law. \( P_2(X) \) is a Lie 2-groupoid since thin homotopy classes of bigons are invertible in both ways and the functions that assign every class to its vertical and horizontal inverses are both smooth.

**Remark 3.3.42.** In the definition of vertical composition (3.3.39), we can always choose representatives of Γ and Δ so that δ = δ′ and we can ignore Σ for all practical purposes of this paper. Therefore, we will always write the
vertical composition as
\[
\Gamma \colon (t, s) := \begin{cases} 
\Gamma(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\
\Delta(t, 2s - 1) & \text{for } \frac{1}{2} \leq s \leq 1,
\end{cases}, \quad t \in [0, 1].
\tag{3.3.43}
\]

**Definition 3.3.44.** Let \( Gr \) be a Lie 2-groupoid, \( T \) be a 2-category, \( i : Gr \to T \) a 2-functor, and \( M \) a smooth manifold. Fix a surjective submersion \( \pi : Y \to M \). A \( \pi \)-local \( i \)-trivialization of a 2-functor \( F : \mathcal{P}_2(M) \to T \) is a pair \((\text{triv}, t)\) of a strict 2-functor \( \text{triv} : \mathcal{P}_2(Y) \to Gr \) and a pseudonatural equivalence

\[
\begin{array}{ccc}
\mathcal{P}_2(M) & \xrightarrow{\pi^*} & \mathcal{P}_2(Y) \\
F & \downarrow & \downarrow \text{triv} \\
T & \xleftarrow{i} & Gr
\end{array}
\tag{3.3.45}
\]

meaning that there exist a weak inverse \(i_t\) along with modifications (see Definition A.8. in [ScWa] or Definition A.125 in Appendix A of this thesis) \( i_t : \frac{t}{\pi} \Rightarrow \text{id}_{\pi^*F} \) and \( j_t : \text{id}_{\text{triv}} \Rightarrow \frac{i}{t} \) satisfying the zig-zag identities (see Definition 7. of [BaLa04] and particularly their discussion on string diagrams).

The 2-groupoid \( Gr \) is called the **structure 2-groupoid** for \( F \).

2-functors \( F : \mathcal{P}_2(M) \to T \) equipped with \( \pi \)-local \( i \)-trivializations \((\text{triv}, t)\) form the objects, written as triples \((F, \text{triv}, t)\), of a 2-category denoted by \( \text{Triv}^2_{\pi}(i) \).

**Definition 3.3.46.** A **1-morphism of \( \pi \)-local \( i \)-trivializations**

\[
\alpha : (F, \text{triv}, t) \to (F', \text{triv}', t')
\tag{3.3.47}
\]
in $\text{Triv}_i^2$ is a pseudo-natural transformation $\alpha : F \Rightarrow F'$. A 2-morphism $\alpha \Rightarrow \alpha'$ is a modification.

**Definition 3.3.48.** Let $\text{Gr}$ be a Lie 2-groupoid, $T$ a 2-category, $i : \text{Gr} \to T$ a 2-functor and $\pi : Y \to M$ a surjective submersion. A descent object is a quadruple $(\text{triv}, g, \psi, f)$ consisting of a strict 2-functor $\text{triv} : \mathcal{P}_2(Y) \to \text{Gr}$, a pseudonatural equivalence

$$
\begin{align*}
\mathcal{P}_2(Y) & \xrightarrow{\pi_1^*} \mathcal{P}_2(Y^{[2]}) \\
\text{triv}_i & \downarrow g \quad \downarrow \pi_2^* \\
T & \xrightarrow{\text{triv}} \mathcal{P}_2(Y)
\end{align*}
$$

and invertible modifications

$$
\begin{align*}
f & : \pi_{12}^*g \Rightarrow \pi_{13}^*g \\
& \pi_{23}^*g
\end{align*}
$$

and

$$
\psi : \text{id}_{\text{triv}_i} \Rightarrow \Delta^*g.
$$

These modifications must satisfy the coherence conditions which are explicitly given in Definition 2.2.1. of [ScWa] (in the examples of this current paper, the above modifications will actually be trivial and the coherence conditions will automatically be satisfied, which is why we leave them out).

Descent objects form the objects of a 2-category denoted by $\mathcal{D} \text{es}_i^2$. Morphisms and 2-morphisms are defined as follows.
Definition 3.3.52. A **descent 1-morphism** from \((\text{triv}, g, \psi, f)\) to \((\text{triv}', g', \psi', f')\) is a pair \((h, \epsilon)\) with \(h\) a pseudo-natural transformation \(h : \text{triv}_i \Rightarrow \text{triv}'_i\) and \(\epsilon\) an invertible modification

\[
\epsilon : \frac{g}{\pi^*_2 h} \Rightarrow \frac{\pi^*_1 h}{g'}.
\]  

(3.3.53)

These must satisfy certain identities explained in Definition 2.2.2. of [ScWa].

Definition 3.3.54. Let \((h, \epsilon)\) and \((h', \epsilon')\) be two descent 1-morphisms from \((\text{triv}, g, \psi, f)\) to \((\text{triv}', g', \psi', f')\). A **descent 2-morphism** from \((h, \epsilon)\) to \((h', \epsilon')\) is a modification \(E : h \Rightarrow h'\) satisfying a certain identity explained in Definition 2.2.3. of [ScWa].

There is a 2-functor \(\text{Ex}^2_\pi : \text{Triv}^2_\pi(i) \longrightarrow \text{Des}^2_\pi(i)\) that extracts descent data from trivialization data. At the level of objects, this functor is defined as follows. Let \((F, \text{triv}, t)\) be an object in \(\text{Triv}^2_\pi(i)\). For the quadruple \((\text{triv}, g, \psi, f)\), take \(\text{triv}\) to be exactly the same. For \(g\) take the composition \(g := \frac{\pi^*_2 t}{\pi^*_2 t}\) coming from the composition in the diagram

\[
P_2(Y) \xleftarrow{\pi^*_1} P_2(Y^{[2]})
\]

\[
\text{Gr} \xleftarrow{\pi_*} P_2(M) \xrightarrow{\pi_*} P_2(Y)
\]

\[
\text{Gr} \xrightarrow{\text{triv}} P_2(Y)
\]

\[
\text{Gr} \xleftarrow{\text{triv}} P_2(Y^{[2]})
\]

(3.3.55)
just as before but this time $\tilde{t}$ is a weak (vertical) inverse to $t$. By definition $f$ should be a modification $f : \pi^*_{12}g \Rightarrow \pi^*_{13}g$. Using our definition of $g$, this means that we can decompose it as follows

$$f : \pi^*_{12}g = \pi^*_{12} \circ \pi^*_{23} \Rightarrow \pi^*_{13}g,$$  \hspace{1cm} (3.3.56)

where all equalities hold by commutativity of certain diagrams and the left-over $\Rightarrow$ is specified by the following sequence of modifications

\[
\begin{align*}
\begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t} \\
\pi^*_{3t}
\end{pmatrix}
& \xrightarrow{\text{associators}}
\begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t} \\
\pi^*_{3t}
\end{pmatrix}
\xrightarrow{\text{id} \circ \pi^*_{1t}}
\begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t} \\
\pi^*_{3t}
\end{pmatrix}
\xrightarrow{\text{id} \circ \pi^*_{2t}}
\begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t} \\
\pi^*_{3t}
\end{pmatrix}
\xrightarrow{\text{id} \circ \pi^*_{3t}}
\begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t} \\
\pi^*_{3t}
\end{pmatrix},
\end{align*}
\]

where $i_t$ is part of the pseudo-natural equivalence from $t$ and $\tilde{t}$, and $l$ is a left unifier.

Finally, by definition $\psi$ should be a modification $\psi : \text{id}_{\text{triv}} \Rightarrow \Delta^*g$. Using our definition of $g$, we can decompose it as follows

$$\psi : \text{id}_{\text{triv}} = \Delta^* \pi^* \text{id}_{\text{triv}} \Rightarrow \Delta^* \left( \begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t}
\end{pmatrix} \right) = \Delta^* g$$  \hspace{1cm} (3.3.58)

and such a modification can be achieved by

$$\Delta^* \pi^* \text{id}_{\text{triv}} \xrightarrow{\Delta^* \pi^* \Delta^*} \Delta^* \pi^* \left( \begin{pmatrix}
\pi^*_{1t} \\
\pi^*_{2t}
\end{pmatrix} \right),$$  \hspace{1cm} (3.3.59)

where $j_t$ is the other part of the pseudo-natural equivalence from $t$ and $\tilde{t}$.

This indeed defines a descent object and that this assignment of descent
data to trivialization data extends to a 2-functor $\text{Ex}^2_\pi : \text{Triv}^2_\pi(i) \to \text{Des}^2_\pi(i)$ to include 1-morphisms and 2-morphisms (see Lemma 2.3.1., Lemma 2.3.2., and Lemma 2.3.3. of [ScWa]).

**Definition 3.3.60.** Let $(F, \text{triv}, t)$ be a $\pi$-local $i$-trivialization of a 2-functor $F : \mathcal{P}_2(M) \to T$, i.e. an object of $\text{Triv}^2_\pi(i)$. The *descent object associated to the $\pi$-local $i$-trivialization* is $\text{Ex}^2_\pi(F, \text{triv}, t)$. A similar definition is made for 1- and 2-morphisms.

### 3.3.4 Transport 2-functors

We now wish to discuss smoothness for descent data. However, to do this is not so simple as it was for ordinary functors. We will have to make a detour to describe how to think of natural transformations as functors and modifications as natural transformations by altering the source and target categories.

For the purposes of this document, we will make stricter assumptions than those in [ScWa13] that are sufficient for our purposes and simplify several of the arguments and constructions.

Let $\mathcal{C}$ and $\mathcal{D}$ be two strict 2-categories. Let $\mathcal{C}_{0,1}$ denote the category whose objects and morphisms are the objects and 1-morphisms of $\mathcal{C}$ respectively. Because $\mathcal{C}$ is strict, this defines a category. Let $\Lambda \mathcal{D}$ be the category whose objects are morphisms $X_f \xrightarrow{f} Y_f$ of $\mathcal{D}$. The set of morphisms in $\Lambda \mathcal{D}$ from...
$X_f \xrightarrow{f} Y_f$ to $X_g \xrightarrow{g} Y_g$ are pairs of morphisms $(x : X_f \xrightarrow{f} X_g, y : Y_f \xrightarrow{g} Y_g)$ along with a 2-morphism $\varphi : g \circ x \Rightarrow y \circ f$ as in the diagram

\[
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}
\xrightarrow{\varphi} 
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}
\xrightarrow{f} 
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}.
\tag{3.3.61}
\]

The composition is given by stacking

\[
\begin{array}{c}
X_h \xleftarrow{x'} X_g \xleftarrow{x} X_f \\
\downarrow h \\
Y_h \xleftarrow{y'} Y_g \xleftarrow{y} Y_f \\
\end{array}
\xrightarrow{\varphi} 
\begin{array}{c}
X_h \xleftarrow{x' \circ x} X_f \\
\downarrow h \\
Y_h \xleftarrow{y' \circ y} Y_f \\
\end{array}.
\tag{3.3.62}
\]

One can check that under our assumptions, this forms a category.

Notice that $\Lambda D$ has a bit more structure. It also has a partially defined operation on objects and 1-morphisms given by “stacking vertically.” Suppose that $X_f \xrightarrow{f} Y_f$ and $Y_f \xrightarrow{f'} Z_f$ are two 1-morphisms in $\mathcal{D}$ then one can compose them and this gives a partially defined associative and unital operation on objects of $\Lambda D$. Similarly, given morphisms in $\Lambda D$ which can be vertically stacked as in the diagram

\[
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}
\xrightarrow{\varphi} 
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}\xrightarrow{f'} 
\begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array} = \begin{array}{c}
X_g \xleftarrow{x} X_f \\
\downarrow g \\
Y_g \xleftarrow{y} Y_f \\
\end{array}\xrightarrow{f' \circ f}.
\tag{3.3.63}
\]
This additional partially defined composition is written as $\otimes$ in [ScWa13] so we stick with this notation.

Associated to a pseudo-natural transformation $\rho$ as in

$$D \xrightarrow{\rho} C$$

is a functor $\mathcal{F}(\rho) : \mathcal{C}_{0,1} \to \Lambda D$ defined by

$$F X \xrightarrow{\mathcal{F}(\rho)} G X$$

on objects $X$ in $\mathcal{C}_{0,1}$, i.e. objects in $\mathcal{C}$, and

$$F Y \xleftarrow{\mathcal{F}(\rho)} F X$$

$$G Y \xleftarrow{G f} G X$$

on morphisms in $\mathcal{C}_{0,1}$, i.e. 1-morphisms in $\mathcal{C}$. One can check this defines a functor.

Associated to a modification $A$ as in

$$D \xrightarrow{\rho} C$$

is a natural transformation $\mathcal{F}(A) : \mathcal{F}(\rho) \Rightarrow \mathcal{F}(\sigma)$ defined by

$$X \xrightarrow{\mathcal{F}(A)} F X \xrightarrow{\sigma(X)} F X$$

$$G X \xrightarrow{id_{G X}} G X$$
This defines a functor $\mathcal{F} : \text{Hom}(F, G) \longrightarrow \text{Funct}(\mathcal{C}_0, \Lambda\mathcal{D})$, where $\text{Hom}(F, G)$ is the category whose objects are pseudonatural transformations and morphisms are modifications while $\text{Funct}(\mathcal{E}, \mathcal{E}')$ (between two ordinary categories $\mathcal{E}$ and $\mathcal{E}'$) is the category whose objects are functors from $\mathcal{E}$ to $\mathcal{E}'$ and whose morphisms are natural transformations.

Separately, notice also that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a 2-functor then there is a functor $\Lambda F : \Lambda\mathcal{C} \longrightarrow \Lambda\mathcal{D}$ defined by

$$
\begin{array}{ccc}
X_f & \xrightarrow{\Lambda F} & FX_f \\
\downarrow f & & \downarrow Ff \\
Y_f & \xleftarrow{\Lambda F} & FY_f
\end{array}
$$

on objects and

$$
\begin{array}{ccc}
X_g & \xleftarrow{x} & X_f \\
g & \downarrow f & \leftarrow \Lambda F \\
Y_g & \xleftarrow{\sigma} & Y_f \\
\downarrow y & & \downarrow Fg \\
FY_g & \xleftarrow{FY} & FY_f \\
\end{array}
$$

on morphisms.

**Definition 3.3.71.** A descent object $(\text{triv}, g, \psi, f)$ as in Definition 3.3.48 is said to be **smooth** if

i) the 2-functor $\text{triv} : \mathcal{P}_2(Y) \longrightarrow \text{Gr}$ is smooth,

ii) the functor $F(g) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda T$ is a transport functor with $\Lambda\text{Gr}$-structure, and
iii) the natural transformations $\mathcal{F}(\psi) : \mathcal{F}(\text{id}_{\text{triv}}) \Rightarrow \Delta^* \mathcal{F}(g)$ and $\mathcal{F}(f) : \pi_{23}^* \mathcal{F}(g) \otimes \pi_{12}^* \mathcal{F}(g) \Rightarrow \pi_{13}^* \mathcal{F}(g)$ are morphisms between transport functors.

Smooth descent objects form the objects of a 2-category denoted by $\text{Des}^2_{\pi}(i)^\infty$ and form a sub-2-category of $\text{Des}^2_{\pi}(i)$. Smoothness of descent 1-morphisms and descent 2-morphisms is discussed in [ScWa13] following Definition 3.1.2.

**Definition 3.3.72.** A $\pi$-local $i$-trivialization $(F, \text{triv}, t)$ is said to be **smooth** if the associated descent object $\text{Ex}^2_{\pi}(F, \text{triv}, t)$ is smooth. The same can be said of 1-morphisms and 2-morphisms.

Smooth local trivializations, 1-morphisms, and 2-morphisms form a sub-2-category denoted by $\text{Triv}^2_{\pi}(i)^\infty$ of $\text{Triv}^2_{\pi}(i)$. Furthermore, $\text{Ex}^2_{\pi}$ restricts to an equivalence of 2-categories of smooth data (Lemma 3.2.3. of [ScWa13]).

After all this formalism, it should be more or less clear now what the definition of a transport 2-functor is by just abstracting what we did for the one-dimensional case (Definition 3.2.1. of [ScWa13]).

**Definition 3.3.73.** Let $\text{Gr}$ be a Lie 2-groupoid, $T$ a 2-category, $i : \text{Gr} \rightarrow T$ a 2-functor, and $M$ a smooth manifold. A **transport 2-functor on $M$ with values in a 2-category $T$ and with $\text{Gr}$-structure** is a 2-functor $\text{tra} : \mathcal{P}_2(M) \rightarrow T$ such
that there exists a surjective submersion \( \pi : Y \to M \) and a smooth \( \pi \)-local \( i \)-trivialization \((\text{triv}, t)\).

Transport 2-functors over \( M \) with values in \( T \) with Gr-structure form the objects of a 2-category \( \text{Trans}^2_{\text{Gr}}(M, T) \). A 1-morphism of transport functors is a pseudo-natural transformation of 2-functors for which there exists a common surjective submersion \( \pi \) and smooth \( \pi \)-local \( i \)-trivializations of both 2-functors so that the associated descent 1-morphism is smooth. A similar definition exists for 2-morphisms.

As a short summary, in the past two sections we introduced three categories for describing transport 2-functors. These were \( \mathcal{D} \text{es}^2_\pi(i) \), \( \text{Triv}^2_\pi(i) \), and \( \text{Trans}^2_{\text{Gr}}(M, T) \). The category \( \text{Triv}^2_\pi(i) \) was used to describe local triviality of transport 2-functors and their morphisms in \( \text{Trans}^2_{\text{Gr}}(M, T) \). We then constructed a 2-functor \( \text{Ex}^2_\pi : \text{Triv}^2_\pi(i) \to \mathcal{D} \text{es}^2_\pi(i) \) that allowed us to describe smoothness via the subcategory \( \mathcal{D} \text{es}^2_\pi(i)^\infty \subset \mathcal{D} \text{es}^2_\pi(i) \) from which we defined \( \text{Triv}^2_\pi(i)^\infty \subset \text{Triv}^2_\pi(i) \).

### 3.3.5 The reconstruction 2-functor: from local to global

The 2-functor \( \text{Ex}^2_\pi : \text{Triv}^2_\pi(i) \to \mathcal{D} \text{es}^2_\pi(i) \) is an equivalence of 2-categories (Proposition 4.1.1. of [ScWa]). To construct a (weak) inverse

\[
\text{Rec}^2_\pi : \mathcal{D} \text{es}^2_\pi(i) \to \text{Triv}^2_\pi(i),
\] (3.3.74)
we need to enhance the Čech path groupoid so that it includes more data.

We do not require the full general definition of $\mathcal{P}_2^\pi(M)$ in Section 3.1 of [ScWa] for our purposes, but briefly the general definition is obtained by keeping the same objects and morphisms but replacing the relations that we imposed by 2-morphisms and setting relations on those. There are also additional 2-morphisms given by thin bigons, thin paths on intersections, and other formal 2-morphisms such as associators, unitors, and 2-morphisms relating the formal product to the usual composition of paths. We therefore warn the reader that although the following definition is not the same as that in [ScWa], we use their general results and theorems which in fact rely on their more general definition.

**Definition 3.3.75.** Let $M$ be a smooth manifold and let $\pi : Y \to M$ be a surjective submersion. The **Čech path 2-groupoid of $M$** is the 2-category $\mathcal{P}_2^\pi(M)$ whose set of objects and 1-morphisms are the objects and morphisms of $\mathcal{P}_1^\pi(M)$, respectively. The set of 2-morphisms are freely generated by

i) thin bigons $\Gamma$ in $Y$;

ii) thin paths $\Theta : \alpha \to \beta$ in $Y^{[2]}$ with sitting instants thought of as 2-
isomorphisms

\[
\begin{align*}
\pi_1(\beta) \xrightarrow{(\Theta)} & \pi_1(\alpha) \\
\beta \downarrow & \downarrow \Theta \\
\pi_2(\beta) \xrightarrow{(\Theta)} & \pi_2(\alpha)
\end{align*}
\] (3.3.76)

(one should think of this as weakening the first relation in Definition 3.2.46 of \(\mathcal{P}_1^\pi(M)\)—see Figure 3.10 for a visualization of this),

![Diagram of isomorphisms](image)

Figure 3.10: Thinking in terms of an open cover as a submersion, condition ii) above says that if a thin path \(\Theta : \alpha \to \beta\) (with chosen representative) is in a double intersection, there is a relationship between going along the path first and then jumping versus jumping first and then going along the path. The two need not be equal.

iii) points \(\Xi\) in \(Y^{[3]}\) thought of as 2-isomorphisms

\[
\begin{align*}
\pi_3(\Xi) & \xrightarrow{\pi_3(\Xi)} \Xi \\
\pi_2(\Xi) & \xrightarrow{\pi_2(\Xi)} \Xi \\
\pi_1(\Xi) & \xrightarrow{\pi_1(\Xi)} \Xi
\end{align*}
\] (3.3.77)

(one should think of this as weakening the second relation in Definition 3.2.46 of \(\mathcal{P}_1^\pi(M)\)),

\(\Xi\) thought of as 2-isomorphisms.
iv) points $a$ in $Y$ thought of as 2-isomorphisms ($\text{id}_a^*$ is the formal identity)

\[ \begin{tikzcd}
 a & a \\
 & \Delta(a)
\end{tikzcd} \]

(one should think of this as weakening part of the third relation in Definition 3.2.46 of $\mathcal{P}^r_1(M)$),

v) and several other more technical generators that will not be discussed here.

There are several relations imposed on the set of 1-morphisms and 2-morphisms. We will not discuss any of them, and the reader is referred to Section 3.1 of [ScWa] for the details. As before, the compositions will be written with $*$ and will be drawn vertically or horizontally when dealing with 2-morphisms.

As before, we associate to every object $(\text{triv}, g, \psi, f)$ in $\mathfrak{D}es^2_\pi(i)$ a functor $R_{(\text{triv}, g, \psi, f)} : \mathcal{P}^r_2(M) \rightarrow T$ defined as follows. It sends $y \in Y$ to $\text{triv}_i(y)$, thin paths $\gamma$ in $Y$ to $\text{triv}_i(\gamma)$, and jumps $\alpha \in Y^{[2]}$ to $g(\alpha) : \text{triv}_i(\pi_1(\alpha)) \rightarrow \text{triv}_i(\pi_2(\alpha))$.

For the basic 2-morphisms, it makes the following assignments

\[ \begin{tikzcd}
 y & x \\
 & \text{triv}_i(y)
\end{tikzcd} \]
for thin bigons $\Gamma : \gamma \Rightarrow \delta$ in $Y$,

\[
\begin{array}{c}
\pi_2(\alpha) \xrightarrow{\alpha} \pi_1(\alpha) \\
\pi_2(\Theta) \xleftarrow{\Theta} \pi_1(\Theta) \\
\pi_2(\beta) \xleftarrow{\beta} \pi_1(\beta)
\end{array}
\xrightarrow{R_{\text{triv}, g, \psi, f}}
\begin{array}{c}
\text{triv}_i(\pi_2(\alpha)) \xrightarrow{g_1(\alpha)} \text{triv}_i(\pi_1(\alpha)) \\
\text{triv}_i(\pi_2(\Theta)) \xrightarrow{g(\Theta)} \text{triv}_i(\pi_1(\Theta)) \\
\text{triv}_i(\pi_2(\beta)) \xrightarrow{g_1(\beta)} \text{triv}_i(\pi_1(\beta))
\end{array}
\quad (3.3.80)
\]

for thin paths $\Theta : \alpha \to \beta$ in $Y^2$,

\[
\begin{array}{c}
\pi_{23}(\Xi) \\
\pi_{2}(\Xi) \xrightarrow{\pi_{23}(\Xi)} \Xi \\
\pi_{3}(\Xi) \xleftarrow{\pi_{13}(\Xi)} \pi_{1}(\Xi)
\end{array}
\xrightarrow{R_{\text{triv}, g, \psi, f}}
\begin{array}{c}
\text{triv}_i(\pi_{23}(\Xi)) \\
\text{triv}_i(\pi_2(\Xi)) \xrightarrow{g(\pi_{23}(\Xi))} \text{triv}_i(\pi_{12}(\Xi)) \\
\text{triv}_i(\pi_{3}(\Xi)) \xrightarrow{f(\Xi)} \text{triv}_i(\pi_{1}(\Xi))
\end{array}
\quad (3.3.81)
\]

for points $\Xi$ in $Y^3$, and

\[
\begin{array}{c}
a \\
\Delta(a) \xleftarrow{\Delta(a)} a
\end{array}
\xrightarrow{R_{\text{triv}, g, \psi, f}}
\begin{array}{c}
\text{triv}_i(a) \\
\text{triv}_i(a) \xrightarrow{\psi(a)} \text{triv}_i(a) \\
\text{triv}_i(a) \xrightarrow{g(\Delta(a))} \text{triv}_i(a)
\end{array}
\quad (3.3.82)
\]

for points $a$ in $Y$. This defines a 2-functor $R : \text{Des}^2_{\pi}(i) \to \text{Funct}(\mathcal{P}_2^\pi(M), T)$ at the level of objects. The rest of this 2-functor is defined in Proposition 3.3.2. of [ScWa].

There is a canonical projection functor $p^\pi : \mathcal{P}_2^\pi(M) \to \mathcal{P}_2(M)$ defined in the same way as $p_\pi : \mathcal{P}_1^\pi(M) \to \mathcal{P}_1(M)$ on the level of objects and morphisms. On the level of 2-morphisms, $p^\pi$ sends a thin bigon $\Gamma$ in $Y$ to a thin bigon $\pi(\Gamma)$ in $M$. It sends a thin path $\Theta$ in $Y^2$ to the identity thin bigon $\text{id}_{\pi(\Theta)}$ (the vertical identity) in $M$ and it sends a point $\Xi$ in $Y^3$ to the constant thin bigon at the point $\pi(\Xi)$ in $M$. Finally, it sends a point $a$ in $Y$ to the
constant thin bigon at the point $\pi(a)$ in $M$. We now move on to defining, as before, a weak inverse $s^\pi : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ of the canonical projection functor. To define $s^\pi$, we will constantly use the following important fact (Lemma 3.2.2. of [ScWa]).

**Lemma 3.3.83.** Let $\gamma : x \to x'$ be a thin path in $M$ and let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be two lifts of $\gamma$ as 1-morphisms in $\mathcal{P}_2^\pi(M)$ (the existence follows from our choices above when we defined $s^\pi : \mathcal{P}_1(M) \to \mathcal{P}_1^\pi(M)$). Then there exists a unique 2-isomorphism $A : \tilde{\gamma} \Rightarrow \tilde{\gamma}'$ in $\mathcal{P}_2^\pi(M)$ such that $p^\pi(A) = \text{id}_\gamma$.

We will use this to define $s^\pi : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ on thin bigons (we have already defined $s^\pi$ near (3.2.53) on objects and 1-morphisms). Let $\Gamma : \gamma \Rightarrow \delta$ be any thin bigon in $M$ as in Figure 3.11.

![Figure 3.11: A representative of a thin bigon $\Gamma$ in $M$ drawn as a map of a square into $M$. The $s = 0$ line is drawn on top in the figure on the right while the $s = 1$ line is drawn on the bottom. The entire $t = 0$ line gets mapped to the source point and the $t = 1$ line gets mapped to the target point.](image)

As in the case of a path, because the domain is compact, there exists
a decomposition of the bigon $\Gamma$ (we abuse notation and write $\Gamma$ to mean a bigon and its thin homotopy class relying on context to distinguish them) into smaller bigons $\{\Gamma_j\}_j$, as in Figure 3.12, each of which fits into an open set $U_j$. We use the same notation $s_j : U_j \rightarrow Y$ as before for our local sections.

Figure 3.12: A decomposition of a representative of a thin bigon $\Gamma$ in $M$ with a single sub-bigon $\Gamma_j$ highlighted. $s^\pi(\Gamma)$ will be defined as a composition of several $s^\pi(\Gamma_j)$. Of course, a general decomposition would not necessarily look like this, but such a decomposition always exists by a thin homotopy so that the decomposed pieces are bigons.

Therefore, it suffices to define $s^\pi(\Gamma_j)$ for a single one of the associated thin bigons provided that we match up all sources and targets for the individual ones. Denote the thin bigon by

$$
\begin{align*}
\gamma_j & \quad \delta_j & \quad x_j' & \quad \Gamma_j & \quad x_j.
\end{align*}
$$

(3.3.84)
Then the image of this under \( s^\pi \) is defined as the composition

\[
\begin{align*}
    \xymatrix{
        s^\pi(x_j') & s_j(x_j') \ar[r] & s_j(x_j) \ar[r] & s^\pi(x_j) \ar[r] & s^\pi(b_j) \ar[l] & s_j(b_j) \ar[l] & s^\pi(\gamma_j) \ar[l] & s_j(\gamma_j) \ar[l] & s^\pi(\gamma_j) \ar[l] & s_j(\gamma_j) \ar[l] & s^\pi(\gamma_j) \ar[l]
    }.
\end{align*}
\]

(3.3.85)

In other words, we have lifted \( \Gamma_j \) using the section \( s_j : U_j \to Y \), but to make sure that this image matches up with how \( s^\pi \) was already defined on objects and 1-morphisms, we use the obvious jumps and the unique 2-isomorphisms from Lemma 3.3.83 to match everything (these are the unla-
beled 1-morphisms and 2-morphisms). The image of the entire thin bigon \( \Gamma \) is then defined by vertical and horizontal compositions of all the \( s^\pi(\Gamma_j) \) so that \( s^\pi \) respects compositions.

The 2-functor \( s^\pi \) is a weak inverse to \( p^\pi \) as in the case for the path groupoid (Proposition 3.2.1. of [ScWa]). However, a weak inverse in 2-
category theory in this case means (see Definition A.127 of Appendix A) that there exists a pseudo-natural equivalence \( \zeta : s^\pi \circ p^\pi \Rightarrow \text{id}_{P_M^2} \) since \( p^\pi \circ s^\pi = \text{id}_{P_M^2} \). This means there exists a weak inverse to \( \zeta \) which is written as \( \xi : \text{id}_{P_M^2} \Rightarrow s^\pi \circ p^\pi \). “Weak” means that there are invertible modifications \( i_\xi : \xi \circ \zeta \Rightarrow \text{id}_{s^\pi \circ p^\pi} \) and \( j_\xi : \text{id}_{p^\pi \circ s^\pi} \Rightarrow \zeta \circ \xi \) that satisfy
the zig-zag identities. The details are irrelevant for our purposes but can be found in Section 3.2 of [ScWa]. An important consequence of \( s_\pi \) being a weak inverse to \( p_\pi \) is the following (general categorical) fact reproduced here for convenience (Corollary 3.2.5. of [ScWa] and Lemma A.128 in Appendix A).

**Corollary 3.3.86.** Any two weak inverses \( s_\pi, s_1\pi : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M) \) of \( p_\pi \) are pseudo-naturally equivalent.

We can define such a pseudo-natural equivalence \( \eta : s_\pi \Rightarrow s_1\pi \) by the following assignment \( M \ni x \mapsto \) the jump from \( s_\pi(x) \) to \( s_1\pi(x) \) and \( P^1M \ni \gamma \mapsto \) the unique 2-isomorphism \( \gamma \Rightarrow s_1\pi(\gamma) \) specified by Lemma 3.3.83. We will exploit this fact when discussing examples of higher holonomy in Section 3.5.

As before, the 2-functor \( s_\pi : \mathcal{P}_2(M) \longrightarrow \mathcal{P}_2^\pi(M) \) induces a 2-functor \( s_\pi^* : \text{Funct}(\mathcal{P}_2^\pi(M), T) \rightarrow \text{Funct}(\mathcal{P}_2(M), T) \), the pullback along \( s_\pi \). Similarly, \( \text{Rec}^2_\pi \) is defined as the composition in the diagram

\[
\begin{array}{ccc}
\text{Funct}(\mathcal{P}_2(M), T) & \xrightarrow{\text{Rec}^2_\pi} & \mathcal{D}\text{es}_\pi^2(i) \\
\downarrow{s_\pi^*} & & \downarrow{\zeta} \\
\text{Funct}(\mathcal{P}_2^\pi(M), T) & & \\
\end{array}
\]  

(3.3.87)

As before, the image of \( \mathcal{D}\text{es}_\pi^2(i) \) under \( \text{Rec}^2_\pi \) lands in \( \text{Triv}_\pi^2(i) \) and the definition is the same as it was before, only this time \( \zeta \) is a pseudo-natural equivalence between 2-functors between 2-categories.
As a short summary, in this section we introduced a weak inverse functor \( \text{Rec}_T^2 : \text{Des}_T^2(i) \to \text{Triv}_T^2(i) \) for \( \text{Ex}_T^2 : \text{Triv}_T^2(i) \to \text{Des}_T^2(i) \) by using the 2-groupoid \( \mathcal{P}_T^\pi(M) \) associated to the surjective submersion \( \pi : Y \to M \) to lift points, thin paths, and thin bigons in \( M \) to points, thin paths and/or jumps, and thin bigons and/or jumps in \( \mathcal{P}_T^\pi(M) \), respectively.

### 3.3.6 Differential cocycle data

In this section, we will give a brief review of an equivalence between differential forms and smooth 2-functors following Section 2 of [ScWa11]. This will allow us to describe parallel transport locally in terms of differential cocycle data. We will leave out several proofs but will provide pictures that we find illustrate the necessary ideas behind the statements. We first remind the reader of the “Lie algebra” of a Lie crossed module.

Given a Lie crossed module \( (H, G, \tau, \alpha) \) (recall Definition 3.3.1) there is an associated \textit{differential Lie crossed module} \( (H, G, \tau, \alpha) \), where \( \tau : H \to G \) is the differential of \( \tau : H \to G \), \( \alpha : G \to \text{Der}(H) \) is the differential of the associated action (given the same name) \( \alpha : G \times H \to H \) (“Der” stands for derivations). The differential Lie crossed module data satisfy

\[
\alpha_{\tau}(B')(B) = [B', B] \tag{3.3.88}
\]
and
\[ \tau(\alpha_A(B)) = [A, \tau(B)] \] 
(3.3.89)
for all \( A \in G \) and \( B, B' \in H \).

Note that by restricting the action \( \alpha \) to \( \{g\} \times H \) for any \( g \in G \) and differentiating with respect to the second coordinate, we obtain a Lie algebra homomorphism \( \alpha_g : H \to H \). Both \( \alpha \) and \( \alpha_g \) are important for understanding the differential cocycle data of Section 3.3.6. A more thorough review can be found in [BaHu11] and the Appendix of Chapter 2 of this thesis.

**From 2-functors to 2-forms**

Let \( B\mathfrak{G} \) be a Lie 2-group and \( (H, G, \tau, \alpha) \) its corresponding crossed module. Given a strict smooth 2-functor \( F : \mathcal{P}_2(X) \to B\mathfrak{G} \), we will obtain differential forms \( A \in \Omega^1(X; G) \) and \( B \in \Omega^2(X; H) \). These will form the objects of a 2-category \( Z^2_X(\mathfrak{G}) \). By our previous discussion and since our 2-categories \( \mathcal{P}_2(X) \) and \( B\mathfrak{G} \) are strict and the 2-functor \( F \) is strict, the restriction of \( F \) to \( \mathcal{P}_1(X) \) is smooth. Therefore, we obtain a differential form \( A \in \Omega^1(X; G) \) by the results of Section 3.2.6. To obtain the differential form \( B \in \Omega^2(X; H) \) we will “differentiate” the composition

\[ H \xrightarrow{\Phi} H \times G \xrightarrow{F_2} P^2 X, \] 
(3.3.90)
where $p_H$ is the projection onto the $H$ factor and $F_2$ is $F$ restricted to 2-morphisms.

Infinitesimally, a bigon is determined by a point and the two tangent vectors that begin to span it. Therefore, let $x \in X$ and $v_1, v_2 \in T_x X$ and let $\Gamma : \mathbb{R}^2 \to X$ be a smooth map such that

$$
\Gamma((0, 0)) = x, \quad \left. \frac{\partial}{\partial s} \right|_{s=0} \Gamma(s, t = 0) = v_1, \quad \text{and} \quad \left. \frac{\partial}{\partial t} \right|_{t=0} \Gamma(s = 0, t) = v_2.
$$

(3.3.91)

Let $\Sigma : \mathbb{R}^2 \to P^2 \mathbb{R}^2$ be the (smooth) map that sends $(s, t)$ to the thin homotopy class of the bigon in Figure 3.13. This is unambiguously defined after modding out by thin homotopy because a thin bigon in $\mathbb{R}^2$ is determined by its source and target thin paths in $\mathbb{R}^2$.

![Figure 3.13: A point $(s, t)$ in $\mathbb{R}^2$ gets mapped to the bigon in $\mathbb{R}^2$ shown on the right under the map $\Sigma$.](image)

Then we use this to define a smooth map $F_\Gamma$ by the composition of smooth
maps

\[
H \xrightarrow{\rho} H \times G \xrightarrow{F_2} P^2 X \xrightarrow{\Gamma^*} P^2 \mathbb{R}^2 \xrightarrow{\Sigma_2} \mathbb{R}^2.
\] (3.3.92)

This gives an element of the Lie algebra $H$ by taking derivatives

\[
B_x(v_1, v_2) := -\frac{\partial^2 F_{\Gamma}}{\partial s \partial t} \bigg|_{(0,0)} \in H.
\] (3.3.93)

Furthermore, this element is independent of the choice of $\Gamma$ provided that equation (3.3.91) still holds. In fact, we get a smooth differential form $B \in \Omega^2(X; H)$.

Now let $\Gamma : \gamma \Rightarrow \delta$ be a thin bigon between two thin paths. The \textit{source-target matching condition}, which says $\tau(p_H(F(\Gamma)))F(\gamma) = F(\delta)$, implies

\[
dA + \frac{1}{2}[A, A] = \tau(B).
\] (3.3.94)

All of these claims are proved in Section 2.2.1 of [ScWa11]. These results are what allowed us to assume the infinitesimal forms in (2.3.31)–(2.3.33) in Section 2.3.2. We have also supplied an argument for the vanishing of the fake curvature preceding (2.3.40).

\textbf{From 2-forms to 2-functors}

Starting with a $G$-valued 1-form $A \in \Omega^1(X; G)$ on $X$ and a $H$-valued 2-form $B \in \Omega^2(X; H)$ on $X$ we want to define a smooth functor $\mathcal{P}_2(X) \rightarrow \mathcal{B}\mathcal{G}$. From Section 3.2.6, we have already defined the functor at the level of objects and
thin paths. What remains is to define $F_2 : P^2 X \longrightarrow H \times G$. To do this, we will define a function $k_{A,B} : BX \rightarrow H$ on bigons in $X$ (we do not mod out by thin homotopy). Given a bigon $\Sigma : [0,1] \times [0,1] \rightarrow X$, we can pull back the 1-form $A$ and the 2-form $B$ to $[0,1] \times [0,1]$, obtaining $\Sigma^*(A) \in \Omega^1([0,1] \times [0,1]; G)$ and $\Sigma^*(B) \in \Omega^2([0,1] \times [0,1]; H)$.

To define $k_{A,B}$, we first introduce an $H$-valued 1-form $A_{\Sigma} \in \Omega^1([0,1]; H)$ obtained by integrating over one of the directions for the bigon. It is defined by

$$
(A_{\Sigma})_s \left( \frac{d}{ds} \right) := - \int_0^1 dt \frac{\alpha_{F_1(\Sigma^*_s \gamma_{s,t})^{-1}}}{\Sigma^*_s B}_{(s,t)} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \quad (3.3.95)
$$

where $\gamma_{s,t}$ is defined to be the straight vertical path from $(s,0)$ to $(s,t)$ in $[0,1] \times [0,1]$ as in Figure 3.14. Note that in expression $(3.3.95)$, it is assumed that $\Sigma^*_s \gamma_{s,t}$ refers to the thin homotopy class of the path (otherwise, applying the function $F_1$ would not make sense). Therefore, the parametrization of $\gamma_{s,t}$ is irrelevant.

Besides the path-ordered integral expression from the term $F_1(\Sigma^*_s(\gamma_{s,t}))$, the expression for $A_{\Sigma}$ is an ordinary integral. Also note that $A_{\Sigma}$ depends on $\Sigma$. In particular, it is not invariant under thin homotopy.

**Remark 3.3.96.** Incidentally, although Schreiber and Waldorf in [ScWa11] made their own arguments for how to obtain such a formula for $A_{\Sigma}$, this
formula appears in a special case as early as 1977 in the work of Goddard, Nuyts, and Olive on magnetic monopoles [GoNuOl77] on the right-hand side of equation (2.9) and it may have been known earlier [Ch75]. The special case considered is the case of the crossed module $(G, G, \text{id}, \alpha)$ with $\alpha$ being the ordinary conjugation action.

Finally, to every bigon $\Sigma : \gamma \Rightarrow \delta$, we define

$$k_{A,B}(\Sigma) := \alpha_{F_1(\gamma)} \left( \mathcal{P} \exp \left\{ - \int_0^1 A_{\Sigma} \right\} \right).$$  \hspace{1cm} (3.3.97)

In Figure 3.15, this integral is schematically drawn as a power series of graphs with marked points and paths analogous to Figure 3.5. Each of the paths drawn has a path-ordered integral expression attached to it, and therefore each expression has an additional power series of the form we discussed for the ordinary path-ordered integral.
Figure 3.15: The path-ordered integral \( \mathcal{P} \exp \left\{ -\int_0^1 A_\Sigma \right\} \) is depicted schematically as an infinite sum of terms expressed by placing \( B \) at the endpoints of the paths, along which we’ve computed parallel transport using \( A \) making sure to keep the later \( s \)-valued terms on the right. The picture is to be interpreted similarly to the one-dimensional case once we’ve integrated along the \( t \) direction (vertical) to obtain \( A_\Sigma \).

**Definition 3.3.98.** The group element \( k_{A,B}(\Sigma) \) is called the **surface transport** associated to the bigon \( \Sigma \) and the differential forms \( A \) and \( B \).

\[ k_{A,B} \text{ only depends on the thin homotopy class of } \Sigma \text{ and therefore factors through a smooth map } F_2 : P^2X \longrightarrow H \text{ on thin homotopy classes of paths.} \]

This map together with \( F_1 \) define a strict smooth 2-functor \( F : \mathcal{P}_2(X) \rightarrow \mathcal{B} \mathcal{G} \) (Proposition 2.17. of [ScWa11]).

**Remark 3.3.99.** Historically, understanding the appropriate generalization of the path-ordered integral to surfaces was a difficult task. It was not obvious which formulas were correct or even what the criteria for correctness should be. The language of functors allows one to make this precise. The criteria for correctness is that surface holonomy should be expressed in terms of a trans-
port 2-functor. Any formula that satisfies these functorial properties, has the local constraint given by equation (3.3.94), and changes appropriately under gauge transformations (which we have so far only discussed locally but will discuss differentially soon), can be rightfully called surface transport. The specific formula in equation (3.3.97) is only one such formula that works. However, there could be many other, potentially simpler formulas, that also describe 2-holonomy. In Section 3.4 for instance, we prove that for certain structure 2-groups, the formula (3.3.97) agrees with one that is easily computable in terms of homotopy classes of paths. A more direct derivation of (3.3.97) from infinitesimal data was given in Theorem 2.3.79 in Section 2.3.2.

Local differential cocycles for transport 2-functors

By similar considerations to the previous sections, we can differentiate transport functors and use their properties to obtain relations among all the differential data. All the information in this section is discussed in more detail in [ScWa13]. In particular, the functions, differential forms, and their relations are all derived. We merely reproduce the results here for use in later calculations.

**Definition 3.3.100.** Let $Z^2_X(\mathfrak{g})^\infty$ be the category defined as follows. An object of $Z^2_X(\mathfrak{g})^\infty$ is a pair $(A, B)$ of a 1-form $A \in \Omega^1(X; G)$ and a 2-form
$B \in \Omega^2(X; H)$ satisfying

$$\tau(B) = dA + \frac{1}{2}[A, A].$$

(3.3.101)

A 1-morphism from $(A, B)$ to $(A', B')$ is a pair $(h, \varphi)$ of a smooth map $h : X \to G$ and a 1-form $\varphi \in \Omega^1(X; H)$ satisfying

$$A' + \tau(\varphi) = \text{Ad}_h(A) - h^*\theta$$

(3.3.102)

and

$$B' + \alpha_{A'}(\varphi) + d\varphi + \frac{1}{2}[\varphi, \varphi] = \alpha_g(B).$$

(3.3.103)

The composition is defined by

$$(A'', B'') \xleftarrow{(h', \varphi')} (A', B') \xleftarrow{(h, \varphi)} (A, B) := (A'', B'') \xleftarrow{(h'h, \alpha_{g'}(\varphi) + \varphi')} (A, B).$$

(3.3.104)

A 2-morphism from $(h, \varphi)$ to $(h', \varphi')$, which are both 1-morphisms from $(A, B)$ to $(A', B')$, is a smooth map $f : X \to H$ satisfying

$$h' = \tau(f)h$$

(3.3.105)

and

$$\varphi' + (R_f^{-1} \circ \alpha_f)(A') = \text{Ad}_f(\varphi) - f^*\theta.$$  

(3.3.106)

The vertical composition is defined by

$$(A', B') \xleftarrow{(h', \varphi')} (A, B) := (A', B') \xleftarrow{f'f} (A, B).$$

(3.3.107)
The horizontal composition is defined by

\[
\begin{array}{c}
(A'', B'') \xrightarrow{(h_1, \varphi_1)} (A', B') \xrightarrow{(f_1, (g, \varphi))} (A, B) := (A'', B'') \xrightarrow{(h_2, \psi)} (A, B).
\end{array}
\]

(3.3.108)

As in Section 3.2.6, these arguments define 2-functors

\[
Z^2_X(\mathfrak{g})^\pi \xrightarrow{\mathfrak{p}_X} \text{Funct}^\times(X, \mathcal{B}\mathfrak{g}),
\]

(3.3.109)

which turn out to be strict inverses of each other (Theorem 2.21 of [ScWa11]).

As before, this was for globally defined differential data corresponding to globally trivial transport 2-functors. Transport 2-functors on \( M \) are not necessarily of this type, but they are locally trivializable via some surjective submersion \( \pi : Y \longrightarrow M \) and a \( \pi \)-local \( i \)-trivialization. By similar arguments to the discussion in Section 3.2.6, we are led to the following, rather long and complicated, definition.

**Definition 3.3.110.** Let \( \pi : Y \longrightarrow M \) be a surjective submersion. Define the 2-category \( Z^2_\pi(\mathfrak{g})^\pi \) of differential cocycles subordinate to \( \pi \) as follows. An object of \( Z^2_\pi(\mathfrak{g})^\pi \) is a tuple \(((A, B), (g, \varphi), \psi, f)\), where \((A, B)\) is an object in \( Z^2_Y(G)\), \((g, \varphi)\) is a 1-morphism from \( \pi^*_1(A, B) \) to \( \pi^*_2(A, B) \) in \( Z^2_{Y[2]}(\mathfrak{g})\), \( \psi \) is a 2-morphism from \( \text{id}_{(A, B)} \) to \( \Delta^*(g, \varphi) \) in \( Z^2_Y(\mathfrak{g})\), and \( f \) is a 2-morphism from...
\[
\pi^*_2(g, \varphi) \circ \pi^*_1(g, \varphi) \to \pi^*_1(g, \varphi). \]
A 1-morphism from \(((A, B), (g, \varphi), \psi, f)\) to \(((A', B'), (g', \varphi'), \psi', f')\) is tuple \(((h, \phi), \epsilon)\), where \((h, \phi)\) is a 1-morphism from \((A, B)\) to \((A', B')\) in \(Z^2_Y(\mathfrak{G})\) and \(\epsilon\) is a 2-morphism from \(\pi^*_2(h, \phi) \circ (g, \varphi)\) to \((g', \varphi') \circ \pi^*_1(h, \phi)\) in \(Z^2_Y(\mathfrak{G})\). A 2-morphism from \(((h, \phi), \epsilon)\) to \(((h', \phi'), \epsilon')\) is a 2-morphism \(E\) from \((h, \phi)\) to \((h', \phi')\) in \(Z^2_Y(\mathfrak{G})\).

The above generalizations produce functors

\[
Z^2_\pi(\mathfrak{G})^\infty \xrightarrow{\pi^*_2} \mathcal{D}_{\pi^*_2}(i)^\infty \quad (3.3.111)
\]

exhibiting an equivalence of 2-categories whenever \(i : \mathcal{B} \mathfrak{G} \to T\) is an equivalence.

### 3.3.7 Direct limits

In this section, we get rid of the dependence on the surjective submersion in the categories introduced in the prequel. Several of our 2-categories depended on the choice of a surjective submersion. These 2-categories were \(\text{Triv}^2_\pi(i)^\infty, \mathcal{D}_{\mathfrak{G}}^2(i)^\infty\), and \(Z^2_\pi(\mathfrak{G})^\infty\). One can take a limit over the collection of surjective submersions in this case. This will be a slight generalization of what was done in Section 3.2.7. However, there are subtle issues in terms of defining the many compositions.

The general construction proceeds as follows. Let \(S_\pi\) be a family of 2-categories parametrized by surjective submersions \(\pi : Y \to M\) and let \(F(\zeta) :\)
$S_{\pi} \longrightarrow S_{\pi \circ \zeta}$ be a family of 2-functors for every refinement $\zeta : Y' \longrightarrow Y$ of $\pi$ satisfying the condition that for any iterated refinement $\zeta' : Y'' \longrightarrow Y'$ and $\zeta : Y' \longrightarrow Y$ of $\pi : Y \longrightarrow M$ then $F(\zeta' \circ \zeta) = F(\zeta') \circ F(\zeta)$. In this case, an object of $S_M := \lim_{\pi} S_{\pi}$ is given by a pair $(\pi, X)$ of a surjective submersion $\pi : Y \longrightarrow M$ and an object $X$ of $S_{\pi}$. A 1-morphism from $(\pi_1, X_1)$ to $(\pi_2, X_2)$ consists of a common refinement

\begin{equation}
\begin{array}{c}
Z \\
\downarrow \downarrow \\
Y_1 \xleftarrow{\zeta} Y_2 \\
\downarrow \downarrow \\
M
\end{array}
\end{equation}

(3.3.112)

together with a 1-morphism $f : (F(y_1))(X_1) \longrightarrow (F(y_2))(X_2)$ in $S_{\zeta}$. It is written as a pair $(\zeta, f)$. The composition

\begin{equation}
(\pi_3, X_3) \xleftarrow{\zeta_{23}, g} (\pi_2, X_2) \xleftarrow{\zeta_{12}, f} (\pi_1, X_1)
\end{equation}

(3.3.113)

consists of the pullback refinement

\begin{equation}
\begin{array}{c}
Z_{13} \\
\downarrow \downarrow \\
Z_{12} \xleftarrow{\zeta_{12}} Z_{23} \\
\downarrow \downarrow \\
Y_1 \xleftarrow{\zeta_1} Y_2 \xleftarrow{\zeta_2} Y_3 \\
\downarrow \downarrow \\
M \xleftarrow{\pi_4}
\end{array}
\end{equation}

(3.3.114)

along with the composition $(F(pr_{23}))(g) \circ (F(pr_{12}))(f)$. A 2-morphism from $(\zeta, f)$ to $(\zeta', f')$ consists of an equivalence class of pairs $(\omega, \alpha)$, where $\omega$ is a
common refinement of $\zeta$ and $\zeta'$ as in the following diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
W \\
\downarrow z' \\
Y' \\
\downarrow y' \\
M \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Z' \\
\downarrow y_2 \\
Z \\
\downarrow y_1 \\
Y_1 \\
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\downarrow \pi_2 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\downarrow \pi_1 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
(3.3.115)

and $\alpha$ is a 2-morphism $\alpha : F(z)(f) \Rightarrow F(z')(f')$. Two such pairs $(\omega_1, \alpha_1)$ and $(\omega_2, \alpha_2)$ are equivalent if they agree on the pullback.

After getting rid of the specific choices of the surjective submersions, we can take the limits of all the categories we have introduced. We make the following notation, slightly differing from that of [ScWa13]:

\[
\begin{align*}
\text{Triv}_M^2(i)^\infty & := \lim_{\pi} \text{Triv}_\pi^2(i)^\infty \\
\text{Des}_M^2(i)^\infty & := \lim_{\pi} \text{Des}_\pi^2(i)^\infty \\
Z^2(M; \mathfrak{G})^\infty & := \lim_{\pi} Z^2_\pi(\mathfrak{G})^\infty.
\end{align*}
\]

Then from our previous discussions, we collect the functors we have introduced relating all these categories to $\text{Trans}_{\mathfrak{B}\mathfrak{G}}^2(M, T)$ after taking such limits over surjective submersions:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Z^2(M; \mathfrak{G})^\infty \\
\rightarrow \text{Des}_M^2(i)^\infty \\
\rightarrow \text{Triv}_M^2(i)^\infty \\
\rightarrow \text{Trans}_{\mathfrak{B}\mathfrak{G}}^2(M, T)
\end{array}
\end{array}
\end{array}
\end{array}
(3.3.119)
Under the conditions that \( i : B\mathfrak{G} \to T \) is an equivalence of categories, all of the above 2-functors are equivalence pairs. Without the smoothness assumptions, a simpler version of some of these equivalences is proven in Proposition 4.2.1. and Theorem 4.2.2. of [ScWa] while the equivalences in (3.3.119) are proven in Theorem 3.2.2., Lemma 3.2.3., and Lemma 3.2.4. of [ScWa13]. Completely analogous versions of comments regarding the assumptions on \( i \) made before (3.2.83) apply here as well.

### 3.3.8 Surface transport, 2-holonomy, and gauge invariance

In Section 3.2.8, we described a procedure that began with a transport functor and produced a group-valued parallel transport operator for thin loops with markings. We discovered that the value of holonomy changed by conjugation depending on the markings for the loops, the choice of a local trivialization procedure, and by using an isomorphic transport functor. In this section, we will analyze holonomy along surfaces in an analogous manner. The main difference is that bigons have source and target paths so that a closed surface has a marking of one lower dimension, and is therefore not in general just a point as it was for loops. For the examples we give later in this chapter, we specialize to spheres with a point marking. Such a surface is depicted as a bigon from the constant loop at a point \( x \) to itself (see
Figure 3.16 below and \cite{ChTs93}). However, a sphere can be more generally described as a bigon from a loop to itself, so we analyze parallel transport for such bigons to cover these extra cases. This analysis is completely independent of what types of Lie 2-groups $B\mathfrak{G}$ we use. For simplicity, we assume that $i : B\mathfrak{G} \to T$ is a full and faithful 2-functor. This will differ from the presentation in Section 5 of \cite{ScWa13}, where surface holonomy was defined using the reduced 2-group. We will not be making this restriction.

**Definition 3.3.120.** A 2-group-valued transport extraction is a composition of functors (starting at the left and moving clockwise)

\[
\begin{array}{c}
\text{Trans}^2_{B\mathfrak{G}}(M,T) \xrightarrow{\epsilon} \text{Triv}^2(i)^{\infty} \xrightarrow{\text{Ex}^2} \\
\text{Rec}^2 \xrightarrow{v \circ} \text{Triv}^2(i)^{\infty} \xrightarrow{\text{Des}^2(i)^{\infty}}
\end{array}
\]  \hspace{1cm} (3.3.121)

We write the composition (3.3.121) as $\epsilon$. By the reconstruction procedure of Section 3.3.5, $\epsilon$ assigns $G$-valued elements to thin paths for every
transport functor $F$ as well as $H$-valued elements to thin bigons (more on this below). Technically, thin bigons will be assigned elements in $H \rtimes G$ but as is discussed in Section 3.3.1, particularly after the proof of Theorem 3.3.4, such elements are completely determined by their source, an element of $G$, and their projection in $H$. $\mathcal{L}$ will also assign $G$-valued and $H$-valued gauge transformations for every 1-morphism $\eta : F \rightarrow F'$ of transport functors. In addition, $\mathcal{L}$ will assign $H$-valued 2-gauge transformations for every 2-morphism $A : \eta \Rightarrow \eta'$. A pseudo-natural equivalence $\triangleright : \text{id} \Rightarrow \mathcal{L}$ describes how to relate the transport functor to the locally trivialized one. Although modifications of pseudo-natural transformations are allowed, we will not analyze them here. Such modifications are to be interpreted as relating the two different ways of choosing the pseudo-natural transformations that relate the transport functor to the locally trivialized one.

Just as before, we briefly review what the composition of 2-functors defining $\mathcal{L}$ are. For a transport 2-functor $F$, we choose a local trivialization $c(F) = (\pi, F, \text{triv}, t)$. Then we extract the local descent object $\text{Ex}^2(\pi, F, \text{triv}, t) = (\pi, \text{triv}, g, \psi, f)$. Then, we reconstruct a transport 2-functor $\text{Rec}^2(\pi, \text{triv}, g, \psi, f)$ and then forget the trivialization data keeping just the 2-functor

$$v(\text{Rec}^2(\pi, \text{triv}, g, \psi, f)).$$

(3.3.122)
The resulting transport 2-functor, written as $\mathcal{F}$, is defined by (see Section 3.3.5)

$$
\mathcal{P}_2(M) \xrightarrow{\mathcal{F}} T
$$

$$
M \ni x \mapsto i(\bullet) =: \text{triv}_i(s^\pi(x)), \quad (3.3.123)
$$

$$
P^1 M \ni \gamma \mapsto R_{\text{Ex}^2(c(F))}(s^\pi(\gamma)), \text{ and}
$$

$$
P^2 M \ni \Sigma \mapsto R_{\text{Ex}^2(c(F))}(s^\pi(\Sigma)).
$$

Points in $M$ get sent to $i(\bullet)$ by construction. Because $i$ is full and faithful, the 1-morphisms $R_{\text{Ex}^2(c(F))}(s^\pi(\gamma)) : i(\bullet) \longrightarrow i(\bullet)$ determine unique elements of $G$. Similarly, the 2-morphisms $R_{\text{Ex}^2(c(F))}(s^\pi(\Sigma))$ determine unique elements in $H$.

The interested reader can explicitly define the compositor and the unitor for the 2-functor $\mathcal{F}$. We will not need the precise details for our analysis when studying surface holonomy. All we need to know is that the 2-functors defining $\mathcal{F}$ are (weakly) invertible.

We would like to restrict surface holonomy to thin homotopy classes of marked spheres for the purpose of this paper (in general, one would like to restrict to the more general space of thin homotopy classes of marked closed surfaces) and eventually thin free spheres. First we make a definition of the thin marked sphere space, which should be thought of as analogous to the thin marked loop space.
Definition 3.3.124. The marked sphere space of $M$ is the set

$$\mathcal{S}M := \{\Sigma \in BM \mid s(\Sigma) = t(\Sigma) \text{ and } s(s(\Sigma)) = t(t(\Sigma))\}$$

(3.3.125)

equipped with the subspace smooth structure. Elements of $\mathcal{S}M$ are called marked spheres. Similarly, the thin marked sphere space of $M$ is the smooth space

$$\mathcal{S}^2M = \{\Sigma \in P^2M \mid s(\Sigma) = t(\Sigma) \text{ and } s(s(\Sigma)) = t(t(\Sigma))\}.$$ 

(3.3.126)

Elements of $\mathcal{S}^2M$ are called thin marked spheres.

Remark 3.3.127. Note that elements of $\mathcal{S}^2M$ need not look like embedded spheres in $M$. Indeed, they might look like pinched croissants as Figure 3.17 indicates (or worse). This won’t matter in any of our calculations or proofs.

Figure 3.17: A pinched croissant is an example of a thin marked sphere.

Definition 3.3.128. The $\ell$-holonomy of $F$, written as $\text{hol}_\ell F$, is defined as the projection to $H$ of the restriction of parallel transport of a transport 2-functor $F$ to the thin marked sphere space of $M$:

$$\text{hol}_\ell F := p_H \circ [\ell_F|_{\mathcal{S}^2M}] : \mathcal{S}^2M \longrightarrow H.$$  

(3.3.129)
Remark 3.3.130. Note that $\text{hol}_F^\Sigma$ is the same notation used for thin loops with values in $G$. This should cause no confusion because thin loops are always written using lower case Greek letters such as $\gamma, \delta$, etc. while thin spheres are written using upper case Greek letters such as $\Sigma, \Gamma$, etc.

We now pose three questions analogous to those for 1-holonomy.

i) How does $\text{hol}_F^\Sigma$ depend on the choice of a thin marked sphere? Namely, suppose that two thin marked spheres $\Sigma$ and $\Sigma'$, with possibly different markings, are thinly homotopic without preserving the marking (see Definition 3.3.131). Then, how is $\text{hol}_F^\Sigma(\Sigma)$ related to $\text{hol}_F^{\Sigma'}(\Sigma')$?

ii) How does $\text{hol}_F^\Sigma$ depend on $F$? Namely, suppose that $\eta : F \longrightarrow F'$ is a morphism of transport functors. How is $\text{hol}_F^\Sigma$ related to $\text{hol}_{F'}^\Sigma$ in terms of $\eta$?

iii) How does $\text{hol}_F^\Sigma$ depend on $\mathcal{L}$, the choice of trivialization? Namely, suppose that $\mathcal{L}'$ is another trivialization. Then how is $\text{hol}_F^\Sigma$ related to $\text{hol}_{F'}^\Sigma$?

Due to the fact that we are restricting ourselves to marked spheres instead of arbitrary surfaces, the answer will be closely related to the 1-holonomy case and will be given by a generalized version of conjugation. As before, we need
to define what we mean by thin free sphere space and then we will proceed to answer the above questions. Denote the smooth space of spheres in $M$ by $SM = \{ \Sigma : S^2 \to M \mid \Sigma \text{ is smooth} \}$.

**Definition 3.3.131.** Two smooth spheres $\Sigma$ and $\Sigma'$ in $M$ are **thinly homotopic** if there exists a smooth map $h : S^2 \times [0, 1] \to M$ such that

i) there exists an $\epsilon > 0$ with $h(t, s) = \Sigma(t)$ for $s \leq \epsilon$ and $h(t, s) = \Sigma'(t)$ for $s \geq \epsilon$ and for all $t \in S^2$ and

ii) the smooth map $h$ has rank $\leq 2$.

The space of equivalences classes is denoted by $S^2M$ and is called the **thin free sphere space of $M$**. Elements of $S^2M$ are called **thin spheres**.

**Definition 3.3.132.** Define a function $f : \mathcal{S}M \to SM$ by sending a marked sphere $\Sigma : [0, 1] \times [0, 1] \to M$ to the associated smooth map $f(\Sigma) : S^2 \to M$ obtained from identifying the top and bottom of the second interval and then pinching the two ends (see Figure 3.18). $f$ is called the **forgetful map**.

**Lemma 3.3.133.** There exists a unique map $f^2 : \mathcal{S}^2M \to S^2M$ such that

the diagram

$$
\begin{array}{ccc}
\mathcal{S}M & \longrightarrow & \mathcal{S}^2M \\
f \downarrow & & \downarrow f^2 \\
SM & \longrightarrow & S^2M
\end{array}
$$

(3.3.134)
Figure 3.18: The definition of $f : \mathcal{G}M \to SM$. This definition makes sense even when $y \neq x$. $y = x$ is a special case.

commutes (the horizontal arrows are the projections onto thin homotopy classes).

Proof. The proof is analogous to the case of loops. One chooses a representative, applies $f$, and then projects. The map is well-defined by the thin homotopy equivalence relation on $S^2M$.

Note that there is also a function $\ev_1 : \mathcal{G}^2M \to \mathcal{L}^1M$ given by evaluating a thin marked sphere at its source/target. This function forgets the sphere and remembers only the source thin marked loop.

**Definition 3.3.135.** A **marking of thin spheres** is a section $\mathbf{m} : S^2M \to \mathcal{G}^2M$ of $f^2 : \mathcal{G}^2M \to S^2M$.

**Lemma 3.3.136.** A marking of thin spheres exists.

Proof. Let $[\Sigma] \in \mathcal{G}^2M$ be a thin sphere and choose representative $\Sigma : S^2 \to M$ in $SM$. Pick a point $\bullet$ on the equator viewed as a loop $\ell : \bullet \to \bullet$. The image of $\ell$ under $\Sigma$ defines a loop, $\gamma : x \to x$, where $x := \Sigma(\bullet)$. There exists
a thin homotopy $h : S^2 \times [0, 1] \to M$ from $\Sigma$ to a smooth map $\Sigma_\ell : S^2 \to M$ such that the family of loops in Figure 3.19 on the domain of $\Sigma_\ell$ define a marked sphere $\tilde{\Sigma} : \gamma \Rightarrow \gamma$. Projecting to thin marked spheres defines $m([\Sigma])$. To see that this is well-defined, let $\Sigma' \in SM$ be another representative. Then there exists a thin unmarked homotopy $\tilde{h} : \Sigma' \Rightarrow \Sigma$. Composing this with the thin homotopy $h$ gives $h \circ \tilde{h} : \Sigma' \Rightarrow \Sigma_\ell$. By the thin homotopy equivalence relation on $S^2M$, this defines a section of $f^2$. $\blacksquare$

![Figure 3.19](image_url)

Figure 3.19: By a thin homotopy, the region around the equator is made to sit at the loop $\ell$ around the equator so that the nearby loops drawn in the shaded region agree with $\ell$. The family of all these loops define a marking.

We now proceed to answering the above questions in order.

i) Let $m, m' : S^2M \to \mathcal{S}^2M$ be two markings for thin spheres in $M$. Let $[\Sigma] \in S^2M$ be a thin sphere and let $\Sigma : \gamma \Rightarrow \gamma$ with $\gamma : x \to x$ be a representative of $m([\Sigma])$ and $\Sigma' : \gamma' \Rightarrow \gamma'$ with $\gamma' : x' \to x'$ be a representative of $m'([\Sigma])$. Note that these representatives need not have associated marked loops that lie on some common image. Figure 3.20
Figure 3.20: Two different representatives $\Sigma$ (the ‘inner’ sphere in green extending left) and $\Sigma'$ (the ‘outer’ sphere in purple extending right) of two markings of a thin sphere are shown. The extensions do not enclose any volume so that both spheres are thinly homotopic. Their respective sources are $\gamma : x \to x$ and $\gamma' : x' \to x'$, neither of which lie on the other’s image. Compare this to Figure 3.23 where the two marked loops do lie on a common sphere.

As in the case of loops, we can use thin homotopy to draw both marked loops on the same sphere (a more precise statement will be given shortly).

First notice that there is a thin homotopy $h : S^2 \times [0,1] \to M$ with $h(\cdot,s) = \Sigma$ for $s \leq \epsilon$ and $h(\cdot,s) = \Sigma'$ for $s \geq 1-\epsilon$ for some $\epsilon > 0$. Such a homotopy allows us to choose a sphere $\tilde{\Sigma} \in SM$, a path $\gamma_{x'x} : x \to x'$, and three bigons $\Sigma_{\gamma x} : \text{id}_x \Rightarrow \gamma$, $\Sigma_{x'\gamma'} : \gamma' \Rightarrow \text{id}_{x'}$, and $\Delta : \gamma_{x'x} \circ \gamma \circ \gamma_{x'x}$ with the following properties. First $\tilde{\Sigma}$ can be expressed
as either of the compositions

\[
\begin{align*}
    f \left( \frac{\Sigma_{\gamma'x'} \circ \id_{\gamma'x'}}{\Delta} \right) & \quad \text{or} \quad f \left( \frac{\id_{\gamma'x'} \circ \Delta \circ \id_{\gamma'x'}}{\Sigma_{\gamma x}} \right)
\end{align*}
\]

(3.3.137)

(in either order vertically). Second, the composition of bigons

\[
\begin{align*}
    \id_{\gamma'x'} \circ \Delta \circ \id_{\gamma'x'} \\
    \id_{\gamma'x'} \circ \Sigma_{\gamma x'} \circ \id_{\gamma'x'} \\
    \Sigma_{\gamma x}
\end{align*}
\]

is thinly homotopic to \(\Sigma\) preserving the marked loop \(\gamma : x \rightarrow x\). Third, the composition of bigons

\[
\begin{align*}
    \Sigma_{\gamma'x'} \\
    \id_{\gamma'x'} \circ \Sigma_{\gamma x'} \circ \id_{\gamma'x'} \\
    \Delta
\end{align*}
\]

is thinly homotopic to \(\Sigma'\) preserving the marked loop \(\gamma' : x' \rightarrow x'\). This is depicted in Figures 3.21 and 3.22.

These last two equations let us write the bigon \(\Sigma\) in terms of \(\Sigma'\) and vice versa. In fact, we have

\[
\Sigma' = \id_{\gamma'x'} \circ \Sigma \circ \id_{\gamma'x'}
\]

(3.3.140)

up to thin homotopy preserving the marked loop \(\gamma' : x' \rightarrow x'\). There is also a similar expression for \(\Sigma\) preserving the marked loop \(\gamma : x \rightarrow x\).
Figure 3.21: The domain of the homotopy $h : S^2 \times [0, 1] \to M$ is drawn as a solid ball with a smaller solid ball removed from the center. It depicts $\Sigma$ as the inner sphere and $\Sigma'$ as the outer sphere. The marked loop $\gamma : x \to x$ of $\Sigma$ is drawn on the northern hemisphere while the marked loop $\gamma' : x' \to x'$ of $\Sigma'$ is drawn on the southern hemisphere (by a thin homotopy, one can always position the marked loops in this way). The homotopy $h$ allows us to choose a sphere $\tilde{\Sigma}$, drawn somewhat in the middle (in orange), that contains both based loops $\gamma$ and $\gamma'$ and is thinly homotopic to both $\Sigma$ and $\Sigma'$. As a result, there exists a path $\gamma_{x'x} : x \to x'$ on $\tilde{\Sigma}$. We continue this analysis in Figure 3.22.

The above argument says that given two marked spheres, with possibly different markings, that are thinly homotopic without preserving the markings, one can always choose a representative of such a thin sphere in $M$ with two marked loops so that the associated two marked spheres (coming from starting at either marking) are thinly homotopic to the original two with a thin homotopy that preserves the marking. More precisely, we proved the following.
Lemma 3.3.141. Let \( m, m' : S^2M \to \mathcal{G}^2M \) be two markings. Let \([\Sigma] \in S^2M\) be a thin sphere in \( M \) and write \([\gamma] : x \to x\) for \( ev_1(m([\Sigma]))\) and \([\gamma'] : x' \to x'\) for \( ev_1(m'([\Sigma]))\). Then, there exists representatives \( \gamma \) and \( \gamma' \) of \([\gamma]\) and \([\gamma']\), respectively, a path \( \gamma_{x'x} : x \to x' \) with sitting instants and three bigons \( \Sigma_{\gamma x} : id_x \Rightarrow \gamma, \Sigma_{x'\gamma'} : \gamma' \Rightarrow id_{x'}, \) and \( \Delta : \gamma_{x'x} \circ \gamma \circ \gamma_{x'x} \Rightarrow \gamma' \), such that the following three properties hold (see Figure 3.23).

i) The vertical composition of \( \Sigma_{\gamma'x'}, id_{\gamma_{x'x}} \circ \Sigma_{\gamma x} \circ id_{\gamma_{x'x}} \), and \( \Delta \) in the order given (or a cyclic permutation of this order) and forgetting the marking is a representative of \([\Sigma]\).

\[
\begin{pmatrix}
  id_{x'x} \\
  id_{\gamma_{x'x}} \\
  id_{\gamma_{x'x}} \\
\end{pmatrix}
\begin{pmatrix}
  \Sigma_{\gamma x} \\
  \Sigma_{x'\gamma'} \\
  \Delta \\
\end{pmatrix}
\]

is a representative of \( m([\Sigma]) \) as a bigon.

\[
\begin{pmatrix}
  id_{x'x} \\
  id_{\gamma_{x'x}} \\
  id_{\gamma_{x'x}} \\
\end{pmatrix}
\begin{pmatrix}
  \Sigma_{\gamma x} \\
  \Sigma_{x'\gamma'} \\
  \Delta \\
\end{pmatrix}
\]

is a representative of \( m'([\Sigma]) \) as a bigon.

Therefore, without loss of generality, we can choose a single representa-
Figure 3.23: For every thin sphere and two markings, there exists a representative with a decomposition as in Lemma 3.3.141. On the left is a bigon $\Sigma : \gamma \Rightarrow \gamma$ with $\gamma : x \rightarrow x$. The shaded region depicts the surface swept out between $s = 0$ and some small $s$. In the middle is another bigon $\Sigma' : \gamma' \Rightarrow \gamma'$ with $\gamma' : x' \rightarrow x'$ and a path $\gamma_{x'x} : x \rightarrow x'$ with sitting instants. On the right is a bigon $\Delta : \gamma_{x'x} \circ \gamma \circ \gamma_{x'x} \Rightarrow \gamma'$ relating the two marked loops as in (3.3.143).

We use $\Sigma$ to denote the bigon in ii) of Lemma 3.3.141 and $\Sigma'$ to denote the bigon in iii). The two are related by

\[
\tilde{\Sigma} \circ \Sigma = \Sigma' \circ \tilde{\Sigma}
\]

i.e.

\[
\Sigma_y = \operatorname{id}_{\gamma_{x'x}} \circ \tilde{\Sigma} \circ \operatorname{id}_{\gamma_{x'x}} \circ \Delta.
\]
By functoriality of the transport 2-functor $\mathcal{F}$, we have

$$\text{hol}_F^\Sigma' = p_H(\mathcal{F}(\Sigma')) = p_H \left( \left( \begin{array}{c} \mathcal{F}(\Delta) \\ \mathcal{C}^{-1} \\ \mathcal{C} \end{array} \right) \right).$$

(3.3.144)

where $C : \mathcal{F}(\gamma_{x'x}) \mathcal{F}(\gamma_x) \mathcal{F}(\overline{\gamma_{x'x}}) \Rightarrow \mathcal{F}(\gamma_{x'x} \circ \gamma_x \circ \overline{\gamma_{x'x}})$ is a combination of compositors and associators. Writing out this composition in the 2-group $\mathcal{B}G$ gives

$$((p_H(\mathcal{F}(\Delta)))^{-1}, \mathcal{F}(\gamma')) \quad (p_H(C)^{-1}, \mathcal{F}(\gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}})) \quad (e, \mathcal{F}(\gamma_{x'x})) \quad (\text{hol}_F^\Sigma, \mathcal{F}(\gamma))(e, \mathcal{F}(\overline{\gamma_{x'x}})) \quad (p_H(C), \mathcal{F}(\gamma_{x'x}) \mathcal{F}(\gamma) \mathcal{F}(\overline{\gamma_{x'x}})) \quad (p_H(\mathcal{F}(\Delta)), \mathcal{F}(\gamma_{x'x} \circ \gamma \circ \overline{\gamma_{x'x}}))$$

Multiplying these results out using the rules of 2-group multiplication (see equations (3.3.6) and (3.3.7)) and taking the $H$ component gives

$$\text{hol}_F^\Sigma' = p_H(\mathcal{F}(\Delta)) p_H(C) \alpha_{\mathcal{F}(\gamma_{x'x})} \left( \text{hol}_F^\Sigma \right) p_H(C)^{-1} p_H(\mathcal{F}(\Delta))^{-1}$$

$$= \alpha_{p_H(\mathcal{F}(\Delta)) p_H(C) \mathcal{F}(\gamma_{x'x})} \left( \text{hol}_F^\Sigma \right).$$

(3.3.146)

This result says that the 2-holonomy changes by $\alpha$-conjugation under a change of marking for a thin sphere.

ii) Now suppose that $\eta : F \longrightarrow F'$ is a 1-morphism of transport 2-functors. Then, for every thin path $\gamma : x \longrightarrow y$ we have a 2-isomorphism (remember
that \( \mathcal{F} \) \( x = \{ \bullet \} \) and \( \mathcal{F} \) \( x = \{ \bullet \} \) for all \( x \in M \)

\[
\begin{array}{c}
\{ \bullet \} \\
\{ \bullet \}
\end{array}
\]

\[
\mathcal{L}_q(x) \quad \mathcal{L}_q(y) \quad \mathcal{L}_q(\gamma) \quad \mathcal{L}_q(\delta)
\]

satisfying the condition that for any thin bigon \( \Sigma : \gamma \Rightarrow \delta \), with \( \delta : x \rightarrow y \)

another path, the diagram

\[
\begin{array}{c}
\{ \bullet \} \\
\{ \bullet \}
\end{array}
\]

\[
\mathcal{L}_q(x) \quad \mathcal{L}_q(y) \quad \mathcal{L}_q(\gamma) \quad \mathcal{L}_q(\delta)
\]

commutes. In this diagram, the \( \mathcal{L}_q(\delta) \) in the back is not shown. This diagram commuting means that

\[
\mathcal{L}_q(\gamma) \circ \mathcal{L}_q(\Sigma) \mathcal{L}_q(\delta) = \mathcal{L}_q(\delta) \circ \mathcal{L}_q(\Sigma) \mathcal{L}_q(\gamma)
\]

(3.3.149)

and writing this out using group elements gives

\[
\begin{align*}
(p_H(\mathcal{L}_q(\gamma)), \mathcal{L}_q(y) \mathcal{L}_q(\gamma)) &= (e, \mathcal{L}_q(y)) (p_H(\mathcal{L}_q(\Sigma)), \mathcal{L}_q(\gamma)) \\
(p_H(\mathcal{L}_q(\Sigma)), \mathcal{L}_q(\gamma))(e, \mathcal{L}_q(x)) &= (p_H(\mathcal{L}_q(\delta)), \mathcal{L}_q(y) \mathcal{L}_q(\delta))
\end{align*}
\]

(3.3.150)

which after evaluating both sides and projecting to \( H \) yields

\[
p_H(\mathcal{L}_q(\Sigma)) p_H(\mathcal{L}_q(\gamma)) = p_H(\mathcal{L}_q(\delta)) \mathcal{L}_q(y) (p_H(\mathcal{L}_q(\Sigma))).
\]

(3.3.151)
Solving for $p_H(\mathcal{C}_v(\Sigma))$ gives

$$p_H(\mathcal{C}_v(\Sigma)) = p_H(\mathcal{C}_v(\delta))\alpha_{\mathcal{C}_v(y)}(p_H(\mathcal{C}_v(\Sigma)))p_H(\mathcal{C}_v(\gamma))^{-1}. \quad (3.3.152)$$

Now, after specializing to the case where the source and targets of $\Sigma$ are all the same, i.e. $y = x$ and $\delta = \gamma$, so that we are comparing this transport along thin marked spheres, this reduces to

$$\text{hol}^{F'}(\Sigma) = p_H(\mathcal{C}_v(\gamma))\alpha_{\mathcal{C}_v(x)}(\text{hol}^{F}(\Sigma))p_H(\mathcal{C}_v(\gamma))^{-1}$$

$$= \alpha_{\tau(p_H(\mathcal{C}_v(\gamma)))}\alpha_{\mathcal{C}_v(x)}(\text{hol}^{F}(\Sigma)). \quad (3.3.153)$$

This says that $\text{hol}^{F'}$ when restricted to thin marked spheres changes under $\alpha$-conjugation when the functor $F$ is changed to a gauge equivalent one $F'$.

iii) Suppose that another 2-group transport extraction procedure $\mathcal{C}'$ was chosen. Any two such procedures are pseudo-naturally equivalent, i.e. if $\mathcal{C}'$ was another such choice, then there exists a weakly invertible pseudonatural transformation $\mathcal{J} : \mathcal{C}' \Rightarrow \mathcal{C}$. This follows from the fact that each 2-functor in the composition of 2-functors that define $\mathcal{C}$ is an equivalence of 2-categories and weak inverses are unique up to pseudonatural equivalences. Therefore, for every transport 2-functor $F$ we have a 1-morphism of transport functors $\mathcal{A}_F : \mathcal{C}_v' \to \mathcal{C}_v$. Of course, we also have a map assigning to every 1-morphism of transport functors $\eta : F \to F'$. 
a 2-morphism $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$ satisfying naturality, but we will not need this fact for the following observation because we are dealing with strict Lie 2-groups. The 1-morphism of transport functors $\mathcal{F}$ assigns to every point $x \in M$ a morphism $\mathcal{F}_x(x) : \mathcal{E}_x(x) \rightarrow \mathcal{E}_y(x)$ and to every path $\gamma : x \rightarrow y$ a 2-isomorphism

$$
\begin{array}{ccc}
i(\bullet) & \xrightarrow{\mathcal{F}_\gamma(x)} & i(\bullet) \\
\downarrow & & \downarrow \\
\mathcal{F}_\gamma(\gamma) & \xleftarrow{\mathcal{F}'_\gamma(\gamma)} & \mathcal{F}'_\gamma(\gamma)
\end{array}
$$

(3.3.154)

satisfying the condition that for a thin bigon $\Sigma : \gamma \rightarrow \delta$ between two thin paths $\gamma, \delta : x \rightarrow y$ the diagram

$$
\begin{array}{ccc}
i(\bullet) & \xrightarrow{\mathcal{F}_\gamma(x)} & i(\bullet) \\
\downarrow & & \downarrow \\
\mathcal{F}_\gamma(\Sigma) & \xleftarrow{\mathcal{F}'_\gamma(\Sigma)} & \mathcal{F}'_\gamma(\Sigma)
\end{array}
$$

(3.3.155)

commutes. This result is very similar to the previous one and is given by

$$
\text{hol}_x^F(\Sigma) = \alpha_{\tau(p_H(\mathcal{F}_\gamma(\gamma)))\mathcal{F}_\gamma(x)} \left( \text{hol}_x^F(\Sigma) \right),
$$

(3.3.156)

which is again just $\alpha$-conjugation.

In conclusion, when restricted to a sphere, 2-holonomy changes under $\alpha$-conjugation in each of the three situations described above. This should
therefore also be called gauge covariance as in the case for loops. This motivates the following definition.

**Definition 3.3.157.** Let \((H, G, \tau, \alpha)\) be a crossed module. The \(\alpha\)-conjugacy classes in \(H\), denoted by \(H/\alpha\), is defined to be the quotient of \(H\) under the equivalence relation

\[ h \sim h' \iff \text{there exists a } g \in G \text{ such that } h = \alpha_g(h'). \] (3.3.158)

Denote the quotient map by \(q : H \longrightarrow H/\alpha\).

As before, we have a similar theorem for gauge-invariance of 2-holonomy.

**Theorem 3.3.159.** Let \(M\) be a smooth manifold, \(\mathcal{B}G\) a Lie 2-group, \(T\) a 2-category, and suppose that \(i : \mathcal{B}G \longrightarrow T\) is a full and faithful 2-functor. Let \(F\) be a transport 2-functor and \(\mathcal{T}\) a 2-group-valued transport extraction. Let \(S^2M, \mathcal{S}^2M, m, \text{hol}_F^\mathcal{T}\) and \(q\) be defined as above. Then the composition

\[ H/\alpha \xleftarrow{q} H \xleftarrow{\text{hol}_F^\mathcal{T}} \mathcal{S}^2M \xleftarrow{m} S^2M \] (3.3.160)

is

i) independent of \(m\),

ii) independent of the equivalence class of \(F\),

iii) and independent of the equivalence class of \(\mathcal{T}\).
This theorem lets us make the following definition.

**Definition 3.3.161.** Let $[F]$ be an equivalence class of transport 2-functors.

The *gauge invariant 2-holonomy* of $[F]$ is defined to be the smooth map in the previous theorem, namely

$$\text{hol}^{[F]} := q \circ \text{hol}^F \circ m : S^2M \rightarrow H/\alpha$$  \hspace{1cm} (3.3.162)

where $F$ is a representative of $[F]$, $\mathcal{C}$ is a group-valued transport extraction, and $m : S^2M \rightarrow \mathcal{G}^2M$ is a marking for thin spheres in $M$. Let $\Sigma \in S^2M$. If $\text{hol}^{[F]}(\Sigma)$ is such that $q^{-1}(\text{hol}^{[F]}(\Sigma))$ is a single element, we will say that $\text{hol}^{[F]}(\Sigma)$ is *gauge invariant* and abusively write $\text{hol}^{[F]}(\Sigma)$ instead of $q^{-1}(\text{hol}^{[F]}(\Sigma))$.

**Remark 3.3.163.** A result analogous to Theorem 3.3.159 was obtained in the context of a cubical category approach to 2-bundles in [MaPi11].

We now compare this result to that in [ScWa13], where the *reduced group* associated to a 2-group was introduced in order to obtain a well-defined 2-holonomy independent of the marking as well as the representative of the transport functor used.

**Definition 3.3.164.** Let $\mathcal{B}\mathcal{G}$ be a 2-group with associated crossed module $(H, G, \tau, \alpha)$. The *reduced group of $\mathcal{B}\mathcal{G}$* is $\mathcal{G}_{\text{red}} := H/[G, H]$, where $[G, H] = \ldots$
\[ \langle h^{-1} \alpha_g(h) \mid g \in G, h \in H \rangle, \text{i.e. the subgroup of } H \text{ generated by elements of } \text{the form } h^{-1} \alpha_g(h). \]

The analogue of the reduced 2-group in the case of ordinary holonomy for principal \( G \) bundles with connection is \( G/[G,G] \), the abelianization of \( G \). Recall, \( [G,G] = \langle gg'g^{-1}g'^{-1} \mid g, g' \in G \rangle \) is a normal subgroup, called the **commutator subgroup**, of \( G \) so the quotient is an abelian group, in fact in a universal sense.

**Lemma 3.3.165.** Let \( G \) be a group, \( [G,G] \) its commutator subgroup, and \( G/\text{Inn}(G) \) conjugacy classes in \( G \). The map \( G/\text{Inn}(G) \rightarrow G/[G,G] \) given by taking a conjugacy class \( [g] \), choosing a representative, and projecting to the quotient \( G/[G,G] \), is

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

**Proof.**

i) The map \( G/\text{Inn}(G) \rightarrow G/[G,G] \) is well-defined because if \( g' \) was another representative of \( [g] \), then there would be a \( \tilde{g} \in G \) such that \( \tilde{g}g\tilde{g}^{-1} = g' \),
and under the quotient map, the difference between $g$ and $g'$ is $g'g^{-1} = \tilde{g}g\tilde{g}^{-1}g^{-1} \in [G, G]$.

ii) Since $G \rightarrow G/[G, G]$ is surjective, and the map $G/\text{Inn}(G) \rightarrow G/[G, G]$ defined by choosing a representative is well-defined, the map

$$G/\text{Inn}(G) \rightarrow G/[G, G] \quad (3.3.166)$$

is surjective.

iii) To see why the map $G/\text{Inn}(G) \rightarrow G/[G, G]$ is, in general, not injective, consider the following example [DuFo04]. Let $S_n$ be the symmetric group on $n$ letters, i.e. it is the permutation group of $n$ elements. Let $A_n$ be the alternating group on $n$ letters. This group is defined as the kernel of the homomorphism $S_n \rightarrow \{-1, 1\}$ given by taking the sign of the permutation. It turns out this kernel is also the commutator subgroup of $S_n$. Furthermore, its index is $[S_n/[S_n, S_n]] = [S_n/A_n] = [S_n : A_n] = 2$.

On the other hand, let us compute the conjugacy classes of $S_n$ for some small $n$. The simplest case actually suffices, although we’ll quote some results for higher $n$ to indicate that the difference between conjugacy classes and abelianization gets bigger. For $n = 3$, the set of conjugacy classes in $S_3$ is given by the following elements. The identity element, written as ( ), is in its own class. The elements $(1, 2), (1, 3), \text{ and } (2, 3)$.
are in their own class. Finally, the elements \((1, 2, 3)\) and \((1, 3, 2)\) are in their own class. Therefore, the set of conjugacy classes for \(S_3\) is given by a 3-element set whereas the abelianization is a 2-element group. For \(S_4\), the set of conjugacy classes is a set of 5 elements. For \(S_5\), the set of conjugacy classes is a set of 7 elements. The abelianization, however, is always of order 2.

Therefore, conjugacy classes contain at least as much information about ordinary holonomy as do elements of the abelianization, and they are exactly the elements needed to define holonomy in a gauge invariant way due to Theorem 3.2.117.

In a similar way, the reduced group \(\mathcal{G}_{\text{red}}\) of a 2-group \(\mathcal{B}\) is analogous to the abelianization and does not contain the full information of 2-holonomy in general. One needs an analogue of conjugacy classes for 2-holonomy. The candidate, for spheres at least, is \(\alpha\)-conjugacy classes, \(H\{r\}\). In fact, we have a similar fact concerning \(\alpha\)-conjugacy classes and the reduced group.

**Lemma 3.3.167.** Let \((H, G, \tau, \alpha)\) be a crossed module, \(\mathcal{B}\) the associated 2-group, \(\mathcal{G}_{\text{red}} := H/[G, H]\) the reduced group of \(\mathcal{B}\), and \(H/\alpha\) the \(\alpha\)-conjugacy classes in \(H\). The map \(H/\alpha \longrightarrow \mathcal{G}_{\text{red}}\) given by taking a conjugacy class \([h]\),

\[\text{map: } H/\alpha \rightarrow \mathcal{G}_{\text{red}} \]
choosing a representative, and projecting to the quotient $H/[G,H]$, is

i) well-defined,

ii) surjective,

iii) and need not be injective in general.

Proof.

i) Let $h$ and $h'$ be two representatives. Then there exists a $g \in G$ such that
    \[ \alpha_g(h) = h' \]
    and so the difference between $h$ and $h'$ is $h^{-1}h' = h^{-1}\alpha_g(h) \in [G,H]$.

ii) Since $H \rightarrow H/[G,H]$ is surjective, and the map $H/\alpha \rightarrow \mathcal{G}_{\text{red}}$ defined by choosing a representative is well-defined, the map $H/\alpha \rightarrow \mathcal{G}_{\text{red}}$ is surjective.

iii) To see why the map $H/\alpha \rightarrow \mathcal{G}_{\text{red}}$ is, in general, not injective, consider the special case where $H = G$, $\tau = \text{id}$, and $\alpha$ is the ordinary conjugation. Then this case reduces to the previous case of Lemma 3.3.165.

Although the previous example suffices to show why $\alpha$-conjugacy classes $H/\alpha$ contain more information than the reduced group in general, holonomy along spheres takes values in $\ker \tau \leq H$ by the source-target matching condition. Therefore, it is also important to find an example of a
crossed module \((H, G, \tau, \alpha)\) such that \(\ker \tau = H\) and the map \(H/\alpha \to \mathfrak{G}_{\text{red}}\) is not injective.

Take \(H := \mathbb{Z}_p\), the (additive) cyclic group of order \(p\), where \(p \geq 3\) is prime. Set \(G := \text{Aut}(\mathbb{Z}_p)\), the automorphism group of \(\mathbb{Z}_p\). Let \(\tau\) be the trivial map and \(\alpha := \text{id}\) be the identity map. \((\mathbb{Z}_p, \text{Aut}(\mathbb{Z}_p), \tau, \alpha)\) defines a crossed module.

Every element of \(\text{Aut}(\mathbb{Z}_p)\) is of the form \(\sigma_k\) with \(k \in \{1, 2, \ldots, p-1\}\) and is determined by where it sends the generator: \(\sigma_k(1 \mod p) := k \mod p\). For this proof, denote the \(\alpha\)-conjugacy class of an element \(m \in \mathbb{Z}_p\) by \([m]\).

For all \(k\), \(\sigma_k(0 \mod p) = 0 \mod p\) so that \(0 \mod p\) is fixed under the \(\alpha\) action. However, since \(\sigma_k(1) = k \mod p\), the set of \(\alpha\)-conjugacy classes of \((\mathbb{Z}_p, \text{Aut}(\mathbb{Z}_p), \tau, \alpha)\) is \(\mathbb{Z}_p/\alpha = \{[0], [1]\}\), which is just a 2-element set.

However, the reduced group is trivial. To see this, consider generators of \([\text{Aut}(\mathbb{Z}_p), \mathbb{Z}_p]\), which are of the form \((\sigma_k(m) - m) \mod p\) with \(k \in \{1, 2, \ldots, p-1\}\) and \(m \in \{0, 1, 2, \ldots, p-1\}\). Set \(m = 1\) and \(k = 2\). Then \((\sigma_k(m) - m) \mod p = 1 \mod p\). Therefore, the generator of \(\mathbb{Z}_p\) is in the subgroup \([\text{Aut}(\mathbb{Z}_p), \mathbb{Z}_p]\) which means \([\text{Aut}(\mathbb{Z}_p), \mathbb{Z}_p] = \mathbb{Z}_p\). Thus \(\mathbb{Z}_p/([\text{Aut}(\mathbb{Z}_p), \mathbb{Z}_p]) = \mathbb{Z}_p/\mathbb{Z}_p \cong \{e\}\). 

\[\blacksquare\]
In this case, one can make sense of gauge-invariant quantities coming from 2-holonomy without passing to the reduced group as is done in [ScWa13]. In the case of the examples considered in Section 3.5, one even gets a fixed point under the $\alpha$ action, in which case one does not need to pass to the $\alpha$-conjugacy classes.

**Definition 3.3.168.** Let $(H, G, \tau, \alpha)$ be a crossed module. Denote the fixed points of $H$ under the $\alpha$ action by

$$\text{Inv}(\alpha) := \{ h \in H \mid \alpha_g(h) = h \text{ for all } g \in G \}. \quad (3.3.169)$$

**Lemma 3.3.170.** In the notation of Definition 3.3.168, $\text{Inv}(\alpha)$ is a central subgroup of $H$.

**Proof.** Let $h, h' \in \text{Inv}(\alpha)$. Then

$$\alpha_g(hh') = \alpha_g(h)\alpha_g(h') = hh' \quad (3.3.171)$$

for all $g \in G$. Thus, $\text{Inv}(\alpha)$ is closed. $\alpha_g(e) = e$ for all $g \in G$ says $e \in \text{Inv}(\alpha)$.

Let $h \in \text{Inv}(\alpha)$, then $\alpha_g(h^{-1}) = (\alpha_g(h))^{-1} = h^{-1}$ showing that $h^{-1} \in \text{Inv}(\alpha)$.

Finally, $\text{Inv}(\alpha)$ is central because

$$hh^{-1} = \alpha_{\tau(h)}(k) = k \quad (3.3.172)$$

for all $h \in H$ and $k \in \text{Inv}(\alpha)$. ■
This will have physical relevance when discussing monopoles, which, as we will show, take values in $\text{Inv}(\alpha)$.

### 3.4 The path-curvature 2-functor associated to a transport functor

In this section, given a principal $G$-bundle with connection and a choice of a subgroup of $\pi_1(G)$, we construct a principal 2-bundle with connection whose structure 2-group is a covering 2-group obtained from $G$ and the subgroup of $\pi_1(G)$. This assignment is functorial. We describe it on all levels introduced in the review, namely as a globally defined transport functor, in terms of descent data, and via differential cocycle data. These constructions respect all of the functors relating these different levels.

#### 3.4.1 The path-curvature 2-functor

The transport 2-functor defined later in this section is motivated by the study of magnetic monopoles in gauge theories as described in [ChTs93]. Some of the earlier accounts of similar descriptions can be found in the work of Wu and Yang in [WuYa75] under the name ‘total circuit’ and also in the work of Goddard, Nuyts, and Olive in [GoNuOl77]. Of course, several others worked on understanding the “topological quantum number” due to a magnetic charge in terms of just the magnetic charge alone, but the three references men-
tioned are the ones that have influenced us. We argue in Section 3.5 that in the case where the base space is a 3-manifold, this transport 2-functor has 2-holonomy along a sphere which is given by the magnetic flux through that sphere. Therefore, we give a mathematically rigorous description of non-abelian flux for magnetic monopoles in a non-abelian gauge theory. A more detailed description of the physics will be given in that section, but first we explain the mathematical structure.

The starting data consist of (i) a principal $G$-bundle, where $G$ is a connected Lie group, with connection over a smooth manifold $M$, and (ii) a subgroup $N$ of $\pi_1(G)$. By the main theorem of [ScWa09], the first part of the data corresponds to a transport functor $\text{tra} : \mathcal{P}_1(M) \to G$-Tor with $BG$ structure. From this data, we will construct a transport 2-functor which we call the path-curvature 2-functor. We will discuss two interesting cases for the choice of $N$ although other choices are important for applications in physics so we keep this generality for future applications. When $N = \pi_1(G)$, the path-curvature 2-functor coincides with the curvature 2-functor of Schreiber and Waldorf [ScWa13]. The choice $N = \{1\}$, the trivial group, will be more appropriate in the context of gauge theory and computing invariants. This is the case we focus on for all our computations in Section 3.5.

To set up this example, we introduce the following Lie 2-group associated
to any connected Lie group $G$. Let $\tilde{G}$ be the universal over of $G$ (we will describe what happens for arbitrary covers later) and denote the covering map by $\tau : \tilde{G} \to G$. An explicit construction of $\tilde{G}$ in terms of homotopy classes of paths will be useful for our purposes

$$\tilde{G} := \{ h : [0, 1] \to G \mid h(0) = e \text{ and } h \text{ is continuous} \}/\sim \quad (3.4.1)$$

where $h \sim h'$ if $h(1) = h'(1)$ and there exists a homotopy $h \Rightarrow h'$ relative the endpoints. $\tilde{G}$ naturally acquires a topology as the quotient space of a subspace of paths. Denote the equivalence class representing a path with square brackets as in $[h]$ or $[t \mapsto h(t)]$, where it is understood that $t$ takes values in $[0, 1]$. The multiplication in $\tilde{G}$ is defined by choosing representatives and multiplying them pointwise (later we will show that this multiplication can be described in another way that is sometimes more convenient for our examples). Let $\alpha : G \to \text{Aut}(\tilde{G})$ be the conjugation map $\alpha_g([h]) := [ghg^{-1}]$, meaning

$$\alpha_g([h]) := [t \mapsto gh(t)g^{-1}] \quad (3.4.2)$$

where the concatenation means multiplication in $G$. Define $\tau : \tilde{G} \to G$ to be evaluation at the endpoint,

$$\tau([h]) := h(1). \quad (3.4.3)$$
Proposition 3.4.4. \((\tilde{G}, G, \tau, \alpha)\) defined in the previous paragraph is a Lie crossed module.

Proof. It is useful to recall the definition of a crossed module (Definition 3.3.1) at this point. Since the equivalence relation involves homotopy relative endpoints, \(\tau\) is well-defined. \(\alpha\) is well-defined because \(h \sim h'\) implies \(ghg^{-1} \sim gh'g^{-1}\). The topological space \(\tilde{G}\) has a unique smooth structure making the map \(\tau\) a homomorphism and a smooth covering map, i.e. a smooth surjective submersion with the property that for every \(g \in G\), there exists an open neighborhood \(U\) containing \(g\) such that each component of \(\tau^{-1}(U)\) maps to \(U\) diffeomorphically. This follows from some basic differential topology (see for example Theorem 2.13 of [Le03]). Conjugation in \(G\) is a smooth map, and because \(\alpha\) is well-defined, \(\alpha\) is therefore smooth. The only things left to check are the crossed module identities. First, let \([h], [h'] \in \tilde{G}\) and let \(h\) and \(h'\) be representatives of \([h]\) and \([h']\) respectively. Then the map

\[
[0, 1] \times [0, 1] \ni (s, t) \mapsto h\left((1 - s) + st\right)h'(t)h\left((1 - s) + st\right)^{-1} \tag{3.4.5}
\]

is a homotopy (relative endpoints) from the path \(t \mapsto h(1)h'(t)h(1)^{-1}\) (when
s = 0) to the path \( t \mapsto h(t)h'(t)h(t)^{-1} \) (when \( s = 1 \)). Therefore,

\[
\alpha_{\tau([h])}([h']) = \left[ t \mapsto h(1)h'(t)h(1)^{-1}\right]
\]

\[
= [t \mapsto h(t)h'(t)h(t)^{-1}]
\]

\[
= [h][h'][h^{-1}],
\]

which is the first identity (3.3.2). For the second identity, let \( g \in G \) and \([h] \in \hat{G}\) with a representative \( h \). Then

\[
\tau(\alpha_g([h])) = \tau[t \mapsto gh(t)g^{-1}] = gh(1)g^{-1} = g\tau([h])g^{-1},
\]

which proves the other identity (3.3.3). ■

**Definition 3.4.8.** The Lie crossed module \((\hat{G}, G, \tau, \alpha)\) defined above is called the *universal cover crossed module* associated to a Lie group \( G \). The associated Lie 2-group, denoted by \( \mathcal{G}_{[1]} \), is called the *universal cover 2-group* associated to the Lie group \( G \).

In fact, the only way to give a smooth covering map a Lie crossed module structure is the way we have done so above. This follows from the following fact.

**Lemma 3.4.9.** Let \((H, G, \tau, \alpha)\) be a crossed module (not necessarily Lie) with \( \tau : H \to G \) a surjective homomorphism. Then \( \alpha \) is conjugation in \( H \) by a choice of lift, namely

\[
\alpha_g(h') = hh'h^{-1}, \quad \text{for all } g \in G, h' \in H
\]
for some \( h \) with \( \tau(h) = g \).

**Proof.** First we prove that conjugating by a lift is well-defined. Let \( \bar{h} \in H \) be another lift with \( \tau(\bar{h}) = g \). Then

\[
\begin{align*}
hh'h^{-1} \left( \bar{h}h'h^{-1} \right)^{-1} &= hh'h^{-1} \bar{h}h'h^{-1} \\
&= \alpha_{\tau(h)}(h') \alpha_{\tau(\bar{h})}(h'^{-1}) \quad \text{by (3.3.2)} \\
&= \alpha_g(h') \alpha_g(h'^{-1}) \\
&= \alpha_g(h'h'^{-1}) \\
&= \alpha_g(e) \\
&= e
\end{align*}
\]

since \( \alpha_g : H \rightarrow H \) is a homomorphism. The claim that \( \alpha_g(h') = hh'h^{-1} \) for a choice of lift \( h \) of \( g \) then follows from the identity (3.3.2) since \( \alpha_g(h') = \alpha_{\tau(h)}(h') = hh'h^{-1} \) for some \( h \) because \( \tau \) is surjective and a lift always exists.

\[\blacksquare\]

**Lemma 3.4.12.** Let \((H, G, \tau, \alpha)\) be a Lie crossed module with \( \tau : H \rightarrow G \) a smooth covering map. Then \( \alpha \) is conjugation in \( H \) by a choice of lift, namely

\[
\alpha_g(h') = hh'h^{-1}, \quad \text{for all } g \in G, h' \in H
\]

for some \( h \) with \( \tau(h) = g \).
Proof. The claim holds even if $\tau$ is just surjective. The proof follows from Lemma 3.4.9 viewing $H$ and $G$ as groups (ignoring smooth structure) and using the identity $\alpha_g(h') = \alpha_{\tau(h)}(h')$ for some lift $h$ of $g$. ■

Given any subgroup $N \subseteq \pi_1(G)$, we can construct another Lie 2-group in a similar way but by using a different equivalence relation. Define

$$\tilde{G}_N := \{ h : [0, 1] \to G \mid h(0) = e \text{ and } h \text{ is continuous} \}/\sim_N,$$  \hspace{1cm} (3.4.14)

where $h \sim_N h'$ if $h(1) = h'(1)$ and $\left[ \begin{array}{c} h \\ \overline{h'} \end{array} \right] \in N$, where $\overline{h'}$ denotes the reverse path and we use a vertical representation for the concatenation of paths in this context

$$h_\overline{h'}(t) := \begin{cases} h(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'(2-2t) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \hspace{1cm} (3.4.15)$$

**Definition 3.4.16.** An equivalence class of paths under the $\sim_N$ equivalence relation in equation (3.4.14) will be denoted by $[h]_N$ or $[t \mapsto h(t)]_N$ and will be called an $N$-class.

**Proposition 3.4.17.** Let $G$ be a connected Lie group, $N \subseteq \pi_1(G)$ a subgroup, and $\tilde{G}_N$ as in (3.4.14). Then for $[h]_N \in \tilde{G}_N$, the function $\tau : \tilde{G}_N \to G$ given by

$$\tau([h]_N) := h(1),$$  \hspace{1cm} (3.4.18)

with $h$ a choice of a representative of $[h]_N$, is a well-defined homomorphism.
Furthermore, $\tilde{G}_N$ has a unique smooth structure so that $\tau$ is a smooth covering map. Finally, $(\tilde{G}_N, G, \tau, \alpha)$ with $\alpha : G \longrightarrow \text{Aut}(\tilde{G}_N)$ defined by

$$\alpha_g([h]_N) := [t \mapsto gh(t)g^{-1}]_N$$

is a Lie crossed module.

Proof. $\tau$ is well-defined by definition of the equivalence relation $\sim_N$. $\tau$ is a homomorphism because $\tau([h]_N[h']_N) = h(1)h'(1) = \tau([h]_N)\tau([h']_N)$. $\tilde{G}_N$ has a natural topology coming from the quotient space of a subspace of paths in $G$. Because $\pi_1(G)$ is abelian, the conjugacy class of $N$ is $N$ itself. Therefore, by a standard theorem of constructing covering spaces (see for instance Chapter 3 of [Ma99]), $\tau : \tilde{G}_N \longrightarrow G$ is a covering map. By another standard result in differential topology (see Proposition 2.12 of [Le03]), there is a unique smooth structure on $\tilde{G}_N$ making $\tau$ a smooth covering map. By construction, $\tilde{G}_N$ has a continuous multiplication making it a topological group. The only things left to prove is that the multiplication and inversion maps in $\tilde{G}_N$ are smooth. This can be done locally using the smoothness of multiplication and inversion in $G$ and the fact that $\tau$ is a local diffeomorphism. Therefore, $\tilde{G}_N$ is a Lie group. Since $\tau$ is smooth, $\tau$ is a Lie group homomorphism. $\alpha_g$ is a well-defined group homomorphism for all $g \in G$ because it can be described as conjugation. It is smooth because for any $g \in G$, there exists an open
neighborhood $U$ around $g$, a diffeomorphism $\varphi : U \to V$, with $V$ a component of $\tau^{-1}(U)$, so that $U \ni g' \mapsto \alpha_{g'}$ coincides with conjugation by $\varphi(g')$ by the proof of Lemma 3.4.12. Since conjugation is smooth for any Lie group, $\alpha$ is smooth. Therefore, $(\tilde{G}_N, G, \tau, \alpha)$ is a Lie crossed module. ■

Note: We use the same notation $\tau$ and $\alpha$ for the maps instead of $\tau_N$ and $\alpha_N$ since we typically fix $N$ in any given context.

**Definition 3.4.20.** Let $G$ be a Lie group and $N$ a subgroup of $\pi_1(G)$. Then $(\tilde{G}_N, G, \tau, \alpha)$ as described in Proposition 3.4.17 is called the $N$-cover crossed module of $G$. Its associated 2-group is called the $N$-covering 2-group and is denoted by $B\tilde{G}_N$. We sometimes abusively say covering crossed module or covering 2-group without referring to $N$ explicitly.

Let $N \leq \pi_1(G)$ be a subgroup of the fundamental group of a Lie group $G$. We will now construct a 2-category $\tilde{G}$-Tor whose underlying 1-category $(\tilde{G}$-Tor$_N)_{0,1}$ (recall the notation from the beginning of Section 3.3.4) is $G$-Tor. Although the category $G$-Tor is not a Lie groupoid, notice that the set of morphisms between any two $G$-torsors is isomorphic to $G$ and therefore has a unique smooth structure. Furthermore, the composition is a smooth map and is modeled by the group multiplication map $G \times G \to G$. By this we mean that by choosing basepoints $a, b$, and $c$ in $G$-Torsors $A, B$, and $C$ respectively,
the composition

$$G\text{-Tor}(B, C) \times G\text{-Tor}(A, B) \to G\text{-Tor}(A, C) \quad (3.4.21)$$

agrees with the multiplication $G \times G \to G$ under the isomorphisms specified by the choice of basepoints. Therefore, the composition is smooth. Thus, $G\text{-Tor}$ is enriched in smooth manifolds. Using this fact, we can extend $G\text{-Tor}$ to an interesting 2-category $\underline{G\text{-Tor}}_N$ in a non-trivial way.

Let $A$ and $B$ be two $G$-torsors and let $\varphi, \psi : A \to B$ be two morphisms of $G$-torsors. We define the set of 2-morphisms from $\varphi$ to $\psi$, drawn as

$$B \xleftarrow{\varphi} A \xrightarrow{\psi}$$

(3.4.22)

to be the set of $N$-classes of paths from $\varphi$ to $\psi$ in $G\text{-Tor}(A, B)$. This means the following.

**Definition 3.4.23.** Let $N \leq \pi_1(G)$ be a subgroup. Two paths $\Sigma : \varphi \to \psi$ and $\Sigma' : \varphi \to \psi$ in $G\text{-Tor}(A, B)$, drawn as

$$B \xleftarrow{\varphi} A \xrightarrow{\psi}$$

(3.4.24)

are said to be $N$-equivalent if under the diffeomorphism defined by

$$G\text{-Tor}(A, B) \to G$$

$$\varphi \mapsto e,$$

(3.4.25)
the homotopy class of the loop \( \overset{\Sigma}{\circ} : \varphi \longrightarrow \varphi \), which gets sent to an element of \( \pi_1(G) \) under this diffeomorphism, is an element of \( N \). The class associated to \( \Sigma \) is called an \( N \text{-class of paths} \) and is denoted by \([\Sigma]_N\).

The choice of diffeomorphism (3.4.25) where \( \varphi \mapsto e \) is merely for convenience. In particular, the element \([\overset{\Sigma}{\circ} : \varphi \longrightarrow \varphi]\) is independent of this diffeomorphism. To see this, if any other diffeomorphism was chosen, say sending some other morphism \( \varphi' : A \longrightarrow B \) to \( e \in G \), then there exists a unique \( g \in G \) so that \( \varphi \circ g = \varphi' \) so that \( \varphi \mapsto g^{-1} \). In this case, one gets a loop based at \( g^{-1} \). To get one at \( e \), we merely multiply by \( g \) to obtain a loop based at \( e \in G \). This loop is exactly the same as \( \overset{\Sigma}{\circ} \) under the diffeomorphism defined by \( \varphi \mapsto e \). Therefore, the homotopy class is independent of the diffeomorphism chosen.

Vertical composition is defined on representatives as concatenation of paths. Horizontal composition can be defined using the \( G \times G \longrightarrow G \) multiplication. More explicitly, for two composable 2-morphisms as in

\[
\begin{array}{c}
C \xrightarrow{[\Sigma'\!]} B \xleftarrow{[\Sigma]} A,
\end{array}
\]

choose representatives of such paths so that \( \Sigma : [0,1] \longrightarrow G\text{-Tor}(A,B) \) and \( \Sigma' : [0,1] \longrightarrow G\text{-Tor}(B,C) \) with \( \Sigma(0) = \varphi, \Sigma(1) = \psi, \Sigma'(0) = \varphi', \) and \( \Sigma'(1) = \psi' \)
ψ'. Define the horizontal composition to be the $N$-class of the path $\Sigma' \circ \Sigma$ defined by

\[ s \mapsto (\Sigma' \circ \Sigma)(s) := \Sigma'(s) \circ \Sigma(s) \quad \text{for } s \in [0, 1], \quad (3.4.27) \]

where the composition on the right-hand-side is the usual composition of morphisms in $G$-Tor. We check that horizontal composition is well-defined. Suppose that $\Sigma \sim_N \Omega$ and $\Sigma' \sim_N \Omega'$. We must show that $\Sigma' \circ \Sigma \sim_N \Omega' \circ \Omega$, i.e.

\[ \left[ \frac{\Sigma' \circ \Sigma}{\Omega' \circ \Omega} \right] \in N \quad (3.4.28) \]

but a representative of this is given by

\[
\Sigma' \circ \Sigma \left( \frac{\Omega'}{\Omega} \right) (s) = \begin{cases} 
\Sigma'(2s) \circ \Sigma(2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\Omega'(2 - 2s) \circ \Omega(2 - 2s) & \text{for } \frac{1}{2} \leq s \leq 1 
\end{cases}
\]

\[ = \left( \frac{\Sigma'}{\Omega'} \right) (s) \circ \left( \frac{\Sigma}{\Omega} \right) (s) \quad (3.4.29) \]

which gives two elements of $N$ (after taking the homotopy class) and since $N$ is a subgroup the result is also an element of $N$. A similar argument is used to show that the interchange law holds. Therefore, $G$-$\text{Tor}_N$ defines a strict 2-category. We summarize this as a definition.

**Definition 3.4.30.** Let $G$ be a Lie group and $N \leq \pi_1(G)$ a subgroup of the fundamental group. The 2-category $G$-$\text{Tor}_N$ has objects and 1-morphisms that of $G$-Tor. The composition of 1-morphisms is the same as that in $G$-Tor.
The set of 2-morphisms between $G$-torsor morphisms $\varphi$ and $\psi$ in $G$-$\text{Tor}(A, B)$ are $N$-classes of paths from $\varphi$ to $\psi$. The vertical composition of 2-morphisms is concatenation of representative paths. The horizontal composition of 2-morphisms is the pointwise composition of $G$-torsor morphisms after choosing representatives.

**Remark 3.4.31.** When $N = \pi_1(G)$ the 2-categories $\widetilde{G}$-$\text{Tor}_N$ and $\widetilde{G}$-$\text{Tor}$ of $[\text{ScWa13}]$ are equivalent because there is a unique $\pi_1(G)$-class of paths between any two morphisms of $G$-torsors (since every loop is $\pi_1(G)$-equivalent to every other loop).

We will now start describing the path-curvature 2-functor, the structure 2-groupoid, and prove that it is indeed a transport 2-functor in the sense of Definition 3.3.73.

**Lemma 3.4.32.** Let $\text{tra} \in \text{Trans}^1_{BG}(M, G$-$\text{Tor})$ be a transport functor and let $N \leq \pi_1(G)$ be a subgroup. Let $K_N(\text{tra}) : \mathcal{P}_2(M) \to \widetilde{G}$-$\text{Tor}_N$ be the following assignment. At the level of objects and 1-morphisms $K_N(\text{tra})$ agrees with $\text{tra} : \mathcal{P}_1(M) \to G$-$\text{Tor}$. For every thin bigon $\Gamma : \gamma \Rightarrow \delta$ in $\mathcal{P}_2(M)$, choose a representative bigon, also denoted by $\Gamma$, and let

$$K_N(\text{tra})(\Gamma) := [s \mapsto \text{tra}(\Gamma(\cdot, s))]_N, \quad (3.4.33)$$
i.e. the $N$-class of the path from $\text{tra}(\gamma)$ to $\text{tra}(\delta)$ going along $\text{tra}(\Gamma(\cdot, s))$ as a function of $s \in [0,1]$. The notation means that $\Gamma(\cdot, s)$ is a thin path with respect to the first coordinate for each fixed $s$, and is depicted as a one-parameter family of $G$-torsor morphisms

$$B \xrightarrow{\varphi} \text{tra}(\Gamma(\cdot, s)) \xrightarrow{\psi} A. \quad (3.4.34)$$

This assignment is well-defined, i.e. the function $s \mapsto \text{tra}(\Gamma(\cdot, s))$ defines a continuous path and $K_N(\text{tra})(\Gamma)$ is independent of the choice of representative bigon.

$K_N$ is called the path-curvature 2-functor associated to $\text{tra}$ and $N \leq \pi_1(G)$.

Proof. The assignment in (3.4.33) is well-defined since ordinary homotopy is a special case of thin homotopy. More explicitly first notice that for a given bigon $\Gamma : \gamma \Rightarrow \delta$ the function $s \mapsto \text{tra}(\Gamma(\cdot, s))$ is smooth because $\text{tra}$ is a transport functor (this follows for instance from Theorem 3.12 of [ScWa09] and the fact that $G$-$\text{Tor}(\text{tra}(x), \text{tra}(y))$ is diffeomorphic to $G$). Now, suppose that $\Gamma'$ is another representative bigon for the thin bigon $\Gamma$. Then there exists a thin homotopy $H : [0,1] \times [0,1] \times [0,1] \rightarrow M$ with $H(t, s, 0) = \Gamma(t, s)$ and $H(t, s, 1) = \Gamma'(t, s)$. Thus $(s, r) \mapsto \text{tra}(H(\cdot, s, r))$ is a smooth
homotopy from \( s \mapsto \text{tra}(\Gamma(\cdot, s)) \) to \( s \mapsto \text{tra}(\Gamma'(\cdot, s)) \), which in particular is a homotopy. Thus \( K_N(\text{tra})(\Gamma) \) is well-defined. Similar arguments show that vertical and horizontal compositions are respected under this assignment. Therefore, \( K_N(\text{tra}) \) defines a strict 2-functor.

We construct a 2-functor \( i_N : \mathcal{B}G_N \longrightarrow \hat{G}\text{-Tor}_N \) as follows. By definition, a 2-morphism in \( \mathcal{B}G_N \) is of the form

\[
\begin{array}{c}
\bullet \\
\downarrow \text{[h]_N} \\
\downarrow h(1)g
\end{array}
\xleftarrow{g}
\begin{array}{c}
\bullet \\
\downarrow \text{[h]_N}g \\
\downarrow h(1)g
\end{array}
\] (3.4.35)

where \([h]_N\) is viewed as an \( N \)-class of a path \( h \) in \( G \) starting at the identity \( e \) in \( G \) and ending at a point written as \( \tau([h]_N) \equiv h(1) \). The image of this under \( i_N \) is defined to be

\[
\begin{array}{c}
G \\
\downarrow \text{[s\rightarrow L_{h(s)}g]_N} \\
\downarrow L_{h(1)g}
\end{array}
\xleftarrow{L_g}
\begin{array}{c}
G \\
\downarrow \text{[s\rightarrow L_{h(s)}g]_N} \\
\downarrow L_{h(1)g}
\end{array}

\] (3.4.36)

where \( s \mapsto L_{h(s)}g \) is the path in \( G\text{-Tor}(G, G) \cong G \) corresponding to the path \( s \mapsto h(s)g \) in \( G \) under this isomorphism. At this point it is not immediately clear why the vertical composition is respected under \( i_N \).

Lemma 3.4.37. Let \((H, G, \tau, \alpha)\) be a covering crossed module with elements of \( H \) thought of as certain equivalence classes of paths in \( G \) starting at the
Let \( h \) and \( h' \) be two representatives of elements \([h], [h'] \in H\).

Denote the targets of \( h \) and \( h' \) by \( g \) and \( g' \), respectively. Then

\[
[h'][h] = \begin{bmatrix} h & g \end{bmatrix},
\]

(3.4.38)

where \((h'g)(t) := h'(t)g\) for all \( t \in [0, 1]\), and the vertical composition is the composition of paths starting with the one on top.

**Proof.** A homotopy is given by

\[
(t, s) \mapsto \begin{cases} h'(st)h((2-s)t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ h'((2-s)t-1+s)h(st+1-s) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}
\]

(3.4.39)

with \( s = 0 \) projecting to \( \begin{bmatrix} h \\ h'g \end{bmatrix} \) and \( s = 1 \) projecting to \([h'h] = [h'][h]\). \(\blacksquare\)

We now come to one of our main theorems.

**Theorem 3.4.40.** The path-curvature 2-functor \( K_N(\text{tra}) \) defined above is a transport 2-functor with \( B\mathcal{G}_N \)-structure.

**Proof.** To prove this, we must provide a \( \pi_N \)-local \( \text{i}_N \)-trivialization of \( K_N(\text{tra}) \) and show that the associated descent object is smooth. This will be done in several steps, outlined as follows.

i) Define \( \text{triv}_N : \mathcal{P}_2(Y) \longrightarrow B\mathcal{G}_N \) and show it is a smooth strict 2-functor.

ii) Define a natural equivalence \( t_N : \pi^*K_N(\text{tra}) \Rightarrow \text{i}_N \circ \text{triv}_N \).
iii) Explicitly construct the associated descent object \((\text{triv}_N, g_N, \psi_N, f_N)\).

iv) Show that the descent object is smooth.

i) To start, \(\text{tra} : \mathcal{P}_1(M) \to G\)-Tor is assumed to be a transport functor, so there exists a \(\pi\)-local \(i\)-trivialization \((\text{triv} : \mathcal{P}_1(Y) \to BG, t : \pi^i_{\text{triv}_i} \Rightarrow \pi^i_{\text{triv}_i})\), where \(\pi : Y \to M\) is a surjective submersion, and whose associated descent object \(\text{Ex}_N^1(\text{tra}, \text{triv}, t)\) is smooth. We first define \(\pi_N : Y \to M\) to be \(\pi\). Then we define \(\text{triv}_N : \mathcal{P}_2(Y) \to BG_N\) by making it agree with \(\text{triv}\) on the 1-category \(\mathcal{P}_1(Y)\) inside \(\mathcal{P}_2(Y)\). For a thin bigon \(\Gamma : \gamma \Rightarrow \delta\) in \(Y\) we define

\[
\text{triv}_N(\Gamma) := \left( [s \mapsto \text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}]_N, \text{triv}(\gamma) \right) \in \tilde{G}_N \rtimes G. \tag{3.4.41}
\]

Note that \([s \mapsto \text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}]_N\) makes sense as an element of \(\tilde{G}_N\) because \(\tilde{G}_N\) is precisely defined to be the set of \(N\)-classes of paths in \(G\) starting at the identity of \(G\). This is well-defined because thin homotopy factors through ordinary homotopy (see the proof of Lemma 3.4.32).

We first prove that \(\text{triv}_N\) as defined is a strict 2-functor. It is already a strict 2-functor at the level of objects and 1-morphisms. We first check that vertical composition of bigons goes to vertical composition of
bigons. Consider two bigons $\Gamma : \gamma \Rightarrow \delta$ and $\Delta : \delta \Rightarrow \epsilon$. Their respected images under the assignment above gives

\[
\begin{align*}
\text{triv}(y) &\xleftarrow{\text{triv}^\theta(\Gamma)} \text{triv}(x) = \text{triv}(y) \xrightarrow{\text{triv}^\eta(\Delta)} \text{triv}(x),
\end{align*}
\]

which, after taking the $\tilde{G}_N$ component, gives

\[
\left[ s \mapsto \text{triv}(\Delta(\cdot,s))\text{triv}(\delta)^{-1}\text{triv}(\Gamma(\cdot,s))\text{triv}(\gamma)^{-1} \right]_N
\]

while first composing in $P_2(Y)$ and then applying $\text{triv}_N$ gives

\[
\text{triv}_N\left(\frac{\Gamma}{\Delta}\right) = \left[ s \mapsto \text{triv}\left(\frac{\Gamma}{\Delta}(\cdot,s)\right)\text{triv}(\gamma)^{-1}\right]_N, \text{triv}(\gamma).
\]

A homotopy between these two representatives is given by $H(s,r) :=$

\[
\begin{cases}
\text{triv}(\Gamma(\cdot,(r+1)s))\text{triv}(\gamma)^{-1} & \text{for } 0 \leq s \leq \frac{r}{2} \\
\text{triv}(\Delta(\cdot,(r+1)s-r))\text{triv}(\delta)^{-1} \times \text{triv}(\Gamma(\cdot,(r+1)s))\text{triv}(\gamma)^{-1} & \text{for } \frac{r}{2} \leq s \leq 1 - \frac{r}{2} \\
\text{triv}(\Delta(\cdot,(r+1)s-r))\text{triv}(\gamma)^{-1} & \text{for } 1 - \frac{r}{2} \leq s \leq 1
\end{cases}
\]

which indeed satisfies

\[
H(s,0) = \text{triv}(\Delta(\cdot,s))\text{triv}(\delta)^{-1}\text{triv}(\Gamma(\cdot,s))\text{triv}(\gamma)^{-1}
\]

and

\[
H(s,1) = \begin{cases}
\text{triv}(\Gamma(\cdot,2s))\text{triv}(\gamma)^{-1} & \text{for } 0 \leq s \leq \frac{1}{2} \\
\text{triv}(\Delta(\cdot,2s-1))\text{triv}(\gamma)^{-1} & \text{for } 1 - \frac{1}{2} \leq s \leq 1
\end{cases}
\]

\[\text{Technically, } \Delta : \delta' \Rightarrow \epsilon \text{ and there is a thin homotopy } \Sigma : \delta \Rightarrow \delta' \text{ but this means } \text{triv}(\delta) = \text{triv}(\delta') \text{ so the above statement still holds.}\]

\[\text{Again, this is technically not correct. One has to use a thin homotopy } \Sigma : \delta \Rightarrow \delta' \text{ but the reader can check that the proof follows through with a slightly modified homotopy.}\]
This proves more than what we needed since all we had to show was that the two elements are in the same $N$-class. Showing that the two representatives are homotopic is stronger and implies they are in the same $N$-class.

Now consider the horizontal composition of $\Gamma : \gamma \Rightarrow \delta$ and $\Pi : \alpha \Rightarrow \beta$ written as $\Pi \circ \Gamma : \alpha \circ \gamma \Rightarrow \beta \circ \delta$. First composing the thin bigons and then applying the map $\text{triv}_N$ gives

$$\text{triv}_N(\Pi \circ \Gamma) = \left( [s \mapsto \text{triv}((\Pi \circ \Gamma)(\cdot, s))\text{triv}(\alpha \circ \gamma)^{-1}]_N, \text{triv}(\alpha \circ \gamma) \right)$$

while first applying the map $\text{triv}$ to each thin bigon and then multiplying in $BG_N$ gives

$$p_{BG_N}(\text{triv}_N(\Pi)\text{triv}_N(\Gamma))$$

$$= [s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\alpha)^{-1}\text{triv}(\alpha)\text{triv}(\Gamma(\cdot, s))\text{triv}(\gamma)^{-1}\text{triv}(\alpha)^{-1}]_N$$

$$= [s \mapsto \text{triv}(\Pi(\cdot, s))\text{triv}(\Gamma(\cdot, s))\text{triv}(\alpha \circ \gamma)^{-1}]_N$$

$$= [s \mapsto \text{triv}((\Pi \circ \Gamma)(\cdot, s))\text{triv}(\alpha \circ \gamma)^{-1}]_N$$

because for every fixed $s$, parallel transport of paths is a homomorphism. Therefore, $\text{triv}_N$ defines a strict 2-functor.

We now show that $\text{triv}_N$ is a smooth 2-functor. We already know $\text{triv}_N$ is smooth at the level of objects and 1-morphisms. We must therefore
show $\text{triv}_N : P^2 Y \to \tilde{G}_N \times G$ is smooth. At this point, the reader should review the Appendix on smooth spaces in this chapter because we will recall several facts in the proof of this claim. By Definition 3.5.70, $\text{triv}_N$ is smooth if and only if for every plot $\varphi : U \to P^2 Y$, the composition $\text{triv}_N \circ \varphi : U \to \tilde{G}_N \times G$ is a plot. By Example 3.5.71, $\text{triv}_N \circ \varphi$ is a plot if and only if it is smooth. By Example 3.5.75, $\text{triv}_N \circ \varphi$ is smooth if and only if both projections $p_G \circ \text{triv}_N \circ \varphi$ and $p_{\tilde{G}_N} \circ \text{triv}_N \circ \varphi$ are smooth.

Since we already showed that $p_G \circ \text{triv}_N \circ \varphi = \text{triv} \circ s \circ \varphi$ is smooth (here $s$ is the source of a thin bigon), it remains to show that $p_{\tilde{G}_N} \circ \text{triv}_N \circ \varphi$ is smooth.

For convenience for this proof, set $f := p_{\tilde{G}_N} \circ \text{triv}_N$. By definition, $f \circ \varphi$ is given by

$$U \ni u \mapsto \left[ s \mapsto \text{triv}(\varphi(u)(\cdot, s)) \text{triv}(\varphi(u)(\cdot, 0))^{-1} \right]_N,$$

where we have chosen a representative bigon $\varphi(u) : [0, 1] \times [0, 1] \to Y$, fixed $s$ to get a thin path $\varphi(u)(\cdot, s)$, and then applied triv (unfortunately, there is a lot of abuse of notation to avoid an overabundance of brackets and symbols). The problem with this is that although we know we can always choose bigons $\varphi(u)$, these choices need not form a smooth family of bigons in an open neighborhood of $u \in U$. Therefore, proving
smoothness this way will not work.

Instead, we use the smooth structures we have defined to construct such a smooth family of bigons. \( P^2Y \) is the quotient of \( BY \), bigons in \( Y \), under thin homotopy and its smooth structure was defined as such. Therefore, by Example 3.5.73, \( \varphi : U \rightarrow P^2Y \) is a plot if and only if there exists an open cover \( \{U_j\}_{j \in J} \) of \( U \) and plots \( \{\varphi_j : U_j \rightarrow BY\}_{j \in J} \) such that

\[
\begin{array}{ccc}
BY & \xleftarrow{\varphi_j} & U_j \\
q & & \downarrow \\
P^2Y & \xleftarrow{\varphi} & U
\end{array}
\]

(3.4.51)

commutes for all \( j \in J \). For the purposes of this proof, \( q \) is the quotient map.

Now, \( BY \) itself is a subspace of the space of smooth squares \( Y^{[0,1]^2} \) in \( Y \). Denote the inclusion of \( BY \) into \( Y^{[0,1]^2} \) by \( k \). By Example 3.5.72, \( \varphi_j : U_j \rightarrow BY \) is a plot if and only if \( k \circ \varphi_j : U_j \rightarrow Y^{[0,1]^2} \) are plots. By Example 3.5.76, \( k \circ \varphi_j : U_j \rightarrow Y^{[0,1]^2} \) is a plot if and only if the associated function \( \widetilde{k \circ \varphi_j} : U_j \times [0, 1]^2 \rightarrow Y \) defined by \( \widetilde{k \circ \varphi_j}(u, t, s) := (k(\varphi_j(u)))(t, s) \) is smooth. This gives us our first desired fact: the plot \( \varphi : U \rightarrow P^2Y \) gives a smooth family of bigons \( \varphi_j : U_j \rightarrow BY \) such that \( q \circ \varphi_j = \varphi|_{U_j} \). Furthermore, since \( \widetilde{k \circ \varphi_j} \) is a smooth map of finite-dimensional manifolds, it is continuous and therefore the smooth family
of bigons is also continuous.

By using another adjunction, the smooth map \( \widetilde{k \circ \varphi_j} \) can be turned into a plot \( \widetilde{k \circ \varphi_j : U_j \times [0, 1] \to Y^{[0,1]} \) that factors through paths with sitting instants and is defined by \( \widetilde{k \circ \varphi_j(u, s)(t) := \left( k(\varphi_j(u)) \right)(t, s) \). Using this, we get a smooth map \( U_j \times [0, 1] \to G \) given by

\[
(u, s) \mapsto \text{triv}\left( \widetilde{k \circ \varphi_j(u, s)} \right) \text{triv}\left( \widetilde{k \circ \varphi_j(u, 0)} \right)^{-1}
\]

(3.4.52)

because \( \text{triv} \) is smooth on thin paths (we have taken the thin homotopy classes of the paths \( \widetilde{k \circ \varphi_j(u, s)} \) and \( \widetilde{k \circ \varphi_j(u, 0)} \) in the arguments of \( \text{triv} \)).

For each fixed \( u \in U_j \subset U \), this gives a path in \( G \) starting at \( e \) and the \( N \)-class of this path coincides with \( f(\varphi(u)) \) by commutativity of the diagram in (3.4.51). By continuity (which we proved in the previous paragraph), for each \( u \) there exists a (sufficiently small) contractible open neighborhood \( V \) of \( u \) with \( u \in V \subset U_j \) together with a (sufficiently small) contractible open neighborhood \( W \) of \( f(\varphi(u)) \) in \( \tilde{G}_N \) such that \( f(\varphi(V)) \subset W \) and \( W \) maps diffeomorphically to \( \tau(W) \subset G \) under the smooth covering map \( \tau \). But we just showed that the projection \( \tau \circ f \circ \varphi|_V : V \to G \) is smooth and since all neighborhoods are small and contractible, a lift is uniquely specified, is smooth, and agrees with \( f \circ \varphi|_V \). Therefore, \( f \circ \varphi \) is smooth at the point \( u \in U \). By applying this
argument to all plots at all points, this proves that $f = p_{\tilde{G}_N} \circ \text{triv}_N : P^2 Y \to \tilde{G}_N$ is smooth.

ii) Our goal now is to define a natural equivalence $t_N : \pi^* K_N(\text{tra}) \Rightarrow i_N \circ \text{triv}_N$. Note that since $\text{tra}$ is a transport functor, we have a natural isomorphism $t : \pi^* \text{tra} \Rightarrow i \circ \text{triv}$. Therefore, on points $y \in Y$, i.e. objects of $\mathcal{P}_2(Y)$, define $t_N(y) := t(y)$. For $\gamma \in P^1 Y$, since $t$ was a natural transformation for ordinary functors, the required diagram already commutes so we set $t_N(\gamma) := \text{id}$.

iii) Because of our definition of $\text{triv}_N$ and $t$ and since our target category is a strict 2-category, the associated descent data will not be too different from the ordinary transport functor case. Namely, the modifications $\psi_N$ and $f_N$ are both trivial, i.e. they are the identity 2-morphisms on objects. $g_N$ is also completely specified by $g$ since $t_N$ is specified by $t$.

iv) As mentioned above, $\text{triv}_N$ is smooth. What is left to show is that $\mathcal{F}(g_N) : \mathcal{P}_1(Y^{[2]}) \to \Lambda G\text{-Tor}_N$ is a transport functor with $\Lambda B\mathcal{G}_N$-structure. First let us explicitly describe $\Lambda B\mathcal{G}_N$ and $\Lambda G\text{-Tor}_N$. The objects of $\Lambda B\mathcal{G}_N$ are 1-morphisms of $B\mathcal{G}_N$ which are precisely elements of $G$. A morphism from $g_1$ to $g_2$ in $\Lambda B\mathcal{G}_N$ is a pair of elements $g_3$ and $g_4$ of $G$ along with
an element $h \in H$ fitting into the diagram

\[
\begin{array}{c}
\bullet & \xrightarrow{g_3} & \bullet \\
\downarrow{g_2} & \downarrow{(h,g_2g_3)} \quad & \downarrow{g_1} \\
\bullet & \xleftarrow{g_4} & \bullet
\end{array}
\]  

(3.4.53)

Similarly an object of $\Lambda\tilde{G}\text{-Tor}_N$ is a pair of objects $P$ and $P'$ in $\tilde{G}\text{-Tor}_N$ and a $G$-equivariant map $P \xrightarrow{f} P'$. A morphism from $P \xrightarrow{f} Q$ to $P' \xrightarrow{g} Q'$ in $\Lambda\tilde{G}\text{-Tor}_N$ is a pair of $G$-equivariant maps $p : P \longrightarrow P'$ and $q : Q \longrightarrow Q'$ along with an $N$-class of a path $\alpha : g \circ p \Rightarrow q \circ f$ as in the diagram

\[
\begin{array}{c}
P' & \xleftarrow{p} & P \\
\downarrow{g} & \downarrow{\alpha} \quad & \downarrow{f} \\
Q' & \xleftarrow{q} & Q
\end{array}
\]  

(3.4.54)

By applying the general definition of $F(g_N)$, we have

\[
\begin{array}{c}
Y^{[2]} \ni y \\
\xrightarrow{F(g_N)} \\
\downarrow{L_g(y)} \\
\xrightarrow{i(\text{triv}(\pi_1(y)))} G
\end{array}
\]

(3.4.55)

and

\[
\begin{array}{c}
P^1Y^{[2]} \ni \left( y' \xleftarrow{\gamma} y \right) \\
\xrightarrow{F(g_N)} \\
\downarrow{L_g(y')} \\
\xleftarrow{G} \\
\xrightarrow{L_g(y)} G \\
\xleftarrow{G} \\
\xrightarrow{\text{id}} \\
\xrightarrow{G}
\end{array}
\]

(3.4.56)

Now, since $g$ is part of the smooth descent object for the functor $\text{tra}$, there exists a smooth natural isomorphism $\tilde{g} : \pi_1^*\text{triv} \Rightarrow \pi_2^*\text{triv}$ such that
Using this fact, one can define $\tilde{g}_N : \pi_1^* \text{triv}_N \Rightarrow \pi_2^* \text{triv}_N$ in an analogous way to how $g_N$ was defined from $g$ but this time using $\tilde{g}$. Furthermore, $\mathcal{F}(g_N)$ factors through $\Lambda(i_N)$ via $\mathcal{F}(g_N) = \Lambda(i_N) \circ \mathcal{F}(\tilde{g}_N)$ since $g = \text{id}_i \circ \tilde{g}$.

Therefore, this defines a global trivialization with the identity surjective submersion $\text{id} : Y^{[2]} \longrightarrow Y^{[2]}$ with the trivialization functor being $\mathcal{F}(\tilde{g}_N) : \mathcal{P}_1(Y^{[2]}) \longrightarrow \Lambda \mathcal{B} \mathcal{G}_N$. This functor is smooth since $\tilde{g}$ is smooth.

Furthermore, the descent object associated to this transport functor is trivial because of the global trivialization. Thus $\mathcal{F}(g_N)$ defines a transport functor.

Thus $K_N(\text{tra})$ defines a transport 2-functor with $\mathcal{B} \mathcal{G}_N$ structure.

**Definition 3.4.57.** Let $\text{tra} : \mathcal{P}_1(M) \longrightarrow \mathcal{G} \text{-Tor}$ be a transport functor over $M$ with $\mathcal{B} \mathcal{G}$ structure and values in $\mathcal{G} \text{-Tor}$ and let $N \leq \pi_1(G)$ be a subgroup. Then the transport 2-functor $K_N(\text{tra}) : \mathcal{P}_2(M) \longrightarrow \mathcal{G} \text{-Tor}_N$ defined by

$$y \xleftarrow{\gamma} r \xrightarrow{\delta} x \mapsto \text{tra}(y) \left( \begin{array}{c} \text{tra}(\gamma) \\ \text{tra}(\delta) \end{array} \right) \xrightarrow{[s \mapsto \text{tra}(\gamma)(s)]_N} \text{tra}(x) \quad (3.4.58)$$

is called the *path-curvature transport 2-functor* associated to $\text{tra}$ and $N$.

More can be said, although we will not prove the details since the proofs are simple. The above construction is functorial. Namely, for any morphism
of parallel transport functors \( h : \text{tra} \Rightarrow \text{tra}' \) with \( BG \)-structure with values in \( G \)-Tor, there is a corresponding 1-morphism of parallel transport 2-functors \( h_N : K_N(\text{tra}) \Rightarrow K_N(\text{tra}') \) with \( B\mathcal{G}_N \)-structure with values in \( \widehat{G\text{-Tor}}_N \). By viewing \( \text{Trans}^1_{BG}(M, G\text{-Tor}) \) as a 2-category whose 2-morphisms are all identities, this defines a 2-functor

\[
K_N : \text{Trans}^1_{BG}(M, G\text{-Tor}) \longrightarrow \text{Trans}^2_{B\mathcal{G}_N}(M, \widehat{G\text{-Tor}}_N). \tag{3.4.59}
\]

In fact, in the above proof, in steps i) and ii), we have also outlined the definition of a 2-functor (see equation (3.4.41) and surrounding text)

\[
K^\text{Triv}_N : \text{Triv}^1_{\pi}(i)^\times \longrightarrow \text{Triv}^2_{\pi}(i_N)^\times \tag{3.4.60}
\]

given by the assignment

\[
(\text{tra}, \text{triv}, t) \mapsto (K_N(\text{tra}), \text{triv}_N, t_N := t) \tag{3.4.61}
\]
on objects (see Definitions 3.2.25 and 3.3.44) and

\[
\alpha \mapsto \alpha_N := \alpha \tag{3.4.62}
\]
on morphisms (see Definitions 3.2.27 and 3.3.46). In these two assignments, we are viewing a natural transformation as a pseudonatural transformation by assigning the identity 2-morphism to every 1-morphism.

In steps iii) and iv) we have also outlined the definition of a 2-functor

\[
K^\text{Des}_N : \text{Des}^1_{\pi}(i)^\times \longrightarrow \text{Des}^2_{\pi}(i_N)^\times \tag{3.4.63}
\]
given by the assignment

\[(\text{triv}, g) \mapsto (\text{triv}_N, g_N := g, \psi_N := 1, f_N := 1)\]  \hspace{1cm} (3.4.64)

on objects (see Definitions 3.2.31 and 3.3.48) and

\[h \mapsto (h_N := h, \epsilon_N := 1)\]  \hspace{1cm} (3.4.65)

on morphisms (see Definitions 3.2.35 and 3.3.52).

By definition, both squares in the diagram

\[
\begin{array}{ccc}
\mathcal{D}es^1(i)^\infty & \xleftarrow{\text{Ex}^1} & \text{Triv}^1(i)^\infty \\
\downarrow & & \downarrow \\
\mathcal{D}es^2(i_N)^\infty & \xleftarrow{\text{Ex}^2} & \text{Triv}^2(i_N)^\infty \\
\downarrow & & \downarrow \\
\mathcal{D}es^1 (i_N) & \xleftarrow{\text{Ex}^1} & \text{Triv}^1 (i_N) \\
\end{array} \\
\xleftarrow{\mathcal{K}_N^\text{ex}} \hspace{2cm} \xleftarrow{\mathcal{K}_N^\text{triv}} \hspace{2cm} \xleftarrow{\mathcal{K}_N}
\]

commute (on the nose).

The path-curvature 2-functor associated to a transport functor is flat. To explain this, we first define a modified version of the thin path 2-groupoid.

**Definition 3.4.67.** Let \(X\) be a smooth manifold. If one drops condition ii) from Definition 3.3.37, then one obtains a 2-groupoid \(\Pi_2(X)\) that has points of \(X\) as objects, thin paths for 1-morphisms, and (ordinary) homotopy classes of bigons for 2-morphisms.

[ScWa13] call this 2-groupoid the **fundamental 2-groupoid**. Although we prefer to use that terminology for the usual fundamental 2-groupoid (whose
1-morphisms are also ordinary homotopy classes of paths), we use this terminology here to avoid confusion.

**Definition 3.4.68.** A transport 2-functor $F : \mathcal{P}_2(M) \to T$ with Gr-structure is said to be flat if it factors through the fundamental 2-groupoid $\Pi_2(M)$.

The curvature 2-functor $K(\text{tra}) \equiv K_{\pi_1(G)}(\text{tra})$ introduced in [ScWa13] is completely determined on bigons by the boundary of the bigon. It is therefore obviously flat, but it is even more restrictive than just that. Not only does it not depend on the homotopy class of the bigon, it does not depend on the bigon at all. On the other hand, the path-curvature 2-functor $K'_N(\text{tra})$ introduced here depends on the homotopy class of the bigon.

**Corollary 3.4.69.** The path-curvature 2-functor $K'_N(\text{tra})$ is flat.

**Proof.** Let $\Gamma$ and $\Gamma'$ be two bigons that are smoothly homotopic (as opposed to just thinly homotopic). Let $H : [0,1]^3 \to Y$ be a smooth homotopy from $\Gamma$ to $\Gamma'$ so that $H(t, s, 0) = \Gamma(t, s)$ and $H(t, s, 1) = \Gamma'(t, s)$. By compactness of $[0,1]^3$, one can choose $H$ so that it has sitting instants around its boundary so that $\text{tra}(H(\cdot, s, r))$ is well-defined for each $s, r \in [0,1]$. Then

$$
[0,1] \times [0,1] \to G
$$

$$(s, r) \mapsto \text{tra}(H(\cdot, s, r))$$

(3.4.70)
is a smooth homotopy from the path \( s \mapsto \text{tra}(\Gamma(\cdot, s)) \) to the path \( s \mapsto \text{tra}(\Gamma'(\cdot, s)) \). Therefore, since \( N \)-classes of paths is a quotient of the universal cover \( \tilde{G} \), the \( N \)-classes of these paths agree.

3.4.2 A description in terms of differential form data

In this section, we prove several important and useful facts. The first theorem says that \( locally \) transport functors whose structure 2-group is a covering 2-group can be described in terms of the path-curvature 2-functor. The second part of this section contains a discussion about the relationship between the path-curvature 2-functor specifically and its differential cocycle data. As before, let \( \pi : Y \to M \) denote a surjective submersion, \( G \) a connected Lie group, \( N \leq \pi_1(G) \) a subgroup, and \( \tau : \tilde{G}_N \to G \) the cover of \( G \) defined by \( N \).

It is important to note that \( \tau : \tilde{G}_N \to G \), the induced map of Lie algebras, is an isomorphism of Lie algebras because \( \tau \) is a local diffeomorphism. Denote the 2-group associated to the Lie crossed module \((\tilde{G}_N, G, \tau, \alpha)\) by \( B\mathcal{G}_N \).

First, we define a 2-functor \( K_N^Z : Z_1^2(G) \to Z_2^2(\mathcal{G}_N) \) by

\[
(A, g) \mapsto \left( \left( A, B := \tau^{-1} \left( dA + \frac{1}{2}[A, A] \right) \right), (g, \varphi := 1), (\psi := 1, f := 1) \right)
\]

\[
h \mapsto (h, \varphi := 0)
\]

on objects and morphisms, respectively.

Second, notice that specifically for the path-curvature 2-functor \( K_N(\text{tra}) \),
and particularly its associated descent object $K_N^{\text{Des}}(\text{tra})$, the analysis in Section 3.3.6 gives the following differential cocycle data associated to $K_N(\text{tra})$.

The assignment on thin paths induces a 1-form $A$ with values in $G$ since the functor $K_N(\text{tra})$ agrees precisely with $\text{tra}$ on thin paths. On thin bigons, the assignment induces a 2-form $B$ with values in $\tilde{G}$ satisfying $dA + \frac{1}{2}[A, A] = \tau(B)$. Since $\tau$ is an isomorphism, $B$ is determined by this condition and is given by $B = \tau^{-1}(dA + \frac{1}{2}[A, A])$. Therefore, the associated differential cocycle data to the path-curvature 2-functor $K_N(\text{tra})$ is

$$\mathcal{D}(K_N^{\text{Des}}(\text{triv}, g)) = \left( A, B := \tau^{-1}\left( dA + \frac{1}{2}[A, A] \right), g, \varphi = 0, f = 1, \psi = 1 \right).$$

(3.4.72)

Therefore, the two descriptions agree showing that the diagram

$$
\begin{array}{ccc}
Z^1_\pi(G) & \xrightarrow{D} & \mathcal{D} \mathcal{S}_\pi^1(i)^\infty \\
K^2_N & \xrightarrow{} & K_N^{\text{Des}} \\
Z^2_\pi(G_N) & \xrightarrow{D} & \mathcal{D} \mathcal{S}_\pi^2(i_N)^\infty
\end{array}
$$

(3.4.73)

commutes. This analysis is actually a bit more general as the following theorem shows.

**Theorem 3.4.74.** Let $X$ be a smooth manifold and $F_N : \mathcal{P}_2(X) \to BG_N$ be any smooth strict 2-functor. Then there exists a unique smooth functor $F : \mathcal{P}_1(X) \to BG$ such that $F_N = K_N(F)$. 

Proof. The functor $D_X : \text{Funct}^X(\mathcal{P}_2(X), \mathcal{BG}_N) \rightarrow Z^2_X(\mathcal{G}_N)^{\infty}$ (defined around (3.3.109) in Section 3.3.6) produces $(A \in \Omega^1(X; \mathcal{G}), B \in \Omega^2(X; \mathcal{G}))$ satisfying $dA + \frac{1}{2}[A, A] = \tau(B)$. Since $\tau : \tilde{\mathcal{G}}_N \rightarrow \mathcal{G}$ is an isomorphism, $B = \tau^{-1}(dA + \frac{1}{2}[A, A])$. Restricting $F_N$ to $\mathcal{P}_1(X)$ gives a unique $F : \mathcal{P}_1(X) \rightarrow \mathcal{B}G$ that satisfies $D_X(F) = A$. By the the same token, we have $D_X(K_N(F)) = (A, \tau^{-1}(dA + \frac{1}{2}[A, A]))$ which coincides with $D_X(F_N)$. Since $\mathcal{P}_X : Z^2_X(\mathcal{G}_N)^{\infty} \rightarrow \text{Funct}^X(\mathcal{P}_2(X), \mathcal{BG}_N)$ is a strict inverse to $D_X$ by Theorem 2.21 of [ScWa11], we conclude that $F_N = K_N(F)$.  

This theorem implies the following interesting and simple explicit formula for local 2-holonomy for transport 2-functors with covering 2-groups as their structure 2-groupoids. This is another one of our main results.

**Corollary 3.4.75.** The formula for local parallel transport for any bigon under any smooth 2-functor $F_N : \mathcal{P}_2(X) \rightarrow \mathcal{B}G_N$ is given by the formula

$$F_N\left(\begin{array}{c} y \\ \delta \end{array}\right) = \bullet \left(\begin{array}{c} F(\gamma) \\ F(\delta) \end{array}\right)$$

where $F$ is the 2-functor $F_N$ restricted to 1-morphisms.

Finally, by Corollary 4.9 of [ScWa09], Theorem 2.21 of [ScWa11], and
Proposition 4.1.3 of [ScWa13], the functors $P$ in each row of

\[
\begin{array}{ccc}
Z^1_n(G)^{\infty} & \xrightarrow{P} & \mathcal{D} \mathfrak{es}_{\pi}^1(i)^{\infty} \\
\kappa_{\mathcal{N}} \downarrow & & \downarrow \kappa_{\mathcal{R}^{\pi}} \\
Z^2_n(G_N)^{\infty} & \xrightarrow{P} & \mathcal{D} \mathfrak{es}_{\pi}^2(i_N)^{\infty}
\end{array}
\]

are (weak) inverses to $\mathcal{D}$ so this diagram commutes weakly.

3.5 Examples and magnetic monopoles

As briefly mentioned above, the path-curvature transport 2-functor is motivated by constructions in physics. In 1931, Dirac studied the charge of a magnetic monopole in $\mathbb{R}^3$ and found it to be quantized and proportional to $\int_{S^2} R$, where $S^2$ is a sphere enclosing the magnetic monopole and $R$ is the curvature of the $U(1)$ bundle with connection over $\mathbb{R}^3 \setminus \{\ast\}$ where $\{\ast\} \subset \mathbb{R}^3$ is the location of the monopole [Di31]. Of course, the language of bundles and connections was not around at the time, but the ingredients were there. Because $R$ is well-defined globally, the integral $\int_{S^2} R$ is unambiguously defined. Furthermore, it is a topological invariant in the sense that it only depends on the homotopy class of the sphere in $\mathbb{R}^3 \setminus \{\ast\}$. However, for a non-abelian principal $G$-bundle with connection, $R$ is not globally defined so it was not clear how to define the magnetic charge. In [WuYa75], [ChTs93], and [GoNuOl77] the authors define the charge of a magnetic monopole in terms of a
magnetic flux through a sphere by calculating the holonomy along a family of loops as in Figure 3.16. This defines a loop at the identity of the group. Taking the homotopy class of this loop was the definition of the magnetic charge in the physics literature. Goddard, Nuyts, and Olive were closer to defining this flux as a double-path-ordered integral, but stopped short and used other means to analyze it [GoNuOl77].

We want to point out here that it is not obvious that the methods described in the literature make sense. For instance, is it necessary to begin with the constant loop? What should this loop have anything to do with a magnetic flux, which was defined in the abelian case to be $\int_{S^2} R$. Is the resulting quantity gauge invariant? What does gauge invariance even mean? And how does one know that these concepts are even correct?

As we show in this section, the path-curvature transport 2-functor introduced in the previous section describes magnetic flux in terms of surface holonomy. Furthermore, since this magnetic flux is defined using surface holonomy, for which we have proven gauge covariance in Section 3.3.8 (specifically Theorem 3.3.159), we can meaningfully ask if the magnetic flux is a gauge invariant quantity. This would be the case if it is invariant under $\alpha$-conjugation. We review the interesting cases considered in the physics literature, those of $U(1)$ monopoles, $SO(3)$ monopoles, and $SU(n)/Z(n)$ for all
for all $n$. We also consider the cases $U(n)$ for all $n$. For all of these examples, we take the subgroup $N \leq \pi_1(G)$ to be $N = \{1\}$, the trivial subgroup of $\pi_1(G)$. This case is interesting in its own right as the examples will illustrate.

We do this in two ways. We first start with a transport functor, described in terms of its differential cocycle data, and use the methods of Section 3.3.5 and Section 3.3.8 to reconstruct a transport functor with group-valued holonomies. We then construct the path-curvature 2-functor and compute surface holonomy. The other method we use, which is equivalent by Theorem 3.4.74 and Corollary 3.4.75, is to use the surface-ordered integral of equation (3.3.97) from [ScWa11] and the definition of the differential cocycle data of the path-curvature 2-functor discussed in Section 3.4.2. This is unnecessary due to Corollary 3.4.75 but we do it anyway for the reader’s convenience. In the process, we must choose weak inverses $s^\pi : \mathcal{P}_2(M) \to \mathcal{P}_2^\pi(M)$ to the projections $p^\pi : \mathcal{P}_2^\pi(M) \to \mathcal{P}_2(M)$ associated to some surjective submersion $\pi : Y \to M$. We will define the 2-functor $s^\pi$ for the paths and bigons of interest to us (rather than defining it for all paths and bigons) in the case of the first example of $U(1)$ monopoles. We then use the same 2-functor $s^\pi$ for all other examples.

For the following discussions, we will be using the following conventions depicted in Figure 3.24 for describing coordinates on the sphere.
3.5.1 Abelian $U(1)$ monopoles

First, we will give an explicit example coming from abelian magnetic monopoles. Let $P[n] \rightarrow S^2$ be the principal $U(1)$-bundle described by the following local trivialization. Denote the northern and southern hemispheres by $U_N$ and $U_S$, respectively. We assume that $U_N$ extends a little bit to the southern hemisphere so that $U_{NS} \neq \emptyset$ (and similarly for $U_S$ to the northern hemisphere).

Let $Y := U_N \coprod U_S$ and $\pi : Y \rightarrow S^2$ be the projection. Let $s_N : U_N \rightarrow Y$ and $s_S : U_S \rightarrow Y$ be the obvious sections. Define the transition function $g_{NS} : U_{NS} \simeq S^1 \rightarrow U(1)$ along the equator to be

$$S^1 \ni \phi \mapsto g_{NS}(\phi) := e^{i n \phi}, \quad (3.5.1)$$
where $\phi$ is the azimuthal angle and $n$ is an integer. Equip this bundle with a connection $A_N \in \Omega^1(U_N; \underline{U(1)})$ and $A_S \in \Omega^1(U_S; \underline{U(1)})$ given by

$$A_N = \frac{n}{2\pi}(1 - \cos \theta)d\phi \quad \& \quad A_S = -\frac{n}{2\pi}(1 + \cos \theta)d\phi. \quad (3.5.2)$$

These forms satisfy the condition

$$A_N = g_{NS}A_Sg_{NS}^{-1} - dg_{NS}g_{NS}^{-1} \quad (3.5.3)$$
on $U_{NS}$ so that $g_{NS}, A_N,$ and $A_S$ are the local differential cocycle data of a principal $U(1)$-bundle with connection. Since $i : BU(1) \longrightarrow U(1)$-Tor is an equivalence of categories, this differential cocycle data corresponds to a global transport functor (recall (3.2.83)).

We now consider the path-curvature 2-functor where $N = \{1\} \leq \pi_1(S^1) \cong \mathbb{Z}$ so that the associated covering 2-group is $(\mathbb{R}, U(1), \tau, \alpha)$ with $\tau : \mathbb{R} \longrightarrow U(1)$ the universal covering map defined by $\phi \mapsto e^{2\pi i\phi}$. The functor $\mathcal{P} : Z^1_\pi(G) \longrightarrow \mathcal{O} \mathcal{E} \mathcal{S}_\pi(i)$ sends the differential cocycle object $(g, A)$ to $\text{triv} : \mathcal{P}_1(U_N \sqcup U_S) \longrightarrow B\mathcal{G}$ defined by the path-ordered exponential and the natural transformation $g : \pi^*_1(\text{triv}_i) \Rightarrow \pi^*_2(\text{triv}_i)$ defined on components $\phi \in S^1$ by $i(g_{NS}(\phi))$. We partially define $s^\pi : \mathcal{P}_2(S^2) \longrightarrow \mathcal{P}^\pi_2(S^2)$ as follows. We first make the choice

$$s^\pi(x) := \begin{cases} s_N(x) & \text{if } x \in U_N \\ s_S(x) & \text{if } x \in S^2 \setminus U_N \end{cases} \quad (3.5.4)$$

for objects. We will be a little sloppy now and define a lift of thin paths and thin bigons on representatives of thin homotopy classes. We only lift paths,
labelled as $\gamma_\theta$, of the form depicted in Figure 3.25. The reason for this is

Figure 3.25: A loop on the sphere is made to always start at the equator at the point $\bullet$. In this figure, the loop is drawn for some $\theta$ in the range $\frac{\pi}{2} < \theta < \pi$.

because we will consider a sequence of such loops starting at the constant loop at the point $\bullet$ on the equator (so that $s^\pi(\bullet) = (\bullet, N)$) enclosing the sphere going from $U_N$ to $U_S$ and finally ending on the constant loop at the point $\bullet$ as depicted in Figure 3.26. Therefore, we define the assignment on

Figure 3.26: Loops along the $\phi$ direction on the sphere of constant $\theta$ are drawn for $\theta = \frac{\pi}{2}$ and two intermediate values in the range $0 < \theta < \frac{\pi}{2}$. However, each loop is made to start at the point $\bullet$ so that the sphere is thought of as a bigon $S^2 : \text{id}_\bullet \Rightarrow \text{id}_\bullet$. 
these loops to be

\[
\begin{cases}
    s_{N\ast}(\gamma_\theta) & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
    \alpha_{NS}(\bullet) \ast s_{S\ast}(\gamma_\theta) \ast \alpha_{SN}(\bullet) & \text{if } \frac{\pi}{2} < \theta \leq \pi.
\end{cases}
\]

(3.5.5)

We now define the lift on two bigons. The first bigon \(\Sigma_N\) is given by

\[
[0, 2\pi] \times [0, \pi/2] \ni (\phi, \theta) \mapsto \Sigma_N(\phi, \theta) := \gamma_\theta(\phi)
\]

(3.5.6)

and is a bigon \(\gamma_{\pi/2}\) which lands in \(U_N\) and covers the northern hemisphere. We send this bigon to \(s^\pi(\Sigma_N) := s_{N\ast}(\Sigma_N)\) in \(P_2^\pi(S^2)\) because our prescription (3.3.85) says

\[
\begin{align*}
    s^\pi(\Sigma_N) & := s^\pi(\bullet) \xrightarrow{id} s_N(\bullet) \xrightarrow{s_{N\ast}(\Sigma_N)} s_N(\bullet) \xrightarrow{id} s^\pi(\bullet) \\
    & \xrightarrow{s_{N\ast}(\gamma_{\pi/2})} \xrightarrow{id} \xrightarrow{s^\pi(\gamma_{\pi/2})}
\end{align*}
\]

(3.5.7)

We do a similar thing for the bigon \(\Sigma_S\) given by

\[
[0, 2\pi] \times (\pi/2, \pi] \ni (\phi, \theta) \mapsto \Sigma_S(\phi, \theta) := \gamma_\theta(\phi)
\]

(3.5.8)

which is a bigon \(\gamma_{\pi/2} \Rightarrow \text{id}_\bullet\) that lands in \(U_S\). This is a bigon covering the southern hemisphere. However, our boundary data need to match up so that
we can compose in $\mathcal{P}_2^n(S^2)$. Again, following (3.3.85)), this is given by

$$s^n(\Sigma_S) := s^n(\bullet) \xrightarrow{\alpha_{SN}(\bullet)} s_S(\bullet) \xrightarrow{s_{S*}(\Sigma_S)} s_S(\bullet) \xrightarrow{\alpha_{SN}(\bullet)} s^n(\bullet), \quad (3.5.9)$$

where the $!$ signifies the unique 2-isomorphisms from Lemma 3.3.83. For the full bigon $\Sigma : \text{id}_\bullet \Rightarrow \text{id}_\bullet$ depicting the full sphere as the composition $\Sigma_N : \text{id}_\bullet \Rightarrow \gamma_{\pi/2} \Rightarrow \text{id}_\bullet$, we break it up into the two pieces defined above and compose vertically. The result of this is

$$s^n(\text{id}_\bullet) = \alpha_{SN}(\text{id}_\bullet)$$

We rescale our angle $\theta$ to $s = \frac{\theta}{\pi}$ to be consistent with our earlier notation. Going from $Z_2^2(\mathcal{BG}_{(1)})^x$ to $\text{Trans}^2_{\mathcal{BG}_{(1)}}(M, G^{\text{Tor}_{(1)}})$ from above to define the global transport functor applied to the sphere, we obtain the following dia-
gram in $G$-Tor$_{(1)}$

\[
\begin{array}{cccc}
G & \overset{id_G}{\Rightarrow} & G & \overset{id_G}{\Rightarrow} G \\
\downarrow & & \downarrow & & \downarrow \\
G & \overset{id_G}{\Rightarrow} & G & \overset{id_G}{\Rightarrow} G \\
\end{array}
\]

\[\left(3.5.11\right)\]

since $i_{(1)} \circ \text{triv}(y) = G$ for all $y$ and so on for paths and bigons (see the definition of $R_{(\text{triv},g,\psi,f)}$ in Section 3.3.5) and $g_{NS}(\phi = 0) \equiv g_{NS}(\bullet) = 1$.

Furthermore, $g_{NS}$ on paths is the identity since $g_{NS}$ came from a natural transformation of ordinary functors between ordinary categories. With these simplifications, the composition in (3.5.11) is given by

\[
\left[s \mapsto \begin{cases} 
L_{\text{triv}}(\Sigma_N(\cdot,2s)) & \text{for } 0 \leq s \leq \frac{1}{2} \\
L_{\text{triv}}(\Sigma_S(\cdot,2s-1)) & \text{for } \frac{1}{2} \leq s \leq 1 
\end{cases} \right],
\]

\[\left(3.5.12\right)\]

which reduces to a computation on the group level. Therefore, all we have to do is compute the homotopy class of the path

\[
\left[s \mapsto \begin{cases} 
\text{triv}(\Sigma_N(\cdot,2s)) & \text{for } 0 \leq s \leq \frac{1}{2} \\
\text{triv}(\Sigma_S(\cdot,2s-1)) & \text{for } \frac{1}{2} \leq s \leq 1 
\end{cases} \right]
\]

\[\left(3.5.13\right)\]
in the group $U(1)$ thanks to Lemma 3.4.37. This is easily calculable

$$\text{triv}(\Sigma_N(\cdot, 2s)) = \text{triv}\left(\Sigma_N\left(\cdot, \frac{2\theta}{\pi}\right)\right)$$

$$= e^{\frac{n}{\pi} \int_0^{2\pi} (1 - \cos \theta) d\phi}$$

$$= e^{-in\pi (1 - \cos \theta)}$$

(3.5.14)

since the paths going along $\theta$ do not contribute to the parallel transport since
the connection form only has a $d\phi$ contribution. Similarly,

$$\text{triv}(\Sigma_S(\cdot, 2s - 1)) = \text{triv}\left(\Sigma_S\left(\cdot, \frac{2\theta}{\pi} - 1\right)\right) = e^{in\pi(1 + \cos \theta)}. \quad (3.5.15)$$

As a sanity check, notice that

$$e^{-in\pi (1 - \cos \frac{\pi}{2})} = e^{-in\pi} = e^{in\pi} = e^{in\pi (1 + \cos \frac{\pi}{2})}$$

(3.5.16)

showing that the matching condition (so that our path is continuous) is satisfied. This matching condition was the one used, for instance, in [WuYa75] (see equation (47)).

Notice that $1 - \cos \theta$ is a monotonically increasing function of $\theta$ for
$0 \leq \theta \leq \frac{\pi}{2}$ starting at 0 when $\theta = 0$ and ending at 1 when $\theta = \frac{\pi}{2}$. Therefore,
$e^{-in\pi(1 - \cos \theta)}$ winds around the circle starting at 1 and ending at $e^{-in\pi} = (-1)^n$
winding around monotonically $\frac{n}{2}$ times clockwise if $n$ is positive and counterclockwise otherwise. Now, the function $1 + \cos \theta$ is a monotonically decreasing function of $\theta$ for $\frac{\pi}{2} \leq \theta \leq \pi$ starting at $(-1)^n$ when $\theta = 0$ and ending
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at 1 when $\theta = \pi$. Therefore, $e^{in\pi(1+\cos \theta)}$ winds around the circle starting at $e^{in\pi} = (-1)^n$ and ending at 1 winding around monotonically $\frac{n}{2}$ times clockwise if $n$ is positive and counterclockwise otherwise. In other words, the loop goes a total of $n$ times around clockwise if $n$ is positive and $n$ times counterclockwise if $n$ is negative and the 2-holonomy along $S^2$ is given by (using the notation of Definition 3.3.161)

$$\text{hol}^{[n]}(S^2) = -n. \quad (3.5.17)$$

If we wanted to, we could have also computed this using differential forms and the formula for 2-transport (3.3.97) of Schreiber and Waldorf [ScWa11] locally and pasted the group elements together vertically as above. Of course, by the equivalence between local smooth functors and differential forms, our formula in terms of ordinary holonomy bypasses the rather (a-priori) complicated surface holonomy formula (3.3.97) due to Corollary 3.4.75. It will actually turn out that the surface holonomy formula (3.3.97) is not so complicated in this particular case due to our choice of bigon representing the sphere and the differential forms representing the connection, i.e. our choice of gauge. We will subsequently do this analysis strictly in terms of the differential forms associated to the path-curvature 2-functor discussed in Section 3.4.2.
The curvature is given by

\[ R_N = \frac{n}{2i} \sin \theta d\theta \wedge d\phi \in \Omega^2(U_N; U(1)) \]  

(3.5.18)

and similarly for \( R_S \in \Omega^2(U_S; U(1)) \). Therefore, the connection 2-form is given by

\[ B_N = \tau^{-1}(R_N) = \frac{1}{2\pi i} R_N = -\frac{n}{4\pi} \sin \theta d\theta \wedge d\phi \]  

(3.5.19)

and similarly for \( B_S \). The 1-form \( A_{\Sigma_N} \) (see equation (3.3.95)) is given by

\[ (A_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) = -\int_0^{2\pi} d\phi \, B_{(\theta,\phi)} \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right) = \frac{n}{2} \sin \theta \]  

(3.5.20)

and the 2-transport along \( \Sigma_N \) is given by

\[ k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{ -\int_0^{\pi/2} d\theta \, (A_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) \right\} \]  

(3.5.21)

\[ = -\int_0^{\pi/2} d\theta \, \frac{n}{2} \sin \theta \]

because the exponential map \( \mathbb{R} \rightarrow \mathbb{R} \) is the identity. The 2-transport along \( \Sigma_S \) is done similarly and is given by

\[ k_{A,B}(\Sigma_S) = -\int_{\pi/2}^\pi d\theta \, \frac{n}{2} \sin \theta. \]  

(3.5.22)

Vertically composing these results yields

\[ k_{A,B}(\Sigma_S) + k_{A,B}(\Sigma_N) = -\int_{\pi/2}^\pi d\theta \, \frac{n}{2} \sin \theta - \int_0^{\pi/2} d\theta \, \frac{n}{2} \sin \theta \]

\[ = -\int_0^\pi d\theta \, \frac{n}{2} \sin \theta \]  

(3.5.23)

\[ = -n \]
because the group operation in $\mathbb{R}$ is addition. Therefore, the result obtained in terms of the path-curvature 2-functor in terms of homotopy classes of paths in $G$ agrees with the double path-ordered exponential formula (3.3.97) of Schreiber and Waldorf [ScWa11] from the differential cocycle data, which is what we expect due to Corollary 3.4.75.

### 3.5.2 SO(3) monopoles

Now we will give examples for non-abelian magnetic monopoles. The first example will be similar to the abelian case since we will consider the following principal $SO(3)$ bundle over $S^2$ defined by the two open sets $U_N$ and $U_S$ with transition function $g_{NS} : U_{NS} \simeq S^1 \to SO(3)$ to be

$$g_{NS}(\phi) := e^{-\phi J_3}$$

where

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.5.25)$$

form a set of generators for the Lie algebra $SO(3)$. One can give explicit connection forms $A_N$ and $A_S$ on $U_N$ and $U_S$, respectively, by

$$A_N := \frac{J_3}{2}(1 - \cos \theta) d\phi \quad \& \quad A_S := -\frac{J_3}{2}(1 + \cos \theta) d\phi. \quad (3.5.26)$$
These define local curvature 2-forms $R_N$ and $R_S$. Indeed, the gauge transformation defined above shows that
\[ g_{NS}A_s g_{NS}^{-1} - d g_{NS} g_{NS}^{-1} = A_S + J_3 d\phi \]
\[ = -\frac{J_3}{2} (1 + \cos \theta) d\phi + J_3 d\phi \]
\[ = \frac{J_3}{2} (1 - \cos \theta) d\phi \]

(3.5.27)
because all elements commute. The curvature 2-form is given by
\[ R_N = dA_N + \frac{1}{2} [A_N, A_N] = \frac{J_3}{2} \sin \theta \, d\theta \wedge d\phi \] (3.5.28)
again because the elements commute. Since $R_N = R_S$ on $U_{NS}$, this defines a $SO(3)$-valued closed 2-form on $S^2$. Let $\tau : SU(2) \to SO(3)$ be the double cover map so that $N = \{1\} \leq \pi_1(SO(3)) \cong \mathbb{Z}_2$. Recall that the induced map on the level of Lie algebras $\tau : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is an isomorphism and is given by
\[ \tau \left( \frac{1}{2i} \sigma_i \right) = J_i, \] (3.5.29)
where the $\sigma_i$ are the Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \& \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (3.5.30)
As in the general case, define $B_N := \tau^{-1}(R_N)$ and $B_S := \tau^{-1}(R_S)$, or explicitly
\[ B = \frac{\sigma_3}{4i} \sin \theta \, d\theta \wedge d\phi \] (3.5.31)
since $B_N = B_S$ on $U_{NS}$. By our analysis in Section 3.4.2, this defines the differential cocycle data of the path-curvature 2-functor. We will compute the 2-holonomy in two different ways. We will follow the same procedure as in the $U(1)$ case and compute 2-holonomy in terms of homotopy classes of paths and then we will use formula (3.3.97).

To help us with the first task, we first recall how $SU(2)$, described above in terms of the Pauli spin matrices, is isomorphic to the universal cover of $SO(3)$ described in terms of homotopy classes of paths starting at the identity in $SO(3)$. An isomorphism $\tilde{SO}(3) \cong SU(2)$ from the universal cover of $SO(3)$ to $SU(2)$ can be given by using the universal property and the fact that $SU(2)$ is simply connected. Given any path $\gamma : [0, 1] \rightarrow SO(3)$ starting at $\gamma(0) = I_3$, the $3 \times 3$ identity matrix, there exists a unique lift $\tilde{\gamma} : [0, 1] \rightarrow SU(2)$ starting at $\tilde{\gamma}(0) = I_2$ and such that the diagram

$$
\begin{array}{ccc}
SU(2) & \longrightarrow & \rightarrow \\
\downarrow & & \\
[0, 1] & \xrightarrow{\gamma} & SO(3)
\end{array}
$$

(3.5.32)

commutes. In this way, we can define a map

$$
\tilde{SO}(3) \longrightarrow SU(2)
$$

(3.5.33)

By using the universal property one more time, one can show that this map is well-defined. Finally, it is a smooth diffeomorphism of covering spaces.
We can now check what the value of the path-curvature transport 2-functor is on the sphere by doing the same computations as above but using the new $SO(3)$-valued differential forms. The result for the bigon describing the northern hemisphere is given by

$$\text{triv}(\Sigma_N(\cdot, 2s)) = \text{triv} \left( \Sigma_N \left( \cdot, \frac{2\theta}{\pi} \right) \right)$$

$$= e^{\int \frac{J_3}{2} (1 - \cos \theta) d\phi}$$

$$= e^{\pi J_3(1 - \cos \theta)}$$

(3.5.34)

since the paths going along $\theta$ do not contribute to the parallel transport since the connection form only has a $d\phi$ contribution. The path-ordered exponential is reduced to an ordinary exponential of an integral because only $J_3$ is involved and $J_3$ commutes with itself. Similarly, the southern hemisphere gives

$$\text{triv}(\Sigma_S(\cdot, 2s - 1)) = \text{triv} \left( \Sigma_S \left( \cdot, \frac{2\theta}{\pi} - 1 \right) \right) = e^{-\pi J_3(1 + \cos \theta)}.$$

(3.5.35)

Again, as a sanity check we show that the boundary values match up between the two hemispheres along the equator:

$$e^{\pi J_3(1 - \cos \frac{\theta}{2})} = e^{\pi J_3} = -I_3 = e^{-\pi J_3} = e^{-\pi J_3(1 + \cos \frac{\theta}{2})}.$$

(3.5.36)

Now we can compute the homotopy class of the path as $\theta$ ranges from 0 to $\pi$. Using similar arguments, namely that $1 - \cos \theta$ is a monotonically increasing
function of $\theta$ for $\theta$ between 0 and $\frac{\pi}{2}$, we see that this defines a nontrivial loop in $SO(3)$ at the identity which agrees with our previous calculation. Therefore, the 2-holonomy along the sphere is
\[
\text{hol}(S^2) = -I_2. \tag{3.5.37}
\]

Now we will use the differential cocycle data and integrate using formula (3.3.97). First, we compute $A_{\Sigma_N}$ for the northern hemisphere bigon. Because only $\sigma_3$ is involved in the computation, everything commutes and conjugation is trivial. Therefore,
\[
(A_{\Sigma_N})_{\theta} \left( \frac{d}{d\theta} \right) = -\int_{0}^{2\pi} d\phi \ B(\theta, \phi) \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right) = -\frac{\pi \sigma_3}{2i} \sin \theta \tag{3.5.38}
\]
and the 2-transport along $\Sigma_N$ is given by
\[
k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{ -\int_{0}^{\theta=\pi/2} \left( A_{\Sigma_N} \right)_{\theta} \left( \frac{d}{d\theta} \right) \right\} = \exp \left\{ \int_{\theta=0}^{\theta=\pi/2} \frac{\pi \sigma_3}{2i} \sin \theta \right\}. \tag{3.5.39}
\]
The 2-transport along $\Sigma_S$ is done similarly and is given by
\[
k_{A,B}(\Sigma_S) = \exp \left\{ \int_{\theta=\pi/2}^{\theta=\pi} \frac{\pi \sigma_3}{2i} \sin \theta \right\}. \tag{3.5.40}
\]
Vertically composing these results yields
\[
k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) = \exp \left\{ \int_{\theta=0}^{\theta=\pi} \frac{\pi \sigma_3}{2i} \sin \theta \right\} = e^{\pi i \sigma_3} = -I_{2 \times 2} \tag{3.5.41}
\]
because every term commutes. We will discuss what these group elements mean after we finish a few more examples.
3.5.3 $SU(n)/Z(n)$ monopoles

Another collection of non-abelian examples arise from the Lie group $SU(n)$. The center of $SU(n)$ is $Z(n)$ where, in the fundamental representation, elements in $Z(n)$ are of the form

$$\exp \left\{ \frac{2\pi ki}{n} \right\} I_n, \quad (3.5.42)$$

where $k \in \{0, 1, \ldots, n - 1\}$ and $I_n$ is the $n \times n$ unit matrix. $SU(n)/Z(n)$ is a Lie group with fundamental group $\pi_1(SU(n)/Z(n))$ isomorphic to $Z(n)$. To see this, recall that the universal cover $\widetilde{SU(n)/Z(n)}$ constructed via paths in $SU(n)/Z(n)$ and modding out by homotopy is naturally isomorphic to $SU(n)$, which is simply connected, by the universal property of universal covers. The isomorphism preserves the fibers over the identity in $SU(n)/Z(n)$ and restricts to the isomorphism between $\pi_1(SU(n)/Z(n))$ and $Z(n)$. The previous example was the special case $n = 2$.

The equivalence relation on $SU(n)/Z(n)$ says that two elements $A$ and $B$ of $SU(n)$ are equivalent if there exists a $k \in \{0, 1, \ldots, n - 1\}$ such that

$$AB^{-1} = \exp \left\{ \frac{2\pi ki}{n} \right\} I_n. \quad (3.5.43)$$

We denote the elements of equivalence classes with square brackets such as $[A]$. 
The possible $SU(n)/Z(n)$ principal bundles over the sphere are determined by the clutching function along the equator, which is a homotopy class of a loop $S^1 \to SU(n)/Z(n)$ which by the isomorphism above is precisely an element of $Z(n)$. The quotient map is written as $\tau : SU(n) \to SU(n)/Z(n)$ and is a covering map of Lie groups. Therefore, it defines a Lie 2-group.

Let us first consider the case for $n = 3$, which is relevant in the theory of quarks and gluons (see Section 1.4 of [ChTs93]). We fix $k \in \{0, 1, 2\}$. Define $X$ to be the element in the Lie algebra of $SU(3)$ to be

$$X := \frac{i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (3.5.44)$$

The exponential of this matrix is unitary. We define transition functions by

$$g_{NS}(\phi) := \exp \{-k\tau(X)\}$$

$$= [\exp\{-k\phi X\}]$$

$$= \begin{pmatrix} e^{-\frac{k\phi}{3}} & 0 & 0 \\ 0 & e^{-\frac{k\phi}{3}} & 0 \\ 0 & 0 & e^{2\frac{k\phi}{3}} \end{pmatrix}. \quad (3.5.45)$$

The element $X$ is a scalar multiple of the Gell-Mann matrix $\lambda_8$. Note we have

$$g_{NS}(0) = g_{NS}(2\pi) = g_{NS}(4\pi) = [\mathbb{I}_3] \in SU(3)/Z(3). \quad (3.5.46)$$

The transition function defines a map $\phi \mapsto g_{NS}(\phi)$ whose homotopy class determines a principal $SU(3)/Z(3)$ bundle characterized by the integer $k \in \{0, 1, 2\}$. 
We define a connection on this bundle analogously to the $SO(3)$ case by setting

$$A_N := \frac{k_{\mathbb{T}}(X)}{2}(1 - \cos \theta) d\phi \quad \& \quad A_S := -\frac{k_{\mathbb{T}}(X)}{2}(1 + \cos \theta) d\phi. \quad (3.5.47)$$

A similar computation shows that this collection of 1-forms is consistent with the transition function. The connection 2-form is similarly given by

$$B_N = \frac{kX}{2} \sin \theta \ d\theta \wedge d\phi \quad (3.5.48)$$

and likewise for $B_S$. This defines an $SU(3)/Z(3)$-valued closed 2-form on $S^2$.

Again, we can do the computation for the 2-holonomy in the two ways described earlier. The first case is done by computing the homotopy class of the path of holonomies using the definition of the path-curvature 2-functor of Definition 3.4.57. The second way is via the differential forms associated to the path-curvature 2-functor described in Section 3.4.2 and equation (3.3.97). The computation is completely analogous to the previous two examples.

For the first case, we have

$$\text{hol}^k(S^2) = \left[ \begin{array}{c} \theta \\ \rightarrow \\ \begin{array}{c} e^{\frac{k}{\mathbb{T}} X \int_{\theta}^{\frac{\pi}{2}} (1 - \cos \theta) d\phi} \\ \text{if } 0 \leq \theta \leq \frac{\pi}{2} \end{array} \\ e^{-\frac{k}{\mathbb{T}} X \int_{\frac{\pi}{2}}^{\theta} (1 + \cos \theta) d\phi} \\ \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{array} \right]$$

$$= \left[ \begin{array}{c} \theta \\ \rightarrow \\ \begin{array}{c} e^{k\pi X (1 - \cos \theta)} \\ \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ e^{-k\pi X (1 + \cos \theta)} \\ \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{array} \right]$$

$$= e^{\frac{2\pi ik}{3} \mathbb{I}_3}. \quad (3.5.49)$$
As for the computation in terms of differential forms, also by analogous computations to previous cases,

\[
(\mathcal{A}_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) = -\int_0^{2\pi} d\phi \frac{kX}{2} \sin \theta = -k\pi X \sin \theta \tag{3.5.50}
\]

and likewise for \((\mathcal{A}_{\Sigma_S})_\theta \left( \frac{d}{d\theta} \right)\). Also

\[
k_{A,B}(\Sigma_N) = \exp \left\{ \int_0^{\pi/2} k\pi X \sin \theta \, d\theta \right\} \tag{3.5.51}
\]

and finally the 2-holonomy along the sphere is

\[
\text{hol}^{[k]}(S^2) = k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N) = \exp\{2\pi kX\} = e^{\frac{2\pi ik}{n}} \mathbb{I}_3. \tag{3.5.52}
\]

For the general case of \(SU(n)\), by using the matrix

\[
X := \frac{i}{n} \left( \begin{array}{cccc}
1 \\
& 1 \\
& & \ddots \\
& & & 1 \\
& & & & 1 - n
\end{array} \right) \tag{3.5.53}
\]

the formulas for the transition function, connection 1-forms, and connection 2-forms are all the same with this new \(X\) replacing the old one. Completely analogous computations lead to a 2-holonomy along the sphere given by

\[
\text{hol}^{[k]}(S^2) = e^{\frac{2\pi ik}{n}} \mathbb{I}_n, \tag{3.5.54}
\]

where \(k \in \{0, 1, \ldots, n - 1\}\). The result is the magnetic charge of a magnetic monopole computed as a non-abelian flux in \(SU(n)/Z(n)\) gauge theories.
3.5.4 \( U(n) \) monopoles

We now discuss yet another collection of examples generalizing the \( U(1) \) case. Consider the group \( U(n) \) of unitary \( n \times n \) matrices. The Lie algebra, \( U(n) \) consists of Hermitian matrices. The universal cover of \( U(n) \) is \( SU(n) \times \mathbb{R} \). The covering map \( \tau : SU(n) \times \mathbb{R} \longrightarrow U(n) \) is defined by \( \tau(A, t) := Ae^{2\pi it} \). The image of \( \tau \) is a \( U(1) \) subgroup of \( U(n) \). The fiber of this covering map is given by the kernel which is

\[
\ker \tau = \left\{ (A, t) \bigg| A = e^{-2\pi it} \quad \text{and} \quad \det A = e^{-2\pi int} = 1 \iff t = \frac{k}{n}, \quad k \in \mathbb{Z} \right\}
\]

\[
= \left\{ \left( e^{\frac{2\pi ik}{n}} I_n, \frac{k}{n} \right) \bigg| k \in \mathbb{Z} \right\}
\]

\[
\cong \mathbb{Z}.
\]  

(3.5.55)

Consider the Lie algebra element along this real line

\[
X := (0_n, 1), \quad (3.5.56)
\]

where \( 0_n \) is the \( n \times n \) zero matrix. Then its image in \( U(n) \) under \( \tau \) is

\[
\tau(X) = 2\pi i I_n.
\]  

(3.5.57)

With this, for every integer \( k \), we define the transition function, connection 1-forms, and connection 2-forms completely analogously to the previous examples (specifically the \( \mathbb{R} \longrightarrow U(1) \) example), namely

\[
g_{NS}(\phi) = e^{ik\phi}I_n,
\]  

(3.5.58)
\[ A_N = \frac{k}{2i} (1 - \cos \theta) I_n d\phi \quad \text{&} \quad A_S = -\frac{k}{2i} (1 + \cos \theta) I_n d\phi, \quad (3.5.59) \]

and

\[ B = \tau^{-1} \left( \frac{k}{2i} \sin \theta I_n d\theta \wedge d\phi \right) = -\frac{k}{4\pi} \sin \theta (0_n, 1) d\theta \wedge d\phi. \quad (3.5.60) \]

In terms of the path of holonomies via the path-curvature 2-functor, the surface holonomy is

\[ \text{hol}^{[k]}(S^2) = \begin{cases} 
\theta \mapsto \left\{ 
eq \frac{k}{2} I_n \int_0^{\theta} (1 - \cos \phi) d\phi & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
\neq \frac{k}{2} I_n \int_0^{\theta} (1 + \cos \phi) d\phi & \text{if } \frac{\pi}{2} \leq \theta \leq \pi
\end{cases} \]

\[ \begin{align*}
\text{hol}^{[k]}(S^2) & = \begin{cases} 
\theta \mapsto \left\{ e^{\frac{k}{2} I_n \int_0^{\theta} (1 - \cos \phi) d\phi} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\
\neq \frac{k}{2} I_n \int_0^{\theta} (1 + \cos \phi) d\phi & \text{if } \frac{\pi}{2} \leq \theta \leq \pi
\end{cases} \\
& = -k \in \mathbb{Z}.
\end{align*} \quad (3.5.61) \]

If we want to compute the surface holonomy in terms of formula (3.3.97), we first compute

\[ (A_{\Sigma_N})_\theta \left( \frac{d}{d\theta} \right) = \int_0^{2\pi} d\phi \frac{k}{4\pi} \sin \theta (0_n, 1) = \frac{k}{2} \sin \theta (0_n, 1) \quad (3.5.62) \]

so that we get

\[ k_{A,B}(\Sigma_N) = \mathcal{P} \exp \left\{ -\int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta (0_n, 1) \right\} \]

\[ = \left( \mathbb{I}_n, -\int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta \right) \quad (3.5.63) \]
and the 2-holonomy along the sphere is
\[
\text{hol}^k(S^2) = k_{A,B}(\Sigma_S)k_{A,B}(\Sigma_N)
\]
\[
= \left( \mathbb{I}_n, -\int_{\pi/2}^{\pi} d\theta \frac{k}{2} \sin \theta \right) \left( \mathbb{I}_n, -\int_0^{\pi/2} d\theta \frac{k}{2} \sin \theta \right) 
\]
\[
= \left( \mathbb{I}_n, -\int_0^{\pi} d\theta \frac{k}{2} \sin \theta \right) 
\]
\[
= (\mathbb{I}_n, -k).
\]

3.5.5 Magnetic flux is a gauge-invariant quantity

In this section we state a theorem that is trivial to prove in the formalism presented above but gives an interesting physical interpretation. As mentioned earlier, the definition of the magnetic flux in the literature [ChTs93] is given as the homotopy class of a loop of holonomies. However, it was not known [GoNuOl77] how to define it as a surface-ordered integral except in the abelian case. The constructions in this paper use the theory of transport 2-functors as models for 2-bundles with 2-connections to describe this loop of holonomies in terms of a transport 2-functor. The equivalence between this description and the definition in terms of surface holonomy is made precise. This motivates the following definition.

**Definition 3.5.65.** Let \( P \rightarrow M \) be a principal \( G \)-bundle with connection over \( M \) and denote the associated transport functor by \( \text{tra} \). Let \( \Sigma : S^2 \rightarrow M \) be the map of a smooth sphere in \( M \). Let \( N \leq \pi_1(G) \) be a subgroup, \( \tilde{G}_N \rightarrow G \)
the associated $N$-cover, $\mathcal{B}G_N$ the associated Lie 2-group, and $K_N(\text{tra})$ the associated path-curvature transport 2-functor. The 2-holonomy $\operatorname{hol}^{[K_N(\text{tra})]}(\Sigma)$ is the magnetic flux of any magnetic monopole enclosed by $\Sigma$ associated to $\text{tra}$ and $N$.

All the previous examples relied on choices for the open cover, paths and bigons used to describe the sphere, and choices of lifts of paths and bigons. It is not immediately clear that the surface holonomy computed is independent of these choices. Theorems 3.3.159 and 3.4.74 give us two important results, the first of which tells us the magnetic flux is indeed independent of these choices.

**Corollary 3.5.66.** Under the assumptions of Definition 3.5.65, the magnetic flux is a gauge-invariant quantity (in terms of the notation of Definition 3.3.168)

$$\operatorname{hol}^{[K_N(\text{tra})]}(\Sigma) \in \operatorname{Inv}(\alpha).$$

(3.5.67)

*Proof.* Choose a marking for the thin sphere as a thin bigon $\Sigma : \gamma \to \gamma$ from a thin loop to itself. Then $K_N(\text{tra})(\Sigma) \in \ker \tau$ by the source-target matching condition (recall comment preceding (3.3.94)). By Theorem 3.3.159, 2-holonomy along a sphere for any gauge 2-group is well-defined up to $\alpha$-conjugation. But $\alpha$-conjugation for covering 2-groups agrees with ordinary
conjugation by a lift by Lemma 3.4.12. Therefore, the $\alpha$-conjugation action restricted to $G \times \ker \tau$ is trivial because $\ker \tau$ is a central subgroup of $\tilde{G}_N$ by Lemma 3.3.8.

A corollary of this and Theorem 3.4.74 is the following which relates the magnetic flux to a surface integral of the magnetic field. This is more of a physics corollary than a math corollary.

**Corollary 3.5.68.** The magnetic flux (Definition 3.5.65) can be computed as a surface integral by using (3.3.97) locally. This surface integral, which lands in the covering group, is the analogue of $\int_{S^2} R$ where in electromagnetism $R$ is the electromagnetic field strength due to the local potential $A$.

Therefore, the surface holonomies of transport 2-functors give a mathematically rigorous explanation for the topological quantum number (the magnetic charge) associated to magnetic monopoles for gauge theories with any structure/gauge group in terms of magnetic flux. It is topological in the sense that it only depends on the homotopy class of the sphere by Corollary 3.4.69. Furthermore, it expresses this quantity as a group element in the center of the universal cover of the gauge group. We emphasize that no Higgs field was introduced to do these computations. This therefore gives a rigorous mathematical result first mentioned by Goddard, Nuyts, and Olive at
the end of Section 2 of their paper [GoNuOl77] by using the notion of trans- 
port 2-functors introduced by Schreiber and Waldorf in [SeWa13] to describe 
magnetic flux generalizing the notion from the theory of electromagnetism 
to non-abelian gauge theories.

Appendix on smooth spaces

We will briefly state important definitions and smooth structures needed in 
this paper. The category of finite-dimensional manifolds is not suitable for 
our purposes, nor is the category of certain infinite-dimensional manifolds. 
This section reviews diffeological spaces, which constitute one candidate for 
a notion of smooth spaces. For a review of smooth spaces that also compares 
several other candidates, please refer to [BaHo11].

Definition 3.5.69. A smooth space is a set $X$ together with a collection of 
plots $\{\varphi : U \to X\}$, called its smooth structure, where each $U$ is an open set 
in some $\mathbb{R}^n$ ($n$ can vary) satisfying the following conditions.

i) If $\varphi : U \to X$ is a plot and $\theta : V \to U$, where $V$ is an open set of some 
$\mathbb{R}^m$, is a smooth map, then $\varphi \circ \theta : V \to X$ is a plot.

ii) Every map $\mathbb{R}^0 \to X$ is a plot.

iii) Let $\varphi : U \to X$ be a function and let $\{U_j \}_{j \in I}$ be a collection of open
sets covering $U$ with $i_j : U_j \to U$ denoting the inclusion. Then if $\varphi \circ i_j : U_j \to X$ is a plot for all $j \in I$, then $\varphi : U \to X$ is a plot.

**Definition 3.5.70.** A function $f : X \to Y$ between two smooth spaces is smooth if for every plot $\varphi : U \to X$ of $X$, $f \circ \varphi : U \to Y$ is a plot of $Y$.

**Example 3.5.71.** Let $M$ be a smooth manifold. The manifold smooth structure has as its collection of plots all infinitely differentiable functions $\varphi : U \to M$ for various open sets $U$ in Euclidean space. $M$ with this collection of plots forms a smooth space. With this smooth structure, for any two manifolds $M$ and $N$, a function $M \to N$ is smooth if and only if it is differentiable in the usual sense.

**Example 3.5.72.** Let $A$ be a subset of a smooth space $X$ and denote the inclusion by $i : A \hookrightarrow X$. The subspace smooth structure on $A$ has as its collection of plots all functions $\varphi : U \to A$ such that $i \circ \varphi : U \to X$ are plots of $X$. With this smooth structure, the inclusion $i : A \to X$ is smooth.

**Example 3.5.73.** Let $X$ be a smooth space, $\sim$ an equivalence relation on $X$, and $q : X \to X/\sim$ the quotient map. The quotient smooth structure on $X/\sim$ has as its collection of plots all functions $\varphi : U \to X/\sim$ such that there
exists an open cover \( \{U_j\}_{j \in J} \) along with plots \( \varphi_j : U_j \to X \) for \( X \) such that
\[
\begin{array}{ccc}
X & \xleftarrow{\varphi_j} & U_j \\
\downarrow & & \downarrow \\
X/\sim & \xleftarrow{\varphi} & U
\end{array}
\]
commutes for all \( j \in J \). With this smooth structure, the quotient map \( q : X \to X/\sim \) is smooth.

**Example 3.5.75.** Let \( X \) and \( Y \) be smooth spaces. The *product smooth structure* on \( X \times Y \) has as its collection of plots all functions \( \varphi : U \to X \times Y \) such that \( \pi_X \circ \varphi : U \to X \) and \( \pi_Y \circ \varphi : U \to Y \) are both plots of \( X \) and \( Y \), respectively. Here \( \pi_X : X \times Y \to X \) and \( \pi_Y : X \times Y \to Y \) are the projection maps and are smooth with respect to this smooth structure.

**Example 3.5.76.** Let \( X \) and \( Y \) be two smooth spaces. The *mapping smooth structure* on the set of functions \( Y^X \) of \( X \) into \( Y \) is defined as follows. A function \( \varphi : U \to Y^X \) is a plot if and only if the associated function \( \tilde{\varphi} : U \times X \to Y \), defined by \( \tilde{\varphi}(u, x) := \varphi(u)(x) \), is smooth. With this smooth structure and the smooth structure on a product, the adjunction \( Z^{X \times Y} \cong (Z^Y)^X \) is an isomorphism in the category of smooth spaces for all \( X, Y, Z \).
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Chapter 4
Convex categories

The current chapter is preliminary work on convex categories and related structures. In particular, the definitions have not been completely settled. Their final form will depend on claims and conjectures listed in Sections 4.4.3 and 4.5.1.

4.1 Introduction, motivation, and outline

4.1.1 The many forms of entropy and information in science

The concept of entropy appears in several contexts in physics and mathematics. However, it is still not understood in any universal sense. There are several constructions of entropy beginning with the work of Boltzmann in statistical mechanics in the late 1860’s to the early 1870’s [Bo77]. Gibbs then later in 1878 [Gi78] gave a probabilistic interpretation providing a notion of
entropy for probability measure spaces by the formula

\[ S = -k_B \sum_i p_i \ln p_i, \]  

(4.1.1)

where \( k_B \) is Boltzmann’s constant and \( p_i \) is the probability of an event \( i \) taking place. In 1948, Shannon developed a notion of entropy in information theory interpreting entropy in terms of information loss [Sh48]. Gibbs entropy was shown by Jaynes in 1957 to be a special case of Shannon’s entropy, thus deriving thermodynamics and statistical mechanics from information theory [Ja57].

In 1932, von Neumann described the quantum wave function collapse (due to a measurement) as an irreversible process and introduced the density matrix \( \rho \) [vN32]. Landau had also independently introduced the density matrix but for a different purpose. Although less well-known, the root of Einstein, Rosen, and Podolsky’s (EPR) argument showed that something more general than a wavefunction is required in quantum mechanics [EPR35] and Cantrell and Scully confirmed this in 1978 by realizing that EPR’s initial argument could be resolved via the introduction of density matrices into quantum mechanics [CaSk78].

\footnote{This of course is not the reason we celebrate the EPR paper as a society, but it is nevertheless another good motivation for the introduction of the density matrix. I would like to thank Jonathan Ben-Benjamin for bringing my attention to this reference and for discussions regarding it.} An excellent and accessible review of a mathematically
rigorous explanation of where the density matrix comes from in terms of measurement can be found in chapter 19 of Hall’s book on quantum theory [Ha13]. The von Neumann entropy associated to a density matrix is given by the formula

\[ S(\rho) := -\text{tr}(\rho \ln \rho). \]  

Narnhofer and Thirring generalized this definition to the context of $C^*$-algebras [NaTh85], where states still make sense even if density matrices corresponding to states may not exist (which occurs for instance in quantum field theory) [Wa94], or if they do exist, they might not be unique [BdQV13].

Entropy has again become an area of interest among quantum field theorists, and although generalizations have been made in this context (see [CaCa04] for example), it is still not completely understood what the appropriate notion of entropy is. Entropy has also been introduced in the surprising context of black hole thermodynamics beginning with the work of Bekenstein and Hawking around 1973 [Be73] and 1975 [Ha75], respectively. Bekenstein showed that the entropy of a Schwarzschild black hole is proportional to the area of its horizon and Hawking used quantum field theory in curved spacetime to obtain the proportionality constant as well as to show that black holes radiate. The formula for the entropy in this case was shown
to be\footnote{The subscript “BH” was probably used for “Bekestein-Hawking” but it can equally be used for “black hole.”}

\[ S_{BH} = \frac{k_B A}{4\ell_P^2}, \tag{4.1.3} \]

where \( \ell_P \) is the Planck length defined to be

\[ \ell_P = \sqrt{\frac{G\hbar}{c^3}}. \tag{4.1.4} \]

This formula seems drastically different from the previous examples. It is a formula coming from purely geometric considerations. Furthermore, the fact that it is proportional to the area of the black hole instead of the volume is thought to be counter-intuitive. Understanding this and what it means for physics, especially quantum gravity, is still an unresolved issue to this day.

4.1.2 Motivation for a categorical framework for entropy

Although some of these notions of entropy are related, or are special cases of one another, it is important to stress that they are indeed different. They all have similar properties, but seem to be defined in completely different categories. The purpose of this article is to precisely identify these categories, and exactly show in what way these notions of entropy are related in terms of functors relating these categories. Of crucial importance is the notion of
convexity. All of the examples mentioned above have convex structures on their appropriate categories.

Entropy is a convex function in all of its manifestations. The reason is because if two ensembles are combined (via a convex linear combination), one loses information that distinguishes from which ensemble a particular sample comes from [We78]. However, not every convex function is proportional to the appropriate notion of entropy—additional properties are needed. Much of our work is motivated by recent characterizations based on categories and functors. The key insight, initiated in the work of Baez, Fritz, and Leinster [BFL11] in 2011 in the context of finite probability spaces and Shannon entropy, is that although not every convex function on finite probability spaces is proportional to entropy, every convex functor is. Viewing entropy as a functor in the context of finite probability spaces places the perspective of entropy in terms of information loss associated to a process instead of the entropy associated to a state. In addition, viewing entropy as a functor immediately shows that entropy is an invariant, which has been utilized in several ways, for example in the context of dynamical systems. For instance, Kolmogorov used entropy to first prove that certain dynamical systems, known as Bernoulli shifts, are not isomorphic by calculating their entropies [Bo14]. The ubiquity of convexity of entropy is probably well known to the reader.
However, the functoriality property is *intuitively obvious* and says simply that the information loss associated to a sequence of processes is the sum of the information losses associated to each process in the sequence.

Motivated by this work [BFL11], and more recent work of Baez and Fritz [BF14] for relative entropy, we will *define* entropy as a *convex functor* on a *convex category*. Uniqueness or lack of uniqueness results for such functors may aid our understanding of entropy in terms of information loss associated to processes or perhaps may offer new insight. To do this, we have to abstract the convex structure utilized in the proof of uniqueness of convex functors in [BFL11] and [BF14]. It is therefore one of the main purposes of this chapter is to set up a theory aimed towards a definition of entropy robust enough to include all examples mentioned above as well as many new and interesting ones.

The definition of a convex category will be a categorified version of a convex space, which we learned about through Fritz’s work [Fr09], which itself is actually not the first abstract definition written down for a convex space. It was known as early as *Świrszcz’s work* [Św74] though the earliest reference we currently have access to is Flood’s work [Fl80]. A convex space is an abstract set equipped with a collection of operations to be thought of as convex linear combinations. These binary operations need to satisfy
several properties motivated by convex subspaces of vector spaces. However, as shown in [Fr09], there are examples of convex spaces that are not subspaces of vector spaces. Nevertheless, under certain additional assumptions, one can show that all convex spaces can be embedded as convex subspaces of vector spaces [Fl80]. The definition of a convex space can be internalized in any cartesian monoidal category. Furthermore, our definition of convex category can also be internalized in any cartesian monoidal 2-category, though such abstractions will not be needed in this introductory work. However, there are interesting examples that become more apparent, when one formulates the Gelfand-Naimark-Segal construction as an adjunction, which is done in Chapter 5 of this thesis.

Both entropy and information loss become special cases of convex functors between convex categories. Our definition is robust enough to allow for notions of information loss whose values are not just non-negative numbers but could be morphisms in some convex category. Similarly, entropy is allowed to be quantified in terms of objects of a convex category, or more precisely a cone category, which we also define.
4.1.3 General results and outline of chapter

In Section 4.2, we review the notion of an abstract convex space, which gives an algebraic description of convex sets. Afterwards, we “categorify” this notion and define a convex category. Normally, in the process of categorification, one replaces conditions with isomorphisms. We find that this is too restrictive and find that most of our examples require ordinary morphisms. This will be related to the fact that in some situations one cannot duplicate or erase information as freely as one might wish. Our definition is formulated purely arrow-theoretically enabling it to be internalized into certain 2-categories.

Following the abstract (and dense) definition of convex category, we proceed in Section 4.3 to give an enormous number of examples of convex categories. These examples are the domains on which entropy shall be defined. Most of the examples we cover are those arising in classical probability theory. These include finite probability spaces and measure-preserving maps, finite probability spaces and measure-preserving stochastic maps, and general (possibly infinite) probability spaces with and without probability density functions. In the near future, we will examine density matrices in quantum mechanics and states on $C^*$-algebras. It is also possible (though this

---

$^3$The latter examples were worked out together with Brian Dressner.
is speculative), that dynamical systems, states in quantum field theory, and Riemannian (and pseudo-Riemannian) geometry, particularly in the context of black hole thermodynamics, have categorical descriptions fitting them into this framework and only lack of time has prevented us from exploring these examples further. We defer these latter examples to future work.

In Section 4.4, we define several different notions of convex functors (convex, concave, affine, and so on). Because we are dealing with categories as opposed to sets, functors need not preserve the convex structure on the nose, and we indeed find several examples where this happens, particularly in formulating the Gelfand-Naimark-Segal (GNS) construction categorically, the topic of Chapter 5. Unfortunately, we will not describe how convexity plays a role in this thesis in relation to the GNS construction and will save this for future work. We provide only a few examples of convex functors here. The reason for this is that technical issues have prevented us from presenting a larger number in this thesis. However, significant progress is being made towards convex functors between classical and quantum probabilities. Ideally, there will be a sequence of convex embeddings

\[
\text{FinProb} \hookrightarrow \text{DenMa} \hookrightarrow \text{states}
\]  

(4.1.5)

describing how ordinary (classical) finite probability is a special case of the
category of density matrices, which itself is a special case of states on $C^*$-algebras. While this result in some form has been known, it is important to phrase it categorically to make sense of entropy categorically in future work.

In Section 4.5, we begin formalizing the notion of entropy as information loss. In order to accomplish this, we introduce cone categories, which are closely related to convex categories. This is needed to make sense of classification theorems for entropy that state results of the form “any convex functor from a [specific] convex category $\mathcal{C}$ to real numbers” is proportional to a well-known entropy formula. It is the concept of proportionality which must be made precise, and this is done using cone categories and convex functors from convex categories to cone categories. In the process, we show that every cone category is naturally a convex category, which is just a categorified version of results known in [Fl80]. Many examples of entropy arise as convex functors from a convex category to real numbers, though we only discuss the one that has motivated this work.

The Appendix of this chapter reviews some basic category theory that is needed to understand the results in this chapter. There, we introduce categories, functors, natural transformations, and all their compositions. In addition, we review symmetric semigroupal categories, symmetric monoidal categories, cartesian monoidal categories, and their associated structure-
preserving functors and natural transformations. Several proofs use “higher-dimensional algebra,” which was described in some detail in Chapter 2.

4.1.4 Acknowledgements

Firstly, we would like to thank Brian Dressner who has contributed to some of the examples in this paper (probability measure spaces and probability density functions). We are also indebted to Tobias Fritz who explained several aspects of his work with John Baez and Tom Leinster [BFL11] and who provided insight and additional references. We have also benefited from conversations with Jonathan Ben-Benjamin, Lewis Bowen, Brian Hall, Azeemul Hassan, Mark Hillery, Manas Kulkarni, Jamie Lennox, William Mayer, V. P. Nair, George Poppe, Xing Su, Josiah Sugarman, Dennis Sullivan, Scott O. Wilson, Cody Youmans, and Lai-Sang Young. This work was partially supported by NSF grant PHY-1213380 and the Capelloni Dissertation Fellowship.

4.2 Convex categories

4.2.1 Preliminary on convex spaces

The following definition was obtained from [Fr09] but goes back to as early as Świrszcz’s work in 1974 [Św74].
**Definition 4.2.1.** A **convex set** is a set $C$ together with a family of functions known as **convex linear combinations** $F_\lambda : C \times C \rightarrow C$ indexed by $\lambda \in [0,1]$ satisfying the following axioms:

\[ F_0(x, y) = y \]  \hspace{1cm} \text{(unit law)} \hspace{1cm} (4.2.2)

\[ F_\lambda(x, x) = x \]  \hspace{1cm} \text{(idempotency)} \hspace{1cm} (4.2.3)

\[ F_\lambda(x, y) = F_{1-\lambda}(y, x) \]  \hspace{1cm} \text{(parametric commutativity)} \hspace{1cm} (4.2.4)

\[ F_\lambda(F_\mu(x, y), z) = F_{\lambda,\mu}(x, F_{\lambda,\mu}(y, z)) \]  \hspace{1cm} \text{(deformed parametric associativity)} \hspace{1cm} (4.2.5)

for all $x, y, z \in C$ and $\lambda, \mu \in [0,1]$. Here

\[ \lambda_\mu := \lambda \mu \quad \& \quad \lambda_\mu := \begin{cases} \frac{\lambda(1-\mu)}{1-\lambda \mu} & \text{if } \lambda \mu \neq 1 \\ \text{arbitrary} & \text{if } \lambda = \mu = 1 \end{cases} \]  \hspace{1cm} (4.2.6)

Here “arbitrary” means that one can assign any value to the quantity.

It is convenient to use the notation

\[ \lambda x + (1 - \lambda)y := F_\lambda(x, y) . \]  \hspace{1cm} (4.2.7)

In this case, the laws take a more familiar form

\[ 0x + 1y = y \]

\[ \lambda x + (1 - \lambda)x = x \]

\[ \lambda x + (1 - \lambda)y = (1 - \lambda)y + \lambda x \]

\[ \lambda\left(\mu x + (1 - \mu)y\right) + (1 - \lambda)z = (\lambda,\mu)x + (1 - \lambda,\mu)\left((\lambda,\mu)y + (1 - \lambda,\mu)z\right) \]
Examples of convex spaces are abundant. They are motivated by convex subspaces of vector spaces (see Theorem 4.1 in [Fr09]).

**Example 4.2.8.** Let $V$ be a real vector space and $C \subseteq V$ a convex subset. Then the vector space structure gives $C$ the structure of a convex space.

Even though convex spaces are motivated by the previous example, there are “non-geometric” examples, i.e. convex spaces that cannot be realized as convex subspaces of vector spaces. Plenty examples are given in [Fr09] and we will be content with just knowing that such examples exist. One of the nice things about Świrszcz’s definition is that it can easily be defined internally in any cartesian monoidal category. Because we will be heavily using diagramatic notation in this work and also because we will categorify this definition, it will be helpful to provide the definition of such convex objects.

**Definition 4.2.9.** A **convex object** in a cartesian monoidal category (see Definition 4.5.129 for notation and definitions used) $(C, \otimes, I, a, l, r, \gamma, \pi_1, \pi_2, e)$ consists of an object $C$ in $C$ together with a family of morphisms $F_\lambda : C \otimes C \to C$ indexed by $\lambda \in [0, 1]$ such that the following axioms hold.
(a) The diagram
\[ C \otimes C \xrightarrow{\pi_{2,C}} C \]
commutes.

(b) The diagram
\[ \begin{array}{ccc}
C \otimes C & \xrightarrow{\Delta_C} & C \\
\downarrow{F_\lambda} & & \downarrow{F_\lambda} \\
C & \xrightarrow{id_C} & C \\
\end{array} \]
commutes for all \( \lambda \in [0, 1] \).

(c) The diagram
\[ \begin{array}{ccc}
C \otimes C & \xrightarrow{\gamma} & C \otimes C \\
\downarrow{F_\lambda} & & \downarrow{F_{1-\lambda}} \\
C & & C \\
\end{array} \]
commutes for all \( \lambda \in [0, 1] \).

(d) The diagram\(^4\)
\[ \begin{array}{ccc}
C \otimes C \otimes C & \xrightarrow{F_\mu \otimes id_C} & C \otimes C \\
\downarrow{id_C \otimes F_{\lambda,\mu}} & & \downarrow{F_\lambda} \\
C \otimes C & \xrightarrow{F_{\lambda,\mu}} & C \\
\end{array} \]
commutes for all \( \lambda, \mu \in [0, 1] \).

\(^4\)Technically, there is an associator isomorphism from \((C \otimes C) \otimes C\) to \(C \otimes (C \otimes C)\) which we have not written for visual clarity. This is not a serious issue because the isomorphism is unique since \(C\) is cartesian (see Remark 4.5.143 in the Appendix of this chapter).
A convex object as above is typically denoted by \((C, \{F_\lambda\})\) or sometimes abusively as \(C\).

In the particular cartesian monoidal category of Sets (with a choice for the product of sets as the monoidal structure), a convex object is precisely a convex set as in Definition 4.2.1.

**Remark 4.2.14.** It is known that convex sets satisfying a cancellative property can be embedded into vector spaces and can therefore be viewed as convex subsets of vector spaces [Fl80]. It is not known to us whether analogous geometric-type characterizations exist for convex objects, but we merely leave this as a question for the reader—its answer is not our immediate concern here.

There is something quite important to point out regarding convex sets and the above definitions that the categorical language makes apparent. This is the fact that in order for the idempotency axiom to be defined, one needs a cartesian structure on a symmetric monoidal category. Such a structure is used to duplicate/copy (using the diagonal \(\Delta\)) and delete information (using the erase \(e\)), which is not possible in all physically interesting examples such as quantum mechanics [Ba06]. Nevertheless, all of the other axioms make sense with a slight modification to the unit law. The way the unit law has
been presented makes it also seem like the cartesian structure is necessary
but the unit law can be described using the monoidal unit from just the
monoidal structure as follows.

Definition 4.2.15. A quantum convex object in a symmetric monoidal cat-
egory \((\mathcal{C}, \otimes, I, a, l, r, \gamma)\) consists of an object \(C\) in \(\mathcal{C}\) together with a family
of morphisms \(F_\lambda : C \otimes C \rightarrow C\) indexed by \(\lambda \in [0,1]\) satisfying parametric
commutativity (4.2.12), deformed parametric associativity (4.2.13), and the
unit law, which says that to every “element” \(x : I \rightarrow C\), the diagram
\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{F_0} & C \\
\downarrow{\lambda} \otimes \text{id}_C & & \downarrow{\lambda} \otimes \text{id}_C \\
I \otimes C & \xrightarrow{t} & C
\end{array}
\]
commutes.

4.2.2 The definition of a convex category

The following definition was initially motivated by the structure described
in [BFL11] for the category of finite probability measure spaces (see Section
4.3.1 below). We first provide a “categorification”\(^5\) of Definition 4.2.1 and
then we discuss how to realize a familiar example from basic probability the-
ory. We then describe what the physical meaning of these abstract concepts
are in that case. We will then supply several other examples.

\(^5\)Usually, in categorification, one replaces equalities with isomorphisms. We will later
see that in some examples, this assumption is too strict. This should not cause severe
alarm provided we have enough consistent data for a robust definition.
**Definition 4.2.17.** A **convex category** is a category $C$ equipped with a family of bifunctors $F_\lambda : C \times C \to C$ indexed by $\lambda \in \mathbb{R}$ together with the following data.

(a) Natural isomorphisms\(^6\)

\[
\begin{array}{ccc}
C \times C & \xrightarrow{F_0} & C \\
\downarrow{u_0} & & \downarrow{\pi_{2,C}} \\
\pi_{2,C} & & \& \\
& C \xrightarrow{F_1} C & \downarrow{u_1} \\
& \downarrow{\pi_{1,C}} & & \end{array}
\]

(4.2.18)

called the **left & right unitors**, respectively. Here $\pi_{i,C}$ is the projection bifunctor onto the $i$-th factor.

(b) For every $\lambda \in [0, 1]$, natural isomorphisms

\[
\begin{array}{ccc}
C \times C & \xrightarrow{\gamma} & C \\
\downarrow{F_\lambda} & & \downarrow{F_{1-\lambda}} \\
\phi_\lambda & & \& \\
& C \xrightarrow{\gamma} C & \downarrow{F_{1-\lambda}} \\
& \downarrow{\pi_{2,C}} & & \end{array}
\]

(4.2.19)

called **parametric commutors**. Here $\gamma : C \times C \to C \times C$ is the swapping functor that sends objects $(x, y)$ to $(y, x)$ and similarly for morphisms.

(c) For every pair $\lambda, \mu \in [0, 1]$, natural isomorphisms

\[
\begin{array}{ccc}
C \times C \times C & \xrightarrow{F_\mu \times \text{id}_C} & C \times C \\
\downarrow{\text{id}_C \times F_{\lambda,\mu}} & & \downarrow{F_{\lambda}} \\
C \times C & \xrightarrow{\alpha_{\lambda,\mu}} & C \\
\downarrow{F_{\lambda,\mu}} & & \end{array}
\]

(4.2.20)

\(^6\)We will describe what all this means in case the reader is unfamiliar with natural transformations after we make the definition. Also, you can refer to Appendix 4.5.3.
called **convex associators**.

These data must satisfy the following compatibility conditions.

i) The left and right unitors together with parametric commutors satisfy

\[
\begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \ar[r]^{\gamma} \ar[dr]_{F_1} & \mathcal{C} \\
\mathcal{C} \ar[ur]_{F_0} & \mathcal{C} }
\end{array}
= \begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \ar[r]^{\gamma} \ar[dr]_{u_1} & \mathcal{C} \\
\mathcal{C} \ar[ur]_{\pi} & \mathcal{C} }
\end{array}
\] (4.2.21)

ii) For each \( \lambda \in [0, 1] \),

\[
\begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \ar[r]^{\gamma} \ar[dr]_{F_\lambda} & \mathcal{C} \\
\mathcal{C} \ar[ur]_{F_\lambda} & \mathcal{C} }
\end{array}
= \begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \ar[r]^{\gamma} \ar[dr]_{F_\lambda} & \mathcal{C} \\
\mathcal{C} \ar[ur]_{F_\lambda} & \mathcal{C} }
\end{array}
\] (4.2.22)

iii) For each \( \lambda \in [0, 1] \),

\[
\begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \ar[r]^{F_1 \times id_C} \ar[dr]_{id_C \times F_0} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \times \mathcal{C} \ar[ur]_{F_\lambda} & \mathcal{C} }
\end{array}
\] = \begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \ar[r]^{F_1 \times id_C} \ar[dr]_{id_C \times F_0} \ar[dl]_{id_C \times u_0} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \times \mathcal{C} \ar[ur]_{id_C} \ar[ur]_{id_C \times F_\lambda} & \mathcal{C} }
\end{array}
\] (4.2.23)

and

\[
\begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \ar[r]^{F_\lambda \times id_C} \ar[dr]_{id_C \times F_1} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \times \mathcal{C} \ar[ur]_{F_\lambda} & \mathcal{C} }
\end{array}
= \begin{array}{c}
\xymatrix{ \mathcal{C} \times \mathcal{C} \times \mathcal{C} \ar[r]^{F_\lambda \times id_C} \ar[dr]_{id_C \times F_1} \ar[dl]_{id_C \times u_1} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \times \mathcal{C} \ar[ur]_{id_C} \ar[ur]_{id_C \times F_\lambda} & \mathcal{C} }
\end{array}
\] (4.2.24)
Here $\pi_{ij,C} : C \times C \times C \rightarrow C \times C$ is the projection onto the $i$ and $j$-th factors.

iv) For each $\nu, \lambda, \mu \in [0, 1]$,

\[
\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
C \times C \times C \times C & & C \times C \times C \\
F_{\mu} \times \text{id}_{C \times C} & \text{id}_{C \times C} \times F_{\nu,\lambda} & \\
F_{\lambda} \times \text{id}_{C} & \text{id}_{C} \times F_{\nu,\lambda} & F_{\mu} \times \text{id}_{C} \\
\alpha_{\nu,\lambda} & \text{id} & \alpha_{\nu,\lambda,\mu} \\
F_{\nu} & F_{\nu,\lambda} & F_{(\nu,\lambda),\mu} \\
C & C & C \\
\end{array}
\end{array}
\end{align*}
\]

where

\[
(\dagger) := a_{\lambda,\mu} \times \text{id}_{C}, \quad (*) := \text{id}_{C \times C} \times F_{(\nu,\lambda,\mu),\lambda,\mu}, \quad \& \quad (*) := \text{id}_{C \times C} \times a_{\nu,\lambda,\mu,\lambda,\mu}
\]

(4.2.26)
v) For each \( \lambda, \mu \in [0, 1] \),

\[
\begin{align*}
\lambda\gamma \ast \mu & : C \times C \times C \\
\ast \gamma \ast \mu & : C \times C \times C \quad \text{(†)} \\
\mu \gamma \ast \lambda & : C \times C \times C \\
\ast \mu \ast \gamma & : C \times C \times C \quad \text{(‡)}
\end{align*}
\]

where

\[
(\dagger) := \text{id}_C \times F_{(1-\lambda,\mu), (\lambda, \mu)}, \quad (\ast) := a_{1-\lambda, \mu, \lambda, \mu}, \quad \& \quad (\ast) := \text{id}_C \times F_{1-\lambda, (1-\mu)}
\]

It will sometimes be convenient to write

\[
\lambda a \oplus (1-\lambda)b := F_\lambda(a, b) \quad \& \quad \lambda f \oplus (1-\lambda)g := F_\lambda(f, g)
\]

for objects \( a, b \) in \( C \) and similarly for morphisms \( f : a \longrightarrow a' \) and \( g : b \longrightarrow b' \) in \( C \). In this case, \( \lambda a \oplus (1-\lambda)b \) or \( \lambda f \oplus (1-\lambda)g \) are called convex linear combinations of objects and morphisms, respectively. A convex category as above will be often denoted abusively by \((C, F)\) instead of \((C, F, u_0, u_1, \phi, a)\).

**Remark 4.2.30.** Because this definition is expressed diagrammatically in terms of objects, 1-morphisms, and 2-morphisms, it can also be used to

---

7 Note that on the right side of each diagram, \( F_{(1-\lambda, \mu), (\lambda, \mu)} = F_{1-\lambda, (1-\mu)} \) and \( F_{(1-\lambda, \mu), (\lambda, \mu)} = F_{\lambda, (1-\mu)} \) since the subscripts are equal.
define \textit{convex category objects} in symmetric monoidal 2-categories (however, the diagrams become more complicated when the monoidal structure is not strict). Technically, the way the left and right unitors were described above use projection functors which use a cartesian structure. However, this can be dropped by assuming the existence of a monoidal unit $1$ in the 2-category with which “objects,” a.k.a. “elements,” of $C$ can be chosen via morphisms $x : 1 \rightarrow C$ and then a left unitor, for instance, would be a natural isomorphism $l_x$ of the form

\[
\begin{array}{c}
\xymatrix{
1 \times C & & C \\
& C \times C \ar[ur]^{x \times \text{id}_C} \ar[ul]_{\text{id}_C} \ar[dr]_{l_x} & \ar[dl]^{l_x} & F_0 \\
& & C
}\end{array}
\]

(4.2.31)

where $l_C$ is the natural isomorphism from the monoidal structure. We would then require additional assumptions for “morphisms” $f$ in $C$, which are described by 2-morphisms of the form

\[
\begin{array}{c}
\xymatrix{
1 & & C \\
& \ar @/^/[ur]^x & \ar @/_/[dr]^y & \\
& & C
}\end{array}
\]

(4.2.32)

We will not need this more abstract notion in the current chapter though it does appear in the context of the Gelfand-Naimark-Segal construction briefly described in Section 4.5.3.

It is useful to spell out what all the data present in Definition 4.2.17 are
explicitly on objects. The data consists of functors and natural transformations.

(a) The unitors \( u_0 : F_0 \Rightarrow \pi_{2,C} \) and \( u_1 : F_1 \Rightarrow \pi_{1,C} \) assign to every pair of objects \( x, y \) in \( C \) isomorphisms

\[
0x + 1y \xrightarrow{u_0(x,y)} y \quad \& \quad 1x + 0y \xrightarrow{u_1(x,y)} x \tag{4.2.33}
\]

such that for every pair of morphisms \( f : x \rightarrow v \) and \( g : y \rightarrow w \) in \( C \), the diagrams

\[
\begin{array}{ccc}
0x + 1y & \xrightarrow{u_0(x,y)} & y \\
\downarrow{0f \oplus 1g} & \ & \downarrow{g} \\
0v + 1w & \xrightarrow{u_0(v,w)} & w
\end{array} \quad \& \quad 
\begin{array}{ccc}
1x + 0y & \xrightarrow{u_1(x,y)} & x \\
\downarrow{1f \oplus 0g} & \ & \downarrow{f} \\
1v + 0w & \xrightarrow{u_1(v,w)} & v
\end{array} \tag{4.2.34}
\]

commute.

(b) The parametric commutors \( \phi_\lambda \) assign to every pair of objects \( (x, y) \) in \( C \times C \), an isomorphism

\[
(1 - \lambda)y + \lambda x \xrightarrow{\phi_\lambda(x,y)} \lambda x + (1 - \lambda)y \tag{4.2.35}
\]

such that for any pair of morphisms \( f : x \rightarrow v \) and \( g : y \rightarrow w \) the diagram

\[
\begin{array}{ccc}
(1 - \lambda)y + \lambda x & \xrightarrow{\phi_\lambda(x,y)} & \lambda x + (1 - \lambda)y \\
\downarrow{(1-\lambda)y \oplus \lambda f} & & \downarrow{\lambda f \oplus (1-\lambda)g} \\
(1 - \lambda)w + \lambda v & \xrightarrow{\phi_\lambda(v,w)} & \lambda v + (1 - \lambda)w
\end{array} \tag{4.2.36}
\]

commutes.
(c) The convex associators $a_{\lambda,\mu}$ assign to every triple of objects $(x, y, z)$ in $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ an isomorphism

$$
\lambda \left( \mu x \oplus (1-\mu)y \right) \oplus (1-\lambda)z \xrightarrow{a_{\lambda,\mu}(x,y,z)} (\lambda,\mu)x \oplus (1-\lambda,\mu) \left( (\lambda,\mu)y \oplus (1-\lambda,\mu)z \right)
$$

(4.2.37)

such that for any triple of morphisms $f : x \to u$, $g : y \to v$, and $h : z \to w$, the necessary diagram (which should by now be obvious) commutes.

The conditions on these natural isomorphisms are meant to be a sufficient set of relations needed so that any two ways of reducing one object to another via these natural isomorphisms, the two ways are equal. However, there could be some redundancies in these relations as this is a preliminary definition. Regardless, these conditions are given as follows.

i) For all objects $x$ and $y$ in $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
0x \oplus 1y & \xrightarrow{\phi_0(x,y)} & u_0(x,y) \\
1y \oplus 0x & \xrightarrow{u_1(y,x)} & y
\end{array}
$$

commutes. In other words, the right unitor is determined by the left one and the parametric commutors.

---

\footnote{For those familiar with monoidal categories, which are reviewed in the Appendix of this chapter, these are analogous to the coherence conditions, which Mac Lane proved give minimal and sufficient criteria so that the isomorphism associated to any change of parentheses of tensor products of objects is unique [Ma63]. We do not check whether our axioms are enough to prove such a general theorem.}
ii) For each $\lambda$ and each pair of objects $x$ and $y$ of $\mathcal{C}$, the diagram

$\begin{align*}
(1 - \lambda)y \oplus \lambda x \\
\phi_{1-\lambda}(y, x) & \rightarrow \phi_{\lambda}(x, y) \\
\lambda x \oplus (1 - \lambda)y & \xrightarrow{id_{\lambda x \oplus (1-\lambda)y}} \lambda x \oplus (1 - \lambda)y
\end{align*}$

commutes, i.e.

$\phi_{\lambda}(x, y)^{-1} = \phi_{1-\lambda}(y, x).$  \hfill (4.2.40)

iii) For every triple of objects $x, y, z$ in $\mathcal{C}$ and every $\lambda \in [0, 1]$, the diagrams

$\begin{align*}
\lambda(1x \oplus 0y) \oplus (1 - \lambda)z & \xrightarrow{a_{\lambda, 1}(x, y, z)} \lambda x \oplus (1 - \lambda)(0y \oplus 1z) \\
\lambda u_1(x, y, y) \oplus (1 - \lambda)id_z & \xrightarrow{\lambda id_z \oplus (1 - \lambda)u_0(y, z)} \lambda x \oplus (1 - \lambda)z \\
\lambda x \oplus (1 - \lambda)z & \xrightarrow{id_{\lambda x \oplus (1-\lambda)z}} \lambda x \oplus (1 - \lambda)z
\end{align*}$

and

$\begin{align*}
1(\lambda x \oplus (1 - \lambda)y) \oplus 0z & \xrightarrow{a_{1, \lambda}(x, y, z)} \lambda x \oplus (1 - \lambda)(1y \oplus 0z) \\
\lambda u_1(\lambda x \oplus (1 - \lambda)y, z) & \xrightarrow{\lambda id_z \oplus (1 - \lambda)u_1(y, z)} \lambda x \oplus (1 - \lambda)y \\
\lambda x \oplus (1 - \lambda)y & \xrightarrow{id_{\lambda x \oplus (1-\lambda)y}} \lambda x \oplus (1 - \lambda)y
\end{align*}$

commute.

iv) For each triple of numbers $\nu, \lambda, \mu \in [0, 1]$ and for each quadruple of
objects $w, x, y, z$ in $C$, the diagram\(^9\)

\[
\begin{align*}
F_{\nu}(F_{\lambda}(F_{\mu}(w, x), y), z) \\
F_{\lambda}(F_{\mu}(w, x), F_{\nu}\lambda(y, z)) \\
F_{\nu}(F_{\lambda}(F_{\mu}(w, x), F_{\nu}(y, z))) \\
F_{\nu}(F_{\lambda}(F_{\mu}(w, x), F_{\nu}(y, z)), F_{\nu}(\lambda, \mu)(F_{\lambda}(x, y), z))
\end{align*}
\]

\[(4.2.43)\]

commutes.

v) For every pair $\lambda, \mu \in [0, 1]$ and every triple of objects $x, y, z$ in $C$, the diagram

\[
\begin{align*}
F_{\lambda}(F_{\mu}(x, y), z) \\
F_{\lambda}(F_{1-\mu}(y, x), z) \\
F_{\lambda}(F_{1-\mu}(y, x), z) \\
F_{\lambda}(F_{1-\mu}(y, x), z)
\end{align*}
\]

\[(4.2.44)\]

commutes.

\(^9\)We occasionally switch between the notations $F_{\mu}(x, y) \equiv \mu x \oplus (1 - \mu)y$ to fit such diagrams as neatly as possible.
The above definition is a categorified version of a quantum convex object (see Definitions 4.2.9 and 4.2.15). There are actually different types of categorified versions of convex objects which we discuss now.

**Definition 4.2.45.** A *convex category with idempoters* is a convex category together with natural transformations

\[
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \times C \\
\downarrow F_\lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\]

(4.2.46)

for every \( \lambda \in [0, 1] \) called *idempoters*. Here \( \Delta : C \to C \times C \) is the diagonal functor. These natural transformations satisfy the following additional conditions

i) The idempoters and unitors satisfy

\[
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \times C \\
\downarrow F_0 \\
\downarrow id \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\]

(4.2.47)

and

\[
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \times C \\
\downarrow F_1 \\
\downarrow id \\
\end{array}
\begin{array}{c}
\Delta \\
\downarrow \lambda \\
\downarrow id \\
\end{array}
\begin{array}{c}
C \\
\downarrow \\
C \\
\end{array}
\]

(4.2.48)
ii) For each $\lambda \in [0, 1]$,

$$
\begin{array}{ccc}
\Delta & \gamma & \phi_{1-\lambda} \\
\downarrow \iota_{1-\lambda} & \downarrow \phi_{1-\lambda} & \downarrow F_{1-\lambda} \\
C & C \times C & C \\
\downarrow \text{id}_C & \downarrow F_{\lambda} & \downarrow \text{id}_C \\
C & C \\
\end{array}
= \begin{array}{ccc}
\Delta & \iota_{\lambda} & \phi_{\lambda} \\
\downarrow \iota_{\lambda} & \downarrow \phi_{\lambda} & \downarrow F_{\lambda} \\
C & C \times C & C \\
\downarrow \text{id}_C & \downarrow F_{\lambda} & \downarrow \text{id}_C \\
C & C \\
\end{array}
$$

(4.2.49)

iii) For every $\lambda, \mu \in [0, 1]$,

$$
\begin{array}{ccc}
\Delta & \text{id}_C \times \Delta & \Delta \\
\downarrow \iota_{\lambda,\mu} & \downarrow \text{id}_C \times \iota_{\lambda,\mu} & \downarrow \text{id}_C \times \Delta \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \text{id}_C & \downarrow \text{id}_C \times \text{id}_C & \downarrow \text{id}_C \times \text{id}_C \\
C \times C \times C & C \times C \times C & C \times C \times C \\
\downarrow \iota_{\lambda,\mu} \times \text{id}_C \times \Delta & \downarrow \Delta \times \iota_{\lambda,\mu} \times \text{id}_C \times C \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \Delta & \downarrow \Delta \times \text{id}_C \times \iota_{\lambda,\mu} \times C \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \Delta & \downarrow \Delta \times \text{id}_C \times \text{id}_C \times \Delta \\
C & C \times C \times C & C \times C \times C \\
\end{array}
= \begin{array}{ccc}
\Delta & \iota_{\lambda} & \phi_{\lambda} \\
\downarrow \iota_{\lambda} & \downarrow \phi_{\lambda} & \downarrow F_{\lambda} \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \text{id}_C & \downarrow \text{id}_C \times \text{id}_C & \downarrow \text{id}_C \times \text{id}_C \\
C \times C \times C & C \times C \times C & C \times C \times C \\
\downarrow \iota_{\lambda} \times \Delta & \downarrow \Delta \times \iota_{\lambda} \times \Delta \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \Delta & \downarrow \Delta \times \text{id}_C \times \iota_{\lambda} \times C \\
C & C \times C \times C & C \times C \times C \\
\downarrow \text{id}_C \times \Delta & \downarrow \Delta \times \text{id}_C \times \text{id}_C \times \Delta \\
C & C \times C \times C & C \times C \times C \\
\end{array}
$$

(4.2.50)
iv) For every \( \mu, \lambda \in [0, 1] \),

\[
\begin{array}{ccc}
\Delta \times \Delta & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^4 & \longrightarrow & C^2
\end{array}
\]

\[
\begin{array}{ccc}
F \mu \times F \mu & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^2 & \longrightarrow & C^2
\end{array}
\]

\[
\begin{array}{ccc}
\Delta \times \Delta & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^4 & \longrightarrow & C^2
\end{array}
\]

\[
\begin{array}{ccc}
F_\mu \times F_\mu & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^2 & \longrightarrow & C^2
\end{array}
\]

\[
\begin{array}{ccc}
\Delta \times \Delta & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^4 & \longrightarrow & C^2
\end{array}
\]

\[
\begin{array}{ccc}
F_\mu \times F_\mu & \longrightarrow & C^2 \\
\downarrow & & \downarrow \operatorname{id}_C \\
C^2 & \longrightarrow & C^2
\end{array}
\]

(4.2.51)

where

\[
\tau : = \operatorname{id}_C \times F_{1-\nu} \times \operatorname{id}_C,
\quad
\star : = \operatorname{id} \times \phi_{1-\nu} \times \operatorname{id},
\quad
\ast : = \operatorname{id} \times a_{\mu, \lambda, \mu, 1-\nu}
\]

(4.2.52)

and we have used shorthand notation \( C^n := C \times \cdots \times C \) \( n \) times Furthermore,

\[
\nu := \begin{cases} 
\frac{\lambda(1-\mu)}{\mu+\lambda-2\mu\lambda} & \text{if } \mu \lambda \neq 1 \\
\text{arbitrary} & \text{if } \lambda = \mu = 1
\end{cases}
\]

(4.2.53)

Note that the subdiagram with no 2-morphism filling in the space commutes.
Explicitly, the idempoters $i_{\lambda}$ assign to every object $y$ in $\mathcal{C}$ morphisms

$$\lambda y \oplus (1 - \lambda)y \xrightarrow{i_{\lambda}(y)} y$$

(4.2.54)

such that for every morphism $f : y \rightarrow z$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc}
\lambda y \oplus (1 - \lambda)y & \xrightarrow{i_{\lambda}(y)} & y \\
\downarrow{\lambda f \oplus (1 - \lambda)f} & & \downarrow{f} \\
\lambda z \oplus (1 - \lambda)z & \xrightarrow{i_{\lambda}(z)} & z
\end{array}$$

(4.2.55)

commutes. The conditions relating them to the other structure of a convex category are given as follows.

i) For each object $y$ in $\mathcal{C}$, the diagrams

$$\begin{array}{ccc}
0y \oplus 1y & \xrightarrow{i_0(y)} & y \\
\downarrow{u_0(y,y)} & & \downarrow{u_1(y,y)} \\
1y \oplus 0y & \xrightarrow{i_1(y)} & y
\end{array}$$

(4.2.56)

both commute, i.e. $i_0(y) = u_0(y,y)$ and $i_1(y) = u_1(y,y)$.

ii) For each $\lambda$ and object $y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc}
(1 - \lambda)y \oplus \lambda y & \xrightarrow{\phi_{1-\lambda}(y,y)} & \lambda y \oplus (1 - \lambda)y \\
\downarrow{i_{1-\lambda}(y)} & & \downarrow{i_\lambda(y)} \\
\lambda y \oplus (1 - \lambda)y & \xrightarrow{i_\lambda(y)} & y
\end{array}$$

(4.2.57)

commutes.
iii) For every pair $\lambda, \mu \in [0, 1]$ and every object $x$ in $\mathcal{C}$ the diagram

$$
\begin{align*}
\lambda \left( \mu x \oplus (1 - \mu)x \right) \oplus (1 - \lambda)x \\
a_{\lambda, \mu}(x,x,x) \\
\end{align*}
$$

commutes.

iv) For every pair $\lambda, \mu \in [0, 1]$ and every pair of objects $x, y$ in $\mathcal{C}$ the diagram

$$
\begin{align*}
F_\lambda(F_\mu(x, x), F_\mu(y, y)) \\
F_\lambda(F_\mu(F_\lambda(x, y), F_\lambda(x, y))) \\
F_\lambda(F_{\lambda\mu}(x, F_{\mu\lambda}(y, F_\lambda(x, y)))) \\
F_{\lambda\mu}(x, F_{\mu\lambda\nu}(y, F_\lambda(x, y))) \\
F_{\lambda\mu}(x, F_{\mu\lambda\nu}(y, F_\lambda(x, y))) \\
\end{align*}
$$

commutes.
Definition 4.2.60. A \textit{convex category with memory} is a convex category $\mathcal{C}$ equipped with an additional family of natural transformations

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \\
\downarrow{\delta_\lambda} & & \downarrow{\delta_\lambda} \\
\mathcal{C} & \xrightarrow{\text{id}_\mathcal{C}} & \mathcal{C}
\end{array}
\quad (4.2.61)
\]

parametrized by $\lambda \in [0,1]$ and called \textit{diagonal rectifiers}, such that $\delta_\lambda$ is a section of $i_\lambda$ for all $\lambda$, i.e.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \times \mathcal{C} \\
\downarrow{\delta_\lambda} & & \downarrow{\text{id}_\mathcal{C}} \\
\mathcal{C} & \xrightarrow{i_\lambda} & \mathcal{C}
\end{array}
= 
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{id}_\mathcal{C}} & \mathcal{C} \\
\downarrow{\delta_\lambda} & & \downarrow{\text{id}_\mathcal{C}} \\
\mathcal{C} & \xrightarrow{\text{id}_\mathcal{C}} & \mathcal{C}
\end{array}
\quad (4.2.62)
\]

Furthermore, the diagonal rectifiers satisfy analogous conditions to the ones that idempoters satisfy (see Definition 4.2.45).

The diagonal rectifiers $\delta_\lambda$ assign to every object $y$ in $\mathcal{C}$ morphisms

\[
y \xrightarrow{\delta_\lambda(y)} \lambda y \oplus (1 - \lambda)y
\quad (4.2.63)
\]

such that for every morphism $f : y \rightarrow z$ in $\mathcal{C}$, the obvious diagram commutes. Furthermore, the condition that $\delta_\lambda$ is a section of $i_\lambda$ means that for each object $x$ in $\mathcal{C}$ the diagram

\[
\begin{array}{ccc}
\lambda y \oplus (1 - \lambda)y & & \lambda y \oplus (1 - \lambda)y \\
\downarrow{\delta_\lambda(y)} & & \downarrow{\text{id}_y} \\
\lambda y \oplus (1 - \lambda)y & & \lambda y \oplus (1 - \lambda)y
\end{array}
\quad (4.2.64)
\]

commutes.
**Definition 4.2.65.** A convex category with perfect memory is a convex category with memory $\mathcal{C}$ whose idempoters $i_\lambda$ and diagonal rectifiers $\delta_\lambda$ are natural inverses of each other for all $\lambda \in [0, 1]$.

Due to the several slightly different notions of convex categories, it will be occasionally helpful to simply call them all convex categories and when necessary we will indicate which ones we mean if a certain type has to be specified.

**Definition 4.2.66.** The type of a convex category will refer to any of the following: convex category, convex category with idempoters, convex category with memory, and convex category with perfect memory.

Morphisms of convex categories will be discussed in Section 4.4 after several examples are given first in Section 4.3.

**Remark 4.2.67.** Convex categories alone only require a symmetric monoidal structure to be internalized. However, convex categories with idempoters and/or diagonal rectifiers need a cartesian structure since they use diagonals and projections.
4.3 Examples of convex categories

The idea of abstracting the notion of a convex category began from the example of finite sets with probability measures and measure-preserving functions in Baez, Fritz, and Leinster’s work [BFL11]. However, for the purposes of our examples and to fit it within the framework we have established, we will have to actually use a slightly different category than the one in [BFL11]. The slight difference is that we must use “equal almost everywhere” equivalence classes of measure-preserving functions. For those familiar with probability theory and measure theory in general, this subtle difference would seem like the natural thing to consider. However, because of this slight difference, we will have to be careful about using any of their results, since we may need to readjust some statements and proofs. Incidentally, our adjustments, which are natural from the categorical perspective, take care of some puzzles regarding 0 in later work of Baez and Fritz [BF14].

We therefore begin with this example in Section 4.3.1. Afterwards, we discuss in great detail other examples. These include finite sets with probability measures and stochastic maps, convex sets themselves, non-negative real numbers, probability measure spaces, probability density functions, and Hilbert spaces. We speculate that an enormously large number of examples
exist beyond the ones studied here and future work will consider examples from quantum (non-commutative) probability theory, geometry, particularly in the context of black hole thermodynamics, quantum field theory (and conformal field theory), metric spaces, dynamical systems, and so on.

4.3.1 Finite probability measure spaces

We first recall some basic definitions.

**Definition 4.3.1.** A **finite probability measure space** consists of a finite set $X$ together with a function

$$ p : \mathcal{P}(X) \longrightarrow \mathbb{R}_{\geq 0}, \quad (4.3.2) $$

where $\mathcal{P}(X)$ is the power set of $X$ (the set of all subsets of $X$) satisfying

$$ p(\emptyset) = 0, \quad (4.3.3) $$

$$ p \left( E_1 \cup \cdots \cup E_n \right) = p(E_1) + \cdots + p(E_n) \quad (4.3.4) $$

for any finite set $\{E_1, \ldots, E_n\}$ of disjoint subsets of $X$, i.e. $E_i \cap E_j = \emptyset$ for all $i \neq j$, and

$$ p(X) = 1. \quad (4.3.5) $$

Such a finite probability space will be written as a pair $(X, p)$. 
CONVEX CATEGORIES

We interpret the quantity $p(E)$ for $E \subset X$ as the probability of realizing the set of states/events described by $E$. Note that the disjoint union axiom tells us that the measure (probability) of any subset $E$ is determined by the measure of the singleton subsets.

**Definition 4.3.6.** Let $(X, p)$ and $(X', p')$ be two finite probability measure spaces. A measure-preserving function from $(X, p)$ to $(X', p')$ is a function $f : X \rightarrow X'$ satisfying

$$p'(x') = \sum_{x \in f^{-1}(x')} p(x) \quad (4.3.7)$$

for all $x' \in X'$.

Below, we depict several examples of measure preserving-functions and also examples of functions that are not measure-preserving. The bullets • represent distinct elements of the sets. The numbers next to the elements represent the probabilities associated with them. The arrows represent a particular function. An example of a measure-preserving function is

$$\begin{array}{c}
1/5 \bullet \\
1/5 \bullet \\
1/5 \bullet \\
1/5 \bullet \\
0 \bullet \\
0 \bullet \\
\end{array} \xrightarrow{\text{✓}} \begin{array}{c}
2/5 \\
3/5 \\
0 \\
\end{array} $$

(4.3.8)
while one that is not is given by

In [BFL11], the morphisms above defined the category of finite probability measure spaces. We will not take this as our definition. Instead, we will first define an equivalence relation on measure-preserving functions.

**Definition 4.3.10.** Two measure-preserving functions \( f, g : X \rightarrow X' \) are said to be *equal almost everywhere* (or equal a.e. for short) if the set

\[
\{ x \in X \mid f(x) \neq g(x) \} \subset X
\]

has measure zero, i.e.

\[
p\left( \{ x \in X \mid f(x) \neq g(x) \} \right) = 0,
\]

The following figure shows two measure-preserving functions that are
equal almost everywhere yet are not equal as functions.

However, the following two are not equal a.e.

Being equal almost everywhere defines an equivalence relation. We write such an equivalence class of measure-preserving functions from \((X, p)\) to \((X', p')\) as \([f] : (X, p) \rightsquigarrow (X', p')\).

**Lemma 4.3.15.** Let \([f] : (X, p) \rightsquigarrow (Y, q)\) and \([g] : (Y, q) \rightsquigarrow (Z, r)\) be two a.e. equivalence classes of measure-preserving functions of finite probability spaces. Then the composition

\[ [g] \circ [f] := [g \circ f] \]
using representatives and then taking the a.e. equivalence class is well-defined.

Although this fact is standard, we prove it here since we will use similar arguments later for stochastic maps.

**Proof.** Let $f': (X, p) \rightarrow (Y, q)$ and $g': (Y, q) \rightarrow (Z, r)$ be two other representatives. Let $x \in X$ be an element such that $g(f(x)) \neq g'(f'(x))$. Then there are two possibilities. (i) If $f(x) \neq f'(x)$ then since $f =_{a.e.} f'$, $p(x) = 0$. (ii) If $f(x) = f'(x)$, call this element $y$. Then $q(y) = 0$, but since $f$ and $f'$ are measure-preserving, $p(x) = 0$. Thus $g \circ f =_{a.e.} g' \circ f'$.

**Definition 4.3.17.** Let $\text{FinProb}$ be the category whose objects are finite probability measure spaces $(X, p)$ and whose morphisms are equivalence classes of equal a.e. measure-preserving functions.

Again, we remind the reader that our definition of $\text{FinProb}$ is not the one of [BFL11]. Note that with this definition of morphisms, an isomorphism $[f]: (X, p) \rightarrow (X', p')$ need not be a bijection of sets. We are allowed to add on sets of measure zero without changing the isomorphism class of an object. In other words, there are fewer isomorphism classes of objects in this category than there would be if we just used measure-preserving functions as our morphisms.
Our goal now is to define a convex structure on \textbf{FinProb}, which was used in [BFL11] but not explicitly phrased in this way.

Let \( \lambda \in [0, 1] \). Define the convex linear combinations of objects by

\[
\lambda (X, p) \oplus (1 - \lambda) (Y, q) := (X \sqcup Y, \lambda p \oplus (1 - \lambda)q),
\]

where

\[
(\lambda p \oplus (1 - \lambda)q)(z) := \begin{cases} 
\lambda p(z) & \text{if } z \in X \\
(1 - \lambda)q(z) & \text{if } z \in Y 
\end{cases}.
\]

Let \([f] : (X, p) \to (X', p')\) and \([g] : (Y, q) \to (Y', q')\) be two morphisms in \textbf{FinProb} and let \(f\) and \(g\) be representative measure-preserving maps. Then for any \( \lambda \in [0, 1]\), the convex linear combination

\[
\lambda [f] \oplus (1 - \lambda)[g] : (X \sqcup Y, \lambda p \oplus (1 - \lambda)q) \to (X' \sqcup Y', \lambda p' \oplus (1 - \lambda)q')
\]

of \([f]\) with \([g]\) is defined by the a.e. equivalence class associated to the map

\[
(\lambda f \oplus (1 - \lambda)g)(z) := \begin{cases} 
f(z) & \text{if } z \in X \\
g(z) & \text{if } z \in Y 
\end{cases}.
\]

It should be clear that the map in (4.3.21) is well-defined and measure-preserving. The left unitor \(u_0((X, p), (Y, q)) : 0(X, p) \oplus 1(Y, q) \to (Y, q)\) is the a.e. equivalence class associated to the function

\[
\left(u_0((X, p), (Y, q))\right)(z) := \begin{cases} 
\text{arbitrary} & \text{if } z \in X \\
z & \text{if } z \in Y
\end{cases}.
\]
The right unitor \( u_1((X, p), (Y, q)) : 1(X, p) \oplus 0(Y, q) \to (X, p) \) is similarly defined as the a.e. equivalence class associated to

\[
(u_1((X, p), (Y, q)))(z) := \begin{cases} 
  z & \text{if } z \in X \\
  \text{arbitrary} & \text{if } z \in Y.
\end{cases}
\]

(4.3.23)

Because of the fact that morphisms are defined as a.e. equivalence classes, the left and right unitors are isomorphisms for all \((X, p)\). Otherwise, they would not have been—this is one difference between our viewpoint and that of [BFL11].

The idempoters \( i_\lambda(X, p) : \lambda(X, p) \oplus (1 - \lambda)(X, p) \to (X, p) \) are defined to be the a.e. equivalence class associated to

\[
(i_\lambda(X, p))(x) := x.
\]

(4.3.24)

There is a slight abuse of notation here. Namely, we have not distinguished between the first factor \((X, p)\) and the second factor \((X, p)\). We hope this does not cause too much confusion.

The parametric commutors \( \phi_\lambda((X, p), (Y, q)) : (1 - \lambda)(Y, q) \oplus \lambda(X, p) \to \lambda(X, p) \oplus (1 - \lambda)(Y, q) \) are defined in the obvious way by swapping factors.

The convex associators \( a_{\lambda, \mu}((X, p), (Y, q), (Z, r)) \) are also defined in the obvious way using the canonical set-theoretic isomorphism \((X \amalg Y) \amalg Z \to X \amalg (Y \amalg Z)\).
Proposition 4.3.25. \textit{FinProb} with the convex linear combinations and structure natural transformations defined in this section is a convex category with idempoters.

\textit{Proof.} The convex structure has been specified above. Checking that all the axioms in Definitions 4.2.17 and 4.2.45 hold is tedious, but not difficult. ■

4.3.2 Finite probability measure spaces with measure-preserving stochastic maps

\textit{FinProb} is only a convex category and specifically not a convex category with memory because diagonal rectifiers $\delta_\lambda((X, p)) : (X, p) \rightarrow \lambda(X, p) \oplus (1 - \lambda)(X, p)$ cannot exist. To see this, we would need to send $x$ to the two elements $x$ in each disjoint union. However, this is not a function! Functions take in one input and spit out one output. In order to include diagonal rectifiers, we will need a generalization of functions that assign multiple elements to a single element. Furthermore, this “spread” of an element over the codomain should be a probability distribution. Such functions are called stochastic maps. We will follow [BF14] closely.

Definition 4.3.26. Let $X$ and $Y$ be two finite sets. Let $\text{PrM}(Y)$ denote the set of probability measures on $Y$.\footnote{Note that $\text{PrM}(Y) \cong \Delta^{[Y]} - 1$, where $\Delta^n$ is the $n$-simplex. This is important in formulating continuity of entropy.} A \textit{stochastic map} from $X$ to $Y$ is a...
function

\[ X \xrightarrow{f} \text{PrM}(Y) \]  
\[ x \mapsto f(x) \]  
which we write as

\[ Y \ni y \xmapsto{f(x)} \langle y, x \rangle_f \in \mathbb{R}_{\geq 0}. \]  

(4.3.27)

(4.3.28)

Note that by definition of \( f(x) \) being a probability distribution on \( Y \), this means that

\[ \sum_{y \in Y} \langle y, x \rangle_f = 1 \]  

for all \( x \in X \). We will denote stochastic maps as

\[ f : X \rightarrow^{\sim} Y, \]  

(4.3.30)

following the convention of [BF14], to distinguish them from ordinary functions.

We think of a stochastic map \( f \) from \( X \) to \( Y \) as a “weak function” in the sense that instead of assigning a unique element in \( Y \) to each element \( x \in X \), it assigns a “spread” of elements in \( Y \) but that “spread” is controlled by a probability distribution.

**Example 4.3.31.** Every function \( f : X \longrightarrow Y \) between finite sets determines a stochastic map from \( X \) to \( Y \) by setting

\[ \langle y, x \rangle_f := \delta_{yf(x)} \equiv \begin{cases} 1 & \text{if } y = f(x) \\ 0 & \text{otherwise} \end{cases}. \]  

(4.3.32)
Example 4.3.33. Let $\{\bullet\}$ be a one-element set and $X$ any finite set. Then a stochastic map $p : \{\bullet\} \to X$ is the same as a probability distribution $p$ on $X$.

Stochastic maps can be composed.

Definition 4.3.34. Let $X, Y,$ and $Z$ be finite sets. The composition of two stochastic maps $f : X \to Y$ and $g : Y \to Z$, written as $g \circ f : X \to Z$ is the stochastic map $g \circ f : X \to \text{PrM}(Z)$ defined by sending $x$ to the probability measure defined by

$$Z \ni z \xrightarrow{(g \circ f)(x)} \langle z, x \rangle_{g \circ f} := \sum_{y \in Y} \langle z, y \rangle_{g} \langle y, x \rangle_{f}. \quad (4.3.35)$$

This is indeed a stochastic map because

$$\sum_{z \in Z} \langle z, x \rangle_{g \circ f} = \sum_{y \in Y} \sum_{z \in Z} \langle z, y \rangle_{g} \langle y, x \rangle_{f} = \sum_{y \in Y} \langle y, x \rangle_{f} = 1. \quad (4.3.36)$$

Now suppose that we do not just have finite sets and stochastic maps between them, but suppose our sets are equipped with probability measures. We would like a definition of a stochastic map preserving this measure. Thankfully, we know now how to view a probability measure space as a set with a stochastic map into it from a single element set. We use this to define measure-preserving stochastic maps.
Definition 4.3.37. Let $X$ and $Y$ be finite sets with probability measures $p : \{\bullet\} \rightarrow X$ and $q : \{\bullet\} \rightarrow Y$. A measure-preserving stochastic map from $(X, p)$ to $(Y, q)$ is a stochastic map $f : (X, p) \rightarrow (Y, q)$ such that the diagram

$$
\begin{array}{ccc}
\{\bullet\} & \xrightarrow{p} & X \\
\downarrow & & \downarrow \uparrow f \\
\{\bullet\} & \xrightarrow{q} & Y
\end{array}
$$

commutes. Explicitly, this means that

$$
\langle y, \bullet \rangle_q = \sum_{x \in X} \langle y, x \rangle_f \langle x, \bullet \rangle_p
$$

(4.3.39)

for all $y \in Y$.

Example 4.3.40. A measure-preserving function (see Definition 4.3.6) is the special case of a measure-preserving stochastic map with $f$ a function (recall Example 4.3.31) because

$$
\sum_{x \in X} \langle y, x \rangle_f \langle x, \bullet \rangle_p = \sum_{x \in X} \delta_{y,f(x)} \langle x, \bullet \rangle_p = \sum_{x \in f^{-1}(y)} \langle x, \bullet \rangle_p = \langle y, \bullet \rangle_q.
$$

(4.3.41)

Again, it is reasonable to put an equivalence relation on measure-preserving stochastic maps so that sets of measure zero become irrelevant.

Definition 4.3.42. Let $(X, p)$ and $(Y, q)$ be two probability measure spaces. Two measure-preserving stochastic maps $f, g : (X, p) \rightarrow (Y, q)$ are said to be equal almost everywhere (or equal a.e. for short) if the set

$$
\{ x \in X \mid f(x) \neq g(x) \} \subset X
$$

(4.3.43)
has measure zero. Here \( f(x) \) and \( g(x) \) refer to the probability distributions on \( Y \) associated with the element \( x \). In this case, we use the notation \( f = \text{a.e.} \ g \).

To be clear at times, we will denote the a.e. equivalence class associated to \( f \) by \([f]\). However, we may sometimes abuse notation and drop the square brackets.

**Lemma 4.3.44.** Let \((X,p),(Y,q)\), and \((Z,r)\) be finite probability measure spaces and let \([f] : (X,p) \rightarrow (Y,q)\) and \([g] : (Y,q) \rightarrow (Z,r)\) be two a.e. equivalence classes of measure-preserving stochastic maps. Then the composition

\[
[g] \circ [f] := [g \circ f] \tag{4.3.45}
\]

is well-defined. Furthermore, with this definition, the collection of finite probability measures and a.e. equivalence classes of measure-preserving stochastic maps is a category, denoted by \textbf{FinProbStoch}.

**Proof.** Let \( f' \) and \( g' \) be other representatives of \([f]\) and \([g]\), respectively. Then

\[
p\left( \{ x \in X \mid \langle y, x \rangle_f \neq \langle y, x \rangle_{f'} \text{ for some } y \in Y \} \right) = 0 \tag{4.3.46}
\]

and

\[
q\left( \{ y \in Y \mid \langle z, y \rangle_g \neq \langle z, y \rangle_{g'} \text{ for some } z \in Z \} \right) = 0. \tag{4.3.47}
\]
The set on which \( g \circ f \) and \( g' \circ f' \) differ is

\[
\left\{ x \in X \mid \sum_{y \in Y} \langle z, y \rangle_g \langle y, x \rangle_f \neq \sum_{y \in Y} \langle z, y \rangle_{g'} \langle y, x \rangle_{f'} \text{ for some } z \in Z \right\}. \tag{4.3.48}
\]

Let \( x \) be an element in this set. Then there exists a \( y \in Y \) such that

\[
\langle z, y \rangle_g \langle y, x \rangle_f \neq \langle z, y \rangle_{g'} \langle y, x \rangle_{f'}.
\]

This implies that at least \( \langle y, x \rangle_f \neq \langle y, x \rangle_{f'} \) or \( \langle z, y \rangle_g \neq \langle z, y \rangle_{g'} \). In the first case, this implies \( p(x) = 0 \) since \( f = \text{a.e. } f' \). In the second case, this implies \( q(y) = 0 \) since \( g = \text{a.e. } g' \). But since \( f \) and \( f' \) are measure-preserving, \( p(x) = 0 \). Thus \( g \circ f = \text{a.e. } g' \circ f' \). \qed

The convex combinations on objects of \( \text{FinProbStoch} \) are the same as in \( \text{FinProb} \). Let \([f] : (X, p) \sim (X', p')\) and \([g] : (Y, q) \sim (Y', q')\) be two morphisms in \( \text{FinProbStoch} \) and let \( f \) and \( g \) be representative measure-preserving stochastic maps. Then for any \( \lambda \in [0, 1] \), the convex linear combination

\[
\lambda[f] \oplus (1-\lambda)[g] : (X \sqcup Y, \lambda p \oplus (1-\lambda)q) \sim (X' \sqcup Y', \lambda p' \oplus (1-\lambda)q') \tag{4.3.49}
\]

of \([f]\) with \([g]\) is defined by the a.e. equivalence class associated to the stochastic map

\[
(\lambda f \oplus (1-\lambda)g)(z) := \begin{cases} f(z) & \text{if } z \in X, \\ g(z) & \text{if } z \in Y' , \end{cases} \tag{4.3.50}
\]
i.e. for each $z' \in X' \sqcup Y'$,

$$
\langle z', z \rangle_{\lambda f \oplus (1 - \lambda) g} :=
\begin{cases}
\langle z', z \rangle_f & \text{if } z \in X \text{ and } z' \in X' \\
\langle z', z \rangle_g & \text{if } z \in Y \text{ and } z' \in Y' \\
0 & \text{otherwise}
\end{cases}
$$

(4.3.51)

It will be convenient to drop the “otherwise case” and implicitly remember that $\langle z', z \rangle_f$ and $\langle z', z \rangle_g$ are zero when they do not make sense. Therefore, we will abusively write this as

$$
\langle z', z \rangle_{\lambda f \oplus (1 - \lambda) g} :=
\begin{cases}
\langle z', z \rangle_f & \text{if } z \in X \\
\langle z', z \rangle_g & \text{if } z \in Y 
\end{cases}
$$

(4.3.52)

to save space. The fact that the resulting map in (4.3.50) on representatives is a measure-preserving stochastic map is not immediately obvious. We will prove this first, then we will prove the a.e equivalence class is indeed well-defined.

**Lemma 4.3.53.** The map in (4.3.50) is a measure-preserving stochastic map. Furthermore, $\lambda[f] \oplus (1 - \lambda)[g]$ is well-defined.

**Proof.** First, the map is stochastic because for each $z \in X \sqcup Y$,

$$
\sum_{z' \in X' \sqcup Y'} \langle z', z \rangle_{\lambda f \oplus (1 - \lambda) g} =
\begin{cases}
\sum_{z' \in X'} \langle z', z \rangle_f & \text{if } z \in X \\
\sum_{y' \in Y'} \langle y', z \rangle_g & \text{if } z \in Y
\end{cases}
\begin{cases}
1 & \text{if } z \in X \\
1 & \text{if } z \in Y
\end{cases}
= 1.
$$

(4.3.54)
Second, it is measure-preserving because for each \( z' \in X' \sqcup Y' \),
\[
\sum_{z \in X' \sqcup Y'} \langle z', z \rangle_{\lambda f \oplus (1 - \lambda) g} \delta(x, \cdot)_{\lambda p \oplus (1 - \lambda) q} = \sum_{z \in X'} \langle z', z \rangle_{\lambda f} \delta(x, \cdot)_{\lambda p}
\]
\[
+ \sum_{z \in Y'} \langle z', z \rangle_{g} (1 - \lambda) \delta(x, \cdot)_{\lambda q}
\]
\[
= \begin{cases} 
\lambda p'(z') & \text{if } z' \in X' \\
(1 - \lambda) q'(z') & \text{if } z' \in Y'
\end{cases} 
\]
\[
= \langle z', \cdot \rangle_{\lambda f' \oplus (1 - \lambda) g'}.
\]
\[(4.3.55)\]

Finally, to see that the map is well-defined, let \( f' \) and \( g' \) be two other representatives of \( f \) and \( g \), respectively. Let \( z \in X \sqcup Y \) be such that
\[
(\lambda f \oplus (1 - \lambda) g)(z) \neq (\lambda f' \oplus (1 - \lambda) g')(z),
\]
\[(4.3.56)\]
i.e. either \( z \in X \) and \( f(z) \neq f'(z) \) or \( z \in Y \) and \( g(z) \neq g'(z) \). In the first case, \( p(z) = 0 \) and in the second \( q(z) = 0 \). Hence \( (\lambda p \oplus (1 - \lambda))(z) = 0 \) and the two functions are equal a.e.

The left and right unitors \( u_0((X, p), (Y, q)) : 0(X, p) \oplus 1(Y, q) \to (Y, q) \) and \( u_1((X, p), (Y, q)) : 1(X, p) \oplus 0(Y, q) \to (X, p) \) are defined in the same way as in \textbf{FinProb}. The idempoters \( i_{\lambda}(X, p) : \lambda(X, p) \oplus (1 - \lambda)(X, p) \to (X, p) \) are also defined similarly. In addition, there are diagonal rectifiers \( \delta_{\lambda}(X, p) : (X, p) \to \lambda(X, p) \oplus (1 - \lambda)(X, p) \) defined by
\[
\langle x', x \rangle_{\delta_{\lambda}(X, p)} := \begin{cases} 
\lambda p(x') \delta_{xx'} & \text{if } x' \text{ is in the 1st factor} \\
(1 - \lambda) p(x') \delta_{xx'} & \text{if } x' \text{ is in the 2nd factor}
\end{cases}
\]
\[(4.3.57)\]

The convex associators \( a_{\lambda, \mu}((X, p), (Y, q), (Z, r)) \) are also defined in the obvious way using the obvious set-theoretic isomorphism \( (X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z) \).
Proposition 4.3.58. FinProbStoch with the convex linear combinations and structure natural transformations defined in this section is a convex category with memory.

Note that FinProbStoch is not a convex category with perfect memory because \( \lambda(X, p) \oplus (1 - \lambda)(X, p) \) is not isomorphic to \((X, p)\) for any \(\lambda \in (0, 1)\).

4.3.3 Convex sets

Every convex set is a convex category with perfect memory when viewed as a discrete category. All natural transformations are identities. Fritz has several examples of these so we refer to his list in [Fr09]. For reference, if \(C\) is a convex set, we denote its associated category by \(\mathcal{D}(C)\).

4.3.4 Non-negative real numbers

Let \(\mathbb{R}_{\geq 0}\) be the set of all non-negative real numbers equipped with the usual topology as a subspace of \(\mathbb{R}\). Let \(\mathbb{R}_{\geq 0}\) be the one-object category whose morphisms are the elements of \(\mathbb{R}_{\geq 0}\) (this is the usual way of viewing a monoid as a one-object category) and with composition given by addition of real numbers

\[
a \circ b := a + b
\]

(4.3.59)
for all $a, b \in \mathbb{R}_{\geq 0}$. Because there is only a single object, denoted by $\bullet$, any convex linear combination is just the object itself

$$\lambda \bullet \oplus (1 - \lambda) \bullet := \bullet. \quad (4.3.60)$$

For two morphisms, i.e. non-negative numbers $a, b \in \mathbb{R}_{\geq 0}$, the convex linear combination is defined to be the convex linear sum

$$F_\lambda(a, b) \equiv \lambda a \oplus (1 - \lambda)b := \lambda a + (1 - \lambda)b. \quad (4.3.61)$$

This is a functor because for two pairs of composable morphisms $\bullet \xrightarrow{a} \bullet \xrightarrow{a'} \bullet$ and $\bullet \xrightarrow{b} \bullet \xrightarrow{b'} \bullet$, we have

$$\lambda(a' \circ a) \oplus (1 - \lambda)(b' \circ b) := \lambda(a' + a) + (1 - \lambda)(b' + b)$$

$$= (\lambda a' + (1 - \lambda)b') + (\lambda a + (1 - \lambda)b) \quad (4.3.62)$$

All the unitors, idempoters, diagonal rectifiers, parametric commutors, and parametric associators are taken to be identities.

**Proposition 4.3.63.** $\mathbb{BR}_{\geq 0}$ is a convex category with perfect memory.

$\mathbb{BR}$ is similarly defined and is also a convex category with perfect memory.

### 4.3.5 More general probability measure spaces

The current example was worked out mostly by Brian Dressner. $\text{FinProb}$ can be extended to more general probability spaces that allow for infinitely
many possible events. These are described in terms of measure theory. As a result, we include a brief review of some of the main concepts. A reference for this section is [LiLo01].

**Definition 4.3.64.** A *measure space* is a triple \((X, \Sigma, \mu)\), consisting of a set \(X\), a collection of subsets \(\Sigma \subseteq \mathcal{P}(X)\) of \(X\), and a function \(\mu : \Sigma \to \mathbb{R}_{\geq 0}\) subject to the following conditions. First, the collection \(\Sigma\) must satisfy

i) \(X \in \Sigma\),

ii) if \(A \in \Sigma\), then \(A^c := \{x \in X \mid x \notin A\}\), the complement of \(A\), is also in \(\Sigma\), and

iii) if \(A_1, A_2, \ldots\) is a countable family of sets, their union \(\bigcup_{i=1}^{\infty} A_i\) is also in \(\Sigma\).

In this case, \(\Sigma\) is called a *sigma algebra* (on \(X\)) and \(A \in \Sigma\) is said to be *measurable*. The pair \((X, \Sigma)\) on its own is called a *measurable space*. Second, the function \(\mu\) must satisfy

i) \(\mu(\emptyset) = 0\) and

ii) for every countable family of disjoint subsets \(\{A_1, A_2, \ldots\}\) of \(X\),

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{4.3.65}
\]
The function $\mu$ is called a \textit{(positive) measure} on $(X, \Sigma)$. 

All measures in this article will be positive. As an example, every topological space $(X, \tau)$ gives rise to a measurable space.

**Definition 4.3.66.** Let $(X, \tau)$ be a topological space. The \textit{Borel sigma algebra}, written as $\mathcal{B}$ or $\mathcal{B}(\tau)$, on $(X, \tau)$ is the smallest sigma-algebra containing all open sets, i.e. all elements of $\tau$.

The Borel sigma algebra is constructed by taking countable unions, countable intersections, and relative complements of all open sets. There is no natural useful measure on (the Borel sigma algebra of) a topological space. One needs additional data, something we will discuss in detail later.

**Definition 4.3.67.** A \textit{probability space} is a measure space $(X, \Sigma, \mu)$ such that $\mu(X) = 1$. In this case, $\mu$ is often said to be a \textit{probability measure}.

**Definition 4.3.68.** Let $(X, \Sigma)$ and $(X', \Sigma')$ be two measurable spaces. A \textit{measurable function} from $(X, \Sigma)$ to $(X', \Sigma')$ is a function $f : X \rightarrow X'$ such that for every $A' \in \Sigma'$, $f^{-1}(A') \in \Sigma$.

**Definition 4.3.69.** Let $(X, \Sigma, \mu)$ and $(X', \Sigma', \mu')$ be two measure spaces. A \textit{measure-preserving function} from $(X, \Sigma, \mu)$ to $(X', \Sigma', \mu')$ is a measurable function $f : X \rightarrow X'$ such that for every $A' \in \Sigma'$, $\mu'(A') = \mu(f^{-1}(A'))$. 

The equivalence relation of two measure-preserving functions being equal
almost everywhere is very similar to the case in FinProb.

**Definition 4.3.70.** Two measure-preserving functions

\[
f, g : (X, \Sigma, \mu) \rightarrow (X', \Sigma', \mu')
\]  

are said to be *equal almost everywhere* (or equal a.e. for short) if the set

\[
\{ x \in X \mid f(x) \neq g(x) \} \subset X
\]

has measure zero (in particular, this set must be measurable), i.e.

\[
\mu\left( \{ x \in X \mid f(x) \neq g(x) \} \right) = 0,
\]

Just as in the FinProb case, being equal almost everywhere defines an
equivalence relation. Furthermore, composition of such equivalence classes is
well-defined. Therefore, we can define the analogue of FinProb.

**Definition 4.3.74.** Let MeasProb be the category whose objects are prob-
ability measure spaces \((X, \Sigma, \mu)\) and whose morphisms are equivalence classes
of equal a.e. measure-preserving functions.

The convex structure on MeasProb is defined similarly to FinProb. Let
\(\lambda \in [0, 1]\). Define the convex linear combinations of objects by

\[
\lambda(X, \Sigma, \mu) \oplus (1 - \lambda)(Y, \Omega, \nu) := (X \sqcup Y, \Sigma \oplus \Omega, \lambda \mu \oplus (1 - \lambda)\nu),
\]
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where \( \Sigma \oplus \Omega \) is the sigma algebra defined by\(^{11}\)

\[
\Sigma \oplus \Omega = \left\{ E \subset X \sqcup Y \mid i_X^{-1}(E) \in \Sigma \text{ and } i_Y^{-1}(E) \in \Omega \right\},
\]

(4.3.76)

where \( i_X : X \hookrightarrow X \sqcup Y \) and \( i_Y : Y \hookrightarrow X \sqcup Y \) are the inclusion functions, and where

\[
(\lambda \mu \oplus (1 - \lambda)\nu)(E) := \lambda \mu \left( i_X^{-1}(E) \right) + (1 - \lambda)\nu \left( i_Y^{-1}(E) \right)
\]

for all measurable \( E \in \Sigma \oplus \Omega \). Let \([f] : (X, \Sigma, \mu) \longrightarrow (X', \Sigma', \mu')\) and \([g] : (Y, \Omega, \nu) \longrightarrow (Y', \Omega', \nu')\) be two morphisms in \( \text{MeasProb} \) and let \( f \) and \( g \) be representative measure-preserving maps. Then for any \( \lambda \in [0, 1] \), the convex linear combination

\[
(X \sqcup Y, \Sigma \oplus \Omega, \lambda \mu \oplus (1 - \lambda)\nu) \xrightarrow{\lambda[f] \oplus (1 - \lambda)[g]} (X' \sqcup Y', \Sigma' \oplus \Omega', \lambda \mu' \oplus (1 - \lambda)\nu')
\]

(4.3.78)

of \([f]\) with \([g]\) is defined by the a.e. equivalence class associated to the map

\[
(\lambda f \oplus (1 - \lambda)g)(z) := \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in Y \end{cases}
\]

(4.3.79)

Right and left unitors, idempoters, parametric commutors, and convex associators are defined exactly as in \( \text{FinProb} \) and have the same properties (namely, all are isomorphisms except the idempoters).

\(^{11}\Sigma \oplus \Omega \) is equivalently the sigma algebra generated by \( \Sigma \sqcup \Omega \).
Proposition 4.3.80. MeasProb with the convex linear combinations and structure natural transformations defined in this section is a convex category with idempoters.

4.3.6 Probability density functions

The current example was worked out mostly by Brian Dressner. We briefly review some basics about integration theory over measure spaces [LiLo01] and if the notation is unfamiliar, please review the previous example. By convention, when we refer to a measurable function $f : X \rightarrow \mathbb{R}$, we mean a measurable function $f : (X, \Sigma) \rightarrow (\mathbb{R}, \mathcal{B})$, where $\mathbb{R}$ is equipped with the usual topology and $\mathcal{B}$ is its associated set of Borel measurable subsets. Equivalently, a measurable function $f : X \rightarrow \mathbb{R}$ is one such that the level set

$$S_f(t) := \{ x \in X \mid f(x) \geq t \} \quad (4.3.81)$$

is measurable for every $t \in \mathbb{R}$. Denote the set of measurable functions on $(X, \Sigma)$ by $M(X)$. Measurable functions can be integrated.

Definition 4.3.82. Let $f : (X, \Sigma, \mu) \rightarrow (\mathbb{R}_{\geq 0}, \mathcal{B})$ be a non-negative measurable function. The Lebesgue integral of $f$ over $X$ is defined by

$$\int_X f(x) d\mu(x) := \int_0^\infty \mu(S_f(t)) dt. \quad (4.3.83)$$

Occasionally, the left-hand-side is written as $\int_X f d\mu$. 
In order to define the entropy in an analogous way to the finite probability space, we need a probability density function with respect to a measure \( \mu \).

**Definition 4.3.84.** Let \((X, \Sigma, \mu)\) be a measure space. A **probability density function** on \((X, \Sigma, \mu)\) is a function \( p : X \to \mathbb{R}_{\geq 0} \) such that

\[
\int_X p \, d\mu = 1. \tag{4.3.85}
\]

A measure space together with a probability density function is called an **mspdf** (short for “measure space with probability density function”).

**Definition 4.3.86.** Let \((X, \Sigma, \mu, p)\) and \((X', \Sigma', \mu', p')\) be two mspdf’s. A **pdf-preserving morphism** from \((X, \Sigma, \mu, p)\) to \((X', \Sigma', \mu', p')\) is an a.e. equivalence class of measure-preserving maps \( f : (X, \Sigma, \mu) \to (X', \Sigma', \mu') \) such that

\[
p' \circ f = =_{a.e.} p.
\]

Notice that if \( f = =_{a.e.} f' \) and \( p' \circ f = =_{a.e.} p \) in the previous definition, then

\[
p' \circ f' = =_{a.e.} p' \circ f = =_{a.e.} p. \tag{4.3.87}
\]

**Definition 4.3.88.** Let \( \text{MSPDF} \) be the category whose objects are mspdf’s and whose morphisms are pdf-preserving morphisms.

The convex structure on \( \text{MSPDF} \) is more similar to that of \( \text{FinProb} \) than that of \( \text{MeasProb} \). It is given by

\[
\lambda(X, \Sigma, \mu, p) \oplus (1-\lambda)(Y, \Omega, \nu, q) := (X\oplus Y, \Sigma \oplus \Omega, \mu \oplus \nu, \lambda p \oplus (1-\lambda)q), \tag{4.3.89}
\]
where all of the individual components have already been defined and \( \mu \oplus \nu \) is the usual measure on \( X \sqcup Y \), i.e.

\[
\Sigma \oplus \Omega \ni E \mapsto (\mu \oplus \nu)(E) := \mu(i^{-1}_X(E)) + \nu(i^{-1}_Y(E)).
\] (4.3.90)

The definition for convex combinations on morphisms is the same as in \textbf{FinProb} and \textbf{MeasProb}. Similarly, convex associators, right and left uni-
tors, idempoters, and parametric commutors are defined as in \textbf{FinProb} and \textbf{MeasProb}.

\textbf{Proposition 4.3.91.} \textit{MSPDF with the convex linear combinations and structure natural transformations defined in this section is a convex category with idempoters.}

\textbf{4.3.7 Hilbert spaces}

Let \( \textbf{Hilb} \) be the category whose objects are (complex) Hilbert spaces and whose morphisms are bounded linear operators. Let \( (\mathcal{H}, \langle \cdot , \cdot \rangle_{\mathcal{H}}) \) and \( (\mathcal{V}, \langle \cdot , \cdot \rangle_{\mathcal{V}}) \) be two Hilbert spaces. For each \( \lambda \in (0,1) \), define the convex linear combination of these two

\[
\lambda (\mathcal{H}, \langle \cdot , \cdot \rangle_{\mathcal{H}}) \oplus (1 - \lambda) (\mathcal{V}, \langle \cdot , \cdot \rangle_{\mathcal{V}})
\] (4.3.92)

to be the Hilbert space \( \mathcal{H} \oplus \mathcal{V} \) completed with respect to the inner product

\[
\langle (w_1, v_1), (w_2, v_2) \rangle_{\lambda \mathcal{H} \oplus (1- \lambda) \mathcal{V}} := \lambda \langle w_1, w_2 \rangle_{\mathcal{H}} + (1 - \lambda) \langle v_1, v_2 \rangle_{\mathcal{V}}
\] (4.3.93)
for all $w_1, w_2 \in \mathcal{H}$ and $v_1, v_2 \in \mathcal{V}$. When $\lambda = 0$ or $\lambda = 1$, set

$$0\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right) \oplus 1\left(\mathcal{V}, \langle \cdot, \cdot \rangle_\mathcal{V}\right) := \left(\mathcal{V}, \langle \cdot, \cdot \rangle_\mathcal{V}\right) \quad (4.3.94)$$

and

$$1\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right) \oplus 0\left(\mathcal{V}, \langle \cdot, \cdot \rangle_\mathcal{V}\right) := 1\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right). \quad (4.3.95)$$

The reason for this definition when $\lambda = 0$ or $\lambda = 1$ is because the inner product must be non-degenerate. By these last two definitions, we can set the unitors to be the identity maps. The convex linear combination of bounded linear operators is just the direct sum. Parametric commutors and parametric associators are somewhat obvious. For each $\lambda \in [0, 1]$, set the idempoter to be

$$\lambda\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right) \oplus (1 - \lambda)\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right) \overset{\lambda\left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right)}{\longrightarrow} \left(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}\right) \quad (4.3.96)$$

$$(w_1, w_2) \longrightarrow \lambda w_1 + (1 - \lambda)w_2.$$
It is clear the idempoters are linear. The idempoters are bounded operators because addition is a bounded operation. In more detail,
\[
\|i_\lambda(w_1, w_2)\|^2 = \|\lambda w_1 + (1 - \lambda) w_2\|^2
\]
\[
= \lambda^2 \|w_1\|^2 + (1 - \lambda)^2 \|w_2\|^2 + \lambda(1 - \lambda)(\langle w_1, w_2 \rangle + \langle w_2, w_1 \rangle)
\]
\[
\leq \lambda^2 \|w_1\|^2 + (1 - \lambda)^2 \|w_2\|^2 + 2\lambda(1 - \lambda)\|w_1\|\|w_2\|
\]
\[
= (\lambda\|w_1\| + (1 - \lambda)\|w_2\|)^2
\]
\[
= \|(w_1, w_2)\|^2
\]  
(4.3.97)
where the second line comes from the polarization identity, the fourth line follows from the Cauchy-Schwarz inequality, and the sixth line comes from the definition of the norm with respect to the inner product we have defined in (4.3.93). Hence \(\|i_\lambda\| \leq 1\) and \(i_\lambda\) is bounded. There are diagonal rectifiers \(\delta_\lambda\) as well and are defined by
\[
(H, \langle \cdot, \cdot \rangle_H) \stackrel{\delta_\lambda(H, \langle \cdot, \cdot \rangle_H)}{\longrightarrow} \lambda(H, \langle \cdot, \cdot \rangle_H) \oplus (1 - \lambda)(H, \langle \cdot, \cdot \rangle_H)
\]
\[
w \quad \longrightarrow (w, w).
\]  
(4.3.98)
This map is clearly linear and is bounded because
\[
\|\delta_\lambda(w)\| = \|(w, w)\| = \lambda\|w\| + (1 - \lambda)\|w\| = \|w\|.
\]  
(4.3.99)

Proposition 4.3.100. Hilb with the convex structure described above is a convex category with memory.
4.3.8 Other examples

Technical details have prevented us from including several almost complete examples in the context of quantum (non-commutative) probability theory. These include density matrices on Hilbert spaces and states on $C^*$-algebras. These will be examined in full detail in future work. It would be interesting to see if there are also many more examples of convex structures on categories coming from metric spaces and dynamical systems [AKM65], [Bo71], [Bo73], [Bo14], Riemannian geometry, in particular black hole thermodynamics [Be73], and quantum field theory [CaCa04] to name a few. We hope to address these examples in future work.

Although we are a bit skeptical, we also believe that if convex structure is indeed a necessary one for information theory, entropy, information loss, etc., then category theory itself should be phrased in this language. This is because functors themselves lose information as they are merely invariants of structure.
4.4 Convex functors and natural transformations

4.4.1 Abstract convex functions

From the algebraic definition of a convex set, or more generally a convex object (see Definitions 4.2.1, 4.2.9, and 4.2.15), a natural definition for a morphism from one convex object \((C, F)\) to another \((D, G)\) (both internal in the same category) would be one that preserves the structure.

**Definition 4.4.1.** Let \((C, \otimes, I, a, l, r, \gamma, \pi_1, \pi_2, e)\) be a cartesian monoidal category (see Definition 4.5.129) and let \((C, F)\) and \((D, G)\) be two convex objects in \(C\). An affine function from \((C, F)\) to \((D, G)\) is a morphism \(S : C \rightarrow D\) in \(C\) such that the diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{S \otimes S} & D \otimes D \\
F_\lambda \downarrow & & \downarrow G_\lambda \\
C & \xrightarrow{S} & D
\end{array}
\]

(4.4.2)

commutes.

Written as an equation and in the case that \(C\) is the category of sets, commutativity of this diagram says that

\[
S(\lambda x \oplus (1 - \lambda)y) = \lambda S(x) \oplus (1 - \lambda)S(y)
\]

(4.4.3)

for all elements \(x, y \in C\). The diagram also makes sense for quantum convex objects. Notice that from this algebraic definition of a convex set, it is not
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apparent how to allow for convex or concave functions. On the other hand, convex functors naturally allow for such possibilities due to the existence of natural transformations.

4.4.2 Convex functors and natural transformations

Definition 4.4.4. Let \((C, F)\) and \((D, G)\) be two convex categories (of the same type).

A convex functor from \(C\) to \(D\) consists of a functor \(S : C \to D\) together with a family of natural transformations \(\eta_\lambda\)

\[
\begin{array}{ccc}
C \times C & \overset{S \times S}{\longrightarrow} & D \times D \\
\downarrow{F_\lambda} & & \downarrow{G_\lambda} \\
C & \overset{S}{\longrightarrow} & D \\
\end{array}
\]

(4.4.5)

indexed by \(\lambda \in [0, 1]\) satisfying the following conditions.

i) The unitors must satisfy

\[
\begin{array}{ccc}
C & \overset{S}{\longrightarrow} & D \\
\downarrow{F_0} & & \downarrow{G_0} \\
C \times C & \overset{S \times S}{\longrightarrow} & D \times D \\
\end{array}
\]

\[
\begin{array}{ccc}
C \times C & \overset{S \times S}{\longrightarrow} & D \times D \\
\downarrow{F_0} & & \downarrow{G_0} \\
C & \overset{S}{\longrightarrow} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
C \times C & \overset{S \times S}{\longrightarrow} & D \times D \\
\downarrow{F_0} & & \downarrow{G_0} \\
C & \overset{S}{\longrightarrow} & D \\
\end{array}
\]

(4.4.6)

and a similar equality for \(u_1\).

There are more data in these definitions such as unitors, idempoters/diagonal rectifiers, parametric commutators, and associators (see Definitions 4.2.17 and 4.2.45). These are written using the same notation for both \(C\) and \(D\).
ii) If $C$ and $D$ have idempoters, then for every $\lambda \in [0, 1]$,

\[
\begin{array}{c}
\xymatrix{
C \times C \ar[r]^{S} & \mathcal{D} \\
\Delta \ar[ru]^{F_{\lambda}} & & \Delta \ar[ru]^{i_{\lambda}} \\
C \ar[r]_{S} & \mathcal{D} \\
\end{array}
\quad = \quad
\begin{array}{c}
\xymatrix{
\mathcal{C} \ar[r]^{S} & \mathcal{C} \\
\Delta \ar[ru]^{F_{\lambda}} & & \Delta \ar[ru]^{i_{\lambda}} \\
\mathcal{C} \ar[r]_{S} & \mathcal{C} \\
\end{array}
\]  \tag{4.4.7}

A similar condition must hold if $C$ and $D$ have diagonal rectifiers.

iii) For every $\lambda \in [0, 1]$,

\[
\begin{array}{c}
\xymatrix{
\mathcal{C} \times \mathcal{C} \ar[r]^{S \times S} \ar[rd]_{\gamma} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[ru]^{G_{1-\lambda}} & & \mathcal{D} \\
\mathcal{C} \ar[r]_{S} & \mathcal{D} \\
\end{array}
\quad = \quad
\begin{array}{c}
\xymatrix{
\mathcal{C} \times \mathcal{C} \ar[r]^{S \times S} \ar[rd]_{\gamma} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[ru]^{G_{1-\lambda}} & & \mathcal{D} \\
\mathcal{C} \ar[r]_{S} & \mathcal{D} \\
\end{array}
\]  \tag{4.4.8}

iv) For every $\lambda, \mu \in [0, 1]$,

\[
\begin{array}{c}
\xymatrix{
\mathcal{C} \times \mathcal{C} \ar[r]^{F_{\mu} \times \text{id}_{\mathcal{C}}} \ar[rd]_{\eta_{\lambda} \times \text{id}_{\mathcal{S}}} & \mathcal{C} \times \mathcal{C} \ar[r]^{S \times S \times S} \ar[rd]_{G_{\mu}} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[r]_{F_{\lambda}} & \mathcal{D} \times \mathcal{D} \ar[r]_{\text{id}_{\mathcal{D}} \times G_{\lambda, \mu}} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[r]_{S} & \mathcal{D} \times \mathcal{D} \\
\end{array}
\quad = \quad
\begin{array}{c}
\xymatrix{
\mathcal{C} \times \mathcal{C} \ar[r]^{F_{\mu} \times \text{id}_{\mathcal{C}}} \ar[rd]_{\eta_{\lambda} \times \text{id}_{\mathcal{S}}} & \mathcal{C} \times \mathcal{C} \ar[r]^{S \times S \times S} \ar[rd]_{G_{\mu}} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[r]_{F_{\lambda}} & \mathcal{D} \times \mathcal{D} \ar[r]_{\text{id}_{\mathcal{D}} \times G_{\lambda, \mu}} & \mathcal{D} \times \mathcal{D} \\
\mathcal{C} \ar[r]_{S} & \mathcal{D} \times \mathcal{D} \\
\end{array}
\]  \tag{4.4.9}

We write $(S, \eta)$ to denote a convex functor as above.
Let us spell out what the definition of a convex functor is explicitly. For each \( \lambda \in [0, 1] \) the family of natural transformations \( \eta_\lambda \) assigns a morphism

\[
S(\lambda x \oplus (1 - \lambda)y) \xrightarrow{\eta_\lambda(x,y)} \lambda S(x) \oplus (1 - \lambda)S(y)
\]

for all objects \( x, y \) in \( \mathcal{C} \) satisfying naturality with respect to morphisms \( f : x \to x' \) and \( g : y \to y' \). The extra conditions say the following.

i) For every pair of objects \( x, y \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
0S(x) \oplus 1S(y) & \xrightarrow{\eta_0(x,y)} & u_0(S(x),S(y)) \\
\downarrow & & \downarrow \\
S(0x \oplus 1y) & \xrightarrow{\eta_1(x,y)} & S(y)
\end{array}
\]

(and a similar one for \( u_1 \)) commutes.

ii) If \( \mathcal{C} \) and \( \mathcal{D} \) have idempoters, for every \( \lambda \in [0, 1] \) and object \( x \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
S(\lambda x \oplus (1 - \lambda)x) & \xrightarrow{\eta_\lambda(x,x)} & \lambda S(x) \oplus (1 - \lambda)S(x) \\
\downarrow & & \downarrow \\
S(\lambda(x)) & \xrightarrow{\iota_\lambda(S(x))} & S(x)
\end{array}
\]

commutes.

iii) For every \( \lambda \in [0, 1] \) and pair of objects \( x, y \) in \( \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
S((1 - \lambda)y \oplus \lambda x) & \xrightarrow{\eta_{1-\lambda}(y,x)} & (1 - \lambda)S(y) \oplus \lambda S(x) \\
\downarrow & & \downarrow \\
S(\phi_\lambda(x,y)) & \xrightarrow{\phi_\lambda(S(x),S(y))} & S(\lambda S(x) \oplus (1 - \lambda)S(y))
\end{array}
\]

commutes.
commutes.

iv) For every triple $x, y, z$ of objects in $C$ and every pair $\lambda, \mu \in [0, 1]$ the diagram

\[
S \left( \lambda \left( \mu x \oplus (1-\mu)y \right) \oplus (1-\lambda)z \right) \xrightarrow{\eta_{\lambda} (\mu x \oplus (1-\mu)y) \oplus (1-\lambda)z} \lambda S \left( \mu x \oplus (1-\mu)y \right) \oplus (1-\lambda)S(z)
\]

\[
S(\alpha_{\lambda, \mu}(x, y, z)) \xrightarrow{\lambda \eta_{\mu, \mu}(x, y) \oplus (1-\lambda)\text{id}_{S(z)}} \lambda \eta_{\mu}(x, y) \oplus (1-\lambda)\text{id}_{S(z)}
\]

\[
S \left( \lambda, \mu x \oplus (1-\lambda, \mu) \left( \lambda, \mu y \oplus (1-\lambda, \mu)z \right) \right) \xrightarrow{\lambda, \mu S(x) \oplus (1-\mu, \mu)S(y)} \lambda, \mu S(x) \oplus (1-\lambda, \mu) \left( \lambda, \mu S(y) \oplus (1-\lambda, \mu)S(z) \right)
\]

\[
\eta_{\lambda, \mu, \mu}(x, \lambda, \mu y) \oplus (1-\lambda, \mu) \lambda, \mu \text{id}_{S(z)} \oplus (1-\lambda, \mu) \eta_{\lambda, \mu, \mu}(y, z)
\]

commutes.

Example 4.4.15. Recall the example from Section 4.3.4. A convex functor $\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a linear map $\mathbb{R} \longrightarrow \mathbb{R}$ with positive slope restricted to $\mathbb{R}_{\geq 0}$.

Example 4.4.16. Let $C_1$ and $C_2$ be convex sets with total orderings $\geq$ (such as $\mathbb{R}$ with its usual ordering) and let $(C_1, \geq)$ and $(C_2, \geq)$ be the convex categories whose set of morphisms from an element $a$ to $b$ consists of a single element if $a \leq b$ or is empty otherwise. A convex functor $F : (C_1, \geq) \longrightarrow (C_2, \geq)$ is equivalent to a convex function $F : C_1 \longrightarrow C_2$, i.e. a function satisfying

\[
F(\lambda a + (1-\lambda)b) \leq \lambda F(a) + (1-\lambda)F(b)
\]
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for all $\lambda \in [0, 1]$ and all $a, b \in C_1$. This is the motivating example for the terminology in Definition 4.4.4 (compare (4.4.17) with (4.4.10)). If we had demanded $\eta_\lambda$ to be an isomorphism for all $\lambda \in [0, 1]$, then we could not obtain ordinary convex analysis in our categorical generalization.

Example 4.4.18. Let $C_1$ and $C_2$ be convex sets and $\mathcal{D}(C_1)$ and $\mathcal{D}(C_2)$ be the associated discrete convex categories as in Section 4.3.3. A convex functor $F : \mathcal{D}(C_1) \to \mathcal{D}(C_2)$ is the same as an affine function $F : C_1 \to C_2$, i.e. a function satisfying (see [Fr09])

$$F(\lambda a + (1 - \lambda)b) = \lambda F(a) + (1 - \lambda)F(b)$$  \hspace{1cm} (4.4.19)

for all $\lambda \in [0, 1]$ and $a, b \in C_1$. This is because $\eta_\lambda$ can only be the identity since $\mathcal{D}(C_2)$ is a discrete category.

There are several variants of convex functors, which we mention now for completeness.

Definition 4.4.20. Let $(\mathcal{C}, F)$ and $(\mathcal{D}, G)$ be two convex categories (of the same type) as in Definition 4.4.4. A **concave functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $S : \mathcal{C} \to \mathcal{D}$ together with a family of natural transformations $\eta_\lambda$

$$\begin{array}{c}
\mathcal{C} \times \mathcal{C} \xrightarrow{S \times S} \mathcal{D} \times \mathcal{D} \\
\mathcal{D} \xrightarrow{\eta_\lambda} \mathcal{D} \\
\eta_\lambda \\
\mathcal{C} \xrightarrow{S} \mathcal{D}
\end{array}$$  \hspace{1cm} (4.4.21)
indexed by $\lambda \in [0, 1]$ and satisfying completely analogous conditions to those
of Definition 4.4.4. $S$ together with the family $\{\eta_\lambda\}$ is called an
\textit{affine functor} if $\eta_\lambda$ is a natural isomorphism for all $\lambda \in [0, 1]$. It is a
\textit{strictly affine functor} if $\eta_\lambda$ is the identity natural transformation for all $\lambda \in [0, 1]$.

\textbf{Remark 4.4.22.} Because of the subtle variants of these definitions, we will
use the abusive terminology and refer to any of these functors as “convex” (or
say, less abusively, “morphism of convex categories”) since any statements
made here for convex functors will hold true for concave, affine, and strictly
affine. However, when we wish to emphasize the type of functor, such as in
examples, we will explicitly say so.

\textbf{Definition 4.4.23.} Let $(\mathcal{C}, F)$ and $(\mathcal{D}, G)$ be two convex categories and
let $(S, \eta), (T, \kappa) : \mathcal{C} \rightarrow \mathcal{D}$ be two convex functors. Then a \textit{convex natural
transformation} from $(S, \eta)$ to $(T, \kappa)$ consists of a natural transformation $\sigma : S \Rightarrow T$ such that

\begin{equation}
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow F_\lambda \\
\mathcal{C}
\end{array}
\xrightarrow{\eta_\lambda}
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\downarrow G_\lambda \\
\mathcal{D}
\end{array}
\xrightarrow{T \times T}
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow F_\lambda \\
\mathcal{C}
\end{array}
\xrightarrow{\sigma}
\begin{array}{c}
\mathcal{C} \\
\downarrow \eta_\lambda \\
\mathcal{D}
\end{array}
\xrightarrow{\sigma \times \sigma}
\begin{array}{c}
\mathcal{D} \\
\downarrow G_\lambda \\
\mathcal{D}
\end{array}
\end{array}
\quad (4.4.24)

\end{equation}

Completely analogous definitions hold when the functors are concave, affine,
and strictly affine.
Explicitly, a convex natural transformation as above assigns to every object \( x \) in \( C \), a morphism \( \sigma_x : S(x) \rightarrow T(x) \) in \( D \) while the diagram in (4.4.24) says that for any two objects \( x, y \) in \( C \), the diagram

\[
\begin{align*}
S(\lambda x \oplus (1 - \lambda)y) & \xrightarrow{\eta_{\lambda x}(x,y)} \lambda S(x) \oplus (1 - \lambda)S(y) \\
\sigma_{\lambda x \oplus (1 - \lambda)y} & \downarrow \downarrow \\
T(\lambda x \oplus (1 - \lambda)y) & \xrightarrow{\kappa_{\lambda x}(x,y)} \lambda T(x) \oplus (1 - \lambda)T(y)
\end{align*}
\]  

(4.4.25)

commutes.

**Remark 4.4.26.** The axioms for convex functors and natural transformations have been described diagrammatically and hence make sense in cartesian monoidal 2-categories.

**Example 4.4.27.** Let \( C_1 \) and \( C_2 \) be convex sets with total orderings \( \geq \) as in Example 4.4.16 and let \((C_1, \geq)\) and \((C_2, \geq)\) be their associated convex categories. Let \( F, G : (C_1, \geq) \rightarrow (C_2, \geq)\) be two convex functors. A natural transformation \( \sigma : F \Rightarrow G \) (not necessarily a priori convex) says that \( F \leq G \). In particular, \( \sigma \) is automatically a convex natural transformation.

### 4.4.3 Some results about convex categories

**Lemma 4.4.28.** Let \((C, F), (D, G),\) and \((E, H)\) be convex categories and \((S, \eta) : C \rightarrow D\) and \((T, \kappa) : D \rightarrow E\) convex functors. The composition
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T \circ S : \mathcal{C} \rightarrow \mathcal{E}, together with the family \( \kappa \circ \eta \) defined by the composition

\[
\begin{align*}
\mathcal{C} \times \mathcal{C} & \xrightarrow{S \times S} \mathcal{D} \times \mathcal{D} \xrightarrow{T \times T} \mathcal{E} \times \mathcal{E} \\
\mathcal{C} & \xrightarrow{S} \mathcal{D} \xrightarrow{T} \mathcal{E}
\end{align*}
\]

for \( \lambda \in [0, 1] \), is a convex functor from \( \mathcal{C} \) to \( \mathcal{E} \).

**Proof.** All the axioms from Definition 4.4.4 follow from the interchange law for compositions of natural transformations.

\[\blacksquare\]

**Lemma 4.4.30.** Let \( (\mathcal{C}, F) \) and \( (\mathcal{D}, G) \) be two convex categories, \( (R, \theta), (S, \eta), (T, \kappa) : \mathcal{C} \rightarrow \mathcal{D} \) three convex functors, and \( \tau : R \Rightarrow S \) and \( \sigma : S \Rightarrow T \) two convex natural transformations. Then the vertical composition \( \bar{\tau} \circ \bar{\sigma} \) is convex.

**Proof.** The required condition from Definition 4.4.23 holds by the following sequence of equalities

\[
\begin{align*}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\kappa_\lambda} \mathcal{D} \times \mathcal{D} \\
\mathcal{C} & \xrightarrow{\theta} \mathcal{D} \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{\eta_\lambda} \mathcal{D} \times \mathcal{D} \\
\mathcal{C} & \xrightarrow{T} \mathcal{E}
\end{align*}
\]

and the fact that \( \bar{\tau} \circ \bar{\sigma} = \bar{\tau \times \tau} \circ \bar{\sigma \times \sigma} \).

\[\blacksquare\]

**Lemma 4.4.32.** Let \( (\mathcal{C}, F), (\mathcal{D}, G), \) and \( (\mathcal{E}, H) \) be convex categories, \( (Q, \varphi), (S, \eta) : \mathcal{C} \rightarrow \mathcal{D} \) and \( (R, \theta), (T, \kappa) : \mathcal{D} \rightarrow \mathcal{E} \) be convex functors, and \( \sigma : Q \Rightarrow S \) and
PROPOSITION 4.4.34. The collection of convex categories, convex functors, and convex natural transformations form a 2-category, which we denote by ConvexCat.

Proof. This follows from the previous Lemmata.
Note that since convex functors have additional data, ConvexCat is not a subcategory of Cat, the 2-category of categories, but there is a forgetful functor from ConvexCat to Cat.

**Proposition 4.4.35.** Let \( C \) be a category and \( D \) a convex category. Then \( \text{Fun}(C, D) \), the category of functors from \( C \) to \( D \), is a convex category with pointwise convex structure. Namely, for every \( \lambda \in [0, 1] \), the convex combination of two functors \( S, T : C \to D \) is given by

\[
\lambda \mapsto \lambda S(x) \oplus (1 - \lambda)T(x)
\]

\[
\left( x \xrightarrow{f} y \right) \mapsto \left( \lambda S(x) \oplus (1 - \lambda)T(x) \xrightarrow{(\lambda S(f) \oplus (1 - \lambda)T(f))} \lambda S(y) \oplus (1 - \lambda)T(y) \right).
\]  

(4.4.36)

Also, for every \( \lambda \in [0, 1] \) the convex combination of two natural transformations \( \sigma : Q \Rightarrow S \) and \( \tau : R \Rightarrow T \), where \( Q, R, S, T : C \to D \) are all functors, is given by

\[
\lambda Q \oplus (1 - \lambda)R \xrightarrow{\lambda \sigma \oplus (1 - \lambda)\tau} \lambda S \oplus (1 - \lambda)T
\]

\[
C_0 \ni x \mapsto \left( \lambda Q(x) \oplus (1 - \lambda)R(x) \xrightarrow{(\lambda \sigma(x) \oplus (1 - \lambda)\tau(x))} \lambda S(x) \oplus (1 - \lambda)T(x) \right).
\]

(4.4.37)

Furthermore, the following data are specified (whenever \( D \) has them):

(a) The left and right unitors \( u_0 \) and \( u_1 \) assign to every pair of functors
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\( S, T : \mathcal{C} \longrightarrow \mathcal{D} \) the natural isomorphisms

\[
0S \oplus 1T \xrightarrow{u_0(S,T)} T
\]

\[
C_0 \ni x \mapsto \left( 0S(x) \oplus 1T(x) \xrightarrow{u_0(S(x),T(x))} T(x) \right) \tag{4.4.38}
\]

and

\[
1S \oplus 0T \xrightarrow{u_1(S,T)} S
\]

\[
C_0 \ni x \mapsto \left( 1S(x) \oplus 0T(x) \xrightarrow{u_1(S(x),T(x))} S(x) \right), \tag{4.4.39}
\]

respectively.

(b) For every \( \lambda \in [0, 1] \), the idempoters (if they exist in \( \mathcal{D} \)) assign to every

functor \( T : \mathcal{C} \longrightarrow \mathcal{D} \) the natural transformation

\[
\lambda T \oplus (1 - \lambda)T \xrightarrow{i_\lambda(T)} T
\]

\[
C_0 \ni x \mapsto \left( \lambda T(x) \oplus (1 - \lambda)T(x) \xrightarrow{i_\lambda(T(x))} T(x) \right) \tag{4.4.40}
\]

and the diagonal rectifiers (if they exist in \( \mathcal{D} \)) assign

\[
T \xrightarrow{\delta_\lambda(T)} \lambda T \oplus (1 - \lambda)T
\]

\[
C_0 \ni x \mapsto \left( T(x) \xrightarrow{\delta_\lambda(T(x))} \lambda T(x) \oplus (1 - \lambda)T(x) \right). \tag{4.4.41}
\]

(c) For every \( \lambda \in [0, 1] \), the parametric commutors assign to every pair of

functors \( S, T : \mathcal{C} \longrightarrow \mathcal{D} \) the natural isomorphism

\[
(1 - \lambda)T \oplus \lambda S \xrightarrow{\phi_\lambda(S,T)} \lambda S \oplus (1 - \lambda)T
\]

\[
C_0 \ni x \mapsto \left( (1 - \lambda)T(x) \oplus \lambda S(x) \xrightarrow{\phi_\lambda(S(x),T(x))} \lambda S(x) \oplus (1 - \lambda)T(x) \right) \tag{4.4.42}
\]
(d) For every pair of numbers $\lambda, \mu \in [0, 1]$, the convex associators assign to every triple of functors $R, S, T : \mathcal{C} \rightarrow \mathcal{D}$ the natural isomorphism

$$a_{\lambda, \mu}(R, S, T)$$

$$\lambda \left( \mu R \oplus (1 - \mu) S \right) \oplus (1 - \lambda) T \Rightarrow (\lambda \mu) R \oplus (1 - \lambda \mu) \left( (\lambda \mu) S \oplus (1 - \lambda \mu) T \right)$$

$$\mathcal{C}_0 \ni x \mapsto a_{\lambda, \mu}(R(x), S(x), T(x)) \tag{4.4.43}$$

Proof. This is tedious, but not difficult, to prove. Hence, we omit the proof. 

\[\blacksquare\]

**Lemma 4.4.44.** Let $(\mathcal{C}, F)$ be a convex category. Then the following conditions hold.

i) For every pair of objects $x, y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc}
\phi_{1}(y, x) & x \oplus 1y & u_{0}(x, y) \\
\downarrow & \downarrow & \downarrow \\
1y \oplus 0x & y & \downarrow u_{1}(y, x)
\end{array}$$

$$\tag{4.4.45}$$

commutes.

ii) For every triple of objects $x, y, z$ in $\mathcal{C}$ and every $\lambda \in [0, 1]$, the diagram

$$\begin{array}{ccc}
\lambda(0x \oplus 1y) \oplus (1 - \lambda)z & a_{\lambda, 0}(x, y, z) & 0x \oplus 1(\lambda y \oplus (1 - \lambda)z) \\
\downarrow \lambda u_{0}(x, y) \oplus (1 - \lambda) id_{z} & \downarrow u_{0}(x, \lambda y \oplus (1 - \lambda)z) & \downarrow \lambda y \oplus (1 - \lambda)z
\end{array}$$

$$\tag{4.4.46}$$

commutes.
iii) For every triple of objects $x, y, z$ in $\mathcal{C}$ and every $\nu \in [0, 1]$, the diagram

$$
\begin{array}{ccc}
1z \oplus 0(\nu x \oplus (1 - \nu)y) & \xrightarrow{\alpha_{1,1}(z,x,y)^{-1}} & 1(1z \oplus 0x) \oplus 0y \\
\downarrow u_1(z,\nu x \oplus (1 - \nu)y) & & \downarrow u_1(1z \oplus 0x,y) \\
z & \xleftarrow{u_1(x,x)} & 1z \oplus 0x
\end{array}
$$

(4.4.47)

iv) For every triple of objects $x, y, z$ in $\mathcal{C}$ and every $\nu \in [0, 1]$, the diagram

$$
\begin{array}{ccc}
0(\nu x \oplus (1 - \nu)y) \oplus 1z & \xrightarrow{\alpha_{0,\nu}(x,y,z)} & 0x \oplus 1(0y \oplus 1z) \\
\downarrow u_0(\nu x \oplus (1 - \nu)y,z) & & \downarrow 0id_1 \oplus 1u_0(y,z) \\
z & \xleftarrow{u_0(x,z)} & 0x \oplus 1z
\end{array}
$$

(4.4.48)

commutes.

**Proof.** i) This follows immediately from (4.2.40) and (4.2.38).
ii) This follows from commutativity of the sub-diagrams in

\[
\lambda(0x \oplus 1y) \oplus (1 - \lambda)z \quad \xrightarrow{a_{\lambda,0}(x,y,z)} \quad 0x \oplus 1(\lambda y \oplus (1 - \lambda)z)
\]

where the equation number at the center of a sub-diagram indicates the reason why it commutes.
iii) This follows from commutativity of the sub-diagrams in

\[
\begin{array}{c}
1z \oplus 0\left(\nu x \oplus (1 - \nu)y\right) \xrightarrow{\phi_0(y, 1z \oplus 0x)} 1z \oplus 0y \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
1z \oplus 0y \xrightarrow{\phi_0(y, z)} 0y \oplus 1z \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
0y \oplus 1z \xrightarrow{u_1(1z \oplus 0x, y)} 1z \oplus 0x \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
1z \oplus 0 \nu x \oplus (1 - \nu)y \xrightarrow{a_{1,1}(z, x, y)^{-1}} 1(1z \oplus 0x) \oplus 0y \\
\downarrow \quad \downarrow
\end{array}
\]

where the equation number at the center of a sub-diagram indicates the reason why it commutes. Note that \( \nu \) can be taken arbitrary in the expression \( 1z \oplus 0\left(\nu x \oplus (1 - \nu)y\right) \). In particular, we have conveniently chosen \( \nu = 0 \).
iv) This follows from commutativity of the sub-diagrams in

\[
\begin{array}{c}
\nu x \oplus (1 - \nu) y \oplus 1 \mathbf{z} \\
\phi_1 (z, \nu x \oplus (1 - \nu) y) \rightarrow \phi_0 (x, z) \oplus \nu y \\
\phi_0 (x, z) \oplus \nu y \rightarrow (1, z) \oplus 0 y \\
\phi_0 (x, z) \oplus 0 y \rightarrow 0 x \oplus 1 (0 y \oplus 1 z)
\end{array}
\] (4.4.44)

where the equation number at the center of a sub-diagram indicates the reason why it commutes.

The following says that the convex sum of convex functors is again a convex functor, completely analogous to the result from ordinary convex analysis that the convex sum of convex functions is convex.\(^{13}\)

\(^{13}\)The proof that the sum of convex functions is vastly simpler than the proof for functors. It helps to draw a comparison. Let \(f, g\) be two convex functions and \(\lambda, \mu \in [0, 1]\). Then for any \(x, y\) in the domain of \(f\) and \(g\),

\[
(\lambda f + (1 - \lambda) g)(\mu x + (1 - \mu) y) = \lambda f(\mu x + (1 - \mu) y) + (1 - \lambda) g(\mu x + (1 - \mu) y) \\
\leq \lambda \mu f(x) + \lambda (1 - \mu) f(y) + (1 - \lambda) \mu g(x) + (1 - \lambda) (1 - \mu) g(y) \\
= \mu \lambda f(x) + \mu (1 - \lambda) g(x) + (1 - \mu) \lambda f(y) + (1 - \mu)(1 - \lambda) g(y) \\
= \mu (\lambda f + (1 - \lambda) g)(x) + (1 - \mu)(\lambda f + (1 - \lambda) g)(y).
\]
**Proposition 4.4.52.** Let $\mathcal{C}$ and $\mathcal{D}$ be convex categories and let $(S, \eta), (T, \kappa) : \mathcal{C} \to \mathcal{D}$ be convex functors. Then for every $\lambda \in [0, 1]$, the functor (see Proposition 4.4.35 for the definition) $\lambda S \oplus (1 - \lambda)T : \mathcal{C} \to \mathcal{D}$ together with the family of natural transformations

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{(\lambda S \oplus (1-\lambda)T) \times (\lambda S \oplus (1-\lambda)T)} & \mathcal{D} \times \mathcal{D} \\
\downarrow F_\mu & & \downarrow G_\mu \\
\mathcal{C} & \xrightarrow{\lambda S \oplus (1-\lambda)T} & \mathcal{D}
\end{array}
\]

(4.4.53)

indexed by $\mu \in [0, 1]$ defined by sending the pair of objects $(x, y) \in \mathcal{C}_0 \times \mathcal{C}_0$ to

Notice that strict associativity and commutativity of $+$ were used throughout.
the composition of morphisms

\[
\lambda S \left( \mu x \oplus (1-\mu)y \right) \oplus (1-\lambda) T \left( \mu x \oplus (1-\mu)y \right)
\]

\[
\lambda \eta_{\mu}(x,y) \oplus (1-\lambda) \kappa_{\mu}(x,y)
\]

\[
\lambda \left( \mu S(x) \oplus (1-\mu)S(y) \right) \oplus (1-\lambda) \left( \mu T(x) \oplus (1-\mu)T(y) \right)
\]

\[
\alpha_{\lambda,\mu}(S(x),S(y),\mu T(x) \oplus (1-\mu)T(y))
\]

\[
\lambda,\mu S(x) \oplus (1-\lambda,\mu) \left( \lambda,\mu S(y) \oplus (1-\lambda,\mu) \left( \mu T(x) \oplus (1-\mu)T(y) \right) \right)
\]

\[
\lambda,\mu \text{id}_{S(x)} \oplus (1-\lambda,\mu) \alpha_{\mu,\lambda,\mu}(S(y),T(x),T(y))^{-1}
\]

\[
\lambda,\mu S(x) \oplus (1-\lambda,\mu) \left( (\mu,\lambda+\lambda,\mu)(\nu S(y) \oplus (1-\nu)T(x)) \oplus (1-(\mu,\lambda+\lambda,\mu))T(y) \right)
\]

\[
\lambda,\mu \text{id}_{S(x)} \oplus (1-\lambda,\mu)(\nu,\mu+\lambda,\mu)\phi_{1-\nu}(T(x),S(y)) \oplus (1-(\mu,\lambda+\lambda,\mu))\text{id}_{T(y)}
\]

(4.4.54)

\[
\lambda,\mu S(x) \oplus (1-\lambda,\mu) \left( (1-\nu)T(x) \oplus \nu S(y) \right) \oplus (1-(\mu,\lambda+\lambda,\mu))T(y)
\]

\[
\lambda,\mu \text{id}_{S(x)} \oplus (1-\lambda,\mu) \alpha_{\mu,\lambda,\mu,1-\nu}(T(x),S(y),T(y))
\]

\[
\lambda,\mu S(x) \oplus (1-\lambda,\mu) \left( (\mu,\lambda+\lambda,\mu)(1-\nu)T(x) \oplus (1-\mu,\lambda) \left( \lambda S(y) \oplus (1-\lambda)T(y) \right) \right)
\]

\[
\mu,\lambda S(x) \oplus (1-\mu,\lambda) \left( \mu,\lambda T(x) \oplus (1-\mu,\lambda) \left( \lambda S(y) \oplus (1-\lambda)T(y) \right) \right)
\]

\[
\mu \left( \lambda S(x) \oplus (1-\lambda)T(x) \right) \oplus (1-\mu) \left( \lambda S(y) \oplus (1-\lambda)T(y) \right)
\]

is a convex functor. Here

\[
\nu := \begin{cases} 
\frac{\lambda(1-\mu)}{\mu+\lambda-2\mu\lambda} & \text{if } \mu \lambda \neq 1 \\
\text{arbitrary} & \text{if } \lambda = \mu = 1
\end{cases}
\]

(4.4.55)
Proof. This is not a difficult proof but it consists of several long diagram chases. We will prove the first two axioms of Definition 4.4.4.

i) For every pair of objects $x, y$ in $\mathcal{C}$, we must show the diagram

\[
\begin{array}{c}
\lambda S_p \xrightarrow{1} \lambda q T \xrightarrow{p_0} x, y \xrightarrow{s} u_0 \xrightarrow{\lambda S_p (1-\lambda) T} (\lambda S_p (1-\lambda) T)(y) \\
\lambda S_p (1-\lambda) T(0x \oplus 1y) \xrightarrow{\lambda S_p (1-\lambda) T} (\lambda S_p (1-\lambda) T)(x) (u_0(x, y)) \xrightarrow{\lambda S_p (1-\lambda) T} (\lambda S_p (1-\lambda) T)(y)
\end{array}
\]

(4.4.56)

commutes for all $\lambda \in [0, 1]$. This will follow from commutativity of several diagrams and the claim will be shown after pasting together all of these diagrams. The overall diagram looks like

![Diagram](image_url)
Diagram I is given by

\[
\lambda \left( (0S(x) \oplus 1S(y)) \oplus (1-\lambda)(0T(x) \oplus 1T(y)) \right) \xrightarrow{\lambda u_0 \left( (S(x), S(y)) \oplus (1-\lambda)\text{id} \right)} \lambda S(y) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right)
\]

\[
\lambda (0S(x) \oplus 1S(y)) \oplus (1-\lambda) u_0 (x,y)
\]

\[
\lambda (0S(x) \oplus 1y) \oplus (1-\lambda) u_0 (x,y)
\]

\[
\lambda S(0x \oplus 1y) \oplus (1-\lambda) T(0x \oplus 1y) \xrightarrow{\left( \lambda S \oplus (1-\lambda) T \right) \left( u_0 (x,y) \right)} \lambda S(y) \oplus (1-\lambda) T(y)
\]

(4.4.58)

and commutes by (4.4.11). Diagram II is given by

\[
0S(x) \oplus 1 \left( \lambda S(y) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right) \right)
\]

\[
\xrightarrow{\lambda S(y) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right)}
\]

\[
\xrightarrow{\lambda \left( 0S(x) \oplus 1S(y) \right) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right)}
\]

(4.4.59)

and commutes by (4.4.46). Diagram III is given by

\[
0S(x) \oplus 1 \left( \lambda S(y) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right) \right)
\]

\[
\xrightarrow{\lambda \left( 0S(x) \oplus 1S(y) \right) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right)}
\]

\[
\xrightarrow{\lambda S(y) \oplus (1-\lambda) \left( 0T(x) \oplus 1T(y) \right)}
\]

(4.4.60)
and commutes by (4.2.34). Diagram IV is given by

\[
0S(x)\oplus 1\left(\lambda \left(1S(y)\oplus 0T(x)\right)\oplus (1-\lambda)T(y)\right)
\]

\[
\xrightarrow[\text{id}_{S(x)}\oplus 1\lambda_{1,1}]\left(S(y),T(x),T(y)\right)^{-1}
\]

\[
0S(x)\oplus 1\left(\lambda S(y)\oplus (1-\lambda)T(y)\right)
\]

\[
0S(x)\oplus 1\left(\lambda S(y)\oplus (1-\lambda)(0T(x)\oplus T(y))\right)
\]

(4.4.61)

and commutes by (4.2.41). Diagram V is given by

\[
0S(x)\oplus 1\left(\lambda \left(0T(x)\oplus 1S(y)\right)\oplus (1-\lambda)T(y)\right)
\]

\[
\xrightarrow[\text{id}_{S(x)}\oplus 1\lambda_{0}]\left(T(x),S(y)\right)\oplus (1-\lambda)\text{id}_{T(y)}
\]

\[
0S(x)\oplus 1\left(\lambda S(y)\oplus (1-\lambda)T(y)\right)
\]

\[
0S(x)\oplus 1\left(\lambda \left(1S(y)\oplus 0T(x)\right)\oplus (1-\lambda)T(y)\right)
\]

(4.4.62)

and commutes by (4.2.38). Diagram VI is given by

\[
0S(x)\oplus 1\left(0T(x)\oplus 1\left(S(y)\oplus (1-\lambda)T(y)\right)\right)
\]

\[
\xrightarrow[\text{id}_{S(x)}\oplus 1\lambda_{0,0}]\left(T(x),S(y),T(y)\right)
\]

\[
0S(x)\oplus 1\left(\lambda S(y)\oplus (1-\lambda)T(y)\right)
\]

\[
0S(x)\oplus 1\left(\lambda \left(0T(x)\oplus 1S(y)\right)\oplus (1-\lambda)T(y)\right)
\]

(4.4.63)
and commutes by (4.4.46). Diagram VII is given by

\[
0S(x) \oplus 1 \left( 0T(x) \oplus 1 \left( \lambda S(y) \oplus (1-\lambda)T(y) \right) \right)
\]

\[
0\text{id}S(x) \oplus 1 \left( u_0 \left( T(x), \lambda S(y) \oplus (1-\lambda)T(y) \right) \right)
\]

\[
0\left( \lambda S(y) \oplus (1-\lambda)T(x) \right) \oplus 1 \left( \lambda S(y) \oplus (1-\lambda)T(y) \right)
\]

ii) If \( C \) and \( D \) have idempoters, for every pair of objects \( x, y \) in \( C \) and every pair \( \lambda, \mu \in [0, 1] \) the diagram

\[
\frac{\lambda S(x) \oplus (1-\lambda)T(x)}{\mu x \oplus (1-\mu)x} \xrightarrow{\lambda S(x) \oplus (1-\lambda)T(x)} \frac{\mu \left( \lambda S(x) \oplus (1-\lambda)T(x) \right) \oplus \lambda S(x) \oplus (1-\lambda)T(x)}{\lambda S(x) \oplus (1-\lambda)T(x)}
\]

commutes by (4.4.12) and (4.2.59).

\[\text{Remark 4.4.66.}\] It is possible that our axioms of a convex category need to be modified for the above proposition to be true (time has prevented us from verifying the other two axioms). In fact, axioms (4.2.42) and (4.2.59) were
added so that i) and ii) in the above proposition are true. Similar modifications can be made to the definition, if necessary, to ensure the statement is true. Such results will be explored in future work.

**Remark 4.4.67.** One could have also defined the composition in Proposition 4.4.52 at least one other way by using the associators in a different order. The coherence conditions guarantee that the resulting morphism is equal to the one above.

Lack of time has prevented us from proving the following two reasonable claims. They will be checked in future work.

**Conjecture 4.4.68.** Let $C$ and $D$ be two convex categories, \( (Q, \varphi), (R, \theta), (S, \eta), (T, \kappa) : C \rightarrow D \) \hspace{1cm} (4.4.69)

four convex functors, and $\sigma : Q \Rightarrow S$ and $\tau : R \Rightarrow T$ two convex natural transformations. Then the natural transformation (see Proposition 4.4.35 for the definition) $\lambda \sigma \oplus (1 - \lambda)\tau : \lambda Q \oplus (1 - \lambda)R \Rightarrow \lambda S \oplus (1 - \lambda)T$ is convex.

**Conjecture 4.4.70.** Let $C$ and $D$ be convex categories (of the same type). Then $\text{ConvexCat}(C, D)$ is a convex category (with the structure described in Proposition 4.4.35 and Conjectures 4.4.52 and 4.4.68).
4.4.4 From probability density functions to probability measures

**Lemma 4.4.71.** Let \((X, \Sigma, \mu, p)\) be an mspd (Definition 4.3.84). The assignment

\[
\begin{align*}
\Sigma & \xrightarrow{\mu_p} \mathbb{R}_{\geq 0} \\
E & \mapsto \int_E p \, d\mu
\end{align*}
\]  

(4.4.72)

is a probability measure on \((X, \Sigma)\).

**Proposition 4.4.73.** Using the notation of Lemma 4.4.71, the assignment

\[
\text{MSPDF} \longrightarrow \text{MeasProb}
\]

\[
(X, \Sigma, \mu, p) \mapsto (X, \Sigma, \mu_p)
\]  

(4.4.74)

is a strictly affine functor.

**Proof.** First let \(f\) be a representative of \([f]\). We will show well-definedness afterwards. The first claim to check is that \((X, \Sigma, \mu_p) \xrightarrow{f} (Y, \Omega, \nu_q)\) preserves the measure, i.e. \(\mu_p(f^{-1}(E)) = \nu_q(E)\) for all \(E \in \Omega\). This follows from
the following list of equalities

\[
\begin{align*}
\mu_p(f^{-1}(E)) &= \nu_q(E) \\ (4.4.72) &\text{ for } \mu \text{ and } p \\
\int_{f^{-1}(E)} p \, d\mu &= \int_E q \, d\nu \\
&\text{since } p=q \iff \int_{f^{-1}(E)} (q \circ f) \, d\mu = \int (f^{-1}(E)) = \nu_q(E) \\
&\text{since } \mu(f^{-1}(E)) = \nu_q(E) \\
&\text{(4.4.75)}
\end{align*}
\]

Let \((X, \Sigma, \mu, p)\) and \((Y, \Omega, \nu, q)\) be two mspdf’s and \(\lambda \in [0, 1]\). We must show that the two resulting measures \((\mu \oplus \nu)_{\lambda \mu \oplus (1-\lambda)\nu}\) and \(\lambda \mu_p \oplus (1-\lambda)\nu_q\) are equal. This follows from evaluating both measures on an arbitrary measurable set \(E \in \Sigma \oplus \Omega\) as in

\[
\begin{align*}
(\mu \oplus \nu)_{\lambda \mu \oplus (1-\lambda)\nu}(E) &= (\lambda \mu_p \oplus (1-\lambda)\nu_q)(E) \\
\int_E (\lambda p \oplus (1-\lambda)q)(z) \, d(\mu \oplus \nu)(z) &= \lambda \mu_p(i_X^{-1}(E)) + (1-\lambda)\nu_q(i_Y^{-1}(E)) \\
&\text{since } \mu(f^{-1}(E)) = \nu_q(E) \\
&\text{(4.4.76)}
\end{align*}
\]

The other conditions are easy to check. In particular, the assignment (4.4.74) is well-defined by the comment after Definition 4.3.86. \(\blacksquare\)
4.5 Entropy and information loss

4.5.1 Cone categories

A closely related notion to a convex category is a cone category. While cone categories are not needed to explain what entropy is, they are needed to explain the notion of proportionality between entropy functors which can be used to classify entropy when it is indeed classified as a convex functor.\footnote{We will review one example where entropy is classified as a convex functor. Whether this is true in greater generality is not the subject of this work, but will hopefully be addressed in future work.}

Proportionality is an equivalence relation and a choice of a representative is essentially a choice of units. The following definition is based on the notion of semicones in [Fl80]. Therefore, we review the definition first before giving the (semi-) categorified version.

**Definition 4.5.1.** A \emph{semicone} consists of a commutative monoid \((C, +, 0)\), with unit written as 0, together with functions \(k_\lambda : C \to C\) for each \(\lambda \in \mathbb{R}_{\geq 0}\).
satisfying

\[ k_{\lambda\mu}(x) = k_{\lambda}(k_{\mu}(x)) \quad \text{(left action axiom)} \] (4.5.2)

\[ k_1(x) = x \quad \text{(unit axiom)} \] (4.5.3)

\[ k_\lambda(x + y) = k_\lambda(x) + k_\lambda(y) \quad \text{(distributivity over addition in } C) \] (4.5.4)

\[ k_{\lambda+\mu}(x) = k_\lambda(x) + k_\mu(x) \quad \text{(distributivity over addition in } \mathbb{R}_{\geq 0}) \] (4.5.5)

\[ k_\lambda(0) = 0 \quad \text{(0 is scale invariant)} \] (4.5.6)

\[ k_0(x) = 0 \quad \text{(apex axiom)} \] (4.5.7)

for all \( \lambda, \mu \in \mathbb{R}_{\geq 0} \) and all \( x, y \in C \).

We will occasionally write \( k_\lambda(x) =: \lambda x \).

**Remark 4.5.8.** A commutative monoid is not in general cancellative (which means that \( x + y = x + z \implies y = z \) for all \( x, y, z \in C \)) so that (4.5.6) and (4.5.7) do not follow from the other axioms. Furthermore, (4.5.3) does not follow from the other conditions. For example, a counterexample to show the unit axiom is not automatic is given by the function \( k_\lambda \) that sends every element to 0 for all \( \lambda \in \mathbb{R}_{\geq 0} \) even though it satisfies all the other axioms.

Before categorifying semicones, we internalize the notion as we did for convex objects in Section 4.2.1.

**Definition 4.5.9.** A **semicone object** in a cartesian monoidal category
(\mathcal{C}, \otimes, I, a, l, r, \gamma, \pi_1, \pi_2, e) \) with diagonal denoted by \( \Delta \) (see the comments following Definition 4.5.129) consists of a commutative monoid object\(^{15}\) \((C, +, 0)\) in \( \mathcal{C} \) together with a family of morphisms \( k_\lambda : C \rightarrow C \) indexed by \( \lambda \in \mathbb{R}_{\geq 0} \) such that the following axioms hold.

i) First, the collection \( \{k_\lambda\} \) defines a left action of \( \mathbb{R}_{\geq 0} \) on \( C \), i.e.

\[
k_\lambda \circ k_\mu = k_{\lambda\mu}
\] (4.5.10)

ii) \( 1 \in \mathbb{R}_{\geq 0} \) is a unit for the action, i.e.

\[
k_1 = \text{id}_C
\] (4.5.11)

iii) \( \{k_\lambda\} \) is distributive over \( + \), i.e. the diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{k_\lambda \times k_\mu} & C \otimes C \\
+ & \downarrow & + \\
C & \xrightarrow{k_\lambda} & C
\end{array}
\] (4.5.12)

commutes.

iv) \( \{k_\lambda\} \) is distributive over the addition in \( \mathbb{R}_{\geq 0} \), i.e. the diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{k_\lambda \times k_\mu} & C \otimes C \\
\Delta & \downarrow & \oplus \\
C & \xrightarrow{k_{\lambda+\mu}} & C
\end{array}
\] (4.5.13)

commutes.

\(^{15}\) The notion of a commutative monoid object is immediately extractable from Definitions 4.5.110 and 4.5.113.
v) 0 is a fixed point, i.e. the diagram

\[
\begin{array}{ccc}
I & \xrightarrow{k_\lambda} & C \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

(4.5.14)

commutes.

vi) The apex axiom, i.e. the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{k_0} & C \\
e_C & \downarrow & \downarrow \\
I & \rightarrow & 0
\end{array}
\]

(4.5.15)

commutes.

In the definition of a semicone and semicone object, we assumed the existence of an element that acts as a 0 for the addition. It turns out that not all examples of categorified semicones have such an object. Nevertheless, we might expect that equations such as \(0x + y = y\) and \(x + 0y = x\) for all \(x, y \in C\). This motivates the following definition of a cone category.

**Definition 4.5.16.** A **cone category** consists of a symmetric semigroupal category (see Definition 4.5.115) \((\mathcal{C}, \oplus, a, \phi)\) together with a family of functors \(k_\lambda : \mathcal{C} \rightarrow \mathcal{C}\) for each \(\lambda \in \mathbb{BR}_{>0}\), written as \(k_\lambda(x) =: \lambda x\) on objects \(x\) and similarly for morphisms, and natural isomorphisms

\[
\begin{align*}
\begin{array}{ccc}
C \times C & \xrightarrow{k_0 \times \text{id}_C} & C \times C \\
\downarrow & \ominus & \downarrow \\
C \times C & \rightarrow & C
\end{array}
\end{align*}
\quad
\begin{align*}
\begin{array}{ccc}
C \times C & \xrightarrow{\text{id}_C \times k_0} & C \times C \\
\downarrow & \ominus & \downarrow \\
C \times C & \rightarrow & C
\end{array}
\end{align*}
\quad
\begin{align*}
\begin{array}{ccc}
0 \oplus y & \xrightarrow{\beta_{x,y}} & y \\
& \ominus & \\
x & \oplus 0y & \xrightarrow{\beta_{x,y}} x
\end{array}
\end{align*}
\]

(4.5.17)
satisfying the following conditions.\(^{16}\)

i) First, \(\{k_\lambda\}\) defines an action of \(\mathbb{R}_{\geq 0}\) on \(\mathcal{C}\), i.e.

\[
k_\lambda \circ k_\mu = k_{\lambda\mu} \quad \& \quad k_1 = \text{id}_\mathcal{C}
\]

(4.5.18)

ii) \(\{k_\lambda\}\) is distributive over \(\oplus\), i.e. the diagram

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_\lambda \times k_\lambda} & \mathcal{C} \times \mathcal{C} \\
\oplus & \searrow & \oplus \\
\mathcal{C} & \xrightarrow{k_\lambda} & \mathcal{C}
\end{array}
\]

(4.5.19)

commutes, which says that for every pair of objects \(x, y\) (and morphisms) in \(\mathcal{C}\),

\[
\lambda(x \oplus y) = \lambda x \oplus \lambda y.
\]

iii) The symmetric unit law

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\gamma} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{\phi} & \mathcal{C} \times \mathcal{C} \\
\mathcal{C} & \xrightarrow{\beta} & \mathcal{C} \\
\mathcal{C} & \xrightarrow{\phi \oplus \phi} & \mathcal{C} \times \mathcal{C}
\end{array}
\]

(4.5.20)

holds, which says that the diagram

\[
\begin{array}{ccc}
0x \oplus y & \xrightarrow{\phi_y \oplus \phi_x} & y \oplus 0x \\
\downarrow \text{id} & & \downarrow \text{id} \\
y & \xrightarrow{\phi_y} & y
\end{array}
\]

commutes for all pairs of objects \(x, y\) in \(\mathcal{C}\).

\(^{16}\)Although one could imagine vastly generalizing this definition by weakening more axioms, we do not see a reason for this since all our examples satisfy these conditions.
iv) The triangle law

\[
\begin{array}{c}
\array{ccc}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{id \times \pi_{2, \mathcal{C}}} & \mathcal{C} \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{id \times \oplus} & \mathcal{C} \\
\end{array}
\]

\[
\begin{array}{c}
\array{ccc}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{id \times \pi_{2, \mathcal{C}}} & \mathcal{C} \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{\oplus} & \mathcal{C} \\
\end{array}
\]

(4.5.21)

holds, which says that the diagram

\[
\begin{array}{c}
\array{ccc}
(x \oplus 0y) \oplus z & \xrightarrow{a_{x, 0y, z}} & x \oplus (0y \oplus z) \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{\oplus} & \mathcal{C} \\
\end{array}
\]

(4.5.22)

commutes for all \(x, y, z\) in \(\mathcal{C}\).

A cone category may sometimes be written as \((\mathcal{C}, \oplus, a, \phi, \{k_{\lambda}\}, l^\oplus, r^\oplus)\) to illustrate all the data.

Explicitly, a cone category \(\mathcal{C}\) consists of the usual data associated with a symmetric semigroupal or monoidal category along with functors that associates to objects \(x\) and morphisms \(x \xrightarrow{f} y\) a \(\lambda\)-scaled version

\[
k_\lambda(x) =: \lambda x \quad \& \quad k_\lambda \left( x \xrightarrow{f} y \right) =: \lambda x \xrightarrow{\lambda f} \lambda y.
\]

(4.5.23)

**Remark 4.5.24.** Even though we do not assume the existence of a zero object 0 and hence no monoidal structure on the cone category, we assume \(k_0(x)\) acts as a zero object under the \(\oplus\) operation. Furthermore, even if there
is a zero object 0, it need not be that 0x is isomorphic to 0. While this seems pedantic, there are examples where this occurs, which will be discussed later.

This motivates the following definition.

**Definition 4.5.25.** A pointed cone category consists of a symmetric monoidal category (see Definition 4.5.124) \((\mathcal{C}, \oplus, a, \varphi, 0, l, r)\) together with a family of functors \(k_\lambda : \mathcal{C} \to \mathcal{C}\) satisfying the following conditions.

i) First, the collection \(\{k_\lambda\}\) defines a left action of \(\mathbb{R}_{\geq 0}\) on \(\mathcal{C}\), i.e.

\[
k_\lambda \circ k_\mu = k_{\lambda \mu}. \tag{4.5.26}
\]

ii) \(1 \in \mathbb{R}_{\geq 0}\) is a unit for the action, i.e.

\[
k_1 = \text{id}_\mathcal{C}. \tag{4.5.27}
\]

iii) \(\{k_\lambda\}\) is distributive over \(\oplus\), i.e. the diagram

\[
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow \oplus \\
\mathcal{C}
\end{array}
\begin{array}{c}
\xrightarrow{k \lambda \times k \lambda} \\
\kappa
\end{array}
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow \oplus \\
\mathcal{C}
\end{array} \tag{4.5.28}
\]

commutes.

iv) 0 is a fixed point, i.e. the diagram

\[
\begin{array}{c}
0 \\
\downarrow k \lambda
\end{array}
\begin{array}{c}
\mathcal{C}
\end{array}
\begin{array}{c}
1 \\
\downarrow 0
\end{array} \tag{4.5.29}
\]

commutes.
v) The apex axiom, i.e. the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{k_0} & C \\
\downarrow{e_C} & & \downarrow{0} \\
1 & \xrightarrow{l_0} & 0
\end{array}
\]  (4.5.30)

commutes.

A pointed cone category may sometimes be written as \((C, \oplus, a, \phi, 0, l, r, \{k_\lambda\})\) to accurately list all the data.

Note that the definition of a pointed cone category seems exactly the same as the definition of a semicone object besides the distributivity over the addition in \(\mathbb{R}_{\geq 0}\). This is misleading because there are natural isomorphisms built into the definition of a symmetric monoidal category. This will be explored in more detail now as we construct a cone category from a pointed cone category.

**Proposition 4.5.31.** Let \((C, \oplus, a, \phi, 0, l, r, \{k_\lambda\})\) be a pointed cone category.

Let \(l^\oplus\) and \(r^\oplus\) be the compositions in the following two diagrams

\[
\begin{array}{ccc}
C \times C & \xrightarrow{k_0 \times \text{id}_C} & C \\
\downarrow{\text{id}_C \times 0} & & \downarrow{l_0} \\
1 \times C & \xrightarrow{l_0} & C
\end{array}
\quad \&
\quad
\begin{array}{ccc}
C \times C & \xrightarrow{\text{id}_C \times 0} & C \\
\downarrow{\text{id}_C \times 0} & & \downarrow{r_0} \\
1 \times C & \xrightarrow{0 \times \text{id}_C} & C
\end{array}
\]  (4.5.32)

respectively. Then \((C, \oplus, a, \phi, \{k_\lambda\}, l^\oplus, r^\oplus\) is a cone category.

**Proof.** The proof is tedious but straightforward.
We now discuss how the distributivity over addition in $\mathbb{R}_{\geq 0}$ can be categorified. The result will be similar to how the axiom $\lambda x + (1 - \lambda)x = x$ was categorified for convex categories.

**Definition 4.5.33.** A cone category with idempoters consists of a cone category $(\mathcal{C}, \oplus, a, \phi, \{\kappa\lambda\}, \ell^\oplus, r^\oplus)$ together with a family of natural transformations

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_\lambda \times k_\mu} & \mathcal{C} \times \mathcal{C} \\
\Delta & \downarrow & \oplus \\
\mathcal{C} & \xrightarrow{k_\lambda + k_\mu} & \mathcal{C} \\
\end{array}
$$

(4.5.34)

called idempoters indexed by $\lambda, \mu \in \mathbb{R}_{\geq 0}$ satisfying the following conditions.

i) For each pair $\lambda, \mu \in \mathbb{R}_{\geq 0}$,

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_\mu \times k_\lambda} & \mathcal{C} \times \mathcal{C} \\
\Delta & \downarrow & \oplus \\
\mathcal{C} & \xrightarrow{k_\lambda + k_\mu} & \mathcal{C} \\
\end{array} =
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_\mu \times k_\lambda} & \mathcal{C} \times \mathcal{C} \\
\Delta & \downarrow & \oplus \\
\mathcal{C} & \xrightarrow{k_\lambda + k_\mu} & \mathcal{C} \\
\end{array}
$$

(4.5.35)

ii) The coherence between the idempoters and $\ell^\oplus$ and $r^\oplus$ says

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_0 \times k_1} & \mathcal{C} \times \mathcal{C} \\
\Delta & \downarrow & \oplus \\
\mathcal{C} & \xrightarrow{k_1} & \mathcal{C} \\
\end{array} =
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_0 \times \text{id}_C} & \mathcal{C} \times \mathcal{C} \\
\Delta & \downarrow & \oplus \\
\mathcal{C} & \xrightarrow{\text{id}_C} & \mathcal{C} \\
\end{array}
$$

(4.5.36)
and

\[
\begin{array}{ccc}
C \times C & \stackrel{k_1 \times k_0}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C & \stackrel{k_1}{\longrightarrow} & C
\end{array}
\quad =
\begin{array}{ccc}
C \times C & \stackrel{id \times k_0}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C & \stackrel{id}{\longrightarrow} & C
\end{array}
\]

which make sense because \(k_1 = id_C\).

iii) The coherence between the idempoters and associators says that for each triple \(\lambda, \mu, \nu \in \mathbb{R}_{\geq 0}\),

\[
\begin{array}{ccc}
C \times C & \stackrel{id \times \Delta}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C \times C \times C & \stackrel{id \times id}{\longrightarrow} & C \times C \times C
\end{array}
\quad =
\begin{array}{ccc}
C \times C & \stackrel{id \times \Delta}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C \times C \times C & \stackrel{id \times id}{\longrightarrow} & C \times C \times C
\end{array}
\]

\[
\begin{array}{ccc}
C \times C & \stackrel{id \times \Delta}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C \times C \times C & \stackrel{id \times id}{\longrightarrow} & C \times C \times C
\end{array}
\]

\[
\begin{array}{ccc}
C \times C & \stackrel{id \times \Delta}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C \times C \times C & \stackrel{id \times id}{\longrightarrow} & C \times C \times C
\end{array}
\]

\[
\begin{array}{ccc}
C \times C & \stackrel{id \times \Delta}{\longrightarrow} & C \times C \\
\Delta & \downarrow & \oplus \\
C \times C \times C & \stackrel{id \times id}{\longrightarrow} & C \times C \times C
\end{array}
\]

(4.5.38)

**Remark 4.5.39.** The above definition of cone category is preliminary. One needs the analogue of the fourth axiom in Definition 4.2.45. This will be addressed in future work.

The idempoters assign to every object \(x\) in \(C\), a morphism

\[
\lambda x \oplus \mu x \xrightarrow{i_{\lambda,\mu}(x)} (\lambda + \mu)x.
\]

(4.5.40)

The coherence conditions say the following.
i) For every object $x$ in $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
\mu x \oplus \lambda x & \xrightarrow{\phi_{\lambda x,\mu x}} & \lambda x \oplus \mu x \\
\downarrow{\mu,\lambda(x)} & & \downarrow{\mu,\lambda(x)} \\
(\lambda + \mu)x & & (\lambda + \mu)x
\end{array}
$$

(4.5.41)

commutes.

ii) For every object $x$ in $\mathcal{C}$,

$$
i_{0,1}(x) = l_{x,x}^\oplus \quad \& \quad i_{1,0}(x) = r_{x,x}^\oplus.
$$

(4.5.42)

iii) For every triple $\lambda, \mu, \nu \in [0,1]$ and every object $x$ in $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
(\lambda x \oplus \mu x) \oplus \nu x & \xrightarrow{a_{\lambda x,\mu x,\nu x}} & \lambda x \oplus (\mu x \oplus \nu x) \\
\downarrow{\lambda,\mu(x) \oplus \id_{\nu x}} & & \downarrow{\id_{\lambda x} \oplus \id_{\mu,\nu(x)}} \\
(\lambda + \mu)x \oplus \nu x & \xrightarrow{i_{\lambda+\mu,\nu(x)}} & \lambda x \oplus (\mu + \nu)x \\
\downarrow{\mu,\lambda+\nu(x)} & & \downarrow{\mu,\lambda+\nu(x)} \\
(\lambda + \mu + \nu)x
\end{array}
$$

(4.5.43)

commutes.

**Definition 4.5.44.** A **cone category with diagonal rectifiers** consists of a cone category $(\mathcal{C}, \oplus, a, \phi, \{k_\lambda\}, l^\oplus, r^\oplus)$ together with a family of natural transformations

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{k_\lambda \times k_\mu} & \mathcal{C} \times \mathcal{C} \\
\downarrow{\Delta} & & \downarrow{\delta_{\lambda,\mu}} \\
\mathcal{C} & & \mathcal{C}
\end{array}
$$

(4.5.45)
called \textit{diagonal rectifiers} indexed by $\lambda, \mu \in \mathbb{R}_{\geq 0}$ satisfying completely analogous conditions to those in the Definition 4.5.33.

\textbf{Definition 4.5.46.} Similar definitions are made for cone categories with memory and perfect memory (see Definitions 4.2.60 and 4.2.65, respectively).

\textbf{Conjecture 4.5.47.} Let $\mathcal{C}$ be a cone category with the data described in Definition 4.5.16. Then $\mathcal{C}$ equipped with the family of functors

$$F_\lambda : \mathcal{C} \times \mathcal{C} \xrightarrow{k_\lambda \times k_{1-\lambda}} \mathcal{C} \xrightarrow{\oplus} \mathcal{C}$$

\begin{equation}
(4.5.48)
\end{equation}

indexed by $\lambda \in [0, 1]$ is a convex category whose unitors, parametric commutors, and convex associators are given as follows.

(a) The left $u_0$ and right $u_1$ unitors are given by $l^\oplus$ and $r^\oplus$, respectively.

(b) For each $\lambda \in [0, 1]$, the parametric commutors $\phi_\lambda$ are defined by the composition

\begin{equation}
(4.5.49)
\end{equation}

(c) For each pair $\lambda, \mu \in [0, 1]$, the convex associators $a_{\lambda, \mu}$ are defined by the
Furthermore, if $C$ has idempoters or diagonal rectifiers, then the idempoters $i_\lambda$ and diagonal rectifiers $\delta_\lambda$ for the convex category are given by the compositions

\[
\begin{align*}
C \times C & \xrightarrow{k_\lambda \times k_{1-\lambda}} C \times C \\
& \xrightarrow{\oplus \times \text{id}_C} C \times C \\
& \xrightarrow{k_\lambda \times k_{1-\lambda}} C \\
C \times C & \xrightarrow{k_{\lambda, \mu} \times k_{1-\lambda, \mu}} C \times C \\
& \xrightarrow{\oplus \times \text{id}_C} C \times C \\
& \xrightarrow{k_{\lambda, \mu} \times k_{1-\lambda, \mu}} C \\
\end{align*}
\]

respectively.

Lack of time has prevented us from proving this assertion. Proofs will be provided elsewhere.

A convex category $C$ is said to be of the same type as a cone category $D$ if the associated convex category of $D$ has the same type as $C$. Although we can define appropriate 1-morphisms and 2-morphisms of cone categories,
we find it sufficient to consider the following notion of a morphism from a convex category to a cone category.

**Definition 4.5.52.** Let $\mathcal{C}$ be a convex category and $\mathcal{D}$ a cone category of the same type. A *convex functor* from $\mathcal{C}$ to $\mathcal{D}$ is a convex functor in the usual sense (see Definition 4.4.4, Definition 4.4.20, and Remark 4.4.22) with $\mathcal{D}$ equipped with the convex structure from Proposition 4.5.47. A *convex natural transformation* of convex functors is defined similarly.

**Proposition 4.5.53.** Let $\mathcal{C}$ be a convex category and $\mathcal{D}$ a cone category. Then $\text{Fun}(\mathcal{C}, \mathcal{D})$, the category of functors from $\mathcal{C}$ to $\mathcal{D}$ is a cone category with pointwise cone structure.

*Proof.* The proof is similar to that of Proposition 4.4.35. ■

**Conjecture 4.5.54.** Let $(\mathcal{C}, F)$ be a convex category, $(\mathcal{D}, \{k_\lambda\})$ a cone category with convex linear combinations as defined in Conjecture 4.5.47, $(T, \eta) : \mathcal{C} \to \mathcal{D}$ a convex functor, and $c$ a positive real number. Then $cT : \mathcal{C} \to \mathcal{D}$, together with the natural transformations $(c\eta)_\lambda$ defined by the composition

\[
\begin{array}{c}
\mathcal{C} \times \mathcal{C} \\
\downarrow F_\lambda \\
\mathcal{C} \\
\end{array}
\xrightarrow{T \times T}
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\downarrow \eta_\lambda \\
\mathcal{D} \\
\end{array}
\xrightarrow{k_\lambda \times k_{1-\lambda}}
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\downarrow \text{id} \\
\mathcal{D} \\
\end{array}
\xrightarrow{k_\lambda \times k_{1-\lambda}}
\begin{array}{c}
\mathcal{D} \times \mathcal{D} \\
\downarrow \text{id} \\
\mathcal{D} \\
\end{array}
\xrightarrow{k_c}
\begin{array}{c}
\mathcal{D} \\
\end{array}
\]

(4.5.55)
is a convex functor.

**Definition 4.5.56.** Let $\mathcal{C}$ be a convex category, $\mathcal{D}$ a cone category, and $S, T : \mathcal{C} \to \mathcal{D}$ two convex functors. A *proportionality* from $S$ to $T$ consists of a constant $c \in \mathbb{R}_{\geq 0}$ together with a convex natural isomorphism $\zeta : S \Rightarrow cT$. Such a proportionality will be written as $S \xrightarrow{c} T$.

**Remark 4.5.57.** Using the notation of Conjecture 4.5.54, the definition of $cT : \mathcal{C} \to \mathcal{D}$ can be extended to include the case $c = 0$ provided that $\mathcal{D}$ is a pointed cone category (see Definition 4.5.25). In this case, everything is sent to the monoidal unit 0 (or an object canonically and naturally isomorphic to it).

**Conjecture 4.5.58.** Let $\mathcal{C}$ be a convex category, $\mathcal{D}$ a cone category, and $(S, \eta), (T, \kappa) : \mathcal{C} \to \mathcal{D}$ two convex functors. If $S \xrightarrow{\zeta} cT$ is a proportionality from $S$ to $T$ with $c \neq 0$, then $T \xrightarrow{\frac{1}{c} \kappa^{-1}} \frac{1}{c}S$ is a proportionality from $T$ to $S$.

**Definition 4.5.59.** Let $\mathcal{C}$ be a convex category, $\mathcal{D}$ a cone category, and $S, T : \mathcal{C} \to \mathcal{D}$ two convex functors. $S$ and $T$ are said to be *proportional* if there exists a proportionality from $S$ to $T$ and is written as $S \propto T$ (or equivalently, by Conjecture 4.5.58, $T \propto S$).

**Lemma 4.5.60.** Proportionality of convex functors into cone categories with positive coefficients is an equivalence relation.
We now finally come to the definition of entropy.

**Definition 4.5.61.** Let $(\mathcal{C}, F)$ be a convex category and $(\mathcal{D}, \{k_\lambda\})$ a cone category. A **$\mathcal{D}$-valued entropy** on $\mathcal{C}$ is a convex functor $(S, \eta) : \mathcal{C} \rightarrow \mathcal{D}$.

**Warning.** The previous definition of entropy is preliminary. First of all, one needs to include a proper notion of continuity for convex functors, which, in particular, requires the notion of topology on convex and cone categories. Secondly, this definition is inadequate for all kinds of convex combinations of interest and alone cannot possibly classify all forms of entropy. We will address these issues in future work and will therefore only sketch some ideas using this preliminary definition.

**Remark 4.5.62.** Standard notions of entropy are obtained by letting $\mathcal{D} = \mathbb{R}$ or $\mathcal{D} = \mathbb{B}\mathbb{R}$. The case of $\mathbb{R}$ associates entropy to objects (states) while $\mathbb{B}\mathbb{R}$ associates numbers to morphisms (processes) and is sometimes interpreted as information loss or gain. We will discuss this in the following sections.

### 4.5.2 Entropy for finite probability measure spaces

In [BFL11], Baez-Fritz-Leinster prove a theorem that characterize entropy functors on the category **FinProb** of finite sets with probability measures. Our notion of convex categories, functors, and natural transformations above
will make such a theorem more precise and allow for generalizations. The main slogan to emphasize, due to the work of [BFL11], is the following:

While not every convex function on finite-probability spaces is the Shannon entropy, every convex functor is!

We first recall the definition of Shannon entropy.

**Definition 4.5.63.** Let \((X, p)\) be an object of \(\text{FinProb}\). The Shannon entropy of \((X, p)\) is

\[
H_{\text{Sh}}((X, p)) := - \sum_{x \in X} p(x) \ln p(x)
\]

with the convention \(0 \ln 0 := 0\).

**Lemma 4.5.65.** Shannon entropy is a convex functor

\[
H_{\text{Sh}} : \text{FinProb} \rightarrow (\mathbb{R}, \geq),
\]

with \(\mathbb{R}\) viewed as a totally ordered convex category as in Example 4.4.16.

A slight variant of this is the information content, described in more detail in [Ki57].

**Definition 4.5.67.** Let \((X, p)\) be an object of \(\text{FinProb}\). The information content of \((X, p)\) is

\[
I((X, p)) = -H_{\text{Sh}}((X, p)).
\]
We can use the Shannon entropy to define an entropy functor on \(\text{FinProb}\) to be interpreted as information loss.

**Proposition 4.5.69.** The assignment

\[
\begin{align*}
\text{FinProb} & \xrightarrow{S_{\text{Sh}}} \mathbb{B} \mathbb{R} \\
(X, p) & \mapsto \bullet \\
\left( (X, p) \xrightarrow{f} (X', p') \right) & \mapsto H_{\text{Sh}} \left( (X, p) \right) - H_{\text{Sh}} \left( (X', p') \right),
\end{align*}
\] (4.5.70)

where \(H_{\text{Sh}}\) is defined in Definition 4.5.63. Then \(S_{\text{Sh}}(f) \in \mathbb{R}_{\geq 0}\) for all morphisms \(f\) in \(\text{FinProb}\) and \(S_{\text{Sh}} : \text{FinProb} \longrightarrow \mathbb{B} \mathbb{R}_{\geq 0}\) is a strictly affine functor.

**Remark 4.5.71.** The reason information content was introduced was to make the assignment (4.5.70) more intuitive as the difference of the information content

\[
S_{\text{Sh}} \left( (X, p) \xrightarrow{f} (X', p') \right) = I \left( (X', p') \right) - I \left( (X, p) \right).
\] (4.5.72)

In this case, information is generally lost (we will prove this) so we view the functor \(S_{\text{Sh}}\) as quantifying the information loss associated to a process.

**Proof of Proposition 4.5.69.** This is basically proved in Section 3 of [BFL11] without the formal introduction of convex structures on the category \(\text{FinProb}\). It is clear that \(S_{\text{Sh}}\) is well-defined on morphisms because it only depends on the source and target. The fact that \(S_{\text{Sh}}(f)\) is always non-negative for (a.e.
equivalence classes of) measure-preserving morphisms \((X, p) \overset{f}{\rightarrow} (X', p')\) is because

\[
\sum_{x' \in X'} p'(x') \ln p'(x') = \sum_{x' \in X'} \left[ \left( \sum_{x \in f^{-1}(x')} p(x) \right) \ln \left( \sum_{\overline{x} \in f^{-1}(x')} p(\overline{x}) \right) \right]
\]

\[
= \sum_{x' \in X'} \left[ \sum_{x \in f^{-1}(x')} p(x) \ln \left( \sum_{\overline{x} \in f^{-1}(x')} p(\overline{x}) \right) \right]
\]

\[
= \sum_{x' \in X'} \left[ \sum_{x \in f^{-1}(x')} p(x) \ln \left( p(x) + \sum_{\overline{x} \in f^{-1}(x') \setminus \{x\}} p(\overline{x}) \right) \right] \quad (4.5.73)
\]

\[
\geq \sum_{x' \in X'} \left[ \sum_{x \in f^{-1}(x')} p(x) \ln p(x) \right]
\]

\[
= \sum_{x \in X} p(x) \ln p(x)
\]

since the logarithm is a monotonically increasing function. Functoriality of \(S_{Sh}\) is easy to see since

\[
S_{Sh}\left((X, p) \overset{f}{\rightarrow} (X', p') \overset{g}{\rightarrow} (X'', p'')\right) = I\left((X'', p'')\right) - I\left((X, p)\right)
\]

\[
= I\left((X'', p'')\right) - I\left((X', p')\right) + I\left((X', p')\right) - I\left((X, p)\right)
\]

\[
= S_{Sh}(f') + S_{Sh}(f). \quad (4.5.74)
\]

The fact that \(S_{Sh}\) is strictly affine follows from the fact that for any \(\lambda \in [0, 1]\)
and any pair \((X, p)\) and \((Y, q)\),

\[
H_{Sh} \left( \lambda(X, p) \oplus (1 - \lambda)(Y, q) \right) = \sum_{x \in X} \lambda p(x) \ln \left( \lambda p(x) \right) + \sum_{y \in Y} (1 - \lambda) q(y) \ln \left( (1 - \lambda) q(y) \right)
\]

\[
= \lambda \sum_{x \in X} p(x) \ln p(x) + \lambda \sum_{x \in X} p(x) \ln \lambda + (1 - \lambda) \sum_{y \in Y} q(y) \ln q(y) + (1 - \lambda) \sum_{y \in Y} q(y) \ln(1 - \lambda)
\]

\[
= \lambda \sum_{x \in X} p(x) \ln p(x) + (1 - \lambda) \sum_{y \in Y} q(y) \ln q(y) + \lambda \ln \lambda + (1 - \lambda) \ln(1 - \lambda).
\]  

(4.5.75)

Therefore, for any pair of morphisms \((X, p) \xrightarrow{f} (X', p')\) and \((Y, q) \xrightarrow{g} (Y', q')\), the constant terms above cancel in the difference so that

\[
S_{Sh}(\lambda f \oplus (1 - \lambda)g) = \lambda S_{Sh}(f) + (1 - \lambda) S_{Sh}(g). 
\]  

(4.5.76)

Finally, \(S_{Sh}\) is continuous in a certain sense which will be addressed in a later version of this work. In the special case of \(\text{FinProb}\), this continuity condition is described in [BFL11].

The entropy of a fixed probability space \((X, p)\) can be obtained from just the functor \(S_{Sh}\) in the following way. The following fact is simple to check.

**Lemma 4.5.77.** The finite probability space \((\{\ast\}, 1)\), consisting of a single element with probability distribution given by just 1, is a terminal object in
Using this fact, we can define the entropy of $(X, p)$ to be the information loss associated to the unique map to the terminal object, i.e.

$$S_{\text{Sh}}((X, p) \xrightarrow{(X, p)} ([\ast], 1)).$$

(4.5.78)

Baez-Fritz-Leinster prove (Theorem 2 of [BFL11]) that the Shannon entropy functor is the unique such (non-trivial) entropy functor up to proportionality. More precisely, they proved the following.

**Theorem 4.5.79.** Let $S : \text{FinProb} \to \mathbb{B}_{\geq 0}$ be a continuous strictly affine functor. Then there exists a constant $c \in \mathbb{R}_{\geq 0}$ such that

$$S(f) = cS_{\text{Sh}}(f)$$

(4.5.80)

for all morphisms $f$ in $\text{FinProb}$.

Of course, our category $\text{FinProb}$ is slightly different than that of [BFL11] so one should reprove this result in our more general setting. We will not do this here. One of our goals in setting up our formalism is to see how general this result is for arbitrary convex categories equipped with an entropy functor. In terms of physics, we are most interested in states on $C^*$-algebras for describing quantum theory.
4.5.3 Future directions

There are many issues that we have not addressed in this work ranging from subtle issues to general open questions. We conclude by briefly commenting on a few.

- Many of the conjectures still need to be proved in Sections 4.4.3 and 4.5.1. These facts will shed more light on the proper definitions of convex categories and such.

- There is an additional continuity assumption used in characterizing entropy [Ki57], [BFL11], [BF14], [Oc75], [OrWe07]. To make sense of these conditions categorically, the notions of convex and cone categories need to be internalized into some topological category. This is indicated in [BF14] but we wish to explore this more generally in our framework.

- Ideally, one would like to prove characterization theorems for entropy analogous to the ones for FinProb as in [BFL11] and for related notions of relative entropy as in [BF14]. Such a characterization theorem would be simple, emphasize the important role of processes, and have a chance of being applied to other areas due to its categorical formulation. However, characterization theorems may also fail in some convex
categories. For instance, it is known that if one assumes a fixed basis \( \{ \psi_i \}_{i=1,\ldots,n} \) on a finite-dimensional Hilbert space \( \mathcal{H} \), then a type of “Shannon” entropy can be defined for a density matrix \( \rho \) by

\[
- \sum_{i=1}^{n} \langle \psi_i, \rho \psi_i \rangle.
\]

(4.5.81)

It is known that this quantity is in general greater than or equal to the von Neumann entropy, which does not use the extra data given by the choice of basis (this fact is interesting in its own right). Hence, there are two entropies one can define in this context and they are not proportional. Nevertheless, the notion is useful as it pertains to certain ensembles of pure states. This then leads to a question as to what additional constraints are needed to obtain characterization theorems for entropy viewed as a functor. Perhaps a general characterization exists in a general categorical setting. If this is the case, it might allow one to define entropy in contexts where it is either undefined or the currently known construction is ad hoc and deserves more justification. If this is still not the case, then one obtains another notion of entropy that might be used to quantify information content.

- Although we have not discussed it in this thesis, there is a lot of evidence (though the proofs are not complete) that there are convex struc-
tures on categories of density matrices of quantum mechanics and more generally states on $C^*$-algebras analogous to those on finite probability spaces. In addition, one can consider the functor that associates to every $C^*$-algebra its convex space of states. The convex space of states can itself be viewed as a category with no nontrivial morphisms. Then, one can study convex morphisms

$$C^*\text{-Alg}^{op} \overset{\text{States}}{\longrightarrow} \text{Cat}, \quad (4.5.82)$$

where $\mathbb{BR}$ is viewed as a constant prestack, provided that one views $\text{States}$ and $\mathbb{BR}$ as convex categorical objects in the 2-category of functors $\text{Fun}(C^*\text{-Alg}^{op}, \text{Cat})$. This is a current project that is being explored. The categorical viewpoint expressed in the above diagram is described in more detail in Chapter 5.

- Convex analysis has been a fruitful area of study in its own right. One might then wonder what are the consequences of a convex analysis for categories and what other possible applications exist.

- Is there a use for the more abstract notions of $\mathcal{D}$-valued entropy when $\mathcal{D}$ is not a category related to real numbers? The fact that morphisms $\rightarrow$ replace $\leq$ seems like a vast and potential useful generalization for the
concept of convexity. There exists one example of this in the context of
the GNS construction, which itself can be viewed as a form of entropy
satisfying some similar properties. However, this is not described in
this thesis. It would be interesting to find more examples.

Appendix: Monoidal categories

Because this chapter contains several examples, we will be content with sim-
ply giving the most basic definitions from category theory: categories, iso-
morphisms, functors, natural transformations, and various types of monoidal
categories. The mathematical objects that we study always contain some
“data.” These data can be subject to further conditions (constraints) or
satisfy certain properties.

Definition 4.5.83. A category \( \mathcal{C} \) consists of the following data.

(a) A collection \( \mathcal{C}_0 \), elements of which are called objects.

(b) For every pair of objects \( a, b \in \mathcal{C}_0 \), a set \( \mathcal{C}_1(a,b) \), elements of which are
called morphisms from \( a \) to \( b \) and written typically as \( f : a \to b \) or \( a \xrightarrow{f} b \).

(c) For every triple of objects \( a, b, c \), a function \( \mathcal{C}_1(a,b) \times \mathcal{C}_1(b,c) \to \mathcal{C}_1(a,c) \)
called composition and written as

\[
\left( a \xrightarrow{f} b , b \xrightarrow{g} c \right) \mapsto \left( a \xrightarrow{gf} c \right). \tag{4.5.84}
\]
(d) For every object $a$ in $C_0$, an element $\text{id}_a \in C_1(a, a)$ called the identity at $a$.

These data are subject to the following conditions.

i) For every pair of objects $a, b$ and morphism $f : a \rightarrow b$,

\[ f \circ \text{id}_a = f \quad \& \quad \text{id}_b \circ f = f. \]  \hspace{1cm} (4.5.85)

ii) For every quadruple of objects $a, b, c, d$ and triple of morphisms $f : a \rightarrow b$,

\[ g : b \rightarrow c, \ h : c \rightarrow d, \]  

\[ (h \circ g) \circ f = h \circ (g \circ f). \]  \hspace{1cm} (4.5.86)

**Definition 4.5.87.** A morphism $f : a \rightarrow b$ in a category $C$ is called an isomorphism if there exists a morphism $g : b \rightarrow a$ such that $g \circ f = \text{id}_a$ and $f \circ g = \text{id}_b$.

**Definition 4.5.88.** A functor $F$ from a category $C$ to a category $D$ consists of assignments $F_0 : C_0 \rightarrow D_0$ and $F_1 : C_1 \rightarrow D_1$ (both of which are often abusively written as $F$) subject to the following conditions.

i) For every pair of objects $a, b$ in $C$,

\[ F(C_1(a, b)) \subseteq D_1(F(a), F(b)) \]  \hspace{1cm} (4.5.89)

or more visually, if $f : a \rightarrow b$ then $F(f) : F(a) \rightarrow F(b)$. 

ii) For every object $a$ in $\mathcal{C}$, $F(\text{id}_a) = \text{id}_{F(a)}$.

iii) For every triple of objects $a, b, c$ and pair of morphisms $f : a \to b$, $g : b \to c$, $F(g \circ f) = F(g) \circ F(f)$. \hfill (4.5.90)

It is common to depict such a functor as $F : \mathcal{C} \to \mathcal{D}$.

**Definition 4.5.91.** A *natural transformation* $\sigma$ from a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor $G : \mathcal{C} \to \mathcal{D}$ consists of an assignment $\sigma : \mathcal{C}_0 \to \mathcal{D}_1$ subject to the following conditions.

i) For every object $a$, $\sigma(a) : F(a) \to G(a)$. One also occasionally writes $\sigma_a$ instead of $\sigma(a)$.

ii) For every morphism $f : a \to b$, the diagram $\hfill (4.5.92)$

\[
\begin{array}{c}
F(a) \\ F(f) \downarrow \end{array} \longrightarrow \begin{array}{c}
\sigma_a \\ G(f) \downarrow \end{array} \longrightarrow \begin{array}{c}
G(a) \\ F(b) \quad \sigma_b \quad G(b) \end{array}
\]

commutes.

Typically, one writes such a natural transformation as $\hfill (4.5.93)$

\[
\begin{array}{c}
\mathcal{C} \\ G \downarrow \sigma \downarrow \end{array} \longrightarrow \begin{array}{c}
\mathcal{D} \\ F \uparrow \end{array}
\]

or $\sigma : F \Rightarrow G$ when the categories $\mathcal{C}$ and $\mathcal{D}$ are clear from context.
Functors can be composed and natural transformations can be composed in two ways. Both types are used in this paper, which is why we include their definitions.

**Definition 4.5.94.** The composition of two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is a functor $G \circ F : \mathcal{C} \to \mathcal{E}$ defined by

$$
(G \circ F)_0(a) := G(F(a))
$$

for all objects $a$ in $\mathcal{C}$ and

$$
(G \circ F)_1(f) := G(F(f))
$$

for all morphisms $f$ in $\mathcal{C}$.

**Definition 4.5.97.** The *vertical composition* of two natural transformations

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow \sigma & & \downarrow G \\
\mathcal{C} & \xrightarrow{\tau} & \mathcal{D}
\end{array}
\quad \& \quad
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow \sigma & & \downarrow G \\
\mathcal{C} & \xrightarrow{\tau} & \mathcal{D}
\end{array}
\]

(4.5.98)

is a natural transformation

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow \sigma & & \downarrow G \\
\mathcal{C} & \xrightarrow{\tau} & \mathcal{D}
\end{array}
\]

(4.5.99)

defined by

$$
\sigma_{\tau}(a) := \tau_a \circ \sigma_a
$$

(4.5.100)

for all objects $a$ in $\mathcal{C}$. 
Definition 4.5.101. The *horizontal composition* of two natural transformations

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sigma} & \mathcal{D} \\
\downarrow{\sigma} & & \downarrow{\eta} \\
\mathcal{G} & & \mathcal{E}
\end{array}
\quad \& \quad
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\eta} & \mathcal{E} \\
\downarrow{\eta} & & \downarrow{\eta} \\
\mathcal{F} & & \mathcal{H}
\end{array}
\]

(4.5.102)

is a natural transformation

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\eta \circ \sigma} & \mathcal{E} \\
\downarrow{\eta \circ \sigma} & & \downarrow{\eta \circ \sigma} \\
\mathcal{G} & & \mathcal{H}
\end{array}
\]

(4.5.103)

defined by either of the (equal) compositions

\[
(\eta \circ \sigma)_a := K(\sigma_a) \circ \eta_{F(a)} = \eta_{G(a)} \circ J(\sigma_a)
\]

(4.5.104)

for all objects \(a\) in \(\mathcal{C}\).

Definition 4.5.105. A natural transformation as in (4.5.93) is a *natural isomorphism* if there exists a natural transformation \(\tau : G \Rightarrow F\) such that

\[
\begin{aligned}
\sigma \circ \tau &= \text{id}_F \\
\tau \circ \sigma &= \text{id}_G
\end{aligned}
\]

(4.5.106)

where \(\text{id}_F : F \Rightarrow F\) is the identity natural transformation defined by

\[
\text{id}_F(a) = \text{id}_{F(a)}
\]

(4.5.107)

for all objects \(a\) in \(\mathcal{C}\) and similarly for \(\mathcal{G}\).

Alas, in this article, certain functors have additional properties. For completeness, we define such functors.
Definition 4.5.108. A functor $F : \mathcal{C} \to \mathcal{D}$ is \textit{faithful} if for every pair of objects $a, b$ in $\mathcal{C}$, the function

$$F_1 : \mathcal{C}_1(a, b) \to \mathcal{D}_1(F(a), F(b))$$

is injective. $F$ is \textit{full} if \eqref{eq:faithful} is surjective. $F$ is \textit{fully faithful} if \eqref{eq:faithful} is bijective.

The following definitions are meant to serve as reference and we include them for completeness. They should prepare the reader for the definitions of monoidal categories and their variants.

Definition 4.5.110. A \textit{commutative semigroup} consists of a set $C$ and a function $\mu : C \times C \to C$, written as $\mu(x, y) =: x + y$ on elements $x, y \in C$, satisfying commutativity of the following diagrams (with their more standard expressions in terms of elements of $C$ on the right)

$$
\begin{align*}
C \times C \times C & \xrightarrow{\mu \times id_C} C \times C \\
\downarrow{id_C \times \mu} & \downarrow{id_C} \\
C \times C & \xrightarrow{\mu} C
\end{align*}
$$

(associativity)$$x + (y + z) = (x + y) + z \quad (4.5.111)$$

$$
\begin{align*}
C \times C & \xrightarrow{\gamma} C \times C \\
\mu & \downarrow{\mu} \\
C & \xleftarrow{\mu}
\end{align*}
$$

$$x + y = y + x \quad (4.5.112)$$
(commutativity). Here $\gamma$ is the function that swaps entries, namely $\gamma(x, y) = (y, x)$ for all $x, y \in C$. We will occasionally write $(C, \mu)$ for the commutative semigroup to indicate all the data.

Definition 4.5.113. A **commutative monoid** consists of a commutative semigroup $(C, \mu)$ and an element $0 : \{\ast\} \rightarrow C$ (here $\{\ast\}$ is a set with a single element) satisfying $0 + x = x$ (left identity axiom). (4.5.114)

We will often write $(C, \mu, 0)$ for the commutative monoid to indicate all the data.

The “categorification” of commutative semigroups and commutative monoids are called symmetric semigroupal category and symmetric monoidal category, respectively, whose definition we provide now.

Definition 4.5.115. A **symmetric semigroupal category** consists of a category $\mathcal{C}$ together with a functor $\mu : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (whose value on objects and morphisms is written as $\mu(x, y) =: x \oplus y$, etc.) and the following data (along with the assignments on objects written on the right).
(a) A natural isomorphism

\[
\begin{align*}
\mu \times \text{id}_C & : C \times C \times C \to C \\
\text{id}_C \times \mu & : C \times C \to C \\
\mu & : C \to C
\end{align*}
\]

\[\text{asso} \quad (x \oplus y) \oplus z \quad (4.5.116)\]

called the associator.

(b) A natural isomorphism

\[
\begin{align*}
\phi \times \gamma & : C \times C \to C \\
\gamma & : C \times C \to C \\
\mu & : C \to C \\
\phi & : \text{id}_C \to C
\end{align*}
\]

\[\text{braid} \quad y \oplus x \quad (4.5.117)\]

called the braiding.

These data must satisfy the following conditions.

i) The symmetric condition

\[
\begin{align*}
\gamma \times \phi & : C \times C \to C \\
\phi \times \gamma & : C \times C \to C \\
\mu & : C \to C \\
\text{id}_C \times \mu & : C \times C \to C
\end{align*}
\]

\[\text{sym} \quad C \times C = C \times C \quad (4.5.118)\]

which says the diagram

commutes for all object \(x, y\) in \(C\).
ii) The associahedron condition

\[ C \times C \times C \times C \quad \xrightarrow{\mu \times \text{id}_{C \times C}} \quad \xrightarrow{\text{id}_{C \times C} \times \mu} \quad C \times C \times C \times C \]

\[ C \times C \times C \quad \xrightarrow{\mu \times \text{id}_C} \quad \xrightarrow{\text{id}_C \times \mu} \quad C \times C \times C \]

\[ C \quad \xrightarrow{\mu} \quad C \quad \xrightarrow{\mu} \quad C \]

\[ C \]

\[ C \times C \times C \quad \xrightarrow{\varphi} \quad C \times C \times C \]

\[ C \times C \times C \quad \xrightarrow{\mu \times \text{id}_C} \quad \xrightarrow{\text{id}_C \times \mu} \quad C \times C \times C \]

\[ C \times C \quad \xrightarrow{\mu} \quad C \times C \quad \xrightarrow{\mu} \quad C \]

\[ C \]

(4.5.119)

where

\[(\ast) := \text{id}_{\text{id}_C} \times a, \quad (4.5.120)\]

which says the diagram

\[ (x \oplus y) \oplus (z \oplus w) \]

\[ ((x \oplus y) \oplus z) \oplus w \]

\[ (x \oplus (y \oplus z)) \oplus w \]

\[ x \oplus (y \oplus (z \oplus w)) \]

\[ x \oplus ((y \oplus z) \oplus w) \]

commutes for all objects \(x, y, z, w\) in \(C\).
iii) The hexagon condition

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\gamma \times \text{id}_\mathcal{C}} & \mathcal{C} \times \mathcal{C} \\
\mu \times \text{id}_\mathcal{C} & \downarrow & \mu \\
\mathcal{C} & \xrightarrow{\sigma} & \mathcal{C} \\
\end{array}
\]

which says the diagram

\[
\begin{array}{ccc}
(x \oplus y) \oplus z & \xrightarrow{a_{x,y,z}} & x \oplus (y \oplus z) \\
\phi_{y \oplus z,x} & \downarrow & \phi_{y,z} \oplus \text{id}_x \\
(y \otimes z) \oplus x & \xrightarrow{a_{y,z,x}} & y \oplus (x \oplus z) \\
\phi_{y,z,x} & \downarrow & \text{id}_y \oplus \phi_{x,z} \\
y \oplus (z \oplus x) & \xrightarrow{a_{y,z,x}} & y \oplus (x \oplus z) \\
\end{array}
\]

commutes for all objects \(x, y, z\) in \(\mathcal{C}\).

A symmetric semigroupal category as above will be written as a quadruple 
\((\mathcal{C}, \mu, a, \phi)\) or \((\mathcal{C}, \oplus, a, \phi)\) when it is not too confusing.

**Definition 4.5.124.** A **symmetric monoidal category** is a symmetric semigroupal category \((\mathcal{C}, \mu, a, \phi)\) together with an object \(0 : 1 \rightarrow \mathcal{C}\) and natural
isomorphisms

\[
\begin{align*}
&\begin{array}{c}
1 \times \mathcal{C} \xrightarrow{\pi_\mathcal{C}} \mathcal{C} \\
\end{array} & \begin{array}{c}
\mu \\
\mu^{-1} \\
\end{array} & \begin{array}{c}
\mathcal{C} \times \mathcal{C} \xrightarrow{l} \mathcal{C} \\
\mathcal{C} \xrightarrow{r} \mathcal{C} \\
\end{array} \\
&\begin{array}{c}
0 \oplus x \xrightarrow{l_x} x \\
\end{array} & \begin{array}{c}
\phi_{x,0} \\
\phi_{x,0}^{-1} \\
\end{array} & \begin{array}{c}
x \oplus 0 \xrightarrow{r_x} x \\
\end{array}
\end{align*}
\]

(4.5.125)

called \textit{left} and \textit{right unitors}, respectively. These data must satisfy the following conditions.

i) The symmetric unit law

\[
\begin{align*}
&\begin{array}{c}
1 \times \mathcal{C} \xrightarrow{\pi_\mathcal{C}} \mathcal{C} \\
\end{array} & \begin{array}{c}
\mu \\
\mu^{-1} \\
\end{array} & \begin{array}{c}
\mathcal{C} \times \mathcal{C} \xrightarrow{\gamma} \mathcal{C} \\
\mathcal{C} \xrightarrow{\gamma^{-1}} \mathcal{C} \\
\end{array} \\
&\begin{array}{c}
0 \oplus x \xrightarrow{l_x} x \\
\end{array} & \begin{array}{c}
\phi_{x,0} \\
\phi_{x,0}^{-1} \\
\end{array} & \begin{array}{c}
x \oplus 0 \xrightarrow{r_x} x \\
\end{array}
\end{align*}
\]

(4.5.126)

which says that the diagram

\[
\begin{align*}
&\begin{array}{c}
0 \oplus x \xrightarrow{\phi_{x,0}} x \oplus 0 \\
\end{array} \\
&\begin{array}{c}
\phi_{x,0}^{-1} \\
\phi_{x,0} \\
\end{array} \\
&\begin{array}{c}
l_x \\
r_x \\
\end{array}
\end{align*}
\]

commutes for all objects \(x\) in \(\mathcal{C}\).
ii) The triangle law

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{C} \times 1 \times \mathcal{C} & \xrightarrow{id \times \pi_2 \times \mathcal{C}} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{id \times \mu} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{\mu} & \mathcal{C} \\
\end{array}
\end{array}
\]

which says that the diagram

\[
(x \oplus 0) \oplus y \xrightarrow{a_{x,0,y}} x \oplus (0 \oplus y)
\]

commutes for all \(x, y\) in \(\mathcal{C}\).

A symmetric monoidal category will be written as \((\mathcal{C}, \mu, a, \phi, 0, l, r)\) or as \((\mathcal{C}, \oplus, a, \phi, 0, l, r)\) to keep track of all the data.

We now provide the definition of a cartesian (symmetric) monoidal category. Unfortunately, we do not have a clear reference for the following definition and have attempted to provide a reasonable definition suitable for our purposes basing our ideas off the nLab.\(^{17}\)

**Definition 4.5.129.** A **cartesian monoidal category** consists of a symmetric monoidal category \((\mathcal{C}, \otimes, a, \phi, I, l, r)\) together with natural transformations\(^{18}\)

\(^{17}\)https://ncatlab.org/nlab/show/cartesian+monoidal+category. We would also like to thank Josiah Sugarman for discussions that led to the definition used here.

\(^{18}\)We will comment on the choice of this definition afterwards.
(with their explicit form on objects on the right)

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\
\pi_i & \downarrow & \pi_i \downarrow \\
\pi_{i,C} & \downarrow & \pi_{i,C} \downarrow \\
x \oplus y & \rightarrow & x \oplus y \\
\end{array}
$$

$$\pi_{i(x,y)} & \rightarrow & \pi_{2(x,y)}$$

(4.5.130)

and

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{id_C} & \mathcal{C} \\
\circ & \downarrow & \circ \downarrow \\
1 & \downarrow & 1 \\
x & \rightarrow & x \\
\end{array}
$$

\begin{array}{c}
e \\
t
\end{array}

$$\circ_e, e.$$ (4.5.131)

Here \( \pi_{i,C} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \) is the projection bifunctor onto the \( i \)-th factor and \(! \) is the unique map from any category \( \mathcal{C} \) to the category consisting of a single object and morphism \( 1 \). These natural transformations must satisfy the following two universal properties.

i) For any category \( \mathcal{D} \) with functors

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\xi} & \mathcal{C} \times \mathcal{C} \\
\mathcal{D} & \xrightarrow{\zeta} & \mathcal{C}
\end{array}
$$

and natural transformations

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\zeta} & \mathcal{C} \\
\xi & \downarrow & \pi_{i,C} \\
\mathcal{C} \times \mathcal{C} & \downarrow & \mathcal{C} \\
\zeta(z) & \rightarrow & \pi_i(\xi(z)) \\
\rho_i & \downarrow & \pi_{i,C} \downarrow \\
\zeta(z) & \rightarrow & \pi_i(\xi(z)),
\end{array}
$$

(4.5.133)

there exists a (single) unique natural transformation

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\zeta} & \mathcal{C} \\
\xi & \downarrow & \otimes \\
\mathcal{C} \times \mathcal{C} & \downarrow & \mathcal{C} \\
\zeta(z) & \rightarrow & \pi_1(\xi(z)) \otimes \pi_2(\xi(z)) \\
\h & \downarrow & \pi_{i,C} \downarrow \\
\xi(z) & \rightarrow & \pi_1(\xi(z)) \otimes \pi_2(\xi(z))
\end{array}
$$

(4.5.134)
such that

\[
\begin{array}{c}
\overset{\zeta}{\longrightarrow} \\
\overset{\zeta}{\longrightarrow}
\end{array}
\]

\[\mathcal{D} \xrightarrow{\xi} \mathcal{C} \times \mathcal{C} \xrightarrow{\pi_i} \mathcal{C} \]

for both \(i = 1, 2\), i.e.

\[
\begin{array}{c}
\overset{\zeta(z)}{\longrightarrow} \\
\overset{\zeta(z)}{\longrightarrow}
\end{array}
\]

\[\mathcal{D} \xrightarrow{\xi} \mathcal{C} \times \mathcal{C} \xrightarrow{\pi_i} \mathcal{C} \]

(4.5.135)

commutes for all objects \(z\) in \(\mathcal{D}\).

ii) For any category \(\mathcal{D}\) with functors

\[
\begin{array}{c}
\overset{\zeta}{\longrightarrow} \\
\overset{\zeta}{\longrightarrow}
\end{array}
\]

\[\mathcal{D} \xrightarrow{\xi} \mathcal{C} \quad \& \quad \mathcal{D} \xrightarrow{\xi} \mathcal{C} \]

(4.5.137)

and a natural transformation

\[
\begin{array}{c}
\overset{\zeta(z)}{\longrightarrow} \\
\overset{\zeta(z)}{\longrightarrow}
\end{array}
\]

\[\mathcal{D} \xrightarrow{\xi} \mathcal{C} \xrightarrow{\rho} \mathcal{C} \]

(4.5.138)

there exists a unique natural transformation

\[
\begin{array}{c}
\overset{\zeta(z)}{\longrightarrow} \\
\overset{\zeta(z)}{\longrightarrow}
\end{array}
\]

\[\mathcal{D} \xrightarrow{\xi} \mathcal{C} \]

(4.5.139)
such that

\[
\begin{array}{ccc}
D & \xrightarrow{\xi} & C \\
\downarrow{h} & & \downarrow{\mathrm{id}_C} \\
\downarrow{\rho} & & \downarrow{\mathrm{id}_C} \\
! & \xrightarrow{!} & I \\
\end{array}
\]

\[
= \begin{array}{ccc}
D & \xrightarrow{\xi} & C \\
\downarrow{e} & & \downarrow{e} \\
\downarrow{\rho} & & \downarrow{\rho} \\
! & \xrightarrow{!} & I \\
\end{array} \tag{4.5.140}
\]

i.e.

\[
\begin{array}{c}
\zeta(z) \\
\downarrow{h(z)} \\
\xi(z) \\
\downarrow{e(\xi(z))} \\
I
\end{array}
\quad \quad \quad
\begin{array}{c}
\zeta(z) \\
\downarrow{\rho(z)} \\
I
\end{array} \tag{4.5.141}
\]

commutes for all objects \(z\) in \(D\).

Let \((C, \otimes, a, \phi, I, l, r, \pi, e)\) be a cartesian monoidal category. From the universal property of the product and projections, we can construct a diagonal map as follows. Let \(D := C\), \(\xi := \Delta_C : C \to C \times C\) be the diagonal for categories, and \(\zeta := \otimes \circ \Delta_C : D \to C\). Let \(\rho_i := \mathrm{id}_C\). Then the universal property applies and \(h := \Delta : \mathrm{id}_C \Rightarrow \otimes \circ \Delta_C\) is the diagonal natural transformation which when applied to an object \(x\) in \(C\) gives

\[
x \xrightarrow{\Delta_x} x \otimes x. \tag{4.5.142}
\]

**Remark 4.5.143.** Many of the data in a cartesian monoidal category

\((C, \otimes, a, \phi, I, l, r, \pi, e)\) are redundant. For instance, the associator \(a\) can be uniquely constructed as follows. Let \(D := C \times C \times C\). Set \(\zeta : D \to C\) to be \(\zeta := \otimes \circ (\otimes \times \mathrm{id}_C)\), i.e.

\[
\zeta(x, y, z) := (x \otimes y) \otimes z \tag{4.5.144}
\]
on objects $x, y, z$ of $C$. Set $\xi : D \rightarrow C \times C$ to be $\xi := \text{id}_C \times \otimes$, i.e.

$$\xi(x, y, z) := (x, y \otimes z) \quad (4.5.145)$$

on objects $x, y, z$ of $C$. Set $\rho_1$ and $\rho_2$ to be

$$\rho_1 := \pi_1 \circ (\text{id}_\otimes \times \text{id}_C) \quad (4.5.146)$$

and

$$\rho_2 := \pi_2 \otimes \text{id}_C \quad (4.5.147)$$

i.e.

$$\rho_1(x, y, z) := \pi_1(x) \circ \pi_1(x \otimes y, z) \quad (4.5.148)$$

and

$$\rho_2(x, y, z) := \pi_2(x, y) \otimes \text{id}_z \quad (4.5.149)$$

on objects $x, y, z$ of $C$. Then the universal property gives an $h$ that is exactly the associator.

**Remark 4.5.150.** Typically, a cartesian category is defined as a category with finite limits [Ma78], which is a property and not additional structure. The definition we have chosen therefore seems significantly more complicated. In fact, one can construct $\otimes, a, \phi, I, l, r, \pi_1, \pi_2, e, \ldots$ all from the existence of limits and a form of Zorn’s lemma. Furthermore, all such choices are natu-
rally equivalent. However, for presentation purposes and an easier comparison to monoidal categories, we have avoided this option. In addition, we find our formulation of universal properties in Definition 4.5.129 quite non-standard in that they are formulate purely arrow-theoretically. Note that we could not have stated the universal property for the \( \pi_i \)'s, for instance, as “for any functor \( \zeta : C \times C \to C \) and natural transformations \( \rho_i : \zeta \Rightarrow \pi_i \), there exists a unique \( h : \zeta \Rightarrow \otimes \) such that \( \rho_i = \frac{h}{\delta_i} \)” because then we could not construct the diagonal \( \Delta \) nor could we construct the other natural transformations such as the associator (see the previous remark).
Chapter 5

The GNS construction as an adjunction

5.1 Introduction and outline

There is a familiar construction whose input consists of a representation of a \( C^* \)-algebra on a Hilbert space together with a vector and whose output is a state on the \( C^* \)-algebra via restriction. Namely, given an algebra \( \mathcal{A} \), a representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) to bounded operators on a Hilbert space \( \mathcal{H} \), and a unit vector \( \psi \in \mathcal{H} \), one obtains a state on \( \mathcal{A} \) given by the expectation values of observables in \( \mathcal{A} \) sending \( a \in \mathcal{A} \) to \( \langle \psi, \pi(a)\psi \rangle \). We show that this construction, denoted by \( \text{rest} \), can be expressed categorically as a natural transformation

\[
\begin{array}{ccc}
\text{C}^*\text{-Alg}^{\text{op}} & \overset{\text{States}}{\longrightarrow} & \text{Rep}^* \\
\downarrow \text{rest} & & \downarrow \\
\text{Cat} & \overset{\text{rest}}{\longrightarrow} & \text{Cat}
\end{array}
\]  

(5.1.1)
Here, \textbf{Cat} is the category of categories, \textbf{C*-Alg} is the category of \( C^* \)-algebras, \textbf{States} is the functor that associates a category of states to every \( C^* \)-algebra, and \textbf{Rep} is the functor that associates the category of representations of \( C^* \)-algebras (the \( \bullet \) is to denote the additional choice of a vector).

The main purpose of this chapter is to prove that the natural transformation \( \text{rest} \) has a left-adjoint

\[
\begin{array}{ccc}
\text{C*-Alg}^{\text{op}} & \xrightarrow{\text{GNS}^*} & \text{Cat} \\
\text{States} \downarrow & & \downarrow \text{rest} \\
\text{Rep}^* & \xleftarrow{\text{GNS}^*} & \text{Cat}
\end{array}
\]

(5.1.2)

denoted by \( \text{GNS}^* \) since its ingredients are composed of constructions due to Gelfand, Naimark, and Segal [GeNe43], [Se47]. However, there are subtleties in this description. First, the GNS construction is a 2-categorical natural transformation (utilizing the fact that \textbf{Cat} is a 2-category) instead of a natural transformation in the usual sense of ordinary category theory. Second, the category of states is not the naive one that one might think of—one must view the states of a \( C^* \)-algebra as a discrete category. Third, for a robust statement with physical applications, the morphisms in the representation category associated to a \( C^* \)-algebra must include all intertwiners that are isometries and not only the unitary equivalences.

The GNS construction has many useful and interesting properties. We
isolate the key properties that can be used to characterize it as the left adjoint to the restriction map from representations to states. By the essential uniqueness of adjoints, this offers a definition of the GNS construction so that one can now view the GNS construction as exhibiting the existence of such an adjoint. Several of the ingredients used in this characterization were known for a long time. Here, however, we offer a categorical perspective together with interpretations of all results in physical terms.

The outline of this chapter is as follows. Section 5.2 defines all relevant notions from $C^*$-algebras as well as the states functor and the representation functor. Section 5.3 describes the GNS construction as is usually found in the literature but from a more categorical perspective. For simplicity, we ignore the cyclic vector and focus only on the fact that the GNS construction produces a representation. In particular, we prove that the GNS construction is a semi-pseudo-natural transformation (though not a natural transformation) in Theorem 5.3.22. Section 5.4 explains why the category of states (introduced in Section 5.2) must have no non-trivial morphisms for our purposes. Section 5.5 properly accounts for the fact that the GNS construction produces a cyclic representation. The statement that the GNS construction is left-adjoint to the restriction to states natural transformation is proved in Theorem 5.5.57. In Section 5.6, we illustrate several of the constructions and
results in terms of a simple example of a bipartite system familiar (to physicists) from the EPR setup. Throughout, we provide physical interpretations of most results. Although we assume the reader is familiar with the basics of categories, we include a short appendix on 2-categories and 2-categorical adjunctions (but refer the reader to Appendix A of this thesis for how to compose pseudo-natural transformations and modifications).

5.2 States and representations of $C^*$-algebras

For more details on $C^*$-algebras, the reader is referred to [Di69].

**Definition 5.2.1.** A unital Banach algebra is a vector space $\mathcal{A}$ together with

i) a binary multiplication operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$,

ii) a norm $\| \cdot \| : \mathcal{A} \to \mathbb{R}_{\geq 0}$,

iii) and an element $1_\mathcal{A} \in \mathcal{A}$.

The multiplication must be distributive over vector addition, the element $1_\mathcal{A}$ must satisfy the condition $a1_\mathcal{A} = 1_\mathcal{A}a = a$ for all $a \in \mathcal{A}$, and finally, all Cauchy sequences must converge.

**Definition 5.2.2.** A unital $C^*$-algebra is a unital Banach algebra $\mathcal{A}$ with an involution $*: \mathcal{A} \to \mathcal{A}$ that is an anti-homomorphism for the multiplication
and satisfies $\|aa^*\| = \|a\|^2$ for all $a \in A$. An element $a \in A$ is \textit{self-adjoint} if $a^* = a$, an \textit{isometry} if $a^*a = 1_A$, and \textit{unitary} if $a^*a = 1_A = aa^*$.

**Definition 5.2.3.** A \textit{map/morphism of C*-algebras} is a (bounded) linear map $f : A \rightarrow A'$ from a C*-algebra $A$ to another one $A'$ such that $f(a^*) = f(a)^*$, $f(a_1a_2) = f(a_1)f(a_2)$, and $f(1_A) = 1_{A'}$ for all $a, a_1, a_2 \in A$.

**Definition 5.2.4.** Let $\text{C*-Alg}$ be the category of unital C*-algebras, namely an object of $\text{C*-Alg}$ consists of a unital C*-algebra $A$ and a morphism $f : A \rightarrow A'$ is a map of unital C*-algebras.

Throughout this article, \textit{all} C*-algebras will be assumed unital and we will avoid overuse of this adjective unless it is necessary to stress it.

**Definition 5.2.5.** Given a C*-algebra $A$, a \textit{state} on $A$ is a bounded linear function $\omega : A \rightarrow \mathbb{C}$ such that $\omega(1_A) = 1$ and $\omega(a^*a) \geq 0$ for all $a \in A$.

Denote the set of states on a C*-algebra $A$ by $S(A)$.

**Definition 5.2.6.** Let $\text{Rep}(A)$ be the category of representations of the C*-algebra $A$ on Hilbert spaces. This means the objects are pairs $(\pi, \mathcal{H})$ with $\mathcal{H}$ a Hilbert space and $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ a map of C*-algebras (the involution on $\mathcal{B}(\mathcal{H})$ is taking the adjoint). Morphisms $(\pi, \mathcal{H}) \rightarrow (\pi', \mathcal{H}')$ are intertwiners, i.e. bounded linear operators $L : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$L \circ \pi(a) = \pi'(a) \circ L \quad \text{for all } a \in A.$$  \hfill (5.2.7)
Remark 5.2.8. It is very important that we assume our morphisms in $\text{Rep}(\mathcal{A})$ are intertwiners and not just unitary isomorphisms. We will explain why later.

Physics 5.2.9. We think of a $C^*$-algebra $\mathcal{A}$ as the algebra of observables of a physical system.\footnote{Actually, $\mathcal{A}$ contains un-observable operators because it contains elements that are not self-adjoint. Examples include creation and annihilation operators. In fact, it contains observables that are self-adjoint but need not be things we can actually measure in a lab (such as momentum to the $8^{th}$ power). Nevertheless, we call $\mathcal{A}$ the algebra of observables by slight abuse of terminology.} An example to relate to is the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. However, the main point of this abstract perspective is to place the emphasis on the observables rather than the Hilbert space of vector states or the particular realization of an abstract observable as an operator. Indeed, we can think of angular momentum being defined in different ways on different Hilbert spaces (or even classically on phase space), but when we think of angular momentum, we do not think of which Hilbert space it acts on—we just think angular momentum!

Furthermore, we do not measure vectors in a Hilbert space. What we measure are expectation values. This is precisely the meaning of a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ as defined above. A state assigns an expectation value to each physical observable. That is what a physical state is: a sequence of expectation values for all our observables (satisfying reasonable postulates). For
instance, if $a$ is self-adjoint, then $\omega(a)$ is the expectation value of $a$ and
$\omega(a^2) - (\omega(a))^2$ is the variance. Therefore, the definition of state includes
not only expectation values of observables, but also their moments.

Of course, technically thinking of observables as an algebra is an ideal-
ization because observables (as described by the working physicist) are not
always bounded operators and therefore they do not form an algebra in the
strict sense. We will ignore this issue and assume all our observables corre-
spond to bounded operators.

The above definitions of $S(A)$ and $\text{Rep}(A)$ extend to functors.

**Proposition 5.2.10.** The assignment\(^2\)

$$C^*\text{-Alg}^{\text{op}} \xrightarrow{S} \text{Set}$$

$A \mapsto S(A)$ \hspace{1cm} (5.2.11)

$\left( A' \xrightarrow{f} A \right) \mapsto \left( S(A') \xleftarrow{S(f)} S(A) \right)$,

where $S(f)$ is defined by

$$S(A) \ni \omega \mapsto \omega \circ f \in S(A')$$ \hspace{1cm} (5.2.12)

is a functor, henceforth referred to as the states pre-sheaf.

---

\(^2\)For any category $\mathcal{C}$, the opposite category $\mathcal{C}^{\text{op}}$ has the same objects as $\mathcal{C}$ but a morphism from an object $a$ to an object $b$ in $\mathcal{C}^{\text{op}}$ is a morphism from $b$ to $a$ in $\mathcal{C}$. Also, $\text{Set}$ is the category of sets.
Proof. First, \( \omega \circ f \) is a state on \( \mathcal{A} \) because

\[
\omega(f(1_{\mathcal{A}})) = \omega(1_{\mathcal{A}}) = 1
\]  
(5.2.13)

and

\[
\omega(f(a^* a')) = \omega(f(a')^* f(a')) \geq 0
\]  
(5.2.14)

for all \( a' \in \mathcal{A}' \). \( \mathcal{S} \) is functorial because the identity \( \text{id}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \) gets sent to the identity and the composition of \( C^* \)-algebra maps \( \mathcal{A}^* f' \mathcal{A}' f \mathcal{A} \) gets sent to \( \mathcal{S}(f \circ f') = \mathcal{S}(f') \circ \mathcal{S}(f) \).

\[ \blacksquare \]

Physics 5.2.15. The meaning of this functor physically can be seen by considering a special case, which will be used throughout this work. Suppose \( \mathcal{A}_0 \) is a subalgebra of physical observables of \( \mathcal{A} \). Let \( i : \mathcal{A}_0 \hookrightarrow \mathcal{A} \) be the inclusion map. The functor \( \mathcal{S}(i) \) takes a state \( \omega : \mathcal{A} \twoheadrightarrow \mathbb{C} \) that gave expectation values for all observables in \( \mathcal{A} \) and it restricts that state to only give expectation values for a smaller collection of observables, mathematically described by \( \mathcal{A}_0 \).

In thermodynamic or statistical-mechanical terminology, one can imagine \( \mathcal{A} \) as describing the algebra of observables for microstates and \( \mathcal{A}_0 \) as describing the set of observables for some macrostates.\(^4\) In fact, Jaynes used a closely related idea, that is actually more physically reasonable, by assuming that

\(^3\)The flipping of the order of morphism composition in the equation \( \mathcal{S}(f \circ f') = \mathcal{S}(f') \circ \mathcal{S}(f) \) is why we use \( \text{op} \) in \( C^* \text{-Alg}^{\text{op}} \).

\(^4\)I would like to thank V. P. Nair for pointing this out.
\(A_0\) is just a subset of \(A\) and develops thermodynamics from it [Ja57]. In this process, one therefore loses some information about the state—we only know fewer of its expectation values.

There is a functor \(D : \text{Set} \to \text{Cat}\) from the category of sets to the category of categories given by sending a set to the discrete category with only identity morphisms. Thus, since the composition of functors is a functor, this gives a functor

\[
\text{C}^{*}\text{-Alg}^{\text{op}} \xrightarrow{S} \text{Set} \xrightarrow{D} \text{Cat},
\]

which we denote by \(\text{States}\) and call it the \textit{states pre-stack}. The categorically-minded reader will immediately point out that \(\text{Cat}\) is actually a 2-category, and we will indeed use this fact in a crucial way when we describe the GNS construction. But for now, let us put this aside.

\textbf{Proposition 5.2.17.} The assignment

\[
\text{C}^{*}\text{-Alg}^{\text{op}} \xrightarrow{\text{Rep}} \text{Cat}
\]

\[
\mathcal{A} \mapsto \text{Rep}(\mathcal{A})
\]

\[
(\mathcal{A} \xrightarrow{f} \mathcal{A}) \mapsto \left( \text{Rep}(\mathcal{A}') \xrightarrow{\text{Rep}(f)} \text{Rep}(\mathcal{A}) \right).
\]

is a functor. Here \(\text{Rep}(f)\), sometimes written as \(f^*\), is the functor defined by sending a representation \((\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \mathcal{H})\) to the representation \((\pi \circ f : \mathcal{A}' \to \mathcal{B}(\mathcal{H}), \mathcal{H})\) and by sending an intertwiner \((\pi, \mathcal{H}) \xrightarrow{L} (\rho, \mathcal{V})\) to the
intertwiner \((\pi \circ f, \mathcal{H}) \xrightarrow{L} (\rho \circ f, \mathcal{V})\).\(^5\) The \textbf{Rep} is also called the representation pre-stack.

\textit{Proof.} Let us first make sure \textbf{Rep}(f) itself is indeed a functor. For \(L\) to be an intertwiner in \textbf{Rep}(\mathcal{A}') it must be that

\[ L \circ \pi(f(a')) = \rho(f(a')) \circ L \tag{5.2.19} \]

for all \(a' \in \mathcal{A}'\), but this is true because \(f(a') \in \mathcal{A}\) and \(L\) is an intertwiner in \textbf{Rep}(\mathcal{A}).\) It is not difficult to see that id\(_\mathcal{A}\) gets sent to id\(_{\textbf{Rep}(\mathcal{A})}\) and the composition of \(\mathcal{A}'' \overset{f'}{\rightarrow} \mathcal{A}' \overset{f}{\rightarrow} \mathcal{A}\) gets sent to \(\textbf{Rep}(f') \circ \textbf{Rep}(f)\).

\textbf{Physics 5.2.20.} The meaning of the functor (5.2.18) is as follows. With each abstract algebra of observables, there is a collection of Hilbert spaces on which we can realize these observables. This collection is not just a \textit{set}\(^6\) but a \textit{category} because there are intertwiners between representations. If you think you do not care about intertwiners, think again. Every tensor operator is an intertwiner. For instance, the angular momentum for particles in three-dimensional space is a \textit{vector} of operators. This vector of operators is precisely an intertwiner [Ha03]. Other examples of intertwiners are unitary equivalences of representations. These are (some of the) symmetries of

\(^{5}\) The same notation \(L\) is used because it is the same operator \(L : \mathcal{H} \longrightarrow \mathcal{V}\) at the level of Hilbert spaces.

\(^{6}\) Technically, it is not even a set in the strict sense, but that is not the point I am trying to make.
quantum mechanics, and one would certainly not want to just ignore them. For instance, different observers might associate a slightly different Hilbert space to a collection of observables. In particular, the observables themselves might be expressed differently. The position and momentum representations of basic quantum mechanics provide one example. The unitary map defined by the Fourier transform is an intertwiner (a unitary equivalence) of representations. That is why we care about the full category of representations, and not just the objects.

5.3 The GNS construction: from observables and states to Hilbert spaces

We will split the GNS construction into three parts. First, we will describe the construction as is common in the literature. Then we will describe something that is less commonly illustrated, but is described nicely for physicists in [BGdQRL13], which is what the GNS construction gives for $C^*$-algebra morphisms (and specifically not necessarily $C^*$-algebra isomorphisms). The GNS construction was first introduced by Segal in [Se47] and we will utilize many of the facts proved in this work. At the end of this section, we state our first theorem which says that the GNS construction is a semi-pseudo-natural transformation (see Definition 5.6.21 in the Appendix of this Chapter) be-
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 tween the functors introduced in the previous section.

Construction 5.3.1. Let $\omega : A \rightarrow C$ be a state for a unital $C^*$-algebra $A$.

Then the function

$$A \times A \rightarrow C$$

$$(b, a) \mapsto \omega(b^*a)$$

is a bilinear map that is skew-conjugate in the first variable. Furthermore, it satisfies

$$\omega(b^*a) = \overline{\omega(a^*b)} \quad \forall \, a, b \in A$$

(5.3.3)

and

$$|\omega(b^*a)|^2 \leq \omega(b^*b)\omega(a^*a) \quad \forall \, a, b \in A.$$  

(5.3.4)

Define the set of null-vectors by

$$\mathcal{N}_\omega := \{ a \in A \mid \omega(a^*a) = 0 \}.$$  

(5.3.5)

---

7Proof: By assumption $\omega((aa + \beta b)^*(aa + \beta b)) \geq 0$ for all $\alpha, \beta \in C$ and $a, b \in A$, which in particular implies that $\omega((aa + \beta b)^*(aa + \beta b))$ is real. Equating this expression with its conjugate gives $\overline{\alpha\beta}\omega(a^*b) + \overline{\alpha}\overline{\beta}\omega(b^*a) = \alpha\overline{\beta}\omega(a^*b) + \overline{\alpha}\overline{\beta}\omega(b^*a)$. Setting $a = 1$ and $b = 1$ gives $-\omega(a^*b) + \omega(b^*a) = \omega(a^*b) - \overline{\omega(b^*a)}$ while setting $a = 1$ and $b = 1$ gives $\omega(a^*b) + \omega(b^*a) = \overline{\omega(a^*b)} + \omega(b^*a)$. Adding these two gives $2\omega(b^*a) = 2\overline{\omega(a^*b)}$ which proves the claim.

8Proof (this is more or less a standard proof of the Cauchy-Schwarz inequality): This splits up into two cases. First, if $\omega(b^*a) = 0$, then the claim is true. In the other case, suppose that $\omega(b^*a) \neq 0$. As in the previous footnote, consider the inequality $\omega((aa + \beta b)^*(aa + \beta b)) \geq 0$ valid for all $\alpha, \beta \in C$ and $a, b \in A$. Choose $\alpha = \frac{\omega(b^*a)}{\omega(b^*b)} \sqrt{\omega(b^*b)}$ and $\beta = -\sqrt{\omega(a^*a)}$. Then, $\omega((aa + \beta b)^*(aa + \beta b)) = 2\omega(b^*b)\omega(a^*a) - 2|\omega(b^*a)|\sqrt{\omega(b^*b)\omega(a^*a)}$ using (5.3.3) along the way to cancel some terms. Rearranging and canceling the factor of 2 gives $|\omega(b^*a)|\sqrt{\omega(b^*b)\omega(a^*a)} \leq \omega(b^*b)\omega(a^*a)$. Squaring both sides and canceling the common terms proves the claim.
Then $\mathcal{N}_\omega$ is a left ideal inside $\mathcal{A}$, meaning that $ab \in \mathcal{N}_\omega$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{N}_\omega$. To see this, first notice that (5.3.4) implies

$$|\omega(ab)|^2 \leq \omega(a^*a) \omega(b^*b) = 0 \implies \omega(ab) = 0 \forall a \in \mathcal{A}, b \in \mathcal{N}_\omega. \quad (5.3.6)$$

Using this fact,

$$\omega((ab)^*(ab)) = \omega(b^*a^*ab) = \omega((a^*ab)^*) = 0 \quad (5.3.7)$$

since $a^*ab \in \mathcal{A}$. Furthermore, (5.3.3) and (5.3.6) imply

$$\omega(b^*a) = 0 \quad \forall b \in \mathcal{N}_\omega, a \in \mathcal{A}. \quad (5.3.8)$$

Denote the quotient vector space by

$$\mathcal{H}_\omega := \mathcal{A}/\mathcal{N}_\omega, \quad (5.3.9)$$

write the equivalence class of $a \in \mathcal{A}$ as $[a]$, and define an inner product

$$\mathcal{H}_\omega \times \mathcal{H}_\omega \xrightarrow{\langle \cdot, \cdot \rangle_\omega} \mathcal{H}_\omega \quad (5.3.10)$$

by choosing lifts of the equivalence classes. Note that this is well-defined because for any other choice $b'$ and $a'$ of $[b]$ and $[a]$, respectively, which
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means that $b - b', a - a' \in \mathcal{N}_\omega$, then

$$\omega(b^* a') = \omega\left( (b - (b'))^* (a - (a')) \right)$$

$$= \omega(b^* a) - \omega\left( (b - b')^* a \right) - \omega\left( b^* (a - a') \right) + \omega\left( (b - b')^* (a - a') \right)$$

$$= \omega(b^* a) - 0 - 0 + 0 \quad \text{by (5.3.6) and (5.3.8)}$$

$$= \omega(b^* a). \quad (5.3.11)$$

Complete $\mathcal{H}_\omega$ with respect to the norm induced by $\langle \cdot , \cdot \rangle_\omega$ and use the same notation $\mathcal{H}_\omega$ to denote this Hilbert space. Furthermore, there is a natural representation $\pi_\omega$ of $\mathcal{A}$ on $\mathcal{H}_\omega$ given by

$$\pi_\omega(a)[b] := [ab] \quad (5.3.12)$$

for all $a \in \mathcal{A}$ and $[b] \in \mathcal{H}_\omega$. Thus, to every state $\omega : \mathcal{A} \rightarrow \mathbb{C}$, we have constructed a representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathcal{A}$. We denote this assignment by $\text{GNS}_\mathcal{A} : \text{States}(\mathcal{A}) \rightarrow \text{Rep}(\mathcal{A})$, i.e. $\text{GNS}_\mathcal{A}(\omega) := (\pi_\omega, \mathcal{H}_\omega)$. It is trivially a functor because $\text{States}(\mathcal{A})$ has no non-trivial morphisms. This construction is called the GNS construction.

Physics 5.3.13. As we discussed earlier, $\omega$ is a list of expectation values of all the observables of interest described by $\mathcal{A}$. As a particular example, consider again the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ with inner

---

9This is well-defined because for any other $b' \in \mathcal{A}$ with $[b'] = [b]$, then $ab' - ab = a(b' - b) \in \mathcal{N}_\omega$ because $b' - b \in \mathcal{N}_\omega$ and $\mathcal{N}_\omega$ is an ideal by the comment preceding (5.3.6).
product $\langle \cdot , \cdot \rangle$. Then, there is actually a one-to-one correspondence between states $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ satisfying certain conditions\footnote{If $\dim \mathcal{H} < \infty$, no such additional conditions are necessary. However, in the case $\dim \mathcal{H} = \infty$, one needs stronger continuity assumptions on the state.} and density matrices, i.e. bounded linear operators $\rho \in \mathcal{B}(\mathcal{H})$ that are self-adjoint and $\text{tr}(\rho) = 1$ (see Proposition 19.8 and Theorem 19.9 of [Ha13]). The correspondence is obtained by the map that sends a density matrix $\rho$ to the state $\omega_\rho$, defined by $\omega_\rho(a) := \text{tr}(\rho a)$ for all $a \in \mathcal{A}$. Therefore, we will think of an abstract state $\omega : \mathcal{A} \to \mathbb{C}$ as being equivalent to a density matrix.\footnote{Though in many cases of interest, such a density matrix need not exist. This occurs for instance in the Unruh effect whereupon restricting the algebra of observables to a Rindler observer does not lead to a density matrix, but rather an abstract state satisfying the KMS condition (see Section 5.1 of Wald [Wa94]).} This example will help us interpret the GNS construction physically. The meaning of the function $(b, a) \mapsto \omega(b^* a)$ for two observables $a$ and $b$ in $\mathcal{A}$ is less mysterious if we focus on the case $b = a$ and think of $a$ and $a^*$ as annihilation and creation operators, respectively. Then $a^* a$ is the number operator and $\omega(a^* a)$ is the expectation value of the particle number for the state $\omega$.

The meaning of the null-space $\mathcal{N}_\omega$ can be interpreted as the set of observables that annihilate the state $\omega$ for all observable purposes. If we go back to the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and the special case of $\rho = P_\psi$ (written as $|\psi\rangle \langle \psi|$ in Dirac notation), the projection operator onto the subspace spanned by a unit vector $\psi \in \mathcal{H},$ then an observable $a \in \mathcal{N}_\omega$ would mean that $\text{tr}(P_\psi a^* a) = \langle a\psi, a\psi \rangle = 0$
which, since $\langle \cdot, \cdot \rangle$ is an inner product, would mean that $a\psi = 0$, i.e. $a$ annihilates $\psi$. So now consider two observables $b, c \in \mathcal{A}$ such that $b - c \in \mathcal{N}_\omega$. This means that $(b - c)\psi = 0$, i.e. $b\psi = c\psi$, which means that the observables $b$ and $c$ cannot be distinguished by the particular state $\psi$. Therefore, to summarize, if we fix a state $\omega$ on a set of observables $\mathcal{A}$, it may be that with respect to that particular state, there are some observables that are indistinguishable in terms of their expectation values. That is why we consider the quotient $\mathcal{A}/\mathcal{N}_\omega$ where we have identified these equivalent observables. Therefore, this construction tells us that the associated Hilbert space is just equivalence classes of observables of $\mathcal{A}$ distinguished by the state $\omega$.

**Construction 5.3.14.** Let $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ be a morphism of $C^*$-algebras and let $\omega : \mathcal{A} \to \mathbb{C}$ be a state on $\mathcal{A}$. Then, as discussed in Proposition 5.2.10, $\omega \circ f : \mathcal{A}' \to \mathbb{C}$ is a state on $\mathcal{A}'$. By applying the previous construction, we get two representations $\pi_{\omega \circ f} : \mathcal{A}' \to \mathcal{B}(\mathcal{H}_{\omega \circ f})$ and $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$. There is a canonical map $L_f : \mathcal{H}_{\omega \circ f} \to \mathcal{H}_\omega$ obtained from the diagram

$$
\begin{align*}
\mathcal{A}' & \xrightarrow{f} \mathcal{A} \\
\downarrow & & \downarrow \\
\mathcal{A}'/\mathcal{N}_{\omega \circ f} & \xrightarrow{L_f} \mathcal{H}_{\omega \circ f} - \longrightarrow \mathcal{H}_\omega = \mathcal{A}/\mathcal{N}_\omega
\end{align*}
\tag{5.3.15}
$$

given by

$$
L_f([a']) := [f(a')] \tag{5.3.16}
$$
for all \([a'] \in \mathcal{H}_{\omega f}\). This is well-defined because for any other representative \(b'\) of \([a']\),

\[
(\omega \circ f) \left( (b' - a')^* (b' - a') \right)
\]

since \(b' - a' \in \mathcal{N}_\omega\) and hence \(f(b')\) is equivalent to \(f(a')\) in \(\mathcal{A}\). Furthermore, the map \(L_f\) is an intertwiner \(\pi_{\omega f}, \mathcal{H}_{\omega f} \rightarrow (\pi_{\omega} \circ f, \mathcal{H}_\omega)\) of representations of \(\mathcal{A}'\), which means that the diagram

\[
\begin{array}{ccc}
\mathcal{H}_{\omega f} & \xrightarrow{L_f} & \mathcal{H}_\omega \\
\pi_{\omega f}(a') \downarrow & & \downarrow \pi_{\omega}(f(a')) \\
\mathcal{H}_{\omega f} & \xrightarrow{L_f} & \mathcal{H}_\omega
\end{array}
\]  

(5.3.18)

commutes for all \(a' \in \mathcal{A}'\). This is because for any \([b'] \in \mathcal{H}_{\omega f}\) we have

\[
L_f \left( \pi_{\omega f}(a') ([b']) \right) = L_f ([a'b'])
\]

\[
= [f(a'b')]
\]

\[
= [f(a') f(b')]
\]

\[
= \pi_{\omega} \left( f(a') \right) \left( [f(b')] \right)
\]

\[
= \pi_{\omega} \left( f(a') \right) \left( L_f ([b']) \right)
\]

(5.3.19)

\[\text{Diagrams such as (5.3.17) are read from top to bottom in either clockwise or counterclockwise order to replicate the argument in the order in which it was originally conceived.}\]
for all $a' \in \mathcal{A}'$. The assignment sending a state $\omega : \mathcal{A} \to \mathbb{C}$ and a morphism $f : \mathcal{A}' \to \mathcal{A}$ of $C^*$-algebras to the intertwiner $L_f : (\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}) \to (\pi_{\omega}, \mathcal{H}_\omega)$ therefore defines a natural transformation\footnote{In the present situation, the definition of natural transformation only reduces to an assignment on objects of $\text{States} (\mathcal{A})$ because $\text{States} (\mathcal{A})$ is a discrete category.}

associated to every morphism $f : \mathcal{A}' \to \mathcal{A}$ of $C^*$-algebras. We denote the intertwiner $L_f$ by $\text{GNS}_f (\omega)$ to indicate explicitly what it depends on.

\textbf{Physics 5.3.21.} Let us go back to the case $i : \mathcal{A}_0 \hookrightarrow \mathcal{A}$ of restricting ourselves to a subalgebra of observables and let $\omega$ be a state on $\mathcal{A}$. Let $\omega_0 := \omega \circ i$ be the state pulled back to $\mathcal{A}_0$. Then, since $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$, there are fewer experiments we can perform on the state and we conclude $\mathcal{N}_{\omega_0} \subset \mathcal{N}_\omega$, which means there are fewer indistinguishable observables for the state $\omega_0$. This therefore alters what the state is because a state is characterized by its expectation values for some set of observables.

The associated intertwiner $L_i : (\pi_{\omega_0}, \mathcal{H}_{\omega_0}) \to (\pi_{\omega} \circ i, \mathcal{H}_{\omega})$ is an injection because if for any $[a_0] \in \mathcal{H}_{\omega_0}$ such that $L_i ([a_0]) \equiv [i (a_0)] = 0$ in $\mathcal{H}_\omega$, this means that $\omega((i(a_0)^* i(0))) = 0$ but this equals $(\omega \circ i)(a_0^* a_0)$ which exactly
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means \([a_0] = 0\) in \(\mathcal{H}_{0}\). Therefore, physically, the intertwiner describes a subspace \(L_i(\mathcal{H}_{0})\) of the Hilbert space \(\mathcal{H}_{0}\). The act of restricting our view to a subalgebra corresponds to restricting to a subspace of our Hilbert space since the Hilbert space is described as equivalence classes of observables.\(^{14}\)

**Theorem 5.3.22.** The assignment\(^{15}\)

\[
\mathcal{C}^{\ast}\text{-Alg}_{0}^{\text{op}} \xrightarrow{\text{GNS}} \text{Cat}_1
\]

\[\mathcal{A} \mapsto \left(\text{States}(\mathcal{A}) \xrightarrow{\text{GNS}_\mathcal{A}} \text{Rep}(\mathcal{A})\right)\]

from Construction 5.3.1 and

\[
\mathcal{C}^{\ast}\text{-Alg}_{1}^{\text{op}} \xrightarrow{\text{GNS}} \text{Cat}_2
\]

\[\left(\mathcal{A}' \xrightarrow{f} \mathcal{A}\right) \mapsto \left(\text{GNS}_{\mathcal{A}'} \circ \text{States}(f) \xrightarrow{\text{GNS}_f} \text{Rep}(f) \circ \text{GNS}_\mathcal{A}\right)\]

from Construction 5.3.14 defines a semi-pseudo-natural transformation\(^{16}\)

\[
\mathcal{C}^{\ast}\text{-Alg}_{0}^{\text{op}} \xrightarrow{\text{GNS}} \text{Cat}
\]

\[\text{States} \quad \text{Rep}
\]

\(^{14}\)This phrasing is a bit misleading, however, since every \(\mathcal{C}^{\ast}\)-algebra morphism \(f : \mathcal{A}' \rightarrow \mathcal{A}\) will lead to \(L_f\) being injective regardless of whether or not \(f\) is injective since our argument did not depend on this. Nevertheless, for psychological reasons and simplicity for interpretation, we will always use inclusions for explaining the physics.

\(^{15}\)Given a 2-category (or a category) \(\mathcal{C}\), the objects, 1-morphisms, and 2-morphisms are denoted by \(\mathcal{C}_0\), \(\mathcal{C}_1\), and \(\mathcal{C}_2\), respectively.

\(^{16}\)We are viewing \(\text{Cat}\) as a strict 2-category whose 2-morphisms are natural transformations. By also viewing \(\mathcal{C}^{\ast}\text{-Alg}_{0}^{\text{op}}\) as a 2-category (all of whose 2-morphisms are identities), we can view \(\text{States}\) and \(\text{Rep}\) as 2-functors. Because \(\text{GNS}_f\) is not invertible, which is usually required in the definition of a pseudo-natural transformation, we use the more general notion of semi-pseudo-natural transformation described in Definition 5.6.21 of the Appendix in this Chapter.
Proof. There are two things to check (see Definition 5.6.21). First, the GNS construction associated to an identity morphism \( \text{id}_A \) at a C*-algebra \( A \) gives \( GNS_{\text{id}_A} \) which is precisely the identity natural transformation \( GNS_A \circ \text{States}(\text{id}_A) = GNS_A \Rightarrow GNS_A = \text{Rep}(\text{id}_A) \circ GNS_A \). Second, associated to a pair of composable morphisms

\[
A'' \xrightarrow{f'} A' \xrightarrow{f} A
\]

(5.3.26)

there are two diagrams one obtains. On the one hand, applying the GNS construction to the composition \( f \circ f' \) gives \( GNS_{f \circ f'} \). On the other hand, applying GNS to each \( f' \) and \( f \) and then composing gives another natural transformation. These two results look like

\[
\begin{array}{ccc}
\text{States}(A) & \xrightarrow{GNS_A} & \text{Rep}(A) \\
\text{States}(f \circ f') & \xrightarrow{GNS_{f \circ f'}} & \text{Rep}(f \circ f') \\
\text{States}(A'') & \xrightarrow{GNS_{A''}} & \text{Rep}(A'')
\end{array}
\quad \& \quad
\begin{array}{ccc}
\text{States}(A) & \xrightarrow{GNS_A} & \text{Rep}(A) \\
\text{States}(f) & \xrightarrow{GNS_f} & \text{Rep}(f) \\
\text{States}(A') & \xrightarrow{GNS_{A'}} & \text{Rep}(A')
\end{array}
\quad (5.3.27)
\]

respectively. The second condition that \( GNS \) be a pseudo-natural transformation is that the compositions in these two diagrams are equal. This follows from the commutativity of the individual squares and triangles in the
Physics 5.3.29. Semi-pseudo-naturality means the following if we restrict our attention to a subalgebra and then restrict to yet another subalgebra, as in

\[ A_0 \overset{i}{\hookrightarrow} A_1 \overset{j}{\hookrightarrow} A. \]  

(5.3.30)

Equality of the two diagrams in (5.3.27) means that constructing the physical subspace \( H_{\omega_0 \circ \omega^i} \) of \( H_{\omega} \) of quantum configurations for the state \( \omega \) with respect to the subalgebra \( A_0 \) is the same subspace obtained from first restricting to \( A_1 \) and then to \( A_0 \), i.e.

\[ H_{\omega_0 \circ \omega^i} \xrightarrow{L_i} H_{\omega_0 \circ \omega^j} \xrightarrow{L_j} H_{\omega} \]  

(5.3.31)

commutes, where we have used the notation from Construction 5.3.14.

Remark 5.3.32. In current terminology [Br93], this proves that the GNS construction is not only a functor, but it is also a morphism of pre-stacks.
Note that it is not a morphism of pre-sheaves of categories because the diagram in (5.3.20) does not commute (a condition that is required to have a morphism of pre-sheaves). Instead, a natural transformation (which is a 2-morphism in $\text{Cat}$) is needed, and this is why 2-categories play a crucial role.

### 5.4 Some comments on the category of states

One would like to think of $\text{States}(\mathcal{A})$ as a category of states with non-trivial morphisms. Namely, a morphism from $\omega : \mathcal{A} \to \mathbb{C}$ to $\mu : \mathcal{A} \to \mathbb{C}$ consists of a $C^*$-algebra morphism $\phi : \mathcal{A} \to \mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \\
\downarrow{\omega} & & \downarrow{\mu} \\
\mathbb{C} & & 
\end{array}
$$

commutes, i.e. $\mu \circ \phi = \omega$. Let us call this closely related category $\text{states}(\mathcal{A})$.

While one can define a functor $\text{states}(\mathcal{A}) \to \text{Rep}(\mathcal{A})$ basically as a special case of $\text{GNS}_\mathcal{A}$ on objects and $\text{GNS}_\phi$ on morphisms, this is too restrictive and not what we want in general because we also care about mappings of different algebras. Recall from Physics 5.2.15, that an injective $C^*$-algebra map $\mathcal{A}_0 \to \mathcal{A}$ is supposed to be thought of as using macrostate observables described by $\mathcal{A}_0$ instead of microstate observables described by $\mathcal{A}$. To incorporate this, we would therefore still want to think of the different categories
of states as a pre-sheaf of categories on the category of $C^*$-algebras, i.e. a functor $\text{states}: C^*\text{-Alg}^{\text{op}} \to \text{Cat}$. For a morphism $f: \mathcal{A}' \to \mathcal{A}$ this should get mapped to a functor $\text{states}(f): \text{states}(\mathcal{A}) \to \text{states}(\mathcal{A}')$. How is this functor defined? This agrees with $\text{States}(f)$ at the level of objects. However, for a morphism $\phi: \omega \to \mu$ of states in $\mathcal{A}$, all we have is the collection of morphisms

$$
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{f} & \mathcal{A}' \\
\downarrow{f} & & \downarrow{f} \\
\mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \\
\downarrow{\omega} & & \downarrow{\mu} \\
\mathcal{C} & & \mathcal{C} \\
\end{array}
$$

and from this data we are supposed to produce a map of $C^*$-algebras $\phi': \mathcal{A}' \to \mathcal{A}'$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{A}' & \xrightarrow{\phi'} & \mathcal{A}' \\
\downarrow{\omega \circ f} & & \downarrow{\mu \circ f} \\
\mathcal{C} & & \mathcal{C} \\
\end{array}
$$

commutes. One can show that the only such maps $f: \mathcal{A}' \to \mathcal{A}$ of $C^*$-algebras for which we can do this in a functorial manner are $C^*$-algebra isomorphisms. Since we specifically do not want this for physical reasons, we use the discrete category $\text{States}(\mathcal{A})$ instead of the more reasonable, yet naive, category $\text{states}(\mathcal{A})$. 
5.5 An adjoint to the GNS construction

Besides producing a representation \((\pi_\omega, \mathcal{H}_\omega)\) of \(A\) given a state \(\omega\) on \(A\), the GNS construction also produces a cyclic vector on \(\mathcal{H}_\omega\). This fact will let us construct a sort of inverse to the GNS construction provided that we include this extra datum in the definition of the semi-pseudo-natural transformation GNS.

**Definition 5.5.1.** A *cyclic vector* \(\Omega\) for a representation \(\pi\) of a \(C^*\)-algebra \(A\) on a Hilbert space \(H\) is a vector \(\Omega \in H\) such that

\[
\{ \pi(a)\Omega : a \in A \}
\]

is a dense subset in \(H\) (with respect to the norm induced by the inner product on \(H\)). A representation \((\pi, \mathcal{H})\) of \(A\) together with a cyclic vector \(\Omega\) is called a *cyclic representation* and is written as a triple \((\pi, \mathcal{H}, \Omega)\). A representation \((\pi, \mathcal{H})\) of \(A\) together with a vector (not necessarily cyclic) is called a *pointed representation*.

**Physics 5.5.3.** When \(A\) is the algebra of observables for a quantum field theory, the vacuum vector/state is typically a cyclic vector (any particle content state is obtained by creation operators on the ground state). When a representation is irreducible, every non-zero vector is cyclic (by using annihilation operators, one can get to the ground state).
Definition 5.5.4. Let $\text{Rep}^\bullet(\mathcal{A})$ be the category of pointed representations of $\mathcal{A}$. Namely, an object of $\text{Rep}^\bullet(\mathcal{A})$ consists of a pointed representation $(\pi, \mathcal{H}, \Omega)$ of $\mathcal{A}$. A morphism $(\pi, \mathcal{H}, \Omega) \rightarrow (\pi', \mathcal{H}', \Omega')$ is an intertwiner $L : \mathcal{H} \rightarrow \mathcal{H}'$ of representations such that

$$L(\Omega) = \Omega' \quad \& \quad L^* L = \text{id}_{\mathcal{H}}. \tag{5.5.5}$$

Let $\text{Rep}^\circ(\mathcal{A})$ be the sub-category of $\text{Rep}^\bullet(\mathcal{A})$ of cyclic representations of $\mathcal{A}$.

Proof. Some things must be checked so that the above definition is in fact valid. For instance, let

$$(\pi, \mathcal{H}, \Omega) \xrightarrow{L} (\pi', \mathcal{H}', \Omega') \xrightarrow{L'} (\pi'', \mathcal{H}'', \Omega'') \tag{5.5.6}$$

be a pair of composable morphisms. Then the composition $L'L$ satisfies

$$(L'L)^*(L'L) = L^* L'^* L'L = L^* L = \text{id}_{\mathcal{H}}. \tag{5.5.7}$$

Associativity follows from associativity of composition of functions. The other axioms of a category all hold. $\text{Rep}^\circ(\mathcal{A})$ is a fully faithful subcategory of $\text{Rep}^\bullet(\mathcal{A})$ because a vector being cyclic is a property and not additional structure.

Remark 5.5.8. Condition (5.5.5) says that $L$ is an isometry and is, in particular, injective. We do not require $L$ to be unitary, which would require
$LL^* = \text{id}_\mathcal{H}$ as well. Note that if $L : (\pi, \mathcal{H}, \Omega) \rightarrow (\pi', \mathcal{H}', \Omega')$ is a morphism of cyclic representations, then $L$ sends a dense subset of $\mathcal{H}$ to a dense subset of $\mathcal{H}'$ because $L(\Omega) = \Omega'$. Therefore, in this case, $L$ is almost surjective (it is surjective onto a dense subset). Furthermore, the number of morphisms between any two pointed representations is quite small: there is either one or none at all.

**Construction 5.5.9.** Let $(\pi, \mathcal{H}, \Omega)$ be a pointed representation of a $C^*$-algebra $\mathcal{A}$. The vector $\Omega$ defines a state $\omega_\Omega$ on $\mathcal{B}(\mathcal{H})$ by the formula\(^{17}\)

$$\mathcal{B}(\mathcal{H}) \ni B \mapsto \omega_\Omega(B) := \langle \Omega, B\Omega \rangle.$$  

(5.5.10)

Pulling this state back to $\mathcal{A}$ along $\pi$ defines a state $\omega_\Omega \circ \pi : \mathcal{A} \rightarrow \mathbb{C}$ on $\mathcal{A}$. This state is also more appropriately denoted by $\text{rest}_\mathcal{A}((\pi, \mathcal{H}, \Omega))$ for “restriction.”

**Lemma 5.5.11.** Let $L : (\pi, \mathcal{H}, \Omega) \rightarrow (\pi', \mathcal{H}', \Omega')$ be a morphism of pointed representations of $\mathcal{A}$. Then,

$$\omega_{\Omega'} \circ \pi' = \omega_\Omega \circ \pi,$$

(5.5.12)

i.e. the two states $\text{rest}_\mathcal{A}((\pi, \mathcal{H}, \Omega))$ and $\text{rest}_\mathcal{A}((\pi', \mathcal{H}', \Omega'))$ are equal.

---

\(^{17}\)Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. 
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Proof. For any \( a \in \mathcal{A} \),

\[
\langle \Omega', \pi'(a)\Omega' \rangle = \langle L(\Omega), \pi'(a)L(\Omega) \rangle \\
= \langle L(\Omega), L\pi(a)\Omega \rangle \\
= \langle \Omega, L^*L\pi(a)\Omega \rangle \\
= \langle \Omega, \pi(a)\Omega \rangle,
\]

since \( L \) is an intertwiner and \( L^*L = \text{id} \).

\[\square\]

Physics 5.5.14. We could imagine a context in which we begin with a Hilbert space and a pure vacuum state \( \Omega \). Given a subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \), the construction \( \text{rest}_\mathcal{A} \) restricts our vacuum state to a state on this subalgebra. This is useful if we can only make measurements of certain observables. For instance, a Rindler observer has a restricted algebra of observables so that restricting a Minkowski vacuum state to their algebra results in a thermal state, a phenomenon known as the Unruh effect [Wa94]. If we change our representation in such a way that the two are still related by an intertwiner satisfying (5.5.5), then we get the same state. \( \text{rest}_\mathcal{A} \) is an “obvious” construction from the physics perspective since every normalized vector in \( \mathcal{H} \) gives a state on any \( C^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \). What is not obvious is that there is a canonical way to go back—the purpose of this section is to make this statement precise and prove that the GNS construction achieves this.
Proposition 5.5.15. Let \( A \) be a \( C^* \)-algebra. The assignment
\[
\text{Rep}^*(A)_0 \ni (\pi, \mathcal{H}, \Omega) \mapsto \omega_\Omega \circ \pi \in \text{States}(A)_0
\]
from Construction 5.5.9 defines a functor \( \text{rest}_A : \text{Rep}^*(A) \rightarrow \text{States}(A) \).

\[
\text{Rep}^*(A)_1 \ni \left( (\pi, \mathcal{H}, \Omega) \xrightarrow{L} (\pi', \mathcal{H}', \Omega') \right) \mapsto \text{id}_{\omega_\Omega \circ \pi} \in \text{States}(A)_1
\]

Proof. This follows directly from Construction 5.5.9 and Lemma 5.5.11. \( \blacksquare \)

Construction 5.5.17. Let \( A' \xrightarrow{f} A \) be a morphism of \( C^* \)-algebras. The induced functor \( \text{Rep}(f) : \text{Rep}(A) \rightarrow \text{Rep}(A') \) extends to a functor \( \text{Rep}^*(f) : \text{Rep}^*(A) \rightarrow \text{Rep}^*(A') \) as follows. Let \( (\pi, \mathcal{H}, \Omega) \) be a pointed representation of \( A \). Then this gets sent to \( (\pi \circ f, \mathcal{H}, \Omega) \). Note that even if \( (\pi, \mathcal{H}, \Omega) \) is a cyclic representation, \( (\pi \circ f, \mathcal{H}, \Omega) \) is not necessarily a cyclic representation of \( A' \) since
\[
\{ \pi(f(a')) \Omega : a' \in A' \}
\]
is not necessarily dense in \( \mathcal{H} \). Nevertheless, \( (\pi \circ f, \mathcal{H}, \Omega) \) is a pointed representation. A morphism of pointed representations of \( A \) gets sent to a morphism of pointed representations of \( A' \) under the functor \( \text{Rep}^*(f) \) using the same intertwiner. In fact, the diagram
\[
\begin{array}{ccc}
\text{Rep}^*(A) & \xrightarrow{\text{rest}_A} & \text{States}(A) \\
\text{Rep}^*(f) \downarrow & & \downarrow \text{States}(f) \\
\text{Rep}^*(A') & \xrightarrow{\text{rest}_{A'}} & \text{States}(A')
\end{array}
\]
commutes.
This proves the following fact.

**Proposition 5.5.20.** rest, as defined in Construction 5.5.9, is a natural transformation\(^{18}\)

\[
\begin{array}{ccc}
\text{States} & \xrightarrow{\text{rest}} & \text{Cat} \\
\text{C}^*\text{-Alg}^{\text{op}} & \xrightarrow{\text{rest}} & \text{Rep}^* \\
\end{array}
\]

(5.5.21)

We will now modify the GNS construction to include the construction of a cyclic vector. Due to the similarity of this construction and that of Constructions 5.3.1 and 5.3.14, we will skip many details and only focus on the new ones.

**Construction 5.5.22.** For every \(C^*\)-algebra \(\mathcal{A}\), define a functor \(\text{GNS}^*_\mathcal{A} : \text{States}(\mathcal{A}) \rightarrow \text{Rep}^*(\mathcal{A})\) by the following assignment. To a state \(\omega : \mathcal{A} \rightarrow \mathbb{C}\), assign the cyclic representation\(^{19}\) \(\text{GNS}^*_\mathcal{A}(\omega) := (\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}])\). Because \(\text{States}(\mathcal{A})\) has no non-trivial morphisms, this defines a functor. Furthermore, the image of this functor actually lands in the subcategory \(\text{Rep}^\odot(\mathcal{A})\).

To every morphism \(\mathcal{A} \xrightarrow{f} \mathcal{A}'\) of unital \(C^*\)-algebras, define a natural transfor-

\(^{18}\)This is special case of a pseudo-natural transformation since \(\text{rest}_f = \text{id}\) in (5.5.19).

\(^{19}\)[1_\mathcal{A}] is a cyclic vector because \(\{\pi_\omega(a)[1_\mathcal{A}] = [a] \mid a \in \mathcal{A}\} = \mathcal{A}/\mathcal{N}_\omega\) is dense in \(\mathcal{H}_\omega\) by definition.
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as follows. To every state $\omega : A \to \mathbb{C}$ on $A$ define the morphism

$$
\begin{array}{c}
\text{States}(A) \xrightarrow{\text{GNS}^*_\omega} \text{Rep}^*(A) \\
\text{States}(f) \xrightarrow{\text{GNS}^*_f} \text{Rep}^*(f) \\
\text{States}(A') \xrightarrow{\text{GNS}^*_{A'}} \text{Rep}^*(A')
\end{array}
$$

(5.5.23)

of pointed representations to be exactly the same as $L_f$ as in (5.3.16) and simply note that a property of this linear map is that

$$
L_f([1_{A'}]) = [f(1_{A'})] = [1_A]
$$

(5.5.24)

(5.5.25)

since $f$ is a morphism of \textit{unital} $C^*$-algebras. We need to check that $L_f$ also satisfies $L_f^*L_f = \text{id}_{\mathcal{H}_{\omega,f}}$. This follows from the calculation

$$
\begin{align*}
\langle L_f([a']), L_f([b']) \rangle_{\omega} & \quad \text{by (5.3.16)} \\
\langle [a'], L_f^*L_f([b']) \rangle_{\omega,f} & \quad \text{by (5.3.10)} \\
\langle [a'], [f(a')], [f(b')] \rangle_{\omega} & \quad \text{by (5.3.10)} \\
\langle [a'], [b'] \rangle_{\omega,f} & \quad \omega(f(a')^*f(b')) \\
\omega(f(a^*b')) & \quad \text{by Def'n 5.2.3}
\end{align*}
$$

(5.5.26)
for all \( a', b' \in \mathcal{A}' \). Going clockwise from the top proves that \( L_f \) preserves the inner-product while going counterclockwise from the top is simply the definition of the adjoint \( L_f^* \) of the operator \( L_f \). By Riesz’s theorem (see Theorem A.52 in [Ha13] for instance) and the fact that \( \{ [a'] : a' \in \mathcal{A}' \} \) is a dense subset of \( \mathcal{H}_{\omega f} \), the right-hand-side of the inner product is unique, i.e.

\[
L_f^*L_f([b']) = [b'].
\] (5.5.27)

**Remark 5.5.28.** Note that although \((\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}])\) and \((\pi_{\omega f}, \mathcal{H}_{\omega f}, [1_{\mathcal{A}'}])\) are cyclic representations of \( \mathcal{A} \) and \( \mathcal{A}' \), respectively, the pointed representation \((\pi_\omega \circ f, \mathcal{H}_\omega, [1_\mathcal{A}])\) obtained by restriction by \( f \) is *not* necessarily cyclic. This is why the target of the GNS functor was chosen to be the category of pointed representations instead of cyclic representations. This is analogous to the fact that the restriction of an irreducible representation to a subalgebra need not be irreducible.

**Theorem 5.5.29.** The assignment

\[
\begin{align*}
\mathbb{C}^*-\text{Alg}_0^{\text{op}} & \xrightarrow{\text{GNS}^*} \text{Cat}_1 \\
\mathcal{A} & \mapsto \left( \text{States}(\mathcal{A}) \xrightarrow{\text{GNS}^*_\mathcal{A}} \text{Rep}^*(\mathcal{A}) \right)
\end{align*}
\] (5.5.30)

and

\[
\begin{align*}
\mathbb{C}^*-\text{Alg}_1^{\text{op}} & \xrightarrow{\text{GNS}^*} \text{Cat}_2 \\
\left( \mathcal{A}' \xrightarrow{f} \mathcal{A} \right) & \mapsto \left( \text{GNS}^*_\mathcal{A} \circ \text{States}(f) \xrightarrow{\text{GNS}^*_f} \text{Rep}^*(f) \circ \text{GNS}^*_\mathcal{A} \right)
\end{align*}
\] (5.5.31)
defined in Construction 5.5.22 defines a semi-pseudo-natural transformation

\[
\begin{array}{c}
\text{States} \\
\text{C}^*\text{-Alg}^{\text{op}} \xrightarrow{\text{GNS}^*} \text{Cat} \\
\text{Rep}^* \downarrow \downarrow \text{rest} \\
\text{States}
\end{array}
\]  

(5.5.32)

**Proof.** The proof is not much different than what it was for \text{GNS}. 

There is one last construction we must confront. This involves relating the composition of semi-pseudo-natural transformations \text{rest} and \text{GNS}^* with the identity natural transformation.

**Lemma 5.5.33.** The vertical composition

\[
\begin{array}{c}
\text{States} \\
\text{C}^*\text{-Alg}^{\text{op}} \xrightarrow{\text{GNS}^*} \text{Cat} \\
\text{Rep}^* \downarrow \downarrow \text{rest} \\
\text{States}
\end{array}
\]  

(5.5.34)

of pseudo-natural transformations is equal to the identity natural transformation.

**Proof.** Let \( \mathcal{A} \) be a \( C^* \)-algebra. The composition acting on a state \( \omega : \mathcal{A} \longrightarrow \mathbb{C} \) gives

\[
\begin{array}{ccc}
\text{States}(\mathcal{A}) & \xrightarrow{\text{GNS}^*} & \text{Rep}^*(\mathcal{A}) \\
\text{rest} & \xrightarrow{\text{rest}_\mathcal{A}} & \text{States}(\mathcal{A})
\end{array}
\]

\[
\omega \longmapsto (\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}]) \longmapsto \langle [1_\mathcal{A}], \pi_\omega(\cdot)[1_\mathcal{A}] \rangle_\omega,
\]

which agrees with \( \omega \) because

\[
\langle [1_\mathcal{A}], \pi_\omega(a)[1_\mathcal{A}] \rangle_\omega = \langle [1_\mathcal{A}], [a] \rangle = \omega(1_\mathcal{A}^*a) = \omega(a) \quad \forall \ a \in \mathcal{A}.
\]  

(5.5.36)
There are no non-trivial morphisms in $\text{States}(\mathcal{A})$ so the composition is the identity functor. To every morphism $f : \mathcal{A}' \to \mathcal{A}$ of $C^*$-algebras, the composition of natural transformations

\[
\begin{array}{ccc}
\text{States}(\mathcal{A}) & \xrightarrow{\text{GNS}^*_A} & \text{Rep}^*(\mathcal{A}) & \xrightarrow{\text{rest}_A} & \text{States}(\mathcal{A}) \\
\text{States}(f) & \xrightarrow{\text{GNS}^*_f} & \text{Rep}^*(f) & \xrightarrow{\text{id}=\text{rest}_f} & \text{States}(f) \\
\text{States}(\mathcal{A}') & \xrightarrow{\text{GNS}^*_{A'}} & \text{Rep}^*(\mathcal{A}') & \xrightarrow{\text{rest}_{A'}} & \text{States}(\mathcal{A}')
\end{array}
\] (5.5.37)

must equal the identity natural transformation. This follows from Lemma 5.5.11 and the fact that $\text{States}(\mathcal{A}')$ has no non-trivial morphisms: both of the outer functors give the same state $\omega \circ f : \mathcal{A}' \to \mathbb{C}$ from the state $\omega : \mathcal{A} \to \mathbb{C}$. ■

However, the composition in the order

\[
\begin{array}{ccc}
\text{C}^*\text{-Alg}^{\text{op}} & \xrightarrow{\text{States}} & \text{Cat} \\
\text{GNS}^* & \xrightarrow{\text{Rep}^*} & \text{GNS}^* \\
\text{rest} & \xrightarrow{\text{rest}} & \text{rest}
\end{array}
\] (5.5.38)

is certainly not the identity. In the following, we construct the required modification (see Definition 5.6.30 in the Appendix of this Chapter).

**Construction 5.5.39.** Let $\mathcal{A}$ be a unital $C^*$-algebra and consider the dia-
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The diagram

\[
\begin{array}{c}
\text{States}(\mathcal{A}) \\
\text{rest}_{\mathcal{A}} \\
\text{Rep}^*(\mathcal{A}) \xrightarrow{\text{id}_{\text{Rep}^*(\mathcal{A})}} \text{Rep}^*(\mathcal{A}) \\
\end{array}
\]

\[
\xrightarrow{\text{GNS}_{\mathcal{A}}^*}
\]

of functors. Recalling the notation from Constructions 5.5.9 and 5.3.1, observe what happens to a pointed representation \((\pi, \mathcal{H}, \Omega)\) of \(\mathcal{A}\) along the two functors

\[
(\pi, \mathcal{H}, \Omega) \xrightarrow{\text{rest}_{\mathcal{A}}} \langle \Omega, \pi(\cdot)\Omega \rangle \equiv \omega_{\Omega} \xrightarrow{\text{GNS}_{\mathcal{A}}^*} (\pi_{\omega_{\Omega}}, \mathcal{H}_{\omega_{\Omega}}, [1_{\mathcal{A}}])
\]

Therefore, we have two pointed representations of \(\mathcal{A}\) satisfying

\[
\langle \Omega, \pi(a)\Omega \rangle = \langle [1_{\mathcal{A}}], \pi_{\omega_{\Omega}}(a)[1_{\mathcal{A}}] \rangle_{\omega_{\Omega}} \quad \text{for all } a \in \mathcal{A}.
\]

If \((\pi, \mathcal{H}, \Omega)\) was also a cyclic representation, then it was already known by Segal that any other cyclic representation restricting to the same state is unitarily equivalent to it \([\text{Se}47]\). For reference, we illustrate Segal’s proof for our special case. Define the unitary intertwiner

\[
(\pi_{\omega_{\Omega}}, \mathcal{H}_{\omega_{\Omega}}, [1_{\mathcal{A}}]) \xrightarrow{m_{\mathcal{A}}(\pi, \mathcal{H}, \Omega)} (\pi, \mathcal{H}, \Omega).
\]

\[
[\pi(a)\Omega] \xrightarrow{\pi(\cdot)\Omega} (\pi(a)\Omega).
\]

To see that this is well-defined, let \(a' \in \mathcal{A}\) be another representative of \([a]\). Then for all \(b \in \mathcal{A}\),

\[
\langle \pi(b)\Omega, \pi(a - a')\Omega \rangle = \omega_{\Omega}\left(b^*(a - a')\right) = 0.
\]
Since \( \{ \pi(b)\Omega \mid b \in \mathcal{A} \} \) is dense in \( \mathcal{H} \) (since \( (\pi, \mathcal{H}, \Omega) \) is assumed cyclic for now) and \( \langle \cdot, \cdot \rangle_{\omega\Omega} \) is non-degenerate, \( \pi(a-a')\Omega = 0 \), i.e. \( \pi(a')\Omega = \pi(a)\Omega \), which proves well-definedness. Next, \( m_A((\pi, \mathcal{H}, \Omega)) \) is an intertwiner if the diagram

\[
\begin{array}{ccc}
\mathcal{H}_{\omega\Omega} & \xrightarrow{m_A((\pi, \mathcal{H}, \Omega))} & \mathcal{H} \\
\pi_{\omega\Omega}(a') & \downarrow & \pi(a') \\
\mathcal{H}_{\omega\Omega} & \xrightarrow{m_A((\pi, \mathcal{H}, \Omega))} & \mathcal{H}
\end{array}
\]  
(5.5.45)

commutes for all \( a' \in \mathcal{A} \). Following the image of an element \( [a] \in \mathcal{H}_{\omega\Omega} \) along both of these morphisms gives\(^{20}\)

\[
\begin{array}{c}
m_A(\pi_{\omega\Omega}(a')[a]) \quad \pi(a)m_A([a]) \\
m_A([a'a]) \quad \pi(a')\pi(a)\Omega \\
\pi(a'a)\Omega
\end{array}
\]  
(5.5.46)

proving that the diagram indeed commutes. To see that \( m_A((\pi, \mathcal{H}, \Omega)) \) is unitary, notice that it is isometric from a dense subset of \( \mathcal{H}_{\omega\Omega} \) onto a dense subset of \( \mathcal{H} \) (by Remark 5.5.8) because

\[
\langle m_A([a]), m_A([b]) \rangle = \langle \pi(a)\Omega, \pi(b)\Omega \rangle = \omega\Omega(a^*b) = \langle [a], [b] \rangle_{\omega\Omega} \quad (5.5.47)
\]

\(^{20}\)We have abusively written \( m_A([a]) \) instead of \( m_A((\pi, \mathcal{H}, \Omega))([a]) \) because the notation would be too difficult to read otherwise. Since our representation \( (\pi, \mathcal{H}, \Omega) \) is fixed for now, this should cause no confusion. This same abuse of notation is done in (5.5.47).
for all \([a], [b] \in \mathcal{H}_{\omega \Omega}\). In more detail, since \(m_A\) is bounded (since it is an isometry, its norm is one) on a dense domain, there exists a unique extension to the completion. Since \(m_A\) is an isometry, this extension is also an isometry. Since the image of \(m_A\) is dense, this extension is unitary.

Unfortunately, however, \((\pi, \mathcal{H}, \Omega)\) is in general \textit{not} a cyclic representation but is only a pointed representation of \(\mathcal{A}\). As a result, \(\{\pi(b)\Omega : b \in \mathcal{A}\}\) is not dense in \(\mathcal{H}\) and the above argument fails. Fortunately, there is another (simpler) argument that does not require cyclicity. The map (5.5.43) is still a well-defined intertwiner satisfying (5.5.5) even if \((\pi, \mathcal{H}, \Omega)\) is not cyclic. To see this, let \(a'\) be another representative of \([a]\). Then \(a - a' \in \mathcal{N}_{\omega \Omega}\), which means, by definition, that \(\omega_{\Omega}((a - a')^*(a - a')) = 0\). Meanwhile,

\[
\omega_{\Omega}((a - a')^*(a - a')) = \langle \Omega, \pi(a - a')^* \pi(a - a') \Omega \rangle
\]

\[= \langle \pi(a - a')\Omega, \pi(a - a')\Omega \rangle.\]

Since \(\langle \cdot, \cdot \rangle\) is an inner product, this holds if and only if \(\pi(a - a')\Omega = 0\) and so \(\pi(a)\Omega = \pi(a')\Omega\) and well-definedness of \(m_A((\pi, \mathcal{H}, \Omega))\) in (5.5.43) still holds. The same argument as above proves that \(m_A((\pi, \mathcal{H}, \Omega))\) is an intertwiner of \(\mathcal{A}\)-representations. Although the image of \(m_A((\pi, \mathcal{H}, \Omega))\) is no longer necessarily dense, \(m_A((\pi, \mathcal{H}, \Omega))\) is still an isometry onto its image by the same argument as in (5.5.47). Hence,

\[
m_A((\pi, \mathcal{H}, \Omega))^* m_A((\pi, \mathcal{H}, \Omega)) = \text{id}_{\mathcal{H}_{\omega \Omega}}\] (5.5.49)
proving that $m_A((\pi, \mathcal{H}, \Omega))$ is a morphism in $\text{Rep}'(\mathcal{A})$.

**Physics 5.5.50.** The map $m_A((\pi, \mathcal{H}, \Omega))$ tells us that if we start with an arbitrary representation $(\pi, \mathcal{H})$ of the algebra of observables $\mathcal{A}$ together with a normalized vector state $\Omega \in \mathcal{H}$ (our representation need not be irreducible because our vector state need not be cyclic), then if we forget about our Hilbert space, remember only the algebra of observables and our state, then we might not be able to recover our exact Hilbert space back, but we can get close (in an optimal way that will be discussed in more detail later). The best we can do from the GNS construction is to get a new Hilbert space that embeds into the Hilbert space we started with. Furthermore, in this subspace, the vector state we started with becomes cyclic with respect to the algebra of observables. In other words, we lose some information, namely the vectors orthogonal to this subspace, but we keep many of the essential features of our initial state.

**Lemma 5.5.51.** $m$ from Construction 5.5.39 defines a modification (recall Definition 5.6.30)
via the assignment

\[
\begin{align*}
\text{C}^*-\text{Alg}_0 & \ni \mathcal{A} \mapsto \\
\begin{tikzcd}
\text{States}(\mathcal{A}) & \text{GNS}^*_\mathcal{A} \\
\text{Rep}^*(\mathcal{A}) \ar{r}{\text{id}_{\text{Rep}^*(\mathcal{A})}} & \text{Rep}^*(\mathcal{A}) \ar{u}{m_\mathcal{A}}
\end{tikzcd}
\end{align*}
\tag{5.5.53}
\]

Furthermore, for each C*-algebra \( \mathcal{A} \), when restricted to the subcategory \( \text{Rep}^\mathcal{O}(\mathcal{A}) \), the natural transformation \( m_\mathcal{A} \) is vertically invertible.

**Proof.** In order for \( m \) to be a modification, for every morphism \( f : \mathcal{A}' \longrightarrow \mathcal{A} \) of C*-algebras, the following equality must hold (see equation (5.6.29))

\[
\begin{align*}
\begin{tikzcd}
\text{Rep}^*(\mathcal{A}) & \text{States}(\mathcal{A}) \\
\text{Rep}^*(\mathcal{A}') \ar{u}{\text{id}_{\text{Rep}^*(\mathcal{A})}} & \\
\text{Rep}^*(\mathcal{A}) \ar{r}{\text{id}} & \text{Rep}^*(\mathcal{A}) \ar{u}{} \ar{r}{\text{GNS}^*_\mathcal{A}} & \text{Rep}^*(\mathcal{A}') \ar{u}{}
\end{tikzcd}
\end{align*}
\tag{5.5.54}
\]
i.e. for every object \((\pi, \mathcal{H}, \Omega)\) of \(\text{Rep}^\star(\mathcal{A})\), the diagram

\[
\begin{array}{ccc}
\text{GNS}^\star_{(\omega_\Omega)}(\omega_\Omega) & \xrightarrow{(\pi_\omega \circ f, \mathcal{H}_\omega, [1_A])} & f^*(m_A((\pi, \mathcal{H}, \Omega))) \\
(\pi_{\omega_\Omega \circ f}, \mathcal{H}_{\omega_\Omega \circ f}, [1_{A'}]) & \xleftarrow{m_A'((\pi \circ f, \mathcal{H}, \Omega))} & (\pi \circ f, \mathcal{H}, \Omega)
\end{array}
\]

(5.5.55)

of intertwiners of pointed representations of \(\mathcal{A}'\) must commute. The image of a vector \([a'] \in \mathcal{H}_{\omega_\Omega \circ f}\) under the top two linear maps is \(\pi (f(a')) \Omega\) while the image under the bottom map is \((\pi \circ f)(a') \Omega\) which are equal elements in \(\mathcal{H}\). Because the maps agree on a dense domain, the diagram (5.5.55) commutes.

Finally, when \(m_A\) is restricted to the subcategory \(\text{Rep}^{\ominus}(\mathcal{A})\), it was shown in Construction (5.5.39) that \(m_A\) is unitary and hence \(m_A\) is a vertically invertible natural transformation.

\[\blacksquare\]

Everything we have done up to this point leads to the following theorem encompassing the GNS construction. To state it, we introduce the functor 2-category (see Definition 5.6.43).

**Definition 5.5.56.** Let \(\text{Fun}(\text{C}^\star\text{-Alg}^{\text{op}}, \text{Cat})\) be the 2-category whose objects are functors from \(\text{C}^\star\text{-Alg}^{\text{op}}\) to \(\text{Cat}\), 1-morphisms are semi-pseudo-natural transformations, and 2-morphisms are modifications (see the Appendix of this Chapter for definitions).
Theorem 5.5.57. The semi-pseudo-natural transformation $\text{GNS}^*$ : $\text{States} \Rightarrow \text{Rep}^*$ is left-adjoint to $\text{rest}$. In fact, the quadruple $(\text{GNS}^*, \text{rest}, \text{id}, m)$ is an adjunction in $\text{Fun}(C^*\text{-Alg}^{\text{op}}, \text{Cat})$.

Proof. The only thing left to check are the zig-zag identities from Lemma 5.6.44. For us, $F := \text{States}$, $G := \text{Rep}^*$, $\sigma := \text{GNS}^*$, $\rho := \text{rest}$, $\eta := \text{id}$, and $\epsilon := m$. By Remark 5.6.48, it suffices to prove

\[
\text{Rep}^*(A) \xrightarrow{\text{id}_{\text{Rep}^*(A)}} \text{id}_{\text{Rep}^*(A)} \xrightarrow{\text{id}_{\text{Rep}^*(A)}} \text{Rep}^*(A) \xrightarrow{\epsilon_A} \text{States}(A) \xrightarrow{\text{rest}_A} \text{States}(A) \xrightarrow{\rho_A} \text{Rep}^*(A) \xrightarrow{\eta_A} \text{States}(A) = \text{GNS}^*_A \xrightarrow{\text{GNS}^*_A} \text{Rep}^*(A) \xrightarrow{\epsilon_A} \text{States}(A) \xrightarrow{\text{rest}_A} \text{States}(A) \xrightarrow{\rho_A} \text{Rep}^*(A) \xrightarrow{\eta_A} \text{States}(A)
\]

for each object $A$ of $C^*\text{-Alg}^{\text{op}}$. Fortunately, these identities are essentially tautologous. For (5.5.58), since $\text{States}(A)$ has no non-trivial morphisms,
the equality holds. For (5.5.59), it suffices to check what happens to a state \( \omega \). Under the composition in (5.5.59), \( \omega \) gets sent to

\[
\omega \mapsto (\pi_\omega, \mathcal{H}_\omega, [1_A]) \mapsto \langle [1_A], \pi_\omega(\cdot) [1_A] \rangle = \omega \mapsto (\pi_\omega, \mathcal{H}_\omega, [1_A]) \quad (5.5.60)
\]

which is exactly the same representation as in the second step. Furthermore, 
\( m_A((\pi_\omega, \mathcal{H}_\omega, [1_A])) \) is the identity intertwiner.

\[\textbf{Physics 5.5.61.}\] In particular, by Remark 5.6.48, this theorem says that for every \( C^* \)-algebra \( \mathcal{A} \), the quadruple \((\text{GNS}^*_A, \text{rest}_A, \text{id}, m_A)\) is an adjunction (in the usual sense). This means that for every state \( \omega \in \text{States}(\mathcal{A})_0 \) and pointed representation \((\pi, \mathcal{H}, \Omega) \in \text{Rep}^*(\mathcal{A})_0\), there is a natural bijection of morphisms\(^{21}\)

\[
\text{Rep}^*(\mathcal{A})\left(\text{GNS}^*_A(\omega), (\pi, \mathcal{H}, \Omega)\right) \cong \text{States}(\mathcal{A})\left(\omega, \text{rest}_A(\pi, \mathcal{H}, \Omega)\right), \quad (5.5.62)
\]

which illustrates in what sense the GNS construction \( \text{GNS}^*(\omega) \) is optimal: for every other choice of representation \((\pi, \mathcal{H}, \Omega)\) on which to realize the state \( \omega \) as a vector state, there is always a (unique) isometric intertwiner from the GNS Hilbert space to \( \mathcal{H} \). In particular, the GNS Hilbert space is the \textit{smallest} space on which one can represent states as vector states. If the states do not agree, this result also says that there is \textit{no} such intertwiner (since \( \text{States}(\mathcal{A}) \)).

\(^{21}\)This is how one remembers that \( \text{GNS}^* \) is \textit{left} adjoint to \( \text{rest} \).
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is a discrete category). This can change if we have an isomorphism of our algebra.

A special case of the adjunction \((\text{GNS}_A^*, \text{rest}_A, \text{id}, m_A)\) occurs when restricted to the category of cyclic representations of \(A\). In this case, it is an adjoint equivalence (meaning, an equivalence of categories). In other words, in the cyclic case, the categories \(\text{Rep}^\otimes(A)\) and \(\text{States}(A)\) are equivalent and the restriction functor exhibits this equivalence with a canonical inverse given by the GNS construction. In particular, this reproduces the well-known result [Se47] that there is a one-to-one correspondence between isomorphism classes of cyclic representations of \(A\) and states on \(A\).

However, our general result also extends the adjunction in a functorial manner incorporating \(C^*\)-algebra morphisms (such as restrictions to subalgebras). For instance, recall from Construction 5.3.14 where \(\text{GNS}_f\) was defined given a morphism \(A' \xrightarrow{f} A\) of \(C^*\)-algebras. If we start with a state \(\omega : A \longrightarrow \mathbb{C}\) on a \(C^*\)-algebra, we have an optimal way to obtain a Hilbert space representation \(\text{GNS}_A^*(\omega) = (\pi_\omega, \mathcal{H}_\omega, [1_A])\) with a realization of our state as a vector state. If we restrict this state to \(A'\), then we get a representation \((\pi \circ f, \mathcal{H}_\omega, [1_A])\) in which the vector might no longer be cyclic.

On the one hand, we can look at the subspace \(\{[f(a')] : a' \in A'\} \subset \mathcal{H}_\omega\) in which the vector \([1_A]\) is cyclic. On the other hand, we can apply the
GNS construction to the restricted state $\omega \circ f$ to get a new representation $\text{GNS}^\ast_{\mathcal{A}'}(\omega \circ f) = (\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}, [1_{\mathcal{A}'}, \mathcal{A}'])$ in which the state is also realized as a cyclic vector. By earlier observations, there exists a unique map $\mathcal{H}_{\omega \circ f} \longrightarrow \mathcal{H}_\omega$ sending $[1_{\mathcal{A}'}]$ to $[1_{\mathcal{A}}]$ defined by $[a'] \mapsto [f(a')]$ because it is densely defined.

What is not immediately obvious is that this map is an isometry (see Physics 5.3.21 for a proof). Therefore, the two spaces are canonically isomorphic.

Our results can be summarized by saying that we can now provide a definition instead of a construction that produces, in a functorial manner, cyclic representations from states on $C^\ast$-algebras.

**Definition 5.5.63.** The GNS construction is a left-adjoint to rest.

### 5.6 Examples

The authors of [BGdQRL13] include several examples, and we will go through the simplest one to illustrate the meaning of our constructions and theorems.

**Example 5.6.1.** Let $\mathcal{A} = \mathcal{B}(\mathbb{C}^2)$, $2 \times 2$ matrices with complex coefficients. This is the algebra of observables for a spin $\frac{1}{2}$ system, i.e. a qubit. Label an orthonormal basis by $\{\uparrow, \downarrow\}$—this basis refers to the spin of a particle along a particular axis. Let $\mathcal{A}$ act on $\mathbb{C}^2$ by the identity representation, meaning that the representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathbb{C}^2)$ is just the identity map.
Let \( \omega : B(\mathbb{C}^2) \to \mathbb{C} \) be the state corresponding to a pure state with spin up, i.e., \( \omega(a) = \langle \uparrow | a \uparrow \rangle \) for all \( a \in B(H) \). Applying the restriction functor \( \text{rest}_A \) to the pointed representation \( (\pi, \mathbb{C}^2, | \uparrow \rangle) \) gives \( \omega_\uparrow \). Next, apply the GNS construction \( \text{GNS}^* \) to the state \( \omega_\uparrow \). As a vector space, \( B(\mathbb{C}^2) \) is a four-dimensional vector space, with a basis given by

\[
e_{\uparrow \uparrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_{\uparrow \downarrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_{\downarrow \uparrow} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_{\downarrow \downarrow} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\] (5.6.2)

The expectation values for these operators are given by

\[
\omega_\uparrow(e_{\uparrow \uparrow}) = 1, \quad \omega_\uparrow(e_{\uparrow \downarrow}) = 0, \quad \omega_\uparrow(e_{\downarrow \uparrow}) = 0, \quad \omega_\uparrow(e_{\downarrow \downarrow}) = 0.
\] (5.6.3)

Notice that \( e_{\uparrow \downarrow}^\dagger e_{\downarrow \uparrow} = e_{\downarrow \downarrow} \) and \( e_{\downarrow \uparrow}^\dagger e_{\uparrow \downarrow} = e_{\uparrow \uparrow} \) so that \( \omega_\uparrow(e_{\uparrow \downarrow}^\dagger e_{\downarrow \uparrow}) = 0 \) and \( \omega_\uparrow(e_{\downarrow \uparrow}^\dagger e_{\uparrow \downarrow}) = 0 \). In fact,

\[
\mathcal{N}_{\omega_\uparrow} = \text{span}(e_{\uparrow \downarrow}, e_{\downarrow \uparrow}).
\] (5.6.4)

Then, \( H_{\omega_\uparrow} = B(\mathbb{C}^2)/\mathcal{N}_{\omega_\uparrow} \) consists of equivalence classes of matrices

\[
a = \begin{pmatrix} a_{\uparrow \uparrow} & a_{\uparrow \downarrow} \\ a_{\downarrow \uparrow} & a_{\downarrow \downarrow} \end{pmatrix}
\] (5.6.5)

where \( a_{ij} \in \mathbb{C} \) with \( i, j \in \{ \uparrow, \downarrow \} \) and \( a \sim b \) if and only if

\[
b - a = \begin{pmatrix} 0 & b_{\uparrow \downarrow} - a_{\uparrow \downarrow} \\ 0 & b_{\downarrow \uparrow} - a_{\downarrow \uparrow} \end{pmatrix}.
\] (5.6.6)

---

\textsuperscript{22} We are using Dirac bra-ket notation for the examples.

\textsuperscript{23} To avoid confusion with the physics literature, for the purposes of this section, we will use \( ^\dagger \) to denote the adjoint instead of \( * \).
The associated cyclic representation from the GNS construction applied to this state is \((\pi_{\omega_1}, \mathcal{H}_{\omega_1}, [1_A])\), where \(1_A\) is the \(2 \times 2\) identity matrix and \(\pi_{\omega_1}(a)([b]) = [ab]\) is obtained from ordinary matrix multiplication. The intertwiner \(m_A\) from (5.5.43) applied to our representation \((\pi, \mathbb{C}^2, |\uparrow\rangle)\) is the map \([a] \mapsto a|\uparrow\rangle\). Since our representation was cyclic to begin with, this intertwiner is unitary. This map compares our original Hilbert space representation to the one obtained from the GNS construction in a canonical way.

**Example 5.6.7.** Let \(\mathcal{A} = \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)\) and let \(\omega\) be the state corresponding to the pure state

\[
|\Psi\rangle := \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
\]

(5.6.8)

where \(|\uparrow\downarrow\rangle\) is short for \(|\uparrow\rangle \otimes |\downarrow\rangle\). Using the same notation as in the previous example, this means

\[
(a \otimes b) \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) = a|\uparrow\rangle \otimes b|\downarrow\rangle - a|\downarrow\rangle \otimes b|\uparrow\rangle
\]

\[
= \left( a_{\uparrow\uparrow}|\uparrow\rangle + a_{\downarrow\uparrow}|\downarrow\rangle \right) \otimes \left( b_{\uparrow\uparrow}|\uparrow\rangle + b_{\downarrow\downarrow}|\downarrow\rangle \right)
\]

\[
- \left( a_{\uparrow\downarrow}|\uparrow\rangle + a_{\downarrow\downarrow}|\downarrow\rangle \right) \otimes \left( b_{\uparrow\uparrow}|\uparrow\rangle + b_{\downarrow\downarrow}|\downarrow\rangle \right)
\]

(5.6.9)

\[
= (a_{\uparrow\downarrow}b_{\uparrow\downarrow} - a_{\downarrow\downarrow}b_{\uparrow\uparrow})|\uparrow\uparrow\rangle + (a_{\uparrow\downarrow}b_{\downarrow\downarrow} - a_{\downarrow\downarrow}b_{\uparrow\uparrow})|\uparrow\downarrow\rangle
\]

\[
+ (a_{\downarrow\uparrow}b_{\uparrow\uparrow} - a_{\downarrow\downarrow}b_{\downarrow\downarrow})|\downarrow\uparrow\rangle + (a_{\downarrow\uparrow}b_{\downarrow\downarrow} - a_{\downarrow\downarrow}b_{\downarrow\uparrow})|\downarrow\downarrow\rangle.
\]
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Hence,

\[
\omega(a \otimes b) = \frac{1}{2} \left( \langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) (a \otimes b) \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
\]

\[
= \frac{1}{2} \left( a_{\uparrow\downarrow} b_{\downarrow\uparrow} - a_{\downarrow\uparrow} b_{\uparrow\downarrow} + a_{\downarrow\uparrow} b_{\uparrow\downarrow} - a_{\uparrow\downarrow} b_{\downarrow\uparrow} \right). 
\]

(5.6.10)

It is not important for us to calculate \( H_\omega \) explicitly. All that is important is that there is a unitary intertwiner

\[
(\pi_\omega, H_\omega, [1, A]) \overset{m_{B(C^2 \otimes C^2)}}{\longrightarrow} (\pi, C^2 \otimes C^2, |\Psi\rangle)
\]

(5.6.11)

extended linearly. Now, let \( i_1 : B(C^2) \longrightarrow B(C^2 \otimes C^2) \) be the map defined by

\[
i_1(a) := a \otimes 1. \quad (5.6.12)
\]

Physically, such a map corresponds to an observer \( O_1 \) only being able to make measurements on the observables \( B(C^2) \) corresponding to a single particle. It is convenient to denote the first \( C^2 \) by \( H_1 \) and the second by \( H_2 \). This situation occurs, for instance, in an EPR-like experiment, where a particle decomposes into two with a corresponding state given by (5.6.8). The two particles fly off in opposite directions and observers far away are waiting to measure the spin.

\[
O_1 \leftarrow \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \rightarrow O_2
\]

Observer \( O_1 \) cannot measure the observables \( B(H_2) \) and vice versa. There-
fore, the state that $O_1$ sees is given by the restriction

$$\omega_1 := \omega \circ i_1 : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \longrightarrow \mathbb{C}. \quad (5.6.13)$$

This state corresponds to the density matrix

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad (5.6.14)$$

using our basis \{\ket{\uparrow}, \ket{\downarrow}\}. What is the GNS construction applied to such a state and how is it related to the original Hilbert space on which $\Psi$ is defined?

Let $a \in \mathcal{B}(\mathcal{H}_1)$. Then

$$(a^\dagger a)_{ik} = \sum_{j \in \{\uparrow, \downarrow\}} (a^\dagger)_{ij} a_{jk} = \sum_{j \in \{\uparrow, \downarrow\}} \bar{a}_{ji} a_{jk} \quad (5.6.15)$$

implies

$$\omega_1(a^\dagger a) = \text{tr}(\rho_1 a^\dagger a) = \frac{1}{2} \left( \langle \uparrow | a^\dagger a | \uparrow \rangle + \langle \downarrow | a^\dagger a | \downarrow \rangle \right)$$

$$= \frac{1}{2} \sum_{j \in \{\uparrow, \downarrow\}} \left( |a_{ji}|^2 + |a_{j\dagger}|^2 \right) = \frac{1}{2} \sum_{j, k \in \{\uparrow, \downarrow\}} |a_{jk}|^2 \quad (5.6.16)$$

so that $\omega_1(a^\dagger a) = 0$ if and only if $a = 0$. Therefore, $N_{\omega_1} = 0$ and hence $\mathcal{H}_{\omega_1} = \mathcal{B}(\mathcal{H}_1)$ as a Hilbert space. Furthermore, the associated GNS representation $\pi_{\omega_1}$ acts as

$$\pi_{\omega_1}(a)b = ab = \sum_{i,j,k \in \{\uparrow, \downarrow\}} a_{ij} b_{jk} e_{ik} \quad (5.6.17)$$

The induced map $L_{i_1} : \mathcal{H}_{\omega_1} \longrightarrow \mathcal{H}_{\omega}$ corresponding to (5.3.15) is given by

$$\mathcal{B}(\mathcal{H}_1) \equiv \mathcal{H}_{\omega_1} \xrightarrow{L_{i_1} = \text{GNS}^*_{i_1} (\omega)} \mathcal{H}_{\omega} \quad (5.6.18)$$

$${a} \longrightarrow [a \otimes 1]$$
Using this with the intertwiner \( m_{B(H_1 \otimes H_2)} \) from (5.6.11), gives a canonical intertwiner of \( B(H_1) \)-representations to our original Hilbert space

\[
\begin{array}{ccc}
\mathcal{H}_{\omega_1} & \xrightarrow{\text{GNS}^*_{\omega}} & \mathcal{H}_\omega \\
\downarrow a & & \downarrow [a \otimes 1] & \downarrow (a \otimes 1)|\Psi\rangle.
\end{array}
\]

(5.6.19)

This canonical map can also be thought of as the top arrow in the diagram (5.5.55). This exhibits our Hilbert space \( \mathcal{H}_{\omega_1} \), which was the Hilbert space from the GNS construction associated to the EPR density matrix for observer \( O_1 \), as a subspace of our original Hilbert space \( H_1 \otimes H_2 \) for the entangled EPR state \( |\Psi\rangle \). In fact, the map (5.6.19) is a unitary map. This is because

\[
(a \otimes 1)|\Psi\rangle = -a_{\uparrow\downarrow}|\uparrow\uparrow\rangle + a_{\uparrow\downarrow}|\uparrow\downarrow\rangle - a_{\downarrow\uparrow}|\downarrow\uparrow\rangle + a_{\downarrow\uparrow}|\downarrow\downarrow\rangle.
\]

(5.6.20)

illustrating that the map is also surjective.

**Appendix: Semi-pseudo-natural transformations, modifications, and 2-categorical adjunctions**

In this paper, we use semi-pseudo-natural transformations, which are different from the pseudo-natural transformations that appear in the literature [Bé67] and in Appendix A of this thesis. Fortunately, the difference is minor. For completeness, we include this definition along with the notion of modifications.
Definition 5.6.21. Let $\mathcal{C}$ and $\mathcal{D}$ be two (strict)$^{24}$ 2-categories and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two 2-functors. A **semi-pseudo-natural transformation** $\rho$ from $F$ to $G$, written as $\rho : F \Rightarrow G$, consists of the following data:

i) a function $\rho : C_0 \rightarrow D_1$ assigning a 1-morphism to an object $x$ in the following manner

$$
\begin{array}{ccc}
  x & \xrightarrow{\rho} & F(x) \\
  & \downarrow^{\rho(x)} & \\
  & & G(x)
\end{array}
$$

(5.6.22)

ii) and a function $\rho : C_1 \rightarrow D_2$ assigning a 2-morphism$^{25}$ to every 1-morphism $y \xleftarrow{\alpha} x$ in the following manner

$$
\begin{array}{ccc}
  y & \xleftarrow{\alpha} & x & \xrightarrow{\rho} & F(y) \xrightarrow{F(\alpha)} F(x) \\
  & \downarrow^{\rho(y)} & \downarrow^{\rho(\alpha)} & \downarrow^{\rho(x)} & \\
  & & G(y) \xleftarrow{G(\alpha)} G(x)
\end{array}
$$

(5.6.23)

These data must satisfy the following conditions.

(a) For every object $x$ in $\mathcal{C}$,

$$
\rho(\text{id}_x) = \text{id}_{\rho(x)}. 
$$

(5.6.24)

---

$^{24}$A definition exists for weak 2-categories and weak 2-functors but such a definition is not needed here.

$^{25}$In definitions in the literature, one often requires this 2-morphism to be vertically invertible, motivated by the fact that equations should replace isomorphisms upon categorification [BaDo95]. However, we see no good reason to force ourselves to this requirement if examples exist where no such isomorphism is available. Absolutely nothing else in the definition changes.
(b) For every pair $(z \xrightarrow{\alpha} y, y \xleftarrow{\beta} x)$ of composable 1-morphisms in $C$, the diagram

$$
\begin{array}{c}
\rho(z) \circ F(\alpha) \circ F(\beta) \\
\downarrow \text{id} \\
\rho(z) \circ F(\alpha \beta)
\end{array}
\quad
\begin{array}{c}
\xrightarrow{\rho(\alpha) \circ \text{id}_F(\beta)} \\
\downarrow \text{id}_{G(\alpha) \circ \rho(\beta)} \\
G(\alpha \beta) \circ \rho(x)
\end{array}
\quad
\begin{array}{c}
\xleftarrow{\id} \\
\downarrow \\
\id \circ G(\alpha) \circ G(\beta) \circ \rho(x)
\end{array}
$$

\[(5.6.25)\]

commutes, i.e.

$$
\begin{array}{c}
F(z) \xleftarrow{\rho(z)} F(y) \xleftarrow{\rho(\alpha)} F(x) \\
G(z) \xleftarrow{\rho(z)} G(y) \xleftarrow{\rho(\alpha)} G(x)
\end{array}
\quad
\begin{array}{c}
F(z) \xleftarrow{\rho(z)} F(y) \xleftarrow{\rho(\alpha) \circ \text{id}_F(\beta)} F(x) \\
G(z) \xleftarrow{\rho(z)} G(y) \xleftarrow{\rho(\alpha) \circ \text{id}_F(\beta)} G(x)
\end{array}
$$

\[(5.6.26)\]

(c) For every 2-morphism

$$
\begin{array}{c}
\xymatrix{ & \Sigma \ar[dl]_{\alpha} \ar[dr]^{\beta} & \\
y & & x, \\
& \gamma \ar[ul]_{\gamma} \ar[ur]_{\gamma} &}
\end{array}
$$

\[(5.6.27)\]

the diagram

$$
\begin{array}{c}
G(\alpha) \circ \rho(x) \xleftarrow{\rho(\alpha)} \rho(y) \circ F(\alpha) \\
\downarrow \text{id}_{G(\Sigma) \circ \rho(x)} \quad \quad \downarrow \text{id}_{\rho(\gamma) \circ F(\Sigma)} \\
G(\gamma) \circ \rho(x) \xleftarrow{\rho(\gamma)} \rho(y) \circ F(\gamma)
\end{array}
$$

\[(5.6.28)\]
commutes, i.e.

\[
\begin{array}{ccc}
F(y) & \xrightarrow{F(\alpha)} & F(x) \\
\rho(y) \downarrow & & \downarrow \rho(x) \\
G(y) & \xrightarrow{\rho(\gamma)} & G(x)
\end{array}
\]

\[
\begin{array}{ccc}
F(y) & \xrightarrow{F(\alpha)} & F(x) \\
\rho(y) \downarrow & & \downarrow \rho(x) \\
G(y) & \xrightarrow{G(\alpha)} & G(x)
\end{array}
\]

\[
\begin{array}{ccc}
F\left(\frac{x}{y}\right) & \xrightarrow{F(\alpha)} & F\left(\frac{x}{q}\right) \\
\rho\left(\frac{y}{x}\right) \downarrow & & \downarrow \rho(x) \\
G\left(\frac{x}{y}\right) & \xrightarrow{G(\alpha)} & G\left(\frac{x}{q}\right)
\end{array}
\]

(5.6.29)

The definition of a modification does not change if one uses semi-pseudo-natural transformations instead of pseudo-natural transformations.

**Definition 5.6.30.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two 2-categories, \( F, G : \mathcal{C} \to \mathcal{D} \) be two 2-functors, and \( \rho, \sigma : F \Rightarrow G \) be two semi-pseudo-natural transformations.

A **modification** \( m \) from \( \sigma \) to \( \rho \), written as \( m : \sigma \Rightarrow \rho \) and drawn as

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{m} & \mathcal{C} \\
\rho & \downarrow & \sigma \\
\mathcal{H} & \xleftarrow{m} & \mathcal{C}
\end{array}
\]

(5.6.31)

consists of a function \( m : C_0 \to D_2 \) assigning a 2-morphism to an object \( x \) in the following manner

\[
x \xrightarrow{m} \begin{array}{c}
F(x) \\
\rho(x) \xrightarrow{m(x)} \sigma(x)
\end{array}
\]

(5.6.32)

This assignment must satisfy the condition that for every 1-morphism \( y \xleftarrow{\alpha} x \),
the diagram
\[
\begin{align*}
G(\alpha) \circ \sigma(x) & \xleftarrow{\sigma(\alpha)} \sigma(y) \circ F(\alpha) \\
\text{id}_{G(\alpha)m(x)} & \cong m(y)\text{id}_{F(\alpha)} \\
G(\alpha) \circ \rho(x) & \xleftarrow{\rho(\alpha)} \rho(y) \circ F(\alpha)
\end{align*}
\]
(5.6.33)
commutes, i.e.
\[
\begin{align*}
F(y) & \xleftarrow{F(\alpha)} F(x) \\
\sigma(y) & \xleftarrow{\rho(y)} \rho(x) \\
G(y) & \xleftarrow{G(\alpha)} G(x)
\end{align*}
\]
(5.6.34)
Compositions of semi-pseudo-natural transformations and modifications are not changed as a result of these alterations to the usual definitions and therefore we refer the reader to Appendix A in this thesis for the definitions of compositions.

**Definition 5.6.35.** Let \( \mathcal{C} \) be a (strict) 2-category. An **adjunction** in \( \mathcal{C} \) consists of a pair of objects \( x, y \) in \( \mathcal{C} \), a pair of morphisms
\[
x \xrightarrow{f} y \xleftarrow{g}
\]
(5.6.36)
and a pair of 2-morphisms
\[
\begin{align*}
x & \xrightarrow{id_x} x \\
\eta & \xrightarrow{\epsilon} \\
f & \xrightarrow{g}
\end{align*}
\]
\[
\begin{align*}
x & \xrightarrow{f} \\
g & \xrightarrow{id_y}
\end{align*}
\]
(5.6.37)
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satisfying

\[
\begin{align*}
\begin{array}{c}
\text{id}_x \\
\eta \\
\text{id}_y \\
\end{array}
\end{align*}
\xymatrix{
\ar [rr]^f & & y \\
\ar [rr] & & x \\
\ar [rr] & & y
}
\begin{align*}
\begin{array}{c}
f \\
\epsilon \\
g \\
\end{array}
\end{align*}
\xymatrix{
x \\
\ar [rr]^g & & x \\
\ar [rr] & & x \\
\ar [rr] & & y \\
\ar @/_/[rrr]_{\text{id}} & & y
}
\] \tag{5.6.38}

and

\[
\begin{align*}
\begin{array}{c}
\text{id}_x \\
\eta \\
\text{id}_y \\
\end{array}
\end{align*}
\xymatrix{
\ar [rr] & & \ar [rr]_x & & y \\
\ar [rr] & & \ar [rr]_y & & x \\
\ar [rr] & & \ar [rr] & & g
}
\begin{align*}
\begin{array}{c}
g \\
\epsilon \\
f \\
\end{array}
\end{align*}
\xymatrix{
y & & x \\
\ar [rr]^g & & y \\
\ar [rr] & & x \\
\ar @/_/[rrr]_{\text{id}} & & x
}
\] \tag{5.6.39}

Conditions (5.6.38) and (5.6.39) are known as the zig-zag identities. An adjunction as above is typically written as a quadruple \((f, g, \eta, \epsilon)\) and we say \(f\) is left-adjoint to \(g\).

The reason adjunctions are important is because they satisfy a certain universal property.

**Lemma 5.6.40.** Let \(\mathcal{C}\) be a (strict) 2-category and let \(x\) and \(y\) be two objects in \(\mathcal{C}\). Let \(x \xrightarrow{g} y\) be a 1-morphism and let \((f, g, \eta, \epsilon)\) and \((f', g, \eta', \epsilon')\) be adjunctions in which which \(f\) and \(f'\) are both left-adjoint to \(g\). Then there exists a vertically invertible 2-morphism \(\sigma : f \Rightarrow f'\) such that

\[
\begin{align*}
\begin{array}{c}
f' \\
\eta' \\
g \\
\end{array}
\end{align*}
\xymatrix{
x & & y \\
\ar [rr] & & x \\
\ar [rr] & & y \\
\ar @/_/[rrr]_{\text{id}} & & x
}
\begin{align*}
\begin{array}{c}
f \\
\eta \\
g \\
\end{array}
\end{align*}
\xymatrix{
x & & y \\
\ar [rr] & & x \\
\ar [rr] & & y \\
\ar @/_/[rrr]_{\text{id}} & & x
}
\] \tag{5.6.41}
and

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow g & \sigma & \downarrow f \\
  y & \xrightarrow{id_y} & y
\end{array}
\]

In this paper, we focus on an adjunction in a particular 2-category obtained from functors between 2-categories.

**Definition 5.6.43.** Let $\mathcal{C}$ and $\mathcal{D}$ be two (strict) 2-categories. Let $\text{Fun}(\mathcal{C}, \mathcal{D})$ be the 2-category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, 1-morphisms are semi-pseudo-natural transformations, and 2-morphisms are modifications.

We spell out what it means to have an adjunction in this 2-category explicitly.

**Lemma 5.6.44.** Let $\mathcal{C}$ and $\mathcal{D}$ be two (strict) 2-categories and let $\text{Fun}(\mathcal{C}, \mathcal{D})$ be the functor 2-category described in Definition (5.6.43). An adjunction in $\text{Fun}(\mathcal{C}, \mathcal{D})$ consists of two (strict) 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, two semi-pseudo-natural transformations $\sigma : F \Rightarrow G$ and $\rho : G \Rightarrow F$, and two modifications $\eta : \text{id}_F \Rightarrow \sigma \rho$ and $\epsilon : \rho \sigma \Rightarrow \text{id}_G$ such that the diagrams

\[
\begin{array}{ccc}
  \begin{array}{ccc}
    \rho & \xrightarrow{id_{id_{\rho}}} & \rho \\
    \downarrow \text{id}_F & \sigma & \downarrow \text{id}_F
  \end{array}
  \quad & \& 
  \begin{array}{ccc}
    \sigma & \xrightarrow{id_{\sigma}} & \sigma \\
    \downarrow \text{id}_\sigma & \epsilon & \downarrow \text{id}_G
  \end{array}
\end{array}
\]

(5.6.45)
both commute, i.e.

\[
\begin{array}{c}
G \\
\rho \\
F & \Rightarrow & \id_G \\
\sigma \\
\rho \\
\id_F & \Rightarrow & G \\
\eta \\
\epsilon \\
\end{array}
\]

\[
= \rho \begin{array}{c}
G \\
\id \rho \\
F \\
\rho \\
\end{array}
\] \tag{5.6.46}

and

\[
\begin{array}{c}
F \\
\sigma \\
\rho \\
\id_F & \Rightarrow & G \\
\eta \\
\epsilon \\
\end{array}
\]

\[
= \sigma \begin{array}{c}
F \\
\id \sigma \\
G \\
\sigma \\
\end{array}
\] \tag{5.6.47}

respectively.

**Remark 5.6.48.** Because the zig-zag identities only involve the equality of modifications, and since the datum of a modification consists only of an
assignment of 2-morphisms in \( D \) to objects of \( C \), they can be re-expressed as

\[
\begin{align*}
G(x) &\xleftarrow{\rho(x)} F(x) \xrightarrow{\epsilon(x)} G(x) \\
\Downarrow^{\sigma(x)} &\Downarrow^{\rho(x)} &\Downarrow^{\epsilon(x)} \\
G(x) &\xleftarrow{\rho(x)} F(x)
\end{align*}
\]

(5.6.49)

and

\[
\begin{align*}
F(x) &\xleftarrow{\sigma(x)} G(x) \xrightarrow{\epsilon(x)} F(x) \\
\Downarrow^{{\eta(x)}} &\Downarrow^{\rho(x)} &\Downarrow^{{\epsilon(x)}} \\
F(x) &\xleftarrow{\sigma(x)} G(x)
\end{align*}
\]

(5.6.50)

for each object \( x \) of \( C \). In other words, for every object \( x \) in \( C \), the quadruple \((\sigma(x), \rho(x), \eta(x), \epsilon(x))\) is an adjunction, i.e. \( \sigma(x) \) is left-adjoint to \( \rho(x) \).

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Appendix A

Compositions in 2-category theory

We define 2-categories, functors, pseudo-natural transformations, modifications, and all of their compositions. We also define equivalences and the many levels of inverses for all compositions. In particular, we spell out what an equivalence between 2-categories is and what a pseudo-natural equivalence between 2-functors is. We prove many (though not all) statements in order to be somewhat self-contained. The following is a table of some basic definitions along with their locations.
To set some notation, the pullback of two morphisms $f : X \to Z$ and $g : Y \to Z$ will be written as

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_Y} & Y \\
\downarrow{\pi_X} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}
$$

(A.1)

**Definition A.2.** A *(small)* 2-category $C$ consists of the following data:

i) a set $C_0$, elements of which are called objects,

ii) a set $C_1$, elements of which are called 1-morphisms,

iii) a set $C_2$, elements of which are called 2-morphisms,
iv) functions

\[ C_2 \xrightarrow{s} C_1 \xrightarrow{s} C_0, \quad (A.3) \]

where \( s, t, \) and \( i \) stand for \textit{source}, \textit{target}, and \textit{identity-assignment}, respectively.

v) functions

\[ C_1, s \times t C_1 \longrightarrow C_1, \quad C_2, s \times t C_2 \longrightarrow C_2, \quad C_2, \circ \times t C_2 \longrightarrow C_2 \quad (A.4) \]

called (ordinary) \textit{composition} of 1-morphisms, \textit{vertical composition} of 2-morphisms, and \textit{horizontal composition} of 2-morphisms, respectively, and drawn as\(^1\)

\[ \begin{align*}
  z \xrightarrow{\alpha} y \xrightarrow{\beta} x & \quad \mapsto \quad z \xrightarrow{\alpha \circ \beta} x, \quad (A.5) \\
  y \xrightarrow{\gamma} x & \quad \mapsto \quad y \xrightarrow{\gamma} x, \quad (A.6) \\
  z \xrightarrow{\alpha} y \xrightarrow{\beta} x & \quad \mapsto \quad z \xrightarrow{\alpha \circ \beta} x \quad (A.7)
\end{align*} \]

respectively.

\(^1\)These drawings place restrictions on the above mentioned functions, which are automatically imposed by the pullbacks in (A.4).
vi) for every triple \((\alpha, \beta, \gamma)\) of composable 1-morphisms, a 2-morphism
\[
\begin{align*}
(\alpha \circ \beta) \circ \gamma & \qquad \downarrow a_{\alpha, \beta, \gamma} \\
\alpha \circ (\beta \circ \gamma) & \quad \downarrow a_{\alpha, \beta, \gamma}
\end{align*}
\] (A.8)
called the **associator**.

vii) and finally, for every morphism \(y \leftarrow x\), two 2-morphisms
\[
\begin{align*}
\alpha \circ \text{id}_x & \quad \downarrow \beta_\alpha \\
\text{id}_y \circ \alpha & \quad \downarrow \beta_\alpha
\end{align*}
\] (A.9)
called the **left** and **right unifiers**, respectively. Here we write \(\text{id}_x\) instead of \(i(x)\).

These data must satisfy the following conditions.

(a) The functions \(s, t, \) and \(i\) have to satisfy the following equalities
\[
s \circ i = \text{id}_{C_0} = t \circ i, \quad s \circ i = \text{id}_{C_1} = t \circ i, \quad \text{(A.10)}
\]
\[
s \circ s = s \circ t, \quad \& \quad t \circ s = t \circ t. \quad \text{(A.11)}
\]

(b) Vertical composition is associative and the identity-assigning map gives units with respect to this composition. The latter are drawn as
\[
\begin{align*}
y & \xleftarrow{\gamma} x \quad \overset{\text{id}_y \circ \Sigma}{\downarrow} \quad \overset{\Sigma}{\downarrow} \quad y \xleftarrow{\gamma} x \quad \overset{\Sigma \circ \text{id}_x}{\downarrow} \quad \overset{\Sigma}{\downarrow} \quad y \xleftarrow{\gamma} x. \quad \text{(A.12)}
\end{align*}
\]
(c) For every quadruple \((\alpha, \beta, \gamma, \delta)\) of composable 1-morphisms, the diagram

\[
\begin{array}{c}
\alpha \circ (\beta \circ (\gamma \circ \delta)) \\
\alpha \circ ((\beta \circ \gamma) \circ \delta)
\end{array}
\]

\[(\alpha \circ \beta) \circ (\gamma \circ \delta)
\]

\[(\alpha \circ (\beta \circ \gamma)) \circ \delta
\]

commutes. This is called the pentagon axiom.

(d) For every pair \((z \xleftarrow{\alpha} y, y \xleftarrow{\beta} x)\) of composable 1-morphisms, the diagram

\[
\begin{array}{c}
\alpha \circ (\id_y \circ \beta) \\
\alpha \circ \beta
\end{array}
\]

\[
(\alpha \circ \id_y \circ \beta)
\]

\[
(\alpha \circ \beta)
\]

\[(\alpha \circ \beta) \circ (\gamma \circ \delta)
\]

\[(\alpha \circ (\beta \circ \gamma) \circ \delta)
\]

commutes. Furthermore,

\[
\id_{\alpha} \circ \id_{\beta} = \id_{\alpha \circ \beta}.
\]

(e) For every triple

\[(\alpha' \circ \beta') \circ \gamma'
\]

\[
(\alpha' \circ \beta') \circ (\Sigma \circ \Omega) \circ \Gamma
\]

\[
(\Sigma \circ \Omega) \circ \Gamma
\]

of horizontally composable 2-morphisms, the diagram

\[
\begin{array}{c}
(\alpha' \circ \beta') \circ \gamma' \\
(\alpha \circ \beta) \circ \gamma
\end{array}
\]

\[
\begin{array}{c}
(\Sigma \circ \Omega) \circ \Gamma \\
(\Sigma \circ \Omega) \circ \Gamma
\end{array}
\]

\[
\alpha' \circ (\beta' \circ \gamma')
\]

\[
\alpha \circ (\beta \circ \gamma)
\]

\[
(\alpha \circ \beta) \circ (\gamma \circ \delta)
\]

\[(\alpha \circ (\beta \circ \gamma) \circ \delta)
\]

\[(\alpha \circ \beta \circ \gamma) \circ \delta
\]
(f) For every quadruple

\[
\begin{align*}
\begin{array}{c}
\text{commutes.}
\end{array}
\end{align*}
\]

of 2-morphisms composable in the fashion indicated above,

\[
\begin{align*}
(\Sigma \circ \Omega) & = \left( \begin{array}{c} \Sigma \\ \circ \end{array} \right) \circ \left( \begin{array}{c} \Omega \\ \circ \end{array} \right), \\
(\Sigma' \circ \Omega') & = \left( \begin{array}{c} \Sigma' \\ \circ \end{array} \right) \circ \left( \begin{array}{c} \Omega' \\ \circ \end{array} \right).
\end{align*}
\]

This is called the \textit{interchange law}. Because of this, it is common to simply write this composition unambiguously as

\[
\begin{align*}
\begin{array}{c}
\Sigma \Omega \\
\Sigma' \Omega'
\end{array}
\end{align*}
\]

(g) All associators and unifiers are vertically invertible 2-morphisms in the following sense. A 2-morphism

\[
\begin{align*}
\begin{array}{c}
\text{is said to be} \emph{vertically invertible} \text{ if there exists a 2-morphism}
\end{array}
\end{align*}
\]
such that

\[
\begin{align*}
\Sigma & = \Sigma' \\
\Sigma' & = \id_\alpha \& \id_\beta = \Sigma
\end{align*}
\]  

(A.23)

We shall write \( \Sigma^{-1}_v \) for the vertical inverse of \( \Sigma \).

(h) For every 2-morphism

\[
\begin{tikzcd}
& y \ar[ld, \alpha] \ar[rr, \Sigma] & & x \ar[ld, \beta] \\
\Sigma & & & & \Sigma
\end{tikzcd}
\]

(A.24)

the 2-morphisms \( \id_{\id_x} \) and \( \id_{\id_y} \) act as right and left identities, respectively, using the left and right unifiers, namely

\[
\begin{align*}
\Sigma \circ \id_{\id_x} & = \Sigma \\
\id_{\id_y} \circ \Sigma & = \Sigma
\end{align*}
\]  

(A.25)

or equivalently, the diagrams

\[
\begin{tikzcd}
\Sigma \ar[rr, \Sigma \circ \id_{\id_x}] & & \Sigma \ar[rr, \id_{\id_y} \circ \Sigma] \\
\id_x \ar[rr, \id_x] \\
\beta \ar[rr, \id_x] & & \beta \circ \id_x
\end{tikzcd}
\]

(A.26)

commute.

**Definition A.27.** Let \( \mathcal{C} \) be a 2-category and

\[
\begin{tikzcd}
y \ar[rr, \alpha] & & x
\end{tikzcd}
\]

(A.28)

a 1-morphism in \( \mathcal{C} \). An \underline{inverse} to \( \alpha \) consists of a 1-morphism

\[
\begin{tikzcd}
x \ar[rr, \pi] & & y
\end{tikzcd}
\]

(A.29)
and 2-morphisms
\[ i_\alpha : \text{id}_y \Rightarrow \alpha \circ \overline{\alpha} \quad \& \quad e_\alpha : \overline{\alpha} \circ \alpha \Rightarrow \text{id}_x \quad \text{(A.30)} \]
satisfying
\[ \begin{array}{c}
\alpha \circ (\overline{\alpha} \circ \alpha) \\
\downarrow \text{id}_y \circ e_\alpha
\end{array} \xRightarrow{\text{eq.}} \begin{array}{c}
(\alpha \circ \overline{\alpha}) \circ \alpha \\
\downarrow \alpha \circ \text{id}_x
\end{array} \xRightarrow{i_\alpha \circ \text{id}_{\alpha}} \text{id}_y \circ \alpha \quad \text{(A.31)} \]
and
\[ \begin{array}{c}
(\overline{\alpha} \circ \alpha) \circ \overline{\alpha} \\
\downarrow e_\alpha \circ \text{id}_{\alpha}
\end{array} \xRightarrow{\text{eq.}} \begin{array}{c}
\overline{\alpha} \circ (\alpha \circ \overline{\alpha}) \\
\downarrow \overline{\alpha} \circ \text{id}_x
\end{array} \xRightarrow{\text{id}_{\overline{\alpha}} \circ i_{\alpha}} \overline{\alpha} \circ \text{id}_y \quad \text{(A.32)} \]

These last equalities are known as the \textit{zig-zag identities} (this is explained in more detail in [BaLa04]). \( \alpha \) together with its weak inverse will be written as a quadruple \((\alpha, \overline{\alpha}, i_\alpha, e_\alpha)\).

**Definition A.33.** Let \( \mathcal{C} \) be a 2-category and

\[ \begin{array}{c}
y \\
\downarrow \gamma
\end{array} \xrightarrow{\alpha} \begin{array}{c}
\Sigma \\
\downarrow \text{id}_x
\end{array} \xrightarrow{\alpha} \begin{array}{c}
x \\
\downarrow \text{id}_y
\end{array} \quad \text{(A.34)} \]

a 2-morphism in \( \mathcal{C} \). A \textit{horizontal inverse} of \( \Sigma \) consists of inverses \((\alpha, \overline{\alpha}, i_\alpha, e_\alpha)\)
and \((\gamma, \overline{\gamma}, i_\gamma, e_\gamma)\) and a 2-morphism

\[
\begin{array}{c}
\xymatrix{
\gamma \ar@{=>}[r]^{\pi} & y \\
\Sigma^{-1} & x \\
\gamma \ar@{=>}[u]^\tau & 
}
\end{array}
\tag{A.35}
\]

such that the diagrams

\[
\begin{array}{c}
\xymatrix{
\gamma \circ \overline{\gamma} \ar@{=>}[r]^{\Sigma \Sigma^{-1}} & \alpha \circ \overline{\alpha} \\
i_\gamma \ar@{=>}[u]^i_y & 
\ar@{=>}[d]_{i_x} & 
\ar@{=>}[l]^{-\alpha} & 
\ar@{=>}[d]^{\overline{\alpha}} & 
\ar@{=>}[u]_{\overline{\alpha}} & 
\ar@{=>}[l]^e_x
}
\end{array}
\tag{A.36}
\]

commute.

**Definition A.37.** A **strict 2-category** is a 2-category whose associators and unifiers are all identity 2-morphisms.

**Definition A.38.** Let \(C\) be a strict 2-category and

\[
\begin{array}{c}
\xymatrix{
y \ar@{=>}[r]^{\alpha} & x
}
\end{array}
\tag{A.39}
\]

a 1-morphism in \(C\). A **strict inverse** to \(\alpha\) is a weak inverse \((\alpha, \overline{\alpha}, i_\alpha, e_\alpha)\), where\(^2\)

\[
i_\alpha = \text{id}_{id_y} \quad \& \quad e_\alpha = \text{id}_{id_x}
\tag{A.40}
\]

Similarly, a **strict inverse** to a 2-morphism

\[
\begin{array}{c}
\xymatrix{
y \ar@{=>}[r]^{\alpha} & x \\
\Sigma \ar@{-}[u] & \\
\gamma \ar@{=>}[u]_{\tau}
}
\end{array}
\tag{A.41}
\]

\(^2\)These equalities only make sense because \(C\) is strict.
in $C$ consists of an inverse, which consists of strict inverses to $\alpha$ and $\gamma$ and a 2-morphism $\Sigma^{-1}$ as in Definition A.33 satisfying

$$
\Sigma^{-1} \circ \Sigma = \text{id}_{id_y} \quad \& \quad \Sigma \circ \Sigma^{-1} = \text{id}_{id_y}.
$$

(A.42)

**Definition A.43.** A *strict 2-groupoid* is a strict 2-category where all 1-morphisms and 2-morphisms are strictly invertible.

**Definition A.44.** Let $C$ and $D$ be two 2-categories. A *2-functor* $F$ from $C$ to $D$, written as $F : C \rightarrow D$, consists of

i) functions

$$
F_i : C_i \rightarrow D_i
$$

(A.45)

for $i = 0, 1, 2$, that assign objects, 1-morphisms, and 2-morphisms in the following manner

$$
\begin{array}{ccc}
F_0(y) & \overset{F_1(\alpha)}{\rightarrow} & F_0(x) \\
\downarrow_{F_1(\beta)} & & \downarrow_{F_1(\beta)} \\
F_0(x) & \overset{F_1(\beta)}{\rightarrow} & F_0(y)
\end{array}
$$

(A.46)

ii) for every pair $(\alpha, \beta)$ of 1-morphisms in $C$, a 2-morphism

$$
\begin{array}{ccc}
F_1(\alpha) & \circ & F_1(\beta) \\
\downarrow_{\alpha, \beta} & & \downarrow_{\alpha, \beta} \\
F_1(\alpha \circ \beta)
\end{array}
$$

(A.47)

called the *compositor*,


iii) and for every object $x$ in $C$, a 2-morphism

$$F_1(id_x) \xrightarrow{u_x} id_{F_0(x)}$$

(A.48)

called the unitor.

These data must satisfy the following conditions.$^3$

(a) For every triple $(\alpha, \beta, \gamma)$ of composable 1-morphisms in $C$, the diagram

$$F\left((\alpha \circ \beta) \circ \gamma\right) \xleftarrow{c_{\alpha,\beta,\gamma}} F(\alpha \circ \beta) \circ F(\gamma) \xrightarrow{c_{\alpha,\beta,\gamma} \circ id_{F(\gamma)}} \left(F(\alpha) \circ F(\beta)\right) \circ F(\gamma)$$

$$\xrightarrow{a_{F(\alpha), F(\beta), F(\gamma)}} F\left(\alpha \circ (\beta \circ \gamma)\right) \xleftarrow{c_{\alpha,\beta,\gamma}} F(\alpha) \circ (\beta \circ \gamma) \xrightarrow{id_{F(\alpha)} \circ c_{\beta,\gamma}} \left(F(\alpha) \circ (\beta \circ F(\gamma))\right)$$

(A.49)

commutes.

(b) For every 1-morphism $y \xleftarrow{\alpha} x$ in $C$, the diagrams

$$\xrightarrow{F(l_{\alpha})} F(\alpha) \xleftarrow{id_{F(y)} \circ F(\alpha)} F(id_y) \circ F(\alpha) \xrightarrow{c_{id_y,\alpha}}$$

$$\xrightarrow{F(r_{\alpha})} F(\alpha) \xleftarrow{id_{F(y)} \circ F(\alpha)} F(id_y \circ \alpha)$$

(A.50)

and

$$\xrightarrow{F(r_{\alpha})} F(\alpha) \xleftarrow{id_{F(x)} \circ id_{F(\alpha)}} F(\alpha) \circ id_x \xrightarrow{c_{\alpha, id_x}}$$

$$\xrightarrow{F(l_{\alpha})} F(\alpha) \xleftarrow{id_{F(x)} \circ id_{F(\alpha)}} F(\alpha \circ id_x)$$

(A.51)

both commute.

$^3$Just as in ordinary category theory we now write $F$ instead of $F_0, F_1,$ or $F_2$ since it will be clear from the context which one is used depending on the input.
(c) For every pair \((\Sigma, \Omega)\) of vertically composable 2-morphisms in \(\mathcal{C}\)

\[
F\left(\frac{\Sigma}{\Omega}\right) = \frac{F(\Sigma)}{F(\Omega)}. \tag{A.52}
\]

(d) For every 1-morphism \(\alpha\) in \(\mathcal{C}\),

\[
F(\text{id}_\alpha) = \text{id}_{F(\alpha)}. \tag{A.53}
\]

(e) For every pair

\[
\begin{array}{ccc}
\Sigma & \xleftarrow{\alpha} & y \\
\downarrow & & \downarrow \\
\gamma & \xleftarrow{\beta} & \Omega \\
\downarrow & & \downarrow \\
\delta & \xleftarrow{\delta} & x
\end{array}
\]

of horizontally composable 2-morphisms in \(\mathcal{C}\), the diagram

\[
\begin{array}{ccc}
F(\gamma) \circ F(\delta) & \xleftarrow{F(\Sigma) \circ F(\Omega)} & F(\alpha) \circ F(\beta) \\
\downarrow_{c_{\gamma,\delta}} & & \downarrow_{c_{\alpha,\beta}} \\
F(\gamma \circ \delta) & \xleftarrow{F(\Sigma \circ \Omega)} & F(\alpha \circ \beta)
\end{array}
\]

commutes.

**Definition A.56.** Let \(\mathcal{C}, \mathcal{D},\) and \(\mathcal{E}\) be 2-categories and let \(F : \mathcal{D} \to \mathcal{E}\) and \(G : \mathcal{C} \to \mathcal{D}\) be two 2-functors. The **composition** of \(F\) and \(G\), written as \(F \circ G : \mathcal{C} \to \mathcal{E}\), is the 2-functor defined as follows.

i) The functions \((F \circ G)_i : C_i \to E_i\) are defined to be

\[
(F \circ G)_i := F_i \circ G_i. \tag{A.57}
\]
ii) For every pair \((\alpha, \beta)\) of composable 1-morphisms in \(C\), the compositor 

\[ c^{F \circ G}_{\alpha, \beta} \] 

is defined to be the vertical composite of the 2-morphisms

\[
\begin{array}{c}
(F \circ G)(\alpha) \circ (F \circ G)(\beta) \\
\Downarrow c^{E}_{G(\alpha), G(\beta)}
\end{array}
\]

\[
F\left(G(\alpha) \circ G(\beta)\right)
\]

\[
\Downarrow F(c^{G}_{\alpha, \beta})
\]

\[
(F \circ G)(\alpha \circ \beta)
\]

where superscripts are used to distinguish the compositors for the two 2-functors.

iii) For every object \(x\) in \(C\), the unitor \(u^{F \circ G}_{x}\) is defined to be the vertical composite of the 2-morphisms

\[
\begin{array}{c}
(F \circ G)(\text{id}_x) \\
\Downarrow F(u^G_x)
\end{array}
\]

\[
F(\text{id}_{G(\beta)})
\]

\[
\Downarrow F(u^G_{\beta})
\]

\[
\text{id}_{(F \circ G)(x)}
\]

It is not immediately clear from this definition that the data defines a 2-functor \(F \circ G : C \rightarrow \mathcal{E}\). A proof is therefore included to check the necessary axioms.

**Proof.** The properties are checked one at a time.

(a) Let \((\alpha, \beta, \gamma)\) be a triple of composable 1-morphisms in \(C\). The outer part
of the following diagram must commute.\(^4\)

\[
\begin{array}{cccc}
(FG)\left( (\alpha\beta) \right) & (FG)\left( \gamma \right) \\
\downarrow \alpha & \downarrow \beta \\
F(G(\alpha)G(\beta)) & F(G(\alpha)G(\gamma)) \\
\downarrow \gamma & \downarrow \gamma \\
(FG)\left( (\alpha\beta) \gamma \right) & F\left( (G(\alpha)G(\beta))G(\gamma) \right) & \left( (FG)(\alpha)(FG)(\beta) \right) & (FG)(\gamma) \\
\downarrow \gamma & \downarrow \gamma & \downarrow \gamma & \downarrow \gamma \\
F\left( G(\alpha)(G(\beta)G(\gamma)) \right) & F\left( (G(\beta)G(\gamma)) \right) & \left( (FG)(\beta)G(\gamma) \right) & (FG)(\alpha) \\
\downarrow \gamma & \downarrow \gamma & \downarrow \gamma & \downarrow \gamma \\
\left( (FG)(\alpha) \right) & \left( (FG)(\beta) \gamma \right) & \left( (FG)(\alpha)(FG)(\beta) \right) & (FG)(\gamma) \\
\end{array}
\]

(A.60)

The right hexagon commutes by condition (a) for the 2-functor \(F\) applied to the three 1-morphisms \(G(\alpha), G(\beta), \) and \(G(\gamma)\). The left hexagon commutes by condition (c) for the 2-functor \(F\), associativity of vertical composition, and by condition (a) for the 2-functor \(G\) applied to the three 1-morphisms \(\alpha, \beta, \) and \(\gamma\). The top square commutes because \(\text{id}_{(FG)\gamma} = F(\text{id}_G)\) by condition (d) for the 2-functor \(F\) and by condi-

\(^4\)We have temporarily removed the composition notation \(\circ\) and will continue to do so when we feel it is convenient.
tion (e) applied to the pair \((c^G_{\alpha,\beta}, \text{id}_{G(\gamma)})\). The bottom square commutes by condition (d) again and condition (e) applied to the pair \((\text{id}_{G(\alpha)}, c^G_{\beta,\gamma})\). Therefore, the outer part of the diagram commutes.

(b) Let \(y \xrightarrow{\alpha} x\) be a 1-morphism in \(\mathcal{C}\). The outer part of the following diagram must commute.

\[
\begin{array}{cccccc}
\text{id}_{(FG)(y)}(FG)(\alpha) & \overset{u_{FG}(y\cdot\text{id}_{G(y)})}{\xRightarrow{\sim}} & (F(\text{id}_{G(y)}))(\alpha) & \overset{F(u^G_{\text{id}_{G(y)}})}{\xRightarrow{\sim}} & (FG)(\text{id}_{y})(FG)(\alpha) \\
\downarrow{\text{id}_{(FG)(\alpha)}} & & \downarrow{c^F_{\text{id}_{G(y)}\cdot G(\alpha)}} & & \downarrow{c^F_{G(\text{id}_{y})\cdot G(\alpha)}} \\
(FG)(\alpha) & \overset{F(\text{id}_{G(y)})(\alpha)}{\xRightarrow{\sim}} & F(\text{id}_{G(y)\cdot G(\alpha)}) & \overset{F(u^G_{\text{id}_{G(\alpha)}})}{\xRightarrow{\sim}} & F(G(\text{id}_{y})\cdot G(\alpha)) \\
\downarrow{\text{id}_{(FG)(\alpha)}} & & \downarrow{F(l_{G(\alpha)})} & & \downarrow{F(c^G_{\text{id}_{y}\cdot \alpha})} \\
(FG)(\alpha) & \overset{(FG)(l_{\alpha})}{\xRightarrow{\sim}} & (FG)(\text{id}_{y\alpha}) & \overset{(FG)(\alpha)}{\xRightarrow{\sim}} & (FG)(\text{id}_{y\alpha})
\end{array}
\]

(A.61)

The top right corner commutes by condition (e) for the 2-functor \(F\) applied to the pair \((u^G_{y}, \text{id}_{G(\alpha)})\) of horizontally composable 2-morphisms.

The left corner commutes by condition (b) for the 2-functor \(F\) applied to the 1-morphism \(G(y) \xleftarrow{G(\alpha)} G(x)\). The bottom corner commutes by condition (c) for the 2-functor \(F\), associativity of vertical composition, and by condition (b) for the 2-functor \(G\) applied to the 1-morphism \(y \xleftarrow{\alpha} x\). Therefore, the outer part of the diagram commutes.

A similar argument shows that the other required diagram also commutes.
(c) Let \((\Sigma, \Omega)\) be a pair of vertically composable 2-morphisms in \(C\). Then

\[
(FG) \left( \frac{\Sigma}{\Omega} \right) = F \left( \frac{G(\Sigma)}{G(\Omega)} \right) = \frac{F(G(\Sigma))}{F(G(\Omega))} = \frac{(FG)(\Sigma)}{(FG)(\Omega)}; \tag{A.62}
\]

(d) Let \(\alpha\) be a 1-morphism in \(C\). Then

\[
(FG)(\text{id}_\alpha) = F\left( G(\text{id}_\alpha) \right) = F(\text{id}_{G(\alpha)}) = \text{id}_{F(G(\alpha))} = \text{id}_{(FG)(\alpha)}. \tag{A.63}
\]

(e) Let

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\Sigma \\
\downarrow \\
y \\
\beta \\
\downarrow \\
\Omega \\
\downarrow \\
x
\end{array}
\]

be a pair of horizontally composable 2-morphisms. The outer part of the following diagram must commute.

\[
F(\gamma) F(\delta) = F(\gamma) G(\delta) = (FG)(\gamma\delta) \tag{A.65}
\]

The top square commutes by condition (e) for the 2-functor \(F\) applied to the pair \((G(\Sigma), G(\Omega))\). The bottom square commutes by condition (c) for the 2-functor \(F\) and by condition (e) for the 2-functor \(G\) applied to the pair \((\Sigma, \Omega)\). Therefore, the outer part of the diagram commutes.
This verifies all the axioms of a 2-functor.

At this point, a natural question to ask is whether the composition of 2-functors is associative. It is also not immediately obvious whether or not the composition with the identity 2-functor does not change the 2-functor that it is composed with. However, in order to properly discuss this, pseudo-natural transformations and pseudo-natural equivalences must be introduced.

**Definition A.66.** Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two 2-functors. A *pseudo-natural transformation* $\rho$ from $F$ to $G$, written as $\rho : F \Rightarrow G$, consists of the following data:

i) a function $\rho : C_0 \rightarrow D_1$ assigning a 1-morphism to an object $x$ in the following manner

$$
\begin{array}{ccc}
x & \xrightarrow{\rho} & F(x) \\
\downarrow \rho(x) & & \downarrow \rho(x) \\
G(x) & & 
\end{array}
$$

(A.67)

ii) and a function $\rho : C_1 \rightarrow D_2$ assigning a vertically invertible 2-morphism to every 1-morphism $y \xrightarrow{\alpha} x$ in the following manner

$$
\begin{array}{ccc}
y & \xleftarrow{\alpha} & x \\
\xrightarrow{\rho} & & \xrightarrow{\rho} \\
y & \xrightarrow{\rho(y)} & G(y) \\
\xrightarrow{\rho(x)} & & \xrightarrow{\rho(x)} \\
F(y) & \xleftarrow{F(\alpha)} & F(x) \\
\downarrow \rho(y) & & \downarrow \rho(y) \\
G(y) & \xleftarrow{G(\alpha)} & G(x)
\end{array}
$$

(A.68)

These data must satisfy the following conditions.
(a) For every pair \((z \xleftarrow{\alpha} y, y \xrightarrow{\beta} x)\) of composable 1-morphisms in \(\mathcal{C}\), the diagram

\[
\begin{array}{c}
\rho(z)\left(F(\alpha)F(\beta)\right) \\
\downarrow \rho(\alpha\beta) \\
G(\alpha\beta)\rho(x)
\end{array}
\begin{array}{c}
\rightarrow \rho(\gamma)\rho(y)F(\alpha) \\
\downarrow \rho(\gamma) \rho(y)F(\gamma) \\
G(\gamma)\rho(x)
\end{array}
\]

commutes.

(b) For every 2-morphism

\[
\begin{array}{c}
y \\
\downarrow \Sigma \\
x
\end{array}
\begin{array}{c}
\alpha \\
\gamma
\end{array}
\]

the diagram

\[
\begin{array}{c}
G(\alpha)\rho(x) \\
\downarrow G(\Sigma)\rho(x) \\
G(\gamma)\rho(x)
\end{array}
\begin{array}{c}
\rho(\alpha) \\
\rho(y)F(\alpha) \\
\rho(y)F(\gamma)
\end{array}
\]

commutes.

**Remark A.72.** There was no condition on \(\rho\) in the previous definition for the identity 1-morphism \(x \xleftarrow{\text{id}_x} x\) for an object \(x\) of \(\mathcal{C}\). This condition would
require that the diagram
\[
\begin{align*}
\rho(x) \text{id}_{F(x)} & \quad \text{id}_{\rho(x) \cdot u^F(x)} \\
\rho(x) & \quad \rho(x) \text{id}_x \\
\rho(x) & \quad \rho(x) \text{id}_x \\
\rho(x) & \quad \rho(x) \text{id}_x \\
\rho(x) & \quad \rho(x) \text{id}_x
\end{align*}
\]
commute. This, however, follows from the axioms. We leave this verification to the enthusiast (see Lemma A.7. in [ScWa] for a proof).

**Definition A.74.** Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and let $F, G, H : \mathcal{C} \to \mathcal{D}$ be three 2-functors and let $\rho : F \Rightarrow G$ and $\sigma : G \Rightarrow H$ be two pseudo-natural transformations. The *vertical composition* of $\rho$ with $\sigma$, written as $\rho \circ \sigma : F \Rightarrow H$ is defined as follows.

i) To every object $x$ in $\mathcal{C}$, assign
\[
\begin{array}{ccc}
F(x) & \xrightarrow{\rho} & G(x) \\
\downarrow{\rho(x)} & & \downarrow{\sigma(x)} \\
H(x) & \xleftarrow{\sigma(x)} & G(x)
\end{array}
\]

the composition $\sigma(x) \rho(x)$.

ii) To every 1-morphism $y \xleftarrow{\alpha} x$ in $\mathcal{C}$, assign the 2-morphism
\[
\left( \sigma(y) \rho(y) \right) F(\alpha) \xRightarrow{\rho(x)} H(\alpha) \left( \sigma(x) \rho(x) \right)
\]

(A.76)
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defined by the vertical composition of the following 2-morphisms in \( \mathcal{D} \)

\[
\begin{align*}
\sigma(y)\rho(y) & \xrightarrow{\alpha_{\sigma(y),\rho(y),F(\alpha)}} \sigma(y)(\rho(y)F(\alpha)) \xrightarrow{\text{id}_{\sigma(y)\rho(\alpha)}} \sigma(y)(G(\alpha)\rho(x)) \\
H(\alpha)(\sigma(x)\rho(x)) & \xleftarrow{\alpha_{H(\alpha),\sigma(x),\rho(x)}} (H(\alpha)\sigma(x))\rho(x) \xleftarrow{\sigma(\alpha)\text{id}_{\rho(x)}} \sigma(y)G(\alpha)\rho(x)
\end{align*}
\]

(A.77)

Again, it is not obvious that this definition of vertical composition of pseudo-natural transformations results in another pseudo-natural transformation. We leave it to the reader to check that conditions (a) and (b) from Definition A.66 hold. Instead, we move on to discussing the horizontal composition of pseudo-natural transformations.

**Definition A.78.** Consider a collection of 2-categories, 2-functors, and pseudo-natural transformations fitting into a diagram of the form

\[
\begin{array}{ccc}
\mathcal{E} & \xleftarrow{F} & \mathcal{D} \\
\downarrow{H} & & \downarrow{\sigma} \\
\mathcal{C} & \xleftarrow{\rho} & \mathcal{J}
\end{array}
\]

(A.79)

The *horizontal composition* of \( \rho \) with \( \sigma \), written as \( \rho\sigma : FG \Rightarrow HJ \) is defined as follows.

i) To every object \( x \) in \( \mathcal{C} \), assign the composition

\[
\begin{align*}
x & \xrightarrow{\rho\sigma} F(J(x)) \xrightarrow{F(\sigma(x))} F(G(x)) \\
& \xleftarrow{\rho(J(x))} H(J(x))
\end{align*}
\]

(A.80)
ii) To every 1-morphism \( y \xleftarrow{\alpha} x \) in \( C \), assign the 2-morphism
\[
\left( (\rho\sigma)(y) \right) \left( (FG)(\alpha) \right) \xrightarrow{(\rho\sigma)(\alpha)} \left( (HJ)(\alpha) \right) \left( (\rho\sigma)(x) \right)
\]
(A.81)
defined by the vertical composition of the following 2-morphisms in \( D \)
\[
\begin{array}{c}
\rho(J(y)) \left( F(\sigma(y))(FG)(\alpha) \right) \\
\downarrow \text{id}_{\rho(J(y))} \cdot e_{\sigma(J(y)),G(\alpha)}
\end{array}
\xRightarrow{\alpha_{\rho(J(y)),F(\sigma(y)),(FG)(\alpha)}}
\begin{array}{c}
\rho(J(y))F(\sigma(y)G(\alpha)) \\
\downarrow \text{id}_{\rho(J(y))} \cdot F(\sigma(\alpha))
\end{array}
\xRightarrow{\alpha_{\rho(J(y)),(FJ)(\alpha),F(\sigma(x))}}
\begin{array}{c}
\rho(J(y))F(J(\alpha)\sigma(x)) \\
\downarrow \rho(J(\alpha)) \cdot \text{id}_{F(\sigma(x))}
\end{array}
\xRightarrow{\alpha_{(HJ)(\alpha),\rho(J(x)),F(\sigma(x))}}
\begin{array}{c}
(HJ)(\alpha) \left( \rho(J(x))F(\sigma(x)) \right) \\
\downarrow \rho(G(x))
\end{array}
\]
(A.82)

**Remark A.83.** There are actually two natural choices for the composition of pseudo-natural transformations. The other one involves assigning to every object \( x \) of \( C \)
\[
x \xrightarrow{\rho\sigma} \quad \quad \quad F(G(x)) \xrightarrow{\rho(G(x))} \quad \quad \quad H(J(x)) \xrightarrow{H(\sigma(x))} \quad \quad \quad H(G(x))
\]
(A.84)
and a similar adjustment for the assignment on morphisms. By the existence of the vertical isomorphism \( \sigma(\rho(x)) : \rho(J(x))F(\sigma(x)) \Rightarrow H(\sigma(x))\rho(G(x)) \), these two results are isomorphic. We will not discuss in more detail the relationship between the two and we will stick with the first definition.
As before, one should check that the definition above indeed defines a pseudo-natural transformation. Again, we skip the proof.

**Definition A.85.** Let $C$ and $D$ be two 2-categories, $F, G : C \rightarrow D$ be two 2-functors, and $\rho, \sigma : F \Rightarrow G$ be two pseudo-natural transformations. A modification $A$ from $\sigma$ to $\rho$, written as $A : \sigma \Rightarrow \rho$ and drawn as

$$
\begin{array}{ccc}
F & \cong & \sigma \\
\downarrow & & \downarrow \\
\rho & \cong & \sigma \\
\end{array}
$$

(A.86)

consists of a function $A : C_0 \rightarrow D_2$ assigning a 2-morphism to an object $x$ in the following manner

$$
\begin{array}{c}
x \overset{A}{\longrightarrow} \\
F(x) \overset{\sigma(x)}{\leftarrow} \sigma(x) \overset{\rho(x)}{\leftarrow} G(x)
\end{array}
$$

(A.87)

This assignment must satisfy the condition that for every 1-morphism $y \overset{\alpha}{\leftarrow} x$, the diagram

$$
\begin{array}{ccc}
G(\alpha)\sigma(x) & \overset{\sigma(\alpha)}{\leftarrow} & \sigma(y)F(\alpha) \\
\downarrow & & \downarrow \\
\overset{id_{G(\alpha)}A(x)}{G(\alpha)\rho(x)} & \overset{A(y)id_{F(\alpha)}}{\leftarrow} & \rho(y)F(\alpha) \\
\end{array}
$$

(A.88)

commutes.

Modifications have three types of compositions.
**Definition A.89.** Consider 2-categories, 2-functors, pseudo-natural transformations, and modifications as in the following diagram

\[ \begin{array}{ccc}
\mathcal{D} & \xleftarrow{\sigma} & \mathcal{C} \\
\lambda & \xleftarrow{\rho} & \mathcal{A} \\
\mathcal{G} & \xleftarrow{\sigma} & \mathcal{F}
\end{array} \]

The *internal composition* of \( \mathcal{A} \) and \( \mathcal{B} \), written as \( \mathcal{B} \bullet \mathcal{A} : \sigma \Rightarrow \lambda \), is the modification defined by the assignment

\[ x \xrightarrow{\mathcal{B} \bullet \mathcal{A}} x, \quad \text{i.e.} \quad (\mathcal{B} \bullet \mathcal{A})(x) := \frac{\mathcal{A}(x)}{\mathcal{B}(x)}. \quad (A.91) \]

The internal composition of two modifications is indeed a modification.

**Proof.** Let \( y \xleftarrow{\alpha} x \) be a 1-morphism in \( \mathcal{C} \). The outer part of the following diagram should commute.

\[ \begin{array}{ccc}
G(\alpha)\sigma(x) & \xleftarrow{\sigma(y)} & \sigma(y)F(\alpha) \\
\frac{\text{id}_{G(\alpha)}A(x)}{A(x)A(\alpha)} & \xleftarrow{\lambda(\alpha)} & \lambda(\alpha)F(\alpha) \\
\frac{\text{id}_{G(\alpha)}B(x)}{B(\alpha)B(\alpha)} & \xleftarrow{\lambda(\alpha)} & \lambda(\alpha)F(\alpha)
\end{array} \]  

Each square commutes by the only condition for \( \mathcal{A} \) and \( \mathcal{B} \) being modifications and by the interchange law in \( \mathcal{D} \).

\[ \square \]
**Definition A.93.** Consider 2-categories, 2-functors, pseudo-natural transformations, and modifications as in the following diagram

\[
\begin{array}{c}
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow \sigma & & \downarrow \rho \\
A & \xleftarrow{\epsilon} & B \\
\downarrow \lambda & & \downarrow \epsilon \\
H & \xleftarrow{\epsilon} & \gamma
\end{array}
\end{array}
\]

\[
(A.94)
\]

The **vertical composition** of \( A \) and \( B \), written as \( A \sigma \circ \lambda \Rightarrow B \rho \), is the modification defined by the assignment

\[
 x \quad \mapsto \quad \begin{array}{c}
 A \\
 \sigma(x) \quad A(x) \quad \rho(x) \\
 B \\
 \lambda(x) \quad B(x) \quad \epsilon(x)
\end{array}
\]

i.e.

\[
 A \circ_{x} B(x) := B(x)A(x), \quad (A.95)
\]

the horizontal composition of 2-morphisms in \( \mathcal{D} \).

**Remark A.96.** This notation is unfortunately confusing but is essentially the same abusive notation as in Definition A.74. The modification was defined using vertical compositions of natural transformations but the actual definition involved horizontal composition in the 2-category \( \mathcal{D} \). The reader is encouraged to draw more pictures of diagrams to avoid further confusion.
**Definition A.97.** Consider 2-categories, 2-functors, pseudo-natural transformations, and modifications as in the following diagram

\[
\begin{array}{c}
\xymatrix{
\mathcal{E} \ar[d]_{\rho} \ar[r]_{\alpha} & \mathcal{D} \ar[d]_{\beta} \\
\mathcal{C} \ar[r]_{\lambda} & \mathcal{C}
}
\end{array}
\]  

(A.98)

The **horizontal composition** of \( \mathcal{A} \) and \( \mathcal{B} \), written as \( \mathcal{A} \circ \mathcal{B} : \sigma \circ \lambda \Rightarrow \rho \circ \epsilon \), is the modification defined by the assignment

\[
\begin{array}{c}
\xymatrix{
\mathcal{X} \ar[r]^{AB} & \mathcal{X}
}
\end{array}
\]  

(A.99)

i.e.

\[
(\mathcal{A} \circ \mathcal{B})(x) := \left( \mathcal{A}(J(x)) \right) \left( \mathcal{B}(\mathcal{J}(x)) \right)
\]  

(A.100)

for all objects \( x \) in \( \mathcal{C} \).

We now come back to answering the question posed about the associativity, or lack thereof, of composition of 2-functors and pseudo-natural transformations.

**Lemma A.101.** Consider the following sequence of 2-categories and 2-functors

\[
\begin{array}{c}
\xymatrix{
\mathcal{F} & \mathcal{E} & \mathcal{D} & \mathcal{C}
}
\end{array}
\]  

(A.102)
Then \((FG)H = F(GH)\), i.e. the composition of 2-functors is associative, or equivalently the associator (a-priori a nontrivial pseudo-natural transformation) for 2-functor composition is the identity.

**Proof.** We prove this by checking all the data that specify the 2-functors \((FG)H\) and \(F(GH)\) are equal (see Definition A.56).

i) Because ordinary composition of functions is associative,

\[
((FG)H)_i = (FG)_i H_i = (F_i G_i) H_i = F_i (G_i H_i) \quad \text{(A.103)}
\]

\[
= F_i (GH)_i = (F(GH))_i.
\]

ii) For every pair \((\alpha, \beta)\) of 1-morphisms in \(\mathcal{C}\), we have the following list of equalities

\[
\begin{align*}
&\xymatrix{c^{FG}_{H(\alpha),H(\beta)}_{c^H_{\alpha,\beta}} \\
&c^{FG}_{H(\alpha),H(\beta)}} \\
&\xymatrix{c^F_{G(H(\alpha)),G(H(\beta))} \\
&c^F_{G(H(\alpha)),G(H(\beta))}} \\
&\xymatrix{c^F_{G(H(\alpha)),G(H(\beta))} \\
&c^F_{G(H(\alpha)),G(H(\beta))}} \\
&\xymatrix{F^{(GH)}_{c^H_{\alpha,\beta}} \\
&F^{(GH)}_{c^H_{\alpha,\beta}}}
\end{align*}
\]

\[
\text{(A.104)}
\]

because vertical composition is associative and because 2-functors respect vertical composition.
iii) For every object $x$ in $\mathcal{C}$, we have the following list of equalities

\[
\begin{align*}
(FG)(u_x^H) & \quad F(u^H_x) \quad u^G_{H(x)} \\
F(G(u^H_x)) & \quad G(u^H_x) \\
F \quad (u^G_{H(x)}) & \quad u^G_{H(x)} \\
F & \quad u^G_{(GH)(x)}
\end{align*}
\]  

(A.105)

because vertical composition is associative and because 2-functors respect vertical composition.

\[\square\]

**Definition A.106.** Let $\mathcal{C}$ be a 2-category. The **identity 2-functor** on $\mathcal{C}$, written as $\text{id}_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C}$, is defined as follows.

i) The assignment on objects, 1-morphisms, and 2-morphisms is given by the identity functions

\[ (\text{id}_\mathcal{C})_j := (\text{id}_{C_j} : C_j \rightarrow C_j) \]  

(A.107)

for all $j = 0, 1, 2$.

ii) For every pair $(\alpha, \beta)$ of composable 1-morphisms in $\mathcal{C}$, the compositor is

\[ c_{\alpha,\beta}^{\text{id}_\mathcal{C}} := \text{id}_{\alpha\beta}, \]  

(A.108)

the identity 2-morphism on $\alpha\beta$. 
COMPOSITIONS IN 2-CATEGORY THEORY

iii) For every object \( x \) in \( C \), the unitor is

\[
u^{id_x}_x := id_{id_x}
\]

the identity 2-morphism on \( id_x \).

Lemma A.110. Let \( F : C \to D \) be a 2-functor between two 2-categories \( C \) and \( D \). Then \( F id_C = F = id_D F \), i.e. the left and right unifiers for the composition of 2-functors are both equal to the identity.

Proof. This is proved similarly to the previous Lemma.

i) Because ordinary composition of functions by an identity function results in that same function,

\[
(F id_C)_j = F_j (id_C)_j = F_j = (id_D)_j F_j = (id_D F)_j
\]

(A.111)

for all \( j = 0, 1, 2 \).

ii) For every pair \((\alpha, \beta)\) of 1-morphisms in \( C \), we have the following list of equalities

\[
\begin{align*}
(F id_C)_j & = F_j (id_C)_j = F_j = (id_D)_j F_j = (id_D F)_j \\
F^F_{\alpha,\beta} & = F_{\alpha,\beta} \\
id_{F(\alpha)} & = F(\alpha) \\
id_{F(\beta)} & = F(\beta) \\
id_{id_D} & = id_D \\
id_{c_{\alpha,\beta}} & = c_{\alpha,\beta}
\end{align*}
\]

(A.112)
because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.

iii) For every object $x$ in $\mathcal{C}$, we have the following list of equalities

$$
\begin{array}{cccc}
F(u^F_{id_c(x)}) & F(id_{id_c(x)}) & id_{id_{F(x)}} & u^F_{id_{D(x)}} \\
F(id_{id_c(x)}) & id_{id_{F(x)}} & u^F_{id_{D(x)}} & id_{id_{D(x)}} \\
u^F_{id_{D(x)}} & id_{id_{D(x)}} & id_{id_{D(x)}} & F(id_{id_c(x)}) \\
u^F_{id_{D(x)}} & id_{id_{D(x)}} & id_{id_{D(x)}} & F(id_{id_c(x)})
\end{array}
$$

(A.113)

because the vertical identity 2-morphism is an identity for vertical composition and because 2-functors respect vertical identities.

\[\blacksquare\]

Although composition of 2-functors has no surprises, vertical composition of pseudo-natural transformations is a bit more complicated. In particular, there are associators and unifiers.

**Lemma A.114.** Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories, $F, G, H, J : \mathcal{C} \to \mathcal{D}$ be four 2-functors, and $\rho : F \Rightarrow G$, $\sigma : G \Rightarrow H$, and $\lambda : H \Rightarrow J$ be three pseudo-natural transformations. Then the assignment

$$
x \mapsto a_{\lambda,\sigma,\rho}(x) := a_{\lambda(x),\sigma(x),\rho(x)},
$$

(A.115)

the associator in the category $\mathcal{D}$, for any object $x$ in $\mathcal{C}$, defines a modification

$$
a_{\lambda,\sigma,\rho} : \left(\begin{array}{c}
\rho \\
\sigma \\
\lambda
\end{array}\right) \Rightarrow \left(\begin{array}{c}
\rho \\
\sigma \\
\lambda
\end{array}\right).
$$

(A.116)
Furthermore, \(a_{\rho,\sigma,\lambda}\) is invertible and satisfies the pentagon axiom of condition (c) in Definition A.2.

**Definition A.117.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be two 2-categories and let \(F : \mathcal{C} \to \mathcal{D}\) be a 2-functor. The *identity pseudo-natural transformation* \(id_F : F \Rightarrow F\) is defined as follows.

i) The assignment
\[
x \mapsto \left( id_{F(x)} : F(x) \longrightarrow F(x) \right)
\] (A.118)
defines the map \(id_F : C_0 \to D_1\).

ii) The assignment
\[
\left( y \xleftarrow{\alpha} x \right) \mapsto \left( r_{F(\alpha)}^{F(x)} : \text{id}_{F(y)} F(\alpha) \Rightarrow F(\alpha) \Rightarrow F(\alpha) \text{id}_{F(x)} \right)
\] (A.119)
defines \(id_F : C_1 \to D_2\).

**Lemma A.120.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be two 2-categories, \(F,G : \mathcal{C} \to \mathcal{D}\) be two 2-functors, and \(\rho : F \Rightarrow G\) a pseudo-natural transformation. Then the assignments
\[
x \mapsto l_\rho(x) := l_{\rho(x)},
\]
(A.121)
the left unifier in \(\mathcal{D}\), and
\[
x \mapsto r_\rho(x) := r_{\rho(x)},
\]
(A.122)
the right unifier in $\mathcal{D}$, define modifications

$$l_\rho : \frac{\text{id}_F}{\rho} \Rightarrow \rho$$

(A.123)

and

$$r_\rho : \frac{\rho}{\text{id}_F} \Rightarrow \rho$$

(A.124)

respectively. Furthermore, $l_\rho$ and $r_\rho$ are invertible and satisfy condition (d) in Definition A.2.

Finally, after giving the numerous definitions that arise in basic 2-category theory, we come to the notion of a psuedonatural equivalence of 2-functors between two 2-categories.

**Definition A.125.** Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ two 2-functors. A *pseudo-natural equivalence* from $F$ to $G$ is a quadruple $(\rho, \sigma, i, j)$ consisting of pseudo-natural transformations $\rho : F \Rightarrow G$, $\sigma : G \Rightarrow F$, and invertible modifications $i : \rho \Rightarrow \text{id}_F$ and $j : \text{id}_G \Rightarrow \sigma$ such that the diagrams

\[
\begin{array}{ccc}
\rho & \xleftarrow{i} & \text{id}_F \\
\rho \downarrow & & \downarrow \text{id}_F \\
\sigma & \xleftarrow{j} & \text{id}_G
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\sigma & \xrightarrow{\text{id}_\rho} & \text{id}_F \\
\sigma \downarrow & & \downarrow \text{id}_F \\
\rho & \xrightarrow{\text{id}_\sigma} & \text{id}_G
\end{array}
\]

\[
\begin{array}{ccc}
\rho & \xrightarrow{id} & \sigma \\
\rho \downarrow & & \downarrow \sigma \\
\rho & \xrightarrow{l_\rho} & \rho
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\sigma & \xrightarrow{id} & \rho \\
\sigma \downarrow & & \downarrow \rho \\
\sigma & \xrightarrow{r_\sigma} & \sigma
\end{array}
\]

(A.126)
both commute. In these diagrams, \( l \) and \( r \) are the left and right unifiers, respectively, of Lemma A.120 and \( a \) is the associator of Lemma A.114. The above two diagrams are known as \textit{zig-zag identities} (since they are closely related to those of Definition A.27). We abusively say that \( \rho : F \Rightarrow G \) is a pseudo-natural equivalence without writing out the other pieces of data. \( F \) and \( G \) are said to be \textit{pseudo-naturally equivalent} if there exists a pseudo-natural equivalence between them.

**Definition A.127.** Let \( S \) and \( T \) be two 2-categories and \( F : S \rightarrow T \) a 2-functor. \( F \) is called an \textit{equivalence of 2-categories} if there exists a functor \( G : T \rightarrow S \) together with pseudo-natural equivalences \( \rho_S : GF \Rightarrow id_S \) and \( \rho_T : FG \Rightarrow id_T \). The functor \( G : T \rightarrow S \) along with the pseudo-natural equivalences is called a \textit{weak inverse} of \( F \).

**Lemma A.128.** Let \( S \) and \( T \) be two 2-categories. Two weak inverses to a 2-functor \( F : S \rightarrow T \) are pseudo-naturally equivalent.

\begin{proof}
By assumption, there exist \( G, G' : T \rightarrow S \) with pseudo-natural equivalences

\begin{align}
(\rho_S : GF \Rightarrow id_S, \sigma_S : id_S \Rightarrow GF, i_S : \sigma_S^\rho \Rightarrow id_{GF}, j_S : id_{id_S} \Rightarrow \sigma_S^{id_S}, i_{id_S} \Rightarrow \rho_S^{i_S}), \tag{A.129}\end{align}

\begin{align}
(\rho_T : FG \Rightarrow id_T, \sigma_T : id_T \Rightarrow FG, i_T : \sigma_T \Rightarrow id_{FG}, j_T : id_{id_T} \Rightarrow \sigma_T^{id_T}, i_{id_T} \Rightarrow \rho_T^{j_T}), \tag{A.130}\end{align}
\end{proof}
\[
(\rho'_S : G'F \Rightarrow id_S, \sigma'_S : id_S \Rightarrow G'F, i'_S : \varphi'_S^S \Rightarrow id_{G'F}, j'_S : id_{id_S} \Rightarrow \sigma'_S^S)\), (A.131)
\]
and
\[
(\rho'_T : FG' \Rightarrow id_T, \sigma'_T : id_T \Rightarrow FG', i'_T : \varphi'_T^T \Rightarrow id_{FG'}, j'_T : id_{id_T} \Rightarrow \sigma'_T^T)\). (A.132)
\]

We define a pseudo-natural transformation \(\rho : G \Rightarrow G'\) by taking the vertical composition of pseudo-natural transformations
\[
G = id_S G \xrightarrow{id_G \rho_T} (G'F)G = G'(FG) \xrightarrow{id_G \rho_T} G' id_T = G'\] (A.133)
and a psuedonatural transformation \(\sigma : G' \Rightarrow G\) by
\[
G' = G' id_T \xrightarrow{id_G \sigma_T} G'(FG) = (G'F)G \xrightarrow{id_G \rho_T} id_S G = G, \] (A.134)
both of which have been simplified by Lemma A.101 and Lemma A.110. We define a modification \(i : \rho \Rightarrow id_G\) by the internal composition

\[
(A.135)
\]
and a modification \( j : \text{id}_G \Rightarrow \sigma \rho \) by the composition

\[
\sigma \rho = \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) a \sigma G \sigma_T \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) = \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right)
\]

\[
\text{id}_G \equiv \text{id}_G \cdot \text{id}_T = \text{id}_G \sigma_T \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right) = \left( \begin{array}{c} \text{id}_G \sigma_T \\ \sigma G \sigma_T \end{array} \right)
\]

We leave it to the reader to check that all the required diagrams from Definition A.125 commute.

Recall that a functor \( F : C \rightarrow D \) of (ordinary) categories is \textit{full and faithful} or \textit{fully faithful} if for any two objects \( x \) and \( y \) in \( C \), the induced map of sets

\[
\text{Hom}_C(y, x) \rightarrow \text{Hom}_D(F(y), F(x))
\]

\[
\alpha \mapsto F(\alpha)
\]

is a bijection. The analogous property for 2-categories and 2-functors is a bit more subtle.

First note that for any two objects \( x \) and \( y \) of a 2-category \( C \), one can define a \textit{category} \( \text{Hom}_C(y, x) \) by setting

\[
\text{Hom}_C(y, x)_0 := \{ \alpha \in C_1 \mid s(\alpha) = x \text{ and } t(\alpha) = y \}
\]
and
\[ \text{Hom}_C(y, x)_1 := \{ \Sigma \in C_2 \mid ss(\Sigma) = x \text{ and } tt(\Sigma) = y \}. \] (A.139)

One can define the source, target, and identity-assigning maps by restricting the ones from \( C \). Composition in \( \text{Hom}_C(y, x) \) is the restriction of the vertical composition in \( C \). It is associative and unital by condition (b) of Definition A.2.

**Definition A.140.** Let \( F : C \rightarrow D \) be a 2-functor between two 2-categories. \( F \) is said to be **fully faithful** if the restriction of \( F \) to the induced functor
\[ \text{Hom}_C(y, x) \rightarrow \text{Hom}_D(F(y), F(x)) \] (A.141)
is an equivalence of categories, or equivalently, if the above functor on Hom-categories is both essentially surjective and fully faithful, for all objects \( x \) and \( y \) in \( C \).

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