On the Derivative of 2-Holonomy for a Non-Abelian Gerbe

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On the Derivative of 2-Holonomy for a Non-Abelian Gerbe

by

Cheyne Miller

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

On the Derivative of 2-Holonomy for a Non-Abelian Gerbe

by

Cheyne Miller

Advisor: Thomas Tradler

The local 2-holonomy for a non abelian gerbe with connection is first studied via a local zig-zag Hochschild complex. Next, by locally integrating the cocycle data for our gerbe with connection, and then glueing this data together, an explicit definition is offered for a global version of 2-holonomy. After showing this definition satisfies the desired properties for 2-holonomy, its derivative is calculated whereby the only interior information added is the integration of the 3-curvature. Finally, for the case when the surface being mapped into the manifold is a sphere, the derivative of 2-holonomy is extended to an equivariant closed form in the spirit of the construction of Tradler-Wilson-Zeinalian for abelian gerbes.
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Introduction

In recent years, gerbes with connection have been studied as a higher analog of bundles with connection. This notion of higher can be expressed in the way that abelian gerbes with connection were the geometric representation of higher Deligne cocycles in [BM], 2-holonomy for abelian gerbes with connection were modeled by a higher hochschild complex in [TWZ], non abelian gerbes with connection being defined by a parallel transport 2-functor in [SWIII], etc. This project was born out of the desire to generalize the work done in [TWZ], where 2-holonomy for an abelian gerbe with connection produced, via Hochschild methods, an equivariantly closed extension of 2-holonomy.

The first attempt at generalizing [TWZ], and the content of Chapter 1, was to build a Hochschild Complex which would come equipped with an iterated integral (see [C]) yielding 2-holonomy for a non-abelian gerbe. Section 1.2 introduces the solution to the non-abelian problem of constructing
such a Hochschild Complex, whereby we retain the order or elements in the
our algebra $\mathcal{A}$ by using a Zig Zag version of the bar-construction. In section
1.3 the Chen Iterated Integral out of our zigzag complex is given and is
shown to be a chain map. Since 2-holonomy requires integrating along a
2-dimensional surface, rather than a 1-dimensional path, Section 1.4 shows
how the zigzag complex will be extended to a 2-dimensional complex. Since,
in the case of non-abelian gerbes as shown by Schreiber and Waldorf, the
2-form, $B$ which is integrated along the square, is acted on by a one form,
$A$, a curved version of the zigzag Hochschild differential is needed. Thus
section 1.5 adds the curving information to our zigzag Hochschild Complex
and finally Section 1.6 shows that the Chen map out of our complex produces
2-Holonomy over an open set $U_i$.

The limitation with using Chapter 1 to describe 2-Holonomy is two-fold:
(a) it is assumed that the square $\Sigma : I^2 \to M$ on which 2-Holonomy is
evaluated, lives entirely in one open set $U \subset M$ of our open cover on which
the local data for a non-abelian gerbe is defined and (b) a special case for the
structure 2-group of the non-abelian gerbe in question was used. The goal of
Chapter 2 then, is to calculate $d(Hol)$ in the most general setting, where $Hol$
stands for the 2-Holonomy of a square which might require several open sets
of a given cover in order to compute. Furthermore, for the crossed module
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(H \rightarrow G), we remove the assumption of G being a matrix group. While in this work we keep the assumption of H being a matrix group (H \subset \text{Mat}, where \text{Mat} = \mathbb{R}^{n,n} or \text{Mat} = \mathbb{C}^{n,n} for suitable n), we believe that most if not all of the arguments will hold true even without assuming this restriction on H. Using the works of Schreiber and Waldorf we review local transport data for bigons in section 2.1.4 and then define a version of local transport data for squares in 2.1.5. Considering a square in M which might require several open sets to cover, it is shown how to glue the local transport data together in Section 2.2. Finally, Section 2.3 spends a considerable amount of detail to prove Theorem 2.3.1 which says that, on the interior of \Sigma, d(\text{Hol}) amounts to replacing one \( B_i \) with \( H_i \) and summing over all choices. In fact, the theorem can be written

\[
d(\text{Hol}) = -(\alpha_{\text{Hol}})_*(A_i) + \text{Hol} \cdot \int_{S^2} H + \text{Hol} \cdot \left( \int_{\partial S^2} (B) + \sum a_{i,j} \right).
\]

Finally, in Section 3.1, we come back to the goal of this project and extend \( d(\text{Hol}) \) to a closed form in a complex of differential forms. In the case when \( \Sigma \) is a sphere mapped into \( M \) we obtain the following characterization where the boundary terms have vanished:

\[
d(\text{Hol}) = -(\alpha_{\text{Hol}})_*(A_i) + \text{Hol} \cdot \int_{S^2} H.
\]
In section 3.2 we consider the case where $\alpha$ is inner, $\alpha : G \to Inn(H)$, which applies, for example, to abelian gerbes. Here $Hol \in \Omega^0(M^{S^2}, Mat)$ is a globally defined 0-form and can be extended to a closed element $\mathcal{H}ol$ in the complex $\Omega^\bullet(M^{S^2}, Mat^\mathcal{C})$ (with differential $d + i \frac{\partial}{\partial t}$), very much in the spirit of [TWZ]. In section 3.3 we no longer restrict ourselves and instead consider any $\alpha$ action. Since $Hol$ is now only defined on certain open sets of the mapping space $M^{S^2}$, and does not agree on the intersection of these open sets, $Hol$ no longer is a well-defined global form with values in $H$. However, there are transition functions for $Hol$ between these open sets in $M^{S^2}$ which we can used to analogously define a closed form in a suitable complex. Note that in this context, $d(Hol)$ lives in the tangent space $T_{Hol}H$, so a translation (via $L_{Hol^{-1}}$, i.e. left-multiplication by $Hol^{-1}$) of the complex is made to ensure we are working in the tangent space $T_eH \simeq \mathfrak{h}$. After translation, the equivariant-extension of the function $Hol$ is reformulated as an extension of the constant function 1, where the new complex involves the transition functions for where the forms take their values and involves the left-translation in the differential. Producing such a closed form as an equivariant extension of the constant function 1 is the conclusion of section 3.3.
Chapter 1

A Local ZigZag Hochschild Complex

1.1 The ZigZag Hochschild Complex: An Illustrated Introduction

The motivation for this chapter is born out of modeling iterated integrals \( [C] \) in a non-commutative setting. While it is not an issue to simply compute an iterated integral in a non-abelian setting, finding the correct algebraic structure which respects the product and differential in that setting is a problem of interest to not only the study of 2-holonomy for non-abelian gerbes, but also other fields such as, for example, in the study of Quantum Control Theory and the study of Multiple Dedekind Zeta Values. While the latter two fields may find some tools in this paper useful, we will focus our attention on differential forms on smooth spaces with values in some not-necessarily-commutative-algebra – so that we generally restrict ourselves to
Matrix Lie Algebras – which we will denote \( \Omega_{\mathfrak{g}}(M) \) for a manifold, \( M \).

In a commutative setting, suppose we would like to consider the wedge product of two iterated integrals (each thought of as a differential form on the path space \( M^I \) of a manifold \( M \)) in \( \Omega(M^I) \). Then we are able to write

\[
\int_{\Delta^n} a_1(\tau_1) \cdots a_n(\tau_n) d\tau_1 \cdots d\tau_n \wedge \int_{\Delta^m} a_{n+1}(\tau_{n+1}) \cdots a_{n+m}(\tau_{n+m}) d\tau_{n+1} \cdots d\tau_{n+m} = \sum_{\sigma} \pm \int_{\Delta^{n+m}} a_{\sigma^{-1}(1)}(\tau_{\sigma^{-1}(1)}) a_{\sigma^{-1}(n+m)}(\tau_{\sigma^{-1}(n+m)}) d\tau_{\sigma^{-1}(1)} \cdots d\tau_{\sigma^{-1}(n+m)}
\]

where the sum is over order-preserving shuffles, \( \sigma \). The picture corresponding to this wedge product is the illustration for the shuffle product on the interval Hochschild complex:

\[
\begin{array}{ccccccccccc}
\otimes & a_1 & a_2 & a_3 & a_4 & a_5
\end{array}
\quad
\begin{array}{ccccccccccc}
\otimes & b_1 & b_2 & b_3 & b_4
\end{array}
\]

\[
= \sum_{\text{shuffles}}
\begin{array}{ccccccccccc}
 & a_1 & b_1 & a_2 & b_2 & a_3 & a_4 & b_3 & b_4 & a_5
\end{array}
\]

However, when we are in the non-commutative setting, the best we can do when writing the wedge product out is, modulo signs:

\[
\int_{\Delta^n} a_1(\tau_1) \cdots a_n(\tau_n) d\tau_1 \cdots d\tau_n \wedge \int_{\Delta^m} b_1(\tau_{n+1}) \cdots b_m(\tau_{n+m}) d\tau_{n+1} \cdots d\tau_{n+m} = \sum_{\sigma} \int_{\Delta^{n+m}} a_{\sigma^{-1}(1)}(\tau_{\sigma^{-1}(1)}) a_{\sigma^{-1}(n+1)}(\tau_{\sigma^{-1}(n+1)}) \cdots b_m(\tau_{\sigma^{-1}(n+m)}) d\tau_{n+1} \cdots d\tau_{n+m}
\]

We would like an illustration for this wedge product in some Hochschild
CHAPTER 1. A LOCAL ZIGZAG HOCHSCHILD COMPLEX

complex. Note that the order of the differential forms is preserved but the coordinates are shuffled around. Moreover, by integrating all of the $a$’s we have passed over the interval (from left to right) once and then we pass over the interval a second time while integrating all of the $b$’s. The corresponding picture below is our shuffle product in the zigzag Hochschild complex which preserves the order of the differential forms but shuffles the coordinates:

$$a_1a_2a_3a_4a_5 \circ b_1b_2b_3b_4 = \sum_{\text{shuffles}} a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ b_1 \ b_2 \ b_3 \ b_4$$

In the above diagram our zigzags, although they require some vertical space to draw, are actually passing back and forth over the same interval. With this motivation for our zigzags in hand, Section 1.2 provides the formal details for the zigzag Hochschild complex, $CH_{ZZ}(A)$, of an associative (unital) algebra, $A$. Section 1.2.2 deals with certain special cases and the compatibility between the interval Hochschild complex and our zigzag Hochschild
complex for these cases. Section 1.3 deals with an iterated integral map
\[ CH_{ZZ}(\Omega_{\boxplus}(M)) \xrightarrow{I_t} \Omega_{\boxplus}(M^I). \]

In Section 1.4 we explore the higher (2-dimensional) Hochschild complexes. In a commutative setting it is sufficient to use a bi-simplicial model, 
\[ CH^{Sw}(A), \]
where \( Sq := I \times I \), consisting of monomials of the form

Here, the variables \( t_1, t_2, s_1, \) and \( s_2 \) describe the coordinates by which a 2-dimensional iterated integral map \( CH^{Sq}(\Omega_{\boxplus}(M)) \rightarrow \Omega_{\boxplus}(M^{Sq}) \) has to be evaluated. The differential for \( CH^{Sq}(A) \) consists of two components: one which “collapses” vertically and one which “collapses” horizontally. We see below that \( D^2 = 0 \) is not possible in the non-abelian case, where we have highlighted the significant areas of the pictures:
Instead of using just one row per $s_i$-coordinate, we adopt a complex \(^1\) $C H^{{\text{Rec}}} _{ZZ}(A)$ where we start with $k$-many zigzags at each $s_i$ and then allow our differential forms to be placed at the intersection-points of $t_i$’s and these

\(^1\)Of course, one could use a "square complex" or even $C H_{ZZ}(C H_{ZZ}(\Omega_{\mathfrak{M}}(M)))$, but we adopt the rectangular complex out of personal preference and comment on the relationships later.
zigzags. The corresponding picture looks as follows:

This complex resolves the issues involved when $\Omega_{\boxtimes}(M)$ is non-abelian, since a collapse of rows only stacks zigzags together without changing the order in which the forms appear. After defining $CH_{\text{Rec}}^{\boxtimes}(A)$, we proceed in Section 1.4 to define an iterated integral $It : CH_{\boxtimes}^{\text{Rec}}(\Omega_{\boxtimes}(M)) \to \Omega_{\boxtimes}(M^{S_q})$, for the space $M^{S_q}$ of squares $[0, 1]^2 \to M$.

As mentioned at the beginning, our goal is to have an element in our Hochschild complex which maps to holonomy under our iterated integral maps. In the commutative setting, such an element representing 2-holonomy in $CH^{S_q}(\Omega(M))$ is given by an infinite sum (the exponential of some element)
of monomials of the form, see \cite{TWZ},

\[
\begin{array}{cccccc}
  & t = 0 & t_1 & t_2 & t_3 & t = 1 \\
 s = 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
 s_1 & & 1 & B & 1 & \\
 s_2 & & B & 1 & 1 & \\
 s = 1 & & & & & \\
\end{array}
\]

where \( B \in \Omega^2(U) \) is a connection 2-form on an open set \( U \subset M \). Through the works of \cite{BaSc}, \cite{MP1}, \cite{MP2}, \cite{SWII} et al we observe that the 2-holonomy we are interested in (for the non-commutative setting) can be expressed loosely as

\[
\iint \alpha_*(B)
\]

where \( \alpha_* \) is the action data coming from a crossed module \( (\mathfrak{h} \overset{t}{\longrightarrow} \mathfrak{g} \overset{\alpha_*}{\longrightarrow} Der(\mathfrak{h})) \). The formula is similar to our abelian situation but is more accu-
rately written as

$$P \exp \left( \int_0^t \text{hol}(t',s) \cdot (B)(s,t') \cdot \text{hol}^{-1}(t',s) dt' \right),$$

involving a path ordered exponential and parallel transport, \(\text{hol}(t',s)\), from 
(0,0) to \((t',s)\), via a given connection 1-form, \(A \in \Omega^1_{\text{hol}}(U)\). All of this means we are interested in integrating over a sum of monomials as illustrated in Figure 1.1 on page 13. In all of the above structures, we obtain our differentials on the various Hochschild complexes by requiring the iterated integral to be a chain map. For example, when working in the 1-dimensional non-curved zigzag Hochschild Complex, \(CH_{ZZ}(A)\), we require our differential to be defined \(D = (d+b)\) where \(b\) and \(d\) come from the terms in Stokes’ formula,

$$d \int_F \omega = (-1)^{\dim F} \int_F d\omega + (-1)^{\dim(F)-1} \int_{\partial F} \omega,$$

which differentiate the integrand and take the boundary of the fiber, respectively. For the 2-dimensional non-curved zigzag Hochschild Complex, \(CH_{ZZ}^{Rec}(A)\), the boundary term in Stokes’ formula also gives a vertical boundary, which we will call \(*\), so that our differential becomes \(D = (d+b+*)\). If we use the above figure as a guide to what happens in the curved 2-dimensional case, where we apply parallel transport between the forms originally placed along our zigzags, the boundary term in Stokes’ also yields terms of the form
Figure 1.1: An illustration of our curved iterated integral, $It^A$, applied to one term of $\exp(B)$. 
A \wedge A
differentiable the integrand adds terms \( dA \). Putting these terms together we see that \( d_{DR} \) of our Iterated Integral will add terms \( R := dA + A \wedge A \) in between our differential forms on the zigzags. For this reason, the curved zigzag (both 1-d and 2-d) Hochschild complex has a \( c \) component in the differential, which precisely shuffles in these \( R \)'s. In addition, one can see that the boundary term in Stokes’ results in terms \( A \wedge \omega \) and \( \omega \wedge A \), where \( \omega \) is some differential form originally placed on the zigzag. These terms require us to replace the \( d \) component of our differential with a component \( \nabla = d + [A, -] \). Since \( \nabla^2 = [R, -] \neq 0 \) in this case, we call these complexes \textit{curved} and we use the differentials:

- \( D = (\nabla + b + c) \) for the 1-dimensional curved zigzag Hochschild complex, \( CH_{\text{ZZ}}(A) \), and

- \( D = (\nabla + b + c + \star) \) for the 2-dimensional curved zigzag Hochschild complex, \( CH_{\text{ZZ}}^{\text{Rec}}(A) \).

Chapter 5 deals with our zigzag Hochschild complexes in this curved setting, complete with iterated integral maps.

Finally, in Section 1.6 we verify that we have elements in \( CH_{ZZ}(\Omega^\otimes(M)) \) which map (in their limit) to holonomy, and elements in \( CH_{ZZ}^{\text{Rec}}(\Omega^\otimes(M)) \) which map to 2-holonomy under the curved iterated integral.
1.2 The One-Dimensional ZigZag Hochschild Complex

In this chapter, we define the complex $CH_{\text{ZZ}}(A)$ along with a product making it into a DGA.

1.2.1 For a (Non-Abelian) DGA

Let $(A, \cdot, d)$ be a (possibly non-commutative) associative, unital DGA over a commutative ring $S$. We have the well-known interval Hochschild complex\footnote{Also known as the two-sided bar construction where the modules are chosen to be $A$ in this case}

$CH^I(A) := \bigoplus_{n \geq 0} A \otimes (A[1])^\otimes n \otimes A$, with its usual differential, $D$, given by

$D(\omega_0 \otimes \omega_1 \otimes \ldots \otimes \omega_n \otimes \omega_{n+1}) := \sum_{i=0}^{n+1} (-1)^{n+\beta_i} \omega_0 \otimes \ldots \otimes d(\omega_i) \otimes \ldots \otimes \omega_{n+1}$

$\sum_{i=0}^n (-1)^i \omega_0 \otimes \ldots \otimes (\omega_i \cdot \omega_{i+1}) \otimes \ldots \otimes \omega_{n+1}$

However under the usual shuffle product, $D$ is not a derivation; that would require $A$ to be a commutative algebra. In this section, we define a new Hochschild complex, $(CH_{\text{ZZ}}(A), D)$ with an associative shuffle product, $\odot$, for which our differential $D$ is a derivation.

Before providing a formal definition, we will give the underlying ideas for its definition. For our underlying vector space, which will consist of tensor
products of elements in $\mathcal{A}$, monomials can easily be represented by diagrams

where we’ve replaced the interval with $k$-many zigzags going back and forth over the same interval. An odd-numbered trip over the interval (left-to-right) is referred to as a “zig” and an even-numbered trip over the interval (right-to-left) is called a “zag”. The elements of $\mathcal{A}$ are placed at the points where the zigzags cross the $n$-many columns$^3$ Consider the following monomial

$^3$These columns represent the “time-slots” $0 \leq t_1 \leq \ldots \leq t_n \leq 1$ which we will eventually integrate over when considering differential forms and an iterated integral.
with $n = 2$ and $k = 4$ for further clarification:

The differential $D$ will consist of two components: $d$, which applies the differential coming from $\mathcal{A}$ to each element in the tensor product

and $b$ which identifies two columns.

**Definition 1.2.1.** Let $(\mathcal{A}, \cdot, d)$ be a (possibly non-commutative) associative, unital DGA over a fixed commutative ring $S$. We define the zigzag Hochschild complex, $CH_{ZZ}(\mathcal{A}) = \bigoplus_{n,k \geq 0, k \text{ even}} (\mathcal{A} \otimes (\mathcal{A}^\otimes n \otimes \mathcal{A}^\otimes k))[n]$, where $[n]$ denotes a total shift down by $n$, with a differential defined below. Monomials in
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$\text{CH}_{\text{ZZ}}(\mathcal{A})$ will be written as

$$x = x^L \otimes (x_{(1,1)} \otimes \cdots \otimes x_{(1,n)} \otimes x_1^R) \otimes$$

$$\cdots \otimes (x_{(i,1)} \otimes \cdots \otimes x_{(i,n)} \otimes x_i^\dagger) \otimes$$

$$\cdots \otimes (x_{(k,1)} \otimes \cdots \otimes x_{(k,n)} \otimes x_k^L)$$

where $^\dagger_i$ is $\mathcal{R}$ if $i$ is odd (i.e. on a “zig”) and is equal to $\mathcal{L}$ if $i$ is even (i.e. on a “zag”). The differential $D : \text{CH}_{\text{ZZ}}(\mathcal{A}) \to \text{CH}_{\text{ZZ}}(\mathcal{A})$ is given by

$$D(x) := (d + b)(x)$$

The two components $d$ and $b$ are defined below:

$$d(x) := (-1)^n d(x^L) \otimes \cdots \otimes (x_{(i,1)} \otimes \cdots \otimes x_{(i,n)} \otimes x_i^\dagger) \otimes \cdots \otimes x_k^L$$

where

$$\beta_i^\dagger := |x^L| + \cdots + |x_{(i,1)}| + \cdots |x_{(i,n)}|$$

and

$$\beta_{i,p} := |x^L| + \cdots + |x_{(i,1)}| + \cdots + |x_{(i,p-1)}|$$

for $p > 1$ and

$$\beta_{i,1} := |x^L| + \cdots + |x_{(i-1,1)}| + \cdots |x_{(i-1,n)}| + |x_i^\dagger|.$$

$$b(x) := (-1)^n (x^L \cdot x_{(1,1)} \otimes (x_{(1,2)} \otimes \cdots \otimes x_{(1,n)} \otimes x_1^R) \otimes$$

$$\cdots \otimes (x_{(i,1)} \otimes \cdots \otimes x_{(i,n)} \otimes (x_{i,n} \cdot x_i^L \cdot x_{(i+1,1)})) \otimes$$

$$\cdots \otimes (x_{(k,1)} \otimes \cdots \otimes x_{(k,n-1)} \otimes (x_{k,n} \cdot x_k^L))$$
\begin{align*}
+ \sum_{p=1}^{n-1} & (-1)^{n+p} x^L \otimes (\ldots \otimes (x_{(1,p)} \cdot x_{(1,p+1)}) \otimes \ldots \otimes x^R_1) \otimes \\
& \ldots \otimes (\ldots \otimes (x_{(i,p)} \cdot x_{(i,p+1)}) \otimes \ldots \otimes x^R_i) \otimes \\
& \ldots \otimes (\ldots \otimes (x_{(k,p+1)} \cdot x_{(k,p)}) \otimes \ldots \otimes x^L_k) \\
& + x^L \otimes (x_{(1,1)} \otimes \ldots \otimes x_{(1,n-1)} \otimes (x_{(1,n)} \cdot x^R_1 \cdot x_{(2,1)})) \otimes \\
& \ldots \otimes (x_{(i,1)} \otimes \ldots \otimes x_{(i,n-1)} \otimes (x_{(i,n)} \cdot x^R_i \cdot x_{(i+1,1)})) \otimes \\
& \ldots \otimes (x_{(k,2)} \otimes \ldots \otimes x_{(k,n)} \otimes x^L_k)
\end{align*}

For a pictorial representation of a simple monomial, see the figure on page 17. For a pictorial representation of \(d\) and \(b\), see the figures on page 17.

In the definition above we are applying the following sign convention: When \(d\) is applied, each summand has a sign of \((-1)^{n+\beta}\) where \(n\) is the number of columns between endpoints in our zigzags and \(\beta\) is the sum of the degrees of elements preceding the current element which \(d\) is being applied to. When \(b\) is applied, each summand has a sign of \((-1)^{n+p}\) where we keep track of the fact that \(b\) had to move over \(p-1\) many columns and the \(n-1\) is motivated by the eventual use of Stokes’ formula for \(\mathcal{A} = \Omega(M)\),

\[
d \int_F \omega = (-1)^{\dim F} \int_F d\omega + (-1)^{\dim(F)-1} \int_{\partial F} \omega.
\]

**Proposition 1.2.2.** \(D^2 = 0\)

**Proof.** \(D^2 = D \circ D = (d+b) \circ (d+b) = d^2 + d \circ b - b \circ d + b^2 = 0\). We analyze
each term independently:

- $d^2 = 0$ is a consequence of $\mathcal{A}$ being a DGA.

- For $b^2 = 0$, we note that by associativity of "·" in $\mathcal{A}$, we need only to check that we get opposite signs for the two ways in which the columns $j - 1$, $j$, and $j + 1$ come together. It is easy to check that for when a collapse of the $j$-th and $j + 1$-th columns is followed by a collapse of the $j - 1$-th and $j$-th columns, we get a total sign of $(-1)^0$. But when we collapse the $j - 1$-th column with the $j$-th column, followed by collapsing the $j$-th and $j + 1$-th columns, we obtain a sign of $(-1)^1$. Similarly, terms in $b^2$ vanish when the columns being collapsed are separated.

- For $d \circ b + b \circ d = 0$, we consider an element and focus on certain tensor factors\footnote{The calculation is shown along a "zig", the same result holds along a "zag" but the multiplication would be ordered differently.} of that element, namely:

$$x_{k,n} = \ldots \otimes x_{(i,p-1)} \otimes x_{(i,p)} \otimes x_{(i,p+1)} \otimes x_{(i,p+2)} \otimes \ldots .$$

Looking first at $d \circ b$ we obtain terms

$$d \circ b(x_{k,n})$$

$$= d \circ b(\ldots \otimes x_{(i,p-1)} \otimes x_{(i,p)} \otimes x_{(i,p+1)} \otimes x_{(i,p+2)} \otimes \ldots).$$
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\[= d(\ldots + (-1)^{n+p}(\ldots \otimes x_{i,p-1} \otimes x_{i,p} \cdot x_{i,p+1} \otimes x_{i,p+2} \otimes \ldots) + \ldots)
\]

\[= \ldots + (-1)^{\beta_{i,p-1}}+p-1(\ldots \otimes d(x_{i,p-1}) \otimes x_{i,p} \cdot x_{i,p+1} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p}}+p-1(\ldots \otimes d(x_{i,p}) \cdot x_{i,p+1} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p+1}}+p-1(\ldots \otimes x_{i,p} \cdot x_{i,p+1} \otimes d(x_{i,p+2}) \otimes \ldots) + \ldots\]

After applying the fact that \( d \) is a derivation with respect to \( \cdot \), along with \( \beta_{i,p} + |x_{i,p}| = \beta_{i,p+1} \) we get

\[d \circ b(x_{k,n})\]

\[= \ldots + (-1)^{\beta_{i,p-1}}+p-1(\ldots \otimes d(x_{i,p-1}) \otimes x_{i,p} \cdot x_{i,p+1} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p}}+p-1(\ldots \otimes d(x_{i,p}) \cdot x_{i,p+1} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p+1}}+p-1(\ldots \otimes x_{i,p} \cdot x_{i,p+1} \otimes d(x_{i,p+2}) \otimes \ldots) + \ldots\]

Next we consider \( b \circ d \) and observe

\[b \circ d(x_{k,n}) = b \circ d(\ldots \otimes x_{i,p-1} \otimes x_{i,p} \otimes x_{i,p+1} \otimes x_{i,p+2} \otimes \ldots)\]

\[= b(\ldots + (-1)^{\beta_{i,p-1}}+n(\ldots \otimes d(x_{i,p-1}) \otimes x_{i,p} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p}}+n(\ldots \otimes d(x_{i,p}) \otimes x_{i,p+1} \otimes \ldots) + \ldots + (-1)^{\beta_{i,p+1}}+n(\ldots \otimes x_{i,p} \otimes d(x_{i,p+1}) \otimes \ldots) + \ldots + (-1)^{\beta_{i,p+2}}+n(\ldots \otimes x_{i,p+1} \otimes d(x_{i,p+2}) \otimes \ldots) + \ldots + (-1)^{\beta_{i,p-1}}+p(\ldots \otimes d(x_{i,p-1}) \otimes x_{i,p} \cdot x_{i,p+1} \otimes \ldots) + \ldots\]
... + (−1)^{β(i,p)+p} (\ldots \otimes d(x(i,p)) \cdot x(i,p+1) \otimes \ldots ) +
... + (−1)^{β(i,p+1)+p} (\ldots \otimes x(i,p) \cdot d(x(i,p+1)) \otimes \ldots ) +
... + (−1)^{β(i,p+2)+p} (\ldots \otimes x(i,p+1) \cdot x(i,p+1) \otimes d(x(i,p+2)) \otimes \ldots ) + \ldots

By comparing the corresponding terms we see that \( d \circ b + b \circ d = 0 \).

Similar arguments apply when considering other tensor-factors as well as when \( b \) collapses the first or last columns; in which case one has to apply the derivation \( d \) over a product of three tensor-factors.

Next we define a shuffle product \( \otimes : CH_{ZZ}(A) \otimes CH_{ZZ}(A) \to CH_{ZZ}(A) \).

Recall that for \( CH^I(A) \), we could define a shuffle product, but it would not be compatible with the usual Hochschild differential \( D \) unless \( A \) was a commutative DGA. For \( CH_{ZZ}(A) \), the idea is to concatenate the two monomials while shuffling in 1’s like so:
Remark 1.2.3 (Some comments on shuffles and their signatures). Consider the set $S_{n,m}$ of $(n, m)$ shuffles: $S_{n,m} = \{ \sigma \in S_{n+m} | \sigma(1) < \ldots < \sigma(n), \ \text{and,} \ \sigma(n+1) < \ldots < \sigma(n+m) \}$. We can use these shuffles to get maps $\sigma : A^\otimes n \times A^\otimes m \to A^\otimes n+m$ via

$$
\sigma((a_1 \otimes \ldots \otimes a_n), (a_{n+1} \otimes \ldots \otimes a_{n+m})) := a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n+m)}.
$$

Next, since we will be shuffling in 1’s on zigs and zags in the opposite order, we offer a formal convention to describe that process. For an element $E = e_1 \otimes \ldots \otimes e_p \in A^\otimes p$, we define $\overline{E} := e_p \otimes \ldots \otimes e_1 \in A^\otimes p$. Now, for a shuffle $\sigma \in S_{n,m}$ interpreted as $\sigma : A^\otimes n \times A^\otimes m \to A^\otimes n+m$, we can define $\sigma^{\text{III}} \in S_{n,m}$ giving the induced map $\sigma^{\text{III}} : A^\otimes n \times A^\otimes m \to A^\otimes n+m$ to be

$$
\sigma^{\text{III}}(L, R) := \overline{\sigma(L, R)}.
$$

For a given $i \in \mathbb{N}$ we will define $\sigma^{\text{III}_i}$ to be $\sigma$ if $i$ is odd (i.e. on a “zig”) and $\sigma^{\text{III}}$ if $i$ is even (i.e. on a “zag”). Finally, recall that for a shuffle (or permutation of any type) we can define the signature of that shuffle by $\text{sgn}(\sigma) = (-1)^{N(\sigma)}$ where $N(\sigma)$ is the number of transpositions needed to write $\sigma$ as a product purely of transpositions. Note that $N(\sigma)$ is well defined $\text{mod} \ 2$. By our convention, for reasons having nothing to do with zigs and zags, we will need to consider $\text{sgn}(\sigma^{\text{III}})$. It is straightforward to prove that $\text{sgn}(\sigma^{\text{III}}) = (-1)^{nm} \text{sgn}(\sigma)$ where $\sigma$ shuffles an $n$-tuple with an $m$-tuple.
Definition 1.2.4. The shuffle product $\odot$ defined for $CH_{\ZZ}(A)$ is defined on monomials by

$$\bar{x}_{k,n} \odot \bar{y}_{l,m} := \left( x^{\mathcal{L}} \otimes \ldots \otimes \left( (x_{(i,1)} \otimes x_{(i,2)} \otimes \ldots \otimes x_{(i,n)}) \otimes x_{i}^{\mathcal{L}} \right) \otimes \ldots \otimes x_{k}^{\mathcal{L}} \right) \odot \left( y^{\mathcal{L}} \otimes \ldots \otimes \left( (y_{(j,1)} \otimes y_{(j,2)} \otimes \ldots \otimes y_{(j,m)}) \otimes y_{j}^{\mathcal{L}} \right) \otimes \ldots \otimes y_{l}^{\mathcal{L}} \right)$$

:= \sum_{\sigma \in S_{n,m}} (-1)^{\epsilon} x^{\mathcal{L}} \otimes \ldots \otimes \left( \sigma^{\mathcal{III}}((x_{(i,1)} \otimes \ldots \otimes x_{(i,n)}), (1,\ldots,1)) \otimes x_{i}^{\mathcal{L}} \right) \otimes \ldots \otimes (x_{k}^{\mathcal{L}} \cdot y^{\mathcal{L}}) \otimes \ldots \otimes \left( \sigma^{\mathcal{III}}((1,\ldots,1), (y_{(j,1)} \otimes \ldots \otimes y_{(j,m)})) \otimes y_{j}^{\mathcal{L}} \right) \otimes \ldots \otimes y_{l}^{\mathcal{L}}$$

and extended linearly to all of $CH_{\ZZ}(A)$, where $\epsilon := |x_{k,n}| \cdot m + N(\sigma^{\mathcal{III}}) = (|x_{k,n}| + n) \cdot m + N(\sigma)$, using the abbreviation $|x_{k,n}| := |x^{\mathcal{L}}| + |x_{(1,1)}| + \ldots + |x_{k}^{\mathcal{L}}|$, and the shuffles are happening at all $i,j$ simultaneously for a given $\sigma$. 

Proposition 1.2.5. The shuffle product $\odot$ is associative and $D$ is a derivation with respect to $\odot$.

Proof. Associativity comes from the fact that placing zigzags on top of one another is an associative operation. We wish to show that $D(x \odot y) = D(x) \odot y + (-1)^{|x|+n} x \odot D(y)$. We note that $D(x) \odot y$ amounts to applying the differential only to the top zigzag where as $x \odot D(y)$ applies the differential only to the bottom zigzag. $D(x \odot y)$ applies first $d$ to each term, in which case you are either on the top or the bottom zigzag and so recovering
those terms is straightforward. Applying $b$ to $x \odot y$ has some terms which vanish and some terms which cancel with $x \odot D(y)$ or $D(x) \odot y$ as we will now demonstrate. Consider the two diagrams below:

The diagram on the left is a term in the shuffle product of two zigzags where two columns in the top zigzag remain adjacent. After applying the $b$-component of our differential $D$ to this new zigzag where we collapse the columns of the elements $c$ and $d$, we obtain the diagram on the right. This term can will cancel with the term coming from $D(x) \odot y$ which first collapses that column in the zigzag $x$ and then shuffles the result in with $y$ in such a way that the zigzag on the right (above) is obtained. A similar argument would show were we see cancelation with terms in $x \odot D(y)$ On the other hand, consider the three diagrams below:
where the left and middle diagrams are zigzags coming from different shuffles (off by a transposition and thus off by exactly \((-1)\)) and the right diagram is a common term shared in the result of the \(b\)-component of the differential \(D\) applied to these zigzags. Thus, cancelation occurs.

\[\square\]

Remark 1.2.6. Note that \((CH_{ZZ}(A), D, \circ)\) is a DGA, whose unit is the monomial with 0-many zigzags and a 1 in the only tensor-factor, and thus we can iterate the \(CH_{ZZ}\) functor to define the DGA \(CH_{ZZ}(CH_{ZZ}(\ldots(CH_{ZZ}(A))\ldots))\).

### 1.2.2 Special Cases

We begin by showing the Hochschild complex \(CH_{ZZ}(A)\) is compatible with \(CH^I(A)\) in the case where \(A\) is a commutative DGA. The other special case we are interested in is when our DGA comes from a commutative DGA tensored with an associative algebra. This case is motivated by considering
matrix-valued differential forms $\Omega^\bullet(M, \text{Mat}) \cong \Omega^\bullet(M, \mathbb{R}) \otimes \text{Mat}$.

**Definition 1.2.7.** If $\mathcal{C}$ is a commutative DGA, we have a column-collapse map $Col : CH_{ZZ}(\mathcal{C}) \to CH^1(\mathcal{C})$ defined as follows: Let $x_{(k,n)} \in CH_{ZZ}(\mathcal{C})$.

Then

$$Col(x_{(k,n)}) = (-1)^\epsilon Col^C(x_{(k,n)}) \otimes Col^1(x_{(k,n)}) \otimes \ldots \otimes Col^n(x_{(k,n)}) \otimes Col^R(x_{(k,n)})$$

where

$$Col^C(x_{(k,n)}) := x^C_1 \cdot x^C_2 \cdot \ldots \cdot x^C_k$$

$$Col^p(x_{(k,n)}) := \prod_{i=1 \atop i \; \text{odd}}^{k-1} x_{(i,p)} \cdot \prod_{i=2 \atop i \; \text{even}}^k x_{(i,n-p+1)}$$

$$Col^R(x_{(k,n)}) := x^R_1 \cdot x^R_3 \cdot \ldots \cdot x^R_{k-1}$$

Here by $\prod$ we refer to the ordered product induced by the algebra $\mathcal{C}$ and $\epsilon$ comes from the usual Koszul rule of changing the order of elements $x_{(i,p)}$.

**Proposition 1.2.8.** Let $\mathcal{C}$ be a commutative differential graded algebra. Then $Col : CH_{ZZ}(\mathcal{C}) \to CH^1(\mathcal{C})$ is a chain map and an algebra map.

**Proof.** It is straightforward to check that the differentials are compatible because of Leibniz and because we always collapse full rows. Similarly the shuffle products are compatible since the insertion of 1’s in an entire column amounts to the usual shuffle after collapsing. \(\square\)
Let $\mathcal{C}$ be a commutative DGA and $\mathcal{B}$ an associative algebra. Recall we have the associative DGA $\mathcal{C} \otimes \mathcal{B}$ generated by monomials $c \otimes b \in \mathcal{C}^n \otimes \mathcal{B}$ with differential $d : \mathcal{C}^n \otimes \mathcal{B} \to \mathcal{C}^{n+1} \otimes \mathcal{B}$ given by $d(c \otimes b) := d(c) \otimes b$ and associative product $(c \otimes b) \cdot (c' \otimes b') := (-1)^{|b|-|c'|}(c \cdot c' \otimes b \cdot b')$ which yields the product $(\mathcal{C} \otimes \mathcal{B})^{\otimes n} \times (\mathcal{C} \otimes \mathcal{B})^{\otimes n} \to (\mathcal{C} \otimes \mathcal{B})^{\otimes n}$. With all of this in mind we can consider the Hochschild complex $CH^I(\mathcal{C} \otimes \mathcal{B})$ and define a special shuffle product for it, where the idea is to shuffle the commutative part and push all of the information from $\mathcal{B}$ to the end, while preserving the order.

**Definition 1.2.9.** For $\mathcal{C}$ a commutative DGA and $\mathcal{B}$ an associative algebra, $CH^I(\mathcal{C} \otimes \mathcal{B})$ has the shuffle product:

$$(x \otimes \omega) \odot (y \otimes \nu) := \sum_{\sigma \in S_{n,m}} (-1)^{\epsilon_\sigma} \sigma ((x \otimes 1), (y \otimes 1)) \cdot ((1 \otimes 1) \otimes \ldots \otimes (1 \otimes \omega^L \cdot \ldots \cdot \omega_i \cdot \ldots \cdot \omega^R \cdot \nu^L \cdot \ldots \cdot \nu_i \cdot \ldots \cdot \nu^R))$$

where

$$(x \otimes \omega) := (x^L \otimes \omega^L) \otimes (x_1 \otimes \omega_1) \otimes \ldots \otimes (x_n \otimes \omega_n) \otimes (x^R \otimes \omega^R) \in (\mathcal{C} \otimes \mathcal{B})^{\otimes n}$$

and

$$(y \otimes \nu) := (y^L \otimes \nu^L) \otimes (y_1 \otimes \nu_1) \otimes \ldots \otimes (y_n \otimes \nu_n) \otimes (y^R \otimes \nu^R) \in (\mathcal{C} \otimes \mathcal{B})^{\otimes n}$$

**Proposition 1.2.10.** For $\mathcal{C}$ a commutative DGA and $\mathcal{B}$ an associative alge-
bra, the differential $D := d + b$ on $CH^I(C \otimes B)$ is a derivation of the shuffle product defined in Definition 1.2.9.

**Proof.** The proof is similar to the proof for Proposition 1.2.5 \(\square\)

The special case for $C \otimes B$ we are considering can also fall into the case where we consider it as a single (non-commutative) DGA with unit, and thus we could also define $CH_{ZZ}(C \otimes B)$. The column-collapse map in this case is almost the same as in Definition 1.2.7 but this time we push all of the elements from $B$ to the end, preserving order.

**Definition 1.2.11.** If $C$ is a commutative DGA and $B$ is an associative algebra, we have a column-collapse map $Col_{Mat} : CH_{ZZ}(C \otimes B) \to CH^I(C \otimes B)$ defined as follows:

\[
Col_{Mat}(x \otimes m_{(k,n)}) = (-1)^s Col^C_{Mat}(x \otimes m_{(k,n)}) \otimes \ldots \otimes Col^R_{Mat}(x \otimes m_{(k,n)})
\]

where

\[
Col^C_{Mat}(x \otimes m_{(k,n)}) := (x^C_1 \cdot x^C_2 \cdot \ldots \cdot x^C_k) \otimes 1
\]

\[
Col^R_{Mat}(x \otimes m_{(k,n)}) := \left( \prod_{i=1, \text{odd}}^{k-1} x_{(i,p)} \cdot \prod_{i=2, \text{even}}^{k} x_{(i,n-p+1)} \right) \otimes 1
\]

\[
Col^L_{Mat}(x \otimes m_{(k,n)}) := (x^R_1 \cdot x^R_3 \cdot \ldots \cdot x^R_{k-1}) \otimes (m^L \cdot \prod m_{(i,p)})
\]
where again by \( \prod \) we mean the ordered product induced by \( C \) or \( B \) and \( \epsilon \) again comes from the Koszul rule.

**Proposition 1.2.12.** Given a commutative DGA, \( C \), and an associative algebra \( B \), the column-collapse map \( \text{Col}_{\text{Mat}} : CH_{ZZ}(C \otimes B) \to CH^{I}(C \otimes B) \) is a chain map and an algebra map.

**Proof.** The proof is similar to the one for Proposition 1.2.8. \( \square \)

### 1.3 A Chen Map out of the ZigZag Hochschild Complex

We use \( CH_{ZZ}(A) \) to study non-abelian differential forms on the path space and so for the remainder of the paper, when we are considering matrix-valued differential forms, we write \( \Omega_{\otimes}(M) := \Omega(M, \text{Mat}) \), so as to distinguish from our real-valued forms \( \Omega(M) := \Omega(M, \mathbb{R}) \). First let us recall the usual interval Hochschild complex. Let \( \omega_n \in CH^{I}(\Omega(M)) \). We have an evaluation map

\[
\Delta^n \times M^I \overset{ev}{\longrightarrow} M^{n+2},
\]

\[
((t_1, \ldots, t_n), \gamma) \overset{ev}{\longmapsto} (\gamma(0), \gamma(t_1), \ldots, \gamma(t_n), \gamma(1))
\]

and so we use the composition

\[
\Omega(M)^{\otimes n+2} \to \Omega(M^{n+2}) \overset{ev^*}{\longrightarrow} \Omega(\Delta^n \times M^I) \overset{f_{\Delta^n}}{\longrightarrow} \Omega(M^I)
\]
to define $It(\omega) := \int_{\Delta^n} ev^*(\omega)$. Similarly we can use the evaluation map governed by placing differential forms along a zigzag at each time-slot and can define an iterated integral map out of $CH_{\ZZ}(\Omega_{\boxtimes}(M))$:

**Definition 1.3.1.** Let $\omega_{(n,k)} \in CH_{\ZZ}(\Omega_{\boxtimes}(M))$. We have an evaluation map

$$\Delta^n \times M^I \xrightarrow{ev} M^{nk+k+1},$$

$$\left( (t_1, \ldots, t_n), \gamma \right) \xrightarrow{ev} (\gamma(0), \gamma(t_1), \ldots, \gamma(t_n), \gamma(1),$$

$$\gamma(t_n), \ldots \gamma(t_1), \gamma(0),$$

$$\ldots \gamma(t_n), \ldots \gamma(t_1), \gamma(0) \right)$$

and so we use the composition

$$\Omega_{\boxtimes}(M)^{\otimes nk+k+1} \to \Omega_{\boxtimes}(M^{nk+k+1}) \xrightarrow{ev^*} \Omega_{\boxtimes}(\Delta^n \times M^I) \xrightarrow{\int_{\Delta^n}} \Omega_{\boxtimes}(M^I)$$

to define $It(\omega) := \int_{\Delta^n} ev^*(\omega)$. The evaluation map here can be clarified by the figure on page 16.

When applying the differential in the image of $CH_{\ZZ}(\Omega_{\boxtimes}(M)) \xrightarrow{H} \Omega_{\boxtimes}(M^I)$, we encounter the situation where two time-slots in $\Delta^n$ come together. For this reason, we recall the following Lemma which will be applied below without further reference:

**Lemma 1.3.2.** The map $\Omega_{\boxtimes}(M) \otimes \Omega_{\boxtimes}(M) \xrightarrow{EZ} \Omega_{\boxtimes}(M \times M) \xrightarrow{\Delta^*} \Omega_{\boxtimes}(M)$, where $EZ$ is the Eilenberg-Zilber map and $\Delta : M \times M \to M$ is the diagonal,
is given by the wedge product of forms.

We now arrive at our first important result and continue by establishing the relationship between $CH^{I}(\Omega(M))$, $CH_{HZ}(\Omega_{\mathbb{R}}(M))$, and their Chen maps.

**Proposition 1.3.3.** The Iterated Integral $I^{I} : CH_{HZ}(\Omega_{\mathbb{R}}(M)) \to \Omega_{\mathbb{R}}(M^{I})$ is a chain map.

**Proof.** We have $d_{DR} \int_{\Delta^{n}} e^{*}(\omega) = (-1)^{n} \int_{\Delta^{n}} d_{DR}e^{*}(\omega) + (-1)^{n-1} \int_{\partial\Delta^{n}} e^{*}(\omega)$.

Now we use: (a) the chain map $\Omega_{\mathbb{R}}(M)^{\otimes k} \to \Omega_{\mathbb{R}}(M^{k})$ along with the exterior derivative acting as a derivation and (b) the commutative diagram

\[
\begin{array}{ccc}
\Delta^{n-1} \times M^{I} & \xrightarrow{e^{*}} & M^{(n-1)k+k+1} \\
\downarrow d_{i} \times \text{id} & & \downarrow d_{i} \\
\Delta^{n} \times M^{I} & \xrightarrow{e^{*}} & M^{nk+k+1}
\end{array}
\]

where $d_{i} : \Delta^{n-1} \to \Delta$ is the map $(t_{1}, \ldots, t_{i}, \ldots t_{n-1}) \mapsto (t_{1}, \ldots, t_{i}, t_{i}, \ldots t_{n-1})$ and $d_{i}$ on the right arrow is, by abuse of notation, the diagonal making the diagram commute. So then the above equation can be expressed as:

\[
d_{DR} \int_{\Delta^{n}} e^{*}(\omega) = (-1)^{n} \int_{\Delta^{n}} d_{DR}e^{*}(\omega) + (-1)^{n-1} \sum_{i} \int_{\Delta^{n-1}} (d_{i} \times \text{id})^{*}(e^{*}(\omega))
\]

\[
= (-1)^{n} \sum_{p} (-1)^{\beta_{p}} \int_{\Delta^{n}} e^{*}(\ldots \otimes d_{DR}(\omega) \otimes \ldots)
\]

\[
+ (-1)^{n-1} \sum_{i} (-1)^{i-1} \int_{\Delta^{n-1}} e^{*}(\ldots \otimes (\omega \cdot \omega') \otimes \ldots)
\]
\[
\int_{\Delta^n} ev^*(d(\omega)) + \int_{\Delta^{n-1}} ev^*(b(\omega)) = It(D(\omega))
\]

Remark 1.3.4. By a similar calculation, we have that \( It : CH^I(\Omega(M)) \to \Omega(M^I) \) is a chain map for any coefficients since we never have to commute forms in the differential of \( CH^I(\Omega(M)) \).

We want to show that \( It : CH_{ZZ}(\Omega(\mathbb{M})(M)) \to \Omega(\mathbb{M})(M^I) \) is also an algebra map. As a warm-up, we first recall why in the abelian case, \( It : CH^I(\Omega(M, \mathbb{R})) \to \Omega(M^I, \mathbb{R}) \) is an algebra map.

**Proposition 1.3.5.** The Iterated Integral \( CH^I(\Omega(M)) \to \Omega(M^I) \) is a map of algebras.

**Proof.** We have degeneracy maps \( s_i : \Delta^{r+1} \to \Delta^r \), given by

\[
(t_1, \ldots, t_r) \mapsto (t_1, \ldots, \hat{t_i}, \ldots, t_r)
\]

yielding

\[
\begin{array}{ccc}
\Delta^{r+1} & \xrightarrow{ev} & M^{r+1}+2 \\
\downarrow{s_i} & & \downarrow{s_i} \\
\Delta^r & \xrightarrow{ev} & M^{r+2}
\end{array}
\]

where \( s_i \) on the right arrow is the induced projection making the diagram commute. So by composing these degeneracy maps (and again abusing no-
tation) we obtain for a fixed shuffle $\sigma \in S_{n,m}$ the commutative diagram:

$$
\begin{array}{ccc}
\Delta^{n+m} \times M^I & \xrightarrow{ev_{n+m}} & M^{m+n+2} \\
\downarrow{\beta \times id} & & \downarrow{\rho^\sigma} \\
\Delta^m \times \Delta^n \times M^I & \xrightarrow{(ev_n, ev_m)} & M^{m+2} \times M^{n+2}
\end{array}
$$

Here $\beta^\sigma : \Delta^{n+m} \to \Delta^n \times \Delta^m$ is the unshuffle map

$$(t_1, \ldots, t_{n+m}) \mapsto ((t_{\sigma(1)}, \ldots, t_{\sigma(n)}), (t_{\sigma(n+1)}, \ldots, t_{\sigma(n+m)}))$$

and $\rho^\sigma$ on the right is the map which makes the diagram commute. Now we claim that

$$\int_{\Delta^m} ev^*(\omega) \wedge \int_{\Delta^n} ev^*(\nu) = \int_{\Delta^{m+n}} ev^*(\omega \circ \nu)$$

We begin proving this claim by noting that we can evaluate the left hand side

$$\int_{\Delta^n} ev^*_n(\omega) \wedge \int_{\Delta^n} ev^*_m(\nu)$$

$$= (-1)^{|\omega|+n-m} \int_{\Delta^n \times \Delta^n} (ev_n, ev_m)^*(\omega \otimes \nu)$$

$$= (-1)^{|\omega|+n-m} \prod_{\sigma \in S_{n,m}} (ev^*_n, ev^*_m)(\omega \otimes \nu)$$

$$= \sum_{\sigma \in S_{n,m}} (-1)^{|\omega|+n-m} \cdot sgn(\sigma) \int_{\Delta^{m+n}} (\beta^\sigma \times id)^*(ev^*_n, ev^*_m)(\omega \otimes \nu)$$

$$= \int_{\Delta^{m+n}} ev^*_n(\sum_{\sigma \in S_{n,m}} (-1)^{|\omega|+n-m} \cdot sgn(\sigma)(\rho^\sigma)^*(\omega \otimes \nu))$$

$$= \int_{\Delta^{m+n}} ev^*_n(\omega \circ \nu)$$
Remark 1.3.6. In the above proof, the reason why we need abelian coefficients can be seen by the fact that the map $\rho^\sigma : M^{m+n+2} \to M^{m+2} \times M^{n+2}$ switches the order of coordinates. In the following proposition and proof, no coordinates are switched, which is exactly why we can correctly model the wedge product of two Iterated Integrals with our shuffle product.

**Proposition 1.3.7.** The Iterated Integral $\text{CH}_{\text{ZZ}}(\Omega_{\boxplus}(M)) \to \Omega_{\boxplus}(M^I)$ is a map of algebras.

**Proof.** The proof is analogous to the above proposition, so we include the commutative diagram in this case for clarity:

\[
\begin{array}{ccc}
\Delta^{n+m} \times M^I & \overset{\text{ev}_{n,m,k}}{\longrightarrow} & M^{(m+n+1)(k_n+k_m)+1} \\
\downarrow^{\beta^\sigma \times \text{id}} & & \downarrow^{\pi^\sigma} \\
\Delta^m \times \Delta^n \times M^I & \overset{\text{ev}_{m,k_m}, \text{ev}_{n,k_n}}{\longrightarrow} & M^{(m+1)k_m+1} \times M^{(m+1)k_m+1}
\end{array}
\]

where $\beta^\sigma$ is the same as in Proposition 1.3.5, $\pi^\sigma$ is the map which makes the diagram commute, and the evaluation maps $\text{ev}_{\bullet, \bullet}$ are the ones defined in Definition 1.3.1. Then

\[
\int_{\Delta^n} \text{ev}_{n,k_n}^*(\omega) \wedge \int_{\Delta^m} \text{ev}_{m,k_m}^*(\nu) = \sum_{\sigma \in S_{n,m}} (-1)^{(|\omega|+n)-m} \cdot \text{sgn}(\sigma) \int_{\Delta^{n+m}} (\beta^\sigma \times \text{id})^*(\text{ev}_{n,k_n}, \text{ev}_{m,k_m})^*(\omega \otimes \nu)
\]
= \int_{\Delta^{n+m}} \sum_{\sigma} (-1)^{(|\omega|+n) \cdot m} \cdot \text{sgn}(\sigma)(\pi^\sigma)^*(\omega \otimes \nu)
\]

Lemma 1.3.8. The collapse map \(\text{Col} : CH_{ZZ}(\Omega(M, R)) \to CH^I(\Omega(M, R))\) from Definition 1.2.7 is given by \(zz^*\) where \(zz : M^{n+2} \to M^{nk+k+1}\) is a particular diagonal map. In other words, we have the commutative diagram:

\[
\begin{array}{ccc}
CH_{ZZ}(\Omega(M)) & \xrightarrow{EZ} & \Omega(M^{nk+k+1}) \\
\text{Col} \downarrow & & \downarrow zz^* \\
CH^I(\Omega(M)) & \xrightarrow{EZ} & \Omega(M^{n+2})
\end{array}
\]

Proof. This fact is another consequence of Lemma 1.3.2. \(\square\)

Proposition 1.3.9. If \(\Omega(M) = \Omega(M, \mathbb{R})\) then we have the following commutative diagram of DGAs.

\[
\begin{array}{ccc}
CH_{ZZ}(\Omega(M)) & \xrightarrow{\text{ev}^Z} & \Omega(M^I) \\
\downarrow & & \downarrow \\
CH^I(\Omega(M))
\end{array}
\]

Proof. We observe the following commutative diagram

\[
\begin{array}{ccc}
M^{nk+k+1} & \xrightarrow{\text{ev}_{ZZ}} & \Delta^n \times M^I \\
zz \downarrow & & \downarrow \text{ev}^I \\
M^{n+2}
\end{array}
\]
combined with the lemma above we obtain our result from the diagram

\[
\begin{align*}
CH_{ZZ}(\Omega(M)) & \rightarrow \Omega(M^{n+k+1}) \xrightarrow{(ev_{ZZ})^*} \Omega(\Delta^n \times M^I)^{\Delta_n} \rightarrow \Omega(M^I) \\
& \downarrow Col \downarrow \quad \downarrow zz^* \\
CH^I(\Omega(M)) & \rightarrow \Omega(M^{n+2}) \xrightarrow{(ev^I)^*} \Omega(\Delta^n \times M^I)^{\Delta_n} \rightarrow \Omega(M^I)
\end{align*}
\]

\[\square\]

**Remark 1.3.10.** In the case of non-abelian coefficients, \(\Omega_{\infty}(M) = \Omega(M, Mat)\), both \(It : CH_{ZZ}(\Omega_{\infty}(M)) \rightarrow \Omega_{\infty}(M^I)\) and \(It : CH^I(\Omega_{\infty}(M)) \rightarrow \Omega_{\infty}(M^I)\) are chain maps, while \(Col : CH_{ZZ}(\Omega_{\infty}(M)) \rightarrow CH^I(\Omega_{\infty}(M))\) is not a chain map. However, in this case the diagram from Propostition 1.3.9 does not commute either.

### 1.4 The Two-Dimensional ZigZag Hochschild Complex(es)

Given a commutative DGA, \((A, d, \cdot)\), we have the Hochschild complex of the Hochschild complex, the rectangular Hochschild complex, and the square Hochschild complex, denoted \(CH^I(CH^I(A))\), \(CH^I_{Rec}(A)\), and \(CH^I_{Sq}(A)\), respectively. Note that for a non-commutative DGA, \((A, d, \cdot)\), the 2-dimensional (both simplicial and bisimplicial) higher Hochschild structures do not form a complex \([GTZ]\). In this section we give two related 2-d Hochschild complexes using the ideas from \(CH_{ZZ}(A)\).
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Given an associative DGA, \((A,d,\cdot)\), we have the Hochschild complex of the zigzag Hochschild complex, denoted \(CH^I(CH_{ZZ}(A))\), and below we define the rectangular zigzag Hochschild complex and the square zigzag Hochschild complex, denoted \(CH^{Rec}_{ZZ}(A)\) and \(CH^{Sq}_{ZZ}\), respectively.

**Definition 1.4.1.** Let \((A,d,\cdot)\) be a DGA. The rectangular zigzag Hochschild complex has underlying vector space

\[ CH^{Rec}_{ZZ}(A) := \bigoplus_{m,n \geq 0} (A \otimes (A \otimes \ldots \otimes A) \otimes \ldots \otimes (A \otimes \ldots \otimes A) \otimes \ldots \otimes (A \otimes \ldots \otimes A) \otimes k_{m+1})[n+m] \]

Monomials, \((x_{i,j,n})^{m+1}_{j=0}\), are to be thought of as \(m+2\)-many rows of elements
\( \mathfrak{x}^j_{i, n} \in CH_{ZZ}(A), \)

with differential

\[
D((\mathfrak{x}^0_{k_0, n}) \otimes \ldots \otimes (\mathfrak{x}^{m+1}_{k_{m+1}, n})) \\
:= \sum_{r=0, p=0}^{m+1, n+1} (-1)^{n+m+\beta_{r,p}}(\mathfrak{x}^0_{k_0, n}) \otimes \ldots \otimes d_p((\mathfrak{x}^r_{k_r, n})) \otimes \ldots \otimes (\mathfrak{x}^{m+1}_{k_{m+1}, n}) \\
+ \sum_{p=0}^{n} (-1)^{m+n+p} b_p((\mathfrak{x}^0_{k_0, n}) \otimes \ldots \otimes (\mathfrak{x}^{m+1}_{k_{m+1}, n}))
\]
where \( d_p \) is the differential coming from \( A \) applied to exactly the \( p \)-th slot of \((x_{k,n})\) and \( b_p \) is collapsing/multiplying the slots in the \( p \)-th and \((p + 1)\)-th columns of \((x_{k,n})\). The operation \( \star \) is simply a concatenation of the two zigzags:

\[
a_{k,n} \star b_{l,n} := a^L \otimes (a_{(1,1)} \otimes \ldots \otimes a_{(1,n)} \otimes a^R_{1}) \otimes \\
\ldots \otimes (a_{(i,1)} \otimes \ldots \otimes a_{(i,n)} \otimes a^L_i) \otimes \\
\ldots \otimes a_{(k,1)} \otimes \ldots \otimes a_{(k,n)} \\
\otimes (a^C_k \cdot b^C_{k}) \otimes (b_{(1,1)} \otimes \ldots \otimes b_{(1,n)} \otimes b^R_1) \otimes \\
\ldots \otimes (b_{(i,1)} \otimes \ldots \otimes b_{(i,n)} \otimes b^C_i)
\]

**Proposition 1.4.2.** Let \((A,d,\cdot)\) be an associative DGA, then \( CH^{Rec}_{ZZ}(A)\) forms a complex.

**Proof.** The components \(d\) and \(b\) of \(D\) are just as in Proposition 1.2.2. For the new \(\star\) part, it is straightforward to check that \(\star d + d\star = 0\), \(\star^2 = 0\), and \(\star b + b\star = 0\). 

**Definition 1.4.3.** Let \((\omega^j_{k,n})_{j=0}^{m+1} \in CH^{Rec}_{ZZ}(\Omega_\boxplus(M))\). We define the iterated
integral, $CH_{ZZ}(\Omega(M)) \xrightarrow{It} \Omega(M^{Sq})$, by

$$It((\omega_{k_j,n}^{j+1})_{j=0}^{m+1}) := \int_{\Delta^n \times \Delta^m} ev_{k,n}^{\ast} ((\omega_{k_j,n}^{j+1})_{i=0}^{m+1})$$

where we use the evaluation map, with $k := (k_0, k_1, \ldots, k_{m+1})$,

$$\Delta^n \times \Delta^n \times M^{Sq} \xrightarrow{ev_{k,n}} M \sum_{i=0}^{m+1} n_{k_i + k_i + 1}$$

$$(t_1, \ldots, t_n, s_1, \ldots, s_m, \Sigma)$$

$$\mapsto (\Sigma(0, 0), \Sigma(0, t_1), \ldots, \Sigma(0, t_n), \Sigma(0, 1), \Sigma(0, t_n), \ldots, \Sigma(0, 0), \ldots,$$

$$\Sigma(s_1, 0), \Sigma(s_1, t_1), \ldots, \Sigma(s_1, t_n), \Sigma(s_1, 1), \ldots, \Sigma(s_1, 0), \ldots,$$

$$\vdots$$

$$\Sigma(s_m, 0), \Sigma(s_m, t_1), \ldots, \Sigma(s_m, t_n), \Sigma(s_m, 1), \ldots, \Sigma(s_m, 0), \ldots,$$

$$\Sigma(1, 0), \Sigma(1, t_1), \ldots, \Sigma(1, t_n), \Sigma(1, 1), \ldots, \Sigma(1, 0))$$

While the $d_p$ and $b_p$ above will be similar to the pieces of the 1-d zigzag story, something should be said about the $\ast$ component of the differential.

Consider a very simple monomial (pictured below) on which we’d like to observe the star operation, say two zig zags with $n = 2$, each having $k = 1$:

i.e. $1 \otimes \mathcal{Z}_{(1,2)}^1 \otimes \mathcal{Z}_{(1,2)}^2 \otimes 1$ with $a = 1$, $\mathcal{Z}_{(1,2)} = b \otimes c \otimes \ldots \otimes h$, $\mathcal{Z}_{(1,2)}' = i \otimes j \otimes \ldots \otimes o$, and $p = 1$. 
When applying star to the two no-trivial zigzags above, \( 1 \otimes (x_{(1,2)} \ast x'_{(1,2)}) \otimes 1 \), the picture above simply becomes
The maps which describe this $\star$ operation include the diagonal

$$\Delta^1 \times \Delta^2 \times M^{Sq} \to \Delta^2 \times \Delta^2 \times M^{Sq}$$

given by $(s, (t_1, t_2), \Sigma) \mapsto ((s, s), (t_1, t_2), \Sigma)$

and

$$M^{15} \to M^{16}$$

given by $(x_1, \ldots, x_{15}) \mapsto (x_1, \ldots, x_7, x_8, x_8, x_9, \ldots, x_{15})$
yielding the commutative diagram

$$
\Delta^1 \times \Delta^2 \times M^{Sq} \xrightarrow{ev(0,2,2,0,2)} M^{15} \\
\Delta^2 \times \Delta^2 \times M^{Sq} \xrightarrow{ev(0,4,0,2)} M^{16}
$$

So in our figures above, the multiplication of exactly $h$ and $i$ comes from the fact that the map $M^{15} \to M^{16}$ had exactly one diagonal map built into it.

Also recall that before the star operation, we had $k_1 = 2$ and $k_2 = 2$, and after the star operation we had $m = 1$ and $k_1 = 4$. In general, the evaluation maps in the commutative diagram need to keep track of the $k_j$'s in order for $CH^{Rec}_\ZZ(\Omega_\oplus(M))$ to be relevant to, let alone be a model for, $\Omega_\oplus(M^{Sq})$. In general, when we want to investigate

$$
It \left( (x^0_{(k_0,n)}) \otimes \ldots \otimes (x^r_{(k_r,n)} \star (x^{r+1}_{(k_{r+1},n)})) \otimes \ldots (x^{m+1}_{(k_{m+1},n)}) \right)
$$

we use the commutative diagram with $\phi_j = nk_j + k_j + 1$,

$$
\Delta^{m-1} \times \Delta^n \times M^{Sq} \xrightarrow{ev(k_0,\ldots,k_{r-1},k_r+k_{r+1},k_{r+2},\ldots,k_{m+1}),n} M^{\sum_{j=0}^{m+1} \phi_j - 1} \\
\Delta^m \times \Delta^n \times M^{Sq} \xrightarrow{ev(k_0,\ldots,k_{m+1}),n} M^{\sum_{j=0}^{m+1} \phi_j}
$$

and see that the $\star$ operation corresponds to the wedge product during concatenation as shown in the figures above. In the calculation of $d(\int_{\Delta^m \times \Delta^m} ev^*(\omega))$
we then obtain precisely the new term $\int_{\Delta^n \times \partial \Delta^n} ev^{\ast}(\omega)$ (see Proposition 1.3.3). We therefore have:

**Proposition 1.4.4.** The iterated integral $CH_{ZZ}^{Rec}(\Omega_{\mathbb{H}}(M)) \xrightarrow{I} \Omega_{\mathbb{H}}(M^{Sq})$ is a chain map.

Below we briefly mention a variation of the 2-d Rectangular Hochschild complex and use the fact that since $CH_{ZZ}(A)$ is a DGA, we can take $CH^I(CH_{ZZ}(A))$.

**Definition 1.4.5.**

$$CH_{ZZ}^{Sq}(A) := \bigoplus_{n \geq 0 \atop k_0, \ldots, k_{n+1} \geq 0} (A \otimes (A^\otimes n \otimes A)^{\otimes k_0} \otimes \cdots \otimes A \otimes (A^\otimes n \otimes A)^{\otimes k_{n+1}})[n]$$

is to be thought of as $(n+2)$-many rows of elements $((\omega_i^{(k_i,n)}) \in CH_{ZZ}(A)$, with differential

$$D((\underline{z}^0_{(k_0,n)}) \otimes \cdots \otimes (\underline{z}^n_{(k_{n+1},n)}))$$

$$:= \sum_{r=0}^{n} (-1)^{n+r} b_r \left( (\underline{z}^0_{(k_0,n)}) \otimes \cdots \otimes (\underline{z}^r_{(k_r,n)}) \star (\underline{z}^{r+1}_{(k_{r+1},n)}) \otimes \cdots \otimes (\underline{z}^n_{(k_{n+1},n)}) \right)$$

$$+ \sum_{r=1, \atop p=1}^{n,n} (-1)^{n+\beta_{r,p}} (\underline{z}^0_{(k_0,n)}) \otimes \cdots \otimes d_p((\underline{z}^r_{(k_r,n)})) \otimes \cdots \otimes (\underline{z}^{n+1}_{(k_{n+1},n)})$$

where $d_p$ is the differential coming from $A$ applied to exactly the $p$-th slot of $(\underline{z}_{(k,n)})$ and $b_r$ is collapsing/multiplying the $r$-th and $(r+1)$-th slots of $(\underline{z}_{(k,n)})$. Note that we use the fact that collapsing two columns commutes with the $\star$ product of two rows coming from elements in $CH_{ZZ}(A)$. 


Proposition 1.4.6. If $A$ is a commutative DGA, then using our map from Proposition 1.2.8 there exists a commutative diagram of chain complexes:

\[
\begin{array}{ccc}
CH^I(CH_{ZZ}(A)) & \longrightarrow & CH^I(CH^I(A)) \\
\downarrow & & \downarrow \\
CH_{Rec}^I(A) & \longrightarrow & CH_{Rec}^{I\times I}(A) \\
\downarrow & & \downarrow \\
CH_{ZZ}^{S_q}(A) & \longrightarrow & CH_{S_q}^{I\times I}(A)
\end{array}
\]

where the horizontal arrows come from collapsing zigzags and the vertical arrows come from adding in degeneracies (see for example [GTZ], Corollary 2.4.4, for the maps on the right).

1.5 The ZigZag Hochschild Complex for a Curved DGA

1.5.1 The One-Dimensional Case

We have in mind integrating differential forms along our zigzags as before, but now we would like to apply parallel transport between the variables at which the differential forms are sitting. When we take the De Rham differential after integration, we will have some extra terms show up. For this reason we define the curved zigzag Hochschild complex, using the same underlying vector space as in the “1-d” case, $CH_{ZZ}(A)$, but with an additional component added to its differential.
Definition 1.5.1. Let \((A, \cdot, d)\) be a DGA and let \(A \in A\) be an element of degree 1. Then denote by \(\nabla : A \to A\), \(\nabla(x) := dx + [A, x]\), with \(\nabla_R(x) := dx + (-1)^{|x}|x \cdot A\) and \(\nabla_L(x) := dx + A \cdot x\). The curved zigzag Hochschild complex is defined as

\[CH_{ZZ}(A) = \bigoplus_{n,k \geq 0, k \text{ even}} (A \otimes (A^{\otimes n} \otimes A)^{\otimes k})[n],\]

with differential \(D : CH_{ZZ}(A) \to CH_{ZZ}(A)\) given by \(D(x_{k,n}) := (\nabla + b + c)(x_{k,n})\). We define these three components below:

\[
\nabla(x_{k,n}) := (-1)^n \nabla_R(x^L) \otimes \ldots \otimes x^L_k \\
+ \sum_{p=1}^{n} \sum_{i=1}^{k} (-1)^{n+\beta_{i,p}} x^L \otimes \ldots \otimes \nabla(x_{i,p}) \otimes \ldots \otimes x^L_k \\
+ \sum_{i=1}^{k-1} (-1)^{n+\beta^L_{i,i}} x^L \otimes \ldots \otimes \nabla(x^L_i) \otimes \ldots \otimes x^L_k \\
+(-1)^{n+\beta^L_k} x^L \otimes \ldots \otimes \nabla_L(x^L_k)
\]

\(b\) is defined in exactly the same way as in Definition 1.2.1, and

\[c(x_{k,n}) := \sum_{\sigma \in S_{n,1}} c^R_\sigma(x_{k,n})\]

\[= \sum_{\sigma \in S_{n,1}} \sum_{i=1}^{k} (-1)^{n+\sigma(n+1)} x^L \otimes \ldots \otimes \nabla((x_{i,1} \otimes \ldots \otimes x_{i,n}), R) \otimes x^L_{i+} \otimes \ldots \otimes (\sigma^{III_i}((x_{j,1} \otimes \ldots \otimes x_{j,n}), 1) \otimes x^L_{j+})\ldots
\]

The term \(c^R_\sigma\) shuffles in exactly one new column in \(x_{k,n}\) which consists of all
1’s except for exactly one $R$, and whose placement is determined by $\sigma(n+1)$ (see Remark 1.2.3 for the definition of $\sigma$ and $\sigma^{III}$):

Moreover, we have the same shuffle product, $\circ$, for the curved zigzag Hochschild complex as we do in the zigzag Hochschild complex, defined in Definition 1.2.4.

**Proposition 1.5.2.** For a unital DGA, $(A, \cdot, d)$ with $A \in \mathcal{A}^1$, the curved differential $D$ on $CH_{ZZ}(A)$ satisfies $D^2 = 0$. Furthermore, $D$ is a (graded) derivation of $\circ$.

**Proof.** To see that $D^2 = 0$, we observe $\nabla^2 + cb + bc = 0$, $b^2 = c^2 = 0$, $\nabla b + b \nabla = 0$, and $\nabla c + c \nabla = 0$. The proof is straightforward but the reader should note that the terms $\nabla_L$ and $\nabla_R$ correspond to $R$ being multiplied to the left or the right of the endpoints, respectively, when considering $b \circ c$. The proof that $D$ is a derivation of $\circ$ is comparable to that of Proposition 1.2.5 if one replaces all instances of $d$ with $\nabla$. □
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The setup we actually have in mind for this curved algebra case is when $\mathcal{A} = \Omega_\heartsuit(M) := \Omega(M, \text{Mat})$ is the DGA of matrix-valued forms on $M$, or more generally, $\Omega_\heartsuit(M) = \Omega(M, g)$ where $g$ is a Lie-Algebra whose bracket comes from an underlying product. Now we fix a 1-form $A \in \Omega^1_\heartsuit(M)$. We wish to define a map $It^A : CH_{\text{ZZ}}(\Omega_\heartsuit(M)) \rightarrow \Omega_\heartsuit(M^I)$. We illustrate the map on a simple monomial in $CH_{\text{ZZ}}(\Omega_\heartsuit(M))$. So consider the following element of $\Omega_\heartsuit(M)^{\otimes 7}[2] \subset CH_{\text{ZZ}}(\Omega_\heartsuit(M))$, where the evaluation map normally used in the non-curved case $\Delta^2 \times M^I \rightarrow M^7$ should be evident from the picture,

\[
\omega^L \otimes \omega_{(1,1)} \otimes \omega_{(1,2)} \otimes \omega^R_1 \otimes \omega_{(2,1)} \otimes \omega_{(2,2)} \otimes \omega^L_2 \in \Omega_\heartsuit(M)^{\otimes 7}[2]
\]

which represents an element
Next, we apply a map of degree zero\(^3\),

\[
Ins^A_{(q_1, q_2, \ldots, q_6)} : \Omega_{\otimes 7}(M)^{\otimes 7}[2] \to \Omega_{\otimes 7}(M)^{\otimes 7+\sum q_i[2 + \sum q_i]}
\]

which will insert some \(A\)'s as prescribed by the \(q_i \in \mathbb{Z}_{\geq 0}\). So applying

\[
Ins^A_{(2,3,1,2,0,5)} : \Omega_{\otimes 7}(M)^{\otimes 7}[2] \to \Omega_{\otimes 20}(M)^{\otimes 20}[15]
\]

to our element in \(CH_{ZZ}(\Omega_{\otimes 7}(M))\) above yields

resulting in an element

\[
\omega^c \otimes A^{\otimes 2} \otimes \omega_{(1,1)} \otimes A^{\otimes 3} \otimes \omega_{(1,2)} \otimes A \otimes \omega^R_1 \otimes A^{\otimes 2} \otimes \omega_{(2,1)} \otimes A^{\otimes 5} \otimes \omega^c_2
\]

in \(\Omega_{\otimes 20}(M)^{\otimes 20}[15]\) where, for example \(\Delta^{q_5} = \ast\) and

\[
\Delta^{q_6} = \{(\tau_1, \ldots, \tau_5) \in \mathbb{R}^5 | t_1 \geq \tau_1 \geq \tau_2 \geq \ldots \geq \tau_5 \geq 0\}.
\]

\(^3\)While we are inserting arbitrarily many 1-forms \(A\), they will all be shifted down by one-degree, since they each will be integrated along some interval at some \(\tau\). This is consistent with the rest of the shifts in the paper where we always shift a monomial by the dimension of the fiber we integrate over.
We consider the bounded convex polytope:

\[ E = \{ (\tau_1^1, \tau_1^2, \tau_2^2, \tau_3^1, \tau_1^3, \tau_2^4, \tau_1^4, \tau_4^6, \ldots, \tau_6^5, t_1, t_2) \mid 0 \leq t_1 \leq t_2 \leq 1, \]

\[ 0 \leq \tau_1^1 \leq \tau_1^2 \leq \tau_2^2 \leq \tau_3^1 \leq \tau_3^2 \leq \tau_4^2 \leq \tau_4^3 \leq \tau_5^1 \leq \tau_5^3 \leq \tau_6^1 \leq \ldots \leq \tau_6^5 \geq 0 \}. \]

Once again, we use the picture as our guide for the evaluation map, and define a summand of our iterated integral using the following diagram (ignoring degree-shifts):

\[
\begin{array}{ccc}
\Omega_{\otimes}(M_1 \times E) & \xrightarrow{exp^*} & \Omega_{\otimes}(M^{20}) \\
\downarrow f_E & & \downarrow \Omega_{\otimes}(M)^{\otimes 20} \\
\Omega_{\otimes}(M_I) & \xleftarrow{\text{Ins}^A} & \Omega_{\otimes}(M)^{\otimes 7}
\end{array}
\]

where \( q = (2, 3, 1, 2, 0, 5) \) in this case and then we can finally define the iterated integral map \( \Omega_{\otimes}(M)^{\otimes 7}[2] \xrightarrow{It^A} \Omega_{\otimes}(M_I) \) by

\[
It^A(\omega) := \sum_{q_1, \ldots, q_6 \geq 0} (-1)^{\sum |q_i| \sum_{i} q_i} It^A_{(q_1, \ldots, q_6)}(\omega),
\]

where here by \( \omega_i \) we mean the \( i \)-th tensor-factor in \( \omega = \omega_0 \otimes \cdots \otimes \omega_{(n+1)k+1} \) and we define \( Q_i := q_{i+1} + \ldots + q_{(n+1)k} \). In the same way we can now define the iterated integral in general. In the definition below, we fix a choice of 1-form, \( A \), for our curving.

**Definition 1.5.3.** Consider some monomial \( \omega_{k,n} \in \Omega_{\otimes}(M)^{\otimes \Phi}[n] \subset CH_{ZZ}(\Omega_{\otimes}(M)) \)
where $\phi = nk + k + 1$. For each choice $q_i \geq 0$, for $i = 1, \ldots, (n+1) \cdot k$ we have the corresponding diagram

$$\begin{align*}
\Omega^*(M^I \times E) &\xrightarrow{\exp^*} \Omega^*(M^{\phi_A}) \hookleftarrow \Omega^*(M^{\otimes \phi}) \xleftarrow{Ins^A} \Omega^*(M^{\otimes \phi}) \\
\Omega^*(M^I) &\xrightarrow{f_E} \Omega^*(M^I) \xleftarrow{I t^A} \Omega^*(M^I)
\end{align*}$$

where $\phi^A = \phi + \sum q_i$ and $q = (q_1, \ldots, q_{\phi-1})$. Then $\Omega^*(M^{\otimes [n]} \xrightarrow{I t^A} \Omega^*(M^I)$ is defined by $I t^A := \sum q \cdot I t^A$.

A point should be made about this infinite sum. In the case when we were considering $I t : \Omega^*(M^{\otimes 7}) \rightarrow \Omega^*(M^I)$ we were considering a cartesian product of 6 different infinite-sums of choices: $q_1, \ldots, q_6$. However, for a fixed path $\gamma$ and $(t_1, t_2)$, we want to simply compute the parallel transport between two points on $\gamma$ using a connection 1-form, $A$. In particular, since the image of $\gamma$ is compact in $M$, $|A|$ is bounded on $\gamma$ by some element $\rho \in \mathbb{R}$ and so we consider

$$|1 + \int_{\Delta^1} A + \int_{\Delta^2} AA + \ldots|$$

$$\leq |1| + |\int_{\Delta^1} A| + |\int_{\Delta^2} AA| + \ldots$$

$$\leq 1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \ldots$$

$$= \exp(\rho) \in \mathbb{R}.$$ 

This shows that the infinite sum in $I t^A$ indeed converges.
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Proposition 1.5.4. The map $I^A : CH_{\text{ZZ}}(\Omega_{\text{zz}}(M)) \to \Omega_{\text{zz}}(M^I)$ as defined above is a chain map and an algebra map, where $CH_{\text{ZZ}}$ has the differential $D = \nabla + b + c$.

Proof. Contrasting with Proposition 1.3.7 we now have inserted $A$'s. However, the insertion of $A$'s produces parallel transport functions which will not interact significantly with the wedge product. Hence, up to sign, the proof of $I^A$ being an algebra map similar to Proposition 1.3.7 since we have a diagram using the evaluation map $M^I \times E \xrightarrow{ev} M^\delta^A$ where no factors of $M$ have to be switched. The computations involving keeping track of the choices of $q_i$, although not trivial, are more straightforward than those for the proof that we have a chain map and so we leave those details to the reader.

We now show that $I^A$ is a chain map by applying $d_{DR}$ and focusing on what occurs in two scenarios: at some $\omega_{(-,-)}$ and at some inserted $A$, respectively. We have:

$$d \circ I^A(\omega) = \sum q d \circ I^A_q(\omega) = \sum q (-1)^{\dim E} \int_E d(ev^*(\text{Ins}_q^A(\omega))) + \sum q (-1)^{\dim E - 1} \int_{\partial E} ev^*(\text{Ins}_q^A(\omega)).$$

(1.5.1)

First recall that for our bounded polytope

$$E = \{ \ldots \tau^1_a \leq \ldots \leq \tau^a_{\delta_a} \leq t_i \leq \tau^1_b \leq \cdots \leq \tau^b_{\delta_b} \leq t_{i+1} \ldots \},$$
taking the place of $\Delta^n$, is a subspace of $\prod_{q_i} \Delta^{q_i} \times \Delta^n$ given by elements that satisfy conditions such as $t_i \leq \tau^1$ and $\tau^b \leq t_{i+1}$, etc. Using this as a guide, we note that $\partial E$ has components of the form

$$E|_{t_i=t_{i+1}} \ E|_{\tau^a=\tau^a+1} \ E|_{t_i=\tau^b} \ E|_{\tau^a=t_i}.$$ 

Each component of the boundary comes with a well-defined induced orientation coming from Stokes’ Theorem (using outward pointing normal-vectors of each component). We have boundary maps $\partial_{(-,-)}$ and $\partial_-$ which take adjacent coordinates and identify them. For example, we consider the adjacent coordinates

$$\tau^1_a \leq \cdots \leq \tau^{q_a}_a \leq t_i \leq \tau^1_b \leq \cdots \leq \tau^{q_b}_b \leq t_{i+1} \cdots$$

in $E$. Then we have maps of the form

$$\partial_{(\tau^a_q, t_i)} : E|_{\tau^a_q=t_i} \hookrightarrow E$$
$$\partial_{(\tau^a, \tau^a+1)} : E|_{\tau^a=\tau^a+1} \hookrightarrow E$$
$$\partial_{(t_i, \tau^b)} : E|_{t_i=\tau^b} \hookrightarrow E$$
$$\partial_{t} : E|_{t_i=\tau^b=\cdots=\tau^{q_b}=t_{i+1}} \hookrightarrow E$$

Note that for any zig-zag diagram having $n$ columns and $k$ zigzags, consisting of information $(n,k,q)$, we can associate its corresponding information $\partial_{(-,-)}(n,k,q)$ and $\partial_{i}(n,k,q)$, coming from the zigzag diagrams of the bound-
aries $\partial_{(-,-)}$ and $\partial_1$, respectively. We have corresponding maps $b_{(-,-)}$ and $b_i$ which multiply the appropriate differential-forms in our monomial $\text{Ins}_q^A(\omega)$. We can rewrite the term on the right in equation (1.5.1) above as

$$\sum_q (-1)^{\epsilon_q + \epsilon_{\partial E}} \int_{\partial E} ev^* (\text{Ins}_q^A(\omega))$$

(1.5.2)

$$= \sum_q (-1)^{\epsilon_q + \epsilon_{\partial E}} \sum_i (-1)^{\epsilon_{\partial_i E}} \int_{\partial_i E} (\partial_1 \times id)^* (ev^*_{(n,k,q)}(\text{Ins}_q^A(\omega)))$$

$$+ \sum_q (-1)^{\epsilon_q + \epsilon_{\partial E}} \sum_{(-,-)} (-1)^{\epsilon_{\partial_{(-,-)} E}} \int_{\partial_{(-,-)} E} ((\partial_{(-,-)} \times id)^* (ev^*_{(n,k,q)}(\text{Ins}_q^A(\omega))))$$

(1.5.3)

where $\epsilon_{\partial E} := n + 1 + \sum q_r$, $\epsilon_q := \sum |\omega_r|Q_r$, $\epsilon_{\partial_i E} := i + 1 + \sum q_i$, and $\epsilon_{\partial_{(-,-)} E}$ is defined as well by the orientation of $E$. We also use the fact that the pullback along $\partial_1$ amounts to pulling back along a diagonal and so we wedge the adjacent forms $\omega_{(i,p)}$ and $\omega_{(i,p+1)}$. Note, however, that we have essentially dropped all of the inserted $A$'s between the two adjacent forms.

This is only because if we were to follow through with the iterated integral, we would be integrating along a 0-dimensional subspace (i.e. a point) and so the integral over that point of $1 + A + A \wedge A + \ldots$ would equal 1. This is the same as saying that parallel transport along the constant path must equal

---

6or $\omega_{(i,n-p)}$ and $\omega_{(i,n-p+1)}$ depending upon whether or not the $i$-th level is a “zig” or a “zag”}
the identity. When we use the pullback along the \( \partial_{(-,-)} \) we simply wedge the adjacent forms \((A \wedge \omega, A \wedge A, \text{ or } \omega \wedge A)\) and no further identifications are used. Next we rewrite the first term on the right side of (1.5.1)

\[
\sum_{q} (-1)^{q+\epsilon} \int_{E} d(ev^*(Ins_{q}^{A}(\omega))) = \sum_{q} (-1)^{q+\epsilon} \int_{E} ev^*(d(Ins_{q}^{A}(\omega)))
\]

\[
= \sum_{q} (-1)^{q+\epsilon} \sum_{i=1}^{k} \sum_{\tau_{i}} \int_{E} ev^*(d(\partial_{(\tau_{i},\tau_{i})}^{}(Ins_{q}^{A}(\omega))))
\]

\[
+ \sum_{q} (-1)^{q+\epsilon} \sum_{\tau_{i}} \int_{E} ev^*(d_{\tau_{i}}(Ins_{q}^{A}(\omega)))
\]

where \( d_{(i,l)} \) (in a slight abuse of notation) applies the DeRham differential to a form \( \omega^{L}, \omega_{(i,l)}, \text{ or } \omega_{l}^{h} \) and \( d_{\tau_{i}} \) applies \( d \) to an inserted \( A \). We now show where our term \( \nabla(\omega_{(i,k)}) \) comes from. From equation (1.5.3) we get our suggestively-labeled “\( A \wedge \omega \)” and “\( \omega \wedge A \)” terms, without sign:

\[
\int_{\partial_{(\tau_{i},\tau_{i})}} ev_{\partial_{(\tau_{i},\tau_{i})}}^{*}(n,k,q)(b_{(\tau_{i},\tau_{i})})(Ins_{q}^{A}(\omega))) (A \wedge \omega)
\]

\[
\int_{\partial_{(\tau_{i},\tau_{i})}} ev_{\partial_{(\tau_{i},\tau_{i})}}^{*}(n,k,q)(b_{(\tau_{i},\tau_{i})})(Ins_{q}^{A}(\omega))) (\omega \wedge A)
\]

and from equation (1.5.5) we get another suggestively-labeled term.

\textsuperscript{7}For the rest of this proof, we will proceed without sign as it is mostly straightforward to check that our sign conventions work out, but the details would unnecessarily obfuscate the ideas.
\[
\int_{E} ev^* (d_{(i,k)}(Ins_{q/2}^A(\omega)))
\]

(d\omega)

Notice that for these three terms, all of the changes to the fiber E involved removing a single \(\tau\) or none at all. Thus, when we sum over all choices of \(q\) we will recover any removed \(\tau\) slots. Now we can write

\[
\sum_{q} \pm \int_{E} ev^* \partial_{(i,\tau)}^-(n,k,q) (b_{(i,\tau)}^-(n,k,q)(\omega) (A \wedge \omega))
\]

\[
\sum_{q} \pm \int_{E} ev^* \partial_{(i,\tau)}^+(n,k,q) (b_{(i,\tau)}^+(n,k,q)(\omega) (\omega \wedge A))
\]

\[
\sum_{q} \pm \int_{E} ev^* (d_{(i,l)}(Ins_{q/2}^A(\omega)))
\]

\[
= It^A (\nabla_{(i,l)}(\omega)), \quad (d\omega + [A, \omega])
\]

where \(\nabla_{(i,l)}\) applies \(\nabla\) to the corresponding component of \(\omega\). Next we show where shuffling in the \(R\)'s comes from. From equations (1.5.3) and (1.5.5) we obtain

\[
\int_{\partial_{(\tau,\tau+1)}^+ E} (ev^* \partial_{(\tau,\tau+1)}^(n,k,q) (b_{(\tau,\tau+1)}^+(n,k,q)(\omega))) (A \wedge A)
\]

\[
\int_{E} ev^* (d_{\tau}(Ins_{q/2}^A(\omega)))
\]

\[
= It^A (c_{(i,l)}(\omega)) \quad (R = dA + A \wedge A)
\]
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Note that the sum of these kinds of terms gives $\text{It}^A(c(\omega))$ since these terms insert exactly one $R = dA + A \wedge A$ in all positions and 1’s in the corresponding positions $\tau_a^i$ of the other zig zags. Finally we note that the first term on the right hand side of (1.5.3) gives

$$
\int_{\partial_i E} (ev_{\partial_i(n,k,q)})^*(b_i(Ins^A_{\partial_i(q)}(\omega)))
\quad (\omega \wedge \omega')
$$

where $b_i$ collapses the $i^{th}$ and $i + 1^{th}$ columns in $\omega$. This provides us with the $b_i$ part of our differential again after summing over all configurations $q$ to give:

$$
\sum_{q} \pm \int_{\partial_i E} (ev_{\partial_i(n,k,q)}^*(b_i(Ins^A_{\partial_i(q)}(\omega)))) = \text{It}^A(b_i(\omega))
\quad (\omega \wedge \omega')
$$

Similar calculations can be made at the endpoints. In particular, if we focus our attention to $\omega^L$, when we apply $d$ we obtain $d\omega^L$, considering the boundary $\partial_{(0,\tau_a^1)}$ yields a term of the form $\omega^L \wedge A$, and $\partial_0$ yields $\omega^L \wedge \omega_{(1,1)}$. Similar terms arise when we focus our attention to $\omega^R_k$. Focusing our attention to $\omega^R_i$, when we apply $d$ we obtain $d\omega^R_i$, considering the boundary $\partial_{(\tau_{a,n}^1)}$ we obtain $A \wedge \omega^R_i$, applying the boundary $\partial_{(0,\tau_a^1)}$ yields $\omega^R_i \wedge A$, and the boundary $\partial_{(n,1)}$ yields $\omega^R_i \wedge \omega'. Similar terms arise when we focus our attention to $\omega^L_i$ when $i$ is even and $1 < i < k$. If we sum over all choices of $i, k, q, \tau_a^i$, etc, we obtain all terms in $D(\omega)$. Thus we have shown that

$$
(d_{DR} \circ \text{It}^A)(\omega) = (\text{It}^A \circ D)(\omega).
$$

$\Box$
1.5.2 The Two-Dimensional Case

For the two dimensional curved case, we can follow the transition from the non-curved 1-d case to the non-curved 2-d case with one small addition. Although one can proceed by collapsing the vertical left and right boundaries of our squares, and work on bigons, we will keep the square un-identified and so we need to account for parallel transport along the vertical paths moving from one zigzag to the next. See Figure 1.1 on page 13 for the idea.

Definition 1.5.5. The curved rectangular zigzag Hochschild complex, \( \text{CH}_{\text{Rec}}^{ZZ}(A) \), has the same underlying vector space as in Definition 1.4.1 with differential \( D = * + b + c + \nabla \). Here, \( * \) and \( b \) are the same as in the non-curved \( \text{CH}_{\text{Rec}}^{ZZ}(A) \) and we use the two-dimensional analog of our \( \nabla \) and \( c \) defined in the curved \( \text{CH}_{ZZ}(A) \) where now \( c \) may also add a curvature term, \( R \), at a vertical path on the left of the square. In particular

\[
D((x_{k_0,n}) \otimes \ldots \otimes (x_{k_{m+1},n})) \\
:= \sum_{r=1}^{m-1} (-1)^{m+r} ((x_{k_1,n}) \otimes \ldots \otimes (x_{k_r,n}) \star (x_{k_{r+1},n}) \otimes \ldots \otimes (x_{k_m,n})) \\
+ \sum_{j=1}^{m} \sum_{p=1}^{n-1} (-1)^{n+m+p} \ldots \otimes b_p(x_{k_{j_1},n}) \otimes \ldots \otimes b_p(x_{k_{j_p},n}) \otimes \ldots \\
+ \sum_{j=1}^{m} \sum_{p=1}^{n-1} (-1)^{n+\beta_{j,p}} \ldots \otimes \nabla_{p}(x_{k_j,n}) \otimes \ldots \\
+ c((x_{k_1,n}) \otimes \ldots \otimes (x_{k_m,n}))
\]
where

\[ c((x_{k_1, n}) \otimes \ldots \otimes (x_{k_m, n})) \]
\[ = \sum_{j=1}^{m} (-1)^m \sum_{\sigma \in S_{n,1}} \ldots \otimes c_{\sigma}^1(x_{k_i, n}) \otimes \ldots \otimes c_{\sigma}^R(x_{k_j, n}) \otimes \ldots \]
\[ + \sum_{j=1}^{m-1} (-1)^{m+j} \ldots \otimes (x_{k_j, n}) \otimes R \otimes (x_{k_{j+1}, n}) \otimes \ldots \]

Here we used \( c_{\sigma}^R \) to represent the usual component of our differential, \( c \), which inserts an \( R \) into one zig or zag and inserts 1’s everywhere else in that new column. The \( c_{\sigma}^1 \) mimics \( c^R \) except that it only inserts 1’s along the entire column. Note also that the sign on the first line is only an \( m \) since \( c^R \) will be a sum of terms inserting \( R \) between different columns, each term having an additional sign of \((-1)^{n+\sigma(n+1)}\) just as in Definition 1.5.1.

For the sake of having a complete figure without all of the \( A \)’s inserted for the moment, we recall the following figure to work through the definition.
of our Iterated Integral in this case.

First, notice that we have 15 sections in which to insert an arbitrary number of $A$’s. So we consider those choices $q = (q_1, \ldots, q_{15})$ and using our previous notation we have $(\phi = 16)$-many forms on $M$, namely $a, \ldots, p$ and $\phi^A = 16 + \sum q_i$. Next, it becomes important to pull-back our form $A$ along a 1-path rather than a 2-path so that there is a single 1-path along which parallel transport is performed. So for a 2-path $\Gamma(s, t)$ we mean define the 1-paths $\gamma_s(t) = \gamma^t(s) = \Gamma(t, s)$. Note that our fiber $E$ will be quite cumbersome to write down in general. There is certainly a formula, but the diagram gives that formula more easily than symbols. For the particular diagram we are
considering above, we have \(t_1, t_2, s_1, s_2\) and 15 different choices of inserting \(A\)'s, labeled \(q_1, \ldots, q_{15}\). We then have

\[
E \subset \left( \prod \Delta^q \right) \times \Delta^n \times \Delta^m
\]

where

\[
E = \{ (\sigma_1, \tau_2, \tau_3, \ldots, \tau_7, \sigma_8, \tau_9, \tau_{10}, \ldots, \tau_{14}, \sigma_{15}, t_1, t_2, s_1, s_2) \}
\]

\(0 \leq t_1 \leq t_2 \leq 1\) and \(0 \leq s_2 \leq s_2 \leq 1\)

\[0 \leq \sigma_{(1,1)} \leq \ldots \leq \sigma_{(1,q_1)} \leq s_1\]

\[0 \leq \tau_{(2,1)} \leq \ldots \leq \tau_{(2,q_2)} \leq t_1 \leq \tau_{(3,1)} \leq \ldots \leq \tau_{(3,q_3)} \leq t_2 \leq \tau_{(4,q_4)} \leq 1\]

\[0 \leq \tau_{(7,q_7)} \leq \ldots \leq \tau_{(7,1)} \leq t_1 \leq \tau_{(6,q_6)} \leq \ldots \leq t_2 \leq \tau_{(5,q_5)} \leq \ldots \leq \tau_{(5,1)} \leq 1\]

\[s_1 \leq \sigma_{(8,1)} \leq \ldots \leq \sigma_{(8,q_8)} \leq s_2\]

\[0 \leq \tau_{(9,1)} \leq \ldots \leq \tau_{(9,q_9)} \leq t_1 \leq \ldots \leq t_2 \leq \tau_{(11,1)} \leq \ldots \leq \tau_{(11,q_{11})} \leq 1\]

\[0 \leq \tau_{(14,q_{14})} \leq \ldots \leq \tau_{(14,1)} \leq t_1 \leq \ldots \leq t_2 \leq \tau_{(12,q_{12})} \leq \ldots \leq \tau_{(12,1)} \leq 1\]

\[s_2 \leq \sigma_{(15,1)} \leq \ldots \leq \sigma_{(15,q_{15})} \leq 1\]

using the convention \(\sigma_i = (\sigma_{(1,1)}, \ldots, \sigma_{(1,q_i)})\) and \(\tau_i = (\tau_{(1,1)}, \ldots, \tau_{(1,q_i)})\). Observe that to each choice of \(n, m, k\), and \(q\), where \(k \coloneqq (k_0, \ldots, k_{m+1})\) and \(q \coloneqq (q_1, \ldots, q_{n+1}(k_0+\ldots+k_{m+1})+(m+1))\), we have a uniquely determined zigzag diagram with \(A\)'s inserted. Our fiber \(E\) is determined by these choices as well. With this in mind, the evaluation map for the above element can be
written:

\[ M^{Sq} \times E \xrightarrow{ev} M^{\phi_A} \]

\[ (\gamma, \tau_1, \tau_2, \tau_3, \ldots, \tau_7, \tau_8, \tau_9, \ldots, \tau_{14}, \tau_{15}, t_1, t_2, s_1, s_2) \]

\[ \mapsto (\Gamma(0, 0), (\gamma^0(\sigma_{(1,i)}))_{i=1}^{q_1}, \Gamma(0, s_1), (\gamma_{s_1}(\tau_{(2,i)}))_{i=1}^{q_2}, \Gamma(t_1, s_1), (\gamma_{s_1}(\tau_{3,i}))_{i=1}^{q_3}, \ldots, (\gamma_{s_1}(\tau_{5,i}))_{i=1}^{q_{14}}, \ldots, (\gamma_{s_2}(\tau_{14,i}))_{i=1}^{q_{15}}, \Gamma(0, s_2), (\gamma^0(\sigma_{(15,i)}))_{i=1}^{q_1}, \Gamma(0, 1)) \]

**Definition 1.5.6.** We define the curved iterated integral \( It : CH^{Rec}_{ZZ}(\Omega \bowtie (M)^{\phi}) \to \Omega \bowtie (M^{Sq}) \) by

\[ It := \sum_{(q)} (-1)^{\sum |\omega_j|Q} It^A_{(q)} \]

where the component of the iterated integral \( It^A_{(q)} : \Omega \bowtie (M)^{\otimes \phi} \to \Omega \bowtie (M^{Sq}) \) is given by the diagram

\[ \Omega \bowtie (M^{Sq} \times E) \xrightarrow{ev^*} \Omega \bowtie (M^{\phi_A}) \xrightarrow{Ins^A_{(q)}} \Omega \bowtie (M)^{\otimes \phi} \]

Since our differential is simply a mix of terms from the 2-d non-curved case and the 1-d curved case, we can combine all of those arguments to obtain the following theorem.

**Theorem 1.5.7.** The curved iterated integral \( It : CH^{Rec}_{ZZ}(\Omega \bowtie (M)) \to \Omega \bowtie (M^{Sq}) \) is a chain map.
1.6 Holonomy

We first remark on elements in $CH_{\text{ZZ}}(\Omega_{\boxplus}(M))$ which map to 1-dimensional holonomy. Given a 1-form $A \in \Omega^1_{\boxplus}(M)$ we denote by $P_\gamma(t)$ the parallel transport along a path $\gamma$ from 0 to $t$. By the construction of our zigzag Hochschild complex and its shuffle product we note that

$$P_\gamma(t) = \sum_{n \geq 0} It(\tilde{A}^\otimes n)$$

where $\tilde{A} := (1 \otimes A \otimes 1 \otimes 1 \otimes 1)$ has $n = 1$ and $k = 2$ and each $\tilde{A}^\otimes n \in CH_{\text{ZZ}}(\Omega_{\boxplus}(M))$.

Finally, we show that we have elements in the completion of $CH_{\text{ZZ}}^{\text{Rec}}(\Omega_{\boxplus}(M))$ which map to 2-dimensional holonomy. In the remainder of this chapter we restrict our space $M^{SQ}$ to the subspace of $\{\Sigma : [0, 1]^2 \to M\}$ where for each $\Sigma(t, s) : [0, 1]^2 \to M$ there exist $x, y \in M$ so that $\Sigma(0, s) = x$ and $\Sigma(1, s) = y$ for all $s \in [0, 1]$ (i.e. “bigons”). Let $B \in \Omega^2_{\boxplus}(M)$ be a matrix-valued 2-form. We define an element

$$\exp(B) := \sum_{m \geq 0} B^{\otimes m},$$

via Figure 1.2 below, so that each $B^{\otimes m} \in CH_{\text{ZZ}}^{\text{Rec}}(\Omega_{\boxplus}(M))$, as in the completed zigzag Hochschild complex, and show that the curved iterated integral of this element is the well-known 2-holonomy as defined in [BaSc], [MP1].
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Note that although $I^A(\exp(B)) := \sum_{m \geq 0} I^A(B^{\tilde{\circ} m})$ is an infinite sum of forms in $\Omega_{\oplus}(M^{Sq})$, a simple boundedness condition on $B$ guarantees that this sum converges in $\Omega_{\oplus}(M^{Sq})$; compare the argument for the well-definedness of $I^A$ above Proposition 1.5.4. Following Baez, Martins, Picken, Schreiber, and Waldorf ([MP1], [MP2], [SWII], and [BaSc]), we show that $I^A(\exp(B))$ solves the differential equation which governs 2-holonomy. We do so using more familiar notation, suppressing the evaluation pullback notation used previously in this paper.

**Proposition 1.6.1.** Let $A \in \Omega^1_{\oplus}(M)$ and $B \in \Omega^2_{\oplus}(M)$, then

$$
\frac{\partial}{\partial s}(I^A(\exp(B(t,s)))) = I^A(\exp(B(t,s))) \wedge \int_0^1 \text{hol}_{(t',s)}(B(t',s))\text{hol}_{(t,s)}^{-1} dt.
$$

where $\text{hol}_{(t,s)}$ is the 1-holonomy obtained from $A$ via the path $\gamma_s|_{[0,t]} \circ \gamma_0|_{[0,s]}$.

**Proof.** Let $\text{tra}(B)(a,b) := \text{hol}_{(a,b)}(B(a,b))\text{hol}_{(a,b)}^{-1}$, then

$$
\frac{\partial}{\partial s}(I^A(\exp(B(t,s)))) = \sum_{m \geq 0} \frac{\partial}{\partial s} I^A(B(t,s)^{\tilde{\circ} m})
$$

$$
= \sum_{m \geq 0} \sum_{\sigma \in S_m} \text{sgn} (\sigma) \frac{\partial}{\partial s} \int_{\Delta^m \times \Delta^m} \text{tra}(B)(t_{\sigma^{-1}(1)}, s_1) \ldots \text{tra}(B)(t_{\sigma^{-1}(m-1)}, s_{m-1}) \text{tra}(B)(t_{\sigma^{-1}(m)}, s_m) dt_1 \ldots dt_m ds_1 \ldots ds_m
$$

$$
= \sum_{m \geq 0} \sum_{\sigma \in S_m} \text{sgn} (\sigma) \int_{\Delta^m \times \Delta^{m-1}} \text{tra}(B)(t_{\sigma^{-1}(1)}, s_1) \ldots \text{tra}(B)(t_{\sigma^{-1}(m-1)}, s_{m-1}) \text{tra}(B)(t_{\sigma^{-1}(m)}, s_m) dt_1 \ldots dt_m ds_1 \ldots ds_{m-1}
$$
\[
= \sum_{m \geq 0} \sum_{\sigma \in S_{m-1}} sgn(\sigma) \int_{\Delta \times \Delta} tra(B)(t_{\sigma^{-1}(1)}, s_1)
\]
\[
\ldots \cdot tra(B)(t_{\sigma^{-1}(m-1)}, s_{m-1}) dt_1 \ldots dt_{m-1} ds_1 \ldots ds_{m-1}
\]
\[
\wedge \int_{0}^{t} tra(B)(t', s) dt'
\]
\[
= It^A(\exp(B)) \wedge \int_{0}^{t} hol(t', s)(B(t', s))hol^{-1}(t', s) dt'
\]

\[\square\]

Figure 1.2: Our element \(\exp(B)\). Here the signs come from the permutation governing where the \(B\)'s appear along the zigzags.
Corollary 1.6.2. For a gerbe \( \mathcal{G} \) with local data given by a 1-form \( A \in \Omega^1_\mathcal{G}(U) \) and \( B \in \Omega^2_\mathcal{G}(U) \) on an open set \( U \subset M \) of a manifold \( M \), the form \( \text{It}^A(\exp(B)) \in \Omega^0_\mathcal{G}(BU) \) represents the local 2-holonomy on \( U \); see equation (2.1.33).
Chapter 2

The Derivative of Two-Holonomy on Squares

The purpose of this chapter is to offer a formula for the differential of 2-holonomy, even in the case where we are working over multiple open sets. Using the local data for a non-abelian gerbe, without connection, was shown in [NW] to be equivalent to the other well-known notions of a non-abelian gerbe. For a non-abelian gerbe with connection, Schreiber and Waldorf offer a definition based on a parallel transport 2-functor. Within such an approach section 4 of [SWIII] explains that, given a nice open cover, one can take a (normalized) differential $G$-cocycle in $Z^2_{\pi}(G)^{\infty}$, i.e. our local differential data, and obtain transport 2-functors on $M$. This chapter assumes that a non-abelian gerbe with connection is given in terms of the familiar local data, which we recall below in Section 2.1.3, and then proceeds in section 2.1.4 by recalling, and making use of, the properties of the associated local
transport data from [SWII] and [SWIII]. Since the structure group of our non-abelian gerbe will be given by a crossed module \((G \to H)\), where this paper will assume that \(H\) is a matrix-group \((H \subset Mat, \text{ where } Mat := \mathbb{R}^{n,n} \text{ or } Mat := \mathbb{C}^{n,n} \text{ for suitable } n)\), we will first review the relevant basics of this crossed module structure. The definition of a non-abelian gerbe is given in Definition 2.1.34 using local differential cocycle data a la section 4.1.3 of [SWIII]. Following the section From Forms to Functors of [SWII], we review definitions and some properties of local transport data (2-holonomy) of bigons. In section 2.1.5 the data for bigons is reconfigured into functions \(\text{Hol}_i, \text{Hol}_{ij}, \text{and } \text{Hol}_{ijkl}\) serving as our local transport data for 2-holonomy of a square. This collection of group-valued functions will be glued together in Section 2.2 over multiple open sets \(U_i \subset M\), providing a function out of open sets \(\mathcal{N} \subset M^{Sq}\), where \(Sq := I \times I\) is the square, forming an expression, \(\text{Hol}^{\mathcal{N}} : \mathcal{N} \to H\), for two-holonomy on \(\mathcal{N}\). Propositions 2.2.3-2.2.6 will deal with justifying this definition of 2-holonomy; i.e. showing that it satisfies properties desired for 2-Holonomy. The work in Section 2.2 is analogous to that of [MP2] and the stated properties for 2-holonomy can also be found in [MP2], where, however, the setup is different than the one used in this work. Finally, in Section 2.3 we state and prove Theorem 2.3.1 which provides an explicit formula for the total derivative of 2-holonomy, \(d(\text{Hol})\). To the best
of my knowledge, this formula for $d(Hol)$ is new.

## 2.1 Conventions, Notation, and Setup

In order to arrive at a definition for 2-Holonomy, Definition 2.2.1, it is necessary to introduce some preliminary definitions and conventions.

### 2.1.1 Diffeological Spaces

Since we will be working on the space of smooth maps $M^{Sq} := \{\Sigma : Sq \to M\}$ where $Sq := [0, 1] \times [0, 1]$ is the standard square and $M$ is a smooth manifold, it is convenient to use the language of diffeological spaces as described in [IZ].

**Definition 2.1.1.** A **diffeology** of $X$ is a set $\mathcal{D} = \{U \xrightarrow{P} X\}$ of parametrizations of $X$ where $U$ is an open subset of $\mathbb{R}^n$ for some $n$, satisfying

- **D1 (Covering):** For each $x \in X$ and $n \in \mathbb{N}$, the set $\mathcal{D}$ contains the constant parametrizations.

- **D2 (Locality):** Let $P : U \to X$ be a parametrization. If for every point $r \in U$, there exists an open neighborhood $V \subset U$ of $r$ such that $P|_V \in \mathcal{D}$ then $P \in \mathcal{D}$.

- **D3 (Smooth Compatibility):** For every $(P : U \to X) \in \mathcal{D}$, and for every open subset, $V \subset \mathbb{R}^m$, if $f \in C^\infty(V, U)$ then $P \circ f \in \mathcal{D}$.
We call the parametrizations in $\mathcal{D}$ the plots of $\mathcal{D}$.

**Definition 2.1.2** (Generating Family). Let $X$ be a set. Pick a set $\mathcal{F}$ of parametrizations of $X$. There exists a finest diffeology containing $\mathcal{F}$, this diffeology is called the diffeology generated by $\mathcal{F}$, denoted $\langle \mathcal{F} \rangle$. Conversely, let $X$ be a diffeological space and $\mathcal{D}$ be its diffeology. A family $\mathcal{F}$ of plots of $X$ which generates $\mathcal{D}$ is called a generating family.

**Lemma 2.1.3.** A parametrization $P : U \to X$ is a plot for $\langle \mathcal{F} \rangle$ if and only if for all $r \in U$ there exists an open neighborhood $V \subset U$ of $r$ such that $P|_V$ is the constant parametrization or there exists $(F : W \to X) \in \mathcal{F}$ and $Q : V \to W$ so that $P|_V = F \circ W$.

**Remark 2.1.4.** In particular, if the outputs of the $F$’s cover $X$ then we drop the constant parametrization condition.

**Example 2.1.5.** Every manifold is a diffeological space with $\mathcal{F}$ being the set of charts.

**Example 2.1.6.** Given two diffeologies, $X$ and $Y$, $P : U \to C^\infty(X,Y)$ is a plot in the standard functional diffeology if and only if the map $(r,x) \mapsto P(r)(x)$ is smooth. In other words, for every plot $Q : V \to X$, $P \circ Q : (r,s) \mapsto P(r)(Q(s))$ is a plot of $Y$. 

**Definition 2.1.7.** A differential $k$-form on a diffeological space, $X$, is a map $w$ which associates with every plot $P$ of $X$ a smooth $k$-form, $\omega(P)$ defined on the domain of $P$, such that for every smooth parametrization $F$ in the domain of the plot, $P$, $\omega(P \circ F) = F^*(\omega(P))$.

In particular, for all integers $n$, and for all $n$-plots, $P : U \to X$, $\omega(P)$ is a smooth $k$-form on $U \subset \mathbb{R}^n$. It should be noted that in [IZ], along with all of these definitions and remarks, we see that smooth $k$-forms pull-back, are determined by $k$-plots, can be glued together to yield global forms, and have an exterior derivative given by $(d\omega)(P) = d(\omega(P))$.

**Covering $M^{Sq}$ With Open Sets $\mathcal{N}$**

By Example 2.1.6 we can define our mapping space $M^{Sq}$, where $Sq := I \times I$:

**Definition 2.1.8.** The diffeological space, $M^{Sq}$ is defined by the functional diffeology of plots $P : U \to C^\infty([0,1]^2, M)$.

**Definition 2.1.9.** For a fixed open cover $\mathcal{U}$ of $M$, and a choice of grid $I = \{1, \ldots, n\} \times \{1, \ldots, m\}$ and open sets $\{U_{i(p,q)}\}_{(p,q) \in I}$, define

$$\mathcal{N}_I := \{\Sigma \in M^{Sq} \mid \text{for each } (p,q) \in I, \Sigma(Sq_{(p,q)}) \subset U_{i(p,q)}\},$$

where $Sq_{(p,q)} := \left[\frac{p-1}{n}, \frac{p}{n}\right] \times \left[\frac{q-1}{m}, \frac{q}{m}\right] \subset [0,1] \times [0,1]$. When the indexing set, $I$, is understood, we will simply refer to the open set as $\mathcal{N}$. 
It will be useful later in section 2.2.2 to have the notion of a grid on $\Sigma \in \mathcal{N}_I$.

**Definition 2.1.10.** For $\Sigma \in \mathcal{N}_I$, define the grid on $\Sigma$ by the information:

- **Faces:** For each $i = (p, q) \in I$ define the $i$-face by $\Sigma_i := \Sigma|_{S_{pq}}$.

- **Vertical Edges:** For each $i = (p, q), j = (p+1, q)$ define the $ij$-vertical edge $\gamma_{ij}^v$ by

  \[
  \gamma_{ij}^v := \Sigma_i|_{\{\frac{p}{n}\} \times \left[\frac{q-1}{m}, \frac{q}{m}\right]} = \Sigma_j|_{\{\frac{p}{n}\} \times \left[\frac{q-1}{m}, \frac{q}{m}\right]} = \Sigma|_{\{\frac{p}{n}\} \times \left[\frac{q-1}{m}, \frac{q}{m}\right]}
  \]

- **Horizontal Edges:** For each $i = (p, q), j = (p, q+1)$ define the $ij$-horizontal edge $\gamma_{ij}^h$ by

  \[
  \gamma_{ij}^h := \Sigma_i|_{\left[\frac{p-1}{n}, \frac{p}{n}\right] \times \{\frac{q}{m}\}} = \Sigma_j|_{\left[\frac{p-1}{n}, \frac{p}{n}\right] \times \{\frac{q}{m}\}} = \Sigma|_{\left[\frac{p-1}{n}, \frac{p}{n}\right] \times \{\frac{q}{m}\}}
  \]

- **Vertices:** For each $ijkl$ where $i = (p, q), j = (p, q+1), k = (p+1, q), l = (p+1, q+1)$ define the $ijkl$-vertex $x_{ijkl}$ by

  \[
  x_{ijkl} := \Sigma|_{\left\{\frac{p}{n}\right\} \times \{\frac{q}{m}\}}
  \]

- **Boundary Edges:** For each $i = (p, 1)$, define the northern boundary edge $\gamma_i^N$ by

  \[
  \gamma_i^N := \Sigma_i|_{\left[\frac{p-1}{n}, \frac{p}{n}\right] \times \{0\}}
  \]
For each \( i = (p, m) \), define the southern boundary edge \( \gamma^S_i \) by

\[
\gamma^S_i := \Sigma_i \left[ \frac{p-1}{m}, \frac{p}{m} \right] \times \{1\}.
\]

For each \( i = (1, q) \), define the western boundary edge \( \gamma^W_i \) by

\[
\gamma^W_i := \Sigma_i \left[ 0 \times \left[ \frac{q-1}{m}, \frac{q}{m} \right] \right].
\]

For each \( i = (n, q) \), define the eastern boundary edge \( \gamma^E_i \) by

\[
\gamma^E_i := \Sigma_i \left[ 1 \times \left[ \frac{q-1}{m}, \frac{q}{m} \right] \right].
\]
Every diffeology on $X$ induces a topology on $X$, called the $\mathcal{D}$-topology. Page 54 of [IZ] states gives the following characterization of the $\mathcal{D}$-topology:

**Proposition 2.1.11.** A $A \subset X$ is open for the $\mathcal{D}$-topology if and only if for every plot, $\rho : U \to X$, $\rho^{-1}(A)$ is open in $U$.

We now show that these sets $\mathcal{N}$ are open in the diffeology $M^{sq}$ under its $\mathcal{D}$-topology.

**Proposition 2.1.12.** Consider the diffeological space $M^Y := \{ f : Y \to M \mid f \text{ is smooth} \}$ with plots $\rho : U \to M^Y$ given by smooth maps $\tilde{\rho} : U \times Y \to M$ where $(r, y) \mapsto \rho(r)(y)$. If $K \subset Y$ is compact and $U \subset M$ is open, then

$$\mathcal{N}(K, U) := \{ f \in M^Y \mid f|_K \subset U \}$$

is an open set in $M^Y$.

**Proof.** Let $\rho : U \to M^Y$ be a plot. The claim is that the subset

$$\rho^{-1}(\mathcal{N}(K, U)) := \{ r \in U \subset \mathbb{R}^n \mid \rho(r)|_K \subset U \} =: V$$

of $U$ is open. Let $r_0 \in V$. We will show there is a neighborhood $B \subset V$ of $r_0$ so that $\rho(r')(K) \subset U$ for every $r' \in B$. Since $\tilde{\rho} : U \times Y \to M$ is a continuous map we have the open set

$$\tilde{\rho}^{-1} = \{(r, y) \in U \times Y \mid \rho(r)(y) \in U \}.$$
For each \( k \in K \), \( \rho(r_0)(k) \in \tilde{\rho}^{-1}(U) \), so cover the compact set \( \{r_0\} \times K \) with finitely many open neighborhoods \( B_k \times C_k \subset \tilde{\rho}^{-1}(U) \), by the product topology. Finally, we prove that \( B := \cap B_k \) is the desired open neighborhood of \( \rho_0 \). Let \( r' \in B \), then for each \( k \in K \), \((r', k) \in \cap B_k \times K \) and so \( k \in C_k \) meaning \((r', k) \in B_k \times C_k \). This means that each \((r', k) \in \cup B_k \times C_k \subset \tilde{\rho}^{-1}(U) \) yielding \( \rho(r')(k) \in U \) for each \( k \), or \( \rho(r')(K) \subset U \) as desired.

**Corollary 2.1.13.** Each \( \mathcal{N}_I \) is an open subset of \( M^{Sq} \).

**Proof.** Note that for each \( \mathcal{N}_I \), we can write:

\[
\mathcal{N}_I = \bigcap_{k=1,...,n} \bigcap_{l=1,...,m} \mathcal{N}(Sq_{(k,l)}, U_{(i,k,l)})
\]

**Proposition 2.1.14.** For a fixed open cover \( \mathcal{U} \), the open sets \( \mathcal{N}_I \) cover \( M^{Sq} \), where \( n, m \) in \( I \) range over all natural numbers in \( \mathbb{N} \).

**Proof.** Let \( \Sigma \in M^{Sq} \), \( \Sigma(Sq) \) is a compact subset of \( M \) and so can be covered by finitely many \( U_i \). The claim is that there exists \( p, q \in \mathbb{N} \) so that for each \( k = 1, \ldots, n \) and \( l = 1, \ldots, m \) we have \( Sq_{(k,l)} \subset \Sigma^{-1}(U_{i(k,l)}) \) for some \( U_{i(k,l)} \). Suppose not. Then given any \((n, m)\) there would exist a \( Sq_{(k,l)} \) which did not sit completely inside \( \Sigma^{-1}(U_i) \) for any \( U_i \). Then there would be a sequence of \( Sq_j \), with \(|Sq_j| \to 0\) as \( j \to \infty \) where \( Sq_j \) does not sit completely
inside any single $\Sigma^{-1}(U_i)$. However, we would then have a sequence of points $x_j$ converging to $x$ which must sit inside some $U_i$. This implies there are infinitely-many $x_j \in U_i$. However, $x_j \in Sq_j \not\subset \Sigma^{-1}(U_i)$, and so there exists $x'_j \in Sq_j$ such that $x'_j \in Sq_j$ such that $x'_j \not\in U_i$. However, $x'_j, x_j$ must converge to the same $x \in U_i$ yielding the desired contradiction.

2.1.2 Crossed Module Conventions and Relations

Following [GiPf] as a reference, we review the definition of crossed modules, and some related properties that will be essential later on.

**Definition 2.1.15.** A crossed module of Lie Groups is a pair of Lie groups, $(H, G)$, with a smooth group homomorphism, $(H \rightarrow G)$, called the target, and an action $\alpha : G \rightarrow \text{Aut}(H)$, written $\alpha_g(h)$. Since $t$ is a homomorphism

\begin{align*}
t(hh') &= t(h)t(h') \quad (2.1.1) \\
t(1) &= 1. \quad (2.1.2)
\end{align*}

The map $\alpha$ is a group action of $G$ on $H$ i.e.

\begin{align*}
\alpha_{gg'}(h) &= \alpha_g(\alpha_{g'}(h)) \quad (2.1.3) \\
\alpha_1(h) &= h. \quad (2.1.4)
\end{align*}

For a fixed $g \in G$, the map $h \mapsto \alpha_g(h)$ is a homomorphism yielding

\begin{align*}
\alpha_g(hh') &= \alpha_g(h)\alpha_g(h') \quad (2.1.5)
\end{align*}
\( \alpha_g(1) = 1. \) \hspace{1cm} (2.1.6)

Finally, \( t \) and \( \alpha \) are required to satisfy the following compatibility conditions

\[ t(\alpha(g)(h)) = gt(h)g^{-1} \] \hspace{1cm} (2.1.7)

\[ \alpha(t(h))(h') = hh'h^{-1} \] \hspace{1cm} (2.1.8)

**Example 2.1.16.** If \( H \) is a Lie group, then \( G := \text{Aut}(H) \) induces a crossed module \((H \xrightarrow{t} \text{Aut}(H))\) via the target \( t : H \rightarrow \text{Aut}(H) \) given by

\[ t(h)(h') := hh'h^{-1} \]

and the action \( \alpha : G \rightarrow \text{Aut}(H) \) given by the identity automorphism.

**Example 2.1.17.** Define the crossed module \( BS^1 := (S^1 \xrightarrow{\{\ast\}} \{\ast\}) \) given by the trivial target and identity action \( \alpha \). This is the crossed module often used in the study of abelian gerbes.

Associated to such a crossed module is a crossed module of Lie algebras:

**Definition 2.1.18.** A crossed module of Lie algebras is a pair of Lie algebras, \((\mathfrak{h}, \mathfrak{g})\), with a Lie algebra map, \((\mathfrak{h} \xrightarrow{t} \mathfrak{g})\), called the target, and a map \( \alpha : \mathfrak{g} \rightarrow \text{der}(\mathfrak{h}) \), written \( \alpha_A(B) \). By \( t \) being a map of Lie algebras, we mean

\[ t([Y_1, Y_2]) = [t(Y_1), t(Y_2)]. \]

The map \( \alpha \) is an action of \( \mathfrak{g} \) on \( \mathfrak{h} \) by derivations, meaning,

\[ \alpha_{[X_1, X_2]}(Y) = \alpha_{X_1}(\alpha_{X_2}(Y)) - \alpha_{X_2}(\alpha_{X_1}(Y)). \] \hspace{1cm} (2.1.9)
For any $X \in \mathfrak{g}$, the map $\alpha_X$ is a derivation on $\mathfrak{h}$, meaning

$$\alpha_X [Y_1, Y_2] = [\alpha_X Y_1, Y_2] + [Y_1, \alpha_X Y_2]. \quad (2.1.10)$$

Finally, these two maps $t$ and $\alpha$ must satisfy the following compatibility conditions:

$$t(\alpha_X Y) = [X, t(Y)] \quad (2.1.11)$$

$$\alpha_{t(Y_1)}(Y_2) = [Y_1, Y_2] \quad (2.1.12)$$

In [BaSc], [SWII], and [SWIII] crossed modules arise as the example of a structure-2-category for a parallel transport 2-functor out of the path 2-category. The structure group for a non-abelian gerbe is considered to be a strict 2-group. We use the following definition to connect crossed modules and 2-categories, which can be found, for example, in [BaLa]:

**Definition 2.1.19.** A strict 2-group is a 2-category with one object where all morphisms and 2-morphisms are invertible.

**Example 2.1.20.** Every strict Lie 2-group is equivalent to the strict 2-group given by a crossed module of Lie groups. Below, a (one-way) dictionary from the language of crossed modules to that of a strict Lie 2-group is provided in the table 2.1

---

1See the appendix of [SWIV] for a treatment of 2-categories. This paper will simply provide reference to and brief examples of any relevant notions.
A useful proposition regarding the center of $H$ is as follows:

**Proposition 2.1.21.** The kernel of the target map, $t : H \to G$, is in the center, $Z(H)$, of $H$. Similarly, the kernel of the target map, $t : \mathfrak{h} \to \mathfrak{g}$ is in the center, $Z(\mathfrak{h})$, of $\mathfrak{h}$.

**Proof.** If $t(h) = 1$, then for any $h' \in H$, $h' = \alpha_1(h') = \alpha_{t(h)}(h') = hh'h^{-1}$, thus $h \in Z(H)$. A similar proof can be applied to the statement for $X \in \mathfrak{h}$ with $t(X) = 0$:

$$0 = \alpha_0(Y) = \alpha_{t(X)}(Y) = [X, Y].$$
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An important and well-known lemma, [KN1, Proposition 1.4] in calculating the derivative of the \( \alpha \) map is provided now for later reference.

**Lemma 2.1.22.** For the function \( \alpha : G \times H \to H \) defined above, given functions \( g(t) : \mathbb{R} \to G \) and \( h(t) : \mathbb{R} \to H \), we have

\[
\left. \frac{\partial}{\partial t} \right|_{t=t_0} (\alpha_{g(t)}(h(t))) = (\alpha_{g(t_0)})^* \left( \left. \frac{\partial}{\partial t} \right|_{t=t_0} h(t) \right) + (\alpha_{h(t_0)})^* \left( \left. \frac{\partial}{\partial t} \right|_{t=t_0} g(t) \right)
\]

(2.1.13)

as an equality in \( T_{\alpha_{g(t_0)}(h(t_0))}H \), the tangent space of \( H \) at \( \alpha_{g(t_0)}(h(t_0)) \). We will sometimes write

\[
\alpha \left( \left. \frac{\partial}{\partial t} \right|_{t=t_0} g(t) \right) h(t_0) := (\alpha_{h(t_0)})^* \left( \left. \frac{\partial}{\partial t} \right|_{t=t_0} g(t) \right)
\]

and refer to these as path terms.

**Lie Algebra Valued Forms**

In the next section, a gerbe will be defined via local differential data with values in a crossed module of Lie Algebras. As such, it is useful to review some important definitions and properties within this context, which was adapted from [GiPf].

**Definition 2.1.23.** A Lie algebra valued \( p \)-form \( \omega \in \Omega^p(M, g) \) is a sum \( \omega^a \cdot T_a \) where \( T_a \) is a basis for the Lie Algebra \( g \).

**Definition 2.1.24.** For a \( p \)-form \( \psi \in \Omega^1(M, g) \) and a \( q \)-form \( \omega \in \Omega^p(M, g) \)
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Define

\[ [\psi \wedge \omega] := \psi^a \wedge \omega^b \cdot [T_a, T_b] \]

**Proposition 2.1.25.** Note that for \( A \in \Omega^1(M, \mathfrak{g}) \), where \( \mathfrak{g} \subset \text{Mat} \),

\[ \frac{1}{2} [A \wedge A] = A \wedge A \]

**Proof.**

\[ [A \wedge A] = A^a \wedge A^b [T_a, T_b] = A^a \wedge A^b T_a T_b - A^a \wedge A^b T_b T_a \]

\[ = A^a \wedge A^b T_a T_b + A^b \wedge A^a T_b T_a = 2A \wedge A \]

where \( A \wedge A \) is the wedge product of matrix-valued forms using the matrix product.

**Definition 2.1.26.** Define the curvature 2-form of the connection 1-form, \( A \), by

\[ R := dA + \frac{1}{2} [A \wedge A] \]

which in the case of \( \mathfrak{g} \subset \text{Mat} \) can be written

\[ R = dA + A \wedge A = dA + \frac{1}{2} [A \wedge A] \]

**Definition 2.1.27.** Define the exterior covariant derivative

\[ \nabla(\omega) := d_A(\omega) := d\omega + [A \wedge \omega]. \]
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Note that $R$ in general is **not** equal to $\nabla(A)$; however it is in the case of matrix valued forms.

**Proposition 2.1.28** (First Bianchi Identity). *The first Bianchi Identity holds for differential forms with values in $\mathfrak{g}$:*

$$\nabla^2(\psi) = [R, \psi]$$

**Proof.**

$$\nabla_A \nabla_A(\psi) = (d + [A \wedge -])(d\psi + [A \wedge \psi])$$

$$= d^2\psi + d[A \wedge \psi] + [A \wedge d\psi] + [A \wedge [A \wedge \psi]]$$

$$= [dA \wedge \psi] - [A \wedge d\psi] + [A \wedge [A \wedge \psi]]$$

$$= [dA \wedge \psi] + [A \wedge [A \wedge \psi]] = [dA \wedge \psi] + \frac{1}{2}[[A \wedge A] \wedge \psi]$$

$$= [R \wedge \psi]$$

$\square$

**Proposition 2.1.29** (Second Bianchi Identity). *The second Bianchi Identity holds:*

$$\nabla(R) = dR + [A \wedge R] = 0$$

**Proof.**

$$\nabla(R) = dR + [A \wedge R] = d(dA + \frac{1}{2}[A \wedge A]) + [A \wedge (dA + \frac{1}{2}[A \wedge A])]$$
where we use the fact that $d$ is a derivation of $[- \wedge -]$ and $[A \wedge [A \wedge A]] = 0$ follows from the Jacobi Identity via

$$[A \wedge [A \wedge A]] = A^a \wedge A^b \wedge A^c [T_a, [T_b, T_c]] = A^a \wedge A^b \wedge A^c (-[T_b, [T_c, T_a]] - [T_c, [T_a, T_b]])$$

$$= - A^a \wedge A^b \wedge A^c [T_b, [T_c, T_a]] - A^a \wedge A^b \wedge A^c [T_c, [T_a, T_b]]$$

$$= - A^b \wedge (A^c \wedge A^a) [T_b, [T_c, T_a]] - A^c \wedge (A^a \wedge A^b) [T_c, [T_a, T_b]]$$

$$= - 2[A \wedge [A \wedge A]].$$

We can analogously define a type of wedge-action in the context of our crossed module:

**Definition 2.1.30.** For a $p$-form $\psi \in \Omega^1(M, \mathfrak{g})$ and a $q$-form $\omega \in \Omega^q(M, \mathfrak{h})$ define

$$\alpha_\psi(\omega) := \psi^a \wedge \omega^b \cdot \alpha_{T_a} (T_b')$$

Since the wedge-action involves real-valued forms multiplied by constants (degree 0), the following properties are easily obtained:

**Proposition 2.1.31.** The wedge-action $\alpha$ defined above satisfies

1. $d(\alpha_\psi(\omega)) = \alpha_{d(\psi)}(\omega) + (-1)^p \alpha_\psi(d\omega)$ for $\psi \in \Omega^p(M, \mathfrak{g})$ and $\omega \in \Omega^q(M, \mathfrak{h})$. 
2. $\alpha_t(\omega') = [\omega \wedge \omega']$

3. $t(\alpha_\psi(\omega)) = [\psi \wedge t(\omega)]$

**Definition 2.1.32.** Fix a 1-form $A \in \Omega^1(M, g)$. For a $p$-form $\omega$ with values in $\mathfrak{h}$, define

$$\nabla(\omega) := d\omega + \alpha_A(\omega)$$

**Proposition 2.1.33** (First Bianchi Identity for wedge-action). The First Bianchi Identity for the covariant derivative $\nabla$ from Definition 2.1.32 holds for differential forms with values in $\mathfrak{h}$:

$$\nabla^2(\omega) = \alpha_R(\omega)$$

**Proof.**

$$\nabla^2(\omega) = \nabla(d\omega + \alpha_A(\omega)) = d^2\omega + d(\alpha_A(\omega)) + \alpha_A(d\omega) + \alpha_A(\alpha_A(\omega))$$

$$= \alpha_{dA}(\omega) - \alpha_A(d\omega) + \alpha_A(d\omega) + \alpha_A(\alpha_A(\omega))$$

$$= \alpha_{dA}(\omega) + \alpha_{\frac{1}{2}[A\wedge A]}(\omega) = \alpha_R(\omega)$$

\[\square\]

### 2.1.3 Local Differential Data for a Gerbe

In [NW] it is shown that the cocycle description of a non-abelian gerbe is equivalent to the other three common formulations: classifying maps,
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groupoid bundle gerbes, and principal 2-bundles. In [SWIII] they go on to provide their formulation for a connection on, and associated parallel transport for, a given non-abelian gerbe. Following [SWIII], and since we assume we are working on a nice open cover, we use the following for our local data of a gerbe:

Definition 2.1.34. Given a smooth manifold, $M$, an open cover $\mathcal{U} = \{U_i\}$ of $M$, and a crossed module of Lie groups $\mathcal{G} = (H \xrightarrow{i} G)$, a $\mathcal{G}$-gerbe with a connection on $M$, subordinate to the cover $\mathcal{U}$, is defined by the following local cocycle data:

- On each open set, $U_i$ a pair
  
  \[(A_i \in \Omega^1(U_i, g), B_i \in \Omega^2(U_i, h)).\]

- On each intersection, $U_{ij} := U_i \cap U_j$ a pair
  
  \[(g_{ij} \in \Omega^0(U_{ij}, G), a_{ij} \in \Omega^1(U_{ij}, h)).\]

- On each triple intersection, $U_{ijk} := U_i \cap U_j \cap U_k$ a function
  
  \[f_{ijk} \in \Omega^0(U_{ijk}, H).\]

satisfying the relations,
1. On each open set,
\[ R_i := dA_i + \frac{1}{2} [A_i \wedge A_i] = t(B_i) \]
\[ g_{ii} = 1 \]
\[ a_{ii} = 0. \]

2. On each intersection,
\[ A_j = g_{ij} A_i g^{-1}_{ij} - dg_{ij} g^{-1}_{ij} - t(a_{ij}) \]
\[ B_j = \alpha_{g_{ij}} (B_i) - \nabla_j (a_{ij}) - \frac{1}{2} [a_{ij} \wedge a_{ij}] \]
\[ f_{ii} = f_{jj} = 1. \]

3. On each triple intersection,
\[ g_{ik} = t(f_{ijk}) g_{jk} g_{ij} \]
\[ f_{ijk} a_{ik} f^{-1}_{ijk} = (\alpha_{g_{ijk}}) (a_{ij}) + a_{jk} + ((\alpha_{g_{ijk}})(A_k)) f^{-1}_{ijk} + df_{ijk} f^{-1}_{ijk}. \]

4. On each quadruple intersection,
\[ f_{ikl} \alpha_{g_{ikl}} (f_{ijkl}) = f_{ijkl} f_{jkl}. \]

where \( \nabla_j (\omega) := d\omega + \alpha_{A_j} (\omega), g \cdot A := (L_g)_*(A), \) and \( A \cdot g := (R_g)_*(A). \)

Remark 2.1.35. In [SWIII], the above data describes a differential \( G \)-cocycle in \( Z^2_\pi(G)^\infty \). In Section 4.1.4 of [SWIII], the above definition is used to pro-
vide, given a suitable cover, a representative of an element in the degree two
differential non-abelian cohomology, \( \hat{H}^2(M, \mathcal{G}) \), of \( M \) with values in \( \mathcal{G} \):

\[
\hat{H}^2(M, \mathcal{G}) := \lim_{\pi} \pi_0(\mathcal{G}(\infty)).
\]

Dropping all of the differential data would result in a representative of an
element in non-abelian cohomology, \( H^2(M, \mathcal{G}) \).

For the case when \( \mathcal{G} = BS^1 \) is abelian (see Example 2.1.17), \( \hat{H}^2(M, \mathcal{G}) = H^2(M, D(2)) \) agrees with the degree-two Deligne Cohomology. In the case
where \( \mathcal{G} = Aut(H) \) (see Example 2.1.16), the definition provides a one-to-
one correspondence with Breen-Messing Gerbes, assuming vanishing fake-
curvature.

Pfeiffer, Baez, and Schreiber define the curvature 3-form of a gerbe \( H_i \in \Omega^3(U_i, \mathfrak{g}) \) by

\[
H_i := \nabla_i(B_i) := dB_i + \alpha_{A_i}(B_i)
\]

where \( B_i \in \Omega^2(U_i, \mathfrak{g}) \) and \( (A_i, B_i) \) define the 2-connection of our gerbe on \( U_i \).

**Proposition 2.1.36.** The curvature 3-form, \( H_i \), of a gerbe, satisfies

1. \( t(H_i) = 0 \)

2. \( \nabla_i(H_i) = 0. \)

3. On each intersection of open sets, \( U_{ij} \), we have \( H_j = \alpha_{g_{ij}}(H_i) \)
Proof. To prove the first claim, we use the \textit{vanishing fake curvature} condition $t(B_i) = R_i$ of a gerbe, and apply Proposition \ref{prop:vanishing-fake-curvature} to observe that

\[ t(H_i) = t(dB_i + \alpha_{A_i}(B_i)) = d(t(B_i)) + [A_i \wedge t(B_i)] = d(R_i) + [A_i \wedge R_i] = 0 \]

To prove the second claim, we use the first Bianchi identity for the wedge-action (Proposition \ref{prop:bianchi-identity}), along with the vanishing fake-curvature to write,

\[ \nabla_i(H_i) = \nabla_i(\nabla_i(B_i)) = \alpha R_i(B_i) = \alpha t(B_i)(B_i) = [B_i, B_i] = 0. \]

For the third claim, we use the transition of $B_i$ to write

\[ H_j = dB_j + \alpha_{A_j}(B_j) \]

\[ = d(\alpha_{g_{ij}}(B_i) - \alpha_{A_i}(a_{ij}) - da_{ij} - \frac{1}{2} [a_{ij} \wedge a_{ij}]) + \alpha_{A_j}(\alpha_{g_{ij}}(B_i) - \alpha_{A_i}(a_{ij}) - da_{ij} - \frac{1}{2} [a_{ij} \wedge a_{ij}]) \]

\[ = \alpha_{g_{ij}}(dB_i) + \alpha_{g_{ij}}(dA_i(a_{ij}) + \alpha_{A_j}(da_{ij}) - \frac{1}{2} d[a_{ij} \wedge a_{ij}] + \alpha_{A_j}(\alpha_{g_{ij}}(B_i) - \alpha_{A_i}(a_{ij}) - da_{ij} - \frac{1}{2} \alpha_{A_j} [a_{ij} \wedge a_{ij}] \]

\[ = \alpha_{g_{ij}}(dB_i) + \alpha_{g_{ij}}(\alpha_{A_i}(B_i)) - \alpha R_j(a_{ij}) - \frac{1}{2} \nabla_j([a_{ij} \wedge a_{ij}]) - a_{ij} \wedge \alpha_{g_{ij}}(B_i) \]

\[ = \alpha_{g_{ij}}(H_i) - \alpha R_j(a_{ij}) - [\nabla_j(a_{ij}) \wedge a_{ij}] - [a_{ij} \wedge \alpha_{g_{ij}}(B_i)] \]

\[ = \alpha_{g_{ij}}(H_i) - [B_j \wedge a_{ij}] - [\nabla_j(a_{ij}) \wedge a_{ij}] - [a_{ij} \wedge \alpha_{g_{ij}}(B_i)] \]

\[ = \alpha_{g_{ij}}(H_i) - [B_j \wedge a_{ij}] + [\alpha_{g_{ij}}(B_i) \wedge a_{ij}] - [\nabla_j(a_{ij}) \wedge a_{ij}] \]

\[ = \alpha_{g_{ij}}(H_i) + [\alpha_{A_j}(a_{ij}) \wedge a_{ij}] + [da_{ij} \wedge a_{ij}] - [\nabla_j(a_{ij}) \wedge a_{ij}] \]

\[ = \alpha_{g_{ij}}(H_i). \]
2.1.4 Local Transport Data for Bigons

In this section we recall the local transport data for bigons as given in [SWII]. Since this paper is not concerned with assertions within the realm of categories and 2-categories, we will simply extract properties from the works of Schreiber and Waldorf as they pertain to being used as a starting point. Following the notation and definitions in Section 2.1 of [SWII], the functions (along with their properties) relevant to this paper are recalled below in Definition 2.1.38 (and then in Propositions 2.1.39, 2.1.40, and 2.1.42). Thus in this section we merely summarise results, while the full details can be found in [SWII,SWIII]. For the reader who would like an informal reminder of the categorical setting, remark 2.1.43 follows. All of the assertions in this section are a consequence of Theorem 4.1.1 of [SWIII] stating that there is an isomorphism

$$Funct^\infty(P_2(M),BG) \rightarrow Z^2_M(G)^\infty$$

from the 2-category of smooth 2-functors, pseudonatural transformations, and modifications (i.e. two-dimensional parallel transport), to the descent 2-category of smooth functions and differential forms (i.e. local cocycle data). While Theorem 4.1.1. of [SWIII] asserts an isomorphism, here we essentially
use the more constructive approach provided in the section *Forms to Functors* of [SWII].

**Remark 2.1.37.** The diffeological spaces used both in this paper and in [SWI, SWII, SWIII] involve maps $\gamma : [0,1] \to M$ and $\Sigma : [0,1] \times [0,1] \to M$. However, sometimes it is necessary to restrict $[0,1]$ to a closed sub-interval before evaluating on that path, bigon, square, etc. In such cases we will assume a canonical re-parametrization to $[0,1]$ wherever necessary.

**Definition 2.1.38.** Given a $G$-gerbe with connection on $M$ via Definition 2.1.34, define (see [SWII])

1. On each open set, $U_i \subset M$, a function $F_{A_i} : P^1(U_i) \to G$, mapping paths in $U_i$ to $G$, which is the solution to the initial-value problem
   \[
   \frac{\partial}{\partial t}(F_{A_i}(\gamma_{|[0,t]})) = -A_i \left( \frac{\partial}{\partial t} \right) \cdot F_{A_i}(\gamma_{|[0,t]}), \quad F_{A_i}(\gamma_{|[0,0]}) = 1 \in G
   \]
   where given a path $\gamma$, we obtain the one parameter family of paths $\gamma_{|[0,t]} : \gamma(0) \to \gamma(t)$, re-parametrized to a path $[0,1] \xrightarrow{\gamma} M$.

2. On each open set, $U_i \subset M$, a function $\kappa_{A_i,B_i} : B^2(U_i) \to H$, mapping bigons, $\gamma_0 \xrightarrow{\Sigma} \gamma_1$, in $U_i$ to $H$, defined by $\kappa_{A_i,B_i} := \alpha_{F_{A_i}(\gamma_0)}(f_{\Sigma}^{-1})$ where

---

\[2\text{This function corresponds to ordinary 1-d parallel transport.}\]
f_\Sigma is the solution to the initial value problem
\[
\frac{\partial}{\partial s}(f_\Sigma_s) = -\int_{[0,1]} (\alpha F_{A_i}(\gamma_s|_{[0,t]})^{-1})_* \left( B_i \left( \frac{\partial}{\partial t} \frac{\partial}{\partial s}\right) \right) dt \cdot f_\Sigma_s,
\]
\[f_{\Sigma_0} = 1 \in H\]

where \(\Sigma_s\) is the one-parameter family of bigons \(\gamma_0 \xrightarrow{\Sigma_s} \gamma_s\), induced by a given bigon, \(\Sigma\).

(3) On each intersection, \(U_{ij} \subset M\), a function \(h_{ij} : P^1(U_{ij}) \to H\), mapping paths in \(U_{ij}\) to \(H\), solving the initial value problem [SWII equation (2.38)],
\[
\frac{\partial}{\partial t} h_{ij}(\gamma|_{[0,t]} = -a_{ij} \left( \frac{\partial}{\partial t} \right) \cdot h_{ij}(\gamma|_{[0,1]}) - (\alpha h_{ij}(\gamma|_{[0,t]}))_*(A_j \left( \frac{\partial}{\partial t} \right)),
\]
\[h_{ij}(\gamma|_{[0,0]}) = 1 \in H.\]

Due to the functorial origins of the above functions, the following propositions are a summary of details recalled from [SWII] and [SWIII] where, again, the full details (e.g. definitions such as composition of bigons) can be found in Section 2.1 of [SWII].

**Proposition 2.1.39.** Given the local transport data in Definition 2.1.38,

(P1) The function \(F_{A_i}\) provides a 1-morphism in the 2-group associated to \((H \xrightarrow{1} G)\)

\[
\bullet \xrightarrow{F_{A_i}(\gamma_0)} \bullet \quad (2.1.14)
\]
Furthermore $F_{A_i}(\gamma_1 \circ \gamma_2) = F_{A_i}(\gamma_1) \cdot F_{A_i}(\gamma_2)$ (See path composition in Table 2.1).

(P2) The function $\kappa_{A_i,B_i}$ provides a 2-morphism in $(H \xrightarrow{\ell} G)$,

$$F_{A_i}(\gamma_0) \xrightarrow{\kappa_{A_i,B_i}} F_{A_i}(\gamma_1) = t(\kappa_{A_i,B_i}) \cdot F_{A_i}(\gamma_0) \quad (2.1.15)$$

Furthermore,

$$\kappa_{A_i,B_i}(\Sigma_2 \circ v \Sigma_1) = \kappa_{A_i,B_i}(\Sigma_2) \cdot \kappa_{A_i,B_i}(\Sigma_1) \quad (2.1.16)$$

$$\kappa_{A_i,B_i}(\Sigma_2 \circ h \Sigma_1) = \kappa_{A_i,B_i}(\Sigma_2) \cdot \alpha_{F_{A_i}(\gamma_2)}(\kappa_{A_i,B_i}(\Sigma_1)) \quad (2.1.17)$$

(See source-target matching condition, vertical multiplication, and horizontal multiplication, respectively, in Table 2.1).

(P3) The function $h_{ij}^{-1}$ provides a 2-morphism in $(H \xrightarrow{\ell} G)$,

$$\text{meaning } t(h_{ij}^{-1}) = \text{hol}_j(\gamma)g_{ij}(x)\text{hol}_i(\gamma)^{-1}g_{ij}(y)^{-1}. \quad (2.1.18)$$
(P4) The function $f_{ijk}$ from Definition 2.1.34 provides a 2-morphism in $(H \to G)$,

\[ g_{jk}g_{ij} = t(f_{ijk}) \cdot g_{jk}g_{ij} \]  

(2.1.19)

We need two more equalities which follow from [SWII,SWIII]:

**Proposition 2.1.40.** Let $\gamma : [0, 1] \to M$ be a path in $M$ from $x$ to $y$. The local transport data for bigons satisfy

\[ \alpha_{F_{A_k}(\gamma)}(f_{ijk}(\gamma(0))) \cdot h^{-1}_{jk}(\gamma)\alpha_{g_{jk}(\gamma(1))}(h^{-1}_{ij}) = h^{-1}_{ij}(\gamma) \cdot f_{ijk}(\gamma(1)) \]

which we express diagrammatically as

\[ (2.1.20) \]

**Proof.** Via A.3 of [SWII] and Definition 2.2.2 of [SWII] we have the above
diagram. We now show that this diagram amounts to the stated equation.

First compose the $h^{-1}_{ij}$ and $h^{-1}_{jk}$ squares by considering them as bigons and use the conventions from Table 2.1

$$\begin{align*}
\frac{g_{ij}(y)F_{A_i}^{-1}}{F_{A_i}g_{ij}(x)} \quad \frac{F_{A_i}^{-1}}{h^{-1}_{ij}} \quad \frac{F_{A_i}^{-1}}{h^{-1}_{jk}} \quad \frac{g_{jk}(y)F_{A_j}}{F_{A_j}g_{jk}(x)F_{A_j}^{-1}}
\end{align*}$$

(2.1.21)

$$\begin{align*}
\frac{g_{ij}(y)F_{A_i}^{-1}}{h^{-1}_{ij}} \quad \frac{F_{A_i}g_{ij}(x)}{g_{jk}(y)} \quad \frac{g_{jk}(y)F_{A_j}}{h^{-1}_{jk}}
\end{align*}$$

(2.1.22)

$$\begin{align*}
\frac{g_{jk}(y)g_{ij}(y)F_{A_i}}{h^{-1}_{jk} \alpha_{g_{jk}(y)}(h^{-1}_{ij})} \quad \frac{F_{A_i}g_{jk}(x)g_{ij}(x)}{}
\end{align*}$$

(2.1.23)

Similar computations yield the equality in $H$ that is stated.

\[\square\]

**Definition 2.1.41.** Given $(t, s) \in \mathbb{R}^2$ parametrize the standard bigon $\rho_{(t,s)}$

$$\begin{align*}
(0,0) \xrightarrow[\gamma(-,0)]{\gamma(\cdot,-)} (t,0)
\end{align*}$$

(2.1.24)

by the parametrization $(u,v) \mapsto \rho(\tilde{u}, \tilde{v})$ where $\rho : [0,1] \times [0,1]$ is the map

$$\rho_{t,s}(u,v) := \begin{cases}
(0,3uv \cdot s) & \text{if } u < 1/3 \\
((3u - 1) \cdot t, v \cdot s) & \text{if } 1/3 \leq u \leq 2/3 \\
(t, (3(1 - v)u + (3v - 2)) \cdot s) & \text{if } 2/3 < u
\end{cases}$$

(2.1.25)
which can be pictured as

\begin{equation}
\begin{array}{c}
0 \\
\downarrow \\
\vdots \\
\downarrow \\
\end{array}
\begin{array}{c}
t \\
\downarrow \\
\vdots \\
\downarrow \\
0 < 1/3 \\
\downarrow \\
1/3 \leq u \leq 2/3 \\
\downarrow \\
2/3 < u \\
\end{array}
\end{equation}

and \( \tilde{u} \) is the smooth transition function \( \psi : \mathbb{R} \to [0, 1] \) which has a sitting instant of size \( \epsilon \).

**Proposition 2.1.42.** For the standard bigon, \( \Sigma \) we have

\[
\kappa_{A_j,B_j}(\gamma) \cdot \alpha_{F_{A_i}(t,-)}(h_{ij}^{-1}(-,0) \cdot h_{ij}^{-1}(t,-)) \\
= \alpha_{F_{A_j}(-,s)}(h_{ij}^{-1}(0,-)) \cdot h_{ij}^{-1}(-,s) \cdot \alpha_{g_{ij}(t,s)}(\kappa_{(A_i,B_i)}^{-1})
\]

given by the diagram on page 26 of [SWII]:

\[
(2.1.27)
\]
where $\kappa_{(A_i,B_i)}$ is evaluated on the standard bigon in $\mathbb{R}^2$. Note that, via [SWII, SWIII], the output of $\kappa_{(A_i,B_i)}$ is independent of parametrization (in fact, invariant under thin homotopy) and so the parametrization of the standard bigon is not given. By $F_{A_i}(-,t)$, and similar terms, we mean to evaluate on the path $\gamma$ associated to $(-,t)$ from the standard bigon. The vertices, $f(-,-)$ come from the part of the transport functor of [SWII] which associated to each point the object in our 2-group.

**Sketch of Proof.** The equation is obtained by calculating the compositions just as in the proof for Proposition 2.1.40. \qed

**Remark 2.1.43.** The above propositions follow from the fact that $(g_{ij}, h_{ij})$ is a smooth pseudonatural transformation $(F_{A_i}, \kappa_{A_i,B_i}) \to (F_{A_j}, \kappa_{A_j,B_j})$ and $f_{ijk}$ is a modification $(g_{jk}, h_{jk}) \circ (g_{ij}, h_{ij}) \Rightarrow (g_{ik}, h_{ik})$. The transport data consists of functions taking values in the crossed module $(H \xrightarrow{i} G)$ which, when assembled properly, build a 2-functor. The domain of this two-functor for Schreiber and Waldorf is the path 2-space, $\mathcal{P}_2(M)$ whose

- objects are elements $x \in M$,
- 1-morphisms are (thin homotopy equivalence classes of) paths $(\gamma : I \to M) \in P^1(M)$,
and 2-morphisms are (thin homotopy equivalence classes of) bigons \( \Sigma \in B^2(M) \) which are maps \( \text{Sq} \xrightarrow{\Sigma} M \) having the property that the vertical edges of \( \text{Sq} \) become “pinched” in \( M \).

The advantage of using bigons, with sitting instants, is that they glue together smoothly, and they model the common 2-group diagrams nicely.

2.1.5 Local Transport Data for Squares

While the referenced works of Schreiber and Waldorf focused on parallel transport as it pertains to bigons, this paper will follow an approach based on squares. The reason such a translation is easily available is due to the fact that associated to each square in \( \mathbb{R}^2 \) is the standard bigon (see equation (2.5) of \textit{SWII}). In this section, we define first local transport data for squares, which is basically the transport data for bigons translated to squares, and then re-express their properties (recall Definition 2.1.38 - Proposition 2.1.42) for this square data.

Recall the standard bigon from Definition 2.1.41. We now recall how to associate a particular bigon to a square \( \Sigma : \text{Sq} \rightarrow M \), thus yielding an element in \( H \) for a square, \( \Sigma \), given the element \( \kappa_{(A_i, B_i)}(\Sigma) \) for a bigon.

Thus, given a square mapped into \( U_i \), \( \Sigma : \text{Sq} \rightarrow U_i \), the standard bigon provides an associated bigon in \( B^2(U_i) \), that we have an associated element
\( \kappa_{(A_i, B_i)} \) from Definition 2.1.38

The Local Data

Below we provide definitions and properties of our local transport data for squares, \( \Sigma : Sq \rightarrow M \), which uses essentially the local transport data for bigons by Schreiber and Waldorf:

**Definition 2.1.44.** Given a gerbe on \( M \) as defined in Definition 2.1.34, equipped with the local transport data for bigons, and given a point \( x \in M \), a path \( \gamma : I \rightarrow M \) in \( M \), and a square \( \Sigma : Sq \rightarrow M \) in \( M \), we define

\[
\begin{align*}
\text{hol}_i(\gamma) &:= F_{A_i}(\gamma) \\
\text{Hol}_i(\Sigma) &:= f_{\Sigma}^{-1} = (\alpha_{\text{hol}_i(\gamma(0,-))^{-1} \text{hol}_i(\gamma(-,s))^{-1}}) (\kappa_{A_i, B_i}(\rho_{1,1} \circ \Sigma)) \\
\text{Hol}_{ij}(\gamma) &:= \alpha_{g_{ij}^{-1}(\gamma(0))} h_{ij}^{-1}(\gamma)) \\
\text{Hol}_{ijkl}(x) &:= \alpha(g_{ik}^{-1}(x)) \left( \alpha_{g_{kl}^{-1}(x)} (f_{jkl}(x)) \cdot f_{ijkl}^{-1}(x) \right)
\end{align*}
\]

**Remark 2.1.45.** Note that \( \text{hol}_i, \text{Hol}_i, \) and \( \text{Hol}_{ij} \) is essentially given by the local data \( F_{A_i}, \kappa_{(A_i, B_i)}, \) and \( h_{ij}^{-1} \), respectively, from Definition 2.1.38. The main difference is that the source of \( \text{Hol}_i \) and \( \text{Hol}_{ij} \) has been converted to a point; compare (2.1.15) with (2.1.35) and (2.1.18) with (2.1.36). Similarly, the quadruple intersection functions, \( \text{Hol}_{ijkl} \) are just a combination of the triple intersection functions \( f_{ijk} \) and \( f_{jkl} \) from Definition 2.1.34.
The local transport data for bigons by Waldorf and Schreiber can be assembled to a transport 2-functor which satisfies various functorial properties. We will use these functorial properties as needed. While one could translate these functorial properties to their analogs from Definition 2.1.44, we will not do so in this paper, as it is not necessary for the purposes of this paper. Below we only include some properties of the functions $\text{hol}_i, \text{Hol}_i, \text{Hol}_{ij}$, and $\text{Hol}_{ijkl}$ which are necessary for this paper.

The local data for bigons solved the differential equations from Definition 2.1.38. This translates to $\text{hol}_i, \text{Hol}_i,$ and $\text{Hol}_{ij}$ as follows:

**Proposition 2.1.46 (Differential Equations).** The collection of local transport data for squares $(\text{hol}_i, \text{Hol}_i, \text{Hol}_{ij})$ satisfy the differential equations:

1. **(D1)** For a one-parameter family of paths, $\gamma_t$, induced by the path, $\gamma(t)$, we have
   \[
   \frac{\partial}{\partial t} \text{hol}_i(\gamma_t)^{-1} = \text{hol}_i \cdot A_i \left( \frac{\partial}{\partial t} \right) \tag{2.1.32}
   \]

2. **(D2)** For a one-parameter family of squares, $\Sigma_s$, induced by the square, $\Sigma(t,s)$, we have
   \[
   \frac{\partial}{\partial s} \text{Hol}_i(\Sigma_s) = \text{Hol}_i(\Sigma_s) \cdot \int_0^1 (\alpha_{\text{hol}_i(-,s)})_*(B_i) \ dt \left( \frac{\partial}{\partial s} \right) \tag{2.1.33}
   \]

3. **(D3)** For a one-parameter family of paths, $\gamma_t$, induced by the path, $\gamma(t)$, we
have
\[
\frac{\partial}{\partial t}
\bigg|_{t=0}(\text{Hol}_{ij}(\gamma_t)) = \alpha_{g_{ij}^{-1}(\gamma(0))}
\left(a_{ij}\big|_{\gamma(0)}\left(\frac{\partial}{\partial t}\bigg|_{t=0}\right)\right)
\]
(2.1.34)

\textbf{Proof.} To prove (D1), observe that

\[
\frac{\partial}{\partial \text{hol}^{-1}} = -\text{hol}^{-1} \cdot \frac{\partial}{\partial t} \text{hol} \cdot \text{hol}^{-1} = -\text{hol}^{-1} \cdot \frac{\partial}{\partial t} F_{A_i} \cdot \text{hol}^{-1}
\]

\[
= -\text{hol}^{-1} \cdot (-A_i \cdot F_{A_i}) \cdot \text{hol}^{-1} = \text{hol}^{-1} \cdot A_i \cdot \text{hol} \cdot \text{hol}^{-1} = \text{hol}^{-1} \cdot A_i
\]

To prove (D3), since we are taking the derivative at \(t = 0\), we simply apply \(\frac{\partial}{\partial t}\) to \(\text{Hol}_{ij}\) and use the fact that \(g_{ij}\) is independent of \(t\) and \(h_{ij}\) evaluated on a path of length 0 is equal to 1:

\[
\frac{\partial}{\partial t}
\bigg|_{t=0}\text{Hol}_{ij} = \frac{\partial}{\partial t}
\bigg|_{t=0}\left(\alpha_{g_{ij}^{-1}(\text{hol}^{-1}(h_{ij}^{-1}))}\right) = \alpha_{g_{ij}^{-1}}\left(\frac{\partial}{\partial t}
\bigg|_{t=0}\text{hol}^{-1}\right)
\]

\[
= -\alpha_{g_{ij}^{-1}}\left(h_{ij}^{-1} \cdot \frac{\partial}{\partial t}
\bigg|_{t=0}\text{hol} \cdot \text{hol}^{-1}\right)
\]

\[
= -\alpha_{g_{ij}^{-1}}\left(-a_{ij}\left(\frac{\partial}{\partial t}
\bigg|_{t=0}\right) - (\alpha_1)_{\ast}\left(A_j \left(\frac{\partial}{\partial t}
\bigg|_{t=0}\right)\right)\right)
\]

\[
=\alpha_{g_{ij}^{-1}}\left(a_{ij}\left(\frac{\partial}{\partial t}
\bigg|_{t=0}\right)\right)
\]

\(\square\)

Similarly, the targets from Proposition 2.1.39 translate as follows. The target of our local data for squares follows algebraically based on how we defined our local data.

\textbf{Proposition 2.1.47} (Target Conditions). The collection of local transport
data for squares $\text{Hol}_i$ and $\text{Hol}_{ij}$ have the following targets:

- For $\text{Hol}_i$, we have target

\[
t(\text{Hol}_i) = t(\alpha_{\text{hol}_i(0,-)}^{-1}\text{hol}_i(-,s)^{-1}(\kappa_{A_i,B_i})
\]

\[
= \text{hol}_i(-,0)^{-1}\text{hol}_i(t,-)^{-1}\text{hol}_i(-,s)\text{hol}_i(0,-)
\]

which we will picture diagrammatically as (see (2.1.15)):

• For $\text{Hol}_{ij}$ we have target

\[
t(\text{Hol}_{ij}) = t(\alpha_{g_{ij}(x)}^{-1}\text{hol}_j^{-1}(h_{ij}^{-1})) = \text{hol}_i^{-1}g_{ij}^{-1}(y)\text{hol}_jg_{ij}(x)
\]

which we will picture diagrammatically as (see (2.1.18)):

Proposition 2.1.46 and 2.1.47 are the translations of Definition 2.1.38 and Proposition 2.1.39 to the new square data $\text{Hol}_i$, $\text{Hol}_{ij}$, and $\text{Hol}_{ijkl}$. We also
translate Propositions 2.1.40 and 2.1.42 to these new square data below in Propositions 2.1.49 and 2.1.50. Before we do so, we explain how square-diagrams are composed horizontally and vertically.

**Glueing-Paths**

Here we lay out some of the conventions and computations necessary to translate between the 2-functor axioms in [SWII] and the local transport data for squares defined above. As a medium of discussion, we prove the interchange law for our “glueing paths”. At the end of this section, we use the conventions established to prove Propositions 2.1.49 and 2.1.50, which are the statements from Propositions 2.1.40 and 2.1.42 written for $\text{Hol}_i$, $\text{Hol}_{ij}$, and $\text{Hol}_{ijkl}$.

If we want to glue two (horizontally) adjacent squares, we use the following “zip/unzip” procedure:

\[
\begin{array}{c}
\vdots & a & \vdots \\
\vdots & d & Hol_1 & b & Hol_2 & f \\
\vdots & c & \vdots & & g & \vdots \\
\end{array}
\]  

(2.1.37)

We can conceive this as a horizontal glueing via the following diagram of bigons (where the convention is momentarily switched to that of 2-categories
and thus the order of the arrows is reversed)

\[
\begin{array}{cccc}
\text{id} & d^{-1}c^{-1}b & \text{id} & b^{-1}cd \\
\downarrow & \uparrow & \downarrow & \uparrow \\
Hol_1 & \| & \| & Hol_2 \\
a^{-1}b^{-1}cd & d^{-1}c^{-1}b & e^{-1}f^{-1}gb & b^{-1}cd \\
\end{array}
\]  

(2.1.38)

which simplifies to (similar to the proof of Proposition 2.1.40)

\[
\begin{array}{c}
\text{id} \\
\downarrow \\
\text{Hol}_1 \cdot \alpha(d^{-1}c^{-1}b)(\text{Hol}_2) \\
\downarrow \\
a^{-1}e^{-1}f^{-1}gcd \\
\end{array}
\]  

(2.1.39)

and so we can finally obtain the horizontal path-glueing as

\[
\begin{array}{cccc}
& a & b \\
\downarrow & & \downarrow \\
\text{Hol}_1 & \| & \| & \text{Hol}_2 \\
& d & f \\
\end{array}
\]  

(2.1.40)

Similarly, for vertically adjacent squares, we use the following “zip/unzip”
procedure:

\[
\begin{array}{c}
\begin{array}{c}
\text{Hol}_2 \quad \text{Hol}_1
\end{array}
\end{array}
\]

We can conceptualize this vertical glueing via the following diagram of bigons (where remember we switch the convention and thus the order of the arrows)

\[
\begin{array}{c}
\begin{array}{c}
\text{id} \quad a^{-1} \quad \text{id} \quad a
\end{array}
\end{array}
\]

which simplifies to

\[
\begin{array}{c}
\begin{array}{c}
\text{id} \quad \alpha(a^{-1})(\text{Hol}_2)
\end{array}
\end{array}
\]

(2.1.43)
and so we can finally obtain the vertical path-glueing as

\[
\begin{align*}
\text{Hol}_2 \cdot \alpha(c^{-1}f^{-1}d) & \quad \text{Hol}_4 \cdot \alpha(h^{-1}k^{-1}i) \\
\text{Hol}_3 \cdot \alpha(h^{-1}k^{-1}i) & \quad \text{Hol}_2 \cdot \alpha(c^{-1}f^{-1}d)
\end{align*}
\]

(2.1.44)

Proposition 2.1.48. The procedures above for composing 2-squares satisfy the interchange law. Moreover, any grid of squares glues together to a single square providing a unique element in \( H \) associated to the grid.

Proof. We exhibit the proof in diagram-form. When first composing horizontally and then composing vertically we obtain
whereas if we compose vertically and then horizontally we obtain

\[
\begin{align*}
\text{Hol}_1 & \cdot \alpha(c^{-1})(\text{Hol}_3) \cdot \alpha((hc)^{-1}k^{-1}(\text{id})) \cdot \text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4) \\
= & \text{Hol}_1 \cdot \alpha(c^{-1})(\text{Hol}_3) \cdot \alpha((hc)^{-1}k^{-1}(\text{id})) \cdot \text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4)
\end{align*}
\]

We conclude the proof by showing the two expressions we obtained in \( H \) are equal. We start with the latter term (vertical then horizontal) and show it is equal to the former term (horizontal then vertical):

\[
\begin{align*}
\text{Hol}_1 & \cdot \alpha(c^{-1})(\text{Hol}_3) \cdot \alpha((hc)^{-1}k^{-1}(\text{id})) \cdot \text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4) \\
= & \text{Hol}_1 \cdot \alpha(c^{-1})(\text{Hol}_3) \cdot \alpha((hc)^{-1}k^{-1}(\text{id})) \cdot \text{Hol}_2 \cdot \alpha(d^{-1})(\text{Hol}_4)
\end{align*}
\]
We are now ready to rewrite Proposition 2.1.40.

**Proposition 2.1.49** (Vertex Cube). For a path $\gamma$ in $U_{ijkl}$, with $x = \gamma(0)$ and $y = \gamma(1)$, the local data for squares of Definition 2.1.44 satisfies

$$
\text{Hol}_{ijkl}(x) = \text{Hol}_{ij} \cdot \alpha_{g_{ij}^{-1}(x)\text{hol}_{ijkl}^{-1}\text{hol}_{ijkl}(y)} (\text{Hol}_{ijkl}(y))
$$

expressed by the following cube-diagram

![Cube Diagram](2.1.45)

**Proof.** Using the diagram (2.1.20), we can vertically glue in the left hand
and finally glue horizontally to obtain:

\[ \alpha(h_{lk}^{-1})(f_{ijk}(x)) \cdot h_{jk}^{-1} \cdot \alpha(g_{jk}(y))(h_{ij}^{-1}) = h_{ik}^{-1} \cdot f_{ijk}(y) \]  

(2.1.47)

Solving for \( f_{ijk}^{-1}(x) \) above yields

\[ f_{ijk}^{-1}(x) = \alpha(h_{lk}^{-1})(h_{jk}^{-1} \cdot \alpha(g_{jk}(y))(h_{ij}^{-1} \cdot f_{ijk}^{-1} \cdot h_{ik}) \]  

(2.1.48)

while rewriting the expression above via exchanging indices \( ijk \) with \( jkl \) and solving for \( \alpha^{-1}(f_{jkl}(x)) \) yields

\[ \alpha(g_{kl}^{-1})(f_{jkl}(x)) = \alpha(g_{kl}^{-1} h_{li}^{-1})(h_{jl}^{-1} f_{jkl}(y) \alpha(g_{kl}(y))(h_{jk}) \cdot h_{kl}) \]  

(2.1.49)

meaning we can rewrite \( Hol_{ijkl} \) using some standard computations as follows:

\[ Hol_{ijkl} = \alpha(g_{ik}^{-1}) \left( \alpha(g_{kl}^{-1})(f_{jkl}(x)) \cdot f_{ijk}^{-1}(x) \right) \]

\[ = \alpha(g_{ik}^{-1}) \left[ \alpha(g_{kl}^{-1} h_{li}^{-1})(h_{jl}^{-1} f_{jkl}(y) \alpha(g_{kl}(y))(h_{jk}) \cdot h_{kl}) \cdot h_{kl} \right] \cdot \alpha(h_{lk}^{-1})(h_{jk}^{-1} \cdot \alpha(g_{jk}(y))(h_{ij}^{-1} \cdot f_{ijk}^{-1} \cdot h_{ik}) \right] \]
\[= \alpha(g_{ik}^{-1}) \left[ \alpha(g_{kl}^{-1}(x)\text{hol}_{t}^{-1}(h_{jkt})f_{ijkl}(y)h_{kl}) \right. \\
\quad \cdot \alpha(\text{hol}_{k}^{-1})(\alpha(g_{jk}(y))(h_{ij}^{-1}f_{ijk}(y)h_{ik}) \left] , \right.
\]

which in turn, after repeatedly using the identity \(hh'h^{-1} = \alpha_t(h)(h')\) to get the \(h_{ij}^{-1}\)'s in the correct order, yields

\[\text{Hol}_{ijkl}(x) = \text{Hol}_{ij} \cdot \alpha_{g_{ij}^{-1}(x)\text{hol}_{j}g_{ij}(y)\text{hol}_{i}(\text{Hol}_{ik}^{-1})} \cdot \alpha_{g_{ij}^{-1}(x)\text{hol}_{i}^{-1}g_{ij}(y)\text{hol}_{k}(\text{Hol}_{kl}^{-1})} \]

Finally, we translate Proposition 2.1.42 to our current setup.

**Proposition 2.1.50 (Edge Cube).** Given a square, \(\Sigma\), we have the equation at each intersection \(U_{ij}\)

\[\text{Hol}_{ij}(-, 0) = \text{Hol}_{ij}(0, -) \cdot \alpha_{g_{ij}(0, 0)\text{hol}_{j}(0, -)\text{hol}_{i}(0, -)}(\text{Hol}_{i}(\Sigma)) \]

\[\cdot \alpha_{g_{ij}(0, 0)}^{-1}\text{hol}_{j}(0, -)\text{hol}_{i}(0, -)(\text{Hol}_{ij}(-, s))\alpha_{g_{ij}(0, 0)}^{-1}(\text{Hol}_{ij}^{-1}(\Sigma)) \]

\[\cdot \alpha_{g_{ij}(0, 0)}^{-1}\text{hol}_{j}(0, -)\text{hol}_{i}(t, -)g_{ij}(t, s)\text{hol}_{i}(t, -)(\text{Hol}_{ij}^{-1}(t, -))\]
expressed by the following cube-diagram

\[ \begin{array}{ccc}
\bullet & \text{hol}_i(-,0) & \bullet \\
\text{hol}_j(-,0) & \text{Hol}_{ij}(-,0) & \text{Hol}_i(-,0) \\
(0,0),g_{ij} & \text{Hol}_i(-,0) & \text{Hol}_i(-,0) \\
(0^{-1},0) & \text{Hol}_j(-,s) & \text{Hol}_j(-,s) \\
\end{array} \]

\[ = \begin{array}{ccc}
\bullet & \text{hol}_i(-,0) & \bullet \\
\text{hol}_j(-,0) & \text{Hol}_{ij}(-,0) & \text{Hol}_i(-,0) \\
(0,0),g_{ij} & \text{Hol}_i(-,0) & \text{Hol}_i(-,0) \\
(0^{-1},0) & \text{Hol}_j(-,s) & \text{Hol}_j(-,s) \\
\end{array} \]

\[ \text{Hol}_{ij}(\Sigma) \]

\[ \text{Hol}_i^{-1}(\Sigma) \]

\[ \text{Hol}_j^{-1}(\Sigma) \]

\[ (2.1.50) \]

Proof. Using the diagram (2.1.27), and the same ideas from the proof for the \(ijkl\) vertex cube above, we obtain the desired equation. \(\square\)

### 2.2 Glueing Together Local 2-Holonomy

In this section, we give a definition for 2-holonomy of a square mapped into \(M, \Sigma \in \mathcal{N} \subset M^{sq}\) (Definition 2.1.9), which lands in multiple open sets \(U_i \subset M\).
2.2.1 The Basic Idea

Following the ideas in [TWZ], in Section 2.1.1 we gave an open cover of $M^{Sq}$ by sets

$$N_I = \{ \Sigma \in M^{Sq} \mid \text{for each } (pq) \in I, \Sigma(Sq_{(p,q)}) \subset U_{i(p,q)} \}. $$

To explain the basic idea for our 2-holonomy, we assume a $2 \times 2$ grid of open sets $U_i$, $U_j$, $U_k$, and $U_l$. We first assemble our functions $\text{Hol}_i, \text{Hol}_{ij}$, and
\( \text{Hol}_{ijkl} \) in the following grid,

\[
\begin{array}{cccc}
\text{Hol}_i & \text{Hol}^{-1}_{ik} & \text{Hol}_k \\
\text{Hol}_{ij} & \text{Hol}_{ijkl} & \text{Hol}_{kl} \\
\text{Hol}_j & \text{Hol}^{-1}_{ji} & \text{Hol}_l \\
\text{Hol}_j & \text{Hol}_{jkl} & \text{Hol}_{kl} \\
\text{Hol}_i & \text{Hol}_{ijkl} & \text{Hol}_{kl} \\
\end{array}
\]

and we glue the terms together using the multiplication rules coming from
the horizontal and vertical gluing-conventions established in (2.1.40) and
(2.1.44). Even though the order is interchangeable, we have a preferred
order of first glueing vertically and then glueing horizontally.
In this case, we would define \( \text{Hol} := \text{Hol}^N : \mathcal{N} \to H \) by

\[
\text{Hol} := \text{Hol}_i \cdot \overline{\text{Hol}_{ij}} \cdot \overline{\text{Hol}_j} \cdot \text{Hol}_{ik}^{-1} \cdot \overline{\text{Hol}_{ijkl}} \cdot \overline{\text{Hol}_{jl}} \cdot \text{Hol}_k \cdot \text{Hol}_{kl} \cdot \text{Hol}_l. \quad (2.2.2)
\]

Here the bar notation, "\( \overline{X} \)" , is used to denote an \( \alpha \)-action of a path applied to the given "\( \text{Hol} \)"-term. As stated in Proposition 2.1.48 the grid determines the element in \( H \) but the order we use to glue together will determine the path-action. Since the default convention is to glue vertically and then horizontally, then we see such a path action will always be the one where we use the path going down, right, and then up:

\[
(2.2.3)
\]

For example, for \( \text{Hol}_{jl}^{-1} \) and \( \text{Hol}_{ijkl} \) in the diagram above, we write:

\[
\overline{\text{Hol}_{jl}} := \alpha((\text{hol}_i(x)_a)^{-1} \cdot (g_{ij}(x,b))^{-1} \cdot (\text{hol}_j(x)_c)^{-1} \cdot (\text{hol}_j(e)_d)^{-1} \cdot (\text{hol}_j(y)_e)^{-1} \cdot (\text{hol}_{ijkl})^{-1});
\]

\[
\overline{\text{Hol}_{ijkl}} := \alpha((\text{hol}_i(x)_a)^{-1} \cdot (g_{ij}(x,b))^{-1} \cdot (\text{hol}_j(x)_c)^{-1} \cdot (\text{hol}_j(e)_d)^{-1} \cdot (\text{hol}_j(y)_e)^{-1} \cdot \text{hol}_{ijkl} \cdot g_{ij}(y,b)^{-1} \cdot (\text{Hol}_{ijkl})).
\]

### 2.2.2 The Formal Definition

Fix an open cover \( \mathcal{U} = \{U_i\} \) of \( M \) and an open set \( \mathcal{N} \subset M^{S^q} \) as in Definition 2.1.9

**Definition 2.2.1.** Given local transport data for squares, \( \{\text{Hol}_i\}, \{\text{Hol}_{ij}\}, \)
\{ \text{Hol}_{ijkl} \}, define \text{Hol}^N : \mathcal{N} \to H on a square \Sigma \in \mathcal{N} \subset M^\text{Sq} by assembling the local data on the grid described in Definition \textbf{2.1.10} and included here:

\begin{align*}
\text{Hol}_a & \quad \text{Hol}_a^{-1} \quad \text{Hol}_b \quad \text{Hol}_b^{-1} \quad \text{Hol}_c \quad \cdots \quad \text{Hol}_d \\
\text{Hol}_{ac} & \quad \text{Hol}_{aef} \quad \text{Hol}_{bf} \quad \text{Hol}_{bfg} \quad \text{Hol}_{cg} \quad \cdots \quad \text{Hol}_{dh} \\
\text{Hol}_{ae} & \quad \text{Hol}_{ef} \quad \text{Hol}_f \quad \text{Hol}_f^{-1} \quad \text{Hol}_g \quad \cdots \quad \text{Hol}_h \\
\text{Hol}_{ei} & \quad \text{Hol}_{efj} \quad \text{Hol}_j \quad \text{Hol}_j^{-1} \quad \text{Hol}_k \quad \cdots \quad \text{Hol}_l \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\
\text{Hol}_m & \quad \text{Hol}_m^{-1} \quad \text{Hol}_m \quad \text{Hol}_m^{-1} \quad \text{Hol}_o \quad \cdots \quad \text{Hol}_p \\
\end{align*}

(2.2.4)

Using the multiplication conventions (diagrams \textbf{(2.1.40)} and \textbf{(2.1.44)}) for squares, we glue together first vertically and then horizontally to obtain the following expression

\[
\text{Hol}^N := \text{Hol}_a \cdot \text{Hol}_{ae} \cdot \text{Hol}_e \cdot \cdots \cdot \text{Hol}_m
\]

(2.2.5)
\[
\cdot \text{Hol}^{-1}_{ab} \cdot \text{Hol}^{-1}_{abef} \cdot \text{Hol}^{-1}_{ef} \ldots \cdot \text{Hol}^{-1}_{mn} \\
\vdots \\
\cdot \text{Hol}^{-1}_d \cdot \text{Hol}^{-1}_{dh} \cdot \text{Hol}^{-1}_h \ldots \cdot \text{Hol}^{-1}_p
\]

where \(\text{Hol}_i\) is evaluated on the face \(\Sigma_i\), \(\text{Hol}_{ij}\) is evaluated on the edge \(\gamma_{ij}\), and \(\text{Hol}_{ijkl}\) is evaluated on the vertex \(x_{ijkl}\) as described in Definition \(2.1.10\), and the "\(\overline{X}\)" is described in the following convention. Note that each face, \(\Sigma_a\), and edge, \(\gamma_{ab}\), technically is a subspace of \(I \times I\) or subinterval of \(I\). However, reparametrization does not change the output since these functions are based on the thin-homotopy-invariant functions of [SWII].

**Convention 2.2.2.** In general, the \(\overline{X}\) is given by the rules for glueing horizontally and vertically (diagrams \(2.1.40\) and \(2.1.44\)) once the order of the squares is chosen. Informally, the convention of first glueing vertically and then horizontally means we will act on \(X\) by the following path-holonomy: start at the upper left corner of the grid, then move down the far left edge of the grid until you reach the bottom, then move right until you are at the left side of the column of \(X\), then move up until you end at the upper left corner of the \(X\)-square. An example of this is given in Section \(2.2.1\).
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2.2.3 Properties of $\text{Hol}^N$ for Squares

Many of the desired properties of 2-holonomy can be obtained by interpreting 2-holonomy as a 2-functor from some 2-path space to a 2-group. Since we would like our 2-path space to have a vertical and horizontal composition, our holonomy must respect that composition with the product in the 2-group. Moreover, due to associativity in the 2-path space, along with requiring that holonomy respect composition, we must have invariance of thin-homotopy. Since the local transport data on squares defined here is just a variation of the transport data for bigons, these desired properties of thin-homotopy-invariance and respecting composition are satisfied quite readily on a local level. As this section as a whole deals with providing a definition for holonomy of a square which spans multiple open sets, similar properties for holonomy are verified below justifying it is a suitable definition. Note that the propositions here are original in so far as $\text{Hol}^N$ is an original description of 2-holonomy, despite using the local data of Schreiber and Waldorf and borrowing ideas from Martins and Picken in our definition.

Proposition 2.2.3. $\text{Hol}^N$ is invariant under thin homotopy.

Sketch of Proof. In this proof we will appeal to the expression for $d(\text{Hol}^N)$ included in the next section. This is not circular reasoning since none of the
following statements about $\text{Hol}^N$ on $\mathcal{N}$ are needed in computing $d(\text{Hol}^N)$. So then we consider a plot in $M^{Sq}$, giving us a map $[0,1]^3 \to M$, and at each $r \in [0,1]$ we have a different $\Sigma_r \in M^{Sq}$. The point is that the derivative (i.e. $d(\text{Hol}^N)$) on such a thin plot will result in zero derivative and thus a constant function yielding $\text{Hol}^N$ being invariant under thin homotopy.

**Proposition 2.2.4.** The target of $\text{Hol}^N$, assuming the grid (2.2.4), is

$$t(\text{Hol}^N) = \text{hol}_a^{-1} \cdot g_{ab}^{-1} \cdot \text{hol}_b^{-1} \cdot \ldots \cdot \text{hol}_d^{-1} \cdot g_{de}^{-1} \cdot \ldots \cdot \text{hol}_e^{-1} \cdot g_{ae}^{-1} \cdot \text{hol}_a.$$ 

Furthermore, $\text{Hol}^N$ respects composition of squares.

**Proof.** The expression for $t(\text{Hol}^N)$ follows from the conventions for glueing squares. For composition of squares, let $\Sigma, \Sigma' \in M^{Sq}$ such that $\Sigma(1,s) = \Sigma'(0,s)$, for all $s \in I$. Then we have a horizontal composition of squares $\Sigma \circ_h \Sigma' \in M^{Sq}$ given by

$$\Sigma \circ_h \Sigma'(t,s) = \begin{cases} 
\Sigma(2t,s) & \text{if } 0 \leq t \leq 1/2 \\
\Sigma'(2t-1) & \text{if } 1/2 \leq t \leq 1 
\end{cases}$$

By an argument similar to that in the proof of Proposition 2.1.14 there exists an open set which we suggestively label $\mathcal{N}_{I_0 I'}$, the open set in $M^{Sq}$ where the open sets $U_a \subset M$ being used on either side of the edge (i.e. boundary path) that $\Sigma$ and $\Sigma'$ share are the same. The indexing set is given by $I \circ_h I' = \{1, \ldots, 2n\} \times \{1, \ldots, m\}$ so that $\Sigma \circ_h \Sigma' \in \mathcal{N}_{I_0 I'}$, and for each
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\[ q \in \{1 \ldots m \} \]

\[ \Sigma \circ_h \Sigma' \bigg|_{\left[\frac{n-1}{2m}, \frac{n+1}{2m}\right] \times \left[\frac{q-1}{m}, \frac{q}{m}\right]} \subset U_{n,q}. \]

Now there are induced open sets \( \mathcal{N}_I \) and \( \mathcal{N}_I' \) to be though of as the left and right halves of \( \mathcal{N}_{Ioh'} \). Thus we have a notion of horizontal composition written

\[ \text{Hol}^{\mathcal{N}_{Ioh'}}(\Sigma \circ_h \Sigma') = \text{Hol}^{\mathcal{N}_I}(\Sigma) \cdot \overline{\text{Hol}^{\mathcal{N}_{I'}}(\Sigma)} \]

where the overline is given by equation (2.1.40); compare also Convention 2.2.2. Similarly we have a compatibility for vertical composition:

\[ \text{Hol}^{\mathcal{N}_{Iov'}}(\Sigma \circ_v \Sigma') = \text{Hol}^{\mathcal{N}_I}(\Sigma) \cdot \overline{\text{Hol}^{\mathcal{N}_{I'}}(\Sigma)} \]

**Proposition 2.2.5.** \( \text{Hol}^\mathcal{N} \) is invariant under subdivision.

*Proof.* Subdivision induces a new grid, on which the added transition data will be shown to be the identity. Consider the following vignette of what
subdivision might produce near an $ijkl$ vertex:

\[
\begin{array}{ccc}
\text{Hol}_i & \text{Hol}_{ij}^{-1} & \text{Hol}_j \\
\text{Hol}_{ij} & \text{Hol}_{ijij} & \text{Hol}_{ijij}^{-1} \\
\text{Hol}_j & \text{Hol}_{jij}^{-1} & \text{Hol}_j \\
\end{array}
\]

\[
= \begin{array}{ccc}
\text{Hol}_i & \text{Hol}_{iij} & \text{Hol}_{iij}^{-1} & \text{Hol}_i \\
\text{Hol}_{iij} & \text{Hol}_{iijjj} & \text{Hol}_{iijjj}^{-1} & \text{Hol}_{iijjj} \\
\text{Hol}_{iij} & \text{Hol}_{iijij}^{-1} & \text{Hol}_{iijij}^{-1} & \text{Hol}_{iijij}^{-1} \\
\end{array}
\]

(2.2.6)

where equality comes from $\text{Hol}_{ii}$, $\text{Hol}_{iii}$, and $\text{Hol}_{ijij}$ all being the identity in $H$ which we will briefly explain to conclude the proof. From Definition 2.1.34 we have

\[
\alpha_{ii} = 0 \quad (2.2.7)
\]

\[
f_{ij} = f_{ij} = 1 \quad (2.2.8)
\]

\[
f_{ikl} \alpha_{g_{kl}} (f_{ijk}) = f_{ij} f_{jkl} \quad (2.2.9)
\]

Equation (2.2.7) yields $\text{Hol}_{ii}$ by the differential equation and initial condition satisfied by $\text{Hol}_{ij}$. For $\text{Hol}_{iijj}$, we use (2.2.9), with $i = k$, $j = l$, to write

\[
\alpha_{g_{ij}}^{-1} (f_{ij}^{-1} f_{ij}) = \alpha_{g_{ij}}^{-1} (f_{ij}) f_{ij}^{-1} \\
\Rightarrow \text{Hol}_{iijj} := \alpha_{g_{ij}}^{-1} (\alpha_{g_{ij}}^{-1} (f_{ij}) f_{ij}^{-1}) \\
= \alpha_1 (\alpha_{g_{ij}}^{-1} (f_{ij}^{-1} f_{ij})) = \alpha_1 (\alpha_{g_{ij}}^{-1} (1)) = 1.
\]
Similarly, and much more simply,
\[ \text{Hol}_{ikk} = \alpha_{g_{kk}}^{-1}(\alpha_{g_{kk}}^{-1}(f_{ikk}) \cdot f_{ikk}^{-1}) = 1 \]
follows from (2.2.8) and, thus also, \(\text{Hol}_{i_1i_2i_3} = 1\). □

**Proposition 2.2.6.** \(\text{Hol}^N\) transforms across open sets \(N_i\) and \(N'_i\) by \(\alpha_{g_{ii}'}\) at the base point and by \(\text{Hol}_{jj'}\) and \(\text{Hol}_{ij'i'}\) along the boundary. In particular we can write
\[ \text{Hol}^{N_{i'i'}} = \alpha_{g_{ii}'}(\text{Hol}^{N_i}) \cdot \prod_{\partial \Sigma} \text{Hol}_{jj'} \cdot \prod_{\partial \Sigma} \text{Hol}_{jkj'} \]

**Proof.** Let \(\Sigma \in N_{i_0} \cap N'_{i_0}\). By Proposition 2.2.5, there exists a subdivision using open sets \(N_i\) and \(N'_i\), which both use a grid of size \(n\) (t-direction) by \(m\) (s-direction) such that
\[ \text{Hol}^{N_{i_0}} = \text{Hol}^{N_i} \quad \text{and} \quad \text{Hol}^{N'_{i_0}} = \text{Hol}^{N'_i}. \]

Consider the grid of \(\text{Hol}^{N_i}\) analogous to that from Definition 2.2.1 but re-
place each face $\text{Hol}_i'$ with a cube (Proposition 2.1.50)

(2.2.10)

replace each horizontal edge $ij$ with a cube (Proposition 2.1.49)

(2.2.11)
and similarly each vertical edge $ik$ with a cube

$$
\begin{align*}
&\begin{array}{c}
(0,0) \\
\downarrow
\end{array} \quad \begin{array}{c}
(t,0)
\end{array} \\
\end{align*}
\end{align*}
$$

$$
\begin{align*}
&\begin{array}{c}
(0,s)
\end{array} \quad \begin{array}{c}
(s,t)
\end{array}
\end{align*}
$$

(2.2.12)

Although it has not been laid out in a previous proposition, we can divide
an $ijkl'j'k'l'$ cube into $i_1i_2i_3i_4$-tetrahedra and use similar arguments to the
proofs of Proposition 2.1.49 and Proposition 2.1.50, this time using the iden-
tities for $f_{ijk}$ on $ijkl$ from Definition 2.1.34 to build a cube:

$$
\begin{align*}
&\begin{array}{c}
(0,0) \\
\downarrow
\end{array} \quad \begin{array}{c}
(t,0)
\end{array} \\
\end{align*}
\end{align*}
$$

$$
\begin{align*}
&\begin{array}{c}
(0,s)
\end{array} \quad \begin{array}{c}
(s,t)
\end{array}
\end{align*}
$$

(2.2.13)

Replacing each function in the grid of $\text{Hol}^{N\nu}$ with their associated cube from
above, as shown in the following glimpse near an \(ijkl\) vertex,

\[
\begin{align*}
\text{Hol}_{ii'}(t_0, -) & \quad \text{Hol}_{ik'}(t_1, s_0) \quad \text{Hol}_{kk'}(t_2, -) \\
\text{Hol}_{i} & \quad \text{Hol}_{ik}^{-1} & \quad \text{Hol}_{k} \\
\text{Hol}_{ii'}(-, s_1) & \quad \text{Hol}_{ik'}(t_1, s_1)^{-1} & \quad \text{Hol}_{kk'}(-, s_1) \\
\text{Hol}_{ij'}(t_0, s_1) & \quad \text{Hol}_{ijkl} & \quad \text{Hol}_{kl} \\
\text{Hol}_{ij'}(-, s_1) & \quad \text{Hol}_{ijkl'}(t_1, s_1) & \quad \text{Hol}_{kl'}(-, s_1) \\
\text{Hol}_{ij'}(t_1, s_1) & \quad \text{Hol}_{ijkl'}(t_1, s_1)^{-1} & \quad \text{Hol}_{kl'}(t_2, s_1) \\
\text{Hol}_{ij'}(-, s_2) & \quad \text{Hol}_{ijkl'}(t_1, s_2) & \quad \text{Hol}_{kl'}(-, s_2) \\
\text{Hol}_{ij'}(t_1, s_2) & \quad \text{Hol}_{ijkl'}(t_1, s_2)^{-1} & \quad \text{Hol}_{kl'}(t_2, s_2) \\
\text{Hol}_{ij'}(t_2, -) & \quad \text{Hol}_{ijkl'}(t_2, -) & \quad \text{Hol}_{kl'}(-, -) \\
\end{align*}
\]

we see that all of the interior transition data cancels, leaving only transition data at the boundary. Note that the \(\alpha_{g_{ii'}}\) from the statement comes from changing the basepoint from \(U_i\) to \(U_{i'}\) in the upper left corner of the square. This proves the statement of the proposition. \(\square\)
2.3 \( d(Hol) \) for Squares

In this section, we prove the following theorem, and the main result of this chapter, which says that the deRham differential applied to our global 2-holonomy amounts to replacing one \( B_i \) in any summand with one \( H_i \) in addition to some terms on the boundary of \( \Sigma \).

**Theorem 2.3.1.** For the function \( Hol = Hol^N : N \to H \subset Mat \) defined on the open set \( N \subset M^{Sq} \), we can compute its total derivative \( d(Hol) \in \Omega^1(M^{Sq}, Mat) \) as follows:

\[
d(Hol) = -(\alpha_{Hol})^*(A_{i(1,1)}) + Hol \cdot \int_{Sq} H + Hol \cdot \left( \int_{\partial Sq} B + \sum a \right) \quad (2.3.1)
\]

where \( i_{(1,1)} \) is the index of the upper left open set \( U_{i_{(1,1)}} \), and

\[
\int_{Sq} H := \sum_{k=1}^n \sum_{l=1}^m \int_{Sq(k,l)} (\alpha_{path_{k,l}})^*(H_{i(k,l)}) \quad (2.3.2)
\]

While the terms in the parenthesis on the right in (2.3.1) are boundary terms:

\[
\int_{\partial Sq} B := -\sum_{k=1}^n \int_{\gamma_N} (\alpha_{path_{k,1}})^*(B_{i(k,1)}) - \sum_{l=1}^m \int_{\gamma_{E(n,l)}} (\alpha_{path_{E(n,l)}})^*(B_{i(n,l)}) \quad (2.3.3)
\]

\[
+ \sum_{k=n-1}^n \int_{\gamma_S} (\alpha_{path_{S(k,m)}})^*(B_{i(k,m)}) + \sum_{l=m-1}^1 \int_{\gamma_{W(1,l)}} (\alpha_{path_{W(1,l)}})^*(B_{i(1,l)}) \quad (2.3.4)
\]

\[
\sum a := -\sum_{k=1}^{n-1} (\alpha_{path_{N(k+1,1)}})^*(a_{i(k,1)i(k+1,1)}) - \sum_{l=1}^{m-1} (\alpha_{path_{E(n,l)}})^*(a_{i(n,l)i(n,l+1)}) \quad (2.3.5)
\]
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\[ \sum_{k=n,...,2} (\alpha \text{path}^S_{(k-1,m)(k,m)}) \cdot (a_{i(k-1,m)}i(k,m)) \]  \hspace{1cm} (2.3.7)

\[ \sum_{l=m,...,2} (\alpha \text{path}^W_{(1,l-1)(1,l)}) \cdot (a_{i(1,l-1)}i(1,l)) \]  \hspace{1cm} (2.3.8)

The proof of the above theorem, taking up the rest of this section, will proceed in the following steps:

Step 1 First, we look at the locally defined holonomy functions \( \text{Hol}_i(s,t) \), where \( \text{Hol}_i \) depends on the size of the square \([0,s] \times [0,t] \subset [0,1] \times [0,1] \).

For a 1-parameter family of squares, parametrized by a parameter \( r \), we prove in Proposition 2.3.3 of Section 2.3.1 that

\[ \left. \frac{\partial}{\partial t} \right|_{t=0} \frac{\partial}{\partial s} \left| \frac{\partial}{\partial r} \right|_{r=0} (\text{Hol}_i) = H_i, \]

where \( H_i \) is the 3-curvature on \( U_i \). Next, Lemma 2.3.4 computes the more general expression \( \left. \frac{\partial}{\partial r} \right|_{r=0} (\text{Hol}_i) \).

Step 2 In Section 2.3.2 we derive an equation that will be used in the calculation for \( d(\text{Hol}) \). More precisely, we take the derivative of the Vertex Cube equation (Proposition 2.3.36 placed into \( \text{Hol} \); see (2.3.36)) and then the Edge Cube (Proposition 2.1.50 placed into \( \text{Hol} \)), both vertical (2.3.69) and horizontal (2.3.65), to obtain relations to be used in the final section.

Step 3 Finally, in Section 2.3.3 we lay out the remaining pieces of the argu-
ment. The basic idea is that after taking $\frac{\partial}{\partial r}(\text{Hol})$, where we write $\text{Hol}$ as a product of $\text{Hol}_i$’s, $\text{Hol}_{ij}$’s, and $\text{Hol}_{ijkl}$’s, we can use the relations from Step 2 to see that the only terms left are path terms, which will cancel, boundary terms which we will leave as-is, and $H_r$-terms, yielding the result.

2.3.1 A Local Lemma

Here we constructively prove a very useful lemma, Lemma 2.3.4, which calculates $d(\text{Hol}_i)$, the derivative of 2-holonomy over one open set, $U_i$. First, we recall an argument for the 1-dimensional analog of Lemma 2.3.4 and then later prove, in full detail, the main Lemma of this section.

Motivational Warmup: The 1-dimensional case

In the 1-dimensional case, taking “local” paths in one open set, $U$, recall the holonomy function $\text{hol} : M^{[0,1]} \rightarrow G$. Then $\text{hol}$ satisfies:

$$d(\text{hol}) \left( \frac{\partial}{\partial r} \right) = -A_a \left( \frac{\partial}{\partial r} \right) \cdot \text{hol}_a^b + \text{hol}_a^b \cdot A_b \left( \frac{\partial}{\partial r} \right) + \int_a^b \text{hol}_a^t \cdot R_t \left( \frac{\partial}{\partial r} \right) \cdot \text{hol}_a^b dt$$

This fact can be found in Proposition 2.1 of [TWZ], among other places. The idea of the proof we wish to adapt below, consists of first rewriting $\text{hol}$ as a product of parallel transports of $n$-many squares and two edge terms, at a
and \( b \), respectively:

$$\text{hol}(\gamma_r)$$

Then we take the limit as \( n \to \infty \) of the derivative with respect to \( r \) of this picture. As \( n \to \infty \), the area of each box goes to 0 and the derivative of the holonomy around \( c_i \) goes to \( R \). The two bookends provide the \( A \) terms while the rest of the cubes approach \( \text{hol}(0) \). Taking this limit yields the formula for \( d(\text{hol}) \) stated above.

**The 2-dimensional case**

We wish to consider \( d(Hol) \left( \frac{\partial}{\partial r} \right) \) for some vector \( \frac{\partial}{\partial r} \) in the domain of a plot \( \rho : U \to M^{Sq} \). Since we are only dealing with the \( r \) direction, this amounts to focusing on a one dimensional plot.

Fix a plot \( \rho : \mathbb{R} \to M^{Sq} \) i.e. a map

\[
\tilde{\rho} : Sq \times \mathbb{R} \to M
\]

\[(t, s, r) \mapsto \Sigma_r(t, s)\]
where \((\Sigma_r := \tilde{\rho}(r) : [0, 1]^2 \rightarrow M)\). We wish to compute \(d(Hol)|_{\rho}(\frac{\partial}{\partial r})\) via a product of cubes. The first step is to introduce cubes analogous to the squares, “\(c_i\)” in the 1-dimensional case.

Consider a sub-square \(S_{q_\alpha} \subset Sq := [0, 1]^2\), of \(\Sigma_0 : [0, 1]^2 \rightarrow M\),

\[
(2.3.10)
\]

For each \(r_0 > 0\), the plot \(\rho\) gives a corresponding cube, for which we only care about the boundary (i.e. faces, edges, and vertices), shown here:

\[
(2.3.11)
\]

It will be necessary to glue such cubes together and so we follow a convention involving a possible reorientation of each face and then taking \(Hol\) of the resulting square, which we now explain. The top face, \(Hol^T_{\alpha}\) will be given by

\footnote{Here the index \(\alpha\) is unrelated to the \(\alpha\)-map for a crossed module.}
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\( \text{Hol} \left( \Sigma_r \big|_{Sq_\alpha} \right) \), or 2-holonomy of the square at height \( r \):

\[
\text{Hol}^T (t + \epsilon_t)
\]

where we include the target/boundary of \( \text{Hol}^T_\alpha \) as an orientation of the square, and the target of its 2-holonomy, for the reader’s convenience. The bottom face is defined simply via \( \text{Hol}_\alpha := \text{Hol} \left( \Sigma_0 \big|_{Sq_\alpha} \right) \). The western face is defined, \( \text{Hol}^W_\alpha := \text{Hol}(\Gamma_W)^{-1} \), where \( \Gamma_W(a, b) := \Sigma_b \big|_{Sq_\alpha}(t, a) \), visualized as

\[
\text{Hol}^W
\]

Similarly, we define the eastern face \( \text{Hol}^E_\alpha := \text{Hol}(\Gamma_E) \) where \( \Gamma_E(a, b) := \Gamma_b \big|_{Sq_\alpha}(t + \epsilon_t, a) \), the northern face \( \text{Hol}^N_\alpha := \text{Hol}(\Gamma_N) \) where \( \Gamma_N(a, b) := \Sigma_b \big|_{Sq_\alpha}(a, 0) \), and the southern face \( \text{Hol}^S_\alpha := \text{Hol}(\Gamma_S)^{-1} \) where \( \Gamma_S(a, b) := \Sigma_b \big|_{Sq_\alpha}(a, s + \epsilon_s) \). The above conventions can be summarized in the following
picture, which will be useful throughout the section.\footnote{In fact, this idea is used repeatedly throughout this paper.}

For any such subsquare, $S_{q_\alpha}$, we define the associated cube

$$C_\alpha := \hat{\text{Hol}}_{T_\alpha} \cdot \text{Hol}_{N_\alpha} \cdot \text{Hol}_{W_\alpha} \cdot \text{Hol}_{E_\alpha} \cdot \text{Hol}_{S_\alpha} \cdot \text{Hol}_{T_\alpha}^{-1}$$

where $\hat{X}$ and $X$ are used in this paper to represent an action on $X$ by the 1-holonomy of a particular path not involving $r$ or not involving $r$, respectively.

\textbf{Convention 2.3.2.} Both $\hat{X}$ and $X$ denote $\alpha_{\text{path}}(X)$ for an implicit choice of path, determined by the ordering of the squares and the horizontal and vertical gluing conventions. We write $\hat{X}$ if the path depends on a chosen parameter, and $X$ if it does not depend on the parameter. As an example, in the above case, where we are concerned with dependence on $r$, only $\text{Hol}_{T_\alpha}$ depends on $r$:

$$\hat{\text{Hol}}_{T_\alpha} := \alpha_{\text{hol}_{(t,s,-)}}^{-1} (\text{Hol}_{T_\alpha}),$$
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\[ \overline{\text{Hol}}^S_\alpha := \alpha_{\text{hol}^{-1}_t,0} (\text{Hol}^S_\alpha), \]

\[ \overline{\text{Hol}}^E_\alpha := \alpha_{\text{hol}^{-1}_t,0,0} (\text{Hol}^E_\alpha). \]

**Proposition 2.3.3.** For a cube, \( C_\alpha \), as defined above we have

\[ \frac{\partial}{\partial r} \bigg|_{r=0} \frac{\partial}{\partial s} \bigg|_{s=0} \frac{\partial}{\partial t} \bigg|_{t=0} (C_\alpha) = H \bigg|_{(t,s,0)} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \]

**Proof.** Note that we are taking the derivative of the 2-holonomy around the cube as the volume of the cube goes to zero. Without loss of generality, we can assume that \( t = s = 0 \) and think of \( C_\alpha \) as a function \( C(u,v,r) \) to simplify some notation. We are interested in computing

\[ \frac{\partial}{\partial r} \bigg|_{r=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial u} \bigg|_{u=0} (C_\alpha(u,v,r)) \]

and since

\[ C(u,v,r) := \overline{\text{Hol}}^T_\alpha(u,v,r) \cdot \overline{\text{Hol}}^W_\alpha(0,v,r) \cdot \overline{\text{Hol}}^S_\alpha(u,v,r) \cdot \overline{\text{Hol}}^E_\alpha(u,v,r) \cdot \overline{\text{Hol}}^N_\alpha(u,0,r) \cdot \overline{\text{Hol}}_\alpha(u,v,o)^{-1} \]

we can apply the Liebniz rule. The only non-zero terms arise from applying all three partial derivatives to a \( \text{Hol}^\bullet \) which depends on \( u,v \), and \( r \) (since applying only one derivative and setting \( u = v = r = 0 \) yields zero via Proposition \( \ref{prop:1.46} \)). Since only \( \text{Hol}^T, \text{Hol}^E, \) and \( \text{Hol}^S \) depend on all three
variables, we obtain:

\[
\begin{align*}
\frac{\partial}{\partial r} \bigg|_{r=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial u} \bigg|_{u=0} (C_\alpha(u,v,r)) &= \frac{\partial}{\partial r} \bigg|_{r=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial u} \bigg|_{u=0} \widehat{\text{Hol}}^T(u,v,r) \\
&+ \frac{\partial}{\partial u} \bigg|_{u=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial r} \bigg|_{r=0} \widehat{\text{Hol}}^E(u,v,r) \\
&+ \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial u} \bigg|_{u=0} \frac{\partial}{\partial r} \bigg|_{r=0} \widehat{\text{Hol}}^S(u,v,r).
\end{align*}
\] (2.3.15)

To compute \( \frac{\partial}{\partial u} \bigg|_{u=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial r} \bigg|_{r=0} \widehat{\text{Hol}}^E(u,v,r) \) we write

\[
\overline{\text{Hol}}^E = \alpha_{\text{path}_E(u,v,r)}(\text{Hol}^E)
\]
and see that

\[
\begin{align*}
\frac{\partial}{\partial r} \bigg|_{r=0} \overline{\text{Hol}}^E(u,v,r) &= (\alpha_{\text{path}_E(u,v,0)})_*(\frac{\partial}{\partial r} \bigg|_{r=0} \text{Hol}^E) \\
&+ (\alpha_{\text{Hol}^E(u,v,0)})_*(\frac{\partial}{\partial r} \bigg|_{r=0} \text{path}_E) \\
&= (\alpha_{\text{path}_E(u,v,0)})_* \left( \int_0^v \alpha_*(B) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) ds \right) \\
&+ 0
\end{align*}
\] (2.3.16)

where the last equality comes from the fact that the path approaching \( \text{Hol}^E \) is constant with respect to \( r \). We continue next by computing

\[
\frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial r} \bigg|_{r=0} \overline{\text{Hol}}^E(u,v,r) = \frac{\partial}{\partial v} \bigg|_{v=0} (\alpha_{\text{path}_E(u,v,0)})_* \left( \int_0^v \alpha_*(B) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) ds \right)
\]

\[
= (\alpha_{\text{path}_E(u,0,0)})_* \left( \int_0^v \alpha_*(B) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) ds \right)
\]

\[
= (\alpha_{\text{path}_E(u,0,0)})_* \left( \int_0^v \alpha_*(B) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) ds \right)
\]

\[
= (\alpha_{\text{path}_E(u,0,0)})_* \left( B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right)
\]
where we abuse some notation but use \( \int_0^0 \alpha_*(B) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) ds = 0 \) and so the pushforward is trivial. Finally we obtain

\[
\frac{\partial}{\partial u} \bigg|_{u=0} \left( \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial r} \right) \left. \right|_{r=0} \text{Holg}(u,v,r)
\]

\[
= \frac{\partial}{\partial u} \bigg|_{u=0} (\alpha_{path_E(u,0,0)})^* \left( B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right)
\]

\[
= (\alpha_{path_E(0,0,0)})^* \left( \frac{\partial}{\partial u} \bigg|_{u=0} B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right) + (\alpha_B(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}))^* \left( \frac{\partial}{\partial u} \bigg|_{u=0} path_E(u,0,0) \right)
\]

\[
= \frac{\partial}{\partial u} \bigg|_{u=0} B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) + (\alpha_B(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}))^* A \left( \frac{\partial}{\partial t} \right)
\]

\[
= \frac{\partial}{\partial t} \left( B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right) + \alpha_A(\frac{\partial}{\partial s}) \left( B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right).
\]

Where again we abused some notation for better readability. By analogous arguments and considerations, we can rewrite all three terms from (2.3.15) as:

\[
\frac{\partial}{\partial r} \bigg|_{r=0} \frac{\partial}{\partial v} \bigg|_{v=0} \frac{\partial}{\partial u} \bigg|_{u=0} (C_\alpha(u,v,r)) = \frac{\partial}{\partial r} B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} + (\alpha_B(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}))^* A \left( \frac{\partial}{\partial t} \right)
\]

\[
+ \frac{\partial}{\partial t} B \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) + (\alpha_B(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}))^* A \left( \frac{\partial}{\partial t} \right)
\]

\[
- \frac{\partial}{\partial s} B \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial r} \right) - (\alpha_B(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}))^* A \left( \frac{\partial}{\partial s} \right)
\]

\[
= (dB + \alpha_A(B)) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right)
\]

\[
= H \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right).
\]

\[\square\]
We are now ready to prove the main Lemma of this section:

**Lemma 2.3.4.** For the local 2-holonomy function $Hol_i : U_i^{Sq} \to H$ as defined in Definition 2.1.44 we have

$$d(Hol_i) = -\left(\alpha_{Hol_i}\right)_\star(A_i(0,0)) + Hol_i \cdot \int_{Sq} \alpha_\star(H_i) + Hol_i \cdot \int_{\partial Sq} \alpha_\star(B_i)$$

(2.3.18)

**Proof.** Assume that we have a 1-parameter family of squares, $\Sigma_r \in U_i^{Sq}$ for $r \in [0, \epsilon)$, with associated holonomies $Hol(\Sigma_r) := Hol_i(\Sigma_r)$. We compute $\frac{\partial}{\partial r}\big|_{r=0} Hol(\Sigma_r)$. Following the above proposition and proof for the 1-dimensional case outlined in the beginning of this section, we first rewrite $Hol(r)$ by an $n \times m$ grid of smaller cubes.

![Diagram](image-url)
The cube at each $Sq_{i,j}$ will be labeled $c_{i,j}$, and the entire cube given by $Sq$ will be labeled, $c$. These cubes glue together and respect composition meaning we can glue them together (using the path-action where all paths run along the $r = 0$ plane) to obtain

$$\prod_{i,j=1}^{n,m} c_{i,j} = c.$$ 

Note that $t(c_{i,j}) = 1$, and so $c_{i,j}$ can commute with any other term via Proposition 2.1.21. Next we attach sides to $c$ so that the resulting product will be $Hol(\Sigma_r)$. We label these sides, $Hol^N$, $Hol^S$, $Hol^E$, $Hol^W$ where each $Hol^\bullet$ is defined analogously to its $Hol^\bullet_\alpha$, but now using the whole square. The following visualization summarizes these conventions, where the orientations on faces allude to how terms will cancel, and the center cube, $c$, is thought to be divided up into cubes, $c_{i,j}$, which glue to the whole cube:

$$\alpha_{hol^{-1}}_{(0,0,\rightarrow)}(Hol(\Sigma_r)) = Hol(\Sigma_0) \cdot c \cdot Hol^N \cdot Hol^E \cdot Hol^S \cdot Hol^W.$$
Since we can glue any number of $c_{i,j}$ together to obtain the cube, $c$, we can write

$$\left. \frac{\partial}{\partial r} \right|_{r=0} (\text{Hol}(\Sigma_r)) \quad (2.3.21)$$
\[
\frac{\partial}{\partial r}
\bigg|_{r=0}
(\alpha_{\text{hol}_{(0,0,-1}}(\text{Hol}(\Sigma_0) \cdot c \cdot \text{Hol}^N \cdot \text{Hol}^E \cdot \text{Hol}^S \cdot \text{Hol}^W))
\]

(2.3.22)

\[
= - (\alpha_{\text{Hol}(\Sigma_0)})_*(A_i(0,0,0) \left( \frac{\partial}{\partial r} \right)) + (\alpha_{1_G})_* \left( \text{Hol} \cdot \frac{\partial}{\partial r} c \right)
\]

(2.3.23)

\[
+ (\alpha_{1_G})_* \left( \text{Hol} \cdot \frac{\partial}{\partial r} \text{Hol}^N \right)
\]

(2.3.24)

\[
+ (\alpha_{1_G})_* \left( \text{Hol} \cdot \frac{\partial}{\partial r} \text{Hol}^E \right)
\]

(2.3.25)

\[
+ (\alpha_{1_G})_* \left( \text{Hol} \cdot \frac{\partial}{\partial r} \text{Hol}^S \right)
\]

(2.3.26)

\[
+ (\alpha_{1_G})_* \left( \text{Hol} \cdot \frac{\partial}{\partial r} \text{Hol}^W \right)
\]

(2.3.27)

\[
= - (\alpha_{\text{Hol}(\Sigma_0)})_*(A_i(0,0,0) \left( \frac{\partial}{\partial r} \right)) + \left( \text{Hol} \cdot \frac{\partial}{\partial r} c \right)
\]

(2.3.28)

\[
\text{We need only to prove that}
\]

\[
\left( \frac{\partial}{\partial r} \bigg|_{r=0} c \right) = \int_{\partial S_q} \alpha_*(H_i)
\]

and the result is proven. Note that \( c = \prod_{i,j=1}^{n,m} c_{ij} = \prod_{i,j=1}^{n,m} c_{(t,s)}(1/n,1/m) \) for all \( n,m \), where \( c_{(t,s)}(u,v) \) is the cube sitting over the square whose upper-left
corner is \((t, s)\), and has \(t\)-width equal to \(u\), and \(s\)-width equal to \(v\). The idea here is to use the fact that we can write the limit of the derivative of the product as a Riemann Sum:

\[
\frac{\partial}{\partial r}\c_j = \lim_{m \to \infty} \lim_{n \to \infty} \frac{\partial}{\partial r}\left(\prod_{i,j} c_{i,j}\right) = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{j=1}^m \sum_{i=1}^n \frac{\partial}{\partial r} c_{i,j} = \lim_{m \to \infty} \sum_{j=1}^m \lim_{n \to \infty} \sum_{i=1}^n \frac{\partial}{\partial r} c_{i,j} = \lim_{m \to \infty} \sum_{j=1}^m \int_{t=0}^{t=1} \frac{\partial}{\partial u} c_{t,s}(u, 1/m) \, dt.
\]

To prove the last equality, we need to show that the limit \(n \to \infty\) converges to the stated integral, i.e., that given \(\epsilon > 0\), there exists \(N_0 \in \mathbb{N}\) so that, for all \(N \geq N_0\), fixing the \(s\)-coordinate and \(s\)-width of the squares we are evaluating on, and thus suppressing the \(s\)-variable,

\[
\left| \sum_{i=1}^N \frac{\partial}{\partial r} c_{t_s}(1/N) - \sum_{i=1}^N \frac{\partial}{\partial u} c_{t_s}(u) \cdot \frac{1}{N} \right| < \epsilon
\]

This follows however from the following inequality, using \(c_{t_s}(0) = 1\) and so \(c_{t_s}(0) = 1\) is constant with respect to \(r\):

\[
\sum_{i=1}^N \left| \frac{\partial}{\partial r} c_{t_s}(1/N) - \frac{\partial}{\partial r} c_{t_s}(0) \right| \frac{1}{N} - \frac{\partial}{\partial u} c_{t_s}(u) \cdot \frac{1}{N} \right| < \epsilon
\]

which is equivalent to:

\[
\frac{1}{N} \sum_{i=1}^N \left| \frac{\partial}{\partial r} c_{t_s}(1/N) - \frac{\partial}{\partial r} c_{t_s}(0) \right| - \frac{\partial}{\partial u} c_{t_s}(u) \cdot \frac{1}{N} \right| < \epsilon.
\]

The last inequality follows from the fact that \(\frac{\partial}{\partial r} c_{t,s}\) is smooth and so, as
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a consequence of Taylor’s theorem, the difference quotient converges to its
derivative with respect to \( u \), uniformly, on any compact subspace. Applying
the same argument in the \( s \)-direction, we can finish the proof of the lemma
since we have

\[
\left. \frac{\partial}{\partial r} \right|_{r=0} (c) = \lim_{m \to \infty} \sum_{j=1}^{m} \int_{t=0}^{t=1} \left. \frac{\partial}{\partial u} \right|_{u=0} \left( \frac{\partial}{\partial r} \right|_{r=0} c_{t,s_j}(u, 1/m) \right) dt.
\]

\[
= \int_{s=0}^{s=1} \int_{t=0}^{t=1} \left. \frac{\partial}{\partial v} \right|_{v=0} \left. \frac{\partial}{\partial u} \right|_{u=0} \left( \frac{\partial}{\partial r} \right|_{r=0} c_{t,s}(u, v) \right) dtds
\]

\[
= \int_{s=0}^{s=1} \int_{t=0}^{t=1} \mathcal{H}_i.
\]

\( \square \)

2.3.2 The Edge and Vertex Relations

Here we prove two lemmas, which give us relations amongst the edges and
vertices for when we later compute \( \frac{\partial}{\partial r}(\text{Hol}) \). Recall our expression for 2-
holonomy, Definition 2.2.1:

\[
\text{Hol}^N = \text{Hol}_a \cdot \overline{\text{Hol}_{ac}} \cdot \overline{\text{Hol}_c} \cdot \ldots \cdot \overline{\text{Hol}_m} \cdot \overline{\text{Hol}_{ab}} \cdot \overline{\text{Hol}_{aebf}} \cdot \overline{\text{Hol}_{ef}} \cdot \ldots \cdot \overline{\text{Hol}_{mn}} \cdot \overline{\text{Hol}_d} \cdot \overline{\text{Hol}_{dh}} \cdot \overline{\text{Hol}_h} \cdot \ldots \cdot \overline{\text{Hol}_p}
\]
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Since we compute \( d(Hol)(\frac{\partial}{\partial r}) \) by considering a one-parameter family of squares, \( \Sigma(r) \), each term which we will be differentiating is of the form

\[
\alpha_{\text{path}(r)}(X(r)) : \mathbb{R} \to G \times H \to H
\]

whether the path does or does not depend on \( r \) (represented by \( \hat{X} \) and \( X \), respectively) and so we apply Lemma 2.1.22. To be more specific, if we write both path and \( X \) as functions of \( r \), we have

\[
\frac{\partial}{\partial r}_{r=0} (\alpha_{\text{path}}(X(r))) = (\alpha_{\text{path}(0)})_* \left( \frac{\partial}{\partial r}_{r=0} (X(r)) \right) + (\alpha_{\text{X}(0)})_* \left( \frac{\partial}{\partial r}_{r=0} (\text{path}(r)) \right) \tag{2.3.34}
\]

where we will sometimes write \( (\alpha_{\text{path}(0)})_* \left( \frac{\partial}{\partial r}_{r=0} (X(r)) \right) =: (\frac{\partial}{\partial r}_{r=0} (X(r))) \) and terms coming from \( (\alpha_{\text{X}(0)})_* \left( \frac{\partial}{\partial r}_{r=0} (\text{path}(r)) \right) \) will be referred to as path terms, which we will sometimes write

\[
(\alpha_{\text{X}(0)})_* \left( \frac{\partial}{\partial r}_{r=0} (\text{path}(r)) \right) =: (\alpha \frac{\partial}{\partial r}_{r=0} (\text{path}(r))) (X(0)). \tag{2.3.35}
\]

The following two technical lemmas, 2.3.5 and 2.3.6, provide relations amongst the terms of \( d(Hol) \) appearing at vertices and edges that are needed in the proof of Theorem 2.3.1.

To state the first lemma, we use the following setup: Assume for the moment that the open set \( \mathcal{N} \) only requires four open sets \( U_i, U_j, U_k, \) and \( U_l \), written in counterclockwise order from the upper left of \( \Sigma \).
Recall from Proposition \textbf{2.1.49} that we have a vertex cube which corresponds to the equation:

\[
\text{Hol}_{ijkl}(x) = \text{Hol}_{ij} \cdot \alpha_{g_{ij}^{-1}(x)g_{ii}(y)g_{kl}^{-1}(x)g_{kl}(y)}(\text{Hol}_{ijkl}(y))
\]

Thus we can replace \text{Hol}_{ijkl} in \text{Hol} to obtain the following equality of local transport data:

\[
\text{(2.3.36)}
\]

Here, \text{Mol} is an uninspired notation standing in for its counterpart \text{Hol} in the \(r\)-direction; see Proposition \textbf{2.1.49} for the full notation. The above diagram can be glued together in the following way:

\[
\text{Hol} = \text{Hol}_{i} \cdot \ldots \cdot \text{Hol}_{ik}^{-1} \cdot (\text{Hol}_{ijkl}) \cdot \text{Hol}_{ji}^{-1} \cdot \ldots \cdot \text{Hol}_{i}
\]

\[
\text{(2.3.37)}
\]

\[
= \text{Hol}_{i} \cdot \ldots \cdot \text{Hol}_{ik}^{-1} \cdot (\text{Mol}_{ij} \text{Mol}_{ik}^{-1} \text{Mol}_{ijkl}^{-1} \text{Mol}_{kl}^{-1}) \cdot \text{Hol}_{ji}^{-1} \cdot \ldots \cdot \text{Hol}_{i}
\]

\[
\text{(2.3.38)}
\]
where the $\hat{X}$ refers to an adjustment in the path-action of $X$, which uses the $r$ direction; i.e. $\hat{X}$ suggests that we had to move “up or down” in the $r$-direction due to a $Mol$ term being placed out of order in the glueing process.

For example, if the path in $\alpha_{path}(Hol_{ik}^{-1})$ was given by the path below (with the target of $Hol_{ik}^{-1}$ added on):

\[
\begin{array}{c}
\text{Hol}_i \\
\text{Hol}_{ij} \\
\text{Hol}_j \\
\end{array}
\quad
\begin{array}{c}
\text{Hol}_{ik}^{-1} \\
\text{Mol}_{ik}^{-1} \\
\text{Hol}_{jkl}^{-1} \\
\end{array}
\quad
\begin{array}{c}
\text{Hol}_k \\
\text{Mol}_{ijkl}^{-1} \\
\text{Hol}_{kl}^{-1} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Hol}_{ij} \\
\text{Mol}_{ijkl} \\
\text{Hol}_{jkt}^{-1} \\
\end{array}
\quad
\begin{array}{c}
\text{Hol}_{ijkl} \\
\text{Mol}_{ijkl} \\
\text{Hol}_{kl}^{-1} \\
\end{array}
\quad
\begin{array}{c}
\text{Hol}_l \\
\text{Mol}_{ijkl} \\
\text{Hol}_{lt}^{-1} \\
\end{array}
\]
we would write $\overrightarrow{\text{Hol}^{-1}_{ik}}$ since there is no $r$-contribution in the path holonomy. But if the path in $\alpha_{\text{path}}(\text{Hol}^{-1}_{ik})$ was given by the path below (with the target of $\text{Hol}^{-1}_{ik}$ added on),

then we would write $\overleftarrow{\text{Hol}^{-1}_{ik}}$ since there is an $r$-contribution to the path. Note that throughout a computation, the $\bullet$ and $\circ$ notation is implicit and might change when terms are rearranged following the crossed module relations.

The rewrite in (2.3.39) demonstrates how the global holonomy is unchanged when we replace the vertex term with the remaining 5 faces of the
vertex cube from Proposition 2.1.49. Now, we can rearrange the terms in (2.3.39), using only the crossed module relation (2.1.8) as follows:

\[
Hol = Hol_i Hol_{ij} Mol_{ij} Hol_j^{-1} Mol_{ik}^{-1} Mol_{ijkl} Hol_{kl} Mol_{l} Hol =: Hol^{ijkl}.
\]

Here, some faces of the vertex cube were moved to a more convenient location, according to the following order: 

\footnote{Convenient in the sense that comparing non-commutative terms will be easier later on.}
Differentiating $\text{Hol}^{ijkl}$ yields the following lemma:

**Lemma 2.3.5** (Vertex Cube Equation, (VCE)). *For a one-parameter family of squares, $\Sigma_r$, at each vertex $x_{ijkl}$, we have the vertex cube equation, denoted by (VCE), where we isolate the terms in the arbitrarily-long product*

$$
\ldots \cdot \text{Hol}_{ik}^{-1} \cdot d(\text{Hol}_{ijkl}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{jl}^{-1} \cdot \ldots
$$

(2.3.41)
\[
= - \ldots \Hol_{ij} \cdot \alpha_{g_{ij}(x_{ijkl})^{-1}} \left(a_{ij}\big|_{x_{ijkl}} \frac{\partial}{\partial r}\right) \cdot \Hol_j \ldots \tag{2.3.42}
\]
\[
- \sum_{\bullet_2} \ldots \left(\alpha_{\Hol_j \cdot (-t(a_{ij})(\frac{\partial}{\partial r}))g_{ij} \cdot \ldots} \right) \left(\Hol_{\bullet_2}\right) \ldots \tag{2.3.43}
\]
\[
- \ldots \Hol_j \cdot \ldots \left(\alpha_{\Hol_j \cdot (-t(a_{ij})(\frac{\partial}{\partial r}))g_{ij} \cdot \ldots} \right) \left(\Hol_{ik}^{-1}\right) \cdot \Hol_{ijkl} \ldots \tag{2.3.44}
\]
\[
+ \ldots \Hol_{ik}^{-1} \cdot \alpha_{g_{ik}^{-1}(x_{ijkl})} \left(a_{ik}\big|_{x_{ijkl}} \frac{\partial}{\partial r}\right) \cdot \Hol_{ijkl} \ldots \tag{2.3.45}
\]
\[
- \ldots \Hol_{ik}^{-1} \cdot \alpha_{\Hol_j \cdot (A_j(\frac{\partial}{\partial r}) + dg_{ij}(\frac{\partial}{\partial r}g_{ij}^{-1}) \cdot g_{ij} \cdot \ldots} \left(\Hol_{ikl} \right) \cdot \Hol_{jk}^{-1} \ldots \tag{2.3.46}
\]
\[
- \ldots \Hol_{ijkl}^{-1} \cdot \alpha_{g_{jl}^{-1}(x_{ijkl})} \left(a_{jl}\big|_{x_{ijkl}} \frac{\partial}{\partial r}\right) \cdot \Hol_{ijkl} \ldots \tag{2.3.47}
\]
\[
- \sum_{\bullet_3} \ldots \left(\alpha_{\Hol_i \cdot (-t(a_{ik})(\frac{\partial}{\partial r}))g_{ik} \cdot \ldots} \right) \left(\Hol_{\bullet_3}\right) \ldots \tag{2.3.48}
\]
\[
- \ldots \Hol_{jl}^{-1} \cdot \left(\alpha_{\Hol_i \cdot (-t(a_{ik})(\frac{\partial}{\partial r}))g_{ik} \cdot \ldots} \right) \left(\Hol_{ik} \right) \cdot \Hol_{kl} \ldots \tag{2.3.49}
\]
\[
+ \ldots \Hol_{k} \cdot \alpha_{g_{kl}^{-1}(x_{ijkl})} \left(a_{kl}\big|_{x_{ijkl}} \frac{\partial}{\partial r}\right) \cdot \Hol_{kl} \ldots \tag{2.3.50}
\]

where recall from Definition 2.1.34 we have
\[
-t(a_{ij}) = A_j + dg_{ij}g_{ij}^{-1} - g_{ij}A_i g_{ij}^{-1}
\]

and where \(\Hol_{\bullet_2}\) is any square in the grid appearing above \(\Hol_{ik}^{-1}\), and \(\Hol_{\bullet_3}\)
is any square in the grid appearing above \(\Hol_k\).

Proof. Differentiating \(\Hol^{ijkl}\) and applying the Leibniz rule yields,
\[
0 = \frac{\partial}{\partial r} \bigg|_{r=0} (\Hol) = \frac{\partial}{\partial r} \bigg|_{r=0} (\Hol^{ijkl}) \tag{2.3.51}
\]
\[
= \Hol_i \cdot \Hol_{ij} \cdot \frac{\partial}{\partial r} \bigg|_{r=0} \Hol_{ij} \cdot \ldots \cdot \Hol_l \tag{2.3.52}
\]
\[
+ \Hol_i \cdot \Hol_{ij} \cdot \Hol_{ij} \cdot \Hol_{ij} \cdot \frac{\partial}{\partial r} \bigg|_{r=0} \Hol_{ik}^{-1} \cdot \Hol_{ik}^{-1} \cdot \ldots \cdot \Hol_l \tag{2.3.53}
\]
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\[ \vdots + \text{Hol}_i \cdot \text{Hol}_{ij} \ldots \cdot \text{Hol}_{jl}^{-1} \cdot \text{Hol}_k \cdot \frac{\partial}{\partial r} \bigg|_{r=0} \text{Mol}_{kl}^{-1} \cdot \text{Hol}_{kl} \cdot \text{Hol}_i \] (2.3.55)

Next, we can evaluate each term using (2.3.34). We show some examples of how these calculations are performed. For example, for the term \( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}) \) above, the straight overline is indicative of the fact that the path which acts on \( \text{Mol}_{ij} \) is constant with respect to \( r \). Hence by (2.3.34), we have

\[
\frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}) = (\alpha_{\text{path}})_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}(r)) \right) \]
(2.3.56)

\[
+ (\alpha_{\text{Mol}_{ij}(0)})_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} \text{(path)} \right) \]
(2.3.57)

\[
= (\alpha_{\text{path}})_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}(r)) \right) \]
(2.3.58)

\[
+ (\alpha_1)_* (0) \]
(2.3.59)

\[
= (\alpha_{\text{path}})_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}(r)) \right) \]
(2.3.60)

\[
= \left( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Mol}_{ij}(r)) \right) \alpha_{g_{ij}^{-1}(x_{ijkl})} \left( a_{ij} \bigg|_{x_{ijkl}} \left( \frac{\partial}{\partial r} \right) \right) \]
(2.3.61)

Next, we move our focus to the term \( \frac{\partial}{\partial r} \bigg|_{r=0} (\text{Hol}_{ik}^{-1}) \). We see that the hat indicates the path depends on \( r \) but notice that \( \text{Hol}_{ik}^{-1} \) does not so we obtain the path term

\[
\frac{\partial}{\partial r} \bigg|_{r=0} (\text{Hol}_{ik}^{-1}) = (\alpha_{\text{Hol}_{ik}^{-1}(0)})_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} \text{(path}(r)) \right) \]
\[ = (\alpha_{\text{hol}} j (A_j (\frac{\partial}{\partial r}) + d g_{ij} (\frac{\partial}{\partial r}) g_{ij}^{-1} - g_{ij} A_j (\frac{\partial}{\partial r}) g_{ij}^{-1}) g_{ij}) (\text{Hol}_{ik}^{-1}). \]

Similar calculations can be performed for each term of (2.3.40) that depends on \( r \); i.e. either has a \( \hat{\text{Mol}} \) or is a \( \text{Mol} \). For example, consider

\[
\begin{align*}
\left. \frac{\partial}{\partial r} \right|_{r=0} (\text{Mol}_{ijkl}) &= (\alpha_{\text{path}(0)})_\ast \left( \left. \frac{\partial}{\partial r} \right|_{r=0} (\text{Mol}_{ijkl}(r)) \right) \\
&\quad + (\alpha_{\text{Mol}_{ijkl}(0)})_\ast \left( \left. \frac{\partial}{\partial r} \right|_{r=0} (\text{path}(r)) \right) \\
&= (\alpha_{\text{path}(0)})_\ast \left( \left. \frac{\partial}{\partial r} \right|_{r=0} (\text{Mol}_{ijkl}(r)) \right) \\
&\quad + (\alpha_{\text{Mol}_{ijkl}(0)})_\ast \left( \left. \frac{\partial}{\partial r} \right|_{r=0} (\ldots \text{hol}_j \cdot (\text{hol}_j(r)^{-1} \cdot g_{ij}(r)) \right) \\
&= \text{d}(\text{Hol}_{ijkl}) \left( \left. \frac{\partial}{\partial r} \right|_{r=0} \right) \\
&\quad + \alpha_{\text{hol}_j(A_j (\frac{\partial}{\partial r}) + d g_{ij} (\frac{\partial}{\partial r}) g_{ij}^{-1}) g_{ij}} (\text{Hol}_{ijkl})
\end{align*}
\]

Applying the above calculations to each line in the more general (i.e. with more than 4 open sets) equation

\[
\text{Hol}^{(ijkl)} := \ldots \cdot \text{Hol}_i \cdot \text{Hol}_j \cdot \text{Mol}_{ij} \cdot \text{Hol}_j \cdot \ldots
\]

\[
\ldots \cdot \text{Hol}_{ik}^{-1} \cdot \text{Mol}_{ik}^{-1} \cdot \text{Mol}_{ijkl} \cdot \text{Hol}_{jl}^{-1} \cdot \ldots
\]

\[
\ldots \cdot \text{Hol}_k \cdot \text{Mol}_{kl}^{-1} \cdot \text{Hol}_{kl} \cdot \text{Hol}_l \cdot \ldots
\]

given by the numbered-diagram above in a general \( n \) by \( m \) subdivision of the square yields the result.

Next, we consider vertical and horizontal edges using the edge cubes from
CHAPTER 2. THE DERIVATIVE OF TWO-HOLONOMY

Proposition [2.1.50] The Lemma below yielding the horizontal and vertical edge cube equations follows from a proof analogous to that of Lemma [2.3.5] where we use the following expressions and corresponding diagrams for horizontal edges:

\[ Hol = \text{Hol}^{(ij)}_{\text{h-edge}} \quad (2.3.65) \]

\[ := \ldots \cdot \text{Hol}^{-1}_{pi} \cdot \text{Hol}_{pqij} \cdot \text{Hol}^{-1}_{qj} \cdot \ldots \quad (2.3.66) \]

\[ \ldots \cdot \text{Hol}_{i} \cdot \text{Mol}_{ij}(x) \cdot \text{Mol}_{i} \cdot \text{Mol}_{ij}(r) \cdot \text{Mol}_{j}^{-1} \cdot \text{Mol}_{ij}^{-1}(y) \cdot \text{Hol}_{j} \cdot \ldots \quad (2.3.67) \]

\[ \ldots \cdot \text{Hol}_{ik}^{-1} \cdot \text{Hol}_{ijkl} \cdot \text{Hol}_{ji}^{-1} \cdot \ldots \quad (2.3.68) \]
Similarly, we have the following expressions and diagram for vertical edges:

\[
Hol = Hol_{v-edge} := \ldots \overline{Hol_{pi}} \cdot Mol_i^{-1} \cdot \overline{Hol_i} \cdot \overline{Hol_{ik}} \cdot \ldots \tag{2.3.69}
\]

\[
\ldots \cdot \overline{Hol_{piqj}} \cdot \overline{Mol_{ij}}^{-1}(a) \cdot Mol_{ij}^{-1}(r) \cdot Mol_{ij}(b) \cdot \overline{Hol_{ijkl}} \cdot \ldots \tag{2.3.70}
\]
Differentiating these equations gives the following lemma:

**Lemma 2.3.6** (Edge Cube Equations, (ECEh) and (ECEv)). For a one-parameter family of squares, $\Sigma_r$, we have the following horizontal edge cube equation at $\gamma^h_{ij}$, which we denote by (ECEh), where we isolate the terms in
the arbitrarily-long product

\[
\cdots \text{Hol}_i \cdot d(\text{Hol}_{ij}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_j \cdots
\]  

(2.3.72)

\[
= - \cdots \text{Hol}_i \cdot \alpha_{g_{ij} (\gamma^h_{ij}(0))^{-1}} \left( a_{ij} \bigg|_{\gamma^h_{ij}(0)} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}_{ij} \cdots
\]  

(2.3.73)

\[
- \cdots \text{Hol}_i \cdot \int_{\gamma^h_{ij}} \alpha_*(B_i) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ij} \cdots
\]  

(2.3.74)

\[
- \cdots \text{Hol}_i \cdot \left( \alpha_{A_i \bigg|_{\gamma^h_{ij}(0)} \left( \frac{\partial}{\partial r} \right) + d_{g_{ij}} \bigg|_{\gamma^h_{ij}(0)} \left( \frac{\partial}{\partial r} \right) g_{ij}^{-1}} \right) (\text{Hol}_{ij}) \cdot \text{Hol}_j \cdots
\]  

(2.3.75)

\[
+ \cdots \text{Hol}_{ij} \cdot \int_{\gamma^h_{ij}} \alpha_*(B_j) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_j \cdots
\]  

(2.3.76)

\[
+ \cdots \text{Hol}_{ij} \cdot \alpha_{g_{ij} (\gamma_{ij}^k(1))^{-1}} \left( a_{ij} \bigg|_{\gamma^h_{ij}(1)} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}_j \cdots
\]  

(2.3.77)

Similarly, we have the following vertical edge cube equation at \( \gamma^v_{ij} \), which we denote by (ECEv),

\[
\cdots \text{Hol}_{piqj} \cdot d(\text{Hol}^{-1}_{ij}) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_{ikjl} \cdots
\]  

(2.3.78)

\[
= + \cdots \text{Hol}_{pj} \cdot \int_{\gamma^v_{ij}} \alpha_*(B_i) \left( \frac{\partial}{\partial r} \right) \cdot \text{Hol}_i \cdots
\]  

(2.3.79)

\[
- \sum_{\bullet_2} (\alpha_{d_{\text{hol}_{i}} \bigg|_{\gamma^v_{ij}} \left( \frac{\partial}{\partial r} \right) \text{hol}, g_{pi} \cdots}) (\text{Hol}_{\bullet \bullet}) \cdots
\]  

(2.3.80)

\[
- \cdots \text{Hol}_{ik} \cdots (\alpha_{d_{\text{hol}_{i}} \bigg|_{\gamma^v_{ij}} \left( \frac{\partial}{\partial r} \right) \text{hol}, g_{pi} \cdots}) (\text{Hol}_{piqj}) \cdot \text{Hol}^{-1}_{ij} \cdots
\]  

(2.3.81)

\[
+ \cdots \text{Hol}_{piqj} \cdot \alpha_{g_{ij} (\gamma^v_{ij}(0))^{-1}} \left( a_{ij} \bigg|_{\gamma^v_{ij}(0)} \left( \frac{\partial}{\partial r} \right) \right) \cdot \text{Hol}^{-1}_{ij} \cdots
\]  

(2.3.82)
− \ldots \mathcal{H}_{\mathcal{L}_{ij}} \cdot (\alpha \mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) + d\mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) \mathcal{H}_{\mathcal{L}_{ij}}^{-1}) \cdot \mathcal{H}_{\mathcal{L}_{kl}} \ldots \hspace{1cm} (2.3.33)

− \ldots \mathcal{H}_{\mathcal{L}_{ij}}^{-1} \cdot \alpha_{ij}^{\gamma} \left( \frac{\partial}{\partial r} \right) \cdot \mathcal{H}_{\mathcal{L}_{klj}} \ldots \hspace{1cm} (2.3.34)

− \sum_{i, j, k, l} \ldots (\alpha \mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) \mathcal{L}_{ij}^{(n)} \cdot \mathcal{H}_{\mathcal{L}_{qj}} \cdot \mathcal{H}_{\mathcal{L}_{lj}} \ldots \hspace{1cm} (2.3.35)

− \ldots \mathcal{H}_{\mathcal{L}_{ij}} \cdot (\alpha \mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) \mathcal{L}_{ij}^{(n)} \cdot \mathcal{H}_{\mathcal{L}_{qj}} \cdot \mathcal{H}_{\mathcal{L}_{lj}} \ldots \hspace{1cm} (2.3.36)

− \ldots \mathcal{H}_{\mathcal{L}_{ij}} \cdot \int_{\gamma_{ij}}^{\gamma_{ij}} \alpha_{ij} \left( \frac{\partial}{\partial r} \right) \cdot \mathcal{H}_{\mathcal{L}_{lj}} \ldots \hspace{1cm} (2.3.37)

where we write

\mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) := A \left( \frac{\partial}{\partial r} \right) + d\mathcal{L}_{ij}^{(n)} \left( \frac{\partial}{\partial r} \right) \mathcal{H}_{\mathcal{L}_{ij}}^{-1} \cdot \mathcal{H}_{\mathcal{L}_{ij}}^{-1}

and where \mathcal{H}_{\mathcal{L}_{ij}} is any square in the grid appearing above \mathcal{H}_{\mathcal{L}_{pqij}}, and \mathcal{H}_{\mathcal{L}_{ij}} is any square in the grid appearing above \mathcal{H}_{\mathcal{L}_{adj}},

2.3.3 Proof of Theorem [2.3.1]

In this section, we will show that \( d(Hol) \) does not have differential information at the interior edges or vertices, while also gathering the boundary terms into a convenient expression. It is important to note that \( d(Hol) \), applied to
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\[ \frac{\partial}{\partial r} \bigg|_{r=0} \text{can be written, using the Liebniz rule as,} \]

\[ \frac{\partial}{\partial r} \bigg|_{r=0} \text{Hol} := \frac{\partial}{\partial r} \bigg|_{r=0} \left( \ldots \cdot \text{Hol}_i \cdot \text{Hol}_{ij} \cdot \text{Hol}_j \cdot \text{Hol}_{ik}^{-1} \ldots \cdot \text{Hol}_l \cdot \ldots \right) \]

\[ = \ldots + \ldots \left. \frac{\partial}{\partial r} \right|_{r=0} \text{Hol}_i \cdot \text{Hol}_{ij} \cdot \text{Hol}_j \cdot \text{Hol}_{ik}^{-1} \ldots \cdot \text{Hol}_l \cdot \ldots \]

\[ + \ldots \left. \text{Hol}_i \cdot \frac{\partial}{\partial r} \right|_{r=0} \text{Hol}_{ij} \cdot \text{Hol}_j \cdot \text{Hol}_{ik}^{-1} \ldots \cdot \text{Hol}_l \cdot \ldots \]

\[ : \]

\[ + \ldots \left. \text{Hol}_i \cdot \text{Hol}_{ij} \cdot \text{Hol}_j \cdot \text{Hol}_{ik}^{-1} \ldots \right. \left. \frac{\partial}{\partial r} \right|_{r=0} \text{Hol}_l \cdot \ldots \]

\[ + \ldots \]

(2.3.89)

(2.3.90)

(2.3.91)

(2.3.92)

(2.3.93)

(2.3.94)

This expression has the following types of terms

1. **3-curvature terms**: namely the terms in \( \overline{d(\text{Hol}_i)} \) from Lemma 2.3.4 where we replace one \( B_i \) with an \( H_i \)

2. **Boundary terms**: these are terms which occur on the boundary of \( \Sigma \).

3. **Edge and Vertex terms**:

   (a) The remaining 2 parts/terms coming from \( \overline{d(\text{Hol}_i)} \): four edge terms, written as one integral around the four sides in Lemma 2.3.4 and a corner term where \( A_i \) is applied to the upper left corner of the square.
(b) $d(Hol_{ij})$ and $d(Hol_{ijkl})$: These are the de Rham differentials applied to any (vertical or horizontal) edges or vertices, see Lemmas \ref{lemma:hol_edge} and \ref{lemma:hol_edge_cube}.

(c) Path terms: These are the terms which appear from differentiating the $\bullet$ of any $Hol_{\bullet}$ term in $d(Hol)$ (see Lemma \ref{lemma:hol_vertex}.

Summary of Cancelation

In the next subsection, details and arguments will be provided. For the moment, we summarize all of the terms which end up canceling. We begin by using Lemmas \ref{lemma:hol_edge} and \ref{lemma:hol_edge_cube} to replace any $\frac{\partial}{\partial r} \big|_{r=0} Hol_{ij}$ or $\frac{\partial}{\partial r} \big|_{r=0} Hol_{ijkl}$ with the corresponding expression and then showing that the $Mol_{ij}$ (edge) terms cancel with each other. In other words, for each term $\overline{Hol}_{\bullet}$ or $\overline{Hol}_{\bullet}$, applying $\frac{\partial}{\partial r}$ to such terms and using Lemma \ref{lemma:hol_vertex} we obtain terms of the form

\[ \frac{\partial}{\partial r} \bigg|_{r=0} Hol_{\bullet}. \]

The following table gives a summary of all instances where two such terms will appear in $d(Hol)$ with opposite sign. Note that for the reader’s convenience we label the Local Lemma \ref{lemma:hol_edge} as “(LL)”, the Vertex Cube Equation from Lemma \ref{lemma:hol_vertex} as “(VCE)”, and the vertical and horizontal edge cube equations from Lemma \ref{lemma:hol_edge_cube} as “(ECEv)” and “(ECEh)”, respectively. While those
lemmas introduce such terms via relations, there were also the original path terms coming from differentiating the $\bar{\mathbf{\bullet}}$ in the expression for $\text{Hol}$ which we label as “(path-$d(\text{Hol})$)”.

The other terms that appear from applying $\frac{\partial}{\partial r}|_{r=0}$ to each term $\text{Hol}_{\mathbf{\bullet}}$ or $\widetilde{\text{Hol}}_{\mathbf{\bullet}}$ are the path terms from Lemma 2.1.22.

$$\alpha \left( \frac{\partial}{\partial r} \bigg|_{r=0} \right)^n (\text{Hol}_{\mathbf{\bullet}}).$$

The two tables below summarize all of the instances where path terms show up an even number of times, with opposite signs. In the first table, the focus is on path terms resulting from $d(\text{Hol})$; i.e. they show up one time in differentiating the path approaching each term in the expression (2.2.5) for $\text{Hol}^N$. The second table deals with path terms which cancel amongst the
other relations.

<table>
<thead>
<tr>
<th>Label</th>
<th>Term</th>
<th>Found in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \ldots \cdot \alpha_{\ldots -d\text{hol}(\ldots)}(\text{Hol}_z) \cdot \ldots \cdot )</td>
<td>(path-(d(Hol))) (ECEv)</td>
</tr>
<tr>
<td>(C1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(C2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(C3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(C4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \ldots \cdot \alpha_{\ldots -d\text{hol}(\ldots)}(\text{Hol}_{yz}) \cdot \ldots \cdot )</td>
<td>(path-(d(Hol))) (ECEv)</td>
</tr>
<tr>
<td>(D1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(D4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that in (C3) we can have \( wx = ix \) and in (D4) we can have \( ij = wx \).

In (D2) we can have \( ij = yz \) but then the cancellation is due to (path-\(d(Hol)\)) and (ECEh).
Note that in (E3) we can have $wx = iq$, in (E4) we can have $wxyz = wiqy$, and in (F1) we can have $z = i$.

The only terms that are left, for the interior of $\Sigma$, after all of this cancellation are the 3-curvature terms, $H_i$ integrated over the square. We now provide the details for some of this cancellation and then show how to deal with the boundary terms in more detail.


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Canceling $\text{Mol}_{ij}$ terms

As an example, we discuss (B3) from the interior side terms table on page 157. Consider a horizontal-intersection-edge, $(ij)$. Without loss of generality, we assume that none of its edges lie on the boundary of the grid and that we have the indices $p, q, i, and j$, arranged as in the figure of the vertex cube below:

Note that there are exactly two terms in which $\frac{\partial}{\partial r} \mid_{r=0} (\text{Mol}_{ij}(x))$ will appear: namely, when we differentiate $\text{Hol}^{(pqij)}$, and how we’ve already laid out in (VCE), Lemma 2.3.5, and when we differentiate $\text{Hol}^{(ij)}_{h\text{-edge}}$ (refer to Figure 2.2), as we have laid out in (ECEh), Lemma 2.3.6. In either case, the term (B3) will appear by differentiating,

$$\frac{\partial}{\partial r} \mid_{r=0} \text{Mol}_{ij}(x) = \alpha_{g_{ij}^{-1}(\gamma_{ij}(0))} \left( a_{ij} \mid_{\gamma_{ij}(0)} \left( \frac{\partial}{\partial r} \right) \right).$$
Thus \((B3)\) is the same as \([2.3.50]\) in (VCE) and \([2.3.73]\) in (ECEh). In \(Hol^{(pqij)}\) you can see that the term will appear with a negative sign and in the derivative of \(Hol^{(ij)}_{\text{hor}}\) it will appear with a positive sign.

**Canceling \(\int \alpha_{\ast}(B)\) terms**

Next, we show that the *edge terms* coming from \(d(Hol_i)\) (see Lemma 2.3.4 (LL)) also vanish using \((A3)\) from the table on page 157 as an example.

Notice that if \(Hol_j\) is the face to the right of \(Hol_i\), then between the two we have a vertical edge, \(Hol_{ij}^{-1}\) (*see vertical edge cube*, Figure 2.3). In this setup, we see that we have a \(Mol_i\) term in exactly the right place. In particular, from \(d(Hol)\) we will obtain the term

\[
\ldots \cdot Hol_{pi} \cdot \left( \int_{\gamma_{ij}} \alpha_{\ast}(B_i(t, \sigma))d\sigma \right) \cdot Hol_i \cdot \ldots
\]

which is \([2.3.79]\) in (ECEv). Here, in the (ECEv), the \(\circ\) for this \(B_i\)-term refers to the path along the northern edge of \(Hol_i\). By the crossed module relations, we can write this alternatively as

\[
\ldots \cdot Hol_{pi} \cdot \left( \int_{\gamma_{ij}^c} \alpha_{\ast}(B_i(t, \sigma))d\sigma \right) \cdot Hol_i \cdot \ldots
\]

where now the \(\circ\) for the \(B_i\)-term refers to the path along the western, southern, then eastern edges of \(Hol_i\). On the other hand, in the *Local Lemma*...
(LL), Lemma 2.3.4, we have the term (2.3.30),

\[ \ldots \cdot \text{Hol}_i \cdot \left( \int_{\gamma_i^v} \alpha_*(B_i(t, \sigma)) d\sigma \right) \cdot \ldots \]

where the \( \overline{\bullet} \) for the \( B_i \) term, as explained in the proof, refers to the path along the western, southern, then eastern edges of \( \text{Hol}_i \). Note that these terms agree with opposite signs.

The other three edges follow the same ideas. The fact is that we have already arranged all of the terms in our \( \text{Hol}^{(ij)} \) so that these terms would match up seamlessly, and the signs would be opposite. Again, the table on page 157 summarizes all such pairs that cancel.

**Interior path terms**

Next, we come to the *path terms*. In particular, we end up considering expressions of the sort: \( (\alpha_h)_*(p_1 \cdot A \cdot p_2) \) where \( h \in H \) and \( p_1, p_2 \in G \) are the parts of the path coming before and after we see an \( A \in g \). Then we note

\[
(\alpha_h)_*(p_1 \cdot A \cdot p_2) = (\alpha_h)_*((L_{p_1})_* \circ (R_{p_2})(A)) \tag{2.3.95}
\]

\[
= (\alpha_{p_1})_* (\alpha_{\alpha_{p_2}(h)})_* (A) \tag{2.3.96}
\]

Suppose you consider any path term involved with the path approaching any \( \text{Hol}_\bullet \). Based on our convention, we will only be concerned with the north-south portions of the paths (i.e. some vertical edge \( \gamma^u_{\nu,\nu} \)) approaching
Hol\(\bullet\) (where the Hol\(\bullet\) could refer to a Hol\(_z\), Hol\(_{yz}\), Hol\(_{wz}\), or Hol\(_{wxyz}\)). The options for the path terms then are always of the form, following (2.3.95),

\[
(\alpha_{p_1})_*(\alpha_{\alpha p_2(Hol)_\bullet})_*(\frac{\partial}{\partial r} \bigg|_{r=0} g_{kl}(r)) \quad \text{or} \quad (\alpha_{p_1})_*(\alpha_{\alpha p_2(Hol)_\bullet})_*(\frac{\partial}{\partial r} \bigg|_{r=0} hol_k(r))
\]

which we have referred to as

\[
\alpha_{(...d_{g_{kl}}...)}(Hol\bullet) \quad \text{and} \quad \alpha_{(...d_{hol_k}...)}(Hol\bullet),
\]

respectively, in the Interior path terms table on page 158.

Notice that any edge-path-term (i.e. a \(g\) or a \(hol\)) is adjacent to either a vertex or a vertical edge; see (2.2.1). Furthermore, we recall that faces don’t have cube equations and horizontal edges, Figure 2.2, never place Mol terms out of order like the vertex, Figure 2.1, and vertical edge expressions do, Figure 2.3. The point is that the only places where these edge-path-terms can arise in the cube equations are vertex or vertical edge expressions. If you want to account for a \((\alpha_{p_1})_*(\alpha_{\alpha p_2(Hol)_\bullet})_*(\frac{\partial}{\partial r} \bigg|_{r=0} g_{kl}(r))\) term, namely any of the terms (D1) - (D4), you simply find the vertex corresponding to it:
In particular, this term would be the middle summand of the path-action, “$dg_{kl}$”, of (2.3.48) in the *vertex cube equation*, (VCE).

Similarly, to account for a term $(\alpha_{p_1})_*(\alpha_{p_2}(Hol_*))_* \left( \frac{\partial}{\partial r} \bigg|_{r=0} hol_k(r) \right)$, such as any of the (C1) - (C4) terms, you simply find the edge corresponding to it:
In particular, this term would be the “$d\text{hol}_j$” summand in (2.3.85) from (ECEv).

These terms will always appear twice, as described in the *Interior path terms* table on page 158, with opposite sign, and so they cancel.

**Interior corner terms**

Next, we consider the terms from the *Interior corner terms* table on page 159. The argument for (E1)-(F4) is very similar to the one we had for (C1)-(D4). We once again use a path to a $Hol_\bullet$ where $Hol_\bullet$ could represent some $Hol_z$, $Hol_{yz}$, $Hol_{wx}^{-1}$, or $Hol_{wxyz}$. In particular, consider the path terms coming
from

\[ \frac{\partial}{\partial r} \bigg|_{r=0} \mathring{\mathfrak{r}}(\bullet) \]

where “•” could be any Hol or Mol term from any Hol\(^{-}\) expression. One of the path terms will involve an \(A_k\) coming down (in the sense that the cube rises) to a particular vertex of the cube. As mentioned before, however, such path terms can only come from vertex cubes or vertical edges\(^6\). As we show in the picture below, you can see that we will have exactly one \(A_k\) term coming from a vertex cube and one \(A_k\) coming from a vertical edge cube; one will be coming from a \(hol_k\) going “up” the cube and one will be coming from \(hol_k\) going “down” the cube and so they will have opposite signs and cancel. For example, the situation for (F3), or (F4), is depicted in the following figure

\(^6\)Again, the reason being that no cubes occur at the faces Hol\(_i\) and the faces of the cube places at a horizontal cube stay together and so do not cause any interesting hat-terms.
The figure on the left describes the “$g_{ij}A_i g_{ij}^{-1}$” summand in (2.3.43) from (VCE), while the figure on the right describes the “$A_i$” summand in (2.3.80) from (ECEv).

Next, consider the term (G1); i.e. $(\alpha_{\text{Hol}_i})_*(A_i)$ from Lemma 2.3.4 in $d(\text{Hol}_i)$. In our edge and vertex relations, we will obtain an $A_i$ term whenever
we differentiate a $\text{hol}_i$ term going up/down an edge of a vertex cube. Consider the picture below, where we note that at the upper left corner of $\text{Hol}_i$, which is where the $A_i$ is being pushed forward by $\text{Hol}_i$, is adjacent to $\text{Hol}_{ji}$, $\text{Hol}_{pqij}$, $\text{Hol}_{qi}^{-1}$, and $\text{Hol}_i$.

$\begin{array}{ccc}
\text{Hol}_{pq} & \text{Hol}_{pqij} & \text{Hol}_{ji} \\
\text{Hol}_{q} & \text{Hol}_{qi}^{-1} & \text{Hol}_i \\
\end{array}$

In order to get a term from the edge and vertex relations, we would need a cube to be touching this corner. Such a cube never occurs at $\text{Hol}_i$; there are no “face-cubes”. But we have a $pqij$-vertex, a $qi$-vertical edge, and a $ji$-horizontal edge all intersecting at this corner, each of which can provide a cube-equation. It turns out that, as described in the vertical edge cube, Figure 2.3, the $\widehat{\text{Mol}}_i$ term comes after the $\widehat{\text{Hol}}_i$ term in the expression $\text{Hol}_{v\text{-edge}}^{(qi)}$ and so that hat above the $\text{Hol}_i$ is in fact telling us that we must differentiate the path approaching $\text{Hol}_i$, going around $\text{Mol}_i$. In going around $\text{Mol}_i$, the last term in the path before reaching $\text{Hol}_i$ is a $\text{hol}_i$, as shown here:
Differentiating this term with respect to $r$ and then setting $r = 0$ results in exactly an $A_i$ term coming before $\text{Hol}_i$. Since we are acting on $\text{Hol}_i$ by this $\text{hol}_i$ term, we end up pushing forward $A_i$, resulting in the desired term; i.e. it is the "$\text{hol}_j A_j \text{hol}_j^{-1}$" summand in \((2.3.87)\) from (ECEv). On the other hand, cubes induced by $\text{Hol}_{pqij}$ and $\text{Hol}_{ji}$ will not contribute a corner term for $\text{Hol}_i$ so that the above description includes the only two ways (G1) appears; and with opposite signs.

**Getting the Boundary Right**

Modulo the boundary, we have thus far shown that $d(\text{Hol}) = \text{Hol} \cdot \int_{S_q} H$. In order to get the boundary terms to work out properly, we need to check that all terms that accumulate at the boundary of $\Sigma$, due to not being able to cancel with a missing adjacent square, either cancel or are of the form $B_i$.
or \( a_{ij} \) as described in the statement of Theorem 2.3.1. In particular, consider again the general grid from Definition 2.2.1. Note that when we replace \( d(Hol_{ij}) \) with its corresponding cube equation at the northern boundary of the square, \( \Sigma \), we end up differentiating a \( Hol_{ij} \) as it collapses to the northern boundary, placing an \( a_{ij} \) at that spot. In such cases, we would like to be able to write any term

\[
Hol_a \cdot \ldots \cdot Hol_m \cdot a_{ab} \left( \frac{\partial}{\partial r} \right) \cdot Hol_{ab}^{-1} \cdot \ldots \cdot Hol_p
\]

(2.3.97)
as \( \overline{a_{ab}} \cdot Hol \) simply by changing the path-action, \( \overline{\cdot} \), for the \( a_{ij} \) term. The easier terms to deal with will be on the Northern and Eastern boundaries. By using the algebra of the crossed module, we can rewrite the equation \( (2.3.97) \) as desired. However, explaining this algebra is a lot easier by recalling \( hh' = \alpha_t(h)(h')h \) and observing the equality as a picture:
CHAPTER 2. THE DERIVATIVE OF TWO-HOLONOMY

In a similar fashion, based on the ordering of the Edge Cube equations (Lemmas 2.3.5 and 2.3.6) and the Local Lemma 2.3.4 all of the $B_i$ and $a_{ij}$ terms appearing along the Northern and Eastern boundary can be factored outside of $Hol$.

For the Western and Southern boundaries, there is one extra tool needed. Considering again an $a_{ij}$ term, let us consider the term

$$Hol_a \cdot Hol_{ae} \cdot Hol_e \cdot Hol_{ei} \cdot a_{ei} \left(\frac{\partial}{\partial r}\right) \cdot Hol_i \cdots Hol_p$$

(2.3.99)

which we would like to rewrite as $Hol \cdot \alpha$. The tool here is to realize there are leftover terms on the boundary which assemble precisely to $\left[\alpha, -\right]$. To see this, we now finally use one last type of term coming from $d(Hol)$ which he have yet to tap into: the derivative of the path-action terms along the Western and Southern boundaries coming from each $\mathbf{v}$ in the expression for $Hol$. Note that these terms did not appear for the Northern and Eastern boundaries since our convention is to use the path approaching a term going along the Western boundary, then along the Southern boundary, and then up towards that term through the interior. In other words, we can write (2.2.5) as

$$Hol_a \cdot Hol_{ae} \cdot Hol_e \cdot Hol_{ei} \cdots Hol_i \cdots Hol_p$$

(2.3.100)

$$=Hol_a \cdots Hol_{ei} \cdot \alpha_{hol^{-1}} hol_{ae}^{-1} \cdot hol_e^{-1} \cdot g_{e^{-1}}(Hol_i) \cdots \cdot Hol_p$$

(2.3.101)
which we will momentarily write as

\[- \text{Hol}_1 \cdot \text{Hol}_{ei} \cdot \alpha_{\text{hol}_{a^{-1} \text{hol}_{ac^{-1} \text{hol}_{c^{-1} \text{Hol}_i} \text{Hol}_j}} \cdot \text{Hol}_2. \]

(2.3.102)

Using all of the various terms occurring at this Southern \( ei \) boundary-corner, we can combine them in a useful way, where the reference to where the term comes from is listed in place of an equation label:

\[- \text{Hol}_1 \cdot \text{Hol}_{ei} \cdot \alpha_{(g_{ei}^{-1} (A_i + d_{ei} g_{ei}^{-1}))} \alpha_{g_{ei}} (\text{Hol}_{ei}) \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

(ECEh)

\[- \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} A_i} \alpha_{g_{ei}} (\text{Hol}_i) \cdot \text{Hol}_2 \]

(LL)

\[ \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} (a_{ei})} \text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

(ECEh)

\[ + \text{Hol}_1 \cdot \alpha_{(\ldots A_e)} (\text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2) \]

(d(Hol)-path)

\[ - \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} (a_{ei})} \text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

(d(Hol)-path)

\[ = - \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} (a_{ei})} \text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

\[ + \text{Hol}_1 \cdot \alpha_{(\ldots (A_e - d_{ei} g_{ei}^{-1} - g_{ei}^{-1} A_i, g_{ei}))} \alpha_{g_{ei}} (\text{Hol}_{ei}) \cdot \text{Hol}_j \cdot \text{Hol}_2 \]

\[ = - \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} (a_{ei})} \text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

\[ + \text{Hol}_1 \cdot \alpha_{(\ldots (A_e - d_{ei} g_{ei}^{-1} - g_{ei}^{-1} A_i, g_{ei}))} \alpha_{g_{ei}} (\text{Hol}_{ei}) \cdot \text{Hol}_j \cdot \text{Hol}_2 \]

\[ = - \text{Hol}_1 \cdot \alpha_{g_{ei}^{-1} (a_{ei})} \text{Hol}_{ei} \cdot \text{Hol}_i \cdot \text{Hol}_2 \]

\[ + \text{Hol}_1 \cdot \alpha_{(\ldots (a_{ei})} \alpha_{g_{ei}} (\text{Hol}_{ei}) \cdot \text{Hol}_j \cdot \text{Hol}_2 \]

\[ = \text{Hol} \cdot a_{ei} \]
A similar technique is applied to the $B_i$ integrated along the Western and Southern boundaries using the vanishing fake curvature condition $t(B_i) = R_i$, which we will now show below. Just as we did above, first write (2.2.5) as

$$Hol_1 \cdot \overline{Hol_n} \cdot \alpha_{(\cdots \cdot \alpha_{(R_n)}^{-1})}(Hol_2),$$

After differentiating, we obtain side terms along the path $\gamma^S_n$ which we can rewrite in the desirable fashion:

$$- Hol_1 \cdot \overline{Hol_n} \cdot \int_{\gamma^S_n} \alpha_*(B_n) \cdot Hol_2$$  \hspace{1cm} (2.3.103)

$$+ Hol_1 \cdot \overline{Hol_n} \cdot \alpha_{(\cdots \cdot \int_{\gamma^S_n} \alpha_*(R_n))}(Hol_2)$$  \hspace{1cm} (2.3.104)

$$= - Hol_1 \cdot \overline{Hol_n} \cdot \int_{\gamma^S_n} \alpha_*(B_n) \cdot Hol_2$$  \hspace{1cm} (2.3.105)

$$+ Hol_1 \cdot \overline{Hol_n} \cdot \alpha_{(\cdots \cdot \int_{\gamma^S_n} \alpha_*(t(B_n)))}(Hol_2)$$  \hspace{1cm} (2.3.106)

$$= - Hol_1 \cdot \overline{Hol_n} \cdot \int_{\gamma^S_n} \alpha_*(B_n) \cdot Hol_2$$  \hspace{1cm} (2.3.107)

$$+ Hol_1 \cdot \overline{Hol_n} \cdot \left( \int_{\gamma^S_n} \alpha_*(B_n), Hol_2 \right)$$  \hspace{1cm} (2.3.108)

$$= Hol \cdot \int_{\gamma^S_n} \alpha_*(B_n)$$  \hspace{1cm} (2.3.109)
Chapter 3

Extension of Two-Holonomy on Spheres

Throughout this chapter, fix a smooth manifold, $M$, and a nonabelian gerbe with connection as in Definition 2.1.34. This chapter deals with a non-abelian analog of the equivariant extension of 2-holonomy constructed in [TWZ]. In that paper, a square Hochschild complex was used to model the iterated integral corresponding to our $Hol^N$ in their abelian setting. By calculating $dHol^N$ using Hochschild methods, an equivariantly closed extension of $Hol^N$ was constructed as an invariant analogous to Bismut's equivariant extension of the Chern character for a vector bundle with connection.

3.1 The Derivative of Two-Holonomy on Spheres

While [TWZ] dealt with 2-holonomy of a torus, we restrict our attention in this chapter to $S^2$, having the advantage of 2-holonomy taking values in the
center of $H$, which will have considerably favorable consequences for dealing with an equivariant extension of 2-holonomy.

**From Squares to Spheres**

Previously, we have defined $\text{Hol}$ on squares in $M^{Sq}$. While the torus has two natural actions on it, the sphere will have one natural $S^1$-action on it coming from rotating the sphere, holding the north and south poles fixed. Note that this action can similarly be translated to any $S^1$ action on $S^2$ by rotation.

**Convention 3.1.1.** Consider the map $\psi : Sq \to S^2$ given by

$$(t, s) \mapsto (\rho \cos(2\pi t), \rho \sin(2\pi t), 2s - 1)$$

where $\rho := \sqrt{1 - (2s - 1)^2}$. Note that $t$ corresponds to the angle in the $xy$-plane. The map $\psi$ yields a map $M^{S^2} \to M^{Sq}$ which in turn allows us to use $\text{Hol}$ defined on $M^{Sq}$ as a map $M^{S^2} \xrightarrow{\text{Hol}} H$. Within this context, we mean
by the natural $t$-action:

$$(\rho \cos(2\pi t'), \rho \sin(2\pi t'), 2s - 1) \overset{t}{\mapsto} (\rho \cos(2\pi (t' + t)), \rho \sin(2\pi (t' + t)), 2s - 1)$$

In [TWZ] the natural $t$-action on the torus results in an associated vector field $\left(\frac{\partial}{\partial t}\right)$ which is used during integration. Since we are working in the context of diffeological spaces here, we instead wish to consider $\left(\frac{\partial}{\partial t}\right)$ as the derivative of an induced one-parameter family.

**Definition 3.1.2.** Given the $t$ action on $\Sigma \in M^{S^2}$ as described above, we define the $t$-family of spheres, $\Sigma_t$ associated to $\Sigma$, given by

$$\Sigma_t(t', s) := \Sigma(t' + t, s)$$

We now modify our definition of $N$ to take full-advantage of the setting we are in.

**Setting 3.1.3.** For the case of $M^{S^2}$, for a choice of grid $I = \{1, \ldots, n\} \times \{1, \ldots, m\}$, we require the open sets $\{U_{i(p,q)}\}_{(p,q) \in I}$ to satisfy

$$U_{i(p,0)} = U_{i(p',0)}, \quad U_{i(p,m)} = U_{i(p',m)}, \quad U_{i(1,q)} = U_{i(n,q)}.$$

In other words, $\text{Hol}^N$, in the case of $M^{S^2}$ can always be seen as the following
### Properties of $\text{Hol}^N$ for Spheres

We now briefly record some useful propositions for $\text{Hol}^N$ for the case when $\Sigma \in M^{S^2}$.

**Proposition 3.1.4.** The 2-holonomy of a sphere takes its values in the center
CHAPTER 3. EXTENSION OF TWO-HOLONOMY ON SPHERES

of the Lie group, \( H \).

**Proof.** The target of \( Hol^N \) is equal to the 1-holonomy along the boundary of the square; see Proposition 2.2.4. Given that we map the square into the sphere by pinching the north and south edges of the square and mapping the west and east edges to the same meridian in \( S^2 \), the 1-holonomy along the edge of the square is equal to the identity in \( H \) under \( Hol^N \), and thus \( t(Hol^N) = 1 \in H \). The claim follows from Proposition 2.1.21. \( \square \)

**Proposition 3.1.5.** The transformation of 2-holonomy between open sets in \( M^S^2 \) is given by

\[
Hol^{N_i'}(\Sigma) = \alpha_{g_{ii'}(0,0)}(Hol^{N_i})
\]

for \( \Sigma \in N_i \cap N_{i'} \subset M^S^2 \).

**Proof.** Apply Proposition 2.2.6 to the case where we map the square to the sphere using Convention 3.1.1 and note that the only terms on along the boundary cancel. Along the northern and southern boundaries, we have \( Hol_{ii} = Hol_{iii'} = 1 \), and along the western and eastern boundaries, the terms that appear on the western boundary have a matching inverse term on the eastern boundary. Since we can collect all of this edge transition data by traveling around the boundary the result follows. \( \square \)

In the case of \( M^S^2 \) we can simplify Theorem 2.3.1 as follows.
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Proposition 3.1.6. The total derivative of 2-holonomy, $d(Hol)$, can be written:

\[ d(Hol) = -(\alpha_{Hol})_\ast(A_{i(1,1)}) + Hol \cdot \int_{S^2} H \]  \hspace{1cm} (3.1.2)

where $Hol = Hol^N : N \to H \subset Mat$ is defined on the open set $N \subset M^{S^2}$.

Proof. Recall the general formula for $d(Hol)$ given by Theorem 2.3.1. Following the grid-diagram (3.1.1) for the sphere-case, we note that along the northern and southern boundaries, we will have data $a_{ii} = 0$, and $g_{ii} = 1$. Moreover, the terms $\int \alpha_\ast(B)$ will be integrated over an edge of length zero and so we see all of the terms from Theorem 2.3.1 occurring on the northern and southern boundaries vanish. For the western and eastern boundaries, we simply note that each term on the western boundary will have a matching term with opposite sign coming from the eastern boundary.  

Note that $\int_{S^2} H \in \Omega^1(N, \mathfrak{h})$ is a one-form on the open subset $N \subset M^{S^2}$; where $\mathfrak{h}$ is the Lie algebra of $H$, as usual. For two open subsets $N_i, N_{i'} \subset M^{S^2}$, we can write the transformation of $\int_{S^2} H$ in $N_i \cap N_{i'} \subset M^{S^2}$ as follows.

Proposition 3.1.7. The integral of the 3-curvature over a sphere transforms in the following way:

\[ \int_{S^2} H^{N_i} = \alpha_{g_{ii}(0,0)}(\int_{S^2} H^{N_{i'}}) \]
CHAPTER 3. EXTENSION OF TWO-HOLONOMY ON SPHERES

where by $H^N_\nu$ we mean to use the local 3-curvature as defined by the local data on $N_\nu$.

**Proof.** Consider two open sets $N_\nu, N_\nu' \subset M^{S^2}$. Without loss of generality, we assume that these open sets use a common subdivision for the square $Sq$.

For a grid $N_\nu'$ on $S^2$, the integral of the 3-curvature is defined

\[
\int_{S^2} H^{N_\nu'} = \int_{(t,s) \in S^2} \alpha(\text{hol}_a(\gamma_1^W)^{-1}g_a \gamma_1^W(1))^{-1}\text{hol}_b(\gamma_2^W)^{-1} \cdot \ldots \cdot \text{hol}_s(\gamma_s(t))^{-1}(H^{\nu'}_\nu |_{(t,s)}).
\]

By using the target conditions

\[
t(H_{ij}) = \text{hol}_i(\gamma)^{-1}g_{ij}(\gamma(1))^{-1}\text{hol}_j(\gamma)g_{ij}(\gamma(0))
\]

\[
t(H_{ijkl}) = g_{ik}^{-1}g_{kj}g_{ij}
\]

we can write

\[
\text{hol}_a(\gamma_1^W)^{-1}g_a \gamma_1^W(1))^{-1}\text{hol}_b(\gamma_2^W)^{-1} \cdot \ldots \cdot \text{hol}_s(\gamma_s(t))^{-1}
\]

\[
= g_a(\gamma_1^W(0))t(H_{aa}) \gamma_1^W)^{-1}\text{hol}_a(\gamma_2^W)^{-1}g_a(\gamma_1^W)^{-1}
\]

\[
\cdot g_a \gamma_1^W(1))^{-1}\text{hol}_b(\gamma_2^W)^{-1} \cdot \ldots \cdot \text{hol}_s(\gamma_s(t))^{-1}
\]

\[
= g_a(\gamma_1^W(0))t(H_{aa}) \gamma_1^W)^{-1}\text{hol}_a(\gamma_2^W)^{-1}g_a(\gamma_1^W)^{-1}
\]

\[
\cdot g_a \gamma_1^W(1))^{-1}\text{hol}_b(\gamma_2^W)^{-1} \cdot \ldots \cdot \text{hol}_s(\gamma_s(t))^{-1}
\]

\[
= g_a(\gamma_1^W(0))t(H_{aa}) \gamma_1^W)^{-1}\text{hol}_a(\gamma_2^W)^{-1}g_a(\gamma_1^W)^{-1}
\]

\[
\cdot g_a \gamma_1^W(1))^{-1}g_b(\gamma_2^W(0))t(H_{bb} \gamma_2^W(0))^{-1}
\]
meaning we can write the integrand above as
\[ \cdot \text{hol}_b(\gamma_{2,1}^W)^{-1}g_{bb'}(\gamma_{2,1}^W(1))^{-1} \cdot \ldots \cdot \text{hol}_{t'}(\gamma_s(t))^{-1} \]
\[ = g_{aa'}(\gamma_{1,1,1}(0))t(\text{Hol}_{aa'})((\gamma_{1,1,1}^W)^{-1} \text{hol}_a(\gamma_{1,1,1}^W)^{-1} g_{aa'}(\gamma_{1,1,1})^{-1} \cdot \ldots \cdot \text{hol}_{t'}(\gamma_s(t))^{-1} \]
\[ \cdot g_{a'a'}(\gamma_{1,1,1}^W(1))^{-1}g_{bb'}(\gamma_{1,1,1}^W(1))g_{bb'}(\gamma_{1,1,1}^W(1))^{-1} \cdot \ldots \cdot \text{hol}_{t'}(\gamma_s(t))^{-1} \]
\[ = g_{aa'}(\gamma_{1,1,1}(0))t(\text{Hol}_{aa'})((\gamma_{1,1,1}^W)^{-1} \text{hol}_a(\gamma_{1,1,1}^W)^{-1} t(\text{Hol}_{ab'a'}(\gamma_{1,1,1}(0)))^{-1} g_{bb'}(\gamma_{1,1,1}^W(1))^{-1} \]
\[ \cdot t(\text{Hol}_{bb'}(\gamma_{2,1,1}^W)^{-1}) \text{hol}_b(\gamma_{2,1,1}^W)^{-1} g_{bb'}(\gamma_{2,1,1}^W(1))^{-1} \cdot \ldots \cdot \text{hol}_{t'}(\gamma_s(t))^{-1} \]

Now, since \( t(H_i) = 0 \) by Proposition 2.1.36 and hence \( H_i \) takes its values in the center of \( \mathfrak{h} \) by Proposition 2.1.21 we have
\[ \alpha_{t(h)}(H_i) = h \cdot H_i \cdot h^{-1} = H_i, \]
meaning we can write the integrand above as
\[ \alpha(\text{hol}_{a'}(\gamma_{1,1,1}^W)^{-1}g_{a'a'}(\gamma_{1,1,1}^W(1))^{-1} \text{hol}_{a'}(\gamma_{1,1,1}^W)^{-1}) \bigg( H_{t'} \bigg|_{(t,s)} \bigg) \]
\[ = \alpha g_{a'a'}(\gamma_{1,1,1}(0)) \text{hol}_a(\gamma_{1,1,1}^W)^{-1}g_{bb'}(\gamma_{1,1,1}^W(1))^{-1} \text{hol}_b(\gamma_{2,1,1}^W)^{-1}g_{bb'}(\gamma_{2,1,1}^W(1))^{-1} \cdot \ldots \cdot \text{hol}_{t'}(\gamma_s(t))^{-1} \bigg( H_{t'} \bigg|_{(t,s)} \bigg) \]
and then applying the same idea to \( ii' \) and using \( H_{t'} = \alpha_{i'i'}(H_i) \)
\[ = \alpha g_{a'a'}(\gamma_{1,1,1}(0)) \text{hol}_a(\gamma_{1,1,1}^W)^{-1}g_{bb'}(\gamma_{1,1,1}^W(1))^{-1} \text{hol}_b(\gamma_{2,1,1}^W)^{-1}g_{bb'}(\gamma_{2,1,1}^W(1))^{-1} \cdot \ldots \cdot \text{hol}_{i}(\gamma_s(t))^{-1} \bigg( H_{i} \bigg|_{(t,s)} \bigg). \]

Applying the integral yields precisely
\[ \int_{(t,s) \in S^2} H^{N_{t'}} = \alpha g_{a'a'}(\gamma_{1,1,1}(0)) \left( \int_{(t,s) \in S^2} H^{N_i} \right) \]
\[ \square \]
3.2 \( \alpha : G \rightarrow \text{Inn}(H) \)

In this section we assume the following special case, which will allow for considerable simplifications.

**Setting 3.2.1.** Suppose that \( \alpha \) factors through the inner automorphisms of \( H \):

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & \text{Aut}(H) \\
\downarrow & & \downarrow \\
& \xrightarrow{\text{Inn}(H)} & \\
\end{array}
\]  

(3.2.1)

**Remark 3.2.2.** Note that if \( Y \in Z(\mathfrak{h}) \) then in Setting 3.2.1 we have \( \alpha_g(Y) = Y \) for any \( g \in G \). Similarly, if \( h \in Z(H) \), then \( \alpha_X(h) = 0 \) for any \( X \in \mathfrak{g} \), and for \( Y \in Z(\mathfrak{h}) \), \( X \in \mathfrak{g} \) then \( \alpha_X(Y) = 0 \).

Recall from Proposition 3.1.4 that our 2-holonomy, a function on \( \mathcal{N} \subset M^{S^2} \), has trivial target yielding \( \text{Hol}^\mathcal{N} \in Z(H) \), where \( Z(H) \) is the center of \( H \). We have shown in Proposition 3.1.5 that \( \text{Hol} \) transforms between open subsets of \( M^{S^2} \) via \( \alpha_{g_{ij}} \) but since we are working in \( M^{S^2} \) and our action factors through an inner automorphism we can see that our \( \text{Hol}^\mathcal{N} \) functions agree on overlaps:

**Proposition 3.2.3.** The function \( \text{Hol} : M^{S^2} \rightarrow H \) given by \( \text{Hol}|_{\mathcal{N}} := \text{Hol}^\mathcal{N} \) is well-defined (globally).
Proof. For two open sets $\mathcal{N}_I, \mathcal{N}_J \subset M^{S^2}$ we have on $\mathcal{N}_I \cap \mathcal{N}_J$

$$\text{Hol}_{\mathcal{N}_I} = \alpha_{g_{ij}}(\text{Hol}_{\mathcal{N}_J}) = h(\text{Hol}_{\mathcal{N}_J})h^{-1} = \text{Hol}_{\mathcal{N}_J}$$ (3.2.2)

where the equalities are given by Proposition 3.1.5, Setting 3.2.1, and Proposition 3.1.4 respectively. \hfill \square

**Proposition 3.2.4.** In the case of Setting 3.2.1, the local 3-curvature forms, $H_i$, glue together to a global 3-form, $H \in \Omega^3(M, \mathfrak{g})$.

Proof. Recall that $t(H_i) = 0$ by Proposition 2.1.36 and so $H_i \in Z(\mathfrak{g})$ by Proposition 2.1.21. Thus by Remark 3.2.2, we have that

$$H_i = \alpha_{g_{ij}}(H_{ij}) = H_{ij}.$$ \hfill \square

**Proposition 3.2.5.** The total derivative of 2-holonomy, $d(\text{Hol}) = \text{Hol} \cdot \int_{S^2} H \in \Omega^1(M^{S^2}, \text{Mat})$, is globally defined.

Proof. Recall from Proposition 3.1.6 that

$$d(\text{Hol}) = - (\alpha_{\text{Hol}})_*(A_{i(1,1)}) + \text{Hol} \cdot \int_{S^2} H$$

which, in the case where $\alpha$ is inner, reduces to

$$d(\text{Hol}) = \text{Hol} \cdot \int_{S^2} H$$

where $(\alpha_{\text{Hol}})_*(A_{i(1,1)}) = 0$ in setting 3.2.1 when $\text{Hol}$ is in the center of $H$. 
(Proposition 3.1.4). Note that $\int_{S^2} H$ is a sum of terms, defined analogously to the corresponding term in Theorem 2.3.1:

$$\int_{S^2} H := \sum_{k=1}^{n} \sum_{l=1}^{m} \int_{S^2} (\alpha_{\text{path}_{k,l}})^*(H_{i(k,l)}).$$

Next, since the 3-curvature, $H_i$, takes its values in the center, then the transformation law for $\int_{S^2} H$, given in Proposition 3.1.7, now states in the inner case

$$\int_{S^2} H^N_i = \alpha_{g(0,0)}(\int_{S^2} H^N_{i'}) = (\int_{S^2} H^N_{i'}).$$

It was already shown that $\text{Hol}$ is globally defined above. Now with $\int_{S^2} H$ being globally defined as well, the Proposition is proven.

We will now introduce a form $\omega$ with $\iota_\omega = \int_{S^2} H$, which we will write as $\omega = \int_{S^2} Hdt$. Note that a differential form is defined on its plots. Moreover any $k$-form is determined by its definition on $k$-plots, and so to define the 2-form below we need only consider 2-plots, i.e. 2-parameter families of squares.

**Definition 3.2.6.** Define the 2-form, for a given 2-parameter family of spheres, $\Sigma_{(r,u)}$, $\int_{S^2} Hdt \in \Omega^2(M^{S^2},\text{Mat})$ given by

$$\left(\int_{S^2} Hdt \right)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial u}\right) := \int_{S^2} H\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial s}, \frac{\partial}{\partial r}\right) dt ds.$$

**Remark 3.2.7.** $\int_{S^2} Hdt$ can also be defined via the extended iterated integral
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\[ GJP \quad TWZ \] via the diagram

\[
\begin{array}{ccc}
I \times I \times (M \times I \times I) & \xrightarrow{\text{ev}} & M \times I \\
\downarrow \rho & & \downarrow \rho \\
(M \times I \times I) & \xrightarrow{\rho} & M \times I \times I
\end{array}
\]

where for \( f : I \times I \rightarrow M \), \( \rho(f) : I \times I \rightarrow M \times I \times I \) is given by \( (\rho(f))(s,t) := (f(s,t),s,t) \) via, modulo signs:

\[
\int_{S^2} H \, dt = \rho^* \left( \int_{I \times I} ev^*(H \wedge dt) \right).
\]

**Proposition 3.2.8.** The interior multiplication of the 2-form defined in 3.2.6 simplifies:

\[
\iota_{\partial_u} \left( \int_{S^2} H \, dt \right) = \int_{S^2} H.
\]

**Proof.** Recall that

\[
\left( \int_{S^2} H \right) \left( \frac{\partial}{\partial u} \right) = \int_{S^2} H \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) \, dt \, ds
\]

and so

\[
\iota_{\partial_u} \left( \int_{S^2} H \, dt \right) \left( \frac{\partial}{\partial u} \right) = \left( \int_{S^2} H \, dt \right) \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right) = \int_{S^2} H \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) \, dt \, ds
\]

\[
= \left( \int_{S^2} H \right) \left( \frac{\partial}{\partial u} \right).
\]

\[
\square
\]

The remainder of this section is dedicated to the definition of an element \( \mathcal{H}ol \), an extension of \( Hol \), and the chain complex \((\Omega^\bullet(M^S, Mat)^C, D)\), a non-abelian analog of \( \Omega^\bullet(M^T, \mathbb{C})^C \) from TWZ.
Definition 3.2.9. Define the chain complex \( \Omega^\bullet(M^{S^2}, \text{Mat})^L, D \) by
\[
\Omega^\bullet(M^{S^2}, \text{Mat})^L := \{ \alpha \in \Omega^\bullet(M^{S^2}, \text{Mat}) \mid (d \circ \iota_{\frac{\partial}{\partial t}} + \iota_{\frac{\partial}{\partial t}} \circ d)(\alpha) = 0 \}
\]
and \( D := d + \iota_{\frac{\partial}{\partial t}} \). Note that \( D^2 = 0 \) since \( D^2 = (d \circ \iota_{\frac{\partial}{\partial t}} + \iota_{\frac{\partial}{\partial t}} \circ d) \) and we have taken the subspace of \( \Omega^\bullet(M^{S^2}, \text{Mat}) \) on which \( D^2 = 0 \).

Definition 3.2.10. Given a gerbe and it’s associated \( \text{Hol} : M^{S^2} \to H \subset \text{Mat} \), define the extension of 2-holonomy,
\[
\text{Hol} := \sum_{n \geq 0} \frac{(-1)^n}{n!} \cdot \text{Hol} \cdot \left( \int_{S^2} H dt \right)^n
\]
which is an element in \( \Omega^\bullet(M^{S^2}, \text{Mat}) \). Notice that \( \text{Hol} \) can be factored \( \text{Hol} = \text{Hol} \cdot \mathcal{H} \) where
\[
\mathcal{H} := \sum_{n \geq 0} \frac{(-1)^n}{n!} \cdot \left( \int_{S^2} H dt \right)^n
\]
\[
= \left( 1 - \left( \int_{S^2} H dt \right) + \frac{1}{2!} \left( \int_{S^2} H dt \right)^2 - \ldots \right) \in \Omega^\bullet(M^{S^2}, \text{Mat}).
\]

Lemma 3.2.11. We have the following identities:

1. \( \iota_{\frac{\partial}{\partial t}} (\text{Hol}) = 0 \)

2. \( d(\text{Hol}) = \text{Hol} \cdot \int_{S^2} H \)

3. \( \iota_{\frac{\partial}{\partial t}} (\int_{S^2} H dt) = \int_{S^2} H \)
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4. \( d(\int_{S^2} H dt) = 0 \)

Proof. Part 1 is for degree reasons: \( Hol \) is degree 0 and \( \iota_{\frac{\partial}{\pi}} \) is a degree-decreasing map so it must map to zero. Part 2 was the Proposition 3.2.5. Part 3 is Proposition 3.2.8. Part 4 follows from the differential applied to the integral:

\[
d \left( \int_{S^2} H dt \right) = \left( \int_{\partial S^2} H dt \right) + \left( \int_{S^2} d(H dt) \right)
= \left( \int_{S^2} dH dt \right) = 0,
\]

where \( dH = 0 \) follows from Proposition 2.1.36, where in Setting 3.2.1, \( d = \nabla \).

\[\square\]

Theorem 3.2.12. The extension of 2-holonomy, \( \mathcal{H}ol \), is a closed element in \( \left( \Omega^\bullet(M^{S^2}, \text{Mat})^\xi, D \right) \).

Proof.

\[
d \left( \frac{(-1)^n}{n!} Hol \cdot \left( \int_{S^2} H dt \right)^n \right) = \frac{(-1)^n}{n!} Hol \cdot \left( \int_{S^2} H \right) \cdot \left( \int_{S^2} H dt \right)^n
\]

whereas

\[
\iota_{\frac{\partial}{\pi}} \left( \frac{(-1)^{n+1}}{(n+1)!} Hol \cdot \left( \int_{S^2} H dt \right)^{n+1} \right)
= (n+1) \cdot \frac{(-1)^{n+1}}{(n+1)!} Hol \cdot \left( \int_{S^2} H \right) \cdot \left( \int_{S^2} H dt \right)^n
= - d \left( \frac{(-1)^n}{n!} Hol \cdot \left( \int_{S^2} H dt \right)^n \right)
\]
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Thus $D(\text{Hol})$ vanishes via:

\[
\begin{align*}
0 & \quad 0 \\
\frac{t}{\Delta} & \quad \frac{t}{\Delta} \\
\frac{\partial}{\partial t} & \quad \frac{\partial}{\partial t} \\
\text{Hol} & \quad \text{Hol} \cdot \int_{S^2} H dt \\
\text{Hol} \cdot \int_{S^2} H dt & \quad \text{Hol} \cdot (\int_{S^2} H dt) \cdot (\int_{S^2} H dt) \\
\frac{1}{2} \text{Hol} \cdot (\int_{S^2} H dt)^2 & \quad \ldots
\end{align*}
\]

(3.2.3)

\[\square\]

3.3 General $\alpha$ Action

This section no longer restricts our $\alpha$ action to the inner case. However, we are still able to provide analogous results in this more general setting.

**Setting 3.3.1.** In this section we remove Setting 3.2.1 and assume that $\alpha : G \to \text{Aut}(H)$ is a general action.

Note that in the previous section, our 2-holonomy functions defined on the open sets $\mathcal{N}_j$ agreed on intersections and so we were able to define a global function. Now we consider the case where, in general, $\text{Hol}^{N_i} \neq \text{Hol}^{N_j}$, $H_i \neq H_j$, $d(\text{Hol}^{N_i}) \neq d(\text{Hol}^{N_j})$, etc. Since we can not define $\text{Hol}$ as a form on $M^{S^q}$ with values in $\text{Mat}$ in this case, we construct a bundle for the values of our differential forms to live in.
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Definition 3.3.2. Define $\mathcal{E}_0 \to M$ to be the vector bundle given locally by $\mathcal{E}_0|_{U_i} := U_i \times Z(\mathfrak{h})$, where $Z(\mathfrak{h})$ is the center of the Lie Algebra $\mathfrak{h}$, with transition functions

$$\mathcal{E}_0|_{U_i} \xrightarrow{\alpha_{g_{ij}}} \mathcal{E}_0|_{U_j}.$$  

Note that the transition functions, $\alpha_{g_{ij}}$, for $\mathcal{E}_0$ are isomorphisms of the fiber $Z(\mathfrak{h})$ with $\alpha_{g_{ij}}(X) = \alpha_{t(f_{ij}^{-1})}(X) = X$, for $X \in Z(\mathfrak{h})$, and indeed satisfy the cocycle condition since on triple intersections

$$\alpha_{g_{ik}} \alpha_{g_{ij}^{-1}} \alpha_{g_{jk}^{-1}}(X) = \alpha_{t(f_{ijk})}(X) = [f_{ijk}, X] = X$$

whenever $X \in Z(\mathfrak{h})$.

Further define the vector bundle $\mathcal{E} \to M^{S^2}$ as the pull-back $\mathcal{E} := ev_0^*(\mathcal{E}_0)$ where $ev_0 : M^{S^2} \to M$ is the evaluation map $\Sigma \mapsto \Sigma(0,0)$.

Definition 3.3.3. Define a connection $\nabla_0$ on $\mathcal{E}_0$ by taking $\nabla_i := (d + \alpha_{A_i})$ on $\mathcal{E}_0|_{U_i}$. We claim that this defines a connection on $\mathcal{E}_0$.

Proof. To check that $\nabla_i$ is indeed a connection on $\mathcal{E}_0$ in the first place, we check that for $f \in C^\infty(U_i, \mathbb{R})$ and $s \in \Gamma(U_i, \mathcal{E}_0)$ we have:

$$\nabla_i(f \cdot s) = df \cdot s + f \cdot ds + f \cdot \alpha_{A_i}(s) = df \cdot s + f \cdot \nabla_i s.$$

We also note that the $\nabla_i$’s transform correctly:

$$\alpha_{g_{ij}} \left( \nabla_0|_{U_i}(s) \right) = \alpha_{g_{ij}}(ds + \alpha_{A_i}(s))$$
and so applying the relation for $A_i, A_j$ on $U_{ij}$ from Definition 2.1.34,

$$= \alpha_{g_{ij}}(ds) + \alpha_{(dg_{ij} + t(a_{ij} + A_j g_{ij}))}(s)$$

and now using the fact that $\alpha_{t(a_{ij})}(\alpha_{g_{ij}}(s)) = 0$ by $\alpha_{g_{ij}}(s) \in Z(\mathfrak{h})$, and that $d$ is a derivation,

$$= d(\alpha_{g_{ij}}(s)) + \alpha_{A_j}(\alpha_{g_{ij}}(s))$$

$$= \nabla_0|_{U_{ij}}(\alpha_{g_{ij}}(s))$$

\[\Box\]

Further define the connection $\nabla$ on $\mathcal{E}$ by the pullback, $\nabla := ev_0^*(\nabla_0)$ on $\mathcal{E}_0$.

**Proposition 3.3.4.** $\nabla_0$ is a flat connection on $M$.

**Proof.**

$$(d + \alpha_{A_i})(d + \alpha_{A_i})(s) = d^2(s) + \alpha_{A_i}(ds) + d(\alpha_{A_i}) + \alpha_{A_i}(\alpha_{A_i}(s))$$

$$= \alpha_{A_i}(ds) + \alpha_{d(A_i)}(s) - \alpha_{A_i}(ds) + \alpha_{\frac{1}{2}[A_i \land A_i]}(s)$$

$$= \alpha_{(dA_i + \frac{1}{2}[A_i \land A_i])}(s) = \alpha_{t(B_i)}(s) = [B_i, s] = 0$$

where in the last line we used the *vanishing fake curvature condition*,

$$R = t(B)$$

from Definition 2.1.34 and the fact that $s \in Z(\mathfrak{h})$. \[\Box\]

**Corollary 3.3.5.** The connection $\nabla$ on $\mathcal{E}$ is flat.
Definition 3.3.6. Define \( \int_{S^2} Hdt \in \Omega^2(N, Z(\mathfrak{h})) \) on an open set \( N \subset M^{S^2} \) by
\[
\int_{S^2} Hdt := \sum_{k,l} \int_{S^2} \alpha_{\text{path}(k,l)} \left( H_{i(k,l)} \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial s}, \frac{\partial}{\partial r} \right) \right) dt ds
\]

Lemma 3.3.7. The 1-form, \( \int_{S^2} H \), and the 2-form, \( \int_{S^2} \), defined above take their values in \( E \). Moreover, we have \( \iota_{\frac{\partial}{\partial t}} \int_{S^2} H dt = \int_{S^2} H \).

Proof. The bundle \( E \) was constructed precisely so that the values on each \( N \) can glue together in \( E \), via Proposition 3.1.7 which also works for \( \int_{S^2} H dt \).

The second statement follows from Proposition 3.2.8. \( \square \)

Finally, while the powers \( (\int H dt)^n \) of \( \text{Hol} \) made sense in \( \text{Mat} \), we now move to the symmetric algebra, \( SE \), in order to be able to formally define products of elements from \( Z(\mathfrak{h}) \):

Definition 3.3.8. Define \( H \in \Omega^\bullet(M^{S^2}, SE) \) by
\[
H := \sum_{n \geq 0} \frac{(-1)^n}{n!} \cdot \left( \int_{S^2} H dt \right)^n = \left( 1 - \left( \int_{S^2} H dt \right) + \frac{1}{2!} \left( \int_{S^2} H dt \right)^2 - \ldots \right) \in \Omega^\bullet(M^{S^2}, SE).
\]

where we use the same indices and \( \alpha_{\text{path}} \)-notation as in Theorem 2.3.1.

While the bundle \( E \) will resolve the issue regarding where our closed form will take its values, we also will need to change the differential, from \( d \) to \( \nabla \).
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Note that the connection, $\nabla$, on $\mathcal{E}$ extends to a connection on $S\mathcal{E}$, which we also denote by $\nabla$ in an abuse of notation. $\nabla$ acts on $S\mathcal{E}$ by derivation.

Furthermore, note that we have shifted our focus from $\mathcal{H}ol$ (Definition 3.2.10), taking values in the tangent space $T_{\mathcal{H}ol}H$, to $\mathcal{H} = \mathcal{H}ol^{-1}\mathcal{H}ol$, taking values, locally, in the Lie Algebra, $\mathfrak{h}$. Accordingly, we must include this left-translation by $\mathcal{H}ol^{-1}$ in the differential as well:

**Motivation 3.3.9.** In the case of Setting 3.2.1, where $\alpha$ is inner, we had the cochain complex $(C^\bullet, D)$ from Definition 3.2.9, where $C^\bullet := \Omega^\bullet(MS^2, Mat)^\mathcal{E}$. Since $\mathcal{H}ol \in C^0$ is an invertible element of the Lie group, $H$, we have a linear isomorphism, $L : C^\bullet \rightarrow C^\bullet$ given by $L(x) := \mathcal{H}ol^{-1} \cdot x$

\[
\begin{array}{ccccccc}
& C^0 & \xrightarrow{D} & C^1 & \xrightarrow{D} & C^2 & \xrightarrow{D} & \cdots \\
\downarrow{L} & \downarrow{L} & \downarrow{L} & & & & & \\
& C^0 & \xrightarrow{\delta} & C^1 & \xrightarrow{\delta} & C^2 & \xrightarrow{\delta} & \cdots \\
\end{array}
\]

and so the differential, $D$, transports to a differential, $\delta$, which is given by $\delta := L \circ D \circ L^{-1}$. Explicitly, we have

\[
\delta(x) := \mathcal{H}ol^{-1}(D(\mathcal{H}ol(x))) = \mathcal{H}ol^{-1}((D(\mathcal{H}ol))(x) + \mathcal{H}ol \cdot D(x))
\]

\[
= (\mathcal{H}ol^{-1}D\mathcal{H}ol)(x) + D(x) = (L_{\mathcal{H}ol^{-1}D(\mathcal{H}ol)} + D)(x).
\]

Here,

\[
\mathcal{H}ol^{-1}D(\mathcal{H}ol) = \mathcal{H}ol^{-1}\left((d + \iota_{\frac{d}{m}})(\mathcal{H}ol)\right) = \mathcal{H}ol^{-1}d(\mathcal{H}ol) = \int_{S^2} H
\]
by Proposition 3.2.5 motivating the following definition/proposition:

**Proposition 3.3.10.** The complex \( \left( \Omega^\bullet \left( M^{S^2}, SE \right)^L, \delta \right) \) satisfies \( \delta^2 = 0 \),

where

\[
\delta := \nabla + \iota \frac{\partial}{\partial t} + L_\tau, \quad \tau := \int_{S^2} H
\]

and the \( L \) refers to forms which vanish under \( \mathcal{L} := \left[ \iota \frac{\partial}{\partial t}, \nabla \right] \).

**Proof.** First note that \( \tau \in \Omega^\bullet (M^{S^2}, SE)^L \). In particular, \( \mathcal{L} (\tau) = 0 \) since \( \nabla (\tau) = 0 \), as in Lemma 3.2.11, and \( \iota \frac{\partial}{\partial t} (\tau) = \iota \frac{\partial}{\partial t} (\iota \frac{\partial}{\partial t} (\int_{S^2} H dt)) = 0 \). Since we are considering forms which vanish under \( \mathcal{L} \) we need only to show that \( \delta^2 = \mathcal{L} \):

\[
\delta^2 = (\nabla + \iota \frac{\partial}{\partial t} + L_\tau)(\nabla + \iota \frac{\partial}{\partial t} + L_\tau)
\]

\[
= \nabla^2 + \nabla \circ \iota \frac{\partial}{\partial t} + \nabla \circ L_\tau + \iota \frac{\partial}{\partial t} \circ \nabla + \left( \iota \frac{\partial}{\partial t} \right)^2
\]

\[
+ \iota \frac{\partial}{\partial t} \circ L_\tau + L_\tau \circ \nabla + L_\tau \circ \iota \frac{\partial}{\partial t} + (L_\tau)^2
\]

\[
= \mathcal{L} \nabla (\tau) + L_{\iota \frac{\partial}{\partial t} (\tau)} + L_\tau^2 + \mathcal{L}.
\]

Since \( \nabla (\tau) = \iota \frac{\partial}{\partial t} (\tau) = 0 \), and \( \tau^2 = 0 \) by degree reasons, we are done. \( \square \)

**Theorem 3.3.11.** The element \( \mathcal{H} \in \left( \Omega^\bullet \left( M^{S^2}, SE \right)^L, \delta \right) \) is closed.

**Proof.** We show that \( \delta (\mathcal{H}) = 0 \) where \( \delta = \nabla + \iota \frac{\partial}{\partial t} + L_\tau \) by noting the following
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Table of equalities:

\[ \mathcal{H} = 1 - \int_{S^2} H dt + \frac{1}{2} (\int_{S^2} H dt)^2 - \frac{1}{3!} (\int_{S^2} H dt)^3 + \cdots \]

\[ \nabla(\mathcal{H}) = 0 - 0 + 0 - 0 + \cdots \]

\[ \iota_{\frac{\omega}{\mathcal{H}}}(\mathcal{H}) = 0 - \int_{S^2} H + \int_{S^2} H \cdot \int_{S^2} H dt - \frac{1}{2} \int_{S^2} H \cdot (\int_{S^2} H dt)^2 + \cdots \]

\[ L_{\tau}(\mathcal{H}) = \int_{S^2} H - \int_{S^2} H \cdot \int_{S^2} H dt + \frac{1}{2} \int_{S^2} H \cdot (\int_{S^2} H dt)^2 - \frac{1}{3!} \int_{S^2} H \cdot (\int_{S^2} H dt)^3 + \cdots \]
Bibliography


