9-2016

Explicit Formulae and Trace Formulae

Tian An Wong
The Graduate Center, City University of New York

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by

Tian An Wong

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2016

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

Carlos Moreno

Date

Chair of Examining Committee

Ara Basmajian

Date

Executive Officer

Carlos Moreno

Brooke Feigon

Burton Randol

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

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Tian An Wong

Advisor: Carlos Moreno

In this thesis, motivated by an observation of D. Hejhal in [Hej76], we show that the explicit formulae of A. Weil for sums over zeroes of Hecke L-functions, via the Maaß-Selberg relation, occur in the continuous spectral terms in the Selberg trace formula over various number fields.

In Part I, we discuss the relevant parts of the trace formulae classically and adelically, developing the necessary representation theoretic background. In Part II, we show how show the explicit formulae intervene, using the classical formulation of [Wei52]; then we recast this in terms of Weil distributions and the adelic formulation of [Wei72].

As an application, we prove a lower bound for these explicit formulae using properties of the trace formula, in the spirit of Weil's criterion for the Riemann hypothesis.

Acknowledgements

First and foremost, to my advisor, Carlos Moreno, for taking me on as your final student, for the many conversations about the richness and history of number theory, for inspiring me to think deeply about the subject.

To Brooke Feigon and Burton Randol, for taking the time to serve on the committee. To Dorian Goldfeld, Peter Sarnak, Henryk Iwaniec, and Ali Altuğ, for taking an interest in this work. To Rob Landsman, for keeping the Mathematics Department from falling apart.

To Natalie Frank, John McCleary, Charlie Steinhorn, Kari Calta, Jan Cameron, Ming-Wen An, Diane Winkler, Alison Keimowitz, Teresa Garrett, and Himadeep Muppidi at Vassar College, it has been a long journey since those days at Rockefeller Hall, floating between mathematics, art, biochemistry, and political science. I'm not finished with learning yet.

To Jorge Florez, Cihan Karabulut, Bren Cavallo, Joe Kramer-Miller, Sajjad Lakzian, Byundo Park, Jorge Basilio, and everyone else who filled the shoebox with more than just mathematics.

To my parents and brothers and in-laws, and to Grace.

Thank you.

Contents

Chapter 1

Introduction

1.1 Motivation

In his groundbreaking 1859 memoir, Riemann introduced an explicit formula to study prime numbers 'less than a given magnitude'. Through this, having better control over the distribution of zeroes of $\zeta(s)$ leads to better estimates on the growth of prime numbers as expected, for example, from the Prime Number Theorem first sought after by Legendre and Gauss. This formula was made rigorous and generalized by von Mangoldt, Delsarte and others.

Turning this idea around, in 1952 Weil [Wei52] wrote down an explicit formula to study the zeroes of $\zeta(s)$ instead, using certain sums over primes. Inspired by his proof of the Riemann hypothesis for curves over finite fields, Weil showed that the positivity of this sum over the zeroes of $\zeta(s)$ is equivalent to the original Riemann hypothesis. Two decades later [Wei72], he did the same for Artin-Hecke L-functions for a global field.

On the other hand, in 1956 Selberg introduced the trace formula [Sel56], which is a character identity expressing the spectrum of invariant differential operators acting on a Riemannian manifold as a sum of orbital integrals. When the operator is the Laplacian acting on a compact Riemann surface, the formula resembles the explicit formulae, and prime closed geodesics behave in analogy with the prime numbers. This led to what is now called the Selberg zeta function, whose zeroes relate to the eigenvalues of the Laplacian.

Could the explicit formulae be trace formulae? The work of others like Connes, Goldfeld, and Meyer have shown that such an interpretation is possible, giving weight to the conjecture of Hilbert and Pólya that the zeroes of $\zeta(s)$ may be interpreted as eigenvalues of a self-adjoint operator, not to mention the approach of random matrix theory initiated by Montgomery. Yet, appreciable progress towards the Riemann hypothesis using these methods remains to be seen.

For a noncompact, finite volume Riemann surface, additional terms appear in the trace formula related to the continuous spectrum on the spectral side, and parabolic conjugacy classes on the geometric side. Their appearance is parallel in the sense that by subtracting one from the other, Selberg was able to show that the result was absolutely convergent. But the additional terms, which we derive later in §2, written in Hejhal [Hej76, 10.2] as

$$g(0)\log(\frac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\Big(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\Big)\ dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n),$$

look structurally similar to the explicit formula of Weil, as in [Hej76, 6.7],

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\frac{1}{4}+\frac{1}{2}ir)dr + h(\frac{i}{2}) + h(-\frac{i}{2}) - g(0)\log\pi - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).$$

Here $h(r)$ is a function with certain analytic conditions, and $g(u)$ its Fourier transform. The functions $\Gamma(s)$ and $\Lambda(n)$ are the usual gamma function and von Mangoldt functions respectively.

Hejhal points out these parallels to shed some light on the question raised in the same paper: Does the Weil explicit formula have anything to do with the Selberg trace formula [Hej76, p.478]? He notes that 'although there are structural similarities, one finds serious obstructions to interpreting [the explicit formula] as a trace formula.' To be certain, the motivation behind Hejhal's question is to 'interpret the zeroes of $\zeta(s)$ as eigenvalues', à la Hilbert and Pólya.

1.2 Main results

In this thesis, I present a different answer to this question of Hejhal: rather than interpret the entire trace formula as an explicit formula, I show that the sum over zeroes of L-functions of the explicit formula appears naturally in the spectral side of the trace formula. In particular, the continuous spectrum is described by Eisenstein series, and in the trace formula one requires the inner product of Eisenstein series. The Maaß-Selberg relation expresses this inner product using the logarithmic derivative of the constant term of the Eisenstein series, which can be written as a quotient of L-functions, and it is here that the sum over zeroes appear. Having achieved this, we are then in a position to study questions related to the explicit formula using our knowledge of the trace formula.

The main result of this thesis can be illustrated as follows:

Theorem A. *Let $g(x)$ be a smooth compactly supported function on $\mathbf{R}_+^\times$, and $\hat{g}$ its Mellin transform. Consider the Selberg trace formula for $PSL_2(\mathbf{R})$ acting on $PSL_2(\mathbf{Z})\backslash\mathbf{H}_2$. The continuous spectral term*

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{m'}{m}(ir)\hat{g}(ir)dr$$

in the trace formula is equal to

$$\sum_\rho \hat{g}(\rho) - \int_0^\infty \left\{g(x)+\frac{1}{4}g^*(x)\right\}dx + \sum_{n=1}^\infty \Lambda(n)g(n)$$
$$+\int_1^\infty \left\{g(x)+g^*(x)-\frac{2}{x}g(1)\right\}\frac{x\ dx}{2(x^2-1)}+\frac{1}{2}(\log 4\pi-\gamma)g(1),$$

where the sum ρ runs over zeroes of $\zeta(s)$ with $0<\mathrm{Re}(\rho)<1$, and $m(s)=\xi(s)/\xi(1+s)$, with $\xi(s)$ the completed Riemann zeta function.

The shape of the formula obtained reflects that derived by Weil in 1952 [Wei52], though we follow closely the derivation of Bombieri [Bom00] (c.f. Theorem 7.2.1 below). We also obtain a similar statement in the adelic language and over a number field, though the formula is more complicated, as there we allow for arbitrary ramification, which we prove as Theorem 7.2.2.

The next theorem shows that we can also express certain continuous spectral terms as distributions, as obtained in Weil's 1972 form of the explicit formula [Wei72]:

Theorem B. *Let F be a number field with r_1 real and r_2 complex embeddings with a Hecke character $\chi_{\mathbf{m}}$, and $g = \prod g_k$ where g_k is a smooth compactly supported function of $\mathbf{R}_+^\times, k = 1, \ldots, r_1 + r_2$. Consider the Selberg trace formula for $PSL_2(\mathbf{R})^{r_1} \times PSL_2(\mathbf{C})^{r_2}$ acting on $PSL_2(\mathcal{O}_F)\backslash\mathbf{H}_2^{r_1} \times H_3^{r_2}$.*

Then the integrals in involving the logarithmic derivative of $m(s, \chi_{\mathbf{m}})$ defined in 4.4.3 can be expressed as a sum of distributions:

$$\frac{h}{2} g(0) \log |d_F| + \frac{1}{2} \sum_{\chi} \left(\int_{W_F} g(|w|)\chi(w) \frac{dw}{|w|} + \sum_{v} pv_0 \int_{W_v} \hat{g}(|w|)\chi_v(w) \frac{|w|}{|1-w|} dw \right)$$

where the sum v runs over all places of F, and h, d_F are the class number and discriminant of F respectively, and W_F, W_v are the global and local absolute Weil groups of F respectively. The sum over χ runs over the h many ways of extending $\chi_{\mathbf{m}}$ to the ideal class group.

The restrictions of the Hecke character and the notation pv_0 is described preceding

Theorem 7.3.3 below. Having shown a relationship between the explicit formulae and trace formulae, one would hope that the connection will shed light on either of the two. As a first step, we show the following bound for the sums over zeroes, which follows as a straightforward corollary to Theorem A above:

Theorem C. *Let $g = g_0 * g_0^*$, for any g_0 in $C_c^\infty(\mathbf{R}_+^\times)$. Then the sum over zeroes of $\zeta(s)$, where $0 < \mathrm{Re}(\rho) < 1$, is bounded below by the following:*

$$\begin{aligned}\sum_\rho \hat{g}(\rho) \geq \int_0^\infty & \left\{g(x) + \frac{1}{4} g^*(x)\right\} dx - \sum_{n=1}^\infty \Lambda(n) g(n) \\ & - \int_1^\infty \left\{g(x) + g^*(x) - \frac{2}{x} g(1)\right\} \frac{x\ dx}{2(x^2-1)} \\ & - g(0)\log T - \frac{1}{4\pi i}\int_{-\infty}^\infty m(it) g(it) \frac{T^{it}}{t} dt.\end{aligned}$$

for any $T > \sqrt{3}/2$.

This thesis is organized as follows: In Part I, we introduce the Arthur-Selberg trace formula for SL_2. Here we do not prove anything that is not already known to experts, though we do supply proofs of certain statements that may not be available easily or at all in the literature, namely, the derivation of the Maaß-Selberg relation for SL_2 in the adelic setting, the expressions of Hejhal discussed earlier in §1.1, which were originally stated without proof, and thirdly, the explicit formula for Hecke L-functions in the style of Bombieri.

Chapter 2 reviews the general setup of the trace formula, and discusses certain aspects of passing between SL_2 and GL_2. Chapter 3 discusses the trace formula for $PSL_2(\mathbf{R})$ acting on $PSL_2(\mathbf{Z})\backslash\mathbf{H}_2$, which is the classical case. Here we introduce the representation theoretic language that will be used throughout the thesis, and derive the expressions of Hejhal. Chapter 4 begins to consider ramification of the representations, but only at real completions of a given number field. This is an intermediate setting before passing to the general case of Chapter 5, which uses the language of the adeles. We develop the continuous spectral terms of the adelic trace formula for SL_2, using the method outlined by Labesse and Langlands [LL79], and prove the Maaß-Selberg relation in this context.

In Part II, we begin to explore the connection to explicit formulae. In Chapter 7, we recount the historical development of explicit formulae from Riemann to Weil, then derive the explicit formula for Hecke L-functions which will set the stage for the next chapter. Finally, we arrive at Chapter 8 where new statements are proved, and which we have described above.

It will become clear to the reader that our methods generalize to continuous spectral terms of trace formulae for any reductive group, granted that G possesses continuous spectrum. In our exposition we have confined ourselves to SL_2, whilst giving some indication of the picture for GL_n. We point out to the reader that while the trace formula is repeatedly mentioned throughout, strictly speaking our method only requires knowledge of the spectral side of the trace formula, and in particular certain facts about Eisenstein series. While one may rightly object that the trace formula per se, that is, an equivalence of two expressions, is not invoked, we argue that the expressions to which we restrict our attention arise from the development of the trace formula.

That there exists a connection between trace formulae and explicit formulae has been known since Selberg introduced his trace formula. But subsequent work has shown that the exact nature of this connection is not well-understood. We hope that this thesis clarifies the situation, and inspires deeper study of old things.

Part I

The Trace Formula

Chapter 2

The pre-trace formula

2.1 Compact quotients

Begin with a locally compact group G and a discrete subgroup Γ. Denote by $\rho(g)$ the representation of G on $L^2(\Gamma\backslash G)$ acting by right translation. It is a unitary operator, and commutes with the action of the invariant differential operators on G. If $G = SL_2(\mathbf{R})$, the only invariant differential operator is the Laplacian Δ.

Definition 2.1.1. Given a smooth, compactly supported function f on G, one defines the *integral operator*

$$\rho(f) = \int_G f(x)\rho(x)dx$$

acting on functions φ in $L^2(\Gamma\backslash G)$ by

$$\begin{aligned}(\rho(f)\varphi)(x) &= \int_G f(y)\varphi(xy)dy \\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).\varphi(y)dy \\ &:= \int_{\Gamma\backslash G} K(x,y).\varphi(y)dy\end{aligned}$$

where $K(x,y)$ is called the *kernel* of the integral operator $\rho(f)$. The sum over γ is finite, taken over the intersection of the discrete Γ with the compact subset $x\mathrm{supp}(f)y^{-1}$.

Remark 2.1.2. (Positivity.) Note that functions that are convolutions of the form

$$f(x) = f_0(x) * \overline{f_0(x^{-1})},$$

are positive definite, where $f_0 \in C_c^\infty(G)$, so that for such functions the operator $\rho(f)$ is self-adjoint and positive definite [GGPS69, §2.4], and as a consequence its restriction to any invariant subspace is also positive definite. This property shall be useful to us in Part 2 on explicit formulae.

If the quotient $\Gamma\backslash G$ compact, the representation $\rho(g)$ decomposes into a countable discrete sum of irreducible unitary representations with finite multiplicity. The trace of the operator exists, and can be computed in two ways, giving the *pre-trace formula*

$$\mathrm{tr}(\rho(f)) = \sum_{\pi} m(\pi)\mathrm{tr}(\pi(f)) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx$$

where the first sum is over irreducible constituents π of ρ appearing with multiplicity $m(\pi)$, while the second sum is over conjugacy classes γ of Γ. We denote by the centralizer of γ in a group by the subscript γ, for example G_γ. The left hand side is referred to as the spectral side, consisting of characters of representations; whereas the right hand side is the geometric side, consisting of orbital integrals.

Example 2.1.3. When $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$, then $\Gamma\backslash G$ is compact and abelian, and the trace formula reduces to the Poisson summation formula. The spectral side is indexed by irreducible representations $e^{2\pi inx}$, the terms being fourier transforms of f, while the geometric side is the discrete sum of f on $\mathbf{Z}$. The Selberg trace formula should be thought of as a non-abelian analogue of the Poisson summation formula.

2.2 Noncompact quotients

If instead the quotient were noncompact but of finite volume, by the theory of Eisenstein series the L^2-spectrum of $\Gamma\backslash G$ decomposes into an orthogonal sum of discrete and continuous parts, and the discrete spectrum decomposes further into a direct sum of cuspidal and non-cuspidal subspaces:

$$L^2_{\text{disc}}(G) \oplus L^2_{\text{cont}}(G) = L^2_{\text{cusp}}(G) \oplus L^2_{\text{res}}(G) \oplus L^2_{\text{cont}}(G) \tag{2.2.1}$$

where for short we have written $L^2(G)$ for $L^2(\Gamma\backslash G)$. The cuspidal spectrum consist of cusp forms, which are certain functions vanishing at cusps of the fundamental domain, while the continuous spectrum is a direct integral of Hilbert spaces associated to certain principal series representations, which we shall discuss below.

Eisenstein series enter into the picture as the continuous spectrum is described by the inner product of Eisenstein series, and the non-cuspidal discrete spectrum is spanned by the residues of Eisenstein series, given as above. To each inequivalent cusp, or equivalently, to each non-conjugate parabolic subgroup of G, one associates a different Eisenstein series.

The representation ρ acting on the discrete spectrum again gives a trace class operator, and the pre-trace formula is valid in the form

$$\sum_{\pi \text{ disc}} m(\pi)\mathrm{tr}(\pi(f)) = \sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx - \int_{\Gamma\backslash G} K_{\mathrm{cont}}(x,x)dx \quad (2.2.2)$$

where the left-hand sum is over irreducible subquotients of ρ occurring in the discrete spectrum, and $K_{\mathrm{cont}}(x,x)$ is the kernel of operator $\rho(f)$ restricted to the continuous spectrum. We shall refer to the contribution of $K_{\mathrm{cont}}(x,x)$ to the trace formula as the *continuous spectral terms.*

The remarkable insight of Selberg is that while the trace $\mathrm{tr}(\rho(f))$ does not converge in general, arranging the trace formula as (2.2.2) the divergent terms on the geometric and spectral sides cancel such that the formula converges. Adapting the method of Selberg, Arthur introduced the truncation operator Λ^T with respect to some parameter T, such that the integral $\Lambda^T K(x,x)$ converges for 'sufficiently regular' T.

In what follows we will want to consider this term on its own, for in the continuous spectral terms we will find the L-functions which we are interested in. In particular, we only consider the noncompact, finite volume setting where the theory of Eisenstein series is required to treat the resulting continuous spectrum.

2.3 Lifting forms

As we will be discussing automorphic forms both as functions on the ground field F and on the group G, it will be important to know how to transition between the two, and also to the adelic setting. We briefly review this. Throughout $\mathbf{A}_F$ will denote the adele ring of global field F, and when the context is clear we will omit the subscript and simply write $\mathbf{A}$.

Any element of $SL_2(\mathbf{R})$ acts on the upper half plane $\mathbf{H}_2$ by linear fractional transformations, and the point i is fixed by the group SO_2. Thus we may consider $\mathbf{H}_2$ as the homogeneous space $G/K = PSL_2(\mathbf{R})/SO_2$ under the action of $\Gamma = SL_2(\mathbf{Z})$. Write $G = NAK$ for the usual Iwasawa decomposition. Now a modular form of even weight $k > 0$ satisfies the transformation rule

$$f(g(z)) = (cz+d)^k (\det g)^{\frac{k}{2}} f(z), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It is lifted to $SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R})$ by defining $F(g) = f(g(i))(ci+d)^{-k}$ such that $F(g)$

is left-invariant under $SL_2(\mathbf{Z})$, satisfies a slow growth condition, is an eigenfunction of the Casimir operator, and

$$F(g\kappa) = e^{-ikt}F(g), \qquad \kappa = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

To lift further to $GL_2(\mathbf{Z})\backslash GL_2(\mathbf{R})$, one sets $f(-z) = f(z)$, and requires $F(zg) = F(g)$ for any z in the center. Furthermore, this lifting works similarly when Γ is a congruence subgroup, or G the Hilbert modular group. We will consider these cases in what follows. Finally, if

$$GL_2(\mathbf{A}) = GL_2(\mathbf{Q})GL_2(\mathbf{R}) \prod_p GL_2(\mathbf{Z}_p),$$

one lifts to $GL_2(\mathbf{Q})\backslash GL_2(\mathbf{A})$ by requiring the same conditions at $GL_2(\mathbf{R})$, and moreover $F(\gamma zgk) = F(g)$ for γ in $GL_2(\mathbf{Q})$, k in $\prod GL_2(\mathbf{Z}_p)$, and z in the center. The product is taken over all finite primes p in $\mathbf{Q}$. Then from this setting one may generalize further to arbitrary reductive groups.

The following will also be useful to us: Fix a positive integer N, and define the maximal compact subgroup K_p at each finite prime to be elements in $SL_2(\mathbf{Z}_p)$ such that the lower left entry c mod N is integral. For all p prime to N, K_v is simply $SL_2(\mathbf{Z}_p)$, and one has

$$SL_2(\mathbf{Q}) \cap SL_2(\mathbf{R}) \prod_p K_p = \Gamma_0(N) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_0(N)$$

and similarly for $\Gamma_i(N)$. In particular, ramification at finite primes is equivalent to considering Γ as congruence subgroups.

2.4 Special, projective, and general linear groups

Finally, we mention the connection between the various projective, special, and general linear groups. We will only consider these groups defined over F a number field or the complex numbers $\mathbf{C}$, or the adele ring of F, and denote by Z_G the center of G. Of course, we can identify the center of $GL_n(F)$, which are the diagonal matrices $\mathrm{diag}(z, \ldots, z)$ with the multiplicative group of the base field $F^\times$. On the other hand, if $F = \mathbf{R}$ or $\mathbf{C}$, the center of SL_n is the group of n-roots of unity.

Then the groups are related by the following commutative diagram:

$$\begin{tikzcd}
Z_{SL_n}(F) \arrow[r] \arrow[d] & Z_{GL_n}(F) \arrow[r, "\det"] \arrow[d] & (F^\times)^n \arrow[d] \\
SL_n(F) \arrow[r] \arrow[d] & GL_n(F) \arrow[r, "\det"] \arrow[d] & F^\times \arrow[d] \\
PSL_n(F) \arrow[r] & PGL_n(F) \arrow[r, "\det"] & F^\times/(F^\times)^n
\end{tikzcd}$$

One sees from this, for example, the relations $GL_n(F) \simeq SL_n(F) \rtimes F^\times$, and also $PGL_n(F) \simeq GL_n(F)/Z_{GL_n}(F)$.

In the classical setting, one considers the action of $SL_2(\mathbf{R})$ acting on the upper-half plane by Möbius transformations, so that the center acts trivially. Thus it is enough to consider the $PSL_2(\mathbf{R})$, and analogously $PSL_2(\mathbf{C})$ actions on the upper-half space $\mathbf{H}_3$. Moreover, in the special case of $SL_2(\mathbf{R})$ the center is contained in the modular group $SL_2(\mathbf{Z})$, which we will often consider quotients by.

On the other hand, in the modern theory of automorphic forms one considers GL_2 instead of SL_2, for the reason that the nontrivial center of GL_2 allows for classical cusp forms of weight k and *nontrivial* conductor ψ. Moreover, the element

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

does not belong to SL_2 for any prime p, hence does not provide a natural theory of Hecke operators.

For us, our interest will be in connections between the trace formulae and explicit formulae. In particular, Hecke operators do not enter into the analysis, and to illustrate more clearly the connection with the classical theory, we focus mainly on SL_2. Of course, the trace formula is sufficiently general so that it holds for any reductive group, so too our methods are expected to hold as long as G has a continuous spectrum.

Chapter 3

The trace formula for $PSL_2(\mathbf{R})$: unramified

3.1 Preliminaries

In this chapter we will consider the trace formula for PSL_2 over $\mathbf{Q}$ without any ramification, where by ramification we mean choosing test functions that are nontrivial on the maximal compact group. The main result of this chapter is a representation theoretic interpretation of the continuous spectral terms in the trace formula; we also provide a derivation of the expression for them stated without proof by Hejhal in [Hej76, p.475].

Let $G = PSL_2(\mathbf{R})$, containing the discrete subgroup $\Gamma = PSL_2(\mathbf{Z})$ and maximal compact subgroup $K = SO_2(\mathbf{R})$, acting on the complex plane by Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

so that K is the stabilizer of i. The quotient space G/K is isomorphic to the upper half plane $\mathbf{H}_2$, and the quotient $\Gamma\backslash\mathbf{H}_2$ is the usual modular surface, a finite volume Riemann surface with a cusp at infinity. That is, we make the identification

$$PSL_2(\mathbf{Z})\backslash PSL_2(\mathbf{R})/SO_2(\mathbf{R}) := \Gamma\backslash G/K \simeq \Gamma\backslash\mathbf{H}_2.$$

We recall that the regular representation ρ commutes with the Laplace operator, hence decomposing the reducible representation ρ is equivalent to the spectral decomposition of Δ. Now the Selberg trace formula in this setting, in the form obtained by Selberg, which follows from expanding the pre-trace formula of (2.2.2) reads:

Theorem 3.1.1 (Selberg's trace formula [Sel56, p.74, 78])**.** *Let $h(r)$ be an even analytic function in the strip $|\mathrm{Im}(r)| \le 1/2 + \delta$ such that $|h(r)| \ll (1 + |r|)^{-2-\delta}$ for some $\delta > 0$. Denote by $g(u)$ its inverse Fourier transform.*

Then the trace formula for $G = SL_2(\mathbf{R})$ and $\Gamma = SL_2(\mathbf{Z})$ is a sum over the eigenvalue spectrum of the Laplace operator indexed by $\lambda_n = 1/4 + r_n^2$,

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}_2)}{4\pi}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)dr + \sum_{\{P\}} \frac{\ln N(P_0)}{N(P)^{\frac{1}{2}} - N(P)^{\frac{1}{2}}} g(\ln N(P))$$

$$+\sum_{\{T\}} \frac{1}{2m\sin\theta}\int_{-\infty}^{\infty} \frac{e^{-2\theta r}}{1+e^{-2\pi r}} h(r)dr + \dots$$

where $\{P\}$ and $\{T\}$ are hyperbolic and elliptic conjugacy classes in $SL_2(\mathbf{Z})$, P_0 is the generator of the centralizer Γ_P, $\ln N(P)$ is the length of the geodesic defined by P, m is the cardinality of Γ_T, plus the difference of parabolic terms and continuous *terms,*

$$\cdots + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{m'}{m}(\frac{1}{2}+ir)dr - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr \tag{3.1.1}$$
$$+\frac{1}{4}(1-m(\frac{1}{2}))h(0) - g(0)\ln 2.$$

where $m(s)$ is a quotient of completed Riemann zeta functions $\xi(2s-1)/\xi(2s)$.

Remark 3.1.2. (Historical note.) While the above formula is generally referred to as the classical Selberg trace formula, that is for $G = SL_2(\mathbf{R})$ and $\Gamma = SL_2(\mathbf{Z})$ or a congruence subgroup, the original work of Selberg derives the trace formula for a much more general class of spaces he called weakly-symmetric Riemannian spaces [Sel56], and as an application obtained estimates for Maaß forms and the traces of Hecke operators.

To reflect the structural similarity to the explicit formulae of Weil (cf. Part II), Hejhal gave the following expression for the continuous spectral terms:

Proposition 3.1.3 ([Hej76, p.475])**.** *The term (3.1.1) can be expressed as*

$$g(0)\log(\frac{\pi}{2}) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\Big(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\Big)\, dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n).$$

where $\Lambda(n)$ is the von Mangoldt function.

We will derive this expression at the end of chapter.

3.2 The continuous spectrum of $PSL_2(\mathbf{R})$

Consider the space of complex-valued, square-integrable functions on $\Gamma\backslash\mathbf{H}_2$, denoted by $L^2(\Gamma\backslash\mathbf{H}_2)$. We will refer to this as the L^2-spectrum of G, which is classified by the discrete series, complementary series, and principal series representations. We focus on the latter, which make up the continuous spectrum.

Definition 3.2.1. Define the height function on the diagonal subgroup A of G,

$$H : \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \mapsto |y|^2.$$

Then given a complex parameter s, the *spherical principal series representation* P_s^+ acts by right translation on smooth, even functions φ_s on G satisfying

$$\varphi_s(nak) = H(a)^{(1+s)/2}\varphi(k),$$

completed to a Hilbert space, which is the induced representation space

$$\mathrm{Ind}_{NA}^{G}(1 \otimes H^{(1+s)/2}),$$

following the usual Iwasawa decomposition. Such a function is determined by its value on the compact group K, hence can be classified according to its K-type $\varphi(k)$, and in the particular case of $\mathbf{H}_2 \simeq G/K$ we consider only the trivial K-type. Moreover, the representation is unitary if s is pure imaginary, sometimes called the unitary principal series (note that this normalization depends on the choice of compatible Haar measures).

Remark 3.2.2. The function $H(a)$ can be viewed as the height function $\mathrm{Im}(\cdot)$ on $\mathbf{H}_2$, for if $g = nak$, then $H(g) = H(a) = y^2$. On the other hand, the action $g(i)$ is computed as

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} k \cdot i = \begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix} \cdot i = x + iy^2$$

since $k \cdot i = i$. Thus $\mathrm{Im}(g(i)) = H(g)$. We also note here the more general formula

$$\mathrm{Im}(g(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Definition 3.2.3. There is an *intertwining operator*

$$(M(s)\varphi)(g) = \int_N \varphi(wng)dn \tag{3.2.1}$$

intertwining the representations P_s^+ and P_{-s}^+, which are unitarily equivalent if s is imaginary, and w is the nontrivial element of the Weyl group of G,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The intertwining operator sends the trivial function 1_s in P_s^+ to the trivial function 1_{-s} in P_{-s}^+, multiplied by the scalar factor [Kna97, p.368]

$$m(s) = \frac{\xi(s)}{\xi(1+s)} = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1+s}{2})} \frac{\zeta(s)}{\zeta(1+s)}, \tag{3.2.2}$$

a quotient of completed Riemann zeta functions. If one considers PSL_2 over a number field, there is an analogous expression with the completed Dedekind zeta function, which we will encounter in the next chapter.

Definition 3.2.4. Given a vector φ_s in the principal series P_s^+, we introduce the Eisenstein series

$$E(g, \varphi, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi_s(\gamma g)$$

where Γ_∞ is the stabilizer of the cusp at ∞, i.e., the set of upper triangular matrices in Γ with ones on the diagonal. The sum converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$ [Bor97, Theorem 10.4(i)]. The constant term of its Fourier expansion at the cusp ∞ is given by

$$\int_{\Gamma_\infty \backslash N} E(ng, \varphi, s) = 2\varphi(s) + 2(m(s)\varphi)(s).$$

We note that this definition descends to the classical constant term of the Fourier expansion of a modular form on $\mathbf{H}_2$, as in [Kna97, p.359], and that the factor of 2 appears due to our choice of normalization of s.

Remark 3.2.5. The Eisenstein series defined as above is not square integrable, and it is here where one needs to calculate the inner product of Eisenstein series using truncation. To calculate the inner product, we must first perform a truncation with respect to the height $T = \mathrm{Im}(z)$ to make the terms square integrable. First introduced by Selberg, this process was made uniform by a truncation operator defined by Arthur. The 'naive' truncation of Selberg is defined to be

$$\tilde{E}(g,\varphi,s)=\begin{cases}E(g,\varphi,s), & y<T\\ E(g,\varphi,s)-\varphi(s)-(m(s)\varphi)(s), & y\geq T\end{cases}$$

whereas the Arthur truncation operator is

$$\Lambda^T E(g,\varphi,s)=E(g,\varphi,s)-\sum_{\Gamma_\infty\backslash\Gamma}E_N(\gamma g,\varphi,s)\chi_T(\log H(\gamma g))$$

where the subscript N denotes taking the constant term, and χ_T is the characteristic function of $[T,\infty)$. The two definitions coincide over the fundamental domain of $\Gamma\backslash\mathbf{H}_2$.

We will be concerned with the L^2-inner product of truncated Eisenstein series

$$(\Lambda^T E(g,\varphi_1,s),\Lambda^T E(g,\varphi_2,-\bar{s}))$$

which is computed to be, up to a scalar,

$$2(\varphi_1,\varphi_2)\log T+(M(s)^{-1}M'(s)\varphi_1,\varphi_2)+\frac{1}{2s}((\varphi_1,M(-\bar{s})\varphi_2)T^{2s}-(M(s)\varphi_1,\varphi_2)T^{-2s}) \tag{3.2.3}$$

This is sometimes referred to as the Maaß-Selberg relation (cf. Corollary 5.4.3), and using this one proves the analytic continuation of Eisenstein series. We will derive this formula in greater generality below.

3.2.1 Characters of induced representations

We now introduce the relevant integral transforms of test functions. Normalize the Haar measure on G, up to a constant, to be given by the decomposition

$$dg = dn\ dk\ h(a)^{-\frac{1}{2}}\frac{dy}{y}$$

where the measure $d^{\times}a$ on A corresponds to the measure dy/y on the multiplicative group $\mathbf{R}_{+}^{\times}$, and a depending implicitly on y. We shall also fix K to have unit measure.

Definition 3.2.6. Let f be in $C_c^{\infty}(\mathbf{H})$, that is to say, f is an element of the spherical Hecke algebra $C_c^{\infty}(K\backslash G/K)$, and define its *Harish transform* [Lan75, V.2] as

$$H(a)^{\frac{1}{2}}\int_N f(an)dn = |H(a)^{\frac{1}{2}} - H(a)^{-\frac{1}{2}}|\int_{A\backslash G} f(x^{-1}ax)dx.$$

where A is the subgroup of diagonal matrices. The transform is invariant under the action of the Weyl group W of G, and A/W can be represented by matrices a in A such that $H(a) \geq 1$. Thus changing variables $y = e^u$ so that $H(a) = e^{2u}$, we may write additively

$$g(u) = |e^u - e^{-u}|\int_{A\backslash G} f(x^{-1}\begin{pmatrix} e^u & 0 \\ 0 & e^{-u}\end{pmatrix}x)dx,$$

and observe that $g(u) = g(-u)$. The Harish transform is an algebra isomorphism from the space $C_c^{\infty}(K\backslash G/K)$ to $C_c^{\infty}(A)^W$, where the superscript indicates functions invariant under the Weyl group W, and the product given by convolution. Over a

p-adic field, its analog is known as the Satake isomorphism.

Definition 3.2.7. Next, we have the *Mellin transform* defined for g in $C_c^\infty(A)^W$,

$$\int_A g(a)H(a)^{\frac{s}{2}} d^\times a,$$

or, written additively it is the Fourier transform

$$\hat{g}(s) = \int_{-\infty}^{\infty} g(u)e^{us} du := h(r)$$

if we set $s = ir$ and identify g as a function of u rather than a. The image of the Mellin transform lies in the Paley-Wiener space, consisting of entire functions f for which there exists positive constants C and N such that

$$|f(x+iy)| \ll C^{|x|}(1+|y|)^{-N},$$

which is to say f has at most exponential growth with respect to x and is uniformly rapidly decreasing in vertical strips. Again, we note that h is an even function since g is even and real valued on $\mathbf{R}$.

Definition 3.2.8. Now, we claim that the Fourier transform can be realized as the trace of the restriction of the regular representation ρ to the continuous spectrum P_s^+, which we will denote by $\rho(\cdot, s)$. The corresponding integral operator acts on a function f in $C_c^\infty(G)$ with trace

$$\operatorname{tr}(\rho(f,s)) = \operatorname{tr}\int_G f(x)\rho(x,s)dx = \int_K\int_N\int_A f(k^{-1}n^{-1}ank)H(a)^{(1+s)/2}da\ dn\ dk;$$

so that if we give K unit measure, this is precisely the composition of the Mellin and Harish transform of f, and in particular $h(r) = \operatorname{tr}(\rho(f, ir))$. We also remind the reader here of the inverse transform

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{iru}dr$$

which will be useful later.

3.3 The spectral expansion

We now present the spectral side of Selberg trace formula. The spectral decomposition of $L^2(\Gamma\backslash\mathbf{H}_2)$ as in (2.2.1) specializes to

$$L^2_{\text{cusp}}(\Gamma\backslash\mathbf{H}_2) \oplus \mathbf{C} \oplus L^2_{\text{cont}}(\Gamma\backslash\mathbf{H}_2), \tag{3.3.1}$$

where the space of constant functions, represented by $\mathbf{C}$ is spanned by the residue of the Eisenstein series at $s = 1$.

This leads to the following eigenfunction expansion for any square integrable function:

$$f(x) = \sum_{n=0}^{\infty}(f, f_n)f_n(x) + \frac{1}{4\pi}\int_{-\infty}^{\infty}(f, E(x,1,\frac{1}{2}+ir))E(x,1,\frac{1}{2}+ir)dr$$

where the f_n form an orthonormal basis for eigenfunctions of $\rho(f)$, or equivalently, the Laplace operator, with the space of constant functions is represented by f_0, and we have also set $s = \frac{1}{2} + ir$. The integral involving the Eisenstein series represents the contribution of the continuous spectrum.

From this the automorphic kernel function takes the form

$$K(x,y) = \sum_{n=0}^{\infty} h(r_n) f_n(x) f_n(y) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) E(x,1,\frac{1}{2}+ir) E(y,1,\frac{1}{2}+ir) dr$$

where $\lambda_n = \frac{1}{4} + r_n^2$ are the eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ of the Laplacian. As mentioned above, the integral of this kernel over the fundamental domain is generally divergent. Rather, one isolates the discrete spectrum on one side of the trace formula, leaving an absolutely convergent expression on the other.

We translate this into representation theoretic language: the representation ρ restricted to the discrete spectrum leads to the decomposition

$$L^2_{\text{disc}}(\Gamma \backslash G/K) = \bigoplus_{\pi \text{ disc}} \pi^{m(\pi)}$$

where π is an irreducible subrepresentation of ρ occurring discretely in $L^2(\Gamma \backslash G/K)$ with multiplicity $m(\pi)$. Then the discrete contribution to the spectral side can be expressed as a sum of characters or representations

$$\sum_{\pi \text{ disc}} m(\pi) \operatorname{tr}(\pi(f)) = \sum_{\pi \text{ disc}} \int_G f(g) \Theta_\pi(g) dg = \sum_{n=0}^{\infty} h(r_n)$$

where in the second sum Θ_π is the character of the irreducible subrepresentation π of ρ occurring discretely, each corresponding to an eigenspace of the Laplacian.

There is one subrepresentation in the discrete spectrum that is not cuspidal, which is the trivial representation, corresponding to the residue of the Eisenstein series, and the summand $\mathbf{C}$ in (3.3.1). In this case, the kernel is computed as

$$\int_G \Lambda^T K_{\mathrm{res}}(x,x)dx = \int_G \mathrm{tr}(\mathbf{1}(f))\{1 - \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi_T(\log H(\gamma g))\}dg = \int_{D(T)} f(x)dx$$

where we have written $D(T)$ for the fundamental domain of $\Gamma\backslash\mathbf{H}_2$ truncated at height T.

For the contribution of the continuous spectrum we compute using the truncated kernel and consider the limit as T tends to infinity:

$$\begin{aligned}\int_{\Gamma\backslash G} \Lambda^T K_{\mathrm{cont}}(x,x)dx = &\log T \int_{-\infty}^{\infty} \mathrm{tr}(\rho(f,ir)dr - \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{m'}{m}(ir)\mathrm{tr}(\rho(f,ir))dr \\ &+ \frac{1}{4\pi i}\int_{-\infty}^{\infty} m(it)\mathrm{tr}(\rho(f,ir))\frac{T^{2ir}}{r}dr \end{aligned} \tag{3.3.2}$$

and note that for T sufficiently large, the last term is $\frac{1}{4}m(0)\mathrm{tr}(\rho(f,0)) + o(1)$, the dependence on T being negligible for T large by the Riemann-Lebesgue lemma. The formula, which follows from the Maaß-Selberg relation (3.2.3) is valid for $T > 1$, and the dependence on T is dominated by the $\log T$ term.

Remark 3.3.1. (The geometric expansion.) On the other hand, the geometric side can be written as the sum

$$\sum_{\{\gamma\}} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx$$

which for the moment we will not concern ourselves with, for now only mentioning that it can be separated into identity, regular elliptic, hyperbolic, and parabolic classes. Only the parabolic class is affected by the truncation operator, in which case its contribution is

$$\frac{1}{4}\mathrm{tr}(\rho(f,0)) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1+ir)\mathrm{tr}(\rho(f,ir)dr + \log\frac{T}{2}\int_{-\infty}^{\infty} \mathrm{tr}(\rho(f,ir)dr.$$

Most importantly, one sees that the $\log \frac{T}{2}$ term above cancels the $\log T$ term in the continuous spectral term (3.3.2).

3.4 Proof of Hejhal's formula

We are now in a position to derive the expression of Hejhal, given in Proposition 3.1.3. Note that Hejhal's normalization of the Eisenstein series is different from ours: the former follows Selberg, while ours is more reflective of the representation theory, though a routine change of variables allows us to pass from one to the other. The difference being whether to locate the center of the functional equation of the Eisenstein series on the critical line of $\zeta(s)$ or the unitary axis. In this section we follow the former. In particular, in this section we take $m(s) = \xi(2-2s)/\xi(2s)$ rather than $\xi(s)/\xi(1+s)$ above.

First, we treat the parabolic term [Hej76, 10.2]. The logarithmic derivative of the intertwining operator at the value $s=\frac{1}{2}+ir$ is

$$2\log\pi-\frac{\Gamma'}{\Gamma}(\frac{1}{2}-ir)-2\frac{\zeta'}{\zeta}(1-2ir)-\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir)-2\frac{\zeta'}{\zeta}(1+2ir).$$

One observes that the terms are integrated along $\mathrm{Re}(s)=0$, and along this line each of the individual terms are nonvanishing. Then using the assumption that $h(r)$ is even, by a change of variables we may combine the corresponding terms in the above to obtain the simplified expression

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}h(r)\frac{m'}{m}(\frac{1}{2}+ir)dr=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\Big(\log\pi-\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir)-2\frac{\zeta'}{\zeta}(1+2ir)\Big)dr.$$

The zeta function is on the edge of the critical strip, and we may view the contour as tracing a small semicircle to the right of the pole at $s=1$. Better yet, we shift the contour slightly to $1+\delta$ for some $\delta>0$, which now places us in the region of absolute convergence. Having absolute convergence, we denote by $\Lambda(n)$ the usual von Mangoldt function and expand the logarithmic derivative of the Euler product, then shift the contour back to $\mathrm{Re}(s)=1$, noting that the local factors have no pole in the region $\mathrm{Re}(s)>0$. Now, integrating term by term one has by Fourier inversion the expression as in Hejhal,

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)2\frac{\zeta'}{\zeta}(1+2ir)\ dr = 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n).$$

Finally, observing that $m(\frac{1}{2})$ is equal to -1 and thus cancels with the parabolic contribution to the geometric side, we arrive at the desired expression

$$g(0)\log\frac{\pi}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\Big(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\Big)\ dr + 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}g(2\log n).$$

It is important to note here that this parabolic term is the difference of the parabolic orbital integrals with the continuous spectral terms. The subtraction is made to isolate the discrete spectrum, and to obtain an absolutely convergent expression. Namely, the terms involving $m(s)$ are those which arise from the spectral theory, and the von Mangoldt function—a key connection to explicit formulae—originates there.

Remark 3.4.1. To emphasize the duality between the functions g and h, we may write the difference of the geometric side with the continuous contribution in terms of the $g(u)$. In particular, we have from [BS07, p.24]

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(\frac{1}{2}+ir)dr =& -\gamma g(0) + \int_0^{\infty}\log(u)g'(u)du\\ &-\int_0^{\infty}\log\Big(\frac{\sinh(\frac{u}{2})}{\frac{u}{2}}\Big)g'(u)du + 2\int_0^{\infty}\log\Big(\frac{\sinh(\frac{u}{4})}{\frac{u}{4}}\Big)g'(u)du\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr =& -\gamma g(0) + \int_0^{\infty}\log(u)g'(u)du\\ &+\frac{1}{4}\int_{-\infty}^{\infty} g(u)du + \int_0^{\infty}\log\Big(\frac{\sinh(\frac{u}{2})}{\frac{u}{2}}\Big)g'(u)du.\end{aligned}$$

Obtaining explicit expression as these are particularly useful for numerical computations.

Chapter 4

The trace formula for $PSL_2(F)$: ramification at real places

4.1 Preliminaries

In this chapter we consider the trace formula for PSL_2 over a number field F, allowing only ramification at real archimedean places. That is to say, allowing for K_v-finite representations for archimedean places v of F, and otherwise trivial at all finite and complex archimedean places. Specifically,

$$K_v = \begin{cases} SU_2(\mathbf{C}) & v \text{ complex} \\ SO_2(\mathbf{R}) & v \text{ real} \\ PSL_2(\mathcal{O}_{F_v}) & v \text{ finite} \end{cases} \tag{4.1.1}$$

where we remind the reader that $\mathcal{O}_{F_v} = \mathbf{Z}_p$ and $F_v = \mathbf{Q}_p$ if $F = \mathbf{Q}$. Throughout we will denote by d the degree of F over $\mathbf{Q}$, with r_1 real and $2r_2$ complex embeddings into $\mathbf{C}$, so that $d = r_1 + 2r_2$.

We thus identify $PSL_2(F)$ on the level of real points with

$$PSL_2(\mathbf{R})^{r_1} \times PSL_2(\mathbf{C})^{r_2}$$

which acts transitively on the symmetric space

$$\mathbf{H}_2^{r_1} \times \mathbf{H}_3^{r_2},$$

a product of copies of the upper half plane and the upper half space. Note that if we view $\mathbf{H}_3$ as a subset of the quaternions, then the action of $PSL_2(\mathbf{C})$ is given again by linear fractional transformations, and the subgroup $SU_2(\mathbf{C})$ is the stabilizer of this action, giving $\mathbf{H}_3 \simeq PSL_2(\mathbf{C})/SU_2(\mathbf{C})$ as desired.

Definition 4.1.1. More precisely, the action on $PSL_2(\mathbf{C})$ on $\mathbf{H}_3$ is as follows: Let a point P in $\mathbf{H}_3$ be written as $z + jw, w > 0$ in the quaternions, that is, with the fourth term being zero. Then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P &= \frac{aP+b}{cP+d} \\ &= \frac{a(z+jw)+b}{c(z+jw)+d} \\ &= \frac{(az+b)(\overline{cz+d}) + a\bar{c}w^2}{|cz+d|^2+|c|^2w^2} + j\frac{w}{|cz+d|^2+|c|^2w^2}. \end{aligned}$$

And we see that the stabilizer of $j = (0,1)$ in $\mathbf{H}_3$ are the matrices with $|c|^2 + |d|^2 = 1$ and $a\bar{c} + b\bar{d} = 0$. (See [EGM98, §1] for an introduction.)

Remark 4.1.2. Throughout, we shall make use of the identification $g = (g_1, \ldots, g_d)$

in $PSL_2(F)$ with $z = (z_1, \ldots, z_d)$ by the fractional linear transformation of g_i acting on the point $(0, 1)$ in $\mathbf{H}_2$, and $(0, 0, 1)$ in $\mathbf{H}_3$ (which leads to the stabilizers $SO_2(\mathbf{R})$ and $SU_2(\mathbf{C})$.

Now, the discrete subgroup

$$\Gamma = PSL_2(\mathcal{O}_F)$$

will be identified with its image under the $r_1 + r_2$ embeddings of F, and is cofinite, i.e., the quotient $\Gamma \backslash G$ has finite volume (but is not compact).

Definition 4.1.3. In this setting, we encounter the appearance of several cusps. Each cusp is stabilized by a parabolic subgroup of Γ, and we say two cusps are *equivalent* with respect to Γ if they are stabilized by parabolic subgroups of Γ that are conjugate to each other. As we shall see, the number of cusps that arise in the quotient in the action of $PSL_2(\mathcal{O}_F)$ on $\mathbf{H}_2^{r_1} \times \mathbf{H}_3^{r_2}$ will be equal to the class number of F. The finiteness of class numbers implies that the number of inequivalent cusps is finite. If h is the number of inequivalent cusps, denote by $\kappa_1, \ldots, \kappa_h$ the inequivalent cusps.

Remark 4.1.4. Then to each inequivalent cusp is associated an Eisenstein series, defined by the associated parabolic subgroup Γ_κ. Here Γ_κ is the subgroup of Γ that stabilizes the cusp κ. For the moment, denote by $E_i(z, s)$ the Eisenstein series for the cusp κ_i, and one may develop the Fourier series for $E_i(z, s)$ at the cusp

κ_j (cf. [Kub73, p.14]), and the constant term of this Fourier expansion is involves intertwining operator $m_{ij}(s)$. We may arrange the Eisenstein series into a column vector,

$$\begin{bmatrix} E_1(z,s) \\ \vdots \\ E_h(z,s) \end{bmatrix},$$

and the functional equation for Eisenstein series is given in terms of a constant term matrix of size $h \times h$, whose the ij-th entry is $m_{ij}(s)$ (see Definition 4.4.3 below, also [Sor02, Theorem 7.2]), which we denote by $\Phi(s)$. The square matrix $\Phi(s)$ is symmetric and invertible, initially defined for $\mathrm{Re}(s)$ large enough, and extends to $\mathbf{C}$ by meromorphic continuation.

From the point of view of representation theory, to each Γ_κ above we have a parabolically induced representation, and we have now a collection of intertwining operators m_{ij} which intertwine the various representations. This matrix $\Phi(s)$ is the intertwining operator $M(s)$ in the notation of [JL70, p.311], also [GJ79, p.223].

In the case of several cusps, the Maaß-Selberg relation which computes the inner product of truncated Eisenstein series involves the term

$$\Phi(s)^{-1}\Phi'(s),$$

and consequently in the trace formula one considers $\mathrm{tr}(\Phi(s)^{-1}\Phi'(s))$. The following lemma shows this is equal to the logarithmic derivative of the determinant of the constant term matrix.

Lemma 4.1.5. *Let $A(s)$ be an invertible $n \times n$ matrix. Then*

$$\frac{d}{ds} \log \det(A(s)) = \operatorname{tr}(A(s)^{-1} \frac{d}{ds} A(s))$$

wherever $A(s)$ is differentiable in s.

Proof. This result follows from the more general Jacobi's formula, which states that

$$\frac{d}{ds} \det A(s) = \operatorname{tr}(\operatorname{ad}(A(s)) \frac{d}{ds} A(s)) \tag{4.1.2}$$

where $\operatorname{ad}(A(s))$ is the adjugate matrix of $A(s)$, i.e., the transpose of the cofactor matrix of $A(s)$. Secondly, in the case where $A(s)$ is invertible, we have the relation

$$A(s)^{-1} = \frac{1}{\det(A(s))} \operatorname{ad}(A(s)).$$

Substituting this for $\operatorname{ad}(A(s))$ in (4.1.2) then yields the desired identity.

We now sketch a proof of the Jacobi formula: the determinant of a matrix can be expressed as the sum over the entries

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot \operatorname{ad}(a)^T_{ij}$$

where $\operatorname{ad}(a)^T_{ij}$ denotes the ij entry in the transpose of the adjugate $\operatorname{ad}(A(s))^T$. This sometimes referred to as Laplace's formula. Then taking the derivative we use the chain rule with respect to each entry $a_{ij}(s)$, which yields

$$\begin{aligned}\frac{d}{ds}\det(A(s)) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{ad}(a(s))_{ij}^{T}\frac{d}{ds}a_{ij}(s)\\ &= \mathrm{tr}(\mathrm{ad}(A(s))\frac{d}{ds}A(s))\end{aligned}$$

where the last equality follows from the relation

$$\mathrm{tr}(A^T B) = \sum_i (A^T B)_{ii} = \sum_i \sum_j a_{ij} b_{ij}$$

where A and B are any $n \times n$ matrices. □

In the following discussion we will be concerned with the term $\det \Phi(s)$, which involves a quotient of products of L-functions. This will yield the logarithmic derivatives as in the previous chapter.

4.2 Over Q

In his original paper, Selberg already considered the case of allowing $SO_2(\mathbf{R})$-finite functions by the usual representation of the compact group $\theta \mapsto e^{in\theta}$ [Sel56, p.457]. Adelically, this amounts to allowing for ramification at the archimedean place of $\mathbf{Q}$. The shape of the constant term depends on the K_v-type of the principal series representation φ_s, where $K_v = SO_2(\mathbf{R})$. If θ corresponds to the angle of rotation of an element k in $SO_2(\mathbf{R})$, we define the K_v-type

$$\varphi(k) = \varphi(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}) = e^{in\theta},$$

and the resulting Eisenstein series is given as

$$E(g,\varphi,s) = \sum_{P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} \varphi_s(\gamma g) = \sum_{\Gamma_\infty\backslash\Gamma} y(\gamma z)^{\frac{1+s}{2}} \left(\frac{\overline{cz+d}}{cz+d}\right)^n$$

where $G_{\mathbf{Q}} = PSL_2(\mathbf{Q})$, $P_{\mathbf{Q}}$ is the set of upper triangular matrices with entries in $\mathbf{Q}$, Γ_∞ is the stabilizer of the cusp at infinity with $z \in \mathbf{H}$, and $y(z)$ denotes the imaginary part of z. Then the constant term of $E(g,\varphi,s)$ is

$$\begin{aligned} m(s)_n &= (-1)^{\frac{n}{2}} \frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1+s}{2}+\frac{n}{2})} \frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1+s}{2}-\frac{n}{2})} \cdot \pi^{\frac{1}{2}} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1+s}{2})} \frac{\zeta(s)}{\zeta(1+s)} \\ &= (-1)^{\frac{n}{2}} \frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1+s}{2}+\frac{n}{2})} \frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1+s}{2}-\frac{n}{2})} m(s) \end{aligned} \tag{4.2.1}$$

which is to say, $m(s)$ times a quotient of Gamma functions (see [Kub73, p.69]), where $m(s)$ is defined as in (3.2.2). As the following proposition shows, the additional terms that appear in the continuous contribution to the trace formula can be made explicit, and do not greatly change the shape of the explicit formula obtained.

Proposition 4.2.1. *Let n be the highest weight representation of $SO_2(\mathbf{R})$, and let $h(r)$ and $g(u)$ be as in Definition 3.2.8. The contribution of*

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{m'}{m}(ir)_n h(r)dr$$

to the terms (3.3.2) in trace formula for $PSL_2(\mathbf{Z})\backslash PSL_2(\mathbf{R})$ *is*

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{m'}{m}(ir)h(r)dr$$

plus the additional terms

$$\frac{1}{2}\int_{-\infty}^{\infty}g(u)du-\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}g(u)(e^{-2mu}-e^{-(2m+1)u})du$$
$$+\sum_{k=1}^{N}\int_{-\infty}^{\infty}g(u)(e^{-2ku}-e^{2ku})\frac{du}{2}$$

if n *is odd, with* $n=2N+1$*, and*

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty}g(u)(e^{-2ku}-e^{(2k-1)u})\frac{du}{2}.$$

if n *is even, with* $n=2N$*.*

Proof. The denominator of $m(s)_n$ can be simplified thus: using the identity

$$\Gamma(z+1)=z\Gamma(z),$$

we have for any positive integer N,

$$\Gamma(z+N)=z(z+1)\dots(z+N-1)\Gamma(z),$$

and if we substitute $z-n$ for z,

$$\Gamma(z)=(z-N)\dots(z-2)(z-1)\Gamma(z-N),$$

so that

$$\begin{aligned}\Gamma(z+N)\Gamma(z-N) &= \frac{z(z+1)\dots(z+N-1)}{(z-1)(z-2)\dots(z-N)}\Gamma(z)^2 \\ &= \prod_{k=1}^{N}\frac{z+k-1}{z-k}\Gamma(z)^2\end{aligned}$$

To continue we must consider when n is even or odd.

Case 1: $n = 2N$. Then setting $z = (s+1)/2$, we have

$$\Gamma\Big(\frac{s+1}{2}+\frac{2N}{2}\Big)\Gamma\Big(\frac{s+1}{2}-\frac{2N}{2}\Big) = \prod_{k=1}^{N}\frac{s+2k}{s-2k+1}\Gamma\Big(\frac{s+1}{2}\Big)^2.$$

But the gamma factors cancel with the numerators in (4.2.1). So the additional terms in the logarithmic derivative here are

$$-\sum_{k=1}^{N}\Big(\frac{1}{s+2k}-\frac{1}{s-2k+1}\Big).$$

Then the contribution of this term to the trace formula is

$$\sum_{k=1}^{N}\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big(\frac{1}{ir+2k}-\frac{1}{ir-2k+1}\Big)\int_{-\infty}^{\infty}g(u)e^{iru}du\ dr.$$

Change orders of integration by Fubini's theorem,

$$\sum_{k=1}^{N}\frac{1}{4\pi}\int_{-\infty}^{\infty}g(u)\int_{-\infty}^{\infty}\Big(\frac{1}{ir+2k}-\frac{1}{ir-2k+1}\Big)e^{iru}dr\ du$$

and applying the residue theorem, we finally obtain

$$\sum_{k=1}^{N}\int_{-\infty}^{\infty}g(u)(e^{-2ku}-e^{(2k-1)u})\frac{du}{2}. \tag{4.2.2}$$

Case 2: $n = 2N + 1$, we have in that case

$$\Gamma\Big(\frac{s}{2} + \frac{2N}{2} + 1\Big)\Gamma\Big(\frac{s}{2} - \frac{2N}{2}\Big) = (\frac{s}{2} + N)\Gamma\Big(\frac{s}{2} + N\Big)\Gamma\Big(\frac{s}{2} - N\Big)$$
$$= \frac{s}{2}\prod_{k=1}^{N}\frac{s+2k}{s-2k}\Gamma\Big(\frac{s}{2}\Big)^2.$$

The gamma factors do not cancel with the denominator in this case. So the additional terms in the logarithmic derivative here are

$$\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big) - \frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) - \sum_{k=1}^{N}\Big(\frac{1}{s+2k} - \frac{1}{s-2k}\Big) - \frac{1}{s}.$$

Then using the Weierstrass product,

$$\frac{\Gamma'}{\Gamma}(s) = \frac{d}{ds}\ln(\frac{e^{-\gamma s}}{s}\prod_{m=1}^{\infty}\frac{e^{s/m}}{1+\frac{s}{m}})$$
$$= -\frac{1}{s} - \gamma + \sum_{m=1}^{\infty}(\frac{1}{m} - \frac{1}{m+s})$$
$$= -\gamma + \sum_{m=0}^{\infty}(\frac{1}{m+1} - \frac{1}{m+s}),$$

where γ is the Euler-Mascheroni constant, and we arrive at the expression

$$-\frac{1}{s} - \sum_{k=1}^{N}\Big(\frac{1}{s+2k} - \frac{1}{s-2k}\Big) + 2\sum_{m=0}^{\infty}(\frac{1}{2m+s} - \frac{1}{2m+s+1}),$$

the infinite sum converging when s is not a negative integer.

Then the contribution of this term to the trace formula is

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big\{\frac{1}{ir}+\sum_{k=1}^{N}\Big(\frac{1}{ir+2k}-\frac{1}{ir-2k}\Big)\Big\}\int_{-\infty}^{\infty}g(u)e^{iru}du\ dr$$
$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}2\Big\{\sum_{m=0}^{\infty}\Big(\frac{1}{2m+ir}-\frac{1}{2m+ir+1}\Big)\Big\}\int_{-\infty}^{\infty}g(u)e^{iru}du\ dr.$$

Again changing orders of integration by Fubini's theorem, and applying the residue theorem, we finally obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}g(u)du-\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}g(u)(e^{-2mu}-e^{-(2m+1)u})du \qquad (4.2.3)$$
$$+\sum_{k=1}^{N}\int_{-\infty}^{\infty}g(u)(e^{-2ku}-e^{2ku})\frac{du}{2}.$$

Note that the integrand in the last term can also be written as $-\sinh(2ku)$.

Thus we conclude that the expressions (4.2.2) and (4.2.3) give the additional terms for even and odd n respectively. □

4.3 Over a totally real number field

In this section let F be a totally real number field of degree d over $\mathbf{Q}$. This case was studied extensively by Efrat [Efr87], in the context of the Selberg trace formula.

By analogy with the previous chapter, we consider the action of $PSL_2(\mathbf{R})^d$ on d-copies of the upper-half plane $\mathbf{H}_2^d$, with stabilizer

$$K_\infty=\prod_{v|\infty}K_v=SO_2(\mathbf{R})^d.$$

We will consider discrete subgroups Γ of $PSL_2(\mathbf{R})^d$,

$$\Gamma = PSL_2(\mathcal{O}_F)$$

known as the Hilbert modular group, given by the d embeddings of F into $\mathbf{C}$. (See [Sie80, §3] for an introduction.) We shall first survey this intermediate case before considering the general setting of an arbitrary number field.

Definition 4.3.1. Now to each $i = 1, \ldots, d$, we associate a principal series representation in the sense of Definition 3.2.1, with a complex parameter s_i. The principal series representation for $PSL_2(\mathbf{R})^d$ is thus parametrized by complex d-tuples $\mathbf{s} = (s_1, \ldots, s_d)$ such that the unramified principal series

$$\varphi_{\mathbf{s}}(g) = \prod_{i=1}^{d} \varphi_{s_i}(g_i) = \varphi_{s_1}(g_1) \ldots \varphi_{s_d}(g_d)$$

satisfies

$$\varphi_{s_1}(g) \ldots \varphi_{s_d}(g) = y_1(z)^{(s_1+1)/2} \ldots y_d(z)^{(s_d+1)/2},$$

where $y_k(z) = y_k(z_1, \ldots, z_d) = y(z_k)$, and $z_k = g_k(i)$ as in §2.3. We will write this more succinctly as $y(z)^{(\mathbf{s}+1)/2}$. For the Eisenstein series to be well-defined in this case, the d-tuple $s_1, \ldots, s_d$ must satisfy the following relation (cf. [Efr87, p.40], [Sor02, p.13]):

Lemma 4.3.2. *Let* $\mathbf{s} = (\mathbf{s_1}, \ldots, \mathbf{s_d})$ *a complex d-tuple associated to a principal series* *representation of* $PSL_2(\mathbf{R})^d$. *Then for any* $k = 1, \dots, d$, *the* s_k*'s can be expressed as*

$$s_k = s + 2\pi i \sum_{j=1}^{d-1} m_j e_{kj} := s + \mathbf{e}_k \tag{4.3.1}$$

where e_{kj} *are entries in the inverse matrix given in (4.3.2). In particular, the consideration of* d *complex variables is reduced to one complex variables and* $d - 1 = \operatorname{rank} \mathcal{O}_F^{\times}$ *integers.*

Proof. This function is invariant under the stabilizer Γ_∞ in Γ of the cusp ∞,

$$\Gamma_\infty = \left\{ \begin{pmatrix} u & x \\ 0 & u^{-1} \end{pmatrix} : u \in \mathcal{O}_F^{\times}, x \in \mathcal{O}_F \right\}$$

being eigenfunctions of the Laplace operator. Then observing that for γ in Γ_∞,

$$y_i(\gamma z) = y_i\Big(\frac{uz + x}{0z + u^{-1}}\Big) = |u_i|^2 y_i(z),$$

as in Remark 3.2.2, we see that Γ_∞-invariance is equivalent to the condition that

$$|u_1|^{s_1+1} \dots |u_d|^{s_d+1} = 1$$

for any u_i in $\mathcal{O}_F^{\times}$, which means

$$(s_1 + 1)\log|u_1| + \cdots + (s_d + 1)\log|u_d| = 2\pi i m$$

for some integer m. It suffices to check this on any basis of $\mathcal{O}_F^{\times}$. Thus we have d complex variables with $d-1$ conditions, which gives one complex parameter s and $d-1$ integral parameters $\mathbf{m} = (m_1, \dots, m_{d-1})$, which we arrange into the following matrix equation:

$$\begin{pmatrix} \log|u_{1,1}| & \dots & \log|u_{1,d}| \\ \vdots & \ddots & \vdots \\ \log|u_{d-1,1}| & \dots & \log|u_{d-1,d}| \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_{d-1} \\ s_d \end{pmatrix} = \begin{pmatrix} 2\pi i m_1 \\ \vdots \\ 2\pi i m_{d-1} \\ rs \end{pmatrix}.$$

Then the e_{kj}'s satisfying (4.3.1) are entries in the invertible $d \times d$ matrix solving our system of equations,

$$\begin{pmatrix} e_{11} & \dots & e_{1,d-1} & 1/d \\ \vdots & \ddots & \vdots & \vdots \\ e_{d,1} & \dots & e_{d,d-1} & 1/d \end{pmatrix} := \begin{pmatrix} \log|u_{1,1}| & \dots & \log|u_{1,d}| \\ \vdots & \ddots & \vdots \\ \log|u_{d-1,1}| & \dots & \log|u_{d-1,d}| \\ 1 & \dots & 1 \end{pmatrix}^{-1} \tag{4.3.2}$$

and s is replaced by $2s$ for $k > r_1$ if F contains a complex archimedean place. □

Definition 4.3.3. Let m be in $\mathbf{Z}^{d-1}$. Following Efrat [Efr87, p.47], we define a Hecke character on F by evaluating at archimedean places $a_1, \dots, a_d$,

$$\chi_{\mathbf{m}}(a) = \prod_{k=1}^{d} |a_k|^{-\mathbf{e}_k} \tag{4.3.3}$$

which is trivial on $\mathcal{O}_F^\times$. Then for s in $\mathbf{C}$, define the *Eisenstein series* over a totally real field, associated to the cusp at infinity as

$$E(g, \varphi, s) = \sum_{P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} \varphi_{\mathbf{s}}(\gamma g) = \sum_{\Gamma_\infty \backslash \Gamma} \chi_{\mathbf{m}}(y(\gamma z)) y(\gamma z)^{(\mathbf{s}+1)/2}$$

where the s_k's satisfy the relation above, and g is related to z as in Remark 4.1.2.

Remark 4.3.4. We interpret the above representation theoretically as follows: the condition that m_j is nonzero in $\mathbf{m} = (m_1, \dots, m_{d-1})$ is equivalent to the representation $\varphi(k)$ being ramified at the archimedean place v_j, since from the definition of $\chi_{\mathbf{m}}$ we may lift it to a character on G by the map

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \chi(a),$$

nontrivial on the intersection $A \cap K$, where A is the subgroup of diagonal matrices, and K the maximal compact subgroup of $PSL_2(F)$ (see [God95, p.25].)

Definition 4.3.5. (Hecke character.) Let F be number field of degree $d = r_1 + 2r_2$, and $\mathfrak{f}$ a nonzero ideal of $\mathcal{O}_F$. Let also

$$\chi_\infty : (\mathbf{R}^\times)^{r_1} \times (\mathbf{C}^\times)^{r_2} \to \mathbf{C}^\times$$

be a continuous character. Then we define a Hecke character on the group of fractional ideals of F coprime to $\mathfrak{f}$ as

$$\chi : I(\mathfrak{f}) \to \mathbf{C}^\times$$

of conductor $\mathfrak{f}$ and infinity type χ_∞ if the restriction of χ to principal fractional ideals (a) congruent to 1 mod $\mathfrak{f}$ is determined by the relation

$$\chi((a)) = \chi_\infty^{-1}(a)$$

where $\mathbf{R} \otimes_{\mathbf{Q}} F$ is identified with $\mathbf{R}^{r_1} \times \mathbf{C}^{r_2} \to \mathbf{C}^\times$.

A Hecke character χ extends to an idèle class character $\prod_v \chi_v$ on $F^\times \backslash \mathbf{A}_F^\times$ as follows: the infinity type is determined by χ_∞, while for $v \nmid \mathfrak{f}$ we define χ_v by the condition

$$\chi_v(\varpi_v \mathcal{O}_v^\times) = \chi(\mathfrak{p}_v).$$

Definition 4.3.6. (Hecke L-function.) The *global* L-function associated to χ is

$$L(s,\chi) = \prod_v L_v(s,\chi_v)$$

the product taken over all places v of F. It satisfies the functional equation

$$L(s,\chi) = \varepsilon(s,\chi,\psi)L(1-s,\overline{\chi})$$

where the epsilon factor is defined as $\varepsilon(s,\chi,\psi) = W(\chi)|N_{F/\mathbf{Q}}(\mathfrak{f}(\chi))d_F|^{s-\frac{1}{2}}$, where $W(\chi)$ is the root number, $\mathfrak{f}(\chi)$ the conductor of χ, and ψ is a fixed additive character of F. We also define the *completed* L-function to be

$$\Lambda(s) = |N_{F/\mathbf{Q}}(\mathfrak{f}(\chi))d_F|^{s/2}L(s,\chi),$$

in which case the functional equation reads

$$\Lambda(s,\chi) = W(\chi)\Lambda(1-s,\overline{\chi}).$$

The local L-factors are given as follows:

$$L_v(s,\chi_v) = \begin{cases} \Gamma_{\mathbf{C}}(s) = (2\pi)^{1-s}\Gamma(s+w+\frac{|n|}{2}) & v \text{ complex} \\ \Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s+w+n}{2}) & v \text{ real} \\ (1-\chi_v(\mathfrak{p})N_{F/\mathbf{Q}}(\mathfrak{p})^{-s})^{-1} & v = \mathfrak{p} \text{ finite}, \chi_v \text{ unramified} \\ 1 & v = \mathfrak{p} \text{ finite}, \chi_v \text{ ramified} \end{cases}$$

where $\mathfrak{p}$ is a prime ideal of F, and the gamma factors are explained in Remark 4.3.8 below.

Given a finite set of places S, we will write $L^S(s,\chi)$ for the *partial* L-function, the product away from places in S. If S contains the archimedean places and ramified primes, then

$$L^S(s,\chi) = \prod_{v \notin S} L_v(s,\chi) = \sum_{(\mathfrak{a},S)=1} \frac{\chi(\mathfrak{a})}{N_{F/\mathbf{Q}}(\mathfrak{a})^s} = \zeta_F^S(s,\chi)$$

where $\zeta_F(s,\chi)$ is the usual L-series of χ, the sum taken over integral ideals $\mathfrak{a}$ prime to S. When χ is the trivial character, $\zeta_F(s,1)$ recovers the Dedekind zeta function of F. In the simplest case of $F = \mathbf{Q}$, we will simply write $\zeta_{\mathbf{Q}}(s) = \zeta(s)$, the Riemann zeta function.

Remark 4.3.7. (Trivial conductor.) Note that in this chapter we consider only characters χ that are unramified at every finite place of F, so the conductor $\mathfrak{f}(\chi)$ is trivial and the local root numbers $W(\chi_v)$ are trivial for v finite [Roh11, p.33].

Remark 4.3.8. (Ramified Gamma factors.) We continue the discussion in Remark (4.3.4), again following [God95, p.25]. The archimedean L-factors occur as follows:

If v is a real place, the field $F_v \simeq \mathbf{R}$ has no nontrivial automorphisms, thus a character χ_v of $F_v^\times$ can be identified with one of $\mathbf{R}^\times$, hence necessarily of the form

$$\chi_v(t) = \mathrm{sgn}(t)^n |t|^w, \qquad n \in \{0,1\}\ w \in \mathbf{C},$$

where $\mathrm{sgn}(t)$ is the usual sign of t, and one has

$$L_v(s, \chi_v) = \Gamma_{\mathbf{R}}(s + w + n).$$

If v is complex, the field $F_v \simeq \mathbf{C}$ has two possible identifications. Choosing one, the character χ_v will be a character of $\mathbf{C}^\times$, necessarily of the form

$$\chi_v(z) = \arg(z)^n |z|^{2w} \qquad n \in \mathbf{Z},\ w \in \mathbf{C},$$

where $\arg(z)$ is the complex argument $z/|z|_{\mathbf{C}} = z^{1/2}\bar{z}^{-1/2}$, giving

$$L_v(s, \chi_v) = \Gamma_{\mathbf{C}}(s + w + |n|/2).$$

We see that if we identify F_v with the complex conjugate, we replace $\chi_v(z)$ with $\chi(\bar{z})$ and thus m with $-m$, and $L_v(s, \chi_v)$ remains well-defined. We mention in passing that the duplication formula gives the pleasant relation

$$\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = \Gamma_{\mathbf{C}}(s).$$

Finally, if v_k is an archimedean place, we will write $\Gamma_k(s)$ for $\Gamma_{\mathbf{R}}(s)$ or $\Gamma_{\mathbf{C}}(s)$ depending on whether the completion F_{v_k} is real or complex.

We return to the setting of F a totally real field.

Definition 4.3.9. The scalar factor of the intertwining operator over F totally real is now the ratio of Hecke L-functions

$$m(s,\chi_{\mathbf{m}}) = \Big(\frac{\pi^d}{d_F}\Big)^{\frac{1}{2}} \prod_{k=1}^{d} \frac{\Gamma(\frac{s_k}{2})}{\Gamma(\frac{1+s_k}{2})} \frac{\zeta_F(s,\overline{\chi}_{\mathbf{m}})}{\zeta_F(1+s,\chi_{\mathbf{m}})}.$$

where the s_k's are as in (4.3.1) (cf. [Efr87, p.49]). Observe that the character $\chi_{\mathbf{m}}$ is trivial at finite places, and at infinite places ramified with $w = -\mathbf{e_k}$ as in Definition 4.3.3.

When $\mathbf{m} = (0,\dots,0)$, the K_∞-type is unramified everywhere, and we have exactly the quotient the completed Dedekind zeta functions, $\xi_F(s)/\xi_F(s+1)$, recalled below. In particular, in the case of only one cusp we see the logarithmic derivative of $\xi_F(2s-1)/\xi_F(2s)$, similar to the case of $\mathbf{Q}$. There is only one pole at $s = 1$, and the computations of the previous section can be applied.

Remark 4.3.10. Throughout we will assume that our test functions are factorizable, that is, $h(r_1,\dots,r_d) = \prod h_k(r_k)$. This will be useful for the adelic setting, and from the point of view of representation theory. As before, we interpret $h(r_k)$ as the local character

$$h(r_k) = \mathrm{tr}(\rho_{v_k}(ir_k,f)) := \hat{g}_k(ir_k), \tag{4.3.4}$$

for a given function f in $C_c^\infty(G)$, and ρ_{v_k} the regular representation on the induced representation at an archimedean completion F_{v_k}.

We have following spectral decomposition into invariant subspaces, in the notation of [Efr87, p.74]:

$$L^2(PSL_2(\mathcal{O}_F)\backslash \mathbf{H}_2^d) = C \oplus R \oplus E$$

where C denotes the cuspidal spectrum, R the finite-dimensional subspace generated by the residues of Eisenstein series, and E the continuous spectrum, which decomposes further into $\oplus E_m$, the direct sum taken over all m in $\mathbf{Z}^{n-1}/\{\pm 1\}$.

The continuous spectral terms in the trace formula for $PSL_2(\mathbf{R})^d$, corresponding to the subspace E, is then given as follows:

Theorem 4.3.11 ([Efr87, p.86]). *Let F be a totally real field of degree d over $\mathbf{Q}$ with class number 1, and $\hat{g}(s)$ as in (4.3.4). Then the contribution of the continuous spectrum to the Selberg trace formula for $PSL_2(\mathcal{O}_F)\backslash PSL_2(F)$ is given by*

$$-\frac{1}{4\pi}\sum_{\mathbf{m}\in\mathbf{Z}^{d-1}}\int_{-\infty}^{\infty}\frac{m'}{m}(ir,\chi_{\mathbf{m}})\prod_{k=1}^{d}\hat{g}_k(ir+\mathbf{e}_k)dr+\frac{1}{4}m(0,\mathbf{0})\hat{g}(0) \tag{4.3.5}$$

plus the terms depending on the truncation parameter $T > 0$, as $T \to \infty$,

$$+\ 2^{d-1}R_F\log T\sum_{\mathbf{m}\in\mathbf{Z}^{d-1}}\prod_{k=1}^{d}g_k(2\log u_{m,k})+o(1)$$

where R_F is the regulator of F and $u_{m,k}$ are units in $\mathcal{O}_F^{\times}$.

As in the previous chapter, we would like to examine the integral involving the intertwining operator in (4.3.5), beginning first with the unramified case, that is, when $\mathbf{m} = (0, \ldots, 0)$, and then followed by the ramified case, when $\mathbf{m}$ is not exactly zero. In particular, if m_k is nonzero, then the representation is ramified at the infinite place v_k. The Weil distributions in [Wei72] will appear through this term, but rather than proving this here, we postpone to the more general case of the next section, which will include this as a special case.

Remark 4.3.12. Finally, before we pass to an arbitrary number field, we should mention the case of $F = \mathbf{Q}(\sqrt{-D})$ an imaginary quadratic field, considered by Efrat and Sarnak [ES85]. In this case we have $PSL_2(\mathbf{C})$, which under $SU_2(\mathbf{C})$ invariance is identified with hyperbolic 3-space $\mathbf{H}_3 \simeq PSL_2(\mathbf{C})/SU_2(\mathbf{C})$, and $\Gamma = PSL_2(\mathcal{O}_F)$ is referred to as the Bianchi modular group.

They prove that the number of inequivalent cusps of $\Gamma\backslash\mathbf{H}_3$ is equal to the class number h of F, and the determinant of the constant term matrix in the unramified setting is given by

$$(-1)^{(h-2^{t-1})/2}(\frac{2}{d_F^{\frac{1}{2}}})^s\frac{\xi_H(s)}{\xi_H(s+1)}$$

where t is the number of prime divisors of F, and $\xi_H(s) = (d_F^{\frac{s}{2}}/(2\pi)^{sh})\Gamma(s)^h\zeta_H(s)$ is the completed Dedekind zeta function of the Hilbert class field H of F. The results were generalized by Sørenson [Sor02], who proves the case of a general number field, which we now turn to.

4.4 Over any number field

Now let us pass to a finite extension F over $\mathbf{Q}$ of degree $d = r_1 + 2r_2$. Then one considers the group $PSL_2(\mathbf{R})^{r_1} \times PSL_2(\mathbf{C})^{r_2}$, acting on

$$\mathbf{H}_2^{r_1} \times \mathbf{H}_3^{r_2}$$

with the discrete subgroup $PSL_2(\mathcal{O}_F)$ and the maximal compact subgroups $K_\infty = SO_2(\mathbf{R})^{r_1} \times SU_2(\mathbf{C})^{r_2}$ stabilizing the points $(0, 1)$ and $(0, 0, 1)$ respectively depending on whether the completion is real or complex. As before, the number of (inequivalent) cusps of $\Gamma \backslash G/K$ is in bijection with the class number h. In particular, this number is finite, and we will now consider general case of h not necessarily equal to one.

Definition 4.4.1. The Hecke character $\chi_{\mathbf{m}}$ given by (4.3.3) is trivial on $\mathcal{O}_F^\times$, and we may extend it to the ideal class group, i.e., principal ideals modulo $\mathcal{O}_F$ in h many ways. Given a cusp κ_i with stabilizer Γ_i, and a Hecke character χ mod $\mathcal{O}_F$ extending $\chi_{\mathbf{m}}$, we define the Eisenstein series as in Sørenson [Sor02, p.15],

$$E_i(z, s, \chi) = \frac{N_{F/\mathbf{Q}}(\mathfrak{a})^{s+1}}{\chi(\mathfrak{a})} \sum_{\gamma \in \Gamma_i \backslash \Gamma} \prod_{k=1}^{r_1+r_2} y_k(\sigma_i^{-1} \gamma z)^{(s_k+1)/2}$$

where s_k's are as before, σ_i is any scaling matrix for κ that sends the cusp ∞ to κ, and $\mathfrak{a}$ is any integral ideal in the ideal class C_κ associated to κ (cf. [Sor02, p.8]).

Here we have written $y_k(z)$, where

$$y_k(z) = y_k(a + ib) = b_k$$

where $a_k \in \mathbf{R}$ or $\mathbf{C}$ depending on whether z_k belongs to $\mathbf{H}_2$ or $\mathbf{H}_3$, and b_k the last coordinate with $b_k > 0$. Note that this definition of Eisenstein series is independent of choice of σ, but does depend on choice of basis of $\mathcal{O}_F^\times$, and differs slightly from that of [Efr87] and [ES85].

Remark 4.4.2. If F has class number h, then the quotient space

$$PSL_2(\mathcal{O}_F)\backslash(\mathbf{H}_2^d \times \mathbf{H}_3^{r_2})$$

is noncompact with h cusps, and to each cusp κ is associated a parabolic subgroup Γ_κ, and for each Γ_κ we have a different Fourier expansion with respect to this Γ_κ. In particular, we see that the s_k's defined in (4.3.1) depend on the choice of stabilizer of cusp. So in the case of h cusps we shall denote $s_k^{(i)} = s + \mathbf{e}_k^{(i)}$ to be the expression for the complex d-tuple associated to the cusp κ_i.

Definition 4.4.3. To each inequivalent cusp is associated an Eisenstein series, which we may put together to form the Eisenstein vector:

$$\vec{E}(z, s, \chi) = \begin{bmatrix} E_1(z, s, \chi) \\ \vdots \\ E_h(z, s, \chi) \end{bmatrix}$$

From this we form the matrix $\Phi(s)$, called the scattering matrix, whose entries $m_{ij}(s)$

are obtained from the constant terms of the Fourier expansion at the cusp κ_j of the Eisenstein series associated to the cusp κ_i. This matrix gives the functional equation for the Eisenstein vector:

$$\vec{E}(z,s,\chi) = \Phi(s,\chi)\vec{E}(z,1-s,\chi)$$

where we note that χ depends on $\mathbf{m}$. One also has the pleasing formula for the determinant of the scattering matrix, for which we use the same notation as before, that is,

$$\begin{aligned} m(s,\chi_{\mathbf{m}}) &:= \det \Phi(s,\chi_{\mathbf{m}}) \\ &= \operatorname{sgn}(C_F) \prod_{i=1}^{h} \frac{L(s,\chi_i)}{L(s+1,\chi_i)} \end{aligned} \tag{4.4.1}$$

where the product runs over Hecke characters mod $\mathcal{O}_F$ extending $\chi_{\mathbf{m}}$; the sign of the inversion map of class group is equal to $(-1)^{(h-h_2)/2}$, where $h_2 = |C_F[2]|$ the size of the two-torsion of the class group. The inversion map on the class group is the map on fractional ideals

$$\mathfrak{a} \mapsto \mathfrak{a}^{-1},$$

and $\operatorname{sgn}(C_F)$ is the sign of the permutation induced on C_F by the inversion. It is this determinant that we shall use in the trace formula.

In particular when $\mathbf{m} = (0, \dots, 0)$, we have

$$m(s, \chi_{\mathbf{0}}) = \operatorname{sgn}(C_F) \frac{\xi_H(s)}{\xi_H(1+s)}$$

where $\xi_H(s)$ is the completed Dedekind zeta function

$$2^{h_F r_2 (1-s)} \Big(\frac{|d_H|}{\pi^{h_F r}}\Big)^{\frac{s}{2}} \Gamma(\frac{s}{2})^{h_F r_1} \Gamma(s)^{h_F r_2} \zeta_H(s)$$

of the Hilbert class field H of F, with $r = r_1 + r_2$, d_H the discriminant, h_F the number of inequivalent cusps, or equivalently, the class number of F. The zeta function of H arises from its Artin factorization into a product of Hecke L-functions associated to characters of $C_F \simeq \mathrm{Gal}(H/F)$.

We record here the logarithmic derivative of $m(s, \chi_{\mathbf{m}})$ in the following form:

Lemma 4.4.4. *Let F be an arbitrary number field. The logarithmic derivative of the intertwining operator, i.e., the determinant of the scattering matrix (4.4.1) is*

$$-h \log |d_F| - \sum_{j=1}^{h} \Big(\sum_{k=1}^{r_1+r_2} \frac{\Gamma_k'}{\Gamma_k}(1-s_k) + \frac{\Gamma_k'}{\Gamma_k}(1+s_k) + \frac{\zeta_F'}{\zeta_F}(1-s, \overline{\chi}_j) + \frac{\zeta_F'}{\zeta_F}(1+s, \chi_j) \Big) \quad (4.4.2)$$

where χ_i runs over Hecke characters mod $\mathcal{O}_F$ extending $\chi_{\mathbf{m}}$, and $s_k = s + \mathbf{e}_k$.

Proof. From [Neu99, p.501] we see that if the archimedean L-factor is

$$L_{v_k}(s, \chi_v) = \Gamma_k(s + w),$$

then the to the conjugate character we have

$$L_{v_k}(s, \bar{\chi}_v) = \Gamma_k(s + \bar{w}),$$

and in particular, each $\chi_j, j = 1, \ldots, h$ extends $\chi_{\mathbf{m}}$ and thus have ramification at infinite places indexed by $w = \mathbf{e}_k$ for each infinite place F_{v_k}, thus we associate $\Gamma_k(s + \mathbf{e}_k)$ with $\Gamma_k(s - \mathbf{e}_k)$, since $\mathbf{e}_k$ is pure imaginary.

Now, using the definitions and functional equation of the completed L-function we have the following expression for $m(s, \chi_{\mathbf{m}})$:

$$\begin{aligned} m(s, \chi_{\mathbf{m}}) &= \operatorname{sgn}(C_F) \prod_{j=1}^{h} \frac{L(s, \chi_j)}{L(s+1, \chi_j)} \\ &= \operatorname{sgn}(C_F) \prod_{j=1}^{h} \frac{W(\chi) L(1-s, \bar{\chi}_j)}{L(s+1, \chi_j)} \\ &= (-1)^{(h-h_2)/2} \prod_{j=1}^{h} \Big(\frac{W(\chi)}{|d_F|^s} \Big) \prod_{k=1}^{d} \Big\{ \frac{\Gamma_k(1-s_k) \zeta_F(1-s, \bar{\chi}_j)}{\Gamma_k(1+s_k) \zeta_F(1+s, \chi_j)} \Big\}, \end{aligned}$$

where the product is taken over all possible extensions $\chi_{\mathbf{m}}$ to C_F, which correspond to twisting by ‘unramified’ characters of the class group

$$C_F \to \mathbf{C}^{\times}.$$

Then the logarithmic derivative gives the formula immediately.

$\square$

We now present the analogue of Theorem 4.3.11 for an arbitrary number field, with possibly ramification at infinity.

Theorem 4.4.5. *Let F be a number field of degree d over $\mathbf{Q}$ with class number h. Denote by $\hat{g}(ir) = \prod \hat{g}_k(ir_k)$ as before the character $\mathrm{tr}(\rho(f, ir))$ of the regular representation restricted to the principal series representation. The contribution of the continuous spectrum to the Selberg trace formula for $G = PSL_2(\mathbf{R})^{r_1} \times PSL_2(\mathbf{C})^{r_2}$ and $\Gamma = PSL_2(\mathcal{O}_F)$ is given by*

$$\sum_{i,j=1}^{h} \sum_{\mathbf{m} \in \mathbf{Z}^{d-1}} -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{m'_{ij}}{m_{ij}}(ir, \chi_{\mathbf{m}}) \prod_{k=1}^{d} \hat{g}_k(ir + \mathbf{e_k}^{(i)}) dr + \frac{1}{4} \mathrm{tr}(\Phi(0, \chi_{\mathbf{0}})) \hat{g}(0) \quad (4.4.3)$$

plus the terms depending on the truncation parameter $Y > 0$, with $\mathbf{e}_k$ defined as before.

Proof. The formula for several cusps is given in [Efr87, p.103] for a totally real field; we see that the derivation holds in the general case through the analysis in [JL70] and [GJ79]. Then the expression for the integral is derived the same way as in the totally real case. □

Remark 4.4.6. An other method using representation theory is as follows. To allow for ramification at any archimedean places for F a degree d extension of $\mathbf{Q}$, let m_j be the highest weight of the representation of the maximal compact at each archimedean place v_j. The intertwining operator is

$$\prod_{j=1}^{d} \prod_{k=0}^{|m_j|-1} \frac{\mu_j(1-s)+k}{\mu_j s + k} m(s)$$

where μ_j is 1 if the completion at v is real, and 2 if it is complex. Certainly this expression is less transparent, and is derived using the theory of highest weight representations, and raising and lowering operators [KM12]. On the other hand, it has the advantage of a simpler expression for the logarithmic derivative.

Chapter 5

The trace formula for $PSL_2(F)$: arbitrary ramification

5.1 Preliminaries

Let now F be an arbitrary number field, and $\mathbf{A}_F$ the ring of adeles of F, which for simplicity we will sometimes simply write $\mathbf{A}$. Here, rather than considering the quotients of the form

$$SL_2(\mathcal{O}_F)\backslash SL_2(F)/K,$$

we consider the adelic quotient

$$SL_2(F)\backslash SL_2(\mathbf{A}_F)/\prod_v{}' K_v.$$

the product taken over all places v of F, and the K_v's are defined in (4.1.1). For the reason that $F^\times$ embeds discretely into $\mathbf{A}_F^\times$, so $SL_2(F)$ is a discrete subgroup of $SL_2(\mathbf{A}_F)$. Then, setting $G = SL_2$, the decomposition of the discrete spectrum of

$L^2(G_{\mathbf{Q}}\backslash G_{\mathbf{A}})$ can be expressed as

$$L^2_{\text{disc}}(G_{\mathbf{Q}}\backslash G_{\mathbf{A}}) = \bigoplus_{\pi} \Big(\bigotimes_{v}{}' \pi_v \Big)$$

where each π corresponds to an eigenfunction f_n, and π_v is a local representation, unramified for almost all primes v of $\mathbf{Q}$. As before, our focus will be on the continuous spectrum, which is the orthogonal complement of the discrete spectrum.

In this chapter, we introduce the adelic trace formula, which is most amenable to considering arbitrary ramification. We encounter ramification at finite places when we allow for non-trivial K_v-representations. In the case of $\mathbf{Q}$, adelically this means nontrivial $SL_2(\mathbf{Z}_p)$-representations, which in the classical case amounts to choosing Γ to be a congruence subgroup of SL_2 (cf. Chapter 2).

The main result of this chapter is a proof of the following statement, which is a special case of the formula stated in [LL79, p.754] without proof:

Theorem 5.1.1. *Let F be a number field. Then the contribution of the continuous spectrum to the trace formula for $PSL_2(\mathbf{A}_F)$ can be written, with definitions given in §5.3, as*

$$\sum_{\eta^2=1} -\frac{1}{4}\text{tr}(M(\eta,0)\rho(f,\eta,0))$$

taken over quadratic characters of $F^\times\backslash\mathbf{A}_F$, and a sum over characters

$$\sum_{\eta} \frac{1}{4\pi} \int_{-i\infty}^{i\infty} m(\eta,s)^{-1} m'(\eta,s) \text{tr}\rho(f,\eta,s)|ds|,$$

and

$$\sum_{\eta}\sum_{v}\frac{1}{4\pi}\int_{-i\infty}^{i\infty}\mathrm{tr}(R^{-1}(\eta_v,s)R'(\eta_v,s)\rho(f_v,\eta_v,s))\prod_{w\neq v}\mathrm{tr}(\rho(f_w,\eta_w,s))|ds|,$$

with only finitely many nonzero terms appearing in each sum.

The proof of this theorem will take as its starting point the adelic formulation of the trace formula, which requires the formulation of the trace formula used by Labesse and Langlands [LL79] who studied the inner forms of SL_2 using the stablilized trace formula. Nonetheless, in the first section we shall review the classical case, where ramification at finite primes is considered through the congruence subgroups Γ.

The advantage of the adelic viewpoint is that it will treat archimedean and nonarchimedean places uniformly, which is better suited to allowing for arbitrary ramification. One possible disadvantage, depending on the reader's viewpoint, is that in the adelic situation the cusps of the resulting quotient space become do not appear explicitly, and the Eisenstein matrix is simply presented as an intertwining operator.

We point out to the reader that the contents of this chapter follow directly from what is already in the literature; the main contribution of this chapter is a description of the continuous spectral terms in the trace formula for $PSL_2(F)$ in the adelic form which, from the point of view of representation theory, is more natural and suitable for analysis. In §5.2 we describe ramification at finite primes in the classical setting, to complement the discussions of the previous chapters. Then in §5.2 we state the continuous spectral terms in the trace formula of Langlands and Labesse, and in §5.3 we prove the Maaß-Selberg relation as promised.

5.2 Over Q

In this section we review the classical trace formula for congruence subgroups. Taking Γ to be a congruence subgroup, the resulting quotient $\Gamma\backslash\mathbf{H}_2$ will have more than one cusp in general, in which case the interpretation of the number of cusps as the class group of the field of the previous chapter no longer applies.

First, consider $\Gamma = \Gamma_0(N)$ with square free $N = p_1 \dots p_r$, then the Eisenstein matrix is computed by Hejhal in [Hej76, 10.3] to be

$$m(s)(N_{p_1}(s) \otimes \cdots \otimes N_{p_r}(s)), \qquad N_p(s) = \frac{1}{p^{2s}-1}\begin{bmatrix} p-1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p-1 \end{bmatrix},$$

and the expression for the parabolic term is given as 2^r times the parabolic term (3.1.1) in the trace formula with $\Gamma = PSL_2(\mathbf{Z})$, minus the additional terms

$$2^r g(0)\log N + 2^r \sum_{p \nmid N} \sum_{n=p^k} \frac{\Lambda(n)}{n} g(2\log n).$$

Similar but much more complicated expressions are also given for $\Gamma_1(N)$ and $\Gamma_2(N)$, where again N is squarefree, which we refer the reader to Hejhal for the precise expression. We point out that the sum that appears here should be thought of as replacing the logarithmic derivative of $\zeta(s)$ in the case of $\Gamma = SL_2(\mathbf{Z})$ with a suitable

Hecke L-function $L(s,\chi)$.

Here is a more uniform presentation by Huxley [Hux84]: for any congruence subgroup $\Gamma_i(N)$, the determinant of the Eisenstein matrix is given by

$$\det\Phi(s) = (-1)^{\frac{k-k_0}{2}}\frac{\Gamma(1-s)^k}{\Gamma(s)^k}(\frac{A}{\pi^k})^{1-2s}\prod_{\chi}\frac{L(2-2s,\bar{\chi})}{L(2s,\chi)}$$

where as before k is the number of inequivalent cusps, and k_0 is the limit $\mathrm{tr}(\Phi(s))$ as s tends to $\frac{1}{2}$, equal to the number of characters χ for which the Dirichlet L-function has a pole at $s=1$, and A is a product of integers related to N. The product runs over Dirichlet characters with modulus dividing N.

Then the contribution of the continuous spectrum to the trace formula

$$\int_{\Gamma\backslash G}\Lambda^T K_{\mathrm{cont}}(x,x)dx$$

with respect to the truncation parameter T is

$$kg(0)T - \frac{1}{4}h(0)\mathrm{tr}(\Phi(\frac{1}{2})) - \frac{1}{4\pi i}\int_{-\infty}^{\infty}h(r)\frac{d}{dr}\log\det\Phi(\frac{1}{2}+ir)dr + O(e^{-T}).$$

From more general work of Arthur, one knows that this expression is polynomial in T, and that we may concern ourselves with only its constant coefficient. This expression can be evaluated by a method similar to that of Hejhal outlined in Chapter 3 to

$$g(0)T^k\log\frac{A}{\pi^k} + \frac{k}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}(-\frac{1}{2}+ir)dr + \frac{1}{4}h(0)k_0 - 2\sum_{\chi}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\chi(n)g(2\log n).$$

The reader should compare this expression to the one for $PSL_2(\mathbf{R})/SO_2$ of Hejhal, taking the parameter $T = 0$. According to Huxley, this matches with the formulae given by Hejhal up to the sign of $\mathrm{tr}(\Phi(s))$.

We note the similarities between the expressions obtained for ramification at archimedean and nonarchimedean places: there is the sign depending on the number of k and k_0, the appearance of additional gamma factors depending on k, and a finite product of Hecke L-functions.

Theorem 5.2.1 ([Hux84, p.154]). *Let $h(r)$ be an analytic function in the strip $|Im(r)| \leq 1/2 + \delta$ and $|h(r)| \ll |r|^{-2-\delta}$, and let $g(u)$ be its Fourier transform. Then the Selberg trace formula for $PSL_2(\mathbf{R})$ with $\Gamma = \Gamma_i(N)$ for any $i = 0, 1, 2$ is*

$$\sum_{n=1}^{\infty} h(r_n) = \frac{\mathrm{vol}(\Gamma\backslash\mathbf{H}_2)}{4\pi} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr + \sum_{\{P\}} \frac{\log NP_0}{NP^{\frac{1}{2}} - NP^{-\frac{1}{2}}} g(NP)$$
$$+ \sum_{\{E\}} \sum_{k=1}^{m-1} \frac{1}{m \sin(\pi k/m)} \int_{-\infty}^{\infty} \frac{e^{-2\pi kr/m}}{1 + e^{-2\pi r}} h(r) dr + \dots$$

plus the difference of the parabolic term and the contribution of the continuous spectrum:

$$g(1) \log(\frac{2^k A}{\pi^k}) + \frac{k}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1 + ir) dr + \frac{1}{4} h(0)(k + k_0) - 2 \sum_{\chi} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \chi(n) g(n^2)$$

where $m(s) = \det \Phi(s)$ and k denotes the number of inequivalent cusps.

We refrain from defining all the terms used in the formula above, as we shall not use this formula in what follows.

5.3 Over a number field

Now let F be an arbitrary number field of degree d. From this point onwards we turn our attention to the adelic form of the trace formula. The first instance of the adelic trace formula was derived by Jacquet and Langlands in [JL70, §16] for the group GL_2, and shortly afterwards Labesse and Langlands [LL79, §5] considered the trace formula for certain subgroups

$$SL_2 \subset G' \subset GL_2$$

which we define below. Their motivation was to extend the Jacquet-Langlands correspondence in [JL70, §16], which relates automorphic forms on division algebras and on GL_2, to other groups and their inner forms. This, and also the study of zeta functions of Shimura varieties, necessitated a stabilized form of the trace formula. That is, the distributions occurring in the trace formula could be written as a sum of stable distributions, i.e., stable under conjugacy over the algebraic closure, plus certain 'error terms' on the geometric side indexed by so-called endoscopy groups.

We first recall the contribution of the continuous spectrum to the trace formula for G'. In order to do so we introduce some notation and definitions.

Definition 5.3.1. We will define the group $G'_{\mathbf{A}}$, for which we will formulate the adelic trace formula. Let $A = \prod' A_v$ be a closed subgroup of the ring of ideles $\mathbf{A}_F^\times$, such that

1. A_v is a closed subgroup of $F_v^\times$,

2. $F^\times A$ is closed in $\mathbf{A}_F^\times$,

3. If B is an open subgroup then $[A : A^2(A \cap F^\times)B]$ is finite.

Then we define the intermediate group

$$G'_{\mathbf{A}} = \{g \in GL_2(\mathbf{A}_F) : \det(g) \in A\},$$

which, as a side remark, may not be the F points of an algebraic group. Furthermore, let Z'_0 be a closed subgroup of the center $Z'_{\mathbf{A}}$ of $G'_{\mathbf{A}}$ with $Z'_0F^\times$ closed and $Z'_0(Z'_{\mathbf{A}} \cap F^\times)\backslash Z'_{\mathbf{A}}$ compact, and let χ be a character of Z'_0 trivial on $Z'_0 \cap F^\times$ and absolute value one.

Definition 5.3.2. Consider measurable functions φ on $G'_{\mathbf{A}}$ modulo $G'_F := G'_{\mathbf{A}} \cap GL_2(F)$, satisfying

1. $\varphi(zg) = \chi^{-1}(z)\varphi(g)$ for all $z \in Z'_0$,

2. $\int_{Z'_0G'_F\backslash G'} |\varphi(g)|^2 dg < \infty$,

and denote the space of such functions $L^2(G'_F\backslash G'_{\mathbf{A}}, \chi)$, and let ρ be the regular representation of $G'_{\mathbf{A}}$ on this space. By the theory of Eisenstein series (cf. [Lan89]), it decomposes into a direct sum of irreducible representations, and a continuous direct integral of irreducible representations constructed by Eisenstein series, that is, the principal series representations.

Definition 5.3.3. To describe the principal series for $G'_{\mathbf{A}}$, we first describe the principal series for GL_2. Given a pair of idèle class characters (μ, ν) of $F^\times \backslash \mathbf{A}^\times$ such that $\mu\nu = \chi$, define the space $B(\mu\alpha^{\frac{s}{2}}, \nu\alpha^{-\frac{s}{2}})$ of functions of complex valued functions φ on $GL_2(\mathbf{A})$ such that

$$\varphi(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} k) = \mu(a)\nu(b)\left|\frac{a}{b}\right|^{\frac{1+s}{2}} \varphi(k)$$

where s is pure imaginary, and $\alpha(x) = |x|$ the adelic norm. Such functions are determined by their value on the maximal compact subgroup K, and independent of s. Note that if for some v, the characters μ_v, ν_v are unramified then $B(\mu\alpha^{\frac{s}{2}}, \nu\alpha^{-\frac{s}{2}})$ contains a unique K_v-invariant function.

Denote by $\rho(g, \eta, s)$ the action of $G_{\mathbf{A}}$ on this space by right translation, and define an equivalence relation between pairs (μ, ν) and $(\mu\alpha^{\frac{s}{2}}, \nu\alpha^{-\frac{s}{2}})$.

The intertwining operator $M(s)$ is a linear transformation from the principal series representation $B(\mu\alpha^{\frac{s}{2}}, \nu\alpha^{-\frac{s}{2}})$ to $B(\mu\alpha^{-\frac{s}{2}}, \nu\alpha^{\frac{s}{2}})$, satisfying the usual relation $M(s)M(-s) = 1$. The operator acts independently on each $B(\mu\alpha^{\frac{s}{2}}, \nu\alpha^{-\frac{s}{2}})$, so it decomposes as

$$M(s) = \oplus_\eta M(\eta, s),$$

where

$$M(\eta, s) = \frac{L(1-s, \nu\mu^{-1})}{L(1+s, \mu\nu^{-1})} \otimes_v R(\eta_v, s)$$

and $R(\eta_v, s)$ is the normalized local intertwining operator. It is holomorphic for

$\mathrm{Re}(s) \geq 0$, and unitary if μ_v, ν_v are.

Now, the condition $\mu\nu = \chi$ implies that the pair must have a representative $(\eta\alpha^{\frac{s}{2}}, \eta^{-1}\chi\alpha^{-\frac{s}{2}})$ which is equivalent to $(\eta, \eta^{-1}\chi)$ and where, by abuse of notation, η is a character of the norm one idèles. The condition of η being ramified as a class is equivalent to $\eta^2 = \chi$, which implies that $\mu\nu^{-1} = 1$. The reader should be careful to distinguish χ being ramified as a *class* with μ, ν being ramified *as characters*. For simplicity, we will often assume the central character is trivial, making the condition of being a ramified class equivalent to η being a quadratic character.

Definition 5.3.4. Now we define the principal series for $G'_{\mathbf{A}}$ by restricting the above definitions. Let A' be the group of diagonal matrices in $G'_{\mathbf{A}}$ and $A'_F = A' \cap GL_2(F)$. We consider the set D^0 of characters of η of $A'_F \backslash A'$ such that

$$\eta\big|_{Z'_0} = \chi^{-1},$$

and each η is again defined by the pair (μ, ν) of idele class characters on $F^\times \backslash \mathbf{A}_F^\times$. We also have the analogous height function

$$H : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \left|\frac{a}{b}\right|,$$

where $|\cdot|$ is taken to be the adelic norm. Putting these together we obtain the representation $\rho(g, \eta, s)$ acting by right translation on the induced representation space

$$\mathrm{Ind}_{NA'}^{G'} 1_N \otimes (\eta \otimes H^{\frac{1+s}{2}})$$

so that the resulting space is that of smooth functions φ_s on $N_{\mathbf{A}} \backslash G'_{\mathbf{A}}$ satisfying

$$\varphi_s(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} k) = \mu(a)\nu(b)\left|\frac{a}{b}\right|^{\frac{1+s}{2}} \varphi(k),$$

where k is an element of $K' = \prod' K'_v$ where K'_v are maximal compact subgroups, taken to be $G'(\mathcal{O}_{F_v})$ for almost all v. By the Iwasawa decomposition

$$G' = NA'K',$$

this space of functions can be identified with those on K'. Moreover, since $A' \backslash G'_{\mathbf{A}} = A_{GL_2} \backslash GL_2$, with A_{GL_2} the diagonal matrices in GL_2, we may regard the space of functions on which $\rho(g, \eta, s)$ acts as a space of functions on GL_2 by extending η trivially to A_{GL_2}.

If η is associated to the pair (μ, ν), then the intertwining operator analogous to (3.2.1), given by

$$(M(\eta, s)\varphi)(g) = \prod_v M(\eta_v) = \prod_v \int_{N_v} \varphi(wn_v g_v) dn_v$$

intertwines the principal series of (μ, ν) with that of (ν, μ). We normalize it as

$$M(\eta, s) = \frac{L(1-s, \mu^{-1}\nu)}{L(1+s, \mu\nu^{-1})} \otimes_v R(\eta_v, s)$$

where $R(\eta_v)$ denotes the normalized local intertwining operator, as in the case of GL_2. We will refer to the quotient of completed L-functions as the scalar factor, and denote it by $m(\eta, s)$. By the functional equation, we may also write

$$m(\eta, s) = \frac{L(s, \mu\nu^{-1})}{\epsilon(s, \mu\nu^{-1}, \psi)L(1+s, \mu\nu^{-1})} \tag{5.3.1}$$

where ψ is a fixed additive character of $F\backslash\mathbf{A}_F$.

Remark 5.3.5. (Specializing to $PSL_2(\mathbf{A}_F)$.) Note that for PSL_2, any character on the diagonal can be identified with the following character

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \mu\nu^{-1}(a) = \eta^2(a)|a|^s$$

of the idele class group. This is an immediate consequence of the definitions above.

In general, for a number field of degree d over $\mathbf{Q}$, the scalar factor of the constant term can be written as the product over all places v of F, given by Godement [God95, p.26] up to a constant factor,

$$\pi^{-d} \prod_{F_v \simeq \mathbb{R}} \frac{\Gamma(1 - s + \frac{1}{2}(|m_v| - in_v))}{\Gamma(s + \frac{1}{2}(|m_v| - i\nu_v))} \prod_{F_v \simeq \mathbb{C}} \frac{\Gamma(2 - 2s + \frac{1}{2}(\rho_v + \mu_v + in_v))}{\Gamma(2s + \frac{1}{2}(\rho_v + \mu_v + i\nu_v))} \frac{L^S(2-2s, \chi)}{L^S(2s, \bar{\chi})}$$

and the representation τ_v of K_v is as follows:

1. For v finite, let S be the finite number of places where τ_v is nontrivial, and there the local L-factor does not appear,

2. For v real, if $\tau_v(k)$ is the representation of $K_v = SO_2$ highest weight n, then

the $\chi_v(a)$ must necessarily be $\mathrm{sgn}^{m_v}(a)|a|^{n_v}$ with $m_v \in \mathbf{Z}, n_v \in \mathbf{R}$,

3. For v complex, if $\tau_v(k)$ is the representation of $K_v = SU_2$ acting on the space of polynomials of degree ρ_v, then $\chi_v(a) = t^{\mu_v}|t|^{-\mu_v+i\nu_v}$ where $|\mu_v| \leq \rho_v$ such that $\mu_v \equiv \rho_v \pmod 2$ and $\mu_v \in \mathbf{Z}, \nu_v \in \mathbf{R}$.

One can check that this specializes to the cases we have considered in the classical setting.

The following lemma shows that the character of the induced representation $\rho(f, \eta, s)$ defines again the Mellin transform of a smooth compactly supported function on $\mathbf{R}_+^\times$. (Compare with Definition 3.2.8.)

Lemma 5.3.6 (Test function)**.** *Let f be a function in $C_c^\infty(Z_0'\backslash G_{\mathbf{A}}')$. Then the character*

$$\mathrm{tr}(\rho(f, \eta, s))$$

is a Mellin transform of a function g in $C_c^\infty(\mathbf{R}_+^\times)$.

Proof. Using the Iwasawa decompostion , the trace $\mathrm{tr}(\rho(f, \eta, s))$ is equal to

$$\int_{K'}\int_{\mathbf{R}_+^\times}\int_{(\mathbf{A}^\times)^1}\int_{N_{\mathbf{A}}} f(k^{-1}a^{-1}na\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}k)|t|^{1+s}dn\; da\; d^\times t\; dk,$$

for if we interchange

$$na = \begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\begin{pmatrix} 1 & na^{-1}b \\ 0 & 1\end{pmatrix} = an'$$

the measure on $N_{\mathbf{A}}$ is multiplied by an element of norm 1, thus remains the same. Doing so, we obtain

$$\mathrm{vol}((\mathbf{A}^\times)^1) \int_K \int_{\mathbf{R}_+^\times} \int_{N_{\mathbf{A}}} f(k^{-1} n \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} k) |t|^{1+s} dn\ d^\times t\ dk.$$

Now denote the integration over the compact set K' by an auxiliary function

$$\Phi(g) = \int_{K'} f(k^{-1} g k) dk$$

which allows us to write

$$\mathrm{vol}((\mathbf{A}^\times)^1) \int_{\mathbf{R}_+^\times} \int_{N_{\mathbf{A}}} \Phi(n \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) |t|^{1+s} dn\ d^\times t$$

and denote by S the Satake transform

$$S\Phi(t) = \prod_v H_v(t_v)^{\frac{1}{2}} \int_{N_v} \Phi(n_v \begin{pmatrix} t_v & 0 \\ 0 & t^{-1} \end{pmatrix}) dn_v$$

with $H_v(t_v)^{\frac{1}{2}} = |t_v|$ the usual modulus character of $P = M \ltimes N$, related to the height function H defined in 5.3.4, so that one has

$$\mathrm{vol}((\mathbf{A}^\times)^1) \int_{\mathbf{R}_+^\times} S\Phi(t) |t|^s d^\times t$$

Note that while we recognize the integral as the Satake transform, we are not in fact using the Satake isomorphism at unramified places. Finally, setting $g = S\Phi$, we arrive at the Mellin transform

$$\hat{g}(s) = \int_{\mathbf{R}_+^\times} g(t)|t|^s d^\times t = \int_0^\infty g(t)t^{s-1}dt,$$

if we normalize measures such that $\mathrm{vol}((\mathbf{A}^\times)^1) = 1$. The condition is not a serious one; indeed, for our purposes it is enough to know that the measure is nonzero and finite, which is certainly true. (This is related to the constant c in [JL70, §16].) So given a test function f, we may view the trace $\mathrm{tr}(\rho(f, \eta, s))$ as the Mellin transform $\hat{g}(s)$ of a smooth compactly supported function defined on the positive real numbers. □

Now consider the regular representation ρ of $G'_{\mathbf{A}}$ on the discrete spectrum

$$L^2_{\mathrm{disc}}(G'_F \backslash G'_{\mathbf{A}}, \chi)$$

and a smooth function $f = \prod f_v$ in $G'_{\mathbf{A}}$ that is compact modulo Z'_0 such that

$$f(zg) = \chi(z)f(g)$$

for any z in Z'_0, and for almost all v, f_v is supported on $G'_F \cap GL_2(\mathcal{O}_F)$. Define a convolution operator $\rho_0(f)$ acting on $L^2_{\mathrm{disc}}(G'_F \backslash G'_{\mathbf{A}}, \chi)$ by

$$\rho_0(f)(\phi(x)) = \int_{Z'_0 \backslash G'_{\mathbf{A}}} f(g)\rho_0(g)\phi(x)dg.$$

It is a Hilbert-Schmidt operator, and in particular trace class. The trace formula now expresses the trace $\mathrm{tr}(\rho_0(f))$ in two ways, first as a sum of characters of representations, and second as a sum of orbital integrals. We examine the portion of the trace formula arising from the noncupisdal spectrum of $L^2(G'_F\backslash G'_{\mathbf{A}}, \chi)$.

Theorem 5.3.7 ([LL79, p.754])**.** *Let f and ρ be defined as above. Then the contribution of the continuous spectrum to the trace formula for G' are the terms (5.5) and (5.6) in [LL79]. In particular, it is (1) a sum of*

$$\sum_{\eta^2=\chi} -\frac{1}{4}\mathrm{tr}(M(\eta,0)\rho(f,\eta,0)) \tag{5.3.2}$$

taken over quadratic characters in D^0 such that $\eta^2 = \chi$, (2) a sum over characters in D^0 of terms involving the logarithmic derivative of the scalar factor of $M(s)$,

$$\sum_{\eta} \frac{1}{4\pi}\int_{-i\infty}^{i\infty} m(\eta,s)^{-1}m'(\eta,s)\mathrm{tr}\rho(f,\eta,s)|ds|, \tag{5.3.3}$$

and (3),

$$\sum_{\eta}\sum_{v} \frac{1}{4\pi}\int_{-i\infty}^{i\infty} \mathrm{tr}(R^{-1}(\eta_v,s)R'(\eta_v,s)\rho(f_v,\eta_v,s)) \prod_{w\neq v}\mathrm{tr}(\rho(f_w,\eta_w,s))|ds| \tag{5.3.4}$$

where $R^{-1}(\eta_v,s)$ is understood as the inverse operator of $R(\eta_v,s)$.

Remark 5.3.8. For comparison, we mention that these terms correspond to (vi), (vii), and (viii) in Jacquet-Langlands [JL70, §16], with the corrections indicated in

[LL79, p.753]. If η is trivial, then the quotient

$$\lim_{s\to 0}\frac{L(1-s,\mu^{-1}\nu)}{L(1+s,\mu\nu^{-1})}=-1$$

and $R(\eta_v,0)=1$ for all v, thus the associated distribution in (5.3.2) is stable. If η is a nontrivial quadratic character, then the scalar factor is equal to 1. Indeed, if $\eta_v=\eta_v^{-1}$, then $R(\eta_v,0)$ intertwines the representation $\rho(g,\eta_v,0)$ with itself, hence is the identity (cf. [LL79, Lemma 3.5]).

Sketch of proof. The kernel of $\rho(f)$ restricted to the continuous spectrum can be expressed as

$$K_{\text{cont}}(g,h)=\frac{1}{8\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\rho(f,it)\varphi_\beta,\varphi_\alpha)E(g,\varphi_\alpha,it)\overline{E}(h,\varphi_\beta,it)dt$$

where the $\varphi_\alpha,\varphi_\beta$ run over an orthonormal basis of the principal series $\rho(g,it)$, viewed as the representation $\rho(g)$ restricted to the continuous spectrum. This expression follows from the spectral decomposition of $L^2(G'_F\backslash G'_{\mathbf{A}},\chi)$ using the Eisenstein series and an orthogonal projection onto the continuous spectrum (cf. [GJ79, pp.232-234], [Kna97, p395]).

Then we compute the trace

$$\begin{aligned}&\int_{Z_0'G_F'\backslash G_{\mathbf{A}}'}\Lambda^T K_{\text{cont}}(g,g)dg\\ =&\frac{1}{8\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\rho(f,it)\varphi_\beta,\varphi_\alpha)\int_{Z_0'G_F'\backslash G_{\mathbf{A}}'}E(g,\varphi_\alpha,it)\overline{\Lambda^T E}(h,\varphi_\beta,it)dg\ dt.\end{aligned}$$

Observe that the inner integral is an inner product of a pair of truncated Eisenstein series, which is given by the Maaß-Selberg relation, which we prove in the following section as Theorem 5.4.3. Assuming this, we use $\varphi_1(s) = \varphi_\alpha(s)$ and $\varphi_2(s) = \varphi_\beta(s)$ in the theorem to obtain

$$
\begin{aligned}
&\int_{Z_0' G_F' \backslash G_{\mathbf{A}}'} \Lambda^T K_{\text{cont}}(g,g)dg \\
&= \frac{\log T}{2\pi} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (\rho(f,it)\varphi_\beta, \varphi_\alpha)(\varphi_\alpha, \varphi_\beta)dt \\
&\quad - \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (\rho(f,it)\varphi_\beta, \varphi_\alpha)(M^{-1}(it)M'(it)\varphi_\alpha, \varphi_\beta)dt \\
&\quad + \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (\rho(f,it)\varphi_\beta, \varphi_\alpha)\{(\varphi_\alpha, M(it)\varphi_\beta)\frac{T^{it}}{it} - (M(it)\varphi_\alpha, \varphi_\beta)\frac{T^{-it}}{it}\}dt.
\end{aligned}
$$

The first term put back together as

$$\frac{\log T}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}(\rho(f,it))dt$$

cancels with a contribution from the geometric side, so it does not appear in the final expression of the trace formula. The second term can be broken into a sum over each principal series attached to η,

$$-\frac{1}{4\pi} \int_{-\infty}^{\infty} M^{-1}(it)M'(it)\rho(f,it)dt = \sum_\eta -\frac{1}{4\pi} \int_{-\infty}^{\infty} M^{-1}(\eta,it)M'(\eta,it)\rho(f,\eta,it)dt.$$

Then we expand the logarithmic derivative

$$M(\eta, it)^{-1}M'(\eta, it) = m^{-1}(\eta, it)m'(\eta, it) \otimes I + \sum_u R^{-1}(\eta_u, it)R'(\eta_u, it) \otimes_{v \neq u} I_v$$

where I_v is the identity operator on the space $B(\mu_v \alpha_v^{it/2}, \nu_v \alpha_v^{it/2})$, and substituting it into the integral gives the terms (5.3.3) and (5.3.4) directly.

The third term can be rewritten as the sum of two terms

$$\frac{1}{4\pi}\sum_\beta \int_{-\infty}^{\infty} (M^{-1}(it)\rho(f, it)\varphi_\beta, \varphi_\beta)\frac{e^{2Tit} - e^{-2Tit}}{2it} dt$$

$$+\frac{1}{4\pi}\sum_\beta \int_{-\infty}^{\infty} \{(\rho(f, it)\varphi_\beta, M(it)\varphi_\beta) - (\rho(f, it)\varphi_\beta, M^{-1}(it)\varphi_\beta, \varphi_\beta)\}\frac{e^{-2Tit}}{2it}\} dt.$$

The second term here is the Fourier transform of an integrable function, so by the Riemann-Lebesgue lemma it tends to zero as T tends to infinity. On the other hand, the first term by [GJ79, Lemma 6.31] tends to

$$\begin{aligned}\frac{1}{4}\sum_\beta (M(0)\rho(f, it)\varphi_\beta) &= \frac{1}{4}\mathrm{tr}(M(0)\rho(f, it)) \\ &= \frac{1}{4}\sum_{\eta^2 = \chi} \mathrm{tr}(M(\eta, 0)\rho(f, \eta, it)),\end{aligned}$$

plus an $O(1/T)$ term. This last sum is taken over characters η that square to χ, for the operator $M(0)$ intertwines representations such that $\eta = \chi\bar{\eta}$, so that $M(0)$ is zero on characters η that do not square to χ. $\square$

5.4 Proof of the Maaß-Selberg relation

We complete the proof of Theorem 5.3.3 reviewing the Maaß-Selberg relation. We first state the following lemma:

Lemma 5.4.1. *Let f be a measurable function on $N_{\mathbf{A}}P'_F\backslash G'_{\mathbf{A}}$ such that $f(zg) = \chi(z)f(g)$ for all $z \in Z'_0$ and*

$$F := \sum_{\gamma \in P'_F \backslash G'_F} f(\gamma g)$$

is square integrable modulo $Z'_0 G'_F$. Also let f' be a function in $L^2(G'_F\backslash G'_{\mathbf{A}}, \chi)$, with constant term f'_N. Then

$$(F, f')_{L^2(G'_F\backslash G'_{\mathbf{A}},\chi)} = (f, f'_N)_{L^2(Z'_0 N_{\mathbf{A}} P'_F\backslash G'_{\mathbf{A}},\chi)}.$$

where the equality depends on the normalization of Haar measures.

Proof. The proof of this statement for $GL_2(\mathbf{A}_F)$ is given in [Kna97, Lemma 6.4] and $SL_2(\mathbf{R})$ in [Bor97, p.125], and one observes that the proof follows in the same manner for $G'_{\mathbf{A}}$ also. Note that choosing different Haar measures will preserve the equality up to a constant. □

Theorem 5.4.2 (Maaß-Selberg relation)**.** *The inner product of two truncated Eisenstein series*

$$(\Lambda^T E(g, \varphi_1, s_1), \Lambda^T E(g, \varphi_2, \bar{s}_2))$$

can be expressed as

$$\frac{2}{s_1+s_2}\{(\varphi_1,\varphi_2)T^{s_1+s_2} - (M(s_1)\varphi_1, M(\bar{s}_2)\varphi_2)T^{-(s_1+s_2)}\}$$
$$+\frac{2}{s_1-s_2}\{(\varphi_1, M(\bar{s}_2)\varphi_2)T^{s_1-s_2} - (M(s_1)\varphi_1,\varphi_2)T^{-(s_1-s_2)}\}$$

for $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) > 0$.

Proof. Given a function φ_s in the principal series $B(\mu\alpha^{s/2}, \nu\alpha^{s/2})$ belonging to $G'_{\mathbf{A}}$, the Eisenstein series in this setting is

$$E(g,\varphi,s) = \sum_{\gamma\in P'_F\backslash G'_F} \varphi_s(\gamma g)$$

where P'_F denotes the set of upper triangular matrices in G'_F. Applying the truncation operator of Arthur (see Chapter 1), we have for $T > 0$,

$$\Lambda^T E(g,\varphi,s) = E(g,\varphi,s) - \sum_{\gamma\in P'_F\backslash G'_F} E_N(\gamma g,\varphi,s)\chi_T(H(\gamma g)),$$

where $E_N(g,\varphi,s)$ is the constant term of the Fourier expansion,

$$\begin{aligned} E_N(g,\varphi,s) &= \int_{F\backslash\mathbf{A}} \varphi(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g)dx \\ &= \varphi_s(g) + \int_{N_{\mathbf{A}}} \varphi_s(wng)dn \\ &= \varphi_s(g) + (M(s)\varphi_s)(g) \end{aligned}$$

(cf. [Kna97, p.392]). Then the truncated Eisenstein series can be written as

$$\Lambda^T E(g,\varphi,s) = \sum_{\gamma\in P'_F\backslash G'_F} \{\varphi_s(\gamma g) - \chi_T(H(\gamma g))(\varphi_s(\gamma g) - (M(s)\varphi_s)(\gamma g))\}.$$

The truncated Eisenstein series being square integrable, we now compute its inner product. Since Λ^T is an orthogonal projection ([GJ79, p.210]), the inner product is equal to

$$\begin{aligned}(\Lambda^T(E(g,\varphi_1,s_1),\Lambda^T(E(g,\varphi_2,\bar{s}_1) &= (\Lambda^T(E(g,\varphi_1,s_1),(E(g,\varphi_2,\bar{s}_2)\\ &= \int_{Z'_0G'_F\backslash G'_{\mathbf{A}}} \Lambda^T(E(g,\varphi_1,s_1)\overline{(E(g,\varphi_2,\bar{s}_2)}dg\\ &= \int_{Z'_0G'_F\backslash G'_{\mathbf{A}}} \Lambda^T(E(g,\varphi_1,s_1)\overline{(E_N(g,\varphi_2,\bar{s}_2)}dg.\end{aligned}$$

Then unfold the sum and apply Lemma 5.4.1 to get

$$\int_{Z'_0N_{\mathbf{A}}P'_F\backslash G'_{\mathbf{A}}} \{\varphi_{1,s_1}(g) - \chi_T(H(g))(\varphi_{1,s_1}(g) - (M(s_1)\varphi_{1,s_1})(g))\}\overline{(E_N(g,\varphi_2,\bar{s}_2)}dg.$$

Decomposing domain of integration into

$$Z'_0N_{\mathbf{A}}P'_F\backslash G'_{\mathbf{A}} \simeq A'_FZ'_0\backslash A' \times K \simeq \mathbf{R}_+^\infty \times (\mathbf{A}_F^\times)^1 \times K,$$

we rewrite the integrand as

$$\begin{aligned}&\{\varphi_{1,s_1}(g)(1-\chi_T(H(g))) - (M(s_1)\varphi_{1,s_1})(g)\chi_T(H(g))\}\overline{(E_N(g,\varphi_2,\bar{s}_2)}\\ &= \{\varphi_{1,s_1}(g)(1-\chi_T(H(g))) - (M(s_1)\varphi_{1,s_1})(g)\chi_T(H(g))\}(\overline{\varphi_{2,s_2}}(g) + (\overline{M(s_2)\varphi_{2,s_2}})(g)),\end{aligned}$$

and then separate into two terms:

$$\varphi_{1,s_1}(g)(1-\chi_T(H(g)))(\overline{\varphi_{2,s_2}}(g)+(\overline{M(s_2)\varphi_{2,s_2}})(g)) \tag{5.4.1}$$

and

$$-(M(s_1)\varphi_{1,s_1})(g)\chi_T(H(g))(\overline{\varphi_{2,s_2}}(g)+(\overline{M(s_2)\varphi_{2,s_2}})(g)). \tag{5.4.2}$$

In both cases we shall use the transformation rule of the principal series,

$$\varphi_{i,s_i}(n\begin{pmatrix} pau & 0 \\ 0 & qbv \end{pmatrix}k) = \chi(bv)|ab^{-1}|^{\frac{1+s_i}{2}}\varphi_i(\begin{pmatrix} uv^{-1} & 0 \\ 0 & 1 \end{pmatrix}k)$$

for $p, q \in F^\times$, $a, b \in \mathbf{R}_+^\times$, and $u, v \in (\mathbf{A}_F^\times)^1$ (cf. [Kna97, p.390]). For simplicity we will write the latter function as $\varphi_i(uv^{-1}, k)$, as the function $\varphi_i(g)$ is determined by its values on $(\mathbf{A}_F^\times)^1 \times K$. The integrals over this product space will be expressed as an inner product in the Hilbert space associated to the principal series representation.

In the first term (5.4.1), the factor $1-\chi_T(H(g))$ is the characteristic function of $(0, T)$, thus truncates the integral over $\mathbf{R}_+^\times$. The integral becomes

$$\int_0^T \int_{(\mathbf{A}_F^\times)^1} \int_K t^{1+(s_1+s_2)/2}\varphi_1(a,k)\overline{\varphi_2}(a,k) + t^{1+(s_1-s_2)/2}\varphi_1(a,k)\overline{M(s_2)\varphi_2}(a,k)dk\ d^\times a\frac{dt}{t^2}$$

and hence

$$\begin{aligned}&\int_0^T t^{1+(s_1+s_2)/2}(\varphi_1,\varphi_2) + t^{1+(s_1-s_2)/2}(\varphi_1, M(\bar{s}_2)\varphi_2)\frac{dt}{t^2} \\ &= \frac{2}{s_1+s_2}T^{(s_1+s_2)/2}(\varphi_1,\varphi_2) + \frac{2}{s_1-s_2}T^{(s_1-s_2)/2}(\varphi_1, M(\bar{s}_2)\varphi_2).\end{aligned}$$

In the second term (5.4.2), the factor $\chi_T(H(g))$ is the characteristic function of (T,∞), and again truncates the integral over $\mathbf{R}_+^\times$. Computing as before, we obtain

$$\begin{aligned}&-\int_T^\infty t^{1-(s_1-s_2)/2}(M(s_1)\varphi_1,\varphi_2)+t^{1-(s_1+s_2)/2}(M(s_1)\varphi_1,M(\bar{s}_2)\varphi_2)\frac{dt}{t^2}\\ =&\ \frac{2}{s_1-s_2}T^{-(s_1-s_2)/2}(M(s_1)\varphi_1,\varphi_2)-\frac{2}{s_1+s_2}T^{-(s_1+s_2)/2}(M(s_1)\varphi_1,M(\bar{s}_2)\varphi_2).\end{aligned}$$

Then combining the two terms together gives the desired expression. □

The following special case of the Maaß-Selberg relation is the one encountered in the trace formula:

Corollary 5.4.3. *Let $s_1=s+h$, and $s_2=-s$ with $h>0$. Then the limit as $h\to 0$, written as*

$$(\Lambda^T E(g,\varphi_1,s),\Lambda^T E(g,\varphi_2,-\bar{s}))$$

is equal to

$$4\log T(\varphi_1,\varphi_2)+2(M^{-1}(s)M'(s)\varphi_1,\varphi_s)+(\varphi_1,M(\bar{s})\varphi_2)\frac{T^s}{s}-(M(s)\varphi_1,\varphi_2)\frac{T^{-s}}{s}.$$

Proof. By hypothesis $s_1+s_2=h$, and $s_1-s_2=2s+h$. The Maaß-Selberg relation specializes in this case to

$$\begin{aligned}&\frac{2T^h}{h}(\varphi_1,\varphi_2)+\frac{2T^{-h}}{h}(M(s+h)\varphi_1,M(-\bar{s})\varphi_2)\\ &+\frac{2T^{2s+h}}{2s+h}(\varphi_1,M(-\bar{s})\varphi_2)-\frac{2T^{-2s-h}}{2s+h}(M(s+h)\varphi_1,\varphi_2)\end{aligned}$$

which is valid so long as $s_1, s_2 \neq 0$.

The third and fourth terms are immediate. For the first two terms, we use the Taylor expansion of T^h at 0 in the variable h,

$$T^h = 1 + h\log T + \frac{1}{2}h^2\log^2 T + \dots,$$

combined with the fact that the adjoint of $M(s)$ is $M(\bar{s})$ and the identity $M(s)M(-s) = 1$ [Kna97, p.367], we write the second inner product as

$$(M(-s)M(s+h)\varphi_1, \varphi_2).$$

Then the limit of the first two terms now can be seen to evaluate to

$$4\log T(\varphi_1, \varphi_2) + 2(M^{-1}(s)M'(s)\varphi_1, \varphi_s)$$

as desired. $\square$

Part II

The Explicit Formula

Chapter 6

The explicit formula of Weil

6.1 Preliminaries

The explicit formulae of number theory relates certain sums over primes to sums over the zeroes of a given L-function. In this chapter we describe the historical development of the explicit formulae, and prove a variant of Weil's explicit formula for Hecke L-functions $L(s,\chi)$ in the classical formulation of [Wei52], following the method of Bombieri [Bom00] for $\zeta(s)$.

In 1859 Riemann was interested in the number of primes less than x, denoted by $\pi(x)$. The first to consider $\zeta(s)$ as a function of a complex variable, he obtained the formula

$$\sum_{n=1}^{\infty} \frac{\pi(x^{\frac{1}{n}})}{n} = \mathrm{Li}(x) - \sum_{\rho} (\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})) + \log \xi(0) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

where the left hand side is obtained by integrating by parts the logarithmic derivative of $\zeta(s)$ against x^s/s, while on the right hand side appear logarithmic integrals $\mathrm{Li}(x)$

and the Riemann $\xi(s)$ function. This is the first instance of an *explicit formula*, sometimes referred to as Riemann's exact formula, where the primes appear on the left, and the sum over the zeores ρ of $\zeta(s)$ appear on the right.

Valuable as it is, this did not yield the Prime Number Theorem that was sought after. Another formula was given by von Mangoldt in 1895, considering the Weierstraß product of $\xi(s)$ on the one hand, and its Euler product expansion on the other. Taking logarithmic derivatives, and integrating against x^s/s then leads to

$$\sum_{n<x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1 - x^{-2})$$

where if $x > 1$ is a prime power then the sums are weighted by a factor of $\frac{1}{2}$. At around this time Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem.

After this period, Guinand in 1942, followed by Delsarte developed explicit formulae reflecting a duality between the zeroes and the primes, similar to the Fourier duality exhibited by the Poisson summation formula. These were placed into a definitive form by Weil, which we will turn to next. We remind the reader that Selberg's trace formula also represents a generalization of the Poisson summation formula; there one has the Fourier transform of orbital integrals, giving characters of representations, thus a kind of duality for distributions.

6.2 Proof of Weil's formula

In 1952, Weil [Wei52] introduced an explicit formula for Hecke L-functions 'mit Grossencharakteren' over a number field.

To state the formula, we fix the following notation: given $f(x)$ a complex-valued function in $C_c^\infty(\mathbf{R}_+^\times)$, we will denote

$$f^*(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$$

and the Mellin transform

$$\hat{f}(s) = \int_0^\infty f(x) x^{s-1} dx.$$

We will say that f is even if $f = f^*$ and odd if $f = -f^*$.

Throughout we will fix a number field F, with r_1 real and $2r_2$ complex embeddings. Define the von Mangoldt function for number fields as follows: for any nonzero integral ideal $\mathfrak{a} \subset \mathcal{O}_F$ we set

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N_{F/\mathbf{Q}}\mathfrak{p} & \text{if } \mathfrak{a} = p^k \text{ for some } k \geq 1 \\ 0 & \text{otherwise} \end{cases},$$

not to be confused with the completed L-function $\Lambda(s,\chi) = (N_{F/\mathbf{Q}}\mathfrak{f}(\chi)|d_F|)^{s/2} L(s,\chi)$ which we shall use simultaneously. To ease notationt, we will write N for the norm map $N_{F/\mathbf{Q}}$, when there is no confusion.

We now derive a version of Weil's explicit formula for $L(s,\chi)$, in the shape given to it by Bombieri for $\zeta(s)$:

Theorem 6.2.1 (Weil's explicit formula [Wei52]). *Let $g \in C_c^\infty(\mathbf{R}_+^\times)$, and χ a Hecke character of a number field F. Then we have*

$$\sum_\rho \hat{g}(\rho) = \delta_\chi \int_0^\infty (g(x) + g^*(x))dx - \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\chi(\mathfrak{a})(g(N(\mathfrak{a})) + g^*(N(\mathfrak{a}))) \qquad (6.2.1)$$
$$- \log(|d_F|N\mathfrak{f}(\chi))g(1) - \sum_{k=1}^{r_1+r_2} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(\frac{1}{2} + it)\Big]\hat{g}(\frac{1}{2} + it)dt,$$

where the sum ρ is taken over zeroes of $L(s, \chi)$ with $0 < \mathrm{Re}(\rho) < 1$, and δ_χ is 1 if χ is trivial and zero otherwise. Moreover, the last sum can be expressed in terms of g,

$$\sum_{k=1}^{r_1+r_2} \Big\{ \int_1^\infty \{g(x) + g^*(x) - \frac{M}{x^{M-a+ib}} g(1)\} \frac{x^{M-1-a+ib}}{x^M - 1} dx + \Big(\gamma + (\frac{2}{M}) \log(\frac{2\pi}{M})\Big) g(1).\Big\}.$$

where $M = 2$ if F_{v_k} is a real completion and 1 if F_{v_k} is complex.

Proof. Consider the integral

$$I(g) = \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s)\hat{g}(s)ds$$

where the integration is taken over the line $(c - i\infty, c + i\infty)$ with $c > 1$. Since $g(x)$ is smooth with compact support, its Mellin transform $\hat{g}$ is an entire function of s of order 1 and exponential type, rapidly decreasing in every fixed vertical strip.

The logarithmic derivative of $\Lambda(s, \chi)$ is holomorphic for $\sigma > 1$, and has logarithmic growth on any vertical line $(c - i\infty, c + i\infty)$ with $c > 1$, hence the integral $I(g)$ is absolutely convergent.

For $\sigma > 1$, we have

$$\frac{\Lambda'}{\Lambda}(s,\chi) = \frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi)) + \sum_{k=1}^{r_1+r_2} \frac{\Gamma_k'}{\Gamma_k}(s) - \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

whence

$$I(g) = \frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi))\frac{1}{2\pi i}\int_{(c)} \hat{g}(s)ds + \sum_{k=1}^{r_1+r_2} \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma_k'}{\Gamma_k}(s)\hat{g}(s)ds$$
$$- \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\chi(\mathfrak{a})\frac{1}{2\pi i}\int_{(c)} \hat{g}(s)N(\mathfrak{a})^{-s}ds,$$

because term-by-term integration is justified by absolute convergence. The inverse Mellin transform being

$$g(x) = \frac{1}{2\pi i}\int_{(c)} \hat{g}(s)x^{-s}ds,$$

we arrive at the first expression

$$I(g) = \frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi))g(1) + \sum_{k=1}^{r_1+r_2} \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma_k'}{\Gamma_k}(s)\hat{g}(s)ds - \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\chi(\mathfrak{a})g(N(\mathfrak{a})). \tag{6.2.2}$$

Now we compute $I(g)$ in a different way: starting with the initial expression, move the line of integration to the left to $(c' - i\infty, c' + \infty)$ for some $c' < 0$. This step is justified by integrating over a rectangle with vertices $(c' \pm iT, c \pm iT)$ and showing that the integral over the horizontal edges tends to 0 as we let T tend to infinity along a well-chosen sequence. This is a well-known method, described in [Ing32, Theorem 29] for $\zeta(s)$, for example.

In our situation, the following bound in [Lan94, Prop 2.4] for completed Hecke L-functions will suffice: the number of zeroes of $L(s,\chi)$ in a box $0 \le \sigma \le 1$ and $T \le |t| \le T+1$ is $O(\log T)$. It follows that there exists constants c and T_m for every integer $m \neq -1, 0, 1$ such that $L(s,\chi)$ has no zeroes on the horizontal strips

$$|t \pm T_m| \le \frac{c}{\log |m|} \qquad m < T_m < m + t.$$

Allowing then T to tend to infinity via the sequence (T_m), the contribution from the horizontal edges also vanish.

Moving the line of integration to the left, we encounter the residues of $L'/L(s,\chi)$ due to the zeroes of $L(s,\chi)$ inside the critical strip $0 \le \sigma \le 1$, and the simple poles of $L(s,\chi)$ at $s = 0, 1$ in the case where χ is trivial. It follows that

$$I(g) = -\delta_\chi(\hat{g}(0) + \hat{g}(1)) + \sum_\rho \hat{g}(\rho) + \frac{1}{2\pi i}\int_{(c')} \frac{\Lambda'}{\Lambda}(s,\chi)\hat{g}(s)ds. \tag{6.2.3}$$

Now we use the functional equation $\Lambda(s,\chi) = W(\chi)\Lambda(1-s,\bar{\chi})$ to obtain the relation

$$\frac{\Lambda'}{\Lambda}(s,\chi) = -\frac{\Lambda'}{\Lambda}(1-s,\bar{\chi}) = \frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi)) - \sum_{k=1}^{r_1+r_2}\frac{\Gamma_k'}{\Gamma_k}(1-s) + \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})\chi(\mathfrak{a})}{N(\mathfrak{a})^{1-s}}.$$

Then substituting this into the integral in (6.2.3), we obtain as before

$$\frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi))\frac{1}{2\pi i}\int_{(c')}\hat{g}(s)ds - \sum_{k=1}^{r_1+r_2}\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma_k'}{\Gamma_k}(1-s)\hat{g}(s)ds$$

$$+\sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\chi(\mathfrak{a})\frac{1}{2\pi i}\int_{(c')} \hat{g}(s)N(\mathfrak{a})^{s-1}ds,$$

which is

$$\frac{1}{2}\log(|d_F|N\mathfrak{f}(\chi))g(1) - \sum_{k=1}^{r_1+r_2}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma_k'}{\Gamma_k}(1-s)\hat{g}(s)ds + \sum_{\mathfrak{a}}\Lambda(\mathfrak{a})\chi(\mathfrak{a})g^*(N(\mathfrak{a})),$$

where again term-by-term integration is justified because we are again in the region of absolute convergence, thanks to the functional equation.

Now equating the two expressions for $I(g)$ we find

$$\sum_{\rho}\hat{g}(\rho) = \delta_\chi(\hat{g}(0)+\hat{g}(1)) - \sum_{\mathfrak{a}}\Lambda(\mathfrak{a})\chi(\mathfrak{a})(g(N(\mathfrak{a}))+g^*(N(\mathfrak{a}))) - \log(|d_F|N\mathfrak{f}(\chi))g(1) \tag{6.2.4}$$

$$+\sum_{k=1}^{r_1+r_2}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma_k'}{\Gamma_k}(s)\hat{g}(s)ds + \sum_{k=1}^{r_1+r_2}\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma_k'}{\Gamma_k}(1-s)\hat{g}(s)ds.$$

And observe that

$$\hat{g}(0) = \int_0^\infty g^*(x)dx, \qquad \hat{g}(1) = \int_0^\infty g(x)dx,$$

thus we have the desired formula, save for the last two terms.

In order to obtain the explicit formula we compute the last two integrals as follows. First, we move the line of integration of both integrals to $c = c' = \frac{1}{2}$, which we may do without encountering any pole of the integrand. Thus the sum of the two integrals becomes

$$\sum_{k=1}^{r_1+r_2} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(\frac{1}{2}+it)\Big]\hat{g}(\frac{1}{2}+it)dt.$$

Note that our $\Gamma_k(s)$ here depends on the ramification of χ, and in fact the $\Gamma_k(1-s)$ appearing in the second integral is associated to $\bar{\chi}$, in the sense of Remark 4.3.8. We now treat the gamma factors in detail.

Case 1: $\Gamma_k(s) = \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(\frac{s+w}{2})$, with ramification $w = a+ib$. The logarithmic derivative is

$$\frac{\Gamma_{\mathbf{R}}'}{\Gamma_{\mathbf{R}}}(s) = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s}{2}),$$

so we have

$$-(\log\pi)g(1) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}(\frac{1}{2}(\frac{1}{2}+it+a+ib))\Big]\hat{g}(\frac{1}{2}+it)dt. \tag{6.2.5}$$

To treat the integral, we follow the exposition of [Bom00, §2] closely. We first use the expressions [Bom00, p.188]

$$\frac{\Gamma'}{\Gamma}(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\Big\{\frac{1}{n} - \frac{1}{n+z}\Big\}$$

and

$$\sum_{n=1}^{N}\frac{1}{n} = \log N + \gamma + O\big(\frac{1}{N}\big),$$

to obtain

$$\frac{\Gamma'}{\Gamma}(z) = \log N - \sum_{n=0}^{N} \frac{1}{n+z} + O\Big(\frac{1+|z|}{N}\Big) \tag{6.2.6}$$

uniformly for $\mathrm{Re}(z) > -\frac{N}{2}$ and z not equal to zero or a negative integer. This gives

$$\begin{aligned}&\mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}\big(\frac{1}{2}(\frac{1}{2}+it+a+ib)\big)\Big]\\ &= \log N - \sum_{n=0}^{N} \frac{2(2n+a+\frac{1}{2})}{(2n+a+\frac{1}{2})^2+(t+b)^2} + O\Big(\frac{1+|t+b|}{N}\Big)\end{aligned}$$

and so the integral in (6.2.5) becomes

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\log N - \sum_{n=0}^{N} \frac{2(2n+a+\frac{1}{2})}{(2n+a+\frac{1}{2})^2+(t+b)^2}\Big)\hat{g}(\frac{1}{2}+it)dt\\ &+O\Big(\int_{-\infty}^{\infty}\frac{1+|t+b|}{N}\cdot\Big|\hat{g}(\frac{1}{2}+it)dt\Big|dt\Big),\end{aligned}$$

Since $\hat{g}$ is rapidly decreasing on any vertical line, the last integral converges and the error term is $O(1/N)$. Apply also Mellin inversion to the first term, whence

$$-\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(2n+a+\frac{1}{2})}{(2n+a+\frac{1}{2})^2+(t+b)^2}\hat{g}(\frac{1}{2}+it)dt + (\log N)f(1) + O\Big(\frac{1}{N}\Big). \tag{6.2.7}$$

We have by Fubini's theorem

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2c}{c^2+(t+b)^2}\int_0^{\infty} g(x)x^{-\frac{1}{2}+it}dx\ dt\\ &= \int_0^{\infty} g(x)x^{-\frac{1}{2}}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big(\frac{c}{t+ic}-\frac{c}{t-ic}\Big)x^{i(t-b)}dt\ dx\end{aligned}$$

after making the change of variables $t \mapsto t-b$. Applying the calculus of residues yields

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big(\frac{c}{t+ic}-\frac{c}{t-ic}\Big)x^{i(t-b)}dt=\min(x,\frac{1}{x})^{c-ib}.$$

hence, taking $c = 2n + a + \frac{1}{2}$, (6.2.7) is then

$$-\int_{1}^{\infty}\Big(\sum_{n=0}^{N}x^{-2n-a+ib}\Big)g(x)\frac{dx}{x}-\int_{0}^{1}\Big(\sum_{n=0}^{N}x^{2n+a-ib}\Big)g(x)dx+(\log N)f(1)+O\big(\frac{1}{N}\big)$$
$$=-\int_{1}^{\infty}\Big(\sum_{n=0}^{N}x^{-2n-a+ib}\Big)(g(x)+g^*(x))\frac{dx}{x}+(\log N)f(1)+O\big(\frac{1}{N}\big).$$

Finally, we write

$$\int_{1}^{\infty}\Big(\sum_{n=0}^{N}x^{-2n-a+ib}\Big)(g(x)+g^*(x))\frac{dx}{x}$$
$$=\int_{1}^{\infty}\Big(\sum_{n=0}^{N}x^{-2n-a+ib}\Big)\Big(g(x)+g^*(x)-\frac{2}{x^{2-a+ib}}g(1)\Big)\frac{dx}{x}+\sum_{m=1}^{N+1}\frac{1}{m}g(1)$$

and substitute back to obtain

$$-\int_{1}^{\infty}\frac{1-x^{-2N-2}}{1-x^{-2}}x^{-a+ib}\Big(g(x)+g^*(x)-\frac{2}{x^{2-a+ib}}g(1)\Big)\frac{dx}{x}+(\log N-\sum_{m=1}^{N+1}\frac{1}{m})f(1)+O\big(\frac{1}{N}\big).$$

Now we take the limit as $N\to\infty$ and deduce for the integral in (6.2.5)

$$-(\gamma+\log\pi)g(1)-\int_{1}^{\infty}\Big\{g(x)+g^*(x)-\frac{2}{x^{2-a+ib}}g(1)\Big\}\frac{x^{1-a+ib}}{x^2-1}dx.$$

where γ is the usual Euler-Mascheroni constant. This concludes the real archimedean case.

Case 2: $\Gamma_k(s) = \Gamma_{\mathbf{C}}(s) = (2\pi)^{1-s}\Gamma(s+w)$, with ramification $w = a+ib$. The computations are almost identical to the first case. The logarithmic derivative is

$$\frac{\Gamma'_{\mathbf{C}}}{\Gamma_{\mathbf{C}}}(s) = -\log(2\pi) + \frac{\Gamma'}{\Gamma}(s),$$

so we have

$$-2g(1)\log(2\pi) + \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}(\frac{1}{2}+it+a+ib)\Big]\hat{g}(\frac{1}{2}+it)dt. \tag{6.2.8}$$

As with (6.2.7) we have the expression for the integral

$$-\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2(n+a+\frac{1}{2})}{(n+a+\frac{1}{2})^2+(t+b)^2}\hat{g}(\frac{1}{2}+it)dt + (\log N)f(1) + O\big(\frac{1}{N}\big). \tag{6.2.9}$$

Observing that the difference between (6.2.9) and (6.2.7) is n and $2n$, we see that the integral is

$$-(\gamma + 2\log(2\pi))g(1) - \int_1^{\infty}\big\{g(x)+g^*(x) - \frac{1}{x^{1-a+ib}}g(1)\big\}\frac{x^{-a+ib}}{x-1}dx,$$

which completes the complex archimedean case. □

Remark 6.2.2. The shape of the distribution arising from the gamma factor is expressed in slightly different forms in [Wei52], and [Lan94, p.339]. By the above, we have generalized the expression given by Bombieri [Bom00, p.186] to Hecke L-functions.

The novelty in Weil's method is the following condition, referred to as Weil's criterion. We will consider test functions in $C_c^\infty(\mathbf{R}_+^\times)$ which are multiplicative convolutions of a function g and its transpose conjugate $\bar{g}^*$, hence

$$g * \bar{g}^* = \int_0^\infty g(xy^{-1})\bar{g}^*(y)\frac{dy}{y} = \int_0^\infty g(xy)\overline{g(y)}dy.$$

We have then

$$\widehat{g * \bar{g}^*}(s) = \hat{g}(s)\hat{\bar{g}}^*(1-s).$$

Bombieri's strengthening of Weil's criterion is now as follows [Bom00, p.191]:

Theorem 6.2.3. *Let $W(g)$ be the linear functional defined by (6.2.1), on the space $C_c^\infty(\mathbf{R}_+^\times)$, so that*

$$W(g) = W(g^*) = \sum_\rho \hat{g}(\rho),$$

the sum taken over complex zeroes of $L(s,\chi)$. Then the Riemann hypothesis for $L(s,\chi)$ is equivalent to the statement that

$$W(g * \bar{g}^*) \geq 0$$

with equality only if g is identically zero. In short, we say that the functional is positive-definite *on such functions.*

Proof. The Riemann hypothesis for $L(s,\chi)$ is the statement that $1-\rho = \bar{\rho}$ for every nontrivial zero ρ of $L(s,\chi)$. Assuming this, we have

$$\sum_\rho \hat{g}(\rho)\hat{\bar{g}}(1-\rho) = \sum_\rho \hat{g}(\rho)\hat{\bar{g}}(\bar{\rho}) = \sum_\rho |\hat{g}(\rho)|^2 \geq 0.$$

It is also easy to show that equality holds only if $g(x)$ is identically zero. Indeed, equality can hold only if $\hat{g}(\rho) = 0$ for every ρ, whence $\hat{g}(s)$ has at least $(1/\pi + o(1))R\log R$ zeroes in a disk $|s| < R$. On the other hand, $\hat{g}(s)$ is an entire function of exponential type, thus if $\hat{g}$ is not identically zero it can have at most $O(R)$ zeroes in the disk $|s| < R$. This proves the first implication.

For the proof of the converse statement, see [Bom00, p.191-193] for the case of $\zeta(s)$, and [Wei52] or [Lan94, p.342] for a general $L(s,\chi)$. We note that all proofs in this direction are by contradiction, namely, assuming that there exists a zero of $L(s,\chi)$ with real part different from $\frac{1}{2}$, then constructing a test function g on which the functional $W(g)$ is negative. $\square$

Remark 6.2.4. (Historical remark.) Weil's criterion was originally proved for functions $g * \bar{g}^*$ where g is a function on $\mathbf{R}$—written additively rather multiplicatively as we have done—such that

1. It is smooth everywhere except for a finite number of points a_i where g_0 and g_0' have at most a discontinuity of the first kind, and $g(a_i) = \frac{1}{2}[g(a_i+0)+g(a_i-0)]$.

2. There exists a $b > 0$ such that g and g' are $O(e^{-\frac{1}{2}+b}|x|)$.

The functional $W(g)$ was placed into a more tractable form by Barner, the new conditions on the test functions referred to as the Barner conditions [Lan94]. Moreover, one can formulate as we have an equivalent condition for smooth, compactly supported functions, and the criterion for $\zeta(s)$ is proved for functions whose support is restricted to a small neighborhood of 0. In particular, Yoshida proved the analogous criterion for smooth, compactly supported, even functions [Yos92], and verified positivity for functions supported on $[-t, t]$ with $t = \log 2/2$; Burnol proves positivity for $t = \sqrt{2}$ [Bur00, Théorème 3.7], and Bombieri for $t = \log 2$ [Bom00, Theorem 12]. Note that each method of proof is different.

In 1972, Weil [Wei72] wrote down a similar explicit formula for 'Artin-Hecke' L-functions, that is to say traces of complex n-dimensional representations of the Weil group. He observes that the positivity of this distribution for smooth, compactly supported functions $g(x)$ on $\mathbf{R}_+^\times$ is equivalent to the Riemann hypothesis and the Artin conjecture for $L(s, \chi)$. Importantly, this expression is also uniform in the sense that in positive characteristic the distribution is known to be positive-definite after the proof of the Weil conjectures. Weil describes this as 'the most serious argument I know in favor of the conjectures in question for number fields.' We will explore this in detail when we return to the adelic trace formula.

Chapter 7

The explicit formulae in trace formulae

7.1 Preliminaries

We are finally in a position to relate the explicit formulae to trace formulae. As mentioned at the end of the previous chapter, Weil recast his explicit formula in an adelic form: Given a finite extension of number fields K/F, we may associate an n-dimensional representation r of the relative Weil group of K/F, a dense subgroup of $\mathrm{Gal}(K/F)$. Then taking the trace of this representation produces a character of F. The associated L-function is what Weil refers to as an Artin-Hecke L-function, and derives an explicit formula for it. It takes the form:

$$\sum_{\rho} \hat{g}(\rho) = g(1)\log|d_F| + \int_{W_F} g(|w|)\chi(w)(|w|^{\frac{1}{2}} + |w|^{-\frac{1}{2}})dw \qquad (7.1.1)$$
$$- \sum_{v} pv_0 \int_{W_v} g(|w|)\chi_v(w)\frac{|w|^{\frac{1}{2}}}{|1-w|}dw$$

where d_F is the discriminant of F, the sum v taken over all places of F, g a smooth, compactly supported function on $\mathbf{R}_+^\times$. The rest of the terms are defined below in §7.3.

The value of this expression is that the contribution of the L-factors at both archimedean and nonarchimedean primes are placed on an equal footing, giving the uniform expression above. This perspective is more in line with the adelic picture, as we have seen with the trace formula.

In this chapter we will show: (i) that the sums of zeroes of certain $L(s, \chi)$ appear in the continuous spectral terms of the trace formula of SL_2, through the logarithmic derivative of the intertwining operator, first in the classical case of the Riemann zeta function and then in the general adelic setting, and (ii) that this latter term can be rewritten as an expression similar to the distributions appearing in Weil's explicit formula for Artin-Hecke L-functions [Wei72].

As a final note, we point out that we have worked out the following special cases more because of their simplicity rather than as a limitation of our methods.

7.2 Sums of zeroes in the continuous spectral terms

To illustrate our method, we first consider the trace formula for $PSL_2(\mathbf{R})$ without ramification. The proof in the adelic setting will be similar. In particular, the continuous spectral terms as given in (3.3.2), which we reproduce here for convenience:

$$\int_{\Gamma\backslash G} \Lambda^T K_{\mathrm{cont}}(x,x)dx = \log T \int_{-\infty}^{\infty} \mathrm{tr}(\rho(f,ir)dr - \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{m'}{m}(ir)\mathrm{tr}(\rho(f,ir))dr$$
$$+ \frac{1}{4\pi i}\int_{-\infty}^{\infty} m(it)\mathrm{tr}(\rho(f,ir))\frac{T^{2ir}}{r}dr \tag{7.2.1}$$

where once again T is the truncation parameter with $T > 0$ large enough, ρ is the regular representation of $PSL_2(\mathbf{R})$ on $L^2(\Gamma\backslash\mathbf{H}_2)$, f is a smooth compactly supported function on $\mathbf{H}_2$, and $m(s)$ is the scalar factor of the intertwining operator defined in (3.2.2).

Let us look at the term containing the logarithmic derivative of $m(s)$ more closely. The following is Theorem A given in the Introduction.

Theorem 7.2.1. *Let $\hat{g}(ir) = \mathrm{tr}(\rho(f,ir))$. Then the continuous spectral term*

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{m'}{m}(ir)\mathrm{tr}(\rho(f,ir))dr \tag{7.2.2}$$

is equal to

$$\sum_{\rho} \hat{g}(\rho) - \int_0^{\infty} \left\{g(x) + \frac{1}{4}g^*(x)\right\}dx + \sum_{n=1}^{\infty} \Lambda(n)g(n)$$
$$+ \int_1^{\infty} \left\{g(x) + g^*(x) - \frac{2}{x}g(1)\right\}\frac{x\ dx}{2(x^2-1)} + \frac{1}{2}(\log 4\pi + \gamma)g(1),$$

where the sum ρ runs over zeroes of $\zeta(s)$ with $0 < \mathrm{Re}(\rho) < 1$.

Proof. First, we set $s = ir$, so that (7.2.2) takes the form

$$-\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\frac{m'}{m}(s)\hat{g}(s))ds.$$

Then by the functional equation, we express the logarithmic derivative of the scalar factor as

$$\begin{aligned}\frac{m'}{m}(s) &= \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(s)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1+s}{2}\Big)-\frac{\zeta'}{\zeta}(1+s)\\ &= \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(s)-\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(-\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(-s).\end{aligned}$$

Then we substitute this into our integral (7.2.2),

$$\begin{aligned}&-\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\Big\{\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(s)-\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(-\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(-s)\Big\}\hat{g}(s)ds\\ &=-\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\Big\{\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+2\mathrm{Re}\frac{\zeta'}{\zeta}(s)-\log\pi\Big\}\hat{g}(s)dr\end{aligned}$$

where we have used the property that $\hat{g}(ir)$ is even. Applying Mellin inversion to the last term, we obtain

$$\frac{1}{2}g(1)\log\pi-\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big]\hat{g}(s)ds-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\mathrm{Re}\Big[\frac{\zeta'}{\zeta}(s)\Big]\hat{g}(s)ds. \tag{7.2.3}$$

To treat the third term, we use a variant of the method used, for example, in Ingham [Ing32, Theorem 29, p.77], and similar to the derivation of (6.2.3): Move the line of integration to the right to $(c - i\infty, c + \infty)$ for some $c > 1$. This step is justified by integrating over a rectangle R with vertices $(c \pm iT, c \pm iT)$ and showing that the integral over the horizontal edges tends to 0 as we let T tend to infinity along a well-chosen sequence $(T_m), m = 2, 3, \ldots$ such that $m < T_m < m + 1$ and

$$\frac{\zeta'}{\zeta}(s) = O(\log^2 T)$$

for any s with $-1 \leq \sigma \leq 2$ and $T = T_m$, and as T tends to infinity (see [Ing32, Theorem 26, p.71]). Allowing then t to tend to infinity via the sequence (T_m), the contribution from the horizontal edges vanish, as the transform $\hat{g}(s)$ of a smooth, compactly supported function has rapid decay along a fixed vertical strip.

Moving the line of integration to the left, we obtain the residues of $\zeta'/\zeta(s)$ due to the zeroes of $\zeta(s)$ inside the critical strip $0 < \sigma < 1$, and the simple poles of $\zeta(s)$ at $s = 1$. Then allowing T to go to infinity we obtain

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\zeta'}{\zeta}(s)\hat{g}(s)ds = \sum_{\gamma} \hat{g}(\rho) - \hat{g}(1) - \frac{1}{2\pi i}\int_{(c)} \frac{\zeta'}{\zeta}(s)\hat{g}(s)ds$$

where the sum $\rho = \beta + i\gamma$ runs over the zeroes of $\zeta(s)$ with $0 < \beta < 1$. The integral now being in the region of absolute convergence, we may integrate term by term and apply Mellin inversion,

$$\begin{aligned} -\frac{1}{2\pi i}\int_{(c)} \frac{\zeta'}{\zeta}(s)\hat{g}(s)ds &= \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i}\int_{(c)} \hat{g}(s) n^{-s} ds \\ &= \sum_{n=1}^{\infty} \Lambda(n) g(n). \end{aligned}$$

So we have the expression for (7.2.3)

$$\sum_{\gamma} \hat{g}(\rho) - \hat{g}(1) + \sum_{n=1}^{\infty} \Lambda(n) g(n) + \frac{1}{2} g(1) \log \pi - \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big] \hat{g}(s) ds. \quad (7.2.4)$$

To treat the last integral, we first observe that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O(\frac{1}{|s|^2})$$

uniformly in any fixed angle $|\arg(s)| < \pi$ as $|s|$ tends to infinity. Now we move the line integration to the line $\mathrm{Re}(s) = \frac{1}{2}$,

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big] \hat{g}(s) ds = -\frac{1}{4} \hat{g}(0) - \frac{1}{4\pi i} \int_{(\frac{1}{2})} \mathrm{Re}\Big[\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big] \hat{g}(s) ds.$$

Here one half the residue of the pole of $\Gamma(s)$ at $s = 0$ is obtained as the initial line of integration is over $s = 0$. Then from the proof of Theorem 6.2.1, with $M = 2$ and $a, b = 0$ we can rewrite this as

$$-\frac{1}{4} \int_0^{\infty} g^*(x) dx + \frac{1}{2}(\log 4 + \gamma) g(1) + \int_1^{\infty} \Big\{ g(x) + g^*(x) - \frac{2}{x} g(1) \Big\} \frac{x \; dx}{2(x^2 - 1)},$$

(see also [Bom00, p.190]). Then substituting this last expression into (7.2.4) proves the claim.

□

Thus we see that the sum over zeroes as in Weil's explicit formula appear in the continuous spectral terms of the trace formula for $PSL_2(\mathbf{R})$. Having treated the basic case, we now discuss the adelic case: for comparison, we recall the continuous spectral terms described in Theorem 5.3.7, taking $G'_{\mathbf{A}} = SL_2(\mathbf{A}_F)$ where F is any number field and $T > 0$ large enough,

$$\begin{aligned}
&\int_{Z'_0 G'_F \backslash G'_{\mathbf{A}}} \Lambda^T K_{\text{cont}}(g,g)dg \\
= &\frac{\log T}{2\pi} \int_{-\infty}^{\infty} \text{tr}(\rho(f,it))dt \sum_{\eta} -\frac{1}{4\pi} \int_{-\infty}^{\infty} M^{-1}(\eta,it)M'(\eta,it)\rho(f,\eta,it)dt \\
&+ \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} (\rho(f,it)\varphi_\beta,\varphi_\alpha)\{(\varphi_\alpha, M(it)\varphi_\beta)\frac{T^{it}}{it} - (M(it)\varphi_\alpha,\varphi_\beta)\frac{T^{-it}}{it}\}dt,
\end{aligned}$$

and in particular we are interested in the term involving the scalar factor $m(s)$ in $M(s)$, given in (5.3.3),

$$\sum_{\eta} \frac{1}{4\pi} \int_{-i\infty}^{i\infty} m(\eta,s)^{-1}m'(\eta,s)\text{tr}\rho(f,\eta,s)ds$$

where the sum η is over idele class characters of F, f is a function in $C_c^\infty(G'_{\mathbf{A}})$, and $m(\eta,s)$ is defined in (5.3.1). As in the previous theorem, we show that this term produces sums over zeroes of Hecke L-functions, providing the connection to explicit formulae.

Theorem 7.2.2. *Fix an idele class character η. With definitions given as above, the integral*

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} m(\eta,s)^{-1}m'(\eta,s)\text{tr}(\rho(f,\eta,s))ds$$

is equal to the sum over zeroes of $L(s,\eta)$ and $L(s,\bar\eta)$ with real part between 0 and 1,

$$\sum_{\rho,\rho'}\frac{1}{2}\big\{\hat g(\rho)+\hat g(-\bar\rho)\big\}$$

plus

$$\frac{1}{2}\sum_{\mathfrak a}\Lambda(\mathfrak a)\big\{\eta(\mathfrak a)g(N(\mathfrak a))+\bar\eta(\mathfrak a)g^*(N(\mathfrak a))\big\}+\log(N\mathfrak f(\eta)|d_F|)g(1)$$
$$-\sum_{a\le 0}\int_0^\infty g(x)(\delta_\eta+x^{w-1})dx-\sum_{k=1}^{r_1+r_2}\frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(\frac{1}{2}+it+w)\Big]\hat g(\frac{1}{2}+it)dt$$

where the sum over a refers to the archimedean places of F_v where χ is ramified with ramification $w=a+ib, a\le 0$. Moreover, the last term can be expressed as a function of $g(x)$ only, the same as in Theorem 6.2.1.

Proof. The method of proof is similar to that of the previous theorem. By the functional equation, we have

$$m(\eta,s)=\frac{L(s,\eta)}{\epsilon(s,\eta,\psi)L(1+s,\eta)}=\frac{\epsilon(-s,\bar\eta,\psi)}{\epsilon(s,\eta,\psi)}\frac{L(s,\eta)}{L(-s,\bar\eta)},$$

where ψ is a fixed additive character of F. Its logarithmic derivative is

$$-\frac{\epsilon'}{\epsilon}(-s,\bar\eta,\psi)-\frac{\epsilon'}{\epsilon}(s,\eta,\psi)+\frac{L'}{L}(s,\eta)+\frac{L'}{L}(-s,\bar\eta).$$

Writing $\hat g(s)=\mathrm{tr}(\rho(f,\eta,s))$, the integral now becomes

$$\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\Big\{\frac{\epsilon'}{\epsilon}(-s,\bar\eta,\psi)+\frac{\epsilon'}{\epsilon}(s,\eta,\psi)-\frac{L'}{L}(s,\eta)-\frac{L'}{L}(-s,\bar\eta)\Big\}\hat g(s)ds. \qquad (7.2.5)$$

We first treat the epsilon factors. Recall from Definition 4.3.6 that

$$\epsilon(s,\eta,\psi) = W(\eta)|N(\mathfrak{f}(\chi))d_F|^{s-\frac{1}{2}}$$

where the global Artin conductor $\mathfrak{f}(\eta) = \prod p_v^{\mathfrak{f}_v(\eta)}$ is a product over all primes p_v which ramify in F, of local conductors $\mathfrak{f}_v(\eta)$ of the local character η_v. Then

$$\begin{aligned} &\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\Big\{\frac{\epsilon'}{\epsilon}(-s,\bar{\eta},\psi) + \frac{\epsilon'}{\epsilon}(s,\eta,\psi)\Big\}\hat{g}(s)ds \\ &= \frac{1}{4\pi i}(\log(N\mathfrak{f}(\eta)|d_F|) + \log(N\mathfrak{f}(\bar{\eta})|d_F|))\int_{-i\infty}^{i\infty}\hat{g}(s)ds \\ &= \log(N\mathfrak{f}(\eta)|d_F|)g(1), \end{aligned} \tag{7.2.6}$$

since $\mathfrak{f}(\eta) = \mathfrak{f}(\bar{\eta})$. Note that the additive character ψ is chosen such that the measure is normalized as usual for Mellin inversion. Later in Lemma 7.3.9 we separate the $N\mathfrak{f}(\eta)$ from d_F and write as a sum over local conductors.

Now for the L-functions. Denoting by w the ramification of η_{v_k} at archimedean places (as in Definition 4.3.8), the logarithmic derivative is

$$\frac{L'}{L}(s,\eta) + \frac{L'}{L}(-s,\bar{\eta}) = \sum_{k=1}^{r_1+r_2}\Big\{\frac{\Gamma_k'}{\Gamma_k}(s+w) + \frac{\Gamma_k'}{\Gamma_k}(-s+\bar{w})\Big\} + \frac{\zeta_F'}{\zeta_F}(s,\eta) + \frac{\zeta_F'}{\zeta_F}(-s,\bar{\eta}).$$

We then separate the integral, into the logarithmic derivatives of $\zeta_F(s)$ and $\Gamma_k(s)$ respectively. Consider first the zeta functions. Following the proof of Theorem 6.2.1

and Theorem 7.2.1, we move the line of integration to $c > 1$,

$$\begin{aligned}
&-\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\Big\{\frac{\zeta_F'}{\zeta_F}(s,\eta)+\frac{\zeta_F'}{\zeta_F}(-s,\bar{\eta})\Big\}\hat{g}(s)ds \qquad (7.2.7)\\
&=\sum_{\rho,\rho'}\frac{1}{2}\big\{\hat{g}(\rho)+\hat{g}(-\rho')\big\}-\frac{1}{4\pi i}\int_{(c)}\Big\{\frac{\zeta_F'}{\zeta_F}(s,\eta)+\frac{\zeta_F'}{\zeta_F}(-s,\bar{\eta})\Big\}\hat{g}(s)ds
\end{aligned}$$

picking up the residues of $\zeta_F(s,\eta)$ and $\zeta_F(-s,\bar{\eta})$ with $0 < \mathrm{Re}(\rho) < 1$. Once again this step is justified by integrating over a box of height T_m, where (T_m) is a suitably-chosen sequence so as to avoid crossing any zeroes. If η is trivial, then so is $\bar{\eta}$ and hence $L(s,1) = \zeta_F(s)$ has a pole at $s = 1$, contributing

$$-\hat{g}(1) = -\int_0^\infty g(x)dx.$$

The integral now being in the region of absolute convergence, we utilize the Euler product expansion of the L-function,

$$\begin{aligned}
&-\frac{1}{4\pi i}\int_{(c)}\Big\{\frac{\zeta_F'}{\zeta_F}(s,\eta)+\frac{\zeta_F'}{\zeta_F}(-s,\bar{\eta})\Big\}\hat{g}(s)ds\\
&=\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\sum_{\mathfrak{a}}\Big\{\frac{\Lambda(\mathfrak{a})\eta(\mathfrak{a})}{N(\mathfrak{a})^s}+\frac{\Lambda(\mathfrak{a})\bar{\eta}(\mathfrak{a})}{N(\mathfrak{a})^{-s}}\Big\}\hat{g}(s)ds\\
&=\frac{1}{2}\sum_{\mathfrak{a}}\Lambda(\mathfrak{a})\big\{\eta(\mathfrak{a})g(N(\mathfrak{a}))+\bar{\eta}(\mathfrak{a})g^*(N(\mathfrak{a}))\big\}.
\end{aligned}$$

Thus our final expression for (7.2.7) is

$$\sum_\rho \frac{1}{2}\{\hat{g}(\rho)+\hat{g}(-\rho')\} - \delta_\eta \int_0^\infty g(x)dx \tag{7.2.8}$$
$$+\frac{1}{2}\sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\big\{\eta(\mathfrak{a})g(N(\mathfrak{a}))+\bar{\eta}(\mathfrak{a})g^*(N(\mathfrak{a}))\big\},$$

where as before δ_η is 1 if η is trivial and zero otherwise.

Finally, for the gamma factors, we begin with:

$$-\sum_{k=1}^{r_1+r_2} \frac{1}{4\pi i}\int_{-i\infty}^{i\infty} \Big\{\frac{\Gamma_k'}{\Gamma_k}(s+w)+\frac{\Gamma_k'}{\Gamma_k}(-s+\bar{w})\Big\}\hat{g}(s)ds$$
$$=-\sum_{k=1}^{r_1+r_2} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(s+w)\Big]\hat{g}(s)ds,$$

since s here is pure imaginary. We now move the line of integration to $\mathrm{Re}(s)=\frac{1}{2}$, encountering residues of poles of the gamma functions in the following scenarios: Recall that if $w=a+ib$, then for real places we have $a=0$ or 1, whereas for complex places we have $a\in\mathbf{Z}[\frac{1}{2}]$. Since the poles of $\Gamma(s)$ occur at $s=0,-1,-2,\ldots$, we see that the integral of $\Gamma_k'/\Gamma_k(s+w)$ passes through a pole only if $a\le 0$. Thus,

$$-\sum_{a\le 0}\hat{g}(s+w)-\sum_{k=1}^{r_1+r_2}\frac{1}{2\pi i}\int_{(\frac{1}{2})}\mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(s+w)\Big]\hat{g}(s)ds,$$
$$=-\sum_{a\le 0}\int_0^\infty g(x)x^{w-1}dx-\sum_{k=1}^{r_1+r_2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\Big[\frac{\Gamma_k'}{\Gamma_k}(\frac{1}{2}+it+w)\Big]\hat{g}(\frac{1}{2}+it)dt, \tag{7.2.9}$$

and the integral in this last expression is exactly the one studied previously in Theorem 6.2.1. Then putting these together with (7.2.6) and (7.2.8) proves the claim. $\square$

7.3 Weil distributions in the continuous spectral terms

We would like to analyze the integral involving the logarithmic derivative of the intertwining operator. To state the result, we recall some definitions related to Weil's explicit formula [Wei72] for Artin L-functions:

Definition 7.3.1. Define the Weil group W_{F_v} of a local field F_v as follows: if F_v is nonarchimedean with residue field $\mathbf{F}_p$, then we have the exact sequence

$$1 \to I_{F_v} \to \mathrm{Gal}(\overline{F_v}/F_v) \to \mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p) \simeq \mathbf{Z} \to 1$$

where the Frobenius element is sent to 1 in $\mathbf{Z}$, and I_{F_v} is the inertia subgroup defined by the exact sequence. Then W_{F_v} is defined to be the inverse image of the identity 1 in $\mathrm{Gal}(\overline{\mathbf{F}_p}/\mathbf{F}_p)$. It is a dense subgroup in the profinite topology, and is isomorphic to $\hat{\mathbf{Z}}$. In the case where F_v is real or complex, the corresponding Weil group W_{F_v} is $\mathbf{C}^\times$ and $\mathbf{C}^\times \cup j\mathbf{C}^\times$ respectively, where $j^2 = -1$.

Denote by $W^1_{F_v}$ the kernel of the homomophism of the modulus $|\cdot| : W_{F_v} \to \mathbf{R}^\times_+$, which is defined as follows: if v is nonarchimedean, then for any w in W_{F_v} there is a number q_v^{-n}, where q_v is the cardinality of the residue field of F_{v_k}, such that for every root of unity μ of order prime to p so that $\mu^w = \mu^{q_v^n}$. Then we set $|w|_v = q_v^{-n}$. If v is archimedean, then we set $|w|_v = w\bar{w}$, in both cases where F_{v_k} is $\mathbf{C}$ or a subset of the quaternions.

Notice that the morphism is surjective if F_v is archimedean, and discrete if F_v is nonarchimedean. In the following theorem we shall identify $W_{F_v}/W_{F_v}^1$ with $\mathbf{R}_+^\times$ or a discrete subgroup thereof.

Finally, the absolute Weil group associated to the global field F will be identified with the idèle class group by its abelianization $W_F^{\mathrm{ab}} \simeq F^\times \backslash \mathbf{A}_F^\times$. The integrals over Weil groups appearing below will be called *Weil-type* distributions, for their similarity to those appearing in [Wei72]; in our theorem there appears a difference of certain exponents of $\frac{1}{2}$ due to our normalization.

Definition 7.3.2. Define the functions $f_0(x) = \inf(x^{\frac{1}{2}}, x^{-\frac{1}{2}})$ and $f_1(x) = f_0(x)^{-1} - f_0(x)$ on $\mathbf{R}_+^\times$, and the principal value

$$pv \int_0^\infty f(x) d^\times x = \lim_{t \to \infty} (\int_0^\infty (1 - f_0(x)^{2t} f(x) d^\times x - 2c \log t)$$

where c is a constant such that $f(x) - cf_1(x)^{-1}$ is an integrable function on $\mathbf{R}_+^\times$. Furthermore, we denote for simplicity

$$pv_0 \int_0^\infty f(x) d^\times x = pv \int_0^\infty f(x) + 2c \log(2\pi).$$

We are now ready to analyze the integrals in (4.4.3). First, we find the following expression for the unramified term in terms of distributions of Weil-type, i.e, distributions similar to those arising in Weil's explicit formula [Wei72, p.18]. To simplify notation, we shall denote $\hat{g}(\mathbf{s}_k) = \prod \hat{g}_k(s+\mathbf{e}_k)$, which remains a function of the single complex variable s.

Theorem 7.3.3. *Let $\hat{g} = \prod \hat{g}_k$ where $\hat{g}_k$ is given as above. Then the integrals in (4.4.3) involving the logarithmic derivative of $m_{ij}(s, \chi_{\mathbf{m}})$ with $i = j$ can be expressed as a sum involving Weil-type distributions:*

$$\frac{h}{2} g(0) \log |d_F| + \frac{1}{2} \sum_{\chi} \Big(\int_{W_F} g(|w|)\chi(w) \frac{dw}{|w|} + \sum_{v} pv_0 \int_{W_v} \hat{g}(|w|)\chi_v(w) \frac{|w|}{|1-w|} dw \Big)$$

where the sum v runs over all places of F.

The rest of this chapter will be devoted to the proof of this theorem. We first collect several lemmas.

Lemma 7.3.4 ([Wei72, p.13])**.** *Let g be a function in $C_c^\infty(\mathbf{R}_+^\times)$, and $\hat{g}$ its Mellin transform. Assume that there exists an $A > \frac{1}{2}$ such that $g(t) = O(t^A)$ as t tends to 0 and $g(t) = O(t^{-A})$ as t tends to infinity. Then for any σ such that $|\sigma - \frac{1}{2}| < A$, the following formula holds:*

$$\int_{\mathbf{R}_+^\times} \hat{g}(\sigma - \frac{1}{2} + it) X^{\sigma+it} dt = 2\pi X^{\frac{1}{2}} g(X)$$

by Mellin inversion.

Proof. The growth assumption on g implies that the transform $\hat{g}(\sigma - \frac{1}{2} + it)$ is holomorphic in the region $|\sigma - \frac{1}{2}| < A$, so that the inversion formula is independent of σ in this range. $\square$

We combine this lemma with the next one:

Lemma 7.3.5. *Let v be a nonarchimedean completion of F, and q_v the cardinality of the residue field of F_v. Then*

$$\frac{d}{ds}\log L_v(s,\chi) = -\log q_v \sum_{n=1}^{\infty} q_v^{-ns} \int_{W_v^1} \chi_v(\mathfrak{f}^n w)dw,$$

where $\mathfrak{f}$ is a Frobenius element in W_{F_v}.

Proof. We first justify interchanging the sum and integral: the character χ_v is assumed to be unitary, that is $|\chi(w)| \leq 1$ for all w in $W_{F_v}^1$, so the sum converges absolutely for $\mathrm{Re}(s) > 1$, then apply Proposition 1 of [Wei72, p.12] $\square$

The following lemmas will allow us to treat the archimedean places:

Lemma 7.3.6. *Let $Re(s) > 0$. Then there is the Gauss-Weil identity*

$$-2\frac{\Gamma'}{\Gamma}(s) = pv\int_0^{\infty} \frac{f_0(x)^{2s-1}}{f_1(x)} d^{\times}x$$

with the definitions as above.

Proof. This statement is given in [Wei72, p.16], proved in [Mor05, p.160-162]. $\square$

The next is a variant of that used in Weil's explicit formula:

Lemma 7.3.7. *The integral*

$$\frac{1}{2\pi i}\int \hat{g}(s-\frac{1}{2})d\log\Gamma_k(\frac{1}{2}+s+a+ib)+\hat{g}(\frac{1}{2}-s)d\log\Gamma_k(\frac{1}{2}+s+a-ib) \quad (7.3.1)$$

with $a \geq 0, b \in \mathbf{R}$, taken over the line $Re(s)=\frac{1}{2}$, can be expressed as

$$-pv_0\int_0^\infty g(\nu)\nu^{\frac{1}{2}+ib}\frac{f_0(\nu)^{2a+1-E}}{f_1(\nu^E)}d^\times\nu \quad (7.3.2)$$

where $E=\dim_{F_{v_k}}\mathbf{C}$, which is to say $E=2$ if $F_{v_k}\simeq\mathbf{R}$ and $E=1$ if $F_{v_k}\simeq\mathbf{C}$. Note that here we have used $a+ib$ for w as in Remark 4.3.8.

Proof. A detailed proof of Weil's statement is supplied in [Mor05, p.162-174]. We sketch the proof, indicating the necessary modifications. The transform $\hat{h}(s)$ used in [Wei72, Mor05] is a shifted Mellin transform:

$$\hat{h}(s)=\int_0^\infty h(\nu)\nu^{\frac{1}{2}-s}d^\times\nu,$$

which relates to our function $\hat{g}(s)$ by the relation

$$\hat{h}(s)=\hat{g}(\frac{1}{2}-s)=\hat{g}(s-\frac{1}{2}).$$

The integral (7.3.1) is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(ir)\frac{\Gamma_k'}{\Gamma_k}(1+ir+a+ib)+\hat{g}(-ir))\frac{\Gamma_k'}{\Gamma_k}(1+ir+a-ib)\ dr$$

We consider the two cases, first where $k = \mathbf{R}$ the integral becomes after a change of variables $r \mapsto r - b$:

$$-g(1)\log\pi + \frac{1}{4\pi}\int_{-\infty}^{\infty}\hat{g}(i(r-b))(\frac{\Gamma'}{\Gamma}(\frac{1+ir+a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{1-ir+a}{2})dr,$$

where the first term follows from Mellin inversion. Considering next the integral, we shift the contour from ir to $-\frac{1}{2}+ir$, noting that $\hat{g}(s)$ and $\Gamma'/\Gamma(s)$ are holomorphic in this range. As a result we obtain

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\hat{g}(-\frac{1}{2}+i(r-b))(\frac{\Gamma'}{\Gamma}(\frac{\frac{1}{2}+ir+a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{\frac{1}{2}-ir+a}{2})dr;$$

also apply the Gauss-Weil formula of Lemma 7.3.6 to get

$$\frac{\Gamma'}{\Gamma}(\frac{\frac{1}{2}+ir+a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{\frac{1}{2}-ir+a}{2}) = -pv\int_{0}^{\infty}\frac{f_0(\nu)^{-\frac{1}{2}+a}}{f_1(\nu)}\nu^{ir/2}d^{\times}\nu.$$

Making the change of variables $\nu \mapsto \nu^2$, the integral then becomes after interchanging the order of integration:

$$\begin{aligned}
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(-\frac{1}{2}+i(r-b))\cdot pv\int_{0}^{\infty}\frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)}\nu^{ir}d^{\times}\nu\ dr\\
=&-pv\int_{0}^{\infty}\frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(-\frac{1}{2}+i(r-b))\nu^{ir}dr\ d^{\times}\nu.
\end{aligned}$$

Then applying Lemma 7.3.4 to the inner integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(-\frac{1}{2}+i(r-b))\nu^{ir}dr = \nu^{\frac{1}{2}+ib}g(\nu)$$

Then putting it together we obtain the desired form of (7.3.2)

$$-pv_0 \int_0^\infty \frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)} \nu^{\frac{1}{2}+ib} g(\nu) d^\times \nu.$$

Now for the second case where $k = \mathbf{C}$, we begin as before with the expression for (7.3.1),

$$-2g(1)\log(2\pi) + \frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}(-\frac{1}{2} + i(r-b))(\frac{\Gamma'}{\Gamma}(\frac{1}{2}+a+ir) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}+a-ir))dr.$$

applying Mellin inversion to get the first term, and shifting contours in the integral as before. Apply again the Gauss-Weil formula of Lemma 7.3.6 to the Gamma factors,

$$\frac{\Gamma'}{\Gamma}(\frac{1}{2}+a+ir) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}+a-ir) = -pv \int_0^\infty \frac{f_0(\nu)^{2a}}{f_1(\nu)} \nu^{ir} d^\times \nu$$

and substitute back to the original expression, then interchange the orders of integration:

$$\begin{aligned} &-\frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}(-\frac{1}{2}+i(r-b)) \cdot pv \int_0^\infty \frac{f_0(\nu)^{2a}}{f_1(\nu)} \nu^{ir} d^\times \nu \ dr \\ =& -pv \int_0^\infty \frac{f_0(\nu)^{2a}}{f_1(\nu)} \cdot \frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}(-\frac{1}{2}+i(r-b))\nu^{ir} dr \ d^\times \nu. \end{aligned}$$

Finally, apply again Lemma 7.3.4 to the inner integral

$$\frac{1}{2\pi}\int_{-\infty}^\infty \hat{g}(-\frac{1}{2}+i(r-b))\nu^{ir} dr = \nu^{\frac{1}{2}+ib} g(\nu),$$

then putting it together we obtain the desired form of (7.3.2)

$$-pv_0 \int_0^\infty \frac{f_0(\nu)^{2a}}{f_1(\nu^2)} \nu^{\frac{1}{2}+ib} h(\nu) d^\times \nu.$$

$\square$

We apply this lemma to obtain an integral over the archimedean Weil group,

Corollary 7.3.8. *Let χ_v be a character of the local archimedean Weil group W_v, then the expression (7.3.2) can be rewritten as*

$$-pv_0 \int_{W_v} g(|w|)\chi_v(w) \frac{|w|_v}{|1-w|_v} dw. \tag{7.3.3}$$

Proof. The proof of this follows immediately from (7.3.2) by Lemma 3 of [Mor05, p.165], with the function φ of the lemma replaced with

$$\varphi(w) = \frac{|w|_v}{|1-w|_v}.$$

In particular, in the numerator we have $|w|_v$ instead of $|w_v|^{\frac{1}{2}}$. $\square$

Lemma 7.3.9 (Contribution of conductor)**.** *Consider the Herbrand distribution H_v on the local Weil groups W_v, which is described in [Wei74, Ch.VIII, §3, XII, §4] and [Mor05, Ch.II, §6]. It is given by*

$$H_v(\chi) = \int_{W_v^0} \chi_v(w_0) dH_v(w_0)$$

where χ_v is a character of the restriction to W_v of a unitary representation of the

Weil group W_F. The contribution of the conductor can be expressed as

$$g(1)\log|\mathfrak{f}(\chi)| = -\sum_v \log q_v \int_{W_v^0} g(|w_0|)\chi_v(x)dH_v(w_0).$$

Proof. We follow [Wei72, p.17] and [Mor05, p.158]. The local Artin conductor $\mathfrak{f}_v(\chi)$ can be written as the integral

$$\int_{W_v^0} \chi(x)dH_v(x),$$

so that the contribution from the conductor for each place v where p_v ramifies is

$$\log p_v \int_{W_v^0} h(|w_0|)\chi(w_0)dH_v(w_0),$$

and zero otherwise.

By the product formula for the conductor, the contribution of the conductor is then

$$\begin{aligned}
g(1)\log|\mathfrak{f}(\chi)| &= \sum_v g(1)\log(q_v^{-H_v(\chi)}) \\
&= -g(1)\sum_v (\log q_v)H_v(\chi) \\
&= -g(1)\sum_v \log q_v \int_{W_v^0} \chi_v(w_0)dH_v(w_0) \\
&= -\sum_v \log q_v \int_{W_v^0} g(|w_0|)\chi_v(w_0)dH_v(w_0).
\end{aligned}$$

as desired. □

Using this expression we are able to combine the archimedean and nonarchimedean integrals.

Definition 7.3.10. Let g be a locally constant function on W_v and suppose that the integral

$$\int_{W-W_v^0} \frac{g(w)}{|1-w|} dw$$

exists. Then define the 'principal value' by

$$pv_0 \int_{W_v} \frac{g(w)}{|1-w|} dw = \int_{W_v^0} \frac{g(w)-g(1)}{|1-w|} dw + \int_{W-W_v^0} \frac{g(w)}{|1-w|} dw.$$

Corollary 7.3.11. *The contribution of the nonarchimedean integrals and the conductor can be combined to obtain*

$$pv_0 \int_{W_v} \hat{g}(|w|)\chi_v(w) \frac{|w|}{|1-w|} dw.$$

where $1-w$ *and* $|1-w|$ *are to have the same sense by the embedding of* W_v *into the division algebra* A_v*, which follows from Shafarevitch (see [Mor83, Part I, Chap. VIII]).*

Finally, we prove the main theorem:

Proof of Theorem 7.3.3. First, from the logarithmic derivative of $m(s, \chi_{\mathbf{m}})$ in (4.4.2), we obtain the first term immediately by Mellin inversion

$$\begin{aligned}\frac{1}{4\pi}\int h\log|d_F|\hat{g}(i\mathbf{r}_k)dr &= -\frac{1}{2}g(0)\log|d_F| \\ &= -\sum_v \log q_v \int_{W_v^0} g(|w_0|)\chi_v(w_0)dH_v(w_0),\end{aligned}$$

where the last equality follows from Lemma 7.3.9.

Second, consider the individual summands involving the gamma factors, which converge as $\prod \hat{g}_k(r)$ has rapid decay at infinity. They take the form

$$\frac{1}{4\pi}\int_{-i\infty}^{i\infty}\Big(\frac{\Gamma_k'}{\Gamma_k}(1+ir+\mathbf{e}_k)+\frac{\Gamma_k'}{\Gamma_k}(1-ir-\mathbf{e}_k)\Big)\hat{g}(i\mathbf{r}_k)dr.$$

As in the case over $\mathbf{Q}$, we use the property of $\hat{g}$ as an even function, and apply Lemma 7.3.7 to obtain

$$-\frac{1}{2}pv_0\int_{W_v}\hat{g}(|w|)\chi_v(w)\frac{|w|}{|1-w|_v}dw,$$

where the constants a, b depending on the ramification are accounted for by the local character χ_v.

Third, for the $\zeta_F(s)$ terms, write $L_v(s,\chi)$ for the local L-factor in the Euler product for $\zeta_F(s,\chi)$. For absolute convergence, we shift the contour slightly to the right of $\mathrm{Re}(s) = 1$, where we may use the Euler product expansion for the logarithmic derivative:

$$\frac{1}{4\pi i}\int_{1-i\infty}^{1+i\infty}\frac{\zeta_F'}{\zeta_F}(s,\chi)\hat{g}(\mathbf{s}_k)ds = \sum_v \frac{1}{4\pi i}\int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty}\frac{L_v'}{L_v}(s,\chi)\hat{g}(\mathbf{s}_k)ds + \frac{\delta_\chi}{4}g(1)$$

for some $\epsilon > 0$, and δ_χ is 1 if χ is the trivial character, contributing half the residue at $\mathrm{Re}(s) = 1$, and zero otherwise. As in [Wei72, p.12], the residue at 1 can be expressed as an integral over the global Weil group:

$$\frac{1}{4}\int_{W_F} g(|w|)\chi(w)\frac{dw}{|w|}.$$

To each local L-factor we apply Lemmas 7.3.4 and 7.3.5 as follows: set $\sigma = \frac{1}{2}$, and write

$$\begin{aligned}
&\frac{1}{4\pi i}\int_{-i\infty}^{i\infty}\hat{g}(s)\frac{d}{ds}\log L_v(1+s,\chi)\\
&= -\frac{\log q_v}{4\pi i}\int_{-\infty}^{\infty}\hat{g}(it)(\sum_{n=1}^{\infty} q_v^{-n(1+it)}\int_{W_v^1}\chi_v(\mathfrak{f}^n w)dw)dt\\
&= -\frac{1}{2}\log q_v\sum_{n=1}^{\infty} g(q_v^{-n})q_v^{-n}\int_{W_v^1}\chi_v(\mathfrak{f}^n w)dw.
\end{aligned}$$

On the other hand, we have

$$\frac{1}{4\pi i}\int_{-\infty}^{\infty}\hat{g}(s)\frac{d}{ds}\log L_v(1-s,\overline{\chi}) = -\frac{1}{2}\log q_v\sum_{n=1}^{\infty} g(q_v^{n})q_v^{-n}\int_{W_v^1}\overline{\chi}_v(\mathfrak{f}^n w)dw,$$

plus the same contribution associated to the residue at 1, since $\overline{\chi}$ is trivial if χ is.

Then using the fact that $\overline{\chi}(w) = \chi(w^{-1})$ and

$$W_v - W_v^0 = \bigcup_{n \in \mathbf{Z} - \{0\}} f_v^n W_v^0$$

we combine the two terms two obtain

$$-\frac{1}{2}\int_{W_v - W_v^1} \hat{g}(|w|)\chi_v(w)\inf(|w|, |w|^{-1})dw = -\frac{1}{2}pv_0 \int_{W_v - W_v^1} \hat{g}(|w|)\chi_v(w)\frac{|w|}{|1-w|}dw$$

(cf. [Mor05, p.158]). Then we apply Corollary 7.3.11 to combine the contribution of the nonarchimedean L-factors and the conductor,

$$-\frac{1}{2}pv_0 \int_{W_v} \hat{g}(|w|)\chi_v(w)\frac{|w|}{|1-w|}dw,$$

which is identical to the contribution of the archimedean factors.

Finally, putting this all together we have for each χ extending the fixed character $\chi_{\mathbf{m}}$,

$$-\frac{1}{2}g(0)\log|d_F| + \frac{1}{2}\int_{W_F} g(|w|)\chi(w)\frac{dw}{|w|} + \frac{1}{2}pv_0 \int_{W_v} \hat{g}(|w|)\chi_v(w)\frac{|w|}{|1-w|}dw$$

as desired. □

7.4 Application: Lower bounds for sums

The presence of the sums over zeroes, or equivalently, the Weil distributions gives us an approach to the zeroes of L-functions using the trace formula. The key to this analysis will be the following positvity result, proved by Arthur following an idea of Selberg, valid for reductive groups G. Though the case we are interested is also covered by Remark 2.1.2. First, we review the truncation operator in earnest, referring to the exposition of [Art05, §13] and onwards for details:

Definition 7.4.1. Let G be a reductive group over $\mathbf{Q}$, and G^1 the elements of norm one. Let ϕ be a locally bounded, measurable function on $G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}}$, and T a suitably regular point in the positive root space $\mathfrak{a}_0^+$ generated by the roots of the maximal torus A of G. Define

$$\Lambda^T\phi = \sum_P (-1)^{\dim A_P/A_G} \sum_{\delta\in P_{\mathbf{Q}}\backslash G_{\mathbf{Q}}} \int_{N_P(\mathbf{Q})\backslash N_P(\mathbf{A})} \phi(n\delta x)\hat{\tau}_P(\log(H_P(\delta x)-T))dn$$

where $\hat{\tau}_P$ is the characteristic function of the positive Weyl chamber associated to the parabolic subgroup P, and $H_P(x)$ is the usual height function, which in the case of GL_2 is

$$H_P(n\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k) = (|a|,|b|).$$

The inner sum is finite, while the integrand is a bounded function of n. In particular, if ϕ is in $L^2_{\text{cusp}}(G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}})$ then $\Lambda^T\phi = \phi$; also $\Lambda^T E(g,\phi,s)$ is square integrable. It is self-adjoint and idempotent, hence an orthogonal projection.

Using this, Arthur shows that the spectral side

$$\operatorname{tr}(\rho(f)) = \sum_\chi J^T_\chi(f) = \sum_\chi \int_{G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}}} \Lambda^T_2 K_\chi(x,x)dx$$

converges absolutely, where the subscript on Λ^T_2 indicates truncation with respect to the second variable, and the index χ corresponds to the η in the case of SL_2 above. This gives what Arthur calls the coarse spectral expansion of the trace formula. We have the expression for the distribution $J_\chi^T(f)$ as

$$\begin{aligned}&\int_{G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}}} \Lambda_2^T K_\chi(x,x)dx \\ &= \sum_P \frac{1}{n_P} \int_{G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}}} \int_{i\mathfrak{a}_P^*} \sum_\phi \Lambda^T E(x,\rho(f,\eta,s)\phi,\lambda)\overline{\Lambda^T E(x,\phi,\lambda)}d\lambda\ dx\end{aligned}$$

where n_P is the number of chambers in $\mathfrak{a}_P$, and ϕ runs over an orthonormal basis of $\rho(f,\eta,s)$.

Example 7.4.2. When $G = GL_2$, there is only one chamber, and the inner integral is one dimensional, that is, $i\mathfrak{a}_P^* = i\mathbf{R}$. Furthermore, by absolute convergence we may interchange the sum over ϕ with the integral, to obtain the expression for GL_2

$$J_\chi^T(f) = \sum_\phi \int_{-\infty}^{\infty} (\Lambda^T E(x,\rho(f,\eta,s)\phi,s), \Lambda^T E(x,\phi,s))d|s|$$

thus we have an absolutely convergent sum-integrals of an inner product. The case of SL_2 is essentially the same.

Now, the following is a consequence of Arthur's method:

Lemma 7.4.3. *$J_\chi^T(f)$ is a positive-definite distribution.*

Proof. Using Arthur's truncation Λ^T with respect to the parameter T and Arthur's general notation, the intertwining operator is given as

$$(M^T_{P,\chi}(\lambda)\phi', \phi) = \int_{G_{\mathbf{Q}}\backslash G^1_{\mathbf{A}}} \Lambda^T E(x, \phi', \lambda)\overline{\Lambda^T E(x, \phi, \lambda)}dx$$

for any vectors ϕ', ϕ in the induced representation space. It is an integral of the usual L^2-inner product of truncated Eisenstein series (it is square integrable after truncation), so it follows that $M^T_{P,\chi}(\lambda)$ is positive-definite, self-adjoint operator.

Then, following the proof of the conditional convergence of J^T_χ in [Art05, §7], we see that for any positive-definite test function $f * f^*$ with $C_c^\infty(G_{\mathbf{A}})$, the resulting double integral is nonnegative, and the integrals can be expressed as an increasing limit of nonnegative functions. The integral converges, and $J^T_\chi(f)$ is positive-definite.

Alternatively, if we use the known absolute convergence of the spectral side, we may interchange the sum and integrals to obtain the inner product expression as above, and observe again that the inner product is positive-definite. □

For GL_n, the parameter T can be set equal to 1 to obtain the constant term of of the polynomial $J^0_\chi(f) =: J_\chi(f)$. Then the positivity of $J_\chi(f)$ immediately gives a lower bound for the sums over zeroes for any class χ and function f.

We now illustrate our method in the most basic case:

Theorem 7.4.4. *Let $g = g_0 * g_0^*$, for any g_0 in $C_c^\infty(\mathbf{R}_+^\times)$. Then the sum over zeroes* *of $\zeta(s)$, where $0 < \mathrm{Re}(\rho) < 1$, is bounded below by the following:*

$$\begin{aligned}\sum_{\rho} \hat{g}(\rho) \geq & \int_0^\infty \left\{ g(x) + \frac{1}{4} g^*(x) \right\} dx - \sum_{n=1}^{\infty} \Lambda(n) g(n) \\ & - \int_1^\infty \left\{ g(x) + g^*(x) - \frac{2}{x} g(1) \right\} \frac{x\ dx}{2(x^2-1)} - \frac{1}{2}(\log 4\pi + \gamma) g(1), \\ & - g(0) \log T - \frac{1}{4\pi i} \int_{-\infty}^{\infty} m(it) g(it) \frac{T^{it}}{t} dt.\end{aligned}$$

for any $T > \sqrt{3}/2$.

Proof. Consider again the action of $PSL_2(\mathbf{R})$ on $L^2(SL_2(\mathbf{Z})\backslash\mathbf{H}_2)$. The continuous spectral terms in the resulting trace formula described in (3.3.2), we write here as

$$\begin{aligned}\int_{\Gamma\backslash G} \Lambda^T K_{\mathrm{cont}}(x,x) dx = & \, g(0) \log T - \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{m'}{m}(it) g(it) dt \\ & + \frac{1}{4\pi i} \int_{-\infty}^{\infty} m(it) g(it) \frac{T^{it}}{t} dt\end{aligned}$$

From the previous lemma, for our choice of g the above expression is nonnegative. The first term described in Theorem 7.2.1,

$$\begin{aligned}& \sum_{\rho} \hat{g}(\rho) + \int_0^\infty \left\{ g(x) - \frac{1}{4} g^*(x) \right\} dx + \sum_{n=1}^{\infty} \Lambda(n) g(n) \\ & + \int_1^\infty \left\{ g(x) + g^*(x) - \frac{2}{x} g(1) \right\} \frac{x\ dx}{2(x^2-1)} + \frac{1}{2}(\log 4\pi + \gamma) g(1).\end{aligned}$$

The requirement that $T > \sqrt{3}/2$ simply follows from the observation that the fundamental domain of $SL_2(\mathbf{Z})\backslash\mathbf{H}_2$ has height at least $\sqrt{3}/2$ in $\mathbf{H}_2$. Then the claim follows immediately from rearranging the terms. $\square$

Chapter 8

Appendix: The trace formula for $GL_n(F)$

Having dealt with the rank one case, we want to set the scene for higher rank groups. The picture for GL_n will make it clear how one should proceed for other groups. We will use Arthur's refined noninvariant trace formula. Leaving the rank one case, the formulae become much more involved. Nonetheless, the case of maximal parabolic Eisenstein series is analogous to that of GL(2). Instead of Hecke L-functions we encounter Rankin-Selberg L-functions in the constant terms of these Eisenstein series.

8.1 Preliminaries

Let G be a reductive group. There is an orthogonal decomposition

$$L^2(G_{\mathbf{Q}}\backslash G_{\mathbf{A}}) = \bigoplus_{\chi\in\mathfrak{X}} L^2_{\chi}(G_{\mathbf{Q}}\backslash G_{\mathbf{A}})$$

into $G_{\mathbf{A}}$-invariant subspaces indexed by equivalence classes (i.e., up to conjugacy) χ of pairs (P, σ) where P is a standard parabolic subgroup of G and σ an irreducible cuspidal automorphic representation of the Levi subgroup $M_P(\mathbf{A})^1$.

Note that we will be concerned mostly with the simpler case in which χ is *unramified*, that is, for every pair (P, σ) the stabilizer of σ in the Weyl group $W(\mathfrak{a}_P, \mathfrak{a}_P)$ is $\{1\}$. In this case, there are no cuspidal intertwining operators, so the residual discrete spectrum associated to χ is automatically zero.

8.1.1 The refined trace formula

The general formula for the spectral side is, for any $f \in C_c^\infty(G_{\mathbf{A}}^1)$,

$$J_\chi^T(f) = \sum_P \frac{1}{n_P} \sum_\sigma \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \mathrm{tr}(\Omega_{\chi,\sigma}^T(P,\lambda)\rho_\chi(\sigma,\lambda,f))d\lambda$$

where the first sum is over 'associated' parabolics P, n_P is the number of Weyl chambers in $\mathfrak{a}_P$, the second sum if over classes of irreducible unitary representations of $M_{\mathbf{A}}$, and the operator

$$(\Omega_{\chi,\sigma}^T(P,\lambda)\phi', \phi) = \int_{Z_{\mathbf{A}}G_F\backslash G_{\mathbf{A}}} \Lambda^T E(x,\phi',\lambda)\overline{\Lambda^T E(y,\phi,\lambda)}dx$$

is a self-adjoint, positive-definite operator, and is asymptotically equal to

$$\omega_{\chi,\sigma}^T(P,\lambda) = \underset{\lambda=\lambda'}{\mathrm{val}} \sum_{Q\in P(M_P)} \sum_{s\in W(\mathfrak{a}_P)} M_{Q|P}(0,\lambda)^{-1} M_{Q|P}(s,\lambda') \frac{e^{(s\lambda'-\lambda)Y_Q(T)}}{\theta_Q(s\lambda'-\lambda)}$$

with error term decaying exponentially in T. Finally, $\rho_\chi(\sigma,\lambda)$ is the induced representation acting on functions ϕ on $G_{\mathbf{A}}$ whose restriction to $M_P(\mathbf{A})^1$ belongs to $L^2_\chi(M_P(\mathbf{Q})\backslash M_P(\mathbf{A}))$ consisting of functions such that

$$(\rho_\chi(\sigma,\lambda,y)\phi)(x) = \phi(xy)e^{(\lambda+\rho_P)H_P(xy)}e^{-(\lambda+\rho_P)H_P(x)}$$

which defines the integral operator

$$(\rho_\chi(\sigma,\lambda,f)\phi)(x) = \int_{G_{\mathbf{A}}} f(y)(\rho_\chi(\sigma,\lambda,y)\phi)(x)dy$$

for any $f \in C_c^\infty(G_{\mathbf{A}})$ and ϕ as above.

Arthur shows that $J^T_\chi(f)$ is a polynomial in T, which a priori is a sufficiently regular vector in $\mathfrak{a}^+$, in the sense that $\alpha(T)$ is large for each root $\alpha \in \Delta_0$, thus it extends to all $i\mathfrak{a}$. For our purposes, it will simplify the formula by evaluating T at the distinguished point T_0, which for $\mathrm{GL}(n)$ is just 0. By definition,

$$J_\chi(f) = J^{T_0}_\chi(f)$$

is the constant term, and is what finally appears in the (coarse) noninvariant trace formula. This is the form we will use for the rank one case.

We now recall Arthur's fine spectral expansion of $J_\chi(f)$. Fix a minimal Levi M_0 of $GL(n)$. If $M \simeq GL_{n_1} \times \cdots \times GL_{n_r}$, the Lie algebra of the split torus A_M is $\mathfrak{a}_M \simeq \mathbf{R}^r$. Fix a minimal Levi M_0 of G. For any pair of Levis $M \subset L$, there is a surjection $\mathfrak{a}_M \to \mathfrak{a}_L$ with kernel $\mathfrak{a}_M^G$. Then

$$J_\chi(f) = \sum_{M \supset M_0} \sum_{L \supset M} \sum_s \sum_\pi \sum_{P \supset M} a_M^L(s) J_{M,P}^L(f,s)$$

where the finite sums are taken over (i) Levi subgroups M containing M_0, (ii) Levi subgroups L containing M, (iii) elements s in $W^L(\mathfrak{a}_M)_{\mathrm{reg}} = \{t \in W^L(\mathfrak{a}_M) : \ker(t) = \mathfrak{a}_L\}$, (iv) equivalence classes of irreducible unitary representations π of $M(\mathbf{A})^1$, and (v) the set of parabolic subgroups $P(M)$ having Levi subgroup M, of

$$a_M^L(s) = |W_0^M| \big(|W_0^G| |\det(s-1)_{\mathfrak{a}_M^L}| |P(M)| \big)^{-1},$$

a constant we will not be very concerned with, and

$$J_{M,P}^L(f,s) = \int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \mathrm{tr}(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0) \rho_{\chi,\pi}(P,\lambda,f)) d\lambda$$

where $M_{P|P}(s,0)$ is the intertwining operator $M_{Q|P}(s,\lambda)$ at $P = Q, \lambda \in i\mathfrak{a}_L^*$ and $\mathfrak{M}_L(P,\lambda)$ is given by its restriction to π,

$$\mathfrak{M}_L(P,\pi,\lambda) = \sum_S \mathfrak{N}_S'(P,\pi,\lambda) \nu_L^S(P,\pi,\lambda)$$

where S runs over parabolic subgroups containing L,

$$\mathfrak{N}_S'(P,\pi,\lambda) = \frac{1}{q!} \sum_{R \supset S} (-1)^q \hat{\theta}_S^R(\Lambda) \Big(\lim_{t \to 0} \Big(\frac{d}{dt} \Big)^q \mathfrak{N}_R(P,\pi,\lambda,t\Lambda) \Big) \theta_R(\Lambda)^{-1}$$

where the important point is that the operator is built out of normalized intertwining operators $N_{Q|P}(\pi,\lambda) = r_{Q|P}(\pi,\lambda)^{-1}M_{Q|P}(\pi,\lambda)$ on the local groups $G(\mathbf{Q}_v)$, and

$$\nu_L^S(P,\pi,\lambda) = \sum_F \mathrm{vol}(\mathfrak{a}_L^S/\mathbb{Z}(F_L^\vee)) \prod_{\alpha\in F} r_{\overline{P}_\alpha|P_\alpha}(\pi,\lambda(\alpha^\vee))^{-1} r'_{\overline{P}_\alpha|P_\alpha}(\pi,\lambda(\alpha^\vee))$$

where F runs over all subsets of reduced roots of S relative to A_M such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^S$.

Consider the following cases by corank of L:

1. (a) $M = L = G$. This is the discrete spectrum $\Pi_{\mathrm{disc}}(G_{\mathbb{A}}^1)$,

$$\sum_\pi \mathrm{tr}(\rho_{\chi,\pi}(G,0,f))$$

which separates further cuspidal and non-cuspidal representations.

   (b) $M \neq L = G$. We have the sum

$$\sum_s \sum_\pi \sum_P a_M^L(s)\mathrm{tr}(M_{P|P}(s,0)\rho_{\chi,\pi}(P,0,f))$$

sometimes referred to as the 'discrete' part of the continuous spectrum.

2. (a) $M = L \neq G$, $\dim \mathfrak{a}_L^G = 1$. Let α be the unique simple root of P, ϖ be the element in $(\mathfrak{a}_L^G)^* = \mathbf{R}$ such that $\varpi(\alpha^\vee) = 1$, and $\bar{P}$ the opposite parabolic. Then

$$-\sum_\pi a_M^L(s)\mathrm{vol}(\mathfrak{a}_L^G/\mathbb{Z}\alpha^\vee) \int_{i\mathbf{R}} \mathrm{tr}(M_{\bar{P}|P}(z\varpi)^{-1}M'_{\bar{P}|P}(z\varpi)\rho_{\chi,\pi}(P,z\varpi,f))d|z|$$

where again $M_{\bar{P}|P}^{-1} M'_{\bar{P}|P}$ separates into

$$r_{P|\bar{P}}(\pi, z)^{-1} r'_{P|\bar{P}}(\pi, z) + \sum_{v \in S} N_{\bar{P}|P}(\pi_v, z\varpi)^{-1} \frac{d}{dz} N_{\bar{P}|P}(\pi_v, z\varpi)$$

(b) $M \neq L \neq G$, $\dim \mathfrak{a}_L^G = 1$. We have the sum over s, π, and P of

$$-a_M^L(s) \mathrm{vol}(\mathfrak{a}_L^G / \mathbb{Z}\alpha^\vee) \int_{i\mathbf{R}} \mathrm{tr}(M_{\overline{P}|P}(z\varpi)^{-1} M'_{\overline{P}|P}(z\varpi) M_P(s, 0) \rho_{\chi,\pi}(P, z\varpi, f)) d|z|$$

3. $M = L \neq G$, $\dim \mathfrak{a}_L^G = k > 1$. First, using the isomorphism $(\mathfrak{a}_L^S)^* \oplus (\mathfrak{a}_S^G)^* \simeq (\mathfrak{a}_L^G)^*$, we may write

$$\lambda = \sum_{\alpha \in F} z_\alpha \varpi_\alpha + \lambda_1$$

where ϖ_α is a basis of $(\mathfrak{a}_L^S)^*$ dual to $F_L^\vee$, $z_\alpha \in i\mathbf{R}$ and $\lambda_1 \in i\mathfrak{a}_S^* / i\mathfrak{a}_G^*$. Then the integral of

$$\sum_F \mathrm{vol}(\mathfrak{a}_L^S / \mathbf{Z}(F_L^\vee)) \prod_{\alpha \in F} r_{\overline{P}_\alpha | P_\alpha}(\pi, z_\alpha)^{-1} r'_{\overline{P}_\alpha | P_\alpha}(\pi, z_\alpha)$$

times

$$\mathfrak{N}'_S(P, \pi, \lambda) \rho_{\chi,\pi}(P, \lambda, f)$$

summed over S, will factor as k integrals over $i\mathbf{R}$ and an integral over $i(\mathfrak{a}_S^G)^*$. Now $\rho_{\chi,\pi}(P, \lambda)$ is an induced representation with complex parameter λ, where the norm will factor immediately as before. Finally, $\rho_{\chi,\pi}(P, \lambda)$ is the regular representation on the subspace of automorphic forms ϕ where $\phi_x(m) = \phi(mx)$ belongs to the π isotypic subspace of $L^2(M_P(\mathbf{Q}) \backslash M_P(\mathbf{A}))$.

8.1.2 Rankin-Selberg L-function

To fix notation, we review Rankin-Selberg L-functions briefly. For GL_n, the Langlands-Shahidi method gives an expression for the local intertwining operator: Let $P_1, P_2 \in P(M)$,

$$I(\sigma_1, \sigma_2) = \{(i,j) : 1 \leq i, j \leq r, \sigma_1(i) < \sigma_1(j), \sigma_2(i) > \sigma_2(j)\}$$

where $n_1 + \cdots + n_r = n$ and $\sigma_i \in S_r$. Let $\pi_v = \pi_{1,v} \otimes \cdots \otimes \pi_{r,v}$ for $\pi_{i,v}$ in $\Pi(GL_{n_i}(\mathbb{Q}_v))$, and $\mathbf{s} = (s_1, \ldots, s_r)$ in $\mathbb{C}^r$. Then

$$r_{P_1|P_2}(\pi_v, \mathbf{s}) = \prod_{I(\sigma_1,\sigma_2)} \frac{L(s_i - s_j, \pi_{i,v} \times \tilde{\pi}_{j,v})}{L(1 + s_i - s_j, \pi_{i,v} \times \tilde{\pi}_{j,v})\epsilon(s_i - s_j, \pi_{i,v} \times \tilde{\pi}_{j,v}, \psi_v)}$$

is the normalizing factor.

If P_α is a maximal parabolic, its only associated parabolic is its opposite $\bar{P}_\alpha$, and the representation π of $M_P(\mathbf{A})^1$ factors as $\pi_1 \otimes \pi_2$, giving

$$r_{P_\alpha|\bar{P}_\alpha}(\pi, s) = \frac{L(s, \pi_1 \times \tilde{\pi_2})}{L(1 + s, \pi_1 \times \tilde{\pi_2})\epsilon(s, \pi_1 \times \tilde{\pi}_2, \psi)} = \frac{L(s, \pi_1 \times \tilde{\pi}_2)\epsilon(-s, \tilde{\pi}_1 \times \pi_2, \psi)}{L(-s, \tilde{\pi}_1 \times \pi_2)\epsilon(s, \pi_1 \times \tilde{\pi}_2, \psi)}$$

where ψ is some additive character, and $s_1 = s/2, s_2 = -s/2$. The associated functional equation is

$$L(s, \pi_1 \times \tilde{\pi_2}) = \epsilon(s, \pi_1 \times \tilde{\pi_2})L(1 - s, \tilde{\pi}_1 \times \pi_2)$$

in which

$$\epsilon(s, \pi_1 \times \tilde{\pi_2}) = W(\pi_1, \tilde{\pi}_2)A^{\frac{1}{2} - s}$$

where $W(\pi_1, \tilde{\pi}_2)$ is the root number satisfying $W(\pi_1, \tilde{\pi}_2)W(\tilde{\pi}_1, \pi_2) = 1$, and $A = q^{f(\pi_1 \times \tilde{\pi}_2)}$ is a positive integer such that $f(\pi_1 \times \tilde{\pi}_2) = f(\tilde{\pi}_1 \times \pi_2)$.

Explicitly, if π_1 is the Langlands subquotient from the induced representation $\mu_1, \ldots, \mu_{n_1}$ on GL_{n_1} and π_2 from $\nu_1, \ldots, \nu_{n_2}$ on GL_{n_2}, then

$$L(s, \pi_1 \otimes \tilde{\pi}_2, r) = L(s, \pi_1 \times \tilde{\pi}_2) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} L(s, \mu_i \nu_j^{-1})$$

More generally, let δ be a discrete series on GL_a, where $n = ab$. The induced representation

$$\mathrm{Ind}_{GL_a \times \cdots \times GL_a}^{GL_n}(\delta[\frac{b-1}{2}] \otimes \delta[\frac{b-3}{2}] \otimes \cdots \otimes \delta[\frac{1-b}{2}])$$

has an irreducible sub-quotient called the Speh representation, $J(\delta, b)$. Here $[t_i]$ indicates a twist by $|\det|^{t_i}$. By inducing in stages, a representation π occurring in the residual spectrum must have local components

$$\pi_v \simeq \mathrm{Ind}_{GL_{a_1 b} \times \cdots \times GL_{a_r b}}^{GL_n}(J(\delta_1, b)[t_1] \otimes \cdots \otimes J(\delta_r, b)[t_r]), \qquad \max |t_i| < \frac{1}{2} - \frac{1}{n^2+1}$$

where $\sum a_i = a$. From this, Rudnick-Sarnak show that for general π given as a Langlands quotient $J(\sigma_1[t_1], \ldots, \sigma_r[t_r])$, and assume $\sigma_i = J(n_i, \rho_i)$

$$L(s, \pi_i \times \tilde{\pi}_j) = \prod_{i,j=1}^{r} L(s+t_i-t_j, \sigma_i \times \tilde{\sigma}_j) = \prod_{i,j=1}^{r} \prod_{k=1}^{\min(n_i,n_j)} L(s+t_i-t_j+\frac{n_i+n_j}{2}-k, \rho_i \times \tilde{\rho}_j)$$

The analytic property we will use is the following:

Theorem 8.1.1. ([JS90, JPSS83]) *Let π_i be cuspidal representations of $GL(n_i)$, and S be a finite set of places. Then $L^S(s, \pi_1 \times \tilde{\pi}_2)$ is entire unless $\pi_1 \simeq \pi_2 \otimes |\cdot|^w$ for some w in $\mathbf{C}$, in which case it is holomorphic except for simple poles at $s = 1 - w$ and $-w$.*

8.2 The spectral expansion

Consider the simplest case $M = L \neq G$, $\dim \mathfrak{a}_L^G = 1$. That is, the class χ is associated to a Levi M of a maximal parabolic subgroup. Then $J_\chi(f)$ is equal to the sum of the 'continuous' term

$$-\sum_{\pi} \frac{|W_0^M|}{|W_0^G|} \mathrm{vol}(\mathbf{R}/\mathbb{Z}\alpha^\vee) \int_{i\mathbf{R}} \mathrm{tr}(M_{\bar{P}|P}(s\varpi)^{-1} M'_{\bar{P}|P}(s\varpi)\rho_{\chi,\pi}(P, s\varpi, f)) d|s| \tag{8.2.1}$$

where $M_{\bar{P}|P}^{-1} M'_{\bar{P}|P}$ separates into

$$r_{P|\bar{P}}(\pi, s)^{-1} r'_{P|\bar{P}}(\pi, s) + \sum_{v \in S} N_{\bar{P}|P}(\pi_v, s\varpi)^{-1} \frac{d}{dz} N_{\bar{P}|P}(\pi_v, s\varpi),$$

and, if χ is ramified, the 'discrete' term

$$\sum_{s} \sum_{\pi} \sum_{P} a_M^L(s) \mathrm{tr}(M_{P|P}(s, 0)\rho_{\chi,\pi}(P, 0, f)). \tag{8.2.2}$$

Note that these generalize the terms (3.1) and (3.3).

Now take the logarithmic derivative $r_{P|\bar{P}}(\pi, s)$ to get

$$\frac{L'}{L}(s, \pi_1 \times \tilde{\pi}_2) + \frac{L'}{L}(-s, \tilde{\pi}_1 \times \pi_2) - \frac{\epsilon'}{\epsilon}(-s, \tilde{\pi}_1 \times \pi_2) - \frac{\epsilon'}{\epsilon}(s, \pi_1 \times \tilde{\pi}_2)$$

The first term produces the integrand

$$-\sum_{\pi} \frac{|W_0^M|}{|W_0^G|} \mathrm{vol}(\mathbf{R}/\mathbb{Z}\alpha^\vee) \int_{i\mathbf{R}} \frac{L'}{L}(s, \pi_1 \times \tilde{\pi}_2) \mathrm{tr}(\rho_{\chi,\pi}(P, s\varpi, f)) d|s|$$

As before, we view $\mathrm{tr}(\rho_{\chi,\pi}(P, s\varpi, f))$ as a Fourier transform on the diagonal:

$$\int_K \int_{M_P} \int_{N_P(\mathbf{A})} f(k^{-1}ank) \Big| \frac{\det m_1}{\det m_2} \Big|^{n_2/n_1+s} dn\ da\ dk$$

where we use the usual character on each $GL(n_i)$ factor of the Levi M

$$H_P(m_i) = \log|\det m_i|^{1/n_i}$$

and following Lemma 3.1.3, we write this trace as a function $h(s)$.

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