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On Sums of Binary Hermitian Forms

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On Sums of Binary Hermitian Forms

by

Cihan Karabulut

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

On Sums of Binary Hermitian Forms

by

Cihan Karabulut

Advisor: Gautam Chinta

In one of his papers, Zagier defined a family of functions as sums of powers of quadratic polynomials. He showed that these functions have many surprising properties and are related to modular forms of integral weight and half integral weight, certain values of Dedekind zeta functions, Diophantine approximation, continued fractions, and Dedekind sums. He used the theory of periods of modular forms to explain the behavior of these functions. We study a similar family of functions, defining them using binary Hermitian forms. We show that this family of functions also have similar properties.

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Chapter 1

Introduction

1.0.1 History and motivation

In [18], Zagier defines a family of functions that are defined as sums of powers of certain quadratic polynomials with integer coefficients and fixed discriminant, and he discovers that these functions have many surprising properties. Specifically, Zagier considers the following family of functions:

For non-square discriminant $D \equiv 0, 1 \pmod{4}$ and k a positive even integer, define a function $F_{k,D} : \mathbb{R} \rightarrow \mathbb{R}$ as follows,

$$F_{k,D}(x) := \sum_{\substack{\text{disc}(Q)=D \\ a < 0 < Q(x)}} Q(x)^{k-1} \quad (1.1)$$

where $Q(X) = aX^2 + bX + c$ with $(a, b, c) \in \mathbb{Z}^3$ and $\text{disc}(Q) = b^2 - 4ac$. When D is a square discriminant, one has to add a simple correction term consisting of the k -th Bernoulli polynomial. But for the sake of simplicity, let us assume that D is a non square discriminant.

For instance, Zagier finds that $F_{2,5}$ is constant with value $F_{2,5}(x) = 2$ for any $x \in \mathbb{R}$ despite the fact that there are infinitely many Q 's contributing to the sum for $F_{2,5}$ when x is irrational. More generally, $F_{2,D}(x) = -5L(-1, \chi_D)$ where $L(-1, \chi_D)$ is the Dirichlet L -series of the character $\chi_D(n) = \left(\frac{D}{n}\right)$ (Kronecker symbol). For $k = 4$, Zagier shows that $F_{4,D}(x) = L(-3, \chi_D)$, but for $k \geq 6$ $F_{k,D}$ is no longer constant because of the existence of cusp forms of weight $2k$ on the modular group $\mathrm{SL}(2, \mathbb{Z})$. More precisely, for $k \geq 6$, the function $F_{k,D}$ is a linear combination of a constant function and the functions $\sum_{n \geq 1} \frac{a_f(n)}{n^{2k-1}} \cos(2\pi nx)$, where f runs over the normalized Hecke eigenforms in the space of cusp forms of weight $2k$ for the modular group and $a_f(n)$ denotes the n -th Fourier coefficient of f .

The function $F_{k,D}(x)$ is related to a family of cusp forms $f_{k,D}$, defined in chapter 2, that Zagier introduced in [17]. These cusp forms were later studied by Kohnen and Zagier ([10], [12]) in connection with Doi-Naganuma lifting from modular forms to Hilbert modular forms and in connection with Shimura-Shintani correspondence between modular forms of integral and half-integral weight. Kohnen and Zagier [11] also used $f_{k,D}$ as an example of modular forms whose periods are rational and arithmetically very interesting. The precise relation between the function $F_{k,D}$ and $f_{k,D}$ is that $F_{k,D}$ is the even part of the Eichler integral, also defined in chapter 2, of $f_{k,D}$ on \mathbb{R} and so gives the even part of the period polynomial of $f_{k,D}$. This relation beautifully explains a natural question that might have occurred to the reader by

now as to why $F_{k,D}$ is constant for $k = 2, 4$ but not for $k \geq 6$.

The function $F_{k,D}$ is also related to continued fraction expansion of real numbers, modular forms of half integral, and Dedekind sums.

1.0.2 Main results

In this thesis, we define a family of functions, analogous to $F_{k,D}$, from \mathbb{C} to \mathbb{R} by replacing the quadratic polynomials with functions coming from binary Hermitian forms over the Gaussian integers. That is, for d a positive integer that is not a norm in $\mathbb{Z}[i]$ with $d \equiv 0, 1, 2 \pmod{4}$ and k a positive even integer, define a function $H_{k,d} : \mathbb{C} \rightarrow \mathbb{R}$ as

$$H_{k,d}(z) := \sum_{\substack{\Delta(h)=d \\ a < 0 < h(z)}} h(z)^{k-1} \quad (1.2)$$

where $h(z) = az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c$, $\Delta(h) = |b|^2 - 4ac$, and $b \in \mathbb{Z}[i]$. The function $h(z)$ in the sum comes from evaluating the binary Hermitian form $h(z, w) := az\bar{z} + \frac{1}{2}(bz\bar{w} + \bar{b}\bar{z}w) + cw\bar{w}$ attached to the binary Hermitian matrix $h = \begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix}$ over $\mathbb{Z}[i]$ at $w = 1$. The reason we choose the functions $h(z)$ to replace Q 's in equation 1.1 is because quadratic polynomials with integer coefficients are just binary quadratic forms with one of the argument equal to one, and essentially one can think of binary Hermitian forms as binary quadratic forms over the rings of imaginary quadratic fields. More importantly, similar to the action of $\mathrm{SL}(2, \mathbb{Z})$ on binary quadratic forms, we also have an action of $\mathrm{SL}(2, \mathcal{O})$, which allows us to prove results similar to Zagier's.

To state our main results, let us define the following numbers, which are the values of $H_{k,d}$ at $z = 0$:

$$\omega_{k,d} := \sum_{\substack{b_1^2 + b_2^2 - 4ac = d \\ a < 0 < c}} c^k \quad (1.3)$$

$$= \sum_{\substack{b_1^2 + b_2^2 < d \\ b_1 + b_2 \equiv d \pmod{2}}} \sigma_k \left(\frac{d - b_1^2 - b_2^2}{4} \right) \quad (1.4)$$

where $b = b_1 + ib_2 \in \mathbb{Z}[i]$.

Theorem 1. *For every positive integer $d \equiv 0, 1, 2 \pmod{4}$ that is not a norm of a Gaussian integer, $H_{2,d}(z)$ and $H_{4,d}(z)$ have constant values $\omega_{2,d}$ and $\omega_{4,d}$ respectively.*

We prove these results using the continuity of $H_{k,d}$ and the functional equations satisfied by $H_{k,d}$. It is straight forward to prove the functional equations and the continuity when $k \geq 4$ but when $k = 2$ the proof of the continuity is non-trivial. To prove the continuity in the case $k = 2$, we use a continued fraction algorithm for complex numbers using Gaussian integers discovered by Hurwitz in [8]. So, our function like $F_{k,D}$ is related to the continued fraction expansion of complex numbers.

Theorem 1 shows that the vector space spanned by $H_{2,d}$ and $H_{4,d}$ as d varies is one dimensional. Using the same ideas as in the proof of 1, we are able to show for $k = 6$ that the space spanned by $H_{6,d}$ is two dimensional and in general finite dimensional for fixed k and varying d . That is,

Theorem 2. $H_{6,d}(z) = \kappa_d H_{6,6}(z) + (\omega_{6,d} - \kappa_d \omega_{6,6})$, where κ_d is a rational number depending on d .

Theorem 3. $H_{k,d}(z)$ is a linear combination of finitely many H_{k,d_j} where d_j is a positive integer that is not a norm in $\mathbb{Z}[i]$ with $d_j \equiv 0, 1, 2 \pmod{4}$ for each j .

1.0.3 Future directions

The main question that arises from this thesis is: Is there a modular explanation for the behavior of $H_{k,d}$? If so, what kind of a modular object is it? We suspect the function $H_{k,d}$ is also related to an automorphic object, and we suspect this automorphic object to be a Bianchi modular form. One can view the Bianchi modular forms as modular forms over imaginary quadratic fields. However, when the field is an imaginary quadratic field some difficulties arise, since the symmetric space one has to consider is the hyperbolic 3-space \mathcal{H}^3 which has no complex structure. Despite this fact, many of the fundamental arithmetical results known for the classical modular forms are believed to be true for Bianchi modular forms. For example, there is computational evidence of the Shimura-Taniyama-Weil type correspondence between elliptic curves and Bianchi modular forms defined over imaginary quadratic fields. For such an example, we refer the reader to [3] and [16].

Specifically, Bianchi modular forms are certain real analytic functions on \mathcal{H}^3 that transform like modular forms with respect to the action of $\mathrm{PSL}(2, \mathcal{O}_K)$ on \mathcal{H}^3 where

\mathcal{O}_K is the ring of integers of the imaginary quadratic field K .

They can also be defined as certain classes in the cohomology of quotients of \mathcal{H}^3 by congruence subgroups of $\mathrm{PSL}(2, \mathcal{O}_K)$. Similar to classical modular forms, there is a version of the Eichler-Shimura isomorphism, called Eichler-Shimura-Harder isomorphism, for Bianchi modular forms. Eichler-Shimura-Harder isomorphism states that

$$H_{cusp}^1(\Gamma, V_{k,k}(\mathbb{C})) \cong S_k(\Gamma) \quad (1.5)$$

where Γ is a congruence subgroup in $\mathrm{PSL}(2, \mathcal{O}_K)$ and $V_{k,k}(\mathbb{C}) := V_k(\mathbb{C}) \otimes_{\mathbb{C}} \overline{V_k(\mathbb{C})}$ with $V_k(\mathbb{C})$ denoting the space of homogeneous polynomials of degree k in two variables and $\overline{V_k(\mathbb{C})}$ is its twist by complex conjugation.

We review the classical Eichler-Shimura isomorphism in chapter 2 and show how this isomorphism relates $F_{k,D}$ to the cusp form $f_{k,D}$. We have done some work on the cocycle property of $H_{k,d}$ for $\mathrm{PSL}(2, Z[i])$ and shown that the cocycle one gets from $H_{k,d}$ is related to the parabolic 1-cocycles, which live in the cuspidal cohomology of $\mathrm{PSL}(2, Z[i])$. Consequently, we are led to believe that there is a connection between $H_{k,d}$ and Bianchi modular forms. This work is still in progress, and at the moment we are trying to compute the dimension of the space of parabolic 1-cocycles of $\mathrm{PSL}(2, Z[i])$. Finding the dimension of this space will also give us the dimension of the space spanned by the functions $H_{k,d}$ for k fixed and varying d . But ultimately, to establish a relationship between $H_{k,d}$ and Bianchi modular forms, we need to adjust

or find analogs of the other two key ingredients, Bol's identity and Eichler integral, which we define in chapter 2, used to relate $F_{k,D}$ and $f_{k,D}$.

A second task is to consider $H_{k,d}$ over the rings of integers of imaginary quadratic fields different than Gaussian integers. The definition of $H_{k,d}$ makes sense over other rings, since binary Hermitian forms are defined over the rings of imaginary quadratic fields.

The organization of this thesis is as follows. In chapter 2, we recall some of Zagier's results and review the $F_{k,D}$'s connection to continued fractions and to modular forms in a fair amount of detail. In chapter 3, after stating the relevant facts about binary Hermitian forms, we define our function $H_{k,d}$ and prove the Theorems 1, 2 and 3. In chapter 4, we discuss the continuity of $H_{k,d}$. We first prove the continuity for $k > 4$. To prove the continuity for $k = 2$, we use the Hurwitz continued fraction algorithm. We first describe the algorithm as it is usually stated, but for our purposes we recast the algorithm in terms of matrix notation. This allows us to describe the functions appearing in the sum for $H_{2,d}$ for a given $z \in \mathbb{C}$ in terms of the action of the matrices built from the continued fraction expansion of z . These functions decay to zero exponentially quickly, which proves the continuity of $H_{2,d}$.

Chapter 2

Zagier's Function

For non-square $D \equiv 0, 1 \pmod{4}$ and k a positive even integer, Zagier [18] defines the following function:

$$F_{k,D}(x) := \sum_{\substack{\text{disc}(Q)=D \\ a < 0 < Q(x)}} Q(x)^{k-1} \quad (2.1)$$

where $Q(X) = aX^2 + bX + c$ with $(a, b, c) \in \mathbb{Z}^3$ and $\text{disc}(Q) = b^2 - 4ac$.

Zagier shows that,

Theorem 4. *Let D be a positive fundamental discriminant. Then the sums defining $F_{2,D}$ and $F_{4,D}$ converge for all $x \in \mathbb{R}$ and have constant values α_D and β_D respectively independent of x where*

$$\alpha_D = -5L(-1, \chi_D) \quad \text{and} \quad \beta_D = L(-3, \chi_D).$$

For $k \geq 6$, $F_{k,D}$ is no longer constant but instead is a linear combination of a finite collection of functions. More precisely, Zagier proves

Theorem 5. *For every positive integer k and every positive non-square discriminant D , the function $F_{k,D}$ is a linear combination, with coefficients depending on D and k , of a finite collection of functions depending only on k .*

Zagier proves these results using the functional equations

$$F_{k,D}(x+1) = F_{k,D}(x), \quad x^{2k-2}F_{k,D}(1/x) - F_{k,D}(x) = P_{k,D}(x) \quad (2.2)$$

where

$$P_{k,D}(x) = \sum_{\substack{b^2-4ac=D \\ a>0>c}} (ax^2 + bx + c)^{k-1}, \quad (2.3)$$

and the continuity of $F_{k,D}$. The functional equations are easy to prove when $x \in \mathbb{Q}$. The continuity of $F_{k,D}$ is also easy to prove when $k \geq 4$. But for $k = 2$, Zagier only shows that $F_{2,D}$ is convergent and states that the continuity can be proved in an elementary way, but he doesn't give the proof. He conjectures that the quadratic functions appearing in the sum when $k = 2$ for a given value of x are related to the continued fraction expansion of x .

2.1 Connection to continued fractions

The convergence of $F_{k,D}$ is not immediate at all. Zagier observes that one has $Q(x) = O(1/a_Q)$ for all Q in (2.1) and the number of contributing Q 's per value of a is bounded. Therefore the series (2.1) converges at most like $\sum_{a=1}^{\infty} \frac{1}{a^{k-1}}$. So, the sum defining $F_{k,D}$ converges uniformly when $k \geq 4$ but this argument fails when

$k = 2$. However, experiments carried out by Zagier for the value $x = \frac{1}{\pi}$ suggest that the sum defining $F_{2,5}$ converges extremely rapidly. Furthermore, Zagier observes a strong pattern among the Q 's appearing in the sum for the value $x = \frac{1}{\pi}$ and $D = 5$ which explains the rapid convergence of $F_{2,5}$. Based on this observation he makes the following conjecture. To state the conjecture we need some notation.

Let x be a real number between 0 and 1 and write x as a continued fraction

$$x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\ddots}}} = [0, n_1, n_2, \dots] \quad (2.4)$$

using integers $n_1, n_2, \dots \geq 1$ which are called partial quotients. The continued fraction expansion terminates for x if and only if $x \in \mathbb{Q}$, so assume x is irrational. Define the real numbers $\delta_0, \delta_1, \delta_2, \dots$ inductively by

$$\delta_0 = 1, \quad \delta_1 = x, \quad \delta_{j+1} = \delta_{j-1} - n_j \delta_j \quad (j \geq 1). \quad (2.5)$$

Zagier observes the following based on numerical evidence:

Conjecture 1. *The functions appearing in the sum for $F_{2,5}\left(\frac{1}{\pi}\right)$ are all of the expressions $\delta_j^2 + \delta_j \delta_{j+1} - \delta_{j+1}^2$ and some of the expressions $\delta_j^2 - \delta_j \delta_{j+1} - \delta_{j+1}^2$ where the linear forms $\delta_j \in \mathbb{Z} + \mathbb{Z}x$ are defined as above by the continued fraction expansion of $x = \frac{1}{\pi}$.*

The conjecture shows why $F_{2,5}$ converges rapidly; δ_j decays exponentially in j .

In fact, we have the following uniform bound on δ_j 's independent of x ; $\delta_j < 2^{\frac{1-j}{2}}$.

Therefore the sum converges uniformly and is a continuous function.

This conjecture was recently proved by P. Bengoechea, who was a PhD student of Zagier, in [1] for any fundamental discriminant D . Her results also provide a proof of continuity when $k = 2$. The same conjecture was also proved by M. Jameson, who was a PhD student of K. Ono, in [9] for any D . But, she assumes the continuity of $F_{2,D}$.

2.2 The modular connection

A natural question to ask is why $F_{k,D}$ is constant when $k = 2, 4$ but not constant for $k \geq 6$. The short answer is that there are cusp forms of weight $2k$ on the full modular group for $k \geq 6$ but not for $k < 6$. We will expand on this answer by exploring the relation between $F_{k,D}$ and modular forms.

2.2.1 Eichler-Shimura Isomorphism

Let $\Gamma = \text{PSL}(2, \mathbb{Z})$ and V_k be the space of polynomials in X of degree less than or equal to $k - 2$ over \mathbb{C} . Let $|_{2-k}$ denote the usual weight $2 - k$ action of Γ on V_k defined by

$$(P|_{2-k}\gamma)(X) = (cX + d)^{k-2} P\left(\frac{aX + b}{cX + d}\right) \quad \left(P(X) \in V, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right). \quad (2.6)$$

Under the action $|_{2-k}$, V_k splits into the direct sum of subspaces V_k^+ and V_k^- of even and odd polynomials respectively. The action $|_{2-k}$ on V_k can be extended by linearity to the action of the group ring $\mathbb{Z}[\Gamma]$.

A 1-cocycle with values in V_k is a map $\varphi : \Gamma \rightarrow V_k$ such that

$$\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1)|_{2-k}\gamma_2 + \varphi(\gamma_2) \quad (2.7)$$

where $\gamma_1, \gamma_2 \in \Gamma$. We let $Z(\Gamma, V_k)$ denote the group of all 1-cocycles with values in V_k . Given fixed $v \in V_k$, it is clear that the map

$$\gamma \mapsto v|_{2-k}(1 - \gamma) \quad (2.8)$$

for all $\gamma \in \Gamma$ defines a cocycle. A cocycle of this form is called a coboundary. We let $B(\Gamma, V_k)$ to denote the group of all coboundaries with values in V_k . A 1-cocyle φ is called cuspidal or parabolic if

$$\varphi(T) = v|_{2-k}(1 - T) \quad (2.9)$$

for some $v \in V_k$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The group of cuspidal 1-cocyles with values in V_k is denoted by $Z^1(\Gamma, V_k)$. Note that the group of all cuspidal 1-cocycles contains the group of coboundaries since if $\varphi \in B(\Gamma, V_k)$ then by definition $\varphi(T) = v|_{2-k}(1 - T)$.

The factor group

$$H^1(\Gamma, V_k) = \frac{Z^1(\Gamma, V_k)}{B(\Gamma, V_k)}$$

is called the Eichler cohomology group. This group sometimes is also called the cuspidal cohomology and denoted by $H_{cusp}^1(\Gamma, V_k)$

Eichler-Shimura isomorphism states that (cf. [13], [15], [5], [11])

$$\mathfrak{S}_k \oplus \overline{\mathfrak{S}}_k \cong H^1(\Gamma, V_k) \quad (2.10)$$

where \mathfrak{S}_k is the space of cusp forms of weight k on Γ and $\overline{\mathfrak{S}}_k$ is the space of antiholomorphic cusp forms consisting of functions $\bar{f}(z) := \overline{f(\bar{z})}$ with $f \in \mathfrak{S}_k$. The map used to show the isomorphism in (2.10) is called the period map. The Eichler-Shimura isomorphism implies that there exists cusp forms all of whose even and odd periods are rational. Kohnen and Zagier [11] give examples of such cusp forms whose periods are arithmetically interesting expressions – relating to Bernoulli numbers, to binary quadratic forms, to zeta functions of real quadratic fields, to modular forms of half-integral weight, and to Hilbert modular forms. $F_{k,D}$ is related to the even period polynomial of one of the cusp forms they study in [11].

2.2.2 Periods of modular forms

In this section we describe two maps; one from \mathfrak{S}_k to V_k and another from \mathfrak{S}_k to $H^1(\Gamma, V_k)$. The Eichler-Shimura isomorphism can be stated using either one of these maps.

Let f be a cusp form of weight k . One defines the n th period of f by

$$r_n(f) = \int_0^\infty f(it)t^n dt \quad (2.11)$$

where $0 \leq n \leq k-2$. The period polynomial of f is defined to be

$$r(f)(X) = \int_0^{i\infty} f(z)(X-z)^w dz = \sum_{n=0}^w i^{-n+1} \binom{w}{n} r_n(f) X^{w-n} \quad (2.12)$$

where $0 \leq w \leq k-2$. It is clear that $r(f)(X) \in V_k$. We also set

$$r^+(f) = \sum_{\substack{0 \leq n \leq w \\ n \text{ even}}} (-1)^{n/2} \binom{w}{n} r_n(f) X^{w-n}, \quad (2.13)$$

$$r^-(f) = \sum_{\substack{0 \leq n \leq w \\ n \text{ odd}}} (-1)^{(n-1)/2} \binom{w}{n} r_n(f) X^{w-n}, \quad (2.14)$$

so that $r^\pm \in V_k^\pm$, $r = ir^+ + r^-$. It is shown in [11] that

$$r(f)|_{2-k}(1+S) = 0, \quad (2.15)$$

$$r(f)|_{2-k}(1+U+U^2) = 0 \quad (2.16)$$

so that $r(f)$ belongs to the subspace W_k of V_k defined by

$$W_k = \ker(1+S) \cap \ker(1+U+U^2) \quad (2.17)$$

$$= \{P \in V_k : P|_{2-k}(1+S) = P|_{2-k}(1+U+U^2) = 0\} \quad (2.18)$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Under the action, W_k also splits as $W_k = W_k^+ \oplus W_k^-$. We thus have two maps $r^\pm : \mathfrak{S}_k \rightarrow W_k^\pm$. In this setting the statement of

Eichler-Shimura isomorphism becomes:

Theorem 6. *The map $r^- : \mathfrak{S}_k \rightarrow W_k^-$ is an isomorphism. The map $r^+ : \mathfrak{S}_k \rightarrow W_k^+ / \langle X^{k-2} - 1 \rangle$ is an isomorphism.*

Alternatively, one can define the period mapping $f \mapsto r(f)$ directly as a map from \mathfrak{S}_k to $H^1(\Gamma, V_k)$ by using the so-called ‘‘Eichler integral’’ of f . If $f(z) = \sum_{n=1}^{\infty} a_n(f)q^n$ ($q = e^{2\pi iz}$) is in \mathfrak{S}_k then the $(k-1)$ -fold integral of f which we denote by \tilde{f} is given by

$$\tilde{f}(z) = \frac{1}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n(f)}{n^{k-1}} q^n. \quad (2.19)$$

We call \tilde{f} the Eichler integral of f . Note that \tilde{f} is unique up to adding some element of V_k .

Let F be a holomorphic function defined on the upper-half plane \mathbb{H} . The Bol’s identity [2] states that,

$$\frac{d^{k-1}}{dz^{k-1}}(F|_{2-k}\gamma) = \left(\frac{d^{k-1}F}{dz^{k-1}} \right) |_{k}\gamma \quad (2.20)$$

where $(F|_k)(z) = (cz + d)^{-k} F(\gamma z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If we let $F = \tilde{f}$ then we have

$$\frac{d^{k-1}}{dz^{k-1}}(\tilde{f}|_{2-k}(1-\gamma)) = \left(\frac{d^{k-1}\tilde{f}}{dz^{k-1}} \right) |_{k}(1-\gamma) = \frac{1}{(2\pi i)^{k-1}}(f - f|_{\gamma}) = 0 \quad (2.21)$$

Thus, the function

$$r_{f,\gamma}(z) := \left(\tilde{f}|_{2-k}(1-\gamma) \right) (z)$$

is a polynomial in z of degree $\leq k-2$. The map $\gamma \mapsto r_{f,\gamma}$ from Γ to V_k is automatically a cocycle since it is a coboundary in the space of functions on \mathbb{H} . Therefore we get $r_{f,\gamma} \in H^1(\Gamma, V_k)$ and this element doesn't depend on the choice of \tilde{f} . To see this, suppose we chose another Eichler integral \tilde{g} of f then the difference of $\tilde{f} - \tilde{g}$ is a polynomial $P \in V_k$ and so

$$r_{f,\gamma} - r_{g,\gamma} = \left(\tilde{f}|_{2-k}(1-\gamma) \right) - \left(\tilde{g}|_{2-k}(1-\gamma) \right) = P|_{2-k}(1-\gamma)$$

is a coboundary. In [5], M. Eichler proves the isomorphism in 2.10 using the Eichler integral.

The Eichler integral of $f \in \mathfrak{S}_k$ also converges on the real line and there it can be split into even and odd parts as follows

$$\tilde{f}^+ = \frac{1}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n(f)}{n^{k-1}} \cos(2\pi n x), \quad \tilde{f}^- = \frac{1}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a_n(f)}{n^{k-1}} \sin(2\pi n x).$$

Using \tilde{f}^+, \tilde{f}^- one can split $r_{f,\gamma}$ into its even and odd parts as

$$r_{f,\gamma}^+(x) := \left(\tilde{f}^+|_{2-k}(1-\gamma) \right) (x), \quad r_{f,\gamma}^-(x) := \left(\tilde{f}^-|_{2-k}(1-\gamma) \right) (x)$$

where $r_{f,\gamma}^+ \in W_k^+ / \langle X^{k-2} - 1 \rangle$ and $r_{f,\gamma}^- \in W_k^-$.

Now, we can finally give the precise explanation for the behavior of $F_{k,D}$. The

function $F_{k,D}$ arises from studying the function

$$f_{k,D}(z) := C_k D^{k-1/2} \sum_{\substack{b^2-4ac=D \\ a,b,c \in \mathbb{Z}}} \frac{1}{(az^2 + bz + c)^k} \quad (z \in \mathbb{H})$$

where C_k is a normalizing factor, $D \equiv 0, 1 \pmod{4}$ and k is a positive even integer. This function is introduced in [17] and shown to be a cusp form of weight $2k$ for the full modular group. It is studied in connection with Doi-Naganuma lifting from elliptic to Hilbert modular forms, and shown in ([10], [12]) to be the D th Fourier coefficient of the holomorphic kernel function for the Shimura-Shintani correspondence between modular forms of integral and half-integral weight. It is also conjectured that for fixed k , as D varies, the $f_{k,D}$'s span the space of cusp forms of weight $2k$. The function $f_{k,D}$ is an example of a cusp form whose periods are arithmetically interesting expressions. The even period polynomials of $f_{k,D}$ are computed in [11] and are given by

$$r^+(f_{k,D})(x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)} (x^{2k-2} - 1) - x^{2k-2} F_{k,D} \left(\frac{1}{x} \right) - F_{k,D}(x).$$

Since $r^+(f_{k,D}) \in W_k^+ / \langle X^{k-2} - 1 \rangle$, this implies that $F_{k,D}$ has the Fourier expansion

$$F_{k,D}(x) = \frac{\zeta_D(1-k)}{2\zeta(1-2k)} + \sum_{n=0}^{\infty} \frac{a_{k,D}(n)}{n^{2k-1}} \cos(2\pi nx)$$

where the $a_{k,D}(n)$ are the Fourier coefficients of $f_{k,D}$ which is of weight $2k$. But since there are no cusp forms below the weight 12, the sum vanishes for $k = 2, 4$ and the

constancy of $F_{k,D}$ follows.

Chapter 3

Sums of binary Hermitian forms

Throughout this chapter, we use the following terminology and notation

- $k > 0$ an even integer
- A^t is transpose of A

3.1 Binary Hermitian forms

Let A be a 2×2 matrix with entries in \mathbb{C} . A is said to be a Hermitian matrix if

$$A = \bar{A}^t \tag{3.1}$$

where \bar{A}^t is obtained from A by applying the complex conjugation to each of the entries and then taking the transpose or vice versa. Let R be a subring of \mathbb{C} which is closed under conjugation, i.e. $R = \bar{R}$. We denote by $\mathcal{H}(R)$ the set of all 2×2 Hermitian matrices with entries in R . Trivially, $h \in \mathcal{H}(R)$ if and only if $h = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ with $a, c \in R \cap \mathbb{R}$ and $b \in R$. Every $h \in \mathcal{H}(R)$ defines a binary Hermitian form with

coefficients in R . If $h = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ the associated binary Hermitian form is the map $h : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ defined by,

$$h(z, w) = \begin{pmatrix} z & w \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} = az\bar{z} + bz\bar{w} + \bar{b}\bar{z}w + cw\bar{w}.$$

We shall often call an element $h \in \mathcal{H}(R)$ a binary Hermitian form with coefficients in R . The discriminant $\Delta(h)$ of $h \in \mathcal{H}(R)$ is defined as

$$\Delta(h) = -4\det(h). \quad (3.2)$$

The study of these forms parallels the study of binary quadratic forms and they are known to be related to Bianchi modular forms (cf. [6],[7]). Similar to the action of the $\mathrm{GL}(2, \mathbb{Z})$ on binary quadratic forms, we have the action of $\mathrm{GL}(2, R)$ on $\mathcal{H}(R)$ given by the formula

$$\sigma(h) = \sigma h \bar{\sigma}^t \quad (3.3)$$

where $\sigma \in \mathrm{GL}(2, R)$ and $h \in \mathcal{H}(R)$. We note that $\Delta(\sigma(h)) = |\det \sigma|^2 \cdot \Delta(h)$. A binary Hermitian form $h \in \mathcal{H}(R)$ is positive definite if $h(u, v) > 0$ for all $(u, v) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ and h is called indefinite if $\Delta(h) < 0$. We let

$$\mathcal{H}^+(R) = \{h \in \mathcal{H}(R) \mid h \text{ is positive definite}\},$$

$$\mathcal{H}^-(R) = \{h \in \mathcal{H}(R) \mid h \text{ is indefinite}\}.$$

Note that $\mathcal{H}^\pm(R)$ is invariant under the action of the group $\mathrm{GL}(2, R)$.

Let $K = \mathbb{Q}(\sqrt{-D})$ be the imaginary quadratic number field of discriminant D and let \mathcal{O} be its ring of integers. The study of the orbits of $\mathcal{H}(\mathcal{O})$ under the action of $SL(2, \mathcal{O})$ is called the reduction theory of binary Hermitian forms.

Definition 1. *Let $d \in \mathbb{Z}$ be an integer. We define*

$$\mathcal{H}(\mathcal{O}, d) = \{h \in \mathcal{H}(\mathcal{O}) \mid \Delta(h) = d\},$$

$$\mathcal{H}^\pm(\mathcal{O}, d) = \{h \in \mathcal{H}^\pm(\mathcal{O}) \mid \Delta(h) = d\}.$$

Similar to the main result of reduction theory of binary quadratic forms we also have the following result for binary Hermitian forms.

Theorem 7. *For any $d \in \mathbb{Z}$ with $d \neq 0$ the sets $\mathcal{H}(\mathcal{O}, d)$ and $\mathcal{H}^\pm(\mathcal{O}, d)$ split into finitely many $SL(2, \mathcal{O})$ orbits.*

Proof. cf. [6] §9.2, Theorem 2.2. □

3.2 Sums of binary Hermitian forms over $\mathrm{PSL}(2, \mathbb{Z}[i])$

Using binary Hermitian forms we define a function analogous to $F_{k,D}$. Let $h \begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix} \in \mathcal{H}(\mathbb{Z}[i])$. We define the function $H_{k,d} : \mathbb{C} \rightarrow \mathbb{R}$ as

$$H_{k,d}(z) := \sum_{\substack{\Delta(h)=d \\ a < 0 < h(z,1)}} h(z, 1)^{k-1} \quad (3.4)$$

$$= \sum_{\substack{|b|^2 - 4ac = d \\ a < 0}} \max \left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c \right)^{k-1} \right) \quad (3.5)$$

where d is a positive integer that is not a norm in $\mathbb{Z}[i]$ with $d \equiv 0, 1, 2 \pmod{4}$. We will discuss the convergence and continuity in the next chapter. Let us assume for now that $H_{k,d}$ converges and is continuous for all k, d and for all $z \in \mathbb{C}$.

One can think of each binary Hermitian form $h(z, 1)$ appearing in (3.4) as a polynomial in z and \bar{z} . There is a natural action of $\mathrm{PSL}(2, \mathbb{Z}[i])$ on the space of such polynomials and this action is compatible with the action in (3.3). It is essentially one variable version of the action on the space of homogeneous polynomials in z, w tensored over \mathbb{C} with the space of homogeneous polynomials in \bar{z}, \bar{w} . This space is used when one generalizes the Eichler-Shimura isomorphism for modular forms over imaginary quadratic fields which are called Bianchi modular forms.

Let \mathcal{V}_k be the space of polynomials in z and \bar{z} of degree at most $k - 1$ in both z

and \bar{z} over \mathbb{C} , i.e.

$$\mathcal{V}_k = \left\{ P(z) = \sum_{0 \leq n, j \leq k-1} a_{n,j} z^n \bar{z}^j : a_{n,j} \in \mathbb{C} \right\}.$$

$\mathrm{PSL}(2, \mathbb{Z}[i])$ acts on \mathcal{V}_k by,

$$(P|_{1-k}\gamma)(z) = (cz + d)^{k-1} \overline{(cz + d)^{k-1}} P\left(\frac{az + b}{cz + d}\right) \quad (3.6)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that the set of all $h(z, 1)$ appearing in (3.4) is a subset of \mathcal{V}_2 . The next proposition gives the relation between the two actions (3.3) and (3.6) when $P \in \mathcal{V}_2$ is a binary Hermitian form.

Proposition 1. *Let $P(z) = az\bar{z} + bz + \bar{b}\bar{z} + c = (z \ 1) \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \in \mathcal{V}_2$. Then*

$$(P|_{-1}\gamma)(z) = (z \ 1) \gamma^t \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \bar{\gamma} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}.$$

Proof. Let $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then

$$\begin{aligned} (P|_{-1}\gamma)(z) &= (rz + s) \overline{(rz + s)} P\left(\frac{pz + q}{rz + s}\right) \\ &= a(pz + q) \overline{(pz + q)} + b(pz + q) \overline{(rz + s)} + \bar{b} \overline{(pz + q)} (rz + s) + c(rz + s) \overline{(rz + s)} \\ &= (pz + q \ rz + s) \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} \overline{pz + q} \\ \overline{rz + s} \end{pmatrix} \\ &= (z \ 1) \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} \bar{p} & \bar{q} \\ \bar{r} & \bar{s} \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \\ &= (z \ 1) \gamma^t \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \bar{\gamma} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}. \end{aligned}$$

□

Using Proposition 1 we can extend the action of $\mathrm{PSL}(2, \mathbb{Z}[i])$ to $H_{k,d}$ by

$$(H_{k,d}|_{1-k}\gamma)(z) = (rz + s)^{k-1} \overline{(rz + s)}^{k-1} H_{k,d} \left(\frac{pz + q}{rz + s} \right) \quad (3.7)$$

$$= \sum_{\substack{\Delta(h)=d \\ a < 0 < \gamma^t(h)(z,1)}} \left((z \ 1) \gamma^t(h) \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \right)^{k-1} \quad (3.8)$$

where $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

Next we prove some properties of $H_{k,d}$ and show that it is invariant under the action of some special elements of $\mathrm{PSL}(2, \mathbb{Z}[i])$.

Lemma 1. 1. $H_{k,d}(-z) = H_{k,d}(z)$, $H_{k,d}(\bar{z}) = H_{k,d}(z)$ and $H_{k,d}(iz) = H_{k,d}(z)$.

2. $H_{k,d}(z+1) = H_{k,d}(z)$ and $H_{k,d}(z+i) = H_{k,d}(z)$.

Proof. We continue to let the $\begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix}$ be the matrix corresponding to the binary Hermitian form h . Then

$$\begin{aligned} H_{k,d}(-z) &= \sum_{\substack{\Delta(h)=d \\ a < 0 < h(z,1)}} h(-z, 1)^{k-1} \\ &= \sum_{\substack{|b|^2 - 4ac = d \\ a < 0}} \max \left(0, \left(az\bar{z} + \frac{1}{2}(-bz - \bar{b}\bar{z}) + c \right)^{k-1} \right) \\ &= \sum_{\substack{|b|^2 - 4ac = d \\ a < 0}} \max \left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c \right)^{k-1} \right) \end{aligned}$$

since if b is a solution to $|b|^2 - 4ac = d$ then so are $-b, -\bar{b}$. The proofs of $H_{k,d}(\bar{z}) = H_{k,d}(z)$ and $H_{k,d}(iz) = H_{k,d}(z)$ are similar since b can be replaced by \bar{b} or $\pm ib$.

As for the proof of $H_{k,d}(z+1) = H_{k,d}(z)$, notice that $H_{k,d}(z+1) = (H_{k,d}|_{1-k}T)(z)$ where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For ease of notation let us omit writing $(z \ 1)$ and $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}$ when we are computing the action of T . So,

$$\begin{aligned} (H_{k,d}|_{1-k}T)(z) &= \sum_{\substack{\Delta(h)=d \\ a < 0 < T^t(h)(z,1)}} \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)^{k-1} \\ &= \sum_{\substack{\Delta(h)=d \\ a < 0 < T^t(h)(z,1)}} \left(\begin{pmatrix} a & a+b/2 \\ a+\bar{b}/2 & a+\bar{b}/2+b/2+c \end{pmatrix} \right)^{k-1}. \end{aligned}$$

Note that $\begin{pmatrix} a & a+b/2 \\ a+\bar{b}/2 & a+\bar{b}/2+b/2+c \end{pmatrix}$ is just another binary Hermitian matrix with discriminant d and negative first entry. Thus $H_{k,d}$ is invariant under the action of T . The fact that $H_{k,d}(z+i) = H_{k,d}(z)$ can be proved in the same way by computing the action of $T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ on $H_{k,d}$. \square

We define the following numbers which are the values of $H_{k,d}$ at $z = 0$:

$$\omega_{k,d} := \sum_{\substack{b_1^2+b_2^2-4ac=d \\ a < 0 < c}} c^k \tag{3.9}$$

$$= \sum_{\substack{b_1^2+b_2^2 < d \\ b_1+b_2 \equiv d \pmod{2}}} \sigma_k \left(\frac{d - b_1^2 - b_2^2}{4} \right). \tag{3.10}$$

where $b = b_1 + ib_2 \in \mathbb{Z}[i]$. In [17] it is shown that, for positive fundamental discrimi-

nant D and b ranging over integers,

$$\sum_{\substack{|b| < \sqrt{D} \\ b \equiv D \pmod{2}}} \sigma_1 \left(\frac{D - b^2}{4} \right) = -5L(-1, \chi_D) \quad (3.11)$$

$$\sum_{\substack{|b| < \sqrt{D} \\ b \equiv D \pmod{2}}} \sigma_3 \left(\frac{D - b^2}{4} \right) = L(-3, \chi_D). \quad (3.12)$$

These two numbers are the values of $F_{k,D}$ at zero when $k = 2, 4$. Using (3.11) and (3.12) we can write the values of $\omega_{2,d}$ and $\omega_{4,d}$ as a sum of the values of the Dirichlet's L -functions whenever $d - b_1^2$ or $d - b_2^2$ is a fundamental discriminant. However, $d - b_1^2$ or $d - b_2^2$ may not be a fundamental discriminant as b_1 and b_2 range over the solutions of the equation $b_1^2 + b_2^2 - 4ac = d$ with $a < 0 < c$. Below are the two examples where $d - b_1^2$ or $d - b_2^2$ stays a fundamental discriminant and where it doesn't as b_1 and b_2 range over the solutions of the equation $b_1^2 + b_2^2 - 4ac = d$ with $a < 0 < c$.

Example 1: Let $d = 14$, then the equation $b_1^2 + b_2^2 - 4ac = d$ has the following quadruplets (a, b_1, b_2, c) as solutions: $(-1, \pm 1, \pm 3, 1), (-1, \pm 3, \pm 1, 1), (-1, \pm 1, \pm 1, 3), (-3, \pm 1, \pm 1, 1)$.

Since $d - b_1^2$ or $d - b_2^2$ is a fundamental discriminant in this case we can write $\omega_{2,14}$

and $\omega_{4,14}$ as:

$$\begin{aligned} \omega_{2,14} &= 2F_{2,13}(0) + 2F_{2,5}(0) \\ &= -10(L(-1, \chi_{13}) + L(-1, \chi_5)), \end{aligned}$$

and

$$\begin{aligned}\omega_{4,14} &= 2F_{4,13}(0) + 2F_{4,5}(0) \\ &= 2(L(-3, \chi_{13}) + L(-3, \chi_5)).\end{aligned}$$

Example 2: Let $d = 46$, then one of the solutions to the equation $b_1^2 + b_2^2 - 4ac = d$ is the quadruplet $(-1, \pm 1, \pm 1, 11)$ and in this case $d - b_1^2$ or $d - b_2^2$ is 45 which is not a fundamental discriminant. So, when $d = 46$ we are not able to write $\omega_{2,14}$ and $\omega_{4,14}$ just as the sum of L -values.

Similar to $F_{k,D}$, $H_{k,d}$ is also constant for $k = 2, 4$. The proof uses similar ideas as in Zagier's case. We first show that for $k = 2, 4$, $H_{k,d}$ is constant when $z \in \mathbb{Q}(i)$ and using the continuity of $H_{k,d}$ we conclude that $H_{2,d}$ and $H_{4,d}$ are constant for all $z \in \mathbb{C}$.

Lemma 2. Let $z \in \mathbb{Q}(i)$ and let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}[i])$. Then

$$(H_{2,d}|_{-1}S)(z) - H_{2,d}(z) = \omega_{2,d}(z\bar{z} - 1). \quad (3.13)$$

Proof. When $z \in \mathbb{Q}(i)$, $H_{2,d}(z)$ is a finite sum. To see this, suppose $h(z, 1) = az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c$ occurs in the sum for $H_{2,d}(z)$ when $z = \frac{p}{q} + i\frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}$. Then using the identity $d = |\bar{b} + 2az|^2 - 4ah(z, 1)$ one gets,

$$dq^2s^2 = (s(2ap + b_1q))^2 + (q(2ar - b_2s))^2 + 4|a|(a(p^2s^2 + r^2q^2) + qs(b_1ps - b_2rq) + cq^2s^2)$$

which bounds a, b_1, b_2, c . Thus only finitely many $h(z, 1)$ appear in the sum for $H_{2,d}(z)$ when $z \in \mathbb{Q}(i)$. On the other hand computing $(H_{2,d}|_{-1}S)(z) - H_{2,d}(z)$ directly gives,

$$\begin{aligned}
&= z\bar{z}H_{2,d}\left(\frac{-1}{z}\right) - H_{2,d}(z) \\
&= z\bar{z}H_{2,d}\left(\frac{1}{z}\right) - H_{2,d}(z) \\
&= z\bar{z} \sum_{\substack{|b|^2-4ac=d \\ a<0}} \max\left(0, \left(\frac{a}{z\bar{z}} + \frac{1}{2}\left(\frac{b}{z} + \frac{\bar{b}}{\bar{z}}\right) + c\right)\right) - \sum_{\substack{|b|^2-4ac=d \\ a<0}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c\right)\right) \\
&= \sum_{\substack{|b|^2-4ac=d \\ a<0}} \max\left(0, \left(a + \frac{1}{2}(b\bar{z} + \bar{b}z) + cz\bar{z}\right)\right) - \sum_{\substack{|b|^2-4ac=d \\ a<0}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c\right)\right) \\
&= \sum_{\substack{|b|^2-4ac=d \\ c<0}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(b\bar{z} + \bar{b}z) + c\right)\right) - \sum_{\substack{|b|^2-4ac=d \\ a<0}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c\right)\right).
\end{aligned}$$

Notice that the terms with a and c both negative in the two previous sums cancel and d not being norm in $\mathbb{Z}[i]$ implies that $a \neq 0$ and $c \neq 0$ which gives

$$= \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(b\bar{z} + \bar{b}z) + c\right)\right) - \sum_{\substack{|b|^2-4ac=d \\ a<0<c}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c\right)\right).$$

Since $\max(0, X) = -\min(0, -X)$ for any $X \in \mathbb{R}$ we get

$$\begin{aligned}
&= \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(b\bar{z} + \bar{b}z) + c\right)\right) + \sum_{\substack{|b|^2-4ac=d \\ a<0<c}} \min\left(0, \left(-az\bar{z} - \frac{1}{2}(bz + \bar{b}\bar{z}) - c\right)\right) \\
&= \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \max\left(0, \left(az\bar{z} + \frac{1}{2}(b\bar{z} + \bar{b}z) + c\right)\right) + \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \min\left(0, \left(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c\right)\right).
\end{aligned}$$

Finally, because $\max(0, X) + \min(0, X) = X$ for any $X \in \mathbb{R}$ we get

$$(H_{2,d}|_{-1}S)(z) - H_{2,d}(z) = \sum_{\substack{|b|^2 - 4ac = d \\ c < 0 < a}} \left(az\bar{z} + \frac{bz}{2} + \frac{\bar{b}\bar{z}}{2} + c \right). \quad (3.14)$$

Since the equation $|b|^2 - 4ac = d$ and the inequality $c < 0 < a$ bound $a, |b|, c$, there are only finitely many functions, independent of z , contributing to the last sum. The coefficient of z and \bar{z} is zero because if b and \bar{b} are solutions to $|b|^2 - 4ac = d$ then so are $-b$ and $-\bar{b}$. It is easy to see the coefficient of $z\bar{z}$ term is $\omega_{2,d}$ and the coefficient of the constant term is $-\omega_{2,d}$ because $c < 0$. In other words, we have

$$(H_{2,d}|_{-1}S)(z) - H_{2,d}(z) = \omega_{2,d}(z\bar{z} - 1).$$

□

Theorem 8. *For every positive integer $d \equiv 0, 1, 2 \pmod{4}$ that is not a norm of a Gaussian integer, $H_{2,d}(z)$ has a constant value $\omega_{2,d}$.*

Proof. Let $H_{2,d}^0(z) := H_{2,d}(z) - \omega_{2,d}$. Then,

$$(H_{2,d}^0|_{-1}T)(z) = H_{2,d}^0(z), \quad (H_{2,d}^0|_{-1}T_i)(z) = H_{2,d}^0(z)$$

and using the definition of $H_{2,d}^0(z)$ and Lemma 2, we have

$$(H_{2,d}^0|_{-1}S)(z) = (H_{2,d}|_{-1}S)(z) - \omega_{2,d}z\bar{z} = H_{2,d}(z) + \omega_{2,d}(z\bar{z} - 1) - \omega_{2,d}z\bar{z} = H_{2,d}^0(z).$$

Since $\mathbb{Q}(i)$ has an Euclidean algorithm, any $z \in \mathbb{Q}(i)$ can be reduced to zero by

applying S, T , and T_i in finitely many steps. So, we see that

$$H_{2,d}^0(z) = H_{2,d}^0(0) = 0$$

for all $z \in \mathbb{Q}(i)$ which implies that $H_{2,d}(z) = \omega_{2,d}$ for all $z \in \mathbb{Q}(i)$. Assuming the continuity, we conclude that $H_{2,d}(z) = \omega_{2,d}$ for all $z \in \mathbb{C}$. \square

In Lemma 2, we showed that $H_{2,d}|_{-1}(1 - S)$ is a polynomial in z and \bar{z} . Based on the proof of that lemma and equation (3.14) it is natural to define the following polynomials:

Definition 2. For a positive integer d that is not a norm in $\mathbb{Z}[i]$ with $d \equiv 0, 1, 2 \pmod{4}$, we define

$$\mathcal{P}_{k,d}(z) := H_{k,d}|_{1-k}(S - 1)(z) \tag{3.15}$$

$$= \sum_{\substack{|b|^2 - 4ac = d \\ c < 0 < a}} \left(az\bar{z} + \frac{bz}{2} + \frac{\bar{b}\bar{z}}{2} + c \right)^{k-1}. \tag{3.16}$$

Proposition 2. The polynomial $\mathcal{P}_{k,d}$ satisfies

1. $\mathcal{P}_{k,d}(z) = \mathcal{P}_{k,d}(-z) = \mathcal{P}_{k,d}(\bar{z}) = \mathcal{P}_{k,d}(iz)$,
2. $\mathcal{P}_{k,d}|_{1-k}(S + 1) = 0$ and
3. $\mathcal{P}_{k,d}(z + 1) - \mathcal{P}_{k,d}(z) = |z|^{2k-2} \mathcal{P}_{k,d}\left(\frac{z + 1}{z}\right)$.

Proof. Part 1 is clear from the equation (3.16) once we notice that if b is a solution to $|b|^2 - 4ac = d$ then so are $-b, \bar{b}, ib$.

For part 2 one has

$$\mathcal{P}_{k,d}|(S+1) = H_{k,d}|(S-1)(S+1) = H_{k,d}|(S^2-1) = 0$$

since $S^2 = 1$.

As for part 3 we have

$$\begin{aligned} \mathcal{P}_{k,d}(z+1) - \mathcal{P}_{k,d}(z) &= |z+1|^{2k-2} H_{k,d}\left(-\frac{1}{z+1}\right) - H_{k,d}(z+1) - |z|^{2k-2} H_{k,d}\left(-\frac{1}{z}\right) + H_{k,d}(z) \\ &= |z+1|^{2k-2} H_{k,d}\left(-\frac{1}{z+1} + 1\right) - H_{k,d}(z) - |z|^{2k-2} H_{k,d}\left(\frac{1}{z}\right) + H_{k,d}(z) \\ &= |z+1|^{2k-2} H_{k,d}\left(\frac{z}{z+1}\right) - |z|^{2k-2} H_{k,d}\left(\frac{1}{z} + 1\right) \\ &= |z|^{2k-2} \mathcal{P}_{k,d}\left(\frac{z+1}{z}\right) \end{aligned}$$

□

Lemma 3. *We have*

$$(H_{4,d}|_{-3}S)(z) - H_{4,d}(z) = \mathcal{P}_{4,d}(z) = \omega_{4,d}(z^3 \bar{z}^3 - 1). \quad (3.17)$$

Proof. By Definition 2 we have

$$(H_{4,d}|_{-3}S)(z) - H_{4,d}(z) = \mathcal{P}_{4,d}(z) = \sum_{\substack{|b|^2 - 4ac = d \\ c < 0 < a}} \left(az\bar{z} + \frac{bz}{2} + \frac{\bar{b}\bar{z}}{2} + c \right)^3.$$

Using the multinomial theorem we can write $\mathcal{P}_{4,d}(z)$ in the form

$$\mathcal{P}_{4,d}(z) = \sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \left(\sum_{k_1+k_2+k_3+k_4=3} \binom{3}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2+k_3}} |z|^{2k_1} z^{k_2} \bar{z}^{k_3} \right).$$

We now use Proposition 2 to deduce restrictions on the coefficients of $\mathcal{P}_{4,d}$. First,

$$\begin{aligned} \mathcal{P}_{4,d}(-z) &= \sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \left(\sum_{k_1+k_2+k_3+k_4=3} \binom{3}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2+k_3}} |-z|^{2k_1} (-z)^{k_2} (-\bar{z})^{k_3} \right) \\ &= \sum_{k_1+k_2+k_3+k_4=3} \left(\sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \binom{3}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2+k_3}} (-1)^{k_2+k_3} |z|^{2k_1} z^{k_2} \bar{z}^{k_3} \right). \end{aligned}$$

Since $\mathcal{P}_{4,d}(z) = \mathcal{P}_{4,d}(-z)$ we get that the inner sum vanishes unless $2|(k_2 + k_3)$. This implies that $k_2 + k_3 = 0, 2$ since $k_2 + k_3 \leq 3$. Also,

$$\begin{aligned} \mathcal{P}_{4,d}(iz) &= \sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \left(\sum_{k_1+k_2+k_3+k_4=3} \binom{3}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2+k_3}} |iz|^{2k_1} (iz)^{k_2} (\bar{iz})^{k_3} \right) \\ &= \sum_{k_1+k_2+k_3+k_4=3} \left(\sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \binom{3}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2+k_3}} (-1)^{k_3} (i)^{k_2+k_3} |z|^{2k_1} z^{k_2} \bar{z}^{k_3} \right). \end{aligned}$$

Since $\mathcal{P}_{4,d}(z) = \mathcal{P}_{4,d}(iz)$ we have that the inner sum vanishes unless either k_3 is odd and $k_2 + k_3 \equiv 2 \pmod{4}$ or $2|k_3$ and $k_2 + k_3 \equiv 0 \pmod{4}$. Since $k_2 + k_3 \leq 3$, in the first case we get that $k_2 = k_3 = 1$ and in the second case we get that $k_2 = k_3 = 0$.

So,

$$\begin{aligned}
\mathcal{P}_{4,d}(z) &= \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \left(\sum_{k_1+k_4=3} \binom{3}{k_1, 0, 0, k_4} a^{k_1} c^{k_4} |z|^{2k_1} + \sum_{k_1+k_4=1} \binom{3}{k_1, 1, 1, k_4} \frac{a^{k_1} |b|^2 c^{k_4}}{4} |z|^{2k_1+2} \right) \\
&= \sum_{\substack{|b|^2-4ac=d \\ c<0<a}} \left(a^3 |z|^6 + \left(\frac{3}{2} a |b|^2 + 3a^2 c \right) |z|^4 + \left(\frac{3}{2} c |b|^2 + 3ac^2 \right) |z|^2 + c^3 \right) \\
&= \omega_{4,d} |z|^6 + \omega'_{4,d} |z|^4 + \omega''_{4,d} |z|^2 - \omega_{4,d}
\end{aligned}$$

for some $\omega'_{4,d}, \omega''_{4,d} \in \mathbb{Q}$ depending on d .

Finally, $\mathcal{P}_{4,d}(1+S) = 0$, we see that $\omega'_{4,d} = -\omega''_{4,d}$ and,

$$\mathcal{P}_{4,d}(z) = \omega_{4,d} |z|^6 + \omega'_{4,d} |z|^4 - \omega_{4,d} |z|^2 - \omega_{4,d}.$$

By part 3 of Proposition 2 we have

$$\mathcal{P}_{4,d}(z+1) - \mathcal{P}_{4,d}(z) = |z|^6 \mathcal{P}_{4,d} \left(\frac{z+1}{z} \right). \quad (3.18)$$

A direct calculation shows that $|z|^6 - 1$ satisfies the equation (3.18) but that $|z|^4 - |z|^2$ does not. Thus, the middle terms must vanish and we get

$$\mathcal{P}_{4,d}(z) = \omega_{4,d} |z|^6 - \omega_{4,d} = \omega_{4,d} (z^3 \bar{z}^3 - 1).$$

□

Theorem 9. $H_{4,d}(z) = \omega_{4,d}$ for all $z \in \mathbb{C}$.

Proof. Let $H_{4,d}^0(z) := H_{4,d}(z) - \omega_{4,d}$. Then, the same argument as in the proof of

theorem 8 shows that $H_{4,d}^0(z) = H_{4,d}^0(0) = 0$ for all $z \in \mathbb{Q}(i)$ and thus $H_{4,d}(z) = \omega_{4,d}$ for all $z \in \mathbb{Q}(i)$. Since $H_{4,d}(z)$ is continuous, we conclude that $H_{4,d}(z) = \omega_{4,d}$ for all $z \in \mathbb{C}$.

□

Remark 2: Theorems 8 and 9 show that the vector space spanned by $H_{k,d}(z)$ as d varies over the positive integers that are not norm in $\mathbb{Z}[i]$ with $d \equiv 0, 1, 2 \pmod{4}$ is one dimensional for $k = 2, 4$. The same argument used to prove Theorems 8 and 9 can be used to show that the vector space spanned by $H_{k,d}(z)$ as d varies is of dimension 2 when $k = 6$ and finite dimensional for $k \geq 8$.

Theorem 10. $H_{6,d}(z) = \kappa_d H_{6,6}(z) + (\omega_{6,d} - \kappa_d \omega_{6,6})$ where κ_d is a rational number depending on d .

Proof. In order to prove the statement, we will need the expanded form of the function $\mathcal{P}_{6,d}(z)$. Again using the multinomial theorem we can write $\mathcal{P}_{6,d}(z)$ in the form

$$\mathcal{P}_{6,d}(z) = \sum_{\substack{|b|^2 - 4ac = d \\ c < 0 < a}} \left(\sum_{k_1 + k_2 + k_3 + k_4 = 5} \binom{5}{k_1, k_2, k_3, k_4} \frac{a^{k_1} b^{k_2} \bar{b}^{k_3} c^{k_4}}{2^{k_2 + k_3}} |z|^{2k_1} z^{k_2} \bar{z}^{k_3} \right).$$

$\mathcal{P}_{6,d}(z) = \mathcal{P}_{6,d}(-z)$ implies that $2|(k_2 + k_3)$. Since $k_2 + k_3 \leq 5$ we get $k_2 + k_3 = 0, 2, 4$. $\mathcal{P}_{4,d}(z) = \mathcal{P}_{4,d}(iz)$ implies that either k_3 is odd and $k_2 + k_3 \equiv 2 \pmod{4}$ or $2|k_3$ and $k_2 + k_3 \equiv 0 \pmod{4}$. Since $k_2 + k_3 \leq 5$, in the first case we get that $k_2 = k_3 = 1$ and in the second case we get that $(k_2, k_3) = (0, 0), (0, 4), (2, 2), (4, 0)$. Thus,

$$\begin{aligned}
\mathcal{P}_{6,d}(z) &= \sum_{\substack{|b|^2-4ac=d \\ c < 0 < a}} \left(\sum_{k_1+k_4=5} \binom{5}{k_1, 0, 0, k_4} a^{k_1} c^{k_4} |z|^{2k_1} + \sum_{k_1+k_4=3} \binom{5}{k_1, 1, 1, k_4} \frac{a^{k_1} |b|^2 c^{k_4}}{4} |z|^{2k_1+2} \right. \\
&+ \sum_{k_1+k_4=1} \binom{5}{k_1, 2, 2, k_4} \frac{a^{k_1} |b|^4 c^{k_4}}{16} |z|^{2k_1+4} + \sum_{k_1+k_4=1} \binom{5}{k_1, 0, 4, k_4} \frac{a^{k_1} \bar{b}^4 c^{k_4}}{16} |z|^{2k_1} \bar{z}^4 \\
&+ \left. \sum_{k_1+k_4=1} \binom{5}{k_1, 4, 0, k_4} \frac{a^{k_1} b^4 c^{k_4}}{16} |z|^{2k_1} z^4 \right) \\
&= \omega_{6,d} |z|^{10} + A_1 |z|^8 + A_2 |z|^6 + A_3 |z|^4 + A_4 |z|^2 + A_5 z \bar{z}^5 + A_6 z^5 \bar{z} + A_7 \bar{z}^4 + A_8 z^4 - \omega_{6,d}
\end{aligned}$$

where A_i 's are in \mathbb{Q} and depend on d .

Since $\mathcal{P}_{6,d}(z) = \mathcal{P}_{6,d}(\bar{z})$, we have that $A_5 = A_6$ and $A_7 = A_8$. Also, using the equation $\mathcal{P}_{6,d}(S+1) = 0$, we have that $A_1 = -A_4$, $A_2 = -A_3$, $A_5 = -A_8$ and $A_6 = -A_7$. Thus

$$\mathcal{P}_{6,d}(z) = \omega_{4,d}(|z|^{10} - 1) + A_1(|z|^8 - |z|^2) + A_2(|z|^6 - |z|^4) + A_5(z\bar{z}^5 + z^5\bar{z} - \bar{z}^4 - z^4). \quad (3.19)$$

By part 3 of Proposition 2 we have

$$\mathcal{P}_{6,d}(z+1) - \mathcal{P}_{6,d}(z) = |z|^{10} \mathcal{P}_{6,d}\left(\frac{z+1}{z}\right). \quad (3.20)$$

Substituting (3.19) in to (3.20) and comparing the coefficients of $z^5 \bar{z}^4$ on both sides of 3.20 gives $3A_1 + A_2 + 2A_5 = 0$, while comparing the coefficients of $z^5 \bar{z}^2$ gives $4A_1 + A_2 + 6A_5 = 0$. These equations yield $A_1 = -4A_5$ and $A_2 = 10A_5$.

So, we get

$$\mathcal{P}_{6,d}(z) = \omega_{4,d}(|z|^{10} - 1) - 4A_5(|z|^8 - |z|^2) + 10A_5(|z|^6 - |z|^4) + A_5(z\bar{z}^5 + z^5\bar{z} - \bar{z}^4 - z^4).$$

We also computed $\mathcal{P}_{6,6}(z)$ which is

$$\mathcal{P}_{6,6}(z) = 4(|z|^{10} - 1) - 20(|z|^8 - |z|^2) + 50(|z|^6 - |z|^4) + 5(z\bar{z}^5 + z^5\bar{z} - \bar{z}^4 - z^4). \quad (3.21)$$

Then we can write $\mathcal{P}_{6,d}(z)$ as

$$\mathcal{P}_{6,d}(z) = \omega_{6,d}(|z|^{10} - 1) + \kappa_d (\mathcal{P}_{6,6}(z) - 4(|z|^{10} - 1)) \quad (3.22)$$

where $\kappa_d = \frac{A_5}{5}$.

Let $H_{6,d}^0(z) = H_{6,d}(z) - \omega_{6,d} - \kappa_d(H_{6,6}(z) - 4)$. We will show that $H_{6,d}^0(z) = 0$ for all $z \in \mathbb{C}$ which is equivalent to the statement of the theorem. It is clear that

$$(H_{6,d}^0|_{-5}T)(z) = H_{6,d}^0(z), \quad (H_{6,d}^0|_{-5}T_i)(z) = H_{6,d}^0(z).$$

Now, let us show that $(H_{6,d}^0|_{-5}S)(z) = H_{6,d}^0(z)$

$$(H_{6,d}^0|_{-5}S)(z) = (H_{6,d}|_{-5}S)(z) - \omega_{6,d}|z|^{10}\kappa_d((H_{6,6}|_{-5}S)(z) - 4|z|^{10}) \quad (3.23)$$

$$= H_{6,d}(z) + \mathcal{P}_{6,d}(z) - \omega_{6,d}|z|^{10} - \kappa_d (H_{6,6}(z) + \mathcal{P}_{6,6}(z) - 4|z|^{10}) \quad (3.24)$$

Substituting 3.22 for $\mathcal{P}_{6,d}(z)$ in 3.24 gives

$$(H_{6,d}^0|_{-5}S)(z) = H_{6,d}(z) - \omega_{6,d} + 4\kappa_d - \kappa_d H_{6,6}(z) \quad (3.25)$$

$$= H_{6,d}(z) - \omega_{6,d} - \kappa_d(H_{6,6}(z) - 4) \quad (3.26)$$

$$= H_{6,d}^0(z). \quad (3.27)$$

Thus, $H_{6,d}^0(z) = H_{6,d}^0(0) = H_{6,d}(0) - \omega_{6,d} - \kappa_d(H_{6,6}(0) - 4)$ for all $z \in \mathbb{Q}(i)$. Since $H_{6,d}(0) = \omega_{6,d}$ and $H_{6,6}(0) = \omega_{6,6} = 4$, we have that $H_{6,d}^0(z) = 0$ for all $z \in \mathbb{Q}(i)$ and by continuity for all $z \in \mathbb{C}$. \square

Let \mathcal{M}_k denote the vector space of all polynomials of degree at most $k-1$ in both z and \bar{z} defined in Definition 2. Let $\dim(\mathcal{W}_k) = n$ with $n \leq k-1$. The argument we have used so far to prove theorems 8, 9, and 10 generalizes for all k . Note that the elements of \mathcal{M}_k are just different \mathcal{P}_{k,d_j} where d_j is a positive integer that is not a norm in $\mathbb{Z}[i]$ with $d_j \equiv 0, 1, 2 \pmod{4}$ for each j . Since \mathcal{M}_k is finite dimensional we can pick a fixed basis $\{\mathcal{P}_{k,d_j}\}_{j=1}^n$ for \mathcal{M}_k and using the same ideas as in the proofs Lemma 2, 3 and Theorem 10 we can write $\mathcal{P}_{k,d}(z)$ as

$$\mathcal{P}_{k,d}(z) = \omega_{k,d}(|z|^{2k-2}-1) + \kappa_{k,d_1}(\mathcal{P}_{k,d_1}(z) - \omega_{k,d_1}(|z|^{2k-2}-1)) + \dots + \kappa_{k,d_n}(\mathcal{P}_{k,d_n}(z) - \omega_{k,d_n}(|z|^{2k-2}-1))$$

where $\kappa_{k,d_j} \in \mathbb{Q}$. We define $H_{k,d}^0(z)$ in the same way as we defined $H_{6,d}^0$:

$$H_{k,d}^0(z) = H_{k,d}(z) - \omega_{k,d} - \kappa_{k,d_1}(H_{k,d_1}(z) - \omega_{k,d_1}) - \dots - \kappa_{k,d_n}(H_{k,d_n}(z) - \omega_{k,d_n}).$$

Applying by now the familiar argument one sees that $H_{k,d}^0(z) = 0$. Thus we have proved the following theorem.

Theorem 11. $H_{k,d}(z)$ is a linear combination of finitely many H_{k,d_j} where d_j is a positive integer that is not a norm in $\mathbb{Z}[i]$ with $d_j \equiv 0, 1, 2 \pmod{4}$ for each j .

Chapter 4

Continuity of $H_{k,d}$ and Hurwitz's algorithm

It is straight forward to see that $H_{k,d}$ is continuous when $k \geq 4$ as we prove it in the next theorem. However, when $k = 2$ we are not able to prove the continuity using the argument and the uniform bound used to prove the continuity for $k \geq 4$ since that bound is not good enough when $k = 2$. Of course, we could search for a better bound but this is also difficult because the functions that appear in the sum (3.4) depend on z which makes it difficult to find a bound independent of z for the summands in a neighborhood of z . For a given z , one needs a more explicit description of the functions appearing in the sum for $H_{2,d}$ and how they are related to z . In the case of $F_{2,D}$, we saw in section 2.1 that the functions appearing in the sum for $F_{2,D}$ for a given x are related to the continued fraction of x , and Conjecture 1 precisely described these functions. So it is natural to ask if something similar is also true in the case of $H_{2,d}$. This is indeed the case and we can relate the functions

appearing in $H_{2,d}$ to the continued fraction of z which allows us to give a proof of the continuity when $k = 2$. We will discuss this in the next section but now let us prove the continuity of $H_{k,d}(z)$ for $k \geq 4$.

Theorem 12. *For $k \geq 4$, $H_{k,d}$ is a continuous function.*

Proof. We can write $H_{k,d}$ as:

$$H_{k,d}(z) = \sum_{a < 0} \left(\sum_{|b|^2 - 4ac = d} \max \left(0, (az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c)^{k-1} \right) \right) \quad (4.1)$$

where $z = x_1 + ix_2 \in \mathbb{C}$.

Claim 1. *For each a the number of functions appearing in the inner sum is finite and is independent of z .*

Proof. Observe that for all z we have the identity

$$d = |\bar{b} + 2az|^2 - 4a(az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c),$$

and the inequalities

$$a < 0 < az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c,$$

which together imply that $|\bar{b} + 2az|^2 < d$. So \bar{b} is a Gaussian integer belonging to a circle of radius d centered at $(2ax_1, -2ax_2)$. The number of Gaussian integers in a circle of fixed radius is finite. Thus, for each a the number of b 's satisfying the conditions of the sum is finite, independent of z which shows that for each a the

number of functions appearing in the inner sum is finite and is independent of z \square

Also, for each a the functions appearing in the inner sum are bounded by $\frac{d}{4|a|}$. So for each a and all z the inner sum is the sum of finitely many continuous functions that are bounded by $\frac{d}{4|a|}$. Hence by Weierstrass M-test, the sum in the definition of $H_{k,d}(z)$ converges uniformly and is a continuous function for $k \geq 4$. \square

When $k = 2$, the above discussion is of no use, since it shows that $H_{2,d}(z)$ converges like $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

4.1 Hurwitz's continued fraction algorithm

In order to show that $H_{2,d}$ is continuous, we use a continued fraction algorithm developed by Hurwitz in [8], which shows that the binary Hermitian forms appearing in the sum for a given $z \in \mathbb{C}$ have exponential decay. Hurwitz's continued fraction algorithm allows one to approximate complex numbers using Gaussian integers. Although the continued fraction algorithm for real numbers is a classical topic in number theory, playing an important role in Diophantine approximation, computation of the fundamental unit of real quadratic fields and various other areas, and has been studied extensively, there has been relatively little work on an analogous continued fraction algorithm for complex numbers. So, it is a nice surprise to uncover a connection between $H_{2,d}$ and continued fractions of complex numbers.

The algorithm we described in section 2.1 is usually called the regular continued fraction of a real number. The nearest integer continued fraction expansion of a real number is obtained from the regular expansion, by allowing n_i 's to take on negative integer values also.

Hurwitz's continued fraction algorithm for complex numbers is a generalization of the nearest integer continued fraction algorithm for real numbers to the complex field. The partial quotients in Hurwitz's algorithm are Gaussian integers. There is no straight forward generalization of regular case to complex field that we were able to find. The best attempt, in some sense, is a construction by Schmidt [14]. Below, we describe Hurwitz's algorithm and give some of its properties.

Let z be a complex number. We denote by $\lfloor z \rfloor$ the Gaussian integer nearest to z with respect to Euclidean distance in complex plane, rounding down, in both the real and imaginary components to break ties. Hurwitz's continued fraction algorithm proceeds by steps of the form

$$z_0 = z, \quad \alpha_n = \lfloor z_n \rfloor, \quad z_{n+1} = \frac{1}{z_n - \alpha_n} \quad (4.2)$$

where $\alpha_n = a_n + ib_n \in \mathbb{Z}[i]$ and $n \geq 0$. If $z = z_0 \in \mathbb{Q}(i)$ the algorithm terminates when, as must eventually occur, $z_n = 0$. If initially $z \notin \mathbb{Q}(i)$ then the algorithm continues indefinitely. Once the partial quotient α_n 's have been computed, the partial

convergents are computed using the usual formula from the real case:

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_n = \alpha_n p_{n-1} + p_{n-2} \quad (4.3)$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_n = \alpha_n q_{n-1} + q_{n-2}. \quad (4.4)$$

These numbers satisfy the equation

$$p_{n+1}q_n - p_nq_{n+1} = (-1)^n \quad (4.5)$$

and $\lim_{n \rightarrow \infty} \frac{p_n}{q_n} = z$. Moreover, in [4] it is shown that for all $n \geq 0$

$$\left| z - \frac{p_n}{q_n} \right| \leq \frac{c}{|q_n|^2} \quad (4.6)$$

and

$$|q_n| \geq \theta^{(n-1)/2} \quad (4.7)$$

where $\theta = \frac{\sqrt{5} + 1}{2}$ and $c = \frac{2\theta^2}{\theta^2 - 1}$. So, q_n 's have exponential growth. Similar to the real case, let us also define the numbers δ_n for Hurwitz's algorithm. The numbers δ_n ($n \geq -1$) defined by

$$\delta_n = (-1)^n (p_{n-1} - q_{n-1}z) \quad (4.8)$$

and for $n \geq 0$ satisfy the recurrence

$$\delta_{-1} = z, \quad \delta_0 = 1, \quad \delta_{n+1} = \delta_{n-1} - \alpha_n \delta_n,$$

and the inequalities $1 = |\delta_0| > |\delta_1| > |\delta_2| > \dots \geq 0$. If $z \in \mathbb{Q}(i)$, then $z = \frac{p_n}{q_n}$ for some n and the recurrence stops with $\delta_{n+1} = 0$. If $z \notin \mathbb{Q}(i)$, then by (4.6) and (4.7) δ_n 's converge to zero with exponential rapidity.

One can also describe the algorithm using the matrix notation as follows. We let $\tilde{\Gamma} = \text{PGL}(2, \mathbb{Z}[i])$. This group acts on \mathbb{C} by linear fractional transformation, i.e.

$\gamma(z) := \frac{az + b}{cz + d}$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}$ and $z \in \mathbb{C}$. As before, we let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then for $n \geq 0$

$$z_{n+1} = \frac{1}{z_n - \alpha_n} = \varepsilon T_i^{-b_n} T^{-a_n}(z_n) \quad (4.9)$$

where $\alpha_n = a_n + ib_n$. Notice that each z_n is an image of z under the linear fractional transformation by an element of $\tilde{\Gamma}$. This can be described as

$$\gamma_{0,z} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{n,z} := \begin{pmatrix} 0 & 1 \\ 1 & -\alpha_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -\alpha_0 \end{pmatrix} \quad (4.10)$$

for $n \geq 1$. Let $\tilde{\Gamma}(z) := \{\gamma_1, \gamma_2, \gamma_3, \dots\} \subset \tilde{\Gamma}$. Notice also that

$$\begin{aligned} \gamma_{0,z} &= \begin{pmatrix} q_{-2} & -p_{-2} \\ -q_{-1} & p_{-1} \end{pmatrix}, \quad \gamma_{1,z} = \begin{pmatrix} q_{-1} & -p_{-1} \\ -q_0 & p_0 \end{pmatrix}, \\ \gamma_{2,z} &= \begin{pmatrix} 0 & -1 \\ -1 & \alpha_0 \end{pmatrix} \begin{pmatrix} q_{-1} & -p_{-1} \\ -q_0 & p_0 \end{pmatrix} = \begin{pmatrix} q_0 & -p_0 \\ -q_1 & p_1 \end{pmatrix}, \quad \gamma_{3,z} = \cdots \end{aligned}$$

and by induction

$$\gamma_{n,z} = \begin{pmatrix} q_{n-2} & -p_{n-2} \\ -q_{n-1} & p_{n-1} \end{pmatrix} \quad (4.11)$$

which gives an explicit description of $\tilde{\Gamma}(z)$ in terms of the convergents of z .

4.2 Continuity of $H_{2,d}$

Using Hurwitz's algorithm we would like to describe, for a fixed $z \in \mathbb{C}$, the functions appearing in the sum for $H_{2,d}(z)$. We denote the set of such functions as

$$\Omega_d(z) = \left\{ aZ\bar{Z} + \frac{1}{2}(bZ + \bar{b}\bar{Z}) + c : |b|^2 - 4ac = d, a < 0 < az\bar{z} + \frac{1}{2}(bz + \bar{b}\bar{z}) + c \right\}$$

and the set of binary Hermitian forms over $\mathbb{Z}[i]$ with discriminant d evaluated at $w = 1$ as

$$\Omega_d = \left\{ aZ\bar{Z} + \frac{1}{2}(bZ + \bar{b}\bar{Z}) + c : |b|^2 - 4ac = d \right\}.$$

Recall that $\tilde{\Gamma}$ acts on Ω_d by

$$\begin{aligned} (P|\gamma)(Z) &= |tZ + v|^2 P\left(\frac{rZ + s}{tZ + v}\right) \\ &= (Z \ 1) \gamma^t \begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix} \bar{\gamma} \begin{pmatrix} \bar{Z} \\ 1 \end{pmatrix} \end{aligned}$$

where $\gamma = \begin{pmatrix} r & s \\ t & v \end{pmatrix}$. Let \mathcal{A} be an equivalence class in Ω_d under the action of $\tilde{\Gamma}$. For a fixed z , we denote

$$\mathcal{A}(\infty) = \{P \in \mathcal{A} : P(0) < 0 < P(\infty)\}, \quad (4.12)$$

$$\mathcal{A}(z) = \{P \in \mathcal{A} : P(\infty) < 0 < P(z)\} \quad (4.13)$$

where $P(\infty)$ is the coefficient of the term $z\bar{z}$ in $P(z)$. Then for $k = 2$ the sum in (3.4) can be written as

$$H_{2,d}(z) = \sum_{\mathcal{A} \in \Omega_d / \tilde{\Gamma}} \sum_{P \in \mathcal{A}(z)} P(z). \quad (4.14)$$

The next lemma gives a description of $\mathcal{A}(z)$, which is enough to prove the continuity of $H_{2,d}(z)$.

Lemma 4. *Let $z \in \mathbb{C}$, \mathcal{A} be an equivalence class in Ω_d under the action of $\tilde{\Gamma}_1$, and*

$$\mathcal{B}(z) = \left\{ (P, \gamma) \in \mathcal{A}(\infty) \times \tilde{\Gamma}(z) : P(\gamma(\infty)) < 0 < P(\gamma(z)) \right\}.$$

Then the map $\Phi : \mathcal{B}(z) \rightarrow \mathcal{A}(z)$ defined by $\Phi((P, \gamma)) = (P|_\gamma)$ is a surjection.

Proof. First we check that the map is well defined. Let

$$P(Z) = aZ\bar{Z} + \frac{1}{2}(bZ + \bar{b}\bar{Z}) + c \in \mathcal{A}(\infty), \quad \gamma = \begin{pmatrix} r & s \\ t & v \end{pmatrix}.$$

Then

$$(P|_\gamma) = \begin{pmatrix} r & t \\ s & v \end{pmatrix} \begin{pmatrix} a & b/2 \\ \bar{b}/2 & c \end{pmatrix} \begin{pmatrix} \bar{r} & \bar{s} \\ \bar{t} & \bar{v} \end{pmatrix}$$

and the coefficient of $Z\bar{Z}$ in $(P|_\gamma)(z)$ is

$$(P|_\gamma)(\infty) = ar\bar{r} + \frac{1}{2}(br\bar{t} + \bar{b}r\bar{t}) + ct\bar{t},$$

which is equal to $|t|^2 P\left(\frac{r}{t}\right) = |t|^2 P(\gamma(\infty))$. But by assumption $P(\gamma(\infty)) < 0$, thus $(P|_\gamma)(\infty) < 0$. Also, $P(\gamma(z)) = P\left(\frac{rz+s}{tz+v}\right) > 0$ implies that $(P|_\gamma)(z) =$

$|tz + v|^2 P\left(\frac{rz + s}{tz + v}\right) > 0$. So, Φ is well defined.

To show the surjectivity, let $P \in \mathcal{A}(z)$. Since

$$\left|z + \frac{\bar{b}}{2a}\right| = \frac{|b|^2 - 4ac}{4a^2} \quad (4.15)$$

we see that the set of zeros of P forms a circle in the complex plane centered at $\frac{\bar{b}}{2a}$ with radius $\frac{\sqrt{d}}{2|a|}$.

Let B be the open ball centered at $\frac{\bar{b}}{2a}$ with radius $\frac{\sqrt{d}}{2|a|}$. From equation (4.15), it follows that $P(Z) > 0$ if $Z \in B$ and $P(Z) < 0$ if $Z \notin B$, in particular $z \in B$, since $P \in \mathcal{A}(z)$ implies that $P(z) > 0$. Because $\frac{p_n}{q_n} \rightarrow z$ as $n \rightarrow \infty$, we can find a $n \gg 0$ such that $\frac{p_{n-1}}{q_{n-1}} \notin B$ but $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}, \frac{p_{n+2}}{q_{n+2}}, \dots \in B$. If no such n exists, we set $n = 0$. Since $P(\infty) < 0$, in both cases we have

$$P\left(\frac{p_{n-1}}{q_{n-1}}\right) < 0 < P\left(\frac{p_n}{q_n}\right), \quad P\left(\frac{p_{n+1}}{q_{n+1}}\right) > 0.$$

Let

$$\gamma = \gamma_{n+1} = \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} \implies \gamma^{-1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

and

$$R = (P|\gamma^{-1}) \implies R(z) = |q_n z + q_{n-1}|^2 P\left(\frac{p_n z + p_{n-1}}{q_n z + q_{n-1}}\right).$$

Then $R(0) = |q_{n-1}|^2 P\left(\frac{p_{n-1}}{q_{n-1}}\right) < 0$ and $R(\infty) = |q_n|^2 P\left(\frac{p_n}{q_n}\right) > 0$. Thus $R \in \mathcal{A}(\infty)$.

Moreover $(R|\gamma) = P$, completing the proof of surjectivity. \square

Remark 2: When $z = 0$, $\tilde{\Gamma}(z) = \{\emptyset\}$. However, Lemma 4 is still true by simply letting $\gamma = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Theorem 13. $H_{2,d}(z)$ is continuous.

Proof. From (4.14) we see that $H_{2,d}(z)$ is the sum of the sums $\sum_{P \in \mathcal{A}(z)} P(z)$ over all the $\tilde{\Gamma}$ -equivalence classes in Ω_d . By Lemma 4, we have that

$$\sum_{P \in \mathcal{A}(z)} P(z) \leq \sum_{P \in \mathcal{A}(\infty)} \sum_{\substack{\gamma \in \tilde{\Gamma}(z) \\ P(\gamma(\infty)) < 0 \\ P(\gamma(z)) > 0}} (P|\gamma). \quad (4.16)$$

There are finitely many P in $\mathcal{A}(\infty)$. So, the outer sum on the right side of inequality (4.16) is finite. Furthermore, the summands in the inner sum are of the form

$$(P|\gamma_n)(z) = a|\delta_{n-1}|^2 + \frac{1}{2} (b\delta_{n-1}\bar{\delta}_n + \bar{b}\bar{\delta}_{n-1}\delta_n) + c|\delta_n|^2$$

for some $n \geq 0$. The series $\delta_n = |p_{n-1} - q_{n-1}z|$ stops if $z \in \mathbb{Q}(i)$ and decreases exponentially to 0 if $z \notin \mathbb{Q}(i)$. So, the double sum on the right side of (4.16) has exponential decay, which implies that $H_{2,d}(z)$ has exponential decay. Thus $H_{2,d}(z)$ is continuous. \square

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