City University of New York (CUNY)
CUNY Academic Works

All Dissertations, Theses, and Capstone Projects
Dissertations, Theses, and Capstone Projects

2008

Rigidity and Stability for Isometry Groups in Hyperbolic 4-Space

Youngju Kim
Graduate Center, City University of New York

How does access to this work benefit you? Let us know!

More information about this work at: https://academicworks.cuny.edu/gc_etds/1664
Discover additional works at: https://academicworks.cuny.edu

This work is made publicly available by the City University of New York (CUNY).
Contact: AcademicWorks@cuny.edu

Rigidity and Stability for Isometry Groups in Hyperbolic 4-Space

by

Youngju Kim

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2008

UMI Number: 3325467

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI®

UMI Microform 3325467
Copyright 2008 by ProQuest LLC
All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

©2008

Youngju Kim

All Rights Reserved

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

Dr. Ara Basmajian

Date — Chair of Examining Committee

Dr. Józef Dodziuk

Date — Executive Officer

Dr. Frederick Gardiner

Dr. Linda Keen

Dr. Peter Storm

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Rigidity and Stability for Isometry Groups in Hyperbolic 4-Space

by

Youngju Kim

Advisor: Professor Ara Basmajian

It is known that a geometrically finite Kleinian group is quasiconformally stable. We prove that this quasiconformal stability cannot be generalized in 4-dimensional hyperbolic space. This is due to the presence of screw parabolic isometries in dimension 4. These isometries are topologically conjugate to strictly parabolic isometries. However, we show that screw parabolic isometries are not quasiconformally conjugate to strictly parabolic isometries. In addition, we show that two screw parabolic isometries are generically not quasiconformally conjugate to each other. We also give some geometric properties of a hyperbolic 4-manifold related to screw parabolic isometries.

A Fuchsian thrice-punctured sphere group has a trivial deformation space in hyperbolic 3-space. Thus, it is quasiconformally rigid. We prove that the Fuchsian thrice-punctured sphere group has a large deformation space in hyperbolic 4-space which is in contrast to lower dimensions. In particular, we prove that there is a 2-dimensional parameter space in the deformation space of the Fuchsian thrice-punctured sphere group for which the deformations are all geometrically finite and generically quasiconformally distinct. In contrast, the thrice-punctured sphere group is still quasiconformally rigid in hyperbolic 4-space.

Along the way, we classify the isometries of hyperbolic 4-space by their isometric sphere decompositions. Our techniques involve using 2×2 Clifford matrix representations of the isometries of hyperbolic 4-space. This is a natural generalization of the classical cases $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{PSL}(2, \mathbb{R})$.

Acknowledgments

This will be a rather short note simply because my debt to everyone is too deep to be expressed. My education would not have been possible if it were not for the help of every single one of them.

I would like to begin by thanking Professor Ara Basmajian for being a wonderful advisor and for having guided me through graduate school. I would also like to thank the committee members for their encouragement and for making my defense enjoyable, and faculty members at the Graduate Center for many helpful discussions; especially to Professor Saeed Zakeri. Thanks are also due to all my friends at CUNY for being around me and for reading my writing. I thank my teachers and friends at the University of Oklahoma for the first years of my graduate studies. The debt of my gratitude to my teachers and friends in Korea is large. Finally, I would like to express my deepest gratitude to my family. This thesis is dedicated to them.

Sing to the Lord a new song.

Y. K.
August of 2008
New York, NY

Contents

Figures and Tables

Chapter 1

Introduction

In this thesis, we study the deformation space of a Möbius group acting on hyperbolic 4-space $\mathbb{H}^4$. The deformation theory of Kleinian groups which are Möbius groups acting on hyperbolic 3-space is well understood and plays an important role in lower dimensional hyperbolic geometry ([8], [9]). However, it is less well understood in dimension 4.

A Möbius group is a finitely generated discrete subgroup of the group of all orientation-preserving hyperbolic isometries, denoted by $\mathrm{Isom}(\mathbb{H}^n)$. The deformation space of the Möbius group is the set of all discrete, faithful and type-preserving representations into $\mathrm{Isom}(\mathbb{H}^n)$ factored by the conjugation action of $\mathrm{Isom}(\mathbb{H}^n)$. Mostow-Prasad rigidity states that for $n \geq 3$ the deformation space of a torsion-free cofinite volume Möbius group acting on hyperbolic n-space is trivial ([18], [19]). Thus there is no deformation theory for such a Möbius group. For a geometrically finite Möbius group, we have Marden quasiconformal stability in $\mathbb{H}^2$ and $\mathbb{H}^3$ ([16]). That is, for a geometrically finite Kleinian group all deformations sufficiently near the identity deformation are quasiconformally conjugate to the identity. The proof uses the fact that a geometrically finite Kleinian group has a finite-sided fundamental domain. However, since a geometrically finite Möbius group acting on $\mathbb{H}^4$ can have an infinite-sided fundamental domain (see [7], [20]), Marden's proof cannot be generalized in hyperbolic 4-space. In this thesis, we prove that quasiconformal stability does not hold in hyperbolic 4-space $\mathbb{H}^4$.

Quasiconformal non-Stability Theorem. *There is a Möbius group acting on hyperbolic* 4*-space* $\mathbb{H}^4$ *which is geometrically finite, but not quasiconformally stable.*

This is mainly due to the presence of screw parabolic isometries in this dimension. We note that hyperbolic 4-space is the lowest dimension hyperbolic space where screw parabolic isometries appear. The screw parabolic isometry is not Möbius-conjugate to a strictly parabolic isometry but topologically conjugate to a strictly parabolic isometry. We show that any screw parabolic isometry is not quasiconformally conjugate to a strictly parabolic isometry. For the proof, first we show that any rational screw parabolic isometry is not quasiisometric to a strictly parabolic isometry in hyperbolic 4-space. Then, using Tukia's extension theorem which states that a quasiconformal map on the boundary at infinity extends to hyperbolic space as a quasiisometry, we show that a rational screw parabolic isometry can not be quasiconformally conjugate to a strictly parabolic isometry on the boundary at infinity. In the case of an irrational screw parabolic isometry, we use the property that any infinite sequence of uniformly quasiconformal maps has a convergent subsequence and the limit becomes a quasiconformal map under mild conditions. We will discuss these screw parabolic isometries and related results in Chapter 2.

A Möbius group is quasiconformally rigid if any quasiconformal deformation is conjugate to the identity deformation by a Möbius transformation. A thrice-punctured sphere group is a Möbius group generated by two parabolic isometries whose product is a parabolic isometry. In $\mathbb{H}^2$ and $\mathbb{H}^3$, thrice-punctured sphere groups are Fuchsian groups corresponding to the fundamental group of a thrice-punctured sphere and hence, they are all conjugate to each other by Möbius transformations. Thus, they have a trivial deformation space. Therefore, they are quasiconformally rigid. In contrast to lower dimensions, a thrice-punctured sphere group has a large deformation space in hyperbolic 4-space $\mathbb{H}^4$. In particular, a thrice-punctured sphere group generated by two strictly parabolic isometries has at least a 2-dimensional parameter space for its deformation space. Specifically we have

Theorem 3.2.5. *There is a 2-dimensional parameter space containing the identity deformation in the deformation space of the Fuchsian thrice-punctured sphere group in $\mathbb{H}^4$ such that*

1. *Each non-trivial deformation in the space is not quasiconformally conjugate to the identity.*
2. *The deformations are all quasiconformally distinct except a measure zero set.*
3. *All images of the deformations are geometrically finite.*
4. *The hyperbolic 4-manifolds obtained as the quotient of $\mathbb{H}^4$ by the images of deformations have the same marked length spectrum.*
5. *There are no simple closed geodesics in their quotient hyperbolic 4-manifolds.*

Theorem 3.2.5. states that any two deformations in the parameter space are generically not quasiconformally conjugate to each other. For this, using a Lie group property (Lemma 2.2.10), we prove that any two irrational screw parabolic isometries are not quasiconformally conjugate to each other unless they are Möbius-conjugate to each other. Thus the quasiconformal non-stability theorem follows immediately from Theorem 3.2.5.

On the other hand, we have

Theorem 3.2.2. *A thrice-punctured sphere group generated by two strictly parabolic elements is quasiconformally rigid in* $\mathbb{H}^4$.

All details about the deformation space and quasiconformal non-stability and rigidity of Möbius groups are in Chapter 3.

It is known that any hyperbolic isometry can be represented as a Clifford matrix which is a 2×2 matrix whose entries are Clifford numbers satisfying some conditions ([4], [23]). These representations have natural extensions from n-dimensions to $(n+1)$-dimensions. This so-called Clifford representation provides us with a good tool for the computation of isometries acting on $\mathbb{H}^4$.

Specifically, we classify the 4-dimensional hyperbolic isometries in terms of the incidence relations of their isometric spheres on the boundary at infinity (See Table 4.1). In particular, the two isometric spheres corresponding to a screw parabolic isometry and its inverse can transversally intersect. This is in contrast to lower dimensions, where every pair of isometric spheres corresponding to a parabolic isometry and its inverse are tangent. This classification of 4-dimensional hyperbolic isometries is in Chapter 4. In the rest of this chapter, we provide basic material about hyperbolic geometry, quasiconformal homeomorphisms and Clifford matrix representations.

1.1 Hyperbolic geometry

Hyperbolic spaces and isometries.

Hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ is the unique complete simply connected $(n+1)$-dimensional Riemannian manifold with constant sectional curvature -1. It has the natural boundary at infinity, denoted by $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, also called the sphere at infinity.

The upper half space model for $\mathbb{H}^{n+1}$ is

$$U = \{(x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

with the metric $ds = \frac{dx}{x_{n+1}}$. Its boundary $\hat{\mathbb{R}}^n = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\} \cup \{\infty\}$ of U represents all the points of the sphere at infinity. The geodesics are the Euclidean semi-circles orthogonal to the boundary and the vertical Euclidean lines in U. The k-dimensional totally geodesic subspaces of hyperbolic space, called *hyperbolic k-planes*, are k-dimensional vertical planes and k-dimensional half-spheres which are orthogonal to the boundary. We denote by $d(\cdot,\cdot)$ the hyperbolic distance and $d_E(\cdot,\cdot)$ the Euclidean distance. For any $x = (x_1, \cdots, x_{n+1})$ and $y = (y_1, \cdots, y_{n+1})$ of $\mathbb{H}^{n+1}$,

$$\cosh d(x, y) = 1 + \frac{d_E^2(x, y)}{2x_{n+1}y_{n+1}}.$$

A *horoball* Σ of $\mathbb{H}^{n+1}$ based at a of $\hat{\mathbb{R}}^n$ is defined to be an open Euclidean ball in $\mathbb{H}^{n+1}$ which is tangent to the sphere at infinity $\hat{\mathbb{R}}^n$ at a. The boundary $\partial\Sigma$ of a horoball is called a *horosphere*. A horosphere $\partial\Sigma$ based at a is also a surface in $\mathbb{H}^{n+1}$ which is orthogonal to all hyperbolic hyperplanes containing the point a. In particular, a horoball based at ∞ is a set of the form $\{\ (x_1, x_2, \cdots x_{n+1}) \in \mathbb{H}^{n+1} \mid x_{n+1} > t\ \}$ for some $t > 0$.

Möbius transformations acting on $\hat{\mathbb{R}}^n$ are finite compositions of reflections in spheres or hyperplanes. Clearly, Möbius transformations map spheres and planes to spheres and planes. Every orientation-preserving Möbius transformation of $\hat{\mathbb{R}}^n$ extends continuously to a homeomorphism of $\mathbb{H}^{n+1} \cup \hat{\mathbb{R}}^n$ which is an orientation-preserving isometry of $\mathbb{H}^{n+1}$. Moreover, every orientation-preserving isometry of $\mathbb{H}^{n+1}$ extends continuously to the boundary at infinity $\hat{\mathbb{R}}^n$ as a Möbius transformation. Therefore, we can identify the group of all orientation-preserving isometries of hyperbolic $(n+1)$-space, denoted by $\mathrm{Isom}(\mathbb{H}^{n+1})$, with the group of all orientation-preserving Möbius transformations of $\hat{\mathbb{R}}^n$, denoted by $\mathrm{M\ddot{o}b}(\hat{\mathbb{R}}^n)$. We assume for simplicity that all isometries and Möbius transformations are orientation-preserving.

Every isometry of $\mathbb{H}^{n+1}$ has at least one fixed point in the closure of $\mathbb{H}^{n+1}$ by Brouwer's fixed point theorem. We classify all isometries into three types with respect to their fixed points. If it has a fixed point in $\mathbb{H}^{n+1}$, then it is *elliptic*. If it is not elliptic and has exactly one fixed point on the boundary at infinity $\hat{\mathbb{R}}^n$, then it is *parabolic*; otherwise it is *loxodromic*. If it fixes more than two points on the boundary, then it has a fixed point in $\mathbb{H}^{n+1}$. Therefore, a loxodromic isometry has exactly two fixed points on the boundary at infinity.

An elliptic isometry is conjugate to an element of $\mathrm{SO(n+1)}$ by a Möbius transformation. An elliptic isometry is called *boundary elliptic* if it has a fixed point on $\hat{\mathbb{R}}^n$; otherwise it is *non-boundary elliptic*. Every elliptic isometry is boundary elliptic in odd dimensional hyperbolic spaces. A non-boundary elliptic isometry exists only in even dimensional hyperbolic spaces.

On the boundary at infinity $\hat{\mathbb{R}}^n$, a parabolic isometry is conjugate to $x \mapsto Ax + a$ with $A \in O(n)$, $a \in \mathbb{R}^n \backslash \{0\}$ by a Möbius transformation. If $A = I$, then it is called *strictly parabolic*; otherwise it is *screw parabolic*. A screw parabolic element is said to be *rational* if some iteration of it becomes strictly parabolic. Otherwise, it is *irrational*. Screw parabolic isometries exist in at least 4-dimensional hyperbolic spaces. In dimensions 2 and 3, every parabolic isometry is conjugate to $x \mapsto x + 1$ by a Möbius transformation.

On the boundary at infinity $\hat{\mathbb{R}}^n$, a loxodromic isometry is conjugate to $z \mapsto rAx$ with $r > 0$, $r \neq 1$ and $A \in O(n)$ by a Möbius transformation. If $A = I$, then f is called *hyperbolic*. All loxodromic isometries are hyperbolic in hyperbolic 2-space $\mathbb{H}^2$.

Representations of Möbius transformations.

Möbius transformations acting on $\hat{\mathbb{R}}^2$ have classical matrix representations, $\mathrm{PSL}(2, \mathbb{C})$. First, we identify $\hat{\mathbb{R}}^2$ with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by $(x, y) \mapsto x + yi$ and $\infty \mapsto \infty$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, where a, b, c and d are complex numbers with $ad - bc = 1$. The matrix A acts on $\hat{\mathbb{C}}$ by $z \mapsto \frac{az+b}{cz+d}$ for any $z \in \hat{\mathbb{C}}$. Note that $\infty \mapsto \frac{a}{c}$, $-\frac{d}{c} \mapsto \infty$ if $c \neq 0$ and $\infty \mapsto \infty$ if $c = 0$. Then, the induced map is a Möbius transformation of $\hat{\mathbb{R}}^2$ and every Möbius transformation acting on $\hat{\mathbb{R}}^2$ has such a matrix representation. Therefore, the group of all Möbius transformation acting on $\hat{\mathbb{R}}^2$, denoted by $\mathrm{Möb}(\hat{\mathbb{R}}^2)$, is isomorphic to $\mathrm{SL}(2, \mathbb{C})/\{\pm I\}$, denoted by $\mathrm{PSL}(2, \mathbb{C})$. Furthermore, the subgroup $\mathrm{PSL}(2, \mathbb{R})$ of $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to the group $\mathrm{Möb}(\hat{\mathbb{R}})$ of all Möbius transformation acting on $\hat{\mathbb{R}}$. In general, any Möbius transformation acting on $\hat{\mathbb{R}}^n$ can be represented as a 2×2 matrix. Section 3 will cover this representations in detail.

The group of isometries.

We embed $\hat{\mathbb{R}}^n$ into $\hat{\mathbb{R}}^{n+1}$ by $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0)$. A reflection in a Euclidean sphere $S \subseteq \mathbb{R}^n$ is extended to a reflection in a Euclidean sphere $\hat{S} \subseteq \mathbb{R}^{n+1}$, where $\hat{S}$ is a Euclidean sphere in $\mathbb{R}^{n+1}$ which is orthogonal to $\mathbb{R}^n$ and has the same center and radius as S. A reflection in a plane

$P \subseteq \hat{\mathbb{R}}^n$ is extended to a reflection in a plane $\hat{P} \subseteq \hat{\mathbb{R}}^{n+1}$, where $\hat{P}$ is the plane in $\hat{\mathbb{R}}^{n+1}$ which is orthogonal to $\hat{\mathbb{R}}^n$ and passes through P. In this way, we can extend each Möbius transformation acting on $\hat{\mathbb{R}}^n$ to $\hat{\mathbb{R}}^{n+1}$ by extending each reflection in spheres or hyperplanes in $\hat{\mathbb{R}}^n$ to $\hat{\mathbb{R}}^{n+1}$. Hence, the group $\text{Möb}(\hat{\mathbb{R}}^n)$ of all Möbius transformations acting on $\hat{\mathbb{R}}^n$ is a subgroup of the group $\text{Möb}(\hat{\mathbb{R}}^{n+1})$ of all Möbius transformations acting on $\hat{\mathbb{R}}^{n+1}$. The isometries of $\mathbb{H}^{n+1}$ can also be identified with the Möbius transformations acting on $\hat{\mathbb{R}}^n$ which preserve the upper half space U.

The group $\text{Möb}(\hat{\mathbb{R}}^n)$ of all Möbius transformations acting on $\hat{\mathbb{R}}^n$ has a natural topology. First, we can identify $\hat{\mathbb{R}}^n$ with the unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ of $\mathbb{R}^{n+1}$ by the stereographic projection. Thus, we pull back the Euclidean metric from the unit sphere $\mathbb{S}^n$ to the chordal metric ρ on $\hat{\mathbb{R}}^n$. For any $f, g \in \text{Möb}(\hat{\mathbb{R}}^n)$, we define the sup metric d as follows

$$d(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in \hat{\mathbb{R}}^n\}. \tag{1.1}$$

Thus, a sequence $\{f_n\}$ of Möbius transformations acting on $\hat{\mathbb{R}}^n$ converges to a Möbius transformation f acting on $\hat{\mathbb{R}}^n$ in this metric if and only if $\{f_n\}$ converges to f uniformly on $\hat{\mathbb{R}}^n$. This topology is equivalent to the topology on their matrix representations. A subgroup Γ of $\text{Möb}(\hat{\mathbb{R}}^n)$ is said to be *non-discrete* if there is an infinite sequence of distinct elements of Γ converging to the identity. Otherwise, it is *discrete*.

Definition 1.1.1. An n-dimensional *Möbius group* is a finitely generated discrete subgroup of $\text{Möb}(\hat{\mathbb{R}}^n)$.

In particular, a 2-dimensional Möbius group is called *Kleinian* and an 1-dimensional Möbius group is called *Fuchsian*. We will also call any n-dimensional Möbius group Kleinian (or Fuchsian) if it is conjugate to a

Kleinian group (or a Fuchsian group) by a Möbius transformation. If an n-dimensional Möbius group has a screw parabolic element, it is neither Kleinian nor Fuchsian.

Let Γ be a group acting on a topological space X. A group Γ is said to act *discontinuously* at a point $x \in X$ if there exists a neighborhood U of x so that $gU \cap U \neq \emptyset$ for at most finitely many $g \in \Gamma$. A group Γ is said to act *properly discontinuously* on X if for any compact subset K of X, $gK \cap K \neq \emptyset$ for at most finitely many $g \in \Gamma$. An n-dimensional Möbius group Γ is discrete if and only if it acts properly discontinuously on $\mathbb{H}^{n+1}$.

Let Γ be an n-dimensional Möbius group. Then the sphere at infinity $\hat{\mathbb{R}}^n$ is divided into two disjoint sets by the action of Γ. We define the *set of discontinuity*, denoted by Ω_Γ, to be the set of all points at which Γ acts discontinuously. The complement of the set of discontinuity is called *the limit set*, denoted by Λ_Γ. The limit set is also the set of accumulation points of some Γ-orbit in $\mathbb{H}^{n+1}$. That is

$$\Lambda_\Gamma = \{x \in \hat{\mathbb{R}}^n \ | f_i(y) \to x \text{ for a sequence } f_i \in \Gamma \text{ and some } y \in \mathbb{H}^{n+1}\}.$$

Both the set of discontinuity Ω_Γ and the limit set Λ_Γ are Γ-invariant and Γ acts properly discontinuously on the set of discontinuity. If the limit set has more than two points, it consists of uncountably many points. The group Γ is said to be *elementary* if its limit set has at most two points. Otherwise, it is said to be *non-elementary.* If a Möbius group Γ is non-elementary, the limit set is the smallest Γ-invariant closed subset of the sphere at infinity. A Möbius group is elementary if and only if it is *virtually abelian* (i.e., it contains an abelian subgroup of finite index).

Definition 1.1.2. A Möbius group G of $M\ddot{o}b(\hat{\mathbb{R}}^n)$ is said to be *parabolic* if $fix(G) = \bigcap_{g \in G} fix(g)$ consists of a single point $p \in \hat{\mathbb{R}}^n$, and if G preserves setwise some horosphere at p in $\mathbb{H}^{n+1}$.

Definition 1.1.3. A Möbius group G of $M\ddot{o}b(\hat{\mathbb{R}}^n)$ is said to be *loxodromic* if G preserves setwise a unique bi-infinite geodesic and contains a loxodromic element.

Lemma 1.1.4 ([7] 3.3.2)**.** *If a Möbius group is virtually abelian, then it is finite, parabolic or loxodromic.*

Hyperbolic manifolds.

A hyperbolic $(n+1)$-manifold is an $(n+1)$-dimensional manifold with a hyperbolic structure which is a complete metric of constant sectional curvature -1. It can be obtained as the quotient $\mathbb{H}^{n+1}/\Gamma$, where Γ is a torsion-free discrete subgroup of $\mathrm{Isom}(\mathbb{H}^{n+1})$.

Theorem 1.1.5 (Margulis' lemma)**.** *Given any positive integer n, there is some positive number ϵ_n such that for any point x in $\mathbb{H}^{n+1}$ and any discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^{n+1})$, the group $\Gamma_{\epsilon_n}(x)$ generated by the set*

$$F_{\epsilon_n}(x) = \{g \in \Gamma : d(x, g(x)) \leq \epsilon_n\} \tag{1.2}$$

is virtually nilpotent (i.e., it contains a nilpotent subgroup of finite index).

We call ϵ_n the *Margulis constant*. The Margulis constant ϵ_n depends only on the dimension. Any discrete nilpotent subgroups of $\mathrm{Isom}(\mathbb{H}^{n+1})$ are virtually abelian ([7] 2.1.10).

Let Γ be a torsion-free discrete subgroup of $\mathrm{Isom}(\mathbb{H}^{n+1})$ and M be a hyperbolic manifold obtained by $\mathbb{H}^{n+1}/\Gamma$. For $\epsilon > 0$, we define a set

$$T_\epsilon(\Gamma) = \{x \in \mathbb{H}^{n+1} : \Gamma_\epsilon(x) \text{ is } \text{ infinite } \}, \tag{1.3}$$

where a subgroup $\Gamma_\epsilon(x)$ is generated by those elements that move the point x a distance at most ϵ (see (1.2)). Then $T_\epsilon(\Gamma)$ is a closed Γ-invariant subset of $\mathbb{H}^{n+1}$ and hence it descends to the manifold M. The *thin* part of a manifold $M = \mathbb{H}^{n+1}/\Gamma$ is defined by

$$thin_\epsilon(M) = T_\epsilon(\Gamma)/\Gamma \subseteq M. \tag{1.4}$$

By Margulis' lemma the geometry of the thin part is relatively clear to understand. Topologically the thin part is a disjoint union of its connected components. Each component has the form $T_\epsilon(G)/G$, where G is either maximal parabolic or maximal loxodromic. The thin part can be defined equivalently as the set of points where the injectivity radius of M is less than or equal to $\frac{\epsilon}{2}$. We call the complement of the thin part of M the *thick part of* M, denoted by $thick_\epsilon(M)$.

Proposition 1.1.6 ([7] 2.2.6)**.** *Let Γ be a discrete subgroup of Euclidean isometries $Isom(\mathbb{R}^n)$. Suppose that Γ acts properly discontinuously on $\mathbb{R}^n$. Then a Γ-invariant subspace $\mu \subseteq \mathbb{R}^n$ is minimal if and only if μ/Γ is compact. Moreover, any two minimal subspaces are parallel.*

Geometrically finite groups.

Suppose that G is a parabolic Möbius group with $fix(G) = \{p\}$ of $\hat{\mathbb{R}}^n$. We may assume $p = \infty$. Then the set of discontinuity Ω_G of G is $\mathbb{R}^n$. By Proposition 1.1.6, there is a G-invariant Euclidean subspace $\mu \subseteq \mathbb{R}^n$ such that μ/G is compact. Let μ_+ be the unique hyperbolic plane in $\mathbb{H}^{n+1}$ whose boundary at infinity is $\mu \cup \{p\}$. Then, μ_+ is also G-invariant and for any horosphere $\partial\Sigma$ based at p, $(\mu_+ \cap \partial\Sigma)/G$ is compact. For any $r > 0$, we define *a standard parabolic region*

$$C(\mu, r) = \{\ x \in \overline{\mathbb{H}}^{n+1}\ :\ d_E(x, \mu) \geq r\ \}. \tag{1.5}$$

The set $C(\mu, r)$ is a G-invariant hyperbolically convex set and $\bigcap_{r\in[0,\infty)} C(\mu, r) = \emptyset$. Clearly, $C(\mu, r)/G \subseteq \big(\mathbb{H}^{n+1} \cup \Omega_G\big)/G = \big(\overline{\mathbb{H}}^{n+1} \backslash \{p\}\big)/G$ has precisely one topological end. Moreover, the collection $\{C(\mu, r)/G : r \geq 0\}$ forms a base of neighborhood for that end.

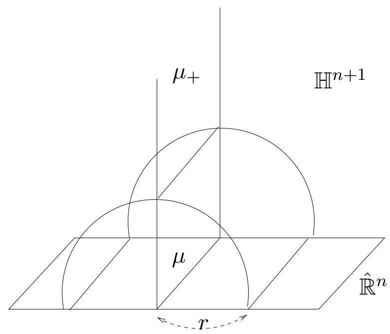

Figure 1.1: A standard parabolic region $C(\mu, r)$.

Now, suppose that Γ is an n-dimensional Möbius group and $p \in \hat{\mathbb{R}}^n$ is a parabolic fixed point of Γ. Since a loxodromic element and a parabolic element cannot share a fixed point in a discrete group, $Stab_\Gamma p$ is a parabolic group and is maximal (i.e., any parabolic subgroup of Γ with a fixed point p is contained in $Stab_\Gamma p$).

Definition 1.1.7. A parabolic fixed point $p \in \hat{\mathbb{R}}^n$ of an n-dimensional Möbius group Γ is said to be *bounded* if $\big(\Lambda \setminus \{p\}\big)/Stab_\Gamma p$ is compact.

In other words, a parabolic fixed point p of Γ is bounded if and only if $d_E(x, \mu)$ is bounded for any $x \in \Lambda\backslash\{p\}$, where μ is some minimal $Stab_\Gamma p$-invariant subspace of $\hat{\mathbb{R}}^n \setminus \{p\}$.

For any $f \in \Gamma$, $Stab_\Gamma(f(p)) = f(Stab_\Gamma)f^{-1}$. Hence, there is a one-to-one correspondence between Γ-orbit of parabolic fixed points and conjugacy classes of maximal parabolic subgroups of Γ. For the subgroup $G = Stab_\Gamma p$, if there is a precisely G-invariant standard parabolic region $C \subseteq \mathbb{H}^{n+1} \cup \Omega_\Gamma$, then the standard parabolic region C descends to a set $E = \big(\cup \Gamma C\big)/\Gamma \subseteq \big(\mathbb{H}^{n+1} \cup \Omega_\Gamma\big)/\Gamma$, called *a standard cusp region.* In this case, the parabolic fixed point p is said to be *associated* to the standard cusp region E. A standard cusp region E is isometric to C/G.

Lemma 1.1.8 ([7])**.** *Let Γ be an n-dimensional Möbius group and $p \in \hat{\mathbb{R}}^n$ be a parabolic fixed point of Γ. Then p is bounded if and only if p is associated to a standard cusp region of $\big(\mathbb{H}^{n+1} \cup \Omega_\Gamma\big)/\Gamma$.*

A point $y \in \hat{\mathbb{R}}^n$ is said to be *a conical limit point* of Γ if for some (hence all) geodesic ray L tending to y and some point (hence all) $x \in \mathbb{H}^{n+1}$, there is an infinite sequence $\{\gamma_i\}$ of element of Γ such that $\gamma_i(x)$ converges to y and $d(\gamma_i(x), L)$ is bounded by a constant.

Theorem 1.1.9 ([25])**.** *A conical limit point of an n-dimensional Möbius group is not a parabolic fixed point.*

The *convex hull* of the limit set Λ_Γ, denoted by $Hull\Lambda_\Gamma$, is the minimal convex subset of $\mathbb{H}^{n+1}$ that contains all the geodesics connecting any two points in the limit set Λ_Γ. By definition, the convex hull is Γ-invariant. The *convex core* of a hyperbolic manifold $M = \mathbb{H}^{n+1}/\Gamma$, denoted by $core(M)$, is the covering map projection of the convex hull of the limit set of Γ. More formally, we can write

$$core(M) = Hull\Lambda/\Gamma \subseteq M. \tag{1.6}$$

Theorem 1.1.10 ([7])**.** *Let Γ be a Möbius group acting on $\mathbb{H}^{n+1}$. Then the followings are equivalent:*

1. *We can write $M_C(\Gamma) = (\mathbb{H}^{n+1} \cup \Omega)/\Gamma$ as the union of a compact set and a finite number of disjoint standard cusp regions.*

2. *The limit set Λ_Γ consists entirely of conical limit points or bounded parabolic fixed points.*

3. *The thick part of the convex core $thick_\epsilon(M) \cap core(M)$ is compact for some $\epsilon \in (o, \epsilon_n)$, where ϵ_n is the Margulis constant.*

4. *There is a bound of the orders of finite subgroups of Γ and a η-neighborhood of $core(M)$, $N_\eta(core(M))$ has finite volume for some $\eta > 0$.*

We have two remarks related to item 4.

Remark 1.1.11. 1. If the dimension of a hyperbolic manifold is less than or equal to 3 or the hyperbolic manifold M itself has finite volume, then the second condition that $N_\eta(core(M))$ has finite volume implies that the first condition that the existence of the bound on the orders of finite subgroups.

2. If Γ is finitely generated, then there is always a bound on the orders of finite subgroups by the Selberg lemma.

Definition 1.1.12. A Möbius group Γ is said to be *geometrically finite* if it satisfies any of the above four conditions. Otherwise, it is said to be *geometrically infinite.*

Also a hyperbolic manifold $M = \mathbb{H}^{n+1}/\Gamma$ is said to be *geometrically finite* if its Möbius group Γ is geometrically finite.

Proposition 1.1.13 ([7])**.** *Any finite volume hyperbolic manifold is geometrically finite.*

If a Möbius group has a finite-sided fundamental domain, it is geometrically finite. In dimensions 2 and 3, every convex fundamental domain for a geometrically finite Möbius group is finite-sided. However, in dimensions 4 or higher, a geometrically finite Möbius group can have an infinite-sided fundamental domain. For example [7], let G be an infinite cyclic group generated by an irrational screw parabolic isometry in $\mathbb{H}^4$. Then it has an infinite-sided Dirichlet domain in $\mathbb{H}^4$. Clearly, the group G is geometrically finite since it is elementary. The good news is that this is the only case.

Proposition 1.1.14 ([7])**.** *Let Γ be a geometrically finite Möbius group of* $\mathrm{Isom}(\mathbb{H}^{n+1})$*. Suppose that Γ has no irrational screw parabolic elements. Then every convex fundamental domain for Γ is finite-sided.*

Main references for general hyperbolic geometry are books by Maskit [17], Beardon [5] and Ratcliffe [20]. For higher dimensional Möbius groups, we can also refer to a survey note by Kapovich [13] and a book by Benedetti-Petronio [6].

1.2 Quasiconformal mappings and quasiisometries

We use [22] to recall the basics of quasiconformal mappings on $\hat{\mathbb{R}}^n$. Let D and D' be domains in $\hat{\mathbb{R}}^n$ and $\phi : D \to D'$ be a homeomorphism.

Suppose that $x \in D$, $x \neq \infty$ and $\phi(x) \neq \infty$. The *linear dilatation* of ϕ at x is

$$K(x, \phi) = \limsup_{r \to 0} \frac{\max_{|y-x|=r} |\phi(y) - \phi(x)|}{\min_{|y-x|=r} |\phi(y) - \phi(x)|}. \tag{1.7}$$

If $x = \infty$ and $\phi(x) \neq \infty$, we define $K(x, \phi) = K(0, \phi \circ u)$ where u is the inversion $u(x) = \frac{x}{|x|^2}$. If $\phi(x) = \infty$, we define $K(x, \phi) = K(x, u \circ \phi)$. Obviously, $1 \leq K(x, \phi) \leq \infty$.

Definition 1.2.1. A homeomorphism $\phi : D \to D'$ is said to be *quasiconformal* if $K(x, \phi)$ is bounded for all $x \in D$. For a real number $K \geq 1$, ϕ is called

K-quasiconformal if $K(x, \phi) \leq K$ for almost all $x \in D$. The dilatation $K(\phi)$ of ϕ is the infimum of K for which ϕ is K-quasiconformal .

For a quasiconformal mapping ϕ, the infimum dilatation $K(\phi)$ of ϕ is greater than or equal to 1. A 1-quasiconformal mapping is conformal.

Theorem 1.2.2 ([22] 15.4)**.** *Let $\phi : D \to D'$ be a diffeomorphism. Then ϕ is K-quasiconformal if and only if for every $x \in D$*

$$\frac{|\phi'(x)|^n}{K} \leq |J(\phi, x)| \leq K l(\phi'(x))^n, \tag{1.8}$$

$$|\phi'(x)| = \limsup_{h \to 0} \frac{|\phi(x+h) - \phi(x)|}{|h|},$$

$$l(\phi'(x)) = \liminf_{h \to 0} \frac{|\phi(x+h) - \phi(x)|}{|h|}.$$

Corollary 1.2.3 ([22] 15.6)**.** *If a quasiconformal mapping ϕ is differentiable at a point x, then either $\phi'(x) = 0$ or the determinant of the Jacobian matrix $J(\phi, x) \neq 0$.*

Since a quasiconformal mapping is almost everywhere differentiable, the inequality (1.8) holds almost everywhere.

Proposition 1.2.4.

1. *If ϕ_1 and ϕ_2 are quasiconformal , then $\phi = \phi_1 \circ \phi_2$ is quasiconformal and $K(\phi) \leq K(\phi_1) K(\phi_2)$.*
2. *$K(\phi) = K(\phi^{-1})$ for any quasiconformal homeomorphism ϕ.*
3. *If ϕ is quasiconformal and f and g are Möbius transformations, then $K(\phi) = K(f \circ \phi \circ g)$.*

We can also define a quasiconformal mapping using curve families. First, we define the modulus of the curve families of a domain. Then the modulus is a conformal invariant. A quasiconformal mapping is defined as a mapping such that the modulus of a curve family can be distorted only by a bounded amount under the mapping. Of course, the two definitions of quasiconformal mappings are equivalent.

Here, we have some examples of quasiconformal mappings. If $\phi : D \to D'$ is a diffeomorphism, the restriction of ϕ to any domain whose closure is a compact subset of D is a quasiconformal mapping. All bijective linear mappings are quasiconformal mappings. Every Möbius transformation is conformal and hence quasiconformal. In fact, by the Liouville's theorem, every conformal mapping of a domain in a Euclidean n-space $\mathbb{R}^n$ with $n \geq 3$ is a restriction of a Möbius transformation.

Example 1.2.5. (The radial stretch mappings) For $K > 0$, define

$$\phi : \mathbb{H}^n \to \mathbb{H}^n \text{ by } \phi(z) = |z|^{K-1} z \text{ for } z \in \mathbb{H}^n.$$

Then ϕ is K-quasiconformal for $K > 1$ and $\frac{1}{K}$-quasiconformal for $0 < K < 1$.

Theorem 1.2.6 ([11])**.** *Suppose that $\mathcal{F}$ is an infinite family of K-quasiconformal self mappings of $\hat{\mathbb{R}}^n$. Then there exists a sequence $\{\phi_j\}$ in $\mathcal{F}$ such that one of the following is true.*

1. *There exits a K-quasiconformal homeomorphism ϕ of $\hat{\mathbb{R}}^n$ such that*

$$\lim_{j \to \infty} \phi_j = \phi \text{ and } \lim_{j \to -\infty} \phi_j = \phi^{-1} \tag{1.9}$$

   uniformly in $\hat{\mathbb{R}}^n$.

2. *There exist points x_0, y_0 in $\hat{\mathbb{R}}^n$ such that*

$$\phi_j \to y_0 \text{ as } j \to \infty \quad \text{and} \quad \phi_j \to x_0 \text{ as } j \to -\infty \tag{1.10}$$

   uniformly on every compact subset of $\hat{\mathbb{R}}^n \setminus \{x_0\}$ and $\hat{\mathbb{R}}^n \setminus \{y_0\}$ respectively. The possibility that $x_0 = y_0$ may occur.

Definition 1.2.7. Let (X, d_X) and (Y, d_Y) be metric spaces. A homeomorphism $\phi : X \to Y$ is said to be a (λ, δ)-*quasiisometry* if there are some constants λ and δ such that

$$\frac{1}{\lambda} d_X(z, w) - \delta \leq d_Y(\phi(z), \phi(w)) \leq \lambda d_X(z, w) + \delta \tag{1.11}$$

for all $z, w \in X$.

When $\delta = 0$, a homeomorphism ϕ is called a *biLipschitz* map. In hyperbolic spaces, a K-quasiconformal mapping is a $(K, Klog4)$-quasiisometry ([24]). However, there is also a quasiisometry which is not a quasiconformal mapping in hyperbolic spaces.

Example 1.2.8. (Radially stretching net map) First, for a real number $K > 1$ we will define a map on the interval $I = [-1, 1]$ of $\mathbb{R}$, which fixes the end points 1 and -1 and is radially stretching by $t \mapsto |t|^{K-1}t$, for $t \in I$. This map fixes the origin and its linear dilatation is K. Now we take infinite copies of the interval maps and paste them together to construct a map from $\mathbb{R}$ to $\mathbb{R}$. Clearly, it is a quasiisometry since it fixes every integer point. However, we will make the linear dilations of each interval blow up so that it cannot be quasiconformal.

Now, we will construct a quasiisometry on 2-dimensional hyperbolic space $\mathbb{H}^2$. First, we tile $\mathbb{H}^2$ with a regular pentagon. Using the above idea, we can define a map which fixes the vertices and centers of pentagons and is a radially stretching map on each pentagon from the center. Note that for a pentagon close to the boundary at infinity, the linear dilatation is arbitrary large. Hence, it is not a quasiconformal mapping. (We are grateful to Noel Brady for telling us about this.)

Proposition 1.2.9 ([20]). *Every quasiisometry $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ extends to a quasiconformal mapping of $\hat{\mathbb{R}}^n$.*

1.3 Möbius transformations and Clifford algebra

In 1984 and 1985, Ahlfors published three papers ([2], [3], [4]) showing that 2×2 matrices whose entries are Clifford numbers can be used to represent Möbius transformations. The idea was introduced by K.T. Vahlen originally in 1901 ([23]). It is a natural generalization of $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ via identifying the real numbers $\mathbb{R}$ with the Clifford algebra C_0 and the complex numbers $\mathbb{C}$ with the Clifford algebra C_1. One advantage of these representations is that it gives us an automatic extension from n-dimensional representations to $(n+1)$-dimensional representations of Möbius transformations.

The Clifford algebra C_{n-1} is the associative algebra over the real numbers generated by n elements $e_1, e_2, \ldots, e_{n-1}$ subject to the relations ${e_i}^2 = -1$ $(i = 1, \cdots, n-1)$ and $e_i e_j = -e_j e_i$ $(i \neq j)$ and no others. An element of C_{n-1} is called a *Clifford number*. A Clifford number a is of the form $\sum a_I I$ where the sum is over all products $I = e_{v_1} e_{v_2} \ldots e_{v_p}$ with $1 \leq v_1 < v_2 < \cdots < v_p \leq n-1$ and $a_I \in \mathbb{R}$. The null product of generators is the real number 1. The Clifford algebra C_{n-1} is a 2^{n-1}-dimensional vector space over the real numbers $\mathbb{R}$. Here are the three involutions in the Clifford algebra C_{n-1}:

1. The main involution $a \mapsto a'$ is an automorphism obtained by replacing each e_i with $-e_i$. Thus, $(ab)' = a'b'$ and $(a+b)' = a' + b'$.

2. Reversion $a \mapsto a^*$ is an anti-automorphism obtained by replacing each $e_{v_1} e_{v_2} \ldots e_{v_p}$ with $e_{v_p} e_{v_{p-1}} \ldots e_{v_1}$. Therefore, $(ab)^* = b^* a^*$ and $(a +$

$b)^* = a^* + b^*$.

3. Conjugation $a \mapsto \overline{a}$ is an anti-automorphism obtained by a composition. Therefore, $\overline{a} = (a')^* = (a^*)'$.

The Euclidean *norm* $|a|$ of $a = \sum a_I I \in C_{n-1}$ is given by $|a|^2 = \sum a_I^2$. For any Clifford number a, we denote by $(a)_{\mathbb{R}}$ the real part of a. Then $a \circ b = \sum a_I b_I = (a\overline{b})_{\mathbb{R}} = (\overline{a}b)_{\mathbb{R}}$ for any two Clifford numbers a and b.

A *vector* is a Clifford number of the form $x = x_0 + x_1 e_1 + \cdots x_{n-1}e_{n-1} \in C_{n-1}$ where the x_i's are real numbers. The set of all vectors form an n-dimensional subspace which we identify with $\mathbb{R}^n$. For any vector x, $x^* = x$ and $\overline{x} = x'$. Every non-zero vector x is invertible with $x^{-1} = \frac{\overline{x}}{|x|^2}$. Since the product of invertible elements is invertible, every product of non-zero vectors is invertible. A *Clifford group* Γ_{n-1} is a multiplicative group generated by all non-zero vectors of C_{n-1}. We note that $\Gamma_{n-1} = C_{n-1} - \{0\}$ is true for only $n = 1, 2, 3$.

Lemma 1.3.1 ([2], [25]). *1. If $a \in \Gamma_{n-1}$, then $|a|^2 = a\overline{a} = \overline{a}a$.*

2. If $a \in \Gamma_{n-1}$, then $|ab| = |a||b|$ for any Clifford number $b \in C_{n-1}$.

In general, the above lemma is not true in C_{n-1}. For example ([25]), let $a = 1 + e_1e_2e_3 \in C_4$. Then $\overline{a} = a$ and $a\overline{a} = \overline{a}a = 2a$ but $|a| = \sqrt{2}$. In addition, $a(e_4a) = 0$ but $(e_4a)a \neq 0$. Hence, $a \notin \Gamma_4$.

Lemma 1.3.2 ([25]). *1. If $a \in C_{n-1}$ and $ae_k a'^{-1} = e_k$, $k = 1, \ldots, n-1$, then $a \in \mathbb{R}$.*

2. For any $a \in \Gamma_{n-1}$, the map $\rho_a : \mathbb{R}^n \to \mathbb{R}^n$ defined by $x \mapsto axa'^{-1}$ is a Euclidean isometry.

3. *For any non-zero vector $v \in \mathbb{R}^n$, let R_v be the reflection in the hyperplane passing through the origin* 0 *and orthogonal to v. Then $\rho_a = R_a R_1$ for a non-zero vector a in $\mathbb{R}^n$.*

4. *The map $\phi : \Gamma_{n-1} \to \mathrm{O}(n)$ given by $\phi(a) \mapsto \rho_a$ is onto* $\mathrm{SO}(n)$ *with the kernel* $\mathbb{R} - \{0\}$.

Proof. Item 1. comes from computation.

2. For any vectors x and y in $\mathbb{R}^n$, $x \circ y = \frac{1}{2}(x\overline{y} + y\overline{x})$.

$$x\overline{y}y + y\overline{x}y = 2(x \circ y)y \tag{1.12}$$

Thus, $y\overline{x}y = 2(x \circ y)y - x|y|^2$ is a vector in $\mathbb{R}^n$. For any $a = y^1 \cdots y^k \in \Gamma_{n-1}$ with each $y^j \in \mathbb{R}^n$,

$$\begin{aligned}\rho_a(x) = axa'^{-1} &= \frac{1}{|a|^2} axa^* \\ =&\frac{1}{|a|^2}(y^1 \cdots (y^k x y^k) \cdots y^1) \in \mathbb{R}^n.\end{aligned} \tag{1.13}$$

$|axa'^{-1}| = |a||x||a'^{-1}| = |a|$ and $(\rho_a)^{-1} = \rho_a^{-1}$. Therefore, ρ_a is an isometry.

3. For any vector $x \in \mathbb{R}^n$,

$$\begin{aligned}R_a : x \mapsto& x - 2(x \circ a)\frac{a}{|a|^2} \\ &= x - (x\overline{a} + a\overline{a})\frac{a}{|a|^2} \text{ by (1.12)} \\ &= -a\overline{x}a'^{-1}.\end{aligned}$$

4. $\rho_a \rho_b = \rho_{ab}$ for any $a, b \in \Gamma_{n-1}$. The homomorphism $\phi : \Gamma_{n-1} \to SO(n)$ is surjective and the kernel is $\{tI : t \in \mathbb{R} \setminus \{0\}\}$. □

The following lemma is useful for calculations.

Lemma 1.3.3 ([2]). *1. For* $I = e_{v_1}e_{v_2}\dots e_{v_p} \in C_{n-1}$

$$e_v I = \begin{cases} -I'e_v, & \text{if } e_v \in I \\ I'e_v, & \text{if } e_v \notin I \end{cases} \tag{1.14}$$

2. For $a, b \in \Gamma_{n-1}$, $ab^{-1} \in \mathbb{R}^n$ *if and only if* $a^*b \in \mathbb{R}^n$.

Proof. 1. comes from a computation.

2. Suppose that ab^{-1} is a vector, say $x \in \mathbb{R}^n$. Then $b^*a = b^*xb$ is a vector by (1.13) and so is $(b^*a) = (b^*a)^* = a^*b$. Now, suppose that a^*b is a vector, say $y \in \mathbb{R}^n$. Then $b^{*-1}yb^{-1} = b^{*-1}a^* = ab^{-1}$ is a vector again. □

Definition 1.3.4. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be a *Clifford matrix* if the following conditions are satisfied:

1. $a, b, c, d \in \Gamma_{n-1} \cup \{0\}$.
2. $ad^* - bc^* = 1$.
3. $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^n$.

A Clifford matrix A acts on $\hat{\mathbb{R}}^n$ by $Ax = (ax+b)(cx+d)^{-1}$ for any vector $x = x_0 + x_1e_1 + \dots + x_{n-1}e_{n-1} \in \mathbb{R}^n$, and $\infty \mapsto \infty$ if $c = 0$ and $\infty \mapsto ac^{-1}$, $-c^{-1}d \mapsto \infty$ if $c \neq 0$. By Lemma 1.3.3. and Condition 3. for a Clifford matrix, ac^{-1} and $-c^{-1}d$ are vectors.

Suppose that $ax + b = cx + d = 0$. Then $x = -c^{-1}d = -d^*c^{*-1}$, which implies $ax + b = a(-d^*c^{*-1} + b) = 0$. We get $ad^* - bc^* = 0$, which is a contradiction. Therefore, $ax + b$ and $cx + d$ cannot be simultaneously zero. If $c \neq 0$, $cx + d = c(x + c^{-1}d)$ is a well-defined element in $\Gamma_{n-1} \cup \{0\}$ since

$c^{-1}d$ is a vector. From $ad^* - bc^* = 1$, $b = ac^{-1}d - c^{*-1}$. Hence

$$\begin{aligned} Ax &= (ax+b)(cx+d)^{-1} \\ &= (ac^{-1}cx + ac^{-1}d - c^{*-1})(cx+d)^{-1} \\ &= ac^{-1} - c^{*-1}(x + c^{-1}d)^{-1}c^{-1} \end{aligned} \tag{1.15}$$

is a vector. If $c = 0$, $ad^* = 1$ and

$$Ax = (ax+b)d^{-1} = axa^* + bd^{-1} \tag{1.16}$$

is a vector since axa^* and bd^{-1} are vectors. We note that $A0 = bd^{-1}$ and $A^{-1}0 = -a^{-1}b$ are vectors. Therefore, the action is well-defined.

A Clifford matrix A has its multiplicative inverse Cliiford matrix $A^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$. Hence the induced mapping is bijective on $\hat{\mathbb{R}}^n$.

Lemma 1.3.5. *All Clifford matrices acting on $\hat{\mathbb{R}}^n$ form a group, denoted by* $\mathrm{SL}(\Gamma_{n-1})$.

A Clifford matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_{n-1})$ can be rewritten as follows: If $c \neq 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{*-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}. \tag{1.17}$$

The first matrix $\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}$ is the translation $x \mapsto x + ac^{-1}$. Similarly, the last matrix is the translation $x \mapsto x + c^{-1}d$. For the second matrix, if $c \in \mathbb{R}$, $\begin{pmatrix} c^{*-1} & 0 \\ 0 & c \end{pmatrix}$ is a dilation $x \mapsto \frac{1}{c^2}x$. In case that $|c| = 1$ and $c \notin \mathbb{R}$, $\begin{pmatrix} c^{*-1} & 0 \\ 0 & c \end{pmatrix}$ is an orthogonal mapping $x \mapsto \rho_{c'}(x)$. Otherwise, $\begin{pmatrix} c^{*-1} & 0 \\ 0 & c \end{pmatrix}$

can be expressed as the product $\begin{pmatrix} |c|^{-1} & 0 \\ 0 & |c| \end{pmatrix} \begin{pmatrix} \frac{c^{*-1}}{|c|} & 0 \\ 0 & \frac{c}{|c|} \end{pmatrix}$. The third matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an orientation-preserving inversion $x \mapsto -x^{-1}$.

Similarly, if $c = 0$, we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}. \tag{1.18}$$

Hence, the group $\mathrm{SL}(\Gamma_{n-1})$ is generated by the following three matrices:

$$\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.19}$$

where $a \in \Gamma_{n-1}$, $b \in \mathbb{R}^n$. In fact, these Clifford matrix induced mappings are orientation-preserving Möbius transformations. Moreover, any orientation-preserving Möbius transformation can be obtained by a composition of these Clifford matrix induced mappings.

Therefore, every Clifford matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_{n-1})$ induces a Möbius transformation of $\hat{\mathbb{R}}^n$ by the formula $A \mapsto f_A$ where

$$f_A(x) = (ax+b)(cx+d)^{-1} \quad \text{for any } x \in \hat{\mathbb{R}}^n.$$

Replacing x with $x + x_n e_n$, we can automatically extend f_A to a Möbius transformation $x + x_n e_n \mapsto (a(x + x_n e_n) + b)(c(x + x_n e_n) + d)^{-1}$ in $\hat{\mathbb{R}}^{n+1}$. The coefficient of the last generator e_n of $f_A(x + x_n e_n)$ is $\frac{x_n}{|cx+d|^2}$. Hence, the extension keeps the upper half space $\mathbb{H}^{n+1}$ invariant.

Theorem 1.3.6 ([23])**.** *The Clifford matrices form a group* $\mathrm{SL}(\Gamma_{n-1})$ *whose quotient modulo* $\pm I$ *is isomorphic to* $M\ddot{o}b(\hat{\mathbb{R}}^n) \cong \mathrm{Isom}(\mathbb{H}^{n+1})$.

Theorem 1.3.7 ([2], [25])**.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_{n-1})$.

1. *The derivative* $A'(x)$ *is the matrix* $|cx+d|^{-2}\rho_{(cx+d)'}$ *with the operator norm* $|A'(x)| = |cx+d|^{-2}$.

2. $|Ax - Ay| = |x-y||A'(x)|^{\frac{1}{2}}|A'(y)|^{\frac{1}{2}}$.

3. $\frac{|A'(z)|}{(Az)_{e_n}} = \frac{1}{x_n}$ *for* $z = x + x_n e_n \in \mathbb{R}^{n+1}$.

Proof. All items come from the following computation. For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} Ax - Ay &= \left[(ax+b) - \{(ay+b)(cy+d)^{-1}\}^*(cx+d)\right](cx+d)^{-1} \\ &= (cy+d)^{*-1}[(cy+d)^*(ax+b) - (ay+b)^*(cx+d)](cx+d)^{-1} \\ &= (cy+d)^{*-1}[y^*(c^*a - a^*c)x + y^*(c^*b - a^*d) \\ &\qquad\qquad + (d^*a - b^*c)x + d^*b - b^*d](cx+d)^{-1} \\ &= (cy+d)^{*-1}(x-y)(cx+d)^{-1}. \end{aligned}$$

□

Since Clifford numbers do not commute in general, the trace of a Clifford matrix is not kept invariant by conjugation. However, we define the *Clifford trace* $(trA)_{\mathbb{R}}$ to be the real part of $a+d^*$ for a Clifford matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_{n-1})$. Then, $(tr)_{\mathbb{R}}$ is a conjugacy invariant.

Lemma 1.3.8 ([25])**.** *The Clifford trace* $(tr)_{\mathbb{R}}$ *is a conjugacy invariant.*

Proof. The group of Clifford matrices $\mathrm{SL}(\Gamma_{n-1})$ is generated by $A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{*-1} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\alpha, \beta \in \Gamma_{n-1} \cup \{0\}$. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any Clifford matrix in $\mathrm{SL}(\Gamma_{n-1})$. Note that

$$
\begin{aligned}
ATA^{-1} &= \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\beta^{*} \\ 0 & \alpha^{*} \end{pmatrix} \\
&= \begin{pmatrix} \alpha a\alpha^{-1} + \beta c\alpha^{-1} & * \\ * & -\alpha^{*-1}c\beta^{*} + \alpha^{*-1}d\alpha^{*} \end{pmatrix}
\end{aligned}
$$

and thus

$$
\begin{aligned}
(trATA^{-1})_{\mathbb{R}} &= [\alpha a\alpha^{-1} + \beta c\alpha^{-1} + (-\alpha^{*-1}c\beta^{*} + \alpha^{*-1}d\alpha^{*})^{*}]_{\mathbb{R}} \\
&= [\alpha a\alpha^{-1} + \beta c\alpha^{-1} - \beta c^{*}\alpha^{-1} + \alpha d^{*}\alpha^{-1}]_{\mathbb{R}} \\
&= [\alpha(a+d^{*})\alpha^{-1} + \beta(c-c^{*})\alpha^{-1}]_{\mathbb{R}} \\
&= [\alpha(a+d^{*})\alpha^{-1}]_{\mathbb{R}} + [\alpha(\alpha^{-1}\beta(c-c^{*}))\alpha^{-1}]_{\mathbb{R}} \\
&= [a+d^{*}]_{\mathbb{R}} \quad \text{since } \alpha^{-1}\beta \in \mathbb{R}^{n} \text{ and hence } [\alpha^{-1}\beta(c-c^{*})]_{\mathbb{R}} = 0.
\end{aligned}
$$

Similarly,

$$
BTB^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}
$$

and

$$
(trBTB^{-1})_{\mathbb{R}} = (d+a^{*})_{\mathbb{R}} = (a+d^{*})_{\mathbb{R}}
$$

since $(x)_{\mathbb{R}} = (x^{*})_{\mathbb{R}}$ for any $x \in \Gamma_{n-1}$. □

Lemma 1.3.9 ([25]). *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is conjugate to* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ *by a Möbius transformation, then* $(trA)_{\mathbb{R}} = (a+d^{*})_{\mathbb{R}} = \alpha + \delta$ *and*

1. $(a+d^{*})_{\mathbb{R}}$ *is a real number and* $((a+d^{*})_{\mathbb{R}})^2 < 4$ *if* A *is elliptic.*

2. $(a+d^{*})_{\mathbb{R}} = \pm 2$ *if* A *is strictly parabolic.*

3. $(a+d^{*})_{\mathbb{R}}$ *is a real number and* $((a+d^{*})_{\mathbb{R}})^2 > 4$ *if* A *is hyperbolic.*

Remark 1.3.10. The converse of Lemma 1.3.9. is not true in general. For example, let $A = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ with $|\lambda| = 1$. If $\lambda \neq \pm 1$, $((trA)_{\mathbb{R}})^2 = (\lambda + \overline{\lambda})^2 = (2\lambda_{\mathbb{R}})^2 < 4$. However, the Möbius transformation A can be screw parabolic or boundary elliptic fixing a point ∞ depending on μ (See Theorem 1.3.14).

Möbius transformations acting on $\hat{\mathbb{R}}^3$.

Now, we consider Möbius transformations acting on $\hat{\mathbb{R}}^3$. In this dimension, the Clifford matrix group representing Möbius transformations is $\mathrm{SL}(\Gamma_2)$. The Clifford algebra C_2 generated by e_1 and e_2 is the *quaternion numbers*:

$$\{x_0 + x_1e_1 + x_2e_2 + x_3(e_1e_2) \mid e_1^2 = e_2^2 = -1,\ e_1e_2 = -e_2e_1 \text{ and } x_l \in \mathbb{R}\}.$$

The Clifford group Γ_2 of C_2 consists of all non-zero quaternion numbers. The three involutions in the quaternions C_2 are $x^{'} = (x_0, -x_1, -x_2, x_3)$, $x^* = (x_0, x_1, x_2, -x_3)$ and $\overline{x} = (x^{'})^* = (x^*)^{'} = (x_0, -x_1, -x_2, -x_3)$ for any $x = (x_0, x_1, x_2, x_3) \in C_2$. We identify the 3-dimensional vector space $\{\ x_0 + x_1e_1 + x_2e_2 \mid x_i \in \mathbb{R}\ \}$ of C_2 with $\mathbb{R}^3$. The upper half space model for 4-dimensional hyperbolic space $\mathbb{H}^4$ is $\{\ x + te_3 \in \mathbb{R}^4 \mid x = x_0 + x_1e_1 + x_2e_2 \in \mathbb{R}^3,\ t > 0\ \}$, where e_1, e_2 and e_3 generate the Clifford algebra C_3.

Proposition 1.3.11. *1. For any vector $x \in \mathbb{R}^3$ and any quaternion $a \in C_2$, $ax - xa' \in \mathbb{R}^3$.*

2. For any vector $x \in \mathbb{R}^3$, $xe_1e_2x = |x|^2e_1e_2$.

Proof. Let $x = x_0 + x_1e_1 + x_2e_2 \in \mathbb{R}^3$ and $a = a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2 \in C_2$.

1. The coefficient of e_1e_2-component of ax is

$$[(a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2)(x_0 + x_1e_1 + x_2e_2)]_{e_1e_2} = a_1x_2 - a_2x_1 + a_3x_0$$

which is the same as $[xa']_{e_1e_2}$.

2. By computation,

$$\begin{aligned}xe_1e_2x &= xe_1e_2(x_0+x_1e_1+x_2e_2) = x(x_0e_1e_2 - x_1e_1e_1e_2 - x_2e_2e_1e_2)\\ &= x(x_0 - x_1e_1 - x_2e_2)e_1e_2 = |x|^2e_1e_2.\end{aligned}$$

□

Lemma 1.3.12 ([10]). *If $\lambda = \cos\theta + \sin\theta\xi e_1e_2 \in \Gamma_2$ with $\xi \in \mathbb{R}^3$ and $|\xi| = 1$, then $\rho_\lambda = \begin{pmatrix}\lambda & 0\\ 0 & \lambda'\end{pmatrix}$ is a rotation around ξ by 2θ and hence $\rho_\lambda \in SO(3)$.*

Proof. For an unit quaternion λ, $\lambda^*\lambda' = 1$. So, $\rho_\lambda : x \mapsto \lambda x\lambda^*$ fixes $t\xi$ for any $t \in \mathbb{R}$. □

Let $f = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ be a Clifford matrix in $\mathrm{SL}(\Gamma_2)$. If f fixes a point ∞, then $c = 0$ and hence $ad^* = 1$. Thus, f is of the form $\begin{pmatrix}\lambda & \mu\\ 0 & \lambda^{*-1}\end{pmatrix}$ for some quaternions λ and μ.

Theorem 1.3.13 ([10]). *Let $f = \begin{pmatrix}\lambda & \mu\\ 0 & \lambda^{*-1}\end{pmatrix}$ be a Clifford matrix in $\mathrm{SL}(\Gamma_2)$. Then f is loxodromic if and only if $|\lambda| \neq 1$.*

Proof. For a vector $x \in \mathbb{R}^3$,

$$\begin{aligned}&(\lambda x + \mu)\lambda^* = x\\ \Longleftrightarrow\ &\lambda x{\lambda'}^{-1}\lambda'\lambda^* + \mu\lambda^* = x \qquad (1.20)\\ \Longleftrightarrow\ &(I - |\lambda|^2\rho_\lambda)(x) = \mu\lambda^*.\end{aligned}$$

Thus, $(I - |\lambda|^2\rho_\lambda)$ has an inverse map and the above equation has a unique solution x if and only if $|\lambda| \neq 1$. In $\mathbb{R}^3$, the fixed point set of a boundary-elliptic element, which includes ∞, is a 1-dimensional subspace. Therefore, f cannot be elliptic in any cases. □

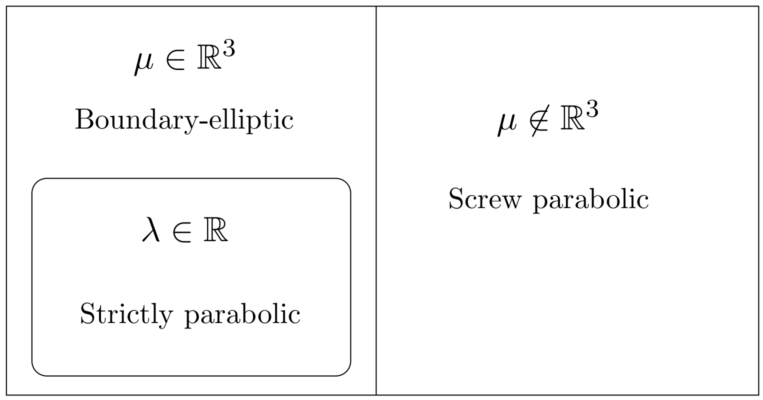

Figure 1.2: Theorem 1.3.14.

Theorem 1.3.14 ([10])**.** $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ *with* $|\lambda| = 1$ *is:*

$$\textit{strictly parabolic if } \lambda \in \mathbb{R},$$
$$\textit{screw parabolic if } \mu \notin \mathbb{R}^3,$$
$$\textit{boundary elliptic otherwise.}$$

Proof. For any vector $x \in \mathbb{R}^3$,

$$\lambda x + \mu = x\lambda' \iff \lambda x - x\lambda' = -\mu \tag{1.21}$$

If λ is a real number, $\lambda \pm 1$ and hence μ is a vector. Thus, it is a strictly parabolic element $x \mapsto x + \mu$.

We note that $\lambda x - x\lambda'$ is a vector (Proposition 1.3.11). If μ is not a vector, λ cannot be a real number and the above equation (1.21) has no solution in $\mathbb{R}^3$. Hence, it is screw parabolic. $\square$

Example 1.3.15. Let u and v be vectors in $\mathbb{R}^3$. The most general Möbius transformation which takes u to 0 and v to ∞ is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix} \begin{pmatrix} 1 & -u \\ (u-v)^{-1} & -(u-v)^{-1}v \end{pmatrix} \tag{1.22}$$

for a non-zero quaternion α ([2]).

Chapter 2

Screw parabolic isometries

2.1 Maximal parabolic subgroups

In this section, we will show that a discrete parabolic group containing an irrational screw parabolic element can only be rank one (see Definition 1.1.2. for a parabolic group).

Let f be a screw parabolic element acting on $\hat{\mathbb{R}}^3$, of the form $\begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} \in$ PSL(Γ_2) where $\alpha = \cos\theta + \sin\theta e_1 e_2 \in \Gamma_2$ with $\theta \in (0, \pi)$. Then $f(x) = \alpha x \alpha^{-1} + 1$ for any $x \in \mathbb{R}^3$. We can see the action as the composition of a rotation around the real axis $< 1 >$ by 2θ and a translation by 1 and hence the real axis is the unique invariant Euclidean line in $\mathbb{R}^3$. We call the unique invariant Euclidean line the *Axis* of f, denoted by $Axis_f$. For a strictly parabolic element fixing a point ∞, there are many invariant Euclidean lines but they are all parallel. So, it is natural to call its translation direction *the Axis*.

Lemma 2.1.1. *Let G_p be an abelian parabolic Möbius group acting on $\hat{\mathbb{R}}^3$*

with a fixed point p. If G_p has an irrational screw parabolic element, then G_p is cyclic.

Proof. We may assume that $p = \infty$ and G_∞ has a normalized irrational screw parabolic element $f = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}$ with $\alpha = \cos\theta + \sin\theta(e_1e_2) \in \Gamma_2$. Suppose that G_∞ has another parabolic element g. Since g fixes ∞, g is of the form $\begin{pmatrix} \beta & v \\ 0 & \beta' \end{pmatrix}$ where β is a unit quaternion and $v \in \Gamma_2$ (Theorem 1.3.12).

$$\begin{aligned} fg &= \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} \beta & v \\ 0 & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\beta & \alpha v + \alpha\beta' \\ 0 & \alpha\beta' \end{pmatrix} \\ gf &= \begin{pmatrix} \beta & v \\ 0 & \beta' \end{pmatrix}\begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \beta\alpha & \beta\alpha + v\alpha \\ 0 & \beta'\alpha \end{pmatrix} \end{aligned} \tag{2.1}$$

Since G_∞ is abelian, $\alpha\beta = \beta\alpha \iff \rho_\alpha(\beta) = \alpha\beta\alpha^{-1} = \beta$ (Lemma 1.3.12), which implies either $\beta \in Axis_f =< 1 >$ or $\beta = \pm\alpha^n$ for some integer n.

If $\beta \in< 1 >$, then v is a vector and g is strictly parabolic. From (2.1), $\alpha v = v\alpha \Leftrightarrow \alpha v\alpha^{-1} = v$ implies $v \in Axis_f =< 1 >$. Thus, $< f, g >$ is a discrete subgroup of G_∞ and keeps $\mathbb{R}$ invariant. On $\mathbb{R}$, f and g are translations along $\mathbb{R}$. Hence, $< f|_\mathbb{R},\ g|_\mathbb{R} >$ is a cyclic group as a discrete group acting on $\mathbb{R}$. We can find $i, j \in \mathbb{Z}$ so that $g^i \circ f^j \in G_\infty$ is a non-trivial irrational rotation around the real axis $< 1 >$ in $\mathbb{R}^3$, and hence G_∞ cannot be discrete. This is a contradiction and hence $\beta = \pm\alpha^n$ for some integer n. Now, we will complete this proof by showing that g belongs to the cyclic group generated by f.

Since G_∞ is abelian, $f(-g) = -gf$ so we may assume $\beta = \alpha^n$. Then $v \notin \mathbb{R}^3$ since g is a screw parabolic element. From (2.1), $\alpha v = v\alpha \Leftrightarrow \alpha v\alpha^{-1} = v$ implies $v = \lambda\alpha^m$ for some $\lambda \in \mathbb{R}$ and some integer m. Since $g(0) = \lambda\alpha^m\alpha^{-n}$

is a vector in $\mathbb{R}^3$, $m = n$. Then again $< f = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}, g = \begin{pmatrix} \alpha^n & \lambda\alpha^n \\ 0 & \alpha^n \end{pmatrix} >$ is a discrete subgroup of G_∞ and keeps $\mathbb{R}$ invariant. Since the group action is discrete on $\mathbb{R}$, λ must be an integer. If $\lambda \neq n$, G_∞ cannot be not discrete because an irrational rotation $f^{-\lambda} \circ g$ exists in G_∞. This is a contradiction. Therefore, $\lambda = n$ and hence $g = f^n$ for some n. $\square$

Corollary 2.1.2. *Let f be an irrational screw parabolic element and g be a parabolic element in* $\text{Möb}(\hat{\mathbb{R}}^3)$. *Then f and g commute if and only if they have the same axis and the same fixed point.*

Suppose that Γ is a torsion-free Möbius group acting on $\hat{\mathbb{R}}^3$. Let G_p be a parabolic subgroup of Γ with a fixed point $p \in \hat{\mathbb{R}}^3$ and $stab_\Gamma p$ be the stabilizer of p in Γ. Since a parabolic element and a loxodromic element cannot share a fixed point in a discrete group, the stabilizer consists of entirely parabolic elements. Hence, the stabilizer $stab_\Gamma p$ is also a parabolic group containing G_p. In fact, $stab_\Gamma p$ is a maximal parabolic subgroup and every parabolic subgroup is contained in a unique maximal parabolic subgroup. A maximal parabolic subgroup is abelian by Margulis' Lemma 1.1.5. Therefore, we have the following corollary.

Corollary 2.1.3. *A maximal parabolic subgroup containing an irrational screw parabolic element of a torsion-free* 3*-dimensional Möbius group is cyclic.*

Corollary 2.1.4. *If a* 3*-dimensional Möbius group* Γ *has cofinite volume, then* Γ *has no irrational screw parabolic elements.*

Proof. If Γ has cofinite volume, then every maximal parabolic subgroup must have full rank, which is 3. $\square$

Corollary 2.1.5. *If a 3-dimensional geometrically finite Möbius group* Γ *has an irrational screw parabolic element, then* Γ *is of the second kind (i.e., the set of discontinuity* $\Omega_\Gamma \neq \emptyset$*).*

Proof. Suppose that Γ has an irrational screw parabolic element f which fixes a point ∞. Then, the maximal parabolic group containing f is rank 1. Since all the parabolic fixed points must be bounded (Definition 1.1.12), the limit set Λ_Γ must be in a bounded cylinder neighborhood of the axis of the screw parabolic element f. Hence, the limit set cannot be everything, which means the set of discontinuity Ω_Γ is not empty. $\square$

Let Γ be a 3-dimensional geometrically finite Möbius group containing an irrational screw parabolic element f which fixes a point ∞. Suppose that the limit set Λ_Γ is a topological sphere. Then $\Lambda \setminus \{\infty\}$ has one end. However, $\Lambda_\Gamma \setminus \{\infty\}$ is invariant under the action of a maximal parabolic group containing f and is embedded in a cylinder neighborhood of $Axis_f$. This is a contradiction. Thus, Λ_Γ cannot be a topological sphere. In fact, this is true for any geometrically finite Möbius group with a rank one maximal parabolic group.

2.2 Conjugacy classes

In this section we consider conjugacy classes of Möbius transformations. Any two elliptic Möbius transformations of $\hat{\mathbb{R}}^2$ that are not Möbius conjugate to each other are also not quasiconformally conjugate. We can prove this using the Poincaré theorem. Poincaré theorem states that if any two homeomorphisms of the circle $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ are topologically conjugate to each other (i.e., they are conjugate to each other by a homeomorphism of

$\mathbb{S}^1$), then their rotation numbers are the same. In the case of a rotation map which is a rigid motion of $\mathbb{R}^2$, the rotation angle is the rotation number. Suppose that two elliptic elements f and g of Möb($\hat{\mathbb{R}}^2$) are quasiconformally conjugate to each other i.e., $\phi f \phi^{-1} = g$ for a quasiconformal mapping ϕ of $\hat{\mathbb{R}}^2$. We may assume that f and g fix 0 and ∞ and hence ϕ fixes 0 and ∞. Otherwise, we can conjugate everything by a Möbius transformation so that each conjugation fixes 0 and ∞. Now f and g are rotations around 0 and keep every circle centered at 0 invariant. If the Euclidean norm $|\phi(1)|$ is not 1, we post compose a dilation which is a Möbius transformation to ϕ so that the norm becomes 1. Hence, we can project all maps to $\mathbb{S}^1$, say $\hat{f}$, $\hat{g}$ and $\hat{\phi}$. Then two projected maps $\hat{f}$ and $\hat{g}$ are topologically conjugate on $\mathbb{S}^1$ by the projected map $\hat{\phi}$. By the Poincaré theorem, the rotation angles of $\hat{f}$ and $\hat{g}$ must be the same. Clearly, $\hat{f}$ and $\hat{g}$ have the same rotation angles as f and g respectively. Therefore, f and g are quasiconformally conjugate to each other if and only if they are Möbius conjugate.

All loxodromic elements (including hyperbolic elements) of Möb($\hat{\mathbb{R}}^2$) are quasiconformally conjugate to each other since their quotients of the set of discontinuity are compact and their limit sets consist of two isolated points. Note that for a quasiconformal mapping, isolated boundary points are removable singularities (Theorem 17.3. [22]). In fact, all loxodromic elements are quasiconformally conjugate to each other in any dimensions for the same reason.

Any parabolic element of Möb($\hat{\mathbb{R}}^2$) is conjugate to $x \mapsto x + 1$ by a Möbius transformation and Möbius transformations are 1-quasiconformal. Therefore, all parabolic elements of are quasiconformally conjugate to each other in $\hat{\mathbb{R}}^2$.

In $\hat{\mathbb{R}}^3$, all strictly parabolic elements are Möbius (hence quasiconformally) conjugate to a Möbius transformation $x \mapsto x+1$. However, the next proposition shows that a strictly parabolic element is not Möbius conjugate to a screw parabolic element.

Proposition 2.2.1. *Two parabolic elements are conjugate to each other by a Möbius transformation if and only if they have the same rotation angle (the rotation angle of a strictly parabolic element is* 0*). In particular, a screw parabolic element cannot be Möbius conjugate to a strictly parabolic element.*

Proof. Any parabolic element fixing a point ∞ is of the form $A = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix}$, where $\lambda = \cos\theta + \sin\theta \xi e_1 e_2$, $\theta \in [0, \pi)$, ξ is a unit vector in $\mathbb{R}^3$ and $\mu \notin \mathbb{R}^3$. The Clifford trace $(tr)_{\mathbb{R}}$ is a Möbius conjugacy invariant and $(trA)_{\mathbb{R}} = 2\cos\theta$ is one-to-one (Lemma 1.3.8). A parabolic element A is strictly parabolic if and only if $(trA)_{\mathbb{R}}$ is 2. Therefore, if two parabolic elements f and g are Möbius conjugate to each other, they have the same rotation angle.

For the converse, suppose that two parabolic elements f and g have the same rotation angle, denoted θ. We can conjugate f and g by a Möbius transformation so that they fix a point ∞. We call their conjugations f and g again. Thus, $f = \begin{pmatrix} \alpha & \mu \\ 0 & \alpha' \end{pmatrix}$, $g = \begin{pmatrix} \beta & \nu \\ 0 & \beta' \end{pmatrix}$ where $\alpha = \cos\theta + \sin\theta \xi e_1 e_2$, $\beta = \cos\theta + \sin\theta \xi' e_1 e_2$, ξ, ξ' are unit vectors in $\mathbb{R}^3$ and $\mu, \nu \notin \mathbb{R}^3$. Let $\rho_\lambda \in SO(3)$ be a Euclidean isometry so that $\rho_\lambda(\xi) = \xi'$ (see Lemma 1.3.12). Then $\rho_\lambda f \rho_\lambda^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} \alpha & \mu \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda^* \end{pmatrix} = \begin{pmatrix} \beta & \lambda\mu\lambda^* \\ 0 & \beta' \end{pmatrix}$. We can further conjugate $\rho_\lambda f \rho_\lambda^{-1}$ by a dilation or a translation Möbius transformation to obtain g. $\square$

Although a screw parabolic element and a strictly parabolic element are not Möbius conjugate to each other, they are topologically conjugate as following ([12]). Let f be a screw parabolic element as the composition of rotation around the real axis $< 1 >$ by a non-zero θ and translation by 1, and T be the translation by 1 in $\mathbb{R}^3 =< 1, e_1, e_2 >$. For any number t, let R_t be a rotation around the real-axis by angle t in $\mathbb{R}^3$ and r_t be the projection map of R_t on the 2-dimensional plane $< e_1, e_2 >$ so that $R_t(s, z) = (s, r_t(z))$ for any $(s, z) \in \mathbb{R} \times \mathbb{R}^2$. Define a homeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $(s, z) \mapsto (s, r_{s\theta}(z))$. Then for any $x = (s, z) \in \mathbb{R}^3$,

$$\begin{aligned} F^{-1}fF(x) &= F^{-1}fF(s, z) = F^{-1}fF(s, r_{s\theta}(z)) \\ &= F^{-1}(s + 1, r_{s\theta+\theta}(z)) = (s + 1, z) = T(x). \end{aligned}$$

Define $F(\infty) = \infty$. Then F is also a homeomorphism of $\hat{\mathbb{R}}^3$. Thus, a screw parabolic f is topologically conjugate to a strictly parabolic T. However, the following calculation shows that F is not a quasiconformal map (Theorem 1.2.2). For any $(s, x, y) \in \mathbb{R}^3$,

$$F(s, x, y) = (s,\ x \cos s\theta - y \sin s\theta,\ x \sin s\theta + y \cos s\theta)$$

is differentiable and

$$DF = \begin{pmatrix} 1 & -\theta x \sin(s\theta) - \theta y \cos(s\theta) & \theta x \cos(s\theta) - \theta y \sin(s\theta) \\ 0 & \cos(s\theta) & \sin(s\theta) \\ 0 & -\sin(s\theta) & \cos(s\theta) \end{pmatrix}.$$

The Jacobian $J(F, (s, x, y)) = 1$ everywhere however, at $(1, x, 0) \in \mathbb{R}^3$

$$\begin{aligned} |D_{(1,x,0)}F(0, 1, 0)|^2 &= |(-\theta x \sin\theta, \cos\theta, -\sin\theta)|^2 \\ &= \theta^2 x^2 \sin^2\theta + 1 \to \infty \end{aligned}$$

as $x \to \infty$.

Now, we ask if a screw parabolic element is quasiconformally conjugate to any strictly parabolic element. The answer is negative. We will divide it into two cases: a rational screw parabolic element and an irrational screw parabolic element.

Let $f = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$, with $\alpha = \cos\theta + \sin\theta e_1 e_2$ and $\theta \in (0, \pi)$ be a screw parabolic element and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ be a strictly parabolic element. We also have their Poincaré extensions to $\mathbb{H}^4$, call them f and T again.

To prove that a rational screw parabolic f is not quasiconformally conjugate to a strictly parabolic T on $\hat{\mathbb{R}}^3$, first we will show that f is not conjugate to T by any quasiisometries in $\mathbb{H}^4$.

We recall that $\mathbb{H}^4 = \{\ x_0 + x_1 e_1 + x_2 e_2 + t e_3 \mid x_i \in \mathbb{R},\ t > 0\ \}$. For $t > 0$, the height t-horosphere at ∞ is $\partial\Sigma_t = \{\ (x, t) \in \mathbb{H}^4 \mid x \in \mathbb{R}^3\ \}$ and the horoball bounded by a height t-horosphere at ∞ is $\Sigma_t = \{\ (x, s) \in \mathbb{H}^4 \mid x \in \mathbb{R}^3,\ s > t\ \}$. On the height t-horosphere $\partial\Sigma_t$, the hyperbolic distance is

$$d(te_3, r + te_3) = d(e_3, \frac{r}{t} + e_3) = 2\ln\frac{\sqrt{r^2 + 4t^2} + |r|}{2t} \qquad (2.2)$$

for any positive numbers t and r.

The image of a horosphere under a quasiconformal map may not be close to a horosphere at all. There are examples of quasiconformal maps whose images of horospheres are not bounded by any horospheres. Let $\phi : \mathbb{H}^4 \to \mathbb{H}^4,\ z \mapsto |z|^{K-1} z$ be the quasiconformal mapping in Example 1.2.5. For $z = (x, 1) \in \partial\Sigma_1 \subseteq \mathbb{H}^4$ with $x \in \mathbb{R}^3$, the last coordinate of $\phi((x, 1))$ is $(|x|^2 + 1)^{\frac{K-1}{2}}$ which is greater than or equal to 1 for $K > 1$ and less than or equal to 1 for $0 < K < 1$. Thus, $\phi(\partial\Sigma_1) \subseteq \Sigma_t$ for some $t > 0$ if and only if $K \geqslant 1$.

Lemma 2.2.2. *If a screw parabolic element f is rational, then f is not conjugate to a strictly parabolic element T by any quasiisometries in $\mathbb{H}^4$.*

Proof. We will prove this by a contradiction. Suppose that there is a (λ, δ)-quasiisometry $\phi : \mathbb{H}^4 \to \mathbb{H}^4$ such that $\phi f \phi^{-1} = T$. So, for any $z \in \mathbb{H}^4$

$$\frac{1}{\lambda} d(z, f(z)) - \delta \leq d(\phi(z), \phi f(z)) \leq \lambda d(z, f(z)) + \delta. \tag{2.3}$$

We will restrict ϕ on the height 1-horosphere $\partial\Sigma_1$, $\phi|_{\partial\Sigma_1} : \partial\Sigma_1 \to \phi(\partial\Sigma_1)$.

Suppose that the image $\phi(\partial\Sigma_1)$ is contained in a horoball at ∞, $\Sigma_{t_0} = \{\ (x,t) \in \mathbb{H}^4 \mid t > t_0\ \}$ for some $t_0 > 0$. From (2.3), for each $z_n = ne_1 + e_3 \in \partial\Sigma_1$ with $n \in \mathbb{Z}$,

$$\frac{1}{\lambda} d(z_n, f(z_n)) - \delta \leq d(\phi(z_n), \phi(f(z_n))). \tag{2.4}$$

The left hand side is

$$\begin{aligned} d(z_n, f(z_n)) &= d\left(ne_1 + e_3, 1 + n(\cos 2\theta e_1 + \sin 2\theta e_2) + e_3\right) \\ &= 2 \ln \frac{\sqrt{F^2(n) + 4} + F(n)}{2} \to \infty \text{ as } n \to \infty, \end{aligned}$$

where $F(n) = \sqrt{1 + 2n^2(1 - \cos\theta)}$.

However, the right hand side of (2.4) is

$$\begin{aligned} d(\phi(z_n), \phi(f(z_n))) &= d(\phi(z_n), T(\phi(z_n))) = d(\phi(z_n), 1 + \phi(z_n)) \\ &\leq d(t_0 e_3, 1 + t_0 e_3) \quad \text{since } \phi(z_n) \in \Sigma_{t_0} \\ &= 2 \ln \frac{\sqrt{4{t_0}^2 + 1} + 1}{2t_0} \end{aligned}$$

is bounded as $n \to \infty$, which is a contradiction.

Therefore, the image $\phi(\partial\Sigma_1)$ is not contained in any horoballs at ∞, so we may have a sequence of points $\{z_k = (x_k, t_k)\} \subset \phi(\partial\Sigma_1)$ such that

$t_k \to 0$ as $k \to \infty$. Since f is rational screw parabolic, there is an integer n_0 so that $f^{n_0}(z) = T^{n_0}(z) = z + n_0$ for $z \in \mathbb{H}^4$. From (2.3), for each $z_k \in \partial\Sigma_{t_k} \cap \phi(\partial\Sigma_1)$

$$d(\phi(z_k), \phi f^{n_0}(z_k)) \leq \lambda d(z, f^{n_0}(z_k)) + \delta = \lambda d\left(\phi(z_k), T^{n_0}(z_k)\right) + \delta$$

$$d\left(z_k, f^{n_0}(z_k)\right) \leq \lambda d\left(\phi^{-1}(z_k), \phi^{-1}T^{n_0}(z_k)\right) + \delta. \tag{2.5}$$

We note that $\phi^{-1}(z_k) \in \partial\Sigma_1$. The right hand side of (2.5)

$$\begin{aligned}
\lambda d(\phi^{-1}(z_k), \phi^{-1}T^{n_0}(z_k)) + \delta &= \lambda d(\phi^{-1}(z_k), f^{n_0}(\phi^{-1}(z_k))) + \delta \\
&= \lambda d(\phi^{-1}(z_k), \phi^{-1}(z_k) + n_0) + \delta, \quad \text{since} \quad f^{n_0} = T^{n_0} \\
&= \lambda 2 \ln \frac{\sqrt{{n_0}^2 + 4} + n_0}{2} + \delta \quad \text{by (2.2)}
\end{aligned}$$

is constant for any k, but the left hand side is

$$\begin{aligned}
d(z_k, f^{n_0}(z_k)) = d(z_k, z_k + n_0) &= d(x_k + t_k e_3, n_0 + x_k + t_k e_3) \\
&= d(t_k i, n_0 + t_k i) \quad \text{applying a Möbius transformation } z \mapsto z - x_k \\
&= 2 \ln \frac{\sqrt{{n_0}^2 + 4{t_k}^2} + n_0}{2t_k} \to \infty \text{ since } t_k \to 0 \text{ as } k \to \infty,
\end{aligned}$$

which contradicts to (2.5). Therefore, f is not conjugate to T by a quasi-isometry in $\mathbb{H}^4$. $\square$

The next corollary follows from the fact that every quasiconformal map is also a quasiisometry in hyperbolic space $\mathbb{H}^n$.

Corollary 2.2.3. *1. Any rational screw parabolic element is not quasiconformally conjugate to a strictly parabolic element in $\mathbb{H}^4$.*

2. Lemma 2.2.2. holds for any hyperbolic space with dimensions $n \geq 4$.

To show that a rational screw parabolic f is not quasiconformally conjugate to a strictly parabolic T on the boundary $\hat{\mathbb{R}}^3$, we need an extension of a quasiconformal map from the boundary at infinity $\hat{\mathbb{R}}^3$ to the hyperbolic space $\mathbb{H}^4$.

Let Γ be an n-dimensional Möbius group. A homeomorphism $\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ is said to be Γ-*compatible* if there is a homomorphism χ from Γ onto another Möbius group Γ' such that

$$\phi \circ g(x) = \chi(g) \circ \phi(x) \tag{2.6}$$

for any $g \in \Gamma$ and any $x \in \hat{\mathbb{R}}^n$.

Theorem 2.2.4 ([21])**.** *For $n \geq 2$, let Γ be an n-dimensional Möbius group and ϕ be a Γ-compatible homeomorphism of $\hat{\mathbb{R}}^n$. Then there is a Γ-compatible continuous extension $\hat{\phi} : \overline{\mathbb{H}}^{n+1} \to \overline{\mathbb{H}}^{n+1}$ of ϕ such that $\hat{\phi}(\mathbb{H}^{n+1}) \subset \mathbb{H}^{n+1}$.*

Furthermore, there are real numbers $M = M(n, K)$ and $L = L(n, K)$ depending on the dimension n and a real number $K \geq 1$ such that if ϕ is K-quasiconformal,

$$\frac{d(z, z')}{L} \leq d(\hat{\phi}(z), \hat{\phi}(z')) \leq L d(z, z') \tag{2.7}$$

for any z, $z' \in \mathbb{H}^{n+1}$ satisfying $d(z, z') \geq M$.

Lemma 2.2.5. *If f is a rational screw parabolic element, then f is not quasiconformally conjugate to a strictly parabolic element T on $\hat{\mathbb{R}}^3$.*

Proof. Suppose that there is a K-quasiconformal homeomorphism $\phi : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ such that $\phi f \phi^{-1} = T$. Then by Theorem 2.2.4, we can extend a K-quasiconformal map ϕ to a continuous map $\hat{\phi} : \mathbb{H}^4 \to \mathbb{H}^4$ such that $\hat{\phi} f = T\hat{\phi}$. From (2.7), for any $z \in \mathbb{H}^4$ satisfying $d(z, f(z)) > M(3, K)$, we have

$$\frac{d(z, f(z))}{L} \leq d(\hat{\phi}(z), \hat{\phi} f(z)) \leq L d(z, f(z)), \tag{2.8}$$

where $M(3, K)$ is a constant depending on K. The same idea used in the proof of Lemma 2.2.2 shows a contradiction here. This proves the lemma. $\square$

For a rational screw parabolic element f, the *order* of f is the order of its rotation or equivalently, the order n of f is the smallest positive integer such that f^n is strictly parabolic.

Corollary 2.2.6. *If two rational screw parabolic elements are quasiconformally conjugate to each other on $\hat{\mathbb{R}}^3$, then they have the same order.*

Proof. Let f and g be two rational screw parabolic element with different orders, say m and n respectively. We may assume $n < m$. Suppose that f and g are quasiconformally conjugate. Then f^k and g^k are also quasiconformally conjugate for any integer k. However, f^n and g^n cannot be quasiconformally conjugate since g^n is strictly parabolic and f^n is screw parabolic by Lemma 2.2.5. Hence f and g are not quasiconformally conjugate on $\hat{\mathbb{R}}^3$. $\square$

Now we will prove that an irrational screw parabolic element is not quasiconformally conjugate to a strictly parabolic element on $\hat{\mathbb{R}}^3$. Since any quasiconformal homeomorphism is also quasiisometry in $\mathbb{H}^n$ and each quasiisometry of $\mathbb{H}^n$ naturally extends to the boundary at infinity $\hat{\mathbb{R}}^{n-1}$ as a quasiconformal homeomorphism, irrational screw parabolic element is not quasiconformally conjugate to a strictly parabolic element in $\mathbb{H}^4$.

Lemma 2.2.7. *If f is irrational, then f is not quasiconformally conjugate to a strictly parabolic element T on $\hat{\mathbb{R}}^3$.*

Proof. We will prove this by a contradiction. Suppose that for some $K > 1$ there is a K-quasiconformal homeomorphism $\phi : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ such that

$\phi f \phi^{-1} = T$. If ϕ does not fix 0, then we will post compose ϕ by a Möbius transformation $m(x) = x - \phi(0)$. Since Möbius transformations are conformal, $m \circ \phi$ is a K-quasiconformal homeomorphism. Since m and T commute, f is again conjugate to T by a K-quasiconformal homeomorphism $m \circ \phi$.

$$m\phi f \phi^{-1} m^{-1} = mTm^{-1} = T.$$

Thus, we may assume that ϕ fixes 0 and hence $\phi f(0) = T(0)$ i.e., ϕ fixes 1. By induction, ϕ fixes all integers.

For each $n \in \mathbb{N}$, we define a Möbius transformation $h_n(x) = \frac{1}{n}x$ for $x \in \mathbb{R}^3$. Then

$$h_n \phi {h_n}^{-1} (h_n f^n {h_n}^{-1}) h_n \phi^{-1} h_n^{-1} = h_n T^n h_n^{-1} = T \tag{2.9}$$

and

$$\begin{aligned} h_n f^n {h_n}^{-1} &= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^n & n\alpha^n \\ 0 & \alpha^n \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^n & \alpha^n \\ 0 & \alpha^n \end{pmatrix} \\ &= \begin{pmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.10}$$

Since f is irrational, there is a subsequence $\{h_{n_k} f^{n_k} h_{n_k}^{-1}\}$ which converges to a rational screw parabolic element $\hat{f}$

$$\hat{f} = \begin{pmatrix} \alpha_0 & \alpha_0 \\ 0 & \alpha_0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

for some unit quaternion $\alpha_0 = \cos\theta_0 + \sin\theta_0 e_1 e_2 \in \Gamma_2$.

For each $k \in \mathbb{N}$, let $\psi_k = h_{n_k} \phi h_{n_k}^{-1}$. Since ϕ is K-quasiconformal and h_{n_k} is a Möbius transformation, each ψ_k is again K-quasiconformal. Thus from (2.9) we have a family $\mathcal{F}$ of K-quasiconformal homeomorphisms

$$\Big\{ \psi_k : k \in \mathbb{N} \;\Big|\; \psi_k^{-1} T \psi_k = h_{n_k} f^{n_k} h_{n_k}^{-1} \Big\}. \tag{2.12}$$

Then $\mathcal{F}$ is a normal family by Theorem 1.2.6. and hence it has a convergent subsequence. Since ϕ fixes all integers, each ψ_k fixes $h_{n_k}(m) = \frac{m}{n_k}$ for any integer m. In particular, each ψ_k fixes three points 0 and 1 and a point ∞. Hence, the limit of the convergent subsequence of $\mathcal{F}$ is a K-quasiconformal homeomorphism, say ψ_0. Then ${\psi_0}^{-1}T\psi_0 = \hat{f}$ by (2.11), which is a contradiction by Lemma 2.2.5. $\square$

Therefore, any screw parabolic element is not quasiconformally conjugate to a strictly parabolic element. We can actually say more about the quasiconformal conjugacy classes of screw parabolic elements. From Corollary 2.2.6., we know when two rational screw parabolic elements are not quasiconformally conjugate. From the proof of the previous lemma, we have the following two corollaries.

Corollary 2.2.8. *Any rational screw parabolic element is not quasiconformally conjugate to an irrational screw parabolic element.*

Proof. Let f be an irrational screw parabolic element and g be a rational screw parabolic element. We may assume that f and g fix ∞. Suppose that there is a quasiconformal map ϕ so that $\phi f \phi^{-1} = g$. Since g is rational, $\phi f^n \phi^{-1} = g^n$ is a strictly parabolic element for some $n \in \mathbb{Z}$. This is a contradiction since f is irrational. $\square$

Let $[[r]] = r - [r]$ denote the fractional part of any real number r. For a fractional number $t \in [0, 1)$, let $f_t = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ with $\alpha = \cos(t\pi) + \sin(t\pi)e_1e_2$, be a parabolic element. The rotational angle of f_t is $2t\pi$.

Corollary 2.2.9. *For $t, s \in [0, 1)$, suppose that two parabolic elements f_t and f_s are quasiconformally conjugate. Then for any sequence $\{n_k\}$ of inte-*

gers such that the sequences of the fractional parts $[[n_k t]]$ and $[[n_k s]]$ converge to t_0 and $s_0 \in [0,1)$ respectively as $k \to \infty$, f_{t_0} and f_{s_0} are quasiconformally conjugate.

Proof. Let $\phi f_t \phi^{-1} = f_s$ for some K-quasiconformal mapping ϕ. Then $\phi f_t^n \phi^{-1} = f_s^n$ for any integer n. $\square$

Lemma 2.2.10. *Let T be the compact abelian Lie group $\mathbb{R}^2/\mathbb{Z}^2$. For any two fractional numbers $t, s \in [0,1)$ with $t < s$, the closure $\overline{H}$ of the additive subgroup H generated by the pair of numbers (t,s) is a submanifold of T. Then the dimension of $\overline{H}$ is either*

$$\begin{cases} 0 & \text{if both } t \text{ and } s \text{ are rational, or} \\ 1 & \text{if } t \text{ or } s \text{ is irrational and the ratio } \frac{t}{s} \text{ is rational, or} \\ 2 & \text{if } \frac{t}{s} \text{ is irrational.} \end{cases}$$

Lemma 2.2.11. *Any two irrational screw parabolic elements that are not Möbius conjugate to each other are not quasiconformally conjugate.*

Proof. For two distinct irrational numbers r, $s \in [0,1)$, assume $r < s$, let f_t and f_s be two irrational screw parabolic elements. Suppose that there is a K-quasiconformal map ϕ of $\hat{\mathbb{R}}^3$ such that $\phi f_t \phi^{-1} = f_s$. Let H be the additive subgroup of the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ generated by (t,s). Then the closure $\overline{H}$ is a submanifold with dimension one or two by Lemma 2.2.10. Using Corollary 2.2.9, for any pair (a,b) in $\overline{H}$, f_a and f_b are quasiconformally conjugate by ϕ. If the dimension of $\overline{H}$ is one, it is an embedded circle in T whose slope $\frac{t}{s} < 1$ is a rational number. If the dimension of $\overline{H}$ is two, then $\overline{H}$ is the whole torus T. In either cases, there is a meridian cut point $(0, s_0) \in \overline{H}$ with non-zero fractional number s_0. This implies that a strictly parabolic f_0 is quasiconformally conjugate to a screw parabolic f_{s_0}, which is a contradiction to Lemma 2.2.5. and Lemma 2.2.7. Therefore, f_t and f_s cannot be quasiconformally conjugate. $\square$

Chapter 3

Deformation spaces

3.1 Introduction

Let Γ be a finitely generated Möbius group acting on $\hat{\mathbb{R}}^n$, generated by $\gamma_1, \ldots, \gamma_m$. A *representation* of Γ is a homomorphism $\rho : \Gamma \to \mathrm{PSL}(\Gamma_{n-1})$.

The *marked length spectrum* for the representation ρ is the function $T_\rho : \Gamma \to \mathbb{R}_{\geq 0}$ given by

$$g \mapsto T_\rho(g) = \inf_{x \in \mathbb{H}^4} d(\rho(g)(x), x) \tag{3.1}$$

for any $g \in \Gamma$. Note that $T_\rho(g) = 0$ if $\rho(g)$ is elliptic or parabolic.

The representation ρ is said to be *discrete* if the image $\rho(\Gamma)$ is discrete and *faithful* if it is injective. The *type-preserving* representation means that $\rho(g)$ is parabolic if and only if $g \in \Gamma$ is parabolic. We call a discrete faithful type-preserving representation *deformation*. The *deformation space* of Γ, denoted by $\mathcal{D}(\Gamma)$, to be the set of all deformations of Γ into $\mathrm{PSL}(\Gamma_{n-1})$ where $\mathrm{PSL}(\Gamma_{n-1})$ acts by conjugation. That is to say

$$\mathcal{D}(\Gamma) = \{\ \rho : \Gamma \to \mathrm{PSL}(\Gamma_{n-1}) \text{ deformation } \}/\mathrm{PSL}(\Gamma_{n-1}), \tag{3.2}$$

where two deformations ρ_1 and ρ_2 are in the same equivalent class if and only if $\rho_2 = A\rho_1 A^{-1}$ for some $A \in \mathrm{PSL}(\Gamma_{n-1})$ (i.e. $\rho_2(g) = A\rho_1(g)A^{-1}$ for any $g \in \Gamma$).

A sequence of representations $\{\rho_k\}_{k=1}^{\infty}$ of Γ converges to a representation ρ_0 if and only if the sequence of Möbius transformations $\{\rho_k(g)\}$ converges to the Möbius transformation $\rho_0(g)$, for any $g \in \Gamma$ (see (1.1)). Two deformations ρ_1 and ρ_2 of Γ are *quasiconformally conjugate* if there exists a quasiconformal map $\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ such that $\phi\rho_1(g)\phi^{-1} = \rho_2(g)$, for any $g \in \Gamma$. If two deformations ρ_1 and ρ_2 of Γ are quasiconformally conjugate, then two Möbius groups $\rho_1(\Gamma)$ and $\rho_2(\Gamma)$ are quasiconformally conjugate. However, the converse is not necessarily true.

For $\epsilon > 0$, *an ϵ-deformation* of Γ (with respect to the set of generators $\gamma_1, \ldots, \gamma_m$) is a deformation $\rho : \Gamma \to \mathrm{PSL}(\Gamma_{n-1})$ satisfying $d(\rho(\gamma_i), \gamma_i) < \epsilon$ for any generator γ_i (see (1.1) for the metric d in $\mathrm{PSL}(\Gamma_{n-1})$).

Definition 3.1.1. A Möbius group Γ is said to be *quasiconformally stable* if given the set of generators $\gamma_1, \ldots, \gamma_m$, there exists $\epsilon_0 > 0$ such that each ϵ-deformation of Γ with $\epsilon < \epsilon_0$ is quasiconformally conjugate to the identity deformation.

The following two theorems are about the deformation space of a Möbius group.

Mostow-Prasad Rigidity Theorem ([18], [19]). *If $n + 1 \geq 3$ and a hyperbolic $(n + 1)$-manifold M is closed or has finite volume, then its hyperbolic structure is unique.*

Marden Quasiconformal Stability Theorem ([16]). *A geometrically finite Kleinian group is quasiconformally stable.*

In particular, Mostow-Prasad Rigidity Theorem tells us that for $n+1 \geq 3$ the deformation space of an n-dimensional cofinite volume Möbius group is trivial and hence quasiconformally stable. Then, we might ask if a geometrically finite Möbius group is quasiconformally stable. In dimension 3, we have the Marden Quasiconformal Stability Theorem. In dimensions 4, we will prove that there is a geometrically finite Möbius group which is not quasiconformally stable (Theorem 3.2.5.) Therefore, a geometrically finite group may not be quasiconformally stable in dimensions 4 and higher. On the other hand, a convex cocompact Möbius group (i.e, a geometrically finite Möbius group with no parabolic elements) is quasiconformally stable in any dimensions ([14]).

A deformation $\rho : \Gamma \to \mathrm{PSL}(\Gamma_{n-1})$ is said to be a *quasiconformal deformation* if there is a quasiconformal map $\phi : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ so that $\phi\rho(g)\phi^{-1} = g$ for any $g \in \Gamma$. The *quasiconformal deformation space* of Γ, denoted by $\mathcal{QD}(\Gamma)$, is the set of all quasiconformal deformations of Γ where $\mathrm{PSL}(\Gamma_{n-1})$ acts by conjugation. That is

$$\mathcal{QD}(\Gamma) = \{\rho : \Gamma \to \mathrm{PSL}(\Gamma_{n-1}) \text{ a quasiconformal deformation }\}/\mathrm{PSL}(\Gamma_{n-1}), \tag{3.3}$$

where two quasiconformal deformations ρ_1 and ρ_2 are in the same equivalence class in $\mathcal{QD}(\Gamma)$ if and only if $\rho_2(g) = A\rho_1(g)A^{-1}$ for some $A \in \mathrm{PSL}(\Gamma_{n-1})$ and for any $g \in \Gamma$. If ρ is a quasiconformal deformation of Γ, then $A\rho A^{-1}$ is also a quasiconformal deformation of Γ for any $A \in \mathrm{PSL}(\Gamma_{n-1})$. Hence, the quasiconformal deformation space $\mathcal{QD}(\Gamma)$ is a subspace of $\mathcal{D}(\Gamma)$.

Definition 3.1.2 ([15])**.** A Möbius group Γ is *quasiconformally rigid* if any quasiconformal deformation is conjugate to the identity deformation by a Möbius transformation. In other words, $\mathcal{QD}(\Gamma)$ is trivial.

3.2 Quasiconformal rigidity and stability

Definition 3.2.1. A *thrice-punctured sphere group* is a Möbius group generated by two parabolic isometries whose product is a parabolic isometry.

In dimensions 2 and 3, thrice-punctured sphere groups are Fuchsian groups corresponding to the fundamental group of a 2-dimensional thrice-punctured sphere. In fact, they are all conjugate to each other by Möbius transformations. Hence, a thrice-punctured sphere group is quasiconformally rigid in $\mathbb{H}^2$ or $\mathbb{H}^3$.

Lemma 3.2.2. *Let Γ be a thrice-punctured sphere group acting on $\mathbb{H}^4$. Suppose that Γ is generated by two strictly parabolic elements. Then Γ is Fuchsian (see Definition 1.1.1).*

Proof. Suppose that Γ is generated by two strictly parabolic elements, f and g. Without loss of generality, we may assume that f, g and fg fix ∞, 0 and 1 respectively. So, $f = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ for some non-zero vectors u and v in $\mathbb{R}^3$. Since $fg = \begin{pmatrix} 1+uv & u \\ v & 1 \end{pmatrix}$ is a parabolic element fixing 1,

$$uv + u = v. \tag{3.4}$$

Now, we conjugate fg by a Möbius transformation $m : 1 \mapsto \infty$ so that $mfgm^{-1}$ fixes ∞:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1+uv & u \\ v & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+v & u \\ 0 & 1-u \end{pmatrix} \tag{3.5}$$

Since u is a vector, $1+v$ and $1-u$ must be real numbers and $|1+v| = 1$ and

$|1-u| = 1$ (see Theorem 1.3.14). Therefore, $u = 2$ and $v = -2$ and hence

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \right\rangle \tag{3.6}$$

is a Fuchsian thrice-punctured sphere group. □

Theorem 3.2.3. *A thrice-punctured sphere group generated by two strictly parabolic elements is quasiconformally rigid in* $\mathbb{H}^4$.

Proof. Let Γ be a thrice-punctured sphere group generated by two strictly parabolic elements f and g and $\rho : \Gamma \to \mathrm{PSL}(\Gamma_2)$ be a deformation. Then if $\rho(f)$ and $\rho(g)$ are strictly parabolic, $\rho(\Gamma)$ is Fuchsian by Lemma 3.2.2. and hence, ρ is conjugate to *id* by a Möbius transformation. If $\rho(f)$ or $\rho(g)$ is a screw parabolic element, $\rho(\Gamma)$ is not quasiconformally conjugate to Γ since Γ does not have any screw parabolic elements. Thus, ρ is not a quasiconformal deformation and hence the quasiconformal deformation space $\mathcal{QD}(\Gamma)$ is trivial. □

Hyperbolic 4-space has much more flexibility than lower dimensions so that there is a non-trivial deformation of a thrice-punctured sphere group. We like to note that conjugating by Möbius transformations does not affect the presence of screw parabolic elements in a Möbius group (Proposition 2.2.1).

Corollary 3.2.4. *Let* Γ *be a thrice-punctured sphere group generated by two strictly parabolic elements. Then every non-trivial deformation in* $\mathcal{D}(\Gamma)$ *is quasiconformally distinct from the identity deformation in* $\mathbb{H}^4$.

We note that in dimensions 2 and 3, a thrice-punctured sphere group is also trivially quasiconformally stable since the deformation space is one point.

Let Γ_t be a Möbius group of $\text{Möb}(\hat{\mathbb{R}}^3)$ generated by two parabolic elements

$$f_t \;=\; \begin{pmatrix} \alpha & 2\alpha \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \tag{3.7}$$

for $\alpha = \cos \pi t + \sin \pi t (e_1 e_2)$ with $t \in [0, 1)$. Note that Γ_0 is the same as Γ (3.6).

The Möbius group Γ_t has a 4-sided fundamental polyhedron FP in the boundary $\hat{\mathbb{R}}^3$ (see Figure 3.1). Since f_t and g keep $\hat{\mathbb{R}} = \mathbb{R}\text{-axis} \cup \{\infty\}$ invariant, the group Γ_t keeps $\hat{\mathbb{R}}$ invariant, too. Hence, the limit set is contained in $\hat{\mathbb{R}}$ and the set of discontinuity is not empty. Therefore, Γ_t is discrete. The product $f_t g$ of the two generators is a parabolic element fixing 1. Thus, Γ_t is a thrice-punctured sphere group of $\hat{\mathbb{R}}^3$.

For each Γ_t, there is a Γ_t-invariant hyperbolic 2-plane P in $\mathbb{H}^4$ whose boundary at infinity is $\hat{\mathbb{R}}$. On P, the action of Γ_t is the same as the group action of Γ. Hence, the limit set Λ_{Γ_t} is exactly $\hat{\mathbb{R}}$. In $\mathbb{H}^4$, the fundamental domain FD of Γ_t is the four-sided domain whose walls are totally geodesic 3-spaces bounded by the sides of the fundamental polyhedron FP of $\hat{\mathbb{R}}^3$ in Figure 3.1. Therefore, Γ_t is geometrically finite. Here, we have a 1-parameter family $\{ \Gamma_t \mid t \in [0, 1) \}$ of thrice-punctured sphere groups of $\text{Möb}(\hat{\mathbb{R}}^3)$. In this 1-parameter family $\{ \Gamma_t \mid t \in [0, 1) \}$, none of two distinct Γ_t and Γ_s are conjugate to each other by a Möbius transformation by Proposition 2.2.1. Hence, none of two distinct quotient hyperbolic manifolds $\mathbb{H}^4/\Gamma_t$ and $\mathbb{H}^4/\Gamma_s$ are isometric to each other. We can also deform the second generator of the thrice-punctured sphere group.

Theorem 3.2.5. *There is a 2-dimensional parameter space containing the identity deformation in the deformation space of the Fuchsian thrice-punctured sphere group in* $\mathbb{H}^4$ *such that*

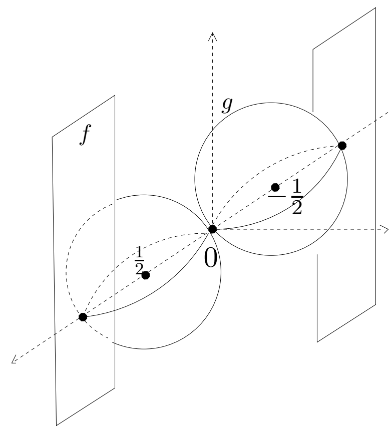

Figure 3.1: The fundamental polyhedron FP of Γ_t in $\hat{\mathbb{R}}^3$

1. *Each non-trivial deformation in the space is not quasiconformally conjugate to the identity.*

2. *The deformations are all quasiconformally distinct except a measure zero set.*

3. *All images of the deformations are geometrically finite.*

4. *The hyperbolic* 4*-manifolds obtained as the quotient of* $\mathbb{H}^4$ *by the images of deformations have the same marked length spectrum.*

5. *There are no simple closed geodesics in their quotient hyperbolic* 4*-manifolds.*

Proof. Let Γ be the Fuchsian thrice-punctured sphere group generated by f and g as in (3.6). For each $t, s \in [0, 1)$, let $\Gamma_{t,s}$ be a group of $\text{Möb}(\hat{\mathbb{R}}^3)$ generated by two parabolic elements

$$f_t = \begin{pmatrix} \alpha & 2\alpha \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad g_s = \begin{pmatrix} \beta & 0 \\ -2\beta & \beta \end{pmatrix} \tag{3.8}$$

where $\alpha = \cos \pi t + \sin \pi t(e_1e_2)$ and $\beta = \cos \pi s + \sin \pi s(e_1e_2)$. We define a representation $\rho_{t,s} : \Gamma \to \mathrm{PSL}(\Gamma_2)$ by $\rho_{t,s}(f) = f_t$ and $\rho_{t,s}(g) = g_s$ as in (3.8). Then each image $\rho_{t,s}(\Gamma) = \Gamma_{t,s}$ is discrete and has an invariant hyperbolic 2-plane P in $\mathbb{H}^4$ whose boundary at infinity is $\hat{\mathbb{R}}$. On P, the group action of $\Gamma_{t,s}$ is the same as the group action of Γ.

Let $\gamma \in \Gamma$ be a parabolic element. Then $\rho_{t,s}(\gamma)$ is not elliptic because it must have infinite order. The fixed point of $\rho_{t,s}(\gamma)$ can only be in $\hat{\mathbb{R}}$ and the action of $\rho_{t,s}(\gamma)$ is the same as γ on $\hat{\mathbb{R}}$. Thus, $\rho_{t,s}(\gamma)$ is parabolic. Similarly, if $\rho_{t,s}(\gamma)$ is parabolic, then γ is parabolic. Hence, $\rho_{t,s}$ is type-preserving. Suppose that $\rho_{t,s}(h)$ is the identity. Since $id = \rho_{t,s}(h)|_{\hat{\mathbb{R}}} = h|_{\hat{\mathbb{R}}}$ and Γ is torsion-free, h is the identity. Thus, $\rho_{t,s}$ is a deformation of Γ.

Therefore, we have a 2-dimensional parameter space of deformations of Γ,

$$\mathcal{P}(\Gamma) = \{\ \rho_{t,s} : \Gamma \to \mathrm{PSL}(\Gamma_2) \mid t, s \in [0, 1)\ \}. \tag{3.9}$$

Note that $\rho_{0,0} = id$. For any non-zero $t, s \in (0, 1)$, each deformation $\rho_{t,s}$ is not quasiconformally conjugate to the identity deformation by Lemma 2.2.5. and Lemma 2.2.7. Hence, Item 1 is verified.

2. For any two distinct irrational numbers t_1 and $t_2 \in (0, 1)$, two deformations $\rho_{t_1,s}$ and $\rho_{t_2,s}$ are not quasiconformally conjugate by Lemma 2.2.11. Hence, $\mathcal{P}(\Gamma)$ has a subfamily $\mathcal{P}' = \{\ \rho_{t,s} \mid t, s :\ \text{irrational numbers in } (0, 1)\ \}$ of all quasiconformally distinct deformations. The complement of $\mathcal{P}'$ has measure zero in the parameter space $\mathcal{P}(\Gamma)$.

3. For any t and $s \in [0, 1)$, the limit set of $\Gamma_{t,s}$ is $\hat{\mathbb{R}}$ and hence the convex core of $\Gamma_{t,s}$ is the hyperbolic plane P whose boundary at infinity is $\hat{\mathbb{R}}$. Thus, $\Gamma_{t,s}$ is geometrically finite

4. and 5. The marked length spectrum for $\rho_{t,s}$ (see (3.1)) is the same

as the marked length spectrum for the identity deformation $\rho_{0,0}$ and hence there is no simple closed geodesics in the quotient hyperbolic 4-manifold $\mathbb{H}^4/\Gamma_{t,s}$. $\square$

Chapter 4

Classification of isometries

4.1 Definitions and Notions

For an n-dimensional Möbius transformation $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_{n-1})$ with non-zero c, the *isometric sphere* I_f of f is the set of points where $|f'(x)| = 1$, i.e., $|cx+d| = 1$. It is an $(n-1)$-dimensional sphere with radius $|c|^{-1}$ centered at $f^{-1}(\infty) = -c^{-1}d$ in $\mathbb{R}^n$.

Every $f \in M\ddot{o}b(\hat{\mathbb{R}}^n)$ with $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$, can be written in the form $f = \psi \circ \tau \circ \sigma$, where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. In fact, $\psi(x) = TAT^{-1}(x)$, $A \in O(n)$ and $T(x) = x + f(\infty)$ for any $x \in \mathbb{R}^n$ (See [20]). We call this *the isometric sphere decomposition* or simply *the decomposition* of f in this paper.

For any $f \in \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}(\mathbb{H}^3)$, the trace has information about the incidence relation between the two isometric spheres I_f and $I_{f^{-1}}$. The incidence relation is enough information to determine the type of a given Möbius transformation (see Lemma 1.3.9.) that is, f is loxodromic if the two isometric spheres I_f and $I_{f^{-1}}$ are disjoint, parabolic if they are tangent, and elliptic if they intersect transversally.

However, in hyperbolic 4-space, the incidence relations of isometric spheres are not conjugacy invariants any more. For example, $f = \begin{pmatrix} e_1 & 0 \\ e_1 + e_1e_2 & -e_1 \end{pmatrix}$ is a screw parabolic fixing 0. The centers of two isometric spheres I_f and $I_{f^{-1}}$ are $f^{-1}(\infty) = \frac{1}{2}(1 - e_2)$ and $f(\infty) = \frac{1}{2}(1 + e_2)$. The distance between two centers is $d(f^{-1}(\infty), f(\infty) = 1$, but the radius of the isometric sphere is $\frac{1}{|e_1+e_1e_2|} = \frac{1}{\sqrt{2}}$. Hence, two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally. However, f is conjugate to $g = \begin{pmatrix} e_1 & 0 \\ e_1e_2 & -e_1 \end{pmatrix}$ by $h = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$. Two isometric spheres $I_g = S(-e_2, 1)$ and $I_{g^{-1}} = S(e_2, 1)$ are tangent to each other.

4.2 Classification of isometries

Here, we provide one way to classify isometries acting on hyperbolic 4-space using the isometric sphere decomposition. We will use the convention that a reflection in a hyperbolic plane is also its reflection plane. From now on, without further mention, we assume that $f \in \mathrm{SL}(\Gamma_2)$ with $f(\infty) \neq \infty$, $f(\infty) \neq f^{-1}(\infty)$ and $f = \psi \circ \tau \circ \sigma$ is the isometric sphere decomposition. We would like to note that parabolic or elliptic fixed points are in the intersection of two isometric spheres in any dimensions ([3]).

Lemma 4.2.1 ([5]). *Let D be an open ball in $\mathbb{R}^n$ and f be a Möbius transformation acting on $\hat{\mathbb{R}}^n$. If $f(\overline{D}) \subset D$, then f is a loxodromic element and* *has a fixed point in $f(\overline{D})$.*

Suppose that two isometric spheres I_f and $I_{f^{-1}}$ are disjoint. Then $f(\overline{D}) = \psi \circ \tau \circ \sigma(\overline{D}) \subsetneq D$, where D is a ball bounded by $I_{f^{-1}}$. Hence f is a loxodromic and has a fixed point in D (Lemma 4.2.1). Now, it suffices to see which type f is when two isometric spheres I_f and $I_{f^{-1}}$ intersect each other.

Suppose that the Euclidean isometry ψ of the isometric sphere decomposition of f is the identity map. Then $f = \tau\sigma$ fixes every point in the intersection of the two isometric spheres I_f and $I_{f^{-1}}$. We conjugate f by a Möbius transformation m which sends an intersection point to a point ∞. Then mfm^{-1} is the composition of two reflections on Euclidean planes $m(\tau)$ and $m(\sigma)$ respectively. If I_f and $I_{f^{-1}}$ are tangent to each other, then $m(\tau)$ and $m(\sigma)$ are parallel and hence mfm^{-1} is a Euclidean translation. If I_f and $I_{f^{-1}}$ intersect each other at more than one point, the intersection of $m(\tau)$ and $m(\sigma)$ is a Euclidean line and hence mfm^{-1} is a Euclidean rotation around the line. Therefore, f is strictly parabolic if the two isometric spheres are tangent to each other and f is boundary elliptic if they intersect transversally at a circle.

From now on, we assume that ψ is not the identity map and the intersection of the two isometric spheres I_f and $I_{f^{-1}}$ is not empty. Then the intersection of the two isometric spheres can be either one point or a circle. First, we will show that if the intersection is exactly one point, f can be either screw parabolic or loxodromic depending on the action of Euclidean isometry ψ.

Lemma 4.2.2. *Let* $f = \begin{pmatrix} \alpha' & \alpha' - \alpha \\ \alpha' & \alpha' \end{pmatrix} \in \mathrm{SL}(\Gamma_2) \leq \mathrm{SL}(\Gamma_3)$ *with a unit* *quaternion $\alpha \in \Gamma_2$, $\alpha \neq \pm 1$ and $\alpha' - \alpha \neq 0$. Then f has no fixed point in the upper half space $\mathbb{H}^4$.*

Proof. Suppose that f has a fixed point $v = x + te_3 \in \mathbb{H}^4$ where $x = x_0 + x_1 e_1 + x_2 e_2 \in \mathbb{R}^3$, t is a positive real number and e_1, e_2 and e_3 generate the Clifford algebra C_3.

$$\begin{aligned}
&f(v) = v \\
&\Leftrightarrow \alpha'(x + te_3) + \alpha' - \alpha = (x + te_3)\alpha'(x + te_3) + (x + te_3)\alpha' \\
&\Leftrightarrow \alpha' x + t\alpha' e_3 + \alpha' - \alpha = x\alpha' x + tx\alpha' e_3 + t\alpha x' e_3 + t^2 e_3 \alpha' e_3 + x\alpha' + t\alpha e_3
\end{aligned}$$

Using (1.14), $t^2 e_3 \alpha' e_3 = t^2 \alpha {e_3}^2 = -t^2 \alpha$. We have

$$\alpha' x + \alpha' - \alpha = x\alpha' x - t^2 \alpha + x\alpha' \tag{4.1}$$

and

$$\begin{aligned}
&t\alpha' e_3 = t(x\alpha' + \alpha x' + \alpha) e_3 \\
&\Rightarrow \alpha' = x\alpha' + \alpha x' + \alpha \quad \text{since } t \neq 0 \\
&\Rightarrow \alpha' - \alpha = x\alpha' + \alpha x'
\end{aligned} \tag{4.2}$$

From (4.2), we have $x \neq 0$ since $\alpha' - \alpha \neq 0$.

$$\begin{aligned}
(4.2) &\Longrightarrow \alpha' - x\alpha' = \alpha + \alpha x' \\
&\Longrightarrow |1 - x| = |x' + 1| \quad \text{since } |\alpha| = 1.
\end{aligned}$$

Therefore, $x_0 = 0$ and hence $x' = -x$. We replace $\alpha' - \alpha$ of equation (4.1) with $x\alpha' + \alpha x'$ to have the following

$$\begin{aligned}
\alpha' x + x\alpha' + \alpha x' &= x\alpha' x - t^2 \alpha + x\alpha' \\
\Leftrightarrow \alpha' x - \alpha x &= (x\alpha' - t^2 \alpha x^{-1}) x
\end{aligned}$$

which implies

$$\begin{aligned}\alpha' - \alpha &= x\alpha' - t^2\alpha x^{-1} \text{ since } x \text{ is invertible} \\ &= x\alpha' - \frac{t^2}{|x|^2}\alpha x' \text{ since } x \neq 0.\end{aligned} \tag{4.3}$$

From (4.2) and (4.3),

$$\begin{aligned} x\alpha' + \alpha x' &= x\alpha' - \frac{t^2}{|x|^2}\alpha x' \\ \Leftrightarrow (1 + \frac{t^2}{|x|^2})\alpha x' &= 0.\end{aligned}$$

Since $1 + \frac{t^2}{|x|^2} \neq 0$ and $\alpha \neq 0$, $x = 0$ which is a contradiction. Therefore, f has no fixed point in $\mathbb{H}^4$. □

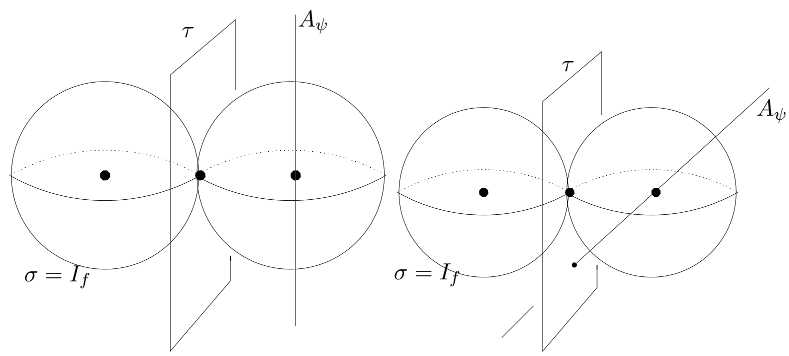

Figure 4.1: $I_f \bigcap I_{f^{-1}} =$ one point : loxodromic

Proposition 4.2.3. *Suppose that I_f and $I_{f^{-1}}$ meet tangentially at a point q in $\mathbb{R}^3$. If ψ does not fix the intersection point q, then f is loxodromic (see Figure 4.1).*

Proof. We may assume that the tangential intersection point q is 0. Then $\tau\sigma$ is strictly parabolic with a fixed point 0. So, it is of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for a non-zero vector $c \in \mathbb{R}^3$. We may conjugate $f = \psi\tau\sigma$ by a dilation so that $c = 1$. Then, the center $f^{-1}(\infty)$ of the isometric sphere I_f is -1 and the center $f(\infty)$ of the isometric sphere $I_{f^{-1}}$ is 1. Since $\psi(x) = TAT^{-1}(x)$ for some $A \in O(3)$ and $T(x) = x + f(\infty)$ for any $x \in \mathbb{R}^3$ in the decomposition of f, we have

$$\psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha' - \alpha \\ 0 & \alpha' \end{pmatrix}$$

for a unit quaternion $\alpha \in \Gamma_2$ with $\alpha \neq \pm 1$. Since the Euclidean isometry ψ does not fix 0 which is the tangential intersection point of the two isometric spheres, $\alpha' - \alpha \neq 0$.

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha' & \alpha' - \alpha \\ \alpha' & \alpha' \end{pmatrix}$$

By Lemma 4.2.2, f has no fixed point in $\mathbb{H}^4$ and hence it has a fixed point $u \in \hat{\mathbb{R}}^3$, i.e., $\alpha' u + \alpha' - \alpha = u(\alpha' u + \alpha')$. The fixed point u cannot be 0 since ψ does not fix 0. We conjugate f by a Möbius transformation which sends u to ∞ to obtain $\widetilde{f}$.

$$\widetilde{f} = \begin{pmatrix} 0 & 1 \\ -1 & u \end{pmatrix} \begin{pmatrix} \alpha' & \alpha' - \alpha \\ \alpha' & \alpha' \end{pmatrix} \begin{pmatrix} u & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha' u + \alpha' & -\alpha' \\ 0 & \alpha' - u\alpha' \end{pmatrix}.$$

Since $\widetilde{f} \in \mathrm{SL}(\Gamma_2)$, $(\alpha' u + \alpha')(\alpha' - u\alpha')^* = 1$. This implies that $|\alpha' u + \alpha'| = |\alpha'||u + 1| = |u + 1|$ and $|\alpha' - u\alpha'| = |1 - u||\alpha'| = |1 - u|$ are simultaneously either 1 or not. However, they cannot be simultaneously 1 since $u \neq 0$. Therefore, $\widetilde{f}$ is loxodromic and so is f (see Lemma 1.3.12). $\square$

Remark 4.2.4. In Proposition 4.2.3, suppose that ψ fixes the tangential intersection point of two isometric spheres i.e. $\psi(0) = 0$. Then $\psi = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}$

for a unit quaternion $\alpha = \cos\theta + \sin\theta e_1 e_2$ and hence

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \alpha' c & \alpha' \end{pmatrix}.$$

Since $\alpha' c$ is not a vector, f is a screw parabolic element (see Theorem 1.3.14).

Now, suppose that two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$. Then the rotational axis A_ψ of ψ intersects τ at a point, say p (p might be ∞). If the intersection point p belongs to the circle C, then f fixes p (See Figure 4.2). In Proposition 4.2.5. We will show that in this case f is screw parabolic When the intersection point p does not belong to the circle C (See Figure 4.3), f is loxodromic or elliptic. That will be the last case.

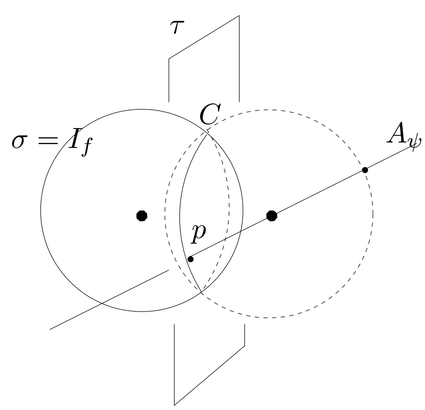

Figure 4.2: $I_f \pitchfork I_{f^{-1}} = C$ and $\tau \cap A_\psi \in C$: Screw parabolic

Proposition 4.2.5. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subseteq \tau \subseteq \mathbb{R}^3$. If ψ fixes a point p in C, then f is screw parabolic (See Figure 4.2).*

Proof. We may assume that $p = 0$. Then $\tau\sigma$ is a boundary elliptic element which fixes every point in C. In particular, $\tau\sigma$ fixes 0, but does not fix ∞. If we conjugate $\tau\sigma$ by $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : 0 \mapsto \infty$, then the conjugation will be a rotation about a line $h(C)$ which does not pass 0. Hence,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix}$$

for a non-zero vector η in $\mathbb{R}^3$ and $\lambda = \cos\theta_\lambda + \sin\theta_\lambda \xi e_1 e_2 \notin \mathbb{R}^3$ for a unit vector ξ in $\mathbb{R}^3$. So we have

$$\tau\sigma = \begin{pmatrix} \lambda' & 0 \\ \lambda\eta - \eta\lambda' & \lambda \end{pmatrix}. \tag{4.4}$$

Note that $\lambda\eta - \eta\lambda'$ is a non-zero vector since $\tau\sigma$ fixes C. If the real part $(\lambda)_{\mathbb{R}} = \cos\theta_\lambda$ of λ is 0, then $\tau\sigma$ has order 2 and $I_{f^{-1}} = I_f$ which means $f(\infty) = f^{-1}(\infty)$. Therefore, $(\lambda)_{\mathbb{R}} \neq 0$.

Let v be a vector $\tau\sigma(\infty) = \lambda'(\lambda\eta - \eta\lambda')^{-1} \in \mathbb{R}^3$ which is the center of $I_{f^{-1}}$. Since ψ fixes 0 and v, it is of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}$, for $\alpha = \cos\theta + \sin\theta \frac{v}{|v|} e_1 e_2$ with $\theta \in (0, 2\pi)$. Therefore, we have

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha\lambda' & 0 \\ \alpha'(\lambda\eta - \eta\lambda') & \alpha'\lambda \end{pmatrix}$$

and we can conjugate f to

$$\begin{pmatrix} \alpha'\lambda & \alpha'(\eta\lambda' - \lambda\eta) \\ 0 & \alpha\lambda' \end{pmatrix} \text{ by } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $|\alpha'\lambda| = 1$ since $|\alpha| = 1 = |\lambda|$. So we know that it is either parabolic or elliptic by Theorem 1.3.12. The $(e_1e_2)^{th}$-coefficient of $\alpha'(\eta\lambda' - \lambda\eta)$ is

$$\begin{aligned}
&\left[\left(\cos\theta + \sin\theta\frac{v'}{|v|}(e_1e_2)'\right)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)} \\
&\quad = \left[\cos\theta(\eta\lambda' - \lambda\eta) - \frac{\sin\theta}{|v|}\lambda(\eta\lambda' - \lambda\eta)'^{-1}(e_1e_2)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)}
\end{aligned}$$

since $\eta\lambda' - \lambda\eta$ is a non-zero vector,

$$\begin{aligned}
&= \left[-\frac{\sin\theta}{|v||\eta\lambda' - \lambda\eta|^2}\lambda(\eta\lambda' - \lambda\eta)^*(e_1e_2)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)} \\
&= \left[-\frac{\sin\theta}{|v|}\lambda(e_1e_2)\right]_{(e_1e_2)}
\end{aligned}$$

since $xe_1e_2x = |x|^2e_1e_2$ for any vector x (See Proposition 1.3.11.),

$$= \frac{\sin\theta}{|v|}(\lambda)_{\mathbb{R}} \neq 0.$$

Therefore, f is screw parabolic. □

Remark 4.2.6. The key of the above proof is $(\lambda)_{\mathbb{R}} \neq 0$. It is also true for higher dimensions. Hence, we can generalize the lemma: in any dimensions $n \geq 3$, if ψ fixes exactly one point in the intersection of the two isometric spheres, then f is screw parabolic.

Ahlfors shows that f is parabolic if and only if f is Möbius conjugate to a matrix of the form $\begin{pmatrix} vc & 0 \\ c & cv \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ with $v \in \mathbb{R}^3$ (see [4]). Then the two isometric spheres are $S(-v, \frac{1}{|c|})$ and $S(v, \frac{1}{|c|})$. The distance between the centers of spheres is $2|v| = \frac{2}{|c|}$ since $|vc| = 1$. Therefore, the two isometric spheres are tangent. We have seen that every parabolic element has a normalized parabolic element in its conjugacy class whose two isometric spheres are tangent. We also have the following corollary.

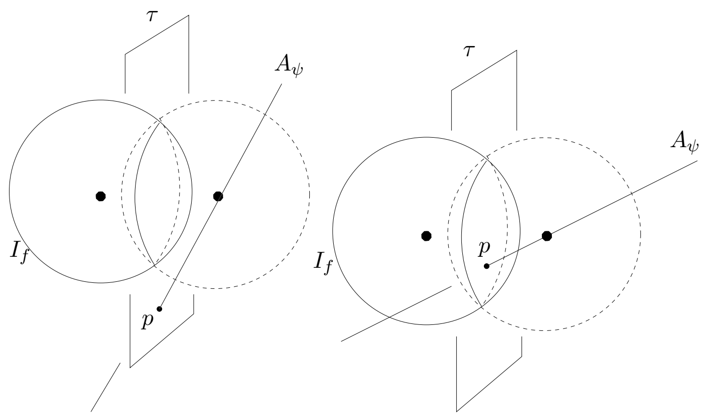

Figure 4.3: Loxodromic or non-boundary elliptic

Corollary 4.2.7. *Let f be a parabolic element. Then f is strictly parabolic if and only if the position of the pair of isometric spheres is tangential for any element of $M\ddot{o}b(\hat{\mathbb{R}}^3)$ conjugate to f.*

The last case is when the intersection point of τ and A_ψ does not belong to the intersection circle of two isometric spheres (see Figure 4.3). In this case, f does not fix any points of the intersection $I_f \cap I_{f^{-1}}$. Hence, f is loxodromic or non-boundary elliptic since a parabolic or elliptic fixed point can only be in the intersection $I_f \cap I_{f^{-1}}$.

Before this case, we will see a characterization of a Kleinian group generated by two Möbius transformations. This characterization will not be generalized into higher dimensions because of the presence of screw parabolic elements.

Lemma 4.2.8 ([1])**.** *Suppose that α, β and $\alpha\beta$ are parabolic, hyperbolic or elliptic Möbius transformations acting on $\mathbb{H}^3$. If α and β do not share a fixed point, then they preserve a common hyperbolic plane in $\mathbb{H}^3$ and the* *group $< \alpha, \beta >$ generated by α and β consists of parabolic, hyperbolic or elliptic elements which preserve this plane.*

Let p be the intersection point of τ and A_ψ. In particular, suppose that the fixed point p is a point ∞. Then, there is a 2-dimensional f-invariant subspace P in $\mathbb{R}^3$ which is perpendicular to A_ψ and passes through the two centers of isometric spheres since the axis A_ψ is parallel to τ. We can think of the restriction of f on P as an element of $M\ddot{o}b(\hat{\mathbb{R}}^2)$. Hence, it is loxodromic because the two isometric circles intersect at two points and it has a non-trivial rotation. This idea can be generalized so that f is loxodromic if p belongs to the exterior of the isometric sphere $Ext(I_f)$.

Proposition 4.2.9. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$ and $\psi\tau$ fixes a point $p \in \mathbb{R}^3$. If $p \in Ext(I_f)$, then f is loxodromic.*

Proof. A Euclidean isometry $\psi\tau$ can be written as a composition of a rotation and a reflection with the same fixed point such that the rotational axis and the reflection plane are orthogonal. Now, without loss of generality we may assume that the axis A_ψ of ψ intersects the plane τ at p in orthogonal.

Since $p \in Ext(I_f)$, there are three possible cases considering the distance from the center of isometric sphere I_f to the plane τ. Let D be the ball bounded by the sphere σ. When τ and σ are disjoint, $\overline{\psi\tau(D)}$ and $\overline{D}$ are disjoint. Hence, $f(\overline{\psi\tau(D)}) \subsetneq \psi\tau(D)$, which means f is loxodromic (Lemma 4.2.1). When τ meet σ tangentially at a point, say q, $\overline{\psi\tau(D)}$ and $\overline{D}$ are disjoint again because ψ does not fixed the point q. Therefore, f is loxodromic as above.

In the last, suppose that τ intersects I_f transversally (see Figure 4.4). When τ passes the center of I_f, then f keeps τ invariant and the two half spaces divided by τ. So, $f|_\tau$ is conjugate to a Möbius transformation of $\hat{\mathbb{R}}^2$. It has a decomposition whose two isometric sphere have two intersection points and non-trivial rotation, therefore $f|_\tau$ is loxodromic and so is f. If τ does not pass the center, then there is also a unique ball B centered at p, whose boundary sphere ∂B is orthogonal to σ since p is in $Ext(I_f)$ (Figure 4.4). Then f keeps $\overline{B}$ invariant, and hence f is conjugate to a Möbius transformation of $\hat{\mathbb{R}}^2$. Since $\psi|_B$ and $\tau\sigma|_B$ are elliptic and there is no common disc preserved by $\psi|_B$ and $\tau\sigma|_B$ whose boundary is a circle in ∂B, $f|_B$ is loxodromic by Lemma 4.2.8. □

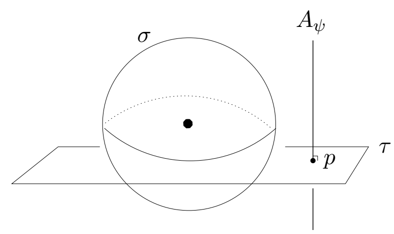

Figure 4.4: $I_f \pitchfork I_{f^{-1}} = C$: Loxodromic

Example 4.2.10. Let C be a unit circle centered at 0 in the $< 1, e_1 >$-plane. An boundary-elliptic element R which fixes every point on C is of the following form:

$$\begin{aligned} R = m\rho_\lambda m^{-1} &= \begin{pmatrix} \frac{1}{2} & e_1 \\ \frac{1}{2}e_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & -e_1 \\ -\frac{1}{2}e_1 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta e_2 \\ -\sin\theta e_2 & \cos\theta \end{pmatrix} \end{aligned} \tag{4.5}$$

where $\lambda = \cos\theta + \sin\theta e_1 e_2$, $\theta \in (0, \pi)$ and m is a Möbius transformation with $m(0) = e_1$, $m(2) = 1$ and $m(\infty) = -e_1$.

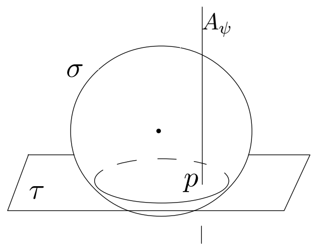

Figure 4.5: $I_f \pitchfork I_{f^{-1}} = C$: Non-boundary elliptic

Proposition 4.2.11. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$ and $\psi\tau$ fixes a point $p \in \mathbb{R}^3$. If $p \in Int(I_f)$, then f is non-boundary elliptic.*

Proof. Without loss of generality we may assume that the axis A_ψ of ψ intersects the plane τ at p in orthogonal and τ is the plane generated by 1 and e_1 (see Figure 4.5). Let C be the circle of the intersection $\tau \cap \sigma$. We may assume that C is the unit circle centered at the origin in the plane $< 1, e_1 >$, and p has a coordinate $(t, 0, 0)$ for $t \in (0, 1)$ on the real axis. The angle between τ and σ is in $(0, \frac{\pi}{2}]$, say θ. Then $\tau\sigma$ is an elliptic element whose fixed point set is the circle C and rotation angle is 2θ. Thus, $\tau\sigma$ is of the form (4.5). Since the axis of ψ is orthogonal to τ and pass through $p = (t, 0, 0)$, ψ is the following form:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & -2t \sin \tau e_1 \\ 0 & \alpha' \end{pmatrix} \tag{4.6}$$

where $\alpha = \cos \tau + \sin \tau e_2(e_1 e_2)$ and $\tau \in (0, \pi)$.

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha & -2t\sin\tau e_1 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta e_2 \\ -\sin\theta e_2 & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\alpha + 2t\sin\theta\sin\tau e_1e_2 & -\sin\theta\alpha e_2 - 2t\cos\theta\sin\tau e_1 \\ -\sin\theta\alpha' e_2 & \cos\theta\alpha' \end{pmatrix}.$$

where $\alpha = \cos\tau + \sin\tau e_1$, $\tau \in (0,\pi)$, $\theta \in (0, \frac{\pi}{2}]$ and $0 < t < 1$.

Suppose that f fixes a point in $\hat{\mathbb{R}}^3$. That is

$$\begin{aligned}(\cos\theta\alpha + 2t\sin\theta\sin\tau e_1e_2)u - \sin\theta\alpha e_2 - 2t\cos\theta\sin\tau e_1 \\ = -\sin\theta u\alpha' e_2 u + \cos\theta u\alpha'\end{aligned} \tag{4.7}$$

for a vector $u = u_0 + u_1e_1 + u_2e_2 \in \mathbb{R}^3$ (clearly, $u \neq \infty$). The e_1e_2-coefficient of (4.7) is

$$2t\sin\theta\sin\tau u_0 - \sin\theta\sin\tau - \sin\theta\sin\tau|u|^2 = 0.$$

Since $\sin\theta \neq 0$ and $\sin\tau \neq 0$ for $\theta \in (0, \frac{\pi}{2}]$ and $\tau \in (0,\pi)$,

$$2tu_0 = 1 + |u|^2$$

which implies $u_0 > 0$ and hence,

$$\begin{aligned}2t = \frac{1 + u_0^2 + u_1^2 + u_2^2}{u_0} &\geq\ u_0 + \frac{1 + u_1^2 + u_2^2}{u_0} \\ &\geq\ 2\sqrt{1 + u_1^2 + u_2^2}\ \geq\ 2.\end{aligned}$$

Thus, $t \geq 1$ which is a contradiction. Therefore, f does not have any fixed point in $\hat{\mathbb{R}}^3$. $\square$

We note that in case $t = 1$ of the above lemma, $u = (1,0,0)$ is the fixed point of f and hence f is parabolic.

Theorem 4.2.12. *$f = \psi\tau\sigma \in M\ddot{o}b(\hat{\mathbb{R}}^3)$ with $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$, where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. Then f is parabolic if and only if ψ fixes a point in $I_f \bigcap I_{f^{-1}}$.*

Corollary 4.2.13. *Let $f = \psi\tau\sigma$ be elliptic. Then f is boundary elliptic if and only if ψ is the identity map.*

We have the following table.

$f = \psi\tau\sigma$, $\psi = id$	
$I_f \bigcap I_{f^{-1}} = \emptyset$	hyperbolic
$I_f \bigcap I_{f^{-1}} =$ one point	strictly parabolic
$I_f \pitchfork I_{f^{-1}} =$ a circle	boundary elliptic

$f = \psi\tau\sigma$, $\psi \neq id$ and $A_\psi \cap \tau = \{p\}$		
$I_f \bigcap I_{f^{-1}} = \emptyset$		loxodromic
$I_f \bigcap I_{f^{-1}} = \{q\}$	$p = q$	screw parabolic
$I_f \bigcap I_{f^{-1}} = \{q\}$	$p \neq q \Leftrightarrow p \in Ext(I_f)$	loxodromic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in C$	screw parabolic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in Ext(I_f)$	loxodromic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in Int(I_f)$	non-boundary elliptic

Table 4.1: Classification of $f \in \mathrm{Isom}(\mathbb{H}^4)$ with $f(\infty) \neq \infty$.

Bibliography

[1] C. Adams, J. Haas and P. Scott, *Simple closed geodesic in hyperbolic 3-manifolds*, Bull. London Math. Soc. 31(1999) 81-86.

[2] L. V. Ahlfors, *Möbius transformations and Clifford numbers*, Differential Geometry and Complex Analysis, H.E. Rauch memorial volume, Springer-Verlag, Belin, (1985), 65–73.

[3] L. V. Ahlfors, *Old and new in Möbius groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 93–105.

[4] L. V. Ahlfors, *On the fixed points of Möbius transformation in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 15–27.

[5] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983. xii+337 pp.

[6] R. Benedetti and C Petronio, *Lectures on hyperbolic geometry.* Universitext, Springer-Verlag, Berlin, 1992.

[7] B. H. Bowditch, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. 113 (1993), no. 2, 245–317.

[8] R. D. Canary, *Introductory Bumponomics: the topology of deformation spaces of hyperbolic 3-manifolds*, preliminary version.

[9] R. D. Canary and D. McCullough, *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups*, Mem. Amer. Math. Soc. 172 (2004), no. 812.

[10] C. Cao and P. L. Waterman, *Conjugacy invariants of Möbius groups*, Quasiconformal Mappings and Analysis, Springer, (1995), 109–139.

[11] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups I*, Proc. London Math. Soc. (3) 55 (1987), no. 2, 331–358.

[12] L. S. Husch, *A homotopy theoretic characterization of the translation in E^n*, Compositio Math. 24 (1972), 55–61.

[13] M. Kapovich, *Kleinian groups in higher dimensions*, Progress in Math, Vol. 265, 485-562, Birkhäuser.

[14] H. Izeki, *Quasiconformal stability of Kleinian groups and an embedding of a space of flat conformal structure*, Conform. Geom. Dyn. 4 (2000), 108–119 (electronic).

[15] C. McMullen, *Renormalization and* 3*-manifolds with fiber over the circle*, Annals of Mathematics Studies 142, Princeton University Press, Princeton, NJ, 1996. x+253 pp.

[16] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) 99 (1974), 383–462.

[17] B. Maskit, *Kleinian groups*, Springer-Verlag, Berlin, 1988.

[18] G. D. Mostow, *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Publ. I.H.E.S. 34 (1968), 53-104.

[19] G. Prasad, *Strong rigidity of $\mathbb{Q}$-rank 1 lattices*, Invent. Math. 21 (1973), 255-286.

[20] J. Ratcliffe, *Foundation of hyperbolic manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[21] P. Tukia, *Quasiconformal extension of quasisymmetric mappings compatible with a Möius group*, Acta Math. 154 (1985), no. 3-4, 153–193.

[22] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, Vol. 229, Springer-Verlag, Berlin-New York, 1971. xiv+144 pp.

[23] K. Th. Vahlen, *Über Bewegungen und complexe Zahlen*, Math. Ann. 55, 1902, 585-593.

[24] M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Mathematics, 1319. Springer-Verlag, Berlin, 1988.

[25] P. L. Waterman, *Möbius transformation in several dimensions*, Advances in Math. 101 (1993), 87-113.