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Splitting of Vector Bundles on Punctured Spectrum of Regular Local Rings

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by

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Abstract

Splitting of Vector Bundles on Punctured Spectrum of Regular Local Rings

by

Mahdi Majidi-Zolbanin

Advisor: Professor Lucien Szpiro

In this dissertation we study splitting of vector bundles of small rank on punctured spectrum of regular local rings. We give a splitting criterion for vector bundles of small rank in terms of vanishing of their intermediate cohomology modules $H^i(U, \mathscr{E})_{2 \leq i \leq n-3}$, where n is the dimension of the regular local ring. This is the local analog of a result by N. Mohan Kumar, C. Peterson, and A. Prabhakar Rao for splitting of vector bundles of small rank on projective spaces.

As an application we give a positive answer (in a special case) to a conjecture of R. Hartshorne asserting that certain quotients of regular local rings have to be complete intersections. More precisely we prove that if $(R, \mathfrak{m})$ is a regular local ring of dimension at least five, $\mathfrak{p}$ is a prime ideal of codimension two, and the ring $\Gamma(V, \widetilde{R/\mathfrak{p}})$ is Gorenstein, where V is the open set $\operatorname{Spec}(R/\mathfrak{p}) - \{\mathfrak{m}\}$, then $R/\mathfrak{p}$ is a complete intersection.

To My Wife Leila

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Without my wife Leila, these long years would feel much longer. I am indebted to her for her continuous support and encouragement. This work is dedicated to her.

The problem and its history: an epitome

Let k be an algebraically closed field. One of the interesting problems in the area of vector bundles on projective spaces $\mathbf{P}^n_k$, and punctured spectrum of regular local rings, is the question of existence of indecomposable vector bundles of small rank on these spaces. Although such bundles exist in lower dimensions, they seem to become extremely rare as the dimension increases, especially in characteristic zero. In Table 1, we have listed some known indecomposable vector bundles of small rank on $\mathbf{P}^n_k$ for $n \geq 4$. Even though this list may not be complete, a complete list would not be much longer!

As the table suggests, for $n \geq 6$ it is not known whether there exist any indecomposable vector bundles $\mathscr{E}$ on $\mathbf{P}^n_k$ with $2 \leq \operatorname{rank} \mathscr{E} \leq n-2$. Also in characteristic zero there are no known examples of indecomposable vector bundles of rank two on $\mathbf{P}^5_k$. For vector bundles of rank two, there is the following

Conjecture 0.1 (R. Hartshorne)**.** [Har74, p. 1030] *If $n \geq 7$, there are no non-split vector bundles of rank 2 on $\mathbf{P}^n_k$.*

It is well-known that vanishing of certain (intermediate) cohomology modules of a vector bundle on projective space or punctured spectrum of a regular local ring, forces that bundle to split. Here are two important splitting criteria of this type:

Theorem 0.1 (G. Horrocks)**.** *If $\mathscr{E}$ is a vector bundle on $\mathbf{P}^n_k$, then $\mathscr{E}$ splits, if and only*

Table 1: Some indecomposable vector bundles of small rank on $\mathbf{P}_k^n$, $n \geq 4$.

Constructed by	**Rank**	**Base Space**	**Characteristic**	**Reference**
G. Horrocks, D. Mumford	2	$\mathbf{P}_{\mathbb{C}}^4$	0	[HM73]
G. Horrocks	2	$\mathbf{P}^4$	2	[Hor80]
H. Tango	2	$\mathbf{P}^5$	2	[Tan76b]
N. Mohan Kumar	2	$\mathbf{P}^4$	$p > 0$	[Kum97]
N. Mohan Kumar C. Peterson A. P. Rao	2 2 3 3	$\mathbf{P}^5$ $\mathbf{P}^4$ $\mathbf{P}^5$ $\mathbf{P}^4$	2 $p > 0$ $p > 0$ all	[KPR02]
H. Abo, W. Decker N. Sasakura	3	$\mathbf{P}^4$	0	[ADS98]
G. Horrocks	3	$\mathbf{P}^5$	$0,\ p > 2$	[Hor78]
H. Tango	$n-1$	$\mathbf{P}^n,\ n \geq 3$	all	[Tan76a]
U. Vetter	$n-1$	$\mathbf{P}^n,\ n \geq 3$	all	[Vet73]

if $H_*^i(\mathscr{E}) = 0$*, for* $1 \leq i \leq n-1$*.*

Theorem 0.2 (E. G. Evans, P. Griffith)**.** [EG81, p. 331, Theorem 2.4] *If* $\mathscr{E}$ *is a vector bundle on* $\mathbf{P}_k^n$ *of rank* $k < n$*, then* $\mathscr{E}$ *splits, if and only if* $H_*^i(\mathscr{E}) = 0$*, for* $1 \leq i \leq k-1$*.*

More recently, a new splitting criterion of this type was proved for vector bundles of small rank on $\mathbf{P}_k^n$ by N. Mohan Kumar, C. Peterson, and A. Prabhakar Rao [KPR03, p. 185, Theorem 1]. For the statement of this result see Theorem II.2. In Theorem II.1, we extend this criterion to vector bundles of small rank on punctured spectrum of regular local rings. The proof is an adapted version of the original proof, that suits the setting of local rings. Chapter II is devoted to the proof of this theorem.

Another interesting question, related to the vector bundle problem, is the question of existence of non-singular subvarieties of $\mathbf{P}_k^n$ (k algebraically closed) of small codimension, which are not complete intersections. Here is the precise statement of this problem:

Conjecture 0.2 (R. Hartshorne)**.** [Har74, p. 1017] *If Y is a nonsingular subvariety of dimension r of $\mathbf{P}_k^n$, and if $r > \frac{2}{3}n$, then Y is a complete intersection.*

Conjecture 0.2 has also a local version, which is in fact stronger than the original conjecture. To see the motivation behind the local conjecture, let Y be a non-singular subvariety of dimension r in $\mathbf{P}_k^n$, defined by a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring $S := k[X_0, \cdots, X_n]$. Let R be the local ring of the origin in $\mathbf{A}_k^{n+1}$, that is, $R = S_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal $(X_0, \cdots, X_n)$ in S. Then Y is a complete intersection in $\mathbf{P}_k^n$, if and only if $R/\mathfrak{p}R$ is a complete intersection in R. Furthermore, R is a regular local ring of dimension $n+1$, the quotient ring $R/\mathfrak{p}R$ has an isolated singularity, and $\dim R/\mathfrak{p}R = r+1$. The inequality $r > \frac{2}{3}n$ of conjecture 0.2 can be rewritten in terms of $\dim R/\mathfrak{p}R$ and $\dim R$, as $\dim R/\mathfrak{p}R - 1 > \frac{2}{3}(\dim R - 1)$, or after simplifying, $\dim R/\mathfrak{p}R > \frac{1}{3}(2\dim R + 1)$. The local version of Conjecture 0.2 is the following:

Conjecture 0.3 (R. Hartshorne)**.** [Har74, p. 1027] *Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R = n$, let $\mathfrak{p} \subset R$ be a prime ideal, such that $R/\mathfrak{p}$ has an isolated singularity, let $r = \dim R/\mathfrak{p}$, and suppose that $r > \frac{1}{3}(2n+1)$. Then $R/\mathfrak{p}$ is a complete intersection.*

There are some affirmative answers to Conjecture 0.3, obtained under additional hypotheses on $\operatorname{depth}(\mathfrak{m}, R/\mathfrak{p})$. We mention some of these results in codimension 2, i.e., when $\dim R/\mathfrak{p} = \dim R - 2$:

Theorem 0.3 (C. Peskine, L. Szpiro)**.** [PS74, p. 294, Theorem 5.2] *Let R be a regular local ring with $\dim R \geq 7$. Let $R/\mathfrak{a}$ be a quotient of codimension 2 of R, which is Cohen-Macaulay, and locally a complete intersection except at the closed point. Then $R/\mathfrak{a}$ is a complete intersection.*

Theorem 0.4 (R. Hartshorne, A. Ogus)**.** [HO74, p. 431, Corollary 3.4] *Let* $(R, \mathfrak{m})$ *be a regular local ring containing its residue field* k *of characteristic* 0*. Let* $\mathfrak{p} \subset R$ *be a prime ideal of codimension* 2 *in* R*, such that* $R/\mathfrak{p}$ *is locally a complete intersection except at the closed point. Assume that* $n = \dim R \geq 7$*, and* $\operatorname{depth}(\mathfrak{m}, R/\mathfrak{p}) > \frac{1}{2}(r+1)$*, where* $r = \dim R/\mathfrak{p}$*. Then* $R/\mathfrak{p}$ *is a complete intersection.*

Theorem 0.5 (E. G. Evans, P. Griffith)**.** [EG81, p. 331, Theorem 2.3] *Let* R *be a regular local ring with* $\dim R \geq 7$*, that contains a field of characteristic zero. Let* $\mathfrak{p}$ *be a prime ideal of codimension* 2 *in* R*, such that* $R/\mathfrak{p}$ *is normal, and has an isolated singularity. Then* $R/\mathfrak{p}$ *is a complete intersection.*

In Theorem III.13, we give an affirmative answer to Conjecture 0.3 in codimension two, under the extra assumption that the ring of sections of $R/\mathfrak{p}$ over the open set $\operatorname{Spec}(R/\mathfrak{p}) - \{\mathfrak{m}\}$ is Gorenstein. The proof of this theorem is based on Theorem II.1, and the method of Serre-Horrocks (Theorem I.21) for constructing vector bundles of rank 2. Chapter III is devoted to proving this theorem.

There are many other interesting problems in the area of vector bundles on projective spaces. We refer the interested reader to [Har79] and [Sch85].

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Chapter I

Splitting of vector bundles: Old results

This chapter contains general definitions and facts about *vector bundles* on *projective spaces* and *punctured spectrum of regular local rings.* In Section I.1 after introducing these spaces, we show thar there is a surjective affine morphism from the punctured spectrum of the local ring of affine $(n+1)-$space at origin onto the projective $n-$space. This result allows us to pass from local to global. Section I.2 contains Horrocks' splitting criterion for vector bundles on punctured spectrum of regular local rings. Also, we prove an important duality theorem for cohomology of such bundles. In Section I.3 we will study the scheme of zeros of a nonzero section of a vector bundle, and introduce the Koszul complex associated with such a section. Finally, in Section I.4 we will present a method for constructing rank two vector bundles, which is due to J.-P. Serre and G. Horrocks.

I.1 Punctured spectrums and projective spaces

We begin with a few definitions:

Definition I.1. *Let* $S := k[X_0, \cdots, X_n]$ *be the polynomial ring in* $n+1$ *variables over a field* k*. The* $n-$*dimensional projective space over* k*, denoted by* $\mathbf{P}^n_k$*, is the scheme* $\mathrm{Proj}(S)$.

Definition I.2. *Let $(R, \mathfrak{m})$ be a local ring, and let U be the open subset* $\mathrm{Spec}(R) - \{\mathfrak{m}\}$ *of* $\mathrm{Spec}(R)$*. Then, the open subscheme $(U, \mathscr{O}_U)$ of* $\mathrm{Spec}(R)$ *is called the punctured spectrum of R.*

Definition I.3. *Let $(X, \mathscr{O}_X)$ be either the projective space $\mathbf{P}^n_k$ over an algebraically closed field k, or the punctured spectrum of a regular local ring. An* algebraic vector bundle *$\mathscr{E}$ on X is a locally free coherent sheaf of $\mathscr{O}_X-$modules. We say that a vector bundle* splits *or is* split*, if it is isomorphic to a (finite) direct sum of line bundles on X. We say that a vector bundle is* decomposable*, if it can be written as a direct sum of vector bundles of smaller rank. Finally, a vector bundle $\mathscr{E}$ over X is said to be of* small rank*, if* $2 \le \operatorname{rank} \mathscr{E} \le \dim X - 1$.

Remark I.4. Projective spaces and punctured spectrum of regular local rings are connected spaces; thus the rank of a vector bundle on these spaces is well-defined.

The next proposition enables one to pass from local to global results:

Proposition I.5. *Let $S := k[X_0, \cdots, X_n]$ be the polynomial ring in $n+1$ variables over a field k. Denote the maximal ideal $(X_0, \cdots, X_n)$ of S by $\mathfrak{m}$, and let $(U, \mathscr{O}_U)$ be the punctured spectrum of the regular local ring $S_{\mathfrak{m}}$. Then, there is a surjective morphism $\pi : U \to \mathbf{P}^n_k$. Moreover π is an* affine *morphism.*

Proof. To give a morphism $\pi : U \to \mathbf{P}^n_k$ is equivalent to give a line bundle $\mathscr{L}$ on U together with global sections $s_0, s_1, \cdots, s_n \in \Gamma(U, \mathscr{L})$ that generate $\mathscr{L}$ at each point of U [Har77, p. 150, Theorem 7.1]. Take $\mathscr{O}_U$ as $\mathscr{L}$. Notice that if $n \ge 1$, this is our only possible choice, because then $S_{\mathfrak{m}}$ is a regular local ring of dimension ≥ 2, and $\mathrm{Pic}(U)$ is trivial [Gro68, Exposé XI, Corollary 3.10]. By Propositions A.2 and A.3,

$\Gamma(U, \mathscr{O}_U) \cong S_{\mathfrak{m}}$ (when $n = 0$, $S_{\mathfrak{m}} \hookrightarrow \Gamma(U, \mathscr{O}_U)$), and we can take $\frac{X_0}{1}, \frac{X_1}{1}, \cdots, \frac{X_n}{1} \in S_{\mathfrak{m}}$ as our global sections; they clearly generate $\mathscr{O}_U$ at each point. Thus, we obtain a morphism $\pi : U \to \mathbf{P}^n_k$.

Looking at construction of the morphism π [Har77, p. 150, proof of Theorem 7.1], one can see that if $x_{\mathfrak{p}} \in U$ is a point corresponding to a prime ideal $\mathfrak{p} \in \operatorname{Spec}(S_{\mathfrak{m}}) - \{\mathfrak{m}S_{\mathfrak{m}}\}$, then $\pi(x_{\mathfrak{p}}) = \oplus_{d\geq 0} (\mathfrak{p} \cap S_d)$, the homogeneous prime ideal associated to $\mathfrak{p}$. This shows that π is surjective. It also shows that $\pi^{-1}\big(D_+(X_i)\big) = D(\frac{X_i}{1})$, $0 \leq i \leq n$. Thus, since $\big(D_+(X_i)\big)_i$ is a covering of $\mathbf{P}^n_k$ with affine open subsets, π is an affine morphism. $\square$

Corollary I.6. *Let $S := k[X_0, \cdots, X_n]$ be the polynomial ring in $n + 1$ variables over a field k. Denote the maximal ideal $(X_0, \cdots, X_n)$ of S by $\mathfrak{m}$, and let $(U, \mathscr{O}_U)$ be the punctured spectrum of the regular local ring $S_{\mathfrak{m}}$. Then, for every graded $S-$module M there are isomorphisms*

$$\bigoplus_{d\in\mathbb{Z}} H^i\big(\mathbf{P}^n_k, \widetilde{M}(d)\big) \xrightarrow{\sim} H^i(U, \widetilde{M_{\mathfrak{m}}}|_U), \quad \text{for } i \geq 0.$$

I.2 Horrocks' splitting criterion

This section provides some important facts about vector bundles on punctured spectrum of regular local rings. In particular, we present a splitting criterion (Theorem I.8) due to G. Horrocks, and a duality theorem (Corollary I.11), which will be extensively used in chapter II. Most of the results of this section can be found in [Hor78], although the proofs may be different. We have not included the analogous results for vector bundles on projective spaces, in this section. For these results we refer the reader to [OSS80].

Theorem I.7. *Let $(R, \mathfrak{m})$ be a Noetherian local domain with* $\operatorname{depth}(\mathfrak{m}, R) \geq 2$*. Denote the punctured spectrum of R by $(U, \mathscr{O}_U)$, and let $\mathscr{E}$ be a vector bundle on U. Let E be the $R-$module $\Gamma(U, \mathscr{E}) = \Gamma(X, \imath_*\mathscr{E})$, where $(X, \mathscr{O}_X)$ is the affine scheme* $\operatorname{Spec}(R)$*, and $\imath$ is the inclusion morphism $U \hookrightarrow X$. Then*

(i) *E is a finite $R-$module,*

(ii) *$\imath_*\mathscr{E} \cong \widetilde{E}$; in particular, $\imath_*\mathscr{E}$ is a coherent $\mathscr{O}_X-$module,*

(iii) $\operatorname{depth}(\mathfrak{m}, E) \geq 2$*,*

(iv) *For every point $x \in U$, E_x is free.*

Proof. (i): R is Noetherian and $\mathscr{E}$ is a coherent $\mathscr{O}_U-$module. Under these conditions, there is a coherent $\mathscr{O}_X-$module $\mathscr{E}'$ such that $\mathscr{E}'|_U = \mathscr{E}$ [Gro60, p. 93, corollary I,1.5.3]. Write E' for $\Gamma(X, \mathscr{E}')$, and $T(E')$ for the torsion submodule of E'. Then E' is a finite $R-$module. At each point $x \in U$, $E'_x \cong \mathscr{E}'_x \cong \mathscr{E}_x$ is free. Thus, $\operatorname{Supp} T(E') \subseteq \{\mathfrak{m}\}$, and

$$\mathscr{E} \cong \mathscr{E}'|_U \cong \widetilde{E'}|_U \cong \widetilde{E'/T(E')}|_U.$$

Since R is a domain, and $E'/T(E')$ is torsion-free and finite, it is a submodule of a free module R^d. Hence $\mathscr{E} \cong \widetilde{E'/T(E')}|_U$ is a subsheaf of $\bigoplus_{i=1}^d \mathscr{O}_U$. Taking global sections, one sees that E is a submodule of $\Gamma(U, \bigoplus_{i=1}^d \mathscr{O}_U)$. But $\Gamma(U, \bigoplus_{i=1}^d \mathscr{O}_U) \cong R^d$, because $\operatorname{depth}(\mathfrak{m}, R) \geq 2$ by assumption. Therefore E is a submodule of R^d, hence finite, since R is Noetherian.

(ii): $(X, \mathscr{O}_X)$ is a Noetherian scheme. Thus, every (open or closed) subscheme of X is Noetherian [Gro60, p.141, Proposition I,6.1.4]. In particular $(U, \mathscr{O}_U)$ is a Noetherian scheme. Hence, $\imath_*\mathscr{E}$ is a quasi-coherent $\mathscr{O}_X-$module [Har74, p. 115, Proposition 5.8].

Since $\Gamma(X, \imath_*\mathscr{E}) = E$, one has $\imath_*\mathscr{E} \cong \widetilde{E}$. In addition, since R is Noetherian, and E is a finite $R-$module, $\widetilde{E}$ is coherent [Har77, p. 113, Corollary 5.5].

(iii): R is Noetherian, and E is a finite $R-$module. The restriction homomorphism $\Gamma(X, \imath_*\mathscr{E}) \to \Gamma(U, \imath_*\mathscr{E})$ is bijective. Since $\widetilde{E} \cong \imath_*\mathscr{E}$, the restriction homomorphism $E \cong \Gamma(X, \widetilde{E}) \to \Gamma(U, \widetilde{E})$ is also bijective. From Propositions A.2 and A.3 it follows that $\operatorname{depth}(\mathfrak{m}, E) \geq 2$. (See also [Gro68, p. 37, Exposé III, Example 3.4].)

(iv): This follows from the isomorphism $\imath_*\mathscr{E} \cong \widetilde{E}$ of (ii). □

Theorem I.8 (Horrocks)**.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$. Let $(U, \mathscr{O}_U)$ be the punctured spectrum of R, and let $\mathscr{E}$ be a vector bundle on U. Then $\mathscr{E}$ splits if and only if $H^i(U, \mathscr{E}) = 0$ for $1 \leq i \leq n-2$.*

Proof. Let $E := \Gamma(U, \mathscr{E})$. Suppose $\mathscr{E}$ is split, say, $\mathscr{E} \cong \bigoplus_{i=1}^r \mathscr{O}_U$, where $r = \operatorname{rank} \mathscr{E}$. Since $\operatorname{depth}(\mathfrak{m}, R) = n \geq 2$, one has $E \cong R^r$, and by Proposition A.3, $H^i_{\mathfrak{m}}(\widetilde{E}) = 0$ for $0 \leq i \leq n-1$. The result follows from the fact (Proposition A.2) that for $i \geq 1$

$$H^i(U, \mathscr{E}) \cong H^{i+1}_{\mathfrak{m}}(\widetilde{E}).$$

Conversely, suppose that $H^i(U, \mathscr{E}) = 0$ for $1 \leq i \leq n-2$. Then

$$H^i_{\mathfrak{m}}(\widetilde{E}) = 0, \quad \text{for} \quad 2 \leq i \leq n-1.$$

By Theorem I.7 (iii), $H^0_{\mathfrak{m}}(\widetilde{E})$ and $H^1_{\mathfrak{m}}(\widetilde{E})$ also vanish. Thus, by Proposition A.3, one has $\operatorname{depth}(\mathfrak{m}, E) = n$. From Auslander-Buchsbaum formula [Mat89, p. 155, Theorem 19.1]

$$\operatorname{proj.dim}_R E + \operatorname{depth}(\mathfrak{m}, E) = \operatorname{depth}(\mathfrak{m}, R),$$

we see that the projective dimension of E is zero, i.e., E is free. Notice that since R is regular, E has finite projective dimension. □

Theorem I.9. [Hor78, p. 697] *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$. Let $(U, \mathscr{O}_U)$ be the punctured spectrum of R, and let $\mathscr{E}$ be a vector bundle on U. Also, let $E := \Gamma(U, \mathscr{E})$. Then*

$$H^i(U, \mathscr{E}) \cong \operatorname{Ext}^i_R(E^\vee, R) \quad (0 \leq i < n-1),$$

where we write $E^\vee$ for $\operatorname{Hom}_R(E, R)$.

Corollary I.10. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$. Let $(U, \mathscr{O}_U)$ be the punctured spectrum of R, and let $\mathscr{E}$ be a vector bundle on U. Then the cohomology modules $H^i(U, \mathscr{E})$ are $R-$modules of finite length for $0 < i < n-1$.*

Proof. Let $E := \Gamma(U, \mathscr{E})$. The modules $\operatorname{Ext}^i_R(E^\vee, R)$ are finite for all i. For $i > 0$ they are also supported on $\{\mathfrak{m}\}$. □

Corollary I.11 (Duality Theorem)**.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$, let $(U, \mathscr{O}_U)$ be its punctured spectrum, and let $\mathscr{E}$ be a vector bundle on U. Let $D(\,\cdot\,)$ be a dualizing functor for R, as discussed in Appendix A. Then*

$$H^i(U, \mathscr{E}) \cong D(H^{n-i-1}(U, \mathscr{E}^\vee)) \quad (1 \leq i \leq n-1),$$

where we write $\mathscr{E}^\vee$ for $\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}, \mathscr{O}_U)$.

Proof. This follows from Theorems I.9 and A.13. More precisely, for $i \geq 1$ one has

$$H^i(U, \mathscr{E}) \cong H^i(U, \widetilde{E}|_U) \cong H^{i+1}_{\mathfrak{m}}(\widetilde{E}) \cong D(\operatorname{Ext}^{n-i-1}_R(E, R)),$$

where $E = \Gamma(U, \mathscr{E})$. Now since by Theorem I.9, $E^{\vee\vee} \cong E$,

$$D(\operatorname{Ext}^{n-i-1}_R(E, R)) \cong D(\operatorname{Ext}^{n-i-1}_R(E^{\vee\vee}, R)) \cong D(H^{n-i-1}(U, \mathscr{E}^\vee)),$$

the last isomorphism being valid for $0 \leq n-i-1 \leq n-2$, i.e., $1 \leq i \leq n-1$. □

Proposition I.12. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 3$, let $(U, \mathscr{O}_U)$ be its punctured spectrum, and let $\mathscr{E}$ be a vector bundle on U. Let $x \in \mathfrak{m} - \mathfrak{m}^2$ be a parameter element. Let A be the quotient ring R/xR, and denote the punctured spectrum of A by $(V, \mathscr{O}_V)$. Also, let $\jmath$ be the closed immersion $V \hookrightarrow U$. Then*

(i) *$\jmath^*\mathscr{E}$ is a vector bundle on V with $\operatorname{rank} \jmath^*\mathscr{E} = \operatorname{rank} \mathscr{E}$;*

(ii) *$\mathscr{E}$ splits if and only if $\jmath^*\mathscr{E}$ is split.*

Proof. (i): By [Gro60, p. 141, Proposition I,6.1.4], $(U, \mathscr{O}_U)$ and $(V, \mathscr{O}_V)$ are Noetherian schemes. Also, $\mathscr{E}$ is a coherent $\mathscr{O}_U-$module. Thus $\jmath^*\mathscr{E}$ is a coherent $\mathscr{O}_V-$module (see [Har77, p. 115, Proposition 5.8].)To show that $\jmath^*\mathscr{E}$ is locally free, let $z \in V$ be a point, and let r be the rank of $\mathscr{E}$. Then

$$(\jmath^*\mathscr{E})_z = (\jmath^{-1}\mathscr{E} \otimes_{\jmath^{-1}\mathscr{O}_U} \mathscr{O}_V)_z \cong \mathscr{E}_z \otimes_{\mathscr{O}_{U,z}} \mathscr{O}_{V,z} \cong \mathscr{O}^r_{U,z} \otimes_{\mathscr{O}_{U,z}} \mathscr{O}_{V,z} \cong \mathscr{O}^r_{V,z},$$

which is what we wanted to show.

(ii): Since $\mathscr{E}$ is locally free and of finite rank, one has an isomorphism [Gro60, p. 52, 0,5.4.10]

$$\jmath_*(\mathscr{O}_V \otimes_{\mathscr{O}_V} \jmath^*\mathscr{E}) \cong \jmath_*\mathscr{O}_V \otimes_{\mathscr{O}_U} \mathscr{E}.$$

Thus, $\jmath_*\jmath^*\mathscr{E} \cong \jmath_*\mathscr{O}_V \otimes_{\mathscr{O}_U} \mathscr{E}$. Let $\mathscr{I}$ be the ideal sheaf $\imath^{-1}(\widetilde{xR})$ of $\mathscr{O}_U$, where $\imath$ is the open immersion $U \hookrightarrow \operatorname{Spec}(R)$. Then $\mathscr{I}$ is the ideal sheaf defining V as a closed subscheme of U, and one can verify that $\jmath_*\mathscr{O}_V \cong \mathscr{O}_U/\mathscr{I}$. Thus $\jmath_*\jmath^*\mathscr{E} \cong \mathscr{E}/x\mathscr{E}$. Let E be equal to $\Gamma(U, \mathscr{E})$. Multiplication by x gives a *complex*

$$0 \longrightarrow E \xrightarrow{x} E \longrightarrow E/xE \longrightarrow 0,$$

which is exact over U, because E is locally free over U by Theorem I.7, and x is an $R-$regular element. Thus, sheafifying this complex, and restricting it to U, one obtains a short exact sequence of $\mathscr{O}_U-$modules

$$0 \longrightarrow \mathscr{E} \xrightarrow{x} \mathscr{E} \longrightarrow \mathscr{E}/x\mathscr{E} \longrightarrow 0. \tag{I.1}$$

Consider the long exact sequence of cohomology associated with (I.1):

$$\cdots \longrightarrow H^i(U,\mathscr{E}) \xrightarrow{x} H^i(U,\mathscr{E}) \longrightarrow H^i(U,\mathscr{E}/x\mathscr{E}) \longrightarrow H^{i+1}(U,\mathscr{E}) \xrightarrow{x} \cdots$$

Now, every closed immersion is an affine morphism [Gro61a, p. 14, Proposition II,1.6.2]. Hence, for all $i \geq 0$ there are isomorphisms [Gro61b, p. 88, Corollary III,1.3.3]

$$H^i(U,\mathscr{E}/x\mathscr{E}) \cong H^i(U,\jmath_*\jmath^*\mathscr{E}) \xrightarrow{\sim} H^i(V,\jmath^*\mathscr{E}),$$

and we obtain a long exact sequence relating the cohomology of $\mathscr{E}$ over U to the cohomology of $\jmath^*\mathscr{E}$ over V:

$$\cdots \longrightarrow H^i(U,\mathscr{E}) \xrightarrow{x} H^i(U,\mathscr{E}) \longrightarrow H^i(V,\jmath^*\mathscr{E}) \longrightarrow H^{i+1}(U,\mathscr{E}) \xrightarrow{x} \cdots \tag{I.2}$$

Now suppose $\mathscr{E}$ is split. Then by Theorem I.8, $H^i(U,\mathscr{E}) = 0$ for $1 \leq i \leq n-2$. Using the above exact sequence we see that $H^i(V,\jmath^*\mathscr{E}) = 0$ for $1 \leq i \leq n-3$, which means that $\jmath^*\mathscr{E}$ is split, because $\dim A = n-1$.

Conversely, assume that $\jmath^*\mathscr{E}$ is split. Then $H^i(V,\jmath^*\mathscr{E}) = 0$ for $1 \leq i \leq n-3$. Therefore the maps $H^i(U,\mathscr{E}) \xrightarrow{x} H^i(U,\mathscr{E})$ are bijective for $1 < i < n-2$, injective for $i = n-2$, and surjective for $i = 1$. By Corollary I.10, the $R-$modules $H^i(U,\mathscr{E})$ are of finite length for $1 \leq i \leq n-2$. Hence they are annihilated by a power of $\mathfrak{m}$, say t. In particular x^t annihilates $H^i(U,\mathscr{E})$ for $1 \leq i \leq n-2$, which is absurd, unless $H^i(U,\mathscr{E}) = 0$, because multiplication by x^t must be injective for $1 < i \leq n-2$, and surjective for $i = 1$. Thus, $H^i(U,\mathscr{E}) = 0$ for $1 \leq i \leq n-2$, that is, $\mathscr{E}$ is split. $\square$

I.3 Scheme of zeros of a section

In this section k will be an algebraically closed field. Let $(X, \mathscr{O}_X)$ be either the projective space $\mathbf{P}^n_k$, or the punctured spectrum $(U, \mathscr{O}_U)$ of a regular local ring $(R, \mathfrak{m})$ of dimension $n+1$. Assume that $n \geq 3$. Let $\mathscr{E}$ be a vector bundle of rank $r < n$ on X. Suppose that $\mathscr{E}$ has a nonzero global section $\sigma \in \Gamma(X, \mathscr{E})$. Then one can define a nonzero morphism

$$\mathscr{O}_X \longrightarrow \mathscr{E},$$

by sending $1 \in \Gamma(X, \mathscr{O}_X)$ to σ. After dualizing, one obtains a morphism

$$\mathscr{E}^\vee \xrightarrow{u} \mathscr{O}_X. \tag{I.3}$$

The image of u is an ideal sheaf $\mathscr{I}$ of $\mathscr{O}_X$. Let $Y := \operatorname{Supp}(\mathscr{O}_X/\mathscr{I})$ and also, let $\mathscr{O}_Y := (\mathscr{O}_X/\mathscr{I})|_Y$. Then $(Y, \mathscr{O}_Y)$ is a closed subscheme of X, which is called *the scheme of zeros of* σ. There is also an exact sequence

$$\mathscr{E}^\vee \xrightarrow{u} \mathscr{O}_X \longrightarrow \jmath_*\mathscr{O}_Y \longrightarrow 0,$$

where $\jmath : Y \hookrightarrow X$ is the corresponding closed immersion.

Theorem I.13 (Koszul complex)**.** *Let $(X, \mathscr{O}_X)$ be either the projective space $\mathbf{P}^n_k$, or the punctured spectrum $(U, \mathscr{O}_U)$ of a regular local ring $(R, \mathfrak{m})$ of dimension $n+1$. Assume that $n \geq 3$. Let $\mathscr{E}$ be a vector bundle of rank $r < n$ on X. Suppose that $\mathscr{E}$ has a nonzero section $\sigma \in \Gamma(X, \mathscr{E})$, whose scheme of zeros Y is equidimensional, and of codimension r in X. Let $\jmath : Y \hookrightarrow X$ be the corresponding closed immersion. Then there is an exact sequence*

$$0 \longrightarrow \textstyle\bigwedge^r \mathscr{E}^\vee \xrightarrow{d} \bigwedge^{r-1} \mathscr{E}^\vee \xrightarrow{d} \cdots \longrightarrow \bigwedge^1 \mathscr{E}^\vee \longrightarrow \mathscr{O}_X \longrightarrow \jmath_*\mathscr{O}_Y \longrightarrow 0. \tag{I.4}$$

Proof. Let $\mathscr{E}^\vee \xrightarrow{u} \mathscr{O}_X$ be the map (I.3) obtained from σ, which we described earlier. The complex (I.4) is the Koszul complex associated with $\mathscr{E}^\vee$ and u as defined in Definition C.1 of Appendix C. We recall, that over each open subset U of X the differential map d is defined as in (C.1):

$$d(x_1 \wedge \cdots \wedge x_t) = \sum_{i=1}^{t} (-1)^{i+1} u|_U(x_i)\ x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_t,$$

where $x_1, \cdots, x_t \in \Gamma(U, \mathscr{E}^\vee)$. Also, by Remark C.2 the cokernel of the map u is $\jmath_* \mathscr{O}_Y$. To show that this complex is exact, it suffices to check that it is locally exact. So, let x be a point in X. If $x \notin Y$, then $\mathscr{O}_{Y,x} = 0$, and the map $u_x : \mathscr{E}_x^\vee \to \mathscr{O}_{X,x}$ is surjective. The result in this case follows from Proposition C.3. Suppose, on the other hand, that $x \in Y$. Let $\{e_1, \cdots, e_r\}$ be a basis of $\mathscr{E}_x^\vee$, and let $x_i := u_x(e_i)$. Then, denoting the ideal subsheaf $u(\mathscr{E}^\vee)$ of $\mathscr{O}_X$, defining Y as a closed subscheme of X by $\mathscr{I}$, the elements $(x_1, \cdots, x_r)$ generate $\mathscr{I}_x$. Since by assumption Y is equidimensional, and of codimension r in X, $\mathscr{I}_x$ has to be of codimension r in $\mathscr{O}_{X,x}$. The local ring $\mathscr{O}_{X,x}$ is regular. Thus, the elements $(x_1, \cdots, x_r)$ form an $\mathscr{O}_{X,x}-$regular sequence [Mat89, p. 135, Theorem 17.4], and the result follows from Proposition C.4. $\square$

In the rest of this section we will restrict our attention to vector bundles on projective spaces $\mathbf{P}_k^n$. Let $\mathscr{E}$ be such a bundle. Assume that $r := \operatorname{rank} \mathscr{E} < n$. It is known that for $d \gg 0$ the scheme of zeros Y of a *general* section σ of $\mathscr{E}(d)$ is *regular*, *irreducible*, and of *codimension* r in $\mathbf{P}_k^n$. We briefly sketch the proof of this statement.

The fact that such a Y is regular and of codimension r follows from Theorem I.14 and Proposition I.15 below:

Theorem I.14 (Kleiman)**.** [Kle69, p. 293, Corollary 3.6] *Let k be an infinite field, X a pure $n-$dimensional algebraic $k-$scheme equipped with an ample sheaf $\mathscr{O}_X(1)$, and $\mathscr{E}$*

a vector bundle of rank r on X. Let σ be a general section of $\mathscr{E}(d)$ for some $d \gg 0$. If $n < r$, then σ has no zeros; if X is smooth, then σ meets the zero section transversally; that is, the scheme of zeros Y of σ is smooth and of codimension r.

Proposition I.15. [Gro03, p. 44, Proposition II,5.4] *Let Y be a scheme of finite type over a field k. If Y is smooth over k, then Y is regular. The converse is true if k is perfect.*

To show that Y is irreducible, we first show it is connected:

Proposition I.16. *Let $\mathscr{E}$ be a vector bundle of rank $r < n$ on $\mathbf{P}^n_k$. Then, for $d \gg 0$ the scheme of zeros Y of a* general *section σ of $\mathscr{E}(d)$ is connected.*

Proof. Let $\mathscr{E}$ be any locally free sheaf on $\mathbf{P}^n_k$. By Serre's theorem, $H^i\big(\mathbf{P}^n_k, \mathscr{E}(d)\big) = 0$ for $d \gg 0$ and $i > 0$ [Gro61b, p. 100, Proposition III,2.2.2]. Since $\mathscr{E}^\vee$ is locally free, by Serre's Duality theorem [Har77, p. 244, Corollary 7.7]

$$H^{n-i}\big(\mathbf{P}^n_k, \mathscr{E}^\vee(-d)\big) \cong H^i\big(\mathbf{P}^n_k, \mathscr{E}(d-n-1)\big)'.$$

Thus, for $d \gg 0$ and $q < n$, $H^q\big(\mathbf{P}^n_k, \mathscr{E}^\vee(-d)\big) = 0$.

Now, let Y be the scheme of zeros of a *general* section σ of $\mathscr{E}(d)$, with $\operatorname{rank} \mathscr{E} < n$. Then by Theorem I.14, Y is smooth and of codimension r, and thus the Koszul complex is exact:

$$0 \to \mathbf{\Lambda}^r\left(\mathscr{E}^\vee(-d)\right) \xrightarrow{d_r} \mathbf{\Lambda}^{r-1}\left(\mathscr{E}^\vee(-d)\right) \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_2} \mathbf{\Lambda}^1\left(\mathscr{E}^\vee(-d)\right) \xrightarrow{u} \mathscr{O}_{\mathbf{P}^n} \to \jmath_*\mathscr{O}_Y \to 0,$$

where $\jmath : Y \hookrightarrow \mathbf{P}^n_k$ is the corresponding closed immersion. Since $\mathscr{E}^\vee$ is locally free,

$$\mathbf{\Lambda}^p\left(\mathscr{E}^\vee(-d)\right) \cong \mathbf{\Lambda}^p\left(\mathscr{E}^\vee \otimes \mathscr{O}_{\mathbf{P}^n}(-d)\right) \cong \mathbf{\Lambda}^p\,\mathscr{E}^\vee \otimes \mathscr{O}_{\mathbf{P}^n}(-pd) \cong (\mathbf{\Lambda}^p\,\mathscr{E}^\vee)(-pd).$$

Thus, by taking d to be large enough, for every $q < n$ and $0 \leq p \leq r$ one can simultaneously achieve:

$$H^q\left(\mathbf{P}_k^n, \mathbf{\Lambda}^p(\mathscr{E}^\vee(-d))\right) = 0. \tag{I.5}$$

Let $\mathscr{G}_{r-1}$ be the image of the map d_{r-1} in the Koszul complex, and consider the short exact sequence

$$0 \to \mathbf{\Lambda}^r\left(\mathscr{E}^\vee(-d)\right) \xrightarrow{d_r} \mathbf{\Lambda}^{r-1}\left(\mathscr{E}^\vee(-d)\right) \xrightarrow{d_{r-1}} \mathscr{G}_{r-1} \longrightarrow 0.$$

Writing the long exact sequence of cohomology for this sequence, and using (I.5), one sees that for $d \gg 0$, and $q < n-1$, $H^q(\mathbf{P}_k^n, \mathscr{G}_{r-1}) = 0$. By further breaking the Koszul complex down into short exact sequences, and proceeding as above, one finally gets to the sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_{\mathbf{P}^n} \longrightarrow \jmath_*\mathscr{O}_Y \longrightarrow 0,$$

where $\mathscr{I}_Y$ is the image of $u : \mathbf{\Lambda}^1\left(\mathscr{E}^\vee(-d)\right) \longrightarrow \mathscr{O}_{\mathbf{P}^n}$. From one step before this exact sequence one knows that for $d \gg 0$ and $q < n-r+1$, $H^q(\mathbf{P}_k^n, \mathscr{I}_Y) = 0$. In particular, since $r < n$, $H^1(\mathbf{P}_k^n, \mathscr{I}_Y) = 0$. Now, from the long exact sequence of cohomology

$$0 \longrightarrow H^0(\mathbf{P}_k^n, \mathscr{I}_Y) \longrightarrow H^0(\mathbf{P}_k^n, \mathscr{O}_{\mathbf{P}^n}) \xrightarrow{\pi} H^0(Y, \mathscr{O}_Y) \longrightarrow H^1(\mathbf{P}_k^n, \mathscr{I}_Y) \longrightarrow \cdots$$

one sees that π is surjective. But $H^0(\mathbf{P}_k^n, \mathscr{O}_{\mathbf{P}^n}) \cong k$, and $H^0(Y, \mathscr{O}_Y)$ is a finite dimensional $k-$vector space. Therefore $H^0(Y, \mathscr{O}_Y) = k$, showing that Y is connected. □

Proposition I.17. [Gro60, p. 143, Corollary I,6.1.11] *Let $(Y, \mathscr{O}_Y)$ be a locally Noetherian scheme. Then Y is irreducible, if and only if Y is connected and nonempty, and for all $y \in Y$,* $\mathrm{Spec}(\mathscr{O}_{Y,y})$ *is irreducible.*

Corollary I.18. *Let $\mathscr{E}$ be a vector bundle of rank r on $\mathbf{P}_k^n$ with $r < n$. Then for $d \gg 0$ the scheme of zeros Y of a general section of $\mathscr{E}(d)$ is irreducible.*

Proof. Let Y be the scheme of zeros of a general section of $\mathscr{E}(d)$ for $d \gg 0$. Then by Proposition I.16, Y is connected. By Theorem I.14, for every $y \in Y$ the local ring $\mathscr{O}_{Y,y}$ is regular. Thus, $\operatorname{Spec}(\mathscr{O}_{Y,y})$ is irreducible. The result follows from Proposition I.17. □

I.4 Vector bundles of rank two: method of Serre-Horrocks

Lemma I.19. [Ser60, p. 28, Lemma 9] *Let R be a Noetherian ring, and let M be an $R-$module. Assume that* $\operatorname{proj.dim}_R M \leq 1$, *and* $\operatorname{Ext}^1_R(M, R) = 0$. *Then M is projective.*

Proof. We reduce to the case where R is a local ring. Since $\operatorname{proj.dim}_R M \leq 1$, there is an exact sequence

$$0 \longrightarrow R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0.$$

Since $\operatorname{Ext}^1_R(M, R^s) = 0$, the above extension has to be split exact. Hence, M is a direct factor of R^t and thus is projective. □

Proposition I.20. [Ser60, p. 31, Proposition 5] *Let R be a regular local ring, and let $\mathfrak{a}$ be an ideal of codimension two. The following two conditions are equivalent:*

(i) *$R/\mathfrak{a}$ is a complete intersection,*

(ii) *$R/\mathfrak{a}$ is Gorenstein.*

Theorem I.21 (Serre-Horrocks)**.** [Szp79, p. 63] *Let X be a regular Noetherian connected scheme. Let Y be a closed subscheme of X, which is equidimensional, Cohen-Macaulay, and of codimension two, and let $\jmath : Y \hookrightarrow X$ be the corresponding closed immersion. Let $\mathscr{I}$ be the ideal sheaf of Y in X. Suppose there are two line bundles ω_X and $\mathscr{L}$ on X with following properties*

(a) *If we write* ω_Y *for* $\mathscr{E}xt^2_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \omega_X)$, *then* $\omega_Y \cong \mathscr{L} \otimes_{\mathscr{O}_X} \jmath_*\mathscr{O}_Y$,

(b) $H^2(X, \mathscr{L}^\vee \otimes \omega_X) = 0$.

Then Y *is the scheme of zeros of a section of a rank two vector bundle* $\mathscr{E}$ *on* X, *and*

$$\textstyle\bigwedge^2 \mathscr{E} \otimes_{\mathscr{O}_X} \omega_X \cong \mathscr{L}.$$

Proof. If Y is the scheme of zeros of a section of a rank two vector bundle $\mathscr{E}$, then the Koszul complex

$$0 \longrightarrow \textstyle\bigwedge^2 \mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{O}_X \longrightarrow \jmath_*\mathscr{O}_Y \longrightarrow 0$$

is exact by Theorem I.13. Thus, one has an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 \mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{I} \longrightarrow 0.$$

But we want $\bigwedge^2 \mathscr{E}^\vee$ to be isomorphic to $\mathscr{L}^\vee \otimes \omega_X$.

Thus, the problem is to find an extension of $\mathscr{I}$ by $\mathscr{L}^\vee \otimes \omega_X$, which is a vector bundle, that is we want an element of $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$, which gives a locally free extension. We use the lower term sequence of the spectral sequence

$$E_2^{pq} = H^p\big(X, \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big) \Rightarrow E^{p+q} = \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X),$$

that is the following exact sequence

$$0 \longrightarrow H^1\big(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big) \longrightarrow \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X) \longrightarrow$$

$$H^0\big(X, \mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big) \longrightarrow H^2\big(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big).$$

By assumption Y is equidimensional, Cohen-Macaulay, and of codimension two, and X is regular. Whence for every point $y \in Y$, one has $\dim \mathscr{O}_{X,y} - \dim \mathscr{O}_{Y,y} = 2$, and we have $\mathscr{E}xt^i_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \mathscr{L}^\vee \otimes \omega_X) = 0$ for $i = 0, 1$, because its support is empty. Now, applying the functor $\mathscr{H}om_{\mathscr{O}_X}(\,\cdot\,, \mathscr{L}^\vee \otimes \omega_X)$ to the exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \longrightarrow \jmath_*\mathscr{O}_Y \longrightarrow 0,$$

we see that

$$\mathscr{L}^\vee \otimes \omega_X \cong \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X),$$

and

$$\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X) \cong \mathscr{E}xt^2_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \mathscr{L}^\vee \otimes \omega_X).$$

One also has [Har77, p. 235, Proposition 6.7]

$$\mathscr{E}xt^2_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \mathscr{L}^\vee \otimes \omega_X) \cong \mathscr{E}xt^2_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \omega_X) \otimes \mathscr{L}^\vee.$$

Hence, $\mathscr{E}xt^2_{\mathscr{O}_X}(\jmath_*\mathscr{O}_Y, \mathscr{L}^\vee \otimes \omega_X) \cong \omega_Y \otimes \mathscr{L}^\vee \cong \jmath_*\mathscr{O}_Y$, where the last isomorphism is by assumption (a). Thus, $\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$ is a monogenic $\mathscr{O}_X-$module, that is there is a section σ of $\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$ over X (corresponding to $1 \in \Gamma(X, \jmath_*\mathscr{O}_Y)$), that generates it at every point of X.

On the other hand $H^2\big(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big) \cong H^2(X, \mathscr{L}^\vee \otimes \omega_X) = 0$, where the vanishing is by assumption (b). Hence, the map

$$\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X) \longrightarrow H^0\big(X, \mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)\big)$$

is surjective. So there is an element of $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$, which maps onto σ. Let $\mathscr{E}^\vee$ be the extension corresponding to this element. We will show that $\mathscr{E}^\vee$ is locally free.

We prove the following: Let σ be a section of $\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$ coming from $\mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{I}, \mathscr{L}^\vee \otimes \omega_X)$. Then the corresponding extension $\mathscr{G}$ is free at a point $x \in X$, if and only if the image of σ in $\operatorname{Ext}^1_{\mathscr{O}_{X,x}}\left(\mathscr{I}_x, (\mathscr{L}^\vee \otimes \omega_X)_x\right)$ generates it as an $\mathscr{O}_{X,x}-$module. This will prove our assertion that $\mathscr{E}^\vee$ is locally free.

Assume $\mathscr{G}$ is free at x. Then we have an exact sequence

$$0 \longrightarrow (\mathscr{L}^\vee \otimes \omega_X)_x \cong \mathscr{O}_{X,x} \longrightarrow \mathscr{G}_x \cong \bigoplus_1^n \mathscr{O}_{X,x} \longrightarrow \mathscr{I}_x \longrightarrow 0,$$

which gives rise to an exact sequence

$$\mathscr{O}_{X,x} = \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}, \mathscr{O}_{X,x}) \longrightarrow \operatorname{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{I}_x, \mathscr{O}_{X,x}) \longrightarrow 0.$$

Since image of σ corresponds to image of $1 \in \mathscr{O}_{X,x}$ in $\operatorname{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{I}_x, \mathscr{O}_{X,x})$, we see that image of σ generates it. Conversely, suppose the image of σ generates $\operatorname{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{I}_x, \mathscr{O}_{X,x})$, and let $\mathscr{G}_x$ be the extension corresponding to this generator. From the exact sequence

$$0 \longrightarrow \mathscr{O}_{X,x} \longrightarrow \mathscr{G}_x \longrightarrow \mathscr{I}_x \longrightarrow 0$$

we see that the assumption implies $\operatorname{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{G}_x, \mathscr{O}_{X,x}) = 0$. If $x \notin Y$, then $\mathscr{I}_x \cong \mathscr{O}_{X,x}$ and from the above exact sequence we see that $\operatorname{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{G}_x = 0$, hence $\mathscr{G}_x$ is free.

If $x \in Y$, then since X is regular, $\mathscr{O}_{Y,x}$ has finite projective dimension as a module over $\mathscr{O}_{X,x}$, and since Y is Cohen-Macaulay, $\mathscr{I}_x$ is a *perfect ideal*, that is projective dimension of $\mathscr{O}_{Y,x}$ is equal to grade of $\mathscr{I}_x$. Since Y is in addition equidimensional and of codimension two, we have $\dim \mathscr{O}_{X,x} - \dim \mathscr{O}_{Y,x} = 2$. Therefore grade of $\mathscr{I}_x$ is two, projective dimension of $\mathscr{I}_x$ is one, and from the above exact sequence we see $\operatorname{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{G}_x \leq 1$. Thus by Lemma I.19, $\mathscr{G}_x$ is projective and hence free.

So $\mathscr{E}^\vee$ is locally free. Since X is connected by assumption, $\mathscr{E}^\vee$ has constant rank. Localizing the extension

$$0 \longrightarrow \mathscr{L}^\vee \otimes \omega_X \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{I} \longrightarrow 0$$

at a point $x \notin Y$, we see that $\operatorname{rank} \mathscr{E}^\vee = 2$. We have an exact sequence

$$0 \longrightarrow \mathscr{L}^\vee \otimes \omega_X \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{O}_X \longrightarrow \jmath_* \mathscr{O}_Y \longrightarrow 0.$$

The fact that the Koszul complex

$$0 \longrightarrow \mathbf{\Lambda}^2 \mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{O}_X \longrightarrow \jmath_* \mathscr{O}_Y \longrightarrow 0$$

is also exact implies that $\mathbf{\Lambda}^2 \mathscr{E}^\vee \cong \mathscr{L}^\vee \otimes \omega_X$. So Y is the scheme of zeros of a section of the rank two vector bundle $\mathscr{E}$ with $\mathbf{\Lambda}^2 \mathscr{E} \otimes_{\mathscr{O}_X} \omega_X \cong \mathscr{L}$. □

Corollary I.22. *Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R \geq 4$. Let $(U, \mathscr{O}_U)$ be the punctured spectrum of R. Let $(V, \mathscr{O}_V)$ be a closed subscheme of U, which is equidimensional, Cohen-Macaulay, and of codimension two, and let $\jmath : V \hookrightarrow U$ be the corresponding closed immersion. Assume in addition, that*

$$\mathscr{E}xt^2_{\mathscr{O}_U}(\jmath_* \mathscr{O}_V, \mathscr{O}_U) \cong \jmath_* \mathscr{O}_V. \tag{I.6}$$

Then V is the scheme of zeros of a section of a rank two vector bundle on U.

Proof. We verify that the assumptions of Theorem I.21 hold: $(U, \mathscr{O}_U)$ is a regular scheme. It is a Noetherian scheme [Gro60, Proposition I-6.1.4], and it is connected by Hartshorne's theorem [Gro67, p. 46, Corollary 3.9]. Take both ω_X and $\mathscr{L}$ to be $\mathscr{O}_U$. Notice that since R is a regular local ring of dimension ≥ 4, it is parafactorial [Gro68, Exposé XI, Corollary 3.10], hence $\operatorname{Pic}(U)$ is trivial.

By (I.6) above, condition (a) of the theorem holds. Condition (b) holds because

$$H^2(U, \mathscr{O}_U) \cong H^3_{\mathfrak{m}}(\widetilde{R}) = 0.$$

Hence the corollary follows from Theorem I.21. □

Chapter II

Splitting of vector bundles: New criterion

Our goal in this chapter is to give a detailed proof of the following splitting criterion for vector bundles of small rank on punctured spectrum of regular local rings

Theorem II.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 5$. Let $\mathscr{E}$ be a vector bundle on $(U, \mathscr{O}_U)$, the punctured spectrum of R. If n is odd and if* $\operatorname{rank}(\mathscr{E}) < n - 1$, *then $\mathscr{E}$ splits if and only if $H^i(U, \mathscr{E}) = 0$ for $2 \leq i \leq n - 3$. If n is even and if* $\operatorname{rank}(\mathscr{E}) < n - 2$, *then $\mathscr{E}$ splits if and only if $H^i(U, \mathscr{E}) = 0$ for $2 \leq i \leq n - 3$.*

This theorem is the local version of the following splitting criterion for vector bundles of small rank on $\mathbf{P}^n_k$, $n \geq 4$

Theorem II.2. [KPR03, p. 185, Theorem 1] *Let $\mathscr{E}$ be a vector bundle on $\mathbf{P}^n_k$. If n is even and if* $\operatorname{rank}(\mathscr{E}) < n$, *then $\mathscr{E}$ splits if and only if $H^i_*(\mathscr{E}) = 0$ for $2 \leq i \leq n-2$. If n is odd and if* $\operatorname{rank}(\mathscr{E}) < n-1$, *then $\mathscr{E}$ splits if and only if $H^i_*(\mathscr{E}) = 0$ for $2 \leq i \leq n-2$.*

The proof of Theorem II.1 is essentially the same as the original proof of Theorem II.2, except for the necessary modifications to adapt it to local ring setting.

The presentation of materials in this chapter is in the following order: Section II.1 contains two important examples of exact sequences involving symmetric powers, that one can associate to any given short exact sequence of (locally) free modules. The exact sequence constructed in Example II.5 will be used in proof of Theorem II.1. In Section II.2 we define the notion of a *monad*, and present the proof of a theorem due to Horrocks [Hor80, p. 199], asserting that every vector bundle on punctured spectrum of a regular local ring can be expressed as the homology of a monad with certain nice properties. Finally, Section II.3 is devoted to the proof of Theorem II.1.

II.1 Variations on the Koszul complex

In this section we will present two important examples of exact sequences, which can be derived from Koszul complex (Appendix C). One of these sequences (Example II.5) will play a crucial role in the proof of Theorem II.1, which is one of our main results.

Let L and M be two $R-$modules. To simplify notation, let $B := \mathbf{S}_R\, M$. Then B is a commutative $R-$algebra with unity. A given $R-$module homomorphism $u : L \longrightarrow M$ induces a $B-$liner map $\overline{u} : B \otimes_R L \longrightarrow B$ defined as $\overline{u}(b \otimes x) = u(x) \cdot b$, where we identify M with $\mathbf{S}^1_R\, M$. The Koszul complex $\mathbf{K}^B(\overline{u})$, as defined in Definition C.1 is the following

$$\cdots \to \mathbf{\Lambda}^r_B(B \otimes_R L) \xrightarrow{d_{\overline{u}}} \mathbf{\Lambda}^{r-1}_B(B \otimes_R L) \longrightarrow \cdots \longrightarrow \mathbf{\Lambda}^1_B(B \otimes_R L) \xrightarrow{d_{\overline{u}}} \mathbf{\Lambda}^0_B(B \otimes_R L) \to 0.$$

Let $\imath_L : L \longrightarrow \mathbf{\Lambda}_R\, L$ be the canonical injection. The canonical extension of the $B-$linear map $1_B \otimes \imath_L : B \otimes_R L \longrightarrow B \otimes_R \mathbf{\Lambda}_R\, L$ gives rise to a graded $B-$algebra isomorphism [Bou70, p. 83, Proposition 8]

$$\mathbf{\Lambda}_B(B \otimes_R L) \cong B \otimes_R \mathbf{\Lambda}_R\, L.$$

Using this isomorphism and the complex $\mathbf{K}^B(\overline{u})$, one obtains an isomorphic complex

$$\cdots \longrightarrow B \otimes_R \mathbf{\Lambda}^r_R\, L \xrightarrow{d} B \otimes_R \mathbf{\Lambda}^{r-1}_R\, L \longrightarrow \cdots \longrightarrow B \otimes_R \mathbf{\Lambda}^1_R\, L \xrightarrow{d} B \otimes_R \mathbf{\Lambda}^0_R\, L \longrightarrow 0 \quad \text{(II.1)}$$

where

$$d(b\otimes(x_1\wedge\cdots\wedge x_r))=\sum_{i=1}^{r}(-1)^{i+1}(u(x_i)\cdot b)\otimes(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_r). \quad \text{(II.2)}$$

Since d maps $\mathbf{S}_R^p\, M\otimes_R \mathbf{\Lambda}_R^q\, L$ into $\mathbf{S}_R^{p+1}\, M\otimes_R \mathbf{\Lambda}_R^{q-1}\, L$, this complex (*as a complex of $R-$modules*) decomposes into a direct sum of complexes of following type

$$0\longrightarrow \mathbf{S}_R^0\, M\otimes_R \mathbf{\Lambda}_R^r\, L\longrightarrow \mathbf{S}_R^1\, M\otimes_R \mathbf{\Lambda}_R^{r-1}\, L\longrightarrow\cdots\longrightarrow \mathbf{S}_R^r\, M\otimes_R \mathbf{\Lambda}_R^0\, L\longrightarrow 0,\quad r\in\mathbb{N}.$$

Now, consider a short exact sequence of *free* $R-$modules

$$0\longrightarrow L\xrightarrow{u} M\xrightarrow{v} P\longrightarrow 0. \quad \text{(II.3)}$$

Then, one can associate two *exact complexes* to this sequence. We will describe them in the next two examples:

Example II.3. Let $r_0=\operatorname{rank} L$ and $n_0=\operatorname{rank} M$. Since P is free, the short exact sequence (II.3) splits, that is, there is a section

$$\begin{tikzcd} M \arrow[r, "v"] & P \arrow[l, bend left, "s"] \end{tikzcd}$$

such that $v\circ s=\mathrm{id}_P$. Thus, $P\cong s(P)$, $M=u(L)\oplus s(P)$, and one can pick a basis $\{e_1,\cdots,e_{r_0},e_{r_0+1},\cdots,e_{n_0}\}$ for M, such that $\{e_1,\cdots,e_{r_0}\}$ is a basis for $u(L)$ and $\{e_{r_0+1},\cdots,e_{n_0}\}$ is a basis for $s(P)$. Since M is free, there is an $R-$algebra isomorphism $B(=\mathbf{S}_R\, M)\xrightarrow{\sim} R[X_1,\cdots,X_{n_0}]$, with $e_i\mapsto X_i$ for $1\le i\le n_0$. It is now clear, that the Koszul complex $\mathbf{K}^B(\overline{u})$ can be identified with the ordinary Koszul complex associated to a free module of rank r_0 over the polynomial ring $R[X_1,\cdots,X_{n_0}]$ and the sequence of elements $X_1,\cdots,X_{r_0}$. Since these elements form a regular sequence in the polynomial ring, the complex $\mathbf{K}^B(\overline{u})$ is acyclic, that is, $H_i\big(\mathbf{K}^B(\overline{u})\big) = 0$ for $i > 0$. By Remark C.2, $H_0(\mathbf{K}^B(\overline{u})) = B/\overline{u}(B \otimes_R L)$. We claim that $B/\overline{u}(B \otimes_R L) \cong \mathbf{S}_R\, P$. This follows from the following proposition, and the fact that $\mathrm{Ker}(v) = L$.

Proposition II.4. [Bou70, p. 69, Proposition 4] *If $v : M \longrightarrow P$ is a surjective $R-$linear mapping, then the homomorphism $\mathbf{S}(v) : \mathbf{S}_R\, M \longrightarrow \mathbf{S}_R\, P$ is surjective and its kernel is the ideal of $\mathbf{S}_R\, M$ generated by* $\mathrm{Ker}(v) \subset M \subset \mathbf{S}_R\, M$.

Thus, we have shown that given a short exact sequence (II.3) of free $R-$modules, and any integer $1 \le r \le \mathrm{rank}\, L$, there is an exact sequence of $R-$modules

$$0 \longrightarrow \mathbf{S}^0_R\, M \otimes_R \mathbf{\Lambda}^r_R\, L \xrightarrow{d} \mathbf{S}^1_R\, M \otimes_R \mathbf{\Lambda}^{r-1}_R\, L \xrightarrow{d} \cdots \longrightarrow \mathbf{S}^r_R\, M \otimes_R \mathbf{\Lambda}^0_R\, L \longrightarrow \mathbf{S}^r_R\, P \longrightarrow 0,$$

where the differential d is given by (II.2).

Example II.5. Dualizing the split exact sequence (II.3), we get an exact sequence

$$0 \longrightarrow P^{\vee} \xrightarrow{v^{\vee}} M^{\vee} \xrightarrow{u^{\vee}} L^{\vee} \longrightarrow 0.$$

Let $B := \mathbf{S}_R\, L^{\vee}$ (notice the change in our notation compared with example II.3.) In this example we want to compute the homology of the Koszul complex $\mathbf{K}^B(\overline{u^{\vee}})$. We will see that although these homology modules do not vanish, "their nonzero parts lie in the lowest degree" in a sense that will be clarified later. This property allows us to establish the exactness of an important class of complexes associated to the sequence (II.3).

We have $M^{\vee} = s(L^{\vee}) \oplus v^{\vee}(P^{\vee})$, where s is a section

$$\begin{tikzcd} M^{\vee} \arrow[r, "u^{\vee}"] & L^{\vee} \arrow[l, bend left, "s"] \end{tikzcd}$$

Let $u_1 = u^{\vee}|_{s(L^{\vee})}$ and $u_2 = u^{\vee}|_{v^{\vee}(P^{\vee})}$. Then $u_2 \equiv 0$, $u^{\vee} = u_1 \oplus u_2$, and $u_1 : s(L^{\vee}) \xrightarrow{\sim} L^{\vee}$ is an isomorphism. The canonical isomorphism of *graded* $B-$algebras [Bou70, p. 84, Proposition 10]

$$\mathbf{\Lambda}_B\left(B \otimes_R s(L^{\vee})\right) \otimes_B \mathbf{\Lambda}_B\left(B \otimes_R v^{\vee}(P^{\vee})\right) \xrightarrow{\sim} \mathbf{\Lambda}_B\left(B \otimes_R s(L^{\vee}) \oplus B \otimes_R v^{\vee}(P^{\vee})\right)$$

is an isomorphism of the complex $\mathbf{K}^B(\overline{u_1}) \otimes_B \mathbf{K}^B(\overline{u_2})$ onto the complex $\mathbf{K}^B(\overline{u^{\vee}})$ [Bou80, p. 148, Proposition 2]. We claim that the complex $\mathbf{K}^B(\overline{u_1})$ is acyclic, that is

$$H_i\big(\mathbf{K}^B(\overline{u_1})\big) = 0, \text{ for } i > 0.$$

To see this pick a basis for $L^{\vee}$, and identify the $R-$algebra $B(= \mathbf{S}_R L^{\vee})$ with the polynomial ring $R[X_1, \cdots, X_{r_0}]$, where $r_0 = \operatorname{rank} L^{\vee}$. Then the complex $\mathbf{K}^B(\overline{u_1})$ is identified with the ordinary Koszul complex associated to a free module of rank r_0 over the polynomial ring $R[X_1, \cdots, X_{r_0}]$ and the sequence of elements $X_1, \cdots, X_{r_0}$. Since these elements form a regular sequence in the polynomial ring, this complex is acyclic. Thus

$$\Big(H\big(\mathbf{K}^B(\overline{u_1})\big) \otimes_B H\big(\mathbf{K}^B(\overline{u_2})\big)\Big)_n = \oplus_{p+q=n} H_p\big(\mathbf{K}^B(\overline{u_1})\big) \otimes_B H_q\big(\mathbf{K}^B(\overline{u_2})\big)$$

$$= H_0\big(\mathbf{K}^B(\overline{u_1})\big) \otimes_B H_n\big(\mathbf{K}^B(\overline{u_2})\big).$$

On the other hand, the differential of the complex $\mathbf{K}^B(\overline{u_2})$ is zero. Thus its homology modules are free $B-$modules. This allows us to apply Künneth formula (Theorem D.1). For $n \geq 0$ we get a $B-$module isomorphism

$$\gamma_{0,n} : H_0\big(\mathbf{K}^B(\overline{u_1})\big) \otimes_B H_n\big(\mathbf{K}^B(\overline{u_2})\big) \xrightarrow{\sim} H_n\big(\mathbf{K}^B(\overline{u^{\vee}})\big).$$

Now, we write $\mathbf{K}^B(\overline{u^{\vee}})$ in its isomorphic form as in (II.1)

$$\cdots \to B \otimes_R \mathbf{\Lambda}_R^r M^{\vee} \xrightarrow{d_r} B \otimes_R \mathbf{\Lambda}_R^{r-1} M^{\vee} \to \cdots \to B \otimes_R \mathbf{\Lambda}_R^1 M^{\vee} \xrightarrow{d_1} B \otimes_R \mathbf{\Lambda}_R^0 M^{\vee} \to 0,$$

where the differential d_r is given as in (II.2). We will denote this complex by $\mathscr{C}$. Since d_r maps $\mathbf{S}^p_R L^\vee \otimes_R \mathbf{\Lambda}^r_R M^\vee$ into $\mathbf{S}^{p+1}_R L^\vee \otimes_R \mathbf{\Lambda}^{r-1}_R M^\vee$, the complex $\mathscr{C}$ (*as a complex of* $R-$*modules*) decomposes into a direct sum of complexes $\mathscr{C}_r$ as following

$$\begin{array}{ccccccccc}
 & & & & & & \mathscr{C}_0 : & & \mathbf{S}^0_R L^\vee \otimes_R \mathbf{\Lambda}^0_R M^\vee \to 0 \\
 & & & & & & & & \oplus \\
 & & & & \mathscr{C}_1 : & & \mathbf{S}^0_R L^\vee \otimes_R \mathbf{\Lambda}^1_R M^\vee & \xrightarrow{d_{1,1}} & \mathbf{S}^1_R L^\vee \otimes_R \mathbf{\Lambda}^0_R M^\vee \to 0 \\
 & & & & & & \oplus & & \oplus \\
 & \mathscr{C}_2 : & & & \mathbf{S}^0_R L^\vee \otimes_R \mathbf{\Lambda}^2_R M^\vee & \xrightarrow{d_{2,1}} & \mathbf{S}^1_R L^\vee \otimes_R \mathbf{\Lambda}^1_R M^\vee & \xrightarrow{d_{1,2}} & \mathbf{S}^2_R L^\vee \otimes_R \mathbf{\Lambda}^0_R M^\vee \to 0 \\
 & & & & \oplus & & \oplus & & \oplus \\
\mathbf{S}^0_R L^\vee \otimes_R \mathbf{\Lambda}^3_R M^\vee & \xrightarrow{d_{3,1}} & & & \mathbf{S}^1_R L^\vee \otimes_R \mathbf{\Lambda}^2_R M^\vee & \xrightarrow{d_{2,2}} & \mathbf{S}^2_R L^\vee \otimes_R \mathbf{\Lambda}^1_R M^\vee & \xrightarrow{d_{1,3}} & \mathbf{S}^3_R L^\vee \otimes_R \mathbf{\Lambda}^0_R M^\vee \to 0 \\
\oplus & & & & \oplus & & \oplus & & \oplus \\
\vdots & & & & \vdots & & \vdots & & \vdots
\end{array}$$

and the differential d_r can be written as $d_r = \oplus_{i=1}^{\infty} d_{r,i}$. Thus, as $R-$modules

$$H_n\big(\mathbf{K}^B(\overline{u^\vee})\big) \cong H_n(\mathscr{C}) \cong \oplus_{r=n}^{\infty} H_n(\mathscr{C}_r).$$

Now by looking at the definition of the map $\gamma_{0,n}$ in Appendix D, we see that the image of this map is contained in $H_n(\mathscr{C}_n) = \mathrm{Ker}(d_{n,1})$. Since $\gamma_{0,n}$ is an isomorphism here, we conclude that as $R-$modules

$$H_n\big(\mathbf{K}^B(\overline{u^\vee})\big) \cong H_n(\mathscr{C}) \cong H_n(\mathscr{C}_n), \quad \text{and} \quad H_n(\mathscr{C}_r) = 0 \ \text{ for } \ n < r.$$

By Remark C.2, $H_0\big(\mathbf{K}^B(\overline{u_1})\big) = B/\overline{u_1}\big(B \otimes_R s(L^\vee)\big)$, and since the differential of the complex $\mathbf{K}^B(\overline{u_2})$ is zero, $H_n\big(\mathbf{K}^B(\overline{u_2})\big) = \mathbf{\Lambda}^n_B\left(B \otimes_R v^\vee(P^\vee)\right) \cong B \otimes_R \mathbf{\Lambda}^n_R v^\vee(P^\vee)$. Thus

$$\begin{aligned} H_0\big(\mathbf{K}^B(\overline{u_1})\big) \otimes_B H_n\big(\mathbf{K}^B(\overline{u_2})\big) &\cong \big(B/\overline{u_1}\big(B \otimes_R s(L^\vee)\big)\big) \otimes_B \left(B \otimes_R \mathbf{\Lambda}^n_R v^\vee(P^\vee)\right) \\ &\cong \big(B/\overline{u_1}\big(B \otimes_R s(L^\vee)\big)\big) \otimes_R \mathbf{\Lambda}^n_R v^\vee(P^\vee). \end{aligned}$$

Since u_1 is surjective, we see that as an $R-$module, $B/\overline{u_1}\big(B \otimes_R s(L^\vee)\big) \cong R$. Thus

$$H_n(\mathscr{C}_n) \cong \mathbf{\Lambda}^n_R\, v^\vee(P^\vee) \cong \mathbf{\Lambda}^n_R\, P^\vee.$$

We have shown that given a short exact sequence (II.3) of free $R-$modules and any integer $0 \leq r \leq \operatorname{rank} P$, there is an exact sequence of $R-$modules

$$0 \to \mathbf{\Lambda}^r_R\, P^\vee \to \mathbf{S}^0_R\, L^\vee \otimes_R \mathbf{\Lambda}^r_R\, M^\vee \to \cdots \to \mathbf{S}^{r-1}_R\, L^\vee \otimes_R \mathbf{\Lambda}^1_R\, M^\vee \to \mathbf{S}^r_R\, L^\vee \otimes_R \mathbf{\Lambda}^0_R\, M^\vee \to 0.$$

Since all terms of this exact sequence are free, it splits at each place; thus, its dual is also exact. Dualizing this sequence, one obtains the following exact sequence

$$0 \to \mathbf{S}^r_R\, L \otimes_R \mathbf{\Lambda}^0_R\, M \longrightarrow \mathbf{S}^{r-1}_R\, L \otimes_R \mathbf{\Lambda}^1_R\, M \longrightarrow \cdots \longrightarrow \mathbf{S}^0_R\, L \otimes_R \mathbf{\Lambda}^r_R\, M \longrightarrow \mathbf{\Lambda}^r_R\, P \to 0.$$

II.2 Monads

Monads were introduced by G. Horrocks in [Hor64, p. 698].

Definition II.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $n \geq 2$, and denote the punctured spectrum of R by $(U, \mathscr{O}_U)$. A* monad *over U is a complex*

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{B} \longrightarrow 0$$

of vector bundles on U, which is exact at $\mathscr{A}$ and $\mathscr{B}$, such that $\operatorname{Im}(\alpha)$ is a subbundle of $\mathscr{F}$. The vector bundle $\mathscr{E} := \operatorname{Ker}(\beta)/\operatorname{Im}(\alpha)$ is called the homology of the monad.

Theorem II.7 (G. Horrocks)**.** [Hor80, p. 199] *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 2$, and denote the punctured spectrum of R by $(U, \mathscr{O}_U)$. Let $\mathscr{E}$ be a vector bundle on U. Assume that $H^1(U, \mathscr{E}) \neq 0$ and $H^{n-2}(U, \mathscr{E}) \neq 0$. Then there is a vector bundle $\mathscr{F}$ on U, and a monad*

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{B} \longrightarrow 0, \tag{II.4}$$

where $\mathscr{A}$ and $\mathscr{B}$ are split vector bundles and

(i) $\operatorname{Ker}(\beta)/\operatorname{Im}(\alpha) \cong \mathscr{E}$,

(ii) $H^1(U,\mathscr{F}) = H^{n-2}(U,\mathscr{F}) = 0$,

(iii) $H^i(U,\mathscr{F}) \cong H^i(U,\mathscr{E})$ *for* $1 < i < n-2$.

Proof. Let $\{g_1,\cdots,g_p\}$ be a minimal set of generators for $H^1(U,\mathscr{E})$. Then, one can view the system $(g_1,\cdots,g_p)$ as an element of $H^1(U,\mathscr{B}\otimes_{\mathscr{O}_U}\mathscr{E})$, where

$$\mathscr{B} := \bigoplus\nolimits_1^p \mathscr{O}_U.$$

One has

$$H^1(U,\mathscr{B}\otimes\mathscr{E}) \cong \operatorname{Ext}^1_{\mathscr{O}_U}(\mathscr{O}_U,\mathscr{B}\otimes\mathscr{E}) \cong \operatorname{Ext}^1_{\mathscr{O}_U}(\mathscr{B}^\vee,\mathscr{E}) \cong \operatorname{Ext}^1_{\mathscr{O}_U}(\mathscr{B},\mathscr{E}),$$

the last isomorphism holding because $\mathscr{B}^\vee \cong \mathscr{B}$. The system $(g_1,\cdots,g_p)$ determines therefore an extension

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}_1 \xrightarrow{\pi_1} \mathscr{B} \longrightarrow 0,$$

where $\mathscr{E}_1$ is also a vector bundle on U, because the other two terms are. Consider the cohomology long exact sequence associated with this extension. As the g_i's are generators of $H^1(U,\mathscr{E})$, one has $H^1(U,\mathscr{E}_1) = 0$. On the other hand, $H^i(U,\mathscr{B}) = 0$ for $1 \leq i \leq n-2$. Whence $H^i(U,\mathscr{E}_1) \cong H^i(U,\mathscr{E})$ for $1 < i \leq n-2$. In order to achieve the demonstration, we need to kill $H^{n-2}(U,\mathscr{E}_1)$ which is isomorphic to $D\big(H^1(U,\mathscr{E}_1^\vee)\big)$ by Corollary I.11. Thus, it suffices to kill $H^1(U,\mathscr{E}_1^\vee)$, which we do by repeating the above process for $\mathscr{E}_1^\vee$ this time. As a result we obtain an extension

$$0 \longrightarrow \mathscr{E}_1^\vee \longrightarrow \mathscr{F}_1 \xrightarrow{\alpha^\vee} \mathscr{A} \longrightarrow 0, \tag{II.5}$$

where

$$\mathscr{A} := \bigoplus_1^s \mathscr{O}_U,$$

and s is the cardinality of a minimal set of generators of $H^1(U, \mathscr{E}_1^\vee)$. As before one can see that $H^1(U, \mathscr{F}_1) = 0$, and $H^i(U, \mathscr{F}_1) \cong H^i(U, \mathscr{E}_1^\vee)$ for $1 < i \leq n-2$. Take $\mathscr{F} := \mathscr{F}_1^\vee$. Using duality of Corollary I.11 one has

$$H^{n-2}(U, \mathscr{F}) \cong D\big(H^1(U, \mathscr{F}_1)\big) = 0, \text{ and}$$

$$H^1(U, \mathscr{F}) \cong D\big(H^{n-2}(U, \mathscr{F}_1)\big) \cong D\big(H^{n-2}(U, \mathscr{E}_1^\vee)\big) \cong H^1(U, \mathscr{E}_1) = 0,$$

and for $1 < i < n-2$,

$$H^i(U, \mathscr{F}) \cong D\big(H^{n-i-1}(U, \mathscr{F}_1)\big) \cong D\big(H^{n-i-1}(U, \mathscr{E}_1^\vee)\big) \cong H^i(U, \mathscr{E}_1) \cong H^i(U, \mathscr{E}).$$

It remains to show the existence of the monad (II.4). Dualizing (II.5) with respect to $\mathscr{O}_U$, and noting that $\mathscr{A}^\vee \cong \mathscr{A}$, one obtains an exact sequence

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\pi_2} \mathscr{E}_1 \longrightarrow 0.$$

One can put a 0 on the right, because

$$\mathscr{E}xt^1_{\mathscr{O}_U}(\mathscr{A}, \mathscr{O}_U) \cong \mathscr{E}xt^1_{\mathscr{O}_U}(\mathscr{O}_U, \mathscr{A}^\vee \otimes \mathscr{O}_U) = 0.$$

Taking $\beta := \pi_1 \circ \pi_2$, we obtain a monad

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{B} \longrightarrow 0.$$

Applying the Snake Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{A} \arrow[r, "\alpha"] \arrow[d] & \mathscr{F} \arrow[r, "\pi_2"] \arrow[d, "\beta"] & \mathscr{E}_1 \arrow[r] \arrow[d, "\pi_1"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathscr{B} \arrow[r, equal] & \mathscr{B} \arrow[r] & 0
\end{tikzcd}$$

one can see that $\mathrm{Ker}(\beta)/\mathrm{Im}(\alpha) \cong \mathscr{E}$, as required. $\square$

A Special Case. Let $(R,\mathfrak{m})$ be a regular local ring of dimension $n \geq 2$, and denote its punctured spectrum by $(U,\mathscr{O}_U)$. Let $\mathscr{E}$ be a vector bundle on U with $H^1(U,\mathscr{E}) \neq 0$, $H^{n-2}(U,\mathscr{E}) \neq 0$, and $H^i(U,\mathscr{E}) = 0$ for $2 \leq i \leq n-3$. Then by Theorem I.8, the bundle $\mathscr{F}$ in the monad (II.4) of Theorem II.7 splits, as well, and the maps α and β can be represented by appropriate matrices. In this paragraph we want to show that these matrices can be taken to be *minimal* in the sense that no matrix entry is a unit.

Going back to the proof of Theorem II.7, the extension

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}_1 \xrightarrow{\pi_1} \mathscr{B} \longrightarrow 0,$$

gives rise to an exact sequence

$$H^0(U,\mathscr{E}_1) \xrightarrow{\pi_1} H^0(U,\mathscr{B}) \xrightarrow{\varepsilon_1} H^1(U,\mathscr{E}) \longrightarrow 0 \tag{II.6}$$

in the level of cohomology. By construction ε_1 picks out a minimal set of generators $\{g_1,\cdots,g_p\}$ of $H^1(U,\mathscr{E})$, i.e., there is a basis $\{e_1,\cdots,e_p\}$ of the free $R-$module $H^0(U,\mathscr{B})$ such that $\varepsilon_1(e_i) = g_i$. Equivalently

$$\mathrm{Im}(\pi_1) = \mathrm{Ker}(\varepsilon_1) \subset \mathfrak{m}H^0(U,\mathscr{B}).$$

Let $\mathscr{G}$ be the kernel of the surjection $\mathscr{F} \xrightarrow{\beta} \mathscr{B}$ in the monad (II.4). The exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \xrightarrow{\beta} \mathscr{B} \longrightarrow 0$$

gives rise to an exact sequence

$$H^0(U,\mathscr{F}) \xrightarrow{\beta} H^0(U,\mathscr{B}) \xrightarrow{\varepsilon_2} H^1(U,\mathscr{G}) \longrightarrow 0 \tag{II.7}$$

in the level of cohomology. As seen in the proof of Theorem II.7, β factors as

$$H^0(U,\mathscr{F}) \overset{\pi_2}{\twoheadrightarrow} H^0(U,\mathscr{E}_1) \overset{\pi_1}{\to} H^0(U,\mathscr{B}),$$

with π_2 surjective. Thus, the maps π_1 and β in (II.6) and (II.7) have the same image in $H^0(U, \mathscr{B})$. Whence

$$\mathrm{Im}(\beta) = \mathrm{Im}(\pi_1) \subset \mathfrak{m}H^0(U, \mathscr{B}).$$

This shows that no entry of the matrix β is a unit.

To show that α can be taken to be minimal, as well, consider the extension (II.5) in the proof of Theorem II.7. Again by construction this extension corresponds to a minimal set of generators of $H^1(U, \mathscr{E}_1^\vee)$. In the level of cohomology we obtain an exact sequence

$$H^0(U, \mathscr{F}_1) \xrightarrow{\alpha^\vee} H^0(U, \mathscr{A}) \xrightarrow{\varepsilon} H^1(U, \mathscr{E}_1^\vee) \longrightarrow 0$$

in which ε picks out a minimal set of generators of $H^1(U, \mathscr{E}_1^\vee)$, i.e., there is a basis $\{e_1', \cdots, e_s'\}$ of the free $R-$module $H^0(U, \mathscr{A})$ and a minimal set $\{g_1', \cdots, g_s'\}$ of generators of $H^1(U, \mathscr{E}_1^\vee)$ such that $\varepsilon(e_j') = g_j'$. Equivalently

$$\mathrm{Im}(\alpha^\vee) = \mathrm{Ker}(\varepsilon) \subset \mathfrak{m}H^0(U, \mathscr{A}),$$

which shows that no entry of the matrix $\alpha^\vee$ (and hence its dual α) is a unit.

II.3 Proof of Theorem II.1

We will need the following lemma in the proof:

Lemma II.8. *Let*

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \xrightarrow{\beta} \mathscr{E}'' \longrightarrow 0$$

be an exact sequence of vector bundles on the punctured spectrum U of a regular local ring R of dimension $n \geq 2$. Assume that $H^1(U, \mathscr{E}') = 0$, and $\mathscr{E}''$ is split. Then the map β is split, i.e., there is a map $\gamma : \mathscr{E}'' \longrightarrow \mathscr{E}$ such that $\beta \circ \gamma = \mathrm{id}_{\mathscr{E}''}$.

Proof. Let

$$E' := \Gamma(U, \mathscr{E}'), \text{ and } E := \Gamma(U, \mathscr{E}).$$

Since $\operatorname{depth}(\mathfrak{m}, R) \geq 2$, $\Gamma(U, \mathscr{E}'') \cong R^r$, where r is the rank of $\mathscr{E}''$. Since $H^1(U, \mathscr{E}') = 0$, there is an exact sequence

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{b} R^r \longrightarrow 0,$$

and the map b is split, that is, there is a map $c : R^r \longrightarrow E$ such that $b \circ c = \mathrm{id}_{R^r}$. Thus, after sheafifying, one obtains an exact sequence

$$0 \longrightarrow \widetilde{E'} \longrightarrow \widetilde{E} \xrightarrow{\widetilde{b}} \widetilde{R^r} \longrightarrow 0,$$

and a map $\widetilde{c} : \widetilde{R^r} \longrightarrow \widetilde{E}$ such that $\widetilde{b} \circ \widetilde{c} = \mathrm{id}_{\widetilde{R^r}}$. Restricting everything to U, gives the desired result, in view of Theorem I.7(ii). $\square$

Proof. (Theorem II.1) If $\mathscr{E}$ does not split, then in view of Theorem I.8 the cohomology modules $H^1(U, \mathscr{E})$ and $H^{n-2}(U, \mathscr{E})$ can not vanish at the same time. In fact we are going to show that if $\operatorname{rank}(\mathscr{E}) < n - 1$, then both of these cohomology modules are nonzero: Suppose $\mathscr{E}$ does not split but $H^1(U, \mathscr{E}) = 0$. Then by duality (Corollary I.11) $D(H^1(U, \mathscr{E}^\vee)) \cong H^{n-2}(U, \mathscr{E}) \neq 0$. After killing $H^1(U, \mathscr{E}^\vee)$ as described in the proof of Theorem II.7 and dualizing the extension corresponding to a minimal set of generators of $H^1(U, \mathscr{E}^\vee)$, one obtains a short exact sequence

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow 0$$

where $\mathscr{A}$ and $\mathscr{F}$ are split vector bundles and α is a matrix without any unit entry. As seen in Example II.5, from this short exact sequence and for any $1 \leq r \leq \operatorname{rank} \mathscr{E}$, we obtain an exact sequence

$$0 \to \mathbf{S}^r \mathscr{A} \xrightarrow{d_{r+1}} \mathbf{S}^{r-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{F} \xrightarrow{d_r} \cdots \xrightarrow{d_2} \mathbf{\Lambda}^r \mathscr{F} \xrightarrow{d_1} \mathbf{\Lambda}^r \mathscr{E} \xrightarrow{d_0} 0,$$

where the map d_{r+1}, for example, is obtained from α as

$$a_1a_2\cdots a_r \mapsto \sum_{i=1}^{r}(\pm a_1a_2\cdots\hat{a}_i\cdots a_r\otimes\alpha(a_i)).$$

Take $r = \operatorname{rank}\mathscr{E}$. Then since $\operatorname{Pic}(U)$ is trivial, one has $\mathbf{\Lambda}^r\,\mathscr{E}\cong\mathscr{O}_U$. Let $\mathscr{G}_i$ be the kernel of d_i. For each $1\le i\le r-1$ there is a short exact sequence

$$0\longrightarrow\mathscr{G}_i\longrightarrow\mathbf{S}^{i-1}\,\mathscr{A}\otimes\mathbf{\Lambda}^{r-i+1}\,\mathscr{F}\longrightarrow\mathscr{G}_{i-1}\longrightarrow 0.$$

By induction on i, and using the fact that $\mathbf{S}^{i-1}\,\mathscr{A}\otimes\mathbf{\Lambda}^{r-i+1}\,\mathscr{F}$ and $\mathbf{\Lambda}^r\,\mathscr{E}$ are split, one can see that as long as $i+1<n-1$,

$$H^{i+1}(U,\mathscr{G}_i)=\cdots=H^{n-2}(U,\mathscr{G}_i)=0.$$

In particular, if $r<n-1$, one has $H^{n-2}(U,\mathscr{G}_{r-1})=0$. On the other hand, from the short exact sequence

$$0\to\mathbf{S}^r\,\mathscr{A}\xrightarrow{d_{r+1}}\mathbf{S}^{r-1}\,\mathscr{A}\otimes\mathbf{\Lambda}^1\,\mathscr{F}\longrightarrow\mathscr{G}_{r-1}\longrightarrow 0$$

one can see that for $1\le j<n-2$, the cohomology modules $H^j(U,\mathscr{G}_{r-1})$ are also zero. Hence $\mathscr{G}_{r-1}$ is a split vector bundle. (A similar argument shows that in fact all the $\mathscr{G}_i$'s are split.) But then, by Lemma II.8 the map d_{r+1} is split, which is absurd, because no entry of the matrix α is a unit. Therefore $H^1(U,\mathscr{E})$ can not be zero. Repeating the same argument for $\mathscr{E}^\vee$, we see that $H^{n-2}(U,\mathscr{E})\ne 0$, either.

Now, by Theorem II.7 and the paragraph following its proof, there is a monad

$$0\longrightarrow\mathscr{A}\xrightarrow{\alpha}\mathscr{F}\xrightarrow{\beta}\mathscr{B}\longrightarrow 0 \tag{II.8}$$

where $\mathscr{A}$, $\mathscr{F}$ and $\mathscr{B}$ are split vector bundles, and the maps α and β are represented by *minimal* matrices, i.e., matrices with all entries in the maximal ideal $\mathfrak{m}$.

We claim that it suffices to prove the theorem for n odd. To see this, suppose we knew the result in this case, and let $\mathscr{E}$ be a vector bundle on the punctured spectrum U of a regular local ring $(R, \mathfrak{m})$ of even dimension $n \geq 5$, with $\operatorname{rank} \mathscr{E} < n-2$, and $H^i(U, \mathscr{E}) = 0$ for $1 < i < n-2$. Let $x \in \mathfrak{m} - \mathfrak{m}^2$ be a parameter element, and consider the regular local ring $\overline{R} := R/xR$. Let V be the punctured spectrum of $\overline{R}$. Then, $(V, \mathscr{O}_V)$ is a closed subscheme of $(U, \mathscr{O}_U)$, and if we denote the corresponding closed immersion by $\jmath$, then by Proposition I.12 (i), $\jmath^*\mathscr{E}$ is a vector bundle on V with the same rank as $\operatorname{rank} \mathscr{E}$. Moreover $\dim \overline{R} = n - 1 \geq 5$ is odd, $\operatorname{rank} \jmath^*\mathscr{E} < \dim \overline{R} - 1$, and from the cohomology long exact sequence (I.2) in the proof of Proposition I.12 (ii), one can see that $H^i(V, \jmath^*\mathscr{E}) = 0$, for $1 < i < n-3$. Thus, by our assumption on validity of the theorem for odd dimensions, $\jmath^*\mathscr{E}$ is split. Then by Proposition I.12 (iii), $\mathscr{E}$ is also split.

Suppose now that n is odd with $n = 2k+1$. Let $\mathscr{E}$ be a vector bundle on U with $\operatorname{rank} \mathscr{E} \leq n-2$. By adding copies of $\mathscr{O}_U$ to $\mathscr{E}$ (if necessary), we may assume that $\operatorname{rank} \mathscr{E} = n - 2 = 2k - 1$. We will study $\boldsymbol{\Lambda}^{k-1} \mathscr{E}$ and $\boldsymbol{\Lambda}^{k} \mathscr{E}$. Let $\mathscr{G}$ be the kernel of the map β in the monad (II.8). One has the following short exact sequences of vector bundles

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \xrightarrow{\beta} \mathscr{B} \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{G} \longrightarrow \mathscr{E} \longrightarrow 0.$$

The second sequence shows that $\operatorname{rank} \mathscr{G} = \operatorname{rank} \mathscr{A} + \operatorname{rank} \mathscr{E} > n-2$. We claim that $H^k(U, \boldsymbol{\Lambda}^{k-1} \mathscr{E}) = 0$. To see this, fix $1 \leq i < n-2 < \operatorname{rank} \mathscr{G}$, and consider the following resolution of $\boldsymbol{\Lambda}^{i} \mathscr{G}^\vee$ by split vector bundles

$$0 \to \mathbf{S}^i \mathscr{B}^\vee \longrightarrow \mathbf{S}^{i-1} \mathscr{B}^\vee \otimes \boldsymbol{\Lambda}^1 \mathscr{F}^\vee \xrightarrow{d_i} \cdots \xrightarrow{d_2} \boldsymbol{\Lambda}^i \mathscr{F}^\vee \xrightarrow{d_1} \boldsymbol{\Lambda}^i \mathscr{G}^\vee \xrightarrow{d_0} 0,$$

obtained from the short exact sequence

$$0 \longrightarrow \mathscr{B}^\vee \longrightarrow \mathscr{F}^\vee \longrightarrow \mathscr{G}^\vee \longrightarrow 0,$$

as explained in Example II.5. Denote the kernel of d_j by $\mathscr{G}_j$. For each $1 \le j \le i$ one has a short exact sequence of vector bundles

$$0 \longrightarrow \mathscr{G}_j \longrightarrow \mathbf{S}^{j-1}\mathscr{B}^\vee \otimes \mathbf{\Lambda}^{i-j+1}\mathscr{F}^\vee \longrightarrow \mathscr{G}_{j-1} \longrightarrow 0.$$

Since $\mathbf{S}^{j-1}\mathscr{B}^\vee \otimes \mathbf{\Lambda}^{i-j+1}\mathscr{F}^\vee$ is a split vector bundle, we obtain

$$H^\ell(U, \mathscr{G}_{j-1}) \cong H^{\ell+1}(U, \mathscr{G}_j), \quad 1 \le \ell \le n-3.$$

Thus for $1 \le \ell \le n-i-2$ one has

$$H^\ell(U, \mathbf{\Lambda}^i\mathscr{G}^\vee) \cong H^{\ell+1}(U, \mathscr{G}_1) \cong \cdots \cong H^{\ell+i-1}(U, \mathscr{G}_{i-1}) \cong H^{\ell+i}(U, \mathbf{S}^i\mathscr{B}^\vee) = 0.$$

Whence, by duality (Corollary I.11)

$$H^r(U, \mathbf{\Lambda}^i\mathscr{G}) = 0, \text{ for } i+1 \le r \le n-2. \tag{II.9}$$

Now from the short exact sequence

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{G} \longrightarrow \mathscr{E} \longrightarrow 0$$

we obtain the following exact sequence

$$0 \to \mathbf{S}^{k-1}\mathscr{A} \longrightarrow \mathbf{S}^{k-2}\mathscr{A} \otimes \mathbf{\Lambda}^1\mathscr{G} \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_2} \mathbf{\Lambda}^{k-1}\mathscr{G} \xrightarrow{d_1} \mathbf{\Lambda}^{k-1}\mathscr{E} \xrightarrow{d_0} 0,$$

as explained in Example II.5. Again denote the kernel of d_j by $\mathscr{G}_j$. Then, for each $1 \le j \le k-1$ we obtain a short exact sequence of vector bundles

$$0 \longrightarrow \mathscr{G}_j \longrightarrow \mathbf{S}^{j-1}\mathscr{A} \otimes \mathbf{\Lambda}^{k-j}\mathscr{G} \longrightarrow \mathscr{G}_{j-1} \longrightarrow 0.$$

Since $\mathbf{S}^{j-1}\mathscr{A}\otimes\mathbf{\Lambda}^{k-j}\mathscr{G}$ is a finite sum of copies of $\mathbf{\Lambda}^{k-j}\mathscr{G}$, using the vanishing of cohomology for $\mathbf{\Lambda}^{k-j}\mathscr{G}$ obtained in (II.9) above, we see that

$$H^{k-j+1}(U,\mathscr{G}_{j-1})\cong H^{k-j+2}(U,\mathscr{G}_j).$$

Thus,

$$H^k(U,\mathbf{\Lambda}^{k-1}\mathscr{E})\cong H^{k+1}(U,\mathscr{G}_1)\cong\cdots\cong H^{2k-1}(U,\mathbf{S}^{k-1}\mathscr{A})=0,$$

which proves our claim.

Now, since for any r the multiplication map $\mathbf{\Lambda}^r\mathscr{E}\otimes\mathbf{\Lambda}^{2k-1-r}\mathscr{E}\longrightarrow\mathbf{\Lambda}^{2k-1}\mathscr{E}$ is a perfect pairing, one obtains an isomorphism

$$\mathbf{\Lambda}^{k-1}\mathscr{E}\cong(\mathbf{\Lambda}^k\mathscr{E})^{\vee}\otimes\mathbf{\Lambda}^{2k-1}\mathscr{E}.$$

On the other hand since $\operatorname{rank}\mathscr{E}=2k-1$ and $\operatorname{Pic}(U)$ is trivial, we have $\mathbf{\Lambda}^{2k-1}\mathscr{E}\cong\mathscr{O}_U$, and using duality

$$H^k(U,\mathbf{\Lambda}^k\mathscr{E})\cong D\big(H^k(U,(\mathbf{\Lambda}^k\mathscr{E})^{\vee})\big)\cong D\big(H^k(U,\mathbf{\Lambda}^{k-1}\mathscr{E})\big)=0.$$

Now consider the following resolution of $\mathbf{\Lambda}^k\mathscr{E}$

$$0\longrightarrow\mathbf{S}^k\mathscr{A}\longrightarrow\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{G}\xrightarrow{d_k}\cdots\xrightarrow{d_2}\mathbf{\Lambda}^k\mathscr{G}\xrightarrow{d_1}\mathbf{\Lambda}^k\mathscr{E}\xrightarrow{d_0}0,$$

and let $\mathscr{G}_j$ denote the kernel of d_j. For each $1\le j\le k$ we have a short exact sequence of vector bundles

$$0\longrightarrow\mathscr{G}_j\longrightarrow\mathbf{S}^{j-1}\mathscr{A}\otimes\mathbf{\Lambda}^{k-j+1}\mathscr{G}\longrightarrow\mathscr{G}_{j-1}\longrightarrow 0.$$

Noting that $\mathbf{S}^{j-1}\mathscr{A}\otimes\mathbf{\Lambda}^{k-j+1}\mathscr{G}$ is a finite sum of copies of $\mathbf{\Lambda}^{k-j+1}\mathscr{G}$, and using the vanishing of cohomology for $\mathbf{\Lambda}^{k-j+1}\mathscr{G}$ obtained in (II.9), and that $H^k(U,\mathbf{\Lambda}^k\mathscr{E})=0$, we obtain

$$0=H^{k+1}(U,\mathscr{G}_1)=H^{k+2}(U,\mathscr{G}_2)=\cdots=H^{2k-1}(U,\mathscr{G}_{k-1}).$$

Hence the map

$$H^{2k}(U, \mathbf{S}^k \mathscr{A}) \longrightarrow H^{2k}(U, \mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})$$

is an inclusion, and by duality

$$D\big(H^0(U, (\mathbf{S}^k \mathscr{A})^{\vee})\big) \longrightarrow D\big(H^0(U, (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee}\big)$$

is also an inclusion. Applying the contravariant exact functor $D(\,\cdot\,)$ to this inclusion, one obtains a surjective map

$$DD\Big(H^0\big(U, (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee}\big)\Big) \longrightarrow DD\Big(H^0\big(U, (\mathbf{S}^k \mathscr{A})^{\vee}\big)\Big).$$

By Theorem I.7, $H^0\big(U, (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee}\big)$ and $H^0\big(U, (\mathbf{S}^k \mathscr{A})^{\vee}\big)$ are finite $R-$modules. Thus, applying the functor $DD(\,\cdot\,)$ to them, gives their $\mathfrak{m}-$adic completion. Hence, one has a surjection

$$H^0\big(U, (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee}\big) \otimes_R \widehat{R} \longrightarrow H^0\big(U, (\mathbf{S}^k \mathscr{A})^{\vee}\big) \otimes_R \widehat{R} \longrightarrow 0.$$

Now, since $(R, \mathfrak{m})$ is a local ring, $\widehat{R}$ is a faithfully flat $R-$module. Whence, the map

$$H^0\big(U, (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee}\big) \longrightarrow H^0\big(U, (\mathbf{S}^k \mathscr{A})^{\vee}\big) \tag{II.10}$$

is also surjective. On the other hand, tensoring the short exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow \mathscr{B} \longrightarrow 0$$

with the split vector bundle $\mathbf{S}^{k-1} \mathscr{A}$, and dualizing the result, one obtains the following exact sequence

$$0 \longrightarrow (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{B})^{\vee} \longrightarrow (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{F})^{\vee} \longrightarrow (\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{G})^{\vee} \longrightarrow 0.$$

From this sequence, since $(\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{B})^\vee$ is split, we see that the map

$$H^0\big(U,(\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F})^\vee\big)\longrightarrow H^0\big(U,(\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{G})^\vee\big)$$

is surjective. Therefore, in view of (II.10), the composite map

$$H^0\big(U,(\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F})^\vee\big)\longrightarrow H^0\big(U,(\mathbf{S}^{k}\mathscr{A})^\vee\big) \tag{II.11}$$

is surjective, as well. We claim that this implies that the map

$$0\longrightarrow\mathbf{S}^k\mathscr{A}\longrightarrow\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F}$$

is a split inclusion. But as we saw earlier, this would contradict the fact that all entries of the matrix α are non-units. Thus, verifying the claim will finish the proof.

To prove the claim, notice that the cokernel $\mathscr{W}$ of the above inclusion is locally free. This is because by definition of a monad, $\mathrm{Im}(\alpha)$ is a subbundle of $\mathscr{F}$ in (II.8). Thus the cokernel $\mathscr{Y}$ of α is locally free, and there is an exact sequence of vector bundles

$$0\longrightarrow\mathscr{A}\xrightarrow{\alpha}\mathscr{F}\longrightarrow\mathscr{Y}\longrightarrow 0,$$

from which, as described in Example II.5, one obtains an exact sequence

$$0\longrightarrow\mathbf{S}^k\mathscr{A}\longrightarrow\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F}\xrightarrow{d_k}\cdots\xrightarrow{d_2}\mathbf{\Lambda}^k\mathscr{F}\xrightarrow{d_1}\mathbf{\Lambda}^k\mathscr{Y}\longrightarrow 0.$$

Starting from the right, and breaking down this sequence into short exact sequences, one sees that for $i=1,\cdots,k$, the kernel of d_i is locally free. In particular, $\mathscr{W}$ which is isomorphic to $\mathrm{Ker}(d_{k-1})$ is locally free. Thus, dualizing the exact sequence

$$0\longrightarrow\mathbf{S}^k\mathscr{A}\longrightarrow\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F}\longrightarrow\mathscr{W}\longrightarrow 0$$

one obtains an exact sequence

$$0\longrightarrow\mathscr{W}^\vee\longrightarrow(\mathbf{S}^{k-1}\mathscr{A}\otimes\mathbf{\Lambda}^1\mathscr{F})^\vee\longrightarrow(\mathbf{S}^k\mathscr{A})^\vee\longrightarrow 0.$$

Notice that $(\mathbf{S}^k \mathscr{A})^\vee$ is split, and (II.11) shows that $H^1(U, \mathscr{W}^\vee) = 0$. Thus, one can apply Lemma II.8 to conclude that the map

$$(\mathbf{S}^{k-1} \mathscr{A} \otimes \mathbf{\Lambda}^1 \mathscr{F})^\vee \longrightarrow (\mathbf{S}^k \mathscr{A})^\vee \longrightarrow 0$$

is a split surjection. This proves the claim. $\square$

Chapter III

Application to complete intersections

III.1 On of finiteness of $H^1_{\mathfrak{m}}(\,\cdot\,)$

In this section we present two instances of (quotient) rings A, for which $H^1_{\mathfrak{m}}(A)$ is finite. In both cases we obtain the finiteness by applying Theorem A.15 of Grothendieck.

Proposition III.1. *Let $(R,\mathfrak{m})$ be a regular local ring. Let $\mathfrak{p}$ be a non-maximal prime ideal of R, and denote the quotient ring $R/\mathfrak{p}$ by A. Then $H^1_{\mathfrak{m}}(\widetilde{A})$ is an $A-$module of finite length.*

Proof. By Theorem A.15 it suffices to show that for every $\mathfrak{q} \in \operatorname{Spec}(A) - \{\mathfrak{m}\}$ one has $H^{1-\dim A/\mathfrak{q}}_{\mathfrak{q}A_{\mathfrak{q}}}(A_{\mathfrak{q}}) = 0$. So, let $\mathfrak{q}$ be a non-maximal prime ideal of A. Notice that since $\mathfrak{q} \neq \mathfrak{m}$, $\dim A/\mathfrak{q} > 0$. If $\dim A/\mathfrak{q} > 1$, then there is nothing to show. The only remaining case is when $\dim A/\mathfrak{q} = 1$. In this case we want to show that $H^0_{\mathfrak{q}A_{\mathfrak{q}}}(A_{\mathfrak{q}}) = 0$. This is equivalent to showing $\operatorname{depth}(\mathfrak{q}A_{\mathfrak{q}}, A_{\mathfrak{q}}) \geq 1$, which is clear, because $A_{\mathfrak{q}}$ is an integral domain, as it is a subring of field of fractions of A. $\square$

Lemma III.2. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring. Let $\mathfrak{a}$ be an ideal of R with the property that $R/\mathfrak{a}$ is equidimensional. Then for every prime ideal $\mathfrak{q}$ of R containing $\mathfrak{a}$, the ring $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}$ is also equidimensional.*

Proof. Let $\mathfrak{p}_1 R_{\mathfrak{q}}$ and $\mathfrak{p}_2 R_{\mathfrak{q}}$ be two minimal prime ideals of $\mathfrak{a}R_{\mathfrak{q}}$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal prime ideals of $\mathfrak{a}$ in R. In a Cohen-Macaulay ring R, for every ideal $\mathfrak{b}$, the formula

$$\mathrm{ht}(\mathfrak{b}) + \dim R/\mathfrak{b} = \dim R \tag{III.1}$$

holds. From this formula and the assumption that $R/\mathfrak{a}$ is equidimensional, we see that $\mathrm{ht}_R(\mathfrak{p}_1) = \mathrm{ht}_R(\mathfrak{p}_2)$. Since $\mathrm{ht}_{R_{\mathfrak{q}}}(\mathfrak{p}_1 R_{\mathfrak{q}}) = \mathrm{ht}_R(\mathfrak{p}_1)$, and $\mathrm{ht}_{R_{\mathfrak{q}}}(\mathfrak{p}_2 R_{\mathfrak{q}}) = \mathrm{ht}_R(\mathfrak{p}_2)$, it follows that $\mathrm{ht}_{R_{\mathfrak{q}}}(\mathfrak{p}_1 R_{\mathfrak{q}}) = \mathrm{ht}_{R_{\mathfrak{q}}}(\mathfrak{p}_2 R_{\mathfrak{q}})$. This gives us the desired result, because $R_{\mathfrak{q}}$ is Cohen-Macaulay, as well. $\square$

Lemma III.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Let $\mathfrak{a}$ be an ideal of R with following properties: $R/\mathfrak{a}$ is equidimensional, $\mathfrak{a}$ has no embedded prime ideals, and* $\mathrm{ht}_R(\mathfrak{a}) < \dim R$. *Then* $\mathrm{depth}(\mathfrak{m}, R/\mathfrak{a}) > 0$.

Proof. If $\mathrm{depth}(\mathfrak{m}, R/\mathfrak{a}) = 0$, then every element of the maximal ideal $\mathfrak{m}$ is a zero divisor of $R/\mathfrak{a}$. Thus, $\mathfrak{m} \subseteq \cup_{i=1}^{s} \mathfrak{p}_i$, where $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$ is the set of all *minimal* prime ideals of $R/\mathfrak{a}$ (notice that by assumption $\mathfrak{a}$ has no embedded prime ideals.) But then we must have $\mathfrak{m} = \mathfrak{p}_i$, for some i, which is not possible, because by equidimensionality assumption, and formula (III.1) all minimal prime ideals of $R/\mathfrak{a}$ have equal heights in R, and thus $\mathrm{ht}_R(\mathfrak{p}_i) = \mathrm{ht}_R(\mathfrak{a}) < \dim R = \mathrm{ht}_R(\mathfrak{m})$. $\square$

Corollary III.4. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{a}$ be an ideal of R with following properties:* $\mathrm{ht}_R(\mathfrak{a}) \leq \dim R - 2$, *the quotient ring $A := R/\mathfrak{a}$ is equidimensional, and $\mathfrak{a}$ has no embedded prime ideals. Then $H^1_{\mathfrak{m}}(\widetilde{A})$ is an $A-$module of finite length.*

Proof. We proceed as in the proof of Proposition III.1 up to the point that $\mathfrak{q}$ is a non-maximal prime ideal of A with $\dim A/\mathfrak{q} = 1$, and we want to show $H^0_{\mathfrak{q}A_{\mathfrak{q}}}(A_{\mathfrak{q}}) = 0$

or equivalently $\operatorname{depth}(\mathfrak{q}A_{\mathfrak{q}}, A_{\mathfrak{q}}) \geq 1$. Then we continue as following: by formula (III.1), the assumption $\operatorname{ht}_R(\mathfrak{a}) \leq \dim R - 2$ is equivalent to $\dim A \geq 2$. Since $\dim A/\mathfrak{q} = 1$ and A is equidimensional, $\mathfrak{q}$ is not a minimal prime ideal of A. Thus

$$\dim R_{\mathfrak{q}} - \operatorname{ht}_{R_{\mathfrak{q}}}(\mathfrak{a}R_{\mathfrak{q}}) = \dim R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}} = \dim A_{\mathfrak{q}} = \operatorname{ht}_A(\mathfrak{q}) \geq 1,$$

which shows that $\operatorname{ht}_{R_{\mathfrak{q}}}(\mathfrak{a}R_{\mathfrak{q}}) < \dim R_{\mathfrak{q}}$. By Lemma III.2, $R_{\mathfrak{q}}/\mathfrak{a}R_{\mathfrak{q}}(\cong A_{\mathfrak{q}})$ is equidimensional. It is also clear, that $\mathfrak{a}R_{\mathfrak{q}}$ has no embedded prime ideals. Thus, we can apply Lemma III.3 and conclude that $\operatorname{depth}(\mathfrak{q}A_{\mathfrak{q}}, A_{\mathfrak{q}}) \geq 1$. □

III.2 Some properties of the ring $\Gamma(V, \widetilde{R/\mathfrak{a}})$

Let R be a regular local ring, $\mathfrak{a}$ an ideal of R, and denote the punctured spectrum of the quotient ring $A := R/\mathfrak{a}$ by $(V, \mathscr{O}_V)$. In this section we want to study some properties of the $A-$algebra $B := \Gamma(V, \widetilde{R/\mathfrak{a}})$. We first recall the following result, which we will need later in this section.

Proposition III.5. [Gro60, p. 160, Corollary I,7.2.6.] *Let $(X, \mathscr{O}_X)$ be a reduced scheme, and let $U \subset X$ be a dense open subset. Then, there is a canonical bijective correspondence between sections of $\mathscr{O}_X$ over U, and rational functions f on X, that are defined at every point of U.*

Example III.6. [Gro60, p. 165 ,I,8.2.1] Let $(X, \mathscr{O}_X)$ be an *integral* scheme, and let K be its field of rational functions, which is identical to the local ring of X at its generic point. We know that for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ can be canonically identified with a subring of K, and for every rational function $f \in K$, the domain of definition of f is the open set consisting of all points $x \in X$, such that $f \in \mathscr{O}_{X,x}$. From

Proposition III.5 it follows that for every open set $U \subset X$, we have

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}.$$

Proposition III.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{a}$ be a non-maximal ideal of R. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{a}$ by $(V, \mathscr{O}_V)$ and the $A-$algebra $\Gamma(V, \widetilde{A})$ by B. Then for every non-maximal prime ideal $\mathfrak{q}$ of A there is a canonical $A-$algebra isomorphism $A_{\mathfrak{q}} \xrightarrow{\sim} B_{\mathfrak{q}}$. In other words, $\widetilde{B}|_V$ is a line bundle on $(V, \mathscr{O}_V)$.*

Proof. We write the exact sequence of local cohomology (see Proposition A.2)

$$0 \longrightarrow H^0_{\mathfrak{m}}(\widetilde{A}) \longrightarrow A \longrightarrow B \longrightarrow H^1_{\mathfrak{m}}(\widetilde{A}) \longrightarrow 0. \tag{III.2}$$

By the very definition of local cohomology, one has $\operatorname{Supp} H^i_{\mathfrak{m}}(\widetilde{A}) \subseteq \{\mathfrak{m}\}$, for all $i \geq 0$. Thus, localizing the exact sequence (III.2) at a non-maximal prime ideal $\mathfrak{q}$ of A, one obtains an isomorphism $A_{\mathfrak{q}} \xrightarrow{\sim} B_{\mathfrak{q}}$. $\square$

Corollary III.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{a}$ be an ideal of R. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{a}$ by $(V, \mathscr{O}_V)$, the $A-$algebra $\Gamma(V, \widetilde{A})$ by B, and the punctured spectrum of R by $(U, \mathscr{O}_U)$. Also, let*

$$j : \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(R) \quad \text{and} \quad \jmath : V \hookrightarrow U$$

be the corresponding closed immersions. Then

(i) $\mathscr{O}_V \cong \widetilde{B}|_V$, *as $\mathscr{O}_V-$algebras;*

(ii) $\jmath_* \mathscr{O}_V \cong (j_* \widetilde{B})|_U$, *as $\mathscr{O}_U-$algebras.*

Proof. (i): The exact sequence (III.2) in proof of Proposition III.7 gives an $\widetilde{A}-$algebra homomorphism $\widetilde{A} \to \widetilde{B}$, which is an isomorphism on V by Proposition III.7. Thus,

$$\mathscr{O}_V = \widetilde{A}|_V \xrightarrow{\sim} \widetilde{B}|_V.$$

(ii): From part (i) one has $\jmath_* \mathscr{O}_V \cong \jmath_*(\widetilde{B}|_V)$. It suffices to show $\jmath_*(\widetilde{B}|_V) \cong (j_* \widetilde{B})|_U$, as $\mathscr{O}_U-$algebras. This can be directly verified, by taking an arbitrary open subset $W \subseteq U$, and showing that $\jmath_*(\widetilde{B}|_V)(W) \cong (j_*\widetilde{B})|_U(W)$. □

Proposition III.9. *Let $(R, \mathfrak{m})$ be a regular local ring. Let $\mathfrak{a}$ be either a non-maximal prime ideal, or an ideal of R with following properties:* $\mathrm{ht}(\mathfrak{a}) \leq \dim R - 2$, *the quotient ring $R/\mathfrak{a}$ is equidimensional, and $\mathfrak{a}$ has no embedded prime ideals. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{a}$ by $(V, \mathscr{O}_V)$ and the $A-$algebra $\Gamma(V, \widetilde{A})$ by B. Then there is an injection $i : A \to B$, and B is integral over $i(A)$.*

Proof. First notice that $\mathrm{depth}(\mathfrak{m}, A) > 0$. The reason in the case where $\mathfrak{a}$ is a prime ideal, is that A is an integral domain. In the other case, this follows from Lemma III.3. Therefore $H^0_{\mathfrak{m}}(\widetilde{A}) = 0$, and the exact sequence of local cohomology (Proposition A.2) gives a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow H^1_{\mathfrak{m}}(\widetilde{A}) \longrightarrow 0. \tag{III.3}$$

By Proposition III.4 and Corollary III.1, the $A-$module $H^1_{\mathfrak{m}}(\widetilde{A})$ is of finite length. Therefore in the exact sequence (III.3), B has to be a finite $A-$module, as the other two terms are, that is, B is integral over $i(A)$. □

Proposition III.10. *Let $(R, \mathfrak{m})$ be a regular local ring, and let $\mathfrak{p}$ be a non-maximal prime ideal of R. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{p}$ by $(V, \mathscr{O}_V)$ and the $A-$algebra $\Gamma(V, \widetilde{A})$ by B. Let K be the field of fractions of A. Then*

(i) *B is an integral domain, $A \subset B \subset K$, and B is integral over A;*

(ii) *If Q is a non-maximal prime ideal of B, and if we set $\mathfrak{q} := Q \cap A$, then*

$$A_{\mathfrak{q}} = B_{\mathfrak{q}} = B_Q.$$

Proof. (i): Since $\operatorname{Spec}(A)$ is an integral scheme, $B = \Gamma(V, \widetilde{A})$, is an integral domain. Moreover, as in Example III.6 one has

$$B = \bigcap_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Therefore $A \subset B \subset K$. Finally, by Corollary III.1, the $A-$module $H^1_{\mathfrak{m}}(\widetilde{A})$ is of finite length. Thus, in the short exact sequence (III.3), B must be a finite $A-$module, as the other two terms are. That is, B is integral over A.

(ii): Let Q be a non-maximal prime ideal of B, and set $\mathfrak{q} := Q \cap A$. Since B is integral over A, the ideal $\mathfrak{q}$ is not maximal. By Proposition III.7, we have a canonical isomorphism $A_{\mathfrak{q}} \xrightarrow{\sim} B_{\mathfrak{q}}$. Since A is in addition a subring of B, this isomorphism is in fact, an equality.

It remains to show that $B_{\mathfrak{q}} = B_Q$. Since $A - \mathfrak{q} \subseteq B - Q$, and B is an integral domain, one has $B_{\mathfrak{q}} \subseteq B_Q$. For the reverse inclusion $B_Q \subseteq B_{\mathfrak{q}}$, we first show that if $x \in B$, then there is an element $t \in A - \mathfrak{q}$, such that $xt \in A$. This is because $x/1 \in B_{\mathfrak{q}} = A_{\mathfrak{q}}$, and therefore $x/1 = r/t$ for some $r/t \in A_{\mathfrak{q}}$. Since B is an integral domain, this means that $xt = r \in A$. Now, let $b/s \in B_Q$ be any element. Choose $t \in A - \mathfrak{q}$ such that $st \in A$. Since $B - Q$ is a multiplicatively closed set, we see that in fact $st \in A - \mathfrak{q}$. Thus, $b/s = bt/st \in B_{\mathfrak{q}}$. $\square$

Corollary III.11. *Let $(R, \mathfrak{m})$ be a regular local ring, and let $\mathfrak{p}$ be a prime ideal of R with $\operatorname{ht}_R(\mathfrak{p}) \leq \dim R - 2$. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{p}$*

by $(V, \mathscr{O}_V)$, the $A-$algebra $\Gamma(V, \widetilde{A})$ by B, and the field of fractions of A by K. Assume that for every non-maximal prime ideal $\mathfrak{q}$ of A,

$$\operatorname{depth}(\mathfrak{q}A_{\mathfrak{q}}, A_{\mathfrak{q}}) \geq \min(\operatorname{ht}_A(\mathfrak{q}), 2). \tag{III.4}$$

In addition, assume that A is regular in codimension ≤ 1. Then B is the integral closure of A in its field of fractions K.

Proof. By Proposition III.10 (i) we know that B is an integral domain, $A \subset B \subset K$, and B is integral over A. It remains to show that B is integrally closed. For this we will use Serre's criterion for normality, that is, $R_1 + S_2 \Rightarrow$ normal [Mat89, p. 183, Theorem 23.8]. Proposition III.10 (ii) and the assumption that A is regular in codimension ≤ 1, give the regularity of B in codimension ≤ 1. To establish the condition S_2 for B and complete the proof, we only need to show that for every *maximal* ideal $\mathfrak{n}$ of B, $\operatorname{depth}(\mathfrak{n}B_{\mathfrak{n}}, B_{\mathfrak{n}}) \geq 2$, because for non-maximal prime ideals the result follows from Proposition III.10 (ii) and assumption (III.4). In view of the formula [Gro67, p. 42, Corollary 3.6]

$$\operatorname{depth}(\mathfrak{m}B, B) = \inf_{\mathfrak{n} \in V(\mathfrak{m}B)} \big(\operatorname{depth}(\mathfrak{n}B_{\mathfrak{n}}, B_{\mathfrak{n}})\big), \tag{III.5}$$

it suffices to show $\operatorname{depth}_B(\mathfrak{m}B, B) \geq 2$, or equivalently, $\operatorname{depth}_A(\mathfrak{m}, B) \geq 2$. Since B is an integral domain containing A, one has $\operatorname{depth}_A(\mathfrak{m}, B) \geq 1$. Thus, $H^0_{\mathfrak{m}}(\widetilde{B}) = 0$, and the exact sequence of local cohomology (Proposition A.2) gives a short exact sequence

$$0 \longrightarrow B \xrightarrow{\rho} H^0(V, \widetilde{B}|_V) \longrightarrow H^1_{\mathfrak{m}}(\widetilde{B}) \longrightarrow 0.$$

Now, by Corollary III.8, $H^0(V, \widetilde{B}|_V) \cong H^0(V, \widetilde{A}|_V) = B$. Thus, the map ρ in the above sequence is an isomorphism, $H^1_{\mathfrak{m}}(\widetilde{B}) = 0$, and $\operatorname{depth}_A(\mathfrak{m}, B) \geq 2$. $\square$

III.3 Application to Hartshorne's conjecture

This section contains one of the main results (Theorem III.13) of this dissertation, which is an affirmative answer to a conjecture of Hartshorne (Conjecture 0.3) in a special case. We begin with a characterization of a certain class of semi-local Gorenstein rings, that is similar to the well-known characterization of Gorenstein local rings (Theorem B.3):

Proposition III.12. *Let R be a regular local ring of dimension n. Let $\mathfrak{a}$ be either a non-maximal prime ideal, or an ideal of R with following properties: the quotient ring $A := R/\mathfrak{a}$ is equidimensional, $\mathfrak{a}$ has no embedded prime ideals, and*

$$\mathrm{ht}(\mathfrak{a}) \leq \dim R - 2.$$

Denote the punctured spectrum of A by $(V, \mathscr{O}_V)$ and the $A-$algebra $\Gamma(V, \widetilde{A})$ by B, and let $c := \mathrm{ht}(\mathfrak{a})$. The following conditions are equivalent

(i) *B is Gorenstein (i.e. for every maximal ideal $\mathfrak{n}$ of B, the ring $B_{\mathfrak{n}}$ is Gorenstein);*

(ii) *B is Cohen-Macaulay and $\mathrm{Ext}^c_R(B, R) \cong B$ as $B-$modules.*

Proof. (i) $\Rightarrow$ (ii): Suppose B is Gorenstein. Then B is Cohen-Macaulay, and we only need to verify that $\mathrm{Ext}^c_R(B, R) \cong B$ as $B-$modules. By Proposition III.9 we know that B is integral over the local ring A, hence it is semi-local and every locally free $B-$module of rank one is free. Using this, we only need to show that $\mathrm{Ext}^c_R(B, R)$ with its natural $B-$module structure is locally free of rank one. Let $\mathfrak{n}$ be a maximal ideal of B. Applying Theorem B.4 we see that the canonical module $K_{B_{\mathfrak{n}}} \cong \left(\mathrm{Ext}^c_R(B, R)\right)_{\mathfrak{n}}$. On the other hand, since by assumption $B_{\mathfrak{n}}$ is a local Gorenstein ring, by Proposition B.3

we have $K_{B_{\mathfrak{n}}} \cong B_{\mathfrak{n}}$. Thus by uniqueness of canonical modules (Remark B.2), we obtain $\left(\operatorname{Ext}^c_R(B,R)\right)_{\mathfrak{n}} \cong B_{\mathfrak{n}}$ as $B-$modules.

(ii) $\Rightarrow$ (i): We need to show that for any maximal ideal $\mathfrak{n}$ of B, $B_{\mathfrak{n}}$ is a local Gorenstein ring. For this we use the characterization given in Proposition B.3. Since by assumption $\operatorname{Ext}^c_R(B,R) \cong B$ as $B-$modules, if we localize at $\mathfrak{n}$ and apply Theorem B.4 we see that

$$K_{B_{\mathfrak{n}}} \cong \left(\operatorname{Ext}^c_R(B,R)\right)_{\mathfrak{n}} \cong B_{\mathfrak{n}}.$$

Therefore $B_{\mathfrak{n}}$ is Gorenstein. □

Theorem III.13. *Let $(R,\mathfrak{m})$ be a regular local ring with $n := \dim R \geq 5$. Let $\mathfrak{p} \subset R$ be a prime ideal of codimension 2. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{p}$ by $(V, \mathscr{O}_V)$, and assume that the $A-$algebra $B := \Gamma(V, \mathscr{O}_V)$ is a Gorenstein ring. Then A is a complete intersection.*

Proof. Since A is a domain, $\operatorname{depth}(\mathfrak{m}, A) \geq 1$. If we knew that $\operatorname{depth}(\mathfrak{m}, A) > 1$, we would be done, because in that case by Proposition A.2 and Theorem A.3 we would have $A \cong \Gamma(V, \mathscr{O}_V)$, and A would be Gorenstein and since it is also of codimension two by assumption, it would be a complete intersection by Proposition I.20. Thus, the actual problem is to show that $\operatorname{depth}(\mathfrak{m}, A) > 1$. However, in the following proof we are not going to follow this line of reasoning. The proof that we present consists of two steps. In step 1 we show that V is the scheme of zeros of a section of a rank two vector bundle $\mathscr{E}$ on the punctured spectrum of R. Then, in step 2 we apply Theorem II.1 to show that $\mathscr{E}$ splits. This is equivalent to A being a complete intersection.

Step 1: The scheme $(V, \mathscr{O}_V)$ is a closed subscheme of the punctured spectrum of R, which we will denote by $(U, \mathscr{O}_U)$. Let $\jmath : V \hookrightarrow U$ and $j : \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(R)$ be the corresponding closed immersions. We verify that the conditions of Theorem I.21

are satisfied: V is irreducible since $\mathfrak{p}$ is prime. It is of codimension two by assumption. Since B is Gorenstein, $(V, \mathscr{O}_V)$ is locally a complete intersection by Propositions III.7 and I.20. Furthermore, the assumption that B is a Gorenstein ring implies by Proposition III.12, that $\operatorname{Ext}^2_R(B, R) \cong B$ as $B-$modules. Using this isomorphism and Corollary III.8 and some facts about local and global Ext functors for which we refer the reader to [Har77, p. 234, Proposition 6.2] and [Har77, p. 238, Exercise 6.7], one has

$$\mathscr{E}xt^2_{\mathscr{O}_U}(\jmath_*\mathscr{O}_V, \mathscr{O}_U) \cong \mathscr{E}xt^2_{\mathscr{O}_U}\big((j_*\widetilde{B})|_U, \widetilde{R}|_U\big) \cong \mathscr{E}xt^2_{\widetilde{R}}(j_*\widetilde{B}, \widetilde{R})|_U \cong$$

$$\cong \operatorname{Ext}^2_R(B, R)^{\sim}|_U \cong (j_*\widetilde{B})|_U \cong \jmath_*\mathscr{O}_V.$$

Thus, by Corollary I.22, V is the scheme of zeros of a section σ of a rank two vector bundle $\mathscr{E}$ on U.

<u>Step 2</u>: Let $\mathscr{P} := \widetilde{\mathfrak{p}}|_U$ be the ideal sheaf of V in U. Consider the Koszul complex defined by σ

$$0 \longrightarrow \textstyle\bigwedge^2 \mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{O}_U \longrightarrow \jmath_*\mathscr{O}_V \longrightarrow 0.$$

We break this complex into two short exact sequences

$$0 \longrightarrow \textstyle\bigwedge^2 \mathscr{E}^\vee \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{P} \longrightarrow 0 \tag{III.6}$$

and

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{O}_U \longrightarrow \jmath_*\mathscr{O}_V \longrightarrow 0. \tag{III.7}$$

We have $\bigwedge^2 \mathscr{E}^\vee \in \operatorname{Pic}(U)$, and as mentioned earlier in the proof of Corollary I.22, $\operatorname{Pic}(U)$ is trivial. Hence $\bigwedge^2 \mathscr{E}^\vee \cong \mathscr{O}_U$, and therefore $H^i(U, \bigwedge^2 \mathscr{E}^\vee) = 0$ for $1 \leq i \leq n-2$. Using this vanishing we see from the exact sequence (III.6) that

$$H^i(U, \mathscr{E}^\vee) \cong H^i(U, \mathscr{P}) \ \text{ for } \ 2 \leq i \leq n-3.$$

On the other hand the sequence (III.7) gives us

$$H^i(U, \mathscr{P}) \cong H^{i-1}(U, \jmath_* \mathscr{O}_V) \;\text{ for }\; 2 \leq i \leq n-2.$$

Thus, for $2 \leq i \leq n-3$ we have $H^i(U, \mathscr{E}^\vee) \cong H^{i-1}(U, \jmath_* \mathscr{O}_V)$. Using the fact that $\jmath_* \mathscr{O}_V \cong (j_* \widetilde{B})|_U$ (Corollary III.8) we obtain for $2 \leq i \leq n-3$

$$H^i(U, \mathscr{E}^\vee) \cong H^{i-1}(U, (j_* \widetilde{B})|_U) \cong H^i_{\mathfrak{m}}(\widetilde{B}).$$

By Proposition A.4, for every i there is an $R-$module isomorphism

$$H^i_{\mathfrak{m}}(\widetilde{B}) \cong H^i_{\mathfrak{m}B}(\widetilde{B}).$$

Claim III.14. $H^i_{\mathfrak{m}B}(\widetilde{B}) = 0$ for $i < n-2$.

Let's assume this claim for a moment. Then, $H^i(U, \mathscr{E}^\vee) = 0$ for $2 \leq i \leq n-3$. By Theorem II.1, this means that $\mathscr{E}^\vee$ splits. Hence, A is a complete intersection. To see this, notice that $\operatorname{depth}(\mathfrak{m}, A) \geq 1$, and the *Depth Lemma* [EG85, p. 13, Lemma 1.1] applied to the exact sequence $0 \to \mathfrak{p} \to R \to A \to 0$, gives $\operatorname{depth}(\mathfrak{m}, \mathfrak{p}) \geq 2$. Thus, $\Gamma(U, \widetilde{\mathfrak{p}}) \cong \mathfrak{p}$, and from (III.6) one obtains an exact sequence

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow \mathfrak{p} \longrightarrow 0,$$

which shows that $\mathfrak{p}$ can be generated by two elements. It remains to prove Claim III.14. For this we first recall a couple of definitions, and other results.

Definition III.15. *A Noetherian ring A is called* catenary, *if the following condition is satisfied: for any prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A with $\mathfrak{p} \subset \mathfrak{q}$, there exists a saturated chain of prime ideals starting from $\mathfrak{p}$ and ending at $\mathfrak{q}$, and all such chains have the same length. A Noetherian ring A is called* universally catenary, *if every finitely generated $A-$algebra is catenary.*

Proposition III.16. [Gro65, p. 99, Proposition IV,5.6.4] *Every quotient ring of a regular ring is universally catenary.*

Proposition III.17. [Gro65, p. 101, Proposition IV,5.6.10] *Let A be a Noetherian, integral, local, and universally catenary ring. Let B be an integral domain, which contains A, and is a finite $A-$algebra. Then for every maximal ideal $\mathfrak{n}$ of B, one has* $\dim(B_{\mathfrak{n}}) = \dim(A)$.

To prove Claim III.14, it suffices by Theorem A.3, to show that $\mathrm{depth}(\mathfrak{m}B, B) = n - 2$. By formula (III.5), this is equivalent to showing that for every maximal ideal $\mathfrak{n}$ of B, $\mathrm{depth}(\mathfrak{n}B_{\mathfrak{n}}, B_{\mathfrak{n}}) = n - 2$. Since B is a Gorenstein ring, this follows from Proposition III.17. □

Chapter IV

A different approach

In this chapter we consider the problem addressed in Theorem III.13 from a different point of view. Let $(R, \mathfrak{m})$ be a regular local ring of dimension $n \geq 4$, and let $\mathfrak{p}$ be a prime ideal of codimension 2 in R. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{p}$ by $(V, \mathscr{O}_V)$, the punctured spectrum of R by $(U, \mathscr{O}_U)$, and assume that the $A-$algebra $B := \Gamma(V, \widetilde{A})$ is a Gorenstein ring.

Claim IV.1. As an $R-$module, B has a minimal free resolution of the form

$$0 \longrightarrow R^m \longrightarrow R^{2m} \longrightarrow R^m \longrightarrow B \longrightarrow 0. \tag{IV.1}$$

Moreover, A is a complete intersection, if and only if $m = 1$ in (IV.1).

Proof. Let $\jmath : V \hookrightarrow U$ and $j : \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(R)$ be the corresponding closed immersions. To verify this claim, we first show that $\operatorname{proj.dim}_R B = 2$. By Proposition III.10, B is an integral domain, it contains A, and it is a finite $A-$algebra. Thus, by Proposition III.17, for every maximal ideal $\mathfrak{n}$ of B, $\dim B_{\mathfrak{n}} = \dim A = n - 2$. Therefore

$$\operatorname{depth}_B(\mathfrak{m}B, B) = \inf_{\mathfrak{n} \in V(\mathfrak{m}B)} \big(\operatorname{depth}(\mathfrak{n}B_{\mathfrak{n}}, B_{\mathfrak{n}})\big) = n - 2,$$

the last equality holding because B is Gorenstein (hence, Cohen-Macaulay). From this equality and Proposition A.4, one obtains

$$H^i_{\mathfrak{m}}(\widetilde{B}) \cong H^i_{\mathfrak{m}B}(\widetilde{B}) = 0, \quad \text{for } i < n-2;$$

that is, $\operatorname{depth}_R(\mathfrak{m}, B) = n-2$, or equivalently $\operatorname{proj.dim}_R B = 2$, by M. Auslander and D. Buchsbaum's formula.

Now, consider a minimal projective (= free) resolution of B over R, say

$$0 \longrightarrow R^k \longrightarrow R^\ell \longrightarrow R^m \longrightarrow B \longrightarrow 0. \tag{IV.2}$$

Since A is a domain, $H^0_{\mathfrak{m}}(\widetilde{A}) = 0$, and the local cohomology exact sequence of Proposition A.2 gives us the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\psi"] & H^0\big(U, (j_*\widetilde{A})|_U\big) \arrow[r] \arrow[d, "\wr"] & H^1_{\mathfrak{m}}(\widetilde{A}) \arrow[r] & 0. \\ & & B & & \end{tikzcd} \tag{IV.3}$$

To see that $H^0\big(U, (j_*\widetilde{A})|_U\big) \cong B$, one can first directly verify, that $(j_*\widetilde{A})|_U = \jmath_*\mathscr{O}_V$, by taking an arbitrary open subset $W \subseteq U$, and showing that $(j_*\widetilde{A})|_U(W) = \jmath_*\mathscr{O}_V(W)$. Then, since $\jmath$ is a closed immersion, and every closed immersion is an affine morphism [Gro61a, p. 14, Proposition II,1.6.2], one has an isomorphism

$$H^i(U, \jmath_*\mathscr{O}_V) \xrightarrow{\sim} H^i(V, \mathscr{O}_V),$$

for all $i \geq 0$ [Gro61b, p. 88, Corollary III,1.3.3]. In particular, for $i = 0$

$$H^0(U, \jmath_*\mathscr{O}_V) \cong H^0(V, \mathscr{O}_V) = B.$$

Now, from the short exact sequence (IV.3), and the fact that $\operatorname{Supp}_R H^i_{\mathfrak{m}}(\widetilde{B}) \subseteq \{\mathfrak{m}\}$ for all $i \geq 0$, one obtains

$$\operatorname{Supp}_R B = \operatorname{Supp}_R A = V(\mathfrak{p}).$$

Since R is Cohen-Macaulay, codimension and grade of any proper ideal are equal; hence $\operatorname{grade}_R \mathfrak{p} = 2$, and $\operatorname{Ext}^i_R(B, R) = 0$, for $i = 0, 1$ [Mat89, p. 129, Theorem 16.6]. Now, dualizing the exact sequence (IV.2) (i.e., applying the functor $\operatorname{Hom}_R(\,\cdot\,, R)$ to it) gives the following exact sequence

$$0 \longrightarrow R^m \longrightarrow R^\ell \longrightarrow R^k \longrightarrow \operatorname{Ext}^2_R(B, R) \longrightarrow 0. \tag{IV.4}$$

Since B is a Gorenstein ring by assumption, one has an isomorphism $\operatorname{Ext}^2_R(B, R) \cong B$, by Proposition III.12. Thus, (IV.4) is also a minimal projective resolution of B over R. Comparing (IV.2) to (IV.4), one must have $m = k$. Now, since $\operatorname{Ann}_R(B) \neq 0$, the Euler characteristic $\chi(B) = 0$, by a result of M. Auslander and D. Buchsbaum [Mat89, p. 160, Theorem 19.8]; hence, $\ell = 2m$. To finish the proof of claim IV.1, notice that if A is a complete intersection, then $\operatorname{depth}(\mathfrak{m}, A) = n - 2 \geq 2$. Hence, $B \cong A$, and B can be generated by one element, i.e., in the minimal resolution (IV.1), one has $m = 1$. On the other hand, if $m = 1$ in (IV.1), then B can be generated by one element, say b. In particular, $ab = 1$, for some $a \in A$. Since B is integral over A, b satisfies a *minimal* monic polynomial, say $b^d + a_{d-1}b^{d-1} + \cdots + a_1 b + a_0 = 0$ over A. Multiplying both sides of this equality by a, one sees that b satisfies a monic polynomial of lesser degree over A, which is a contradiction, unless $b \in A$. But then the element 1 is also a generator of B, and the map ψ in (IV.3) is surjective, hence an isomorphism; that is, $A \cong B$, and A has a minimal free resolution of the form $0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \longrightarrow A \longrightarrow 0$. This resolution shows that the ideal $\mathfrak{p}$ can be generated by two elements. Thus, A is a complete intersection. $\square$

Next, let R be a Noetherian ring and M a finite $R-$module with a given presentation

$$R^n \xrightarrow{\varphi} R^m \longrightarrow M \longrightarrow 0.$$

After choosing bases for R^n and R^m, one can represent the map φ by an $m \times n$ matrix. For any integer j we denote the ideal generated by the $j \times j$ minors of φ by $\Delta_j(\varphi)$. We make the convention that $\Delta_j(\varphi) = R$ for $j \leq 0$.

Definition IV.2. *For any integer $0 \leq j < \infty$, the j^{th} Fitting ideal of M, $\mathrm{Fitt}_j(M)$ is defined to be the ideal $\Delta_{m-j}(\varphi)$.*

We recall that $\mathrm{Fitt}_j(M)$ does not depend on the particular presentation of M that is used. Also, a finite $R-$module M can be generated by j elements, if and only if $\mathrm{Fitt}_j(M) = R$.

Proposition IV.3. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension ≥ 3, and let $\mathfrak{p}$ be a prime ideal of codimension 2 in R. Denote the punctured spectrum of the quotient ring $A := R/\mathfrak{p}$ by $(V, \mathscr{O}_V)$, and assume that the $A-$algebra $B := \Gamma(V, \widetilde{A})$ is a Gorenstein ring. Consider a minimal free resolution of B over R of the form* (IV.1)*:*

$$0 \longrightarrow R^m \longrightarrow R^{2m} \xrightarrow{\varphi} R^m \longrightarrow B \longrightarrow 0,$$

where φ is represented by an $m \times 2m$ matrix. Then, $\Delta_{m-1}(\varphi)$ is an $\mathfrak{m}-$primary ideal.

Proof. Since $\Delta_{m-1}(\varphi) = \mathrm{Fitt}_1(B)$, to say that $\Delta_{m-1}(\varphi)$ is $\mathfrak{m}-$primary is equivalent to saying that B is generated by one element over V, or that the $\mathscr{O}_V-$module, $\widetilde{B}|_V$ is a line bundle on V. This was proved in Proposition III.7. $\square$

We don't know the answer to the following more general problem:

Conjecture IV.4. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension ≥ 4. Let M be a finite $R-$module, having a* self-dual *minimal free resolution of the form*

$$0 \longrightarrow R^m \longrightarrow R^{2m} \xrightarrow{\varphi} R^m \longrightarrow M \longrightarrow 0.$$

Here self-dual means that $\operatorname{Ext}^i_R(M,R) = 0$ *for* $i = 0,1$, *and there is an isomorphism* $\operatorname{Ext}^2_R(M,R) \cong M$. *If* $\Delta_{m-1}(\varphi)$ *is an* $\mathfrak{m}-$*primary ideal, then* $\dim R = 4$.

In view of Claim IV.1 and Proposition IV.3, a positive answer to Conjecture IV.4 will provide a new proof for Theorem III.13. In connection with Conjecture IV.4 we prove the following proposition, which is different from the Conjecture, both in assumptions, and in assertions.

Proposition IV.5. *Let* M *be a finite module over a Noetherian ring* R, *with a minimal free resolution of the form*

$$0 \longrightarrow R^m \xrightarrow{\alpha} R^{2m} \xrightarrow{\beta} R^m \longrightarrow M \longrightarrow 0,$$

where α *and* β *are* $2m \times m$ *and* $m \times 2m$ *matrices, respectively, and have the property*

$$V(\Delta_{m-2}(\alpha)) = V(\Delta_{m-2}(\beta)). \tag{IV.5}$$

If $V(\operatorname{Fitt}_1(M)) \neq V(\operatorname{Fitt}_2(M))$, *then* $\operatorname{ht}_R(\operatorname{Fitt}_1(M)) \leq 8$.

Proof. Let $N := \operatorname{Im}(\beta)$. Then one has the exact sequences

$$0 \longrightarrow R^m \xrightarrow{\alpha} R^{2m} \longrightarrow N \longrightarrow 0 \tag{IV.6}$$

and

$$0 \longrightarrow N \longrightarrow R^m \longrightarrow M \longrightarrow 0.$$

Let $\mathfrak{p} \in V(\operatorname{Fitt}_1(M)) - V(\operatorname{Fitt}_2(M))$ be a prime ideal in R. Such a $\mathfrak{p}$ exists, because $V(\operatorname{Fitt}_1(M)) \supset V(\operatorname{Fitt}_2(M))$, and we have assumed that $V(\operatorname{Fitt}_1(M)) \neq V(\operatorname{Fitt}_2(M))$. This $\mathfrak{p}$ may not contain all the minimal prime ideals of $\operatorname{Fitt}_1(M)$, therefore in general $\operatorname{ht}_R(\operatorname{Fitt}_1(M)) \leq \operatorname{ht}_{R_\mathfrak{p}}(\operatorname{Fitt}_1(M_\mathfrak{p}))$. We will show $\operatorname{ht}_{R_\mathfrak{p}}(\operatorname{Fitt}_1(M_\mathfrak{p})) \leq 8$.

Localizing at $\mathfrak{p}$, one obtains $\mathrm{Fitt}_2(M_\mathfrak{p}) = R_\mathfrak{p}$. Hence, by (IV.5) and (IV.6), one has $\mathrm{Fitt}_{m+2}(N_\mathfrak{p}) = R_\mathfrak{p}$, as well; that is, $N_\mathfrak{p}$ can be generated by $m+2$ elements over $R_\mathfrak{p}$. Thus, $M_\mathfrak{p}$ has a presentation of the form

$$\begin{tikzcd}
R_\mathfrak{p}^{m+2} \arrow[rr, "\gamma"] \arrow[dr] & & R_\mathfrak{p}^{m} \arrow[r] & M_\mathfrak{p} \arrow[r] & 0 \\
& N_\mathfrak{p} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where γ is represented by an $m\times(m+2)$ matrix. By Eagon-Northcott's [Mat89, p. 103] inequality for height of determinantal ideals one obtains

$$\mathrm{ht}_{R_\mathfrak{p}}(\mathrm{Fitt}_1(M_\mathfrak{p})) = \mathrm{ht}_{R_\mathfrak{p}}(\Delta_{m-1}(\gamma)) \leq \big(m+2-(m-1)+1\big)\big(m-(m-1)+1\big) = 8,$$

which finishess the proof. $\square$

Appendix A

Local cohomology

In this appendix we present a quick review of local cohomology. For omitted proofs and more details we refer the reader to [Gro67], [Gro68] and [Szp71]. We are mainly interested in *finiteness* properties of local cohomology modules $H^i_{\mathfrak{m}}(\widetilde{M})$, where $(R, \mathfrak{m})$ is a Noetherian local ring, and M is a finite $R-$module.

Let X be a topological space, let Y be a locally closed subspace of X, and let $\mathscr{F}$ be a sheaf of abelian groups on X. Denote the category of all sheaves of abelian groups on X by $\mathscr{C}(X)$, and the category of abelian groups by $(\mathscr{A}b)$. Choose an open subset $V \subseteq X$ such that $Y \subseteq V$, and Y is closed in V. This is possible since Y is locally closed in X. Let $\Gamma_Y(X, \mathscr{F})$ be the subgroup of $\Gamma(V, \mathscr{F})$ consisting of all those sections of $\mathscr{F}$ over V, whose support is contained in Y. One can check that $\Gamma_Y(X, \mathscr{F})$ is independent of the subset V chosen above, and the functor $\mathscr{F} \rightsquigarrow \Gamma_Y(X, \mathscr{F})$ from $\mathscr{C}(X)$ to $(\mathscr{A}b)$ is left-exact.

Definition A.1. [Gro67, p. 2] *The right derived functors of* $\Gamma_Y(X, \cdot)$ *are denoted by* $H^i_Y(X, \cdot)$, $i = 0, 1, 2, \cdots$, *and are called the cohomology groups of* X *with support in* Y.

We recall that to calculate $H^i_Y(X, \mathscr{F})$ for $\mathscr{F} \in \mathscr{C}(X)$, one can take an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$ in the category $\mathscr{C}(X)$, and apply the functor $\Gamma_Y(X, \cdot)$ to the complex $\mathscr{I}^\bullet$ to obtain

$$0 \longrightarrow \Gamma_Y(X, \mathscr{I}^0) \longrightarrow \Gamma_Y(X, \mathscr{I}^1) \longrightarrow \Gamma_Y(X, \mathscr{I}^2) \longrightarrow \cdots \longrightarrow \Gamma_Y(X, \mathscr{I}^i) \longrightarrow \cdots$$

and take the $i-$th cohomology group of this complex. The result, which turns out to be independent of the choice of injective resolution $\mathscr{I}^\bullet$, is $H^i_Y(X, \mathscr{F})$.

Now, let $X = \mathrm{Spec}(R)$ be an affine scheme, let $Y = V(\mathfrak{a})$ be the closed set defined by an ideal $\mathfrak{a}$ of R, and let M be an $R-$module. In this situation the local cohomology groups $H^i_Y(X, \widetilde{M})$ have an extra $R-$module structure. In what follows we will denote these $R-$modules simply by $H^i_{\mathfrak{a}}(\widetilde{M})$.

Proposition A.2. [Gro67, p. 17, Proposition 2.2] *Let R be a commutative ring with identity, and let $\mathfrak{a}$ be an ideal of R. Let $U \subset \mathrm{Spec}(R)$ be the open set $\mathrm{Spec}(R) - V(\mathfrak{a})$. Then for any $R-$module M there is an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(\widetilde{M}) \longrightarrow M \longrightarrow H^0(U, \widetilde{M}|_U) \longrightarrow H^1_{\mathfrak{a}}(\widetilde{M}) \longrightarrow 0,$$

and there are isomorphisms

$$H^i(U, \widetilde{M}|_U) \cong H^{i+1}_{\mathfrak{a}}(\widetilde{M}), \qquad i > 0.$$

Local cohomology is depth sensitive. To be more precise, we have the following

Theorem A.3. [Gro67, p. 44, Theorem 3.8] *Let R be a Noetherian commutative ring with identity, let $\mathfrak{a}$ be an ideal of R, and let M be a finite $R-$module. Also, let n be an integer. Then the following statements are equivalent*

(i) $H^i_{\mathfrak{a}}(\widetilde{M}) = 0$ *for all $i < n$;*

(ii) $\operatorname{depth}(\mathfrak{a}, M) \geq n$.

Local cohomology behaves nicely with respect to base change:

Proposition A.4. [Gro67, p. 74, Corollary 5.7] *Let* $R \xrightarrow{\varphi} R'$ *be a homomorphism of commutative rings with identity, let* $\mathfrak{a}$ *be an ideal in* R*, and let* M *be an* $R'-$*module. Then for every* $i \geq 0$ *there is an isomorphism*

$$H^i_{\mathfrak{a}}(\widetilde{M^R}) \cong H^i_{\varphi(\mathfrak{a})R'}(\widetilde{M})^R,$$

where a superscript R *applied to an* $R'-$*module means that module considered as an* $R-$*module by restriction of scalars.*

One of the powerful tools in local cohomology is the *local duality* theorem. To state this theorem, and for other uses, we introduce the notion of *dualizing functor*.

Proposition A.5. [Gro67, p. 55, Proposition 4.9] *Let* R *be a commutative Noetherian ring with identity. Let* $\mathfrak{a}$ *be an ideal of finite colength in* R *(i.e., such that* $R/\mathfrak{a}$ *is an Artinian ring). Let* $\mathscr{C}^f_{\mathfrak{a}}$ *be the category of finite* $R-$*modules with support in* $V(\mathfrak{a})$*, and denote its opposite category by* $\mathscr{C}^{f\circ}_{\mathfrak{a}}$*. Let* T *be a left-exact, additive, covariant functor from* $\mathscr{C}^{f\circ}_{\mathfrak{a}}$ *to category of abelian groups. Then the following statements are equivalent*

(i) *For all* $M \in \mathscr{C}^f_{\mathfrak{a}}$*,* $T(M)$ *is a finite* $R-$*module, and there is a canonical isomorphism*

$$M \xrightarrow{\sim} T(T(M));$$

(ii) T *is exact, and for each field* k *of the form* $R/\mathfrak{m}$*, where* $\mathfrak{m}$ *is a maximal ideal containing* $\mathfrak{a}$*, there is some isomorphism* $T(k) \cong k$*.*

Definition A.6. *Let R be a commutative Noetherian ring with identity, and let $\mathfrak{m}$ be a maximal ideal. A left-exact, additive, covariant functor T from $\mathscr{C}_{\mathfrak{m}}^{f\circ}$ to category of abelian groups, satisfying the equivalent conditions of proposition A.5, is called a dualizing functor for R at $\mathfrak{m}$. An injective $R-$module E with support in $\{\mathfrak{m}\}$ is called a dualizing module for R at $\mathfrak{m}$, if the functor $T = \operatorname{Hom}_R(\,\cdot\,, E)$ is a dualizing functor for R at $\mathfrak{m}$.*

We now recall the definition of an injective hull:

Definition A.7. *Let R be a commutative ring with identity. An injection $M \hookrightarrow P$ of $R-$modules is an* essential extension*, if whenever N is a submodule of P such that $M \cap N = 0$, then $N = 0$. An* injective hull *of M is an essential extension $M \hookrightarrow E$, where E is injective.*

It was proved in [ES53], that every $R-$module has an injective hull, which is unique up to a (non-unique) isomorphism. One can use this result, to show that a dualizing module for a Noetherian local ring always exists:

Proposition A.8. [Gro67, p. 59, Proposition 4.10] *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then, an $R-$module E is dualizing for R at $\mathfrak{m}$, if and only if it is an injective hull of the residue field k of R.*

We now state and prove a few important facts about dualizing modules for Noetherian local rings. The following lemma is in fact part of Proposition A.8. Nevertheless, we give a proof of it below.

Lemma A.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E be an injective hull of its residue field k. Then* $\operatorname{Supp} E = \{\mathfrak{m}\}$.

Proof. It suffices to show that $\operatorname{Ass} E = \{\mathfrak{m}\}$. If $\mathfrak{p} \in \operatorname{Ass} E$, then E has a submodule E' isomorphic to $R/\mathfrak{p}$. Since $k \hookrightarrow E$ is an essential extension, we have $k \cap E' \neq \{0\}$. Let $s \neq 0$ be an element of this intersection. Since $s \in k$, we have $\operatorname{Ann}_R(s) = \mathfrak{m}$. On the other hand, since $s \in E' \cong R/\mathfrak{p}$, and $R/\mathfrak{p}$ is a domain, we have $\operatorname{Ann}_R(s) = \mathfrak{p}$. This shows that $\mathfrak{p} = \mathfrak{m}$. □

Lemma A.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E be an injective hull of its residue field k. Then $\operatorname{Hom}_R(E, E) \cong \widehat{R}$, the completion of R for the $\mathfrak{m}-$adic topology.*

Proof. By Lemma A.9 we know that every element of E is annihilated by a power of $\mathfrak{m}$. Let $x \in E$, and assume that $\mathfrak{m}^r x = 0$, for $r \geq n$. Then we can define a homomorphism $\varphi_x : R/\mathfrak{m}^n \to E$ by sending 1 to x, and it is easy to check that the correspondence $x \to [\varphi_x]$ defines an isomorphism

$$E \xrightarrow{\sim} \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{m}^n, E).$$

Applying the functor $\operatorname{Hom}_R(\,\cdot\,, E)$ to each side of the above isomorphism we get

$$\operatorname{Hom}_R(E, E) \cong \operatorname{Hom}_R(\varinjlim_n \operatorname{Hom}_R(R/\mathfrak{m}^n, E), E) \cong \varprojlim_n \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{m}^n, E), E),$$

and since $R/\mathfrak{m}^n$ is of finite length and $\operatorname{Hom}_R(\,\cdot\,, E)$ is a dualizing functor for R, we obtain

$$\operatorname{Hom}_R(E, E) \cong \varprojlim_n R/\mathfrak{m}^n = \widehat{R}.$$

□

Lemma A.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let E be an injective hull of its residue field k. Then*

(i) *For any $R-$module M,* $\operatorname{Hom}_R(M, E) = 0 \Leftrightarrow M = 0$.

(ii) *E is an Artinian $R-$module.*

Proof. (i): If $M = 0$, it is clear that $\operatorname{Hom}_R(M, E) = 0$. Conversely, suppose we have $\operatorname{Hom}_R(M, E) = 0$. If $M \neq 0$, let $0 \neq x \in M$. Since $\operatorname{Ann}_R(x) \subseteq \mathfrak{m}$, there is a nonzero homomorphism $Rx \cong R/\operatorname{Ann}(x) \longrightarrow R/\mathfrak{m} \cong k$. Since $k \hookrightarrow E$, we get a nonzero homomorphism $Rx \longrightarrow E$, which lifts to a nonzero homomorphism $M \longrightarrow E$, because E is injective. This is a contradiction.

$$\begin{tikzcd} 0 \arrow[r] & Rx \arrow[r] \arrow[d] & M \arrow[dl] \\ & E & \end{tikzcd}$$

(ii): First notice that since R is Noetherian, so is $\operatorname{Hom}_R(E, E) \cong \widehat{R}$. Let

$$E = M_0 \supset \cdots \supset M_n \supset M_{n+1} \supset \cdots$$

be a descending chain of submodules of E. Consider the exact sequences

$$0 \longrightarrow M_n \longrightarrow E \longrightarrow E_n \longrightarrow 0, \quad n \geq 0$$

where $E_n := E/M_n$. We apply the exact functor $\operatorname{Hom}_R(\,\cdot\,, E)$ to these sequences

$$0 \longrightarrow \operatorname{Hom}_R(E_n, E) \longrightarrow \widehat{R} \longrightarrow \operatorname{Hom}_R(M_n, E) \longrightarrow 0. \tag{A.1}$$

The surjection $E_{n+1} \to E_n$ induces an injection $\operatorname{Hom}_R(E_n, E) \hookrightarrow \operatorname{Hom}_R(E_{n+1}, E)$ for each n, and we get an ascending chain

$$0 \subset \operatorname{Hom}_R(E_1, E) \subset \cdots \subset \operatorname{Hom}_R(E_n, E) \subset \operatorname{Hom}_R(E_{n+1}, E) \subset \cdots \subset \operatorname{Hom}_R(E, E) \cong \widehat{R}$$

of submodules of $\widehat{R}$, which must be stationary, that is, there is an n_0 such that

$$\operatorname{Hom}_R(E_n, E) = \operatorname{Hom}_R(E_{n+1}, E) \ \text{ for } n \geq n_0.$$

Then from exact sequences (A.1) we see that we also have

$$\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) \quad \text{for } n \geq n_0.$$

Now applying the exact functor $\mathrm{Hom}_R(\,\cdot\,, E)$ to the exact sequence

$$0 \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow M_n/M_{n+1} \longrightarrow 0$$

we conclude that $\mathrm{Hom}_R(M_n/M_{n+1}, E) = 0$. Hence by (i), $M_n = M_{n+1}$ for $n \geq n_0$. $\square$

Lemma A.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Fix a dualizing module E for R and let $D(\,\cdot\,)$ be the associated dualizing functor* $\mathrm{Hom}_R(\,\cdot\,, E)$*. Let M be an $R-$module.*

(i) *If M is finite over R, then $D(M)$ is Artinian;*

(ii) *If M is Artinian, then $D(M)$ is finite over $\widehat{R}$;*

(iii) *If M is finite over R, then $D^2(M) = D(D(M)) \cong \widehat{M}$.*

Proof. (i): If M is finite, there is a surjection $R^n \to M \to 0$, from which applying the exact functor D, we obtain an injection $0 \to D(M) \to D(R^n)$. Using the isomorphism $D(R) = \mathrm{Hom}_R(R, E) \cong E$, we see that $D(M) \hookrightarrow E^n$. Since by Lemma A.11, E is Artinian, so is $D(M)$.

(ii): If M is Artinian, there is an injection $M \hookrightarrow E^m$ [Str90, p. 44, Corollary 3.4.11], for some m. Applying the exact functor D to this, gives a surjection $D(E^m) \to D(M) \to 0$. By Lemma A.10, $D(E^m) \cong \widehat{R}^m$. Thus, we get the result.

(iii): Let $R^n \xrightarrow{\varphi} R^m \to M \to 0$ be a presentation of M. Applying the exact functor D^2 to this presentation, and noting that

$$D(R) = \mathrm{Hom}_R(R, E) \cong E, \quad \text{and} \quad D^2(R) \cong D(E) = \mathrm{Hom}_R(E, E) \cong \widehat{R},$$

we obtain an exact sequence

$$\begin{array}{ccccccc} D^2(R^n) & \xrightarrow{D^2(\varphi)} & D^2(R^m) & \longrightarrow & D^2(M) & \longrightarrow & 0 \\ \| \wr & & \| \wr & & \| & & \\ \widehat{R}^n & \xrightarrow{D^2(\varphi)} & \widehat{R}^m & \longrightarrow & D^2(M) & \longrightarrow & 0. \end{array}$$

On the other hand, since passing to the $\mathfrak{m}-$adic completion is an exact functor in the category of finite $R-$modules, we also have an exact sequence

$$\widehat{R}^n \xrightarrow{\widehat{\varphi}} \widehat{R}^m \longrightarrow \widehat{M} \longrightarrow 0.$$

One checks that the map $D^2(\varphi)$ is induced from φ, by passing to the $\mathfrak{m}-$adic completion. This gives the result, because it shows that $D^2(\varphi)$ and $\widehat{\varphi}$ must have isomorphic cokernels. $\square$

Theorem A.13 (Local Duality). [Gro67, p. 85, Theorem 6.3] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n. Fix a dualizing module E for R, and let $D(\,\cdot\,)$ be the associated dualizing functor* $\mathrm{Hom}_R(\,\cdot\,, E)$*. Let M be a finite $R-$module. Then for $i \geq 0$ there are isomorphisms*

$$H^i_{\mathfrak{m}}(\widetilde{M}) \xrightarrow{\sim} D\big(\operatorname{Ext}^{n-i}_R(M, R)\big),$$

and

$$\widehat{\operatorname{Ext}^{n-i}_R(M, R)} \xrightarrow{\sim} D\big(H^i_{\mathfrak{m}}(\widetilde{M})\big).$$

Corollary A.14. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n, and let M be a finite $R-$module. Then $H^i_{\mathfrak{m}}(\widetilde{M})$ is an Artinian $R-$module for $i \geq 0$.*

Proof. This is follows from Local Duality Theorem, and Lemma A.12 (i). □

Theorem A.15. [Gro68, Exposé v, Proposition 3.5] *Let* $(R, \mathfrak{m})$ *be a regular local ring, and let* A *be quotient ring of* R *by an ideal. Let* M *be an* $A-$*module of finite type. The following statements are equivalent*

(i) $H^i_{\mathfrak{m}}(M)$ *is an* $A-$*module of finite length;*

(ii) $H^{i-\dim A/\mathfrak{q}}_{\mathfrak{q}A_{\mathfrak{q}}}(M_{\mathfrak{q}}) = 0$ *for all* $\mathfrak{q} \in \operatorname{Spec}(A) - \{\mathfrak{m}\}$.

Appendix B

Canonical modules

Definition B.1. [Wal71, p. 47, Definition 5.6] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension n, and let $(\widehat{R}, \widehat{\mathfrak{m}})$ be its $\mathfrak{m}-$adic completion. Fix a dualizing module E for $\widehat{R}$ at $\widehat{\mathfrak{m}}$, and let $D_{\widehat{R}}$ be the associated dualizing functor. An $R-$module K_R is called a* canonical module *of R, if the $\widehat{R}-$module $K_R \otimes_R \widehat{R}$ represents the functor $D_{\widehat{R}}(H^n_{\widehat{\mathfrak{m}}}(\,\cdot\,))$, that is, if for every $\widehat{R}-$module M there is a functorial isomorphism*

$$D_{\widehat{R}}(H^n_{\widehat{\mathfrak{m}}}(\widetilde{M})) \cong \operatorname{Hom}_{\widehat{R}}(M, K_R \otimes_R \widehat{R}).$$

Remark B.2. [Wal71, p. 48, Remark 5.7] If a canonical module of a local ring R exists, it is (up to isomorphism) uniquely determined. Therefore we may speak of *the* canonical module K_R.

Proposition B.3. [Wal71, Theorem 5.9, page 48] *For a Cohen-Macaulay local ring R the following statements are equivalent:*

(i) *The canonical module K_R exists and is equal to R;*

(ii) *R is a Gorenstein ring.*

Theorem B.4. [Wal71, Theorem 5.12, p. 51] *Let $(R, \mathfrak{m})$ and S be local rings, with a local homomorphism $\varphi : R \to S$. Suppose there is a ring $B \subseteq S$ with $\varphi(R) \subseteq B$, such* *that B is finite over R and S is the localization of B at a maximal ideal $\mathfrak{n}$ of B. Furthermore suppose* $\dim R = n$, $\dim S = n - c$ *and duality holds for $\widehat{R}$ up to exponent c, that is*

$$\mathrm{Hom}_{\widehat{R}}(H^{n-j}_{\widehat{\mathfrak{m}}}(\ \cdot\ ), E_{\widehat{R}}) = \mathrm{Ext}^j_{\widehat{R}}(\ \cdot\ , K_{\widehat{R}})$$

for $0 \le j \le c$. *($E_{\widehat{R}}$is the injective envelope of the residue field of $\widehat{R}$.)*
Under these conditions, if the canonical module K_R of R exists, then K_S also exists and

$$K_S \cong \left(\mathrm{Ext}^c_R(B, K_R)\right)_{\mathfrak{n}}$$

where we consider $\mathrm{Ext}^c_R(B, K_R)$ *with its natural $B-$module structure.*

Appendix C

Koszul complex

Let R be a commutative Noetherian ring with unity and let L be an $R-$module. Given an $R-$linear form $u : L \to R$, we can define an $R-$linear map $d_u : \bigwedge_R L \to \bigwedge_R L$ as following: for $x_1, \cdots, x_r \in L$

$$d_u(x_1 \wedge \cdots \wedge x_r) = \sum_{i=1}^{r} (-1)^{i+1} u(x_i)\ x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_r. \tag{C.1}$$

One can check that d_u is an antiderivation of degree (-1), and that $d_u \circ d_u = 0$. It is the unique antiderivation of the $R-$algebra $\bigwedge_R L$ that extends $u : \bigwedge_R^1 L \to \bigwedge_R^0 L$. We obtain a complex

$$\cdots \longrightarrow \bigwedge_R^r L \xrightarrow{d_u} \bigwedge_R^{r-1} L \longrightarrow \cdots \longrightarrow \bigwedge_R^1 L \xrightarrow{d_u} \bigwedge_R^0 L \longrightarrow 0,$$

Definition C.1. [Bou80, p. 147, Section 9.1] *The complex* $(\bigwedge_R L, d_u)$ *is called the* Koszul complex *associated with L and u. It will be denoted by* $\mathbf{K}^R(u)$.

Remark C.2. We have $H_0(\mathbf{K}^R(u)) = \operatorname{Coker} u = R/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal $u(L)$ of R.

The next two propositions concern acyclicity of the Koszul complex $\mathbf{K}^R(u)$:

Proposition C.3. [Bou80, p. 148, Corollary 1] *If the $R-$linear form $u : L \to R$ is surjective, then the Koszul complex $\mathbf{K}^R(u)$ is homotopic to zero, and is therefore exact, that is, $H_i(\mathbf{K}^R(u)) = 0$, for $i \geq 0$.*

Proposition C.4. [Bou80, p. 157, Proposition 5] *Let L be a free $R-$module, and let $\{e_1, \cdots, e_n\}$ be an* ordered *basis of L. Let $u : L \to R$ be an $R-$linear form, and $x_i := u(e_i)$. If the elements $(x_1, \cdots, x_n)$ form an $R-$regular sequence, then the Koszul complex $\mathsf{K}^R(u)$ is acyclic, that is, $H_i(\mathsf{K}^R(u)) = 0$, for $i > 0$. In this situation the complex $\mathsf{K}^R(u)$ gives a free resolution of $R/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal $u(L)$ of R.*

Appendix D

Künneth formula

In this appendix we review the Künneth formula. Our main reference for the material presented here is [Bou80, p. 62 and p. 79].

Let B be a commutative ring. Let $(\mathscr{C}, d)$ and $(\mathscr{C}', d')$ be complexes of $B-$modules, and let $(\mathscr{C} \otimes_B \mathscr{C}', D)$ be their tensor product. By definition we have

$$(\mathscr{C} \otimes_B \mathscr{C}')_n = \oplus_{p+q=n} \, (\mathscr{C}_p \otimes_B \mathscr{C}'_q),$$

and

$$D(x \otimes x') = dx \otimes x' + (-1)^p x \otimes d'x', \quad x \in \mathscr{C}_p, x' \in \mathscr{C}'_q, p, q \in \mathbb{Z}.$$

For $x \in Z_p(\mathscr{C})$ and $x' \in Z_q(\mathscr{C}')$ one checks that the element $x \otimes x'$ of $\mathscr{C}_p \otimes_B \mathscr{C}'_q$ belongs to $Z_{p+q}(\mathscr{C} \otimes_B \mathscr{C}')$. Moreover, if $y \in \mathscr{C}_{p+1}$ and $y' \in \mathscr{C}'_{q+1}$, then

$$(x + dy) \otimes (x' + d'y') = x \otimes x' + D(y \otimes x' + (-1)^p (x + dy) \otimes y').$$

By passing to the quotient we get a well-defined $B-$linear mapping

$$\gamma_{p,q}(\mathscr{C}, \mathscr{C}') : H_p(\mathscr{C}) \otimes_B H_q(\mathscr{C}') \longrightarrow H_{p+q}(\mathscr{C} \otimes_B \mathscr{C}').$$

If we equip $H(\mathscr{C}) \otimes_B H(\mathscr{C}')$ with the grading

$$(H(\mathscr{C}) \otimes_B H(\mathscr{C}'))_n = \oplus_{p+q=n} \, H_p(\mathscr{C}) \otimes_B H_q(\mathscr{C}'),$$

then the $\gamma_{p,q}$ define a graded $B-$linear map of degree 0

$$\gamma(\mathscr{C},\mathscr{C}') : H(\mathscr{C}) \otimes_B H(\mathscr{C}') \longrightarrow H(\mathscr{C} \otimes_B \mathscr{C}'),$$

which we call *canonical*.

Theorem D.1 (Künneth formula)**.** [Bou80, p. 79, Corollary 4] *Let B be a commutative ring. Let $(\mathscr{C}, \mathrm{d})$ and $(\mathscr{C}', \mathrm{d}')$ be two complexes of $B-$modules. Assume that $\mathscr{C}$ is bounded from the right and that $\mathscr{C}$ and its homology $H(\mathscr{C})$ consist of flat modules. Then the canonical homomorphism*

$$\gamma(\mathscr{C},\mathscr{C}') : H(\mathscr{C}) \otimes_B H(\mathscr{C}') \longrightarrow H(\mathscr{C} \otimes_B \mathscr{C}')$$

is an isomorphism.

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