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City University of New York, 1989

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ON THE STOCHASTIC SEQUENTIAL AND NON-SEQUENTIAL PRODUCTION PLANNING PROBLEM

by

SHIVAJI RAO

A dissertation submitted to the Graduate Faculty in Business in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

1989

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This manuscript has been read and accepted for the Graduate Faculty in Business in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

ON THE STOCHASTIC SEQUENTIAL AND NON-SEQUENTIAL PRODUCTION PLANNING PROBLEM

by

Shivaji Rao

Adviser: Prof. George O. Schneller IV.

This dissertation examines a stochastic sequential and a non-sequential capacitated production planning problem (Bitran and Yanasse, Operations Research, 32, 5, 1984) where the demand of each period is a continuous random variable. The stochastic non-sequential production planning problem is at first examined with sequence independent and then with sequence dependent set-up costs and the worst case error determined when an approximate solution is obtained by solving the deterministic equivalent. We prove in general that the worst case error is not dependent on the nature of the set-up cost. Based on a result due to Huang, Ziemba and Ben-Tal (Operations Research, 25, 2, 1977) we identify a family of approximations for both the stochastic sequential and the stochastic non-sequential production planning problem. We find a problem which bounds the stochastic sequential problem of two period from above: the upper bound coupled with Bitran and Yanasses' (Operations Research, 32, 5, 1984) lower bound enable us to perform worst-case analysis. Given uniformly distributed demand, this analysis produces results with in 23% of optimality. Finally, we derive conditions such that an order-up-to the service level policy is optimal for the T-period stochastic sequential capacitated production planning problem.

Dedicated to
the subjugated people of South Africa
and their struggle for equality
and human dignity.

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INTRODUCTION

1.1 Nonsequential, Capacitated, Production Planning Problem

In a recent review of lot-sizing problems by (Bahl,Ritzman and Gupta, 1987), the authors note that future research must be directed to solve more realistic lot-sizing problems. One of the issues, they refer to is the incorporation of uncertainty (Stochastic Demand Distribution) and they suggest the development of heuristics and approximations.

Further, they differentiate between the Uncapacitated and the Capacitated Lot-size problem. They contend, that since the Uncapacitated, single product Lot-size problem is not NP-Complete, it is fairly tractable (Bitran and Yanasse, 1982). The Wagner-Whitin Algorithm(1958), and a number of heuristics, such as the, Lot-for-Lot, Modified EOQ, Periodic Order Quantity, Least Unit Cost, Part Period Balancing and the Silver and Meal (1973), perform well computationally.

The difficulty, they note, lies in developing reasonable solutions for the stochastic and the deterministic version of the capacitated lot-size production planning problem, because even the deterministic, capacitated production lot-size problem is NP-Hard (Florian, Lenstra, Rinnooy Kan, 1980). The problem is however solvable in polynomial time for special cost structures (Bitran and Yanasse, 1982). Approximations for the more intractable, single and multiproduct, capacitated, deterministic lot-size problem have recently been reported in the literature (Bitran and Matsuo, 1986a, 1986b).

The deterministic or the stochastic problem solved in the manner cited above, has one major drawback, no effort is made to incorporate the decision making behavior of the rational decision maker. It is implicitly assumed that decisions are nonsequential in nature, which obviously does not conform to the practical situation, where decisions for each and every time period are revised as better forecasts for demand are known. Thus to incorporate a realistic decision making behavior, we differentiate between **non-sequential** and **sequential** production planning problems. To summarize, our effort will focus on solving the most intractable (but the most realistic) problem which is the stochastic, capacitated, sequential, production planning problem.

To this end, its seems likely that no direct optimal solution procedures are available, given the complex nature of these problems. A line of research, which might be fruitful would be to examine the solution of the nonsequential, capacitated, deterministic, production planning problem as an approximation to the sequential, capacitated, stochastic, production planning problem.

1.2 Sequential, Capacitated Production Planning Problem

The literature to date concentrates on solution procedures for the uncapacitated/capacitated, deterministic, production planning problem (Baker et al, 1978, Bitran and Yanasse, 1982, Bitran et al, 1984, Bitran and Matsuo, 1986a, 1986b, Florian and Klein, 1971, Florian et al, 1980, Jagannathan and Rao, 1973, Karmarkar et al, 1987, Korgaonker, 1977, Love, 1973, Swoveland, 1975 and see (Bahl et al,

1987) for other references)) and for the stochastic uncapacitated problem, (see Schwarz (1981) for references). The stochastic capacitated problems themselves have not often been addressed directly. Bitran and Yanasse (1984), have developed good approximations for the stochastic capacitated production problem where period demands are familiar random variables and production decisions are made non-sequentially for T-time periods. For a particular class of sequential T-period problems, Bitran and Yanasse produced tractable related problems whose solution values were a lower and an upper bound to the solution value of their stochastic, capacitated, sequential problem. The upper bound was derived by solving a single period stochastic capacitated problem on a rolling horizon basis, while the lower bound was the solution to a linear deterministic equivalent problem. For a specific typical numerical example, they found that the relative error of the deterministic approximation to the stochastic problem was at most 3.5 %.

Exact algorithms for the sequential two-period and the T-period problems have recently been reported in the literature: Birge (1985), El Agizy (1967), and Everitt and Ziemba (1979): but for all cases at least one of the random variables was assumed to be discrete. Since the sequential problem is a stochastic non-linear programming problem, other bounds and exact algorithms for variants of the problem have recently appeared in the literature: Ben-Tal (1985), Ben-Tal and Teboulle (1986), Ben-Tal and Teboulle (1987), Birge and Wets (1987), For special cost structures of a two-stage stochastic program, distribution-free upper and lower bounds have been found which can be made to converge to the optimal solution by evaluating them over finer and finer partitions of the domains of the random variables, provided that the objective functions are strictly convex on those random variables: Ben-Tal and Hochman (1972), Huang, Vertinsky and Ziemba (1977), Huang, Ziemba and Ben-Tal (1977).

1.3 Worst-Case Analysis and developing performance bounds

Worst-Case Analysis of Heuristic and Approximations is motivated by the recent work of Karp (1972, 1976) and Cook (1971) who crystallized the growing impression that it is difficult if not impossible to devise polynomial time algorithms for most combinatorial optimization problems. A similar analysis is found in Karp (1986) as well.

Devising performance bounds for intractable problems, provides a practical benefit only if they are evaluated for their efficiency. Natural measures of bound efficiency are the relative error which is itself bounded by the relative difference between the upper and the lower bound. An alternative measure is a probability density function of the relative error.

A review of the literature yields, a small sample of studies which develop worst-case results of production planning approximations and heuristics (Axsater, 1982, 1985, Bitran, Magnanti and Yanasse, 1984, Bitran and Yanasse, 1984, Bitran and Matsuo, 1986a, 1986b). Similarly bounds are developed exploiting the property of convexity of of functions, over the domain of familiar linear operators (Huang et al, 1977a, 1977b, Avriel and Williams, 1970, Madansky, 1960, Dantzig and Madansky, 1961).

1.4 Organization of the Dissertation

The purpose of this dissertation is to bridge the gap between the more realistic (Stochastic) and the more tractable (Deterministic), versions of the production planning problem. We shall investigate these problems when they have variable capacity constraints and incorporate both sequential and nonsequential decision making behavior. In the stochastic problem, when we assume decisions are made sequentially, we exclude fixed costs for set-up, and propose to determine a deterministic equivalent. Given Uniform distribution the equivalent is robust and in the worst case is found to produce results within 23% of optimality.

In addition we study a stochastic non-sequential production planning problem, first with sequence independent and then with sequence dependent set-up costs and determine the worst case error if an approximate solution is obtained by solving the deterministic equivalent. The approach is justified by noting that the literature is somewhat sparse, containing a number of heuristics and approximations. Karmarkar, Kekre and Kekre (1987), did produce a number of heuristics and approximations which solve the deterministic capacitated version in polynomial time.

Further, we prove, that the worst case error is not dependent on the nature of the set-up costs. We also prove that the stochastic version of the non-sequential, single product, production planning problem of (Karmakar, Kekre and Kekre, 1987) is an upper bound for a stochastic production planning problem first introduced by Bitran and Yanasse (1984), and the worst case behavior has an identical bound.

Based on a result due to (Huang, Vertinsky and Ziemba, 1977a, Huang, Ziemba and Ben-Tal, 1977b), we identify a family of approximations for the more intractable stochastic sequential, the stochastic non-sequential production planning problem of Bitran and Yanasse (1984) and the Stochastic version of the problem suggested by Karmarkar, Kekre and Kekre (1987). It is conjectured, since the approximations are a consequence of the convexity of the holding cost (convex in $\sum d_r$), that a family of approximations can be developed for problems with more general cost structures with unaltered holding cost.

Finally, we derive conditions such that an order-up-to the service level policy is optimal for the single period stochastic sequential capacitated production planning problem. The nature of the policy is such that it is 'myopic' and does not consider cases when cost situations demand production quantities in excess of the amount which merely satisfy the service level requirements for the planning periods in question. Further by induction, additional conditions are derived, and the results extended to the T-period problem.

THE STOCHASTIC NON-SEQUENTIAL PROBLEM

2.1 Introduction

It is quite common in the classical economic literature to maximize the expected utility or minimize the expected disutility, discounted to the present, which is in effect a non-sequential approximation of a sequential decision making process. In this chapter we concentrate on two members of the familiy of non-sequential decision making production problems.

At first we compare the stochastic version of the non-sequential, single-product, production planning problem of Karmarkar et al (1987) with a stochastic production planning problem first introduced by Bitran and Yanasse (1984).

Further, we determine a deterministic lower bound of the stochastic version of the production planning problem of Karmarkar et al (1987) and analyze the worst case error if the solution is approximated using the solution of the deterministic equivalent. The approach is justified since a review of the literature yields a number of heuristics and approximations which solve the deterministic capacitated version in polynomial time (Karmarkar et al, 1987).

2.2 Notations for Chapter 2

v_t - the production cost per unit in period t.

o_t - production labor overtime cost per hour in period t.

h_t - holding cost per unit in period t.

X_t - Units produced in period t.

I_t - Units held in inventory from period t to t+1 (Stockout if $I_t < 0$)

I_t^+ - $\mathrm{Max}(I_t, 0)$

α_t - Probability that stockout will occur in period t.

O_t - Overtime hours worked in period t.

y_t - Demand in time period t.

C_t - Regular labor hour capacity per period.

m - Labor hours required to produce one unit.

ℓ_t - The number of units produced through period 1 through t to achieve the mininum acceptable service level in period t (non-sequential problem).

d_t - Cumulative demand for time period 1 through t.

f_t - The probability density function of cumulative demand d_t in time period t.

F_t - The cumulative distribution function for cumulative demand d_t.

s_t - The unit set-up cost for period t.

e_t - The sequence dependent reservation cost in period t.

q_t - The start-up cost in time period t.

Y_t - The binary variable which signifies whether the machine is on or off in time period t.

Z_t - The binary variable which signifies whether the machine is turned on or off in period t.

2.3 Problem Statements

2.3.1 Problem Statement I

In our first production planning problem, we make the following assumptions: stochastic demand, non-sequential decision making behavior, variable capacity limits on regular time production and more significantly, the incorporation of ***sequence dependent*** and ***sequence independent*** set-up costs. The assumption on set-up costs is ***significant*** because it allows us to implicitly consider the situation, that it is sometimes cost effective to reserve machines for production for a successive period (Y_t=1) eventhough no production is being undertaken (($\delta(X_t)$=0) which satisfies constraint (6)), because the start-up costs of time period t (q_t) may be significantly greater than the sequence dependent reservation costs of time period t (e_t). Further, we do not include any stockout costs, because stockout costs are difficult to estimate (Bitran and Yanasse, 1984) but instead consider a chance constraint (costraint (2)) in our set of constraints. Thus we state the first of our problems, where the problem and its optimal objective function value are denoted by ($SP^\star$) and $v(SP^\star)$ respectively.

($SP^\star$)

$$v(SP^\star) = \min_{X} E_{y_t} \sum_{t=1}^{T} [(e_t Y_t + q_t Z_t) + h_t(\sum_{k=1}^{t} X_k - \sum_{k=1}^{t} y_k + I_0)^+ + v_t X_t +$$

$$+ o_t O_t)]$$

s.t.

$$mX_t - O_t \leq C_t \qquad t = 1, 2, \ldots, T. \quad (1)$$

$$\sum_{k=1}^{t} X_k + I_0 \geq \ell_t \qquad t = 1, 2, \ldots, T. \quad (2)$$

$$F_t(\ell_t) = 1 - \alpha_t \qquad t = 1, 2, \ldots, T. \quad (2')$$

$$Y_t = \begin{Bmatrix} 1 \text{ if the machine is on in period } t \\ 0 \text{ if the machine is shut off} \end{Bmatrix} \quad (3)$$

$$Z_t = \begin{Bmatrix} 1 \text{ if the machine state changes from off to on in period } t \\ 0 \text{ otherwise} \end{Bmatrix} \quad (4)$$

$$Z_t \geq Y_t - Y_{t-1} \quad (5)$$

$$Y_t \geq \delta(X_t) \quad (6)$$

$$\delta(X_t) = \begin{Bmatrix} 1 \text{ if } X_t > 0 \\ 0 \text{ otherwise} \end{Bmatrix} \quad (7)$$

$$X_t, O_t \geq 0 \quad (8)$$

$$X = \{X_1, X_2, \ldots, X_T\} \quad (8')$$

The deterministic equivalent of (SP^*) is ($\overline{SP}$) and it is obtained by replacing

$E_{d_t} h_t(\sum_{k=1}^{t}(X_k - y_k) + I_0)^+$ by $h_t(\sum_{k=1}^{t} X_k + I_0 - \mu_t)$, where μ_t is equal to $E\sum_{k=1}^{t} y_k = \sum_{k=1}^{t} E(y_k) = E\ (d_t)$.

2.3.2 Problem Statement II (Bitran and Yanasse, 1984)

We consider the problem of determining production plans over T time periods. Production can occur at any time period and demands are assumed to be stochastic with known distribution functions. Any demand that occurs when the system is out of stock is backordered. Moreover, in our second production planning problem we only consider sequence independent set-up costs and do not consider the realistic situation that it is sometimes cost effective not to turn off the machine at the end of a production run, even though a production run is not scheduled for the subsequent time period. Thus we present the second non-sequential problem, where the problem and its optimal objective function value are denoted by (SP) and v(SP) respectively.

$$v(SP) = \underset{X}{\text{Min}}\; E_{y_t} \sum_{t=1}^{T} [\, s_t \delta(X_t) + v_t X_t + o_t O_t + h_t (\sum_{r=1}^{t} (X_r - d_r) + I_0)^+]$$

s.t.

$$mX_t - O_t \le C_t \qquad t = 1, 2, \ldots, T \qquad (1)$$

$$\sum_{k=1}^{t} X_k + I_0 \ge \ell_t \qquad t = 1, 2, \ldots, T \qquad (2)$$

$$F_t(\ell_t) = 1 - \alpha_t \qquad t = 1, 2, \ldots, T \qquad (2')$$

$$\delta(X_t) = \begin{cases} 1 & \text{if } X_t > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

The deterministic equivalent of (SP) is $(\overline{DP})$. It is again obtained by replacing

$E_{d_t} h_t (\sum_{k=1}^{t} (X_k - y_k) + I_0)^+$ by $h_t (\sum_{k=1}^{t} X_k + I_0 - \mu_t)$, where μ_t

is equal to $E \sum_{k=1}^{t} y_k$.

2.4 Relationship between v(SP) and v($SP^{\star}$)

Theorem 1: If $e_t \geq s_t$ then $v(SP^{\star}) \geq v(SP)$.

Proof: Let $(\tilde{X}, \tilde{O})$ be any choice of vectors feasible in (SP). Then it is feasible in ($SP^{\star}$), because of constraints, (1), (2), (2'), (7) and (8). But given a random choice of a feasible solution, and substituting in ($SP^{\star}$) will result in an objective function value greater than in (SP) because $Y_t \geq \delta(X_t)$, which reflects the fact that in ($SP^{\star}$) an additional cost is incurred even though no production is undertaken in period t. Moreover the set up cost has a sequence dependent cost component which is incurred at the time a production run is scheduled and the machine is turned on. Therefore for given $(\tilde{X},\tilde{O})$, feasible in (SP),

$$\sum_{t=1}^{T} (e_t Y_t + q_t Z_t) \geq \sum_{t=1}^{T} s_t \delta(X_t) \text{ for } t = 1, 2, ..., T. \quad \text{Q.E.D.}$$

2.5 Deterministic equivalent of v($SP^{\star}$)

Theorem 2: $v(SP^{\star}) \geq v(\overline{SP})$

Proof: Since $h_t(\sum_{k=1}^{t}(X_k - y_k) + I_0)^+$ is convex in $\sum_{k=1}^{t} y_k$, and $\sum_{k=1}^{t} y_k$ is a random variable, then using Jensen's inequality (Ross, 1976. p. 340)

$$\sum_{t=1}^{T} [E_{d_t} h_t (\sum_{k=1}^{t} X_k + I_0 - \sum_{k=1}^{t} y_k)]^+ \geq \sum_{t=1}^{T} [h_t(\sum_{k=1}^{t} X_k + I_0 - \mu_t)]^+$$

where $\mu_t = \sum_{k=1}^{t} E(y_k)$. Q.E.D.

2.6 Family of Approximations

We develop a sequence of approximations, by adopting a method due to Huang, Ziemba and Ben-Tal (1977), who compute the weighted average of a set of partial means, by subdividing the domain (a,b) of a convex function ,$\phi(p)$, convex on random variable p, at arbitarary points, $\mu_0=E(p)$, and the result is stated as Theorem 3 of the dissertation.

Theorem 3: (Huang, Ziemba and Ben-Tal, 1977)

(a) Suppose (a,b) is subdivided at arbitrary points $d_0, \ldots, d_m$, where $a = d_0 < \ldots < d_m = b$. Let $J^m \equiv \sum_{i=1}^{i=m} \alpha_i \phi(\beta_i)$, denote the m-fold genegeneralized Jensen bound, where $\alpha_i \equiv \int_{d_{i-1}}^{d_i} dF(p) > 0$, $\beta_i \equiv \alpha^{-1} \int_{d_{i-1}}^{d_i} p dF(p)$ $i = 1, 2, \ldots, m$. Then assuming that the partition corresponding to k+1 is at least as fine as that corresponding to k for $k = 1, \ldots, m-1$, we obtain $J^0 \equiv J^1 \leq J^2 \ldots \leq J^m \leq E\,\phi(p)$.

(b) Suppose (a,b) is subdivided n times on the basis of the partial means of the previous subintervals. Let $J_k \equiv \sum_{i=1}^{i=2^k} c_{ki} \phi(\mu_{ki})$, $k= 0,1, \ldots, n$, denote the generalized Jensen bound obtained from the kth subdivision. Then $J_0 \leq J_1 \leq J_2 \leq \ldots \leq J_n \leq E\,\phi(p)$, where $c_{01}=1$, $\mu_{01}=\mu_0$ and the ith interval the kth subdivision is denoted by $[a_{ki}, b_{ki}]$,

$c_{ki} \equiv \int_{a_{ki}}^{b_{ki}} dF(p) > 0$, $\mu_{ki} \equiv \int_{a_{ki}}^{b_{ki}} p dF(p) \,/\, c_{ki}$, where

$c_{k+1,2i-1} \equiv \int_{a_{ki}}^{\mu_{ki}} dF(p) > 0$ and $c_{k+1,2i} \equiv \int_{\mu_{ki}}^{b_{ki}} dF(p) > 0$.

Following Theorem 3 we state a variant of our problem (SP), as $(SP;J_n)$, which is obtained by replacing the expected holding cost in (SP) by its generalized Jensen bound J_n (illustrated in Theorem 4 as well). Thus $(SP;J_n)$ is stated as:

$$(SP;\ J_n) \qquad v(SP;J_n) = \underset{\mathbf{X}}{\mathrm{Min}} \sum_{t=1}^{T} [s_t\delta(X_t) + v_tX_t + o_tO_t + J_n]$$

where J_n is calculated by letting $\phi(p) = h_t(\sum_{k=1}^{t} X_k + I_0 - \sum_{k=1}^{t} y_k)$ and $p = \sum_{k=1}^{t} y_k$.

Theorem 4: $v(SP) \geq v(SP;J_n), \ldots \geq \ldots \geq v(SP;J_1) \geq v(\overline{DP})$

Proof: We simply illustrate our method of proof by evaluating one candidate problem, say $v(SP;J_1)$, because generating relationships for all other terms in the inequality chain is a simple task of subdividing the domain of the random variable p.

If we now calculate J_1 by letting $\phi(p) = h_t(\sum_{k=1}^{t} X_k + I_0 - \sum_{k=1}^{t} y_k)$, $p = \sum_{k=1}^{t} y_k$, and $\sum_{k=1}^{t} X_k + I_0 = Q$ then J_1 is written as:

$$\begin{aligned} J_1 \quad = \quad & [\int_0^{\mu_t} f_t(y_t)dy_t]\ h_t(\ Q - [\int_0^{\mu_t} y_tf_t(y_t)dy_t / \int_0^{\mu_t} f_t(y_t)dy_t] + \\ & [\int_{\mu_t}^{Q} f_t(y_t)dy_t]\ h_t(\ Q - [\int_{\mu_t}^{Q} y_tf_t(y_t)dy_t / \int_{\mu_t}^{Q} f_t(y_t)dy_t] \end{aligned}$$

where $\mu_t = E\sum_{k=1}^{t} y_k = \sum_{k=1}^{t} E(y_k)$.

and

$$v(SP) \geq \min_{X} \sum_{t=1}^{T} [s_t\delta(X_t) + v_tX_t + o_tO_t + J_1] \geq \min_{X} \sum_{t=1}^{T} [s_t\delta(X_t) + v_tX_t + + o_tO_t + h_t(\sum_{k=1}^{t} X_k + I_0 - \mu_t)^+] = v(\overline{DP}) \quad (9)$$

Thus in (9) we have generated a family of approximations which span the continuum of values between the optimal objective function value of problem (SP) and problem ($\overline{DP}$) respectively. What remains is the determination of the worst-case relative error of obtaining approximate solutions to problems (SP) and (SP^*) by solving their deterministic equivalents ($\overline{DP}$) and ($\overline{SP}$) respectively. This is the focus of our attention in section 2.7.

2.7 Relative Error Analysis of (SP^*)

The methodology of this section which will be used again in Chapter 3 is closely related to the analysis of Bitran and Yanasse (1984).

Let ($\tilde{X}$, $\tilde{O}$, $\tilde{Y}$, $\tilde{Z}$) be optimal in v($\overline{SP}$), which is also feasible in v(SP^*). Then

$$v(SP^*) \leq E\left\{\sum_{t=1}^{T} (e_t\tilde{Y}_t + q_t\tilde{Z}_t + h_t\ (\sum_{k=1}^{t}\tilde{X}_k + I_0 - \sum_{k=1}^{t} y_k)^+ + v_t\tilde{X}_t + o_t\tilde{O}_t)\right\}$$

Let $v(SP^*) - v(\overline{SP}) = \Delta$, $\ell_t \geq \mu_t$, then

$$\Delta \leq \sum_{t=1}^{T}\left\{ h_t\ (F_t(\sum_{k=1}^{t}\tilde{X}_k + I_0) - 1)(\sum_{k=1}^{t}\tilde{X}_k + I_0) + h_t\ (\mu_t - \int_0^{\sum_{k=1}^{t}\tilde{X}_k+I_0} d_tf_t(d_t)\ dd_t\right\}$$

From Theorem 2 we have $\Delta \geq 0$; therefore in the event that both problems are solved at the same feasible set points, the relative error (r.e.), one would experience is the objective function value by using the solution of $\overline{SP}$ to approximate the solution of $SP^{\star}$, would be bounded as follows:

$$\text{r.e.} \leq \Delta / v(SP^{\star}) \leq \Delta / v(\overline{SP}). \tag{10}$$

If following Bitran and Yanasse (1984) one assumes, $O_t \geq 0$, $\sum_{k=1}^{t} X_k + I_0 \geq \ell_t$, $\sum_{k=1}^{T} X_k + I_0 = \ell_T$, $e_t Y_t \geq 0$, $s_t Z_t \geq 0$, $v_t = v$ and $h_t = h = r.v$ we obtain (11).

$$\text{r.e.} \quad \leq \quad \frac{r \sum_{t=1}^{T} \left(\mu_t - \int_0^{\ell_t} d_t \, f_t(d_t) \, dd_t \right)}{r \sum_{t=1}^{T} \left(\ell_t - \mu_t \right) + \ell_T - I_0} \tag{11}$$

which is identical to the relative error bound of Bitran and Yanasse (1984).

Remark: It is no coincidence that the relative error bounds of the two stochastic non-sequential problems are identical, because in obtaining Δ the terms containing the set-up costs drop out and thus we make the observation that the worst-case error bound is independent of the nature of the set-up cost.

THE APPROXIMATIONS TO THE STOCHASTIC SEQUENTIAL PROBLEM

3.1 Introduction

As stated in Chapter 2, the stochastic non-sequential problem is an approximation of the sequential decision making process, and an effort is made to obtain optimal lot-sizes (or production quantities), which minimize the long-run total cost.

In this chapter we show that a two-period stochastic capacitated sequential production problem can be approximated by a quadratic program from above and by a linear program from below in the special case where the demand random variables are distributed uniformly and maximum demand bounds maximum capacity and that these approximations yield optimal objective function values and that must be quite close to the optimal objective function values of the optimal.

3.2 Notations for Chapter 3

v_t - The production cost per unit in time period t.

o_t - The production labor overtime cost per hour in period t.

h_t - Holding cost per unit in period t.

X_t - Units produced in time period t.

I_t - Units held in inventory from period t to t+1.

I_t^+ - Max $(I_t, 0)$

α_t - Probability that stockout will occur in time period t.

O_t - Overtime hours worked in time period t.

y_t - Demand during time period t.

C_t - Regular labor hour capacity per period.

c_t - Capacity limit on overtime hours in time period t.

m - Labor hours required to produce one unit.

λ_t - The number of units produced in period t to achieve the minimum acceptable service level in time period t.

ϕ_t - The probability density function of demand for time period t.

F_t - The cumulative distribution function for demand y_t for time period t.

I_{t-1} - Inventory at the beginning of time period t.

ω_t - Maximum inventory in time period t.

3.3 Problem Statement

In problems (SP) and (SP*) (Chapter 2), we had assumed, stochastic demand, nonzero set-up costs (both sequence dependent and sequence independent setup costs), zero stockout costs and variable capacity limits on regular time production. In addition we had assumed non-sequential decision making behavior. We alter some of these assumptions (zero set-up costs, capacity limits on both regular time as well as overtime production levels and sequential decision making behavior) and attempt to solve a sequential production planning problem first presented by Bitran and Yanasse (1984) where the problem and its optimal objective function value are denoted by (S) and v(S) respectively.

$$(S) \quad v(S) = g_1(I_0)$$

$$g_1(I_0) = \min_{X_1} E_{y_1 | I_0} \left\{ v_1 X_1 + h_1 [X_1 - y_1]^+ + o_1 O_1 + g_2(I_1) \right\}$$

$$g_t(I_{t-1}) = \min_{X_t} E_{y_t | I_{t-1}} \left\{ v_t X_t + h_t \Big[\sum_{k=1}^{t} (X_k - y_k)\Big]^+ + o_t O_t + g_{t+1}(I_t) \right\}$$

$$g_{T+1}(I_T) = 0$$

subject to the constraints:

$mX_t - O_t \leq C_t$	$t = 1, 2, \ldots, T.$	(1)
$\text{Prob}[\, I_t < 0 \mid I_{t-1} \,] \leq \alpha_t$	$t = 1, 2, \ldots, T.$	(2)
$O_t \leq c_t$	$t = 1, 2, \ldots, T.$	(3)
$I_t = I_{t-1} + X_t - y_t$	$t = 1, 2, \ldots, T.$	(4)

$$X_t,\ O_t \geq 0 \qquad\qquad t = 1, 2, \ldots, T. \tag{5}$$

Also given $$F_t(\lambda_t) = \int_0^{\lambda_t} f_t(y_t)\, dy_t = 1 - \alpha_t \tag{6}$$

If distribution functions for demand are known, then constraints (2) and (4) take the following form,

$$X_1 + I_0 \geq \lambda_1 \tag{7}$$

$$\sum_{k=1}^{t} X_k \geq \lambda_t + \sum_{k=1}^{t-1} y_k \qquad t = 2, \ldots, T. \tag{8}$$

Since we assume zero stockout costs, an alternative formulation allows us include service level requirements instead. The service level requirements (λ_t) for each period (unlike a cumulative service level requirement (l_t) in the non-sequential problem) implicitly limits the number of backorders which is a feature often desired by managers.

3.4 Deterministic version of problem (S)

Upon substituting the expected values of the random variables (y_t) in problem (S), we obtain its deterministic version. For our analysis in chapter 3 we limit our attention to the two-period problem and thus we consider problem ($\bar{D}$) for T=2:

$$(\bar{D}) \qquad v(\bar{D}) = \operatorname*{Min}_{X} \sum_{t=1}^{2} [v_t X_t + h_t [\sum_{k=1}^{t} (X_k - E(y_k))]^+ + o_t O_t]$$

where $X = \{ X_1, X_2 \}$.

s.t.

$$X_1 \geq \lambda_1 \tag{9}$$

$$X_1 + X_2 \geq \lambda_2 + E(y_1) \tag{10}$$

$$mX_t - O_t \leq C_t \qquad t = 1, 2. \tag{11}$$

$$O_t \leq c_t \text{ and } X_t, O_t \geq 0. \qquad t = 1, 2. \tag{12}$$

The following theorem, proven by Bitran and Yanasse (1984), shows that the optimal value of Problem ($\bar{D}$) bounds Problem (S) from below.

Theorem 5: $v(\bar{D}) \leq v(S)$

In fact, they prove the result for the more general situation of T time periods: in brief the proof strategy relies on the fact that, if each stage of the stochastic program is convex on the random variable y_t, then it is possible to repeatedly use the inequality of Avriel and Williams (1970): ($E_z \operatorname{Min}_X f(X, z) \leq \operatorname{Min}_X E_z f(X, z)$ where all expectations and minima exist).

For the assumptions of our simple formulation for (S), the stage convexity conditions hold. With this, and the well known inequality of Jensen, (given function ϕ, convex in z, $E_z\phi(z) \geq \phi(E(z))$) the theorem is derived. Note that $\bar{D}$ is solvable by linear programming.

An interesting extension is determining an upper bound of problem (S). We consider a two-period problem where in, the expected cost of the second planning period is evaluated with respect to random variable y_1 prior to the minimization with respect of X_2. Thus we consider problem (D^{**}) and it is stated as:

$$(D^{**})\quad v(D^{**}) = \underset{X_1}{\mathrm{Min}} \left\{ v_1X_1 + o_1O_1 + E_{y_1|I_0}\, h_1[X_1 - y_1]^+ \right\} \quad +$$

$$\underset{X_2}{\mathrm{Min}}\, E_{y_1|I_0} E_{y_2|I_1} \left\{ v_2X_2 + o_2O_2 + h_2[X_1 + X_2 - y_1 - y_2]^+ \right\}$$

s.t.

$$X_1 \geq \lambda_1 \tag{13}$$

$$X_1 + X_2 \geq \lambda_2 + \mathrm{Max}\ y_1 \tag{14}$$

$$mX_t - O_t \leq C_t \qquad t = 1, 2. \tag{15}$$

$$O_t \leq c_t,\ X_t,\ O_t \geq 0 \qquad t = 1, 2. \tag{16}$$

Note the implicit assumption that y_1 is bounded. For any practical decision making purpose, y_1 might as well be bounded by the number of units which can be produced in period 1 and 2 at full, regular and over time capacity; i.e., $y_1 \leq (1/m)(C_1 + C_2 + c_1 + c_2)$. Problem (D^{**}) differs from Problem (S) in that, for S, the expectation is taken over two periods, the second of which incorporates a minimization; thus (S) is essentially a stochastic dynamic programming problem. However, in Problem (D^{**}), the expectations are evaluated prior to the minimization, producing a programming problem quadratic in y_t. Further, we obtain constraint (14) by rewriting constraint (10), and substituting $E(y_1)$ by Max y_1. Such a rewrite is theoretically useful because we are able to create an artificial upper bound by making the <u>the set of feasible solutions of D^{**} a subset of the set of feasible solutions of problem (S), in the special case when $T = 2$</u> (See Fig. 1).

3.5 Relationship between $v(D^{**})$ and $v(S)$

Theorem 6: $v(D^{**}) \geq v(S)$

Proof: Problem (S) can be solved theoretically if the nested optimization problem, $g_2(I_1)$, is solved prior to solving the first period problem $g_1(I_0)$. One inefficient way to do this is to solve $g_1(I_0)$ for all possible values of (X_1, O_1). If X_1 (and therefore O_1) is known, then the nested problem reduces to a function of X_2, y_1 and O_2. Further, O_2 is a function of X_2, if X_2 is the independent variable. Hence the nested optimization problem would simply be a function of X_2 and y_1 if X_1 were known. Similar reasoning on problem D^{**} allows us to modify both S and D^{**}: assume that we have fixed on a specific X_1, call it X_{1j}.

From the Avriel-Williams inequality we have,

$$\min_{X_2} E_{y_1} f(X_2, y_1, X_{1j}) \geq E_{y_1} \min_{X_2} f(X_2, y_1, X_{1j}) \tag{17}$$

then,

$$v(D^{**}) = \min_{\text{for all } X_{1j}} \left\{ \min_{X_1} g(X_1, O_1) + \min_{X_2} (E_{y_1} f(X_2, y_1, X_{1j})) \right\} \tag{18}$$

$$v(S) = \min_{\text{for all } X_{1j}} \left\{ \min_{X_1} g(X_1, O_1) + E_{y_1} (\min_{X_2} f(X_2, y_1, X_{1j})) \right\} \tag{19}$$

and from (17), (18) and (19), we have that

$$v(D^{**}) \geq \min_{X} \left\{ \min_{X_1} g(X_1, O_1) + E_{y_1} \min_{X_2} f(X_2, y_1, X_{1j}) \right\} \geq v(S)$$

where the outer minimization is taken over all pairs (X_1, X_2) in the feasible set of D^{**}. This holds since taking a minimum over the larger feasible set of problem (S) which includes that of (D^{**}), can only result in either a smaller minimum or no change. Thus we have arrived at an inequality which defines the range of solution values to problem S; i.e. $v(\bar{D}) \le v(S) \le v(D^{**})$.

As $\bar{D}$ is an LP, and relatively tractable, it is reasonable to wish to know how inaccurate it might be to approximate problem S by problem $\bar{D}$. Towards that end we explore other bounds and examine the relative difference between the upper and the lower bound in sections 3.11, which itself is an <u>upper bound of the relative error</u>.

We next explore some variants on the production problems presented so far. First consider a variant of problem (D^{**}); where in we replace the random variable y_2 by its expected value in the cost expression of the second planning period and evaluate the expression with respect to y_1, prior to the minimization with respect to X_2.

$$(D^{*}) \quad v(D^{*}) = \underset{X_1}{\text{Min}} \Big\{ E_{y_1|I_0}(v_1X_1 + o_1O_1 + h_1[X_1-y_1]^{+}) \quad + \\ \underset{X_2}{\text{Min}}\, E_{y_1|I_0}(v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-E(y_2)]^{+}) \Big\}$$

satisfying (9) - (12), the set of constraints for problem $(\bar{D})$.

3.6 Relationship between ($D^{\star}$) and ($\bar{D}$)

Theorem 7: $v(D^{\star}) \geq v(\bar{D})$

First we need the following lemma:

Lemma 1: The problems below are equivalent:

(A) $\min_{X_1,O_1}[g(X_1,O_1) + \min_{X_2,O_2} f(X_1, X_2, O_2)]$ (X_1 fixed in the evaluation of f). (20)

(B) $\min_{X,O}[g(X_1, O_1) + f(X_1, X_2, O_2)]$ (21)

Proof: If problem (A) is solved for all possible or admissable values of (X_1,O_1), it reduces to a problem of minimizing over a large number of subproblems, each a minimization with respect to decision variable X_2. Problem (B), on the other hand, minimizes globally from the outset over $X = (X_1, X_2)$, and clearly must arrive at the same solution value.

Proof of Theorem 7: Since $h_1[X_1-y_1]^+$ is convex on y_1, we employ Jensen's inequality to obtain:

$$E_{y_1}\{v_1X_1 + o_1O_1 + h_1[X_1-y_1]^+\} \geq \{v_1X_1 + o_1O_1 + h_1[X_1-E(y_1)]^+\} \quad (22)$$

Similarly, as $h_2[X_1+X_2- E(y_2) - y_1]^+$ is convex on y_1, we have

$$E_{y_1}(v_2X_2 + o_2O_2 + h_2[X_1 + X_2 - E(y_2) - y_1]^+) \geq (v_2X_2 + o_2O_2 + h_2[X_1 + X_2 - E(y_2) - E(y_1)]) \quad (23)$$

So the lemma and inequalities (22) and (23) yield,

$$v(D^{\star}) = \min (E_{y_1}(v_1X_1 + o_1O_1 + h_1[X_1-y_1]^+) + E_{y_1}(v_2X_2 + o_2O_2 + h_2[X_1 + X_2 - E(y_2) - y_1]^+) \geq v(\bar{D}) \quad (24)$$

Hence $v(D^{\star})$ is greater than or equal to $v(\bar{D})$.

3.7 Relationship between (D^{**}) and (D^{*})

Theorem 8: $v(D^{**}) \geq v(D^{*})$

Proof: Since $(v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-y_2]^+)$ is convex in y_2, then using Jensen's Inequality yields,

$$E_{y_2|I_1}(v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-y_2]^+) \geq (v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-E(y_2)]^+) \qquad (25)$$

Further

$$E_{y_1|I_0}E_{y_2|I_0}(v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-y_2]^+) \geq E_{y_1|I_0}(v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-E(y_2)]^+) \qquad (26)$$

Minimizing both sides of (26) over pairs (X_1,X_2) in the feasible set of (D^{**}), one gets $v(D^{**})$ on the left and on the right one has the objective function of D^{**} minimized over a subset of D^{*}'s feasible set; hence an expression $\geq v(D^{*})$. Therefore $v(D^{**})$ is greater than or equal to $v(D^{*})$.

3.8 Family of Approximations

An interesting consequence of convexity led us to the above theorems. Based on the observation, other upper bound candidates for Problem (S) seem worthy of examination. One common method of finding distribution free bounds is based on the principle of partitioning the domain of the random variable, operating on each of the resulting subintervals and summing, Huang, Vertinsky and Ziemba(1977), Huang, Ziemba and Ben-Tal (1977). The summed expression, (say J_1), obeys a Jensen-like inequality of the form: $E\phi(y) \geq J_1 \geq \phi(E(y))$, where y is a random variable and $\phi(y)$ is convex on y. Similarly, if $\phi(y)$ is convex and bounded on $y \in [a,b]$ then the classical Edmundson-Madansky inequality (Madansky, 1960; Dantzig and Madansky, 1961) bounds $\bar{\phi} = E(\phi(y))$ from above. E.g., if μ_0 is defined to be $E(y_1)$, then:

$$\left\{[(b - \mu_0)/(b - a)]\phi(a) + [(\mu_0 - a)/(b - a)]\phi(b)\right\} \geq \bar{\phi} \tag{27}$$

Replacing $\phi(y)$ by $h_1[X_1-y_1]^+$ and $h_2[X_1+X_2-y_1-E(y_2)]^+$ in turn, we note that if y_1 is the random variable and each function is convex on $y_1 \in [0,a]$, such that $a \geq \{X_1+X_2\}$ then we obtain problems $\bar{D}^*$ and M, detailed below, for which it is clearly true that $v(M) \geq v(D^*) \geq v(\bar{D}^*) \geq v(\bar{D})$ (See Fig. 2). The solution values for problem M, D^* and $\bar{D}^*$ are upper bound candidates for problem (S) in the special case when T=2, and M and D^* have the virtue of being distribution free LP's.

We derive the first member of our family of approximations by employing the generalized Jensen's bound (J_1) on the holding cost expressions of problem (D^*). Thus we state the first of our problems, where the problem and its optimal objective function value are denoted by ($\bar{D}^*$) and $v(\bar{D}^*)$ respectively.

$$(\bar{D}^*) \quad v(\bar{D}^*) = \underset{x}{\mathrm{Min}}[v_1X_1 + o_1O_1 + c_{11}h_1(X_1-\tilde{y}_{11}) + c_{12}h_1(X_1-\tilde{y}_{12}) + v_2X_2 + o_2O_2 + c_{11}h_2(X_1+X_2-E(y_2)-\tilde{y}_{11}) + c_{12}h_2(X_1+X_2-E(y_2)-\tilde{y}_{12})]$$

where; $c_{11} = \int_0^{E(y_1)} dF(y_1)$ $\qquad c_{12} = \int_{E(y_1)}^{a} dF(y_1)$

$$y_{11} = \int_0^{E(y_1)} y_1 dF(y_1)/c_{11} \quad y_{12} = \int_{E(y_1)}^{a} dF(y_1)/c_{12}$$

In like manner, employing the classical Edmundson-Madansky Inequality on the holding cost expressions of problem $(D^{\star})$ we obtain the second member of our family of expressions. Thus we state problem (M) and its optimal objective function value is denoted by v(M).

$$\text{(M)} \quad v(M) = \underset{X}{\text{Min}} \Big\{ v_1X_1 + o_1O_1 + [(a-E(y_1))/a]h_1X_1 + [E(y_1)/a]h_1(X_1-a) + v_2X_2 + o_2O_2 + [(a-E(y_1))/a][h_2(X_1+X_2-E(y_2)] + [E(y_1)/a]h_2[X_1+X_2-E(y_2)-a] \Big\}$$

Both problems M and $\bar{D}^{\star}$ are subject to the constraint set (9)- (12).

3.9 Detailed Examination of Bounds $v(D^{\star\star})$ and $v(D^{\star})$

We next examine the nature of our upper bound candidate, $v(D^{\star})$, and the proven upper bound, $v(D^{\star\star})$ [Theorem 6], for the particular case where the demand in period two, random variable y_2, is known to be uniformly distributed. Suppose that we are given that:

$$f_2(y_2) = \begin{cases} 1/b & 0 \le y_2 \le b \\ 0 & \text{otherwise} \end{cases}$$

With this p.d.f. the term $E_{y_1|I_0} E_{y_2|I_1} \left\{ v_2X_2 + o_2O_2 + h_2[X_1+X_2-y_1-y_2]^+ \right\}$ in $D^{\star\star}$ is evaluated as

$$h_2 \int_0^{x_1+x_2} [\int_0^{x_1+x_2-y_1} [X_1+X_2-y_1-y_2]\, dF(y_2)\, dF(y_1) \qquad (28)$$

Substituting the p.d.f. of y_2 in (28) and assuming max $y_1 \leq (X_1+X_2) \leq$ max y_2, (in order to ensure $X_1+X_2-y_1-y_2$ convex in y_2), we obtain

$$h_2/2b \int_0^{\max y_1} [X_1+X_2-y_1]^2 \, dF(y_1) \tag{29}$$

Next $v(D^*)$ is evaluated for the special case where both demands, random variables y_1 and y_2, are known to be uniformly distributed. Suppose that we are given that:

$$f_1(y_1) = \begin{cases} 1/a \text{ for } 0 \leq y_1 \leq a \\ 0 \quad \text{otherwise} \end{cases} \qquad f_2(y_2) = \begin{cases} 1/b \text{ for } 0 \leq y_2 \leq b \\ 0 \quad \text{otherwise} \end{cases}$$

With these p.d.f's the term $E_{y_1|I_0} h_2(X_1+X_2-y_1-E(y_2))^+$ becomes $(h_2/2a)[X_1+X_2-(b/2)]^2$ and expression $E_{y_1|I_0} h_1(X_1-y_1)^+$ reduces to $h_1(X_1^{\,2})/2a$. Hence the objective function simplifies to:

$$v(D^*) = \min_X \Big\{ [(h_2+h_1)/2a](X_1^{\,2}) + (h_2/2a)(X_2^{\,2}) + (h_2/2a)(X_1X_2) +$$

$$(v_1-[bh_2/2a])X_1 + (v_2-[bh_2/2a])X_2 + o_1O_1 + o_2O_2 \Big\}$$

Likewise, in (D^{**}), making the assumption that max $y_1 \leq X_1+X_2 \leq$ max y_2, $v(D^{**})$ is written as:

$$v(D^{**}) = \min_X \Big\{ v_1X_1 + o_1O_1 + h_1[X_1\int_0^{X_1} dF(y_1) - \int_0^{X_1} y_1 \, dF(y_1)] + v_2X_2$$

$$+ o_2O_2 + (h_2/2b)[(X_1+X_2)^2 + \int_0^{\max y_1} y_1^{\,2} dF(y_1) - 2(X_1+X_2)\mu_{y_1}] \Big\} \tag{30}$$

which further simplifies to

$$\underset{X}{\mathrm{Min}}\Big\{v_1X_1 + o_1O_1 + h_1[X_1\int_0^{x_1} dF(y_1) - \int_0^{x_1} y_1 dF(y_1)] + v_2X_2 + o_2O_2 +$$

$$(h_2/2b)[(X_1+X_2)^2 + \sigma_{y_1}^{\ 2} + \mu_{y_1}^{\ 2} - 2(X_1+X_2)\mu_{y_1}]\Big\} \qquad (31)$$

If we further assume that the demand for period 1, random variable y_1, is uniformly distributed, between 0 and a and that $K = \max(X_1+X_2) \leq \max y_2 = b$, then $v(D^{**})$ becomes the following:

$$v\,(D^{**}) = \underset{X}{\mathrm{Min}}\Big\{\sum_{t=1}^{2}(v_tX_t + o_tO_t) + \{(h_1/2a)(X_1)^2 +$$

$$(h_2/6b)[3(X_1+X_2)^2 + a^2 - 3a(X_1+X_2)]\}\Big\} \quad (32)$$

again under the constraints (13)- (16) of (D^{**}) listed at the beginning of this chapter. Therefore, we have shown that for uniformly distributed demands, the two problems D^{*} and D^{**} reduce to manageable quadratic forms. To illustrate, suppose without loss of generality, that h_1=h_2=h, v_1=v_2=v and a=b. Then the objective function of problem (D^{**}) that we seek to minimize over the set of constraints, (13)-(16), can be rewritten as:

$$\Big\{\ [X_1(v- h/2) + X_2(v- h/2) + X_1^{\ 2}(h/a) + X_2^{\ 2}(h/2a) +$$

$$X_1X_2(h/a)] + o_1O_1 + o_2O_2 + ha/6\ \Big\} \quad (33)$$

As in Bazaraa and Shetty (1977), this quadratic programming objective function is in the standard matrix notation form:

$Z(X) = (C^tX) + (1/2)\ (X^tHX)$

such that $AX \leq B$, $h \geq 0$, $a > 0$, H is positive definite and is given by:

$$H = \begin{bmatrix} 2h/a & h/a \\ h/a & h/a \end{bmatrix}$$

Therefore a finite optimal solution can be obtained in a finite number of iterations. Finally, we can use the well known Lemke Fixed Point Theorem to transform these quadratic problems in the standard form into linear complementary problems of the form:

$$w - Mz = q, \; w^t z = 0 \text{ where } w, z \geq 0 \text{ and}$$

$$M = \begin{bmatrix} 0 & -A \\ A^t & H \end{bmatrix} \quad q = \begin{bmatrix} b \\ c \end{bmatrix} \quad w = \begin{bmatrix} y \\ v \end{bmatrix} \text{ and } z = \begin{bmatrix} u \\ x \end{bmatrix}$$

3.10 Relative Difference Analysis

In order to determine the efficacy of our bounds (and bound candidates) on problem (S) we investigate the worst case relative difference between $v(D^{*})$ and $v(\bar{D})$ and between $v(D^{**})$ and $v(\bar{D})$. The latter analysis is immediately significant, as Theorems 5 and 6, imply that the relative error incurred by approximating Problems (S) by Problem $(\bar{D})$ is less than or equal to the relative difference between $v(D^{**})$ and $v(\bar{D})$ (if the solutions of $(\bar{D})$ and (S) occur at the same feasible space point). The former will be relevant only if we can confirm our conjecture that $v(D^{*})$ is a sharper upper bound for v(S): if not in general, at least for some large class of problems. At the end of this chapter, we shall discuss the more realistic situation where the solutions of $(\bar{D})$ and (S) do not occur at the same feasible space point. Reversing the order, we look at $v(D^{*})$ and $v(\bar{D})$ first.

Let $(\tilde{X}, \tilde{O})$ be optimal in Problem $(\bar{D})$; then $(\tilde{X}, \tilde{O})$ must be feasible in Problem (D^{*}), and we can write:

$$v(D^{*}) \le v_1\tilde{X}_1 + o_1\tilde{O}_1 + h_1[\tilde{X}_1]F_1[\tilde{X}_1] - h_1\int_0^{\tilde{x}_1} y_1 f(y_1)\,dy_1 + v_2\tilde{X}_2 +$$

$$o_2\tilde{O}_2 + h_2[\tilde{X}_1 + \tilde{X}_2 - E(y_2)]F_2[\tilde{X}_1 + \tilde{X}_2 - E(y_2)] -$$

$$h_2\int_0^{\tilde{x}_1+\tilde{x}_2-E(y_2)} y_1 f(y_1)dy_1 \qquad (34)$$

and so

$$v(D^{\star}) - v(\bar{D}) \leq h_1[\tilde{X}_1](F_1[\tilde{X}_1] - 1) + h_1\{E(y_1) - \int_0^{\tilde{x}_1} y_1 f(y_1)dy_1\} +$$

$$[h_2(\tilde{X}_1 + \tilde{X}_2 - E(y_2))(F_2[\tilde{X}_1 + \tilde{X}_2 - E(y_2)] - 1)] +$$

$$h_2\{E(y_1) - \int_0^{\tilde{x}_1+\tilde{x}_2-E(y_2)} y_1 f(y_1)dy_1\} \tag{35}$$

Define Δ to be the left hand side of (35) and recalling Theorem 7 we have $0 \leq v(D^{\star}) - v(\bar{D}) = \Delta$. Furthermore, the relative difference (denoted r.d) of $D^{\star}$ and $\bar{D}$ obeys the inequality r.d $\leq \Delta/v(D^{\star}) \leq \Delta/v(\bar{D})$. We let $\bar{y}_i = E(y_i)$ i = 1, 2, and evaluate $\Delta/v(\bar{D})$ to determine the worst case relative difference we get:

$$\text{r.d.} = \frac{\Delta}{v_1\tilde{X}_1 + v_2\tilde{X}_2 + o_1\tilde{O}_1 + o_2\tilde{O}_2 + h_1[\tilde{X}_1 - \bar{y}_1] + h_2[\tilde{X}_1 + \tilde{X}_2 - \bar{y}_1 - \bar{y}_2]} \tag{36}$$

Throwing away negative terms in the numerator and positive terms in the denominator, and using the following facts (equations 37-41), we can simplify (36) to (42).

$$F_1[X_1] - 1 \leq 0 \text{ and } F_2[X_1 + X_2 - \bar{y}_2] - 1 \leq 0 \tag{37}$$

$$X_1 \geq \lambda_1 \tag{38}$$

$$\lambda_t \geq \bar{y}_t, \text{ (since, in practice, service levels exceed 50\%)} \tag{39}$$

$$X_1 + X_2 \geq \bar{y}_1 + \lambda_2 \geq \bar{y}_1 + \bar{y}_2 \tag{40}$$

$$O_t \geq 0 \tag{41}$$

Hence we obtain,

$$\text{r.d.} \le \frac{h_1\{\bar{y}_1 - \int_0^{\lambda_1} y_1 f(y_1)dy_1\} + h_2\{\bar{y}_1 - \int_0^{\bar{y}_1} y_1 f(y_1)dy_1\}}{v_1\tilde{X}_1 + v_2\tilde{X}_2 + h_1[\tilde{X}_1] + h_2[\tilde{X}_1+\tilde{X}_2] - h_1\bar{y}_1 - h_2[\bar{y}_1 + \bar{y}_2]} \quad (42)$$

If the two periods are quite similar in cost and service level constraints, we can assume further (following Bitran and Yanasse, 1984), that $v_1=v_2=v$, $h_1=h_2=h=rv$, where r is a carryover cost, $\alpha_1=\alpha_2=\alpha$. These simplify (42) further, and upon cancellation of the common factor v, we get (43):

$$\text{r.d.} \le \frac{r[2\bar{y}_1 - \int_0^{\lambda_1} y_1 f(y_1)dy_1 - \int_0^{\bar{y}_1} y_1 f(y_1)dy_1]}{(\lambda_2 + \bar{y}_1) + r(\lambda_1 - \bar{y}_1) + r(\lambda_2 - \bar{y}_2)} \quad (43)$$

In the event that we know the period demands are distributed uniformly, with distributions as in (29´), then various expressions in (43) simplify still further: namely $\bar{y}_1 = a/2$ and $\bar{y}_2 = b/2$

$$1/a \int_0^{\lambda_1} y_1 dy_1 = (\lambda_1)^2/2a \qquad 1/a \int_0^{(a/2)} y_1 dy_1 = a/8 \quad (44)$$

$$1/a \int_0^{\lambda_1} dy_1 = 1 - \alpha = \lambda_1/a \qquad \lambda_2 = b(1 - \alpha) \quad (45)$$

Thus (43) becomes,

$$\text{r.d.} \le \frac{r[a - a/2(1 - \alpha)^2 - a/8]}{b(1 - \alpha) + a/2 + r[a(1 - \alpha) - a/2 + b(1 - \alpha) - b/2]} \quad (46)$$

From (39) and (45) we see that we can replace $b(1 - \alpha)$ by its lower bound, (b/2), and, after some manipulation, inequality (46) becomes:

$$\text{r.d.} \le \left\{a/(a+b)\right\} \frac{r\,[7/8 - (1-\alpha)^2/2]}{1/2 + r(1/2 - \alpha)} \tag{47}$$

If we follow Bitran and Yanasse (1984), setting $r = .025$/month and $\alpha = .05$, then (47) reduces to r.d. $\le [a/(a+b)](.0207212)$. Since $a \le (a+b)$, then the worst case relative difference cannot be larger than 1.036%.

The analysis for $v(D^{**})$ begins with the observation that if $(\hat{X},\hat{O})$ is optimal in $(\bar{D}')$, which is problem $(\bar{D})$ with constraint (10) replaced by constraint (14) of (D^{**}), and we let $v(\bar{D}')$ indicate the optimal solution to $(\bar{D}')$, and we write the corresponding objective function value as $v(\bar{D}')$, then $(\hat{X},\hat{O})$ is feasible in (D^{**}) (See Fig. 1). Let $v(D^{**};\hat{X},\hat{O})$ denote the objective function value of D^{**} at $(\hat{X},\hat{O})$, and we shall obtain in Theorem 9 the worst case relative difference $v(D^{**};\hat{X},\hat{O})-v(\bar{D}')$. With the aid of several lemmas we shall use Theorem 9 to prove Theorem 10, which will provide an upper bound on $v(D^{**};\hat{X},\hat{O}) - v(\bar{D})$. Since $v(\bar{D}) \le v(S) \le v(D^{**}) \le v(D^{**};\hat{X},\hat{O})$ the relative difference window between $v(D^{**};X,O)$ and $v(D)$ contains the relative difference window of $v(D^{**})$ and $v(\bar{D})$. To this end we present lemmas 2 through 4.

Lemma 2: If $\delta = v\ (\bar{D}') - v(\bar{D})$ then

$$\frac{v(D^{**};\ \hat{X},\hat{O}) - v(\bar{D}')}{v(\bar{D})} + \frac{\delta}{v(\bar{D})} = \frac{v(D^{**};\hat{X},\hat{O}) - v(\bar{D})}{v(\bar{D})}$$

Proof: The proof is immediate from the definition of δ.

Remark: In our analysis of (D^{**}), when y_2 was bounded, we assumed $K = \max(X_1+X_2) \leq \max y_2 = b$, this creates an additional problem in our analysis of $(\bar{D}')$, because the constraints of $(\bar{D}')$ and (D^{**}) are identical, and in $(\bar{D})$, we make no such assumptions. Moreover, if $\max y_2 \leq K$, then the convex set of the set of constraints of $(\bar{D}')$ will not be a subset of the convex set of the set of constraints of $(\bar{D})$. Hence in Lemma 3 and Lemma 4 we add this to our set of conditions.

Lemma 3: $v(\bar{D}') \geq v(\bar{D})$. (See above **remark**).

Proof: If we observe Fig. 1, we notice, that the convex set of the set of constraints of (D^{**}) is a subset of the convex set of the set of constraints of $(\bar{D})$, and hence the result is immediate.

There are two simpler cases with arguments which parallel what follows; these two cases involve $\lambda_2 + \bar{y}_1 \leq \lambda_1 \leq \lambda_2 + \max y_1$ and $\lambda_1 \geq \lambda_2 + \max y_1$ and would only produce worst case relative difference of δ less than or equal to the worst case relative difference produced by our analysis of Fig. 1. So we shall continue to work with the case of Fig. 1.

Lemma 4: $v(D^{**}) \geq v(\bar{D}')$, if $\max y_2 \geq \max (X_1+X_2) = K$.

Proof: The result is a consequence of the repeated use of Jensen's inequality (with identical set of constraints). Where $Ey_1 h_1 [X_1 - y_1]^+ \geq h_1[X_1 - \bar{y}_1]$ and $E_{y_1} E_{y_2} [X_1+X_2-y_1-y_2] \geq [X_1+X_2-\bar{y}_1 - \bar{y}_2]$.

In section 3.9 when y_1 and y_2 were bounded, (D^{**}) had to satisfy the inequality $\max y_1 \le (X_1+X_2) \le \max y_2$. This condition in Theorem 9 is not violated, because of constraint (14) and our remark in the previous page which makes the inequality redundant. Thus we state Theorem 9 without any additional conditions.

Theorem 9: If random variable y_2 is uniformly distributed between $(0,b)$, $h_1=h_2=h$, and $0 \le y_1 \le b$, but we do not know how it is distributed, then the worst case relative difference (r.d.), between $v(D^{**};\hat{X},\hat{O}))$- $v(\bar{D}')$ is bounded by

$$\frac{h[\mu_{y_1} - \int_0^{\lambda_1} y_1 dF(y_1)] + (h/2b)[K^2 - 2K(\mu_{y_1}+b) + \mu_{y_1}{}^2 + 2b\mu_{y_1} + b^2 + \sigma_{y_1}{}^2]}{v(\lambda_2+\mu_{y_1}) + h(\lambda_1-\mu_{y_1}) + h(\lambda_2 - (b/2))}$$

where $K = \max\{X_1 + X_2\} = (1/m)(C_1 + C_2 + c_1 + c_2)$

Proof: We start with $v(D^{**})$ as given in (31) and observe that, if $(\hat{X},\hat{O})$ is optimal in $(\bar{D}')$, which has the constraint set of (D^{**}) then it is feasible in D^{**} as well. Therefore:

$$v(D^{**};\hat{X},\hat{O}) =$$

$$\sum_{t=1}^{2}\left\{v_t\hat{X}_t + o_t\hat{O}_t\right\} + h_1[\hat{X}_1\int_0^{\hat{x}_1} dF(y_1) - \int_0^{\hat{x}_1} y_1 dF(y_1)] +$$

$$(h_2/2b)[(\hat{X}_1+\hat{X}_2)^2 + \sigma_{y_1}{}^2 + \mu_{y_1}{}^2 - 2(\hat{X}_1+\hat{X}_2)\,\mu_{y_1}] \quad (48)$$

and so

$$v(D^{**};\hat{X},\hat{O}) - v(\bar{D}') \leq h_1\hat{X}_1[F_1(\hat{X}_1)-1]-h_1[\int_0^{\hat{x}_1} y_1 dF(y_1)-\mu_{y_1}] +$$
$$(h_2/2b)[(\hat{X}_1+\hat{X}_2)^2+\sigma_{y_1}{}^2+\mu_{y_1}{}^2-2(X_1+\hat{X}_2)(\hat{\mu}_{y_1}+b)+2b(\mu_{y_1}+\bar{y}_2)] \quad (49)$$

Define Δ' to be the right hand side of (49) and we have $0 \leq v(D^{**};\hat{X},\hat{O})-v(\bar{D}') \leq \Delta'$. Furthermore, the relative difference (r.d.) of $(D^{**};\hat{X},\hat{O})$ and $(\bar{D}')$ obeys the inequality r.d. $\leq \Delta'/v(D^{**};\hat{X},\hat{O}) \leq \Delta'/v(\bar{D}') \leq \Delta'/v(\bar{D})$. Thus evaluating $\Delta'/v(D)$,and using the following facts (equations (50)-(57)),to determine the worst case relative difference, we obtain (58):

$$F_1(X_1) - 1 \leq 0 \quad (50)$$
$$X_1 \geq \lambda_1 \quad (51)$$
$$X_1+X_2 \geq \lambda_2 + \mu_{y_1} \quad \text{(Constraint of problem } \bar{D}) \quad (52)$$
$$o_tO_t \geq 0 \quad (53)$$
$$X_1+X_2 \leq K \quad (54)$$
$$\bar{y}_2 = b/2 \quad (55)$$
$$h_1=h_2=h \quad (56)$$
$$v_1=v_2=v \quad (57)$$

$$\text{r.d.} \leq \frac{h[\mu_{y_1}-\int_0^{\lambda_1} y_1 dF(y_1)]+(h/2b)[K^2-2K(\mu_{y_1}+b)+2b\mu_{y_1}+b^2+\sigma_{y_1}{}^2+\mu_{y_1}{}^2]}{v(\lambda_2+\mu_{y_1}) + h(\lambda_1 - \mu_{y_1}) + h(\lambda_2 - (b/2))} \quad (58)$$

Q.E.D.

From Lemma 2 we see that, we have yet to fulfill our objective of determining the bound on our ***relative difference window*** $(v(D^{**};\hat{X},\hat{O})$ - $v(\bar{D}))$. In Theorem 10 we derive a bound on δ, and a sum of the two bounds shall then result in obtaining a bound on our relative difference window.

Theorem 10: Under the assumptions and conditions of Lemma 2 to Lemma 4, the worst case relative diference of δ is bounded by:

$$\text{r.d.} \leq \frac{[\text{Max } y_1 - \bar{y}_1](v + 2h) + 2om(\lambda_2 + \text{Max } y_1) - 2oC}{v(\lambda_2 + \bar{y}_1) + h(\lambda_1 + \lambda_2 - \bar{y}_2 - \bar{y}_1)} \tag{59}$$

where $v=v_1=v_2$, $h=h_1=h_2=rv$, $o=o_1=o_2$ and $C=C_1=C_2$.

Proof: Let $(\tilde{X},\tilde{O})$ be the optimal solution for problem $(\bar{D})$ and recall that $(\hat{X},\hat{O})$ is the optimal solution for problem $(\bar{D}')$. From Fig. 1 depicting the nested convex feasible sets of both these problems, and from the conditions, $\lambda_2 + \bar{y}_1 \geq \lambda_1$, max $y_2 \geq \lambda_2$ + max y_1, we obtain:

$$\delta = v(\bar{D}') - v(\bar{D}) = (\hat{X}_1 - \tilde{X}_1)(v_1 + h_1 + h_2) + (\hat{X}_2 - \tilde{X}_2)(v_2 + h_2) + o_1(\hat{O}_1 - \tilde{O}_1) + o_2(\hat{O}_2 - \tilde{O}_2) \tag{60}$$

Noting that both $(\bar{D}')$ and $(\bar{D})$ are linear programming problems, with identical objective functions, we can conclude from inspecting Fig. 1 that there are exactly two possibilities for $(\hat{X},\hat{O})$ and $(\tilde{X},\tilde{O})$; namely

$$\begin{array}{llll} \text{either} & \text{(Case 1):} & & \hat{X} = L = (\lambda_2 + \max y_1,\ 0) \\ & & \text{and} & \tilde{X} = L' = (\lambda_2 + \bar{y}_1,\ 0) \\ \text{or} & \text{(Case 2):} & & \hat{X} = M = (\lambda_1,\ \lambda_2 - \lambda_1 + \max y_1) \\ & & \text{and} & \tilde{X} = M' = (\lambda_1,\ \lambda_2 - \lambda_1 + \bar{y}_1) \end{array} \qquad (61)$$

Now we note that one or the other of the first two terms in the R.H.S. of (60) is zero. Moreover the remaining term is either $(v+2h)[\max y_1 - \bar{y}_1]$ or $(v+h)[\mathrm{Max}\ y_1 - \bar{y}_1]$, and, since $h > 0$, clearly the first of these is larger.

Moreover,

$$\begin{aligned} \mathrm{Max}\ (\hat{O}_1, \tilde{O}_1) &= m(\mathrm{Max}\ y_1 - \bar{y}_1) - C \\ \mathrm{Max}\ (\hat{O}_2, \tilde{O}_2) &= m\ (\mathrm{Max} y_1 - \bar{y}_1) - C \end{aligned} \qquad (62)$$

Substituting (61) and (62) into (60) and recalling the expression for $v(\bar{D})$ in equation (58)'s denominator, yields (59). //

Summarizing, the relative error in approximating $v(S)$ by $v(\bar{D})$ is $\dfrac{v(S) - v(\bar{D})}{v(\bar{D})}$ and we can find a worst case upper bound for it from Theorems 9 and 10. Fig. 2 summarizes the relationships between optimal objective function values for production problems met in this dissertation, and from it we can deduce that.

$$\text{r.e.} = \frac{v(S) - v(\bar{D})}{v(\bar{D})} \leq \frac{v(D^{**}) - v(\bar{D})}{v(\bar{D})} \leq \frac{v(D^{**};\hat{X},\hat{O}) - v(\bar{D})}{v(\bar{D})} =$$

$$= \frac{v(D^{**};\hat{X},\hat{O}) - v(\bar{D}')}{v(\bar{D})} + \frac{\delta}{v(\bar{D})} \le \frac{\Delta'}{v(\bar{D})} + \frac{\delta}{v(\bar{D})}$$

Given the assumptions of Theorem 9, and $K=\max\{X_1+X_2\}$ then;

$$\begin{aligned} \text{r.e.} \le \{&h[\mu_{y_1} - \int_0^{\lambda_1} y_1 dF(y_1)] + (h/2b)[K^2 - 2K(\mu_{y_1}+b) + \mu_{y_1}^2 + 2b\mu_{y_1} \\ &+ b^2 + \sigma_{y_1}^2 + \mu_{y_1}^2 + (v+2h)[\text{Max}\, y_1 - \mu_{y_1}] \\ &+ 2o[m(\lambda_2+\text{Max}y_1) - C]\} \div \{v(\lambda_2+\mu_{y_1})+h(\lambda_1+\lambda_2-\mu_{y_1}-\mu_{y_2})\} \quad (63) \end{aligned}$$

In the special case where we know both period demands are distributed uniformly with distributions as in (29)', so that equations (44) holds as well, (63) simplifies to (64):

$$\begin{aligned} \text{r.e.} \le \{&(h/2)(a-(\lambda_1^2/a))+(h/2b)[K^2-2K((a/2)+b)+ab+b^2+(a^2/3))] + \\ &(v+2h)(a/2) + 2o(m(\lambda_2+a)-C)\} \div \{v(\lambda_2+(a/2)) + h(\lambda_1+\lambda_2-((a+b)/2))\} \end{aligned} \quad (64)$$

If equation (45) holds also, and we make the realistic assumption that service levels will always be set above 50%, we can further simplify (64) to (65).

$$\begin{aligned} \{&(a/2)[3+(1/r)-(1-\alpha_1)^2] + (1/2b)[K^2-2K((a/2)+b)+ab+b^2+(a^2/3)] + \\ &(2o/h)(m[b(1-\alpha_2)+a]-C)\} \div \{(1/r)(b(1-\alpha_2)+(a/2))+((a+b)/2)-\alpha_1-\alpha_2)\} \end{aligned} \quad (65)$$

Should the two production periods be so similar that a=b, and $\alpha_1=\alpha_2=\alpha$, and when we define $(N=(K/a))$ then (65) becomes;

$$\frac{(a/2)[(1/r) - (1-\alpha)^2+((16/3)+N^2-3N) + (2o/h)(ma(2-\alpha)-C)}{(a/r)[(3/2)-\alpha] + a(1-2\alpha)} \quad (66)$$

Now the assumption made in Lemmas 2-4 that $K \leq b$ implies that $0 \leq N \leq 1$ which means that $10/3 \leq N^2-3N + (16/3) \leq 16/3$. Further, casting out the negative term, $-C$, in the numerator simply enlarges bound (66), and allows cancellation of a in what remains, and we are left with

$$\frac{(1/2)[(1/r) - (1-\alpha)^2 + (10/3)] + (2om/h)(2 - \alpha)}{(1/r)((3/2)-\alpha) + (1 - 2\alpha)} \quad (67)$$

Finally, when we test our bound with the values of the typical numerical example given by Bitran and Yanasse, (1984, p. 1016), [$h=.4$, $o=9.5$, $\alpha=.05$, $m=.2$, $v=19.0$], inserted in (67), we get a value of .4135.

If we reinstate the negative term $-2oC/ha$ which we cast out of equation (66), assume $a=2$ (average monthly demand of 9248 units) and that the limit on overtime hours is 4800 hours, (since Bitran and Yanasse postulate 2400 regular labor hours monthly and we assume 2 more shifts to be the maximum), then the value of our upper bound on the relative error improves to .2373.

It is reasonable to ask how one can be sure (S) and ($\bar{D}$) will take on their optimal objective values at the same point in their common feasible set, of course, the answer is that one cannot be sure (In fact, if their solution points were identical, it would obviate the need for much of the analysis of this Chapter). Thus there is the very real risk that if the decision maker chooses point $(\tilde{X})_{\bar{D}}$, optimal for $(\bar{D})$, to use in problem (S), the result may be an objective function value much greater than v(S). However, the decision maker facing the 2-period version of production problem (S) will face it repeatedly; as a consequence several benefits may arise from considering the relationships we have developed in this chapter.

(1) As $(\bar{D})$ is easily solvable, we can use the optimal value, $v(\bar{D})$ and the bound on the relative error to form a "target window" within which v(S) must reside.

(2) $(\tilde{X})_{\bar{D}}$ may yield an objective function value close enough to this "target window" to satisfy the decision maker outright, and

(3) this chapter has provided several other problems (e.g., the candidate upper bounds, $(\bar{D}^{*})$, (D^{*}), (D^{**}) and $(\bar{D}')$) which offer the decision maker alternative feasible set points to try for problem (S), and experience over time with using them in (S) may lead to useful heuristics such as "Find $(\tilde{X})_{\bar{D}}$ and move from it a certain distance in the direction of $(\hat{X})_{D}^{**}$ to regularly obtain a satisfactory decision outcome for problem (S)."

THE OPTIMAL SOLUTION OF THE STOCHASTIC SEQUENTIAL PROBLEM

4.1 Introduction

In this chapter we discuss methods to determine the optimal solution of a stochastic sequential problem. Our approach hinges on the use of a familiar dynamic recursion scheme, based on a relationship between beginning inventory (I_{t-1}), maximum inventory (ω_t) and the demand incurred during the time period (y_t) in question.

We present conditions when an order- up-to policy (ordering upto the service level requirement, λ_t) is optimal and derive exact expressions for representative real-life stochastic sequential problems. Our method of proof hinges on the use of a dynamic recursion scheme where the objective function from period to period is evaluated on the basis of a combination of the state variable (I_{t-1}) and the decision variable (X_t), ω_t. We follow the notations introduced in chapter 3, in addition to new notations introduced exclusively in this chapter.

4.2 A Procedure to obtain an Optimal Solution of Problem (S)

Problem (S) of chapter 3 in the special case when T=2 is solvable if we can discretize X_1 (first period production) and y_1 (first period demand) and roll back in a decision tree to obtain the optimal cost of production. The decision tree can be solved if the nested optimization problem

$$\underset{X_2}{\text{Min}}\; E_{y_2}[v_2X_2 + o_2\max\{mX_2-C_2,\, 0\} + h_2(X_1+X_2-y_1-y_2)^+] \qquad (1)$$

has a solution.

Further, a solution to (1) can be found, if the term $E_{y_2}(X_1+X_2-y_1-y_2)^+$ has a closed formed solution. Notice this term is a function which considers only the positive component of a linear expression; therefore at first we must specify the condition for which $X_1+X_2-y_1-y_2$ will be positive, it is $y_2 \le X_1 + X_2 - y_1$.

We find optimal values of X_2, for two known demand situations; first assuming y_2 is a triangular distribution and then assuming y_2 has a uniform distribution.

4.3 Period 2 demand (y_2) is a triangular distribution

The p.d.f is given by

$$p(y_2) = \begin{cases} (4/b^2)y_2 & \text{if } 0 \le y_2 \le b/2 \\ (4/b^2)(b-y_2) & \text{if } (b/2) \le y_2 \le b \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

For evaluating $E_{y_2}[X_1+X_2-y_1-y_2]^+$ we have three possible cases. In Case A, we consider $0 \le y_2 \le X_1+X_2-y_1 < b/2$, in Case B, we consider $0 \le X_1+X_2y_1 \le y_2 \le b$, and finally in Case C, we consider $b \le X_1+X_2-y_1 \le y_2$. In addition we assume $X_1+X_2-y_1 \ge 0$, which implies, $X_2 \ge y_1-X_1$. The expected value is obtained by multiplying the solution of the integrals (3)-(5) by $4/b^2$.

Case A:

$$\int_0^{X_1+X_2-y_1} (X_1+X_2-y_1-y_2)\, y_2\, dy_2 \tag{3}$$

Case B:

$$\int_0^{b/2} (X_1+X_2-y_1-y_2)\, y_2\, dy_2 + \int_{b/2}^{X_1+X_2-y_1} (X_1+X_2-y_1-y_2)\, (b-y_2)\, dy_2 \tag{4}$$

Case C:

$$\int_0^{b/2} (X_1+X_2-y_1-y_2)\, y_2\, dy_2 + \int_{b/2}^{b} (X_1+X_2-y_1-y_2)\, (b-y_2)\, dy_2 \tag{5}$$

Let $\omega = X_1+X_2-y_1$ and $g(\omega) = E_{y_2}[\omega - y_2]$ then $g_2(I_1)$ in our sequential problem is written as $\underset{X_2}{\text{Min}}\ [v_2X_2 + o_2O_2 + h_2\ g(\omega)]$. Rewriting our cost component $g_2(I_1)$ as above, allows us to make one observation, that an optimal policy minimizing $g(\omega)$ minimizes $g_2(I_1)$ as well because $g(\omega)$ is the only non-linear cost component in the total cost structure.

$$g(\omega) = \begin{cases} 0 & \text{if} \quad \omega \le 0 \\ (2/3b^2)\ \omega^3 & \text{if} \quad 0 \le \omega \le b/2 \\ 2\omega^2/b - \omega - (2/3b^2)\ \omega^3 + b/6 & \text{if} \quad b/2 \le \omega \le b \\ \omega - b/2 & \text{if} \quad \omega \ge b \end{cases} \qquad (6)$$

4.3.1 Structure of $(2/3b^2)\omega^3$

Assume for notational purposes, that $I_1=X_1-y_1$ then $\omega=I_1+X_2$. Further, if we graph $(2/3b^2)\omega^3$, we find at $\omega=0$ (or $X_2= -I_1$), $g(0)=0$, $\omega=I_1$ (or $X_2=0$) $g(I_1)=2/3b^2 I_1^3$ and $\omega=b/2$ (or $X_2=b/2-I_1$) $g(b/2)=b/12$. Hence $g(\omega)$ for $0 \le \omega \le b/2$ (or $-I_1 \le X_2 \le b/2-I_1$), is a strictly increasing function in ω.

4.3.2 Structure of $2\omega^2/b+b/6-\omega-(2/3b^2)\omega^3$

If we graph $2\omega^2/b+ b/6 - \omega - (2/3b^2)\omega^3$, we find at $X_2=b(1-(1/\sqrt{2}))-I_1$ and $X_2=b(1+(1/\sqrt{2}))-I_1$ the slopes are zero, $g(b(1 - (1/\sqrt{2}))) = b((3-2\sqrt{2})/6)$ and $g(b(1 + (1/\sqrt{2}))) = b((3+2\sqrt{2})/6)$,$g(\omega=b/2)=b/12$, $g(\omega=b)=b/2$. Thus the function $g(\omega)$, in the region $b/2 \le \omega \le b$ or $((b/2)-I_1 \le X_2 \le b-I_1)$; is strictly increasing in ω and X_2.

4.3.3 Structure of ω - (b/2)

(ω - (b/2)) is a linear function in ω and X_2, thus it is for $\omega \geq b$ or $X_2 \geq b-I_1$, strictly increasing in ω and X_2. Further, $g(\omega=b)=b/2$. Finally $g(\omega)$ is continuous with breaks at points, b/2, and b.

4.3.4 Evaluation of the service level constraint λ_2

For evaluating λ_2 we recall (6) which is written as:

$$F_2(\lambda_2) = \int_0^{\lambda_2} dF_2(y_2) = 1 - \alpha_2$$

Substituting the p.d.f. of y_2 (triangular distribution) and assuming $\alpha_2 \leq$ 0.50, (6) from chapter 3 is rewritten as:

$$F_2(\lambda_2) = (4/b^2)\left\{\int_0^{b/2} y_2\, dy_2 + \int_{b/2}^{\lambda_2} (b-y_2)\, dy_2\right\} = 1 - \alpha_2 \qquad (7)$$

After some algebraic manipulations, (7) is rewritten as:

$$\lambda_2^2 - 2b\lambda_2 + b^2(1 - (\alpha_2/2)) = 0 \qquad (8)$$

Upon solving (8), λ_2 is found to equal either $\lambda_2 = b(1 + \sqrt{(\alpha_2/2)})$ or $b(1 - \sqrt{(\alpha_2/2)})$. Since $b(1 + \sqrt{(\alpha_2/2)})$ is greater than b (maximum demand) then $\lambda_2 = b(1 - \sqrt{(\alpha_2/2)})$ is the only admissable value. Using the admissable root of (8) the condition $X_2 \geq \lambda_2 - X_1 + y_1$ becomes:

$$X_2 \geq b(1 - \sqrt{(\alpha_2/2)}) - I_1 \qquad (9)$$

Lemma 5: $-I_1 < b/2 - I_1 \le b(1 - \sqrt{(\alpha_2/2)}) - I_1 \le b - I_1$
Proof: Let $A = -I_1$; $B = b/2 - I_1$; $C = b(1 - \sqrt{(\alpha_2/2)}) - I_1$ and $D = b-I_1$. The proof follows from the observation that $b \ge 0$; $\alpha_2 \le 0.5$ and expression C evaluated in the limits of α_2 (0 and 0.5) has values $b - I_1$ and $b/2 - I_1$ respectively.

4.3.5 Optimal policy

Since $A < B \le C \le D$, for all positive values of C, the optimal policy is an order-up-to the service level (λ_2) policy, and is written as:

$$X_2 = \begin{cases} \lambda_2 - I_1 & \lambda_2 \ge I_1 \\ 0 & \text{elsewhere} \end{cases}$$

Notice the policy is myopic and does not consider cases when cost situations demand production quantities in excess of the amount which merely satisfy the service level requirement (λ_2) for the period in question. However in a two-period problem the optimal order-up-to the service level policy of the second period need not have a look ahead component. Similarly in a T-period problem, it suffices to consider an order-up-to the service level (λ_T) policy as optimal in the Tth period.

4.4 Period 2 demand (y_2) is a uniform distribution

The p.d.f is given by

$$p(y_2) = \begin{cases} 1/b & 0 \le y_2 \le b \\ 0 & \text{otherwise} \end{cases}$$

For evaluating $E_{y_2}[X_1+X_2-y_1-y_2]^+$ we again assume $\omega=X_1+X_2-y_1$ and $I_1=X_1-y_1$. Also define $g(\omega) = E_{y_2}[\omega - y_2]^+$. Further, $g(\omega)$ is equal to zero, if ω is less than zero. This is justified because, no holding costs are incurred if negative inventory is experienced. For evaluating the expectation we have two possible cases. In Case A, we consider $\omega \le b$ and in Case B, we consider $\omega > b$. The expected value is obtained by solving for the integrals (10)-(12).

Case A:

$$(1/b) \int_0^{\omega} (\omega - y_2)\, dy_2 \qquad (10)$$

Case B:

$$(1/b) \int_0^{b} (\omega - y_2)\, dy_2 \qquad (11)$$

Then $g(\omega)$ is written as:

$$g(\omega) = \begin{cases} 0 & \text{if} \quad \omega \le 0 \\ \omega^2/2b & \text{if} \quad 0 \le \omega \le b \\ \omega - b/2 & \text{if} \quad \omega \ge b \end{cases}$$

4.4.1 Structure of $g(\omega)$

When ω is less than the maximum possible demand (b); $g(\omega)$ is a quadratic expression, symmetric around the origin, with minimum at $\omega = 0$ or $X_2 = -I_1$. At $\omega = b$ or $X_2 = b - I_1$, $g(\omega) = b/2$. Further $g(\omega)$ is an increasing function in ω and in X_2 in the region $0 \le \omega \le b$. When

$\omega \geq b$, $g(\omega)$ is a linear expression, increasing in ω, the minimum at $\omega = b$ or $X_2 = b - I_1$ and $g(\omega) = b/2$.

4.4.2 Evaluation of the service level constraint λ_2

For evaluating λ_2 we recall (6) and upon substituting the p.d.f of y_2 (uniform distribution) (6) is rewritten as:

$$F_2(\lambda_2) = (1/b) \int_0^{\lambda_2} dy_2 = 1 - \alpha_2 \tag{13}$$

Upon solving (13), λ_2 is found to equal $b(1 - \alpha_2)$. Thus :

$$X_2 \geq b(1 - \alpha_2) - I_1 \tag{14}$$

Lemma 6: $- I_1 < b(1 - \alpha_2) - I_1 \leq b - I_1$

Proof: The proof follows from the observation that $b \geq 0$ and $0 \leq \alpha_2 \leq 1$.

Let $A = - I_1$; $B = b(1 - \alpha_2) - I_1$ and $C = b - I_1$.

4.4.3 Optimal Policy

Since $A < B \leq C$, and if $A \geq 0$, then it is optimal to order-up-to the service level (λ_2). The policy is written as:

$$X_2 = \begin{cases} \lambda_2 - I_1 & \text{if } \lambda_2 \geq I_1 \\ 0 & \text{elsewhere} \end{cases}$$

4.5 Period T demand (y_T) is represented by any continuous p.d.f.

For solving (S) a T period stochastic sequential problem we undertake in period T to evaluate the expression:

$$\underset{X_2}{\text{Min}}\ E_{y_T}\left[\sum_{k=1}^{T} X_k - \sum_{k=1}^{T-1} y_k - y_T\right]^+ \tag{15}$$

Let $\omega = \sum_{k=1}^{T} X_k - \sum_{k=1}^{T-1} y_k$ and $I_{T-1} = \sum_{k=1}^{T-1} (X_k - y_k)$, then (15) can be

rewritten as $E_{y_T}[\ \omega - y_T]^+$. Further, the expectation is evaluated by solving (16).

$$\int_0^{\omega} (\omega - y_T)\ dF(y_T) = \omega \int_0^{\omega} dF(y_T) - \int_0^{\omega} y_T\, dF(y_T) \tag{16}$$

4.5.1 Structure of $E_{y_T}[\omega - y_T]$

The expectation is an increasing function in ω in the region $\omega \geq 0$. This follows from the fact that:

$$\lim_{\omega \to \infty} \int_0^{\omega} dF(y_T) = 1\ ;\ \lim_{\omega \to \infty} \int_0^{\omega} y_T\, dF(y_T) = \mu_T \text{ and } \lim_{\omega \to \infty} (\omega - \mu_T) = \infty$$

Further, differentiating (16) with respect to ω_T and invoking the Leibnitz rule for differentiation (Protter and Morrey, 1966) (See Appendix C):

$$\frac{d}{d\omega} \int_0^{\omega} (\omega - y_T)\, \phi_T(y_T)\, dy_T = \int_0^{\omega} \phi_T(y_T)\ dy_T = F_T(\omega) \geq 0 \text{ because } \omega \geq 0.$$

4.5.2 Optimal Policy for period T

The optimal policy is determined by one critical variable $\lambda_T - I_{T-1}$, which is also the upper bound on the chance constraint. Since $g(\omega)$ is an increasing function in ω and X_T, then the optimal policy is charaterized by:

Condition	Optimal Policy	
$\lambda_T - I_{T-1} \geq 0$	$X_T = \lambda_T - I_{T-1}$	(17)
elsewhere	$X_t = 0$	

4.6 Optimal Policy for period T-1

In this section we obtain the optimal policy for period T-1, given that the optimal policy in period T is an order-up-to policy (section 4.5). Further we do not impose any restricting conditions on the probability density function (p.d.f.). We assume in general, that the p.d.f. has finite mean and assume capacity constraints on regular and overtime production.

4.6.1 Dynamic Recursion Scheme

The T-period sequential stochastic problem is hard to solve due to the nested optimization structure and the variability of the feasible solutions of the sequential decision problems with the demand observed in previous periods. The solutions for the inner programs are computed assuming known production and demands of previous periods. Hence they are function of these quantities. To this end we define:

$$\omega_t = I_{t-1} + X_t \qquad t = 1, 2, 3, \ldots, T. \tag{18}$$

$$I_{t-1} = \omega_{t-1} - y_{t-1} \qquad t = 1, 2, 3, \ldots, T. \tag{19}$$

$$g_n^{\star}(I_{n-1}) = E_{y_{n-1}} g_n(I_{n-1}) \quad n = 1, 2, 3, \ldots, T. \tag{20}$$

$$g_n(I_{n-1}) = \operatorname*{Min}_{X_n} f_n(I_{n-1}, X_n) \quad n = 1, 2, 3, \ldots, T. \tag{21}$$

$$f_n(I_{n-1}, X_n) = E_{y_n}\{ v_n X_n + o_n O_n + h_n(\omega_n - y_n)^+ + g_{n+1}(I_n)\} \qquad n = 1, 2, 3, \ldots, T. \tag{22}$$

In our dynamic recursion scheme, $g_n(I_{n-1})$ is the minimum expected cumulative cost rolling back from period T to n, where the cumulative expected cost component is denoted by $f_n(I_{n-1}, X_n)$. Since the minimum expected cumulative cost is obtained after minimizing over X_n (decision variable), $g_n(.)$ is a function of only the state variable unlike $f_n(.)$. Thus the dynamic recursion scheme eliminates a single decision variable from our set of T decision variables $(X_1, X_2, X_3, ..., X_T)$ at each step, because the minimization is undertaken with the aid of an optimal policy (a linear transformation) which is known a priori.

Remark: Since $g_n(I_{n-1}) = \operatorname{Min}_{X_n} f_n(I_{n-1}, \hat{X}_n) = f_n(I_{n-1}, \hat{X}_n)$ and $\hat{X}_n$ (optimal X_n) by our optimal policy is either equal to 0 $(\omega_n=I_{n-1})$ or λ_n-I_{n-1} $(\omega_n=\lambda_n)$; thus we represent $g_n(I_{n-1})$ alternatively as either $f_n(I_{n-1}, 0) = g_n(I_{n-1})$ or $f_n(I_{n-1}, \lambda_n-I_{n-1}) = g_n(\lambda_n)$ for n = 1,2,3, ..., T.

Remark: In order to establish the optimal policy in period T-1, we need to discuss the impact the capacitation assumption might have on the optimal solution. Since $mX_t - O_t \leq C_t$ and $O_t \leq c_t$, in equations (25), (26), (28) and (29), values of $X_T = (\lambda_T - \omega_{T-1} + y_{T-1}) \geq (C_T+c_T)/m$ are infeasible and thus in general values of $X_t = (\lambda_t - \omega_{t-1} + y_{t-1}) \geq (C_t + c_t)/m$ are infeasible.

Since we prove in 4.5 that the optimal policy in the Tth period is an order-up-to policy, the optimal X_T has the following structure:

$$X_T = \left\{ \begin{array}{ll} \lambda_T - I_{T-1} & \text{if } \lambda_T > I_{T-1} \\ 0 & \text{elsewhere} \end{array} \right\} \tag{23}$$

and $f_T(I_{T-1}, X_T)$ for n=T is defined as the expected holding and production cost in period T. Where $f_T(I_{T-1}, X_T)$ is written as:

$$
\begin{cases}
v_T X_T + o_T \max\{ mX_T - C_T, 0\} + h_T \int_0^{\omega_T} (\omega_T - y_T)\, \phi_T(y_T) dy_T & \text{if } \omega_T \geq 0 \\
v_T X_T + o_T \max\{mX_T - C_T, 0\} & \text{if } \omega_T \leq 0
\end{cases} \quad (24)
$$

To determine the expected holding cost of period T-1 and the expectation of the sum of the production and holding cost of period T, we define $f_{T-1}(I_{T-1}, X_T)$ just as we did for period T. ω_{T-1} is the sum of the inventory at the beginning of period T-1 (I_{T-2}) and the amount you produce in period T-1 (X_{T-1}). Further as a consequence of ω_{T-1} and y_{T-1}, I_{T-1} can experience a range of values, which will in turn determine the value of X_T and ω_T. Let us then prove three lemmas to help us in our effort.

Lemma 7: Under the condition $\omega_{T-1} \geq \lambda_T$.

$$
\omega_T = \begin{cases} \omega_{T-1} - y_{T-1} & \text{if } 0 \leq y_{T-1} \leq \omega_{T-1} - \lambda_T. \\ \lambda_T & \text{if } y_{T-1} \geq \omega_{T-1} - \lambda_T. \end{cases}
$$

Proof: By definition $I_{T-1} = \omega_{T-1} - y_{T-1}$. If $\omega_{T-1} \geq \lambda_T$ and $y_{T-1}=0$, then $I_{T-1} \geq \lambda_T$ and $X_T=0$ and the inequality is preserved until $y_{T-1} \leq \omega_{T-1} - \lambda_T$, and thereafter $I_{T-1} < \lambda_T$, and as a consequence of (23) $X_T = \lambda_T - I_{T-1}$ and $\omega_T = \lambda_T$.

Lemma 8: Under the condition $0 \leq \omega_{T-1} < \lambda_T$.

$$
\omega_T = \lambda_T \quad \text{if } 0 \leq y_{T-1} \leq \infty.
$$

Proof: By definition $I_{T-1} = \omega_{T-1} - y_{T-1}$. If $y_{T-1} > \omega_{T-1} - \lambda_T$, then $I_{T-1} < \lambda_T$ and as a consequence of (23) $X_T = \lambda_T - I_{T-1}$ and $\omega_T = \lambda_T$.

Lemma 9: $g_T^{\star}(I_{T-1}) = E_{y_{T-1}}\, g_T(I_{T-1}) = E_{y_{T-1}} \min_{X_T} f_T(I_{T-1}, X_T)$.

this expression differs under two cases: when $\omega_{T-1} \geq \lambda_T$ it is equal to:

$$h_T \int_0^{\omega_{T-1}-\lambda_T} \left\{ \int_0^{\omega_{T-1}-y_{T-1}} (\omega_{T-1}-y_{T-1}-y_T)\phi_T(y_T)dy_T \right\} \phi_{T-1}(y_{T-1})dy_{T-1}$$

$$+ \int_{\omega_{T-1}-\lambda_T}^{\infty} [v_T(\lambda_T-\omega_{T-1}+y_{T-1}) + h_T \int_0^{\lambda_T} (\lambda_T-y_T)\phi_T(y_T)dy_T]\phi_{T-1}(y_{t-1})dy_{T-1} +$$

$$+ o_T \int_{(C_T/m)+\omega_{T-1}-\lambda_T}^{\infty} \left\{ m(\lambda_T-\omega_{T-1}+y_{T-1}) - C_T \right\} \phi_{T-1}(y_{T-1})dy_{T-1} \qquad (25)$$

and when $0 \leq \omega_{T-1} < \lambda_T$ it is equal to:

$$\int_0^{\infty} [v_T(\lambda_T-\omega_{T-1}+y_{T-1}) + h_T \int_0^{\lambda_T} (\lambda_T-y_T)\phi_T(y_T)dy_T]\phi_{T-1}(y_{T-1})dy_{T-1} +$$

$$+ o_T \int_{(C_T/m)+\omega_{T-1}-\lambda_T}^{\infty} \{m(\lambda_T-\omega_{T-1}+y_{T-1}) - C_T\}\phi_{T-1}(y_{T-1})dy_{T-1} \qquad (26)$$

<u>Proof</u>: The expressions (25) and (26) are obtained by the repeated use of Lemma 7 and Lemma 8, after substituting (24) in (20) and (21) for n=T.

To illustrate our method we derive (25). From Lemma 7, for $0 \leq y_{T-1} \leq \omega_{T-1}-\lambda_T$, $\omega_T=\omega_{T-1}-y_{T-1}$, and $X_T=0$. Thus substituting the values of ω_T and X_T in (24) and recalling (20) we obtain:

$$h_T \int_0^{\omega_{T-1}-\lambda_T} \left\{ \int_0^{\omega_{T-1}-y_{T-1}} (\omega_{T-1}-y_{T-1}-y_T)\phi_T(y_T)dy_T \right\} \phi_{T-1}(y_{T-1})dy_{T-1}$$

Similarly, for $\omega_{T-1}-y_{T-1} \le y_{T-1} \le \infty$, $\omega_T=\lambda_T$, and $X_T=\lambda_T-\omega_{T-1}-y_{T-1}$, and substituting the values of ω_T and X_T in (24) and recalling (20) we obtain:

$$\int_{\omega_{T-1}-\lambda_T}^{\infty} [v_T(\lambda_T-\omega_T+y_{T-1}) + h_T \int_0^{\lambda_T} (\lambda_T-y_T)\phi_T(y_T)dy_T]\phi_{T-1}(y_{T-1})dy_{T-1} +$$

$$+ o_T \int_{(C_T/m)+\omega_{T-1}-\lambda_T}^{\infty} \left\{m(\lambda_T-\omega_{T-1}+y_{T-1}) - C_T\right\} \phi_{T-1}(y_{T-1})dy_{T-1}$$

Further from (22) $f_{T-1}(I_{T-2}, X_{T-1})$ is written as:

$$v_{T-1}X_{T-1} + o_{T-1}\max\{mX_{T-1}-C_{T-1},0\} +$$

$$h_{T-1}\int_0^{\omega_{T-1}}(\omega_{T-1}-y_{T-1})\ \phi_{T-1}(y_{T-1})dy_{T-1}+ g_T^{*}(I_{T-1}) \qquad (27)$$

Also from our set of constraints of Problem S, $\omega_t \ge \lambda_t$ for t=1, 2, 3, ..., T. As a result, ω_{T-1} must satisfy an additional constraint, $\omega_{T-1} \ge \lambda_{T-1}$. Finally, substituting (25) and (26) in (27) we obtain $f_{T-1}(I_{T-2}, X_{T-1})$, but the expression differs under two cases: when $\omega_{T-1} \ge \lambda_T$, it is equal to:

$$v_{T-1}X_{T-1} + h_{T-1} \int_0^{\omega_{T-1}}(\omega_{T-1}-y_{T-1})\ \phi_{T-1}(y_{T-1})dy_{T-1} +$$

Prod. Cost in Period T-1. *Holding Cost in Period T-1.*

$$o_{T-1}\max\{mX_{T-1}-C_{T-1},0\} +$$

Overtime Costs in Period T-1.

$$h_T \int_0^{\omega_{T-1}-\lambda_T} \left\{ \int_0^{\omega_{T-1}-y_{T-1}} (\omega_{T-1}-y_{T-1}-y_T)\ \phi_T(y_T)\ dy_T \right\} \phi_{T-1}(y_{T-1})\ dy_{T-1} +$$

Holding cost in Period T as a consequence of the decision in Period T-1. $(I_{T-1} \geq \lambda_T)$.

$$\int_{\omega_{T-1}-\lambda_T}^{+\infty} [v_T(\lambda_T-\omega_{T-1}+y_{T-1}) + h_T \int_0^{\lambda_T} (\lambda_T-y_T)\ \phi_T(y_T)\ dy_T]\ \phi_{T-1}(y_{T-1})dy_{T-1}$$

Production Cost in Period T. *Holding cost of Holding* λ_T *units in Period T.*

$$+o_T \int_{(C_T/m)+\omega_{T-1}-\lambda_T}^{\infty} \{m(\lambda_T-\omega_{T-1}+y_{T-1})-C_T\}\ \phi_{T-1}(y_{T-1})dy_{T-1}$$

Overtime costs in period T. (28)

and when $0 \leq \omega_{T-1} \leq \lambda_T$ it is equal to:

$$v_{T-1}X_{T-1} + h_{T-1} \int_0^{\omega_{T-1}} (\omega_{T-1}-y_{T-1})\ \phi_{T-1}(y_{T-1})dy_{T-1} +$$

Prod. cost in period T-1. *Holding cost in period T-1.*

$$+o_{T-1}\max\{mX_{T-1}-C_{t-1},0\} +$$

Overtime cost in period T-1.

$$+\int_0^{\infty} [v_T(\lambda_T-\omega_{T-1}+y_{T-1}) + h_T \int_0^{\lambda_T} (\lambda_T-y_T)\ \phi_T(y_T)dy_T]\ \phi_{T-1}(y_{T-1})dy_{T-1}$$

Prod. cost in Period T. *Holding cost in Period T.*

$$o_T \int_{(C_T/m)+\omega_{T-1}-\lambda_T}^{\infty} \{m(\lambda_T-\omega_{T-1}+y_{T-1}) - C_T\}\ \phi_{T-1}(y_{T-1})dy_{T-1} \tag{29}$$

Overtime cost in Period T.

From (18) and (19), we observe that ω_t is positively related to X_t and I_{t-1} respectively, thus in the minimization of $f_t(I_{t-1}, X_t)$, the first derivative of $f_t(I_{t-1}, X_t)$ shall remain unchanged if it is differentiated with respect to ω_t instead. We define the condition of stationarity to mean that $v_t=v_{t-1}=v$; and $o_t=o_{t-1}=o$ for t = 1, 2, ..., T. Now we state the first of several Theorems leading upto our claim that the optimal policy for any period t is an order-up-to policy.

Theorem 11: $f_{T-1}(I_{T-2}, X_{T-1})$ is an increasing function in ω_{T-1} and the optimal policy in period T-1 is

$$X_{T-1} = \left\{ \begin{array}{ll} \lambda_{T-1} - I_{T-2} & \text{if } \lambda_{T-1} > I_{T-2} \\ 0 & \text{elsewhere} \end{array} \right\}$$

if $|v_T + o_T m| \leq (f_{T-1}{}'(I_{T-2},X_{T-1}))^+$ {Positive component of the Slope}. <u>Note under the assumption of **stationarity**, theorem 11 has no restricting conditions.</u>

<u>Proof</u>: Differentiating (28) with respect to ω_{T-1} and invoking the Leibnitz rule (Protter and Morrey, 1966) we obtain:

$$(v_{T-1}-v_T) + m(o_{T-1}-o_T) + h_{T-1}F_{T-1}(\omega_{T-1}) +$$

$$+ h_T \int_0^{\omega_{T-1}-\lambda_T} F_T(\omega_{T-1}-y_{T-1})\ \phi_{T-1}(y_{T-1})dy_{T-1} +$$

$$v_T F_{T-1}(\omega_{T-1}-\lambda_T) + o_T m F_{T-1}([C_T/m]+\omega_{T-1}-\lambda_T) \tag{30}$$

Similarly differentiating (29) with respect to ω_{T-1} yields:

(31)

$$(v_{T-1}-v_T) + m(o_{T-1}-o_T) + h_{T-1}F_{T-1}(\omega_{T-1}) +$$
$$o_T m F_{T-1}([C_T/m]+\omega_{T-1}-\lambda_T) \qquad (31).$$

Thus from (30) and (31) it is easy to see that for all positive values of ω_{T-1}, $f_{T-1}(I_{T-2}, X_{T-1})$ is an increasing function in ω_{T-1}, and as a consequence the optimal policy is an order-up-to policy in period T-1.

4.7 Optimal Policy in Period t-1 given optimal policy in period t

In this section we prove that if the optimal policy in *any* period t is an order-up-to policy then the structure of the policy remains unchanged in period t-1 (*under certain conditions*). The result is significant, because in essence by mathematical induction it is easily proved that the optimal solution of Problem S is attained at $X_1=\lambda_1$.

4.7.1 Structure of the Optimal policy in period t

Given that the optimal policy in period t is an order-up-to policy, we can make the following assertion:

$$X_t = \left\{ \begin{array}{ll} \lambda_t - I_{t-1} & \text{if } \lambda_t > I_{t-1} \\ 0 & \text{elsewhere} \end{array} \right\}$$

Further, from (22) we can define for any n, $f_n(I_{n-1}, X_n)$ which is again different under two cases: when $\omega_n \geq \lambda_{n+1}$ it is:

$$v_n X_n + o_n \max\{mX_n - C_n, 0\} + h_n \int_0^{\omega_n} (\omega_n - y_n)\, \phi_n(y_n) dy_n +$$
$$+ \int_0^{\omega_n - \lambda_{n+1}} g_{n+1}(\omega_n - y_n)\phi_n(y_n)dy_n + \int_{\omega_n - \lambda_{n+1}}^{\infty} g_{n+1}(\lambda_{n+1})\phi_n(y_n)dy_n \tag{33}$$

and when $0 \leq \omega_n < \lambda_{n+1}$ it is:

$$v_n X_n + o_n \max\{mX_n - C_n, 0\} + h_n \int_0^{\omega_n} (\omega_n - y_n)\phi_n(y_n)dy_n +$$

$$\int_0^{\infty} g_{n+1}(\lambda_{n+1})\phi_n(y_n)dy_n \tag{34}$$

Reasoning as before and substituting $n=t-1$ in (33)-(34), $f_{t-1}(I_{t-2}, X_{t-1})$, which is again different under two cases: when $\omega_{t-1} \geq \lambda_t$, it is written as:

$$v_{t-1}X_{t-1} + h_{t-1}\int_0^{\omega_{t-1}}(\omega_{t-1} - y_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$o_{t-1}\max\{mX_{t-1}-C_{t-1},0\} +$$

$$\int_0^{\omega_{t-1}-\lambda_t} g_t(I_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$\int_{\omega_{t-1}-\lambda_t}^{\infty} g_t(\lambda_t)\ \phi_{t-1}(y_{t-1})dy_{t-1} \tag{35}$$

and when $0 \leq \omega_{t-1} < \lambda_t$ it is written as:

$$v_{t-1}X_{t-1} + h_{t-1}\int_0^{\omega_{t-1}}(\omega_{t-1}-y_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$o_{t-1}\max\{mX_{t-1}-C_{t-1},0\} +$$

$$\int_0^{\infty} g_t(\lambda_t)\phi_{t-1}(y_{t-1})dy_{t-1} \tag{36}$$

Since our primary aim is to minimize $f_{t-1}(I_{t-2}, X_{t-1})$, we can rewrite (35) and (36), from our definitions (20)-(22) and (27), as:

$$v_{t-1}X_{t-1} + o_{t-1}\max\{mX_{t-1}-C_{t-1}, 0\} + h_{t-1}\int_0^{\omega_{t-1}}(\omega_{t-1}-y_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$g_t^{\star}(I_{t-1}) \tag{37}$$

To evaluate $g_t^{\star}(I_{t-1})$, we substitute n=t in (33) and (34) and derive the expression for $f_t(I_{t-1}, X_t)$. On deriving $f_t(I_{t-1}, X_t)$ we assert that $g_t^{\star}(I_{t-1})$ has <u>five alternative formulations</u> which are presented in Lemma 10. (For a detailed account of the proof refer to Appendix A of the dissertation).

Lemma 10: $g_t^{\star}(I_{t-1})$ has five alternative formulations when the relationships among ω_{t-1}, λ_{t+1} and λ_t are outlined.

<u>Case 1</u>: $\omega_{t-1} \geq \lambda_{t+1} > \lambda_t$, $g_t^{\star}(I_{t-1})$ is expressed as:

$$\int_0^{\omega_{t-1}-\lambda_{t+1}} [h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1}-y_{t-1}-y_t)dF(y_t) + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1}-y_t)dF(y_t) +$$

$$\int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$\int_{\omega_{t-1}-\lambda_{t+1}}^{\omega_{t-1}-\lambda_t} [h_t \int_0^{\omega_{t-1}-y_{t-1}}(\omega_{t-1}-y_{t-1}-y_t)dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$\int_{\omega_{t-1}-\lambda_t}^{\infty} [v_t(\lambda_t-\omega_{t-1}+y_{t-1}) + o_t \max\{m(\lambda_t-\omega_{t-1}+y_{t-1})-C_t, 0\} +$$

$$h_t \int_0^{\lambda_t} (\lambda_t - y_t) dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1}) dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \tag{38}$$

<u>Case 2</u>: $\lambda_t \leq \omega_{t-1} < \lambda_{t+1}$, $g_t^{\star}(I_{t-1})$ is expressed as:

$$\int_0^{\omega_{t-1}-\lambda_t} [h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1} - y_{t-1} - y_t) dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1}) dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$\int_{\omega_{t-1}-\lambda_t}^{\infty} [v_t(\lambda_t - \omega_{t-1} + y_{t-1}) + o_t \max\{m(\lambda_t - \omega_{t-1} + y_{t-1}) - C_t, 0\} + h_t \int_0^{\lambda_t} (\lambda_t - y_t) dF(y_t) +$$

$$\int_0^{\infty} g_{t+1}(\lambda_{t+1}) dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \tag{39}$$

<u>Case 3</u>: $0 \leq \omega_{t-1} < \lambda_t < \lambda_{t+1}$

$$\int_0^{\infty} [v_t(\lambda_t - \omega_{t-1} + y_{t-1}) + o_t \max\{m(\lambda_t - \omega_{t-1} + y_{t-1}) - C_t, 0\} +$$

$$+ h_t \int_0^{\lambda_t} (\lambda_t - y_t) dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1}) dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \tag{40}$$

<u>Case 4</u>: $\lambda_{t+1} \leq \lambda_t \leq \omega_{t-1}$

$$\int_0^{\omega_{t-1}-\lambda_t} [\, h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1} - y_{t-1} - y_t) dF(y_t) + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}(\omega_{t-1} - y_{t-1}) dF(y_t) +$$

$$+ \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1}) dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$+ \int_{\omega_{t-1}-\lambda_t}^{\infty} [v_t(\lambda_t-\omega_{t-1}+y_{t-1}) + o_t \max\{m(\lambda_t - \omega_{t-1}+y_{t-1})-C_t, 0\} +$$

$$+ h_t \int_0^{\lambda_t} (\lambda_t-y_t)dF(y_t) + \int_0^{\lambda_t-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1})dF(y_t) +$$

$$+ \int_{\lambda_t-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1}. \tag{41}$$

<u>Case 5:</u> $0 \le \omega_{t-1} < \lambda_t$ and $\lambda_t \ge \lambda_{t+1}$.

$$\int_0^{\infty}[v_t(\lambda_t-\omega_{t-1}+y_{t-1}) + o_t\max\{m(\lambda_t-\omega_{t-1}+y_{t-1}) - C_t, 0\} +$$

$$+ h_t \int_0^{\lambda_t}(\lambda_t- y_t)dF(y_t) + \int_0^{\lambda_t-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1})dF(y_t) +$$

$$+\int_{\lambda_t-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \tag{42}$$

Theorem 12: If the optimal policy in period t is an order-up-to policy or a <u>one bin policy</u> given by

$$X_t = \begin{cases} \lambda_t - I_{t-1} & \text{if } \lambda_t > I_{t-1} \\ 0 & \text{elsewhere} \end{cases}$$

and assuming I_{t-2} as fixed, then $f_{t-1}(I_{t-2}, X_{t-1})$ is an increasing function in ω_{t-1}, and the optimal policy in period $t-1$ is an order upto the service level (λ_{t-1}) policy, which is:

$$X_{t-1} = \left\{ \begin{array}{ll} \lambda_{t-1} - I_{t-2} & \text{if } \lambda_{t-1} > I_{t-2} \\ 0 & \text{elsewhere} \end{array} \right\}$$

under the <u>following set of conditions:</u>

<u>Case 1</u>: If $\lambda_t < \lambda_{t+1} \le \omega_{t-1}$, then the structure of the policy is as stated above if:

$$(v_{t-1}-v_t) + m(o_{t-1}-o_t) + h_{t-1}F_{t-1}(\omega_{t-1}) + v_t F_{t-1}(\omega_{t-1}-\lambda_t) +$$

$$+ o_t m F_{t-1}((C_t/m) + \omega_{t-1} - \lambda_t) +$$

$$+ \int_0^{\omega_{t-1}-\lambda_{t+1}} [h_t F_t(\omega_{t-1}-y_{t-1}) + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}{}'(\omega_{t-1}-y_{t-1}-y_t)dF(y_t) +$$

$$+ \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}{}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$+ \int_{\omega_{t-1}-\lambda_{t+1}}^{\omega_{t-1}-\lambda_t} [h_t F_t(\omega_{t-1}-y_{t-1}) + \int_0^{\infty} g_{t+1}{}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} +$$

$$+ \int_{\omega_{t-1}-\lambda_t}^{\infty} [\int_0^{\infty} g_{t+1}{}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (43)$$

is greater than zero.

<u>Case 2</u>: if $\lambda_t \le \omega_{t-1} < \lambda_{t+1}$, then the structure of the policy is as stated

above if:

$$
\begin{aligned}
&(v_{t-1}-v_t) + m(o_{t-1}-o_t) + h_{t-1}F_{t-1}(\omega_{t-1}) + v_tF_{t-1}(\omega_{t-1}-\lambda_t) + \\
&o_t mF_{t-1}((C_t/m)+\omega_{t-1}-\lambda_t) + \\
&+ \int_0^{\omega_{t-1}-\lambda_t}[h_tF_t(\omega_{t-1}-y_{t-1}) + \int_0^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} + \\
&+ \int_{\omega_{t-1}-\lambda_t}^{\infty}[\int_0^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1}
\end{aligned}
\tag{44}
$$

is greater than zero.

Case 3: If $0 \leq \omega_{t-1} < \lambda_t < \lambda_{t+1}$, then the policy is as stated above if:

$$
\begin{aligned}
&(v_{t-1}-v_t) + m(o_{t-1}-o_t) + o_t mF_{t-1}((C_t/m)+\omega_{t-1}-\lambda_t) + \\
&+ \int_0^{\infty}[\int_0^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1}
\end{aligned}
\tag{45}
$$

is greater than zero.

Case 4: if $\lambda_{t+1} \leq \lambda_t \leq \omega_{t-1}$ then the policy is as above if:

$$
\begin{aligned}
&(v_{t-1}-v_t) + m(o_{t-1}-o_t) + v_tF_{t-1}(\omega_{t-1}-\lambda_t) + o_t mF_{t-1}((C_t/m)+\omega_{t-1}-\lambda_t) + \\
&+ h_{t-1}F_{t-1}(\omega_{t-1}) + \int_0^{\omega_{t-1}-\lambda_t}[h_tF_t(\omega_{t-1}-y_{t-1})]\ \phi_{t-1}(y_{t-1})dy_{t-1} +
\end{aligned}
$$

$$+\int_0^{\omega_{t-1}-\lambda_t}[\phi_t(\omega_{t-1}-y_{t-1}-\lambda_{t+1})\{g_{t+1}(\omega_{t-1}-y_{t-1})-g_{t+1}(\lambda_{t+1})\}+$$

$$+\int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}'(\omega_{t-1}-y_{t-1})dF(y_t)+\int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1}+$$

$$+\int_{\omega_{t-1}-\lambda_t}^{\infty}[\int_0^{\lambda_t-\lambda_{t+1}} g_{t+1}'(\omega_{t-1}-y_{t-1})dF(y_t)+\int_{\lambda_t-\lambda_{t+1}}^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (46)$$

is greater than zero.

<u>Case 5</u>: If $0 \le \omega_{t-1} < \lambda_t$, and $\lambda_t \ge \lambda_{t+1}$ then the policy is as above if:

$$(v_{t-1}-v_t)+m(o_{t-1}-o_t)+h_{t-1}F_{t-1}(\omega_{t-1})+o_t mF_{t-1}((C_t/m)+\omega_{t-1}-\lambda_t)+$$

$$\int_0^{\infty}[\int_0^{\lambda_t-\lambda_{t+1}} g_{t+1}'(\omega_{t-1}-y_{t-1})dF(y_t)+\int_{\lambda_t-\lambda_{t+1}}^{\infty} g_{t+1}'(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (47)$$

(47)

is greater than zero.

<u>Proof</u>: Expressions (43)-(47) are obtained by differentiating expressions (39)-(42), invoking the Leibnitz rule (Appendix C) of differentiation, with respect to ω_{t-1}. (For a detailed account of the proof refer to Appendix B of the dissertation).

Remark: The following terms are always positive from Cases 1 through 5. They are:

v_{t-1} (production cost in period $t-1$), o_{t-1} (marginal overtime labor cost in period $t-1$),, $h_{t-1}F_{t-1}(\omega_{t-1})$ (expected holding cost in period $t-1$), , $v_tF_{t-1}(\omega_{t-1}-\lambda_t)$ (partial expected production cost in period t), and $o_t m$

$F_{t-1}((C_t/m)+\omega_{t-1}\lambda_t)$ (expected overtime cost in period t).

We have in this chapter examined the total cost structure of problem (S), derived an order-up-to the service level policy, which is at first proven to be optimal in the one period problem. Later, the result is extended to determine conditions under which the policy is optimal in period T-1, given that it is optimal in period T. Finally, in our last induction step, we determine conditions under which the policy is optimal in period $t-1$, given that it is optimal in any period t. Notice the order-up-to the service level (λ_t) policy is "*myopic*" and does not have the look ahead capability. If we are to create policies with a look ahead feature we have to then determine order levels (γ_t), for each individual period as we go along in the dynamic recursion scheme, which are greater than λ_t. Devising such an algorithm is an avenue for future research.

CONCLUSION

5.1 Conclusion

In chapter two, we examined the stochastic non-sequential production planning problem, at first with fixed set-up costs and then with sequence dependent set-up costs. We proved in general that the stochastic version of the problem suggested by Karmarkar et al. (1987), bounds the stochastic production problem first introduced by Bitran and Yanasse (1984). and their respective deterministic equivalents exhibit identical worst case behavior. In chapter three, we examined the more intractable, but the more realistic sequential production planning problem. We focused our attention on the two period problem, and assumed zero fixed set-up costs, and using Jensen's inequality, and a result due to Huang et al. (1977a, 1977b), derived a family of approximations which spanned the spectrum of values between the bounds of the problem. We also examined the worst case difference between the lower and the upper bound and obtained a worst case error no greater than 23% of the optimal solution. Finally in chapter four, we obtained an optimal policy for a version of the one period stochastic sequential production planning problem, and extended our analysis by mathematical induction to derive conditions such that an order-up-to the service level is optimal for the T-period stochastic sequential problem.

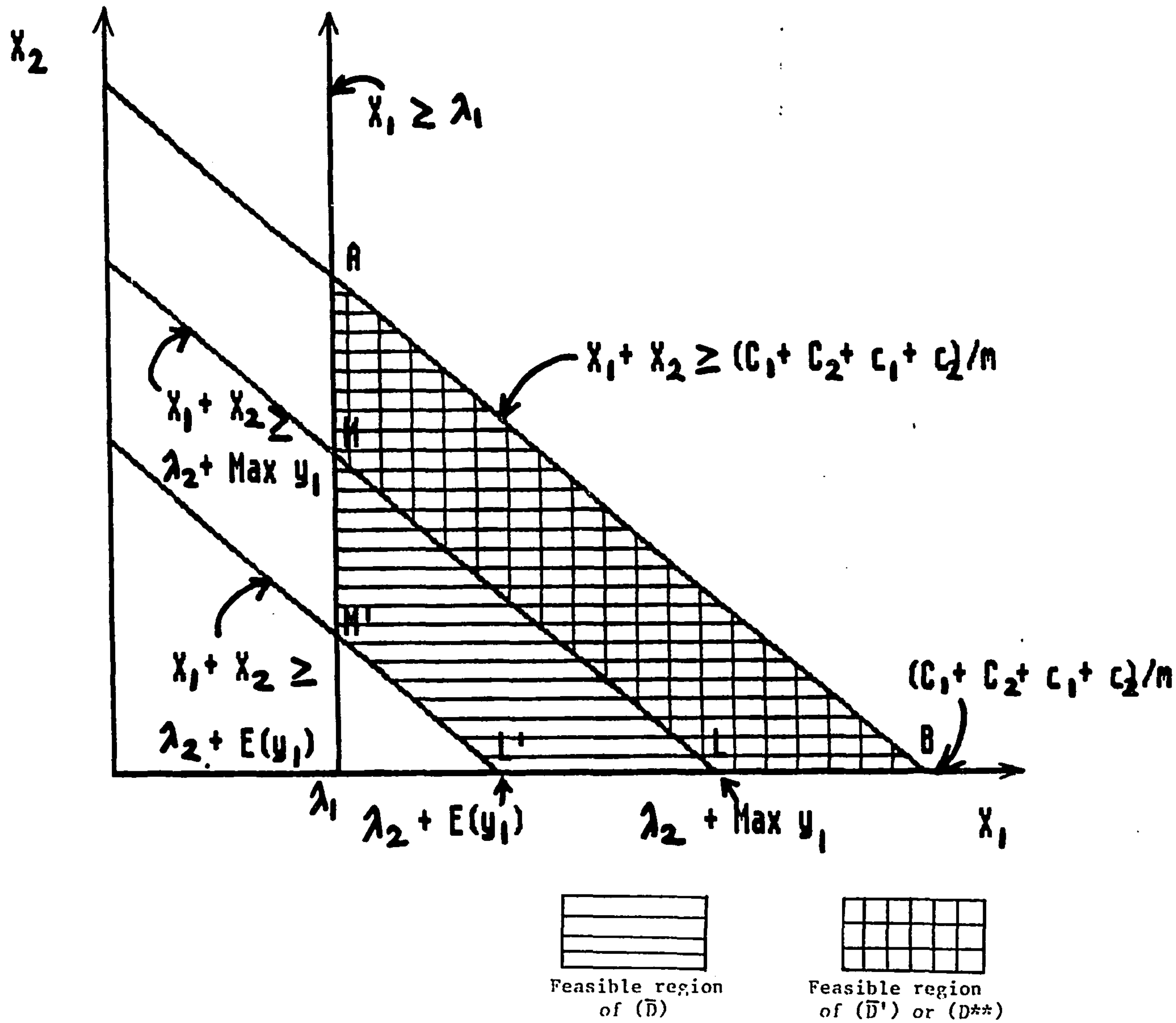
5.2 Opportunities for future research

The next logical step is examining the stochastic sequential production planning problem with non zero fixed set-up costs and monotonically increasing period holding costs as a function of ω_t. It is conjectured, that the optimal policy will be a non-stationary, (s_t, S_t), type of policy. Where the cost expressions from period to period will exhibit K-quasiconvexity (Porteus, 1971).

Another, line of research is examining the types of policies which are optimal when the conditions of Theorem 12 are not satisfied or the slopes in Case 1 through 5 are not positive. Again, it is conjectured, an order-up-to γ_t policy will be optimal, where $\gamma_t \geq \lambda_t$.

Certain problems in finance, for example the cash management problem the pension fund management problem etc., are similar to problem (S). It is conjectured since an order-up-to the service level policy is optimal, a similar policy may be optimal for the cash management and the pension fund management problem as well.

Figure 1



X2
X1 ≥ λ1
A
X1 + X2 ≥ (C1 + C2 + c1 + c2)/m
M
X1 + X2 ≥
λ2 + Max y1
M'
X1 + X2 ≥
λ2 + E(y1)
L'
L
B
(C1 + C2 + c1 + c2)/m
λ1
λ2 + E(y1)
λ2 + Max y1
X1
Feasible region of (D̄)
Feasible region of (D̄') or (D**)

$$
\begin{array}{ccccccc}
 & v(\bar{D}^{*}) & \leq & v(D^{*}) & & & \\
\rotatebox{45}{$\leq$} & & & & \rotatebox{-45}{$\geq$} & & \\
v(\bar{D}) & \leq & v(S) & \leq & v(D^{**}) & \leq & v(D^{**};\hat{X},\hat{O}) \\
 & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & & & \\
 & & v(\bar{D}') & & & &
\end{array}
$$

Fig. 2

Summarizing the relationships
between optimal objective function values
for a collection of production problems.

APPENDIX A

Detailed proof of Lemma 10

Lemma 10 (p. 60): $g_t^{\star}(I_{t-1})$ has five alternative formulations when the relationships among ω_{t-1}, λ_{t+1} and λ_t are outlined.

Remark: In our endeavour we present the derivation of the most detailed expression (Case 1), because all others are easily derived once the method of proof is outlined.

Proof: From (20) and (21) $g_t^{\star}(I_{t-1}) = E_{y_{t-1}} \operatorname{Min}_{X_t} f_t(X_t, I_{t-1})$,

and $f_t(X_t, I_{t-1})$ is derived by substituting n=t in (33) and (34). We also outline in general a **Procedure** to derive the partial expectations and we demonstrate its utility by deriving Case 1, which is a sum of three partial expectations denoted by α, β and γ respectively.

Procedure

Step 1: Identify the breakpoints in the range of y_{t-1} over which the partial expectations of $\operatorname{Min}_{X_t} f_t(X_t, I_{t-1})$ are evaluated.

Step 2: For $y_{t-1} \in [c, d]$, identify the corresponding range of I_{t-1}.

Step 3: On identifying the corresponding range of I_{t-1}, determine the optimal value of X_t, where the optimal policy is given by (1).

$$X_t = \begin{Bmatrix} \lambda_t - I_{t-1} & \text{if } \lambda_t > I_{t-1} \\ 0 & \text{elsewhere} \end{Bmatrix} \tag{1}$$

Step 4: Determine the value of ω_t, by substituting values of I_{t-1} and X_t in the expression $\omega_t = X_t + I_{t-1}$.

Step 5: Choose expression (33) if $\omega_t \geq \lambda_{t+1}$ and (34) if $\omega_t < \lambda_{t+1}$.

Derivation of Case 1

Case 1 of Lemma 10 is obtained if the condition, $\omega_{t-1} \geq \lambda_{t+1} > \lambda_t$ is satisfied. The condition implies, that the expectation is not uniformly evaluated over the range of $y_{t-1} \in [0,\infty]$ but instead, it is a sum of three partial expectations, denoted by α, β and γ where $y_{t-1} \in [0, \omega_{t-1}-\lambda_{t+1}]$, $y_{t-1} \in [\omega_{t-1}-\lambda_{t+1}, \omega_{t-1}-\lambda_t]$ and $y_{t-1} \in [\omega_{t-1}-\lambda_t, \infty]$ respectively.

Derivation of expression α: Since $y_{t-1} \in [0, \omega_{t-1}-\lambda_{t+1}]$, following the procedure, correspondingly $I_{t-1} \in [\omega_{t-1}, \lambda_{t+1}]$. Further, $\lambda_{t+1} > \lambda_t$, hence $X_t = 0$ (step 3) and $\lambda_{t+1} \leq \omega_t \leq \omega_{t-1}$ $(\omega_t = I_{t-1})$ (step 4). Thus we choose expression (33) (step 5) to determine the first of the partial expecations, which is written as:

$$\int_0^{\omega_{t-1}-\lambda_{t+1}} [h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1}-y_{t-1}-y_t)dF(y_t) + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1}-y_t)dF(y_t) + \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (\alpha)$$

Derivation of expression β: Since $y_{t-1} \in [\omega_{t-1}-\lambda_{t+1}, \omega_{t-1}-\lambda_t]$ (step 1), $I_{t-1} \in [\lambda_{t+1}, \lambda_t]$ (step 2), $X_t=0$ (step 3), $\omega_t=I_{t-1}$ (step 4), and we choose expression (34) (step 5) because $\omega_t \leq \lambda_{t+1}$. Thus the second of the partial expectations is written as:

$$\int_{\omega_{t-1}-\lambda_{t+1}}^{\omega_{t-1}-\lambda_t} [h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1}-y_{t-1}-y_t)dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (\beta)$$

Derivation of expression γ: Since $y_{t-1} \in [\omega_{t-1}-\lambda_t, \infty]$ (step 1), $I_{t-1} \in [\lambda_t, -\infty]$ (step 2), $X_t=\lambda_t-I_{t-1}$ (step 3), $\omega_t=\lambda_t$ (step 4), and we choose expression (34) (step 5) to derive expression γ:

$$\int_{\omega_{t-1}-\lambda_t}^{\infty} [v_t(\lambda_t-\omega_{t-1}+y_{t-1}) + o_t\max\{m(\lambda_t-\omega_{t-1}+y_{t-1}) - C_t, 0\} +$$

$$h_t \int_0^{\lambda_t} (\lambda_t-y_t)dF(y_t) + \int_0^{\infty} g_{t+1}(\lambda_{t+1})dF(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (\gamma)$$

And it is easy to see expression (38) (p. 60) is the sum of expressions α, β and γ.

APPENDIX B

Detailed proof of Theorem 12

Theorem 12 (p. 62): If the optimal policy in period t is an order-up-to the service level (λ_t) policy and assuming I_{t-2} as fixed, then $f_{t-1}(I_{t-2}, X_{t-1})$ is an increasing function in ω_{t-1}, and the optimal policy in period t-1 is an order-up-to the service level (λ_{t-1}) policy under <u>five separate set of conditions</u>.

Remark: Just as we derived the most complex of expressions in Appendix A we do the same in Appendix B and obtain the first derivative of Case 1 of Lemma 10, with respect to ω_{t-1} by invoking the Leibnitz rule outlined in Appendix C.

<u>ProoProof</u>: We begin, by making the observation that $g_t^{\star}(I_{t-1})$ in Case1 (Appendix A), is the sum of three different partial expectations (α, β and γ). Further, from (37), $f_{t-1}(I_{t-2}, X_{t-1})$ is written as:

$$v_{t-1}X_{t-1} + o_{t-1}\max\{mX_{t-1}-C_{t-1}, 0\} + h_{t-1}\int_0^{\omega_{t-1}}(\omega_{t-1}-y_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1} + g_t^{\star}(I_{t-1}). \qquad (1)$$

Thus differentiating $f_{t-1}(I_{t-2}, X_{t-1})$ with respect to ω_{t-1} we obtain:

$$v_{t-1} + o_{t-1}m + h_{t-1}\frac{d}{d\omega_{t-1}}\left\{\int_0^{\omega_{t-1}}(\omega_{t-1}-y_{t-1})\phi_{t-1}(y_{t-1})dy_{t-1}\right\} + d(\alpha)/d\omega_{t-1} + d(\beta)/d\omega_{t-1} + d(\gamma)/d\omega_{t-1}. \qquad (2)$$

Differntiating the first of the expressions in (2) under the integral sign and invoking procedure 2 (Leibnitz Rule) of Appendix C, we obtain:

$u_1(x) = \omega_{t-1}$, $u_0(x) = 0$, $x = \omega_{t-1}$, $t = y_{t-1}$,
$f(x, t) = h_{t-1}(\omega_{t-1} - y_{t-1})\phi_{t-1}(y_{t-1})$ (step 1); $f(x, u_1(x))u_1'(x) = 0$ (step 2);

$f(x, u_0(x))u_0'(x) = 0$ (step 3); $\int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)dt = h_{t-1}\int_0^{\omega_{t-1}} \phi_{t-1}(y_{t-1})dy_{t-1}$ (step 4); and $(d/d\omega_{t-1}) \int_{u_0(x)}^{u_1(x)} f(x, t)dt = h_{t-1}F_t(\omega_{t-1})$ (step 5).

Similarly, we invoke procedure 2 (Appendix C), and obtain the first derivative of expressions α, β and γ (Appendix A) respectively.

<u>First derivative of α</u>

$u_1(x) = \omega_{t-1} - \lambda_{t+1}$, $u_0(x) = 0$, $x = \omega_{t-1}$, $t = y_{t-1}$,

$$f(x, t) = [\, h_t \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t + \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})\phi_t(y_t)dy_t]\phi_{t-1}(y_{t-1}) \text{ (step 1);}$$

$$f(x, u_1(x))u_1'(x) = [h_t \int_0^{\lambda_{t+1}} (\lambda_{t+1}-y_t)\phi_t(y_t)dy_t + \int_0^{\infty} g_{t+1}(\lambda_{t+1})\phi_t(y_t)dy_t]\phi_{t-1}(\omega_{t-1} - \lambda_{t+1}) \text{ (Step 2);} \qquad (3)$$

$f(x, u_0(x))u_0'(x) = 0$ (step 3);

$$\int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)dt = \int_0^{\omega_{t-1}-\lambda_{t+1}} [h_t \, (\delta/\delta\omega_{t-1}) \int_0^{\omega_{t-1}-y_{t-1}} (\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t \; +$$
$$(\delta/\delta\omega_{t-1}) \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t \; +$$
$$(\delta/\delta\omega_{t-1}) \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})\phi_t(y_t)dy_t]\phi_{t-1}(y_{t-1})dy_{t-1} \quad \text{(step 3)} \quad (4)$$

In step 4 we observe the partial deriavative of $f(x, t)$ is obtained by differentiating three nested integrals, denoted in the order as they appear as a, b and c respectively.

i) <u>Partial differentiation of integral 'a'</u>

Again invoking procedure 2 of Appendix B we obtain:

$u_1(x) = \omega_{t-1}-y_{t-1}$, $u_0(x) = 0$, $x = \omega_{t-1}$, $t = y_t$,
$f(x, t) = h_t(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)$ (step 1);
$f(x, u_1(x))u_1'(x) = 0$ (step 2); $f(x, u_0(x))u_0'(x) = 0$ (step 3);
$\int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)\, dt = h_t \int_0^{\omega_{t-1}-y_{t-1}} \phi_t(y_t)\, dy_t = h_t\, F_t(\omega_{t-1}-y_{t-1})$ (step 4);

and the final answer is $h_t\, F_t(\omega_{t-1}-y_{t-1})$ (step 5) (5)

<u>Partial differentiation of integral 'b'</u>

Invoking procedure 2 we obtain:

$u_1(x) = \omega_{t-1}-y_{t-1}$, $u_0(x) = 0$, $x = \omega_{t-1}$, $t = y_t$ and
$f(x, t) = g_{t+1}(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)$ (step 1);
$f(x, u_1(x))u_1'(x) = g_{t+1}(\lambda_{t+1})\ \phi_t(\omega_{t-1}-y_{t-1}-y_t)$ (step 2);
$f(x, u_0(x))u_0'(x) = 0$ (step 3);

$$\int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)dt = \int g_{t+1}{'}(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t \text{ and}$$

$$(d/d\omega_{t-1}) \int_{u_0(x)}^{u_1(x)} f(x, t)dt = g_{t+1}(\lambda_{t+1})\phi_t(\omega_{t-1}-y_{t-1}-\lambda_{t+1}) + \int_0^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}{'}(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t \text{ (step 5)} \qquad (6)$$

Partial differentiation of integral 'c'

$u_1(x) = \infty$, $u_0(x) = \omega_{t-1}-y_{t-1}-\lambda_{t+1}$, $x = \omega_{t-1}$, $t = y_t$,
$f(x, t) = g_{t+1}(\lambda_{t+1})\phi_t(y_t)$ (step 1);
$f(x, u_1(x))u_1{'}(x) = 0$ (step 2); $f(x, u_0(x))u_0{'}(x) = g_{t+1}(\lambda_{t+1})\phi_t(\omega_{t-1}-y_{t-1}-\lambda_{t+1})$ (step 3);

$$\int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)dt = \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}{'}(\lambda_{t+1})\phi_t(y_t)dy_t \text{ (step 4);}$$

and $$(d/d\omega_{t-1}) \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}(\lambda_{t+1})\phi_t(y_t)dy_t = - g_{t+1}(\lambda_{t+1})\phi_t(\omega_{t-1}-y_{t-1}-\lambda_{t+1}) + \int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}{'}(\lambda_{t+1})\phi_t(y_t)dy_t \qquad (7)$$

Continuing with our differentiation of expression 'α', and substututing equations (5), (6) and (7) in (4) to complete step 4 of procedure 2, the resulting expression is:

$$\int_{0}^{\omega_{t-1}-\lambda_{t+1}} [h_t \, F_t(\omega_{t-1}-y_{t-1}) + \int_{0}^{\omega_{t-1}-y_{t-1}-\lambda_{t+1}} g_{t+1}''(\omega_{t-1}-y_{t-1}-y_t)\phi_t(y_t)dy_t +$$

$$\int_{\omega_{t-1}-y_{t-1}-\lambda_{t+1}}^{\infty} g_{t+1}''(\lambda_{t+1})\phi_t(y_t)]\phi_{t-1}(y_{t-1})dy_{t-1} \qquad (8)$$

The first derivative of α is obtained by substracting the result obtained from step 3 (which equals zero) from the sum of the result obtained from step 2 (equation (3)) and step 4 (equation (8)).

Similarly, the expressions β and γ are differentiated to obtain the expression of Case 1 (Theorem 12).

APPENDIX C

Leibnitz' Rule for differentiation under the integral sign

Theorem (Protter and Morrey, 1966)

Suppose that f and $\delta f/\delta x$ are continuous in the rectangle R:{a $\leq$ x $\leq$ b, c $\leq$ t $\leq$ d}, and suppose that $u_0(x)$, $u_1(x)$ are continuously differentiable for a $\leq$ x $\leq$ b with the range of u_0 and u_1 in [c, d]. If ϕ is given by

$$\phi(x) = \int_{u_0(x)}^{u_1(x)} f(x, t)\, dt$$

then

$$\phi'(x) = f[x, u_1(x)]u_1'(x) - f[x, u_0(x)].u_0'(x) + \int_{u_0(x)}^{u_1(x)} f_{,1}(x, t)\, dt$$

where $\phi'(x) = \delta\phi(x)/\delta x$; $f_{,1}(x, t) = \delta f(x, t)/\delta x$; $u_1'(x) = \delta u_1(x)/\delta x$ and $u_0'(x) = \delta u_0(x)/\delta x$.

In summary, a procedure is outlined to apply the leibnitz rule:

Procedure 2

Step 1: Identify $u_1(x)$ and $u_0(x)$, x, t and f(x, t).

Step 2: Substitute t by $u_1(x)$ in f(x, t) and multiply the expression by the first derivative of $u_1(x)$ with respect to x.

Step 3: Substitute t by $u_0(x)$ in f(x, t) and multiply the expression by the first derivative of $u_0(x)$ with respect to x.

Step 4: Obtain the partial derivative of f(x,t) with respect to x, and integrate the expression over R $\in$ [$u_0(x)$, $u_1(x)$].

Step 5: Subtract the result obtained from step 3 from the sum of the results obtained from step2 and step4.

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