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Diophantine Approximation and The Atypical Numbers of Nathanson and O'Bryant

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Abstract

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David Seff

Advisor: Kevin O'Bryant

For any positive real number $\theta > 1$, and any natural number n, it is obvious that sequence $\theta^{1/n}$ goes to 1. Nathanson [8] and O'Bryant [9] studied the details of this convergence and discovered some truly amazing properties. One critical discovery is that for almost all n, $\left\lfloor \frac{1}{\{\theta^{1/n}\}} \right\rfloor$ is equal to $\left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor$, the exceptions, when $n > \log_2 \theta$, being termed atypical n (the set of which for fixed θ being named $\mathcal{A}_\theta$), and that for $\log \theta$ rational, the number of atypical n is finite. Nathanson left a number of questions open, and, subsequently, O'Bryant developed a theory to answer most of these questions. He also posed five new unanswered questions of his own [9, Section 7. More Problems] (which are enumerated at the end of this this abstract), of which we completely answer three, and partially answer two.

He constructed infinite families of bounded θ's with rational logarithms, some with no atypical n, and some with infinitely many atypical n. However, he left as an

open problem whether there was some upper bound, θ_0 such that $\{\theta : \theta > \theta_0, \log\theta$ is irrational, and $\mathcal{A}_\theta$ is finite$\}$ is not uncountable, which is his third question. This thesis shows that the restriction of boundedness cannot be removed and is described in detail in Chapter 3. During the course of the development needed to answer that question, this thesis proceeds to answer the fifth question in Section 3.4 and the first question in Section 3.5. Questions 2 and 4 below remain unanswered, but I bring some partial results and suggest methods for further research on these problems in Section 4.1.1 and Section 4.2 respectively. Finally, in Chapter 5, I list some additional open questions. Here is the list of O'Bryant's open questions:

1. Is $\mathcal{A}_{e^e}$ infinite?
2. Are there θ, τ with both $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ infinite, but the symmetric difference $\mathcal{A}_\theta \triangle \mathcal{A}_\tau$ finite?
3. For every θ_0, are there uncountably many $\theta > \theta_0$ with $\mathcal{A}_\theta$ finite?
4. What is the Hausdorff dimension of $\{\theta > 1 : \mathcal{A}_\theta$ is finite$\}$?
5. Is there any algebraic θ for which $\mathcal{A}_\theta$ can be proved finite? Infinite?

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Ann Judith Berenstein
March 28, 1945 - January 24, 2005

Contents

List of Tables

Chapter 1

Introduction

1.1 Fundamental Results of Nathanson and O'Bryant

Nathanson [8] introduced and derived the basic properties of the function

$$M_\theta(n) := \left\lfloor \frac{1}{\{\theta^{1/n}\}} \right\rfloor,$$

where θ is a positive real number, n is an integer, and $\lfloor\cdot\rfloor$ and $\{\cdot\}$ are the floor and fractional part functions, respectively. He also derived the basic properties of this function and identified symmetries that allow one to assume without loss of generality that $\theta > 1$ and that the integer n is positive. He obtained a number of surprising results, among them being that for any real $\theta > 1$ and integer $n > \log_2 \theta$, either $M_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor$ or $M_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$; moreover, if $\log\theta$ is rational, then $M_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor$ for all sufficiently large n. He also mentioned a number of open questions for further research.

Perhaps the most amazing aspect of Nathanson's paper was the fact that he obtained significant results without the use of continued fractions. One would expect to see the use of continued fractions in the development, since the continued fraction algorithm produces a positive integer from a real number $\alpha > 1$ by taking the integer part of the reciprocal of the fractional part of α, which is exactly how the function $M_\theta(n)$ operates on the nth root of a real number $\theta > 1$.

In a subsequent paper [9], O'Bryant gave alternate proofs of some of Nathanson's results, sharpened or refined some of those results, solved most of the unsolved problems presented by Nathanson, and created a list of his own. By using information from continued fractions, O'Bryant's methodology gave additional, and sometimes deeper, insights into Nathanson's results.

O'Bryant showed that the set

$$\left\{ n \in \mathbf{N} : M_\theta(n) \neq \left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor \right\} \tag{1.1}$$

has density 0 for all $\theta > 1$, and for almost all $\theta > 1$ has counting function asymptotic to $\frac{\log \theta}{12} \log n$.

O'Bryant introduced an additional function and a set of atypical numbers: The set of positive integers is denoted $\mathbf{N}$. Throughout, we assume that $\theta > 1$ and that n is a positive integer. If $n > \log_2 \theta$, then $1 < \theta^{1/n} < 2$, and so $\left\{\theta^{1/n}\right\} = \theta^{1/n} - 1$. Set

$$M'_\theta(n) := \left\lfloor \frac{1}{\theta^{1/n} - 1} \right\rfloor,$$

so that $M_\theta(n) = M'_\theta(n)$ if $n > \log_2 \theta$. He called the elements of

$$\mathcal{A}_\theta := \left\{ n \in \mathbf{N} : M'_\theta(n) = \left\lfloor \frac{n}{\log \theta} + \frac{1}{2} \right\rfloor \right\}$$

the *atypical* numbers, meaning this set is relatively small for almost all θ, as explained above and in O'Bryant's Theorem 1 below. Almost all results and open problems are stated in terms of $\mathcal{A}_\theta$. For $n > \log_2 \theta$ and for $\theta < e^6 \approx 400$, he gave criteria for $\mathcal{A}_\theta$ to be finite or infinite in terms of the continued fraction expansions of $1/\log \theta$ and $2/\log \theta$. While Nathanson had proved (1.1) is finite whenever $\theta = e^{p/q}$, where p/q is a rational number, O'Bryant gave another proof that gives an explicit bound on the size in terms of p and q.

Our main results deal with determining when the atypical set will be finite or infinite for irrational $\log \theta$, and are based on certain parity patterns in the continued fraction of $1/\log \theta$.

1.2 O'Bryant's Main Results

The following, with some omissions, are direct citations from O'Bryant's paper that are needed for the following development. All proofs and most discussion are omitted, as the main need is for the statement of the theorems. (O'Bryant did indeed provide proofs, but they are omitted here.) The numbering of lemmas and theorems in this section and the next, follow O'Bryant's numbering. While we use standard results concerning continued fractions, the only new results of O'Bryant that are used directly are his Lemma 7 and 8, the others being included for completeness as well as to give a basic orientation to the concepts used.

We will using the same numbering for O'Bryant's lemmas and theorems as he used, but for our own, every lemma and theorem, as well as every topic in each section, will be given another set of numbers.

Nathanson proved the following result, albeit in different notation.

Theorem 1. *If $n > \log_2 \theta$, then either*

$$M_\theta(n) = \left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor \qquad \text{and} \qquad n \notin \mathcal{A}_\theta$$

or

$$M_\theta(n) = \left\lfloor \frac{n}{\log \theta} + \frac{1}{2} \right\rfloor \qquad \text{and} \qquad n \in \mathcal{A}_\theta.$$

An important consequence of this theorem is that for sufficiently large n, $M'_\theta(n) = M_\theta(n)$, and consequently, most results will be stated in terms of $\mathcal{A}_\theta$.

Theorem 2 (Nathanson)**.** *If $\log \theta = p/q > 1$ is rational, then*

$$\mathcal{A}_\theta \subseteq [1, \frac{p^2}{6q}).$$

Theorem 3. *For all $\theta > 1$, $\mathcal{A}_\theta$ has density 0.*

Theorem 4. *For almost all $\theta > 1$,*

$$|\mathcal{A}_\theta \cap [1, n]| \sim \frac{\log \theta}{12} \log n.$$

Theorem 5. *Let a_i be positive integers with $a_{2k} = 1$ for $k \geq 0$. Set ℓ to be the irrational with simple continued fraction $[a_0; a_1, a_2, \dots]$, and set $\theta = e^{2/\ell}$. Then $\mathcal{A}_\theta = \emptyset$. In particular, if $c \in \mathbf{N}$ and $\theta = e^{-c+\sqrt{c(c+4)}}$, then $\mathcal{A}_\theta$ is empty.*

Theorem 6. *Let a_i be positive integers with $a_0 = 0$, $a_1 = 2$, $a_{2k} = 4$ for all $k \geq 1$. Set ℓ to be the irrational with simple continued fraction $[a_0; a_1, a_2, \dots]$, and set $\theta = e^{2/\ell}$. Then $\mathcal{A}_\theta$ is infinite. In particular, if $c \in \mathbf{N}$ and $\theta = e^{4-c+\sqrt{c(c+1)}}$, then $\mathcal{A}_\theta$ is infinite.*

These last two theorems give explicit uncountable families of θ with $\mathcal{A}_\theta$ empty and infinite, of which the simplest examples are $\mathcal{A}_{e^{\sqrt{5}-1}}$ which is empty and $\mathcal{A}_{e^{2\sqrt{5}}}$ which is infinite. O'Bryant's proofs are based upon inequalities using partial quotients of continued fractions. All his examples consisted entirely of transcendental numbers, and he notes that he did not know whether there were algebraic θ with $\mathcal{A}_\theta = \emptyset$, nor whether there is an algebraic θ with $\mathcal{A}_\theta$ infinite. By extending his methods, we are able to show that both types of algebraic θ exist.

For $t > 0$, he defines the function

$$f(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}. \tag{1.2}$$

Lemma 7. *For $t > 0$, the function $f(t)$ is strictly increasing,* $\lim_{t\to 0^+} f(t) = 0$, *and* $\lim_{t\to\infty} f(t) = 1/2$. *If $0 < t < 1$, then*

$$\frac{t}{12} - \frac{t^3}{720} < f(t) < \frac{t}{12}. \tag{1.3}$$

Lemma 8. *Either*

$$M'_\theta(n) = \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor \text{ or } M'_\theta(n) = \left\lfloor \frac{n}{\log\theta} + \frac{1}{2} \right\rfloor,$$

and $M'_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$ *if and only if*

$$\frac{1}{2} - f\left(\frac{\log\theta}{n}\right) \le \left\{\frac{n}{\log\theta}\right\} < \frac{1}{2}. \tag{1.4}$$

Note that Theorem 1 is a direct consequence of Lemma 8.

O'Bryant's Lemma 11 is also important, but must be preceded by a brief notational introduction as it uses a slightly non-standard notation based upon some notation and theorems of Rockett and Szüsz [10] concerning continued fractions not found in other standard references.

If an irrational real number α has the continued fraction $[a_0; a_1, a_2, \ldots]$, we shall define the *kth convergent* to be the rational number

$$\frac{A_k}{B_k} := [a_0; a_1, a_2, \ldots, a_k]$$

where A_k and B_k are relatively prime positive integers. Also, we define

$$\lambda_k := [0; a_{k-1}, a_{k-2}, \ldots, a_1] + [a_k; a_{k+1}, a_{k+2}, \ldots].$$

The sequence of denominators, sometimes called *continuants*, satisfies $B_k \ge F_{k+1}$, the $(k+1)$th Fibonacci number. Further,

$$\frac{A_{2k-2}}{B_{2k-2}} < \frac{A_{2k}}{B_{2k}} < \alpha < \frac{A_{2k+1}}{B_{2k+1}} < \frac{A_{2k-1}}{B_{2k-1}} \tag{1.5}$$

$$\alpha - \frac{A_k}{B_k} = \frac{(-1)^k}{B_k^2 \lambda_{k+1}}. \tag{1.6}$$

This is often used in conjuction with the trivial bounds

$$a_{k+1} < \lambda_{k+1} < a_{k+1} + 2.$$

If m and n are natural numbers and

$$\left| \alpha - \frac{m}{n} \right| \leq \frac{1}{2n^2}, \tag{1.7}$$

then [10, Theorem II.5.1] there are integers $k \geq 0, c \geq 1$ such that $m = cA_k$ and $n = cB_k$ and $\lambda_{k+1} > 2c^2$.

Lemma 11. *Let $1 < \theta < e^3$ with $\log\theta$ irrational, and a_k, B_k, λ_k be associated to the continued fraction of $2/\log\theta$. For each $n \in \mathcal{A}_\theta$, there exists positive integers c, k such that $n = cB_{2k-1}$ and $\lambda_{2k} > \frac{6c^2}{\log\theta}$.*

Note too, that this result is based upon the boundedness of θ and used to prove O'Bryant's Theorems 5 and 6.

1.3 The Challenges of O'Bryant's Third Unsolved Problem

O'Bryant's third unsolved problem, which is our current problem, is: For every θ_0, are there uncountably many $\theta > \theta_0$ with $\mathcal{A}_\theta$ finite?

His Lemma 8 shows that for $n > \log_2 \theta$ to be atypical, $\{n/\log\theta\}$ must lie in a small interval bounded above by $1/2$, namely,

$$\frac{1}{2} - f\left(\frac{\log\theta}{n}\right) \le \left\{\frac{n}{\log\theta}\right\} < \frac{1}{2},$$

which we call (the standard) atypical interval. We will define $L_\theta(n) := \frac{1}{2} - f\left(\frac{\log\theta}{n}\right)$, which is monotonically increasing to $1/2$ as n tends to infinity, by O'Bryant's Lemma 7. If $\log\theta$ is irrational, then so is $1/\log\theta$, and therefore $\{n/\log\theta\}$ is dense in $[0, 1)$.[1] It is then trivial that there are an infinite number of n such that $\{n/\log\theta\}$ are slightly less than $1/2$. However, the lower bound $L_\theta(n)$ is not fixed, so when n is incremented by 1, two things happen—the value of $\{n/\log\theta\}$ is increased by $1/\log\theta$, and the atypical interval shrinks slightly. Now incrementing n repeatedly by either $\lfloor 1/\log\theta \rfloor$ or $\lceil 1/\log\theta \rceil$ will cause $\{n/\log\theta\}$ to go through a full cycle around the unit circle and land again, slightly to the left of $1/2$, causing the difference to be small again. The question then arises as to whether "slightly to the left" means that it is in the critical atypical interval given by Lemma 8 or not, since the lower bound, $L_\theta(n)$, also increased as n increased, and it is therefore possible that by incrementing n, that the new $\{n/\log\theta\}$ is not inside the new atypical interval but slightly to the left of its lower bound. This is a very delicate question and is one factor that makes O'Bryant's third problem both challenging and interesting.

[1]The standard definition of a set S being dense in an open interval means that for every β in the interval, and for every small ϵ, there is a member of S in an ϵ neighborhood of β. By convention, we consider $[0, 1)$ to mean the unit circle, where it is an open set, and an ϵ neighborhood of 0 is $[0, \epsilon) \cup (1 - \epsilon, 0)$.

A second challenging feature is the boundedness of irrational $\log \theta$, because in the proof Lemma 11, the boundedness of irrational $\log \theta$ plays a critical role, and, as a result, all thetas in the families of θ O'Bryant discovered meeting the hypotheses of Theorems 5 and 6 were also bounded.

1.4 The Statement of the Main Theorem

Definition 1.4.1. Let θ be any real number with an irrational log, with $\theta > 1$, with $\alpha := \frac{1}{\log \theta} = [a_0; a_1, a_2, \ldots]$, and having principal convergents $\{\frac{p_k}{q_k}\}_{k=0}^{\infty}$. The real number α is said to have the "**even property**" if and only if there exists some odd index k such that q_k is odd, q_{k+1} is even, and for every even $j > k+1$, a_j is even. In this case, θ is said to be "**special**." From the recursive formulae for principal convergents (Fact 3 of Section 2.2), θ being special is equivalent to there being at most a finite number of principal continuants of odd-index that are even.

The main theorem is:

Theorem (Main Theorem)**.** *Let θ be any real number with an irrational* log*, with $\theta > 1$, and with $\alpha := \frac{1}{\log \theta} = [a_0; a_1, a_2, \ldots]$.*

(i) If $0 < \log \theta < 3$, then

(i.1) there exist uncountably infinite number of θ with $\mathcal{A}_\theta$ empty[2]*;*

(i.2) there exist uncountably infinite number of θ with $\mathcal{A}_\theta$ finite, but not necessarily empty; and,

(i.3) there exist uncountably infinite number of θ with $\mathcal{A}_\theta$ infinite.

(ii) If $3 < \log \theta < 6$, and θ is not special, then $\mathcal{A}_\theta$ is infinite.

(iii) If $\log \theta > 6$, then $\mathcal{A}_\theta$ is always infinite, even if θ is special.

[2] O'Bryant proved (i.1) and (i.3) in his Theorems 5 and 6, and these facts are repeated here for completeness and because we give other proofs. In this paper we also present proofs of the other parts of this theorem.

Chapter 2

Some Classical Results in Diophantine Approximation

2.1 Homogeneous and Inhomogeneous Diophantine Approximations

Hurwitz's Theorem[1] states that if α is any irrational number then there are an infinite number of reduced fractions p/q such that $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$, and this condition will be met only if the p/q is a principal convergent of the simple continued fraction expansion for α. This inequality is termed a "homogeneous Diophantine approximation," and is often rewritten in another form, obtained by multiplying both sides by q, because this form is sometimes more useful: If α is any irrational number, then there are an infinite number of relatively prime integers, p and q such that $|q\alpha - p| < \frac{1}{\sqrt{5}q}$. While there are irrational numbers α for which the inequality $|q\alpha - p| < \frac{1}{mq}$ is satisfied by an infinite number of g p and q for some $m > \sqrt{5}$, there

[1] This result may be found in any standard reference on continued fractions; however, [2, p. 3] states that it was proved earlier by Korkine and Zolotareff.

is no constant $m > \sqrt{5}$ such that for *every* irrational α there are an infinite number of such pairs. Markoff, however, did prove that there is a strictly increasing sequence of real numbers $\{m_i\}$ whose supremum is 3, and with $m_1 = \sqrt{5}$, such that, for each m_i there exists an infinite set of irrational numbers α, such that for each α in the set corresponding to m_i, the inequality $|q\alpha - p| \leq \frac{1}{mq}$ is satisfied by an infinite number of relatively prime pairs of integers (p, q) if and only if $m \leq m_i$.

The approximation to α stated in Hurwitz's Theorem can be generalized to an "inhomogeneous" approximation: For any pair of real numbers (α, β), where α is irrational, there exist an infinite number of pairs of integers, (p, q) such that

$$|q\alpha - p - \beta| < \frac{1}{4|q|}. \tag{2.1}$$

It is generally assumed that β is non-integral in this inhomogeneous expression, for, if β were integral, then $|q\alpha - p - \beta| = |q\alpha - p'|$ where $p' = p + \beta$ is an integer, causing the expression at hand to be a homogeneous approximation, in which case Hurwitz's Theorem applies and the 4 in the denominator may be replaced with $\sqrt{5}$. Basically this theorem was proven by Minkowski.[2]

Grace made an improvement [5] [3, Vol. 2, p. 99], wherein he showed that the number "4" is sharp (or the "best bound" or "final," based upon what terminology is in vogue in the place or time of writing), meaning the preceding claim, the inequality

[2]Grace [5] cites Minkowski, *Werke*, Vol 1, p. 320, which I did not see. This theorem, is mentioned in other sources, which are sometimes confusing because there are several related theorems called "Minkowski's Theorem," which are cited in various forms, some of which seem to be counter-intuitive. More detailed information is in Appendix B.

of 2.1, is not true when 4 is replaced by a larger numerical constant. I put these two results together in the "Grace-Minkowski" theorem:

Theorem (Grace-Minkowski). *Consider the inequality:*

$$|q\alpha - p - \beta| < \frac{k}{|q|}. \tag{2.2}$$

(i) If $k > 1/4$, then for any pair of real numbers (α, β) where α is irrational and β is not integral, there are an infinite number of pairs of integers (p, q) that satisfy (2.2).

(ii) If $k < 1/4$, then for some choices for (α, β), where α is irrational and β is not integral, (2.2) *cannot be satisfied for an infinite number of pairs of integers (p, q). Specifically, if $\beta = 1/2$ and if the continued fraction expansion of $\alpha = [0; a_1, a_2, \dots]$ where a_1 is odd and for $j > 1$, the a_j's are even and increasing, there are not an infinite number of pairs of integers (p, q) that satisfy* (2.2).

Since this theorem does not deal with the case $k = 1/4$, it is of interest to know what happens in that case, even though this information will not be relevant to our development here. Simply put, for some choices of α there exist an infinite number of pairs of integers (p, q) that satisfy (2.2), and for some choices of α there are only a finite number of such pairs of integers. Grace [5] brought an example of a case, when, for appropriate choice of α, there are an infinite number of pairs of integers (p, q) that satisfy (2.2).

2.2 Properties of Convergents

Other important facts concerning continued fractions and convergents are below, and they are mentioned in pairs, the first of each pair for a principal convergent, and the second for an auxiliary convergent. Some authors call auxilliary convergents intermediate fractions and others use the term "intermediants." Unless stated otherwise, they are found in standard sources such as [7] or [10].

1. Definition of Principal Convergent:

   The j^{th} principal convergent of a positive irrational number α which can be expressed as a continued fraction $\alpha = [a_0; a_1, a_2, \ldots]$ for $j \geq 1$ is $[a_0; a_1, \ldots, a_j]$, which we will indicate by the reduced fraction p_j/q_j, where p_j and q_j are natural numbers.

2. Definition of Auxiliary Convergent:[3]

   Henceforth, let c_j be any integer in the interval $(0, a_j)$, where $a_j > 1$, is the j^{th} partial quotient of α; then for any $j \geq 1$, any number of the form $[a_0; a_1, \ldots, a_{j-1}, c_j]$ is a j^{th} auxiliary convergent or intermediate fraction, which we will indicate by the reduced fraction $p_{j,c}/q_{j,c}$, where $p_{j,c}$ and $q_{j,c}$ are natural numbers.

[3]Based upon the author and the formula used, one or more of the endpoints may be considered valid values for c_j. The case of $c_j = 0$ is excluded here because it does not make sense for 0 to be a partial quotient. However, in cases where it does make sense for $c_j = 0$, such as in, Fact 4 below, the value of $c_j = 0$ will be allowed. Similarly the value of $c_j = a_j$ is sometimes allowed. Whenever $c_j = 0$ or $c_j = a_j$ is allowed, the auxiliary convergent produced is also a principal convergent.

Note: (1) While the j^{th} principal convergent is unique, by definition, there are typically multiple possibilities for a j^{th} auxiliary convergent.

(2) If $c_j = 0$ then we define $p_{j,c}/q_{j,c} = p_{j-2}/a_{j-2}$, which is a principal convergent, and when $c_j = a_j$, then $p_{j,c}/q_{j,c} = p_j/a_j$, which is also a principal convergent. Hence, all principal convegents are also auxiliary convergents, and therefore, whenever we use the term "convergent" or the term "continuant" without being modified by the adjective "principal" or "auxiliary," the meaning will be to include both, and when we use the term "auxiliary convergent," we will mean an auxiliary convergent that is *not* a principal convergent.

3. Recursive Formulae for Principal Convergents:

   If p_j/q_j is the j^{th} principal convergent, then for $j \geq 2$ the recursive formulae for generation of principal convergents are $p_j = a_j p_{j-1} + p_{j-2}$ and $q_j = a_j q_{j-1} + q_{j-2}$. Sometimes it is useful to define two "artificial convergents" by convention to be $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, and $q_{-1} = 0$ (even though there is no real number $1/0$), because with this convention, the recursive formulae now becomes valid for all integral $j \geq 0$.

4. Recursive Formulae for Auxiliary Convergents:

   If p_i/q_i is an i^{th} principal convergent for $i < j$ and $p_{j,c}/q_{j,c}$ is a j^{th} auxiliary convergent, then for $j \geq 2$ and $c_j \in [0, a_j]$ the recursive formulae[4] for generation

[4]These recursive formulae will play an important role in several places later. For more informa-

of a j^{th} auxiliary convergent are $p_{j,c} = c_j p_{j-1} + p_{j-2}$ and $q_{j,c} = c_j q_{j-1} + q_{j-2}$.

5. Nature of Convergence for Principal Convergents:

   The odd indexed principal convergents form a strictly decreasing sequence converging to α, whereas the even-indexed ones form a strictly increasing sequence converging to α, and for all j, $|\alpha - p_{j+1}/q_{j+1}| < |\alpha - p_j/q_j|$.

6. Nature of Convergence for Auxiliary Convergents:

   If c_j is any integer in the interval $[0, a_j]$, then the fraction $p_{j,c}/q_{j,c}$ defined by $p_{j,c} = c_j p_{j-1} + p_{j-2}$ and $q_{j,c} = c_j q_{j-1} + q_{j-2}$ is an auxiliary convergent or intermediate fraction. If $c_j = 0$, then $p_{j,c}/q_{j,c} = p_{j-2}/q_{j-2}$, and if $c_j = a_j$ then $p_{j,c}/q_{j,c} = p_j/q_j$; moreover, if j is even, as c_j increases, the sequence of intermediate fractions increases from p_{j-2}/q_{j-2} to p_j/q_j, and if j is odd, the sequence of intermediate fractions decreases from p_{j-2}/q_{j-2} to p_j/q_j as c_j increases. Moreover, each intermediate fraction is the mediant between the p_{j-1}/q_{j-1} and the previous intermediate fraction. [7, II, 6]

7. Accuracy of Approximation for Principal Convergents:

   If $|\alpha - p/q| \leq 1/2q^2$, then p/q is a principal convergent.

8. Accuracy of Approximation for Auxiliary Convergents:

   If $|\alpha - p/q| \leq 1/q^2$, then p/q is a convergent to α. [4].

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tion, see Appendix A.

9. Definition of λ for Principal Convergents:[5]

   $\lambda_j := [0; a_{j-1}, a_{j-2}, \ldots, a_1] + [a_j; a_{j+1}, a_{j+2}, \ldots].$

10. Definition of λ for Auxiliary Convergents:[6]

    If $j > 1$ and c, is an integer in $[1, j-1]$, then

$$\lambda_{j,c} := [0; c, a_{j-1}, a_{j-2}, \ldots, a_1] + [0; a_j - c, a_{j+1}, a_{j+2}, \ldots].$$

    Note if $c = 0$ or $c = a_j$, then $\lambda_{j,c}$ is not defined.

11. Basic Fact About λ's for Principal Convergents:

    If p_j/q_j is a principal convergent then $\alpha - \frac{p_j}{q_j} = \frac{(-1)^j}{q_j^2 \lambda_{j+1}}$.

12. Basic Fact About λ's for Auxiliary Convergents: [5]

    If $p_{j,c}/q_{j,c}$ is an auxiliary convergent and $c \in [1, a_j - 1]$, then

$$\alpha - \frac{p_{j,c}}{q_{j,c}} = \frac{(-1)^j}{q_{j,c}^2 \lambda_{j+1,c}}.$$

13. Upper and Lower Estimates for λ's for Principal Convergents:

    $a_j < \lambda_j < a_j + 2.$

14. Upper and Lower Estimates for λ's for Auxiliary Convergents:

    (i) If $c \in [2, a_j - 2]$, then $0 < \lambda_{j,c} < 1$. [5]

    (ii) If $c = 1$ or $c = a_j - 1$, then $1 < \lambda_{j,c} < 2$. [5]

[5] Some texts label this expression to be λ_{j-1}.
[6] Some texts use $j - 1$ as the subscript.

2.3 Grace's Construction—proof of second part of Grace-Minkowski Theorem

In the Grace-Minkowski Theorem, in order to prove that "4 is sharp," as defined earlier in 2.1, Grace constructed a family of α's whose continued fractions have special properties causing (2.2) to have only a finite number of integral pairs making it true, and thereby produced an indirect proof that "4 is sharp."

We restate and prove the second part of the Grace-Minkowski Theorem here, but first we define a set relating to the Grace contruction:

Definition 2.3.1. (*Grace's Set*) $G := \{[a_0; a_1, a_2, \dots] : a_0 = 0, a_1$ odd, and for $i \geq 2$, a_i even and increasing$\}$.

Theorem (Grace-Minkowski, Part *ii*)**.** *Let* $\alpha = [0; a_1, a_2, ...] \in G$. *If* $h < 1$, *then the inequality*

$$\left|q\alpha - p - \frac{1}{2}\right| < \frac{h}{4|q|} \tag{2.3}$$

has only finitely many solutions in integers, p *and* q.

Proof. We reproduce Grace's original proof here, and later present some improvements. First, note in the case at hand, we may omit the absolute value sign around the q, because if $q < 0$, we may define $q' = -q$ and and $p' = -p - 1$. In this case $q'\alpha - p' - 1/2 = -q\alpha - (-p - 1) - 1/2$. Accordingly, without loss of generality, we may assume q to be positive by the appropriate change in p, and we may therefore drop the absolute value sign. The inequality (2.3) is then equivalent to

$$\left|\alpha - \frac{2p+1}{2q}\right| < \frac{h}{4q^2} = \frac{h}{(2q)^2}. \tag{2.4}$$

We write

$$\left|\alpha - \frac{r}{s}\right| < \frac{h}{s^2}, \tag{2.5}$$

where $r = 2p+1$ is an odd integer and $s = 2q$ an even one.

If 4 is not sharp (as explained earlier), then there exists a number $h < 1$, such that there would then exist an infinite number of positive integral pairs (p, q) that satisfy (2.4) for that particular $h < 1$.

This result, in turn, means α can be approximated by a rational, $\frac{2p+1}{2q} = \frac{r}{s}$, to accuracy of the square of the denominator. By Fact 8, an approximation of this degree of accuracy, can only be attained by a fraction which is a convergent. [4, 5]

The conditions in the Grace-Minkowski Theorem part (ii), given on the partial quotients of $\alpha = [a_0; a_1, a_2, ...]$ imply that α has no even principal continuant, as is easily seen from Fact 3, the Recursive Formula for Principal Convergents. In such a case, all principal continuants are odd, so there is no principal convergent satisfying (2.5).

Therefore, if there do exist integers p and q satisfying 2.4, the approximating fraction, $\frac{2p+1}{2q}$, must be an auxiliary convergent, and not a principal one, which we will call $p_{n,c}/q_{n,c}$. We will now show that even though there may be infinitely many even auxiliary continuants, it can be arranged that only finitely many of them satisfy (2.4).

From Fact 12, we know that

$$\left|\alpha - \frac{p_{n,c}}{q_{n,c}}\right| = \frac{1}{\lambda_{n+1,c}{q_{n,c}}^2}.$$

Combining that fact with (2.4) yields $\frac{1}{\lambda_{n+1,c}} < h$, i.e., $\lambda_{n+1,c} > \frac{1}{h} > 1$. By the definition of $\lambda_{n,c}$ in Fact 10 (ii) above, we now have

$$1 < \lambda_{n+1,c} := [0; c, a_n, a_{n-1}, \ldots, a_1] + [0; d, a_{n+2}, a_{n+3}, \ldots] < \frac{1}{c} + \frac{1}{d},$$

where $c + d = a_{n+1}$.

From this fact it follows that either $c = 1$ or $d = 1$. Therefore,

$$1 < \lambda_{n+1,c} < \frac{1}{c} + \frac{1}{d} = 1 + \frac{1}{a_{n+1} - 1}$$

where $c + d = a_{n+1}$.

Thus, using n in the place of $n + 1$, since n was any natural number, for any auxiliary convergent that satisfies (2.4) we have the following string of inequalities:

$$1 < \frac{1}{h} < \lambda_{n,c} < 1 + \frac{1}{a_n - 1} \tag{2.6}$$

Since h was fixed and strictly less than 1, if α were to be constructed in such a way that the a's go to infinity, then $\lambda_{n,c} \to 1$, and h would thereby be forced to be 1, resulting in a contradiction. Therefore, (2.6) cannot hold for an infinite number of

n's. Thus, it is possible to construct α in such a way that only a finite number of pairs of integers (p, q) satisfy (2.4). $\square$

It should be noted, as will be seen later, that in applying this theorem to our problem, to construct an α such that there are only a finite number of pairs of integers (p, q) satisfy (2.4), we only need that, at most, a finite number of *odd-indexed* convergents are even, so there are larger sets of α that would work.

Comment: At fist glance, Minkowski's Theorem, as cited by Grace, seems counterintuitive, in that the constant 4 in (2.3) for an inhomogeneous approximation, is larger than the constant for a homogeneous approximation, namely $\sqrt{5}$ in Hurwitz's Theorem. However, this is not the case. In part (*ii*) of the Grace-Minkowski Theorem, β is replaced by $1/2$ in the inhomogeneous inequality (2.2). Subsequently the fraction $1/2$ is combined with p/q in the development from (2.3) to (2.5) changing the denominator of the fraction approximating α from q to $2q$. Consequently, the denominator of the right side is simply the square of the denominator of the fraction approximating α, and not the denominator multiplied by 4 or some other constant. While the of constant for an inhomogeneous approximation is indeed 4 which is greater tnan $\sqrt{5}$, the constant in the denominator of the homogeneous approximation, with the appearance of a new denominator, the "apparent" stronger constant of 4 now becomes 1, which, indeed is not as strong as $\sqrt{5}$, as may be expected, intuitively.[7] It should also be pointed out that (2.5) does not merely define the degree of accuracy of the approximation, but it requires the numerator of the approximating fraction to be odd and the denominator to be even. Thus, while any principal convergent will satisfy inequality (2.5), it may not necessarily satisfy the concomitant parity requirements, thereby enabling a construction of irrational α that cannot be approximated by any principal convergent.

Let us now consider the case when β is rational, but not $1/2$, Let $\beta = a/b$ (where a and b are relatively prime positive integers), $r = pb + a$, and $s = qb$, and we wish to investigate for what values of k, the following analogue of then (2.5) will have an infinite number of solutions in relatively prime integers r and s:

$$\left|\alpha - \frac{r}{s}\right| < \left|\alpha - \frac{pb+a}{qb}\right| < \frac{k}{q^2} \leq \frac{b^2 k}{(bq)^2} = \frac{h}{s^2}, \tag{2.7}$$

where $h = b^2 k$. The Grace-Minkowski Theorem dealt with the case $b = 2$, and $h = 4k$, so as long as $h > 1$, that is, $k > 1/4$, there would be an infinite number of solutions. In the more general case, the minimal value for b is 3, so $h \geq 9k$, and the condition to have an infinite number of solutions, namely $h > 1$, now means $k > 1/b^2 \geq 1/9$ in (2.7). In this sense, $\beta = 1/2$ is the "worst case scenario," in that k has the smallest possible range that guarantees an infinite number of solutions, (r, s) to the inequality. However, in all cases, whenever $h \leq 1$ any reduced fraction satisfying

[7] Even if the constant 4 were incorrect, the main ideas in the theorems here would still be correct, with only the constants changing. More detailed information is in Appendix B.

2.7 must be a convergent. Also, the congruence conditions, namely, $r \equiv a \pmod{b}$ and $s \equiv 0 \pmod{b}$, must be met. When $b = 2$, it is the congruence conditions are simple parity conditions that can easily be controlled by controlling the partial quotients of irrational α. However, when $b \geq 3$, the congruence conditions cannot be controlled in such a trivial fraction through the partial quotients to insure only a finite number of solutions. In addition, when $h > 1$, the congruence conditions make it more difficult to prove there will be an infinite number of solutions. When $h > 1$, there does not seem to be any obvious way to cause or to prevent auxiliary convergents or near-convergents[8] from having the divisibility properties, whenever β is a rational number other than 1/2.

2.4 Extensions of Grace's Construction

We now define the *critical value* of a partial quotient for a given value of h. To obtain the contradiction in the proof of the preceding theorem we needed that, for fixed h, the number of partial quotients that satisfy (2.6) is finite. This leads to the following definition:

Definition 2.4.1. The *critical value* for h is $A(h) := \min\left\{a \in \mathbf{N} : 2|a \quad \& \quad a \geq \dfrac{h}{1-h} + 1\right\}$.

[8]The term "near-convergent" is not clearly defined. As stated earlier, for fixed positive real numbers, α and $h \leq 1$, the existence of a pair of relatively prime integers, p, q that satisfy the inequality $|\alpha - p/q| < \frac{h}{q^2} \Rightarrow p/q$ is a principal convergent or auxiliary convergent to α. If, however, $h > 1$, then there are other fractions that satisfy this inequality, and they are called "near-convergents."

Lemma 2.4.1. $A(h) = 2\lceil \frac{1}{2-2h} \rceil$.

Proof. Noting that $\frac{h}{1-h} + 1 = \frac{1}{1-h}$, if we define $B(h) := \min\{a \in \mathbf{N} : a \geq \frac{h}{1-h} + 1\}$, that is to say, that we do not require $B(h)$ to be even, then $A(h) = 2\lceil \frac{B(h)}{2} \rceil = 2\lceil \frac{1}{2-2h} \rceil$. □

Theorem 2.4.1 (Grace-Minkowski, extended)**.** *Consider the inequality*

$$|q\alpha - p - 1/2| < \frac{h}{4q}, \tag{2.8}$$

If $h < 1$, and if $\alpha = [0; a_1, a_2, ...]$ where a_1 is odd and the other partial quotients are even, and, at most, a finite number of them are less than the criticial value for h, namely $A(h)$, then the inequality (2.8) *has only finitely many integral solutions* (p, q).

Proof. First, the hypothesis $h < 1$ means $1 < 1/h$. Second, in Grace's original work, he obtained a contradiction to (2.6) by making all the partial quotients approach infinity. This condition, however, is stronger than necessary. All that is really needed is that

$$a_n \geq \frac{h}{1-h} + 1$$

is false a finite number of times, which is eqivalent to at most a finite number of $a_n < A(h)$, as stated in the theorem. This extension provides us with a set larger than G for which the inequality (2.3) has a finite number of solutions. □

(It should be noted that when $A(h) = 2$, there will be no partial quotients that are even and less than $A(h)$, implying the inequality (2.8) will have no integral solutions (p,q). When can this situation occur? As will be seen later, we will be most concerned with the case when $0 < \log\theta < 3$, which means that $h = \log\theta/3 < 1$ and that $A(h) = 2\lceil\frac{1}{2-\frac{2}{3}\log\theta}\rceil$. Then, the smallest $A(h)$ will be is 2, which occurs if $\log\theta < 3/2$. For larger values of $A(h)$, the number of possible values for a partial quotient grows since $A(h)$ grows without bound through the even numbers as $\log\theta \nearrow 3$.)

From here it is immediate that if the partial quotients of α are eventually periodic and all greater or equal to $A(h)$, that α will be a quadratic irrational, and hence algebraic, thereby providing us with the following corollary.

Corollary 2.4.1a. *Consider the inequality*

$$\left|q\alpha - p - \frac{1}{2}\right| < \frac{h}{4q}. \tag{2.9}$$

Then for each $h \in (0, 1)$, there exists an infinite set of algebraic irrational α's, namely those for which $\alpha = [0; a_1, a_2, ...]$, where

(1) a_1 is odd,

(2) the other partial quotients are even,

(3) at most a finite number of the $a_n < A(h)$, and

(4) where the partial quotients are eventually periodic.

such that, for each of these algebraic irrational α's, the inequality (2.9) *has only* *finitely many integral solutions* (p, q).

Corollary 2.4.1b. *Consider the inequality (without absolute value signs)*

$$0 < q\alpha - p - \frac{1}{2} < \frac{h}{4q} \tag{2.10}$$

Then for each $h \in (0, 1)$*, there exists an infinite set of irrational* α*'s, namely those for which* $\alpha = [0; a_1, a_2, ...]$*, where*

(1) a_n *is even whenever* n *is odd,*

(2) at most a finite number of the $a_n < A(h)$*, and*

such that, for each of these irrational α*'s, the inequality* (2.10) *has only finitely many integral solutions* (p, q)*. Moreover, if, in addition, the partial quotients are eventually periodic, then for each* $h \in (0, 1)$ *all these* α*'s are algebraic, and for each of these algebraic irrational* α*'s, the inequality* (2.10) *has only finitely many integral solutions* (p, q)*.*

Proof. As seen from Tables 3, 4, 5, and 6 in Appendix A, the recursive formulae for convergents (Facts 3 and 4) guarantee that if the odd-indexed convergents are even, then all even-indexed continuants are odd. Fact 5, the fact that even-indexed convergents are less that the number being approximated, namely α, enable us to delete the absolute value sign in the hypotheses of Theorem 2.4.1 and Corollary 2.4.1a, thereby producing this corollary. (The periodicity of the partial quotients, of course, insures that α is a quadratic irrational and therefore algebraic.) □

Corollary 2.4.1c. *Consider the inequality*

$$0 < -1\Big(q\alpha - p - \frac{1}{2}\Big) < \frac{h}{4q} \tag{2.11}$$

Then for each $h \in (0,1)$, there exists an infinite set of irrational α's, namely those for which $\alpha = [0; a_1, a_2, ...]$, where

(1) a_1 is odd,

(2) for $n > 1$, a_n is even whenever n is even,

(3) at most a finite number of the $a_n < A(h)$,

and each of these irrational α's, inequality (2.11) has only finitely many integral solutions (p, q). Moreover, if, in addition, the partial quotients are eventually periodic, then for each $h \in (0,1)$, there exists infinitely many algebraic irrationals α's, such that inequality (2.11) has only finitely many integral solutions (p, q).

Proof. The recursive formulae for convergents (Facts 3 and 4) guarantee that the parity of the partial quotients as defined in the hypothesis will produce odd continuants whose indices are odd. Fact 5, the fact that odd-indexed convergents are more than the number being approximated, namely α, enable us to replace the absolute value sign by -1 in the hypotheses of Theorem 2.4.1 to produce this corollary. □

It should be noted that in both the last two corollaries, the parity conditions on the partial quotients are sufficient, but not necessary, to produce the desired results.

Chapter 3

The Main Theorems

3.1 The Relationship Between Atypical numbers and the Grace-Minkowski Theorem

In the Grace-Minkowski Theorem and our extensions in Section 2.4, we found conditions on a positive irrational number α and on a positive real number k so that there are either a finite number or infinite number of integer pairs p, q that satisfy the inequality, $\left|q\alpha - p - \frac{1}{2}\right| < \frac{k}{q}$, or the equivalent inequality, $\left|\alpha - \frac{2p+1}{2q}\right| < \frac{h}{(2q)^2}$.

We wish to apply these results to O'Bryant's Lemma 8 which states for fixed θ, that $n \in \mathcal{A}_\theta \Leftrightarrow \left\{\frac{n}{\log\theta}\right\} \in \left[L_\theta(n), \frac{1}{2}\right)$, where $L_\theta(n) = \frac{1}{2} - f\left(\frac{\log\theta}{n}\right)$. To do so, we will call $\left[L_\theta(n), \frac{1}{2}\right)$ the standard atypical interval and introduce two new functions, $L'_\theta(n)$ and $L''_\theta(n)$, which satisfy the inequality $L'_\theta(n) < L_\theta(n) < L''_\theta(n) < \frac{1}{2}$. When $L_\theta(n)$ is replaced by either $L'_\theta(n)$ or $L''_\theta(n)$ in the standard atypical interval, we get what called the extended atypical interval and the contracted atypical interval, respectively. It follows from the construction of the expanded and contracted intervals that if, for fixed θ, there are no n, or at most a finite number of them, such that $\left\{\frac{n}{\log\theta}\right\}$ is in the expanded atypical interval, then *a foritori* there are no n, or at most a finite number of them, with $\left\{\frac{n}{\log\theta}\right\}$ in the standard atypical interval. Therefore, by O'Bryant's Lemma 8, $\mathcal{A}_\theta$ is empty or finite. Similarly, if there are an infinite number of n such that the contracted interval contains an infinite number of $\left\{\frac{n}{\log\theta}\right\}$, then, *a foritori*, there are an infinite number of them in the standard atypical interval, and hence, $\mathcal{A}_\theta$ is infinite. It should be noted that the Grace-Minkowski Theorem cannot be applied directly to the standard atypical interval arising from Lemma 8, because of the complexity of its left endpoint. However, the endpoints of the expanded and contracted intervals are, algebraically speaking, easier to deal with, and the Grace-Minkowski Theorem is applied to them. Since these problems are stated using the letter n to represent a possibly atypical number, we will use n in the place of q, where q appears in the Grace-Minkowki theorem.

Lemma 3.1.1. *Let $\theta > 1$. For each $n \in \mathcal{A}_\theta$ there exists a unique pair of integers p and q such that $p = \left\lfloor \frac{n}{\log\theta} \right\rfloor$, $q = n$ and*

$$\left|\frac{q}{\log\theta} - p - \frac{1}{2}\right| = \left|\frac{q}{\log\theta} - \frac{2p+1}{2}\right| < \frac{\log\theta}{12q}, \tag{3.1}$$

or, equivalently,

$$\left|\frac{1}{\log\theta} - \frac{2p+1}{2q}\right| < \frac{\log\theta}{12q^2} = \frac{\log\theta}{3(2q)^2}.$$

Proof. In O'Bryant's Lemma 8 above, we stated that, for fixed θ, n is atypical, by definition, when $M'_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$ for $n > \log_2\theta$, and this occurs if and only if $\dfrac{1}{2} - f\left(\dfrac{\log\theta}{n}\right) \leq \left\{\dfrac{n}{\log\theta}\right\} < \dfrac{1}{2}$. By using $t = \frac{\log\theta}{n}$ and the upper estimate for $f(t) < t/12$ in O'Bryant's Lemma 7, we can now produce an expanded atypical interval (which contains the atypical interval as a proper subset), namely $\left(\dfrac{1}{2} - \dfrac{\log\theta}{12n}, \dfrac{1}{2}\right)$, meaning $L'_\theta(n) := \dfrac{1}{2} - \dfrac{\log\theta}{12n}$. We now apply Grace's idea to this new interval:

Accordingly, we reformulate what it means for n to be atypical in a fashion that will enable us to apply Grace's idea. Thus, n is atypical means

$$\left\{\frac{n}{\log\theta}\right\} \in \left[\frac{1}{2} - f\left(\frac{\log\theta}{n}\right), \frac{1}{2}\right) \subset \left(\frac{1}{2} - \frac{\log\theta}{12n}, \frac{1}{2}\right).$$

Or, more simply, using the fact that $\left\{\frac{n}{\log\theta}\right\} = \frac{n}{\log\theta} - \left\lfloor\frac{n}{\log\theta}\right\rfloor$, we now have

$$\frac{1}{2} - \frac{\log\theta}{12n} < \frac{n}{\log\theta} - \left\lfloor\frac{n}{\log\theta}\right\rfloor < \frac{1}{2}.$$

Subtracting 1/2 from all three sections yields

$$-\frac{\log\theta}{12n} < \frac{n}{\log\theta} - \left\lfloor\frac{n}{\log\theta}\right\rfloor - \frac{1}{2} < 0. \tag{3.2}$$

We note that $p = \left\lfloor\frac{n}{\log\theta}\right\rfloor$ is equivalent to

$$-\frac{1}{2} \leq \frac{n}{\log\theta} - p - \frac{1}{2} < \frac{1}{2}. \tag{3.3}$$

Since ${M'}_\theta(n)$, by definition, means $n > \log_2\theta$, and since
$n > \log_2\theta > \dfrac{\log\theta}{6} \Rightarrow \dfrac{1}{2} > \dfrac{\log\theta}{12n} \Rightarrow -\dfrac{1}{2} < -\dfrac{\log\theta}{12n}$, it follows that we may replace $\left\lfloor\dfrac{n}{\log\theta}\right\rfloor$ in (3.2) with p.

Setting $q = n$, $\alpha = \dfrac{1}{\log\theta}$, $\beta = \dfrac{1}{2}$, $p = \lfloor q\alpha \rfloor = \left\lfloor \dfrac{n}{\log\theta} \right\rfloor$ in (3.2), and keeping in mind that it is necessary to alternate between these two notations, we now have an inequality in the format used in the Grace-Minkowski Theorem:

$$-\frac{\log\theta}{12q} < q\alpha - p - \beta < 0. \tag{3.4}$$

Multiplying by -1 yields

$$0 < p + \beta - q\alpha = \left(\frac{1}{2} + \left\lfloor \frac{n}{\log\theta} \right\rfloor - \frac{n}{\log\theta}\right) = |q\alpha - p - \beta| < \frac{\log\theta}{12q} = \frac{\log\theta}{12n}. \tag{3.5}$$

Thus, we have

$$0 < \frac{1}{2} - \left\{ \frac{n}{\log\theta} \right\} < \frac{\log\theta}{12n}. \tag{3.6}$$

We have now transformed the notation for atypical n in O'Bryant's Lemma 8 to a notation similar to the one used by Grace (2.2). □

Note: We actually proved a little bit more: The absolute value is used in the Grace-Minkowski Theorem, and, for the purpose of parallelism, in the inequality above (3.1) we also used absolute value signs. However, they are unnecessary in that what we really proved is that if n is atypical we must have $(-1)\left(\dfrac{1}{\log\theta} - \dfrac{2p+1}{2q}\right) < \dfrac{\log\theta}{12q^2}$, meaning the negative branch of the absolute value statement is true.

Corollary 3.1.1a. *If $\theta > 1$ and $n > \log_2\theta$, then for each $n \in \mathcal{A}_\theta$, there exist relatively prime integers $r, s = 2n$, such that*

$$\left|\frac{1}{\log\theta} - \frac{r}{s}\right| < \frac{\log\theta}{3s^2}. \tag{3.7}$$

Proof. Setting $q = n$ in Lemma 3.1.1, we know that if $n \in \mathcal{A}_\theta$, then

$$\left|\frac{1}{\log\theta} - \frac{2p+1}{2n}\right| < \frac{\log\theta}{12n^2} = \frac{h}{(2n)^2}, \tag{3.8}$$

where $h = \frac{\log\theta}{3}$, $r = 2p+1$, and $s = 2n$ give the desired result. Without loss of generality we may assume r and s are relatively prime, for if not, there exist integers c, R, S, N where R and S are relatively prime, $r = cR, s = cS, n = cN$, c is odd, and $S = 2N$. Then inequality (3.7) becomes

$$\left|\frac{1}{\log\theta} - \frac{r}{s}\right| = \left|\frac{1}{\log\theta} - \frac{R}{S}\right| < \frac{\log\theta}{3s^2} < \frac{\log\theta}{3(cS)^2} < \frac{\log\theta}{3S^2}, \tag{3.9}$$

and inequality (3.8) becomes

$$\left|\frac{1}{\log\theta} - \frac{2p+1}{2n}\right| = \left|\frac{1}{\log\theta} - \frac{\frac{2p+1}{c}}{2\frac{n}{c}}\right| < \frac{\log\theta}{12n^2} = \frac{h}{(2n)^2} = \frac{h}{(2cN)^2} < \frac{h}{(2N)^2}. \tag{3.10}$$

□

Corollary 3.1.1b. *If $\theta > 1$and $n \in \mathcal{A}_\theta$, then there exists a reduced fraction, r/s, such that each of the following three conditions hold*

i. The absolute value inequality condition: $\left|\frac{1}{\log\theta} - \frac{r}{s}\right| < \frac{\log\theta}{3s^2}$.

ii. The parity condition: r is odd and s is even, and, in fact, $n = s/2$.

iii. The "overestimate" condition: $\frac{r}{s} - \frac{1}{\log\theta} > 0$,

which we may also call "The negative part of the absolute value condition" because it is equivalent to $\left|\frac{1}{\log\theta} - \frac{r}{s}\right| = (-1)\left(\frac{1}{\log\theta} - \frac{r}{s}\right)$.

Proof. This result follows immediately from the preceding corollary and Lemma 3.1.1.

□

It is important to note several things:

- Also, we could have stated all three conditions more succinctly as "there exists a reduced fraction, $r/2n$, such that $0 < \dfrac{r}{2n} - \dfrac{1}{\log\theta} < \dfrac{\log\theta}{12n^2}$," but we chose not to do so, because the use of three distinct conditions will be more useful for the development that follows.

- From Corollary 3.1.1b, the value of $\log\theta/3$ is seen to be quite important: As previously mentioned, we know from the basic properties of continued fractions that if an irrational number α can be approximated by a fraction r/s so that $\left|\alpha - \dfrac{r}{s}\right| \leq \dfrac{h}{s^2}$ where $h < 1$, then r/s is a (auxiliary) convergent to α, and moreover, if $h \geq 1/\sqrt{5}$ there will an infinite number of approximating fractions, r/s, for any irrational number α. When $h < 1/\sqrt{5}$, the existence of an infinite number of fractions, r/s, approximating α will vary, based upon the value of α, and may be controlled by appropriate choice of partial quotients for α. [7, Theorem II.8.21] From Corollary 3.1.1b, it is clear that the absolute value inequality condition will be satisfied for an infinite number of relatively prime pairs of integers (r, s) if $\log\theta \geq \dfrac{3}{\sqrt{5}} \approx 1.31464$ or $\theta > 3.73$, approximately, and therefore the existence of infinite number of atypical n would depend on whether or not the other two conditions are satisfied. In addition, if $\log\theta < 3$, the fractions satisfying the absolute value inequality condition must be convergents by Fact 8, thereby enabling us to control whether or not $\mathcal{A}_\theta$ is infinite by controlling the continued fraction of $\frac{1}{\log\theta}$. However, when $\log\theta > 3$, the approximating fractions may be near-convergents, whose existence cannot readily be controlled through the continued fraction of $\frac{1}{\log\theta}$.

- The Grace-Minkowski Theorem and the original Grace Construction dealt with two cases, and produced different results depending on whether $h < 1$ or $h > 1$. When $h > 1$ there were always an infinite number of $q = n$ that satisfied the inhomogeneous inequality (2.2). However, when $h < 1$, whether or not there were an infinite number of solutions to the inhomogeneous inequality (2.2) depended upon α, which we could construct so there would be either a finite or an infinite number of solutions based upon our controlling the partial quotients of the continued fraction convergents to α.

- In applying these results to our problem, we will soon consider both the case when $h < 1$, or equivalently in our problem, $\log\theta < 3$, and $h > 1$, or $\log\theta > 3$. It is important to bear in mind that the Grace-Minkowski Theorem and the Grace Construction deal with the existence of a finite or infinite set of numbers that satisfy an absolute-value inequality, but the current problem deals with a direct inequality (as opposed to an absolute-value one), corresponding to the negative value of the absolute value or odd-indexed convergents (as mentioned in Corollary 3.1.1b part *iii*). Thus, the Grace-Minkowski Theorem would insure that only a finite number of solutions to the inequality in the expanded atypical interval would imply that there are only a finite number of solutions to our current problem. This situation is dealt with in Section 3.2. However, the converse is not necessarily true, for an infinite number of solutions to the Grace-Minkowski inequality would not necessarily insure the existence of an infinite number of convergents with odd indices (and whose denominators are even).[1] A partial converse, though, does exist, and is dealt with in Section 3.3, where we introduce the contracted atypical interval.

3.2 A Study of Conditions on θ for $\mathcal{A}_\theta$ to be Finite

Lemma 3.2.1 (Existence of p and q)**.** *If $n > \log_2 \theta$, and if n is atypical, then there exists a pair of integers, p and $q = n$, such that*

$$\left|q\alpha - p - \frac{1}{2}\right| = \left|q\alpha - \frac{2p+1}{2}\right| < \frac{\log\theta}{12q}, \tag{3.11}$$

Proof. The result follows immediately from Lemma 3.1.1 and Corollary 3.1.1a. □

Theorem 3.2.1 (Bounded θ)**.** *Let $0 < \log\theta < 3$ and $\alpha = \frac{1}{\log\theta}$. If $\alpha = [a_0; a_1, a_2, \ldots]$, where $a_0 = \lfloor\alpha\rfloor$, a_1 is an odd natural number, and the rest of the a's are all even and tend to infinity, then $\mathcal{A}_\theta$ is finite, but not necessarily empty.*

[1] Nevertheless, an infinitude of solutions to our current problem would imply an infinitude to the Grace-Minkowski inequality.

Proof. By hypothesis, $\log\theta < 3$, so we have $h < 1$. We now have two cases: In the event $\lfloor\alpha\rfloor = a_0 = 0$, that is to say, $0 < \alpha < 1$, α is one of the numbers produced by the Grace Construction, and we may now apply Lemma 3.1.1 to obtain the result that the expanded atypical interval contains only finite atypical n, and therefore the basic atypical interval does also.

However, since $\log\theta$ may be close to 0, it is certainly possible that $\lfloor\alpha\rfloor \neq 0$. In this case, we write $\alpha = \lfloor\alpha\rfloor + \{\alpha\}$, and therefore (3.11) becomes

$$|q\lfloor\alpha\rfloor + q\{\alpha\} - p - 1/2| < \frac{\log\theta}{12q} \tag{3.12}$$

Since $|p - q\lfloor\alpha\rfloor|$ is an integer we may label it as p', and if we define $\alpha' := \{\alpha\}$, the above can be rewritten as

$$|q\alpha' - p' - 1/2| < \frac{\log\theta}{12q} \tag{3.13}$$

Since $0 < \alpha' < 1$ and $h < 1$, it follows that α' is one of the numbers produced by the oroignal Grace Construction. In this case we can again utilize Lemma 3.1.1 to obtain the result that the expanded atypical interval contains only finite atypical n, and therefore the basic atypical interval does also. $\square$

It should be noted that this theorem will hold for any set of α's produced by an extended version of Grace's Construction, such as one that has only a finite number convergents with even denominators.

The next task is to find conditions for the denominators of odd-indexed convergents to be even in order to determine when $\mathcal{A}_\theta$ is empty or finite but not empty. More specifically, the goal of the next four lemmas is to determine under what conditions:

(1) The number of prinicipal continuants of odd-index that are even is finite.

(2) The number of all continuants of odd-index that are even is finite.

(3) There are no even principal continuants of odd-index.

(4) There are no even continuants of odd-index.

For these lemmas, we refer to the tables and State Diagram in Appendix A. The labels given to the different possibilities and cases delineated in these lemmas come from the cases described in Appendix A.

Using E for even and D for odd, we consider four cases: The previous two denominators will either be EE, ED, DE, or DD, and are labeled cases 1, 2, 3, 4 respectively. Each has two subcases; subcase A occurs when the next partial quotient is even, and subcase B occurs when the next partial quotient is odd, giving 8 cases altogether. We make basic observations from looking at the tables and State Diagram in Appendix A. The following for lemmas are presented without full proof because they follow from these observations using only simple parity arguments. However, some comments relating to these observations and parity arguments are included at the end of Appendix A.

Lemma 3.2.2a (Conditions for number of even principal continuants of odd-index to be finite.). *Let* $\alpha = [a_0; a_1, a_2, \dots]$. *At most a finite number of principal continuants of odd-index are even if and only if any one of three situations occurs:*

A. For some index k, q_{k-2} *and* q_{k-1} *are both odd and for all* $n \geq k$, a_n *is even. (In other words, at some point on, we are always in Case 4A. Hence, we will call this case "Perpetual Case 4.")*

B. For some odd index k *such that the previous two principal continuants have the same parity as their index, and for all even* $n \geq k$, a_n *is even. (From some point on, we are always alternating between Cases 2A and 3. Hence, for short, we will call this the "Alternating Case.")*

C. The partial quotients are such that sometimes we are in the situation of Perpetual Case 4 and sometimes in the situation of Alternating Cases 2A and 3, namely:

Either we are in Perpetual Case 4 and all a_n *are even up to a point, but there exists some odd index* n, *where* a_n *is odd causing us to exit the Perpetual Case 4 and to enter the Alternating Case,*

Or we are in the Alternating Case and for some even index n, a_n *is odd, causing us to exit the Alternating Case and to enter the Perpetual Case 4. (For short, we will call this case, the "Mixed Case.")*

Lemma 3.2.2b (Conditions for number of any even continuants of odd-index to be finite). *Let* $\alpha = [a_0; a_1, a_2, \dots]$. *At most a finite number of all continuants (both* *principal and auxiliary) of odd-index are even if and only if there exists an odd index k such that the previous two principal continuants have the same parity as their index, and for all even $n \geq k$, a_n is even. (This is analogous to 1.B, above.)*

The next two lemmas are parallel to the two preceding ones with the difference being that the preceding ones were concerned with determining the circumstances for which the number of certain continuants is finite, and the following lemmas are concerned with determining the the circumstances for which the number of continuants is zero.

We introduce the next lemma with three observations and define three cases, as was done in introducing Lemma 3.2.2a. For simplicity we use the same terminology, but with slightly different meanings because we are not merely limiting the number of continuants to be finite, but that there should be no continuants at all meeting certain conditions.

Lemma 3.2.2c (Conditions for no even principal continuants of odd-index)**.** *Let $\alpha = [a_0; a_1, a_2, \ldots]$. There are no even principal continuants of odd-index if and only if any one of three situations occurs:*

A. a_1 is odd, and for $n \geq 2, a_n$ is even. (For the purpose of this lemma, we now call this case 'Perpetual Case 4A.")

B. a_1 is odd, and for all even $n \geq 2, a_n$ is even. (For the purposes of this lemma we now call this case "Alternating Cases 2A and 3.")

C. The partial quotients are such that sometimes we are in the situation of Perpetual Case 4 and sometimes in the situation of Alternating Cases 2A and 3, namely,

Either we are in Perpetual Case 4 and all a_n are even up to a point, but there exists some odd index n, where a_n is odd causing us to exit the Perpetual Case 4 and to enter the Alternating Case,

Or we are in the Alternating Case and for some even index n, a_n is odd, causing us to exit the Alternating Case and to enter the Perpetual Case 4.

Lemma 3.2.2d (Conditions for no even continuants of odd-index)**.** *Let $\alpha = [a_0; a_1, a_2, \dots]$. There are no even continuants (either principal and auxiliary) of odd-index if and only if $a_1 = 1$ and and for all even $n \geq 2, a_n$ is even. (This is analogous to 3.B, above.) There are no even continuants (both principal and auxiliary) of odd-index if and only if there exists an odd index k such that the previous two principal continuants have the same parity as their index, and for all even $n \geq k$, a_n is even. (This situation is analogous to situation B in Lemma 3.2.2a above.)*

It should be noted that if a_1 is odd, and all the other partial quotients are even, we will always be in Case 4, and therefore all denominators of all principal convergents will be odd, but it is not possible for all denominators of all auxiliary convegents to be odd—this was the case of the Grace Construction.

Theorem 3.2.2. ($\mathcal{A}_\theta = \emptyset$)**.** *Let $0 < \log\theta < 3$ and $\alpha = \frac{1}{\log\theta}$. If $\alpha = [a_0; a_1, a_2, \dots]$, where $a_1 = 1$, a_2 is odd, and a_n is even whenever $n > 2$ is even, then $\mathcal{A}_\theta = \emptyset$.*

Proof. By Lemma 3.2.1 there exist pairs of integers (p, q) satisfying inequality (3.11). Since any fractions that satisfy (3.11) must be principal or auxiliary convergents by Fact 8. By Lemma 3.2.2a, the necessary conditions of Corollary 3.1.1a are not fulfilled by any convergents, so $\mathcal{A}_\theta$ must be empty. □

Corollary 3.2.2 (Finite $\mathcal{A}_\theta$)**.** *Let* $0 < \log\theta < 3$ *and* θ *is special, then* $\mathcal{A}_\theta$ *is finite, but not necessarily empty.*

Proof. The proof follows immediately from the definition of "special" and Lemma 3.2.2b.

□

Summary: O'Bryant has shown in his Theorem 5 (and Lemma 11 on which it is based), that for $0 < \log\theta < 3$, an appropriate choice of the convergents of $\alpha = 1/\log\theta$ enables us to construct α so that $\mathcal{A}_\theta$ is finite (or even empty). We have extended these results and further shown why the upper bound of 3 for $\log\theta$ is necessary. In particular, we have shown:

(i) If $\log\theta < 3$ *and irrational, then a necessary, but not sufficient, condition for* n *to be atypical is that there are integers* p *and* $q = n$ *such that*

$$\left|q\alpha - p - \frac{1}{2}\right| < \frac{\log\theta}{12q},$$

or, equivalently,

$$\left|\alpha - \frac{2p+1}{2q}\right| < \frac{\log\theta}{3(2q)^2}.$$

(ii) The only possible numbers that could satisfy the above inequality are one-half of the denominators (if they are even) of the principal or auxiliary convergents to $1/\log\theta$.

(iii) There are three necessary conditions for $n > \log_2 \theta$ *to be atypical: There is a fraction approximating* α *that meets the absolute value inequality condition, the parity condition is met (the numerator is odd and the denominator is even), and the estimating fraction must be an over-estimate.*

(iv) The Grace Construction, or any other construction that makes all but a finite number of odd-indexed continuants to be odd, will insure the number of continuants that yield[2] *atypical numbers is finite.*

(v) The denominators of auxiliary convergents can be prevented from being atypical, except in, at most, a finite number of cases, by the Grace Construction or any other construction that insures only a finite number of partial quotients are less than some specified number.

(vii) O'Bryant's Theorem 5 provides another construction for the number of the denominators that are atypical to be finite—in fact, zero. It is not clear how his construction relates to those brought here.

(vii) Also, it is not known if other constructions exist that will cause the number of

[2]Meaning that they are even and one-half of them is the atypical number.

denominators of auxiliary convergents that are atypical to be finite.

3.3 A Study of Conditions on θ for $\mathcal{A}_\theta$ to be Infinite

Our next task is to show that whenever $h > 1$, or equivalently, $\log\theta > 3$, $\mathcal{A}_\theta$ will always be infinite for θ if its log is irrational, with the possible exception of special θ, and even if a θ with irrational log is special, but $\log\theta > 6$, then $\mathcal{A}_\theta$ will always be infinite. To achieve this goal we need some more lemmas.

Lemma 3.3.1. *For any θ and any $\epsilon > 0$, there exists a real number $N = N(\theta, \epsilon)$ such that*

$$n \geq N \Rightarrow \frac{\log\theta}{(12+\epsilon)n} \leq \frac{\log\theta}{12n} - \frac{(\log\theta)^3}{720n^3}.$$

Proof. Consider the inequality

$$\epsilon\left[1 - \frac{(\log\theta)^2}{60n^2}\right] > \frac{(\log\theta)^2}{5n^2}.$$

For fixed θ and ϵ, as n goes to infinity, the left side approaches ϵ and the right side goes to 0. Therefore, we have this sequence of inequalities, each equivalent to the above, and which will be true for sufficiently large n:

$$\epsilon > \frac{(\log\theta)^2}{5n^2} + \frac{\epsilon(\log\theta)^2}{60n^2}$$

Divide by 12 and re-arrange terms:

$$\frac{\epsilon}{12} > \frac{(\log\theta)^2}{60n^2} + \frac{\epsilon(\log\theta)^2}{720n^2}$$

Add 1 and re-arrange:

$$1 < 1 + \frac{\epsilon}{12} - \frac{(\log\theta)^2}{60n^2} - \frac{\epsilon(\log\theta)^2}{720n^2}$$

$$1 < \frac{12+\epsilon}{12} - \frac{(\log\theta)^2}{60n^2} - \frac{\epsilon(\log\theta)^2}{720n^2}$$

Multiplying both sides by

$$\frac{\log\theta}{(12+\epsilon)n}$$

and re-arranging terms, gives the desired result. □

Corollary 3.3.1. *For any θ and any $\epsilon > 0$, there exists a natural number $N = N(\theta, \epsilon)$ such that if $n > N$,then*

$$\left(\frac{1}{2} - \frac{\log\theta}{(12+\epsilon)n}, \frac{1}{2}\right) \subset \left[\frac{1}{2} - f\left(\frac{\log\theta}{n}\right), \frac{1}{2}\right) \subset \left(\frac{1}{2} - \frac{\log\theta}{12n}, \frac{1}{2}\right)$$

Proof. Recall that O'Bryant's Lemma 7 states

$$\frac{t}{12} - \frac{t^3}{720} < f(t) < \frac{t}{12},$$

and therefore

$$\frac{1}{2} - \frac{t}{12} < \frac{1}{2} - f(t) < \frac{1}{2} - \frac{t}{12} + \frac{t^3}{720}$$

Setting $t = \log\theta/n$, and applying Lemma 3.3.1, for sufficiently large n, the previous line now becomes

$$\frac{1}{2} - \frac{\log\theta}{12n} < \frac{1}{2} - f(t) < \frac{1}{2} - \frac{\log\theta}{12n} + \frac{\log\theta^3}{720n^3} < \frac{1}{2} - \frac{\log\theta}{(12+\epsilon)n} < \frac{1}{2}$$

Now recall that for $\log\theta$ irrational, O'Bryant's Lemma 8 states

$$\left\{\frac{n}{\log\theta}\right\} \in \left[\frac{1}{2} - f\left(\frac{\log\theta}{n}\right), \frac{1}{2}\right)$$

implies n is atypical. Therefore, for sufficiently large n, *a fortiori*

$$\left\{\frac{n}{\log\theta}\right\} \in \left[\frac{1}{2} - \frac{\log\theta}{(12+\epsilon)n}, \frac{1}{2}\right) \Rightarrow \left\{\frac{n}{\log\theta}\right\} \in \left[\frac{1}{2} - f\left(\frac{\log\theta}{n}\right), \frac{1}{2}\right)$$

implies n is atypical. □

Thus, for fixed $\epsilon > 0$, we define $L''_{\theta,\epsilon}(n) := \dfrac{1}{2} - \dfrac{\log\theta}{(12+\epsilon)n}$, and we will consider $\left(\dfrac{1}{2} - \dfrac{\log\theta}{(12+\epsilon)n}, \dfrac{1}{2}\right)$ to be a contracted atypical interval. However, we do not want the definition of L'' to be dependent on ϵ, so we wish to define both L'' and the contracted interval without this dependency. To do so we will chose a specific ϵ, dependent only on θ as follows: When $\log\theta > 3$ and $\epsilon = \frac{1}{4}(\log\theta - 3)$, we will use the notation N_θ to mean $N(\theta,\epsilon)$, and an algebraic computation shows this is equal to

$$\frac{1}{30}\frac{\sqrt{15}\log\theta\sqrt{(\log\theta-3)(\log\theta+45)}}{\log\theta-3}.$$

Since we may later need to apply some of the preceding concepts for $0 < \log\theta < 3$, we will define

$$N_\theta := \frac{1}{30}\frac{\sqrt{15}\log\theta\sqrt{|\log\theta-3|(\log\theta+45)}}{|\log\theta-3|},$$

which is defined and bounded in any closed interval in $(0,\infty)$ that does not include 3. We now define $L''_\theta(n) := \dfrac{1}{2} - \dfrac{\log\theta}{(12+\frac{|\log\theta-3|}{4})N_\theta}$ and the contracted interval is defined to be $\left(L''_\theta(n), \dfrac{1}{2}\right)$.

Theorem 3.3.1. *If θ is not special, has an irrational log, and $\log\theta > 3$, then $\mathcal{A}_\theta$ is infinite.*

Proof. By Lemma 3.1.1, we have the inequality for any atypical n:

$$\left|\frac{n}{\log\theta} - p - \frac{1}{2}\right| < \frac{\log\theta}{12n}. \tag{3.14}$$

Furthermore, because we are using n in the place of q and p in the place of $\left\lfloor \frac{n}{\log\theta} \right\rfloor$, it follows that the above, in turn, is equivalent to

$$\left|\left\{\frac{n}{\log\theta}\right\} - \frac{1}{2}\right| < \frac{\log\theta}{12n}. \tag{3.15}$$

Now, since $\log\theta > 3$ means $\frac{\log\theta}{12} > 1/4$, so the Grace-Minkowski Theorem part (i) applies. Accordingly. there are infinitely many such pairs of integers, (p, q) that satisfy 3.14. Specifically, there are pairs of relatively prime integers, r, s, where $r = 2p + 1$ and $s = 2q$ that satisfy 3.14. These fractions r/s can be either principal convergents or auxiliary convergents or near-convergents to α. If θ is not special, then there will an infinite number of pairs of such numbers that are convergents. If θ is special, then, at most a finite number of continuants are even, so the infinite number of pairs will be near-convergents. The following development applies to regular convergents. There is little known about near-convergents, and although it is possible that the following development applies to them as well, it is not clearly known whether that is the case, so we have an exclusion for special θ to the rest of this proof.

Each pair produces an $n = q$, such that $\left\{\frac{n}{\log\theta}\right\}$ is in the expanded atypical interval. In the absence of a proof to the contrary, it is possible that all but a finite number of these $\left\{\frac{n}{\log\theta}\right\}$ are in the expanded part of the expanded interval, and therefore there would be no proof that the number of atypical n is infinite. The truth, however, is just the opposite—all but a finite number of the $\left\{\frac{n}{\log\theta}\right\}$ are in the standard atypical interval, and, at most a finite number, are in the expanded part. Our next task is to prove that claim.

Since there are infinitely many q in the Grace-Minkowski Theorem, which correspond to n, it is clear that n approaches infinity, and therefore, at some point $n > N(\theta, \epsilon)$ for any choice of small positive ϵ. Now, if we choose $\epsilon < \frac{1}{4}(\log\theta - 3)$, and if n is sufficiently large, that is $n > N(\theta, \epsilon)$, then by Corollary 3.3.1, inequality (3.15) becomes

$$\left|\left\{\frac{n}{\log\theta}\right\} - \frac{1}{2}\right| < \frac{\log\theta}{(12+\epsilon)n} = \frac{k}{n}, \tag{3.16}$$

where $k = \frac{\log\theta}{(12+\epsilon)} > 1/4$, because of the restriction on ϵ. Hence, the Grace-Minkowski Theorem part (i) still applies, and there are an infinite number of pairs of relatively prime integers, $r = 2p + 1, s = 2q$, and thus, also a pair of integers (p, q), each producing an $n = q$ so inequality 3.16 is true.

We still do not know, however, that there are an infinite number of atypical n, because this inequality contains an absolute value, which causes the inequality to branch into two portions, a positive one (where the absolute value signs are simply removed), and a negative portion where the removal of the absolute value signs is accompanied by a multiplication by -1. As mentioned before, we are looking for odd-indexed convergents, meaning we are interested in the negative portion of the inequality. Grace only provides us with the information that there are an infinite number of solutions to the absolute value inequality, and, at first glance, they may all lie in the positive portion. Yet, we need that the negative portion of the absolute value in 3.14 has an infinitely number of solutions. However, the solutions are all convergents to α, including the odd-indexed ones. These odd-indexed ones constitute an infinite number of solutions for the negative piece of the absolute value. Thus, the positive portion of the inequality would correspond to the even-indexed convergents, and the odd-indexed ones would correspond to the negative portion of the inequality, thereby satisfying both (3.11) and (3.6) (which does not have an absolute value sign). Furthermore, since θ is not special there are an infinite number of fractions (convergents) r/s where both the index of the convergent is odd and s is even. □

Lemma 3.3.2. *For any positive irrational number α, there are infinite number of reduced fractions p/q with odd numerators satisfying* $0 < \dfrac{p}{q} - \alpha < \dfrac{1}{q^2}$.

Proof. We know that all all odd-indexed principal convergents to α satisfy this inequality. Unless α has the even property, there will be an infinite number of odd-indexed convergents (either principal or auxiliary) that satisfy the preceding inequality and whose denominators are even, and since all convergents are reduced fractions, *per force*, their numerators must be odd.

If, however, α does have the even property, then, according to Lemma 3.2.2b, as seen from the following chart, every even-indexed partial quotient a_n must be even; the denominators must alternate with q_n having the same parity as n, and the numerators of even-indexed convergents must be odd since the denominators are even, and convergents are reduced fractions. The odd-indexed partial quotients are marked with asterisks to indicate, that because of the recursive formula, both odd and even partial quotients of odd-index will produce an odd q_n of odd index.

Table 3.1: Odd Numerators

	$n-2$	$n-1$	n	$n+1$	$n+2$
n	Even	Odd	Even	Odd	Evem
a_n		*	Even	*	Even
p_n	Odd	(Even)	Odd	(Even)	Odd
q_n	Even	Odd	Even	Odd	Even

Given this scenario, we proceed by indirect proof to show that it is not possible for all odd-indexed numerators to be even from some point on. We have indicated the attempt to make them even by putting the word "Even" in parenthesis in the chart above for the numerators.

If we further wish to insure that from some point on, all odd-indexed numerators will be odd, the numerators will then also have an alternating odd-even pattern like the denominators do, except that they will be of opposite parity. This situation can occur if and only if every odd-indexed partial quotient is even, and the even ones are indeterminate ("asterisks"). To be consistent with what already exists, this additional feature is only possible if, from some point on, *all* partial quotients are even. Since every even $a_n \geq 2$, it follows that $a_n - 1 \geq 1$, and therefore 1 is a valid value for c_n (and so is $a_n - 1$ when $a_n > 2$). Therefore we can let c_n be 1 or $a_n - 1$, causing the intermediate fraction (or auxiliary partial quotient) $\frac{p_{n,c}}{q_{n,c}} = \frac{p_{n-2} + c_n p_{n-1}}{q_{n-2} + c_n q_{n-1}}$ to have an odd numerator. Also, when $c_n = 1$ or $a_n - 1$, from Facts 12 and 14, we have both $1 < \lambda_{n,1} < 2$ and $\frac{1}{2{q_{n,c}}^2} < \frac{1}{\lambda_{n,c} q_{n,c}^2} = \frac{p_{n,c}}{q_{n,c}} - \alpha < \frac{1}{{q_{n,c}}^2}$ so an auxiliary convergent also satisfies th inequality $0 < \frac{p}{q} - \alpha < \frac{1}{q^2}$. Thus, while we can arrange it so no odd-indexed continuant (principal or auxiliary) is even, we cannot simultaneously arrange that both the odd-indexed numerators of these convergents will be even for both all principal convergents or auxiliary convergents. Then, there will always be an infinitude of odd-indexed convergent numerators that are odd and satisfies the inequality. □

Theorem 3.3.2. *If* $\log \theta$ *is irrational and* $\log \theta > 6$*, then* $\mathcal{A}_\theta$ *is infinite even if* θ *is special.*

Proof. The preceding theorem shows $\mathcal{A}_\theta$ is infinite if θ is not special. If, however, θ is special, there are only a finite number of convergents that meet the second and third conditions, the parity condition and the over-estimate condition. However, the Grace-Minkowski Theorem does guarantee there are an infinite number of fractions that meet the first condition, the absolute value inequality condition; hence, these must be near convergents. It is our goal to prove that an infinite subset of these near convergents meet the second and third conditions as well whenever $\log\theta > 6$, so that one-half of these denominators will be atypical numbers yielding the result that $\mathcal{A}_\theta$ is infinite.

Using the fact that $\log\theta > 6$, it has already been shown that if the parity condition is met (and s is even), then any reduced fraction r/s (not necessarily a convergent, but possibly a near-convergent) satisfying

$$0 < \frac{r}{s} - \frac{1}{\log\theta} < \frac{\frac{\log\theta}{3}}{s^2} < \frac{2}{s^2} \Rightarrow n := s/2 \in \mathcal{A}_\theta. \tag{3.17}$$

If p/q is an odd-indexed convergent to $\dfrac{2}{\log\theta}$, then $0 < \dfrac{p}{q} - \dfrac{2}{\log\theta} < \dfrac{1}{q^2}$ Dividing all by 2 yields $0 < \dfrac{p}{2q} - \dfrac{1}{\log\theta} < \dfrac{1}{2q^2} = \dfrac{2}{(2q)^2}$. Setting $r = p, n = q, s = 2q = 2n$, we now have

$$0 < \frac{r}{s} - \frac{1}{\log\theta} < \frac{2}{s^2} < \frac{\frac{\log\theta}{3}}{s^2},$$

and therefore the inequality will be satisfied by any odd-indexed convergent, r/s to $1/\log\theta$. If, in addition, $p = r$ is odd, then all the conditions of 3.17 are met, in particular the parity condition, and n is atypical. However, it is necessary to insure that p is odd, lest the 2 in the denominator of $\dfrac{p}{2q}$ cancel with the p in the numerator and the denominator is no longer even. This adverse possibility, though, is excluded by the Lemma 3.3.2, because either there are an infinite number of even denominators (which make the corresponding numerators odd) and which means θ is not special, or if θ is special, there will an infinite number of odd-indexed convergents to $\dfrac{1}{\log\theta}$ with odd numerators, $r = p$. Therefore, the fraction $\dfrac{p}{2q}$ is reduced, and since since $s = 2q$, we now have that $s/2 = q = n \in \mathcal{A}_\theta$. □

Combining the last three theorems produces the main theorem stated in Section 1.4. While it seems likely that $\mathcal{A}_\theta$ is infinite for $\log\theta \in (3, 6)$, even when θ is special, in order that $\mathcal{A}_\theta$ were to be infinite in such a case it would be necessary for an infinite number of denominators of near-convergents r/s to be even and for r/s to be an upper estimate for α (conditions *ii* and *iii* of Corollary 3.1.1b). It is an open problem whether or not such near-convergents exist.

Summary:

(i) By the Grace-Minkowski Theorem, the following inequality will <u>*always*</u> *have an infinite number of integral solutions, p, q, for every irrational, non-special, number α whenever $k > 1/4$. If however, $k \leq 1/4$, it is possible that there are either a finite or an infinite number of such solutions, based upon what α is.*

$$\left|q\alpha - p - \frac{1}{2}\right| < \frac{k}{q}.$$

(ii) By Lemma 3.1 and Theorem 3.3.1, if $\alpha = 1/\log\theta$ where $\log\theta > 3$ and irrational so that $k := \frac{\log\theta}{12} > 1/4$, then for each $n \in \mathcal{A}_\theta$ there exists a unique pair of

integers p, q, *where* $p = \lfloor n/\log\theta \rfloor$ *and* $q = n$ *that satisfy the preceding inequality. In fact, this inequality now becomes*

$$\left|\alpha - \frac{2p+1}{2q}\right| < \frac{\frac{\log\theta}{3}}{q^2}, \tag{3.18}$$

Hence, if θ *is not special,* $\mathcal{A}_\theta$ *is infinite, and every such atypical* n *will be one-half the denominator of some fraction that approximates* α*.*

(iii) If $\log\theta > 3$ *and irrational, the only possible numbers that could be atypical are one-half of even denominators of the principal or auxiliary convergents to* $1/\log\theta$*, or possibly some near-convergents. Unlike the case of* $\log\theta < 3$*, or equivalently* $k < 1/4$*, it is not possible to arrange the continued fraction of* α *so that at most a finite number of approximants satisfy the above inequality, because the Grace-Minkowski inequality insures that there will be an infinite number of solutions to the absolute value inequality—and if they do not come from the denominators of principal or auxiliary convergents, they must come from other fractions, such as near-convergents.*

(iv) If one-half the denominator of a convergent to $1/\log\theta$*, then there are three necessary conditions for it to be atypical, viz., the absolute value inequaltity is met, the denominator must be even, and the convergent has an odd-index.*

(v) There is no known set of conditions that are sufficient for the denominator of a near-convergent to satisfy (3.18) *and therefore to be atypical.*

3.4 Existence of Irrational Algebraic θ for which $\mathcal{A}_\theta$ is Finite or Infinite

In the process of solving O'Bryant's third problem, we have also solved O'Bryant's fifth problem. This problem asks "Is there any algebraic θ for which $\mathcal{A}_\theta$ can be proved finite? Infinite?" The existence of infinite families of algebraic θ for which $\mathcal{A}_\theta$ are finite were proven in Corollaries 2.4.1a, 2.4.1b, and 2.4.1c; and the existence of infinite families of algebraic θ for which $\mathcal{A}_\theta$ are infinite were proven as part of Theorem 3.3.1, since any θ whose log is irrational and greater than 3 will have infinite $\mathcal{A}_\theta$.

3.5 A Study of $\mathcal{A}_{e^e}$

We now provide an affirmative answer for O'Bryants first problems which asks "Is $\mathcal{A}_{e^e}$ infinite?"

Theorem 3.5.1. *If $\alpha = 1/\log\theta$ is irrational and there are an infinite number of pairs of integers p, q, that satisfy*

$$\left|q\alpha - p - \frac{1}{2}\right| < \frac{\frac{\log\theta}{12}}{q}, \tag{3.19}$$

and if there are an infinite number of partial fraction convergents, r/s, to $1/\alpha$ of odd-index and with s even, then $\mathcal{A}_\theta$ is infinite.

Proof. The proof of this theorem is almost identical to that of Theorem 3.3.1. The only difference is that in Theorem 3.3.1, the hypothesis $\log\theta > 3$ was needed to prove that there are an infinite number of relatively prime pairs n, p such that 3.14 is true. The hypothesis of this theorem, using q in the place of n, states that there are an infinite number of such pairs, and therefore, the restriction of $\log\theta > 3$ is no longer necessary. $\square$

Theorem 3.5.2. *$\mathcal{A}_{e^e}$ is infinite, and, in fact,*
$\mathcal{A}_{e^e} = \{q_j/2 : j \equiv 1, 3 \pmod 6, j \neq 1\}$, *where* $\dfrac{p_j}{q_j}$ *is the* j^{th} *principal convergent of* $1/e$.

Proof. It is well known that the simple continued fraction expansion of
$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$ where for $j \geq 1$,

if $j = 3m + 2$, then $a_j = 2(m + 1)$, for $m \geq 0$;

otherwise, $a_j = 1$.

Similarly, if $\theta = e^e$, then $\alpha = \frac{1}{\log\theta} = \frac{1}{e}$, and also the simple continued fraction of
$\alpha = [0; 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$ where for $j \geq 2$,

if $j = 3m$, then $a_j = 2m$, for $m \geq 1$;

otherwise, $a_j = 1$.

The Grace Construction was a successful method for insuring that the number of solutions to (2.5) is finite only because the parity conditions implicit in the inequality enabled one to define partial quotients that did not produce even continuants. In this situation, the recursive relation for convergents, starting with $n = 1$, means that q_n follows the pattern of EDEDDD, which repeats forever, where "D" stands for "odd" and "E" stands for "even." This means we also have a repeating pattern for the cases also, of period 6, namely Case 2, 3, 2, 4, 4, 3. We also have, except for the case of $n = 1$ that a_n follows the pattern of DDEDDE. The end result is that q_n is even for odd-indexed convegents if and only if n is congruent to 1 or 3 mod 6. Hence, there are an infinite number of atypical n by the preceding theorem, namely one-half of these continuants. A computerized check gives the first five atypical numbers as 1, 4, 53, 632, 12973, which are exactly equal to $q_j/2$ for $j = 1, 3, 7, 9, 13$.

While the above argument suffices to solve O'Bryant's first problem by showing $\mathcal{A}_{e^e}$ is infinite, it does not suffice to prove the set of atypical n is precisely these numbers, and no others. When $\theta = e^e, h := \log\theta/3 = e/3 \in (1/\sqrt{5}, 1)$ means the only candidates for an atypical n are one-half odd-indexed denominators (that are even) of principal or auxiliarly convergents; hence, it is possible that one-half of even denominators of auxiliary convergents of odd-index could also be atypical numbers. We will now show that no such numbers exist, that is, if an auxiliary denominator is even, it either has an even index or does not meet the absolute value inequality condition.

First we realize that if $a_j = 1$ that there is no j^{th} auxiliary convergent. Thus, all auxiliary convergents must come from an a_j which is even, that is $j \equiv 1, 3 \pmod 6$. From Facts 10, 12, and 14, we know that if an auxiliary convergent, $\frac{p_{j,c}}{q_{j,c}}$ produces an atypical $n = \frac{q_{j,c}}{2}$ then $\frac{1}{{\lambda_{j,c}}^2} \leq h < 1$, which means $\lambda_{j,c} > 1$, and therefore c or $d = a_j - c = 1$.

Second, if $j \equiv 3 \pmod 6$, the two preceding denominators are even and odd, which means if $c = 1$ and $d = a_j - 1$ will both be odd and therefore $q_{j,c}$ will be odd, which is not divisible by 2, so no atypical number is produced.

Third, if $j \equiv 6 \pmod 6$, then q_j, c is an even-indexed auxiliary continuant, but only odd-indexed ones produce atypical n. Hence, auxiliary continuants, cannot produce atypical n; only principal continuants can. $\square$

Chapter 4

Some Partial results

In the preceding chapter we produced solutions to three of O'Bryants five problems. We do not have a full solution for the other two problems. However we do have some partial results and some indications of what might be a way to approach these problems. This information is presented in this chapter.

4.1 Symmetric Differences

O'Bryant's seond problem is "Does there exist a pair of positive real numbers, (θ, τ), with both $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ infinite, such that the symmetric difference $\mathcal{A}_\theta \triangle \mathcal{A}_\tau$ is finite?" Since the claim is trivially true if $\theta = \tau$, we assume without loss of generally that they are not equal. The following three conditions together are necessary and sufficient to prove the existence of such a pair:

(i) $I := \mathcal{A}_\theta \cap \mathcal{A}_\tau$ is infinite;

(ii) $\mathcal{A}_\theta \backslash I$ is finite; and

(iii) $\mathcal{A}_\tau \backslash I$ is finite.

While we do not have a complete answer to this question, we present two approaches, each with one one theorem giving partial results.

4.1.1 First Approach

Here is a theorem giving a necessary, but not sufficient, condition, for (i) to be true:

Theorem 4.1.1. *If $n \in \mathcal{A}_\theta \cap \mathcal{A}_\tau$, then there exist a real number $h \geq 1$, positive real numbers k and ϕ and a positive integer m, such that the inequality*

$$0 < \frac{m}{n} - \phi \leq \frac{h}{2n^2} \tag{4.1}$$

is satisfied by each $n \in I$. Moreover, if I is infinite, then (4.1) is satisfied by an infinite number of n, where m/n is any odd-indexed convergent to ϕ. When $\log\theta + \log\tau \leq 6$, then $h = 1$, and the only solutions for m/n are the odd-indexed convergents, and when $\log\theta + \log\tau > 6$, then $h < 1$, and there are fractions, m/n, other than convergents, that satisfy (4.1).

Proof. We will use the notation of Section 3.1, and we will also call $\theta' := \tau$ and use primes for the corresponding numerics relating to τ. Using the notation and result of (3.5), we have:

$$n \in \mathcal{A}_\theta \Rightarrow (p + 1/2 - n\alpha) < \frac{\log\theta}{12n} \tag{4.2}$$

Further, if we let $\alpha' = (k+1)\alpha$ for some real number $k \geq 0$[1], then we also have

$$n \in \mathcal{A}_{\theta'} \Rightarrow (p' + 1/2 - n\alpha') = (p' + 1/2 - n(k+1)\alpha) < \frac{\log\theta'}{12n}. \tag{4.3}$$

Adding these two inequalities together produces

$$n \in I \Rightarrow 0 < (p + p' + 1) - n(k + 2\alpha) < \frac{\log\theta + \log\theta'}{12n}. \tag{4.4}$$

Now, since n is atypical for both θ and θ', meaning both $\{n/\log\theta\} < 1/2$ and $\{n/\log\theta'\} < 1/2$, and since we also have $p = \lfloor n\alpha \rfloor = \lfloor n/\log\theta \rfloor$ and $p' = \lfloor n\alpha' \rfloor = \lfloor n/\log\theta' \rfloor$, the sum

$$p + p' = \lfloor n\alpha \rfloor + \lfloor n\alpha' \rfloor = \lfloor n\alpha \rfloor + \lfloor n(k+1)\alpha \rfloor = \lfloor n(k+2)\alpha \rfloor.$$

This means (4.4) now becomes

$$0 < (\lfloor n(k+2)\alpha \rfloor + 1) - n(k + 2\alpha) < \frac{\log\theta + \log\theta'}{12n}. \tag{4.5}$$

Define $m := \lfloor n(k+2)\alpha \rfloor + 1$ and $\phi := n(k + 2\alpha)$, and divide both sides by n, giving

$$0 < \frac{m}{n} - \phi < \frac{\log\theta + \log\theta'}{12n^2}. \tag{4.6}$$

Note that if $k = 0$, then $\log\theta' = \log\theta$ and the previous inequality becomes

$$0 < \frac{m}{n} - \phi < \frac{\log\theta}{6n^2}$$

[1] $k = 0$ is equivalent to $\alpha = \alpha'$, which will be excluded in much of our discussion.

If, also $1 < \theta < e^3$, then $\frac{\log\theta}{6n^2} < \frac{1}{2n^2}$, giving

$$0 < \frac{m}{n} - \phi < \frac{\log\theta}{6n^2} < \frac{1}{2n^2},$$

and implying $\frac{m}{n}$ is an odd-indexed convergent of ϕ [10, Theorem II.5.1], similar to the development in O'Bryant's Lemma 7 [9].

In our case, when $k > 0$, if $0 < \log\theta + \log\theta' \leq 6$ then (4.6) becomes

$$0 < \frac{m}{n} - \phi \leq \frac{1}{2n^2}, \tag{4.7}$$

in which case it follows that $\frac{m}{n}$ is an odd-indexed convergent of ϕ, as just explained. Accordingly, there would be an infinite number of n that satisfy (4.7). Moreover, if $\log\theta + \log\theta' > 6$, the previous inequality would be of the form

$$0 < \frac{m}{n} - \phi \leq \frac{h}{2n^2}, \tag{4.8}$$

where $h > 1$, and this inequality would be satisfied by the odd-indexed convergents of ϕ, as well as by other fractions, again, supplying an infinite number of n.

Thus, for $h \geq 1$, any $n \in I$ will always satisfy (4.1), and therefore, if I is infinite, there will always be an infinite number of solutions to (4.1) as claimed in the theorem. □

If the converse were true, we would be able to produce a number ϕ so that an infinitude of solutions to (4.1) would imply the existence of a pair, θ, ϕ such that (i) is true; we still need to provide proofs for (ii) and (iii) if we wished to prove the conjecture true. Unfortunately, the converse is not true, for it is possible that any n satisfying (4.5) may not necessarily satisfy either of (4.2) or (4.3), so we have not even proven one-third of what must be done. The above work, though, does give some indication of a path for further investigation to provide a proof. The fact that $\{n/\log\theta\}$ is evenly distributed in $(0,1]$, does seem to suggest that almost all sufficiently large choices of θ and τ will produce an infinite intersection, I, in almost all cases. By the same token, it seems that in almost all cases, $\mathcal{A}_\theta \backslash I$ and $\mathcal{A}_\tau \backslash I$ will also be infinite, and therefore, either there are no such pairs of θ, ϕ, or producing such a pair will require a very intricate construction.

4.1.2 Second Approach

We first make use of two known results:

(1) If α is any irrational number, then $\{\{n\alpha\} : n \in \mathbf{N}\}$ is dense in the unit circle $[0,1)$.

(2) For any natural number f, if $\alpha_1, \alpha_2, \ldots, \alpha_f$ are any irrational numbers such that the ratio of any two is also irrational, and if $\beta_1, \beta_2, \ldots, \beta_f$ are any points (not necessarily distinct) on the unit circle, then for any given small positive real number ϵ there are an infinite number of natural numbers n, such that for each $i = 1, 2, \ldots, f$, each $\{n\alpha_i\}$ is in an ϵ neighborhood of β_i [1, Chapter III, Section 5, Theorem IV, page 52]. This theorem is sometimes called Kroenecker's theorem on

simultaneous inhomogeneous approximation. We only need to use it for $f = 2$ and $\beta_1 = \beta_2 = 1/2$.

We conjecture, but cannot prove the following theorem. We do present an outline of what may be an approach to providing a proof.

Theorem 4.1.2. *Let θ and τ be any two positive real numbers with irrational logs the ratio of which is also irrational, and both $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ are infinite. If neither θ nor τ is special then each of the following sets is infinite:*

$I := \mathcal{A}_\theta \cap \mathcal{A}_\tau$

$\mathcal{A}_\theta \backslash I$

$\mathcal{A}_\tau \backslash I.$

Proof. Here is just an outline:

For any irrational number ϕ, define $N_\phi := \frac{1}{30}\frac{\sqrt{15}\log\theta\sqrt{|\log\theta-3|\log\theta+45)}}{|\log\theta-3|}$. Let $N = \max(N_\theta, N_\tau), \alpha_1 = 1/\log\theta, \alpha_2 = 1/\log\tau$. Also let ϵ_1 be less than min and $\beta_1 = 1/2$, and $\beta_2 = 1/2$. If $n > N$, by Kroenecker there exist an infinite number of $n \in I$. If we now make $\beta_2 = 3/4$. there are now an infinite number of n in $\mathcal{A}_\theta$ but not in $\mathcal{A}_\tau$. Switching the two β's around produces an infinite number of n in $\mathcal{A}_\tau$ but not in $\mathcal{A}_\theta$.

The Grace inequality and Kroenecker produce an infinite number of solutions common to both, but it is yet necessary to show that the infinite number of common solutions contains an infinite subset of common even continuants with odd index, even though each α does have by itself and even though there are infinite ones in common that might have even index or odd denominators.

□

4.2 Fractals and The Hausdorf-Besocovitch Dimension of θ's With Finite $\mathcal{A}_\theta$

O'Bryant's Fourth problem is: "What is the Hausdorff dimension of $\{\theta > 1 : \mathcal{A}_\theta \text{ is finite}\}$?" While we do not solve this problem, we introduce two mappings using the continued fraction expansion of $\frac{1}{\log \theta}$, that to the best of our knowledge are not found in the literature, and provide some insight into this problem. Also included are some problems requiring additional research involving the Hausdorf-Besocovitch dimension[2]; other unsolved problems not involving the dimension are in the next chapter.

To define the two maps, we fist introduce some notation. First, whenever we use mod m, we will always use the reduced residue class, $\{0, 1, \ldots, m-1\}$. Second, the m-ary decimal $.d_1 d_2 \cdots := \sum_{i=1}^{\infty} d_i m^{-i}$. Third, for any integer a and any natural number m, let $a(m) = \min\{n \in \mathbf{N} : a \equiv a(m) \pmod{m}\}$.

Our first map will take the continued fraction $\alpha = [a_0; a_1, a_2, \ldots, a_n \ldots]$ to a binary decimal, where even partial quotients are mapped to 0, and odd ones are

[2] In his paper, O'Bryant calls it the "Hausdorf" dimension, but I prefer the fuller name of "Hausdorf-Besocovitch" dimension for the identical concept; for short, we will just use the word "dimension," without any qualifiers to carry the same meaning.

mapped to 1. It is the special case of $\mathcal{F}_m$ where $m = 2$ and we define $\mathcal{F}_m : \alpha \mapsto .d_1d_2\ldots$ where $d_i := a_{i-1}(m)$, and just $\mathcal{F}$ will mean $\mathcal{F}_2$. Alternately, we could define $d_i := a_i(m)$ $n \geq 1$ and, by convention $a_0 = d_0$ is the digit in the "one's" column. This map will be useful in proving the next theorem which follows, and the definition of the second map will be deferred to later. In the Grace construction, we are not concerned about the actual values of the partial quotients, only their parity, since the parity alone is the determining factor as to whether a given convergent will satisfy inequality (2.3) in Section 2.3. Thus, we are not interested in the space of all convergents or all partial quotients, but rather in the space of $\{\mathcal{F}(\alpha) : \alpha \text{ is irrational}\}$.

Theorem 4.2. $\dim G = 0$.

Proof. Recall that G, as defined earlier, was, by the Grace Construction, a set of irrational numbers, α, where $\mathcal{A}_\theta$ was finite where $\alpha := 1/\log\theta$. More specifically, it was the set of all irrational numbers $\alpha = [a_0; a_1, \ldots]$, where $a_0 = 0$, a_1 is odd, and the other were all even and increasing. Since we do not care about the actual values of the individual partial quotients,just their parity, we can investigate $\dim G$, by simply dealing with $\mathcal{F}(\alpha)$. Now G is a subset of all binary decimals that end in an infinite string of zeroes, which is, in turn a subset of all rational numbers, which is countable and therefore $\dim G = 0$. □

While this theorem does not provide a full answer to the problem posed of finding the Hausdorf dimension of $\{\theta > 1 : \mathcal{A}_\theta \text{ is finite}\}$, it does provide a partial answer in that it has now been determined that a prominent subset of the set in question is of zero dimension. All members of G satisfy (2.3), and as is noted before in the proof of Theorem 2.4.1 making all partial quotients (after the first one) even, and thereby making all continuants even, is unnecessary. It is only needed that the odd-indexed continuants be odd. Thus, G is "too small." If we truly wish to find the dim of the set of all θ so that $\mathcal{A}_\theta$ is finite, we now need to consider more deeply the parity considerations. Note that we are looking for convergents where either the numerator is not odd or the denominator is not even. While it is simplest to look for odd-indexed convergents whose denominators (continuants) are odd for completeness purposes, we have included in the tables the possibility that the numerator is (or is not) even.[3]

Using the standard convention $p_{-2}/q_{-2} = 0/1, p_{-1}/q_{-1} = 1/0$, and using $a_1 = 0 = 0/1 = p_1/q_1$, as was used in the Grace construction, we see that we start in Case 2 for computing q_2, and that we start in Case 3 in computing p_2. If we want all but a finite number of the continuants to be odd, the parity of a finite number of partial quotients may be chosen at random. Since the choice of parity of a_n must make p_n odd when n is odd, that means we must follow the arrows in the

[3]See Appendix A.

state diagram through in such a fashion that, after at most a finite number of exceptions, we go to Case 4 or Case 2 every other time. One way would be for all partial quotients to be odd, after reaching Case 4 for the first time. Or, there could be there could alternating even partial quotients (to go from Case 2 to Case 3) with any parity (to go from Case 3 to Case 2.) Still, it would be possible to go from Case 2, with an odd, to Case 4, (possibly have any number of more odds to stay in Case 4, and then an even to Case 3). By integrating these different possibilities, the total number of ways to obtain odd-indexed, odd continuants is uncountable. There does not seem any simple way to compute the dimension of the set of parities $\mathcal{F}$ by this method.

For this reason, we suggest a possible second map that may be of interest, whereby we map the continued fraction to a plane (or higher-dimensional object), possibly enabling the possible use of more tools of analysis and geometry, as well as possibly importing some of fractal methodology to solve this problem. Note that because of the possibility that $\beta = r/s \neq 1/2$, which may be of use in a more general problem than the one mentioned here, we are allowing different modular systems to be used, besides mod 2.

Our second map will take the continued fraction $\alpha = [a_0; a_1, a_2, \ldots, a_n \ldots]$ into a point in the interior of the unit square in $\mathbf{R}^m$. Define $\mathcal{G}_m : \alpha \mapsto (x_1, x_2, \ldots x_m)$ where for $i = 1, 2, ...m-1$ we define x_i to be the m-ary decimal

$.a_{0m+i}(m)a_{1m+i}(m)\ldots a_{(m-1)m+i}(m)$ and $x_m = .a_{0m}(m)a_{1m}(m)\ldots a_{(m-1)m}(m)$. That is to say that for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots$, the j^{th} digit of the i^{th} component is $a_{(j-1)m+i(m)}(m)$. For both maps we make the convention that when the subscript $m = 2$, it will simply be omitted, since $m = 2$ is what is needed in our problem, and, as just mentioned, we allow $m > 2$ to introduce a map that might be useful in more general problems.

Chapter 5

Problems For Additional Research

Nathanson [8, section 5] gives a list of problems concerning $M_\theta(n)$. Several of these problems are solved (explicitly or implicitly) by O'Bryant. O'Bryant's list of unsolved problems was included in the Abstract, and this paper solves three of them. Below are some more problems for additional research. To state these problems with minimum verbiage, we introduce some notation:

Let $\theta > 1$ be a given real number. Let $I = (0, 3)$, $J = (3, 6)$, and p/q be a reduced positive rational number.

Let (A) stand for the inequality $\left|\frac{p}{q} - \frac{1}{\log\theta}\right| < \frac{\frac{\log\theta}{3}}{q^2}$.

Let (B) stand for the inequality $\frac{p}{q} - \frac{1}{\log\theta} < \frac{\frac{\log\theta}{3}}{q^2}$.

Let (C) stand for the inequality $\frac{1}{\log\theta} - \frac{p}{q} < \frac{\frac{\log\theta}{3}}{q^2}$.

Let a subscript of D means q is odd, a subscript of E means q is even, and subscript of F means q is either even or odd. Let (H) be either (A) or (B) or (C). Let G be either D or E or F. Let K be either I or J.

For fixed θ with irrational log, and for each choice of G, H, K, define $S_G(H, K) := \{p/q : (H) \ \& \ \log\theta \in K\}$, thereby producing 18 sets of fractions that satisfy certain inequalities.

1. For each of these 18 sets, define conditions on θ that categorize when each set is empty, finite but not empty, or infinite.

2. Let $T(G, H, K), U(G, H, K), V(G, H, K) :=$ $\{p/q : S_G(H, K)$ is empty, is finite but not empty, is infinite, respectively$\}$.. What is the Hausdorf-Besocovitch dimension of each of these 54 sets?

3. What conditions are there that insure that near-convergents do or do not meet the inequality condition, the parity condition, and/or the upper-estimate condition?

4. We showed that if $\theta > e^6$ and has an irrational log, then $\mathcal{A}_\theta$ is infinite. What is $\inf\{\theta : |\mathcal{A}_\theta| = \infty\}$?

5. We proved that if θ is not special, then when $\log\theta \in (3, 6)$ is irrational $\mathcal{A}_\theta$ is infinite. For $\log\theta \in (3, 6)$ when is the converse true? If there is a contiguous subset of (3,6) for which $\mathcal{A}_\theta$ is infinite (or finite), then the preceding question is very interesting.

6. Since $\log\theta \in (0, 3)$ irrational may have no atypical n, a finite (but non-zero)

number of atypical n, or an infinite number of atypical n the question arises what is $\sup\{\theta : \mathcal{A}_\theta = \emptyset\}$? is it greater than 1? Similarly, is there a sup for those θ whose atypical sets are finite but not empty? If so, what is it?

7. If $\mathcal{A}_\theta$ is finite define $\mu(\theta) = \max\{n : n \in \mathcal{A}_\theta\}$ If $\log\theta = p/q$, then for what values of θ, does $\mu(\theta) = \left\lfloor \frac{p^2}{6q} \right\rfloor$, the maximum possible? Is the number of such θ infinite?

8. If $\log\theta = p/q$, then for what (small) values of a, b, is there an infinite unbounded set of θ, such that $\mu(\theta) = \left\lfloor \frac{ap^b}{6q} \right\rfloor$?

9. If α is a Liouville number, then there are an infinite number of solutions, p/q, to an inequality of the form $|\alpha - p/q| < 1/q^n$ for all natural numbers n. Therefore, there will be many rationals that meet the inequality condition, but it is not clear that they will meet the parity condition or the upper-estimate condition. Under what conditions will they?

Appendices

Appendix A: Recursive Tables

The recursive formula for convergents (Chapter 4, Fact 3), clearly depends on the parity of the partial quotients, which is a matter of concern in a number of places. We summarize the results in the eight cases below, each table dealing with the parity of the previous two numerators or denominators, and the parity of the next partial quotient. We will use the letter “r” to indicate either “p,” or “q,” with n as the index and a_n as the partial quotient The tables are ordered alphabetically according to the last two values of r_n. Using “E” for “even” and “D” for “Odd,” Case I is E E, Case 2 is E D, Case 3 is D E, and Case 4 is D D. Then each case is divided into subcases, Case xA means a_n is even, and Case xB means a_n is odd, and so a_n determines the parity of r_n, and it is explicitly mentioned what the next case will be.

Table 1: **Case 1A**

	$n-2$	$n-1$	n
a_n			Even
r_n	Even	Even	Even

Go to Case 1

Table 2: **Case 1B**

	$n-2$	$n-1$	n
a_n			Odd
r_n	Even	Even	Even

Go to Case 1

Table 3: **Case 2A**

	$n-2$	$n-1$	n
a_n			Even
r_n	Even	Odd	Even

Go to Case 3

Table 4: **Case 2B**

	$n-2$	$n-1$	n
a_n			Odd
r_n	Even	Odd	Odd

Go to Case 4

Table 5: **Case 3A**

	$n-2$	$n-1$	n
a_n			Even
r_n	Odd	Even	Odd

Go to Case 2

Table 6: **Case 3B**

	$n-2$	$n-1$	n
a_n			Odd
r_n	Odd	Even	Odd

Go to Case 2

Table 7: **Case 4A**

	$n-2$	$n-1$	n
a_n			Even
r_n	Odd	Odd	Odd

Go to Case 4

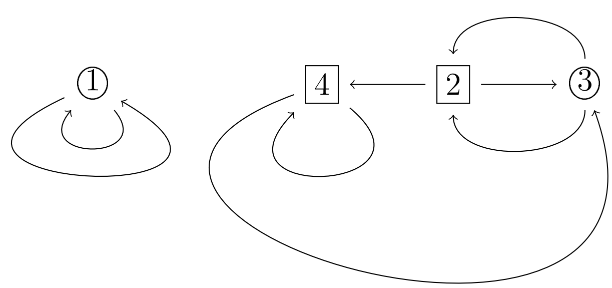
Table 8: **Case 4B**

	$n-2$	$n-1$	n
a_n			Odd
r_n	Odd	Odd	Even

Go to Case 3

The preceding tables may be summed up in the following state diagram, where a clockwise arrow, or one pointed to the right, indicates the next a_n is even, i.e., that we are in the "A" version of the case, and the counter-clockwise arrow, or one pointed to the left, indicates that a_n is odd and we are in the "B" version of the case. Moreover, a number inside a square refers to a case where the last r_n is odd[1] and a circle around the number refers to a case where the last r_n is even.[2]

State Diagram



[1]Needed so q_n odd and convergent does not meet parity condition.
[2]Optional so p_n even and convergent does not meet parity condition.

The essence of these four lemmas is to determine exact conditions will make either all (or almost all, i.e, with a finite number of exceptions) continuants (denominators of continued fractions) for either both principal convergents, viz., q_n and auxiliary convergents, viz., $q_{n,c}$, or just principal convergents, to be odd whenever n is odd. To this end, from the tables and state diagram in Appendix A, we will design a method for constructing such α.

In general, there are two ways to create an odd denominator for an odd-index. We must end up in Case 2 (last two denominators are even and odd respectively), which means the preceding case was Case 3, or we must end up in Case 4 (last two denominators were odd), which means the previous case was either Case 2 or Case 4. As a result we may either have alternating cases of Case 2 and Case 3, or repeated instances of Case 4. From Appendix A we see alternating instances of Case 2 and Case 3 will occur once there exists some n where q_{n-2} even for $n-2$ being even and q_{n-1} odd for $n-1$ being odd. This means we are in Case 2. By making a_n even, we then go to Case 3, and once in Case 3, whatever parity is chosen for a_n or $c_{n,j}$, we will always end up again in Case 2. (This proves A and B of Lemma 3.2.2b.)

Once we initially end up in Case 4 (meaning the last two denominators are both odd) we can stay in Case 4 indefinitely by choosing all subsequent a_n to be even. If any a_n were to be odd, we would end up in Case 3, from where we must go to Case

2, which has the previous two denominators being even and odd. Since we want the odd denominator to belong to a convergent of odd index, the preceding choice of odd a_n leading to Case 3 must be a case where n is even. In other words, once we are in Case 4, an indefinite number of odd a_n will keep us there; if however, we were to have an a_n that is even, which causes us to exit Case 4 and enter Case 3, it is important that n must be even. Similarly, if we ever wish to leave a repeating pattern of alternating Case 2 and Case 3 and still keep odd-indexed denominators to be odd, we may only do so by making a_n odd when n is even (Case 2B), thereby landing in Case 4, which we must exit only when n is odd. (This proves of Lemma 3.2.2b.)

For principal convergents, the above is good, but not for auxiliary convergents, since $0 \leq c_n \leq a_n$, and therefore c_n can be either odd or even as long as $a_n \geq 2$, which it is in Case 4, since staying in Case 4 means all partial quotients are even. If we are in Case 4 and using auxiliary convergents, we could exit Case 4 and go to Case 3 by making c_n odd for some odd index n, making the odd-indexed denominator even, which is to be avoided. Thus, the situation of repeated Case 4 (all previous denominators are odd) is not useful if we also wish to avoid the case auxilary continuants of odd-index being even. Only by being in Case 3, does the choice of c_n and its varying parity, not effect the parity of denominators, because we always go from Case 3 to Case 2. Hence, we do not want to have Case 4 to occur, because it allows for the possibility of an auxiliary fraction of odd-index to have an even denominator.

If Case 4 does occur, but only a finite number of times, and otherwise we are alternating between Case 2 and Case 4, then, at most a finite number of odd-indexed continuants (even auxiliary ones) will be even. (This proves Lemma 3.2.2b.)

However, the existence of a Case 4 situation cannot be avoided entirely. By definition $q_{-1} = 0$ and $q_0 = 1$, so for $n = 1$ we are in Case 2, in that the two preceding denominators are even and odd. From Case 2, we can go either to Case 3 by making a_1 even which also makes q_1 even—which we wish to avoid—or Case 4 by making both a_1 and a_2 odd, causing q_1 and q_2 to be odd and even respectively, putting us in Case 3. Such a scenario will avoid *all* denominators of principal convergents being even, but does allow that for $n = 1$, an auxiliary continuant could be even. If, however, $a_1 = 1$, then there does not exist any auxiliary convergent for $n = 1$ that is not a principal convergent, and therefore the denominator is always odd. If a_2 is also odd, we then move to Case 3 and can then alternate between Case 2 and Case 3. This way it will then end up that *all* denominators of all odd-indexed convergents are odd. This is the only way to guarantee that all denominators of all odd-indexed convergents are odd. (This proves Lemma 3.2.2d.)

If the requirement that $a_1 = 1$ is dropped but still be odd, then it is possible that the denominator of some auxiliary convergent, namely, $q_{1,j}$ will be even. (This proves 3.) Even in this case, it may be for some given α the denominators of auxiliary convergents do not meet condition i of Corollary 3.1.1b, and therefore it is not necessary that all auxiliary denominators be odd. However, as this lemma does not deal with whether or not a given convergent produces an atypical n, but rather just when we can be sure that no denominator of any odd-indexed convergent is even, it is, then, for our purposes at present, necessary to exclude any odd-indexed denominator from being even.

It follows that if there is some index n so that the even property holds for any even $j > n$, that for $k \leq n$ it is possible that some odd-indeed denominator is even, but for $k > n$, no odd-indexed denominator is even—they will all be odd. (This proves Lemma 3.2.2c part B.)

Appendix B: Minkowski's Theorems

Since there are several theorems that are called "Minkowski's Theorem"; each author has a different version, style, and notation; and some of the results may be counter-intuitive, some clarification is in order.

1. Grace's Version

In [5] Grace stated,

"Tchebychef proved that there is an infinite number of integer y's such that

$$|ay - x - b| < \frac{1}{2y},$$

Hermite that the same is true if $\frac{1}{2}$ is replaced by the smaller number $\sqrt{\frac{2}{27}}$, and Minkowski that

$$|ay - x - b| < \frac{1}{4|y|}$$

holds for an infinite number of integer values of y."

2. The Initial Version of Dickson and the Version of Hardy and Wright

Dickson wrote [3, pages 94-96],

"P. L. Tchebychef proved that if a is irrational and b is given, then there exists an *infinitude* [italics are mine] of sets of integers x, y such that there is an infinite number of integer y's such that $y - ax - b$ is numerically $< 2/|x|$.

Hermite proved that in Tchebychef's result, we may replace $2/|x|$ by $1/2|x|$ and in fact by $\sqrt{2/27}/|x|$. ...

H. Minkowski proved that if $\xi = \alpha x + \beta y$ and $\eta = \gamma x + \delta y$ have any real coefficients of determinant $\alpha\delta - \beta\gamma = 1$ and if ξ_0, η_0 are any given real numbers, there exist integers x, y for which $|\xi - \xi_0)(\eta - \eta_0)| \leq \frac{1}{4}$. In particular if a is irrational and b not an integer, there are integers x, y for which $|(y - ax - b)(c - x)| < \frac{1}{4}$; the case $c = 0$ give a better approximation than Hermite's since $1/4 < \sqrt{2/27}$."

From the fact that Dickson no longer mentions "an infinitude of integer y's," just that exist integers x, y is an indication that in this version of Minkowski's Theorem, there need not necessarily be an infinitude of such pairs, just that for any choice of a irrational and b not an integer, there will always be *at least one* pair of integers such that $|(y - ax - b)(c - x)| < \frac{1}{4}$.

Similarly, Hardy and Wright wrote [6, Chap. XXIV **24.7**, p. 534], write,

"We prove next an important theorem of Minkowski concerning non-homogeneous forms

$$\xi - \rho = \alpha x + \eta y - \rho,\ \eta - \sigma = \gamma x + \delta y - \rho$$

Theorem 455. If ξ an η are homogeneous linear forms in x, y with determinant $\triangle \neq 0$, and ρ and σ are real, then there are integral x, y for which

$$|(\xi - \rho)(\eta - \sigma)| \leq \frac{1}{4}\triangle;$$

and this is true with inequality unless..."

The absence of any statement about an *infinite* number of pairs of x, y, is an indication that the version of Minkowski's Theorem mentioned here, only deals with the existence of at least one pair, for any two given homogeneous linear forms.

3. The Version of J. W. S. Cassels

Cassel's however, explicitly states the existence of infinitely many integers

[1, Chapter III, Section 2, Theorem IIA, page 48],

"If θ is irrational and not of the form $\alpha = m\theta + n$ for integers m, n, then there are infinitely many integers q such that

$$|q| \parallel qz - \alpha \parallel < \frac{1}{4}."$$

This statement implies all the other versions of Minkowski's Theorems mentioned previously.

4. A Later Version of Dickson

He [3, p. 99] writes, "J. H. Grace...proved that Minkowski's last result is final, i.e., if $k < \frac{1}{4}$, it is possible to choose a and b such that there is not an infinitude of integers x for which $|y - ax - b| < k/|x|$," thereby implying if $k > 1/4$ there would be an infinitude of integers x.

At first glance, it would appear that the statement that an inhomogeneous approximation cannot be made to the degree of accuracy claimed, namely, $\frac{1}{4q^2}$, for this seems to contract the basic fact that an infinite number of approximations cannot exist for irrationals when that $k = \frac{1}{\sqrt{5}q^2}$ for homogeneous approximations. However, when $\beta = 1/2$ is inserted into the inequality, the denominator becomes $2q$

and not q, so there is no contradiction, because, as mentioned earlier in section 5.1, "This result, in turn, means α can be approximated by a rational, $\frac{2p+1}{2q} = \frac{r}{s}$, to accuracy of the square of the denominator. By Fact 8, an approximation of this degree of accuracy, can only be attained by a fraction which is a principal convergent or an auxiliary convergent."[5]

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