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Diophantine Approximation and The Atypical Numbers of Nathanson and O'Bryant

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Diophantine Approximation and the Atypical Numbers of Nathanson and O’Bryant

by

David Seff

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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THE CITY UNIVERSITY OF NEW YORK
Abstract

Diophantine Approximation and the Atypical Numbers of Nathanson and O’Bryant

by

David Seff

Advisor: Kevin O’Bryant

For any positive real number $\theta > 1$, and any natural number $n$, it is obvious that sequence $\theta^{1/n}$ goes to 1. Nathanson [8] and O’Bryant [9] studied the details of this convergence and discovered some truly amazing properties. One critical discovery is that for almost all $n$, $\left\lfloor \frac{1}{\theta^{1/n}} \right\rfloor$ is equal to $\left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor$, the exceptions, when $n > \log_2 \theta$, being termed atypical $n$ (the set of which for fixed $\theta$ being named $\mathcal{A}_\theta$), and that for $\log \theta$ rational, the number of atypical $n$ is finite. Nathanson left a number of questions open, and, subsequently, O’Bryant developed a theory to answer most of these questions. He also posed five new unanswered questions of his own [9, Section 7. More Problems] (which are enumerated at the end of this this abstract), of which we completely answer three, and partially answer two.

He constructed infinite families of bounded $\theta$’s with rational logarithms, some with no atypical $n$, and some with infinitely many atypical $n$. However, he left as an
open problem whether there was some upper bound, \( \theta_0 \) such that
\[
\{ \theta : \theta > \theta_0, \log \theta \text{ is irrational, and } A_\theta \text{ is finite} \}
\]
is not uncountable, which is his third question. This thesis shows that the restriction of boundedness cannot be removed and is described in detail in Chapter 3. During the course of the development needed to answer that question, this thesis proceeds to answer the fifth question in Section 3.4 and the first question in Section 3.5. Questions 2 and 4 below remain unanswered, but I bring some partial results and suggest methods for further research on these problems in Section 4.1.1 and Section 4.2 respectively. Finally, in Chapter 5, I list some additional open questions. Here is the list of O’Bryant’s open questions:

1. Is \( A_e \) infinite?

2. Are there \( \theta, \tau \) with both \( A_\theta \) and \( A_\tau \) infinite, but the symmetric difference \( A_\theta \triangle A_\tau \) finite?

3. For every \( \theta_0 \), are there uncountably many \( \theta > \theta_0 \) with \( A_\theta \) finite?

4. What is the Hausdorff dimension of \( \{ \theta > 1 : A_\theta \text{ is finite} \} \)?

5. Is there any algebraic \( \theta \) for which \( A_\theta \) can be proved finite? Infinite?
I acknowledge and give thanks, first and foremost, to G-D who has allowed me and helped to reach this stage in my academic development, in spite of many unusual and difficult circumstances. I have recently turned 70, and it probably most unusual for a person my age to submit a doctoral dissertation in mathematics, and perhaps I am one of the oldest, if not the oldest person to earn a doctorate in math from the CUNY Graduate Center, and possibly in America. What may be even more unusual is that I first entered graduate school when I was still a teenager, being at the top of my class in all courses throughout my academic career, including graduate school. Many side problems, often medical or financial, of a severe nature, necessitated taking long breaks, and, as may be expected, my mental abilities have declined considerably over the past 50 years. Accordingly it is with great gratitude that I state, “Baruch shechiyanu, vkiymanyu v’higiyanu lazman hazeh.” “Blessed is He who has kept me alive, sustained me, and enabled me to reach this day.”

In spite of many obstacles, there were many who went out of their way to help me, and it is only proper to acknowledge their help and encouragement. In order
that their efforts can be properly appreciated, it is necessary, somewhat to explain a few of the hardships encountered.

I became extremely ill the day I was born, and doctors did not know what the cause was, and did not think I would live. Only decades later, due to the detective work of many different specialists, doctors have pieced together what they think is the explanation. It is believed that medicine given my mother during pregnancy seriously damaged my immune system, and consequently, within a few minutes after birth, I contracted a very rare form of viral hepatitis that was and still is unknown due to the fact that no person with a normally functioning immune system gets this infection. It has not been studied in adults, and its long-term effects on a newborn infant who contacted it on the day of birth are completely unknown.

I also contracted many other infections or diseases as a child, due to the defective immune system, that no doctor could diagnose at that time—and with an improperly functioning liver that could not metabolize some antibiotics, there was constantly a double threat of contracting many diseases, including ones that were rare or often considered non-communicable and not being able to tolerate the medicines that was used to treat those diseases.

With asthma and allergies, I also had great trouble breathing, and for several years, there were periods of time when I was confined to a room in the house that had special air-filters, and was not allowed out of the room, sometime for months at
a time, even to eat or to go to the bathroom. There were times when I was not even able to attend school.

With the work of good doctors, nutritionists, and other specialists (including those specializing in alternate medicine), a program of diet and exercise was devised that enabled me to go outside and play when I was in the fifth grade. I sometimes did special exercises for hours a day, and I was able to make my high school and college tennis teams. I was no star by a long-shot, but this achievement was a major accomplishment in attaining physical development.

There was a clear silver lining to my problems—being unable to engage in much physical activity my parents and teachers tried to stimulate me intellectually, and hence, I developed intellectually at a very fast pace.

I always wanted to be a mathematics professor for several reasons: Firstly, my father had received a masters in math and had studied for his doctorate, but later switched to optometry in belief he could support his family better. My older sister was also a mathematician. Thus, both parents and my sister either directly or indirectly encouraged my development in this direction. Secondly, my health problems enabled me to spend more time with developing mentally rather than physically, so I liked math, and excelled in it. Thirdly, I was afraid with my medical background a nine-to-five job would be too burdensome for me.

The medical issues thus both thrust me towards an academic career and hindered
my progress due to periodic absences for medical reasons.

There were concomitant legal and financial hindrances to my progress as well. To name just one example, when living in the Washington Heights section of Manhattan, there was a fire in our apartment on a cold day in January, 1978. I returned home one day after studying in graduate school only to find there was no apartment there. It had been nearly completely gutted out. My wife, two daughters, and I were literally homeless. We were knocking on doors to ask for a place to sleep or eat for several days, until some families in the neighborhood raised some money for us and rented us an apartment temporarily until the old one was renovated.

Even though I had been nearing the completion of my doctorate at The Courant Institute of NYU, I had to take a leave of absence (and get a deadline extension to complete my doctorate) and get a job to support my family. Dr. Henry Lisman, chairman of the math department of Yeshiva University, which I had attended, was a fatherly figure who looked after each student as if he were his only son, and got me a job in computers. I had no prior experience or training in computers but he said he could find me a position where they promised to train me. He did—but they did not keep that promise. It was a consulting firm that said they would send me out as a junior programmer to work with and be trained by senior programmers. The senior programmers, though, were on contracts to finish an assignment with tight schedules and could spare no time to train a novice.
My first job in computers was difficult because not only was the consulting company advertising me as an expert in computer languages in which I had no experience—and I was soon sent back by the client as being incapable—but the job was so far away that I had to get up at 5:00 AM to get there, and did not arrive back home until 8:00 PM. I started fainting from lack of sleep.

It was a case of sink or swim, and I was sinking. I therefore got a book on computers, and asked a friend who was a computer-programmer to tutor me, and eventually I began to swim a little. Without formal training in computers it was hard to get the best paying jobs, and without a doctorate in math, it was hard to get a full-time position in academia. Thus, financial problems set in, requiring taking multiple jobs, which in turn did not allow me time to finish my doctorate, and weakened me as well because of my medical conditions. As a result, an unholy triad of no money, no time, and medical problems—all of which created a vicious cycle—made it harder to allocate time to finish my doctoral studies. Legal problems emanating from the fire, added to the mix, creating lawsuits that lasted for years, only made things more difficult. Finishing my doctoral studies could enable me to obtain a full-time teaching position, and then my finances, my schedule, and my health would all improve, but there did not seem any way to complete the doctorate under these circumstances.

This situation was very painful, as this was the third time I had been in grad-
uate school, and dropped from the program for external reasons, in spite of good mathematical performance.

The first time I was in graduate school was immediately after graduation from high school when I was admitted to a special program at Ohio State University, later named the Ross Program, for Gifted Secondary School Students which taught graduate level math to high school students and participated in classes with regular graduate students. While most high school students were dismissed when the program was complete, a small number of us were selected to continue to take more graduate level courses. After additional selections, five of us were selected to get an advisor and write a thesis. While this thesis would be more on the level of a master’s thesis than a doctoral thesis, it did represent original research on an unsolved problem. As we were not formally matriculated, no degree or credits were awarded and were told at the onset that no credit would be awarded. When I complained that we should get credit because at the onset of the program no one expected us to go this far, I was told we were just too young to be awarded a degree or credit in spite of excellence in work exceeding that of matriculated graduate students.

When I enrolled in college, Yeshiva University, I showed Dr. Henry Lisman my notebooks from the work I had done at OSU, and asked to be admitted to Yeshiva University’s Belfer Graduate School of Math and Science (now defunct). Initially he refused to consider my request, saying I had to prove myself first. I soon did and

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then formally matriculated in Belfer Graduate School. It was my hope to earn a
doctorate (or at least a master’s) at the same time I completed college, but these
dreams were dashed when again the complaint I was too young came back to haunt
me, for when a high level university official discovered I had matriculated, I was
retroactively expelled on the grounds the university policy did not allow anyone in
graduate school who had not completed a bachelor’s degree.

The third time in graduate school was after I finally did graduate college and was
enrolled in New York University’s Courant Institute of Mathematical Sciences, and
it was then that I had the devastating fire just mentioned.

In addition to the fire, I have also have been in over a dozen accidents, had at
least ten operations, and have several permanent injuries. I do not think I would
be alive today were it not for the grace of G-D and the timely help of relatives,
friends, neighbors, or teachers who intervened specially to afford assistance at the
times I needed it most. I cannot properly express my gratitude to all, nor do I even
remember all. I present here a short summary of those who I remember have helped
the most.

First, I acknowledge the assistance of my own family, my parents, Dr. and
Mrs. Irving Seff; my wife, Kayla; and my children.

Second, I present in chronological order a brief list of teachers, friends, and others
whose assistance was invaluable.
I was fortunate to attend the Bexley school system in Ohio, the high school having been rated a few years earlier as the number one high school in the nation by the John Hayes Society of the Ford Foundation. Many teachers were supreme, and many students were talented, some becoming world famous in their chosen fields of medicine, law, writing, TV production, and rocket expertise, among others.

I acknowledge here the help of many teachers and study partners who were truly special people:

Elementary school teachers:

Miss Barbara Dugan—my kindergarten teacher, who saw potential in every student and almost miraculously pointed each kindergarten student to the career for which he or she was best suited, and continued to assist them from the day they entered kindergarten until they became parents and grandparents. Initially she had been in a convent, wanting to be a nun, but her superiors thought she, like Maria in the “Sound of Music,” should leave and make a bigger impact elsewhere, and encouraged her to be a kindergarten teacher. She was written up in national newspapers; here is a link to an article by one of her students who became a famous journalist. http://edition.cnn.com/2011/10/09/opinion/greene-miss-barbara/

Miss Thelma Holmes—my fifth grade teacher, who did not think students should be put in a box, and allowed me to advance at my own rate.
High school teachers—my high school math teachers who instituted an honors math program and always encouraged students to learn outside the classroom:

   Mr. Harold Ridenour

   Mr. John F. Schacht

My high school study partners:

   Charles Chittenden, consulting actuary

   Dr. David Mayer PhD., Math (OSU)

Professors in the special program later named the Ross Program, for Gifted Secondary School Students at Ohio State University, in alphabetical order:

   Dr. Harold Brown,

   Dr. Arnold P. Ross, department chairman

   Dr. Andrzej Schinzel

   Dr. Paul Turan

   Dr. Ivo Thomas

   Dr. Hans Zassenhaus

My OSU study partner:

   Dr. Joel Seifras, PhD., Theoretical Computing (MIT)

College Professors at Yeshiva University in alphabetical order:

   Dr. Leopold Flatto

   Dr. Jay Gerber
Dr. Henry Lisman

My college study partner:

Leonard Tribuch, Actuary

Graduate School Professors at Yeshiva University in alphabetical order:

Dr. Leopold Flatto

Dr. Donald Newman

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Dr. Sylvain Cappell

Dr. Kurt Friederichs

Dr. Paul Garabedian

Dr. Harold N. Shapiro

Dr. Peter Ungar

Dr. Earl Glenn Whitehead, Jr.

My NYU study partners:

Dr. David Bassein, PhD., Math (NYU)

Dr. Michael Dalezman, PhD., Math (NYU)

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Dr. Michael Anshel
Dr. Martin Bendersky
Dr. Richard Churchill
Dr. Linda Keen
Dr. Martin Moskowitz
Dr. Janos Pacht
Dr. Burton Randol

Administrators at CUNY:
Dr. Alvany Rocha
Dr. Josef Dodzuik
Dr. Linda Keen
Dr. Ara Basmajian
Robert Landsman, APO
Maria-Helena Reis, APO
Debbie Silverman, APO

My CUNY study partner:
Dr. Phil Williams, PhD., Math (CUNY)

I also wish to extend a special acknowledgment to the administration of the CUNY Graduate Center for allowing me, as an older student, to re-matriculate in graduate school. It is ironic that twice I was expelled for being too young, and now that there were numerous delays in completing my doctorate after the fire, I was con-
sidered too old. It is with deep appreciation that I acknowledge the administration’s special concern in admitting an elderly student to allow the opportunity to fulfill a lifelong dream.

I also wish to extend a special “thank you” to my advisor, Dr. Kevin O’Bryant, and the other committee members, Dr. Richard Bumby and Dr. Melvyn Nathanson, who have not merely advised me mathematically but helped me and given much constructive criticism in many related areas, including, but not limited to how to do research on a topic, how to present and write a certain topic, how to give an oral presentation, and how to use \LaTeX.

Finally, there are two people who offered special help, and to whom this thesis is dedicated. The first of these two people is a neighbor who is a businessman, Mr. Moisha Binick, owner of Moisha’s Kosher Supermarket, who provided much needed financial support and encouragement. The second is my older sister Judy, who was constantly by my side to help me during her entire life and who was clearly my main source of assistance in every way possible as long as she lived.

I lost my job as a mainframe COBOL computer programmer in 2001 when the company hired a new head of IT and outsourced COBOL programming to India. Around the same time there was 9-11 and a downturn in the economy, so I could not find a job. Without my asking for it, Mr. Binick, a neighbor, decided to help, and supported our family since the day I lost my job for several years, until I found
another job. Not only did he provide direly needed assistance at a very high level, but he was the one who encouraged me to go back to graduate school after all these years to complete my much desired doctorate. In addition, I have since discovered he has single-handedly helped hundreds of families when the main bread earner was no longer alive or incapacitated or out of work. This record of human kindness is virtually unparalleled and is much appreciated. Currently, Mr. Binick is not well, and it is our heartfelt prayer that he be cured completely and enjoy many more years of good health with his family.

My older sister Judy was also an excellent student in math, and attended OSU in the doctoral program. She graduated Phi Beta Kappa, and soon entered graduate school. She was already a graduate assistant, teaching freshman calculus at age 19, being younger than many of her students. For several quarters running (they have quarters, not semesters at OSU), of over 80 sections of freshman calculus, her class had the highest average on both the departmental midterm and final exams, and was very well beloved by her students. She was such an understanding person that they often went to her for personal advice as well as for help in math. She got married as a graduate student, and although she had completed both written and oral exams, she found that to care for her children, she could not keep to the regular schedule demanded of the teaching staff. The OSU math department did not want to lose her so they then offered her a position in math advising, giving her more freedom of
hours. She had a heart of gold, and every student was like a beloved son or daughter. She was so popular as an advisor, that the line to her office literally went around the block, like the opening of a new Star Wars movie. In her own house, for months at a time, she often took in homeless people or families where the main breadwinner lost his job, and could not pay their rent. She died young of cancer, and after she passed away one student actually named her daughter after her! Her funeral was one of the largest I have ever seen, with almost the entire OSU math department attending. Since I was always interested in math, and had it in my blood, so to speak, my older sister often encouraged me and taught me more than was taught in school, so I too excelled in math. She was always willing to help me and encourage me in any endeavor and whenever help was needed, and for this reason, she is the main person in whose memory I dedicate my thesis.
Ann Judith Berenstein
March 28, 1945 - January 24, 2005
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Chapter 1

Introduction

1.1 Fundamental Results of Nathanson and O’Bryant

Nathanson [8] introduced and derived the basic properties of the function

\[ M_\theta(n) := \left\lfloor \frac{1}{\{\theta^{1/n}\}} \right\rfloor, \]

where \( \theta \) is a positive real number, \( n \) is an integer, and \( \lfloor \cdot \rfloor \) and \( \{ \cdot \} \) are the floor and fractional part functions, respectively. He also derived the basic properties of this function and identified symmetries that allow one to assume without loss of generality that \( \theta > 1 \) and that the integer \( n \) is positive. He obtained a number of surprising results, among them being that for any real \( \theta > 1 \) and integer \( n > \log_2 \theta \), either \( M_\theta(n) = \lfloor n/\log \theta - 1/2 \rfloor \) or \( M_\theta(n) = \lfloor n/\log \theta + 1/2 \rfloor \); moreover, if \( \log \theta \) is rational, then \( M_\theta(n) = \lfloor n/\log \theta - 1/2 \rfloor \) for all sufficiently large \( n \). He also mentioned a number of open questions for further research.

Perhaps the most amazing aspect of Nathanson’s paper was the fact that he
obtained significant results without the use of continued fractions. One would expect to see the use of continued fractions in the development, since the continued fraction algorithm produces a positive integer from a real number $\alpha > 1$ by taking the integer part of the reciprocal of the fractional part of $\alpha$, which is exactly how the function $M_\theta(n)$ operates on the $n$th root of a real number $\theta > 1$.

In a subsequent paper [9], O’Bryant gave alternate proofs of some of Nathanson’s results, sharpened or refined some of those results, solved most of the unsolved problems presented by Nathanson, and created a list of his own. By using information from continued fractions, O’Bryant’s methodology gave additional, and sometimes deeper, insights into Nathanson’s results.

O’Bryant showed that the set

$$\left\{ n \in \mathbb{N} : M_\theta(n) \neq \left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor \right\}$$

has density 0 for all $\theta > 1$, and for almost all $\theta > 1$ has counting function asymptotic to $\frac{\log \theta}{12} \log n$.

O’Bryant introduced an additional function and a set of atypical numbers: The set of positive integers is denoted $\mathbb{N}$. Throughout, we assume that $\theta > 1$ and that $n$ is a positive integer. If $n > \log_2 \theta$, then $1 < \theta^{1/n} < 2$, and so $\left\{ \theta^{1/n} \right\} = \theta^{1/n} - 1$. Set

$$M'_\theta(n) := \left\lfloor \frac{1}{\theta^{1/n} - 1} \right\rfloor,$$
so that $M_{\theta}(n) = M'_{\theta}(n)$ if $n > \log_2 \theta$. He called the elements of

$$\mathcal{A}_{\theta} := \left\{ n \in \mathbb{N} : M'_{\theta}(n) = \left\lfloor \frac{n}{\log \theta} + \frac{1}{2} \right\rfloor \right\}$$

the *atypical* numbers, meaning this set is relatively small for almost all $\theta$, as explained above and in O’Bryant’s Theorem 1 below. Almost all results and open problems are stated in terms of $\mathcal{A}_{\theta}$. For $n > \log_2 \theta$ and for $\theta < e^6 \approx 400$, he gave criteria for $\mathcal{A}_{\theta}$ to be finite or infinite in terms of the continued fraction expansions of $1/\log \theta$ and $2/\log \theta$. While Nathanson had proved (1.1) is finite whenever $\theta = e^{p/q}$, where $p/q$ is a rational number, O’Bryant gave another proof that gives an explicit bound on the size in terms of $p$ and $q$.

Our main results deal with determining when the atypical set will be finite or infinite for irrational $\log \theta$, and are based on certain parity patterns in the continued fraction of $1/\log \theta$.

### 1.2 O’Bryant’s Main Results

The following, with some omissions, are direct citations from O’Bryant’s paper that are needed for the following development. All proofs and most discussion are omitted, as the main need is for the statement of the theorems. (O’Bryant did indeed provide proofs, but they are omitted here.) The numbering of lemmas and theorems in this section and the next, follow O’Bryant’s numbering. While we use standard results concerning continued fractions, the only new results of O’Bryant that are
used directly are his Lemma 7 and 8, the others being included for completeness as well as to give a basic orientation to the concepts used.

We will using the same numbering for O’Bryant’s lemmas and theorems as he used, but for our own, every lemma and theorem, as well as every topic in each section, will be given another set of numbers.

Nathanson proved the following result, albeit in different notation.

**Theorem 1.** If \( n > \log_2 \theta \), then either

\[
M_\theta(n) = \left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor \quad \text{and} \quad n \notin \mathcal{A}_\theta
\]

or

\[
M_\theta(n) = \left\lfloor \frac{n}{\log \theta} + \frac{1}{2} \right\rfloor \quad \text{and} \quad n \in \mathcal{A}_\theta.
\]

An important consequence of this theorem is that for sufficiently large \( n \), \( M'_\theta(n) = M_\theta(n) \), and consequently, most results will be stated in terms of \( \mathcal{A}_\theta \).

**Theorem 2** (Nathanson). If \( \log \theta = p/q > 1 \) is rational, then

\[\mathcal{A}_\theta \subseteq [1, \frac{p^2}{6q})\].

**Theorem 3.** For all \( \theta > 1 \), \( \mathcal{A}_\theta \) has density 0.

**Theorem 4.** For almost all \( \theta > 1 \),

\[
|\mathcal{A}_\theta \cap [1, n]| \sim \frac{\log \theta}{12} \log n.
\]
Theorem 5. Let $a_i$ be positive integers with $a_{2k} = 1$ for $k \geq 0$. Set $\ell$ to be the irrational with simple continued fraction $[a_0; a_1, a_2, \ldots]$, and set $\theta = e^{2/\ell}$. Then $A_\theta = \emptyset$. In particular, if $c \in \mathbb{N}$ and $\theta = e^{-c+\sqrt{c(c+4)}}$, then $A_\theta$ is empty.

Theorem 6. Let $a_i$ be positive integers with $a_0 = 0$, $a_1 = 2$, $a_{2k} = 4$ for all $k \geq 1$. Set $\ell$ to be the irrational with simple continued fraction $[a_0; a_1, a_2, \ldots]$, and set $\theta = e^{2/\ell}$. Then $A_\theta$ is infinite. In particular, if $c \in \mathbb{N}$ and $\theta = e^{4-c+\sqrt{c(c+1)}}$, then $A_\theta$ is infinite.

These last two theorems give explicit uncountable families of $\theta$ with $A_\theta$ empty and infinite, of which the simplest examples are $A_{e^{\sqrt{5}-1}}$ which is empty and $A_{e^{2\sqrt{5}}}$ which is infinite. O’Bryant’s proofs are based upon inequalities using partial quotients of continued fractions. All his examples consisted entirely of transcendental numbers, and he notes that he did not know whether there were algebraic $\theta$ with $A_\theta = \emptyset$, nor whether there is an algebraic $\theta$ with $A_\theta$ infinite. By extending his methods, we are able to show that both types of algebraic $\theta$ exist.

For $t > 0$, he defines the function

$$f(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}.$$  

(1.2)

Lemma 7. For $t > 0$, the function $f(t)$ is strictly increasing, $\lim_{t \to 0^+} f(t) = 0$, and $\lim_{t \to \infty} f(t) = 1/2$. If $0 < t < 1$, then

$$\frac{t}{12} - \frac{t^3}{720} < f(t) < \frac{t}{12}.$$  

(1.3)
Lemma 8. Either

\[ M'_\theta(n) = \left\lfloor \frac{n}{\log \theta} - \frac{1}{2} \right\rfloor \quad \text{or} \quad M'_\theta(n) = \left\lfloor \frac{n}{\log \theta} + \frac{1}{2} \right\rfloor, \]

and \( M'_\theta(n) = \left\lfloor \frac{n}{\log \theta} + 1/2 \right\rfloor \) if and only if

\[ \frac{1}{2} - f\left(\frac{\log \theta}{n}\right) \leq \left\{ \frac{n}{\log \theta} \right\} < \frac{1}{2}. \]  \hspace{1cm} (1.4)

Note that Theorem 1 is a direct consequence of Lemma 8.

O’Bryant’s Lemma 11 is also important, but must be preceded by a brief notational introduction as it uses a slightly non-standard notation based upon some notation and theorems of Rockett and Szüsz [10] concerning continued fractions not found in other standard references.

If an irrational real number \( \alpha \) has the continued fraction \([a_0; a_1, a_2, \ldots]\), we shall define the \( k \)th convergent to be the rational number

\[ \frac{A_k}{B_k} := [a_0; a_1, a_2, \ldots, a_k] \]

where \( A_k \) and \( B_k \) are relatively prime positive integers. Also, we define

\[ \lambda_k := [0; a_{k-1}, a_{k-2}, \ldots, a_1] + [a_k; a_{k+1}, a_{k+2}, \ldots]. \]

The sequence of denominators, sometimes called continuants, satisfies \( B_k \geq F_{k+1} \), the \((k + 1)\)th Fibonacci number. Further,

\[ \frac{A_{2k-2}}{B_{2k-2}} < \frac{A_{2k}}{B_{2k}} < \alpha < \frac{A_{2k+1}}{B_{2k+1}} < \frac{A_{2k-1}}{B_{2k-1}} \]  \hspace{1cm} (1.5)
\[ \frac{\alpha - A_k}{B_k} = \frac{(-1)^k}{B_k^2 \lambda_{k+1}}. \] 

(1.6)

This is often used in conjunction with the trivial bounds

\[ a_{k+1} < \lambda_{k+1} < a_{k+1} + 2. \]

If \( m \) and \( n \) are natural numbers and

\[ \left| \alpha - \frac{m}{n} \right| \leq \frac{1}{2n^2}, \] 

(1.7)

then [10, Theorem II.5.1] there are integers \( k \geq 0, c \geq 1 \) such that \( m = cA_k \) and \( n = cB_k \) and \( \lambda_{k+1} > 2c^2 \).

**Lemma 11.** Let \( 1 < \theta < e^3 \) with \( \log \theta \) irrational, and \( a_k, B_k, \lambda_k \) be associated to the continued fraction of \( 2/\log \theta \). For each \( n \in A_\theta \), there exists positive integers \( c, k \) such that \( n = cB_{2k-1} \) and \( \lambda_{2k} > \frac{6c^2}{\log \theta} \).

Note too, that this result is based upon the boundedness of \( \theta \) and used to prove O’Bryant’s Theorems 5 and 6.

### 1.3 The Challenges of O’Bryant’s Third Unsolved Problem

O’Bryant’s third unsolved problem, which is our current problem, is: For every \( \theta_0 \), are there uncountably many \( \theta > \theta_0 \) with \( A_\theta \) finite?
His Lemma 8 shows that for \( n > \log_2 \theta \) to be atypical, \( \{n/\log \theta\} \) must lie in a small interval bounded above by 1/2, namely,

\[
\frac{1}{2} - f \left( \frac{\log \theta}{n} \right) \leq \left\{ \frac{n}{\log \theta} \right\} < \frac{1}{2},
\]

which we call (the standard) atypical interval. We will define \( L_\theta(n) := \frac{1}{2} - f \left( \frac{\log \theta}{n} \right) \), which is monotonically increasing to 1/2 as \( n \) tends to infinity, by O’Bryant’s Lemma 7. If \( \log \theta \) is irrational, then so is \( 1/\log \theta \), and therefore \( \{n/\log \theta\} \) is dense in \([0, 1)\).

It is then trivial that there are an infinite number of \( n \) such that \( \{n/\log \theta\} \) are slightly less than 1/2. However, the lower bound \( L_\theta(n) \) is not fixed, so when \( n \) is incremented by 1, two things happen—the value of \( \{n/\log \theta\} \) is increased by \( 1/\log \theta \), and the atypical interval shrinks slightly. Now incrementing \( n \) repeatedly by either \( \lfloor 1/\log \theta \rfloor \) or \( \lceil 1/\log \theta \rceil \) will cause \( \{n/\log \theta\} \) to go through a full cycle around the unit circle and land again, slightly to the left of 1/2, causing the difference to be small again. The question then arises as to whether “slightly to the left” means that it is in the critical atypical interval given by Lemma 8 or not, since the lower bound, \( L_\theta(n) \), also increased as \( n \) increased, and it is therefore possible that by incrementing \( n \), that the new \( \{n/\log \theta\} \) is not inside the new atypical interval but slightly to the left of its lower bound. This is a very delicate question and is one factor that makes O’Bryant’s third problem both challenging and interesting.

\[\text{The standard definition of a set } S \text{ being dense in an open interval means that for every } \beta \text{ in the interval, and for every small } \epsilon, \text{ there is a member of } S \text{ in an } \epsilon \text{ neighborhood of } \beta. \text{ By convention, we consider } [0, 1) \text{ to mean the unit circle, where it is an open set, and an } \epsilon \text{ neighborhood of 0 is } [0, \epsilon) \cup (1 - \epsilon, 0).} \]
A second challenging feature is the boundedness of irrational $\log \theta$, because in the proof Lemma 11, the boundedness of irrational $\log \theta$ plays a critical role, and, as a result, all thetas in the families of $\theta$ O’Bryant discovered meeting the hypotheses of Theorems 5 and 6 were also bounded.
1.4 The Statement of the Main Theorem

Definition 1.4.1. Let $\theta$ be any real number with an irrational log, with $\theta > 1$, with $\alpha := \frac{1}{\log \theta} = [a_0; a_1, a_2, \ldots]$, and having principal convergents $\{\frac{p_k}{q_k}\}_{k=0}^\infty$. The real number $\alpha$ is said to have the “even property” if and only if there exists some odd index $k$ such that $q_k$ is odd, $q_{k+1}$ is even, and for every even $j > k + 1$, $a_j$ is even. In this case, $\theta$ is said to be “special.” From the recursive formulae for principal convergents (Fact 3 of Section 2.2), $\theta$ being special is equivalent to there being at most a finite number of principal continuants of odd-index that are even.

The main theorem is:

Theorem (Main Theorem). Let $\theta$ be any real number with an irrational log, with $\theta > 1$, and with $\alpha := \frac{1}{\log \theta} = [a_0; a_1, a_2, \ldots]$.

(i) If $0 < \log \theta < 3$, then

(i.1) there exist uncountably infinite number of $\theta$ with $A_\theta$ empty;\footnote{O’Bryant proved (i.1) and (i.3) in his Theorems 5 and 6, and these facts are repeated here for completeness and because we give other proofs. In this paper we also present proofs of the other parts of this theorem.}

(i.2) there exist uncountably infinite number of $\theta$ with $A_\theta$ finite, but not necessarily empty; and,

(i.3) there exist uncountably infinite number of $\theta$ with $A_\theta$ infinite.

(ii) If $3 < \log \theta < 6$, and $\theta$ is not special, then $A_\theta$ is infinite.

(iii) If $\log \theta > 6$, then $A_\theta$ is always infinite, even if $\theta$ is special.
Chapter 2

Some Classical Results in Diophantine Approximation

2.1 Homogeneous and Inhomogeneous Diophantine Approximations

Hurwitz’s Theorem\(^1\) states that if \(\alpha\) is any irrational number then there are an infinite number of reduced fractions \(p/q\) such that \(|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}\), and this condition will be met only if the \(p/q\) is a principal convergent of the simple continued fraction expansion for \(\alpha\). This inequality is termed a “homogeneous Diophantine approximation,” and is often rewritten in another form, obtained by multiplying both sides by \(q\), because this form is sometimes more useful: If \(\alpha\) is any irrational number, then there are an infinite number of relatively prime integers, \(p\) and \(q\) such that \(|q\alpha - p| < \frac{1}{\sqrt{5}q}\). While there are irrational numbers \(\alpha\) for which the inequality \(|q\alpha - p| < \frac{1}{mq}\) is satisfied by an infinite number of \(g\) \(p\) and \(q\) for some \(m > \sqrt{5}\), there

\(^1\)This result may be found in any standard reference on continued fractions; however, [2, p. 3] states that it was proved earlier by Korkine and Zolotareff.
is no constant \( m > \sqrt{5} \) such that for every irrational \( \alpha \) there are an infinite number of such pairs. Markoff, however, did prove that there is a strictly increasing sequence of real numbers \( \{m_i\} \) whose supremum is 3, and with \( m_1 = \sqrt{5} \), such that, for each \( m_i \) there exists an infinite set of irrational numbers \( \alpha \), such that for each \( \alpha \) in the set corresponding to \( m_i \), the inequality \( |q\alpha - p| \leq \frac{1}{mq} \) is satisfied by an infinite number of relatively prime pairs of integers \((p,q)\) if and only if \( m \leq m_i \).

The approximation to \( \alpha \) stated in Hurwitz’s Theorem can be generalized to an “inhomogeneous” approximation: For any pair of real numbers \((\alpha, \beta)\), where \( \alpha \) is irrational, there exist an infinite number of pairs of integers, \((p,q)\) such that

\[
|q\alpha - p - \beta| < \frac{1}{4|q|}.
\]  

(2.1)

It is generally assumed that \( \beta \) is non-integral in this inhomogeneous expression, for, if \( \beta \) were integral, then \( |q\alpha - p - \beta| = |q\alpha - p'| \) where \( p' = p + \beta \) is an integer, causing the expression at hand to be a homogeneous approximation, in which case Hurwitz’s Theorem applies and the 4 in the denominator may be replaced with \( \sqrt{5} \). Basically this theorem was proven by Minkowski.\(^2\)

Grace made an improvement [5] [3, Vol. 2, p. 99], wherein he showed that the number “4” is sharp (or the “best bound” or “final,” based upon what terminology is in vogue in the place or time of writing), meaning the preceding claim, the inequality

\footnote{2Grace [5] cites Minkowski, \textit{Werke}, Vol 1, p. 320, which I did not see. This theorem, is mentioned in other sources, which are sometimes confusing because there are several related theorems called “Minkowski’s Theorem,” which are cited in various forms, some of which seem to be counter-intuitive. More detailed information is in Appendix B.}
of 2.1, is not true when 4 is replaced by a larger numerical constant. I put these two results together in the “Grace-Minkowski” theorem:

**Theorem** (Grace-Minkowski). Consider the inequality:

\[ |q\alpha - p - \beta| < \frac{k}{|q|}. \]  \hspace{1cm} (2.2)

(i) If \( k > 1/4 \), then for any pair of real numbers \((\alpha, \beta)\) where \( \alpha \) is irrational and \( \beta \) is not integral, there are an infinite number of pairs of integers \((p, q)\) that satisfy (2.2).

(ii) If \( k < 1/4 \), then for some choices for \((\alpha, \beta)\), where \( \alpha \) is irrational and \( \beta \) is not integral, (2.2) cannot be satisfied for an infinite number of pairs of integers \((p, q)\).

Specifically, if \( \beta = 1/2 \) and if the continued fraction expansion of \( \alpha = [0; a_1, a_2, \ldots] \) where \( a_1 \) is odd and for \( j > 1 \), the \( a_j \)'s are even and increasing, there are not an infinite number of pairs of integers \((p, q)\) that satisfy (2.2).

Since this theorem does not deal with the case \( k = 1/4 \), it is of interest to know what happens in that case, even though this information will not be relevant to our development here. Simply put, for some choices of \( \alpha \) there exist an infinite number of pairs of integers \((p, q)\) that satisfy (2.2), and for some choices of \( \alpha \) there are only a finite number of such pairs of integers. Grace [5] brought an example of a case, when, for appropriate choice of \( \alpha \), there are an infinite number of pairs of integers \((p, q)\) that satisfy (2.2).
2.2 Properties of Convergents

Other important facts concerning continued fractions and convergents are below, and they are mentioned in pairs, the first of each pair for a principal convergent, and the second for an auxiliary convergent. Some authors call auxiliary convergents intermediate fractions and others use the term “intermediants.” Unless stated otherwise, they are found in standard sources such as [7] or [10].

1. Definition of Principal Convergent:

The $j^{th}$ principal convergent of a positive irrational number $\alpha$ which can be expressed as a continued fraction $\alpha = [a_0; a_1, a_2, \ldots]$ for $j \geq 1$ is $[a_0; a_1, \ldots, a_j]$, which we will indicate by the reduced fraction $p_j/q_j$, where $p_j$ and $q_j$ are natural numbers.

2. Definition of Auxiliary Convergent:\(^3\)

Henceforth, let $c_j$ be any integer in the interval $(0, a_j)$, where $a_j > 1$, is the $j^{th}$ partial quotient of $\alpha$; then for any $j \geq 1$, any number of the form $[a_0; a_1, \ldots, a_{j-1}, c_j]$ is a $j^{th}$ auxiliary convergent or intermediate fraction, which we will indicate by the reduced fraction $p_{j,c}/q_{j,c}$, where $p_{j,c}$ and $q_{j,c}$ are natural numbers.

---

\(^3\)Based upon the author and the formula used, one or more of the endpoints may be considered valid values for $c_j$. The case of $c_j = 0$ is excluded here because it does not make sense for 0 to be a partial quotient. However, in cases where it does make sense for $c_j = 0$, such as in, Fact 4 below, the value of $c_j = 0$ will be allowed. Similarly the value of $c_j = a_j$ is sometimes allowed. Whenever $c_j = 0$ or $c_j = a_j$ is allowed, the auxiliary convergent produced is also a principal convergent.
Note: (1) While the \( j^{th} \) principal convergent is unique, by definition, there are typically multiple possibilities for a \( j^{th} \) auxiliary convergent.

(2) If \( c_j = 0 \) then we define \( p_{j,c}/q_{j,c} = p_{j-2}/a_{j-2} \), which is a principal convergent, and when \( c_j = a_j \), then \( p_{j,c}/q_{j,c} = p_j/a_j \), which is also a principal convergent. Hence, all principal convergents are also auxiliary convergents, and therefore, whenever we use the term “convergent” or the term “continuant” without being modified by the adjective “principal” or “auxiliary,” the meaning will be to include both, and when we use the term “auxiliary convergent,” we will mean an auxiliary convergent that is not a principal convergent.

3. Recursive Formulae for Principal Convergents:

If \( p_j/q_j \) is the \( j^{th} \) principal convergent, then for \( j \geq 2 \) the recursive formulae for generation of principal convergents are:

\[
p_j = a_j p_{j-1} + p_{j-2} \quad \text{and} \quad q_j = a_j q_{j-1} + q_{j-2}.
\]

Sometimes it is useful to define two “artificial convergents” by convention to be \( p_{-2} = 0, \ q_{-2} = 1, \ p_{-1} = 1, \ q_{-1} = 0 \) (even though there is no real number \( 1/0 \)), because with this convention, the recursive formulae now becomes valid for all integral \( j \geq 0 \).

4. Recursive Formulae for Auxiliary Convergents:

If \( p_i/q_i \) is an \( i^{th} \) principal convergent for \( i < j \) and \( p_{j,c}/q_{j,c} \) is a \( j^{th} \) auxiliary convergent, then for \( j \geq 2 \) and \( c_j \in [0, a_j] \) the recursive formulae\(^4\) for generation

\(^4\)These recursive formulae will play an important role in several places later. For more informa-
of a $j^{th}$ auxiliary convergent are $p_{j,c} = c_{j}p_{j-1} + p_{j-2}$ and $q_{j,c} = c_{j}q_{j-1} + q_{j-2}$.

5. Nature of Convergence for Principal Convergents:

The odd indexed principal convergents form a strictly decreasing sequence converging to $\alpha$, whereas the even-indexed ones form a strictly increasing sequence converging to $\alpha$, and for all $j$, $|\alpha - p_{j+1}/q_{j+1}| < |\alpha - p_{j}/q_{j}|$.

6. Nature of Convergence for Auxiliary Convergents:

If $c_{j}$ is any integer in the interval $[0, a_{j}]$, then the fraction $p_{j,c}/q_{j,c}$ defined by $p_{j,c} = c_{j}p_{j-1} + p_{j-2}$ and $q_{j,c} = c_{j}q_{j-1} + q_{j-2}$ is an auxiliary convergent or intermediate fraction. If $c_{j} = 0$, then $p_{j,c}/q_{j,c} = p_{j-2}/q_{j-2}$, and if $c_{j} = a_{j}$ then $p_{j,c}/q_{j,c} = p_{j}/q_{j}$; moreover, if $j$ is even, as $c_{j}$ increases, the sequence of intermediate fractions increases from $p_{j-2}/q_{j-2}$ to $p_{j}/q_{j}$, and if $j$ is odd, the sequence of intermediate fractions decreases from $p_{j-2}/q_{j-2}$ to $p_{j}/q_{j}$ as $c_{j}$ increases. Moreover, each intermediate fraction is the mediant between the $p_{j-1}/q_{j-1}$ and the previous intermediate fraction. [7, II, 6]

7. Accuracy of Approximation for Principal Convergents:

If $|\alpha - p/q| \leq 1/2q^{2}$, then $p/q$ is a principal convergent.

8. Accuracy of Approximation for Auxiliary Convergents:

If $|\alpha - p/q| \leq 1/q^{2}$, then $p/q$ is a convergent to $\alpha$. [4].

[7, II, 6]
9. Definition of \( \lambda \) for Principal Convergents:\(^5\)

\[
\lambda_j := [0; a_{j-1}, a_{j-2}, \ldots, a_1] + [a_j; a_{j+1}, a_{j+2}, \ldots].
\]

10. Definition of \( \lambda \) for Auxiliary Convergents:\(^6\)

If \( j > 1 \) and \( c \), is an integer in \([1, j - 1]\), then

\[
\lambda_{j,c} := [0; c, a_{j-1}, a_{j-2}, \ldots, a_1] + [0; a_j - c, a_{j+1}, a_{j+2}, \ldots].
\]

Note if \( c = 0 \) or \( c = a_j \), then \( \lambda_{j,c} \) is not defined.

11. Basic Fact About \( \lambda \)'s for Principal Convergents:

If \( p_j/q_j \) is a principal convergent then

\[
\alpha - \frac{p_j}{q_j} = \frac{(-1)^j}{q_j^2 \lambda_{j+1}}.
\]

12. Basic Fact About \( \lambda \)'s for Auxiliary Convergents: [5]

If \( p_{j,c}/q_{j,c} \) is an auxiliary convergent and \( c \in [1, a_j - 1] \), then

\[
\alpha - \frac{p_{j,c}}{q_{j,c}} = \frac{(-1)^j}{q_{j,c}^2 \lambda_{j+1,c}}.
\]

13. Upper and Lower Estimates for \( \lambda \)'s for Principal Convergents:

\( a_j < \lambda_j < a_j + 2. \)

14. Upper and Lower Estimates for \( \lambda \)'s for Auxiliary Convergents:

(i) If \( c \in [2, a_j - 2] \), then \( 0 < \lambda_{j,c} < 1. \) [5]

(ii) If \( c = 1 \) or \( c = a_j - 1 \), then \( 1 < \lambda_{j,c} < 2. \) [5]

\(^5\)Some texts label this expression to be \( \lambda_{j-1}. \)

\(^6\)Some texts use \( j - 1 \) as the subscript.
2.3 Grace’s Construction—proof of second part of Grace-Minkowski Theorem

In the Grace-Minkowski Theorem, in order to prove that “4 is sharp,” as defined earlier in 2.1, Grace constructed a family of α’s whose continued fractions have special properties causing (2.2) to have only a finite number of integral pairs making it true, and thereby produced an indirect proof that “4 is sharp.”

We restate and prove the second part of the Grace-Minkowski Theorem here, but first we define a set relating to the Grace contraction:

**Definition 2.3.1.** (Grace’s Set) \( G := \{[a_0; a_1, a_2, \ldots] : a_0 = 0, a_1 \text{ odd, and for } i \geq 2, a_i \text{ even and increasing} \} \).

**Theorem** (Grace-Minkowski, Part \( ii \)). Let \( \alpha = [0; a_1, a_2, \ldots] \in G \). If \( h < 1 \), then the inequality

\[
|q\alpha - p - \frac{1}{2}| < \frac{h}{4|q|}
\]

(2.3)

has only finitely many solutions in integers, \( p \) and \( q \).

**Proof.** We reproduce Grace’s original proof here, and later present some improvements. First, note in the case at hand, we may omit the absolute value sign around the \( q \), because if \( q < 0 \), we may define \( q' = -q \) and and \( p' = -p - 1 \). In this case \( q'\alpha - p' - 1/2 = -q\alpha - (-p - 1) - 1/2 \). Accordingly, without loss of generality, we may assume \( q \) to be positive by the appropriate change in \( p \), and we may therefore
drop the absolute value sign. The inequality (2.3) is then equivalent to

\[ \left| \alpha - \frac{2p + 1}{2q} \right| < \frac{h}{4q^2} = \frac{h}{(2q)^2}. \]  

(2.4)

We write

\[ \left| \alpha - \frac{r}{s} \right| < \frac{h}{s^2}, \]  

(2.5)

where \( r = 2p + 1 \) is an odd integer and \( s = 2q \) an even one.

If 4 is not sharp (as explained earlier), then there exists a number \( h < 1 \), such that there would then exist an infinite number of positive integral pairs \((p, q)\) that satisfy (2.4) for that particular \( h < 1 \).

This result, in turn, means \( \alpha \) can be approximated by a rational, \( \frac{2p + 1}{2q} = \frac{r}{s} \), to accuracy of the square of the denominator. By Fact 8, an approximation of this degree of accuracy, can only be attained by a fraction which is a convergent. [4, 5]

The conditions in the Grace-Minkowski Theorem part \((ii)\), given on the partial quotients of \( \alpha = [a_0; a_1, a_2, ...] \) imply that \( \alpha \) has no even principal continuant, as is easily seen from Fact 3, the Recursive Formula for Principal Convergents. In such a case, all principal continuants are odd, so there is no principal convergent satisfying (2.5).

Therefore, if there do exist integers \( p \) and \( q \) satisfying 2.4, the approximating fraction, \( \frac{2p + 1}{2q} \), must be an auxiliary convergent, and not a principal one, which we will call \( p_{n,c}/q_{n,c} \). We will now show that even though there may be infinitely many
even auxiliary continuants, it can be arranged that only finitely many of them satisfy (2.4).

From Fact 12, we know that
\[ |\alpha - \frac{p_{n,c}}{q_{n,c}}| = \frac{1}{\lambda_{n+1,c}q_{n,c}^2}. \]

Combining that fact with (2.4) yields \( \frac{1}{\lambda_{n+1,c}} < h \), i.e., \( \lambda_{n+1,c} > \frac{1}{h} > 1 \). By the definition of \( \lambda_{n,c} \) in Fact 10 (ii) above, we now have
\[ 1 < \lambda_{n+1,c} := [0; c, a_n, a_{n-1}, \ldots, a_1] + [0; d, a_{n+2}, a_{n+3}, \ldots] < \frac{1}{c} + \frac{1}{d}, \]
where \( c + d = a_{n+1} \).

From this fact it follows that either \( c = 1 \) or \( d = 1 \). Therefore,
\[ 1 < \lambda_{n+1,c} < \frac{1}{c} + \frac{1}{d} = 1 + \frac{1}{a_{n+1} - 1} \]
where \( c + d = a_{n+1} \).

Thus, using \( n \) in the place of \( n + 1 \), since \( n \) was any natural number, for any auxiliary convergent that satisfies (2.4) we have the following string of inequalities:
\[ 1 < \frac{1}{h} < \lambda_{n,c} < 1 + \frac{1}{a_n - 1} \]  \hspace{1cm} (2.6)

Since \( h \) was fixed and strictly less than 1, if \( \alpha \) were to be constructed in such a way that the \( a \)'s go to infinity, then \( \lambda_{n,c} \to 1 \), and \( h \) would thereby be forced to be 1, resulting in a contradiction. Therefore, (2.6) cannot hold for an infinite number of
n’s. Thus, it is possible to construct \( \alpha \) in such a way that only a finite number of pairs of integers \( (p, q) \) satisfy (2.4).

It should be noted, as will be seen later, that in applying this theorem to our problem, to construct an \( \alpha \) such that there are only a finite number of pairs of integers \( (p, q) \) satisfy (2.4), we only need that, at most, a finite number of odd-indexed convergents are even, so there are larger sets of \( \alpha \) that would work.

Comment: At first glance, Minkowski’s Theorem, as cited by Grace, seems counterintuitive, in that the constant 4 in (2.3) for an inhomogeneous approximation, is larger than the constant for a homogeneous approximation, namely \( \sqrt{5} \) in Hurwitz’s Theorem. However, this is not the case. In part (ii) of the Grace-Minkowski Theorem, \( \beta \) is replaced by 1/2 in the inhomogeneous inequality (2.2). Subsequently the fraction 1/2 is combined with \( p/q \) in the development from (2.3) to (2.5) changing the denominator of the fraction approximating \( \alpha \) from \( q \) to \( 2q \). Consequently, the denominator of the right side is simply the square of the denominator of the fraction approximating \( \alpha \), and not the denominator multiplied by 4 or some other constant. While the of constant for an inhomogeneous approximation is indeed 4 which is greater than \( \sqrt{5} \), the constant in the denominator of the homogeneous approximation, with the appearance of a new denominator, the “apparent” stronger constant of 4 now becomes 1, which, indeed is not as strong as \( \sqrt{5} \), as may be expected, intu-
It should also be pointed out that (2.5) does not merely define the degree of accuracy of the approximation, but it requires the numerator of the approximating fraction to be odd and the denominator to be even. Thus, while any principal convergent will satisfy inequality (2.5), it may not necessarily satisfy the concomitant parity requirements, thereby enabling a construction of irrational $\alpha$ that cannot be approximated by any principal convergent.

Let us now consider the case when $\beta$ is rational, but not $1/2$. Let $\beta = a/b$ (where $a$ and $b$ are relatively prime positive integers), $r = pb + a$, and $s = qb$, and we wish to investigate for what values of $k$, the following analogue of then (2.5) will have an infinite number of solutions in relatively prime integers $r$ and $s$:

$$\left| \alpha - \frac{r}{s} \right| < \left| \alpha - \frac{pb + a}{qb} \right| < \frac{k}{q^2} \leq \frac{b^2k}{(bq)^2} = \frac{h}{s^2}, \quad (2.7)$$

where $h = b^2k$. The Grace-Minkowski Theorem dealt with the case $b = 2$, and $h = 4k$, so as long as $h > 1$, that is, $k > 1/4$, there would be an infinite number of solutions. In the more general case, the minimal value for $b$ is 3, so $h \geq 9k$, and the condition to have an infinite number of solutions, namely $h > 1$, now means $k > 1/b^2 \geq 1/9$ in (2.7). In this sense, $\beta = 1/2$ is the “worst case scenario,” in that $k$ has the smallest possible range that guarantees an infinite number of solutions, $(r, s)$ to the inequality. However, in all cases, whenever $h \leq 1$ any reduced fraction satisfying

\[7\] Even if the constant 4 were incorrect, the main ideas in the theorems here would still be correct, with only the constants changing. More detailed information is in Appendix B.
2.7 must be a convergent. Also, the congruence conditions, namely, \( r \equiv a \pmod{b} \) and \( s \equiv 0 \pmod{b} \), must be met. When \( b = 2 \), it is the congruence conditions are simple parity conditions that can easily be controlled by controlling the partial quotients of irrational \( \alpha \). However, when \( b \geq 3 \), the congruence conditions cannot be controlled in such a trivial fraction through the partial quotients to insure only a finite number of solutions. In addition, when \( h > 1 \), the congruence conditions make it more difficult to prove there will be an infinite number of solutions. When \( h > 1 \), there does not seem to be any obvious way to cause or to prevent auxiliary convergents or near-convergents\(^8\) from having the divisibility properties, whenever \( \beta \) is a rational number other than \( 1/2 \).

2.4 Extensions of Grace’s Construction

We now define the critical value of a partial quotient for a given value of \( h \). To obtain the contradiction in the proof of the preceding theorem we needed that, for fixed \( h \), the number of partial quotients that satisfy (2.6) is finite. This leads to the following definition:

**Definition 2.4.1.** The critical value for \( h \) is \( A(h) := \min \left\{ a \in \mathbb{N} : 2|a & a \geq \frac{h}{1-h} + 1 \right\} \).

\(^8\)The term “near-convergent” is not clearly defined. As stated earlier, for fixed positive real numbers, \( \alpha \) and \( h \leq 1 \), the existence of a pair of relatively prime integers, \( p, q \) that satisfy the inequality \( |\alpha - p/q| < \frac{h}{q^2} \Rightarrow p/q \) is a principal convergent or auxiliary convergent to \( \alpha \). If, however, \( h > 1 \), then there are other fractions that satisfy this inequality, and they are called “near-convergents.”
Lemma 2.4.1. $A(h) = 2\left\lceil \frac{1}{3-2h} \right\rceil$.

Proof. Noting that $\frac{h}{1-h} + 1 = \frac{1}{1-h}$, if we define $B(h) := \min\{a \in N : a \geq \frac{h}{1-h} + 1\}$, that is to say, that we do not require $B(h)$ to be even, then $A(h) = 2\left\lceil \frac{B(h)}{2} \right\rceil = 2\left\lceil \frac{1}{2-2h} \right\rceil$.

Theorem 2.4.1 (Grace-Minkowski, extended). Consider the inequality

$$|q\alpha - p - 1/2| < \frac{h}{4q}, \quad (2.8)$$

If $h < 1$, and if $\alpha = [0; a_1, a_2, ...]$ where $a_1$ is odd and the other partial quotients are even, and, at most, a finite number of them are less than the critical value for $h$, namely $A(h)$, then the inequality (2.8) has only finitely many integral solutions $(p, q)$.

Proof. First, the hypothesis $h < 1$ means $1 < 1/h$. Second, in Grace’s original work, he obtained a contradiction to (2.6) by making all the partial quotients approach infinity. This condition, however, is stronger than necessary. All that is really needed is that

$$a_n \geq \frac{h}{1-h} + 1$$

is false a finite number of times, which is equivalent to at most a finite number of $a_n < A(h)$, as stated in the theorem. This extension provides us with a set larger than $G$ for which the inequality (2.3) has a finite number of solutions. \qed
(It should be noted that when \( A(h) = 2 \), there will be no partial quotients that are even and less than \( A(h) \), implying the inequality (2.8) will have no integral solutions \((p,q)\). When can this situation occur? As will be seen later, we will be most concerned with the case when \( 0 < \log \theta < 3 \), which means that \( h = \log \theta / 3 < 1 \) and that \( A(h) = 2 \left\lceil \frac{1}{2 - \frac{1}{2} \log \theta} \right\rceil \). Then, the smallest \( A(h) \) will be is 2, which occurs if \( \log \theta < 3/2 \).

For larger values of \( A(h) \), the number of possible values for a partial quotient grows since \( A(h) \) grows without bound through the even numbers as \( \log \theta \nearrow 3 \).)

From here it is immediate that if the partial quotients of \( \alpha \) are eventually periodic and all greater or equal to \( A(h) \), that \( \alpha \) will be a quadratic irrational, and hence algebraic, thereby providing us with the following corollary.

**Corollary 2.4.1a.** Consider the inequality

\[
|q\alpha - p - \frac{1}{2}| < \frac{h}{4q}. \tag{2.9}
\]

Then for each \( h \in (0, 1) \), there exists an infinite set of algebraic irrational \( \alpha \)'s, namely those for which \( \alpha = [0; a_1, a_2, ...] \), where

1. \( a_1 \) is odd,
2. the other partial quotients are even,
3. at most a finite number of the \( a_n < A(h) \), and
4. where the partial quotients are eventually periodic.

such that, for each of these algebraic irrational \( \alpha \)'s, the inequality (2.9) has only
finitely many integral solutions \((p, q)\).

**Corollary 2.4.1b.** Consider the inequality (without absolute value signs)

\[
0 < q\alpha - p - \frac{1}{2} < \frac{h}{4q}
\]

(2.10)

Then for each \(h \in (0, 1)\), there exists an infinite set of irrational \(\alpha\)'s, namely those for which \(\alpha = [0; a_1, a_2, ...]\), where

1. \(a_n\) is even whenever \(n\) is odd,
2. at most a finite number of the \(a_n < A(h)\), and

such that, for each of these irrational \(\alpha\)'s, the inequality (2.10) has only finitely many integral solutions \((p, q)\). Moreover, if, in addition, the partial quotients are eventually periodic, then for each \(h \in (0, 1)\) all these \(\alpha\)'s are algebraic, and for each of these algebraic irrational \(\alpha\)'s, the inequality (2.10) has only finitely many integral solutions \((p, q)\).

**Proof.** As seen from Tables 3, 4, 5, and 6 in Appendix A, the recursive formulae for convergents (Facts 3 and 4) guarantee that if the odd-indexed convergents are even, then all even-indexed continuants are odd. Fact 5, the fact that even-indexed convergents are less that the number being approximated, namely \(\alpha\), enable us to delete the absolute value sign in the hypotheses of Theorem 2.4.1 and Corollary 2.4.1a, thereby producing this corollary. (The periodicity of the partial quotients, of course, insures that \(\alpha\) is a quadratic irrational and therefore algebraic.)
Corollary 2.4.1c. Consider the inequality

\[ 0 < -1 \left( q\alpha - p - \frac{1}{2} \right) < \frac{h}{4q} \quad (2.11) \]

Then for each \( h \in (0, 1) \), there exists an infinite set of irrational \( \alpha \)'s, namely those for which \( \alpha = [0; a_1, a_2, ...] \), where

1. \( a_1 \) is odd,
2. for \( n > 1 \), \( a_n \) is even whenever \( n \) is even,
3. at most a finite number of the \( a_n < A(h) \),

and each of these irrational \( \alpha \)'s, inequality (2.11) has only finitely many integral solutions \((p, q)\). Moreover, if, in addition, the partial quotients are eventually periodic, then for each \( h \in (0, 1) \), there exists infinitely many algebraic irrationals \( \alpha \)'s, such that inequality (2.11) has only finitely many integral solutions \((p, q)\).

Proof. The recursive formulae for convergents (Facts 3 and 4) guarantee that the parity of the partial quotients as defined in the hypothesis will produce odd continuants whose indices are odd. Fact 5, the fact that odd-indexed convergents are more than the number being approximated, namely \( \alpha \), enable us to replace the absolute value sign by \(-1\) in the hypotheses of Theorem 2.4.1 to produce this corollary. \( \square \)

It should be noted that in both the last two corollaries, the parity conditions on the partial quotients are sufficient, but not necessary, to produce the desired results.
Chapter 3

The Main Theorems

3.1 The Relationship Between Atypical numbers and the Grace-Minkowski Theorem

In the Grace-Minkowski Theorem and our extensions in Section 2.4, we found conditions on a positive irrational number $\alpha$ and on a positive real number $k$ so that there are either a finite number or infinite number of integer pairs $p, q$ that satisfy the inequality, $\left| q\alpha - p - \frac{1}{2} \right| < \frac{k}{q}$, or the equivalent inequality, $\left| \alpha - \frac{2p + 1}{2q} \right| < \frac{h}{(2q)^2}$.

We wish to apply these results to O’Bryant’s Lemma 8 which states for fixed $\theta$, that $n \in A_{\theta} \Leftrightarrow \left\{ \frac{n}{\log \theta} \right\} \in \left[ L_{\theta}(n), \frac{1}{2} \right)$, where $L_{\theta}(n) = \frac{1}{2} - f \left( \frac{\log \theta}{n} \right)$. To do so, we will call $\left[ L_{\theta}(n), \frac{1}{2} \right)$ the standard atypical interval and introduce two new functions, $L'_{\theta}(n)$ and $L''_{\theta}(n)$, which satisfy the inequality $L'_{\theta}(n) < L_{\theta}(n) < L''_{\theta}(n) < \frac{1}{2}$.

When $L_{\theta}(n)$ is replaced by either $L'_{\theta}(n)$ or $L''_{\theta}(n)$ in the standard atypical interval, we get what called the extended atypical interval and the contracted atypical interval, respectively. It follows from the construction of the expanded and contracted
intervals that if, for fixed \( \theta \), there are no \( n \), or at most a finite number of them, such that \( \left\{ \frac{n}{\log \theta} \right\} \) is in the expanded atypical interval, then \textit{a foritori} there are no \( n \), or at most a finite number of them, with \( \left\{ \frac{n}{\log \theta} \right\} \) in the standard atypical interval. Therefore, by O’Bryant’s Lemma 8, \( A_\theta \) is empty or finite. Similarly, if there are an infinite number of \( n \) such that the contracted interval contains an infinite number of \( \left\{ \frac{n}{\log \theta} \right\} \), then, \textit{a foritori}, there are an infinite number of them in the standard atypical interval, and hence, \( A_\theta \) is infinite. It should be noted that the Grace-Minkowski Theorem cannot be applied directly to the standard atypical interval arising from Lemma 8, because of the complexity of its left endpoint. However, the endpoints of the expanded and contracted intervals are, algebraically speaking, easier to deal with, and the Grace-Minkowski Theorem is applied to them. Since these problems are stated using the letter \( n \) to represent a possibly atypical number, we will use \( n \) in the place of \( q \), where \( q \) appears in the Grace-Minkowski theorem.

\textbf{Lemma 3.1.1.} Let \( \theta > 1 \). For each \( n \in A_\theta \) there exists a unique pair of integers \( p \) and \( q \) such that \( p = \left\lfloor \frac{n}{\log \theta} \right\rfloor \), \( q = n \) and

\[ \left| \frac{q}{\log \theta} - p - \frac{1}{2} \right| = \left| \frac{q}{\log \theta} - \frac{2p + 1}{2} \right| < \frac{\log \theta}{12q}, \quad (3.1) \]

or, equivalently,

\[ \left| \frac{1}{\log \theta} - \frac{2p + 1}{2q} \right| < \frac{\log \theta}{12q^2} = \frac{\log \theta}{3(2q)^2}. \]

\textit{Proof.} In O’Bryant’s Lemma 8 above, we stated that, for fixed \( \theta \), \( n \) is atypical,
by definition, when \( M'_\theta(n) = \lfloor n / \log \theta + 1/2 \rfloor \) for \( n > \log_2 \theta \), and this occurs if and only if \( \frac{1}{2} - f \left( \frac{\log \theta}{n} \right) \leq \left\{ \frac{n}{\log \theta} \right\} < \frac{1}{2} \). By using \( t = \frac{\log \theta}{n} \) and the upper estimate for \( f(t) < t/12 \) in O'Bryant's Lemma 7, we can now produce an expanded atypical interval (which contains the atypical interval as a proper subset), namely
\[
\left( \frac{1}{2} - \frac{\log \theta}{12n}, \frac{1}{2} \right),
\]
meaning \( L'_\theta(n) := \frac{1}{2} - \frac{\log \theta}{12n} \). We now apply Grace's idea to this new interval:

Accordingly, we reformulate what it means for \( n \) to be atypical in a fashion that will enable us to apply Grace's idea. Thus, \( n \) is atypical means
\[
\left\{ \frac{n}{\log \theta} \right\} \in \left[ \frac{1}{2} - f \left( \frac{\log \theta}{n} \right), \frac{1}{2} \right) \subset \left( \frac{1}{2} - \frac{\log \theta}{12n}, \frac{1}{2} \right).
\]
Or, more simply, using the fact that \( \left\{ \frac{n}{\log \theta} \right\} = \frac{n}{\log \theta} - \left\lfloor \frac{n}{\log \theta} \right\rfloor \), we now have
\[
\frac{1}{2} - \frac{\log \theta}{12n} < \frac{n}{\log \theta} - \left\lfloor \frac{n}{\log \theta} \right\rfloor < \frac{1}{2}.
\]
Subtracting \( 1/2 \) from all three sections yields
\[
-\frac{\log \theta}{12n} < \frac{n}{\log \theta} - \left\lfloor \frac{n}{\log \theta} \right\rfloor - \frac{1}{2} < 0. \tag{3.2}
\]
We note that \( p = \left\lfloor \frac{n}{\log \theta} \right\rfloor \) is equivalent to
\[
-\frac{1}{2} \leq \frac{n}{\log \theta} - p - \frac{1}{2} < \frac{1}{2}. \tag{3.3}
\]
Since \( M'_\theta(n) \), by definition, means \( n > \log_2 \theta \), and since
\[
n > \log_2 \theta > \frac{\log \theta}{6} \Rightarrow \frac{1}{2} > \frac{\log \theta}{12n} \Rightarrow \frac{1}{2} < -\frac{\log \theta}{12n},
\]
it follows that we may replace \( \left\lfloor \frac{n}{\log \theta} \right\rfloor \) in (3.2) with \( p \).
Setting \( q = n, \alpha = \frac{1}{\log \theta}, \beta = \frac{1}{2}, p = \lfloor q\alpha \rfloor = \left\lfloor \frac{n}{\log \theta} \right\rfloor \) in (3.2), and keeping in mind that it is necessary to alternate between these two notations, we now have an inequality in the format used in the Grace-Minkowski Theorem:

\[
-\frac{\log \theta}{12q} < q\alpha - p - \beta < 0. \tag{3.4}
\]

Multiplying by \(-1\) yields

\[
0 < p + \beta - q\alpha = \left(\frac{1}{2} + \left\lfloor \frac{n}{\log \theta} \right\rfloor - \left\lfloor \frac{n}{\log \theta} \right\rfloor\right) = |q\alpha - p - \beta| < \frac{\log \theta}{12q} = \frac{\log \theta}{12n}. \tag{3.5}
\]

Thus, we have

\[
0 < \frac{1}{2} - \left\{ \frac{n}{\log \theta} \right\} < \frac{\log \theta}{12n}. \tag{3.6}
\]

We have now transformed the notation for atypical \( n \) in O'Bryant’s Lemma 8 to a notation similar to the one used by Grace (2.2).

\( \square \)

Note: We actually proved a little bit more: The absolute value is used in the Grace-Minkowski Theorem, and, for the purpose of parallelism, in the inequality above (3.1) we also used absolute value signs. However, they are unnecessary in that what we really proved is that if \( n \) is atypical we must have \((-1)^n \left( \frac{1}{\log \theta} - \frac{2p + 1}{2q} \right) < \frac{\log \theta}{12q^2}\), meaning the negative branch of the absolute value statement is true.

Corollary 3.1.1a. If \( \theta > 1 \) and \( n > \log_2 \theta \), then for each \( n \in A_\theta \), there exist relatively prime integers \( r, s = 2n \), such that

\[
\left| \frac{1}{\log \theta} - \frac{r}{s} \right| < \frac{\log \theta}{3s^2}. \tag{3.7}
\]
Proof. Setting \( q = n \) in Lemma 3.1.1, we know that if \( n \in \mathcal{A}_\theta \), then

\[
\left| \frac{1}{\log \theta} - \frac{2p + 1}{2n} \right| < \frac{\log \theta}{12n^2} = \frac{h}{(2n)^2},
\]

where \( h = \frac{\log \theta}{3} \), \( r = 2p + 1 \), and \( s = 2n \) give the desired result. Without loss of generality we may assume \( r \) and \( s \) are relatively prime, for if not, there exist integers \( c, R, S, N \) where \( R \) and \( S \) are relatively prime, \( r = cR, s = cS, n = cN \), \( c \) is odd, and \( S = 2N \). Then inequality (3.7) becomes

\[
\left| \frac{1}{\log \theta} - \frac{r}{s} \right| = \left| \frac{1}{\log \theta} - \frac{R}{S} \right| < \frac{\log \theta}{3s^2} < \frac{\log \theta}{3(cS)^2} < \frac{\log \theta}{3S^2},
\]

and inequality (3.8) becomes

\[
\left| \frac{1}{\log \theta} - \frac{2p + 1}{2n} \right| = \left| \frac{1}{\log \theta} - \frac{2p+1}{2n} \right| < \frac{\log \theta}{12n^2} = \frac{h}{(2n)^2} = \frac{h}{(2cN)^2} < \frac{h}{(2N)^2}.
\]

\( \square \)

**Corollary 3.1.1b.** If \( \theta > 1 \) and \( n \in \mathcal{A}_\theta \), then there exists a reduced fraction, \( r/s \), such that each of the following three conditions hold

i. The absolute value inequality condition: \( \left| \frac{1}{\log \theta} - \frac{r}{s} \right| < \frac{\log \theta}{3s^2} \).

ii. The parity condition: \( r \) is odd and \( s \) is even, and, in fact, \( n = s/2 \).

iii. The “overestimate” condition: \( \frac{r}{s} - \frac{1}{\log \theta} > 0 \),

which we may also call “The negative part of the absolute value condition”

because it is equivalent to \( \left| \frac{1}{\log \theta} - \frac{r}{s} \right| = (-1) \left( \frac{1}{\log \theta} - \frac{r}{s} \right) \).
Proof. This result follows immediately from the preceding corollary and Lemma 3.1.1.
It is important to note several things:

- Also, we could have stated all three conditions more succinctly as “there exists a reduced fraction, \( r/2n \), such that \( 0 < \frac{r}{2n} - \frac{1}{\log \theta} < \frac{\log \theta}{12n^2} \),” but we chose not to do so, because the use of three distinct conditions will be more useful for the development that follows.

- From Corollary 3.1.1b, the value of \( \log \theta/3 \) is seen to be quite important: As previously mentioned, we know from the basic properties of continued fractions that if an irrational number \( \alpha \) can be approximated by a fraction \( r/s \) so that \( \left| \alpha - \frac{r}{s} \right| \leq \frac{h}{s^2} \) where \( h < 1 \), then \( r/s \) is a (auxiliary) convergent to \( \alpha \), and moreover, if \( h \geq 1/\sqrt{5} \) there will an infinite number of approximating fractions, \( r/s \), for any irrational number \( \alpha \). When \( h < 1/\sqrt{5} \), the existence of an infinite number of fractions, \( r/s \), approximating \( \alpha \) will vary, based upon the value of \( \alpha \), and may be controlled by appropriate choice of partial quotients for \( \alpha \). [7, Theorem II.8.21] From Corollary 3.1.1b, it is clear that the absolute value inequality condition will be satisfied for an infinite number of relatively prime pairs of integers \( (r, s) \) if \( \log \theta \geq \frac{3}{\sqrt{5}} \approx 1.31464 \) or \( \theta > 3.73 \), approximately, and therefore the existence of infinite number of atypical \( n \) would depend on whether or not the other two conditions are satisfied. In addition, if \( \log \theta < 3 \), the fractions satisfying the absolute value inequality condition must be convergents by Fact 8, thereby enabling us to control whether or not \( A_\theta \) is infinite by
controlling the continued fraction of \( \frac{1}{\log \theta} \). However, when \( \log \theta > 3 \), the approximating fractions may be near-convergents, whose existence cannot readily be controlled through the continued fraction of \( \frac{1}{\log \theta} \).

- The Grace-Minkowski Theorem and the original Grace Construction dealt with two cases, and produced different results depending on whether \( h < 1 \) or \( h > 1 \). When \( h > 1 \) there were always an infinite number of \( q = n \) that satisfied the inhomogeneous inequality (2.2). However, when \( h < 1 \), whether or not there were an infinite number of solutions to the inhomogeneous inequality (2.2) depended upon \( \alpha \), which we could construct so there would be either a finite or an infinite number of solutions based upon our controlling the partial quotients of the continued fraction convergents to \( \alpha \).

- In applying these results to our problem, we will soon consider both the case when \( h < 1 \), or equivalently in our problem, \( \log \theta < 3 \), and \( h > 1 \), or \( \log \theta > 3 \). It is important to bear in mind that the Grace-Minkowski Theorem and the Grace Construction deal with the existence of a finite or infinite set of numbers that satisfy an absolute-value inequality, but the current problem deals with a direct inequality (as opposed to an absolute-value one), corresponding to the negative value of the absolute value or odd-indexed convergents (as mentioned in Corollary 3.1.1b part \( iii \)). Thus, the Grace-Minkowski Theorem would insure
that only a finite number of solutions to the inequality in the expanded atypical interval would imply that there are only a finite number of solutions to our current problem. This situation is dealt with in Section 3.2. However, the converse is not necessarily true, for an infinite number of solutions to the Grace-Minkowski inequality would not necessarily insure the existence of an infinite number of convergents with odd indices (and whose denominators are even).¹ A partial converse, though, does exist, and is dealt with in Section 3.3, where we introduce the contracted atypical interval.

### 3.2 A Study of Conditions on \( \theta \) for \( A_\theta \) to be Finite

**Lemma 3.2.1** (Existence of \( p \) and \( q \)). If \( n > \log_2 \theta \), and if \( n \) is atypical, then there exists a pair of integers, \( p \) and \( q = n \), such that

\[
\left| q\alpha - p - \frac{1}{2} \right| = \left| q\alpha - \frac{2p + 1}{2} \right| < \frac{\log \theta}{12q}, \tag{3.11}
\]

**Proof.** The result follows immediately from Lemma 3.1.1 and Corollary 3.1.1a. \qed

**Theorem 3.2.1** (Bounded \( \theta \)). Let \( 0 < \log \theta < 3 \) and \( \alpha = \frac{1}{\log \theta} \). If \( \alpha = [a_0; a_1, a_2, \ldots] \), where \( a_0 = \lfloor \alpha \rfloor \), \( a_1 \) is an odd natural number, and the rest of the \( a \)'s are all even and tend to infinity, then \( A_\theta \) is finite, but not necessarily empty.

¹Nevertheless, an infinitude of solutions to our current problem would imply an infinitude to the Grace-Minkowski inequality.
Proof. By hypothesis, $\log \theta < 3$, so we have $h < 1$. We now have two cases: In the event $|\alpha| = a_0 = 0$, that is to say, $0 < \alpha < 1$, $\alpha$ is one of the numbers produced by the Grace Construction, and we may now apply Lemma 3.1.1 to obtain the result that the expanded atypical interval contains only finite atypical $n$, and therefore the basic atypical interval does also.

However, since $\log \theta$ may be close to 0, it is certainly possible that $|\alpha| \neq 0$. In this case, we write $\alpha = |\alpha| + \{\alpha\}$, and therefore (3.11) becomes

$$|q|A + q\{\alpha\} - p - 1/2| < \frac{\log \theta}{12q} \quad (3.12)$$

Since $|p - q|A|$ is an integer we may label it as $p'$, and if we define $\alpha' := \{\alpha\}$, the above can be rewritten as

$$|q\alpha' - p' - 1/2| < \frac{\log \theta}{12q} \quad (3.13)$$

Since $0 < \alpha' < 1$ and $h < 1$, it follows that $\alpha'$ is one of the numbers produced by the original Grace Construction. In this case we can again utilize Lemma 3.1.1 to obtain the result that the expanded atypical interval contains only finite atypical $n$, and therefore the basic atypical interval does also.

It should be noted that this theorem will hold for any set of $\alpha$’s produced by an extended version of Grace’s Construction, such as one that has only a finite number of convergents with even denominators.
The next task is to find conditions for the denominators of odd-indexed convergents to be even in order to determine when $A_\theta$ is empty or finite but not empty. More specifically, the goal of the next four lemmas is to determine under what conditions:

1. The number of principal continuants of odd-index that are even is finite.
2. The number of all continuants of odd-index that are even is finite.
3. There are no even principal continuants of odd-index.
4. There are no even continuants of odd-index.

For these lemmas, we refer to the tables and State Diagram in Appendix A. The labels given to the different possibilities and cases delineated in these lemmas come from the cases described in Appendix A.

Using E for even and D for odd, we consider four cases: The previous two denominators will either be EE, ED, DE, or DD, and are labeled cases 1, 2, 3, 4 respectively. Each has two subcases; subcase A occurs when the next partial quotient is even, and subcase B occurs when the next partial quotient is odd, giving 8 cases altogether. We make basic observations from looking at the tables and State Diagram in Appendix A. The following for lemmas are presented without full proof because they follow from these observations using only simple parity arguments. However, some comments relating to these observations and parity arguments are included at the end of Appendix A.
Lemma 3.2.2a (Conditions for number of even principal continuants of odd-index to be finite). Let $\alpha = [a_0; a_1, a_2, \ldots]$. At most a finite number of principal continuants of odd-index are even if and only if any one of three situations occurs:

A. For some index $k$, $q_{k-2}$ and $q_{k-1}$ are both odd and for all $n \geq k$, $a_n$ is even. 
(In other words, at some point on, we are always in Case 4A. Hence, we will call this case “Perpetual Case 4.”)

B. For some odd index $k$ such that the previous two principal continuants have the same parity as their index, and for all even $n \geq k$, $a_n$ is even. (From some point on, we are always alternating between Cases 2A and 3. Hence, for short, we will call this the “Alternating Case.”)

C. The partial quotients are such that sometimes we are in the situation of Perpetual Case 4 and sometimes in the situation of Alternating Cases 2A and 3, namely:

Either we are in Perpetual Case 4 and all $a_n$ are even up to a point, but there exists some odd index $n$, where $a_n$ is odd causing us to exit the Perpetual Case 4 and to enter the Alternating Case,

Or we are in the Alternating Case and for some even index $n$, $a_n$ is odd, causing us to exit the Alternating Case and to enter the Perpetual Case 4. (For short, we will call this case, the “Mixed Case.”)

Lemma 3.2.2b (Conditions for number of any even continuants of odd-index to be finite). Let $\alpha = [a_0; a_1, a_2, \ldots]$. At most a finite number of all continuants (both
principal and auxiliary) of odd-index are even if and only if there exists an odd index 
k such that the previous two principal continuants have the same parity as their 
index, and for all even \( n \geq k \), \( a_n \) is even. (This is analogous to 1.B, above.)

The next two lemmas are parallel to the two preceding ones with the difference 
being that the preceding ones were concerned with determining the circumstances for 
which the number of certain continuants is finite, and the following lemmas are con-
cerned with determining the the circumstances for which the number of continuants 
is zero.

We introduce the next lemma with three observations and define three cases, as 
was done in introducing Lemma 3.2.2a. For simplicity we use the same terminology, 
but with slightly different meanings because we are not merely limiting the number 
of continuants to be finite, but that there should be no continuants at all meeting 
certain conditions.

**Lemma 3.2.2c** (Conditions for no even principal continuants of odd-index). Let 
\( \alpha = [a_0; a_1, a_2, \ldots] \). There are no even principal continuants of odd-index if and only 
if any one of three situations occurs:

A. \( a_1 \) is odd, and for \( n \geq 2 \), \( a_n \) is even. (For the purpose of this lemma, we now 
call this case 'Perpetual Case 4A.”)

B. \( a_1 \) is odd, and for all even \( n \geq 2 \), \( a_n \) is even. (For the purposes of this lemma 
we now call this case “Alternating Cases 2A and 3.”)
C. The partial quotients are such that sometimes we are in the situation of Perpetual Case 4 and sometimes in the situation of Alternating Cases 2A and 3, namely,

Either we are in Perpetual Case 4 and all \( a_n \) are even up to a point, but there exists some odd index \( n \), where \( a_n \) is odd causing us to exit the Perpetual Case 4 and to enter the Alternating Case,

Or we are in the Alternating Case and for some even index \( n \), \( a_n \) is odd, causing us to exit the Alternating Case and to enter the Perpetual Case 4.

Lemma 3.2.2d (Conditions for no even continuants of odd-index). Let \( \alpha = [a_0; a_1, a_2, \ldots] \).

There are no even continuants (either principal and auxiliary) of odd-index if and only if \( a_1 = 1 \) and and for all even \( n \geq 2 \), \( a_n \) is even. (This is analogous to 3.B, above.) There are no even continuants (both principal and auxiliary) of odd-index if and only if there exists an odd index \( k \) such that the previous two principal continuants have the same parity as their index, and for all even \( n \geq k \), \( a_n \) is even. (This situation is analogous to situation B in Lemma 3.2.2a above.)

It should be noted that if \( a_1 \) is odd, and all the other partial quotients are even, we will always be in Case 4, and therefore all denominators of all principal convergents will be odd, but it is not possible for all denominators of all auxiliary convergents to be odd—this was the case of the Grace Construction.

Theorem 3.2.2. \( (A_\theta = \emptyset) \). Let \( 0 < \log \theta < 3 \) and \( \alpha = \frac{1}{\log \theta} \). If \( \alpha = [a_0; a_1, a_2, \ldots] \),

where \( a_1 = 1 \), \( a_2 \) is odd, and \( a_n \) is even whenever \( n > 2 \) is even, then \( A_\theta = \emptyset \).
Proof. By Lemma 3.2.1 there exist pairs of integers \((p, q)\) satisfying inequality (3.11). Since any fractions that satisfy (3.11) must be principal or auxiliary convergents by Fact 8. By Lemma 3.2.2a, the necessary conditions of Corollary 3.1.1a are not fulfilled by any convergents, so \(A_\theta\) must be empty.

\[\Box\]

**Corollary 3.2.2** (Finite \(A_\theta\)). Let \(0 < \log \theta < 3\) and \(\theta\) is special, then \(A_\theta\) is finite, but not necessarily empty.

**Proof.** The proof follows immediately from the definition of “special” and Lemma 3.2.2b.

\[\Box\]

**Summary:** O’Bryant has shown in his Theorem 5 (and Lemma 11 on which it is based), that for \(0 < \log \theta < 3\), an appropriate choice of the convergents of \(\alpha = 1/\log \theta\) enables us to construct \(\alpha\) so that \(A_\theta\) is finite (or even empty). We have extended these results and further shown why the upper bound of 3 for \(\log \theta\) is necessary. In particular, we have shown:

(i) If \(\log \theta < 3\) and irrational, then a necessary, but not sufficient, condition for \(n\) to be atypical is that there are integers \(p\) and \(q = n\) such that

\[
|q\alpha - p - \frac{1}{2}| < \frac{\log \theta}{12q},
\]

or, equivalently,

\[
|\alpha - \frac{2p + 1}{2q}| < \frac{\log \theta}{3(2q)^2}.
\]
(ii) The only possible numbers that could satisfy the above inequality are one-half of the denominators (if they are even) of the principal or auxiliary convergents to $1/\log \theta$.

(iii) There are three necessary conditions for $n > \log_2 \theta$ to be atypical: There is a fraction approximating $\alpha$ that meets the absolute value inequality condition, the parity condition is met (the numerator is odd and the denominator is even), and the estimating fraction must be an over-estimate.

(iv) The Grace Construction, or any other construction that makes all but a finite number of odd-indexed continuants to be odd, will insure the number of continuants that yield\textsuperscript{2} atypical numbers is finite.

(v) The denominators of auxiliary convergents can be prevented from being atypical, except in, at most, a finite number of cases, by the Grace Construction or any other construction that insures only a finite number of partial quotients are less than some specified number.

(vii) O’Bryant’s Theorem 5 provides another construction for the number of the denominators that are atypical to be finite—in fact, zero. It is not clear how his construction relates to those brought here.

(vii) Also, it is not known if other constructions exist that will cause the number of

\textsuperscript{2}Meaning that they are even and one-half of them is the atypical number.
denominators of auxiliary convergents that are atypical to be finite.

3.3 A Study of Conditions on $\theta$ for $A_\theta$ to be Infinite

Our next task is to show that whenever $h > 1$, or equivalently, $\log \theta > 3$, $A_\theta$ will always be infinite for $\theta$ if its log is irrational, with the possible exception of special $\theta$, and even if a $\theta$ with irrational log is special, but $\log \theta > 6$, then $A_\theta$ will always be infinite. To achieve this goal we need some more lemmas.

**Lemma 3.3.1.** For any $\theta$ and any $\epsilon > 0$, there exists a real number $N = N(\theta, \epsilon)$ such that

$$n \geq N \Rightarrow \frac{\log \theta}{(12 + \epsilon)n} \leq \frac{\log \theta}{12n} - \frac{(\log \theta)^3}{720n^3}.$$ 

**Proof.** Consider the inequality

$$\epsilon \left[1 - \frac{(\log \theta)^2}{60n^2}\right] > \frac{(\log \theta)^2}{5n^2}.$$ 

For fixed $\theta$ and $\epsilon$, as $n$ goes to infinity, the left side approaches $\epsilon$ and the right side goes to 0. Therefore, we have this sequence of inequalities, each equivalent to the above, and which will be true for sufficiently large $n$:

$$\epsilon > \frac{(\log \theta)^2}{5n^2} + \frac{\epsilon(\log \theta)^2}{60n^2}.$$ 

Divide by 12 and re-arrange terms:

$$\frac{\epsilon}{12} > \frac{(\log \theta)^2}{60n^2} + \frac{\epsilon(\log \theta)^2}{720n^2}.$$ 

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Add 1 and re-arrange:

\[
1 < 1 + \frac{\epsilon}{12} - \frac{(\log \theta)^2}{60n^2} - \frac{\epsilon(\log \theta)^2}{720n^2}
\]

\[
1 < \frac{12 + \epsilon}{12} - \frac{(\log \theta)^2}{60n^2} - \frac{\epsilon(\log \theta)^2}{720n^2}
\]

Multiplying both sides by

\[
\frac{\log \theta}{(12 + \epsilon)n}
\]

and re-arranging terms, gives the desired result. \(\square\)

**Corollary 3.3.1.** For any \(\theta\) and any \(\epsilon > 0\), there exists a natural number \(N = N(\theta, \epsilon)\) such that if \(n > N\), then

\[
\left(\frac{1}{2} - \frac{\log \theta}{(12 + \epsilon)n}, \frac{1}{2}\right) \subset \left[\frac{1}{2} - f\left(\frac{\log \theta}{n}\right), \frac{1}{2}\right] \subset \left(\frac{1}{2} - \frac{\log \theta}{12n}, \frac{1}{2}\right)
\]

**Proof.** Recall that O’Bryant’s Lemma 7 states

\[
\frac{t}{12} - \frac{t^3}{720} < f(t) < \frac{t}{12},
\]

and therefore

\[
\frac{1}{2} - \frac{t}{12} < \frac{1}{2} - f(t) < \frac{1}{2} - \frac{t}{12} + \frac{t^3}{720}
\]

Setting \(t = \log \theta/n\), and applying Lemma 3.3.1, for sufficiently large \(n\), the previous line now becomes

\[
\frac{1}{2} - \frac{\log \theta}{12n} < \frac{1}{2} - f(t) < \frac{1}{2} - \frac{\log \theta}{12n} + \frac{\log \theta^3}{720n^3} < \frac{1}{2} - \frac{\log \theta}{(12 + \epsilon)n} < \frac{1}{2}
\]
Now recall that for log $\theta$ irrational, O’Bryant’s Lemma 8 states
\[
\left\{ \frac{n}{\log \theta} \right\} \in \left[ \frac{1}{2} - f \left( \frac{\log \theta}{n} \right), \frac{1}{2} \right]
\]
implies $n$ is atypical. Therefore, for sufficiently large $n$, \textit{a fortiori}
\[
\left\{ \frac{n}{\log \theta} \right\} \in \left[ \frac{1}{2} - \frac{\log \theta}{(12 + \epsilon)n}, \frac{1}{2} \right] \Rightarrow \left\{ \frac{n}{\log \theta} \right\} \in \left[ \frac{1}{2} - f \left( \frac{\log \theta}{n} \right), \frac{1}{2} \right]
\]
implies $n$ is atypical.

Thus, for fixed $\epsilon > 0$, we define $L''_{\theta,\epsilon}(n) := \frac{1}{2} - \frac{\log \theta}{(12 + \epsilon)n}$, and we will consider
\[
\left( \frac{1}{2} - \frac{\log \theta}{(12 + \epsilon)n}, \frac{1}{2} \right)
\]
to be a contracted atypical interval. However, we do not want the definition of $L''$ to be dependent on $\epsilon$, so we wish to define both $L''$ and the contracted interval without this dependency. To do so we will chose a specific $\epsilon$, dependent only on $\theta$ as follows: When log $\theta > 3$ and $\epsilon = \frac{1}{4}(\log \theta - 3)$, we will use the notation $N_{\theta}$ to mean $N(\theta, \epsilon)$, and an algebraic computation shows this is equal to
\[
1 \frac{\sqrt{15 \log \theta} \sqrt{(\log \theta - 3)(\log \theta + 45)}}{30 \log \theta - 3}
\]
Since we may later need to apply some of the preceding concepts for $0 < \log \theta < 3$, we will define
\[
N_{\theta} := \frac{1}{30} \frac{\sqrt{15 \log \theta} \sqrt{|\log \theta - 3|(\log \theta + 45)}}{|\log \theta - 3|},
\]
which is defined and bounded in any closed interval in $(0, \infty)$ that does not include 3. We now define $L''_{\theta}(n) := \frac{1}{2} - \frac{\log \theta}{(12 + \frac{|\log \theta - 3|}{4})N_{\theta}}$ and the contracted interval is defined to be $\left( L''_{\theta}(n), \frac{1}{2} \right)$.
Theorem 3.3.1. If \( \theta \) is not special, has an irrational \( \log \), and \( \log \theta > 3 \), then \( A_\theta \) is infinite.

Proof. By Lemma 3.1.1, we have the inequality for any atypical \( n \):

\[
\left| \frac{n}{\log \theta} - p - \frac{1}{2} \right| < \frac{\log \theta}{12n}.
\]  

(3.14)

Furthermore, because we are using \( n \) in the place of \( q \) and \( p \) in the place of \( \left\lfloor \frac{n}{\log \theta} \right\rfloor \), it follows that the above, in turn, is equivalent to

\[
\left| \left\{ \frac{n}{\log \theta} \right\} - \frac{1}{2} \right| < \frac{\log \theta}{12n}.
\]  

(3.15)

Now, since \( \log \theta > 3 \) means \( \frac{\log \theta}{12} > 1/4 \), so the Grace-Minkowski Theorem part (i) applies. Accordingly, there are infinitely many such pairs of integers, \( (p, q) \) that satisfy 3.14. Specifically, there are pairs of relatively prime integers, \( r, s \), where \( r = 2p + 1 \) and \( s = 2q \) that satisfy 3.14. These fractions \( r/s \) can be either principal convergents or auxiliary convergents or near-convergents to \( \alpha \). If \( \theta \) is not special, then there will an infinite number of pairs of such numbers that are convergents. If \( \theta \) is special, then, at most a finite number of continuants are even, so the infinite number of pairs will be near-convergents. The following development applies to regular convergents. There is little known about near-convergents, and although it is possible that the following development applies to them as well, it is not clearly known whether that is the case, so we have an exclusion for special \( \theta \) to the rest of this proof.
Each pair produces an \( n = q \), such that \( \left\{ \frac{n}{\log \theta} \right\} \) is in the expanded atypical interval. In the absence of a proof to the contrary, it is possible that all but a finite number of these \( \left\{ \frac{n}{\log \theta} \right\} \) are in the expanded part of the expanded interval, and therefore there would be no proof that the number of atypical \( n \) is infinite. The truth, however, is just the opposite—all but a finite number of the \( \left\{ \frac{n}{\log \theta} \right\} \) are in the standard atypical interval, and, at most a finite number, are in the expanded part. Our next task is to prove that claim.

Since there are infinitely many \( q \) in the Grace-Minkowski Theorem, which correspond to \( n \), it is clear that \( n \) approaches infinity, and therefore, at some point \( n > N(\theta, \epsilon) \) for any choice of small positive \( \epsilon \). Now, if we choose \( \epsilon < \frac{1}{2}(\log \theta - 3) \), and if \( n \) is sufficiently large, that is \( n > N(\theta, \epsilon) \), then by Corollary 3.3.1, inequality (3.15) becomes

\[
\left| \left\{ \frac{n}{\log \theta} \right\} - \frac{1}{2} \right| < \frac{\log \theta}{(12 + \epsilon)n} = \frac{k}{n},
\]

where \( k = \frac{\log \theta}{(12 + \epsilon)} > 1/4 \), because of the restriction on \( \epsilon \). Hence, the Grace-Minkowski Theorem part (i) still applies, and there are an infinite number of pairs of relatively prime integers, \( r = 2p + 1, s = 2q \), and thus, also a pair of integers \( (p, q) \), each producing an \( n = q \) so inequality 3.16 is true.

We still do not know, however, that there are an infinite number of atypical \( n \), because this inequality contains an absolute value, which causes the inequality to branch into two portions, a positive one (where the absolute value signs are simply
removed), and a negative portion where the removal of the absolute value signs is accompanied by a multiplication by $-1$. As mentioned before, we are looking for odd-indexed convergents, meaning we are interested in the negative portion of the inequality. Grace only provides us with the information that there are an infinite number of solutions to the absolute value inequality, and, at first glance, they may all lie in the positive portion. Yet, we need that the negative portion of the absolute value in 3.14 has an infinitely number of solutions. However, the solutions are all convergents to $\alpha$, including the odd-indexed ones. These odd-indexed ones constitute an infinite number of solutions for the negative piece of the absolute value. Thus, the positive portion of the inequality would correspond to the even-indexed convergents, and the odd-indexed ones would correspond to the negative portion of the inequality, thereby satisfying both (3.11) and (3.6) (which does not have an absolute value sign). Furthermore, since $\theta$ is not special there are an infinite number of fractions (convergents) $r/s$ where both the index of the convergent is odd and $s$ is even.

**Lemma 3.3.2.** For any positive irrational number $\alpha$, there are infinite number of reduced fractions $p/q$ with odd numerators satisfying $0 < \frac{p}{q} - \alpha < \frac{1}{q^2}$.

**Proof.** We know that all all odd-indexed principal convergents to $\alpha$ satisfy this inequality. Unless $\alpha$ has the even property, there will be an infinite number of odd-indexed convergents (either principal or auxiliary) that satisfy the preceding
inequality and whose denominators are even, and since all convergents are reduced fractions, *per force*, their numerators must be odd.

If, however, \( \alpha \) does have the even property, then, according to Lemma 3.2.2b, as seen from the following chart, every even-indexed partial quotient \( a_n \) must be even; the denominators must alternate with \( q_n \) having the same parity as \( n \), and the numerators of even-indexed convergents must be odd since the denominators are even, and convergents are reduced fractions. The odd-indexed partial quotients are marked with asterisks to indicate, that because of the recursive formula, both odd and even partial quotients of odd-index will produce an odd \( q_n \) of odd index.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n - 2 )</th>
<th>( n - 1 )</th>
<th>( n )</th>
<th>( n + 1 )</th>
<th>( n + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>Even</td>
<td>Odd</td>
<td>Even</td>
<td>Odd</td>
<td>Even</td>
</tr>
<tr>
<td>( p_n )</td>
<td>Odd</td>
<td>(Even)</td>
<td>Odd</td>
<td>(Even)</td>
<td>Odd</td>
</tr>
<tr>
<td>( q_n )</td>
<td>Even</td>
<td>Odd</td>
<td>Even</td>
<td>Odd</td>
<td>Even</td>
</tr>
</tbody>
</table>

Given this scenario, we proceed by indirect proof to show that it is not possible for all odd-indexed numerators to be even from some point on. We have indicated the attempt to make them even by putting the word “Even” in parenthesis in the chart above for the numerators.

If we further wish to insure that from some point on, all odd-indexed numerators will be odd, the numerators will then also have an alternating odd-even pattern like the denominators do, except that they will be of opposite parity. This situation can
occur if and only if every odd-indexed partial quotient is even, and the even ones are indeterminate ("asterisks"). To be consistent with what already exists, this additional feature is only possible if, from some point on, all partial quotients are even. Since every even \( a_n \geq 2 \), it follows that \( a_n - 1 \geq 1 \), and therefore 1 is a valid value for \( c_n \) (and so is \( a_n - 1 \) when \( a_n > 2 \)). Therefore we can let \( c_n \) be 1 or \( a_n - 1 \), causing the intermediate fraction (or auxiliary partial quotient)

\[
\frac{p_{n,c}}{q_{n,c}} = \frac{p_{n-2} + c_np_{n-1}}{q_{n-2} + c_nq_{n-1}}
\]
to have an odd numerator. Also, when \( c_n = 1 \) or \( a_n - 1 \), from Facts 12 and 14, we have both \( 1 < \lambda_{n,1} < 2 \) and

\[
\frac{1}{2q_{n,c}^2} < \frac{1}{\lambda_{n,c}q_{n,c}^2} = \frac{p_{n,c}}{q_{n,c}} - \alpha < \frac{1}{q_{n,c}^2}
\]

so an auxiliary convergent also satisfies the inequality \( 0 < \frac{p}{q} - \alpha < \frac{1}{q^2} \). Thus, while we can arrange it so no odd-indexed continuant (principal or auxiliary) is even, we cannot simultaneously arrange that both the odd-indexed numerators of these convergents will be even for both all principal convergents or auxiliary convergents. Then, there will always be an infinitude of odd-indexed convergent numerators that are odd and satisfies the inequality. \( \square \)

**Theorem 3.3.2.** If \( \log \theta \) is irrational and \( \log \theta > 6 \), then \( A_\theta \) is infinite even if \( \theta \) is special.

**Proof.** The preceding theorem shows \( A_\theta \) is infinite if \( \theta \) is not special. If, however, \( \theta \) is special, there are only a finite number of convergents that meet the second and third conditions, the parity condition and the over-estimate condition. However, the Grace-Minkowski Theorem does guarantee there are an infinite number of
fractions that meet the first condition, the absolute value inequality condition; hence, these must be near convergents. It is our goal to prove that an infinite subset of these near convergents meet the second and third conditions as well whenever \( \log \theta > 6 \), so that one-half of these denominators will be atypical numbers yielding the result that \( A_\theta \) is infinite.

Using the fact that \( \log \theta > 6 \), it has already been shown that if the parity condition is met (and \( s \) is even), then any reduced fraction \( r/s \) (not necessarily a convergent, but possibly a near-convergent) satisfying

\[
0 < \frac{r}{s} - \frac{1}{\log \theta} < \frac{\log \theta}{s^2} < \frac{2}{s^2} \Rightarrow n := s/2 \in A_\theta. \tag{3.17}
\]

If \( p/q \) is an odd-indexed convergent to \( \frac{2}{\log \theta} \), then \( 0 < \frac{p}{q} - \frac{2}{\log \theta} < \frac{1}{q^2} \). Dividing all by 2 yields \( 0 < \frac{p}{2q} - \frac{1}{2q^2} < \frac{1}{2q^2} = \frac{2}{(2q)^2} \). Setting \( r = p, n = q, s = 2q = 2n \), we now have

\[
0 < \frac{r}{s} - \frac{1}{\log \theta} < \frac{2}{s^2} < \frac{\log \theta}{s^2},
\]

and therefore the inequality will be satisfied by any odd-indexed convergent, \( r/s \) to \( 1/\log \theta \). If, in addition, \( p = r \) is odd, then all the conditions of 3.17 are met, in particular the parity condition, and \( n \) is atypical. However, it is necessary to insure that \( p \) is odd, lest the 2 in the denominator of \( \frac{p}{2q} \) cancel with the \( p \) in the numerator and the denominator is no longer even. This adverse possibility, though, is excluded by the Lemma 3.3.2, because either there are an infinite number of even
denominators (which make the corresponding numerators odd) and which means \( \theta \) is not special, or if \( \theta \) is special, there will be an infinite number of odd-indexed convergents to \( \frac{1}{\log \theta} \) with odd numerators, \( r = p \). Therefore, the fraction \( \frac{p}{2q} \) is reduced, and since \( s = 2q \), we now have that \( s/2 = q = n \in \mathcal{A}_\theta \).

Combining the last three theorems produces the main theorem stated in Section 1.4. While it seems likely that \( \mathcal{A}_\theta \) is infinite for \( \log \theta \in (3, 6) \), even when \( \theta \) is special, in order that \( \mathcal{A}_\theta \) were to be infinite in such a case it would be necessary for an infinite number of denominators of near-convergents \( r/s \) to be even and for \( r/s \) to be an upper estimate for \( \alpha \) (conditions ii and iii of Corollary 3.1.1b). It is an open problem whether or not such near-convergents exist.

Summary:

(i) By the Grace-Minkowski Theorem, the following inequality will always have an infinite number of integral solutions, \( p,q \), for every irrational, non-special, number \( \alpha \) whenever \( k > 1/4 \). If however, \( k \leq 1/4 \), it is possible that there are either a finite or an infinite number of such solutions, based upon what \( \alpha \) is.

\[
\left| q\alpha - p - \frac{1}{2} \right| < \frac{k}{q}.
\]

(ii) By Lemma 3.1 and Theorem 3.3.1, if \( \alpha = 1/\log \theta \) where \( \log \theta > 3 \) and irrational so that \( k := \frac{\log \theta}{12} > 1/4 \), then for each \( n \in \mathcal{A}_\theta \) there exists a unique pair of
integers \( p, q \), where \( p = \lfloor n / \log \theta \rfloor \) and \( q = n \) that satisfy the preceding inequality. In fact, this inequality now becomes

\[
\left| \alpha - \frac{2p + 1}{2q} \right| < \frac{\log \theta}{3q^2},
\]

(3.18)

Hence, if \( \theta \) is not special, \( A_\theta \) is infinite, and every such atypical \( n \) will be one-half the denominator of some fraction that approximates \( \alpha \).

(iii) If \( \log \theta > 3 \) and irrational, the only possible numbers that could be atypical are one-half of even denominators of the principal or auxiliary convergents to \( 1 / \log \theta \), or possibly some near-convergents. Unlike the case of \( \log \theta < 3 \), or equivalently \( k < 1/4 \), it is not possible to arrange the continued fraction of \( \alpha \) so that at most a finite number of approximants satisfy the above inequality, because the Grace-Minkowski inequality insures that there will be an infinite number of solutions to the absolute value inequality—and if they do not come from the denominators of principal or auxiliary convergents, they must come from other fractions, such as near-convergents.

(iv) If one-half the denominator of a convergent to \( 1 / \log \theta \), then there are three necessary conditions for it to be atypical, viz., the absolute value inequality is met, the denominator must be even, and the convergent has an odd-index.

(v) There is no known set of conditions that are sufficient for the denominator of a near-convergent to satisfy (3.18) and therefore to be atypical.
3.4 Existence of Irrational Algebraic $\theta$ for which $A_\theta$ is Finite or Infinite

In the process of solving O’Bryant’s third problem, we have also solved O’Bryant’s fifth problem. This problem asks “Is there any algebraic $\theta$ for which $A_\theta$ can be proved finite? Infinite?” The existence of infinite families of algebraic $\theta$ for which $A_\theta$ are finite were proven in Corollaries 2.4.1a, 2.4.1b, and 2.4.1c; and the existence of infinite families of algebraic $\theta$ for which $A_\theta$ are infinite were proven as part of Theorem 3.3.1, since any $\theta$ whose log is irrational and greater than 3 will have infinite $A_\theta$.

3.5 A Study of $A_{e^e}$

We now provide an affirmative answer for O’Bryant’s first problems which asks “Is $A_{e^e}$ infinite?”

Theorem 3.5.1. If $\alpha = 1/\log \theta$ is irrational and there are an infinite number of pairs of integers $p, q$, that satisfy

$$|q\alpha - p - \frac{1}{2}| < \frac{\log \theta}{12q},$$

and if there are an infinite number of partial fraction convergents, $r/s$, to $1/\alpha$ of odd-index and with $s$ even, then $A_\theta$ is infinite.

Proof. The proof of this theorem is almost identical to that of Theorem 3.3.1. The
only difference is that in Theorem 3.3.1, the hypothesis \( \log \theta > 3 \) was needed to prove that there are an infinite number of relatively prime pairs \( n, p \) such that 3.14 is true. The hypothesis of this theorem, using \( q \) in the place of \( n \), states that there are an infinite number of such pairs, and therefore, the restriction of \( \log \theta > 3 \) is no longer necessary.

**Theorem 3.5.2.** \( \mathcal{A}_e \) is infinite, and, in fact,

\[
\mathcal{A}_e = \{ q_j/2 : j \equiv 1, 3 \pmod{6}, j \neq 1 \}, \text{ where } \frac{P_j}{q_j} \text{ is the } j^{\text{th}} \text{ principal convergent of } 1/e.
\]

**Proof.** It is well known that the simple continued fraction expansion of

\[ e = [2; 1, 2, 1, 4, 1, 1, 6, 1, 1, 8, ...] \text{ where for } j \geq 1, \]

- if \( j = 3m + 2 \), then \( a_j = 2(m + 1) \), for \( m \geq 0 \);
- otherwise, \( a_j = 1 \).

Similarly, if \( \theta = e^e \), then \( \alpha = \frac{1}{\log \theta} = \frac{1}{e} \), and also the simple continued fraction of

\[ \alpha = [0; 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...] \text{ where for } j \geq 2, \]

- if \( j = 3m \), then \( a_j = 2m \), for \( m \geq 1 \);
- otherwise, \( a_j = 1 \).

The Grace Construction was a successful method for insure that the number of solutions to (2.5) is finite only because the parity conditions implicit in the inequality enabled one to define partial quotients that did not produce even
continuants. In this situation, the recursive relation for convergents, starting with $n = 1$, means that $q_n$ follows the pattern of EDEDDD, which repeats forever, where “D” stands for “odd” and “E” stands for “even.” This means we also have a repeating pattern for the cases also, of period 6, namely Case 2, 3, 2, 4, 4, 3. We also have, except for the case of $n = 1$ that $a_n$ follows the pattern of DDEDDE.

The end result is that $q_n$ is even for odd-indexed convergents if and only if $n$ is congruent to 1 or 3 mod 6. Hence, there are an infinite number of atypical $n$ by the preceding theorem, namely one-half of these continuants. A computerized check gives the first five atypical numbers as 1, 4, 53, 632, 12973, which are exactly equal to $q_j/2$ for $j = 1, 3, 7, 9, 13$.

While the above argument suffices to solve O’Bryant’s first problem by showing $\mathcal{A}_e$ is infinite, it does not suffice to prove the set of atypical $n$ is precisely these numbers, and no others. When $\theta = e^e$, $h := \log \theta/3 = e/3 \in (1/\sqrt{5}, 1)$ means the only candidates for an atypical $n$ are one-half odd-indexed denominators (that are even) of principal or auxiliary convergents; hence, it is possible that one-half of even denominators of auxiliary convergents of odd-index could also be atypical numbers. We will now show that no such numbers exist, that is, if an auxiliary denominator is even, it either has an even index or does not meet the absolute value inequality condition.

First we realize that if $a_j = 1$ that there is no $j^{th}$ auxiliary convergent. Thus, all
auxiliary convergents must come from an $a_j$ which is even, that is $j \equiv 1, 3 \pmod{6}$.

From Facts 10, 12, and 14, we know that if an auxiliary convergent, $\frac{p_{j,c}}{q_{j,c}}$ produces an atypical $n = \frac{q_{j,c}}{2}$ then $\frac{1}{\lambda_{j,c}^2} \leq h < 1$, which means $\lambda_{j,c} > 1$, and therefore $c$ or $d = a_j - c = 1$.

Second, if $j \equiv 3 \pmod{6}$, the two preceding denominators are even and odd, which means if $c = 1$ and $d = a_j - 1$ will both be odd and therefore $q_{j,c}$ will be odd, which is not divisible by 2, so no atypical number is produced.

Third, if $j \equiv 6 \pmod{6}$, then $q_j, c$ is an even-indexed auxiliary continuant, but only odd-indexed ones produce atypical $n$. Hence, auxiliary continuants, cannot produce atypical $n$; only principal continuants can.
Chapter 4

Some Partial results

In the preceding chapter we produced solutions to three of O’Bryants five problems. We do not have a full solution for the other two problems. However we do have some partial results and some indications of what might be a way to approach these problems. This information is presented in this chapter.

4.1 Symmetric Differences

O’Bryant’s second problem is “Does there exist a pair of positive real numbers, \((\theta, \tau)\), with both \(A_\theta\) and \(A_\tau\) infinite, such that the symmetric difference \(A_\theta \triangledown A_\tau\) is finite?” Since the claim is trivially true if \(\theta = \tau\), we assume without loss of generally that they are not equal. The following three conditions together are necessary and sufficient to prove the existence of such a pair:

(i) \(I := A_\theta \cap A_\tau\) is infinite;
(ii) \(A_\theta \setminus I\) is finite; and
(iii) $A_r \setminus I$ is finite.

While we do not have a complete answer to this question, we present two approaches, each with one theorem giving partial results.

### 4.1.1 First Approach

Here is a theorem giving a necessary, but not sufficient, condition, for $(i)$ to be true:

**Theorem 4.1.1.** If $n \in A_\theta \cap A_\tau$, then there exist a real number $h \geq 1$, positive real numbers $k$ and $\phi$ and a positive integer $m$, such that the inequality

$$0 < \frac{m}{n} - \phi \leq \frac{h}{2n^2}$$

is satisfied by each $n \in I$. Moreover, if $I$ is infinite, then (4.1) is satisfied by an infinite number of $n$, where $m/n$ is any odd-indexed convergent to $\phi$. When $\log \theta + \log \tau \leq 6$, then $h = 1$, and the only solutions for $m/n$ are the odd-indexed convergents, and when $\log \theta + \log \tau > 6$, then $h < 1$, and there are fractions, $m/n$, other than convergents, that satisfy (4.1).

**Proof.** We will use the notation of Section 3.1, and we will also call $\theta' := \tau$ and use primes for the corresponding numerics relating to $\tau$. Using the notation and result of (3.5), we have:

$$n \in A_\theta \Rightarrow (p + 1/2 - n\alpha) < \frac{\log \theta}{12n}$$

(4.2)
Further, if we let \( \alpha' = (k + 1)\alpha \) for some real number \( k \geq 0 \), then we also have

\[
n \in \mathcal{A}_{\theta'} \Rightarrow (p' + 1/2 - n\alpha') = (p' + 1/2 - n(k + 1)\alpha) < \frac{\log \theta'}{12n}. \tag{4.3}
\]

Adding these two inequalities together produces

\[
n \in I \Rightarrow 0 < (p + p' + 1) - n(k + 2\alpha) < \frac{\log \theta + \log \theta'}{12n}. \tag{4.4}
\]

Now, since \( n \) is atypical for both \( \theta \) and \( \theta' \), meaning both \( \{n/\log \theta\} < 1/2 \) and \( \{n/\log \theta'\} < 1/2 \), and since we also have \( p = \lfloor n\alpha \rfloor = \lfloor n/\log \theta \rfloor \) and \( p' = \lfloor n\alpha' \rfloor = \lfloor n/\log \theta' \rfloor \), the sum

\[
p + p' = \lfloor n\alpha \rfloor + \lfloor n\alpha' \rfloor = \lfloor n\alpha \rfloor + \lfloor n(k + 1)\alpha \rfloor = \lfloor n(k + 2)\alpha \rfloor.
\]

This means (4.4) now becomes

\[
0 < (\lfloor n(k + 2)\alpha \rfloor + 1) - n(k + 2\alpha) < \frac{\log \theta + \log \theta'}{12n}. \tag{4.5}
\]

Define \( m := \lfloor n(k + 2)\alpha \rfloor + 1 \) and \( \phi := n(k + 2\alpha) \), and divide both sides by \( n \), giving

\[
0 < \frac{m}{n} - \phi < \frac{\log \theta + \log \theta'}{12n^2}. \tag{4.6}
\]

Note that if \( k = 0 \), then \( \log \theta' = \log \theta \) and the previous inequality becomes

\[
0 < \frac{m}{n} - \phi < \frac{\log \theta}{6n^2}.
\]

\(^1k = 0\) is equivalent to \( \alpha = \alpha' \), which will be excluded in much of our discussion.
If, also \( 1 < \theta < e^3 \), then \( \frac{\log \theta}{6n^2} < \frac{1}{2n^2} \), giving

\[
0 < \frac{m}{n} - \phi < \frac{\log \theta}{6n^2} < \frac{1}{2n^2},
\]

and implying \( \frac{m}{n} \) is an odd-indexed convergent of \( \phi \) [10, Theorem II.5.1], similar to the development in O’Bryant’s Lemma 7 [9].

In our case, when \( k > 0 \), if \( 0 < \log \theta + \log \theta' \leq 6 \) then (4.6) becomes

\[
0 < \frac{m}{n} - \phi \leq \frac{1}{2n^2},
\]

(4.7)
in which case it follows that \( \frac{m}{n} \) is an odd-indexed convergent of \( \phi \), as just explained. Accordingly, there would be an infinite number of \( n \) that satisfy (4.7). Moreover, if \( \log \theta + \log \theta' > 6 \), the previous inequality would be of the form

\[
0 < \frac{m}{n} - \phi \leq \frac{h}{2n^2},
\]

(4.8)
where \( h > 1 \), and this inequality would be satisfied by the odd-indexed convergents of \( \phi \), as well as by other fractions, again, supplying an infinite number of \( n \).

Thus, for \( h \geq 1 \), any \( n \in I \) will always satisfy (4.1), and therefore, if \( I \) is infinite, there will always be an infinite number of solutions to (4.1) as claimed in the theorem.

If the converse were true, we would be able to produce a number \( \phi \) so that an infinitude of solutions to (4.1) would imply the existence of a pair, \( \theta, \phi \) such that

(i) is true; we still need to provide proofs for (ii) and (iii) if we wished to prove the
conjecture true. Unfortunately, the converse is not true, for it is possible that any $n$ satisfying (4.5) may not necessarily satisfy either of (4.2) or (4.3), so we have not even proven one-third of what must be done. The above work, though, does give some indication of a path for further investigation to provide a proof. The fact that $\{n/\log \theta\}$ is evenly distributed in $(0, 1]$, does seem to suggest that almost all sufficiently large choices of $\theta$ and $\tau$ will produce an infinite intersection, $I$, in almost all cases. By the same token, it seems that in almost all cases, $A_\theta \setminus I$ and $A_\tau \setminus I$ will also be infinite, and therefore, either there are no such pairs of $\theta, \phi$, or producing such a pair will require a very intricate construction.

4.1.2 Second Approach

We first make use of two known results:

(1) If $\alpha$ is any irrational number, then $\{n\alpha : n \in \mathbb{N}\}$ is dense in the unit circle $[0, 1)$.

(2) For any natural number $f$, if $\alpha_1, \alpha_2, \ldots, \alpha_f$ are any irrational numbers such that the ratio of any two is also irrational, and if $\beta_1, \beta_2, \ldots, \beta_f$ are any points (not necessarily distinct) on the unit circle, then for any given small positive real number $\epsilon$ there are an infinite number of natural numbers $n$, such that for each $i = 1, 2, \ldots, f$, each $\{n\alpha_i\}$ is in an $\epsilon$ neighborhood of $\beta_i$ [1, Chapter III, Section 5, Theorem IV, page 52]. This theorem is sometimes called Kroenecker’s theorem on
simultaneous inhomogeneous approximation. We only need to use it for $f = 2$ and $\beta_1 = \beta_2 = 1/2$.

We conjecture, but cannot prove the following theorem. We do present an outline of what may be an approach to providing a proof.

**Theorem 4.1.2.** Let $\theta$ and $\tau$ be any two positive real numbers with irrational logs the ratio of which is also irrational, and both $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ are infinite. If neither $\theta$ nor $\tau$ is special then each of the following sets is infinite:

$I := \mathcal{A}_\theta \cap \mathcal{A}_\tau$

$\mathcal{A}_\theta \setminus I$

$\mathcal{A}_\tau \setminus I$.

**Proof.** Here is just an outline:

For any irrational number $\phi$, define $N_\phi := \frac{1}{30} \sqrt{15 \log \theta \sqrt{\left| \log \theta - 3 \right| \log \theta + 45}}$. Let $N = \max(N_\theta, N_\tau), \alpha_1 = 1/\log \theta, \alpha_2 = 1/\log \tau$. Also let $\epsilon_1$ be less than min and $\beta_1 = 1/2$, and $\beta_2 = 1/2$. If $n > N$, by Kroenecker there exist an infinite number of $n \in I$. If we now make $\beta_2 = 3/4$, there are now an infinite number of $n$ in $\mathcal{A}_\theta$ but not in $\mathcal{A}_\tau$. Switching the two $\beta$’s around produces an infinite number of $n$ in $\mathcal{A}_\tau$ but not in $\mathcal{A}_\theta$.

The Grace inequality and Kroenecker produce an infinite number of solutions common to both, but it is yet necessary to show that the infinite number of common solutions contains an infinite subset of common even continuants with odd
index, even though each $\alpha$ does have by itself and even though there are infinite ones in common that might have even index or odd denominators.

\[ \left\{ \theta > 1 : A_\theta \text{ is finite} \right\} \]

While we do not solve this problem, we introduce two mappings using the continued fraction expansion of $\frac{1}{\log \theta}$, that to the best of our knowledge are not found in the literature, and provide some insight into this problem. Also included are some problems requiring additional research involving the Hausdorf-Besocovitch dimension\(^2\); other unsolved problems not involving the dimension are in the next chapter.

To define the two maps, we first introduce some notation. First, whenever we use mod $m$, we will always use the reduced residue class, $\{0, 1, \ldots, m - 1\}$. Second, the $m$-ary decimal $d_1d_2\cdots := \sum_{i=1}^{\infty} d_im^{-i}$. Third, for any integer $a$ and any natural number $m$, let $a(m) = \min\{n \in \mathbb{N} : a \equiv a(m) \pmod{m}\}$.

Our first map will take the continued fraction $\alpha = [a_0; a_1, a_2, \ldots, a_n \ldots]$ to a binary decimal, where even partial quotients are mapped to 0, and odd ones are

\(^2\)In his paper, O’Bryant calls it the “Hausdorf” dimension, but I prefer the fuller name of “Hausdorf-Besocovitch” dimension for the identical concept; for short, we will just use the word “dimension,” without any qualifiers to carry the same meaning.
mapped to 1. It is the special case of $\mathcal{F}_m$ where $m = 2$ and we define

$\mathcal{F}_m : \alpha \mapsto .d_1d_2\ldots$ where $d_i := a_{i-1}(m)$, and just $\mathcal{F}$ will mean $\mathcal{F}_2$. Alternately, we could define $d_i := a_i(m)$ $n \geq 1$ and, by convention $a_0 = d_0$ is the digit in the “one’s” column. This map will be useful in proving the next theorem which follows, and the definition of the second map will be deferred to later. In the Grace construction, we are not concerned about the actual values of the partial quotients, only their parity, since the parity alone is the determining factor as to whether a given convergent will satisfy inequality (2.3) in Section 2.3. Thus, we are not interested in the space of all convergents or all partial quotients, but rather in the space of $\{\mathcal{F}(\alpha) : \alpha \text{ is irrational}\}$.

**Theorem 4.2.** $\dim G = 0$.

**Proof.** Recall that $G$, as defined earlier, was, by the Grace Construction, a set of irrational numbers, $\alpha$, where $\mathcal{A}_\theta$ was finite where $\alpha := 1/\log \theta$. More specifically, it was the set of all irrational numbers $\alpha = [a_0; a_1, \ldots]$, where $a_0 = 0$, $a_1$ is odd, and the other were all even and increasing. Since we do not care about the actual values of the individual partial quotients, just their parity, we can investigate $\dim G$, by simply dealing with $\mathcal{F}(\alpha)$. Now $G$ is a subset of all binary decimals that end in an infinite string of zeroes, which is, in turn a subset of all rational numbers, which is countable and therefore $\dim G = 0$. \qed
While this theorem does not provide a full answer to the problem posed of finding the Hausdorff dimension of \( \{ \theta > 1 : A_\theta \text{ is finite} \} \), it does provide a partial answer in that it has now been determined that a prominent subset of the set in question is of zero dimension. All members of \( G \) satisfy (2.3), and as is noted before in the proof of Theorem 2.4.1 making all partial quotients (after the first one) even, and thereby making all continuants even, is unnecessary. It is only needed that the odd-indexed continuants be odd. Thus, \( G \) is "too small." If we truly wish to find the dim of the set of all \( \theta \) so that \( A_\theta \) is finite, we now need to consider more deeply the parity considerations. Note that we are looking for convergents where either the numerator is not odd or the denominator is not even. While it is simplest to look for odd-indexed convergents whose denominators (continuants) are odd for completeness purposes, we have included in the tables the possibility that the numerator is (or is not) even.\(^3\)

Using the standard convention \( p_{-2}/q_{-2} = 0/1, p_{-1}/q_{-1} = 1/0 \), and using \( a_1 = 0 = 0/1 = p_1/q_1 \), as was used in the Grace construction, we see that we start in Case 2 for computing \( q_2 \), and that we start in Case 3 in computing \( p_2 \). If we want all but a finite number of the continuants to be odd, the parity of a finite number of partial quotients may be chosen at random. Since the choice of parity of \( a_n \) must make \( p_n \) odd when \( n \) is odd, that means we must follow the arrows in the

\(^3\)See Appendix A.
state diagram through in such a fashion that, after at most a finite number of exceptions, we go to Case 4 or Case 2 every other time. One way would be for all partial quotients to be odd, after reaching Case 4 for the first time. Or, there could be there could alternating even partial quotients (to go from Case 2 to Case 3) with any parity (to go from Case 3 to Case 2.) Still, it would be possible to go from Case 2, with an odd, to Case 4, (possibly have any number of more odds to stay in Case 4, and then an even to Case 3). By integrating these different possibilities, the total number of ways to obtain odd-indexed, odd continuants is uncountable. There does not seem any simple way to compute the dimension of the set of parities $\mathcal{F}$ by this method.

For this reason, we suggest a possible second map that may be of interest, whereby we map the continued fraction to a plane (or higher-dimensional object), possibly enabling the possible use of more tools of analysis and geometry, as well as possibly importing some of fractal methodology to solve this problem. Note that because of the possibility that $\beta = r/s \neq 1/2$, which may be of use in a more general problem than the one mentioned here, we are allowing different modular systems to be used, besides mod 2.

Our second map will take the continued fraction $\alpha = [a_0; a_1, a_2, \ldots, a_n \ldots]$ into a point in the interior of the unit square in $\mathbb{R}^m$. Define $G_m : \alpha \mapsto (x_1, x_2, \ldots x_m)$ where for $i = 1, 2, \ldots m - 1$ we define $x_i$ to be the $m$-ary decimal
That is to say that for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots \), the \( j^{th} \) digit of the \( i^{th} \) component is \( a_{(j-1)m+i}(m) \). For both maps we make the convention that when the subscript \( m = 2 \), it will simply be omitted, since \( m = 2 \) is what is needed in our problem, and, as just mentioned, we allow \( m > 2 \) to introduce a map that might be useful in more general problems.
Chapter 5

Problems For Additional Research

Nathanson [8, section 5] gives a list of problems concerning $M_\theta(n)$. Several of these problems are solved (explicitly or implicitly) by O’Bryant. O’Bryant’s list of unsolved problems was included in the Abstract, and this paper solves three of them. Below are some more problems for additional research. To state these problems with minimum verbiage, we introduce some notation:

Let $\theta > 1$ be a given real number. Let $I = (0, 3)$, $J = (3, 6)$, and $p/q$ be a reduced positive rational number.

Let (A) stand for the inequality $\left| \frac{p}{q} - \frac{1}{\log \theta} \right| < \frac{\log \theta}{q^2}$.

Let (B) stand for the inequality $\frac{p}{q} - \frac{1}{\log \theta} < \frac{\log \theta}{3q^2}$.

Let (C) stand for the inequality $\frac{1}{\log \theta} - \frac{p}{q} < \frac{\log \theta}{3q^2}$.

Let a subscript of D means $q$ is odd, a subscript of E means $q$ is even, and subscript of F means $q$ is either even or odd. Let (H) be either (A) or (B) or (C).

Let G be either D or E or F. Let K be either I or J.
For fixed $\theta$ with irrational log, and for each choice of $G, H, K$, define

$$S_G(H, K) := \{p/q : (H) \& \log \theta \in K\},$$

thereby producing 18 sets of fractions that satisfy certain inequalities.

1. For each of these 18 sets, define conditions on $\theta$ that categorize when each set is empty, finite but not empty, or infinite.

2. Let $T(G, H, K), U(G, H, K), V(G, H, K) :=

$$\{p/q : S_G(H, K) \text{ is empty, is finite but not empty, is infinite, respectively}\}.$$

What is the Hausdorff-Besocovitch dimension of each of these 54 sets?

3. What conditions are there that insure that near-convergents do or do not meet the inequality condition, the parity condition, and/or the upper-estimate condition?

4. We showed that if $\theta > e^6$ and has an irrational log, then $A_\theta$ is infinite. What is $\inf \{\theta : |A_\theta| = \infty\}$?

5. We proved that if $\theta$ is not special, then when $\log \theta \in (3, 6)$ is irrational $A_\theta$ is infinite. For $\log \theta \in (3, 6)$ when is the converse true? If there is a contiguous subset of $(3,6)$ for which $A_\theta$ is infinite (or finite), then the preceding question is very interesting.

6. Since $\log \theta \in (0, 3)$ irrational may have no atypical $n$, a finite (but non-zero)
number of atypical \( n \), or an infinite number of atypical \( n \) the question arises
what is \( \text{sup} \{ \theta : A_\theta = \emptyset \} \)? is it greater than 1? Similarly, is there a sup for
those \( \theta \) whose atypical sets are finite but not empty? If so, what is it?

7. If \( A_\theta \) is finite define \( \mu(\theta) = \max \{ n : n \in A_\theta \} \). If \( \log \theta = p/q \), then for what
values of \( \theta \), does \( \mu(\theta) = \left\lfloor \frac{p^2}{6q} \right\rfloor \), the maximum possible? Is the number of such
\( \theta \) infinite?

8. If \( \log \theta = p/q \), then for what (small) values of \( a, b \), is there an infinite
unbounded set of \( \theta \), such that \( \mu(\theta) = \left\lfloor \frac{ap^2}{6q} \right\rfloor \)?

9. If \( \alpha \) is a Liouville number, then there are an infinite number of solutions, \( p/q \),
to an inequality of the form \( |\alpha - p/q| < 1/q^n \) for all natural numbers \( n \).
Therefore, there will be many rationals that meet the inequality condition,
but it is not clear that they will meet the parity condition or the
upper-estimate condition. Under what conditions will they?
Appendices
Appendix A: Recursive Tables

The recursive formula for convergents (Chapter 4, Fact 3), clearly depends on the parity of the partial quotients, which is a matter of concern in a number of places. We summarize the results in the eight cases below, each table dealing with the parity of the previous two numerators or denominators, and the parity of the next partial quotient. We will use the letter “r” to indicate either “p,” or “q,” with \( n \) as the index and \( a_n \) as the partial quotient The tables are ordered alphabetically according to the last two values of \( r_n \). Using “E” for “even” and “D” for “Odd,” Case I is E E, Case 2 is E D, Case 3 is D E, and Case 4 is D D. Then each case is divided into subcases, Case \( xA \) means \( a_n \) is even, and Case \( xB \) means \( a_n \) is odd, and so \( a_n \) determines the parity of \( r_n \), and it is explicitly mentioned what the next case will be.

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The preceding tables may be summed up in the following state diagram, where a clockwise arrow, or one pointed to the right, indicates the next $a_n$ is even, i.e., that we are in the "A" version of the case, and the counter-clockwise arrow, or one pointed to the left, indicates that $a_n$ is odd and we are in the "B" version of the case. Moreover, a number inside a square refers to a case where the last $r_n$ is odd\(^1\) and a circle around the number refers to a case where the last $r_n$ is even.\(^2\)

---

\(^1\)Needed so $q_n$ odd and convergent does not meet parity condition.

\(^2\)Optional so $p_n$ even and convergent does not meet parity condition.
The essence of these four lemmas is to determine exact conditions will make either all (or almost all, i.e., with a finite number of exceptions) continuants (denominators of continued fractions) for either both principal convergents, viz., \(q_n\) and auxiliary convergents, viz., \(q_{n,c}\), or just principal convergents, to be odd whenever \(n\) is odd. To this end, from the tables and state diagram in Appendix A, we will design a method for constructing such \(\alpha\).

In general, there are two ways to create an odd denominator for an odd-index. We must end up in Case 2 (last two denominators are even and odd respectively), which means the preceding case was Case 3, or we must end up in Case 4 (last two denominators were odd), which means the previous case was either Case 2 or Case 4. As a result we may either have alternating cases of Case 2 and Case 3, or repeated instances of Case 4. From Appendix A we see alternating instances of Case 2 and Case 3 will occur once there exists some \(n\) where \(q_{n-2}\) even for \(n - 2\) being even and \(q_{n-1}\) odd for \(n - 1\) being odd. This means we are in Case 2. By making \(a_n\) even, we then go to Case 3, and once in Case 3, whatever parity is chosen for \(a_n\) or \(c_{n,j}\), we will always end up again in Case 2. (This proves A and B of Lemma 3.2.2b.)

Once we initially end up in Case 4 (meaning the last two denominators are both odd) we can stay in Case 4 indefinitely by choosing all subsequent \(a_n\) to be even. If any \(a_n\) were to be odd, we would end up in Case 3, from where we must go to Case
2, which has the previous two denominators being even and odd. Since we want the odd denominator to belong to a convergent of odd index, the preceding choice of odd $a_n$ leading to Case 3 must be a case where $n$ is even. In other words, once we are in Case 4, an indefinite number of odd $a_n$ will keep us there; if however, we were to have an $a_n$ that is even, which causes us to exit Case 4 and enter Case 3, it is important that $n$ must be even. Similarly, if we ever wish to leave a repeating pattern of alternating Case 2 and Case 3 and still keep odd-indexed denominators to be odd, we may only do so by making $a_n$ odd when $n$ is even (Case 2B), thereby landing in Case 4, which we must exit only when $n$ is odd. (This proves of Lemma 3.2.2b.)

For principal convergents, the above is good, but not for auxiliary convergents, since $0 \leq c_n \leq a_n$, and therefore $c_n$ can be either odd or even as long as $a_n \geq 2$, which it is in Case 4, since staying in Case 4 means all partial quotients are even. If we are in Case 4 and using auxiliary convergents, we could exit Case 4 and go to Case 3 by making $c_n$ odd for some odd index $n$, making the odd-indexed denominator even, which is to be avoided. Thus, the situation of repeated Case 4 (all previous denominators are odd) is not useful if we also wish to avoid the case auxiliary continuants of odd-index being even. Only by being in Case 3, does the choice of $c_n$ and its varying parity, not effect the parity of denominators, because we always go from Case 3 to Case 2. Hence, we do not want to have Case 4 to
occur, because it allows for the possibility of an auxiliary fraction of odd-index to have an even denominator.

If Case 4 does occur, but only a finite number of times, and otherwise we are alternating between Case 2 and Case 4, then, at most a finite number of odd-indexed continuants (even auxiliary ones) will be even. (This proves Lemma 3.2.2b.)

However, the existence of a Case 4 situation cannot be avoided entirely. By definition $q_{-1} = 0$ and $q_0 = 1$, so for $n = 1$ we are in Case 2, in that the two preceding denominators are even and odd. From Case 2, we can go either to Case 3 by making $a_1$ even which also makes $q_1$ even—which we wish to avoid—or Case 4 by making both $a_1$ and $a_2$ odd, causing $q_1$ and $q_2$ to be odd and even respectively, putting us in Case 3. Such a scenario will avoid all denominators of principal convergents being even, but does allow that for $n = 1$, an auxiliary continuant could be even. If, however, $a_1 = 1$, then there does not exist any auxiliary convergent for $n = 1$ that is not a principal convergent, and therefore the denominator is always odd. If $a_2$ is also odd, we then move to Case 3 and can then alternate between Case 2 and Case 3. This way it will then end up that all denominators of all odd-indexed convergents are odd. This is the only way to guarantee that all denominators of all odd-indexed convergents are odd. (This proves Lemma 3.2.2d.)

If the requirement that $a_1 = 1$ is dropped but still be odd, then it is possible
that the denominator of some auxiliary convergent, namely, $q_{1,j}$ will be even. (This proves 3.) Even in this case, it may be for some given $\alpha$ the denominators of auxiliary convergents do not meet condition $i$ of Corollary 3.1.1b, and therefore it is not necessary that all auxiliary denominators be odd. However, as this lemma does not deal with whether or not a given convergent produces an atypical $n$, but rather just when we can be sure that no denominator of any odd-indexed convergent is even, it is, then, for our purposes at present, necessary to exclude any odd-indexed denominator from being even.

It follows that if there is some index $n$ so that the even property holds for any even $j > n$, that for $k \leq n$ it is possible that some odd-indexed denominator is even, but for $k > n$, no odd-indexed denominator is even—they will all be odd. (This proves Lemma 3.2.2c part B.)
Appendix B: Minkowski’s Theorems

Since there are several theorems that are called “Minkowski’s Theorem”; each author has a different version, style, and notation; and some of the results may be counter-intuitive, some clarification is in order.

1. Grace’s Version

In [5] Grace stated,

“Tchebychef proved that there is an infinite number of integer $y$’s such that

$$|ay - x - b| < \frac{1}{2y},$$

Hermite that the same is true if $\frac{1}{2}$ is replaced by the smaller number $\sqrt{\frac{2}{27}}$, and

Minkowski that

$$|ay - x - b| < \frac{1}{4|y|}$$

holds for an infinite number of integer values of $y$.”

2. The Initial Version of Dickson and the Version of Hardy and Wright

Dickson wrote [3, pages 94-96],

“P. L. Tchebychef proved that if $a$ is irrational and $b$ is given, then there exists an infinitude [italics are mine] of sets of integers $x, y$ such that there is an infinite number of integer $y$’s such that $y - ax - b$ is numerically $< 2/|x|$.

Hermite proved that in Tchebychef’s result, we may replace $2/|x|$ by $1/2|x|$ and in
H. Minkowski proved that if \( \xi = \alpha x + \beta y \) and \( \eta = \gamma x + \delta y \) have any real coefficients of determinant \( \alpha \delta - \beta \gamma = 1 \) and if \( \xi_0, \eta_0 \) are any given real numbers, there exist integers \( x, y \) for which \( |\xi - \xi_0)(\eta - \eta_0)| \leq \frac{1}{4} \). In particular if \( a \) is irrational and \( b \) not an integer, there are integers \( x, y \) for which \( |(y - ax - b)(c - x)| < \frac{1}{4} \); the case \( c = 0 \) give a better approximation than Hermite’s since \( \frac{1}{4} < \sqrt{2/27} \)."

From the fact that Dickson no longer mentions “an infinitude of integer y’s,” just that exist integers \( x, y \) is an indication that in this version of Minkowski’s Theorem, there need not necessarily be an infinitude of such pairs, just that for any choice of \( a \) irrational and \( b \) not an integer, there will always be at least one pair of integers such that \( |(y - ax - b)(c - x)| < \frac{1}{4} \).

Similarly, Hardy and Wright wrote [6, Chap. XXIV 24.7, p. 534], write,

“We prove next an important theorem of Minkowski concerning non-homogeneous forms

\[ \xi - \rho = \alpha x + \eta y - \rho, \eta - \sigma = \gamma x + \delta y - \rho \]

Theorem 455. If \( \xi \) an \( \eta \) are homogeneous linear forms in \( x, y \) with determinant \( \Delta \neq 0 \), and \( \rho \) and \( \sigma \) are real, then there are integral \( x, y \) for which

\[ |(\xi - \rho)(\eta - \sigma)| \leq \frac{1}{4} \Delta; \]

and this is true with inequality unless...”

The absence of any statement about an infinite number of pairs of \( x, y \), is an
indication that the version of Minkowski’s Theorem mentioned here, only deals with the existence of at least one pair, for any two given homogeneous linear forms.

3. The Version of J. W. S. Cassels

Cassels’s however, explicitly states the existence of infinitely many integers [1, Chapter III, Section 2, Theorem IIA, page 48],

“If θ is irrational and not of the form α = mθ + n for integers m, n, then there are infinitely many integers q such that

\[ |q| \langle \frac{1}{4}, \]”

This statement implies all the other versions of Minkowski’s Theorems mentioned previously.

4. A Later Version of Dickson

He [3, p. 99] writes, ”J. H. Grace...proved that Minkowski’s last result is final, i.e., if \( k < \frac{1}{4} \), it is possible to choose a and b such that there is not an infinitude of integers x for which \(|y - ax - b| < k/|x|\),” thereby implying if \( k > 1/4 \) there would be an infinitude of integers x.

At first glance, it would appear that the statement that an inhomogeneous approximation cannot be made to the degree of accuracy claimed, namely, \( \frac{1}{4q^2} \), for this seems to contract the basic fact that an infinite number of approximations cannot exist for irrationals when that \( k = \frac{1}{\sqrt{5q^2}} \) for homogeneous approximations. However, when \( \beta = 1/2 \) is inserted into the inequality, the denominator becomes \( 2q \)
and not \( q \), so there is no contradiction, because, as mentioned earlier in section 5.1, “This result, in turn, means \( \alpha \) can be approximated by a rational, \( \frac{2p+1}{2q} = \frac{r}{s} \), to accuracy of the square of the denominator. By Fact 8, an approximation of this degree of accuracy, can only be attained by a fraction which is a principal convergent or an auxiliary convergent.” [5]
Bibliography


