Wave Propagation Inside Random Media

Xiaojun Cheng

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Wave Propagation inside Random Media

by

Xiaojun Cheng

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

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This manuscript has been read and accepted by the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

Wave Propagation inside Random Media

by

Xiaojun Cheng

Advisor: Azriel Genack, Distinguished Professor of Physics

This thesis presents results of studies of wave scattering within and transmission through random and periodic systems. The main focus is on energy profiles inside quasi-1D and 1D random media.

The connection between transport and the states of the medium is manifested in the equivalence of the dimensionless conductance, $g$, and the Thouless number, $\delta$, which is the ratio of the average linewidth and spacing of energy levels. This equivalence and theories regarding the energy profiles inside random media are based on the assumption that LDOS is uniform throughout the samples [1, 2]. We have conducted microwave measurements of the longitudinal energy profiles within disordered samples contained in a copper tube supporting multiple waveguide channels with an antenna moving along a slit on the tube. These measurements allow us to determine the local density of states (LDOS) at a location which is the sum of energy from all incoming channels on both sides. For diffusive samples, the LDOS is uniform and the energy profile decays linearly as expected. However, for localized samples, we find that the LDOS drops sharply towards the middle of the sample and the energy profile does not follow the result of the local diffusion theory where the LDOS is assumed to be uniform. We analyze the field spectra into quasi-normal modes and found that the mode linewidth and the number of modes saturates as the sample length increases. Thus the Thouless number $\delta$ saturates while the dimensionless conductance $g$ continues to fall with increasing length, indicating that the modes are localized near the boundaries. This is in contrast
to the general believing that $g$ and $\delta$ follow the same scaling behavior. Previous measurements show that single parameter scaling (SPS) still holds in the same sample where the LDOS is suppressed [3]. We explore the extension of SPS to the interior of the sample by analyzing statistics of the logarithm of the energy density $lnW(x)$ and found that $\langle lnW(x) \rangle = -x/\ell$ where $\ell$ is the transport mean free path. The result does not depend on the sample length, which is counterintuitive yet remarkably simple. More surprisingly, the linear fall-off of energy profile holds for totally disordered random 1D layered samples in simulations where the LDOS is uniform as well as for single mode random waveguide experiments and 1D nearly periodic samples where the LDOS is suppressed in the middle of the sample.

The generalization of the transmission matrix to the interior of quasi-1D random samples, which is defined as the field matrix, and its eigenvalues statistics are also discussed. The maximum energy deposition at a location is not the intensity of the first transmission eigenchannel but the eigenvalue of the first energy density eigenchannels at that cross section, which can be much greater than the average value. The contrast, which is the ratio of the intensity at the focused point to the background intensity, in optimal focusing is determined by the participation number of the energy density eigenvalues and its inverse gives the variance of the energy density at that cross section in a single configuration. We have also studied topological states in photonic structures. We have demonstrated robust propagation of electromagnetic waves along reconfigurable pathways within a topological photonic metacrystal. Since the wave is confined within the domain wall, which is the boundary between two distinct topological insulating systems, we can freely steer the wave by reconstructing the photonic structure. Other topics, such as speckle pattern evolutions and the effects of boundary conditions on the statistics of transmission eigenvalues and energy profiles are also discussed.
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Contents

Abstract iv

Acknowledgements vi

1 Introduction 1

1.1 Overview 1

1.2 Anderson localization 2

1.3 Speckle 3

1.4 Quasi-normal modes 4

1.5 Transmission matrix 4

1.6 Supersymmetry 5

1.7 Topological insulator 9

1.8 Outline 10

2 Wave transport through random media 12

2.1 Speckle evolution 12

2.1.1 Phase singularity diffusion 12

2.1.2 Motion of intensity maxima in averaged speckle patterns 20

2.2 Transmission eigenvalue distribution in samples with surface reflection 28

3 Wave propagation inside random media 35

3.1 Fields inside random media 35

3.1.1 Field matrix and focusing inside random media 35

3.1.2 Statistics of energy density eigenvalues 44
List of Figures

1.1 Example of a speckle pattern produced by the interference of multiply scattered waves in a random medium. ................................................................. 3

2.1 Singularities in the speckle pattern in microwave experiments. Singularities with positive and negative charges are represented by yellow dots and yellow squares respectively. We show trajectories of two singularities as examples. Note that since singularities are annihilated after a typical decaying frequency, the comparably longer trajectory as in this figure is a relatively rare event. ................................................................. 13

2.2 Average of the square of singularity displacement with frequency shift. Slope of \( \langle R^2 \rangle \) approaches a constant. ................................................................. 14

2.3 Measurement of the velocity auto-correlation function with frequency shift. ................................................................. 15

2.4 Decay of the number \( N \) of phase singularities with frequency shift in microwave experiments. \( N_0 \) is the initial number of singularities. \( N/N_0 \) falls to \( 1/e \) in a frequency shift \( \Delta \nu_N = 20.4 \) MHz. ................................................................. 16

2.5 Comparison of the square displacement of phase singularities in microwave experiments in the frequency domain and simulations in the time domain. The displacement, frequency shift and time are normalized in terms of the singularity density and survival as explained in the text. ................................................................. 17

2.6 Linear decay of intensity averaged over the cross section with depth \( x \) into the sample \( \langle I(x) \rangle \) for sample with \( L=40 \) cm determined from FDTD simulations. \( I_0 \) is the value to which the average intensity within the sample extrapolates at the input surface \( x=0 \). The intensity profile within the sample extrapolates to zero at a length \( z_b=3.85 \) cm beyond the output surface. ................................................................. 18
2.7 The relation between the diffusion coefficient of singularities in the output plane $D'_s$, the diffusion coefficient of photons through the sample $D_p$, and the effective sample length $L_{\text{eff}}$ in FDTD simulations. (a) Results are shown for three values of $L$ with the same structure and so the same value of $D_p$. $D'_s$ is proportional to $L_{\text{eff}}^2$. (b) The relationship between $D'_s$ and $D_p$ at $L = 23$ cm for samples 1, 2 and 3 correspond to filling fractions of alumina spheres of 0.068, 0.1 and 0.15. Here $C = 4D'_s(D_p/L_{\text{eff}}^2)$. 19

2.8 (a) Microwave speckle pattern in transmission at 14.71 GHz. (b)-(d) Speckle patterns averaged over a frequency window of 6.25, 12.25, 18.75 MHz starting at 14.71 GHz. Local maxima are indicated by the yellow circles in each pattern. 22

2.9 Average of square displacements of local maxima vs. frequency shift and corresponding quantity for speckle patterns averaged over a frequency windows of $\Delta \nu_w$. 23

2.10 The diffusion coefficient of local maxima vs. width of frequency window over which speckle patterns are averaged. 24

2.11 Average velocity of position of local maxima determined using different frequency steps. 25

2.12 Average of the number of local maxima in windowed intensity patterns over the frequency range of 14.7 – 15.7 GHz. 26

2.13 Cumulant intensity correlation functions. 26

2.14 Variation of intensity correlation frequency with frequency window. 27

2.15 Variance of the normalized intensity vs. frequency window. 27

2.16 Schematic illustration of the diffusion model. 28

2.17 The analytical theory shows that as the internal reflection on the output surface increases, $\rho(T)$ undergoes an ‘abrupt’ change (‘transition’) at $\zeta = 1$ (lower panel). This transition disappears when the internal reflection is the same on the input and output surfaces (upper panel). The values of $\zeta$ are 0.1, 0.25, 1, 1.2, and 8 from top to bottom. Inset: we re-plot the curves in the main panels in ln-ln scale. 32

2.18 The numerical results and the analytic predictions of are in good agreement. This confirms the transition in transmission eigenvalue statistics. $\zeta$ is treated as the single fitting parameter, found to be 0.16, 0.93, 2.2, 4.0, and 17.2 from top to bottom. 33

2.19 The average conductance obeys Ohm’s law. The data is fit to a straight line whose slope is unity. 34
3.1 The ensemble average of the energy density inside the sample for incident radiation with unit energy density ........................................ 38

3.2 The ensemble average of the highest energy density eigenvalue at \( z \), \( \langle \epsilon_1(z) \rangle \), and the average energy per channel \( \langle u(z)/N \rangle \) for diffusive samples with \( L/\xi = 0.0625 \) .................. 39

3.3 Intensity distribution over the cross sections at \( z = L/4 \) and \( z = L/2 \) for a locally two-dimensional system with \( L/\xi = 0.0625 \) and the focal point at the center of the cross section \( x = 0 \). Here \( x \) is the coordinate in the transverse direction ranges from \([-A/2, A/2]\) with \( A \) the width of the system. \( I_b \) is the average background intensity at \( x \neq 0 \) ................................................................. 40

3.4 Comparison of the contrast in focusing and the eigenchannel participation number along the sample length for (a) diffusive waves with \( g = 16 \), \( N = 166 \) and (b) localized waves with \( g = 0.3 \), \( N = 32 \) respectively ............................................... 41

3.5 \( M(z) \) and \( \text{var}(Nu_a(z)/u(z)) \) for one configuration with \( N = 500 \) ............................................................... 43

3.6 The statistics of the energy density eigenvalues for (a) \( x = L \) and (b) \( x = L/2 \) .... 44

3.7 The statistics of the energy density eigenvalues for (a) small \( \varepsilon \) and (b) large \( \varepsilon \) .... 45

3.8 The statistics of the energy density eigenvalues for (a) small \( \varepsilon \) and (b) large \( \varepsilon \) .... 46

3.9 Microwave measurements of \( <\ln W(x)> \) .................................................. 48

3.10 Microwave measurements of \( \text{Var}(\ln W(x)) \) ............................................. 49

3.11 Microwave measurements of the probability density distribution \( \ln W(x) \) .......... 50

3.12 (a) The average energy density \( \langle W(x) \rangle \) for fixed sample length of \( L = 200 \) layers without boundary reflection at the input for no reflector at the output and the refractive indices of the reflector at the output boundary of \( n_r = 25, 50, 75 \). (b) \( \langle W(x) \rangle \) for sample lengths of 100, 150, 200 and 250 layers and refractive index of the boundary reflector at the output of \( n_r = 50 \). The solid black lines are predictions from Eq. (19). The parameter \( z_b \) is 120 layers for all the samples ............................... 51

3.13 The average intensity profile from 1D simulations with and without boundary reflectivity .................................................. 52
4.1 (a) Contributions of the individual modes in Eq. 4.1 to the DOS. (b) Comparison of the DOS determined from the TM (red curve) by summing spectra of $d\theta_n/d\omega$ and modes found from spectra of the field at the output (black curve) and from spectra of the field inside the sample (gray dotted curve)................... 55

4.2 (a)-(b) Simulation results of $W(x)$ and $\tilde{\rho}(x)$ in 1D layered systems. (c)-(f) Microwave experimental results. (c) and (d) are the intensity profile $W(x)$ and measured in microwave experiment for diffusive waves with sample length $L = 40$ cm. (e) and (f) are the intensity profile $W(x)$ and $\tilde{\rho}(x)$ for sample length $L = 40$ cm and $L = 60$ cm for localized waves. The $\tilde{\rho}(x)$ are obtained from $W(x) + W(L - x)$.................... 57

4.3 Schematic illustration of the experimental setup. The antenna is translated along the tube to measure $W(x)$.......................... 58

4.4 FDTD simulations of $\tilde{\rho}(x)$ for waves with frequency range 14 – 15 GHz for sample lengths of 16, 30, 50, 80, 100, 120 cm..................... 59

4.5 FDTD simulations of $\tilde{\rho}(x)$ for 100 cm sample with the diameter of the spheres fixed ($\Delta R = 0$) and the diameter can vary by 30% ($\Delta R = \pm 30\%$).................. 60

4.6 FDTD simulations of $\tilde{\rho}(x)$ for 100 cm sample with filling fractions of $f = 0.068, 0.3, 0.6$.................. 60

4.7 FDTD simulations of $\tilde{\rho}(x)$ for 100 cm sample with tube diameters of 2, 3 cm............. 61

4.8 Simulation results from 1d. (a) $W(x)$ for different disorder strength of 200 layers. (b) $\tilde{\rho}(x)$ for different disorder strength......................... 62

4.9 Simulation results from 1d. (a) $W(x)$ for different sample lengths with disorder strength of 20%. (b) $\tilde{\rho}(x)$ for different sample lengths......................... 62

4.10 (a) The number of modes as a function of sample length from 1d simulations for two different kinds of disorder strength. FDTD simulations are carried out and (b) the number of modes (c) the average linewidth (d) Thouless number $\delta$ are shown to saturate......................... 63

4.11 Microwave measurements of (a) $lnW(x)$ and (b) $W(x)$. Solid black lines are theoretical predictions......................... 64

5.1 Reconfigurable metacrystal and its bulk band structure......................... 68
5.2 The dispersion for systems a. between \( m > 0 \) and \( m < 0 \). B. between \( m > 0 \) and \( m = 0 \). ................. 68

5.3 The two different kinds of domain walls ........................................... 70

5.4 Experimental demonstration of spin-locked wave-division of an edge mode at a four-port topological junction. ......................... 71

5.5 Dwell time measurement ................................................. 72

5.6 Demonstration of reconfigurable topological switch and its time-resolved dynamics ......................... 73
Introduction

1.1 Overview

We encounter waves every day. We rely on waves to bring us music, cell phone signals, and we even cook with microwave radiation. The wave nature of fundamental particles also explain many phenomena such as tunneling through a barrier which are forbidden for classical particles.

Many real systems are not perfectly ordered. Thus, the study of wave scattering in disordered media is of interest in both quantum and classical systems. In disordered electronic systems, wave can be come localized [4], which originates from coherent back scattering [5 6 7 8 9 10], and the coherence effects also occur for classical waves [11 12 13 14 15].

Recently, there has been a lot of progress in manipulating the pattern of scattered radiation by reconstructing the incident wavefront. Many experiments have successfully seen the control of wavefields for focusing in disordered media [16], the transmission of images across highly scattering samples [17], and also focusing in the time domain [18 19 20]. This may find important applications in medical imaging and diagnosis [21 22].

Many theoretical methods have been developed to study wave scattering in complex media such as random matrix theory (RMT) which treats the Hamiltonian or the transmission matrix to be a large $N \times N$ random matrix [23] including the maximum entropy approach which is implemented through the a maximization of the Shannon entropy [24] and supersymmetry method where anti-commuting variables are utilized in calculations [25]. We will introduce these methods and how they are applied to our problems.
1.2 Anderson localization

Anderson predicted that electrons in random lattices would be exponentially localized by disorder [1]. The scaling theory of localization states that the scaling of $g$, which is the conductance in units of $e^2/h$, depends on only the parameter $g$ itself [26], and a critical value of $g_c \sim 1$ marks the localization threshold. For unbounded three dimensional random systems, localization is predicted to occur when the Ioffe-Regal criterion $k\ell \leq 1$ is satisfied, where $k$ is the wave number and $\ell$ is the transport mean free path [27]. Later, Thouless argued that for finite open systems, the nature of wave propagation can be described in terms of the degree of overlap of quasi-normal modes [1]. This can be expressed by the ratio of the average mode linewidth $\delta \nu$ to the mean spacing between modes $\Delta \nu$ of the medium, which is the Thouless number $\delta$. When the level width exceeds the level spacing, different energy levels overlap spectrally within the medium and the wave becomes extended in space. When the states are isolated spectrally, waves are localized and the transmission decays exponentially with sample length. Therefore, $\delta \sim 1$ gives the threshold of localization.

The localization phenomenon in electronic systems has stimulated interest in the study of localization of classical waves [28, 29]. Geometry is a key factor which determines localization behavior. In 1D and 2D, the system is localized for arbitrary strength of disorder while in 3D there exist a localization transition [4, 26]. Samples with quasi-one-dimensional geometry, with the longitudinal dimension much larger than the transverse dimension, such as wires or random waveguides are widely used in the study of wave transportation and localization [1]. Such samples are three dimensional locally but are confined in the transverse directions. The methods of random matrix theory are well suited for this geometry and the weak localization limit is well described in this framework [23]. The dimensionless conductance $g$, whose classical analogue is the transmittance $g = \langle T \rangle = \sum_a T_a$ where $a$ denotes an incoming channels and $T_a$ the corresponding transmission, is used as a parameter that characterizes statistics of transmission [30, 31, 11]. Here $g \sim N\ell/L$ for diffusive samples and $g \sim e^{-L/\xi}$ for localized samples with $L$ the sample length and $\xi$ the localization length [26]. It is generally believed that $g$ and $\delta$ follow the same scaling behavior with the assumption that the local density of states (LDOS) is uniform throughout a random sample. However, we show in chapter 5 that the LDOS is not uniform and these two parameters differ in random waveguides.
1.3 Speckle

The speckle pattern is the result of the interference of many scattered partial waves that follow different trajectories to arrive at a point at the output, as illustrated in Fig. 1.1.

![Speckle pattern](image)

Figure 1.1: Example of a speckle pattern produced by the interference of multiply scattered waves in a random medium.

The speckle pattern produced by multiply scattered waves averaged over a time much shorter than the correlation time is generic \[32, 33, 34\] and so carries no sample-specific information. However, changes in the speckle pattern with time can be used to characterize and image the internal motion within a sample \[35, 36, 37\]. A variety of approaches have been used to characterize the dynamics of scattered radiation. Among these are the measurement of the temporal correlation function of the intensity \[38, 39, 40\], and the contrast of the speckle pattern which falls as the exposure time increases beyond the correlation time \[37, 41, 42\]. The temporal correlation functions and spatial contrast reflect both the motion of scatterers and wave propagation within the sample.
In this thesis, we develop another way to characterize speckle evolution. We follow the trajectories of the distinctive features in speckle evolution, such as phase singularities and intensity maxima, and find that their motion over the output plane is diffusive. From the 2D diffusion coefficient of these features, we are able to characterize internal dynamics and find the photon diffusion coefficient of the system.

1.4 Quasi-normal modes

Excitations in complex media can be expressed via the eigenstates of the random Hamiltonian, which are called quasi-normal modes for classical waves. The field $E(\vec{r}, \omega)$ at any position $\vec{r}$ and frequency $\omega$ inside an open random medium can be described as

$$E(\vec{r}, \omega) = \sum_n a_n(\vec{r}) \frac{\Gamma_n/2}{\Gamma_n/2 + i(\omega - \omega_n)},$$

(1.1)

where $\omega_n$ and $\Gamma_n$ are the central angular frequencies and linewidths for the $n$th mode respectively. The field can be viewed as a superposition of all these modes. Here $\Gamma_n$ can also be viewed as the imaginary part of the central frequency, indicates a non-Hermitian Helmholtz operator associated with the open boundary conditions and absorption within the medium.

The high transmission due to a single mode is believed to have a spatial shape with a single peak near the center of the sample [45]. The overlap of modes both spectrally and spatially results in large transmission with a relatively broad frequency range [46, 47, 48], which are called the necklace states. The spectrum of the field speckle pattern at the output of a random sample has been decomposed into a sum of speckle patterns due to the modes of the medium [49]. In chapter 5, we show that the number of modes and the mode linewidth saturates for random waveguides as we increase sample length. The spatial profiles of modes $a_n(\vec{r})$ in Eq. 1.1 is not accurately determined when the modes are peaked near the input and will be studied in the future.

1.5 Transmission matrix

Random matrix theory was first proposed by Wigner to be used to describe the statistical properties of excited states of atomic nuclei [50], which later was widely applied to mesoscopic physics [23],
wireless communications \[51\] and finance \[52\]. This theory was further developed by Dyson and Mehta to determine the universal properties only in terms of some symmetry types of the system \[53, 54\]. In the study of the statistics of energy levels of disordered systems, the Hamiltonian \( H \) is treated as a \( N \times N \) random matrix. On the other hand, the statistics of transport properties through disordered systems, such as metal wires or quantum dots with point contacts, the scattering matrix \( S \) or its submatrix, the transmission matrix \( t \) is treated as a large random matrix. The transmitted field is related to the incident field via the field transmission matrix \( E_b = \sum_{a=1}^{N} t_{ba} E_a \), where \( a \) and \( b \) label the incoming and outgoing channels, respectively, and \( N \) is the total number of channels.

The statistics of wave propagation in random media are fully determined by the eigenvalues \( \tau_i \) of the transmission matrix \( tt^\dagger \). The transmission coefficients of the eigenchannels are the eigenvalues of the transmission matrix \( tt^\dagger \) and the transmittance is given by \( T = \sum_i^N \tau_i \). We will discuss in this thesis the effects of boundary reflectivity on the transmission eigenvalues \( \tau \) distribution. We will also investigate the extension of transmission matrix to the interior of the sample in section 3.1.

1.6 Supersymmetry

The supersymmetry approach is an important technique in one way it establishes a connection between different disciplines and in another way it is a powerful method of calculation itself. Even for problems that can be reduced to the random matrix theory, the zero-dimensional \( \sigma \)-model is more convenient for computations and very often is the only way to get explicit results. For example, the calculations of the average Green’s function and its products are made dramatically easier with the tool of supersymmetry, as will be seen in this section. For higher dimensions, the supermatrix \( \sigma \)-model is the only tool available now in most cases \[25\].

Supersymmetric field theory originally dealt with particles with both integer and half integer spin, bosons and fermions. This does not mean that we are dealing with bosons and fermions in our photonic systems. We borrow the mathematical objects, such as the Grassmannians, as discussed below, for calculations of classical wave propagation as well.

Grassmannians are mathematical objects satisfying the anticommuting relation

\[
\chi\chi' = -\chi'\chi. \tag{1.2}
\]
This is similar to the anticommutations of creation and annihilation operators of fermions. This anticommuting relation also tell us that for any integer \( n > 1 \), \( \chi^n = 0 \), similar to creation operators of fermions when \( n > 1 \), \((a^\dagger)^n = 0\). This is the Pauli exclusion principle that two identical fermions cannot be created in the same state.

The integral over a single Grassmannian is defined by

\[
\int d\chi = \int d\chi^* = 0, \quad \int \chi d\chi = \int \chi^* d\chi^* = 1.
\] (1.3)

Consider a non-singular \( N \times N \) Hermitian matrix \( M \). By using the equations above, we find an the identity

\[
\int e^{-\chi^*_i M_{ij} \chi_j} d\chi_1^* d\chi_2^* ... d\chi_n d\chi_n^* = det(M),
\] (1.4)

in contrast to the corresponding Gaussian integration over commuting variables which is proportional to \( det^{-1}(M) \).

With the commuting and anticommuting variables in hand, we are able to construct the supervector space. Suppose \( \phi \) is a supervector of \( n \) anticommuting component vectors \( \chi \) and \( m \) commuting component vector \( S \). The transpose of the vector can be written as

\[
\phi^T = (\chi_1 ... \chi_n S_1 ... S_m).
\] (1.5)

The linear transformation on the supervector space is described by a supermatrix, \( M \) with the structure

\[
M = \begin{pmatrix}
M_{FF} & M_{FB} \\
M_{BF} & M_{BB}
\end{pmatrix}.
\] (1.6)

The subscripts \( F \) and \( B \) refer to the fermionic (anticommuting) and bosonic (commuting) components respectively. The entries in \( M_{BB} \) and \( M_{FF} \) are commuting variables while those in \( M_{FB} \) and \( M_{BF} \) are anticommuting. The transpose of the supermatrix is

\[
M^T = \begin{pmatrix}
M^T_{FF} & -M^T_{BF} \\
M^T_{FB} & M^T_{BB}
\end{pmatrix}.
\] (1.7)
The minus sign comes from the anti-commuting relation. The transpose of $M_{FF}$ is defined in the usual way.

The Hamiltonian of a disordered system is $H$.

$$H \varphi_k = \epsilon_k \varphi_k, \quad H = H_0 + H_1, \quad < H_1 >= 0. \quad (1.8)$$

Here $\epsilon_k$ and $\varphi_k$ are eigenvalues and eigenfunctions of the Hamiltonian $H$. The term $H_0$ stands for the kinetic energy part $\epsilon(\hat{p})$ of a free electron where $\hat{p}$ is the momentum operator. In the simplest case

$$\epsilon(\hat{p}) = \frac{\hat{p}^2}{2m}, \quad (1.9)$$

where $m$ is the electron mass.

The other term $H_1$ describes the random potential that can originate from impurities in the system. Interactions with an external magnetic field $B$ can be obtained by the substitution $\hat{p} \rightarrow \hat{p} - (e/c)A$, where $A$ is the vector potential corresponding to the magnetic field $B$.

The Green’s function for the Schrödinger equation is

$$G_{\epsilon_k}^{R,A}(r, r') = \langle \epsilon |(\epsilon - \hat{H} + i\eta)^{-1}|r'\rangle \quad \eta \rightarrow 0. \quad (1.10)$$

We can express the Green’s function in terms of the eigenfunctions of $H$

$$G_{\epsilon_k}^{R,A}(r, r') = \sum_k \frac{\varphi_k(r)\varphi^*_k(r')}{\epsilon - \epsilon_k + i\eta} \quad \eta \rightarrow 0, \quad (1.11)$$

or

$$G_{\epsilon_k}^{R,A}(r, r') = \sum_k G_{\epsilon_k}^{R,A} \varphi_k(r)\varphi^*_k(r') \quad \eta \rightarrow 0. \quad (1.12)$$

An important property of the Green’s functions is

$$G_{\epsilon_k}^{R}(r, r') = (G_{\epsilon_k}^{A}(r', r))^* \quad (1.13)$$

Generally we assume the disorder potential $H_1 = \nu(r)$ satisfying

$$< \nu(r) >= 0, < \nu(r)\nu(r') >= \frac{1}{2\pi}\delta(r - r'), \quad (1.14)$$
which is the Gaussian potential. It is not possible to solve the Schrodinger equation for any arbitrary potential. However, for many cases such a solution would be unnecessary because what one need to deal with are the statistical properties and has to calculate averages of quantities containing the solutions. One particular important quantity is the local density of states at energy $\epsilon$

$$< \rho(\epsilon, \mathbf{r}) > = \sum_{k, \sigma} \varphi_k(\mathbf{r}, \sigma) \varphi^*_k(\mathbf{r}, \sigma) \delta(\epsilon - \epsilon_k) .$$  \hspace{1cm} (1.15)$$

Now we take the spin variable $\sigma = \pm 1$ into account as well. Use the formular $\frac{1}{x + i\delta} = \frac{1}{\pi} - i\pi \delta(x)$, we obtain the relation between the local density of states and the Green’s functions

$$\rho(\epsilon, \mathbf{r}) = \frac{1}{2\pi} \sum_\sigma (G^A_{\epsilon}(\mathbf{r}, \mathbf{r}) - G^R_{\epsilon}(\mathbf{r}, \mathbf{r})) = \frac{1}{\pi} \sum_\sigma \text{Im} G^A_{\epsilon}(\mathbf{r}, \mathbf{r})$$  \hspace{1cm} (1.16)$$

The LDOS is equivalent to the imaginary part of the Green’s functions. So one of the ways to obtain LDOS is through the Green’s functions. We will introduce other ways later in this thesis.

We can use the formula

$$\frac{\int \Phi_{k}^+ \Phi_k \exp(-\Phi^{+}F\Phi)d\Phi^{+}d\Phi}{\int \exp(-\Phi^{+}F\Phi)d\Phi^{+}d\Phi} = (F^{-1})_{ik}$$  \hspace{1cm} (1.17)$$

for any matrix $F$ to rewrite the Green’s function in terms of integration over auxiliary fields.

$$G^R_{\epsilon ik} = \frac{\mp i \int \Phi_{ak}^+ \Phi_{ak} \exp(-i\Phi^{+}(\pm(\epsilon - \epsilon_k) + i\delta)\Phi)d\Phi^{+}d\Phi)}{\int \exp(-i\Phi^{+}(\pm(\epsilon - \epsilon_k) + i\delta)\Phi)d\Phi^{+}d\Phi}$$  \hspace{1cm} (1.18)$$

A difficulty arises when we are dealing with averaging over the disorder potential due to the term in the denominator. The replica method was used to solve this problem [52]. Within this approach one substitutes the initial disordered system by $n$ identical systems. This makes it possible to average over the disorder in the very beginning. But it is not rigorous due to the fact that one takes the limit $n \rightarrow 0$ to obtain the average values of physical quantities. This is inconsistent with the conjecture of $n$ systems where $n$ is a positive integer.

The supersymmetry field theory approach can be applied to calculate the average of the Green’s functions, which is required to obtain the average LDOS. If we evaluate an integration with the
same number of degrees of fermionic and bosonic components, the integration

\[ \int e^{\phi^+ M \phi} d\phi^* d\phi = 1, \]  
(1.19)
since the integration over the anti-commuting part provides \( \text{det}(M) \) while the commuting part contributes \( \text{det}^{-1}(M) \). This nice property provide us an exact way of evaluating the Green’s function for disordered systems.

With the supermathematics constructed, the denominator becomes unity after integration over the commuting and anti-commuting parts, as in Eq. 1.19. The Green’s function then becomes

\[ G_{\epsilon k}^{R,A} = \mp i \int \Phi \Phi^+ \exp(-i(\epsilon - \epsilon_k) + i\delta) d\Phi^* d\Phi \] 
(1.20)

Introducing the field variables

\[ \Phi(y) = \sum_k \Phi_k \varphi_k(y), \quad \Phi^+(y) = \sum_k \Phi^+_k \varphi_k^*(y) \] 
(1.21)

The Green’s functions \( G_{\epsilon}^{R,A}(y, y') \)

\[ G_{\epsilon}^{R,A}(y, y') = \mp \int \Phi(x) \Phi^+_x \exp[i \int \Phi(x)(\pm(\epsilon - H) + i\delta)\Phi(x) dx] D\Phi^* D\Phi \] 
(1.22)

Here \( x \) and \( y \) denote both for the space and spin variables.

The absence of the denominator enables us to average the Green functions over the random potential exactly for some Hamiltonian \( H \). This is a great advantage of the supersymmetry method.

1.7 Topological insulator

Topological ideas in photonics arises from the development of new phases of matter in solid-states materials [56, 57]. Examples are integer quantum Hall effect in which electrons in a magnetic field form quantized orbits called Landau levels, and the quantum anomalous Hall effect where the same effect is achieved in a periodic system without Landau levels [58].

Topology in mathematics is used to describe geometrical properties. For example, a sphere can
be continuously deformed into the spoon so they are topologically equivalent while a torus can be continuously deformed into a mug. But a sphere cannot be transformed into a torus without making a hole. The objects that can be continuously transformed to each other are called topologically equivalent. Topological invariant is the number that characterize this property, which is the number of holes, or genus, as in this example, and objects of the same topological invariant are in the same topological phase. When the topological invariant changes, that is when a hole is created or removed, the object undergoes a process called topological phase transition [59].

For 2D quantum hall systems, the topological invariant is the Chern number, denoted as C, which is a number, just like momentum or energy, to be used to describe the feature of the entire system. Since the topological invariant is an integer which changes discretely, it is robust against small perturbations such as defects or variations of material parameters. A mirror with zero Chern number below the energy gap, which is generally opened from breaking time reversal symmetry (T), is called topologically trivial and mirrors with non-zero Chern numbers are topologically non-trivial. When two systems with different Chern numbers connect, the gap must be closed and thus ensuring the appearance of gapless states at the interface between the two bulks. In general, the number of the gapless edge states equal the difference between the Chern numbers of the bulks, known as bulk-edge correspondence. One important property of the edge modes is their uni-directionality, which means that the states are not back scattered and is robust again impurities in the bulk. In quantum spin Hall systems, the spin-orbit interaction allow a different topological class when T symmetry is not broken, called $Z_2$ topological insulator [60]. The analogy of these effects in photonic crystals provides many interesting applications and will be explored in chapter 5.

1.8 Outline

This thesis mainly describes wave transport through and within random media. Chapter 1 review key concepts and presents a broad overview of the thesis. In chapter 2 we describe studies from measurements of the output of random media, including work on speckle evolution and the impact of internal reflection on the distribution of transmission eigenvalues. Chapter 3 involves the exploration of energy distribution within random media, including the discussion of the field matrix which is a generalization of transmission matrix to the interior of the media, the statistics of energy distribution
and single parameter scaling inside random media, and the effects of boundary conditions on the energy density profiles. In chapter 4 we present measurements of density of states and local density of states in random waveguides, showing that the LDOS is suppressed in the middle of the sample for localized waves and $\delta$ saturates while $g$ continues to fall. In chapter 5 we address the topologically protected edge states in reconfigurable photonic metacrystals. Our findings and conclusions are summarized in chapter 6.
2

Wave transport through random media

2.1 Speckle evolution

2.1.1 Phase singularity diffusion

Speckle patterns are interference patterns result from multiply scattered waves transmitted through random media. The speckle pattern is generic, with elements of its structural elements and their statistics unchanged as the speckle pattern evolves with changes in the configuration or incident frequency [33, 61, 62, 63, 64]. It is thus possible, in principle, to characterize changes in the interaction of the wave through the sample by following the displacement of specific features of the speckle pattern in time or frequency shifts. The most distinctive features of a speckle pattern are the phase singularities at intensity nulls of the speckle pattern for each polarization component of the wave [62, 65, 66]. Since both the in- and out-of-phase components of a single polarization of the field vanish at the intensity nulls, the phase is ill-defined. The phase is singular with a discontinuity of $\pi$ rad along any line passing through the center of the vortex of flux centered on the singularity. The phase change in a complete counterclockwise circuit around the phase singularity is $\pm 2\pi$ rad, which is associated with a topological charge of $\pm 1$ [65]. Higher order vortices do not appear to arise in speckle patterns of waves transmitted through random media. Phase singularities of opposite signs are created or annihilated in pairs in a process in which charge is conserved. Though much is known about the structure of speckle patterns, their motion has been little studied. The velocity statistics of singularities has been calculated [33] and measured [63, 64] and the variance of singularity velocity
has been shown to be a measure of the impact of weak localization \[63\]. However, the statistics of motion over stretches of time in dynamic samples or over frequency change in static samples have not been characterized.

Figure 2.1: Singularities in the speckle pattern in microwave experiments. Singularities with positive and negative charges are represented by yellow dots and yellow squares respectively. We show trajectories of two singularities as examples. Note that since singularities are annihilated after a typical decaying frequency, the comparably longer trajectory as in this figure is a relatively rare event.

Here, we utilize microwave spectral measurements and computer simulations in the frequency and time domains to demonstrate that singularity motion in the speckle pattern transmitted through a scattering medium is diffusive.

In microwave experiments, the sample is composed of alumina spheres with diameter 0.95 cm and refractive index 3.14 embedded in Styrofoam shells giving an alumina volume fraction of 0.068. Randomly positioned spheres are contained in a copper tube with diameter 7.2 cm and length 61 cm. Speckle patterns are recorded at the output on a 1-mm grid. The full field is recovered with use of the Whittaker-Shannon two-dimensional sampling theorem \[67\]. The frequency is scanned from 14.7 to 15.7 GHz in 1601 frequency steps in 40 random realizations of the sample. New sample
configurations are created by rotating and vibrating the sample tube.

We follow trajectories of singularities as frequency is stepped. An example of singularities in a speckle pattern with the trajectories of two singularities is shown in Fig. 2.1 and the average square displacement $⟨R^2⟩$ vs. frequency shift $Δν$ is shown in Fig. 2.2. $⟨R^2⟩$ increases linearly after a frequency shift of approximately 10 MHz.

![Figure 2.2: Average of the square of singularity displacement with frequency shift. Slope of $⟨R^2⟩$ approaches a constant.](image)

The average of the square displacement is given by the velocity correlation function $[68]$, 

$$⟨R^2(t)⟩ = ⟨(\vec{r}(t) − \vec{r}(0))^2⟩ = \int_0^t dt' \int_0^t dt'' ⟨\vec{u}(t') \cdot \vec{u}(t'')⟩.$$ (2.1)

Here $\vec{r}$ is the position of a singularity. The variable $t$ denoting time can be replaced by $ν$ when considering the variation with frequency shift studied experimentally. The velocity auto-correlation function with frequency shift $C_u(Δν) = ⟨(\vec{u}(ν) \cdot \vec{u}(ν + Δν))⟩/⟨u⟩^2$ shown in Fig. 2.3 decays rapidly. The loss of correlation indicates that the motion of a phase singularity is random. The correlation function in the limit of vanishing frequency step $Δν = 0$, $C_u(0) = ⟨u^2⟩/⟨u⟩^2$ diverges. This can be cal-
culated with the probability distribution of velocities of phase singularities \( P(\bar{u}) = \frac{\bar{u}^2}{(\pi^2 \bar{u}^2 + 4)\pi} \), where \( \bar{u} = u/\langle u \rangle \) \[33, 64\].

![Figure 2.3: Measurement of the velocity auto-correlation function with frequency shift.](image-url)

The number of phase singularities that survive annihilation by oppositely charged singularities as the frequency is shifted is shown to fall exponentially in Fig. 2.4. Here we track the singularities over a limited frequency range with initial positions near the center of the output so that singularities will not disappear over the edge of the speckle pattern. The frequency \( \Delta \nu_N \) in which the number of singularities present at an initial frequency is reduced by a factor of \( e \) provides a characteristic frequency shift in which the speckle pattern changes, which is obtained without a measurement of the field. We expect that when the displacement of singularities \( R \) is normalized by the inverse of the square root of the singularity density to give \( R' \) and the frequency shift is given in units of \( \Delta \nu_N \), \( \Delta \nu' = \Delta \nu/\Delta \nu_N \), the variation of \( \langle R'^2 \rangle \) with \( \Delta \nu' \) will be universal. It is important to normalize \( R \) by the relevant scale of the speckle pattern so the results do not depend upon the degree of enlargement of a speckle pattern and reflect only the evolution of the generic structure of the pattern. Deviations might arise at early times or small frequency shifts due to differences in angular distribution of waves. Singularities also move more rapidly when they are just created or
annihilated and this non-ballistic motion is quite complex, which is out of the scope of this work.

![Graph](image)

Figure 2.4: Decay of the number $N$ of phase singularities with frequency shift in microwave experiments. $N_0$ is the initial number of singularities. $N/N_0$ falls to $1/e$ in a frequency shift $\Delta \nu_N = 20.4$ MHz.

We compare results of frequency domain experiments to random wave simulations in the time domain. Speckle patterns in a plane are generated in the simulation by the superposition of 300 randomly phased waves in the plane, $E(x, y, t) = \sum_i A_i \exp[i k_{ix} x + k_{iy} y - \omega_i t]$, where $k_{ix}$, $k_{iy}$, $k_{iz}$ and $A_i$ are randomly drawn from Gaussian distributions with zero mean and unit variance and $\omega_i = c \sqrt{k_{ix}^2 + k_{iy}^2 + k_{iz}^2}$. We find good agreement in Fig. 2.5 between measurements in the plots of $R' \times \Delta \nu'$ and simulations of $R' \times \frac{t'}{t_N}$. Within $\nu_N$ or $t_N$, singularities will migrate a distance about the average spacing of singularities, or $R' \sim 1$. With the normalized scale, the diffusion behavior is universal, independent of the values of experiment and simulation parameters.

The displacement of phase singularities can be characterized in terms of a two-dimensional diffusion coefficient with displacement normalized to the scale of the speckle pattern $D'_s$, which is obtained from the diffusion of singularities relative to the scale of the speckle pattern, and the photon diffusion coefficient through the random medium $D_p$. The diffusion of singularities in two
Figure 2.5: Comparison of the square displacement of phase singularities in microwave experiments in the frequency domain and simulations in the time domain. The displacement, frequency shift and time are normalized in terms of the singularity density and survival as explained in the text.

dimensions gives

$$\langle R' \rangle = 4D'_s \Delta \nu.$$  \hspace{1cm} (2.2)

The time of flight distribution of waves transmitted through a random medium is the Fourier transform of the field correlation function. As a result the field correlation frequency is proportional to the inverse of the width of the time of flight distribution [69]. For diffusive waves this gives the correlation frequency, $\delta \nu \sim \Delta \nu_N \sim D_p/L_{\text{eff}}^2$. Here, $L_{\text{eff}} = L + 2z_b$ is the effective sample thickness and $z_b$ is the length beyond the sample in which the linear falloff of intensity inside the sample extrapolates to zero. We expect that the field is decorrelated once singularities are displaced on average by a distance comparable to the size of a speckle spot so that $R' \sim 1$ when $\Delta \nu = \Delta \nu_N$. Then at the frequency shift, $\Delta \nu = \Delta \nu_N$, we have $4D'_s(D_p/L_{\text{eff}}^2) = C$, where $C$ is a constant of order unity. Thus $D_p$ can be obtained from observations of the motion of intensity nulls in the speckle
To confirm Eq. 2.3, we carried out finite difference time domain (FDTD) simulations on a structure of spheres with the same dielectric function and diameter as in the experiments using the Omnisim program from Photon Design, Inc. The frequency range in the simulations is 14.49-14.86 GHz over which the wave is diffusive for the lengths and sphere densities studied. The simulations give the intensity profile inside the system as shown shown in Fig. 2.6. The intensity extrapolates to zero at a distance \( z_b = 3.85 \) cm beyond the output face of the sample with density of 0.068 and sample length \( L = 40 \) cm. The intensity at a length is averaged over 1600 points measured on the cross section for 50 configurations. We did not include data points close to the input because the wave is then not totally randomized [70].

Once \( z_b \) is determined, we obtain \( L_{\text{eff}} \) and can check the validity of Eq. 2.3. The dependence
Figure 2.7: The relation between the diffusion coefficient of singularities in the output plane $D'_s$, the diffusion coefficient of photons through the sample $D_p$, and the effective sample length $L_{\text{eff}}$ in FDTD simulations. (a) Results are shown for three values of $L$ with the same structure and so the same value of $D_p$. $D'_s$ is proportional to $L_{\text{eff}}^2$. (b) The relationship between $D'_s$ and $D_p$ at $L = 23$ cm for samples 1, 2 and 3 correspond to filling fractions of alumina spheres of 0.068, 0.1 and 0.15. Here $C = 4D'_s(D_p/L_{\text{eff}}^2)$. 
of $D'_s$ on $L_{\text{eff}}$ is shown in Fig. 2.7(a). $D'_s$ is seen to increase linearly with $L_{\text{eff}}^2$. We have also demonstrated the inverse relationship between $D'_s$ and $D_p$ holds in simulations for three samples with alumina filling fractions of $f = 0.068, 0.1, 0.15$ in a 23-cm-long copper tube with all other simulation parameters the same as described above. $D_p$ is obtained from the average dwell time $\tau = d \varphi / d \omega = L_{\text{eff}}^2 D_p$ [27]. The dimensionless constant $C$ in Eq. 2.3 is found to be $0.11 \pm 0.1$ from the simulation. In conclusion, we found that phase singularities in the speckle pattern of light transmitted through a random medium diffuse with frequency shift or time delay and that their diffusion coefficient is inversely proportional to the photon diffusion coefficient within the medium. Since frequency and time are conjugate variables, analogous results are obtained in the frequency and time domains. The generic diffusion of phase singularities does not depend on the scale of the system so that results obtained in microwave measurements hold at optical wavelengths as well. Applications of this work include measurements of the photon diffusion coefficients in complex systems from radio to optical frequencies as well as the diffusion coefficients of vibrational excitation for sound, ultrasound and acoustic phonons. The variation of singularity motion in speckle patterns of monochromatic light reflected from samples which are inhomogeneous on scales greater than a mean free path may find applications to the analysis of the structure and function in biomedical tissue.

### 2.1.2 Motion of intensity maxima in averaged speckle patterns

For a dynamic sample, the speckle pattern is the average of intensity pattern over a finite exposure time so that phase singularities are washed out. In contrast, local maxima survive in intensity patterns which change in time due to internal motion. Thus intensity maxima can serve as better indicators of wave propagation in dynamic random systems.

Here, we track the motion of local maxima as the frequency is scanned in static samples with and without averaging over different frequency windows. Such windowing is inherent in time domain measurements of dynamic samples and is studied here in the frequency domain in order to explore the impact of windowing on the motion of local maxima. We find that for frequency shifts greater than the field correlation frequency, the average of the square displacement increases linearly with frequency shift. This yields the diffusion coefficient of local maxima in the frequency domain. The diffusion coefficient of maxima in intensity patterns averaged over a frequency window is found to
decrease linearly up to windows of approximately twice the correlation frequency at which point the diffusion coefficient falls to half the value without windowing. Since the statistics of speckle patterns and their change in diffusive systems is universal, we expect these results apply as well to intensity patterns averaged over time. These results, therefore, are useful in considering potential application to imaging dynamic systems.

Microwave measurements of spectra of transmitted field patterns are made with use of a vector network analyzer. The sample is composed of alumina spheres with diameter 0.95 cm and refractive index 3.14 embedded in Styrofoam shells. The alumina sphere filling fraction is 0.068 within a copper tube of inner diameter 7.0 cm and length 61 cm. Measurements are carried out on a 1-mm grid on the output plane, which is dense enough to recover the full transmitted waveform utilizing the Whittaker-Shannon two-dimensional sampling theorem [67]. The frequency is tuned from 14.7 to 15.7 GHz for 40 sample realizations. Random sample configurations are produced after each scan by briefly rotating and vibrating the sample. Wave propagation characteristics such as the degree of long-range intensity correlation hardly change over this frequency range, which lies between Mie resonances of the alumina spheres. Local maxima in intensity patterns are obtained at positions whose intensity is greater than its eight neighboring points.

The speckle pattern in a single configuration at 14.71 GHz is shown in Fig. 1a. The averages of patterns over frequency windows of 6.25, 12.25 and 18.75 MHz are shown in Figs. 1b-d. The positions of local maxima in intensity are indicated in the figure by yellow circles. The field correlation frequency determined from the first zero of the real part of the field-field correlation function $C_E(\Delta \nu) = \langle E^*(\nu)E(\nu + \Delta \nu) \rangle$ is 10 MHz.

The average of the square displacements $\langle R^2 \rangle$ of local maxima in intensity in the output plane is seen in Fig. 2.9 to increase linearly for frequency shifts larger than the field correlation frequency. Thus local maxima in intensity patterns also execute a random walk as phase singularities. The local maxima move slower as we increase the frequency window, and so finer spatial resolution is required to track the maxima.

The diffusion coefficient of intensity maxima $D_m$ in the two dimensional output plane is given via the relation $\langle R^2 \rangle = 4D_m\Delta \nu$. The diffusion coefficient is seen in Fig. 2 to decrease as the frequency window increases. The decrease is seen in Fig. 2.10 to be linear with the width of the window. This provides a way for obtaining $D_m$ of local maxima in the speckle pattern from measurements.
Figure 2.8: (a) Microwave speckle pattern in transmission at 14.71 GHz. (b)-(d) Speckle patterns averaged over a frequency window of 6.25, 12.25, 18.75 MHz starting at 14.71 GHz. Local maxima are indicated by the yellow circles in each pattern.

Of averaged intensity patterns. In principle, measurement of the velocity \( v \) of local maxima in light scattered from dynamic samples could also characterize the evolution of the intensity patterns. In practice, however, this would require accurate measurements of small displacements. The difficulty can be appreciated from measurements of average velocity from the ratio of the displacement and the frequency shift \( v = < \Delta R > / \Delta \nu \) as a function of \( \Delta \nu \). The smallest frequency step in our case is 0.625 MHz which is \( \sim 6\% \) of the correlation frequency. We see in Fig. 2.11 that the velocity obtained this way drops rapidly with \( \Delta \nu \). This appears to be due to the low spatial resolution in the experiment. Measurement of larger displacements at larger frequency shifts or time scales allows the diffusion coefficient to be determined accurately. One factor that might affect the speed of local
The number of local maxima within the output area does not change as the frequency window varies.

Since the field is not defined in the averaged intensity pattern, we consider the cumulant correlation function of the intensity averaged over the frequency window, \( C_I(\Delta \nu) = \langle I_\nu I_{\nu+\Delta \nu} \rangle / \langle I \rangle^2 - 1 \). We take the intensity correlation frequency \( \delta \nu_I \) to be the frequency shift at which \( C_I \) drops to half its initial value. The intensity correlation function for different frequency windows is shown in Fig. 2.13. \( \delta \nu_I \) is seen in Fig. 2.14 to increase as the frequency window \( \delta w \) increases. The value of \( C_I \) at \( \Delta \nu = 0 \), \( \langle I^2 \rangle / \langle I \rangle^2 - 1 = var(I/\langle I \rangle) \). The variation with frequency window of the variance of normalized intensity \( var(I/\langle I \rangle) \) over the cross section and averaged over all configurations is shown in Fig. 2.15. The variance of the normalized intensity \( var(I/\langle I \rangle) \) without averaging is 1.037. This is consistent with \( var(I/\langle I \rangle) = 1 \) in the diffusive limit, with the deviation coming from the long-range correlation in intensity.

One application of this work is to determine the photon diffusion coefficient \( D_p \) of the system from the motion of local maxima. The diffusion coefficient of local maxima in terms of normalized
Figure 2.10: The diffusion coefficient of local maxima vs. width of frequency window over which speckle patterns are averaged.

units is

\[ \langle R'^2 \rangle = 4D'_m \Delta \nu, \]  

(2.4)

where \( R'^2 = R^2/(A/N) \) is the square displacement multiplied by the density, with \( A \) the area in which measurements are made and \( N \) the number of maxima in the area. We expect that the correlation frequency is \( \delta \nu \sim D_p/L_{\text{eff}}^2 \) for diffusive waves, where \( L_{\text{eff}} = L + 2z_b \) is the effective sample length and \( z_b \) is the length beyond the sample at which the intensity extrapolates to zero [43, 72]. Local maxima are expected to move an average distance of \( R' \sim 1 \) in a correlation frequency. At \( \Delta \nu = \delta \nu \) and the expression for \( \delta \nu \), \( 4D'_m \delta \nu = 4D'_m D_p/L_{\text{eff}}^2 = \langle R'^2 \rangle \sim 1 \). This yield the expression \( D_pD'_m/L_{\text{eff}}^2 \) to be a constant. This relation is confirmed in FDTD using the Omnisim program from Photon Design, Inc. The frequency range in the simulation is 14.49–14.86 GHz with 700 frequency points. We have tested this relation for different samples with lengths of 23 cm, 30 cm, 40 cm with a filling fraction of 0.068 and with filling fractions of 0.1, 0.15 for the length of 23 cm, each with 50 configurations. Other parameters, such as the diameter of the tube, the dielectric constants and the size of the scattering balls are set to be the same as in the experiment. The differences between
$D_p D'_m/L_{\text{eff}}^2$ from all these samples are within 4%. Since the diffusion coefficient of local maxima for speckle patterns without averaging can be found from measurements of averaged speckle patterns via Fig. 3, the photon diffusion coefficient can be obtained from measurements of the motion of local maxima in the transmitted intensity patterns averaged over a range of frequency.

In conclusion, we have analyzed the evolution of speckle patterns averaged over a frequency window by tracking the motion of the local intensity maxima. We find that the local maxima diffuse as with a diffusion coefficient that decreases linearly with the frequency window. The photon diffusion coefficient can be obtained from the diffusion coefficient of local maxima. The decrease of the diffusion coefficient of local maxima with time windows provides a method for probing the dynamics. The scale of intensity patterns depends upon the aperture used so that patterns can be enlarged by using a smaller aperture such that the patterns can be measured with adequate resolution. An approach based on the principles explored here may be of use in imaging dynamic systems.
Figure 2.12: Average of the number of local maxima in windowed intensity patterns over the frequency range of $14.7 - 15.7$ GHz.

Figure 2.13: Cumulant intensity correlation functions.
Figure 2.14: Variation of intensity correlation frequency with frequency window.

Figure 2.15: Variance of the normalized intensity vs. frequency window.
2.2 Transmission eigenvalue distribution in samples with surface reflection

As introduced in chapter 1.5, the statistics of transmission eigenvalues can provide a full description of transmission properties through the samples. For diffusive waves in quasi-one-dimensional samples with no boundary reflectivity, the distribution of transmission eigenvalues is the so-called bimodal distribution, \[ \rho_0(T) = \frac{\xi}{2L} \frac{1}{\sqrt{1-T}}, \] (2.5)

which depends only upon the parameter $\xi/L$. Here $L$ is the sample length and $\xi$ is the localization length.

![Figure 2.16: An schematic illustration of the diffusion model.](image)

The impact of internal reflection on the average intensity can be incorporated in a diffusion model where the intensity inside the sample is found by solving the diffusion equation in a region beyond the sample length by an extrapolation length $z_b$ at which the intensity within the sample extrapolates to zero \[ z_b \approx \frac{0.7\ell(1+R)/(1-R)}{\ell} \] with $\ell$ the mean free path. One possibility is that the effect of surface reflection upon the distribution
of transmission eigenvalues could be accounted for by replacing $L$ by the effective sample length $L + 2z_b$ in Eq. (2.5).

We calculate the effect of boundary reflectivity on the distribution of transmission eigenvalues using supersymmetry method.

The distribution of transmission eigenvalues $\rho(\phi)$, where $T = 1/\cosh^2(\phi/2)$ is

$$\rho(\phi) = \Sigma_n \delta(\phi - \phi_n) = \frac{1}{2\pi} [F(\phi + i\pi) + F(\phi - i\pi)],$$

(2.6)

where $\phi_n$ corresponds to the $n$th transmission eigenvalue $T_n = 1/\cosh^2(\phi_n/2)$ and

$$F(\phi) = -\partial_\phi Z|_{\theta=i\phi} = \Sigma_n \frac{-i\sinh\phi}{\cosh\phi_n + \cosh\phi}.$$  

(2.7)

Here $Z(\theta, \phi)$ is the generating function

$$Z(\theta, \phi) = \left\langle \frac{\det(1 - \gamma_1 \gamma_2 t^t)}{\det(1 - \zeta_1 \zeta_2 t^t)} \right\rangle.$$

(2.8)

In the framework of supersymmetry, we introduce a $4 \times 4$ supermatrix field $Q$, which obeys the nonlinear constraint: $Q^2 = 1$. The functions $Z(\theta, \phi)$ and $F$ can be written as integrations of the auxiliary field $Q$.

$$Z(\theta, \phi) = \int D[Q] e^{-\mathcal{F}[Q]}, \quad \mathcal{F} = \frac{i\pi}{2} < tr_{tr}(Q \partial_x Q)_{bb}^{12} > Q |_{\theta=i\phi}.$$  

(2.9)

The supermatrix field is constrained by the boundary conditions arising from internal reflection,

$$(2z_b Q \partial_x Q + [Q, \Lambda])_{|_{x=0}} = 0, \quad (2z_b Q \partial_x Q + [Q, \Gamma])_{|_{x=0}} = 0$$

(2.10)

where $z_b = 0.7\ell \frac{1+R}{1-R}$ depends on the internal reflection coefficient $R$ at the interface, and the $+$ ($-$) sign corresponds to the left (right) interface located at $x = 0$ ($x = L$). The matrix $\Lambda = diag(I^{fb}, -I^{fb})^{ar}$ and $\Gamma$ is

$$\text{diag} \begin{pmatrix} \cos\theta_0 & -i\sin\theta_0 \\ i\sin\theta_0 & -\cos\theta_0 \end{pmatrix}^{ar}, \quad \begin{pmatrix} \cosh\phi_0 & \sinh\phi_0 \\ -\sinh\phi_0 & -\cosh\phi_0 \end{pmatrix}^{ar}.$$
Notice that the boundary conditions has already taken into account of a gauge transformation which takes \( \Lambda \) at the output to be \( \Gamma \) \([81]\). When there is no boundary reflectivity, Eq. 2.10 is equivalent to \([81]\)

\[
\lim_{x \to 0} Q(x) = \Lambda, \quad \lim_{x \to 0} Q(x) = \Gamma,
\]

(2.12)

The solutions of \( Q(x) \) are in forms similar to that in Eq. 2.2 with \( \phi_0 \) and \( \theta_0 \) replaced by position-dependent \( \phi(x) \) and \( \theta(x) \). The parameter \( \phi \) can be expressed in the form \( c_1 x + c_2 \) when subject to the saddle point equation

\[
\partial_x(Q \partial_x Q) = 0.
\]

(2.13)

This gives rise to the equations

\[
z_b C_1 = \sinh C_2, \quad z_b C_1 = -\sinh(C_1 L + C_2 - \phi_0).
\]

(2.14)

for the bosonic components. We replace \( C_1 \sim C_1 L, z_b \sim z_b/L = \zeta \), to make the parameters dimensionless. We eliminate \( C_2 \) to obtain the equation for \( C_1 \)

\[
\zeta C_1 = -\sinh((1 + \zeta_1)C_1 - \phi_0).
\]

(2.15)

When there’s no internal reflection, \( C_1 = \phi_0/L, F \) is obtained from Eq. 2.9 to be \(-i \phi/L \). Here we replace \( \phi_0 \) to \( \phi \) for it is just another name of the variable appeared in the boundary conditions with the same relationship with \( T \). Thus \( \rho(\phi) = constant \) and \( \rho(T) = constant \frac{1}{T \sqrt{1-T}} \), which is the bimodal distribution \([74]\).

When equal internal reflection is present on both sides of the sample, we obtain

\[
\phi = 2 \arccosh \left( \frac{\pi \zeta f}{\cos \frac{\pi f}{2}} \right) + \frac{\sin \frac{\pi f}{2}}{\zeta} \sqrt{\left( \frac{\pi \zeta f}{\cos \frac{\pi f}{2}} \right)^2 - 1},
\]

(2.16)

where \( f(T) \equiv \frac{\rho(T)}{\rho_0(T)} \) is the suppression factor of the new distribution compared with bimodal distribution. This yields a family of curves of \( \rho(T) \) for different values of \( \zeta \), which are shown in the
When only the output surface is reflective, \( f \) is found from

\[
\phi = \text{arccosh} \frac{\pi f \zeta}{\sin \pi f} - \frac{\cos \pi f}{\zeta} \sqrt{\left( \frac{\pi f \zeta}{\sin \pi f} \right)^2 - 1}.
\] (2.17)

The plots for different values of internal reflection are shown in the lower panel of Fig. 2.17. Eqs. 2.16 and 2.17 show that \( f(T) \) is governed by a single parameter \( \zeta \).

We see that a transition of the distribution of transmission eigenvalues appears as the internal reflection at one surface passes through a critical value. Above the critical value, the highest transmission eigenvalue is strictly smaller than unity and decreases with increasing internal reflection. When input and output surfaces are equally reflective, the highest transmission eigenvalue is unity and the transition disappears irrespective of the strength of surface reflection. We find that although the distribution for high transmission is completely suppressed by strong internal reflection at the input surface, surprisingly, it is substantially enhanced by adding an identical reflector to the output.

The results are confirmed in numerical simulations, using the recursive Green’s function method. A scalar wave is launched into a diffusive sample with \( \xi/L \approx 16 \), which is an 1800 × 600 rectangular lattice. The lattice spacing is unity (in units of the inverse wavenumber) with the wave velocity in the surrounding air set to unity. The squared refractive index at each site fluctuates independently around the air background value, taking random values over the interval \([0.7, 1.3]\). To create internal reflection, we add an additional layer of thickness 2 lattice spacings and constant refractive index at the output of the sample. Simulations were carried out for refractive indices of the surface layer of 1.8, 2, 2.1, 2.2, and 2.5. For each of these values, we found the 200 transmission eigenvalues in 3000 disordered configurations. The results, shown in Fig. 2.18, are in good agreement with the analytical results.

We compare our results with diffusion model with respect to the average transmission. Here we only study the most interesting case in which a single surface is reflective. We first rescale the average conductance by \( \xi/L \). The resultant quantity, denoted as \( g \), is given by

\[
g = \frac{1}{2} \int_0^1 dT T \times \frac{f(T)}{T \sqrt{1 - T}} = \frac{1}{2} \int_0^1 dT \frac{f(T)}{\sqrt{1 - T}},
\] (2.18)
Figure 2.17: The analytical theory shows that as the internal reflection on the output surface increases, $\rho(T)$ undergoes an ‘abrupt’ change (‘transition’) at $\zeta = 1$ (lower panel). This transition disappears when the internal reflection is the same on the input and output surfaces (upper panel). The values of $\zeta$ are 0.1, 0.25, 1, 1.2, and 8 from top to bottom. Inset: we re-plot the curves in the main panels in ln-ln scale.
Figure 2.18: The numerical results and the analytic predictions of are in good agreement. This confirms the transition in transmission eigenvalue statistics. \( \zeta \) is treated as the single fitting parameter, found to be 0.16, 0.93, 2.2, 4.0, and 17.2 from top to bottom.

which depends on a single parameter \( \zeta \). We cannot analytically calculate \( g(\zeta) \). Rather, we numerically compute it for a wide range of \( \zeta \) from \( 10^{-1} \) to \( 10^3 \). We find that the numerical results for \( g(\zeta) \) agree well with \( 1/(1 + \zeta) \), as shown in Fig. 2.19. Therefore, the average conductance is

\[
g(\zeta, \xi/L) = \xi/(L + z_b).
\]

It is important that in spite of the transition exhibited by the distribution \( \rho(T) \) at the critical value of \( \zeta = 1 \), the average conductance obeys Ohm’s law for both weak (\( \zeta \ll 1 \)) and strong (\( \zeta \gg 1 \)) internal reflection.
Figure 2.19: The average conductance obeys Ohm’s law. The data is fit to a straight line whose slope is unity.
Wave propagation inside random media

3.1 Fields inside random media

3.1.1 Field matrix and focusing inside random media

The depth of imaging in disordered samples is often limited by the scattering mean free path in which the incident phase is scrambled to create a random speckle pattern of intensity $[82, 37]$. But, because the wave is temporally coherent in static samples so that the random speckle pattern created is stable, the scrambling of the wave may be overcome by manipulating the incident wavefront to control specific characteristics of the transmitted field $[16, 83, 17, 19, 18, 84, 85, 86, 87]$. Vellekoop and Mosk demonstrated that light could be focused at a selected point in the transmitted field by maximizing the intensity at the point via sequentially adjusting the phase of the wave reflected by elements of the spatial light modulator (SLM). When the intensity at the point is maximized, the field from all pixels of the SLM are brought into phase so they interfere constructively to yield the maximal possible intensity at the focal point $[16, 83]$. The incident waveform may also be adjusted to significantly enhance or suppress transmission. The limits of control over the transmitted wave and the full statistics of transmission may be obtained from an understanding of the transmission matrix whose elements are the coupling coefficients relating the fields in incident and outgoing channels $[88, 89, 90, 23, 91, 92, 93, 94, 86]$. Analogously, it might be possible to relate the field inside the sample to the incident waveform. Though such measurements are not possible in general in three dimensional samples, it may nev-
theless be possible to focus at selected points inside the sample when the response to external excitation can be sensed via an opening into or a probe within the sample or via fluorescence or second harmonic generation at a sensitized point [95, 96, 97]. A consideration of the relationship of the fields that are produced inside the sample would enable the characterization and control over the field in the interior of random samples.

In this section, we describe focusing inside scattering systems. We introduce the field matrix $e$ relating the field at a depth $z$ inside the system to the incident field. The energy density is greatly enhanced at $z$ when the incident wavefront is coupled to the highest energy density eigenchannel inside random media. The contrast in optimal focusing is determined by the energy density eigenchannel participation number $M(z)$ of the energy density matrix $ee^\dagger$ and the number of channels $N$. The integrated energy density over the cross section at $z$ for a single incident channel relative to the average over all incident channels is shown to be $M^{-1}(z)$. The statistics of energy density mimic those of transmitted flux even though the eigenvalues of $ee^\dagger$ do not have an absolute maximum as do the eigenvalues of $tt^\dagger$ which cannot exceed unity. The energy density in a given energy density eigenchannel may greatly exceed the incident energy density since wave trajectories can pass many times through a given cross section of the sample. We therefore review the properties of the transmission matrix before discussing the field matrix. The transmittance $T$, which is the sum of the flux transmission coefficients between all of the $N$ incident and outgoing propagation channels, may be expressed as the sum of all the transmission eigenvalues $T = \sum_n^N \tau_n$ [98]. Here $\tau_n$ is the square of the $n$th singular value $\lambda_n$ of $t$ obtained from the singular value decomposition of $t$, $t = U\Lambda V^\dagger$, where $U$ and $V$ are unitary matrices and $\Lambda$ is a diagonal matrix with singular values $\lambda_n$. The transmittance is related to the dimensionless conductance, which is the average conductance in units of the quantum conductance, via $g = \langle T \rangle = \langle \sum_n^N \tau_n \rangle$ [99, 100]. For diffusive samples, $g = \xi/L = N\ell/L$, where $\xi = N\ell$ is the localization length. Transmission in diffusive samples is governed by open channels with eigenvalues $\tau_n > 1/e$ first introduced by Dorokhov to describe the scaling of conductance [73, 98]. The contrast between the peak and background intensity in optimal focusing is essentially the channel participation number $M = (\sum_n^N \tau_n)^2/(\sum_n^N \tau_n^2)$ [101]. However, the transmission matrix only deals with the transmission coefficient and it cannot therefore describe the field inside a random sample which is of interest for focusing and energy depositing inside the medium. The field matrix $e(z)$ with elements $e_{ba}(z)$ relates the field in channel $b$ at a depth $z$
inside a random sample to the fields in an incident channel \( a \) as \( E_\beta(z) = e_{\beta a}(z)E_a \). In practice, \( a \) and \( b \) represent the locations of sources and detectors, respectively, instead of extended transverse propagation channels. The singular value decomposition \( e(z) = U(z)\Lambda(z)V(z)^\dagger \) yields the singular values \( \lambda_n(z) \). The square of the \( \lambda_n(z) \) are the energy density eigenvalues \( \epsilon_n(z) \). The energy density \( u(z) \) at depth \( z \) when all incident channels are excited each with unit energy density, is the sum of the energy density eigenvalues \( u(z) = \sum_{n=1}^{N} \epsilon_n(z) \). Note that at \( z = L \), where \( L \) is the length of the sample, \( e(L) \) is essentially the field transmission matrix \( t \). As \( z \to L \), the energy density eigenvalues \( \epsilon_n(z) \) correspond to the transmission eigenvalues \( \tau_n \) and \( u(x) \) approaches the transmittance \( T \).

The field inside the random sample is found from simulations of wave propagation in disordered media using the recursive Green’s functions method \cite{102}. Simulations are carried out in an ensemble of 300 configurations with \( g = 16 \) and in 2000 configurations with \( g = 0.3 \) and \( g = 0.195 \). The dimensionless conductance \( g \) depends upon \( N \), \( L \) and the transport mean free path \( \ell \). The diffusive samples studied here are composed of \( 800 \times 500 \) units with the dielectric function at each site drawn randomly from a rectangular distribution from \( 0.8 \) to \( 1.2 \). The localized samples are composed of \( 750 \times 100 \) units and \( 2250 \times 100 \) units with the values of the dielectric function drawn randomly from the rectangular distribution between \( 0.5 \) and \( 2.5 \). Waves with wavelength of \( 2\pi \) in units of the lattice spacing are launched from one side and the fields are calculated at all locations. For the diffusive system \( L/\xi = 0.0625 \) while \( L/\xi = 3.3, 10 \) for the two localized samples. \( u(z) \) for the samples supporting diffusive and localized waves is shown in Fig. 3.1. For diffusive waves, the energy density falls off linearly, while for localized waves we see a deviation from linearity. This can be explained in terms of a position-dependent diffusion coefficient \( D(z) \), which reflects the increasing renormalization of transport for points more remote from the sample boundaries \cite{103, 2, 104}. For localized waves, the results are in good agreement with the integrated intensity obtained with the position dependent diffusion coefficient \( D(z)/D(0) = e^{-z(L-z)/L\xi} \). The maximum energy at depth \( z \), which is \( \epsilon_1(z) \) and the average energy per channel is shown in Fig. 3.2. The energy is greatly enhanced at \( z \) for the highest energy density eigenchannel at \( z \) compared to the average energy at \( z \).

The condition for optimal focusing at a target point \( \beta \) can be obtained with the incident wavefront \( e_{\beta a}(z)/\sqrt{I_\beta(z)} \), where \( I_\beta(z) = \sum_a |e_{\beta a}(z)|^2 \). This is similar to the results for focusing obtained at the output \cite{88}. Since \( E_\beta(z) = e_{\beta a}(z)E_a \) for a given \( a \), \( |E_\beta(z)| \leq |e_{\beta a}(z)E_a| \). Only when
Figure 3.1: The ensemble average of the energy density inside the sample for incident radiation with unit energy density.

$E_a \propto e_{\beta a}(z)$ can the maximum of $|E_\beta(z)|$ be reached. We choose $E_a = e_{\beta a}(z)/\sqrt{I_\beta(z)}$ so that the incident flux is unity and the fields are brought into phase at the focal point. The profiles of the focused beam at two depths into the sample in a diffusive system with $g = 16$ are shown for two values of $z$ in Fig. 3.5.

The contrast in focusing is greater at $z = L/4$ than at $z = L/2$ because the participation number of energy density eigenchannels is greater close to the input surface. The contrast at the output is determined by the eigenchannels participation number $M = (\sum_n \tau_n)^2/(\sum_n \tau_n^2)$ [101]. We find similarly that focusing inside a random medium is determined by the energy density eigenchannels participation number at depth $z$

$$M(z) = (\sum_n \epsilon_n(z))^2/(\sum_n \epsilon_n(z)^2), \quad (3.1)$$

which is demonstrated below. The comparisons of the contrast $\mu(z)$ and the energy density eigenchannel participation number $M(z)$ are shown in Fig. 3.4 for diffusive and localized samples. $\mu(z)$
Figure 3.2: The ensemble average of the highest energy density eigenvalue at $z$, $\langle \epsilon_1(z) \rangle$, and the average energy per channel $\langle u(z)/N \rangle$ for diffusive samples with $L/\xi = 0.0625$.

is the ratio of the average over all configurations of the intensity at the focal points at depth $z$ and the average background intensity within the cross section at depth $z$,

$$\mu(z) = \langle I(z) \rangle_\beta / \langle I(z) \rangle_{\beta \neq \beta}. \quad (3.2)$$

The results are in good agreement after a depth as at which the waves are randomized and spread throughout the cross section.

We can express $\mu(z)$ in terms of $M(z)$. We show the key steps below. More details of the calculations can be found from those carried out at the output [101, 86], with all the quantities now inside random media. The intensity at the focal point is $I_\beta(z) = \left| \sum_a |e_{\beta a}(z)|^2 \right|^2 / I_\beta(x) = I_\beta(z)$. For any $\beta$ and $a$, $e_{\beta a}(z)$ can be expressed as $e_{\beta a}(z) = \sum_n \lambda_n(z) u_{n\beta}(z) v_{na}^*(z)$ from the singular value decomposition. The focused intensity can be expressed as

$$I_\beta(z) = \sum_n \epsilon_n(z) |u_{n\beta}(z)|^2, \quad (3.3)$$
Figure 3.3: Intensity distribution over the cross sections at $z = L/4$ and $z = L/2$ for a locally two-dimensional system with $L/\xi = 0.0625$ and the focal point at the center of the cross section $x = 0$. Here $x$ is the coordinate in the transverse direction ranges from $[-A/2, A/2]$ with $A$ the width of the system. $I_B$ is the average background intensity at $x \neq 0$. 
Figure 3.4: Comparison of the contrast in focusing and the eigenchannel participation number along the sample length for (a) diffusive waves with $g = 16$, $N = 166$ and (b) localized waves with $g = 0.3$, $N = 32$ respectively.
where we have used $\sum_a v_{na}^*(z) v_{na}(z) = \delta_{nn'}$ since the energy density eigenchannels are orthonormal. The intensity at $b \neq \beta$ is

$$I_b(z) = \left| \sum_n \epsilon_n(z) u_{nb}(z) u_{n\beta}^* \right|^2 / I_\beta(z). \quad (3.4)$$

Since $\langle |u_{n\beta}|^2 \rangle = 1/N$, this gives

$$\langle I_\beta \rangle = \sum_n \epsilon_n(z)/N. \quad (3.5)$$

$$\langle I_b \rangle_{b \neq \beta} = \frac{1}{N-1} \frac{\sum_b \epsilon_n^2(z)/N}{\sum_b \epsilon_n(z)/N} - \frac{1}{N-1} \sum_b \epsilon_n(z)/N. \quad (3.6)$$

In the limit of $N \gg 1$, we take $N-1 \to N$,

$$\mu(z) = \frac{1}{1/M(z) - 1/N}. \quad (3.7)$$

For $N \gg M(z)$, this expression reduces to $\mu(z) = M(z)$. In contrast to transmission eigenchannels that cannot exceed unity, the energy density eigenvalues can be much larger than unity. From Fig. 3.4, we see that $M(z)$ falls rapidly with increasing depth into the sample. We find that for large $n$, the energy density eigenvalues $\epsilon_n(z)$ fall quickly with increasing $z$ so that fewer incident channels contribute to the energy density as the output boundary of the sample is approached. For localized waves, $M(z)$ approaches a constant close to unity near the sample output since in localized samples transmission is then dominated by the highest eigenchannel.

The statistics of intensity inside the random medium is also determined by $M(z)$. For $N \gg 1$, the variance of the total energy relative to its average $N u_a(z)/u(z)$ at depth $z$, equals $M(z)^{-1}$ within one configuration, which is similar as at the output [101 86]. Here $u_a(z) = \sum_b |e_{ba}(z)|^2$ and $u(z) = \sum_a u_a(z)$. The comparison is shown in Fig. 3.5 for a random sample of length 4500 and width 1500 with all other parameters the same as the diffusive samples in Fig. 3.4. The total energy of an incident channel is $u_a(z) = \sum_n \epsilon(z) |v_{na}(z)|^2$. var($N u_a(z)/u(z)$) = $((N u_a(z)/u(z))^2) - 1$, since
\[ \langle \frac{N u_a(z)}{u(z)} \rangle = 1. \]  

We obtain

\[
\langle \left( \frac{N u_a(z)}{u(z)} \right)^2 \rangle = \sum_n (\epsilon_n(z)/u(z))^2 \langle N^2 |v_{na}(z)|^4 \rangle 
+ \sum_{n \neq n'} \epsilon_n(z) \epsilon_{n'}(z) \langle N^2 |v_{na}(z)|^2 |v_{n'a}(z)|^2 \rangle / u^2(z). 
\]

(3.8)

For \( N \gg 1 \), \( \langle |v_{na}(z)|^2 \rangle = 1/N \), \( \langle |v_{na}(z)|^4 \rangle = 2/N^2 \), so that

\[
\langle \left( \frac{N u_a(z)}{u(z)} \right)^2 \rangle = \frac{\sum_n \epsilon_n^2(z) + (\sum_n \epsilon_n(z))^2}{(\sum_n \epsilon_n(z))^2}. 
\]

(3.9)

This gives

\[ \text{var}(\frac{N u_a(z)}{u(z)})^{-1} = M(z). \]

(3.10)

Note that when \( N \) is not much larger than unity, the relation \( \langle |v_{na}(z)|^4 \rangle = 2/N^2 \) is not satisfied. We therefore carry out the simulations shown in Fig. 3.5 in a sample with \( N = 500 \).
In conclusion, we have described the characteristics of optimal focusing and maximal energy deposition inside disordered systems in terms of the energy density eigenvalues at a depth \( z \). In future work we will consider the probability density of the energy density eigenvalues inside random media, which approaches the bimodal distribution \[23\] towards the output of the sample for diffusive waves.

### 3.1.2 Statistics of energy density eigenvalues

The distribution of the transmission eigenvalues at the output of the sample follow the bimodal distribution \[23\], which is described in section 2.2. This is seen in Fig. 3.6, for a random ensemble with \( g = 16 \), with two peaks near the \( \epsilon \to 0 \) and \( \epsilon \to 1 \) ends. The data used here are the same as in the last section. We can conjecture that the distribution of energy density eigenvalues inside random media does not follow the bimodal distribution, since the energy density eigenvalues can be much greater than 1 inside random media. The distribution in the middle of the sample is shown in Fig. 3.6b.

![Figure 3.6: The statistics of the energy density eigenvalues for (a) \( x = L \) and (b) \( x = L/2 \)](image)

The asymptotic behavior at \( x = L/2 \) near \( \epsilon \to 0 \) is similar to the bimodal distribution with \( \sim 1/\epsilon \), as shown in Fig. 3.7b, which indicates that the small \( \epsilon \) side is not affected by the removal of an upper limit of the energy density eigenvalues. However, for large \( \epsilon \), there is no such peak as in the bimodal distribution of \( 1/\sqrt{1 - \epsilon} \). The asymptotic behavior in Fig. 3.7b shows that the fall-off is exponential.

More analysis regarding the full statistics and a theoretical framework will be developed in the
Figure 3.7: The statistics of the energy density eigenvalues for (a) small $\varepsilon$ and (b) large $\varepsilon$ future, which can provide a better understanding of wavefront shaping to control energy delivery to the interior of random media.

3.2 Single parameter scaling and energy distribution within random media

The statistics of wave transport through 1D and quasi-1D random samples is governed by the single parameter scaling (SPS) theory, which states that the distribution of the logarithm of the transmittance is a Gaussian with variance equal to twice the magnitude of the average transmittance $T$ [23]. We explore in 1D systems the statistics of the logarithm of the energy density, which is a self-averaging quantity, and found an extension of SPS to the interior of random media.

The maximum-entropy approach [24], is a random-matrix theory which gives rise to a Fokker-Planck equation, known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [105, 30], describing the “evolution” with sample length $L$ of the probability distribution function (pdf) $p_L(M)$ of the system transfer matrix $M$, where the the disordered system is assumed to contain a large number of very weak scatterers.

Consider the scattering by a 1D random distribution of scatterers, the intensity at location $x$ between the segments, normalized so that $\mathcal{W}(L) = T$, is

$$\mathcal{W}(x) = |ae^{ikx} + be^{-ikx}|^2. \quad (3.11)$$
Figure 3.8: The statistics of the energy density eigenvalues for (a) small $\varepsilon$ and (b) large $\varepsilon$

The individual transfer matrix for the systems before and after $x$ are denoted as $M_1$ and $M_2$.

$$M_i = \begin{bmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{bmatrix}, \quad i = 1, 2,$$

with the condition $|\alpha_i|^2 - |\beta_i|^2 = 1$, thus satisfying the requirements of time-reversal invariance and flux conservation. When no index $i$ is employed, the various quantities refer to the wire as a whole. And we define $s = L/\ell$, $s_1 = x/\ell$, $s_2 = (L - x)/\ell$.

The energy density given in Eq. (3.11) is expressible as $\text{[106]}$

$$W(x) \equiv W(x; M_1, M_2) = \frac{\alpha_2^* e^{ikx} - \beta_2^* e^{-ikx}}{|\alpha_2 \alpha_1 + \beta_2 \beta_1|^2} \equiv TF_x(M_2),$$

Calculations show that $\text{[107]}$

$$\langle \ln W(x) \rangle = -\frac{x}{\ell}.$$  

For $x = L$, this reduces to $\langle \ln \mathcal{T} \rangle = -\frac{L}{\ell}$.

The linear fall off of $\langle \ln W(x) \rangle$ is unexpected and remarkably simple, since the value as a particular location $x$ only depends on $\ell$ and is not affected by the sample length. As can be seen from 1D scattering matrix simulation results in Fig. 3.8 that the comparison of $\langle \ln W(x) \rangle$ and $\langle W(x) \rangle$ when we vary sample length. It is clear that at $x$, the value of $\langle W(x) \rangle$ is larger since more energy is reflected back when the sample is longer. However $\langle \ln W(x) \rangle$ does not depend on the sample length and is only governed by a single parameter $\ell$. 

46
For the variance of $\ln W(x)$, we can only obtain approximations results,

$$\text{var}[\ln W(x)] = \frac{2x}{\ell} + 2C + \omega(s_2), \quad s, s_2 \gg 1.$$  

(3.15)

where $C = \int_0^\infty \langle T \rangle_s ds = 1.64493 \cdots$, and $\omega_i(s)$ are functions that tend to 0 as $s \to \infty$. For $x = L$,

$$\left(\text{var}(\ln T)\right)_{L/\ell} = \frac{2L}{\ell} - 2C + \omega_1(s), \quad s \gg 1.$$  

(3.16)

The first term of Eq. (3.16) represents the well-known result that the variance of the logarithm of the transmission scales as twice (the absolute value of) its expectation value. The correction $2C$ is a correction to the well-known single parameter scaling.

We have carried out microwave experiments to explore the statistics of $\ln W(x)$ inside a random single-mode waveguide. Since $\ln W(x)$ self-averages, we are able to obtain sufficient sampling to compare the measurements to theoretical predictions in 100 random configurations. Waves are launched from one end of the waveguide and the signal is detected by an antenna just above a slit along the length of the waveguide. The sample is composed of randomly positioned elements contained within a WR-90 rectangular copper waveguide, with width and height of 2.286 cm and 1.016 cm, giving a band cutoff frequency of 6.56 GHz. The sample is a made up of slabs dielectric ceramic material with dielectric constant $\epsilon = 15$, of thickness of 0.66 cm, covering 93% of the waveguide cross section of the waveguide, and a thin U-shaped Teflon which is essentially air. The Each elements in each random configuration are randomly selected with has an equal probability chance of being either a dielectric or air layer. The Thickness of the air layers may have is randomly selected from 1.275, 2.550, or 3.825 cm with equal probability. The incident frequency ranges from 8.50 GHz to 8.59 GHz in 400 frequency steps. The total length of the sample is of length $L=60$ cm.

Absorption is removed by Fourier transforming the spectrum into the time Domain, multiplying by $e^{-\Gamma_a t/2}$ and then transforming back into the frequency domain.; $\Gamma_a = 0.011 \text{ ns}^{-1}$ is the decay rate of energy within the sample due to absorption and leakage through the slot slit along the sample length. It is obtained from the measurement of the linewidth in angular frequency units of the narrowest mode when copper reflectors are placed at the ends of the sample with only a small opening in the reflector onat the LHS of the sample to admit energy from the
source antenna. Absorbers are placed in the waveguide between the source antenna and the sample input and following the sample output to reduce reflection back into the sample.

\[ \langle \ln W(x) \rangle \]

Figure 3.9: Microwave measurements of \( \langle \ln W(x) \rangle \).

The experimental results for \( \langle \ln W(x) \rangle \) shown in Fig. 3.9 are well fit by the line \( 5.06x/L + 0.02 \). This linear behavior is in agreement with the fit \( s = L/\ell \approx 5 \) and \( \ell \approx 12 \text{ cm} \). The \( \operatorname{var}(\ln W(x)) \) is shown in Fig. 3.10: it increases linearly near the beginning of the sample and bends as \( x \) approaches the output boundary.

Here \( \operatorname{var}(\ln W(0)) \sim 5 \) is larger than the predicted value of \( 2C = 3.29 \). This is a consequence of reflection by the source antenna, which will be discussed in the next session. The pdf of \( \ln W(x) \) is shown in Fig. 3.11. At the beginning of the sample the distribution is not symmetric; the fit shown in Fig. 3.11 utilizes different Gaussian functions above and below the maximum value. However, for \( x = L/2 \) and \( x = L \), the pdfs for \( \ln W(x) \) are Gaussian functions as seen in Fig. 3.11.
3.3 Effects of boundary reflectivity on the energy distribution inside random media

3.3.1 Spatial profile of the average energy density based on local diffusion theory

The intensity profile inside random media is governed by the position dependent diffusion theory, where the diffusion coefficient is expressed as $D(x) \sim e^{-\frac{x(L-x)}{\xi}}$ [80].

When there is boundary reflectivity, we obtain that

$$D(x)/D_0 \propto e^{-\frac{(x+z_{b1})(L+z_{b2}-x)}{L+2z_{b1}+2z_{b2}}}.$$  \hspace{1cm} (3.17)

More calculations are in the appendix.

We carried out 1D simulations using the scattering matrix method to explore the impact of surface reflection on the energy density profiles of localized waves inside random media. The sample is composed with binary layers of refractive indices of $n_1 = 1$ and $n_2 = 1.6$. The average thickness of each layer is 1. The thickness of the material with high refractive index $n_2$ is fixed and the thickness
of the $n_1$ layers varies randomly between 0.5 and 1.5. Waves are launched from the left and the incident wavelength is between 1.712 and 1.760. A layer with refractive index $n_r = 25, 50, 75$ and thickness 2 is placed over the output of the sample. The average for 500 configurations of energy density profiles for different boundary reflectors of the same sample length of 200 layers are shown in Fig. 3.12a. We see that by varying the boundary reflectivity, the intensity profiles are no longer symmetric with respect to the center of the sample. Instead, they are a truncated profile of a longer sample. To obtain the value of $z_b$, we fix the refractive index of the reflector at the output to be $n_r = 50$ and obtain the profile of energy density in samples of length 100, 150, 200, 250 layers. We find that the value of $z_b$ at the output surface has the same value of 120 layers for all sample lengths. The simulation results are in excellent agreement with analytical predictions given by Eq. A.20, with $z_{b1}$ set to zero, as shown in Fig. 3.12b.

### 3.3.2 Effects of barriers on the statistics of the logarithm of the energy profile

The relation of $\langle \ln W(x) \rangle = -x/\ell$, as described in section 3.2 depends only on $\ell$ and is independent of the sample length. A reasonable conjecture is that it will also be insensitive to changes of boundary conditions. We test this using transfer matrix simulation methods the same as in section 3.3.1. The
Figure 3.12: (a) The average energy density $\langle W(x) \rangle$ for fixed sample length of $L = 200$ layers without boundary reflection at the input for no reflector at the output and the refractive indices of the reflector at the output boundary of $n_r = 25, 50, 75$. (b) $\langle W(x) \rangle$ for sample lengths of 100, 150, 200 and 250 layers and refractive index of the boundary reflector at the output of $n_r = 50$. The solid black lines are predictions from Eq. (19). The parameter $z_b$ is 120 layers for all the samples.

results are shown in Fig. 3.13. The artificial reflector placed at the output of the sample has a refractive index of $n_r = 75$. We see that the logarithm of the energy profile $\langle \ln W(x) \rangle$ is not affected by the reflectivity over the boundary, thus is very robust against changes after $x$. More analysis such as how a barrier inside random media affects $\langle \ln W(x) \rangle$ and how the full statistics are varied is still under study, which will provide insights to wave interference inside random media.
Figure 3.13: The average intensity profile from 1D simulations with and without boundary reflectivity.
Density of states (DOS) and suppression of local density of states (LDOS)

4.1 Density of states of random media

The statistics of DOS controls emission, absorption and wave localization of a medium \[108, 109\]. The DOS of a bounded open medium for classical waves is the density of quasi-normal modes or resonances of a region per unit angular frequency expressed as \[110\]

\[
\rho(\omega) = \frac{1}{\pi} \frac{\Gamma_n/2}{(\Gamma_n/2)^2 + (\omega - \omega_n)},
\]

(4.1)

where \(\omega_n\) and \(\Gamma_n\) are the central frequency and linewidth of the \(n\)th mode. The integration over frequency of each mode in Eq. (4.1) is unity. Another way to express the DOS is in terms of the scattering matrix \(S\) \[111, 112\],

\[
\rho(\omega) = -i \frac{1}{2\pi} \text{Tr} S^\dagger dS/d\omega.
\]

(4.2)

This is proportional to the integral of the energy stored within the medium for unit incident flux in each channel \[112, 113\],

\[
\rho(\omega) \propto \Sigma_\alpha^{2N} \int_V dV I_\alpha(r, \omega).
\]

(4.3)

The measurements of the scattering matrix is difficult since measuring all channels is hard, especially the reflecting waves. However, the calculations of Brandbyge and Tsukada of the local DOS of
electrons based on the scattering matrix show that the DOS can be determined from measurements of the transmission matrix (TM) \cite{114}. The DOS can be obtained from the summation of the derivative of the composite phase of the transmission eigenchannels with angular frequency. The phase derivative of the $n$th transmission eigenchannel is

$$d\theta_n/d\omega = \frac{1}{i} (u^*_n(du_n/d\omega) - v^*_n(dv_n/d\omega)), \quad (4.4)$$

where $v_n$ and $u_n$ are the input and output wavefront of the $n$th transmission eigenchannel. The DOS is expressed as

$$\rho(\omega) = \frac{1}{\pi} \sum_n^{\eta} d\theta_n/d\omega. \quad (4.5)$$

Each term in the sum is the contribution of a single eigenchannel to the DOS, the eigenchannel density of states (EDOS).

We have conducted microwave experiments to test the agreement between Eq. 4.1 and Eq. 4.5 and results are shown in Fig. 4.1. The DOS found from the modal decomposition involves the analysis of the entire field spectrum and modes can be found from measurements of the TM as well as from measurements of field spectra within the interior of the sample, from which it is impossible to find the transmission eigenchannels. In contrast, the analysis of the transmission eigenchannels requires only the TM at two slightly shifted frequencies so that the derivative of the phase can be found. Thus the DOS determined from an analysis of modes and channels is independent. A plot of the spectrum of the individual modes corresponding to the terms in Eq. 4.1 is shown in Fig. 4.1a. Good agreement is found in Fig. 4.1b between the sums of the contributions to the DOS of all eigenchannels and of all modes determined from the TM and from spectra of the field inside the sample. The analysis of the TM can thus be used to find the DOS in samples with strong modal overlap for modal analysis is not possible.

### 4.2 The vanishing of local density of states

Anderson showed that electrons are exponentially localized in unbounded disordered systems when the range of random electron energies on different lattice sites exceeds a critical level relative to the coupling between sites. Thouless noted that, in bounded media, the amplitudes of localized electron
Figure 4.1: (a) Contributions of the individual modes in Eq. 4.1 to the DOS. (b) Comparison of the DOS determined from the TM (red curve) by summing spectra of $d\theta_n/d\omega$ and modes found from spectra of the field at the output (black curve) and from spectra of the field inside the sample (gray dotted curve).
states at the sample’s edges would be small so that the ratio of the linewidth and spacing of levels, known as the Thouless number, \( \delta \), would fall below unity. Under the widely accepted assumption that the average local density of states (LDOS) is an intensive parameter, \( \delta \) is equivalent to the conductance in units of the quantum of conductance, \( g \). Here we show in microwave measurements and computer simulations that the LDOS in the center of random dielectric quasi-one dimensional waveguides with no structural correlation tends to vanish as the sample length \( L \) increases once waves are localized. Since levels are localized at the sample boundaries, the level width and spacing, and so, \( \delta \), saturate while \( g \) continues to fall exponentially with \( L \). Such boundary localization is distinct from suppressed propagation in traditional localized samples and in the band gaps of periodic or nearly periodic structures. Here the average energy density \( W(x) \), at a fixed depth \( x \) into a random waveguide falls as \( L \) increases, whereas \( W(x) \) increases with \( L \) in samples with uniform LDOS, and is constant in periodic systems. The vanishing of the LDOS decouples the bulk of random samples from the surroundings and so cloaks the interior of the medium. The interplay between the DOS and localization may ease the conditions for localizing waves even in open dielectric systems.

The ability of waves to couple in and out of random systems is determined by the LDOS which can be viewed from Fermi’s golden rule that the rate of emission is \( \frac{2\pi}{\hbar} |H_{ij}|^2 \rho \), where \( H_{ij} \) is the matrix element of the perturbing Hamiltonian. The LDOS at a depth \( x \) into a sample, \( \rho(x) \), is given by the sum of the energy densities produced by all the incident channels that couple to the system with unit incident flux \( \rho(x) = \sum_{a=1}^{2N} I_a(x) \) \[113\]. In quasi-one dimensional samples with constant cross section and reflecting sides, there are \( N \sim 8\pi A/3\lambda^2 \) incident channels on the two ends of the sample, where \( A \) is the sample cross section and \( \lambda \) is the incident wavelength. The average LDOS relative to the value in free space, \( \bar{\rho}(x) = \rho(x)/\rho_0 \), can be written as \( \bar{\rho}(x) = W(x) + W(L-x) \), where \( W(x) \) is the ensemble average of the energy density per unit depth into the medium at depth \( x \) due to unit incident flux. At the output surface, \( W(x) \) equals the average transmission, \( W(x = L) = \langle T \rangle /N \), where \( T \) is the transmittance which is the analogue of \( g \) for classical waves. For diffusive waves, \( \langle T \rangle = N\ell/L > 1 \), where \( \ell \) is the scaling mean free path. \( W(x) \) falls linearly with \( x \), with \( \langle T \rangle /N = -DdW(x)/dx \), where \( D \) is the diffusion coefficient. The sum of energy density over all channels on both sides of the sample and so the LDOS is constant within the sample and independent of \( L \) and \( g \). It has been widely assumed that, in the absence of structural correlation, the LDOS is an intensive parameter for localized samples as well. This
is the basis of analytical calculations \[2\], and has been confirmed in computer simulations \[104\] and optical measurements in random waveguides \[115\]. The uniformity of the LDOS throughout a sample supporting localized waves requires that \(W(x)\) be anti-symmetric about the center of the sample, with \(W(x = L/2)\) to be unity \[2\]. This is seen in simulations in strongly disordered 1D samples shown in Figs. 4.2 a,b.

To measure the LDOS within random quasi-1d samples, we carry out microwave measurements of intensity along a slit in a multimode cylindrical copper waveguide with diameter of 7.3 cm as shown in Fig. 4.3. The sample is made up of randomly positioned alumina spheres with index of refraction \(n = 3.14\) and diameter 0.95 cm contained in Styrofoam shells to produce a filling fraction
Fig. 4.3: Schematic illustration of the experimental setup. The antenna is translated along the tube to measure $W(x)$

of $f=0.068$. Spectra are taken in two frequency ranges. The wave is strongly scattered in the lower frequency range of 10-10.24 GHz just above the first Mie resonance of the spheres at 9.6 GHz. Scattering is significantly weaker in the higher frequency range of 14.7-15.7 GHz. The localization lengths are $\xi=24$ and 340 cm in the low and high frequency ranges.

The intensity measured along the slit in the frequency range of $14.7 - 15.7$ GHz in a sample of length, $L = 0.18\xi$ is seen in Fig. 4.2c to fall linearly, as expected for diffusing waves, giving the expected uniform LDOS within the sample with a constant the sum of the intensity profiles excited from the left and right, as shown in Fig. 4.2d. We can extrapolate the energy density profile in Fig. 4.2c to zero at a distance beyond the sample length, which is called the extrapolation length related to the boundary reflectivities [79]. The extrapolation length obtained is 13 cm, which is in accordance with the previous results obtained from the field-field correlation function. However, for localized waves in samples of length $L = 1.7\xi$ and $2.5\xi$, $W(x = L/2)$ falls substantially below unity in the center of the sample (Fig. 4.2e), with a corresponding drop in $\tilde{\rho}(x)$ in the middle of the sample (Fig. 4.2f). We also see that when the sample is longer, $W(x)$ at the beginning of the sample at a certain depth $x$ is smaller, in contrast to regular random samples where $W(x)$ at $x$ will be larger for longer samples.
One possible explanation for the suppression of LDOS is due to residual periodicity or structural correlations. The DOS vanishes in electronic or photonic band gaps between pass bands in periodic systems, and is also substantially suppressed in the pseudogap of amorphous semiconductors and disordered dielectric structures. A drop in the DOS has been found in simulations of emission in high density collections of randomly positioned dielectric spheres with short range order in the frequency range of Mie resonances and in two-dimensional connected networks of dielectric walls. Though it has been assumed that the DOS is not affected by wave localization, a low value of the DOS fosters localization since $\delta$ is small. John predicted that residual Bragg scattering in the pseudogap of nearly periodic systems would facilitate Anderson localization. But in our sample composed of randomly positioned, monodisperse spheres, $f = 0.068$ is low and a full bandgap would not develop in a periodic arrangement of spheres. To seek an explanation for the suppression of the DOS, we investigate the role of structural correlation and the interplay between geometry and localization in FDTD simulations in samples composed of spheres with the same values of $n$ and $f$ as for the measured sample.
Figure 4.5: FDTD simulations of $\tilde{\rho}(x)$ for 100cm sample with the diameter of the spheres fixed ($\Delta R = 0$) and the diameter can vary by 30% ($\Delta R = \pm 30\%$).

Figure 4.6: FDTD simulations of $\tilde{\rho}(x)$ for 100cm sample with filling fractions of $f = 0.068, 0.3, 0.6$.)
We see in Fig. 4.4 that the suppression of LDOS does not only occur at the strongest scattering frequency range just above the first Mie resonance as in Figs. 4.2a, d, but also appear at higher frequencies when the sample length \( L \) becomes comparable to the localization length and the LDOS continues to fall when \( L \) increases. To further reduce the correlation in the sample, simulations were carried out in samples composed of spheres with diameters randomly drawn from a distribution with diameter values of \( \pm 30\% \) of the sphere diameter in the experiments (Fig. 4.2) and associated simulations (Fig. 4.4). The suppression is almost the same as in samples with fixed sphere diameters as shown in Fig. 4.5. The degree of suppression is also not much affected by the choice of sphere filling fraction as in Fig. 4.6 where we also allow spheres to overlap for \( f = 0.3, 0.6 \) to avoid short range correlations caused by clustering of high density of spheres. The suppression of LDOS depends on the sample diameter as seen in Fig. 4.7 where the LDOS is suppressed to a much lower value when the diameter is decreased. From these results, we see that the suppression of LDOS is robust against variations of local structures of the scattering medium as in Figs 4.5 and 4.6 but is highly sensitive to the global geometry of different sample lengths and diameters as in Figs. 4.4 and 4.7, in contrast to the suppression of LDOS in the pseudogap which is a result of short range correlation.
To further clarify the distinctions between the suppression of LDOS in quasi-1d scattering media and in the pseudogap, we analyze 1D samples similar to those in Figs. 4.2 a,b but with different spatial correlations. In Figs. 4.8 the LDOS is vanishing in a bandgap for periodic samples, and is constant for random samples with strong enough disorder strength. In nearly periodic samples (red solid lines in Fig. 4.8), the LDOS is also suppressed due to residue spatial correlation. However, Figs. 4.9 show that the LDOS is suppressed to a constant level for different sample lengths, which is completely different from Figs. 4.2 e,f and Fig. 4.4. This comparison again shows that short range order is not the causing of the suppression of LDOS in quasi-1d samples.
Figure 4.10: (a) The number of modes as a function of sample length from 1d simulations for two different kinds of disorder strength. FDTD simulations are carried out and (b) the number of modes (c) the average linewidth (d) Thouless number $\delta$ are shown to saturate.
In 1D unbounded random systems, the total number of states, or quasi-normal modes, increases linearly with sample length due to the fact that LDOS is constant and independent of sample length, as the black curve in Fig. 4.10a. In nearly periodic samples, the number of modes still increases linearly with L when L exceeds the initial decay length of 20 layers as in Fig. 4.9 and is shown as the red curve in Fig. 4.10a. For quasi-1D samples where the LDOS in the middle of the sample vanishes, the number of modes saturates as in Fig. 4.10b. Since there are no states in the middle of the sample when the system become sufficiently large, the modes are located near the boundaries and the coupling with the environment is not affected much by varying the sample length. Thus the average linewidth of modes, and therefore the Thouless number δ saturates as in Figs. 4.10c,d.

One surprising comparison is that the system still obeys single-parameter scaling from previous work [3], and g continues to fall even when δ saturates. This is in contrast to the general knowledge that g and δ follow the same scaling behavior for localized samples. The findings mark a previously unknown transport regime where the system is localized but the states are localized only close to the sample boundaries. The local structure is not affecting the suppression much and the interior of the sample is cloaked from the outside.

As in section 3.2, the average of the logarithm of the energy density decays linearly. We found that even when the LDOS is surprised, lnW(x) is still a self-averaging quantity which obeys the generalized single parameter scaling as described in chapter 3.2 and is illustrated in Fig. 4.11. The result of lnW(x) agree with theoretical prediction but < W(x) > does not. Further studies including why the LDOS is suppressed and the interplay between LDOS and localization will be
carried out.
Photonic topological insulator

5.1 Introduction

Topological ideas in photonics arises from the development of new phases of matter in solid-states materials [56, 57]. Different topological phases can be effectively obtained by first obtaining the phase transition boundaries where the bulk frequency spectrum is gapless, such as Dirac cones of linear dispersions in 2D. Then by tuning the system parameters will open gaps belonging to different topological phases, where symmetries are often considered. The 2D quantum spin Hall effect can have two counter-propagating states with different spins, thus preserving time-reversal symmetry (T).

The bianisotropic materials mix electric and magnetic fields as in Eq. 5.1

\[
i \begin{pmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \omega \begin{pmatrix} \epsilon & \chi \\ \chi^\dagger & \mu \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}, \quad (5.1)
\]

where \(\chi \neq 0\), which can achieve topological phases without breaking T.

A bianisotropic metawaveguide is designed for emulating the spins in electronic systems [116, 117]. The metamaterial designed present non-trivial photonic phase with preserved time-reversal symmetry with bi-anistropy playing the role analogous to spin-orbit coupling. Its modes can be classified as TE (with nonzero components \(E_x, E_y, H_x, H_y, H_z\)) and TM (with nonzero components \(E_z, H_x, H_y\)) polarized modes. By adjusting the parameters of the lattice, TE and TM modes are engineered to be degenerate at the Dirac points (K and K')[116]. By breaking the \(\sigma_z\) mirror sym-
metry, TE and TM modes are then mixed. A perturbation approach can be applied to demonstrate the hybridization of the TE and TM modes. Consider a cavity that supports a number of eigenvalues $\omega_m$ with eigenvectors $\mathbf{E}_m, \mathbf{H}_m$. When the volume has shifted by $\Delta V$, the system supports eigenvalues $\omega'_m$ with eigenvectors $\mathbf{E}'_m, \mathbf{H}'_m$. We can expand new states as

$$\mathbf{E}'_m = \Sigma_n a_n \mathbf{E}_n, \quad \mathbf{H}'_m = \Sigma_n a_n \mathbf{H}_n.$$  \hspace{1cm} (5.2)$$

And the coefficients can be cast into a perturbation problem \[116\],

$$\begin{bmatrix} \omega_{TE}(1 + \Delta_{TE,TE}) & \omega_{TE}\Delta_{TE,TM} \\ \omega_{TM}\Delta_{TM,TE} & \omega_{TM}(1 + \Delta_{TM,TM}) \end{bmatrix} \begin{bmatrix} a_{TE} \\ a_{TM} \end{bmatrix} = \omega' \begin{bmatrix} a_{TE} \\ a_{TM} \end{bmatrix},$$ \hspace{1cm} (5.3)$$

where $\Delta_{m,n} = -\int(\epsilon_0 \mathbf{E}_m^s \cdot \mathbf{E}_n - \mu_0 \mathbf{H}_m^s \cdot \mathbf{H}_n)dV$ with $\Delta V$ the deformed part of the cylinder. The eigenvalues and corresponding eigenstates are spin-up, spin-down like states which is a superposition of TE and TM modes.

### 5.2 Robust reconfigurable electromagnetic pathways within a photonic topological insulator

We demonstrate experimentally that electromagnetic radiation can be steered and delivered to any point within a reconfigurable topological metacrystal with no back-reflection by defining arbitrarily shaped pathways for topological edge states. Here, we realize a reconfigurable topological photonic lattice implemented in the parallel plate waveguide shown in Fig. 5.2. The reconfigurable topological structure studied here is formed between two parallel copper plates through which holes are drilled to support a periodic triangular array of copper rods with ring collars. By pushing the rods, the collar can be moved to touch either the upper ('+') or lower copper plate ('-'). This gives us a degree of freedom to construct an arbitrary shaped pathway between the + and – domains.

The key feature of the topologically nontrivial states is that the bulk of the medium is insulating within the energy gap but there exist topologically protected edge modes. The edge states emerge between topologically distinct structures in which the Chern number (or the spin-Chern number) changes across the interface and the number of states equals the difference of the two Chern numbers.
This can take place either at the boundary between topologically nontrivial and trivial systems, or at an interface separating two topological domains with opposite or different topological indices. The edges states along the domain walls between $C = -1$ and $C = 1$ is shown in Fig. 5.2.

We developed an experimental structure to test the robustness of energy transport for the case in Fig. 5.2a, since unlike the case in Fig. 5.2b, the shape of the domain wall can be easily reformed. The experimental metacrystal is an array composed of $35 \times 40$ unit cells built from low-loss copper with a lattice constant of $a_0=1.089$ cm in the parallel copper plates, separated by a distance $H$.
The rods of diameter $d_0=0.3175$ cm have a collar with diameter $d_c=0.6215$ cm and height $h_c=0.358$ cm centred in the rods. Measurements of field spectra in the range $19 \sim 23$ GHz were made using an Agilent vector network analyser (N5242A). Radiation is launched and detected using linear dipole antennas which extend 3 mm beyond the outer conductor. Surface measurements are performed through a hexagonal lattice of 1.1-mm-diameter holes in one of the two copper plates. The probe antenna is introduced into these holes to measure the local amplitude and phase of the electric field inside the metacrystal. Measurements were made along the boundary lines between metacrystals, but also along a line perpendicular to the domain wall extending into the bulk medium. The phase was not significantly affected by the depth into the bulk material in the region of the bandgap in which the wave is evanescent.

Transmission spectra for some of the experimental configurations tested are shown in Fig. 5.3 (left top panels) together with the results of numerical simulations (left bottom panels), which show that the edge modes propagate freely with no dissipation even when the pathway is curved or random. In the right-hand panels of Fig. 5.3, the transmission spectra measured for the three configurations (blue solid curves) are compared against the spectrum of transmission through the bulk of the metacrystal (black dashed curves) with all the collars shifted up. The latter spectrum reveals very low transmission in the frequency range from 20.1 to 21.7 GHz corresponding to the bandgap for the bulk. In contrast to the case of a gapped band structure, the experimental spectra for different domain wall configurations exhibit high transmission over the frequency range of the bandgap due to edge modes. The high transmission observed for all domain wall contours, including an irregularly shaped wall with multiple bends, shows that transport is robust. In contrast, for the case of topologically trivial guided modes, one would expect the wave to be reflected each time when there is a sharp bend in the path. In addition, the presence of multiple bends would lead to the formation of standing waves in intermediate segments. This would give rise to distinct transmission peaks at Fabry-Perot resonances determined by the length of the segment between the bends, which are not observed.

Interestingly, we also find that the transmission along the domain wall in the bandgap frequency range exceeds the transmission in the pass band. This is a consequence of the confinement of the edge modes to the domain wall, the bulk modes where the energy spreads to different directions, which fall off roughly as the inverse of the distance. To confirm that the energy is confined to the
Figure 5.3: The two different kinds of domain walls

propagating direction, we measured the variation of the electric field in the direction perpendicular to the domain wall. This is seen in Fig. 5.3d. to decay exponentially away from the boundary line.

We perform further studies to demonstrate that the spin is locked to the propagating direction as shown in Fig. 5.2. The wave is injected from port 1 and only spin down states are allowed so there should be no energy transmitted to port 3 and energy can be delivered to port 2 and port 4. This is confirmed in numerical simulations as in 5.2 (bottom) and in microwave experiments as in 5.2c.

Topological edge modes are robust since the spin states are locked to their propagating direction. As long as the spin states are preserved, the edge transmission is robust against defects. This is equivalent to the robustness of the edge states in condensed matter topological insulators to all kinds of defects except magnetic impurities giving rise to spinflip processes. A similar robustness is expected for disordered systems where such ‘nonmagnetic’ defects are randomly distributed throughout the structure. Here we consider two cases of disorders. One is a randomly shaped domain wall and the other is a disordered region with collars randomly pushed up or down along the path of the edge state. For the first case, the edge mode confined to the one-dimensional domain wall experiences an effective scattering potential at every bend of the wall. It is known that arbitrary disorder in 1D systems will result in enhanced back scattering and the energy will be localized.
However, this behavior does not appear for topologically protected edge states. These states do not experience backreflection when propagating along a domain wall, regardless of the shape of the path, standing waves and localized modes cannot be formed along the path, as seen in Fig. 5.2. Numerical calculations and experimental results are summarized in Fig. 5.2b for the second case considered: a system with the domain wall interrupted by a region of two-dimensional disorder, where the collars were randomly shifted up and down. These results clearly confirm that, although the edge states may spread into the disordered region, they are immune to backscattering. Thus, for both one- and two-dimensional disorders, topological robustness of the edge states is seen in enhanced transmission over the entire bandgap frequency range.

To further verify the robustness for these two cases, we examined the dwell time of the edge states as they travel through the system. The dwell time at a distances from the source, which equals the derivative of the phase with respect to the angular frequency $d\phi/d\omega$, is a powerful tool for characterizing the nature of transport in disordered systems. For example, if the wave is diffusive, $s^2$ is proportional to the dwell time which is the same the case of a random walk. In the case of ballistic transport without scattering, the dwell time increases linearly with propagation distance $s = vt$. To this end, we measured spectra of the phase $\phi(\omega, s)$ accumulated by the electric field from...
the source at the edge to a point a distance $s$ along the domain wall for the paths shown in Fig. 5.2a,b. In addition, we measured the dwell time in the absence of the disorder (the straight domain wall) and the cases shown in Fig. 5.3. We found in all cases that the dwell time scales linearly with $s$ with nearly the same slope. This confirms that transport is ballistic in the disordered regions. The results of measurements of the phase for different paths are given in Fig. 5.2c. The average over the topological bandgap of the inverse of the slopes of the linear fits is given in the legend in Fig. 5.2c. This average is essentially equal to the group velocity $v_g$ calculated from the dispersion of the edge modes found in numerical calculations shown in Fig. 5.2a. This confirms the ballistic character of transport of topological edge modes along the domain wall, demonstrating the absence of backscattering in the presence of either one- or two-dimensional spin-preserving disorders.

5.3 Device demonstration based on reconfigurable gauge field

To demonstrate the proof-of-concept device operation based on the reconfigurability of the synthetic gauge field and robust edge states, we have implemented a topological switch, in which the domain wall shape is modified by a computer-controlled motorized stage, as illustrated by Fig. 5.3. A subset of the copper posts on one of the sides of the structure was attached and glued to the triangularly shaped plastic sheet, which was moved by a computer-controlled motorized arm, thereby changing the position of the collars inside the metacrystal. As a result, the topological states of the domain could be automatically switched between two distinct topological configurations. Depending on
the configuration selected, up or down, respectively, in the reconfigured area, the electromagnetic radiation was carried by the topological edge states either from Port 1 to Port 2 or from Port 1 to Port 3, as illustrated in the insets to Fig. 5.3b,c. Measurements of topological switching of transmission are shown in Fig. 5.3b,c and confirm the possibility of the steering radiation by changing the configuration of the synthetic gauge field.

Figure 5.6: Demonstration of reconfigurable topological switch and its time-resolved dynamics.

5.4 Conclusion

In conclusion, we have demonstrated the robust propagation of electromagnetic edge states along reconfigurable topological domain walls in a topological metacrystal. This opens up possibilities for manipulations of propagation of electromagnetic radiation. Such a platform provides a versatile and robust topological approach towards controlling and delivering energy along any desired path to a particular location with no back-reflection. In addition, this reconfigurable platform enables the
study of a variety of fascinating physical phenomena such as controllable spin filter and switch. In particular, the pseudo-spin selective wave divider and topological switch demonstrated here prove the versatility of the proposed reconfigurable platform for implementing unique device functionalities. The exploration of the breakdown of robustness in the face of diverse types of disorder and the interplay with Anderson localization in topological systems will be studied in the future.
Summary

We have investigated the statistics of the energy density profiles within random media. We have shown in microwave measurements and FDTD simulations that the average of the LDOS drops sharply towards the center of samples in which the wavelength is comparable to the scale of scattering elements. The uniformity of the LDOS in random samples has been a fundamental assumption of localization theory which lies behind the equivalence of the dimensionless conductance \( g \) and the Thouless number \( \delta \). But in the systems in which the LDOS vanishes, \( \delta \) saturates while \( g \) continues to fall with increasing sample length. The decay of the LDOS is especially puzzling since the samples we have studied satisfy SPS. We have found an extension of SPS to the interior of random 1D samples and disordered single-mode waveguides. The average of the logarithm of the energy density profile \( < \ln W(x) > \) decays linearly with depth into the sample as \( < \ln W(x) >= -x/\ell \). Surprisingly, this linear fall-off holds for random single-mode waveguides where the LDOS tends to vanish. It also holds for nearly periodic 1S samples in which the LDOS is suppressed to a constant level and periodic 1D samples inside the bandgap. Remarkably, the result \( < \ln W(x) >= -x/\ell \) does not depend on sample length \( L \). This is counterintuitive since more energy is reflected back to \( x \) and \( < W(x) > \) does increase as \( L \) increases. It is further surprising that the slope of the fall-off of \( < \ln W(x) > \) is also immune to boundary reflectivity or energy barriers inside random media. The full statistics of \( \ln W(x) \) with energy barriers and the interplay between the LDOS and localization are under study.

We have demonstrated edge transmission along the domain walls between two distinct topological regions with spin-Chern numbers \( C = 1 \) and \( C = -1 \). The waves are shown to be spin-locked and
the robustness of flux along the domain walls was demonstrated in a reconfigurable structure. Further studies will include the analysis of the edge states between topologically trivial and non-trivial domains. We will also explore the robustness of transport in the face of disorder.

We have also discussed the extension of the transmission matrix, which we call the field matrix, to the interior of random samples. The square of its singular values, are the energy density eigenvalues, the highest of which is the maximum energy that can be delivered to at any depth into the sample. The participation number of the energy density eigenchannels determines the contrast of optimal focusing. The construction of the field matrix in experiments will be carried out in the future with random samples in a 2D plane where fields at all locations can be measured.

Besides measuring energy profiles inside random media, we have explored the evolution of speckle patterns at the output of random samples. The motion of phase singularities and local maxima of intensity can be used to determine the photon diffusion coefficient and, in a dynamic sample, the rate of motion of scattering particles.

We have studied the effects of boundary reflection on the distribution of transmission eigenvalues. We show that when reflectivities at the two boundaries differ, there exist a transition in the transmission eigenvalue distribution at a critical reflectivity beyond which perfect transmission cannot be achieved. When two ends of the sample are equally reflecting, there always exist perfect transmission eigenchannels.

In this thesis, we have examined wave scattering inside disordered media from different perspectives. This work may have applications in deep tissue imaging, information transmission, and energy flow in photonic devices.
Appendix A

Appendix

A.1 Calculations of $D(x)$ with boundary reflectivity

We use the supersymmetry method as described in sections 1.6 and 2.2 to calculate the energy profiles within random media.

The super matrix $Q$ can be written as a rotation about its free space form $\Lambda$ as

$$Q(r) = T(r)^{-1}\Lambda T(r). \quad (A.1)$$

We replace $T(r)$ with $1 + iW$ and express $Q$ as [80],

$$Q(r) = (1 + iW)^{-1}\Lambda(1 + iW), W = \begin{pmatrix} 0 & B \\ kB^\dagger & 0 \end{pmatrix}. \quad (A.2)$$

We want to rewrite the boundary conditions of $Q$ described in chapter 2.2 to that of $W$. The original equations are

$$2z_{b1}Q\partial Q + [Q, \Lambda] = 0|_{x=0}$$

$$2z_{b2}Q\partial Q - [Q, \Lambda] = 0|_{x=L}. \quad (A.3)$$
To simplify the calculations, we take $iW \rightarrow W$ and rewrite $Q$ as

$$Q = (1 + W)^{-1} \Lambda(1 + W)$$

$$= (1 - W + W^2 + ...) \Lambda(1 + W)$$

$$= \Lambda(1 + W + W^2 + ...) (1 + W)$$

$$= \Lambda(1 + 2W + 2W^2 + 2W^3 + ...)$$

$$= \Lambda[1 + 2(W + W^3 + ...) + 2(W^2 + W^4 + ...)]$$

$$= \Lambda[1 + \frac{2W}{1 - W^2} + \frac{2W^2}{1 - W^2}]$$

$$= \Lambda\left[\frac{1 + W^2}{1 - W^2} + \frac{2W}{1 - W^2}\right]$$

Since $W^2$ and $\Lambda$ are diagonal matrices, the off-diagonal term here is $\Lambda \frac{2W}{1 - W^2}$ and one of the terms in the boundary conditions \[A.3\] is

$$[Q, \Lambda] = \frac{-4W}{1 - W^2}. \quad (A.5)$$

Now we calculate the other term $Q \partial Q$.

$$Q \partial Q = \Lambda\left[\frac{1 + W^2}{1 - W^2} + \frac{2W}{1 - W^2}\right] \partial \Lambda\left[\frac{1 + W^2}{1 - W^2} + \frac{2W}{1 - W^2}\right]$$

$$= (\frac{1 + W^2}{1 - W^2} - \frac{2W}{1 - W^2}) \partial (\frac{2W^2}{1 - W^2} + \frac{2W}{1 - W^2})$$

$$= \frac{1 - W}{1 + W} \partial \left[\frac{2W(W + 1)}{1 - W^2}\right]$$

$$= \frac{1 - W}{1 + W} \partial \left(\frac{2W}{1 - W}\right)$$

$$= 2\left[\frac{1}{1 + W} \partial W - \frac{1}{1 + W} \partial (W \frac{1}{1 - W})\right]$$

$$= 2\left[\frac{1}{1 + W} \partial W + \frac{1}{1 + W} \partial W \frac{1}{1 - W}\right]$$

$$= 2\left[\frac{1}{1 + W} \partial W \frac{1}{1 - W} + \frac{1}{1 + W} \partial W \frac{1}{1 - W}\right]$$

$$= 2\left[\frac{1}{1 + W} \partial W \frac{1}{1 - W}\right]$$

$$= 2 \frac{1}{1 + W} \partial W \frac{1}{1 - W} \quad (A.6)$$
The boundary conditions are

\[ z_{b1} \partial W - W = 0 |_{x=0}, \quad z_{b2} \partial W + W = 0 |_{x=L} \]  \hfill (A.7)

The correlation function is defined as

\[ \mathcal{Y}(r, r') = \left( \frac{\pi \nu}{\omega} \right)^2 \int D[Q] Q_{bb}^{12}(x) Q_{bb}^{21}(x') e^{-\xi_8 \int_0^L \text{detr}(\partial_x Q)^2 - S_{\text{surface}}}, \]  \hfill (A.8)

where \( S_{\text{surface}} \) is surface term

\[ S^R_{\text{surface}} = - \tilde{N}_d \omega^{d-1} A \frac{1 - R(e^R)}{1 + R(e^R)} \text{str}(\Lambda Q(L)), \]  \hfill (A.9)

and \( \tilde{N}_d = ((4\pi)^{d-1} d\Gamma(d-1)^{-1} \). \( \mathcal{Y} \) can also be expressed in terms of \( W \) and we can obtain boundary conditions. Rigorous calculations are hard and needed to be fulfilled in the future. But a reasonable simplification can be obtained

\[ (z_{b1} - 1) \mathcal{Y}_0(0, x') = 0 |_{x=0}, \quad (z_{b2} + 1) \mathcal{Y}_0(L, x') = 0 |_{x=L}, \]  \hfill (A.10)

which can be tested in simulations. Here \( \mathcal{Y}_0(x, x') \) satisfy the equation

\[ (\gamma - D_0 \partial^2) \mathcal{Y}_0(x, x') = 0 \]  \hfill (A.11)

\( \gamma \) is the absorption rate. There are two kinds of solutions \( sinh \sqrt{\frac{\gamma}{D_0}} \) and \( cosh \sqrt{\frac{\gamma}{D_0}} \). The solutions can be written as

\[ \mathcal{Y}(x, x') = A \text{sinh}(\xi_0 x) + B \text{cosh}(\xi_0 x), \quad x < x' \]  \hfill (A.12)

\[ \mathcal{Y}(x, x') = C \text{sinh}(\xi_0 x) + D \text{cosh}(\xi_0 x), \quad x \geq x' \]

At \( x = x' \),

\[ \mathcal{Y}_0(x'^+, x') = \mathcal{Y}_0(x'^-, x') \]

\[ -D_0[\mathcal{Y}_0(x'^+, x') - \mathcal{Y}_0(x'^-, x')] = 1 \]  \hfill (A.13)
\[ B = z_{b1}/\xi_a A \]

\[ Asinh(x'/\xi_a) + Bcosh(x'/\xi_a) = Csinh(x'/\xi_a) + Dcosh(x'/\xi_a) \]

\[ -D_0/x_ia[Ccosh(x'/\xi_a) + Dsinh(x'/\xi_a) - Acosh(x'/\xi_a) - Bsinh(x'/\xi_a)] = 1 \]

\[ z_{b2}/\xi_a[Ccosh(L/\xi_a) + Dsinh(L/\xi_a)] = -[Csinh(L/\xi_a) + Dcosh(L/\xi_a)] \]

We assume \( z_{b1}, z_{b2} < \xi_a \)

\[ z_{b1}/\xi_a = \tanh\phi_1 \]

\[ z_{b2}/\xi_a = \tanh\phi_2 \]

We obtain

\[ D = -\tanh(\phi_2 + L/\xi) \]

\[ B = \tanh(\phi_1) A \]

\[ A \frac{\sinh(x'/\xi_a + \phi_1)}{\cosh\phi_1} = C \frac{\sinh(x'/\xi_a - \phi_2 - L/\xi_a)}{\cosh(\phi_2 + L/\xi_a)} \]

\[ C = -\xi_a/D_0 \frac{\sinh(x'/\xi_a + \phi_1)\cosh(L/\xi_a + \phi_2)}{\sinh(L/\xi_a + \phi_1 + \phi_2)} \]

Where we have used

\[ \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \quad \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \]

The return probability \( \mathcal{Y}_0(x = x', x') \)

\[ \mathcal{Y}_0(x = x', x') = -\xi_a/D_0 \frac{\sinh(x'/\xi_a + \phi_1)\sinh(x'/\xi_a - \phi_2 - L/\xi_a)}{\sinh(L/\xi_a + \phi_1 + \phi_2)} \]

\[ = \frac{\xi_a}{2D_0} \frac{\cosh(L/\xi_a + \phi_1 + \phi_2) - \cosh(2x'/\xi_a + \phi_1 - \phi_2 - L/\xi_a)}{\sinh(L/\xi_a + \phi_1 + \phi_2)} \]

As long as \( z_{b1}, z_{b2} \ll \xi_a \), we can just take \( L \to L + z_{b1} + z_{b2}, x \to x + z_{b1} \). When there’s no absorption \( \xi_a \to \infty \).

\[ \mathcal{Y}_0(x, x) = \frac{(x + z_{b1})(L + z_{b1} + z_{b2} - x - z_{b1})}{D_0(L + z_{b1} + z_{b2})} \]

From comparison with previous results [80], we can conjecture that

80
\[ D(x)/D_0 \propto e^{-\frac{(x+z_1)(L+z_2-x)}{L(L+z_1+z_2)}}. \] 

(A.20)

### A.2 List of methods used

In this thesis, experimental, numerical and theoretical methods are used to mainly analyze wave scattering in the interior of random samples. We have developed 1) microwave experiments with antenna moving along the longitudinal direction measuring energy from a slit on the waveguide; 2) computer simulations including recursive Green’s function simulations, finite difference time domain simulations (FDTD) and 1D transfer matrix simulations; 3) theoretical approaches including supersymmetry and random matrix calculations.

In microwave experiments, speckle patterns are measured at the output. The analysis of speckle evolution is presented in section 2.1. By measuring the transmission matrix, and analyzing fields into quasi-normal modes, we obtain the DOS as in section 4.1. The energy profile of single mode and quasi-1d samples are presented in section 3.2 and section 4.2. In section 4.2, the LDOS is shown to vanish in the middle of the sample even in samples with no residue periodicity. However, the logarithm of the energy density \( \langle \ln W(x) \rangle \) fall off linearly in both suppressed and uniform LDOS samples. The demonstrating of topological states in photonic crystals is also conducted using microwave as in section 5.

The various ways of computer simulations are widely used in almost every chapter. The recursive Green’s function method provides calculations on a 2D lattice and can obtain fields everywhere inside random media with adjustable boundaries as described in sections 2.2, 3.1. The FDTD simulations can mimic our experimental samples with more flexibility of sample parameters but runs slower and are used in 2.1, 4.2. The simulation method based on transfer matrix approach is a small program which can do simulations fast but currently only restricted in 1d and is utilized in sections 3.2, 3.3.2, 4.2.

The theoretical calculations are not limited to any specific approach. The main approach described in this thesis is supersymmetry approach with an introduction of the basis in 1.6 and is used in 2.2, 3.3.1.

There are many remaining questions in this thesis. In chapter 2, there are questions of the
statistics of phase singularities created and annihilated and the effect of boundary conditions on the
distribution of transmission eigenvalues for localized sample. In chapter 3 the full statistics as
how the bimodal distribution at the output gradually changes to the distribution of energy density
eigenvalues inside is unexplored and the effect of barrier inside random media is still under study.
Chapter 4 presents a new transport regime and many questions such as why the LDOS is suppressed
and how we can interplay this effect with localization and why SPS still holds, which could provide
many insights into wave scattering in open media. In chapter 5 further studies on the role of disorder
and the edge states propagating along domain walls between non-trivial and trivial crystals.
Bibliography


