Turaev Surfaces and Toroidally Alternating Knots

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Turaev surfaces and toroidally alternating knots

by

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Abstract

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In this thesis, we study knots and links via their alternating diagrams on closed orientable surfaces. Every knot or link has such a diagram by a construction of Turaev, which is called the Turaev surface of the link. Links that have an alternating diagram on a torus were defined by Adams as toroidally alternating. For a toroidally alternating link, the minimal genus of its Turaev surface may be greater than one. Hence, these surfaces provide different topological measures of how far a link is from being alternating.

First, we classify link diagrams with Turaev genus one and two in terms of an alternating tangle structure of the link diagram. The proof involves surgery along simple loops on the Turaev surface, called cutting loops, which have corresponding cutting arcs that are visible on the planar link diagram. These also provide new obstructions for a link diagram on a surface to come
from the Turaev surface algorithm. We also show that inadequate Turaev genus one links are almost-alternating.

Second, we give a topological characterization of toroidally alternating knots and almost-alternating knots. In other words, we provide necessary and sufficient topological conditions for a knot to be toroidally alternating or almost-alternating. Our topological characterization extends Howie’s characterization of alternating knots, but is different from Ito’s characterization of almost-alternating knots.
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Chapter 1

Introduction

A link $L \subset S^3$ is surface-alternating if $L$ has an alternating diagram on some closed orientable embedded surface in $S^3$. In particular, a link $L$ is alternating if it has an alternating diagram on $S^2$.

In this thesis, we study subclasses of surface-alternating links. We are particularly interested in Turaev surfaces of knots and links as well as toroidally alternating knots.

Alternating links were the first ones to be studied because many of their topological and geometric features are directly accessible through their alternating diagrams. For example, Tait’s conjecture states that a reduced alternating diagram of a given link realizes the minimal crossing number.

The first higher genus surface-alternating links were toroidally alternating links, introduced by Adams[3], and Hayashi[13] extended this definition to higher genus surfaces, called $F$-alternating links. With certain conditions,
$F$-alternating links preserve various properties of alternating links.

It is natural to ask whether every link is $F$-alternating or not. In [31], Turaev introduced the Turaev surface of a link diagram to use in a new proof of Tait’s conjecture. In [9], Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that every link is $F$-alternating on its Turaev surface $F$. The minimal genus of the Turaev surface of a given link is a link invariant, which measures how far the link is from being alternating.

This thesis divided into 3 chapters. In Chapter 2, we discuss further backgrounds about alternating links and their generalizations. In Chapter 3, we give a diagrammatic classification of Turaev surfaces and its applications to other generalizations of alternating links. In Chapter 4, we give a topological characterization of toroidally alternating knots and almost-alternating knots.
Chapter 2
Alternating links and generalizations

2.1 Alternating links and Turaev surfaces

A link $L$ is a smooth embedding of a disjoint union of circles into $S^3$. If it has only one component, we call it a knot.

Each link $L$ has a regular neighborhood $N(L)$ which is a disjoint union of solid tori. The exterior $E(L)$ of a link $L$ is the closure of $S^3 - N(L)$.

A link diagram $D$ of $L$ on a closed embedded surface $F \subset S^3$ is a 4-valent graph on $F$ with over-under crossing information on each vertex so that we can obtain $L$ by replacing a local neighborhood of each vertex as in Figure 2.2. We call this configuration a crossing-ball configuration of $L$ on $F$.

A link diagram is alternating if as we travel along a diagram, over and under crossings appear alternately. A link is alternating if it admits an
Alternating links have many interesting topological and geometric properties. I want to highlight some of them which motivate our research.

A link diagram $D$ on $S^2$ is reduced if there does not exist a simple loop on $S^2$ which intersects $D$ only in one crossing. Also, $D$ is prime if every simple loop on $S^2$ which intersects $D$ on exactly two points of its edges bounds a disc on $S^2$ which does not contain any crossings of $D$. Lastly, $D$ is split if there exists a simple loop on $S^2 - D$ which separates the component of $D$.

A link $L$ is prime if every embedded $S^2$ in $S^3$ which transversely intersects $L$ in two points bounds a ball which intersects $L$ in an unknotted arc. Also, $L$ is split if there exists an embedded $S^2$ in $S^3 - L$ which separates the component of $D$.

A flype is a move which changes a link diagram as in Figure 2.1.

Figure 2.1: A flype.

**Theorem 2.1.1** (Tait conjectures). [18, 24, 25, 26, 27, 29, 31] Let $K$ be an alternating knot. Then the following holds.

- Every reduced alternating diagram of $K$ realizes the minimal crossing...
• Every reduced alternating diagram of $K$ has equal writhe.

• Any two reduced alternating diagrams of $K$ are related by a sequence of flypes.

The first and the third conjectures are also true for reduced non-split alternating link diagrams. To show the first conjecture Turaev introduced the Turaev surface of a link diagram.

Consider the crossing ball configuration of the link diagram $D$. With this configuration, we can obtain the $A$-smoothing and the $B$-smoothing as shown:

\[ \text{Figure 2.2: A crossing ball.} \]

A state $s$ of $D$ on $S^2$ is a choice of smoothing at every crossing, resulting in a disjoint union of circles on $S^2$. Let $|s|$ denote the number of circles in $s$. Let $s_A$ denote the all-$A$ state, for which every crossing of $D$ is replaced by an $A$-smoothing. Similarly, $s_B$ is the all-$B$ state of $D$.

Now, as we push $s_A$ up and $s_B$ down, then each state circle sweeps out
an annulus. We can glue all such annuli and equatorial discs of each crossing ball to get a cobordism between $s_A$ and $s_B$. Note that each equatorial disc is a saddle of the cobordism.

For any link diagram $D$, the Turaev surface $F(D)$ is obtained by attaching $|s_A| + |s_B|$ discs to all boundary circles of the cobordism above. Note that the crossing ball configuration of $D$ on $S^2$ induces a crossing ball configuration of $D$ on $F(D)$, hence, we can also consider $D$ as a link diagram on $F(D)$.

The Turaev genus of $D$ is defined by

$$g_T(D) = g(F(D)) = (c(D) + 2k(D) - |s_A| - |s_B|)/2. \quad (2.1.1)$$

where $k(D)$ be a number of connected components of $D$. The Turaev genus of any non-split link $L$ is defined by

$$g_T(L) = \min \{ g_T(D) \mid D \text{ is a diagram of } L \}. \quad (2.1.2)$$

The properties below follow easily from the definitions (see [8]).

(i) $F(D)$ is an unknotted closed orientable surface in $S^3$; i.e., $S^3 - F(D)$ is a disjoint union of two handlebodies.

(ii) $D$ is alternating on $F(D)$.

(iii) $L$ is alternating if and only if $g_T(L) = 0$, and if $D$ is a connected sum of alternating diagrams then $F(D) = S^2$. 
(iv) \( D \) gives a cell decomposition of \( F(D) \), for which the 2-cells can be checkerboard colored on \( F(D) \), with discs corresponding to \( s_A \) and \( s_B \) respectively colored white and black.

(v) This cell decomposition is a Morse decomposition of \( F(D) \), for which \( D \) and the crossing saddles are at height zero, and the \( |s_A| \) and \( |s_B| \) 2-cells are the maxima and minima, respectively.

We will say that a link diagram \( D \) on a surface \( F \) is cellularly embedded if \( F - D \) consists of open discs.

Above properties imply that the Turaev genus measures how far a given link is from being alternating. Turaev gave an upper bound of the Turaev genus of a given link \( L \) using the crossing number \( c(L) \) and the span of the Jones polynomial \( \text{span}(V_L(t)) \), which is a difference between the maximal and minimal degree of the Jones polynomial \( V_L(t) \).

**Theorem 2.1.2.** [31] For any non-split link \( L \), \( g_T(L) \leq c(L) - \text{span}(V_L(t)) \).

Lower bounds for the Turaev genus can be obtained from the Khovanov homology, knot Floer homology, signature, Rasmussen \( s \)-invariant and Ozsváth-Szabó \( \tau \) invariant.

**Theorem 2.1.3.** [7] Let \( L \) be a non-split link. Then \( w(Kh(L)) - 2 \leq g_T(L) \) where \( w(Kh(L)) \) be a width of the Khovanov homology of \( L \).
Theorem 2.1.4. \cite{21} Let $K$ be a knot. Then $w(\widehat{HFK}(K)) - 1 \leq g_T(K)$ where $w(\widehat{HFK}(K))$ be a width of the hat version of the knot Floer homology.

Theorem 2.1.5. \cite{10} Let $K$ be a knot. Then

1. $|s(K) + \sigma(K)| \leq 2g_T(K)$,

2. $|\tau(K) + \frac{\sigma(K)}{2} \leq g_T(K)$, and

3. $|\tau(K) - \frac{s(K)}{2} \leq g_T(K)$,

where $s(K)$ be a Rasmussen $s$-invariant, $\tau(K)$ be an Ozsváth-Szabó $\tau$ invariant, and $\sigma(K)$ be a signature of $K$.

However, computing the exact Turaev genus is a difficult problem. The only method to compute exact Turaev genus is using above theorems to compute the lower bound and then finding the diagram which realizes its lower bound.

From the natural Morse decomposition of the Turaev surface $F(D)$, we can obtain two graphs which are dual to each other on the Turaev surface by consider each 0-handle (resp. 2-handle) as a vertex and each 1-handle as an edge. We call this graph an all-$A$ ribbon graph (resp. all-$B$ ribbon graph) of a link diagram $D$. Note that every crossing of $D$ corresponds to an edge of each ribbon graph.
A crossing $c$ of $D$ is called an $A$-loop (resp. $B$-loop) crossing if it corresponds to a loop of an all-$A$ ribbon graph (resp. all-$B$ ribbon graph) of $D$. We say $c$ is a loop crossing if it is an $A$-loop or a $B$-loop crossing. If $c$ is both an $A$-loop crossing and a $B$-loop crossing, then $c$ is called an $AB$-loop crossing. If there are no loop crossings, then $D$ is called an adequate diagram. A diagram with no $A$-loop or no $B$-loop crossings is called a semi-adequate diagram. Otherwise, it is called an inadequate diagram. A link is adequate if it has an adequate diagram. A link is semi-adequate if it has a semi-adequate diagram but does not have an adequate diagram. Otherwise, a link is inadequate.

**Theorem 2.1.6.** [1] Let $D$ be an adequate diagram of $L$. Then $g(F(D)) = g_T(L)$.

By this theorem, we can easily compute the Turaev genus of every adequate link. All remaining unsolved cases are semi-adequate or inadequate links.

A dealternating number $dalt(D)$ of a link diagram $D$ is the minimum number of crossing changes that needed to change $D$ into an alternating diagram. A dealternating number $dalt(L)$ of $L$ is the minimum dealternating number among all possible diagrams of $L$. We say $L$ is almost-alternating if
CHAPTER 2. ALTERNATING LINKS AND GENERALIZATIONS

\[ dalt(L) = 1. \]

**Theorem 2.1.7.** [2] For any non-split link \( L \), \( g_T(L) \leq dalt(L) \).

**Conjecture 2.1.8.** Every non-split Turaev genus one link is almost-alternating.

This conjecture has been proved for non-alternating Montesinos links, and semi-alternating links [1, 2, 22]. We prove this conjecture for inadequate links using our new geometric methods.

In Chapter 3, we discuss a classification of Turaev genus one and two diagrams using new topological tools called cutting arcs and cutting loops. We further discuss its applications to Conjecture 2.1.8 and unknotting problems. Armond and Lowrance [5] proved a similar classification independently at the same time. More recently, Dasbach and Lowrance used this classification theorem to show the following.

**Theorem 2.1.9.** [11] The Jones polynomial of a Turaev genus one link is monic.

### 2.2 Alternating links and Toroidally alternating links

A link \( L \) is hyperbolic if \( S^3 - L \) admits a complete hyperbolic metric. Thurston showed that every link is either hyperbolic, torus link, or satellite. These
three categories are mutually exclusive.

Menasco showed the following by considering properly embedded surfaces in a complement of a link using the crossing ball configuration of an alternating diagram.

**Theorem 2.2.1.** [23] Let $L$ be an alternating link with an alternating diagram $D$. Then the following holds.

- If $D$ is prime, then $L$ is prime.
- If $D$ is non-split, then $L$ is non-split.
- If $L$ is a non-split, prime alternating link which is not a torus link, then $L$ is hyperbolic.

A link $L$ is $F$-alternating if $L$ has a cellularly embedded alternating diagram on some closed unknotted surface $F$ embedded in $S^3$. In particular, if $F$ is an unknotted torus, then $L$ is toroidally alternating.

From the properties of Turaev surface and Theorem 2.1.7, Turaev genus one links and almost-alternating links are toroidally alternating.

Adams introduced toroidally alternating links and showed the following:

**Theorem 2.2.2.** [3] A non-split prime non-torus toroidally alternating knot is hyperbolic.
A *representativity* or a *complexity* $r(D, F)$ of a cellulary embedded, reduced alternating link diagram $D$ on $F$ is defined by

$$r(D, F) = \min\{ |l \cap D|; l \text{ is an essential simple loop on } F - \{\text{crossings}\} \}$$

Note that $r(D, F)$ is always an even integer.

Hayashi used this complexity to show the following:

**Theorem 2.2.3.** [13] Let $L$ be a link which has a cellulary embedded alternating diagram on a closed unknotted surface $F$ embedded in $S^3$ with $g(F) \geq 1$ and $r(D, F) \geq 6$. Then the following holds:

1. $L$ is non-split.

2. If $D$ is prime, i.e., every trivial loop on $F$ which intersects $D$ in two points always bounds a disc with no crossing inside, then $L$ is prime.

3. If $L$ is not a torus link, then it is hyperbolic.

First and second also holds whenever $r(D, F) \geq 4$. This theorem implies that every link which has a complicated enough $F$-alternating diagram behaves in a similar way to alternating links.

Using cutting arcs and cutting loops, we can show that every diagram on some Turaev surface always has low representativity. Hence, the representa-
tivity is an obstruction for a given $F$-alternating diagram to be a diagram on a Turaev surface.

A spanning surface $\Sigma$ of a knot $K$ in $S^3$ is a surface embedded in $S^3$ such that $\partial \Sigma = K$. For $\Sigma$, we define a spanning surface $\Sigma$ in a knot exterior $E(K) = S^3 - \text{int}(N(K))$ by $\Sigma = \bar{\Sigma} \cap E(K)$.

Recently, Greene [12] and Howie [14, 15] independently gave a topological characterization of alternating knots, which answered a long-standing question of Ralph Fox. Below is Howie’s characterization:

**Theorem 2.2.4.** [14, 15] A non-trivial knot is alternating if and only if there exists a pair of connected spanning surfaces $\Sigma$ and $\Sigma'$ in the knot exterior such that

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 2,$$

(2.2.1)

where $i(\partial \Sigma, \partial \Sigma')$ is the minimal intersection number of $\partial \Sigma$ and $\partial \Sigma'$.

Several other generalizations of alternating knots have recently been topologically characterized. In [16], Ito gave a topological characterization of almost-alternating knots, which were defined by Adams in [4]. In [15], Howie defined weakly generalized alternating knots and gave a topological characterization of these knots on the torus. Furthermore, in [17], Kalfagianni gave a characterization of adequate knots in terms of the degree of their colored
Jones polynomial.

In Chapter 4, we give a topological characterization of toroidally alternating knots and almost-alternating knots, extending Howie’s characterization of alternating knots.
Chapter 3

Link diagrams with low Turaev genus

In this chapter, we classify link diagrams with low Turaev genus in terms of an alternating tangle structure on the link diagram. An alternating tangle structure on a diagram $D$ on $S^2$ provides a decomposition of $D$ into maximally connected alternating tangles, defined by Thistlethwaite [28], and below in Section 3.1.

Our main results are the following:

Theorem 3.0.1. [19] Every prime connected link diagram $D$ on $S^2$ with $g_T(D) = 1$ is a cycle of alternating 2-tangles, as shown in Figure 3.1.

Figure 3.1: A cycle of alternating 2-tangles.
CHAPTER 3. LINK DIAGRAMS WITH LOW TURAEV GENUS

**Theorem 3.0.2.** [19] Every prime connected link diagram $D$ on $S^2$ with $g_T(D) = 2$ has one of the eight alternating tangle structures shown below in Figure 3.2.

Green discs represent maximally connected alternating tangles, and black arcs are non-alternating edges of $D$. In Figure 3.2, ribbons denote an even number of linearly connected alternating 2-tangles: 

Armond and Lowrance [5] proved a similar classification independently at the same time. They classified link diagrams with Turaev genus one and two in terms of their alternating decomposition graphs up to graph isomorphism. While their proof is primarily combinatorial, our proof is primarily geometric. Our result is also somewhat stronger; we classify all possible embeddings of alternating decomposition graphs into $S^2$. Their graphs can be obtained from our Figure 3.2 simply by erasing the colors from the ribbons, and contracting the boundaries of the alternating tangles into vertices. Our cases 1, 3, 6 give their case 2, our cases 2, 5 give their case 3, and the other cases correspond bijectively, with our cases 4, 7, 8 giving their cases 1, 4, 5 respectively.

We also prove Conjecture 2.1.8 for inadequate links using our new geometric methods.

**Theorem 3.0.3.** [19] Let $L$ be an inadequate non-split prime link with
$g_T(L) = 1$. Then $L$ is almost-alternating.

### 3.1 Definitions

In this section, we define our main geometric tools, the cutting arc and cutting loop. Throughout this chapter, let $D$ be a connected link diagram on $S^2$ which is checkerboard colored. An edge of $D$, joining two crossings of $D$, is **alternating** if one end is an underpass and the other end an overpass. Otherwise, an edge is **non-alternating**. $D$ is **prime** if every simple loop on $S^2 - \{\text{crossings}\}$ which intersects $D$ in two points bounds a disc on $S^2$ which does not have any crossings inside. Otherwise, $D$ is said to be **composite** and any such simple loop that has crossings on both sides is called a **composite circle** of $D$. We will say that a crossing of $D$ is positive or negative, respectively, as shown: $\begin{array}{c} \bigcirc \bigcirc \end{array}$ In each alternating tangle all crossings have the same sign, so the tangle is either positive or negative.
An alternating tangle structure on a diagram $D$ [28] is defined as follows. For every non-alternating edge of $D$, take two points in the interior. Inside each face of $D$ containing non-alternating edges, pairs of such points are to be joined by disjoint arcs in the following way: Every arc joins two adjacent points on the boundary of the face, and these points are not on the same edge of $D$. Then the union $\Gamma$ of every arc is a disjoint set of simple loops on $S^2$. Let $\Delta$ be the closure of one of the components of $S^2 - \Gamma$ containing at least one crossing of $D$, then each edge of $D$ entirely contained in $\Delta$ is alternating.

We will call the pair $(\Delta, \Delta \cap D)$ a maximally connected alternating tangle of $D$. Let $n$ be the number of all the maximally connected alternating tangles of $D$. We will call $(D, \Delta_1 \cap D, \Delta_2 \cap D, \cdots, \Delta_n \cap D)$ an alternating tangle structure of $D$ and the closure of a component of $S^2 - \{\Delta_1, \cdots \Delta_n\}$ a channel region of $D$.

An alternating tangle structure of $D$ is a cycle of alternating 2-tangles if it satisfies the following properties:

(i) Every maximally connected alternating tangle of $D$ is a pair of a disc and an alternating 2-tangle,

(ii) Any pair of maximally connected alternating tangles is connected with
either two arcs or zero arcs in the channel region.

Our key tools are the cutting loop and the cutting arc. As defined below, a cutting loop is a simple loop on the Turaev surface which is a topological obstruction for a given Heegaard surface with an alternating diagram on it to be the Turaev surface. A cutting arc is a simple arc on $S^2$ which is used to identify a cutting loop.

Let $D$ be a prime diagram. We can isotope $s_A$ and $s_B$ so that $s_A \cap s_B \cap D = \{\text{midpoints of non-alternating edges of } D\}$. A cutting arc $\delta$ is a simple arc in $S^2$ such that $\partial \delta = \delta \cap D \cap \alpha \cap \beta$ for a state circle $\alpha \subset s_A$ and another state circle $\beta \subset s_B$ (see Figure 3.3.)

![Figure 3.3: A cutting arc.](image)

A cutting loop $\gamma$ of a prime non-alternating diagram $D$ is a simple loop on $F(D)$ satisfying the following properties:

1. $\gamma$ is non-separating on $F(D)$,

2. $\gamma$ intersects $D$ twice in $F(D) - \{\text{equatorial discs}\}$,
3. $\gamma$ bounds a disc $U_\gamma$ in one of the handlebodies bounded by $F(D)$ such that $U_\gamma \cap S^2$ is a cutting arc $\delta$. The disc $U_\gamma$ is called a cutting disc of $D$.

Every cutting loop has a corresponding cutting arc. We will prove the converse in Theorem 3.2.1 below.

Let $\tau$ be a simple arc on $S^2 - \{\text{crossings}\}$ such that $\partial \tau = \tau \cap D$. A surgery along $\tau$ is the procedure of constructing a new link diagram $D'$ as follows:

$$D' = (D - (\partial \tau \times [-\epsilon, \epsilon])) \cup (\tau \times \{-\epsilon, \epsilon\}). \quad (3.1.1)$$

Let $\gamma$ be a cutting loop of $D$. A surgery along $\gamma$ is the procedure of constructing a new surface $F'(D)$ as follows:

$$F'(D) = (F(D) - (\gamma \times [-\epsilon, \epsilon])) \cup (U_\gamma \times \{-\epsilon, \epsilon\}) \quad (3.1.2)$$

and constructing a new diagram $D'$ both on $F'(D)$ and on $S^2$

$$D' = (D - (\partial \delta \times [-\epsilon, \epsilon])) \cup (\delta \times \{-\epsilon, \epsilon\}). \quad (3.1.3)$$

More generally, a surgery along any simple loop $\gamma$ on $F(D) - \{\text{equatorial discs}\}$ can be defined similarly if $\gamma$ satisfies conditions (2) and (3) in the definition of cutting loops, with $U_\gamma \cap S^2 = \tau$, where $\tau$ is a simple arc as above (see Figure 3.4(right)).
3.2 Classification of Turaev genus one diagrams

In this section, we prove Theorem 3.0.1, and several related results.

**Theorem 3.2.1.** If $D$ is a prime non-alternating diagram then there exists a cutting arc $\delta$. Moreover, every cutting arc $\delta$ determines a corresponding cutting loop $\gamma$ on $F(D)$. After surgery along $\delta$ and $\gamma$, we get $F(D') = F'(D)$ and $g_T(D') = g_T(D) - 1$.

**Proof.** First, we show the existence of a cutting arc. Consider a state circle $\alpha \subset s_A$ such that $\alpha \cap s_B \neq \emptyset$. Take the outermost bigon in the disc bounded by $\alpha$ which is formed by $\alpha$ and $s_B$. Near this bigon, we have two possible configurations of $D$, $\alpha$ and $\beta \subset s_B$ as in Figure 3.5. If this bigon contains a part of $D$ as in Figure 3.5 (right), then there exists at least one crossing for each side of the bigon. Then the boundary of this bigon is a composite circle, so it contradicts our assumption that $D$ is prime. Therefore, the configuration should be as in Figure 3.5 (left), so we can take a cutting arc.
δ by connecting the two vertices of the bigon as in Figure 3.5 (left).

Next, we prove that each cutting arc δ has a corresponding simple loop γ on $F(D)$ which satisfies conditions (2) and (3) of the definition of cutting loops. By definition, two endpoints of δ lie on α, $α \subset s_A$. Connect the two endpoints with an arc $δ_A$ on the state disk $\alpha$ bounded by α. As in the proof of Lemma 3.1 in [6], $S^2$, crossing balls, and state disks cut $S^3$ into disjoint balls. Among those balls, we can find a ball whose boundary consists of $\alpha$ and a face of D containing δ inside. Therefore, $δ \cup δ_A$ bounds a disc inside that ball. By construction, each ball is contained in one of the handlebodies bounded by $F(D)$, and so does the disc bounded by $δ \cup δ_A$. By the same argument, we can find another arc $δ_B$ in the state disk bounded by $β \subset s_B$, and a loop $δ \cup δ_B$ which bounds a disk in the same handlebody as $δ \cup δ_A$. Then $γ = δ_A \cup δ_B$ is a simple loop on $F(D)$ which satisfies the conditions (2) and (3) of the definition of cutting loops.

Now, we show that $F'(D) = F(D')$. Surgery along δ divides each state circle into two pieces, and each of them is a state circle of $D'$ because the choice of smoothing did not change. By definition, surgery along γ changes
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$D$ into $D'$. So if we consider a copy of the cobordism between $s_A$ and $s_B$ in $F(D)$, surgery along $\gamma$ changes this cobordism into a cobordism between state circles of $D'$. Moreover, surgery along $\gamma$ divides state disks $\overline{\alpha}$ and $\overline{\beta}$ into two disks respectively, so each boundary component of the new cobordism is closed up with a disk. Therefore, $F(D')$ is equal to $F'(D)$. See the last figure of Section 3.1, which describes the cutting loop surgery.

Lastly, we prove that condition (1) of the definition of cutting loops holds. If $\gamma$ is separating, then $F'(D)$ is disconnected, which implies that $D'$ is disconnected since $F'(D) = F(D')$. Therefore, surgery along $\delta$ disconnects $D$, which implies that $D$ is not prime. This is a contradiction, so $\gamma$ is non-separating, hence essential. By this non-separating property, $g_T(D') = g_T(D) - 1$ is obvious. \qed

Lemma 3.2.2. Any two faces of a prime diagram $D$ can share at most one edge.

Proof. Two edges determine a composite circle, contradicting that $D$ is prime. \qed

Proof of Theorem 3.0.1.

Claim 1: A boundary of every face of $D \subset S^2$ which contains a non-alternating edge is an essential loop of $F(D)$. 
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Note that from the proof of Theorem 3.4 of [6], the boundary of every face can be isotoped along \( F(D) \) to intersect any other boundary transversally at the midpoints of non-alternating edges of \( D \). See Figure 3.6.

Figure 3.6: Perturbing a boundary of each face of \( D \) to intersect \( D \) in non-alternating edges

Consider a pair of faces which share a non-alternating edge. By Lemma 3.2.2, this is the only edge shared by those two faces. The boundaries of these two faces can be isotoped to intersect only at the midpoint of such a non-alternating edge. Hence, these curves are essential on \( F(D) \).

By Theorem 3.2.1, we can find a cutting arc \( \delta \) of \( D \) and its corresponding cutting loop \( \gamma \) which is a boundary of a compressing disc of \( F(D) \). Assume that \( \delta \) is in a black face \( B \) of \( D \), and that \( \gamma \) is a meridian of \( F(D) \).

Claim 2: Only two white faces of \( D \) have non-alternating edges of \( D \) on their boundaries.

By Claim 1 and the hypothesis that \( g_T(D) = 1 \), a boundary of every
face which contains non-alternating edges is either a meridian or a longitude. There are only two white faces $W$ and $W'$ which each intersects $\gamma$ once on its boundary. This implies that $\partial W$ and $\partial W'$ are longitudes. Any two faces with the same color are contained in the same handlebody bounded by $F(D)$, so a boundary of every white face is either longitude or trivial on $F(D)$. Since $F(D)$ is a torus, every longitude intersects a meridian, so these are the only two white faces which contain non-alternating edges on their boundaries.

Connect every pair of adjacent midpoints of non-alternating edges with a simple arc entirely in a black face. Then by Claim 2, all such arcs are parallel to $\delta$ in $S^2 - (W \cup W')$, so they cut $D$ into 2-tangles (see Figure 3.7(right)). Furthermore, each 2-tangle is alternating because all edges of the 2-tangle other than the four half edges are alternating. Hence, $D$ is a cycle of alternating 2-tangles. This completes the proof of Theorem 3.0.1. \qed

![Figure 3.7](image)

Figure 3.7: Every simple arc in a black face which connects non-alternating edges is parallel to $\delta$.

Below are corollaries of Theorem 3.2.1. Corollary 3.2.3 was proved by Turaev in [31], but our short proof illustrates the useful features of cutting
loops.

**Corollary 3.2.3.** [31] For a prime non-alternating diagram \( D \subset S^2 \), \( g_T(D) > 0 \).

*Proof.* By Theorem 3.2.1, \( D \) has a cutting arc. Then the corresponding cutting loop is an essential curve of \( F(D) \), hence \( g_T(D) > 0 \). \( \square \)

**Corollary 3.2.4.** Let \( D \) be a connected prime non-alternating link diagram on \( S^2 \). Then \( r(D, F(D)) = 2 \).

For example, Figure 3.8 (left) is an alternating link diagram on a torus. There is no simple loop on the torus which intersects the link diagram twice. Hence, by Corollary 3.2.4, this link diagram on the torus cannot come from the Turaev surface algorithm.

![Figure 3.8](image)

**Figure 3.8:** Each cellularly embedded alternating diagram on a Heegaard torus cannot come from the Turaev surface algorithm.

Note that even if we have a cellularly embedded, reduced alternating diagram \( D \) on some Heegaard surface \( F \) such that \( r(D, F) = 2 \), it might not be a Turaev surface. For example, the connected diagram in Figure 3.8 (right) has four crossings on \( F \), but any connected planar diagram of this
split link has more than four crossings. Hence, this link diagram on the torus also cannot come from the Turaev surface algorithm.

### 3.3 Inadequate links with Turaev genus one

In this section, we prove Theorem 3.0.3 and discuss the unknotting sequence of every Turaev genus one diagram of the trivial knot.

**Lemma 3.3.1.** Let $c$ be a loop crossing and $l(c)$ be a corresponding loop of the ribbon graph. Then a core $\mu$ of $l(c)$ bounds a disc $V$ in one of the handlebodies bounded by $F(D)$. Furthermore, we can perturb $V$ to intersect $S^2$ in a simple arc $\nu$ on $S^2$ such that $\nu \cap D = \partial \nu$.

**Proof.** Both the all-$A$ and all-$B$ ribbon graphs are naturally embedded in $F(D)$, so each core loop is a simple loop on $F(D)$. Then it bounds a disc in one of the handlebodies bounded by $F(D)$. Using the same argument as in Theorem 3.2.1, we can show that $V$ can be isotoped to intersect $S^2$ in a simple arc $\nu$. \qed

**Lemma 3.3.2.** Let $D$ be a prime link diagram with $g_T(D) = 1$. Let $l$ be a longitude of $F(D)$. If a cutting loop of $D$ is a meridian of $F(D)$, then $\min |l \cap D| = \#\{\text{maximally connected alternating tangles of } D\}$.

**Proof.** From the cycle of alternating tangle structure of $D$, the link diagram
on $F(D)$ is as shown in Figure 3.9. In this figure, vertical lines correspond to the cutting loops. Then the longitudes are isotopic to the horizontal lines. Each circle represents an alternating 2-tangle, which has at least one crossing inside. Therefore, the horizontal lines minimize the number of intersections. Thus, $\min |l \cap D| = \# \{\text{maximally connected alternating tangle of } D\}$. □

Figure 3.9: A link diagram on its Turaev surface.

**Lemma 3.3.3.** Let $D$ be a prime link diagram with $g_T(D) = 1$ which is not adequate. Let $c$ be a loop crossing of $D$, and $\mu$ a simple loop on $F(D)$ as in Lemma 3.3.1. Then there exists a cutting loop $\gamma$ of $F(D)$ which is isotopic to $\mu$.

**Proof.** From Lemma 3.3.1, $\mu$ is either meridian or longitude. If the number of maximally connected alternating tangles of $D$ is two then we can find a cutting loop which is isotopic to the meridian, and another cutting loop which is isotopic to the longitude. If the number of maximally connected alternating tangles of $D$ is greater than two, and if $\mu$ is not isotopic to $\gamma$, then by the Lemma 3.3.2, $|\mu \cap D| > 2$. Therefore $\mu$ is isotopic to $\gamma$. □
Remark 3.3.4. Lemma 3.3.3 implies that the cutting arc $\delta$ and the simple arc $\nu$ in Lemma 3.3.1 are parallel, as in Figure 3.10 (left). In other words, if we surger $D$ along $\nu$, it reduces the Turaev genus of $D$ by one.

Figure 3.10: A flype along a cycle of alternating 2-tangles.

Figure 3.11: The all-$A$ state and the all-$B$ state of $D_2$.

Proof of the Theorem 3.0.3. Let $D_1$ be a prime link diagram of $L$ with $g_T(L) = 1$. Assume that $D_1$ has more than two maximally connected alternating 2-tangles and cutting loops are isotopic to the meridian. By Lemma 3.3.3 and Remark 3.3.4, we can flype $D_1$ as in Figure 3.10 to collect all loop crossings into one twist region and reduce all possible pairs of crossings in twist region by Reidemeister-II moves. Note that these flypes and Reidemeister-II moves do not change the Turaev genus.

If the resulting diagram $D_2$ has more than two maximally connected alternating tangles, then the set of all loop crossings of $D_2$ and the set of
crossings in the twist region are the same. Moreover, By Lemma 3.3.2, none of them can be an \( AB \)-loop crossing. All loop crossings have the same sign, hence, \( D_2 \) is a semi-adequate diagram, which contradicts our assumption that \( L \) is inadequate. Hence, \( D_2 \) has two maximally connected alternating tangles, so there are two non-isotopic cutting loops. Therefore, \( D_2 \) can have loop crossings which are not in the twist region above. Then without loss of generality, the configuration of \( D_2 \) is one of the figures in Figure 3.11 (left), in which the crossings in the figures are possible loop crossings. Then we can see from Figure 3.11 (right) that \( D_2 \) has \( B \)-loop crossings if and only if one of the maximally connected alternating 2-tangles contains only one crossing. Therefore, \( D_2 \) is an almost-alternating diagram. \( \square \)

**Corollary 3.3.5.** [19] Let \( D \) be a reduced Turaev genus one diagram of a trivial knot. Then there exists a sequence of Turaev genus one diagrams

\[
D = D_1 \rightarrow \ldots D_k = D'_1 \rightarrow \ldots D'_l = \text{or}
\]

which satisfy the following :

1. \( D_{i+1} \) is obtained from \( D_i \) by a flype or a Reidemeister II-move,

2. Each \( D'_i \) is almost-alternating,

3. \( D'_{i+1} \) is obtained from \( D'_i \) by a flype, an untongue[30] or an untwirl
Proof. Every diagram of a trivial knot is inadequate. The proof of Theorem 3.0.3 implies that every reduced prime diagram $D$ of the trivial knot with $g_T(D) = 1$ can be changed to an almost-alternating diagram by flypes and Reidemeister II-moves. In Theorem 5 of [30], Tsukamoto proved that every almost-alternating diagram of the trivial knot can be changed to one of the two figures in the statement of the theorem by flypes, untongue moves and untiwrl moves via a sequence of almost-alternating diagrams. \qed

3.4 Classification of Turaev genus two diagrams

In this section, we prove Theorem 3.0.2. A set of disjoint simple loops on $S^2$ is said to be concentric if the annular region on $S^2$ bounded by any two curves does not contain a curve which bounds a disc inside the region.

Theorem 3.4.1. Let $\delta$ be a cutting arc of a prime non-alternating diagram $D$ with $g_T(D) = g$. Assume that $\delta$ is in a black face of $D$. If we surger $D$ along $\delta$ to get $D_1$, then $D_1$ satisfies the following:

1. The composite circles of $D_1$ are concentric.

2. Let $D_2$ be a link diagram obtained from $D_1$ by surgery along every arc
which is the intersection of a black face and a composite circle of $D_1$.

Then each component of $D_2$ is prime and the sum of Turaev genera of all components is $g - 1$.

Proof. Let $B$ be a black face of $D$ which contains $\delta$. Let $W$ and $W'$ be white faces of $D$ such that $\partial \delta \cap \partial W \neq \emptyset$ and $\partial \delta \cap \partial W' \neq \emptyset$. Surgery along $\delta$ joins $W$ and $W'$ into $W_1$ and divides $B$ into $B_1$ and $B'_1$ (see Figure 3.12). Every other face of $D$ is not changed by surgery, so it is a face of $D_1$ as well.

Claim 1: Every composite circle of $D_1$ intersects $W_1$. Assume there exists a composite circle of $D_1$ which does not intersect $W_1$. Then there exists a different white face $W'_1$ of $D_1$ which shares two edges with some black face $B'$ of $D_1$. $W'_1$ can be considered as a white face of $D$ and by Lemma 3.2.2, $W'_1$ shares only one edge with other black faces of $D$, so $B'$ is a join of two black faces of $D$. However, surgery along $\delta$ cannot join two black faces, which is a contradiction.

Claim 2: Every black face of $D_1$ intersects at most one composite circle of $D_1$. By Claim 1, every black face which intersects composite circles is adjacent to $W_1$. Every black face of $D$ except $B$ is not changed by surgery, so Lemma 3.2.2 implies each black face intersects at most one composite circle. Now, $B$ shares one edge each with $W$ and $W'$. After surgery, those two edges are changed to two edges $e$ and $e'$ in $D_1$, each on the boundary of different
black faces. Therefore, $W_1$ shares one edge with $B_1$ and $B'_1$, so $B_1$ and $B'_1$ do not intersect with any composite circle of $D_1$. See Figure 3.12.

![Figure 3.12: Surgery along $\delta$.](image)

**Claim 3:** The composite circles of $D_1$ are concentric.

Let $\{\gamma_i\}$ be the set of composite circles of $D_1$. By Claim 2, $\forall i, j, i \neq j$, $\gamma_i \cap \gamma_j \subset W_1$. The number of intersections is even, so we can remove all intersections by perturbing composite circles inside $W_1$. By Lemma 3.2.2, $\partial \gamma_i \cap W_1$ consists of two points one in $\partial W$ and another in $\partial W'$. By the proof of Claim 2, $\gamma_i \cap e = \gamma_i \cap e' = \emptyset$. Then we can connect midpoints of $e$ and $e'$ with a simple arc $\theta$ such that $|\theta \cap \gamma_i| = 1, \forall i$ (see Figure 3.12). If the composite circles are not concentric, then there exists a triple $(\gamma_1, \gamma_2, \gamma_3)$ such that $\gamma_2$ bounds a disc inside an annulus on $S^2$ bounded by $\gamma_1$ and $\gamma_3$. Then $\theta$ intersects $\gamma_2$ an even number of times, which is a contradiction.

Now we will complete the proof by showing (2). The sum of Turaev genera of all components of $D_2$ is $g_T(D) - 1$ by Theorem 3.2.1 and additivity of Turaev genera of diagrams under connected sum. Assume that one of the
components of $D_2$ is composite. Suppose $W_1$ is changed to $W_2$, which is homeomorphic to an $n$-holed disc after surgery. By the same argument as in Claim 2, every black face of $D_1$ which intersects composite circles of $D_1$ is divided into two faces and each face shares exactly one edge which appears after surgery with $W_2$. Therefore, every composite circle of $D_2$ intersects with edges of $D_1$. Now consider each composite circle as a union of two arcs, each of them intersects a face of $D_2$. Using the checkerboard coloring of $D_2$, we can label each arc as a black or white arc. Every face of $D_2$ except $W_2$ is a subset of a face of $D_1$. Therefore, every black and white arc except the one inside $W_2$ is a simple arc inside a face of $D_1$. For the white arc inside $W_2$, we can choose another arc with the same endpoints, which is an arc inside $W_1$ because its endpoints are on the edges of $D_1$. Then the black and white arcs form a composite circle of $D_1$, which contradicts our assumption that we surgered along all composite circles to get $D_2$. This completes the proof of Theorem 3.4.1.

Proof of Theorem 3.0.2. Let $D$ be a prime link diagram on $S^2$ with $g_T(D) = 2$. Choose a cutting arc $\delta$ using an algorithm from the proof of Theorem 3.2.1 and assume that $\delta$ is in a black face of $D$. We surger $D$ along $\delta$ to get $D_1$ with $g_T(D_1) = 1$. $D_1$ has a checkerboard coloring induced by the
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checkerboard coloring of $D$.

Let $D'$ be obtained from $D$ by surgery along an arc $\tau$. We define the
attaching edge $\tau'$ to be midpoint($\tau$) $\times$ $[-\epsilon, \epsilon]$, with ($\tau, \epsilon$) as in the definition
of surgery along a cutting arc, as indicated by a dotted arc. Note that if we
do surgery along $\tau'$, then the attaching edge is $\tau$, and we get $D$ again.

Consider every composite circle of $D_1$. We surger $D_1$ along black arcs to
get $D_2$ which consists of exactly one prime diagram $T$ with $g_T(T) = 1$, and
several prime alternating diagrams. Choose the checkerboard coloring of $T$
that comes from $D$. Note that every attaching edge is in one white face of
$T$. See Figure 3.13.

Now we need to reconstruct $D$ from $T$ and the alternating diagrams.

Theorem 3.4.1 implies components of $D_2$ are pairwise connected by exactly
one attaching edge, if any, and no more than two attaching edges in total.

Below, we consider all possible cases for attaching $T$ and the alternating
components of $D_2$:
Case 1. Every cutting arc of $T$ is inside a black face of $T$.

Every other component of $D_2$ is inside a white face $W$ of $T$, so we have four different sub-cases.

i) $W$ has non-alternating edges on its boundary. See Figure 3.14(a), where $W$ is the yellow face shown.

If two attaching edges are connected to two alternating edges of the same alternating tangle of $T$, then we have an alternating 4-tangle, and the alternating tangle structure of $D$ is shown in Figure 3.2(a). If the two attaching edges are connected to two alternating edges in different alternating tangles of $T$, then we have two alternating 3-tangles, and the alternating tangle structure of $D$ is shown in Figure 3.2(b). If one of the attaching edges is connected to a non-alternating edge of $\partial W \subset T$, then the sign of crossings
of such an alternating diagram is the same as one of the alternating tangles adjacent to such a non-alternating edge. Hence, we can merge the alternating tangles, as shown in Figure 3.15. Therefore, in this case, the alternating tangle structure is the same as one of the above cases.

Figure 3.15: Two alternating tangles with same sign merge into one alternating tangle.

ii) $W$ is contained in one of the alternating tangles, and $W$ is adjacent to a black face $B$ which has a cutting arc inside, as in Figure 3.14(b).

If one of the attaching edges is connected to $\partial B$, then we have two possibilities. First, if the sign of the alternating tangle of $T$ and of the alternating diagram are different, then the alternating tangle structure changes as illustrated in Figure 3.16.

Then we have one alternating 4-tangle, and the alternating tangle struc-

Figure 3.16: The sign of the alternating tangle of $T$ and of the alternating diagram are different.
tecture of $D$ is shown in Figure 3.2(c). If the signs are the same, then we have one alternating 4-tangle which is not simply connected, and the alternating tangle structure is shown in Figure 3.2(d). If there is no attaching edge connected to $\partial B$, then the alternating tangle structure is the same as Figure 3.2(d).

iii) $W$ is contained in one of the alternating tangles, and adjacent to black faces $B$ and $B'$ which each have a cutting arc inside, as in Figure 3.14(c).

If one attaching edge is connected to $\partial B$ and another attaching is connected to $\partial B'$, then we have three possibilities. First, if the sign of the alternating tangle of $T$ and of two alternating diagrams connected to $T$ by two attaching edges are different, then the alternating tangle structure changes as in Figure 3.17(a). Therefore, every maximally connected alternating tangle is a 2-tangle, and the alternating tangle structure of $D$ is shown in Figure 3.2(h).

![Figure 3.17: Possible changes of alternating tangles structures.](image)

If the sign of one of the alternating diagrams is the same as the sign of the alternating tangle of $T$, then we can merge them into one maximally con-
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Connected alternating tangle as in Figure 3.17(b). Then we have one alternating 4-tangle, and the alternating tangle structure of \( D \) is shown in Figure 3.2(c).

If the signs of two alternating diagrams are the same as the sign of the alternating tangle of \( T \), then we can merge them into one maximally connected alternating tangle as in Figure 3.17(c). This maximally connected alternating tangle is not simply connected and the alternating tangle structure of \( D \) is shown in Figure 3.2(d). Other cases are just the same as case ii) above.

iv) A black face adjacent to \( W \) cannot have non-alternating edges on its boundary. This case is the alternating tangle structure shown in Figure 3.2(d).

Case 2. Every cutting arc of \( T \) is inside a white face of \( T \)

i) \( W \) contains a cutting arc of \( T \), as in Figure 3.14(d).

If two attaching edges are connected to alternating edges of \( T \), and the two alternating edges of \( T \) are in different tangles, then we have two alternating 3-tangles and the alternating tangle structure is shown in Figure 3.2(e). If two attaching edges are connected to alternating edges of \( T \), and the two alternating edges of \( T \) are in the same alternating tangle, then we have one alternating 4-tangle and the alternating tangle structure is shown in Figure 3.2(f). If at least one attaching edge is connected to a non-alternating edge
of $T$, then the alternating tangle structure changes as in the figure in the proof of Case(1i), which implies the same alternating tangle structure as in Figure 3.2(e) or 3.2(f).

ii) $W$ does not contain a cutting arc, but is adjacent to two black faces $B$ and $B'$ which have non-alternating edges on their boundaries, as in Figure 3.14(e).

Assume that the two alternating tangles adjacent to $W$ are positive tangles, as in Figure 3.14(e). If two attaching edges are not connected to the edges of $\partial B$ nor $\partial B'$ then the alternating tangle structure is the same as in Figure 3.2(d). If exactly one attaching edge is connected to an edge of either $B$ or $B'$, and an alternating diagram attached to it has negative crossings, then the alternating tangle structure changes as in the figure in the proof of Case(1ii). Therefore, we have one alternating 4-tangle and the alternating tangle structure is shown in Figure 3.2(f). If the alternating diagram has positive crossings, then the alternating diagram and the alternating tangle of $T$ merge. Therefore, it has the same alternating tangle structure as in Figure 3.2(d). If two attaching edges are connected to the edges of $B$ and $B'$, and both alternating diagrams attached to $T$ along them have negative crossings, then the alternating tangle structure changes as in left figure in the proof of Case(1iii). Therefore, every alternating tangle of $D$ is a 2-tangle,
and the alternating tangle structure is shown in Figure 3.2(g). Otherwise, the alternating tangle structure of $D$ can be as in Figure 3.2(d) or Figure 3.2(f).

iii) $W$ is adjacent to exactly one black face $B$ which has non-alternating edges on its boundary as in Figure 3.14(f): If two attaching edges are not connected to $\partial B$, then the alternating tangle structure is shown in Figure 3.2(d). If one attaching edge is connected to $\partial B$, then it is as shown in Figure 3.2(d) or Figure 3.2(f), depending on the sign of the alternating tangle attached to that attaching edge.

iv) A black face adjacent to $W$ cannot have non-alternating edges on its boundary. This is same case as 1(iv), which is the alternating tangle structure in Figure 3.2(d).

To show that we have considered all the possible cases, we need to show all faces of $T$ are used in the proof. First, all faces of $D$ in the channel region are considered in Case(1i) and Case(2i). It remains to show that all the faces in the alternating tangles are used in the proof. From the checkerboard coloring and the cycle of alternating 2-tangle structure, we can show that every face in the alternating tangle can be adjacent to at most two faces in the channel region. Therefore we can categorize every faces in the alternating tangle by the number of adjacent faces in the channel region and the existence of
cutting arcs in adjacent faces. These are considered in the Cases (1ii - 1iv) and Cases (2ii - 2iv).

Lastly, we show that all eight cases are distinct up to isotopy on $S^2$. First, Case 4 is distinct from all others because it has a non-simply connected alternating tangle. If every ribbon contains no alternating tangles, then Cases 1, 3 and 6 have the same alternating tangle structure. Similarly, Cases 2 and 5 have the same alternating tangle structure. Cases 1, 3, 6 have a 4-tangle, and Cases 2, 5 have two 3-tangles, so they are distinct. Cases 7 and 8 are distinct from the others because their alternating tangle structure consists of only 2-tangles. Case 8 has 2-tangles adjacent to four others which Case 7 does not, so Cases 7 and 8 are distinct. We now distinguish Cases 1, 3 and 6. With many alternating tangles in every ribbon, the single 4-tangle is connected to four different alternating 2-tangles. If we orient the boundary of the 4-tangle, non-alternating edges connected to the boundary have a cyclic ordering. If we compare the three cyclic orderings, then they are distinct up to a cyclic permutation. Therefore, Cases 1, 3 and 6 are all distinct. Similarly, Cases 2 and 5 are distinct. This completes the proof of Theorem 3.0.2.
Chapter 4
Toroidally alternating knots

In this chapter, we give a topological characterization of toroidally alternating knots, extending Howie’s characterization of alternating knots.

Several other generalizations of alternating knots have recently been topologically characterized. In [16], Ito gave a topological characterization of almost-alternating knots, which were defined by Adams in [4]. In [15], Howie defined weakly generalized alternating knots and gave a topological characterization of these knots on the torus. Furthermore, in [17], Kalfagianni gave a characterization of adequate knots in terms of the degree of their colored Jones polynomial.

In this chapter, we consider a pair of spanning surfaces satisfying an equation similar to equation (2.2.1). Theorem 4.1.3 shows that in this case, the knot has a "non-trivial" alternating diagram on the torus. Non-triviality is important because every knot has an alternating diagram on the torus.
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boundary of its regular neighborhood. See Figure 4.1.

Figure 4.1: Every knot has an alternating diagram on the torus boundary of its regular neighborhood.

Theorem 4.1.3 also says that if one of the spanning surfaces is free, then we can find an alternating diagram of a knot on an unknotted torus. When the torus is unknotted, it is a Heegaard surface, and this condition plays an important role in defining alternating distances, which measure topologically how far a knot is from being alternating (see [22] for more details.). For example, the alternating genus of a knot is the minimal genus of a Heegaard surface such that the knot has a cellularly embedded alternating diagram on it. The Turaev genus is another interesting alternating distance, which is the minimal genus of a Heegaard surface with a Morse function condition, such that the knot has a cellularly embedded alternating diagram on it. Alternating genus and Turaev genus are both defined for an alternating diagram that is cellularly embedded on the surface. The conditions in Theorem 4.1.3 are not enough to find a cellularly embedded diagram: The alternating diagrams
on the torus that we get from Theorem 4.1.3 may have an annular region and they might not be checkerboard colorable. Note that every cellularly embedded alternating diagram on a closed orientable surface is checkerboard colorable.

In Theorem 4.2.5, we give additional conditions – that the spanning surfaces are relatively separable, and a detachable curve is incident to a bigon (which are defined in Definition 4.2.2 below) – to find a cellularly embedded alternating diagram on a torus. These conditions give a trichotomy for a pair of spanning surfaces:

1. A pair of spanning surfaces is not relatively separable.

2. A pair of spanning surfaces is relatively separable, and every detachable curve on both spanning surfaces is incident to a bigon.
3. A pair of spanning surfaces is relatively separable, but there exists a detachable curve which is not incident to a bigon.

Theorem 4.2.5 shows that a knot is toroidally alternating if and only if there exists a pair of spanning surfaces that satisfies certain conditions and either condition (1) or (2). If every pair of spanning surfaces satisfies condition (3), then we can still find some non-trivial alternating diagram on an unknotted torus by Theorem 4.1.3, but it may or may not be checkerboard colorable.

Finally, in Theorem 4.3.1, we show that for any knot as in Theorem 4.2.5, condition (2) is equivalent to a knot being almost-alternating. In [16], Ito gave a topological characterization of almost-alternating knots, but our characterization is different. He used all-$A$ and all-$B$ state surfaces of an almost-alternating diagram, which are the checkerboard surfaces of the Turaev surface of the almost-alternating diagram. We use a different pair of spanning surfaces to obtain a checkerboard-colorable alternating diagram on an unknotted torus, which is not cellularly embedded. It is an interesting question how the two checkerboard surfaces of this diagram are related to the spanning surfaces used in [16].
CHAPTER 4. TOROIDALLY ALTERNATING KNOTS

4.1 Alternating knots on a torus

Throughout this chapter, we use the following proposition that every alternating knot is both almost-alternating and toroidally alternating.

**Proposition 1.** Let $K$ be an alternating knot. Then $K$ has an almost-alternating diagram and a toroidally alternating diagram.

**Proof.** By [4], every alternating knot has an almost-alternating diagram. By [3], we can find a toroidally alternating diagram from an almost-alternating diagram. 

**Definition 4.1.1.** A spanning surface $\bar{\Sigma}$ of a knot $K$ in $S^3$ is a surface embedded in $S^3$ such that $\partial \bar{\Sigma} = K$. For $\bar{\Sigma}$, we define a spanning surface $\Sigma$ in a knot exterior $E(K) = S^3 - \text{int}(N(K))$ by $\Sigma = \bar{\Sigma} \cap E(K)$. A spanning surface $\bar{\Sigma}$ of a knot $K$ in $S^3$ is **free** if $\pi_1(S^3 - \bar{\Sigma})$ is a free group.

Note that a spanning surface is free if and only if the closure of $S^3 - \bar{\Sigma}$ is a handlebody.

For every pair of spanning surfaces $\Sigma$ and $\Sigma'$, we can isotope them so that their boundaries realize the minimal intersection number, and each such isotopy can be extended to an isotopy of $S^3$. Then we have the following lemma from [14, 15].
Lemma 4.1.2 ([14, 15]). If two spanning surfaces in a knot exterior are isotoped so that their boundaries realize the minimal intersection number, then every intersection arc is standard, as shown in Figure 4.3.

Figure 4.3: A neighborhood of a standard intersection arc of two spanning surfaces in a knot exterior.

Consider $\Sigma \cup \Sigma'$ in $S^3$ as above. If we contract all standard arcs, as in Figure 4.4, then we get an immersed surface in $S^3$ such that every self-intersection is a simple closed curve. We also get a connected 4-valent graph $G_K$ on this immersed surface, coming from $K$, which is away from every self-intersection loop. We will call this immersed surface an almost-projection surface of $\Sigma$ and $\Sigma'$.

Figure 4.4: Contracting a standard arc intersection
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Figure 4.5: An example of an almost-projection surface (left). It is homeomorphic to a Klein bottle which is immersed as in the middle figure, and a 4-valent graph $G_K$ is on the surface as in the right figure.

**Theorem 4.1.3.** [20]

Let $\Sigma$ and $\Sigma'$ be connected spanning surfaces in the knot exterior $E(K)$, such that

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 0, \tag{4.1.1}$$

where $i(\partial\Sigma, \partial\Sigma')$ is the minimal intersection number of $\partial\Sigma$ and $\partial\Sigma'$. Then there exists a torus $T$ embedded in $S^3$ such that $K$ has an alternating diagram $D_K$ on $T$ with $r(D_K, T) \geq 2$. Furthermore, if $\Sigma$ or $\Sigma'$ is free, then $T$ is an unknotted torus (i.e., a Heegaard torus).

**Remark 4.1.4.** Howie [15] considered an alternating diagram on the torus which is checkerboard colorable. To get his characterization of weakly generalized alternating knots, he added certain other conditions. In Theorem 4.1.3, we show that without additional conditions, we can still find a non-trivial alternating diagram of the knot on the torus.

**Proof.** First, we will prove the existence of $D_K$ and $T$. Consider an almost-projection surface of $\Sigma$ and $\Sigma'$. By equation (4.1.1), this almost-projection
surface can be either an immersed torus or an immersed Klein bottle.

**Case 1:** The almost-projection surface is an immersed torus.

For each self-intersection curve, we have two possibilities. First, a self-intersection curve can bound a disc on the immersed torus. Then we can find an innermost self-intersection curve inside the disc. If we surger along the disc bounded by the innermost self-intersection curve as in Figure 4.6, then the resulting surface can be disconnected, or it is an immersed sphere.

![Figure 4.6: Sutery along a disc](image)

If the resulting surface is an immersed sphere, then by [14, 15], $K$ is alternating. Hence, by Proposition 1, $K$ is toroidally alternating. Also in this case, by Corollary 3.2.4, $r(D_K, T) = 2$. If the resulting surface is disconnected, then one component is a torus, and the other component is a sphere. If $G_K$ is on the sphere component, then again, $K$ is alternating. Otherwise, we have reduced the number of self-intersections. We continue until all such inessential self-intersections are eliminated.
On the other hand, the self-intersection curve can be essential on the immersed torus. But we now prove by contradiction that this cannot occur. Let \( f : T' \to S^3 \) be an immersion map, and let \( \sigma \) be a self-intersection loop on \( f(T') \). Then \( f^{-1}(\sigma) \) consists of two essential simple closed curves, so they bound an annulus on the torus as in Figure 4.7. Since \( G_K \) does not intersect with any self-intersection curves, \( f^{-1}(G_K) \) is a connected 4-valent graph on one of the annuli bounded by \( f^{-1}(\sigma) \). Hence, both components of \( f^{-1}(\sigma) \) are in the same region of \( f^{-1}(G_K) \), which implies that \( \sigma \) is a self-intersection of either \( \Sigma \) or \( \Sigma' \), which contradicts the fact that each of them is an embedded surface. Hence, there are no essential self-intersection curves.

![Figure 4.7: Preimages of a self-intersection loop bound an annulus on the torus.](image)

Therefore, \( G_K \) is a 4-valent graph on an embedded torus \( T \) in \( S^3 \). We can recover the diagram \( D_K \) from \( G_K \) by replacing each vertex of \( G_K \) with a neighborhood of a standard arc. If the resulting diagram \( D_K \) is not alternating, there exists a bigon between \( \partial \Sigma \) and \( \partial \Sigma' \), which contradicts the minimality of the intersection number of boundaries. Hence, \( K \) has an al-
ternating diagram on $T$. Also, from the construction, $D_K$ is checkerboard colorable.

We claim that either $r(D_K, T) \geq 2$ or $K$ is alternating. Suppose that for the resulting alternating diagram $D_K$ on the torus $T$, $r(D_K, T) < 2$. Since $D_K$ is checkerboard colorable, every simple closed curve on $T$ intersects $G_K$ transversely in an even number of points. Therefore, $r(D_K, T) = 0$, so we can find a compressing disc of $T$ which does not intersect $D_K$. Then compressing $T$ along this disc yields an embedded $S^2$, so $K$ is an alternating knot. Then, as above, by [19, Corollary 4.6], we can find an alternating diagram $D_K$ of $K$ on some embedded torus $T''$ such that $r(D_K, T'') = 2$. This concludes the proof of Case 1.

**Case 2:** The almost-projection surface is an immersed Klein bottle.

Let $f : B \to S^3$ be an immersion of a Klein bottle $B$. If a simple closed self-intersection curve of $f(B)$ bounds a disc on $f(B)$, then we can surger along this curve as in Figure 4.6 to reduce all such inessential intersections. If $G_K$ is on an immersed sphere, then $K$ is alternating. Hence, we can assume that all preimages of the remaining self-intersections are essential simple loops of $f(B)$. Let $s_1, s_2 \subset B$ be the preimages of an essential self-
intersection \( \sigma \) of \( f(B) \). For \( i = 1, 2 \), we call a regular neighborhood of \( s_i \) 2-sided if it is homeomorphic to an annulus, or 1-sided if it is homeomorphic to a Möbius band. Furthermore, the two regular neighborhoods of \( s_1 \) and \( s_2 \) in \( B \) are homeomorphic, because an annulus and Möbius band embedded in \( S^3 \) cannot intersect only in the core loop. Then we have three subcases to consider, depending on the topology of \( s_1 \) on \( B \):

Subcase 1: \( s_1 \) is a non-separating, 2-sided curve on \( B \).

We prove by contradiction that this subcase cannot occur. The complement of a regular neighborhood of \( s_1 \) in \( B \) is an annulus. Hence, \( s_2 \) is the core of the annulus. Then \( s_1 \) and \( s_2 \) cut \( B \) into two annuli, and \( f^{-1}(G_K) \) is on one of them. Hence, \( s_1 \) and \( s_2 \) are in the same region of \( f^{-1}(G_K) \) on \( B \).
See Figure 4.9. This implies that \( \sigma \) is a self-intersection of \( \Sigma \) or \( \Sigma' \), which
contradicts the assumption that \( \Sigma \) and \( \Sigma' \) are embedded.

Figure 4.9: Two pre-images cut a Klein bottle into two annuli and \( f^{-1}(G_K) \)
is on one of them.

**Subcase 2:** \( s_1 \) is a separating, 2-sided curve on \( B \).

In this case, \( s_1 \) cuts \( B \) into two Möbius bands. Then \( s_2 \) is on one of the Möbius bands, and cuts it into one Möbius band and one annulus. Hence, \( s_1 \) and \( s_2 \) cut \( B \) into two Möbius bands and one annulus.

Figure 4.10: Two pre-images cut a Klein bottle into two Möbius bands and one annulus.

Furthermore, \( f^{-1}(G_K) \) is contained in one of the components. If \( f^{-1}(G_K) \)
is on the Möbius band, then \( \sigma \) is a self-intersection of \( \bar{\Sigma} \) or \( \bar{\Sigma}' \) which is im-
possible. Hence, \( f^{-1}(G_K) \) is on the annulus. In this annulus, every preimage
of an essential self-intersection is isotopic to a core of the annulus. Let $A$ be the annulus which contains $f^{-1}(G_K)$ and does not contain any preimages of self-intersections. Then we can recover $D_K$ from $G_K$, as in the torus case, so that $K$ is alternating on $f(A)$ as follows.

Now, to construct the torus, we consider $B - A$, which consists of two disjoint Möbius bands $M$ and $M'$. The image of each Möbius band under $f$ is a subset of either $\Sigma$ or $\Sigma'$. Furthermore, both Möbius bands cannot be contained in the same spanning surface. Consider $M \cup A$, which is homeomorphic to a Möbius band. Now, $f(M \cup A)$ is embedded in $S^3$ because every self-intersection of $f(B)$ is an intersection of $f(M)$ and $f(M')$. Consider a thickening of the Möbius band $f(M \cup A)$ in $S^3$, which is homeomorphic to a solid torus. Let $T$ be its boundary. Then using the natural projection, we can think of $T - \partial f(M \cup A)$ as a two fold cover of the Möbius band $f(M \cup A)$. Then $f(A)$ is lifted to two annuli with disjoint interiors on the torus. Since $K$ is alternating on $f(A)$, we can choose a lift of the alternating diagram $D_K$ to one of the annuli. Hence, $D_K$ is alternating on $T$.

Now we show that $r(D_K, T) \geq 2$ or $K$ is alternating. By construction, $T$ bounds a solid torus for which the boundary of every compressing disc intersects each lift of $f(A)$ twice. If this boundary curve intersects the diagram less than twice, then this implies that $G_K$ is contained in a disc in $f(A)$.
Figure 4.11: Torus $T$, coming from thickening the Möbius band $f(M \cup A)$, and an example of one of the lifts of $f(A)$ on $T$, denoted by a shaded band.

But then, this implies that $M$ and $M'$ are in the same region of $f^{-1}(G_K)$, which cannot occur because these Möbius bands are not contained in the same spanning surface. Hence, the boundary of every compressing disc of this solid torus intersects $G_K$ at least twice.

Finally, $r(D_K, T) < 2$ may occur for a compressing disc on the other side of $T$. If the 3-manifold on the other side of $T$ has a compressing disc $\Omega$, then it is a solid torus, hence, $T$ is an unknotted torus. Note that $\partial \Omega \cap \partial f(M \cup A) \neq \emptyset$. If $\partial \Omega$ does not intersect the diagram, then just as above, $D_K$ is contained in a disc. Suppose that $\partial \Omega$ intersects the diagram once. This implies that $f(M \cup A)$ is an embedded Möbius band in $S^3$ such that the core is the unknot and its boundary is also the unknot. This implies that we can find an essential arc on $f(A)$ which intersects $G_K$ transversely once. Now, we
need the following lemma.

**Lemma 4.1.5.** Let $K$ be a knot with an alternating diagram $D_K$ on an annulus $A$ embedded in $S^3$. If there exists a properly embedded simple arc $\tau \subset A$ which intersects $D_K$ transversely once, then $K$ is a connected sum of an alternating knot and a knot isotopic to the core of $A$.

*Proof.* Every crossing of $D_K$ is in the complement of $\tau$ in $A$, which is a disc. Hence we can find a decomposing sphere from the boundary of the thickened disc which contains every crossing of $D_K$. This implies that we have an alternating 1-tangle in the decomposing sphere and another 1-tangle outside. Then $K$ is a connected sum of an alternating knot, which is obtained by taking the trivial closure of the alternating 1-tangle, and the trivial closure the other 1-tangle, which is isotopic to the core of $A$. \hfill \square

Then, by Lemma 4.1.5, $K$ is a connected sum of an alternating knot and a knot which is isotopic to the core of $f(A)$, which is an unknot. Hence, $K$ is alternating.

**Subcase 1:** $s_1$ is a 1-sided curve on $B$.

The complement of $s_1$ and $s_2$ in $B$ is an annulus. Hence, $f^{-1}(G_K)$ is on
the annulus. If there is no other self-intersection, then $G_K$ is on the embedded Möbius band $B - s_1$. If there exists another essential self-intersection, then its pre-images are separating 2-sided curves, so $G_K$ is still on the embedded Möbius band $f(M \cup A)$. Hence, the claim follows by the same argument as in the previous subcase.

This completes the proof of Case 2.

To show that the torus $T$ is unknotted, we need following lemmas.

**Lemma 4.1.6.** Let $D_K$ be a knot diagram on the torus $T$ with $r(D_K, T) \geq 2$. Then every region of $T - G_K$ is homeomorphic to a disc, except possibly one region which is homeomorphic to an annulus.

*Proof.* Let $R$ be a region of $D_K$. Since $D_K$ is connected, $|T - R| = 1$. Hence, $\chi(R) \geq -1$. If $\chi(R) = -1$, then $D_K$ is contained in a disc, hence we always can find a compressing disc of $T$ which does not intersect $D_K$. But this violates the condition $r(D_K, T) \geq 2$. Lastly, if there exist two annular regions $R_1$ and $R_2$ of $D_K$, then $|T - (R_1 \cup R_2)| = 2$. Again, since $D_K$ is connected, this is not possible. \qed

**Lemma 4.1.7.** Suppose that a link $L$ has a checkerboard-colorable, connected diagram $D_L$ on a torus $T$ in $S^3$ such that $r(D_L, T) \geq 2$. Then $T$ is unknotted if and only if one of the checkerboard surfaces is free.
Proof. From Lemma 4.1.6, every region is homeomorphic to a disc except possibly one region, which is homeomorphic to an annulus. Let \( \Sigma \) and \( \Sigma' \) be two checkerboard surfaces of \( D_L \). Suppose that \( \Sigma' \) is a checkerboard surface which consists of only disc regions. Since \( \Sigma' - (\Sigma \cap \Sigma') \) is a set of disjoint discs and \( S^3 - T \) has two connected components, \( S^3 - \Sigma \) is homeomorphic to a 3-manifold obtained from connecting the two components of \( S^3 - T \) with 3-dimensional 1-handles, each corresponding to a disc of \( \Sigma' - (\Sigma \cap \Sigma') \) (see Figure 4.12). If \( \Sigma \) contains an annular region of \( D_L \), then \( S^3 - \Sigma' \) can be obtained similarly, except we connect two components with a thickened annulus.

Figure 4.12: A 1-handle correspond to a disc region of \( D_L \).

If \( T \) is an unknotted torus, then \( S^3 - T \) is a disjoint union of two solid tori. If we connect two solid tori with several 3-dimensional 1-handles, then it is still a handlebody. Hence, \( \Sigma \) is free.

Conversely, suppose that \( T \) is knotted. We show that both checkerboard surfaces are not free. We use the fact that compressing a handlebody with
a disjoint set of compressing discs yields a disjoint union of handlebodies. First, we show that $\Sigma$ is not free. We can obtain $S^3 - \Sigma$ from $S^3 - T$ as above. If we compress $S^3 - \Sigma$ along all compressing discs, each corresponding to a disc of $\Sigma' - (\Sigma \cap \Sigma')$, then we get a solid torus and a 3-manifold with boundary, which is not a solid torus, because $T$ is knotted. Hence $\Sigma$ is not free. Lastly, we show that $\Sigma'$ is not free. Consider $S^3 - \Sigma'$ and compress this manifold along all compressing discs each corresponding to a disc of $\Sigma - (\Sigma \cap \Sigma')$. Then we get a 3-manifold which is homeomorphic to a knot exterior, such that the knot is isotopic to a core of the annular region of $D_L$. $T$ is knotted, so the core of the annular region is a non-trivial knot. So, the resulting 3-manifold is not a handlebody. Hence, $\Sigma'$ is not free.

Now we can complete the proof of Theorem 4.1.3. Consider the almost-projection surface of $\Sigma$ and $\Sigma'$. Suppose that the almost-projection surface is an immersed torus. If we surger the almost-projection surface along a disc, the surface might become disconnected or its genus will decrease. If the surgery reduces the genus, then we get an alternating knot by [14, 15]. As mentioned above, using the Turaev surface, such a knot is toroidally alternating. We continue performing the surgery, cutting off spheres until we get an alternating diagram $D_K$ on an embedded torus. As discussed above,
since $D_K$ is checkerboard colorable, $r(D_K, T)$ is even. If $r(D_K, T) = 0$, then $K$ is an alternating knot. Suppose $r(D_K, T) \geq 2$. During the surgery, we cut off spheres, so the resulting checkerboard surfaces of $D_K$ on $T$ are isotopic to $\Sigma$ and $\Sigma'$. By assumption, one of them is free, hence, $T$ is unknotted by Lemma 4.1.7.

On the other hand, suppose that the almost-projection surface is an immersed Klein bottle. From the proof above, we can find an embedded Möbius band, $f(M \cup A)$.

**Lemma 4.1.8.** The core of $f(M \cup A)$ is unknotted.

**Proof.** Every region of $D_K$ on $f(M \cup A)$ is a disc except one annular region in $\Sigma'$ and one Möbius band region in $\Sigma$. Consider the regular neighborhood of $\Sigma$ in $S^3$ as the following:

We first thicken $f(M \cup A)$, and remove every thickened region of $D_K$ on $f(M \cup A)$ that is a subset of $\Sigma'$. The resulting manifold is homeomorphic to a regular neighborhood of $\Sigma$. Now, we compress the complement of $\Sigma$ by filling each thickened disc region of $\Sigma'$. Then, under the assumption that $\Sigma$ is free, the resulting complementary region is still a handlebody. We recover the complement of $f(M \cup A)$, so this handlebody is a solid torus. This implies that the core of the Möbius band is unknotted in $S^3$. □
Hence, by Lemma 4.1.8, the solid torus that we obtained in Case 2 is unknotted. This completes the proof of Theorem 4.1.3. \hfill \Box

## 4.2 Toroidally alternating knots

**Definition 4.2.1.** [15] Let $\Sigma$ and $\Sigma'$ be properly embedded surfaces in general position in $E(K)$. A **bigon** is a disc $B$ embedded in $E(K)$ such that $\partial B = \beta \cup \beta'$, where $\beta \subset \Sigma$ and $\beta' \subset \Sigma'$ are connected arcs, $\beta \cap \beta'$ consists of two distinct points of $\Sigma \cup \Sigma'$ and $B \cap (\Sigma \cup \Sigma') = \partial B$. The arcs $\beta$ and $\beta'$ are called **edges** of $B$, and $\beta \cap \beta'$ are called **vertices** of $B$. A bigon is **inessential** if it can be homotoped to an intersection arc or an intersection loop of $\Sigma$ and $\Sigma'$. Otherwise, it is **essential**.

Here, the homotopy must be such that restricted to the boundary of $B$, $\beta$ and $\beta'$ must remain in $\Sigma$ and $\Sigma'$, respectively, throughout the homotopy.

Let $\Sigma$ and $\Sigma'$ be a pair of spanning surfaces in $E(K)$. A **minimal representative** of a simple loop $\gamma$ in $\Sigma$ is a simple loop in $\Sigma$ which is isotopic to $\gamma$ and intersects $\Sigma'$ minimally. We can define a minimal representative of a simple loop in $\Sigma'$ in the same manner.

**Definition 4.2.2.** [20] Let $\Sigma$ and $\Sigma'$ be a pair of spanning surfaces in $E(K)$. Then $\Sigma$ and $\Sigma'$ are **relatively separable** if there exists an essential 2-sided...
simple loop $\gamma$ in $\Sigma$ or $\Sigma'$ such that its push-off $\gamma'$ does not intersect the other spanning surface. We say such $\gamma$ is detachable. In this case, $\gamma$ is incident to a bigon if for every minimal representative of $\gamma$, there exists an essential bigon whose boundary intersects $\gamma$ transversely in one point.

Figure 4.13: A detachable curve which is incident to a bigon.

**Definition 4.2.3.** Let $\Sigma$ and $\Sigma'$ be spanning surfaces in $E(K)$. We say that $\Sigma$ and $\Sigma'$ are essentially intersecting if their boundaries intersect minimally on $\partial E(K)$ and every intersection loop is essential on both surfaces.

**Lemma 4.2.4.** Let $\Sigma$ and $\Sigma'$ be essentially intersecting spanning surfaces in $E(K)$. Then the almost projection surface $F$ has no self-intersection loops that bounds a disc on $F$.

**Proof.** Suppose that $F$ has a simple loop intersection which bounds a disc in $F$. Take the innermost loop intersection $\sigma$ and consider a disc bounded by $\sigma$. If this disc does not contains $G_K$, then $\Sigma$ and $\Sigma'$ are not essentially intersecting. Suppose that this disc contains $G_K$. Let $f : F' \to S^3$ be an
immersion map such that \( f(F') = F \). Consider the \( f^{-1}(\sigma) \), which consists of two simple loops \( s_1 \) and \( s_2 \) on \( F' \). Then without loss of generality, we can assume that \( s_1 \) bounds a disc on \( F' \) which contain \( f^{-1}(G_K) \), where \( G_K \) is the 4-valent graph on \( F \) induced from \( K \), but does not contains other pre-images of self-intersections of \( F \). Then both pre-images are contained in the same region of \( f^{-1}(G_K) \), which implies that \( \sigma \) is a self-intersection of either \( \Sigma \) or \( \Sigma' \), which contradicts the assumption that \( \Sigma \) and \( \Sigma' \) are embedded. Hence, there is no self-intersection loop which bounds a disc on \( F \).

Since by Proposition 1, every alternating knot is toroidally alternating, we only consider non-alternating knots below.

**Theorem 4.2.5.** [20] A non-alternating knot \( K \) is toroidally alternating if and only if there exists a pair of essentially intersecting connected, free spanning surfaces \( \Sigma \) and \( \Sigma' \) in the knot exterior which satisfy the following:

1. \( \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial \Sigma, \partial \Sigma') = 0 \).

2. If \( \Sigma \) and \( \Sigma' \) are relatively separable, then every detachable curve is incident to a bigon.

**Remark 4.2.6.** In [15, Figure 3.18], Howie gave an example of a weakly generalized alternating projection of the knot 10_{139}, for which one of the regions
is homeomorphic to an annulus. He showed that there is no essential bigon between the two checkerboard surfaces $\Sigma$ and $\Sigma'$. Hence, this pair $\Sigma$ and $\Sigma'$ is an example of a pair of essentially intersecting free spanning surfaces which are relatively separable, but not every detachable curve is incident to a bigon.

**Proof.** First, we show that if the two checkerboard surfaces $\Sigma$ and $\Sigma'$ of a toroidally alternating diagram are relatively separable, then every detachable loop is incident to a bigon. Let $\gamma$ be a minimal representative of a detachable loop on $\Sigma$. Consider the push-off $\gamma'$ of $\gamma$. Let $A$ be the annulus bounded by $\gamma$ and $\gamma'$ such that $A \cap \Sigma = \gamma$. Every essential loop of $\Sigma$ intersects $\Sigma'$, so $A$ intersects with $\Sigma'$. Then every intersection of $\Sigma'$ and $A$ is either an arc which has its endpoints on $\gamma$ or a simple loop isotopic to $\gamma$. We can modify $\gamma'$ so that $A$ only intersects $\Sigma'$ in arcs. Consider an innermost bigon $B$ in $A$ bounded by $\gamma$ and one of the intersection arcs. Then $B$ is an essential bigon, because if $B$ is inessential, we can isotope $\gamma$ and $A$ to remove the intersection arc, which contradicts our hypothesis, $\gamma$ is a minimal representative. Then we can slightly isotope this bigon to intersect $\gamma$ in one point. Hence, every detachable curve is incident to a bigon. This completes the proof of the “only if” part of Theorem 4.2.5.
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Now to show the “if” part, since $\Sigma$ and $\Sigma'$ are essentially intersecting, we can contract every standard arc to get an almost-projection surface. Below, let $F$ denote the almost-projection surface of $\Sigma$ and $\Sigma'$. By Lemma 4.2.4 and the proof of Theorem 4.1.3, $F$ is either an unknotted torus or an immersed Klein bottle with no 2-sided, non-separating self-intersection loop.

First, suppose that $\Sigma$ and $\Sigma'$ are not relatively separable. We will show that $F$ is an unknotted torus and the alternating diagram on the almost-projection surface is cellulary embedded.

Lemma 4.2.7. Let $\Sigma$ and $\Sigma'$ be essentially intersecting spanning surfaces of a knot $K$ which are not relatively separable. Let $F$ be the almost-projection surface of $\Sigma$ and $\Sigma'$. Then $F$ cannot intersect itself in an essential simple loop.

Proof. Suppose that there exists an essential simple loop intersection $\phi$. Then $\phi$ is either 1-sided or 2-sided. Consider one of the components $\psi$ of the boundary of a regular neighborhood of $\phi$ on $\Sigma$.

Claim 1: $\psi$ is detachable, and it is a minimal representative.

From the construction, $\psi$ does not intersect $\Sigma'$, so it is a minimal representative.

If $\phi$ is 2-sided, then $\psi$ is isotopic to $\phi$, hence essential. Furthermore, $\psi$
has a push-off which does not intersect with $\Sigma'$. Hence, $\psi$ is detachable.

If $\phi$ is 1-sided, then $\psi$ bounds a Möbius band on $\Sigma$, which is a regular neighborhood of $\phi$ on $\Sigma$. If $\psi$ bounds a disc on the other side, then we get a closed component, which is homeomorphic to a real projective plane. A real projective plane cannot be embedded in $S^3$, so the boundary does not bound a disc on $\Sigma$, which implies that $\psi$ is essential. Furthermore, $\psi$ has a push-off which does not intersect $\Sigma'$, so, it is detachable.

The existence of a detachable curve contradicts the assumption that $\Sigma$ and $\Sigma'$ are not relatively separable. Hence, there cannot exist an essential simple loop intersection of $F$.

Lemma 4.2.8. Let $\Sigma$ and $\Sigma'$ be essentially intersecting spanning surfaces of a knot $K$ which are not relatively separable. Then $\Sigma$ and $\Sigma'$ are checkerboard surfaces of a cellularly embedded alternating diagram on a closed orientable surface $F$ with Euler characteristic $\chi(F) = \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma')$.

Proof. If $F$ is non-orientable, then it must have a self-intersection. However, by Lemma 4.2.7, $\Sigma$ and $\Sigma'$ are relatively separable. Since this contradicts our hypothesis, $F$ is orientable.

We now show that the alternating diagram on $F$ is cellularly embedded. Suppose that there exists a region which is not homeomorphic to a disc.
Without loss of generality, we can assume that this region is a subset of $\Sigma$. Consider a graph which is a deformation retract of this region. Any loop of this graph is an essential loop of $\Sigma$ because we can find an arc on $\Sigma$ which has both of its endpoints on $K$ and intersects this loop transversely once. Furthermore, we can find a push-off of this loop which does not intersect $\Sigma'$. Hence, it is a detachable curve, so $\Sigma$ and $\Sigma'$ are relatively separable, which contradicts our hypothesis. Therefore, the alternating diagram on $F$ is cellularly embedded. This completes the proof of Lemma 4.2.8.

By Lemma 4.2.8, if $\Sigma$ and $\Sigma'$ are not relatively separable, then the almost-projection surface $F$ is an unknotted torus with a cellularly embedded alternating diagram.

Now, suppose that $\Sigma$ and $\Sigma'$ are relatively separable. If $F$ is an unknotted torus with a cellularly embedded alternating diagram, then we are done. Otherwise, by the proof of Theorem 4.1.3, we have two cases: either $F$ is an unknotted torus with a non-cellularly embedded alternating diagram, or $F$ is an immersed Klein bottle with no 2-sided, non-separating self-intersection loop. We will show in the first case, $K$ is almost-alternating hence toroidally alternating, and that the second case is not possible.
Case 1: $F$ is an unknotted torus with a non-cellularly embedded alternating diagram.

Let $\gamma$ be a core of the the annular region, which is a minimal representative of itself. We showed above that $\gamma$ is detachable. We can assume that $\gamma$ is on $\Sigma$. Now, we show that if $\gamma$ is incident to a bigon, then $K$ is almost-alternating. To show this, we need the following lemma.

**Lemma 4.2.9.** There exists a compressing disc of $F$ which intersects the set of edges of $G_K$ transversely twice.

**Proof.** By assumption, there is an essential bigon $\mathcal{B}$ which intersects $\gamma$ transversely once on its boundary $\partial \mathcal{B}$. After contracting standard arc intersections to get $F$, bigon $\mathcal{B}$ becomes a disc whose interior is embedded in the complement of $F$, and $\partial \mathcal{B}$ is a loop on $F$ which intersects $G_K$ only in its vertices. The loop $\partial \mathcal{B}$ is simple, whenever both vertices of $\partial \mathcal{B}$ are on different standard arc intersections of $\Sigma$ and $\Sigma'$. If $\partial \mathcal{B}$ is simple, we can modify $\partial \mathcal{B}$ to intersect $G_K$ transversely twice on edges of $G_K$. Otherwise, $\partial \mathcal{B}$ on $F$ is a loop which has one self-intersection on some vertex of $G_K$. However, the interior of $\mathcal{B}$ does not intersect itself, so the self-intersection of $\partial \mathcal{B}$ is not transverse. Therefore, we can modify $\partial \mathcal{B}$ to be a simple loop, and intersect $G_K$ transversely twice on its edges. (See Figure 4.14.) Since $\partial \mathcal{B}$ intersects $\gamma$
transversely once, $\partial B$ on $F$ is essential, so $B$ is a compressing disc of $F$. □

Figure 4.14: If $B$ intersect itself in one point on $\partial B$, then we can modify $\partial B$ to remove the self-intersection.

With the assumption that $F$ is unknotted, the fact that $B$ is a compressing disc of $F$ implies that a regular neighborhood of $\gamma$ on $F$ is an annulus whose core is unknotted in $S^3$. Then $G_K$ is also on the annulus with an unknotted core, since $G_K$ is on the complementary region of $\gamma$ on $F$. Now, $\partial B$ intersects $G_K$ transversely twice, hence we can find an essential simple arc on the annulus which contains $G_K$ such that it intersects $G_K$ transversely twice. If we cut $G_K$ along this arc, we get an alternating 2-tangle on a disc as in Figure 4.15.

Since the core of the annulus which contains $G_K$ is unknotted, $K$ can be obtained by taking $n$-full twists on two strands of some alternating knot diagram. This operation yields either an alternating knot diagram or a cycle of two alternating 2-tangles (see Figure 3.1.). By assumption, $K$ is not alternating, so it is a cycle of two alternating 2-tangles. Then by Theorem 3.0.1,
Figure 4.15: If we cut $G_K$ along the red arc, we get an alternating 2-tangle on a disc.

its Turaev genus is one, so $K$ is toroidally alternating.

**Case 1:** $F$ is a Klein bottle.

We showed before that in this case, $\Sigma$ and $\Sigma'$ are relatively separable. We will show that if every detachable curve is incident to a bigon, then $K$ is alternating.

As we discussed above, every self-intersection loop of $F$ is either a 2-sided separating loop or a 1-sided loop. Suppose that there exists a 2-sided self-intersection loop. Consider one of the 2-sided self-intersection loops $\gamma'$ on $\Sigma$, which is adjacent to $G_K$ on the almost-projection surface. Consider a simple loop $\gamma$ on $\Sigma$ which is on the region between $\gamma'$ and $G_K$ on the almost-projection surface, and isotopic to $\gamma'$. By Claim 1 in Lemma 4.2.7, $\gamma$ is detachable and it is a minimal representative of itself. Consider a bigon $B$ which is incident to $\gamma$. Let $\beta$ be the edge of $B$ which is on $\Sigma$ and intersects $\gamma$.
transversely once. Since $\gamma$ is adjacent to $G_K$, one of the vertices of $\beta$ is on $\gamma'$ and the other is on some standard arc intersection as in Figure 4.16(b). This implies that there exists a properly embedded essential arc $\tau$ on the annulus $f(A)$, which intersects $G_K$ once. (Recall that $G_K$ is contained in the annulus $f(A)$ as in the proof of Theorem 4.1.3, Case 2, subcase 2.)

(a) $\Sigma$ and $\Sigma'$ near $K$ and a detachable curve $\gamma$, which is incident to a bigon $B$.

(b) One of the vertices of $\partial B$ is on $\gamma'$ and the other is on a standard arc. Then we can find $\tau$ on the annulus contains $G_K$, which intersects $G_K$ once.

Hence, by Lemma 4.1.5, $K$ is a connected sum of an alternating knot and the other knot, which is isotopic to a core of the annulus which contains $G_K$.

By Lemma 4.1.8, the core of $f(M \cup A)$ is unknotted, so the core of $f(A)$ is a torus knot type $(2, 2q + 1)$, $q \geq 0$ in $S^3$, which is alternating. Therefore, $K$ is alternating.

Instead, suppose that there is no 2-sided intersection loop. Then there exists a 1-sided self-intersection loop, $\eta'$. Note that there is no other 1-sided self-intersection loop because the complement of $\eta'$ is an annulus, so every
other loop is 2-sided. By Claim 1 in Lemma 4.2.7, the boundary $\eta$ of a regular neighborhood of $\eta'$ on $\Sigma$ is detachable and a minimal representative of itself. Consider a bigon $B$ incident to $\eta$. By the same argument, $\partial B$ has one vertex on $\eta'$, and the other vertex on the standard arc intersection. By the same argument as above, $K$ is alternating.

This completes the proof of Theorem 4.2.5.

4.3 Almost-alternating knots

Theorem 4.3.1. [20] A non-alternating knot $K$ is almost-alternating if and only if there exists a pair of essentially intersecting connected, free spanning surfaces $\Sigma$ and $\Sigma'$ in the knot exterior which satisfy the following:

1. $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 0$.

2. $\Sigma$ and $\Sigma'$ are relatively separable and every detachable curve is incident to a bigon.

Proof. We show the “if” part first. In Case 1 of the proof of Theorem 4.2.5, we showed that if $\Sigma$ and $\Sigma'$ are relatively separable and their almost-projection surface is a Klein bottle, then $K$ is alternating. Assuming that $K$ is non-alternating, the almost-projection surface of $\Sigma$ and $\Sigma'$ is an unknotted torus.
Below, let \( F \) denote the almost-projection surface of \( \Sigma \) and \( \Sigma' \). The alternating diagram \( D_K \) on \( F \) may be cellularly embedded or not, which we consider in separate cases.

Suppose that \( D_K \) on \( F \) is not cellularly embedded. Then by Lemma 4.2.9, we can find an essential simple loop on \( F \) which intersects \( D_K \) twice and bounds a compressing disc. Hence, \( K \) has a cycle of two alternating 2-tangles. Then by [2, Proposition A.6], \( K \) can be transformed into an almost alternating diagram.

On the other hand, suppose that the diagram \( D_K \) is cellularly embedded. Let \( \gamma \) be a detachable curve on \( \Sigma \). We will show that we can isotope \( \Sigma \) and \( \Sigma' \) so that after the isotopy, the new almost-projection surface of \( \Sigma \) and \( \Sigma' \) has an alternating diagram with an annular region whose core is \( \gamma \). Let \( \gamma' \) be a push-off of \( \gamma \), and \( A \) be an annulus bounded by \( \gamma \) and \( \gamma' \).

Since \( \gamma \) is on \( \Sigma \) and \( D_K \) is cellularly embedded, \( A \cap \Sigma' \neq \emptyset \). Furthermore, by the relatively separable condition, every intersection of \( A \) and \( \Sigma' \) is an arc which has both its endpoints on \( \gamma \). The innermost intersection arc bounds a bigon \( B \). If \( B \) is inessential, we can isotope \( \gamma \) and \( A \) to remove such an intersection. If \( B \) is essential, then we can isotope the surface along \( B \) to remove the intersection, as in Figure 4.17. After the isotopy, we get a new bigon \( B' \) as in Figure 4.17(right).
Now, we show that after the isotopy, $\Sigma$ and $\Sigma'$ are still essentially intersecting, the new almost-projection surface of $\Sigma$ and $\Sigma'$ is still an embedded unknotted torus. Below, let $F'$ denote the new almost-projection surface of $\Sigma$ and $\Sigma'$ after the isotopy along $\mathcal{B}$.

First, we show that after the isotopy, $\Sigma$ and $\Sigma'$ are still essentially intersecting. Since this isotopy does not change $\Sigma$ and $\Sigma'$ near their boundaries, the number of arc intersections remains minimal. Therefore, if $\Sigma$ and $\Sigma'$ are not essentially intersecting after the isotopy, then there exists an inessential intersection loop. Furthermore, each isotopy can change the number of intersecting components at most once, so there is only one inessential intersection loop $\mu$.

We will show that $\mu$ bounds a disc on both spanning surfaces. Since $\mu$ is inessential, it bounds a disc in one of the spanning surfaces, say $\Sigma$. If $\mu$ does not bound a disc in the other spanning surface, $\Sigma'$, then we can surger $\Sigma'$ along a disc bounded by $\mu$ on $\Sigma$. Let $\Sigma^*$ be the resulting spanning surface.
Then the first condition implies that $\chi(\Sigma) + \chi(\Sigma^*) + \frac{1}{2} i(\partial \Sigma, \partial \Sigma^*) = 2$, so $K$ is alternating. As this contradicts the hypothesis that $K$ is non-alternating, $\mu$ bounds discs in both spanning surfaces.

If we undo the isotopy, then the intersection pattern of $\Sigma$ and $\Sigma'$ changes as in Figure 4.18. More specifically, it changes from the right picture to the left picture. Then the edge of $B$ is an arc on the left picture, which cobounds a disc with a subarc of $\Sigma \cap \Sigma'$. Then each edge can be isotoped onto $\Sigma \cap \Sigma'$, hence $B$ is inessential. This contradicts the assumption that $B$ is essential. Therefore, $\Sigma$ and $\Sigma'$ are essentially intersecting after the isotopy along $B$.

![Figure 4.18: Intersection pattern changes whenever we isotope a surface along a bigon.](image)

Now, we show that $F'$ is an embedded torus. Suppose that $F'$ is a Klein bottle. A Klein bottle cannot be embedded in $S^3$ so there exists at least one self-intersection loop. Since $\Sigma$ and $\Sigma'$ intersect only in standard arcs before the isotopy, the isotopy along $B$ divides one standard arc into a standard arc and an essential intersection loop. Consider a new bigon $B'$ after the isotopy as in Figure 4.17. Then one of the vertices of $B'$ is on the standard
arc intersection and the other is on the essential intersection loop, so \( B' \) is essential. This essential simple intersection loop is a self-intersection loop of \( F' \), and \( F' \) is a Klein bottle, so the self-intersection loop is either 1-sided or 2-sided and separating. Hence, the boundary of a regular neighborhood of the essential intersection loop on each spanning surface is incident to \( B' \) as in Figure 4.16(a). Then, as in Case 1 of the proof of Theorem 4.2.5, we can show that \( K \) is alternating. By hypothesis, \( K \) is non-alternating, so \( F' \) cannot be a Klein bottle. Hence, we still have an alternating diagram on an embedded torus.

We can continue these isotopies to remove all intersections between \( A \) and \( \Sigma' \). Then the almost-projection surface obtained from \( \Sigma \) and \( \Sigma' \) after isotopy is another unknotted torus, such that the alternating diagram on the torus has an annular region whose core is \( \gamma \). We will show that \( \gamma \) is incident to a bigon after isotopies. Then by the same argument as in the previous case (non-cellularly embedded diagram), it follows that \( K \) is almost-alternating.

First, we will show that the new bigon \( B' \) after the isotopy (as in Figure 4.17) is an essential bigon. Suppose \( B' \) is inessential. Then by definition, it can be homotoped to a standard arc intersection. This implies that two vertices of \( B' \) are on the same standard arc intersection and both edges of \( B' \) are homotopic to a subarc of the standard arc intersection on each spanning
surface. Let $\beta$ be an edge of $B'$ on $\Sigma$. Then $\beta$ and the subarc of the standard arc intersection cobound a disc on $\Sigma$ as in Figure 4.18(left). If we undo the isotopy that we performed, we have an inessential loop intersection, which contradicts the assumption that $\Sigma$ and $\Sigma'$ are essentially intersecting. Hence, $B'$ is essential.

Furthermore, by the argument similar to [24, Proposition 2.3], $\partial B'$ is not an inessential curve on $F'$. Hence, $B'$ is a compressing disc of $F'$.

Lastly, we will show that $\gamma$ is incident to $B'$. If $\gamma$ is an inessential simple loop on $F'$, then the diagram is disconnected, which contradicts the assumption that $K$ is a knot. If $\gamma$ is isotopic to $\partial B'$, then we can compress $F'$ along $\gamma$ to get an alternating diagram on a sphere. Hence, $\gamma$ intersects $\partial B'$ at least once. Suppose that $\gamma$ intersects $\partial B'$ more than once. By Lemma 4.2.9, $\partial B'$ intersects the diagram transversely twice. Hence, $\gamma$ intersects $\partial B'$ transversely once or twice.

We will show that $\gamma$ intersects $\partial B'$ once, which is equivalent to saying that $\gamma$ is incident to $B'$. To show this, we first assume that $\gamma$ intersects $\partial B'$ twice and show that $K$ is alternating, which contradicts the assumption that $K$ is non-alternating. If $\gamma$ intersects $\partial B'$ twice, then the annulus $F' - int(N(\gamma))$, which contains the diagram, intersects $\partial B'$ in two essential arcs. The diagram intersects $\partial B'$ in two points, so either two essential arcs intersect the diagram
CHAPTER 4. TOROIDALLY ALTERNATING KNOTS

In one point or one of the essential arcs intersects the diagram in two points. In first case, by Lemma 4.1.5, $K$ is a connected sum of an alternating knot and a core of the annulus. The core is a $(2, q)$ curve on the torus $F'$, because it intersects $\partial B'$ twice. $F'$ is unknotted, hence, the core is a $(2, q)$ torus knot, which is alternating. Hence, $K$ is alternating. In latter case, the diagram is in a disc, so $K$ is also alternating. Since we assumed that $K$ is non-alternating, $\gamma$ intersects $\partial B'$ once, so $\gamma$ is incident to $B'$

Now, to show the “only if” part, suppose that the knot $K$ is almost-alternating. Consider an almost-alternating diagram of $K$ as in Figure 4.19(left). Then we can do a Reidemeister II move as in Figure 4.19(middle) to make the diagram as in Figure 4.19(right). Then $K$ has a checkerboard-colorable, non-cellularly embedded, alternating diagram $D'_K$ on an unknotted torus as in Figure 4.20(left). By Lemma 4.1.6, $D'_K$ has a unique annular region. The core of the annular region is detachable, so the two checkerboard surfaces of $D'_K$ are relatively separable. Now we need to show that every
detachable curve is incident to a bigon. The core of the annular region is incident to the bigon shown in Figure 4.20(right). This bigon is essential because the two vertices of this bigon are contained in different standard arc intersections. If there exists another detachable curve, then it must intersect the standard arc intersection. Hence, as in the proof of the “if” part of this theorem, we can find an essential bigon using the annulus bounded by the detachable curve and its push-off.

This completes the proof of Theorem 4.3.1.

Figure 4.20: An essential bigon between two checkerboard surfaces of $D'_K$. 
Bibliography


