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Quantum Optical Interferometry and Quantum State Engineering

Richard J. Birrittella Jr

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Doctoral Thesis

QUANTUM OPTICAL INTERFEROMETRY AND QUANTUM STATE ENGINEERING

Author:
Richard J. Birritella Jr.

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2017
QUANTUM OPTICAL INTERFEROMETRY AND QUANTUM STATE ENGINEERING

by

Richard J. Birritella Jr.

This manuscript has been read and accepted for the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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We highlight some of our research done in the fields of quantum optical interferometry and quantum state engineering. We discuss the body of work for which our research is predicated, as well as discuss some of the fundamental tenants of the theory of phase estimation. We do this in the context of quantum optical interferometry where our primary interest lies in the calculation of the quantum Fisher information as it has been shown that the minimum phase uncertainty obtained, the quantum Cramér-Rao bound, is saturated by parity-based detection methods. We go on to show that the phase uncertainty one obtains through the quantum Fisher information is in agreement with the error propagation calculus when using parity as a detection observable. We also introduce a technique through which one can generate new non-classical single- and two-mode states of light known as photon catalysis. This involves a projective measurement made at the output of a beam splitter with variable transmittance, for \( l \) photons, where our initial state is a tensor product between a single-mode field state and a number state comprised of \( q \) photons. We close the paper with a discussion on a proposed state-projective scheme for generating pair coherent states using existing and readily available technology.
Declaration of Authorship

I, Richard J. BiritteLLa Jr., declare that this thesis titled, “QUANTUM OPTICAL INTERFEROMETRY AND QUANTUM STATE ENGINEERING” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.

- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

- Where I have consulted the published work of others, this is always clearly attributed.

- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

- I have acknowledged all main sources of help.

- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

________________________________________

Date:

________________________________________
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I would like to thank my supervisor, first and foremost, for his steadfast guidance throughout my time working with him. I would also like to thank my family and friends for putting up with a penniless graduate student for all these years......
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Dedicated to my family, specifically to my parents, for their unwavering support in my academic endeavors......
Chapter 1

Introduction

In this thesis, we present a summary of research done in the field of quantum optical interferometry and quantum state engineering. Chapter 2 begins with a brief review on the basics of phase estimation [114] and the use of the quantum Fisher information, and the corresponding quantum Cramér-Rao bound (qCRB), in calculating the minimum phase uncertainty attainable in a standard interferometric set-up. We discuss the fundamental limits on phase estimation when using classical and non-classical light, the standard quantum limit (SQL) and Heisenberg limit (HL), respectively. Special emphasis is placed throughout this paper on the use of parity-based detection methods. We show that the qCRB is saturated when parity is used as a detection observable, and measurements of the phase uncertainty through error propagation calculus are in agreement with the minimum phase value obtained through the quantum Fisher information. Lastly, we demonstrate how one may calculate the quantum Fisher information in quantum optical interferometry.

In Chapter 3, we explore the properties of the states produced by non-degenerate coherently stimulated parametric down-conversion wherein the signal and idler modes are seeded with coherent states of light and where the nonlinear crystal is driven by a strong classical field as described by the parametric approximation. The states produced are the two-mode squeezed coherent states defined with a specific ordering of operators, namely, the displacement operators of the two modes acting on the double vacuum state followed by the action of the two-mode squeeze operator representing...
the down-conversion process. Though mathematically equivalent to the reverse ordering of operators, but with different displacement parameters, the ordering we consider is closely related to what could most easily be implemented in the laboratory. The statistical properties of the state are studied with an emphasis on how they, and its average photon number, are affected by the various controllable phases, namely, those of the classical pump field of the two input coherent states. We then consider the multi-photon interference effects that arise if the two beams are overlapped on a 50:50 beam splitter, investigating the role of the phases in controlling the statistical properties of the output states. We also study the prospects for the application of the states to quantum-optical interferometry to obtain sensitivities in phase-shift measurements beyond the standard quantum limit. Finally, we close with an analysis of the effects the phases have on the photon statistics of the state in the fully quantum mechanical model, where the pump is no longer a classically prescribed field, but is instead a quantized single mode field state, taken to be coherent light. We study the evolution of the field states for short times using a perturbative approach.

We also study multi-photon quantum-interference effects at a beam splitter and its connection to the prospect of attaining interferometric phase-shift measurements with noise levels below the standard quantum limit in Chapter 4. Specifically, we consider the mixing of the most classical states of light coherent states with the most nonclassical states of light number states at a 50:50 beam splitter. Multi-photon quantum-interference effects from mixing photon-number states of small photon numbers with coherent states of arbitrary amplitudes are dramatic even at the level of a single photon. For input vacuum and coherent states, the joint photon-number distribution after the beam splitter is unimodal, a product of Poisson distributions for each of the output modes but with the input of a single photon, the original distribution is symmetrically bifurcated into a bimodal distribution. With a two-photon-number state mixed with a coherent state, a trimodal distribution is obtained, etc. These distributions are shown to be structured so as to be conducive for approaching Heisenberg-limited sensitivities in photon-number parity-based interferometry. We show that mixing a coherent
state with even a single photon results in a significant reduction in noise over that of the shot-noise limit. Finally, based on the results of mixing coherent light with single photons, we consider the mixing of coherent light with the squeezed vacuum and the squeezed one-photon states and find the latter yields higher sensitivity in phase-shift measurements for the same squeeze parameter owing to the absence of the vacuum state. Hofmann and Ono [69] showed that the mixing of coherent light and single-mode squeezed light at a beam splitter gives good approximation results in a superposition of ph suath-entangled photon number states (so-called N00N states), which can be used for phase-shift measurements by coincident detections at the output of an interferometer. They showed that N00N states for arbitrary photon number $N$ could be produced by this procedure. Afek et. al [73] have implemented the HofmannOno proposal in the laboratory. In this paper, we show that, for a given coherent state amplitude and a given squeezing parameter, the mixing of coherent states and photon-subtracted squeezed vacuum states at the first beam splitter of an interferometer leads to improved phase-shift measurement sensitivity when using the photon-number detection technique on one of the output beams of the device. We also show that the phase-shift measurements will also be super-resolved to a greater degree than is possible by mixing coherent and squeezed vacuum light of the same field parameters.

In Chapter 6 we explore the prospect of using photon catalysis as a means for generating new non-classical states of light. Photon catalysis is a technique by which a readily available Gaussian state of light prepared in one mode is incident upon a beam splitter with a discrete number of photons, $q$, prepared in another mode; the resulting two-mode state is then subjected to single-photon resolving detection for $q$ photons on one of the output modes. By employing beam splitters of different transmissivities and reflectivities, the subsequent single-mode state is shown to possess non-classical properties such as quadrature squeezing and sub-Poissonian statistics. We consider the case in which the input state is the most general of pure single-mode Gaussian states: a squeezed coherent state. Noting the Gaussianity of the initial state, we demonstrate non-Gaussianity
Chapter 1. Introduction

of the photon-catalyzed state by negativity of the Wigner function. We extend this technique to the two-mode squeezed states, whereby we perform photon catalysis on one mode of a two-mode squeezed vacuum state. The resulting correlated two-mode state may have applications in fundamental tests of quantum mechanics such as violations of Bell’s inequalities as the detection loophole can be closed due to the non-Gaussianity of the photon-catalyzed state. We also generalize our method to include state-projective measurements for $l \neq q$ photons. We also explore a state projective scheme for generating the so-called pair coherent states (PCS); a type of two-mode correlated state of light. The pair coherent states of a two-mode quantized electromagnetic field introduced by Agarwal [10] have yet to be generated in the laboratory. The states can mathematically be obtained from a product of ordinary coherent states via projection onto a subspace wherein identical photon number states of each mode are paired. We propose a scheme by which this projection can be engineered. The scheme requires relatively weak cross-Kerr nonlinearities, the ability to perform a displacement operation on a beam mode, and photon detection ability able to distinguish between zero and any other number of photons. These requirements can be fulfilled with currently available technology or technology that is on the horizon.
Chapter 2

Phase Estimation

Interferometry is the art of estimating phase shifts [114]. An interferometer is a physical apparatus that encodes the value of a parameter into a probe state; for the case of optical interferometry the parameter is typically taken to be a phase shift induced by a relative path length difference between the two spatially separated beams of light traveling along the interferometer arms. As we will discuss in later chapters, quantum optical interferometry is most often used to aid in the detection of gravitational waves at observatories such as L.I.G.O. These observatories function as large-scale Michelson interferometers (each arm is around a kilometer in length) where the induced phase shift is caused by passing gravitational waves. These phase shifts are infinitesimal in magnitude: on the order of $\propto 10^{-20}$. The goal is to, in some way, 'measure' this phase shift. It is important to note that phase cannot be measured directly; instead it must be estimated. This is because there is no (Hermitian) quantum mechanical operator corresponding to the phase. In interferometry, the phase is typically treated classically and estimated through the use of a detection observable (estimator) and error propagation calculus.

Before we begin a thorough overview of some of our work done in the field of quantum optical interferometry, it is prudent to first discuss some of the basics of phase estimation. In this chapter, we hope to answer some of the important questions in statistical inference: how precise can a statistical estimation be? Are there any limits on the precision of an estimation? Our goal is to outline some of the work done to answer these questions. Some of the earliest work that endeavored to answer these questions
occurred around the 1940s by Cramér [35], Rao [119], and Fréchet [49]. Their work, independently, placed a lower bound on the variance of an arbitrary estimator. This lower bound was found to be related to the so-called Fisher Information [48] introduced by R. A. Fisher in the 1920s, and is generally known as the Cramér-Rao lower bound. It was later found [23] [122] that maximization of the Fisher information over all possible estimators led to the determination of a quantum lower bound, known as the quantum Cramér-Rao bound. Thus, it is inescapable that the Fisher information plays an important role in the study of phase estimation. With that in mind, the goal of this chapter is to provide the reader with a brief overview of these and other basic concepts behind phase estimation and their use in quantum optical interferometry by demonstrating the Cramér-Rao bound and its relation to different estimation protocols, as well as defining and discussing both the classical and quantum-Fisher information.

2.1 Basic Concepts — Estimators

An interferometer can be taken to be an apparatus that can transform a so-called input ‘probe state’ $\hat{\rho}$ in a manner such that the transformation can be parametrized by a real, unknown, number $\varphi$ [114]. A measurement is then performed on the output state $\hat{\rho}(\varphi)$ from which an estimation of the parameter $\varphi$ takes place. The most general formulation of a measurement (or estimator) in quantum theory [18] is a positive-operator valued measure (POVM). A POVM consists of a set of non-negative Hermitian operators satisfying the unity condition $\sum_\epsilon E(\epsilon) = 1$. Following the work of Pezzé et al. [114]
throughout this chapter, the conditional probability to observe the result $\epsilon$ for a given value $\varphi$, known as the likelihood, is

$$P(\epsilon|\varphi) = \text{Tr}\left[\hat{E}(\epsilon) \hat{\rho}(\varphi)\right].$$

(2.1)

If the input state is made up of $m$ independent uncorrelated subsystems such that

$$\hat{\rho} = \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \hat{\rho}^{(3)} \otimes \hat{\rho}^{(4)} \otimes \ldots \otimes \hat{\rho}^{(m)}$$

(2.2)

and we restrict ourselves to local operations, i.e. $\hat{\rho}(\varphi) = \hat{\rho}^{(1)}(\varphi) \otimes \hat{\rho}^{(2)}(\varphi) \otimes \hat{\rho}^{(3)}(\varphi) \otimes \hat{\rho}^{(4)}(\varphi) \otimes \ldots \otimes \hat{\rho}^{(m)}(\varphi)$ and independent measurements, such that our estimator is given by $\hat{E}(\epsilon) = \hat{E}^{(1)}(\epsilon_1) \otimes \hat{E}^{(2)}(\epsilon_2) \otimes \hat{E}^{(3)}(\epsilon_3) \otimes \hat{E}^{(4)}(\epsilon_4) \otimes \ldots \otimes \hat{E}^{(m)}(\epsilon_m)$, then the likelihood function becomes the product of the single-measurement probabilities

$$P(\epsilon|\varphi) = \prod_{i=1}^{m} P_i(\epsilon_i|\varphi)$$

(2.3)

where $P_i(\epsilon_i|\varphi) = \text{Tr}\left[\hat{E}^{(i)}(\epsilon_i) \hat{\rho}^{(i)}(\varphi)\right]$. For the case of independent measurements, as described in Eq. (2.3), often one considers the log-Likelihood function

$$L(\epsilon|\varphi) \equiv \ln P(\epsilon|\varphi) = \sum_{i=1}^{m} \ln P_i(\epsilon_i|\varphi).$$

(2.4)

We define the estimator $\Phi(\epsilon)$ as any mapping of a given set of outcomes, $\epsilon$, onto parameter space. It is simply a function that associates each set of measurements with an estimation of the phase. The estimator can be characterized by its phase dependent mean value

$$\langle \Phi \rangle_{\varphi} = \sum_{\epsilon} P(\epsilon|\varphi) \Phi(\epsilon),$$

(2.5)

and its variance

$$(\Delta \Phi)^2_{\varphi} = \sum_{\epsilon} P(\epsilon|\varphi) \left[\Phi(\epsilon) - \langle \Phi \rangle_{\varphi}\right]^2.$$
We will now discuss what it means for an estimator to be ‘good’, which in this case, refers to an estimator that provides the smallest uncertainty. These estimators are known as unbiased estimators, and are defined as estimators whose statistical average coincides with the true value of the parameter in question, that is,

\[ \langle \Phi (\epsilon) \rangle_\varphi = \varphi, \]  

is true for all values of the parameter \( \varphi \). Estimators not satisfying Eq. (2.7) are considered biased while estimators that are unbiased for a certain range of the parameter \( \varphi \) is considered locally unbiased. Lastly we define consistent estimators as estimators that asymptotically approach the true value of the parameter, that is,

\[ \lim_{m \to \infty} \langle \langle \Phi (\epsilon) \rangle \rangle_\varphi, \]  

for a given sequence of measurements, \( \epsilon = \{\epsilon_1, ..., \epsilon_m\} \), and a subsequent sequence of estimates \( \Phi (\epsilon_1), \Phi (\epsilon_1, \epsilon_2), ..., \Phi (\epsilon_1, \epsilon_2, ..., \epsilon_m) \).

The principles discussed thus far serve to build the foundation of phase estimation. We now move on to perhaps one of the most important tools in the theory of phase estimation: the Cramér-Rao bound (CRB). In order to properly define and discuss this lower limit on phase estimation, we must first turn our attention to a quantity very closely related to this bound, namely, the Fisher information.

### 2.1.1 The Cramér-Rao Lower Bound

The Cramér-Rao serves to set a lower bound on the variance of any arbitrary estimator. We can derive this lower bound by first considering the quantities
\[
\frac{\partial \langle \Phi \rangle_\varphi}{\partial \varphi} = \frac{\partial}{\partial \varphi} \sum_\epsilon P(\epsilon | \varphi) \Phi(\epsilon) = \sum_\epsilon P(\epsilon | \varphi) \Phi(\epsilon) \frac{\partial L(\epsilon | \varphi)}{\partial \varphi} = \langle \frac{\Phi}{\partial \varphi} \frac{\partial L}{\partial \varphi} \rangle, \tag{2.9}
\]

and

\[
\frac{\partial}{\partial \varphi} \sum_\epsilon P(\epsilon | \varphi) = \sum_\epsilon P(\epsilon | \varphi) \frac{\partial L(\epsilon | \varphi)}{\partial \varphi} = \langle \frac{\partial L}{\partial \varphi} \rangle_\varphi = 0, \tag{2.10}
\]

where we have used the identity \( \partial_\varphi L(\epsilon | \varphi) = P(\epsilon | \varphi)^{-1} \partial_\varphi P(\epsilon | \varphi) \) as well as the unity condition \( \sum_\epsilon P(\epsilon | \varphi) = 1 \). Combining the two equations, Eq. (2.9, 2.10), we obtain

\[
\left( \frac{\partial \langle \Phi \rangle_\varphi}{\partial \varphi} \right)^2 = \langle \left( \Phi - \langle \Phi \rangle_\varphi \right) \frac{\partial L}{\partial \varphi} \rangle_\varphi. \tag{2.11}
\]

Next, we employ the use of the Cauchy-Schwartz inequality \( \langle A^2 \rangle_\varphi \langle B^2 \rangle_\varphi \geq \langle AB \rangle_\varphi \), where \( A \) and \( B \) are real functions of the parameter \( \epsilon \) and the relation is minimized if and only if \( B = \kappa A \). Taking \( A = \Phi - \langle \Phi \rangle_\varphi \) and \( B = \partial_\varphi L \) we arrive at

\[
\langle (\Phi - \langle \Phi \rangle_\varphi)^2 \rangle_\varphi \langle \left( \frac{\partial L}{\partial \varphi} \right)^2 \rangle_\varphi \geq \left( \frac{\partial \langle \Phi \rangle_\varphi}{\partial \varphi} \right)^2. \tag{2.12}
\]

Noting that the quantity \( (\Delta \Phi)^2_\varphi = \langle (\Phi - \langle \Phi \rangle_\varphi)^2 \rangle_\varphi \) we arrive at the CRB, given formally as

\[
(\Delta \Phi)^2_\varphi \geq \left( \frac{\partial \langle \Phi \rangle_\varphi}{\partial \varphi} \right)^2 = (\Delta \Phi_{\text{CR}})^2_\varphi, \tag{2.13}
\]

and where the quantity \( F(\varphi) \) is the classical Fisher information (FI), given by

\[
F(\varphi) \equiv \left( \left( \frac{\partial L(\epsilon | \varphi)}{\partial \varphi} \right)^2 \right) = \sum_\epsilon \frac{1}{P(\epsilon | \varphi)} \left( \frac{\partial P(\epsilon | \varphi)}{\partial \varphi} \right)^2, \tag{2.14}
\]
where the sum extends over all possible values of the measurement values, $\epsilon$. While Eq. (2.13) is the most general form the CRB, it is most useful for the cases of unbiased estimators where the numerator on the right-hand side, $\partial_\varphi \langle \Phi \rangle_\varphi = 1$. For this case, the CRB is simply given as the inverse of the Fisher information $F(\varphi)$. An estimator that saturates the CRB is said to be efficient [114]. The existence, however, of an efficient estimator depends on the properties of the probability distribution. It is important to point out that an estimator $\Phi(\epsilon)$ saturates the CRB at the phase value $\varphi$ when the Cauchy-Schwartz inequality, Eq. (2.12), is saturated. More specifically, this implies

$$\frac{\partial L(\epsilon|\varphi)}{\partial \varphi} = \kappa \left( \Phi(\epsilon) - \langle \Phi \rangle_\varphi \right),$$

(2.15)

for all values of $\epsilon$ and where $\kappa = F(\varphi) / \partial_\varphi \langle \Phi \rangle_\varphi$. In the next section, we will arrive at an upper bound on phase estimation, known formally as the quantum Cramér-Rao bound (qCRB).

### 2.2 Quantum Fisher Information and the Upper Bound

So far we have discussed a lower bound on phase estimation known as the Cramér-Rao bound (CRB), given in terms of the classical Fisher information (FI), which will be dependent on the choice of estimator employed. We now turn our attention to finding an upper bound on phase estimation, known as the quantum Cramér-Rao bound (qCRB), which in turn will be dependent on the quantum Fisher information $F_Q$ (QFI). We obtain this upper bound by maximizing the FI over all possible POVMs,

$$F_Q[\hat{\rho}(\varphi)] \equiv \max_{\{E(\epsilon)\}} F\left[ \hat{\rho}(\varphi), \{ \hat{E}(\epsilon) \} \right],$$

(2.16)

where this quantity is known as the quantum Fisher information. Once again following the work of Pezzé et al. [114], we show that this quantity can be expressed as

$$F_Q[\hat{\rho}(\varphi)] = \text{Tr}\left[ \hat{\rho}(\varphi) \hat{L}_\varphi^2 \right],$$

(2.17)
where $\hat{L}_\varphi$ is known as the symmetric logarithmic derivative (SLD) [45] defined as the solution to the equation

$$\frac{\partial \hat{\rho}(\varphi)}{\partial \varphi} = \hat{\rho}(\varphi) \hat{L}_\varphi + \hat{L}_\varphi \hat{\rho}(\varphi).$$ (2.18)

Our chain of inequalities is now

$$(\Delta \Phi)^2 \geq (\Delta \Phi)^2_{\text{CRB}} \geq (\Delta \Phi)^2_{\text{qCRB}},$$ (2.19)

where it follows that the quantum Cramér-Rao (also known as the Helstrom) [45] bound is given by

$$(\Delta \Phi)^2_{\text{qCRB}} \equiv \left(\frac{\partial \langle \Phi \rangle}{\partial \varphi}\right)^2.$$ (2.20)

It is straightforward enough to prove Eq. (2.17) simply by starting with the definition of the classical Fisher information, Eq. (2.14), and making the substitutions $P(\epsilon|\varphi) = \text{Tr}[\hat{E}(\epsilon) \hat{\rho}(\varphi)]$ with its derivative given by $\partial_\varphi P(\epsilon|\varphi) = \text{Tr}[\hat{E}(\epsilon) \partial_\varphi \hat{\rho}(\varphi)]$. Substituting this into Eq. (2.14) yields

$$F[\hat{\rho}(\varphi), \{\hat{E}(\epsilon)\}] = \sum_\epsilon \frac{\text{Tr}[\hat{E}(\epsilon) \partial_\varphi \hat{\rho}(\varphi)]^2}{\text{Tr}[\hat{E}(\epsilon) \hat{\rho}(\varphi)]}. \quad (2.21)$$

Once again following the work of Pezzé et al. [114], and using the definition of the SLD given in Eq. (2.18), we have

$$\text{Tr}[\hat{E}(\epsilon) \partial_\varphi \hat{\rho}(\varphi)] = \Re \left(\text{Tr}[\hat{\rho}(\varphi) \hat{L}_\varphi \hat{E}(\epsilon)]\right),$$ (2.22)

where $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of the complex number $z$, respectively. In deriving Eq. (2.22), we have made use of the definition and identity.
\begin{align*}
\Re\left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right) &= \frac{\Re\left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right) + \Re\left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right)^*}{2} \\
&= \frac{\Re\left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right) + \Re\left(\text{Tr}\left[\hat{L}_\varphi \hat{\rho} (\varphi) \hat{E} (\epsilon) \right]\right)}{2}, \tag{2.23}
\end{align*}

where in the final line we have taken advantage of the cyclic properties of the trace as well as the Hermiticity of the operators \cite{114}. Considering the inequalities

\begin{align*}
\Re (z)^2 &= |z|^2 - \Im (z)^2 \leq |z|^2, \tag{2.24} \\
|\text{Tr}[\hat{A}^\dagger \hat{B}]|^2 &\leq \text{Tr}[\hat{A}^\dagger \hat{A}] \text{Tr}[\hat{B}^\dagger \hat{B}],
\end{align*}

where the later is once again the Cauchy-Schwartz inequality. Making the designations

\( \hat{A} = \sqrt{\hat{\rho} (\varphi)} \sqrt{\hat{E} (\epsilon)} \) and \( \hat{B} = \sqrt{\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon)} \), we arrive at the chain of inequalities

\begin{align*}
\Re \left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right)^2 &\leq |\text{Tr}[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon)]|^2 \leq \text{Tr}[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon)] \text{Tr}[\hat{E} (\epsilon) \hat{L}_\varphi \hat{\rho} (\varphi) \hat{L}_\varphi] . \tag{2.25}
\end{align*}

The first inequality is saturated provided

\begin{align*}
\Im \left(\text{Tr}\left[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon) \right]\right) &= 0, \quad \forall \epsilon, \tag{2.26}
\end{align*}

while the Cauchy-Schwartz inequality is saturated if and only if

\begin{align*}
\hat{\rho} (\varphi) = \left(1 - \kappa_{\varphi, \epsilon} \hat{L}_\varphi\right) \hat{E} (\epsilon) = 0, \quad \forall \epsilon, \tag{2.27}
\end{align*}

where \( \kappa_{\varphi, \epsilon} = \text{Tr}[\hat{\rho} (\varphi) \hat{E} (\epsilon)] / \text{Tr}[\hat{\rho} (\varphi) \hat{L}_\varphi \hat{E} (\epsilon)] \). Combining Eq. (2.22) and (2.25) we obtain
2.2. Quantum Fisher Information and the Upper Bound

\[
\left( \frac{\text{Tr}[\hat{E}(\epsilon) \partial_\phi \hat{\rho}(\varphi)]}{\text{Tr}[\hat{\rho}(\varphi) \hat{E}(\epsilon)]} \right)^2 \leq \text{Tr}[\hat{E}(\epsilon) \hat{L}_\varphi \hat{\rho}(\varphi) \hat{L}_\varphi] \quad \forall \epsilon. \tag{2.28}
\]

Lastly, using the unity condition \( \sum_\epsilon \hat{E}(\epsilon) = 1 \) in conjunction with Eq. (2.21) and (2.28), we can write

\[
F_Q[\hat{\rho}(\varphi), \{\hat{E}(\epsilon)\}] \leq \sum_\epsilon \text{Tr}[\hat{E}(\epsilon) \hat{L}_\varphi \hat{\rho}(\varphi) \hat{L}_\varphi] = \text{Tr}[\hat{\rho}(\varphi) \hat{L}_\varphi^2], \tag{2.29}
\]

thus concluding the derivation of the QFI. We can identify this result as a maximization over all possible estimators since the right hand side of Eq. (2.29) has no dependence on the POVM. There need only exist one POVM such that the derived inequalities are saturated. Since the qCRB is inversely proportional to the QFI and the QFI itself is a maximization over all possible POVM, it is clear to see how the qCRB serves as an upper bound on phase estimation.

2.2.1 Calculating the QFI for Mixed and Pure States

Our next endeavor is to arrive at a suitable expression for the QFI, using our definition of the SLD given in Eq. (2.18), in terms of the complete basis \( \{|n\}\) , where our density operator is now given generally as \( \hat{\rho}(\varphi) = \sum_n p_n |n\rangle \langle n| \). In this basis the quantum Fisher information can be written as

\[
F_Q[\hat{\rho}(\varphi)] = \text{Tr}[\hat{\rho}(\varphi) \hat{L}_\varphi^2] = \sum_{k, k'} p_k |\langle k | \hat{L}_\varphi | k' \rangle|^2
= \sum_{k, k'} \frac{p_k + p_{k'}}{2} \times |\langle k | \hat{L}_\varphi | k' \rangle|^2. \tag{2.30}
\]

Thus it is sufficient to know the matrix elements of the SLD, \( \langle k | \hat{L}_\varphi | k' \rangle \) in order to calculate the QFI. Using Eq. (2.18) and our general density operator, it is easy to show

\[
\langle k | \hat{L}_\varphi | k' \rangle = \left( \frac{2}{p_k + p_{k'}} \right) \times \langle k | \partial_\phi \hat{\rho}(\varphi) | k' \rangle, \tag{2.31}
\]
which makes Eq. (2.30) [72]

\[
F_Q[\hat{\rho}(\varphi)] = \sum_{k, k'} \frac{2}{p_k + p_{k'}} \times |\langle k | \partial_{\varphi} \hat{\rho}(\varphi) | k' \rangle|^2.
\]  

(2.32)

We proceed further through the use of the definition

\[
\partial_{\varphi} \hat{\rho}(\varphi) = \sum_k (\partial_{\varphi} p_k) |k\rangle \langle k| + \sum_k p_k |\partial_{\varphi} k\rangle \langle k| + \sum_k p_k |k\rangle \langle \partial_{\varphi} k|,
\]

(2.33)

which is a simple application of the chain rule for derivatives. Using the identity \(\partial_{\varphi} \langle k| k' \rangle = \langle \partial_{\varphi} k| k' \rangle + \langle k| \partial_{\varphi} k' \rangle \equiv 0\), the matrix elements in Eq. (2.32) become

\[
\langle k | \partial_{\varphi} \hat{\rho}(\varphi) | k' \rangle = (\partial_{\varphi} \hat{\rho}(\varphi)) \delta_{kk'} + (p_k - p_{k'}) \langle \partial_{\varphi} k| k' \rangle.
\]

(2.34)

The SLD and QFI then become

\[
\hat{L}_\varphi = \sum_k \frac{\partial_{\varphi} p_k}{p_k} \times |k\rangle \langle k| + 2 \sum_{k, k'} \frac{p_k - p_{k'}}{p_k + p_{k'}} \times \langle \partial_{\varphi} k| k' \rangle | k' \rangle \langle k|,
\]

(2.35)

\[
F_Q[\hat{\rho}(\varphi)] = \sum_k \left(\frac{\partial_{\varphi} p_k}{p_k}\right)^2 + 2 \sum_{k, k'} \left(\frac{p_k - p_{k'}}{p_k + p_{k'}}\right)^2 \times |\langle \partial_{\varphi} k| k' \rangle|^2,
\]

respectively. These results, we show next, simplify in the case of pure states where we can write \(\hat{\rho}(\varphi) = |\psi(\varphi)\rangle \langle \psi(\varphi)|\).

For pure states, we can write \(\partial_{\varphi} \hat{\rho}(\varphi) = \partial_{\varphi} \hat{\rho}^2(\varphi) = \hat{\rho}(\varphi) \left[ \partial_{\varphi} \hat{\rho}(\varphi) \right] + \left[ \partial_{\varphi} \hat{\rho}(\varphi) \right] \hat{\rho}(\varphi)\).

Using this, and a cursory glance at Eq. (2.18), it is clear the SLD becomes

\[
\hat{L}_\varphi = 2 \left[ \partial_{\varphi} \hat{\rho}(\varphi) \right] = 2 \left[ \partial_{\varphi} |\psi(\varphi)\rangle \langle \psi(\varphi)| \right]
\]

\[
= 2 |\partial_{\varphi} \psi\rangle \langle \psi| + 2 |\psi\rangle \langle \partial_{\varphi} \psi|,
\]

(2.36)

where in the last step, the \(\varphi\)-dependency of \(|\psi\rangle\) is implicit for notational convenience.

Plugging this directly into the first line of Eq. (2.30) yields
which is the form of the QFI most often used in quantum metrology literature. Next we move on to discussing a specific detection observable (estimator): parity.

2.2.2 Connection To Parity-based Detection

The central theme discussed throughout this paper is the use of the quantum mechanical parity operator as a detection observable in quantum optical interferometry. The use of parity as a detection observable first came about in conjunction with high precision spectroscopy, by Wineland et al. [79] in 1996. It was first adapted for use in quantum optical interferometry by C. C. Gerry [54] in 2000. It was formally introduced and discussed by Gerry et al. [60] in 2010.

A detection observable is said to be optimal for a state if the CRB achieves the qCRB, that is,

\[
(\Delta \Phi)^2 \geq (\Delta \Phi)^2_{\text{CRB}} = (\Delta \Phi)^2_{\text{qCRB}}.
\]

Furthermore, parity detection achieves maximal phase sensitivity at the qCRB for all pure states that are path symmetric [121]. Thus, for the purposes of this paper, it is sufficient to derive the classical Fisher information using Eq. (2.14) [81]. We start with the expression for the classical Fisher information

\[
F(\varphi) = \sum_\epsilon \frac{1}{P(\epsilon|\varphi)} \left( \frac{\partial P(\epsilon|\varphi)}{\partial \varphi} \right)^2,
\]

where \(\epsilon\) represents all the possible outcomes. For parity, \(\epsilon\) can either be positive + or negative −, and satisfies \(P(+|\varphi) + P(-|\varphi) \equiv 1\). The expectation value of the parity operator can then be expressed as a sum over the possible eigenvalues weighed with the probability of that particular outcome, giving us
\[ \langle \Pi \rangle = \sum_i P(i|\varphi)\lambda_i = P(+) - P(-) \]
\[ = 2P(+) - 1 = 1 - 2P(-), \quad (2.40) \]

likewise we can calculate the variance

\[ (\Delta \Pi)^2 = \langle \Pi^2 \rangle - \langle \Pi \rangle^2 = 1 - \langle \Pi \rangle^2 \]
\[ = 1 - (P(+) - P(-))^2 \]
\[ = 1 - (P(+) + P(-))^2 + 4P(+)P(-) \]
\[ = 4P(+)P(-). \quad (2.41) \]

Finally, from Eq. (2.40), it follows that

\[ \frac{\partial P(+)\varphi}{\partial \varphi} = 1 \frac{\partial \langle \Pi \rangle}{\partial \varphi} = -\frac{\partial P(-)\varphi}{\partial \varphi}. \quad (2.42) \]

Combining Eq. (2.41) and (2.42) we find

\[ F(\varphi) = \sum_\epsilon \frac{1}{P(\epsilon|\varphi)} \left( \frac{\partial P(\epsilon|\varphi)}{\partial \varphi} \right)^2 \]
\[ = \frac{1}{(\Delta \Pi)^2} \left| \frac{\partial \langle \Pi \rangle}{\partial \varphi} \right|^2, \quad (2.43) \]

making the CRB / qCRB

\[ \Delta \Phi_{\text{CRB/qCRB}} = \frac{1}{\sqrt{F(\varphi)}} = \frac{1}{\sqrt{1 - \langle \Pi \rangle^2}}. \quad (2.44) \]

Eq. (2.44) is incredibly profound. It tells us that the phase uncertainty obtained via error propagation saturates the qCRB. This result will be utilized throughout the
2.3 Calculating the QFI in Quantum Optical Interferometry

We use the Schwinger realization of the su(2) algebra with two sets of boson operators, discussed in detail in Appendix B, to describe a standard Mach-Zehnder interferometer [14]. In this realization, the quantum mechanical beam splitter can be viewed as a rotation about a given (fictitious) axis, determined by the choice of angular momentum remainder of this paper as we calculate the qCRB in conjunction with the phase uncertainty obtained via error propagation of the parity operator and show the results are in complete agreement. In quite a few of the cases we will discuss, this result is particularly advantageous as the QFI is, in most cases, easier to calculate from a computational standpoint.

It is worth pointing out that the optimal POVM for which the FI is equal to the QFI depends, in general, on $\varphi$. This is somewhat problematic as it requires one to already know the value of the parameter $\varphi$ in order to choose an optimal estimator. Some work has been done to overcome this obstacle [15] which concludes the QFI can be asymptotically obtained in a number of measurements without any knowledge of the parameter.

For all cases considered throughout this paper, we will use parity as our detection observable, which we know saturates the qCRB. We will now move on to discuss how one calculates the QFI in quantum optical interferometry.

2.3 Calculating the QFI in Quantum Optical Interferometry

The standard Mach-Zehnder interferometer where our input state is given as a product of two pure states occupying the $a$- and $b$-modes, respectively; that is $|\text{in}\rangle = |\Psi\rangle_a \otimes |\Lambda\rangle_b$. A phase shift $\varphi$ is induced in the $b$-mode before the state reaches the second beam splitter.

FİGURE 2.2: A standard Mach-Zehnder interferometer where our input state is given as a product of two pure states occupying the $a$- and $b$-modes, respectively; that is $|\text{in}\rangle = |\Psi\rangle_a \otimes |\Lambda\rangle_b$. A phase shift $\varphi$ is induced in the $b$-mode before the state reaches the second beam splitter.
operator, i.e. the choice of a $\hat{J}_1$-operator performs a rotation about the $x$-axis while the choice of a $\hat{J}_2$-operator performs a rotation about the $y$-axis. An induced phase shift, assumed to be in the $b$-mode as detailed in Fig. (2.2), is described by a rotation about the $z$-axis described by the use of the $\hat{J}_3$-operator. The state just before the second beamsplitter is given as

$$|\psi(\varphi)\rangle = e^{-i\varphi \hat{J}_3} e^{-i\frac{\varphi}{2} \hat{J}_1} |\text{in}\rangle,$$  

(2.45)

where we are assuming the beam splitters to be 50:50. This in turn makes the derivative

$$|\psi'(\varphi)\rangle = -ie^{-i\varphi \hat{J}_3} \hat{J}_3 e^{-i\frac{\varphi}{2} \hat{J}_1} |\text{in}\rangle,$$  

(2.46)

leading to

$$\langle \psi'(\varphi) | \psi(\varphi) \rangle = i \langle \text{in} | \hat{J}_2 | \text{in} \rangle,$$  

(2.47)

and

$$\langle \psi'(\varphi) | \psi'(\varphi) \rangle = \langle \text{in} | \hat{J}_2^2 | \text{in} \rangle,$$  

(2.48)

where we have made use of the Baker-Hausdorf identity in simplifying

$$e^{i\frac{\varphi}{2} \hat{J}_1} \hat{J}_3 e^{-i\frac{\varphi}{2} \hat{J}_1} = \hat{J}_2.$$  

(2.49)

Combining Eq. (2.47) and (2.48) into Eq. (2.37) yields for the QFI

$$F_Q[\hat{\rho}(\varphi)] = 4\{\langle \partial_\varphi \psi | \partial_\varphi \psi \rangle - |\langle \partial_\varphi \psi | \psi \rangle|^2\}$$

$$= 4\langle (\Delta \hat{J}_2)^2 \rangle_{\text{in}},$$  

(2.50)

which is simply the variance of the $\hat{J}_2$-operator with respect to the initial input state $|\text{in}\rangle$.

One important thing to notice is that in this case the quantum Fisher information doesn’t
depend on the phase \( \phi \), only the initial state. Next we will demonstrate the fundamental limits on phase uncertainty obtained when using classical or quantum mechanical states of light.

### 2.3.1 Limits on the Phase Uncertainty

Let us first assume an interferometric setup like what is shown in Fig. (2.2) with an input state given by \( |\text{in}\rangle = |\alpha\rangle_a \otimes |0\rangle_b \), where \( |\alpha\rangle \) is a coherent state, the most classical of pure single-mode field states, written as a superposition in number state basis \( \{|n\rangle\} \) as

\[
|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{2.51}
\]

with an average photon number given by \( \bar{n} = |\alpha|^2 \). Assuming our beam splitters are 50:50, that is, \( \theta = \pi/2 \) (see Appendices A and B), and following the convention of Yurke \textit{et. al} [14], then the state after the first beam splitter is

\[
|\text{in}\rangle = |\alpha\rangle_a |0\rangle_b \xrightarrow{\text{BS 1}} \begin{vmatrix} |\alpha\rangle_a \end{vmatrix} \begin{vmatrix} -i\alpha \rangle_b \end{vmatrix}, \tag{2.52}
\]

while the phase shift induced, arbitrarily, in the \( b \)-mode yields

\[
\begin{vmatrix} |\alpha\rangle \end{vmatrix} \begin{vmatrix} -i\alpha \rangle \end{vmatrix} \xrightarrow{\text{Phase } \varphi} \begin{vmatrix} |\alpha\rangle_a \end{vmatrix} \begin{vmatrix} -i\alpha e^{i\varphi} \rangle_b \end{vmatrix}, \tag{2.53}
\]

where it should be noted that the phase \( \varphi \) is defined as \( \varphi = \phi_b - \phi_a \), that is, the difference in path length between the two modes. Finally, the last beam splitter transforms the state to

\[
\begin{vmatrix} |\alpha\rangle \end{vmatrix} \begin{vmatrix} -i\alpha e^{i\varphi} \rangle \end{vmatrix} \xrightarrow{\text{BS 2}} \begin{vmatrix} \frac{\alpha}{\sqrt{2}} (1 + e^{i\varphi}) \rangle_a \end{vmatrix} \begin{vmatrix} \frac{i\alpha}{\sqrt{2}} (1 - e^{i\varphi}) \rangle_b \end{vmatrix} = |\text{out}\rangle. \tag{2.54}
\]

Note that upon setting the phase \( \varphi = 0 \) we end up with our initial input state \( |\text{in}\rangle \), reflecting the unitarity of the transformation. More concisely written, we have performed the transformation
\[ |\text{out}\rangle = e^{\frac{i\pi}{2} \hat{J}_1} e^{-i\varphi \hat{J}_3} e^{-i\frac{\pi}{2} \hat{J}_1} |\text{in}\rangle = e^{-i\varphi \hat{J}_2} |\text{in}\rangle. \]  

(2.55)

The intensities of the two output coherent states are given for the \(a\)- and \(b\)-modes, respectively, as

\[ I_a = \langle \hat{a}^\dagger \hat{a} \rangle = \left| \frac{\alpha}{2} (1 + e^{i\varphi}) \right|^2 = \frac{|\alpha|^2}{2} (1 + \cos \varphi), \]

(2.56)

\[ I_b = \langle \hat{b}^\dagger \hat{b} \rangle = \left| \frac{i\alpha}{2} (1 - e^{i\varphi}) \right|^2 = \frac{|\alpha|^2}{2} (1 - \cos \varphi). \]

We define the difference in intensities as

\[ \langle \hat{\Omega} \rangle = \langle \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \rangle = I_a - I_b = |\alpha|^2 \cos \varphi, \]

(2.57)

where the operator \(\hat{\Omega} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\). Finding the expectation value of the square of this operator is tedious but straightforward, and yields

\[ \langle \hat{\Omega}^2 \rangle = \langle (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 \rangle = |\alpha|^2 (1 - |\alpha|^2 \cos^2 \varphi). \]

(2.58)

Defining the uncertainty as \(\Delta \hat{\Omega} = \sqrt{\langle \hat{\Omega}^2 \rangle - \langle \hat{\Omega} \rangle^2}\), we calculate the phase uncertainty through error propagation as
2.3. Calculating the QFI in Quantum Optical Interferometry

\[ P(n_1, n_2 | N = 5) \]

**Figure 2.3:** Joint photon probability for the 'N00N' state with \( N = 5 \). The uncertainty in photon number is equal to the total number of photons itself.

\[
\Delta \varphi = \frac{\Delta \hat{\Omega}}{|\partial_{\varphi} \langle \Omega \rangle|} = \frac{\sqrt{1 - \bar{n} \cos^2 \varphi}}{\sqrt{\bar{n}} |\sin \varphi|} \approx \frac{1}{\sqrt{\bar{n}}} \quad \varphi \to \pi/2.
\]

(2.59)

This result places a limit on the phase uncertainty obtainable via classical-like light and is known as the Standard Quantum (or Shot Noise) Limit (SQL). Note, however, when our detection observable is simply taking the difference in intensities, the measurement is not optimized at the value \( \varphi = 0 \); in fact, the phase uncertainty is infinite for this value of the phase. We formally define the SQL as

\[
\Delta \varphi_{\text{SQL}} \equiv \frac{1}{\sqrt{\bar{n}}}. \quad (2.60)
\]

It is worth pointing out here the advantage of using parity as a detection observable.
We can define the expectation value of the parity operator for an arbitrary coherent state \(|\lambda\rangle\) as

\[
\langle \hat{\Pi} \rangle = \langle \lambda | \hat{\Pi} | \lambda \rangle \\
= \langle \lambda | \left\{ e^{-\frac{1}{2} |\lambda|^2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \right\} \\
= \langle \lambda | \left\{ e^{-\frac{1}{2} |\lambda|^2} \sum_{n=0}^{\infty} \left( \lambda e^{i\pi} \right)^n |n\rangle \right\} \\
= \langle \lambda | - \lambda \rangle \\
= e^{-2|\lambda|^2}.
\] (2.61)

Applying this result to the \(a\)- and \(b\)-modes of output state in Eq. (2.54) yields

\[
\langle \hat{\Pi} \rangle_a = e^{-|\alpha|^2 (1 + \cos \varphi)},
\]

\[
\langle \hat{\Pi} \rangle_b = e^{-|\alpha|^2 (1 - \cos \varphi)},
\]

which, when used in conjunction with the usual error propagation yields, a SQL optimized for small phase shifts when detection is performed in the output \(b\)-mode, that is

\[
\Delta \varphi_{\hat{\Pi}_b} = \frac{1}{\sqrt{n}} \left\| \right|_{\varphi \to 0},
\]

(2.63)

where we have used L'Hopital’s rule in deriving Eq. (2.63) for the limiting case of \(\varphi \to 0\).

What about a possible bound on the phase uncertainty obtained when using non-classical (quantum) states of light? Consider the so-called N00N states given by

\[
|\text{N00N}\rangle = \frac{1}{\sqrt{2}} \left[ |N\rangle_a |0\rangle_b + e^{i\Phi_N} |0\rangle_a |N\rangle_b \right],
\]

(2.64)
where one may find $N$ photons in the $a$-mode while finding no photons in the $b$-mode or no photons in the $a$-mode and $N$ photons in the $b$-mode. All $N$ photons are found in either one mode or the other. The Joint photon number distribution for the N00N state is shown in Fig. (2.3). This state is maximally entangled between the two modes, violating the so-called Bell Inequality proving the existence of non-local quantum correlations [28] [43]; as such, this state is taken to be the most non-classical state of light. The heuristic phase-number relation is given as $\Delta \varphi \Delta \bar{N} = 1$. For the N00N state described in Eq. (2.64), the uncertainty in the photon number is equal to the total average photon number $\bar{N}$ itself, since all photons can be found in either one mode or the other while the opposite mode will have zero photons. Thus the heuristic relation becomes $\Delta \varphi \bar{N} = 1$. From this we can demonstrate what is known as the Heisenberg limit (HL) on phase uncertainty, given as

$$\Delta \varphi_{HL} = \frac{1}{\bar{n}}. \quad (2.65)$$

thus we have our upper and lower bound on phase uncertainty given by the SQL and HL, respectively

$$\Delta \varphi_{SQL} = \frac{1}{\sqrt{\bar{n}}}, \quad (2.66)$$

$$\Delta \varphi_{HL} = \frac{1}{\bar{n}}.$$

It is worth pointing out that the Heisenberg limit is an improvement of the SQL by a factor of the SQL itself, that is

$$\Delta \varphi_{HL} = \Delta \varphi_{SQL}^2. \quad (2.67)$$

We now move on to discussing some of the work we have done in the field of quantum optical interferometry and quantum state engineering. We begin by discussing one of our more recent works wherein we consider a coherently stimulated two mode
squeezed vacuum state as the input state of the interferometer. Before discussing the state’s use in phase estimation, we discuss the photon statistics of the state itself as well as providing the reader a firm foundation of understanding upon which our work has been built.
Chapter 3

Coherently Stimulated Parametric Down Conversion

For many years now, parametric down-conversion has been a laboratory source of light with strong nonclassical properties [45]. The generated states of light have been used to study a variety of quantum effects and have had applications for fundamental tests of quantum mechanics as in two-photon interference at a beam splitter [29] and to Bell-type inequalities [112], as well as practical applications such as to quantum metrology [113], quantum information processing [40], and quantum imaging [89]. In almost all cases studied so far in the laboratory, the light produced is the result of spontaneous down conversion. That is, a strong classical (UV) pump field drives a nonlinear crystal producing pairs of frequency down-converted (infrared) photons into the signal and idler modes initially in vacuum states. The state produced is the two-mode squeezed vacuum state (TMSVS) [57], which consists of a superposition of products of identical (twin) Fock states of the signal and idler modes, where the photon-number distributions for the reduced density operators for each of the modes is thermal [16]. For low gain, spontaneous down-conversion produces mostly vacuum states in the output signal and idler modes with about 1 in $10^{12}$ pump photons converting to a signal-idler pair of photons. This process was employed in the famous Hong-Ou-Mandel experiement [29], for example.

On the other hand, if the signal and idler modes into the down-converter are initially fed beams of coherent light, the light produced is the two-mode squeezed coherent state
(TMSCS) and the process producing it is called coherently stimulated down-conversion, or sometimes \textit{seeded} parametric down-conversion (CSPDC). The statistical properties of these states were discussed in the literature some years ago by Caes \textit{et al.} \cite{30} and by Selvadoray \textit{et al.} \cite{100}. It is the latter authors who have performed the most complete analysis of the states by considering complex displacement and squeezing parameters. Recently, this light source has been suggested for applications to quantum interferometric photolithography \cite{5} and to quantum optical interferometry \cite{113}.

In this chapter we first reexamine the TMSCS. The effects of the phases of the two input coherent states and of the classical pump field, individually and in combination, are studied as a means of controlling the properties of the output fields. Our motivation comes from the possible applications of such states to photon-number parity-based quantum optical interferometry. Previously, Kolkiran and Agarwal \cite{88} studied quantum optical interferometry using high-gain coherently stimulated down-conversion. In that work, however, they did not study the statistical properties of the states before and after beam splitting, nor did they study the use of photon number parity measurements for interferometry or the related issue of the Cramér-Rao bound based on the quantum Fisher information for optimal sensitivity.

In the literature \cite{30} \cite{100}, the TMSCS are mathematically defined in two ways having to do with the ordering of the two-mode squeeze operator and the displacement operators acting on the double vacuum state. The states generated are mathematically equivalent but differ in their implied methods of physical generation. From an experimental point of view, the natural way to think about the states is to assume coherent light beams are fed into the input signal and idler modes of the down-converter, which then acts to squeeze those input states—hence the states are the result of coherently stimulated down-conversion. As the coherent states may be defined as displace vacuum states, it follows that the TMSCS is mathematically defined by the action of the displacement operators on the vacuum states of each mode followed by the action of the two-mode squeeze operator. However, in the literature, specifically the papers of
Chapter 3. Coherently Stimulated Parametric Down Conversion

Caves et al. [30] and Selvadoray et al. [100] cited above, one finds a definition of the TMSCS with the operators acting in reverse order, i.e., with the two-mode squeeze operator acting on the double vacuum followed by the displacement operator such that the states generated could be called two-mode displaced squeezed vacuum states (TMDSVS). Of course, the definitions are mathematically equivalent with properly chosen displacement parameters, but physically the latter states are generated by performing independent displacements on the two modes of the two-mode squeezed vacuum. This does not appear to be an attractive method for generating the states in the laboratory in view of the fact that displaced vacuum states (coherent states) are readily available from well phase-stabilized lasers. But as will be discussed below, the two definitions can lead to misconceptions, or at least confusion, about the roles that the various phases (pump and coherent state) play in controlling the statistical properties of the states and on how the beams transform upon being incident on a beam splitter. Specifically, in the case of the TMDSVS, the phases are in some sense ‘hidden’. For the purposes of interferometry, it is desirable that the beam splitter create a balanced, well separated, bimodal joint photon-number distribution, and we show that such is possible by judicious choices of the relevant phases and a certain combination of those phases.

This chapter is organized as follows: First, we briefly review the two-mode squeezed states and their production by spontaneous down-conversion. Next we discuss coherently stimulated down-conversion and the statistical properties of the TMSCS produced. We then go on to discuss the multiphoton interference and the consequent transformation of our state by a 50:50 beam splitter and the control of the outcome of the process by the choices of the various phases. We examine, for certain choices of parameters, the efficacy of the states for performing substandard-quantum-limited quantum optical interferometry through the use a photon-number parity measurements. It is known that minimum phase uncertainty for a given input state is given by the corresponding Cramer-Rao bound [67], which in turn is determined by the quantum Fisher information [17]. We have also proven that photon-number parity measurements saturate this bound. Lastly, we discuss the fully quantum mechanical model where we assume a
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quantized pump field. We analyze the short time evolution of the state statistics using a time dependent perturbative method.

3.1 CSPDC and the Two Mode Squeezed Coherent States

3.1.1 The Two-mode Squeezed Vacuum State

We begin by writing down the two-mode squeeze operator

$$\hat{S}(z) = e^{z^* \hat{a}^\dagger - z \hat{a} \hat{b}^\dagger}, \quad z = re^{i\phi},$$

(3.1)

where $r$ is the so-called squeezing parameter, $0 \leq r < \infty$, and where $2\phi$ is the phase of the pump field, treated classically here, driving the down-conversion process. Note, that we have parametrized this phase as $2\phi$ to be consistent with the convention used in the literature. Here $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{b}, \hat{b}^\dagger)$ are the Bose operators representing the signal and idler modes, respectively. For an arbitrary two-mode input state $|\psi_{in}\rangle$, the output state will be given by $|\psi_{out}\rangle = \hat{S}(z) |\psi_{in}\rangle$. The average total photon number of the output state will be

$$\bar{n}_{total} = \langle \psi_{in}| \hat{S}^\dagger(z) \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}\right) \hat{S}(z) |\psi_{in}\rangle$$

$$= \langle \psi_{in}| \left(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}\right) \cosh(2r) - \left(e^{2i\phi} \hat{a}^\dagger \hat{b}^\dagger + e^{-2i\phi} \hat{a} \hat{b}\right) \sinh(2r) + 2 \sinh^2(r) |\psi_{in}\rangle$$

(3.2)

where we have used the operator relations

$$\hat{S}^\dagger(z) \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \hat{S}(z) = \begin{pmatrix} \hat{a} \cosh r - e^{2i\phi} \hat{b}^\dagger \sinh r \\ \hat{b} \cosh r - e^{2i\phi} \hat{a}^\dagger \sinh r \end{pmatrix}. $$

(3.3)

Of course, if the input state is just the pair vacuum state, $|\phi_{in}\rangle = |0\rangle_a \otimes |0\rangle_b$, the output will be the two-mode squeezed vacuum state (TMSVS),
3.1. CSPDC and the Two Mode Squeezed Coherent States

\[ |\psi\rangle_{out} = |\xi\rangle \]

\[ = (1 - |\xi|^2)^{-1/2} \sum_{n=0}^{\infty} \xi^n |n\rangle_a |n\rangle_b \]

\[ = \frac{1}{\cosh (2r)} \sum_{n=0}^{\infty} (-1)^n e^{2i\phi_t} \tanh^n (r) |n\rangle_a |n\rangle_b \] (3.4)

where \( \xi = -e^{2i\phi} \tanh r \), for which the average total photon number is given by

\[ \bar{n}_{total} = 2 \sinh^2 r, \] (3.5)

which, it should be noted, is independent of the pump phase \( 2\phi \). The photon states of each mode are tightly correlated and the state as a whole is highly nonclassical due to the presence of squeezing in one or the other of the superposition quadrature operators of the combined modes. On the other hand, the photon-number statistics are super-Poissonian in each mode. The joint photon-number probability distribution for there being \( n_1 \) photons in mode \( a \) and \( n_2 \) photons in mode \( b \) is

\[ P(n_1, n_2) = |\langle n_1, n_2 |\xi\rangle|^2 = \frac{\tanh^{2n} r}{\cosh^2 r} \times \delta_{n_1,n} \delta_{n_1,n_2}, \] (3.6)

such that only the "diagonal" elements \( n_1 = n = n_2 \) are nonzero. In fact, each mode separately has thermal-like statistics.

The two-mode squeezing operation is realized by the evolution operator given in the interaction picture as

\[ \hat{U}_I(t) = e^{-\frac{i\hat{H}_I t}{\hbar}}, \] (3.7)

where the interaction Hamiltonian under the parametric approximation is given by [120]

\[ \hat{H}_I = i\hbar \left( \gamma \hat{a} \hat{b} - \gamma^* \hat{a}^\dagger \hat{b}^\dagger \right). \] (3.8)
The parameter $\gamma$ is proportional to the second-order nonlinear susceptibility $\chi^{(2)}$ and to the amplitude and phase factor of the driving laser field, here assumed to be a strong classical field such that depletion and fluctuations in the field can be ignored as per the parametric approximation. Writing $\gamma = |\gamma|e^{2i\phi}$, the squeeze operator becomes:

$$\hat{S}(z) = e^{-\frac{i H_I t}{\hbar}} = e^{r(\hat{a}\hat{b}e^{-2i\phi} - \hat{a}^\dagger\hat{b}^\dagger e^{2i\phi})},$$

(3.9)

where the squeeze parameter $r = |\gamma|t$ can be taken as a scaled dimensionless time.

We now turn to a discussion of the TMSCS, which we take to be the output state when the initial state consists of a product of coherent states, i.e., $|\psi\rangle_{in} = |\alpha_1\rangle_a \otimes |\alpha_2\rangle_b$ so that the output becomes $|\psi\rangle_{out} = |z; \alpha_1, \alpha_2\rangle$, where

$$|z; \alpha_1, \alpha_2\rangle = \hat{S}(z)|\alpha_1\rangle_a |\alpha_2\rangle_b = \hat{S}(z) \hat{D}(\alpha_1, \alpha_2)|0\rangle_a |0\rangle_b$$

(3.10)

and where $\hat{D}(\alpha_1, \alpha_2) = \hat{D}_a(\alpha_1) \otimes \hat{D}_b(\alpha_2)$ is the product of the displacement operators of each of the modes:

$$\hat{D}_a(\alpha_1) = \exp \left(\alpha_1 \hat{a}^\dagger - \alpha_1^* \hat{a}\right),$$

$$\hat{D}_b(\alpha_2) = \exp \left(\alpha_2 \hat{b}^\dagger - \alpha_2^* \hat{b}\right).$$

(3.11)

The coherent states are generated from the vacuum by the actions of the displacement operators such that $|\alpha_1\rangle_a |\alpha_2\rangle_b = \hat{D}(\alpha_1, \alpha_2)|0\rangle_a |0\rangle_b$ where

$$|\alpha\rangle = \hat{D}(\alpha) = e^{-\frac{1}{2} |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$  

(3.12)

As we have previously pointed out, these states are readily available from well phase-stabilized lasers, thus making the states described in Eq.(3.10) a more suitable choice of states to work with. We will now draw our attention to the generation of this state and the role played by the individual and joint phases.
3.1. CSPDC and the Two Mode Squeezed Coherent States

3.1.2 Methods of Generation — the “Hidden Phase”

The TMSCS given by Eq. (3.10) are generated by feeding the coherent states $|\alpha_1\rangle_a |\alpha_2\rangle_b$ into the input signal and idler channels, respectively, of the down-converter as illustrated by Fig. (3.1). This process has been coined as coherently stimulated down-conversion. The average total photon number for the state of Eq. (3.10) is given by Eq. (3.2) where the initial states are now taken to be coherent states, is

$$\bar{n}_{\text{total}} = \left( |\alpha_1|^2 + |\alpha_2|^2 \right) \cosh(2r) - \left( e^{2i\phi} \alpha_1^* \alpha_2 + e^{-2i\phi} \alpha_1 \alpha_2 \right) \times \cosh(2r) + 2 \sinh^2(r),$$

where we have used the results of Eq.(3.2) and the unitary transformation of the boson operators

$$\hat{D}^\dagger (\lambda) \hat{a} \hat{D} (\lambda) = \hat{a} + \lambda,$$

$$\hat{D}^\dagger (\lambda) \hat{a} \hat{D} (\lambda) = \hat{a} + \lambda.$$

If we now set $\alpha_1 = |\alpha_1|e^{i\theta_1}$ and $\alpha_2 = |\alpha_2|e^{i\theta_2}$, we have
\[ n = \left( |\alpha_1|^2 + |\alpha_2|^2 \right) \cosh(2r) - 2|\alpha_1||\alpha_2| \cos(\Phi) \sinh(2r) + 2 \sinh^2(r), \]  

(3.15)

where \( \Phi = \theta_1 + \theta_2 - 2\phi \). Evidently, the average photon number for the TMSCS depends on the combination of the phases \( \theta_1, \theta_2, \) and \( 2\phi \) in \( \Phi \). This result is not new [45], but as far as we are aware, the effects of the phases on the average photon number in coherently stimulated parametric down-conversion (CSPDC), as given in Eq. (3.15), has yet to be demonstrated experimentally. The joint photon-number distribution also depends only on the value of \( \Phi \). However, as we demonstrate below, the joint photon-number distribution obtained after the two beams are mixed at a 50:50 beam splitter depends on the individual values of the phases for some particular choices of \( \Phi \), not just on the combination \( \Phi \) itself.

In the literature, one often finds the TMSCS defined according to the reverse ordering of the squeeze and displacement operators operating on the vacuum that was used above. That is, one encounters the definition

\[ |\beta_1, \beta_2; z \rangle \equiv \hat{D}(\beta_1, \beta_2) \hat{S}(z) |0\rangle_a |0\rangle_b \]  

(3.16)

where \( \beta_1 = |\beta_1|e^{i\psi_1} \) and \( \beta_2 = |\beta_2|e^{i\psi_2} \) are, for the moment, arbitrary "coherent" amplitudes with phases \( \psi_1 \) and \( \psi_2 \), respectively. The average total photon number obtained for this representation is given by

\[ \bar{n}_{total} = \langle \beta_1, \beta_2; z \rangle \left( \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \right) |\beta_1, \beta_2; z \rangle \\
a = \langle 0|_b \langle 0|_b \hat{S}^\dagger(z) \hat{D}^\dagger(\beta_1, \beta_2) \left( \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \right) \times \\
\times \hat{S}(z) \hat{D}(\beta_1, \beta_2) |0\rangle_a |0\rangle_b \\
= |\beta_1|^2 + |\beta_2|^2 + 2 \sinh^2(r), \]  

(3.17)

where we have used the results of Eqs. (3.14) and (3.3) in that order. The total photon
number in this case displays no dependence on the phases $\psi_1$, $\psi_2$ and $2\phi$. However, the two representations of the TMSCS are equivalent provided

$$\hat{S}(z) \hat{D}(\alpha_1, \alpha_2) \hat{S}^\dagger(z) = \hat{D}(\beta_1, \beta_2),$$  \hspace{1cm} (3.18)$$

which holds with the choice of displacement amplitudes

$$\beta_1 = \mu \alpha_1 - \nu \alpha_2^*,$$

$$\beta_2 = \mu \alpha_2 - \nu \alpha_1^*,$$  

(3.19)

where $\mu = \cosh r$ and $\nu = e^{2i\phi} \sinh r$. The inverse transformations, needed for later use, are given by

$$\alpha_1 = \mu \beta_1 + \nu \beta_2^*,$$

$$\alpha_2 = \mu \beta_2 + \nu \beta_1^*.$$  

(3.20)

Thus under these conditions $|z; \alpha_1, \alpha_2\rangle$ and $|\beta_1, \beta_2; z\rangle$ are identical states but represent different methods of generation. As mentioned, the former states result from the action of the down-converter on input coherent states while the latter are displaced TMSVS, i.e., they require displacement operations on both modes of a TMSVS.

As just discussed, our result for the average photon number calculated for representation of the state as given by $|\beta_1, \beta_2; z\rangle$ is independent of the phases $\psi_1$, $\psi_2$ and $2\phi$. That is, there is no explicit phase dependence here. However, because of the transformations of Eqs. (3.19) and (3.20), there is an implicit dependence on the phases $\theta_1$, $\theta_2$ and $2\phi$ which show up in the combination $\Phi = \theta_1 + \theta_2 - 2\phi$ in Eq. (3.15). In this sense, the phase dependence of Eq. (3.15) is ‘hidden’. Caves et al. [30] and Selvadoray et al. [100] use the definition of Eq. (3.16) for the TMSCS, though the latter authors, for calculational convenience, also use the definition given by Eq. (3.10). Our results in Eq. (3.17) agree with that of Selvadoray et al. [100], who point out that $\bar{n}$ is insensitive to a certain combination of the phases, that here we shall call $\Psi$, which in our notation has the form $\Psi = \psi_1 + \psi_2 - 2\phi$. 
Chapter 3. Coherently Stimulated Parametric Down Conversion

It is straightforward to derive the relationship between the angles $\Psi$ and $\Phi$ as well as the relationships between the set of angles $(\theta_1, \theta_2)$ and $(\psi_1, \psi_2)$. Consider the quantity $\alpha_1 \alpha_2 e^{-2i\phi}$; by substituting for $\alpha_1$ and $\alpha_2$ from Eq. (3.20) we arrive at the relation

$$\alpha_1 \alpha_2 e^{-2i\phi} = |\alpha_1||\alpha_2| e^{i\Phi} \quad (3.21)$$

equating the real and imaginary parts of Eq. (3.21), we find that

$$\Phi = \tan^{-1} \left[ \frac{2|\beta_1||\beta_2| \sin \Psi}{2|\beta_1||\beta_2| \cosh (2r) \cos \Psi + (|\beta_1|^2 + |\beta_2|^2) \sinh (2r)} \right] \quad (3.22)$$

Conversely, by considering the combination $\beta_1 \beta_2 e^{-2i\phi}$ and substituting from Eq. (3.19) we have

$$\beta_1 \beta_2 e^{-2i\phi} = |\beta_1||\beta_2| e^{i\Psi} \quad (3.23)$$

from which it follows

$$\Psi = \tan^{-1} \left[ \frac{2|\alpha_1||\alpha_2| \sin \Phi}{2|\alpha_1||\alpha_2| \cosh (2r) \cos \Phi + (|\alpha_1|^2 + |\alpha_2|^2) \sinh (2r)} \right] \quad (3.24)$$

Note that the phases $\Phi$ and $\Psi$ as related through the equations above are, in general, nonlinear functions of each other. Selvadoray et al. [100] have identified $\Phi$ as the Gouy phase [63] for the TMSCS.

3.1.3 Photon Statistics

The essential point here is that the phases of the pump field and of the input coherent states, through $\Phi$, can be adjusted so as to exert control over the average photon number of the output field of the down-converter and of the statistics of this field, as will
be discussed below. The dependence on the phases is hidden in the expression for the average photon number as given in Eq. (3.15), though it is carried along through Eq. (3.19). For given values of $|\alpha_1|, |\alpha_2|$ and $r$, the average photon number can vary significantly by adjusting $\Phi$, as we show in Fig. (3.2) for the choices $|\alpha_1| = |\alpha_2| = |\alpha|$ and for $r = 4$. Note that for $\Phi = 0$ the average photon number is essentially independent of the coherent state amplitude $|\alpha|$. It is easy to see why: for the choices of $|\alpha_1| = |\alpha_2| = |\alpha|$ we can rewrite Eq. (3.15) as

$$\bar{n} = 2|\alpha|^2 [\cosh (2r) - \cos (\Phi) \sinh (2r)] + 2 \sinh^2 r,$$

(3.25)

and for $\Phi = 0$ the bracketed term $\cosh (2r) - \sinh (2r) \to 0$ for sufficiently large $r$. We are then left with the dominant contribution $\bar{n} = 2 \sinh^2 r$, which is identical to the average photon number for the squeezed vacuum state. Obviously we can maximize $\bar{n}$ for the choice of $\Phi = \pi$. As we show below, these different choices of $\Phi$ dramatically affect the nature of the photon-number distributions both before and after beam splitting. We also point out that for a fixed value of $\Phi$, difference arrangements and values of the corresponding phase angles $\theta_1, \theta_1$, and $2\phi$ affect the joint photon-number distribution after beam splitting, but not before. We note that Caves et al. [30], who examined the
TMSCS as defined through Eq. (3.16) set $\Phi = 0$ stating this can be done “without loss of generality.” This is misleading as should be clear from the above discussion.

We now proceed to write down the quantum amplitudes and photon distributions associated with the states $|z; \alpha_1, \alpha_2\rangle$, given in terms of the number states as,

$$|z; \alpha_1, \alpha_2\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c(n_1, n_2) |n_1\rangle_a |n_2\rangle_b,$$

(3.26)

At this point it is useful to convert our two-mode number state labeling to angular momentum states $|j, m\rangle$ such that we have the mapping [14]

$$|n_1\rangle |n_2\rangle_b = |j, m\rangle \text{ where } j = \frac{n_1 + n_2}{2} \text{ and } m = \frac{n_1 - n_2}{2}.$$  

(3.27)

Rewriting our two-mode squeezed coherent states in terms of the angular momentum states as

$$|z; \alpha_1, \alpha_2\rangle = \sum_{j=0,1,2...}^{\infty} \sum_{m=-j}^{j} c(j + m, j - m) |j, m\rangle$$

(3.28)

where, adapting and correction a result obtained by Selvadoray et al. [100], the coefficients are given by

$$c(j + m, j - m) = e^{-i\pi(j - |m|)} \left[ \frac{(j - |m|)!}{(j + |m|)!} \right]^{\frac{1}{2}} \left[ \frac{\alpha_1 \alpha_2}{\mu \nu} \right]^{\frac{|m|}{2}} \frac{1}{\mu} \frac{\nu}{\mu} \times$$

$$\times L_{j-|m|}^{2|m|} \left( \frac{\alpha_1 \alpha_2}{\mu \nu} \right) \left( \frac{\alpha_1}{\alpha_2} \right)^{m} e^{-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)} e^{-\frac{\nu^* \alpha_1 \alpha_2}{\mu}},$$

(3.29)

and where it is understood that $n_1 = j + m$ and $n_2 = j - m$ and where, again, $\mu = \cosh r$ and $\nu = e^{2i\phi} \sinh r$. The functions $L_n^k(x)$ are the associated Laguerre polynomials. In terms of the phases $\theta_1, \theta_2$, and $\phi$, the coefficients of Eq. (3.29) can be written as
3.1. CSPDC and the Two Mode Squeezed Coherent States

\[ P(n_1, n_2) \]

**Figure 3.3**: Joint photon number distribution \( P(n_1, n_2) \) versus \( n_1 \) and \( n_2 \) for the two-mode squeezed coherent states with \( r = 1.2 \) and \( \alpha_1 = \alpha_2 = 1 \) for (a) \( \Phi = 0 \), (b) \( \Phi = \pi/2 \) and (c) \( \Phi = \pi \).
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\[ c(j + m, j - m) = e^{-i\pi(j - |m|)} \left[ \frac{(j - |m|)!}{(j + |m|)!} \right] \frac{1}{2} \left[ \frac{2|\alpha_1| |\alpha_2|}{\sinh(2r)} \right]^{\frac{|m|}{2}} \tanh^2 r \times \]
\[ \times e^{i(m|\Phi + 2\phi)} L^{2|m|}_{j-|m|} \left[ \frac{2|\alpha_1| |\alpha_2|}{\sinh(2r)} \right] e^{i\phi} \left( \frac{\alpha_1}{|\alpha_2|} \right)^m e^{im(\theta_1 - \theta_2)} \times \]
\[ \times e^{-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)} e^{|\alpha_1||\alpha_2|e^{i\Phi} \tanh r}. \]  

(3.30)

There are two things to note regarding the dependence of these amplitudes on the various phases. The first is the appearance of the combination \( \Phi = \theta_1 + \theta_2 - 2\phi \). As noted above, the average total photon number for the two beams in this representation depends only on \( \Phi \). This is a reflection of the fact that the joint photon-number statistics depend only on \( \Phi \), as the probability of finding \( n_1 \) photons in mode \( a \) and \( n_2 \) photons in mode \( b \) is given by

\[ P(n_1, n_2) = |c(n_1, n_2)|^2, \]  

(3.31)

as is clear from an examination of the coefficients given by Eq. (3.30).

We now consider the joint photon-number probability distributions for the TMSCS for various values of state parameters. In Fig. (3.3) we plot \( P(n_1, n_2) \) versus \( n_1 \) and \( n_2 \) for the fixed values \( r = 1.2 \) and \( \alpha_1 = \alpha_2 = 1 \) for the choices \( \Phi = 0, \frac{\pi}{2}, \pi \). The distribution is populated about the line \( n_1 = n_2 \) (as is true for the TMSVS), and the effect of the phase \( \Phi \) on the distribution is clear as the peak of the distribution migrates along the aforementioned line in accordance with the change in the total average photon number as the phase angle changes.

We have seen that for fixed values of the squeezing parameter and coherent state amplitudes the average photon number and the shapes of the joint photon-number distribution change with the phase \( \Phi \). This suggests that other quantities change with the phase as well. We consider here only the effect of the phase on the degree of entanglement between the two modes. We use the linear entropy

\[ S = 1 - \text{Tr}_{a(b)} \rho_{a(b)}^2, \]  

(3.32)
where $\hat{\rho}_{a(b)} = \text{Tr}_{b(a)}\hat{\rho}$ is the reduced density operator for the $a$ ($b$) mode. Entanglement becomes maximum for $S = 1$. In Fig. (3.4) we show the linear entropy as a function of the phase $\Phi$ for the fixed values of $r = 1.7$ and $|\alpha_1| = |\alpha_2| = 2$. We see that the entanglement is high for all values of the phase $\Phi$, but it is at the maximum value $S = 1$ for $\Phi = \pi$, the same phase that maximizes the average photon number for a given squeeze parameter and coherent state amplitudes.

As pointed out above, only the choice of the phase angle $\Phi$ affects the distribution; the choices of $\theta_1$, $\theta_2$, and $\phi$ individually do not affect the distribution. They also do not affect the linear entropy. However, as we show in the next section, the individual phases can affect the outcome of mixing the two beams at a beam splitter.

To conclude this section we wish to draw the readers attention to Fig. 2 of the paper by Selvadoray et al. [100] There they plot photon-number distributions for (in our notation) $\beta_1 = \beta_2 = 7$ and $r = 4$, for various values of the phase combination we call $\Psi$. The average photon number of their state is $\bar{n} = 1587.4$, yet they display their distributions only out as far as $n_1 = n_2 = 100$, thus apparently not including most of the distribution, especially near the average $n_1 \approx \bar{n}/2 \approx n_2$. However, the full distribution in such cases
is very broad and also very flat so that the oscillations close to the origin observed are their most interesting features. It turns out that for $\Phi = 0$ the corresponding values of $\alpha_1$ and $\alpha_2$ are both $\sim 380$. On the other hand, for $\Phi = \pi$ the corresponding values of $\alpha_1$ and $\alpha_2$ are $\sim 0.127$, indicating that from the point of view given by the state definition Eq. (3.10), the corresponding state is very close to the two-mode squeezed vacuum, and this explains why it is concentrated along the diagonal precisely as shown in Fig. (2a) of Selvadoray et al. [100] for this choice of the phase $\Phi$. The point here is that the relevant state parameters $(\alpha_1, \alpha_2)$ for a given squeeze parameter $r$ can have remarkably different values than does the set $(\beta_1, \beta_2)$ in representing the same state.

### 3.2 Beam Splitting and Multi-Photon Quantum Interference

We now consider the result of mixing the two output beams of the coherently stimulated down-converter at a 50:50 beam splitter. Of course, to maintain coherences and correlations, the output beams must propagate on equidistant paths to the beam splitter. Equal path lengths can be calibrated experimentally using the Hong-Ou-Mandel effect [29].

For convenience we assume that our beam splitter is balanced (50:50) and thus performs a transformation that can be described as a $\pi/2$ rotation about the "1" axis as given by $\hat{U}_{BS} = \exp \left(-i \pi \hat{J}_1/2 \right)$, where we have followed Yurke et al. [120], who use the Schwinger realization of the angular momentum algebra in terms of a pair of boson operators to describe beam splitters and some other linear optical devices. Our state after beam splitting, in the angular momentum representation, is thus given by

$$\left| \text{out, BS} \right> = \hat{U}_{BS} \left| z; \alpha_1, \alpha_2 \right>$$

$$= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} c (j + m, j - m) i^{m'-m} a_{m',m}^j \left( \frac{\pi}{2} \right) \left| j, m' \right>.$$  (3.33)

The probability after beam splitting that there are $N_1$ photons in mode $a$ and $N_2$ photons in mode $b$ is given by
3.2. Beam Splitting and Multi-Photon Quantum Interference

\[ P(N_1, N_2) = |\langle J, M | \text{out, BS1} \rangle|^2 \]
\[ = \left| \sum_{m=-J}^{J} c(J+m, J-m) i^{M-m} d^J_{M,m} \left( \frac{\pi}{2} \right) \right|^2, \quad (3.34) \]

for \( N_1 = J + M \) and \( N_2 = J - M \), where \( d^J_{m',m}(\beta) \) are the usual Wigner-\( d \) rotation functions, detailed in Appendix C, given as [14] [120]

\[ d^J_{m',m}(\beta) = \langle j, m' | e^{-i\gamma J^2} | j, m \rangle. \quad (3.35) \]

We now consider the effects of mixing the output beams of the down-converter at a beam splitter in the manner as sketched in Fig. (3.5). Again, we first consider the limiting case where the input coherent state amplitudes vanish so that we are dealing only with the TMSVS of Eq. (3.4) as the output. The output after beam splitting is

\[ |\text{out, BS1}\rangle = (1 - |\xi|^2)^{1/2} \sum_{n=0}^{\infty} \left( \frac{i\xi}{2} \right)^n \sum_{k=0}^{n} \left[ \binom{n}{k} \left( \binom{2n}{n-k} \right)^{1/2} |2k\rangle_a |2n-2k\rangle_b, \quad (3.36) \]

where we have used a result from Campos et al. [120] for the action of beam splitting on input twin-Fock states \(|n\rangle_a |n\rangle_b\) with a suitable modification for our choice of representation for the first beam splitter. It has been shown [24] that the output state of Eq. (3.36)
is, in fact, not an entangled state and that the beam splitting transformation causes a factorization of the input TMSVS into a product of single-mode squeezed vacuum states, neither of which has an odd photon-number state populated.

Recall that the TMSVS is a superposition of twin-Fock states $|n\rangle_a |n\rangle_b$ and thus contains perfect photon-number correlations. But this means that there is a very large uncertainty in their relative phases. By overlapping such states on a beam splitter, the phase fluctuations are converted into photon number fluctuations in the sense that there is now a large uncertainty for the beam location of the photons: the probability of finding them in one beam or the other is relatively high. For the twin-Fock state input $|n\rangle_a |n\rangle_b$, the nonzero elements of the output photon-number distribution are given by

$$P(2k, 2n-2k) = \binom{2k}{k} \binom{2n-2k}{n-k} \left(\frac{1}{2}\right)^n,$$

$$k = 0, 1, 2, 3, ..., n,$$  \hspace{1cm} (3.37)

a distribution known in probability theory as the fixed-multiplicative discrete arcsine law of order $n$ [47]. This distribution has the characteristic "U" shape in going from $k = 0$ to $k = n$, where the minimum occurs for $k = n/2$ for $n$ even or $k = (n \pm 1)/2$ for $n$ odd. Twin-Fock states as a resource for sub-standard quantum limited optical interferometry have been discussed by Holland and Burnett [71] and Campos et al. [116], where the latter considered the use of photon-number parity measurements for the detection scheme. The beam-splitter output distribution for input TMSVS is a collection of U” shapes from each of the relevant input states $|n\rangle_a |n\rangle_b$. It is evident that beam splitting results in a large uncertainty in the location of the photons with respect to the two output beams. The application of the TMSVS to interferometry has been studied by Anisimov et al. [113], who showed that sub-Heisenberg limited sensitivity for phase-shift measurements is possible. Later, Gerry and Mimih [59] studied the application to interferometry of yet another state that consists of a superposition of the correlated number state pairs $|n\rangle_a |n\rangle_b$, this being the pair coherent states [10]. In contrast to the TMSVS, the pair coherent states exhibit sub-Poissonian photon statistics in each mode. Recently, Spasibko et al. [83] experimentally examined the interference effects
and photon-number fluctuations from TMSVS whose twin beams fall on opposite sides of a beam splitter.

Because the beam-splitter transformation results in a sum involving the amplitudes $c(j + m, j - m)$ as given in Eq. (3.30), the joint photon-number probability distribution given by Eq. (3.31) will generally depend not only on the angle $\Phi$ but also on the individual phase angles $\theta_1$, $\theta_2$, and $2\phi$. We demonstrate this in Figs. (3.6a) and (3.6b) using the same squeezing and coherent state parameters as before and for the choice $\Phi = \pi/2$ with (a) $\theta_1 = \pi/2$, $\theta_2 = 0$, and $\phi = 0$ and with (b) $\theta_1 = 0$, $\theta_2 = \pi/2$, and $\phi = 0$. In both cases we notice asymmetric distributions with a tendency for the clustering of the photon number states to be populated along the line $n_1 = 0$ for Fig. (3.6a) and $n_2 = 0$ for Fig. (3.6b). In Fig. (3.6c) we displace the case for the choices $\Phi = \pi$ with $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, and $\phi = 0$, which results in a distribution where population is symmetrically clustered along the lines $n_1 = 0$ and $n_2 = 0$. Distributions with this structure are known to be particularly conducive to achieving interferometric phase-shift measurements with sensitivities greater than the standard quantum limit. As discussed previously, if the uncertainty in the photon number is on the order of the number of photons, i.e. $\Delta N \approx N$, which is the case for the state represented in Fig. (3.6c), then the uncertainty in the phase is given by the Heisenberg limit of sensitivity. The essential point is that in a distribution such as in Fig. (3.6c) there is a great uncertainty with regard to the location of most of the photons, an uncertainty created by the beam splitter and certain choices of phases $\theta_1$, $\theta_2$, and $\phi$. It is also worth noting that for particular values of the phase $\Phi$, the joint photon-number distribution after beam splitting does exhibit invariance with respect to the individual phases $\theta_1$, $\theta_2$, and $\phi$. That is to say, for example, when the value $\Phi = \pi/2$ is chosen, the values of the individual phases $\theta_1$, $\theta_2$, and $\phi$ will have a large impact on the resulting distribution, whereas for the choice $\Phi = 0$ or $\Phi = \pi$, the individual phases will not alter the distribution in any way regardless of the values chosen. This could be due to the phase factors in Eq. (3.34) being real for these choices of $\Phi$, as no other value of $\Phi$ exhibits this invariance.
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Figure 3.6: Joint photon number distribution (after beam splitting) $P(n_1, n_2)$ versus $n_1$ and $n_2$ for the two-mode squeezed coherent states with $r = 1.2$ and $\alpha_1 = \alpha_2 = 1$ for (a) $\Phi = \pi/2$ with $\theta_1 = \pi/2$, $\theta_2 = 0$, $\phi = 0$, (b) $\Phi = \pi/2$ with $\theta_1 = 0$, $\theta_2 = \pi/2$, $\phi = 0$ and (c) $\Phi = \pi$ with $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $\phi = 0$. 
3.3 Application to Interferometry

Here we assume the output beams of the down-converter are directed through a Mach-Zehnder interferometer, as indicated in Fig. (3.5). The relative phase shift between the two arms of the interferometer we denote as $\varphi$. We assume that photon-number parity measurements are performed on the output $b$-mode. We follow Yurke et al. [120], taking the input state to the interferometer $|\text{in}\rangle$, which is related to the output state $|\text{out}\rangle$ according to

$$|\text{out}\rangle = e^{i\frac{\varphi}{2}\hat{J}_1}e^{-i\varphi\hat{J}_3}e^{-i\frac{\varphi}{2}\hat{J}_1}|\text{in}\rangle = e^{-i\varphi\hat{J}_2}|\text{in}\rangle,$$  \hspace{1cm} (3.38)

where we have used the relation $e^{i\frac{\varphi}{2}\hat{J}_1}\hat{J}_3e^{-i\frac{\varphi}{2}\hat{J}_1} = \hat{J}_2$. The parity operator for the output $b$-mode is

$$\hat{\Pi}_b = (-1)^{\hat{b}^\dagger\hat{b}} = e^{i\pi\hat{b}^\dagger\hat{b}} = e^{i\pi(\hat{J}_0 - \hat{J}_3)},$$  \hspace{1cm} (3.39)

and the expectation value of the parity operator is given by

$$\langle \hat{\Pi}_b (\varphi) \rangle = \langle \text{out}|\hat{\Pi}_b|\text{out}\rangle = \langle \text{in}|e^{i\varphi\hat{J}_2}e^{i\pi(\hat{J}_0 - \hat{J}_3)}e^{-i\varphi\hat{J}_2}|\text{in}\rangle.$$  \hspace{1cm} (3.40)

For our input state, this becomes

$$\langle \hat{\Pi}_b (\varphi) \rangle = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \langle \text{in}|e^{i\varphi\hat{J}_2}e^{i\pi(\hat{J}_0 - \hat{J}_3)}|J, M\rangle \langle J, M|e^{-i\varphi\hat{J}_2}|\text{in}\rangle$$

$$= \sum_{J=0}^{\infty} \sum_{M=-J}^{J} e^{i\pi(J-M)} \langle \text{in}|e^{i\varphi\hat{J}_2}|J, M\rangle \langle J, M|e^{-i\varphi\hat{J}_2}|\text{in}\rangle$$

$$= \sum_{J=0}^{\infty} \sum_{M=-J}^{J} e^{i\pi(J-M)}|\Gamma_M^{(J)} (\varphi) |^2,$$  \hspace{1cm} (3.41)

where
Figure 3.7: For the choices \( r = 1.2 \) and \( \alpha_1 = 1 = \alpha_2 \), the expectation value of the parity operator \( \hat{\Pi}_b \) as a function of the phase shift \( \varphi \) for \( \Phi = \pi \). To 'center' the plot about \( \varphi = 0 \), we have made the replacement \( \varphi \rightarrow \varphi - \pi/2 \). This shift can be made with appropriate linear optical elements.

\[
\Gamma^{(J)}_M (\varphi) = \langle J, M | e^{-i\varphi \hat{J}_2} | \text{in} \rangle \\
= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c (j + m, j - m) \langle J, M | e^{-i\varphi \hat{J}_2} | j, m \rangle \\
= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c (j + m, j - m) \langle J, M | e^{-i\varphi \hat{J}_2} | J, m \rangle \delta_{J,j} \\
= \sum_{m=-j}^{j} c (J + m, J - m) d_{M,m}^{J} (\varphi). \tag{3.42}
\]

In Fig. (3.7) we plot \( \langle \hat{\Pi}_b (\varphi) \rangle \) versus \( \varphi \) for the case where \( r = 1.2 \) and \( \alpha_1 = 1 = \alpha_2 \). It turns out that without phase-shift adjustments, the expectation values of the parity operators are not centered about \( \varphi = 0 \). To bring about such a centering, we require the phase transformations \( \varphi \rightarrow \varphi + \pi/2 \) for the case where \( \Phi = 0 \) and \( \varphi \rightarrow \varphi - \pi/2 \) for the case where \( \Phi = \pi \), which can be accomplished with simple linear optical elements.

One could determine the uncertainty in the phase shift measurements (the sensitivity) by error propagation calculus according to [Chapter 2].
3.3. Application to Interferometry

Figure 3.8: The phase uncertainty $\Delta \varphi_{\text{min}}$ obtained via the quantum Cramér-Rao bound versus the total average photon number for $\Phi = \pi$ and for a set squeeze parameter of $r = 2.0$. The upper and lower dashed line represent the SQL and HL of phase shift uncertainties, respectively.

$$
\Delta \varphi_{b} = \sqrt{1 - \langle \bar{\Pi}_{b}(\varphi) \rangle^{2}} \left| \partial \langle \bar{\Pi}_{b}(\varphi) / \partial \varphi \rangle \right|.
$$

(3.43)

On the other hand, a more computationally efficient approach is to calculate the minimum achievable uncertainty in the measurement of phase shifts, as we have already discussed, the quantum mechanical parity operator saturates the quantum Cramér-Rao bound [121]. As a reminder to the reader, the quantum Cramér-Rao bound is given by [67]

$$
\Delta \varphi_{\text{min}} = \frac{1}{\sqrt{F_{Q}}}.
$$

(3.44)

where $F_{Q}$ is the quantum Fisher information, given by [17]

$$
F_{Q} = 4 \left[ \langle \psi^{\prime}(\varphi) | \psi^{\prime}(\varphi) \rangle - | \langle \psi^{\prime}(\varphi) | \psi(\varphi) \rangle |^{2} \right].
$$

(3.45)

Making use of Eq. (2.50), our expression for the quantum Fisher information becomes

$$
F_{Q} = 4 \langle (\Delta J_{2})^{2} \rangle_{\text{in}},
$$

(3.46)

where $\langle (\Delta J_{2})^{2} \rangle_{\text{in}}$ is the variance of the operator $\hat{J}_{2}$ with respect to the initial input state.
We have compared our quantum Cramér-Rao bound results with sample error propagation calculus results based on the measurement of photon-number parity and have found complete agreement.

In Fig. (3.8) we plot an example of $\Delta \varphi_{\text{min}}$ versus $\bar{n}$ for the case where $\Phi = \pi$ and $r = 2.0$ and where $|\alpha|$ is being increased. The upper and lower dashed lines on each of the graphs represent the corresponding standard quantum limits (SQL), and Heisenberg limits (HL), respectively. We find that the noise reduction falls almost exactly along the curve for the Heisenberg limit.

### 3.4 Fully Quantum Mechanical Model

So far, we have considered the pump field of the down-converter to be a classically prescribed field, which means we have ignored the effects of photon depletion in the pump field. Next, we shall study the states produced in the case where the pump field is quantized and assumed to be initially in a coherent state or some form of single-mode pure nonclassical state such as a squeezed vacuum. Our goal is to explore the effects of the phases on the evolution of the fully quantized model, especially their effects on the photon-number distributions and on the average photon numbers of the output in the signal and idler modes. For the case where all fields are initially in coherent states and with phase choices such that $\Phi = \pi$, we would expect a more rapid decrease in the average photon number of the pump field as the average photon numbers of the signal and idler modes increase compared to the case when $\Phi = 0$. In fact, we expect that the parametric approximation breaks down after a short interaction time such that the very high average photon numbers appearing in the output signal and idler modes cannot be realized in practice. On the other hand, projective state-reductive measurements performed on the output pump beam at different times could open up the prospects for new forms of non-classical, entangled, two-mode field states.
3.4. Fully Quantum Mechanical Model

3.4.1 Quantizing the Pump

In the semi-classical model, the signal/idler modes are initially prepared in coherent states and the non-linear crystal is driven with a strong classical field, typically composed of UV light. In the fully quantum mechanical model, this classical field is replaced with a coherent state such that the initial state of the three-mode system is $|\alpha\rangle_s \otimes |\beta\rangle_i \otimes |\gamma\rangle_p$, where the mode designations stand for ‘signal’, ‘idler’, and ‘pump’, respectively. With this model, we can account for depletion in the pump over time. We know in the semi-classical case, the average photon number in the output signal/idler beams depends heavily on the phase combination $\Phi = \theta_1 + \theta_2 - 2\phi$, where $\theta_1, \theta_2$ are the phases associated with the coherent states in the signal/idler beam and $2\phi$ is the phase associated with the squeezing operation. The result for the semi-classical case is given in Eq. (3.15). We see a substantial increase in the average photon number in the signal/idler modes as $\Phi \to \pi$. Setting $\alpha \to |\alpha|e^{i\theta_1}$, $\beta \to |\beta|e^{i\theta_2}$, and $\gamma \to |\gamma|e^{-i2\phi}$, we explored the effect of the same phase combination $\Phi$ on the average photon number in the fully quantum mechanical case for short interaction times, with the expectation of finding a similar effect.

We explore the effects this phase has on the average photon number of the three modes through numerical means. We use a third-order Runge Kutta method as our numerical integrator, where the differential equations to be solved for the state coefficients

![Figure 3.9: Average photon numbers in the pump and signal/idler beams with (a) $\Phi = 0$ and (b) $\Phi = \pi$.](image)
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are determined through the time-dependent Schrödinger equation \[44\]. The results are plotted in Fig. (3.9). For short interaction times we see a large increase in average photon number in the signal/idler modes and a corresponding sharp decrease in average photon number in the pump mode for \( \Phi = \pi \). Likewise when \( \Phi = 0 \), for the same time, the average number of photons in the pump far exceeds that in the signal/idler modes.

3.4.2 Results at Short Times — A Perturbative Approach

We have demonstrated that in the semi-classical case, that is, when we have a classically prescribed pump field, that the average photon number in the signal/idler beams, the \( a-b \) modes, drastically increases for a particular choice of the phase \( \Phi \). More specifically, we have arrived at an expression for the average photon number of the two modes as a function of the phase parameter \( \Phi \), given in Eq. (3.15). In the fully quantum mechanical model, where we treat our pump as a quantized field state (more specifically, a coherent state), our initial state is given by

\[
|\psi(0)\rangle = |\alpha\rangle_a \otimes |\beta\rangle_b \otimes |\gamma\rangle_c .
\]  

(3.47)

where the \( c \)-mode denotes our pump field while the \( a-b \) modes are the signal and idler beams, respectively. In the previous section, we have shown a similar phase dependency using numerical integration techniques. In this section we take a perturbative approach in showing this relationship between the phase combination \( \Phi \) and the average photon numbers in the pump and signal+idler modes. The Hamiltonian that drives the interaction between the three field states is

\[
\hat{H}_I = i\hbar \kappa (\hat{a}\hat{b}\hat{c}^\dagger - \hat{a}^\dagger\hat{b}\hat{c}) , \quad \rightarrow \quad \hat{U}(t) = e^{-i\hat{H}_I t/\hbar} ,
\]  

(3.48)

where the parameter \( \kappa \) is a coupling constant proportional to the second order polarizability of the nonlinear medium \[41\] and where the time-evolved state is given by the usual \( |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \). With this we can calculate the average photon number
in the pump as well as the total average photon number in the signal+idler modes as follows:

\[
\bar{n}_{\text{pump}}(t) = \langle \psi(t) | \hat{c}^\dagger \hat{c} | \psi(t) \rangle = \langle \psi(0) | e^{i \hat{H}_{\text{I}} t/\hbar} \hat{c}^\dagger \hat{c} e^{-i \hat{H}_{\text{I}} t/\hbar} | \psi(0) \rangle ,
\]

\[
(3.49)
\]

\[
\bar{n}_{\text{signal+idler}}(t) = \langle \psi(t) | (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) | \psi(t) \rangle = \langle \psi(0) | e^{i \hat{H}_{\text{I}} t/\hbar} (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) e^{-i \hat{H}_{\text{I}} t/\hbar} | \psi(0) \rangle .
\]

We can expand the evolution operator in terms of time

\[
e^{\pm i \hat{H}_{\text{I}} t/\hbar} \simeq 1 \pm i \hat{H}_{\text{I}} t/\hbar - \frac{1}{2!} (\hat{H}_{\text{I}} t/\hbar)^2 - \ldots,
\]

\[
(3.50)
\]

where we assume that \( t \) is small, such that \( t^2 \neq 0 \) but \( t^3 \to 0 \). Under this approximation, the average photon number in the pump field is given by

\[
\bar{n}_{\text{pump}}(t) \simeq \bar{n}_{\text{pump}}^{(0)} + t \bar{n}_{\text{pump}}^{(1)} + t^2 \bar{n}_{\text{pump}}^{(2)} .
\]

\[
(3.51)
\]

Plugging in and calculating each order, we arrive at the result

\[
\bar{n}_{\text{pump}}^{(0)} = |\gamma|^2 ,
\]

\[
(3.52)
\]

\[
\bar{n}_{\text{pump}}^{(1)} = 2 \kappa |\alpha||\beta| |\gamma| \cos \Phi ,
\]

\[
(3.53)
\]

\[
\bar{n}_{\text{pump}}^{(2)} = \kappa^2 (|\alpha|^2 |\beta|^2 - 2 |\gamma|^2 (|\alpha|^2 + |\beta|^2)) .
\]

\[
(3.54)
\]

There are several things worth noting given these results. First, it is not terribly shocking, when considering the Hamiltonian that drives the interaction, that only odd orders of the average photon number produces a phase dependency. Also, we see the trend found in the semi-classical case to continue for short interaction times; as \( \Phi \to \pi \), the
average photon number in the pump decreases, in agreement with Fig. (3.9b). It is also worth pointing out the dependency between each order of the average photon number and the Hamiltonian. It turns out that the first order correction scales as $\bar{n}_{\text{pump}}^{(1)} \propto \langle [\hat{H}_I, \hat{c}^{\dagger} \hat{c}] \rangle$ while the second order correction scales as $\bar{n}_{\text{pump}}^{(2)} \propto \langle [[\hat{H}_I, \hat{c}^{\dagger} \hat{c}], \hat{H}_I] \rangle$, and so forth for higher orders.

Next we turn our attention to the total average photon number in the signal+idler modes. Carrying out the same procedure as for the pump field, we can write

$$\bar{n}_{s+i}(t) \simeq \bar{n}_{s+i}^{(0)} + t \bar{n}_{s+i}^{(1)} + t^2 \bar{n}_{s+i}^{(2)} ,$$

where once again we can plug in the approximation made in equation Eq. (3.50) to find

$$\bar{n}_{s+i}^{(0)} = |\alpha|^2 + |\beta|^2,$$  

$$\bar{n}_{s+i}^{(1)} = -4\kappa|\alpha||\beta||\gamma|\cos \Phi ,$$  

$$\bar{n}_{s+i}^{(2)} = -2\kappa^2 \left(|\alpha|^2|\beta|^2 - 2|\gamma|^2(|\alpha|^2 + |\beta|^2)\right)$$

We see that as $\Phi \to \pi$, the total average photon number in the signal+idler beams increases for short times, as expected. For example, consider the choices $|\alpha|^2 = |\beta|^2 = 5$, $|\gamma|^2 = 30$ and $\kappa t = 0.1$. The average photon number in the pump for $\Phi = 0$ is $\bar{n}_{\text{pump}}|_{\Phi \to 0} \sim 29.7$ while for the signal+idler $\bar{n}_{s+i}|_{\Phi \to 0} = 10.5$. For the choice of $\Phi = \pi$, however, we find $\bar{n}_{\text{pump}}|_{\Phi \to \pi} \sim 18.7$ and $\bar{n}_{s+i}|_{\Phi \to \pi} = 32.4$. So we see that by simply adjusting the combination of phases $\Phi$, we get a considerably larger average photon number in the signal+idler beams.

We note for pedagogical purposes that for an arbitrary observable $\hat{O}$, we can calculate the expectation values for short times,

$$\langle \hat{O}(t) \rangle = \langle \hat{O} \rangle^{(0)} + t \langle \hat{O} \rangle^{(1)} + t^2 \langle \hat{O} \rangle^{(2)} + ...$$
where, up to second order, we find

\[
\langle \hat{O} \rangle^{(0)} = \langle \psi(0) | \hat{O} | \psi(0) \rangle ,
\]

\[
\langle \hat{O} \rangle^{(1)} = -i\hbar \langle \psi(0) \left| \left[ \hat{H}_I, \hat{O} \right] \right| \psi(0) \rangle ,
\]

\[
\langle \hat{O} \rangle^{(2)} = \frac{1}{2\hbar} \langle \psi(0) \left| \left[ \left[ \hat{H}_I, \hat{O} \right], \hat{H}_I \right] \right| \psi(0) \rangle .
\]
Chapter 4

Interferometry mixing $N$ Photons with Coherent Light

The familiar coherent states $|\alpha\rangle$ are the most classical-like of all the pure states of a single-mode quantized electromagnetic field and they represent the light produced by a phase-stabilized laser. They yield field-operator expectation values that behave like classical prescribed fields but with quantum fluctuations at the level of the vacuum. The corresponding Wigner function is Gaussian and positive everywhere in phase space, whereas its corresponding $P$ function is a $\delta$ function. On the other hand, a Fock state, or Number state $|N\rangle$, $N = 1, 2, 3...$, are at the other extreme in that they are the most nonclassical of field states, having highly sub-Poissonian (or amplitude squeezed) photon-number statistics. The Wigner functions of such states are non-Gaussian, oscillatory, and take on negative values in phase space. The corresponding $P$ functions of the number states are highly singular in that they are given as the $2N$th-order derivative of a $\delta$ function.

As we have discussed in Chapter 2, optical interferometry with classical-like light beams only, i.e., with coherent light in the state $|\alpha\rangle$ as one input with the other in the vacuum $|0\rangle$ is known to be limited in sensitivity for phase-shift measurements to the standard quantum limit, or shot-noise limit, given in Eq. (2.67) where the average photon number for a coherent state is given by $\bar{n} = |\alpha|^2$ [57]. The sensitivity of the interferometer can be enhanced by increasing the intensity of the light, that is, increasing
\( \bar{n} \). However, this leads to an increase in radiation pressure fluctuations on the interferometer mirrors, thus, ultimately degrading its sensitivity. A possible way around this problem was proposed by Caves [32] who suggested that a form of nonclassical light, namely, a squeezed vacuum (SV) state, be injected into the previously unused port of the first beam splitter of the interferometer along with coherent light as usual. The mixing of coherent and squeezed light at the first beam splitter results in the increase in the sensitivity of the interferometer to

\[
\Delta \varphi = e^{-r}/\sqrt{\bar{n}}
\]

where \( r \geq 0 \) is the so-called squeezing parameter and \( \bar{n} \) still refers to the average photon number of the input coherent state to a good approximation. The measurement scheme for coherent states alone or for coherent states mixed with squeezed vacuum states is the subtraction of the output photocurrents of the second beam splitter, this being the standard approach for the interferometric measurement of phase shifts.

For linear phase shifts, the ultimate level of sensitivity allowed by quantum mechanics is given by the so-called Heisenberg limit, Eq. (2.67), a reduction in noise over the standard quantum limit by a factor of the SQL itself, Eq. (2.67). There has been much discussion in the literature on the use of so-called N00N states, given by Eq. (2.64) [45], in order to reach this limit. Such states cannot be produced with an ordinary beam splitter; some kind of nonlinear process is required to generate them in lieu of the first beam splitter of the interferometer, and one still requires a number state of high photon number \( N \). But with the appropriate observable, which turns out to be the photon-number parity operator of just one of the output beams of the interferometer [60], Heisenberg-limited sensitivity, in this case \( \Delta \varphi = 1/N \), can be obtained. The use of parity measurements also leads to super-resolution, that is \( \langle \hat{\Pi} \rangle = \cos N \varphi \), which has oscillations in \( A \varphi \) that are \( N \) times “faster” than for the case of one photon or for interferometry with a coherent state. Oscillations with \( N \varphi \) are said to be super-resolved. The necessity of generating number states can be overcome by instead using entangled coherent states of the unnormalized form \( |\alpha\rangle_a |0\rangle_b + |0\rangle_a |\alpha\rangle_b \) which leads to \( \Delta \varphi = 1/\bar{n} \).

Yet another approach is to use an ordinary interferometer, i.e., one requiring no nonlinear elements as part of the interferometer with input twin-Fock states \( |N\rangle |N\rangle \) falling
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on the beam splitter [116]. With parity measurements on one of the output beams, we obtain asymptotically in the limit of large $N$, $\Delta \varphi = 1/(2N)$ [116], which is the Heisenberg limit for this input state. Super-resolved interference fringes in the average of the parity operator are also obtained.

This scheme also has a problem having to do with reliably presenting Fock states of equal photon number simultaneously on opposite sides of the first beam splitter. Thus, superpositions of twin-Fock states have been considered. Anisimov et. al [113] have studied the use of two-mode squeezed vacuum states, whereas, Gerry and Mimih [59] have studied the use of pair coherent states [10]. Both states yield Heisenberg-limited phase uncertainties and, in fact, the former yield phase uncertainties slightly below the Heisenberg limit for small average photon numbers, whereas, the latter yield phase uncertainties that are very similar to those obtained from the pair coherent states. The initial photon-number distributions of the two-mode squeezed vacuum and pair coherent states are very different. The distribution for the former is super-Poissonian and peaks at the two-mode vacuum state (it is a thermal-like distribution), whereas the distribution for the latter is sub-Poissonian and peaks around some twin-Fock state $|N\rangle |N\rangle$ for $\bar{N} \simeq N \gg 0$, $\bar{N}$ being the average photon number in one of the modes.

Some years ago, Ou [111] studied the multiparticle quantum interferences arising in a lossless 50:50 beam splitter with $N$ photons in one mode and a single photon in the other, $|N\rangle_a |1\rangle_b$. The single photon was shown to have a dramatic effect on the joint photon number distribution of the state of the photon beams emerging from the beam splitter. For the input state $|N\rangle_a |0\rangle_b$, the joint distribution of the output state is a binomial (Bernoulli) distribution of the $N$ photons over the two output modes. But with the single-photon input, the output distribution, due to multiparticle quantum interference, has a cancellation in the center of the original binomial distribution. The interference also has the effect of pushing the nonzero elements of the distribution over toward the margins. A similar thing happens with input state $|\alpha\rangle_a |1\rangle_b$ [4]. With only a coherent state and a vacuum as inputs, the output state of a beam splitter is a product of coherent states, as we have shown in Chapter 2, and thus, the joint photon-number distribution
is a double Poisson distribution. But with input state $|N\rangle_a |1\rangle_b$, we once again obtain a dramatic change in the distribution, it now having, as before, a central interference fringe with the bulk of the population distribution migrating along the borders. It is the rearrangement of the output joint photon-number distribution, in light of the above remarks on phase and number uncertainties, that has led us to consider such input states in the context of subshot quantum optical interferometry.

In this chapter, we examine the prospect of performing super-resolved and super-sensitive (i.e., Heisenberg-limited) interferometric measurements with a Mach-Zehnder interferometer (MZI) for input states $|\alpha\rangle_a |N\rangle_b$, $N = 1, 2, ..., $ where it should be noted that we have extended the input number states of the $b$-mode to more than one photon. The multi-photon quantum interference effects resulting from the mixing of $N$-photon number states with coherent states at a beam splitter have not been explored to our knowledge. It turns out that by mixing photon-number states of increasing photon number $N$ along with coherent states, we obtain both increasing sensitivity (sensitivity beyond the standard quantum limit) approaching the Heisenberg limit and increasing resolution. Motivated by our results from mixing coherent states with number states, we then consider the mixing of coherent states with squeezed vacuum and squeezed one-photon states where the latter can be obtained by photon subtraction from the former. The occupation probabilities of these states are heavily weighted for the low photon-number states. We show that mixing coherent light with squeezed one-photon states leads to improved sensitivity over that obtained by mixing coherent light with squeezed vacuum states.

The chapter is organized as follows: first we discuss the mixing of coherent states and number states at a 50:50 beam splitter and examine the resulting joint photon-number probability distributions. Next we discuss the application of these states to phase-shift detection in interferometry. Lastly we extend our considerations to the mixing of squeezed vacuum and squeezed one-photon states.
Chapter 4. Interferometry mixing $N$ Photons with Coherent Light

4.1 Mixing Coherent and Number States at a Beam Splitter

We take as our input state to the MZI $|\text{in}\rangle = |\alpha\rangle_a |N\rangle_b$ as indicated in Fig. (4.1). We can describe the action of a beam splitter, once again, as a rotation [14] [115] by using the well known Schwinger realization of the su(2) algebra, discussed in detail in Appendices A and B, respectively. We can write our input state as

$$|\text{in}\rangle = |\alpha\rangle_a |N\rangle_b = e^{-\frac{1}{2}|\alpha|^2} \sum_{j=N/2,\ldots}^{\infty} \frac{\alpha^{2j-N}}{\sqrt{(2j-N)!}} |j,j-N\rangle,$$  \hspace{1cm} (4.1)

where the summation over $j$ includes all half-odd integers. As shown in Eq. (2.55) [14], the input state $|\text{in}\rangle$ of the MZI is related to the output state $|\text{out}\rangle$ according to

$$|\text{out}\rangle = e^{i\frac{\varphi}{2}\hat{J}_1}e^{-i\varphi\hat{J}_3}e^{-i\frac{\varphi}{2}\hat{J}_1} |\text{in}\rangle = e^{-i\varphi\hat{J}_2} |\text{in}\rangle,$$  \hspace{1cm} (4.2)

where, once again, the factors of $\exp \left[ \pm \frac{\varphi}{2}\hat{J}_1 \right]$ represent the actions of the 50:50 beam splitters and the factor of $\exp \left[ -\varphi\hat{J}_3 \right]$ represents the relative phase shift $\varphi$ between the
two arms of the interferometer. This set of operators constitute a particular choice of beam splitter type; equivalently we can write for the output state

$$|\text{out}\rangle = e^{-i\varphi J_2} |\text{in}\rangle,$$

(4.3)

where we have once again utilized the Baker-Hausdorff identity.

The output state after the first beam splitter can be expressed as

$$|\text{out,BS1}\rangle = e^{-\frac{i}{2}|\alpha|^2} |\text{in}\rangle$$

$$= e^{-\frac{i}{2}|\alpha|^2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{q=0}^{N} (-i)^{n-k+q} \frac{\alpha^n}{n! \sqrt{N!}} \times$$

$$\times \sqrt{(N-q+k)! (n-k+q)!} \binom{n}{k} \binom{N}{q} |N-q+k\rangle_a |n-k+q\rangle_b.\quad (4.4)$$

The probability of detection $m_a$ photons in the $a$-mode and $m_b$ photons in the $b$-mode for a given $N$ is

$$P (m_a, m_b|N) = |\langle m_a, m_b|\text{out,BS1}\rangle|^2$$

$$= e^{-|\alpha|^2} \frac{|\alpha|^{2(m_a+m_b-N)} m_a! m_b!}{2^{m_a+m_b} [(m_a+m_b-N)!]^2 N!} \times$$

$$\times \sum_{q=0}^{N} i^{2q} \binom{N}{q} \binom{m_a+m_b-N}{m_a-N+q}^2.\quad (4.5)$$

For the special cases of $N = 0, 1, \text{ and } 2$, we have

$$P (m_a, m_b|i) = e^{-|\alpha|^2} \frac{|\alpha|^{2(m_a+m_b)}}{2^{m_a+m_b} m_a! m_b!} \times$$

$$\left\{ \begin{array}{ll}
1 & i = 0 \\
|\alpha|^{-2} (m_a - m_b)^2 & i = 1 \\
|\alpha|^{-4} \left( m_a^2 + m_b (m_b - 1) + m_a (2m_b + 1) \right)^2 & i = 2
\end{array} \right.\quad (4.6)$$

and for $N = 3$, we have
Chapter 4. Interferometry mixing N Photons with Coherent Light

\[ P(m_a, m_b|3) = e^{-|\alpha|^2} \frac{|\alpha|^{2(m_a+m_b-3)}}{2^{m_a+m_b}m_a!m_b!} \times \frac{1}{6} \left( m_a (m_a - 1) (m_a - 2) - 3m_a m_b \times \right. \\
\left. (m_a - 1) + 3m_a m_b (m_b - 1) - m_b (m_b - 1) (m_b - 2) \right)^2. \tag{4.7} \]

In Fig. (4.2) we plot the joint photon-number probability distribution \( P(m_a, m_b|N) \) versus \( m_a \) and \( m_b \) for \( |\alpha| = 3 \) and for \( N = 0, 1, 3, \) and \( 3 \). For \( N = 0 \) we obtain the expected distribution for input coherent and vacuum states \( |\alpha\rangle_a |0\rangle_b \) incident on a beam splitter, which results, for our choice of beam splitter type, in the output state (see Appendix A)

\[ |\alpha\rangle_a |0\rangle_b \xrightarrow{\text{BS}} \left| \frac{\alpha}{\sqrt{2}} \right\rangle_a - i\frac{\alpha}{\sqrt{2}} \left\rangle_b. \tag{4.8} \]

For the case of \( N = 0 \), the distribution is unimodal; that is, a composite of Poisson distributions of each of the output coherent states centered near \( \bar{n}_a = \bar{n}_b = |\alpha|^2/2 \). As is well known, no entanglement is generated in this case. For \( N = 1 \), we see that the distribution is bimodal. In fact, we can see from Eq. (4.6) that \( P(m, m|1) = 0 \) for all \( m \) is the result of destructive interference. This is a striking result in that \( |\alpha|^2 \) can be arbitrarily large, yet the appearance of just one photon at the other beam splitter input dramatically alters the distribution obtained with \( n = 0 \), effectively bifurcating it into a bimodal distribution. This is interesting in the context of interferometry because the joint photon-number distribution for the N00N state, given in Eq. (2.64), is also bimodal, although it is nonzero only along the borders where either \( m_a = 0 \) or \( m_b = 0 \). For \( N = 2 \), we obtain a trimodal distribution. Unlike the case for \( N = 1 \), we do not have lines of zeros caused by destructive quantum interference, separating the modes of the distribution, but we do have two lines that contain zeros, these being, from Eq. (4.6), roots of

\[ m_a^2 + m_b (m_b - 1) + m_a (2m_b + 1) = 0. \tag{4.9} \]

The roots of this equation fall along two lines, but there is not a "continuous" line of zeros. For the case of \( N = 3 \), we obtain a quadramodal distribution with separations
4.1. Mixing Coherent and Number States at a Beam Splitter

\[ (A) \]

\[ (B) \]

\[ (C) \]

\[ (D) \]

**Figure 4.2:** Joint photon number distribution \( P(m_a, m_b) \) versus \( m_a \) and \( m_b \) after beam splitting an initial input state \( |\alpha\rangle_a |N\rangle_b \) for \( |\alpha| = 3 \) and (a) \( N = 0 \), (b) \( N = 1 \), (c) \( N = 2 \) and (d) \( N = 3 \).
along the lines obtained from the roots of Eq. (4.7),

\begin{equation}
m_a (m_a - 1)(m_a - 2) - 3m_a m_b (m_a - 1) + 3m_a m_b (m_b - 1) - m_b \times \nonumber \\
\times (m_b - 1)(m_b - 2) = 0. \tag{4.10}
\end{equation}

The case for which \( m_a = m_b = m \) is a solution, that is, \( P(m, m|3) = 0, \forall m \). There are other solutions, but these do not form a line of contiguous zeros. Continuing in this way, it is evident that for a given \( N \), we obtain an \( (N + 1) \)-modal distribution. For all cases where \( N \) is odd, we find that \( P(m, m|N_{\text{odd}}) = 0, \forall m \).

We also note that with increasing photon number \( N \), the distributions become re-arranged symmetrically on an “anti-diagonal” in the \( m_a, m_b \) plane where the modes (peaks) along the edges are highest. These are reminiscent of the kinds of distributions that appear upon mixing twin-Fock states at a 50:50 beam splitter [116] where the output state is what has been called the arcsine states [115], or “bat” states, because of the shape of their joint photon-number distribution across the anti-diagonal [75]. As mentioned earlier, it has long been known that twin-Fock states fed through a Mach-Zehnder interferometer lead to sub-shot-noise sensitivity measurements of phase shifts. The similarity of the joint distributions obtained upon the mixing of coherent and number states and the mixing of twin-Fock states at a beam splitter suggest that the former should also yield sub-SQL phase shift measurements, which we discuss in a later section.

### 4.1.1 Entanglement After Beam Splitting

We have shown in Eq. (4.8) that, upon beam splitting, an initial state \( |\alpha\rangle_a |0\rangle_b \) is separable. That is, the state can be written as a product of two single mode pure states; in this case, two separate coherent states. However, given the distributions in Fig. (4.2), one may not expect this to occur when mixing coherent light with \( N \) photons, \( |\alpha\rangle_a |N\rangle_b \), as destructive quantum interference between the probability amplitudes lead to bifurcations in the distribution. To that end, we utilize the von Neumann entropy in determining how entangled the two modes are after beam splitting.
4.1. Mixing Coherent and Number States at a Beam Splitter

The concept of entropy itself can be understood from thermodynamics as a measurement of the disorder in a system. From the point of view of statistical analysis, it can be thought of as a measure of how much information is ‘missing’ from the system. In the context of quantum optics, for example, the entropy tells us how much information is ‘lost’ if we were to discard one of the modes and make a measurement on the remaining mode. To that end, it can be thought of as a measurement of how much information is lost when making a measurement on one mode as opposed to a measurement on the entire composite system.

We start by considering the well-known von Neumann entropy, given as

\[ S(\hat{\rho}) = -\text{Tr}[\hat{\rho} \ln \hat{\rho}] \]  

(4.11)

where \( \hat{\rho} \) is the density operator for the composite state. If the state is pure, then the entropy will be zero, that is, \( S(\hat{\rho}_{\text{pure}}) = 0 \). This tells us that repeated measurements on the system yields no new information, nor is any information lost when considering simply one mode of the system. For a mixed state, however, \( S(\hat{\rho}_{\text{mixed}}) > 0 \). This can be taken to be proof that the two modes are entangled, as considering a single mode ‘destroys’ information in the mode being measured. It should be noted that the entropy takes on the maximum value of \( S(\hat{\rho}) = 1 \) for a maximally entangled state.
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For the purposes of this chapter, we restrict our attention to what is known as the linear entropy, which as the name suggests, is a linear approximation of the von Neumann entropy, given as

$$S_{\text{Linear}}(\hat{\rho}_a) = 1 - \text{Tr} \left[ \hat{\rho}_a^2 \right], \quad (4.12)$$

where now, $\hat{\rho}_a$ is the reduced density operator for the $a$-mode, and the quantity $\text{Tr} \left[ \hat{\rho}_a^2 \right]$ is known as the state purity. The single mode reduced density operator for the $a$-mode is given by

$$\hat{\rho}_a = \text{Tr}_b [\hat{\rho}], \quad (4.13)$$

where $\hat{\rho} = |\text{out}, \text{BS1} \rangle \langle \text{out}, \text{BS1}|$ and where $|\text{out}, \text{BS1} \rangle$ is given in Eq. (4.4). The linear entropy is plotted against photon number $N$ in Fig. (4.3a) for a set value of $\alpha = 2$. For the case where $N = 0$, the linear entropy is, unsurprisingly zero. This is expected as the state after beam splitting is separable. However, for increasing $N$, the linear entropy asymptotically approaches the maximal value of $S_{\text{Linear}} = 1$, although the largest increase in entanglement occurs when transitioning from $N = 0 \rightarrow N = 1$. Interestingly enough, the linear entropy does not seem to be sensitive to the coherent state amplitude. This is shown in Fig. (4.3b), where for a set value of $N = 2$, the entropy is constant with increasing average photon number initially in the $a$-mode, $|\alpha|^2$. While it may be hard to draw conclusions from this, it may be intuitively explained by the joint photon-number probability distributions in Fig. (4.2) where we see the same bifurcation due to destructive interference in the distributions regardless of the coherent state amplitude.

4.2 Parity-based Phase Shift Detection

The usual way to obtain information on the relative phase shift $\varphi$ is to subtract the output photocurrents after the second beam splitter as indicated in Fig. (4.1a) to obtain the signal (see Eq. (2.57) and (2.58) for reference) $\langle \hat{\Omega}^2 \rangle = \langle (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})^2 \rangle_{\text{out}} = 2 \langle \hat{J}_{3,\text{out}} \rangle$, as shown in Appendix B. Taking the expectation value of $\hat{J}_{3,\text{out}}$ in the Heisenberg picture
4.2. Parity-based Phase Shift Detection

with respect to the initial state $|\text{in}\rangle = |\alpha\rangle_a |N\rangle_b$ we find that $\langle \hat{J}^2 \rangle = (|\alpha|^2 - N) \cos \varphi$. Using the usual error propagation calculus, we may determine the uncertainty in estimating the phase shift or the sensitivity of this measurement scheme according to

$$\Delta \varphi = \frac{\Delta \hat{J}_{3,\text{out}}}{|\frac{\partial \langle \hat{J}_3 \rangle_{\text{out}}}{\partial \varphi}|} = \frac{\sqrt{\langle \hat{J}_0 \rangle - \langle \hat{J}_3 \rangle_{\text{out}}}}{|(|\alpha|^2 - N) \cos \varphi|}. \quad (4.14)$$

If $N = 0$, we obtain the usual result for a coherent state input to an MZI. As we have pointed out in Eq. (2.59) the optimal uncertainty is obtained for a phase shift of $\varphi \to \pi/2$, which yields the SQL in phase uncertainty. By inserting a quarter-wave plate in the upper arm of the interferometer, one can shift the phase by a factor of $\pi/2$ so that, in general, we have

$$\Delta \varphi = \frac{\sqrt{\langle \hat{J}_0 \rangle + N (1 + 2|\alpha|^2) \cos^2 \varphi}}{|(|\alpha|^2 - N) \cos \varphi|}, \quad (4.15)$$

which achieves the phase uncertainty of the SQL for small phase shifts with $N = 0$. Thus, the optimal noise reduction achievable is the SQL and occurs only for the case when $N = 0$. For other values of $N$, the noise level rises to above the SQL. Note that, if the fields are nearly of the same average photon number, i.e., $|\alpha|^2 \simeq N$, the noise level becomes very high.

An alternative method for detecting the phase shift is through the measurement of photon-number parity on just one of the output beams of the MZI [60] as indicated in Fig. (4.1b). We choose this to be the $b$-mode beam for which the parity operator can be written as

$$\hat{\Pi}_b = (-1)^{\hat{b}^\dagger \hat{b}} = e^{i\pi \hat{b}^\dagger \hat{b}} = e^{i\pi (\hat{J}_b - \hat{J}_3)}. \quad (4.16)$$

The expectation value of this operator with respect to the output state is

$$\langle \hat{\Pi}_b \rangle = e^{i\pi \hat{J}_b - \hat{J}_3}.$$
\[
\langle \hat{\Pi}_b (\varphi) \rangle = \langle \text{out} | \hat{\Pi}_b | \text{out} \rangle \\
= \langle \text{in} | e^{i\varphi \hat{J}_2} e^{i\pi (\hat{J}_0 - \hat{J}_3)} e^{-i\varphi \hat{J}_2} | \text{in} \rangle .
\] (4.17)

Writing an arbitrary input state in terms of the angular momentum states \(| j, m \rangle\) as
\[
| \text{in} \rangle = \sum_{j=0,1/2,..}^{\infty} \sum_{m=-j}^{j} C_{j,m} | j, m \rangle ,
\] (4.18)
where the sum over \( j \) includes half-odd integers, we can obtain the general result
\[
\langle \hat{\Pi}_b (\varphi) \rangle = \sum_{j=0,1/2,..}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \sum_{m''=-j}^{j} C_{j,m}^* C_{j,m'} e^{i\pi (j-m')} \times \\
\times d_{m',m''}^j (-\varphi) d_{m'',m'}^j (\varphi) ,
\] (4.19)
where once again \( d_{m',m}^j (\beta) \) are the Wigner-\( d \) matrix elements \([14] [120]\); see Appendix C. For the input state of Eq. (4.2) the have the coefficients
\[
C_{j,m} = e^{-\frac{1}{2} | \alpha |^2} \frac{\alpha^{2j-N}}{\sqrt{(2j-N)!}} \times \delta_{m,j-N} ,
\] (4.20)
so that, after employing some identities detailed in Appendix C, we obtain
\[
\langle \hat{\Pi}_b (\varphi) \rangle_N = (-1)^N e^{-| \alpha |^2} \sum_{j=N/2,..}^{\infty} \frac{| \alpha |^{2(2j-N)}}{(2j-N)!} d_{j-N,j-N}^j (2\varphi) .
\] (4.21)

First we consider the special case \( N = 0 \) for which we can obtain the known result derived in Eq. (2.63),
\[
\langle \hat{\Pi}_b (\varphi) \rangle_{N=0} = e^{-\bar{n} (1-\cos \varphi)}
\] (4.22)
where \( \bar{n} = | \alpha |^2 \). This is the result obtained by Chiruvelli and Lee \([33]\) and discussed by Gao et. al \([137]\) in connection with an application of parity measurements to the
4.2. Parity-based Phase Shift Detection

$$\mathbb{P}$$

$$\alpha$$


\[ (A) \]


\[ (B) \]


\[ (C) \]

\[ (D) \]

Figure 4.4: The expectation value of the parity operator $$\langle \hat{\Pi}_b(\varphi) \rangle_N$$ versus $$\varphi$$ and $$|\alpha|$$ for (a) $$N = 1$$, (b) $$N = 2$$, (c) $$N = 3$$, and (d) $$N = 4$$. 
problem of the quantum laser radar. Note that for small angles, $\varphi \to 0$, we obtain a signal peaking with $\langle \hat{\Pi}_b(\varphi) \rangle_0 = 1$ but which narrows around $\varphi = 0$ for increasing $n$. The signal is not super-resolved in the usual sense of having oscillation frequencies scaling as $M\varphi$ for integer $M > 1$. However, compared with the corresponding result for the output subtraction method $\langle \hat{\Omega} \rangle / \bar{n} = \cos \varphi$, we can see the signal for the parity measurement is much narrower and it is in this sense that Gao et al. [137] interpret the parity result as being super-resolved.

Now we turn to the general case $N > 1$ for which we plot $\langle \hat{\Pi}_b(\varphi) \rangle_N$ against $\varphi$ and $|\alpha|$ in Fig. (4.4). Note that from Eq. (4.21) we have

$$\langle \hat{\Pi}_b(\varphi) \rangle_N \bigg|_{\varphi \to 0} = (-1)^N,$$

(4.23)

and thus the expectation value of the parity operator for the output $b$-mode at $\varphi = 0$ reflects the parity of $N$. We also get oscillations in the signal with $\varphi$ (interference fringes) of the type expected in the usual sense of super-resolution, and furthermore we notice that the central peak of valley at $\varphi = 0$ narrows for increasing $N$. Thus the injection of photon-number states along with coherent states into the MZI apparently leads to enhanced super-resolution because of the narrowing of the central peak or valley and in the increase in the number of oscillations in the signal with changing $\varphi$.

### 4.3 Application to Interferometry

Finally, we consider the effects on the noise reduction with parity-based measurements. From the error propagation calculus for which the parity operator is used as our detection observable we have

$$\Delta \varphi = \frac{\Delta \hat{\Pi}_b}{\left| \partial \langle \hat{\Pi}_b(\varphi) \rangle_N / \partial \varphi \right|}.$$

(4.24)

In Fig. (4.5) we plot $\Delta \varphi$ against the total average photon number $|\alpha|^2 + N$ for $N = 0, 1, 2$ and $3$ in the limit $\varphi \to 0$ where, for computation reasons, we set $\varphi = 10^{-4}$. Included in each graph are the corresponding SQL and HL. It is evident that mixing
coherent light with a number state of $N$ photons allows for sub-SQL noise reduction in the parity-based measurement scheme of detecting phase shifts. The effect is most pronounced for intermediate values of $|\alpha|^2 + N$ where, even for $N = 1$, we see a remarkable reduction in the noise level. It is clear that, overall, the noise reduction approaches the HL for increasing $N$.

The minimal phase uncertainty obtainable for a given state is found by the quantum Cramér-Rao bound \[23\] \[114\], given in Eq. (2.20). More concisely written, it is expressed as

$$\Delta \varphi_{\text{min}} = \frac{1}{\sqrt{F_Q}},$$

(4.25)

where $F_Q$ is the quantum Fisher information given in Eq. (2.50) and again in Eq. (3.46). For our input state $|\alpha\rangle_a |N\rangle_b$, we find that

$$\Delta \varphi_{\text{min}} = \frac{1}{2\sqrt{|\alpha|^2 + N (1 + 2|\alpha|^2) \cos \varphi}}.$$\(\text{(4.26)}\)

Unsurprisingly, this result is in agreement with the phase uncertainty obtained via parity-based detection, as it should be in accordance with Eq. (2.43) and Eq. (2.44).

### 4.4 Coherent Light Mixed with a Single Mode Squeezed State

So far, we have discussed the effects of mixing coherent states with number states at a beam splitter. For a general superposition of number states in the $b$-mode of the form

$$|\psi\rangle_b = \sum_{p=0}^{\infty} C_p |p\rangle_b,$$

(4.27)

the input state is $|\text{in}\rangle = |\alpha\rangle_a |\psi\rangle_b$. After the first beam splitter, we have
The phase uncertainty $\Delta \varphi$ versus $\bar{n} = |\alpha|^2 + N$ with the choice of $\varphi = 10^{-4}$ for (a) $N = 1$, (b) $N = 2$, (c) $N = 3$ and (d) $N = 4$. The upper dashed line represents the SQL while the lower dashed line represents the HL. The dotted line represents the quantum Cramér-Rao bound calculated using the quantum Fisher information. We include the case of $N = 0$ as a reminder to the reader that one achieves the SQL of phase uncertainty when mixing coherent light with a vacuum.

**Figure 4.5**
\[ |\text{out, BS} 1 \rangle = e^{-\frac{i}{2} J} |\text{in} \rangle = \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{q=0}^{\infty} C_p e^{-\frac{1}{2} |\alpha|^2} (-i)^{n-k-p-q} \frac{\alpha^n}{n!} \frac{2^{-(n+p)/2}}{\sqrt{p!}} \times \sqrt{(p - q + k)! (n - k + q)!} \binom{n}{k} \binom{p}{q} |p - q + k\rangle_a |n - k + q\rangle_b. \] (4.28)

The probability of detecting \( m_a \) photons in the \( a \)-mode and \( m_b \) in the \( b \)-mode is given by

\[
P(m_a, m_b | \psi) = e^{-|\alpha|^2} \frac{m_a! m_b!}{2^{m_a + m_b}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} C_p i^{2q - p} \times \frac{\alpha^{m_a + m_b - p}}{(m_a + m_b - p)! \sqrt{p!}} \times \binom{p}{q} \binom{m_a + m_b - p}{m_a - p + q}^2. \] (4.29)

The corresponding expectation value of the parity operator is given by

\[
\langle \hat{\Pi} \rangle_{b(a)} = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} (-1)^{j-(+)^m} F_{j,m} B_{j,m}, \] (4.30)

where

\[
F_{j,m} = e^{-\frac{1}{2} |\alpha|^2} \sum_{p=0}^{2j} \frac{(\alpha^*)^{2p}}{\sqrt{p!}} \frac{C_p}{\sqrt{p!}} d_{j-p,m} \left(-\varphi\right),
\] (4.31)

\[
B_{j,m} = e^{-\frac{1}{2} |\alpha|^2} \sum_{p'=0}^{2j} \frac{(\alpha')^{2p'}}{\sqrt{p'!}} C_{p'} d_{j-p',m} \left(\varphi\right).
\]

Using these generalized results, we can obtain the photon statistics for specific cases where the \( b \)-mode is initially occupied by a single-mode squeezed vacuum or a squeezed one-photon state. We only require the corresponding state coefficients.
4.4.1 The Squeezed Vacuum and Squeezed One-Photon States

In the paper by Caves [32], coherent states are mixed with single-mode squeezed vacuum states at a beam splitter. A single-mode squeezed number state in the $b$-mode is given by [98] [90]

$$|r, M\rangle_b = \hat{S}_b(r) |M\rangle_b,$$  \hspace{1cm} (4.32)

where $\hat{S}_b(r)$ is the squeeze operator, given by

$$\hat{S}_b(r) = e^{\frac{1}{2}r (\hat{b}^2 - \hat{b}^\dagger)} ,$$  \hspace{1cm} (4.33)

and where $r$ is the squeeze parameter $0 \leq r < \infty$. For the squeezed vacuum state $M = 0$, we have

$$C_p = \begin{cases} (-1)^{p/2} \left[ \frac{p!}{2^{p/2}[(p/2)!]^2} \tanh^p r \cosh r \right]^{1/2}, & p \text{ even}, \\ 0, & p \text{ odd}. \end{cases}$$  \hspace{1cm} (4.34)

The average photon number for the squeezed vacuum state is $\bar{n} = \sinh^2 r$. For the case of the one-photon squeezed state $M = 1$, we have
4.4. Coherent Light Mixed with a Single Mode Squeezed State

\[ C_p = \begin{cases} 0, & p \text{ even}, \\ (-1)^{(p-1)/2} \left[ \frac{p^!}{2^{(p-1)}((p-1)/2)!} \frac{\tanh^{(p-1)}r}{\cosh^3 r} \right]^{1/2}, & p \text{ odd}. \end{cases} \]  \hspace{1cm} (4.35)

with an average photon number of \( \bar{n} = \sinh^2 r + \cosh (2r) \).

4.4.2 Joint Photon-Number Distributions

The photon number probability distributions for these states, given by \( P_p = |C_p|^2 \), are plotted against \( p \) in Fig. (4.6) for a squeeze parameter \( r = 1.2 \). For a given value of \( r \), the average number of photons in the squeezed vacuum and squeezed one-photon states is quite different. For \( r = 1.2 \), the average number of photons in the squeezed vacuum state is 2.2278, whereas for the squeezed one-photon state it is 7.835.

Now, by mixing coherent and squeezed vacuum states at a beam splitter, it is possible to choose field-state parameters such that, after beam splitting, the joint photon-number distribution is symmetrically populated along the borders with essentially no population in the interior. In Fig. (4.7), we display such a situation for the case where \( \alpha = \sqrt{1.2} \) and \( r = 0.947 \), which corresponds to beams of equal average photon number: 1.2. We see that the output state consists of a superposition of N00N states for \( N = 2, 3, 4, \) and 5. The \( \alpha \) and \( r \) parameters used for the above graph are those relevant to a recent experiment performed by Afek et al [73] who, working on a suggestion by Hofmann and Ono [69], have performed an interferometry experiment based on the N00N states contained in the superposition of N00N states in which they obtained high sensitivity and super-resolution in the measurement of phase shifts. The idea of the experiment was to use a setup similar to the one pictured in Fig. (4.1a) but to count only the coincident counts where the total numbers counted added up to the selected value of \( N \). In other words, they measured \( \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle _{\text{out}} \) but retained only the counts where, say, if one detector detects \( m \) photons, the other detects \( N - m \) and where all other counts where the total does not add to \( n \) are discarded. This amounts to a projective measurement onto a subspace wherein the photon numbers in the two modes add up to \( N \). In the experiment reported in [73], the total photon numbers \( N = 2 \) through 5 were
studied, and sub-SQL and super-resolved phase shift measurements were performed. However, it seems to be the case that equal intensity input coherent and squeezed vacuum states yield photon-number distributions of the type shown in Fig. (4.7) only for relatively low values of $|\alpha|$. For larger values of $|\alpha|$, many of the states in the plane are populated, as we will show, and one does not have a superposition of N00N states.

With parity measurements performed on one of the output beams, it is not necessary, or even possible, to restrict oneself to a definite $N$-photon N00N state, and that can be an advantage. The total number of photons inside the interferometer for this input state is indeterminate, but the Heisenberg limit is approached in terms of the average total photon number. Seshadressan et. al [80] have already shown that photon-number parity-measurement based interferometry reaches the HL if coherent state and squeezed vacuum light of equal intensity are mixed at a 50:50 beam splitter.

In the case of the squeezed vacuum state, the vacuum state component itself has the highest probability of occupation, the photon-number distribution being thermal-like apart from the fact that only the even photon-number states are populated. However, for the squeezed one-photon state, the vacuum is not present, and it is the one-photon
4.4. Coherent Light Mixed with a Single Mode Squeezed State

The phase uncertainties against total average photon number for coherent light mixed with (a) the squeezed vacuum and (b) the squeezed one-photon states for the choice of $r = 0.3$ and $\varphi = 10^{-4}$.

It seems reasonable, based on the dramatic improvement to sensitivity obtained by mixing the one-photon state with a coherent state, to suspect that the squeezed one-photon state mixed with coherent light might perform better in interferometry than does mixing coherent light with the squeezed vacuum for the same values of $\alpha$ and $r$. The total average photon numbers passing through the interferometer in these cases is

$$\bar{n} = |\alpha|^2 + \sinh^2 r,$$

and

$$\bar{n} = |\alpha|^2 + \cosh (2r) + \sinh^2 r,$$

for the squeezed vacuum and squeezed one-photon states, respectively, mixed with a coherent state.

### 4.4.3 Phase Uncertainty

In Fig. (4.8), we plot the corresponding phase uncertainties against the total average photon number for the mixing of coherent light with the squeezed vacuum, Fig. (4.8a), and squeezed one-photon states, Fig. (4.8b) for the choice $r = 0.3$ and $\varphi = 10^{-4}$. We
repeat for \( r = 0.9 \) in Fig. (4.9). As we expected, the squeezed one-photon state outperforms the squeezed vacuum state, significantly reducing the noise in both examples for a given total average photon number.

An explanation for the improvement in performance by the squeezed one-photon state can be provided by examining the joint photon-number probability distribution after the first beam splitter. In Fig. (4.10a), we plot the joint photon-number probability distribution for the case of \( \alpha = 2 \) and \( r = 0.9 \) where the states are not of equal intensities. The average total photon number for this state is \( \bar{n} = 8.161 \). In Fig. (4.10b), we plot the distribution for an initial squeezed vacuum state with the same parameters. The average total photon number for this state is \( \bar{n} = 5.054 \). In the former case, the distribution is bimodal, populated mainly on the borders with Poisson-like distributions on each axis and with peaks near \( n_{a,b} = 8.161 = \bar{n} \). This distribution resembles that of an entangled coherent state of the form \( |\beta\rangle_a |0\rangle_b + \exp(i\Phi) |0\rangle_a |\beta e^{i\delta}\rangle_b \), the coherent state analog of the N00N state (a superposition of N00N states), and known to be effective in performing HL-limited interferometry in terms of the average total photon number for small phase shifts [56]. In contrast, the distribution involving the squeezed vacuum has some separation along the borders but also has considerable population on the inside. In Fig. (4.11), we plot the expectation value of the parity operator for the mixing of coherent states with the squeezed vacuum and squeezed one-photon states. It is evident
that the resolution obtained for the latter case is enhanced over that of the former.

It is worth noting that the measurement scheme of [73] requires photon counting with resolution at the level of a single photon. Photon counts at the same level of resolution can also be used to perform photon-number parity measurements, so no new technology would be required to perform such measurements, at least for photon numbers that are not too high. On the other hand, quantum non-demolition techniques can be used to measure the parity directly, at least in principle [26].

Lastly, we point out that there is no need to first supply a one-photon state $|1\rangle$, which would then be subjected to the parametric amplifier that performs the squeezing operation to generate the squeezed one-photon state. Instead it has been shown by Biswas and Agarwal [21] that the state obtained by subtracting a single photon from the squeezed vacuum state is identical to the squeezed one-photon state. For completeness, we repeat the demonstration here. The squeezed vacuum and squeezed one-photon states are given, respectively, by

\[
|r, 0\rangle_{b} = \hat{S}_{b}(r) |0\rangle_{b}, \quad |r, 1\rangle_{b} = \hat{S}_{b}(r) |1\rangle_{b},
\]  

(4.38)
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We subtract one photon from the squeezed vacuum state, i.e., we perform the operation \( \hat{b} |r, 0\rangle \), which we can write using the unitarity of the squeezing operator as

\[
\hat{b} |r, 0\rangle = \hat{b} \hat{S}_b (r) |0\rangle_b = \hat{S}_b (r) \hat{S}_b^\dagger (r) \hat{b} \hat{S}_b (r) |0\rangle_b .
\]  

(4.39)

Using the relation

\[
\hat{S}_b^\dagger (r) \hat{b} \hat{S}_b (r) = \hat{b} \cosh r + \hat{b}^\dagger \sinh r ,
\]  

(4.40)

we have

\[
\hat{b} |r, 0\rangle = \sinh r \hat{S}_b (r) |1\rangle_b ,
\]  

(4.41)

from which it follows that

\[
|r, 1\rangle_b = \hat{S}_b (r) |1\rangle_b = \frac{1}{\sinh r} \hat{b} |r, 0\rangle_b .
\]  

(4.42)
Photon-subtracted squeezed vacuum states have already been generated in the laboratory [6] [128] with up to three photons subtracted. The possibilities and benefits for using multiple photon-subtracted squeezed states in interferometry will be discussed in the next chapter, Chapter 5. We will also characterize these states in greater depth; specifically, we will discuss the method by which such states can be generated, the photon statistics of these states, and their application to interferometry.
Chapter 5

Interferometry using the Photon Subtracted Squeezed Vacuum

Over 30 years ago, Caves [32] proposed to mix coherent light with single-mode squeezed vacuum light as a means to reduce quantum mechanical noise in optical interferometers. With quasi-classical light alone, that is, laser light in a coherent state $|\alpha\rangle$ injected into one input of the first beam splitter of the interferometer, and with only the vacuum state at the other, the best one can achieve for the sensitivity of phase shift measurements is the SQL, given by Eq. (2.67). But, as shown by Caves [32], with the introduction of squeezed vacuum light into the previously unused input port, one can achieved the reduction of noise to $\Delta \varphi = e^{-r}/\sqrt{n}$ where now $n = |\alpha|^2 + \sinh^2 r$ and where $r$ $(r > 0)$ is the squeeze parameter. On the other hand, Dowling and collaborators [43] [65] have extensively discussed a different approach to quantum optical interferometry that involves the use of the so-called N00N states of the form in Eq. (2.64), which lead to HL sensitivity, super-sensitivity, in the phase-shift measurements. As we have discussed in Chapter 2, the HL represents the greatest degree of noise reduction allowed by quantum mechanics for linear phase shifts. The traditional approach to phase-shift detection is to subtract the output photocurrents of the final beam splitter of an interferometer, but such an approach cannot work for N00N states and their superpositions. The difference in the output photocurrents vanishes; thus there is no dependence on the phase shift. The same is true for the case of input twin-Fock states [71] injected into an Mach-Zehnder interferometer (MZI). But, as has been shown in a series of papers [54] [58], the
phase shift can be detected for these states if photon-number parity is measured at one of the outputs of the MZI. In fact, continuous superpositions of N00N states [25] and twin-Fock state [116] results are sensitivities that approach the HL. It has been shown [58] [33] that even for states that are not of the type for which the difference in the output photocurrents vanishes, the parity measurement scheme still offers improvements in sensitivity and/or resolution [137]. Gerry and Mimih [60] have reviewed parity-measurement-based optical interferometry.

Recently, Hofmann and Ono [69] showed that the proposal of Caves [32], of mixing coherent and squeezed vacuum light of appropriate intensities on a 50:50 beam splitter, leads, via the resulting multi-photon quantum interference, to the generation of a superposition of N00N states (see Fig. (4.7)) wherein under the appropriate choices of relevant state parameters, the distribution is clustered in one output beam or the other. Because a lossless interferometer conserves photon number, one can consider a particular N00N state of the superposition by a measurement of the joint photon-number at the output of the second beam splitter of the interferometer, \( \langle \hat{a}^\dagger \hat{a}^\dagger \hat{b}^\dagger \hat{b} \rangle_{\text{out}} \), retaining only the counts where the total number of photons adds to a selected \( N \). That is, if one detector counts \( m \) photons, the other detects \( N - m \). All other counts whose total does not add to \( N \) are discarded or binned for use in the cases for whatever other total photon number is determined for those particular joint measurements. These measurements amount to projective measurements onto subspaces wherein the photon numbers in the two modes add up to a fixed number \( N \). An experimental realization of the Hofmann-Ono proposal was implemented by Afek et al [73]. At the first beam splitter of an MZI, they mixed coherent light with squeezed vacuum light obtained from collinear spontaneous parametric down-conversion (SPDC), choosing the field intensities, so that the fidelity of the output states normalized \( N \) photon component was optimized for the corresponding N00N state for each of the cases \( N = 2, 3, 4 \), and 5. Sub-SQL phase-shift measurements were performed by photon-number resolving detection implemented using an array of single-photon resolving counting modules. The phase shift measurements were also
super-resolved, which in this context means that the number of fringes in a given interval scales with $N$, or that the width of a single fringe scales as $N^{-1}$.

It is interesting that the technology required for the Afek experiment [73], that is, of photon detection with resolution at the level of a single photon, is exactly the technology required to perform photon-number parity measurements, at least, for low photon numbers. In fact, it has been shown [80] [121] that the parity-detection scheme achieves Heisenberg-limited sensitivity with states obtained by the mixing of coherent and squeezed vacuum states as per the original proposal of Caves [32]. One advantage of the parity detection scheme is that one need not project out particular $N_{00N}$ states as was done in the Afek experiment. In fact, it was shown some time ago [54] [25] that maximally entangled coherent states, of the form $|\alpha\rangle_a |0\rangle_b + e^{i\Phi} |0\rangle_a |\alpha e^{i\theta}\rangle_b$, a continuous variable analog of the $N_{00N}$ states, which are, in fact, superpositions of $N_{00N}$ states, can lead to Heisenberg-limited phase measurement uncertainties $\Delta \varphi \simeq 1/\bar{n}$, assuming the phase shift is small. Another advantage of the parity-detection approach over that involving coincident counting is that the former generally has a higher signal-to-noise ratio than is the case for coincident detection.

In the present chapter, we consider interferometry performed by mixing coherent light with photon-subtracted squeezed vacuum states (SVS) and show that this approach leads to an improvement in performance over the case where the coherent light and squeezed vacuum light are mixed. Photon-subtracted or added squeezed vacuum states have been intensely studied theoretically and experimentally over that past 15 years or so [97] [128]. This work was an extension of the theoretical work of Agarwal and Tara [11], which showed that excitations on coherent states, i.e., adding photons to coherent states via the operations $\left(\hat{a}^\dagger\right)^m |\alpha\rangle$, produce states of strong nonclassical properties. Zavatta et. al [8] [9] have experimentally studied the addition of photons to coherent states. Our interest in the application of the photon subtracted squeeze vacuum states stems, in part, from the fact that for long interaction times (needed for higher average photon numbers) the fluctuations in the pump beam tend to degrade the purity of the squeezed vacuum state itself as well as the photon-subtracted state. As we point
5.1 Generating the Photon-Subtracted Squeezed Vacuum

Previously, Carranza and Gerry [31] studied the prospect of performing sub-standard-quantum-limit interferometry with states obtained by subtracting identical numbers of photons from both modes of a two-mode squeezed vacuum state. In contrast to that scheme, the current proposal requires photon subtractions from only a single-mode squeezing vacuum before the resulting light is mixed with coherent light at a beam splitter.

![Sketch of the proposed parity-measurement-based interferometric scheme for mixing of coherent and photon-subtracted squeezed vacuum states. A beam splitter with low reflectance and a single-photon resolution photo-detector are used to subtract $p$-photons from the input squeezed vacuum state. The angle $\varphi$ represents the phase shift to be detected, and the photon-number parity measurements are performed on the output $b$-mode.](image)
We consider a Mach-Zehnder interferometer (MZI), as pictured in Fig. (5.1), with input coherent and $p$-photon subtracted squeezed vacuum states, denoted $|\alpha\rangle_a$ and $|r, p\rangle_b$, respectively, where

$$|\alpha\rangle_a = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_a, \quad |r, p\rangle_b \sim \hat{b}^p |r, 0\rangle_b,$$

(5.1)

where $|r, 0\rangle_b$ is the single-mode squeezed vacuum state in the $b$-mode is given by [31] $|r, 0\rangle_b = \hat{S}_b(r) |0\rangle_b$. Here the squeeze operator, $\hat{S}_b(r)$, is given by

$$\hat{S}_b(r) = \exp \left[ \frac{1}{2} r \left( \hat{b}^2 - \hat{b}^\dagger \right) \right],$$

(5.2)

where $r$ is the squeeze parameter satisfying $0 \leq r < \infty$. We can express this in terms of the number states of the $b$-mode as

$$|r, 0\rangle_b = \sum_{N=0}^{\infty} C_N |N\rangle_b$$

(5.3)

where the coefficients in Eq. (5.3) are given by Eq. (4.34) [57]. Note that only the even number states are populated. The average photon number for the squeezed vacuum state is $\bar{n} = \sinh^2 r$. The $p$-photon subtracted SVS we can write in normalized form as

$$|r, p\rangle_b = \sum_{N=0}^{\infty} B_N^{(p)} |N\rangle_b,$$

(5.4)

where

$$B_N^{(p)} = \Omega_p C_{N+p} \sqrt{(N)_p},$$

(5.5)

where $(x)_m = x(x-1)(x-2) \ldots (x-m+1)$ is the falling factorial. The normalization factor is then given by

$$\Omega_p = \left[ \sum_{q=0}^{\infty} |C_{q+p}|^2 (N)_p \right]^{-1/2}.$$
Next we consider the photon statistics of the $p$-photon subtracted squeezed vacuum, initially in the input $b$-mode of the interferometer. We start by discussing the photon number probability distribution.

### 5.1.1 Photon Number Distribution and Projective Measurements

The photon number probability distribution obtained upon the subtraction of $p$ photons from the squeezed vacuum states will be given by $P^p(N) = |D_N^{(p)}|^2$, $N = 0, 1, 2...$. Of course, the average photon number for the photo-subtracted state is given by

\[ \overline{N} = \sum_{N=0}^{\infty} N \cdot P^p(N) \]

**Figure 5.2:** Photon number probability distribution for the $p$-photon subtracted squeezed vacuum state for (a) $p = 0$, (b) $p = 1$, (c) $p = 2$ and (d) $p = 3$. For all cases considered, $r = 0.9$. 

Next we consider the photon statistics of the $p$-photon subtracted squeezed vacuum, initially in the input $b$-mode of the interferometer. We start by discussing the photon number probability distribution.
\[ \bar{N} = \sum_{N=0}^{\infty} |B_N^p|^2 N. \] (5.7)

For the case \( p = 0 \), the average photon number is given by \( \bar{N} = \sinh^2 r \). In Fig. (5.2), we plot distributions for the choice of \( r = 0.9 \) for subtracted photon numbers \( p = 0, 1, 2, \) and 3, and state the corresponding average photon number of the states in the upper right of each graph. Fig. (5.2a) represents the distribution for the squeezed vacuum state with \( r = 0.9 \) for which the average photon number is \( \bar{N} \sim 1.05 \). As photons are subtracted, the parity of the state shifts back and forth between even and odd photon numbers, as we would expect, but we notice that the peak of the distribution shifts towards a higher photon number, the average photon number counter intuitively increasing with increasing numbers of subtracted photons. Fig. (5.2d) contains the distribution after the subtraction of three photons from the squeezed vacuum states, yet the average photon number has increased over eightfold. The fact that the act of photon subtraction, or "photon annihilation", via the \( \hat{a} \) operator can sometimes counter-intuitively lead to the creation of quanta (an increase in average energy) was noted a few years ago by Mizrahi and Dodonov [106] and earlier by Ueda et. al [102]. These authors showed that if \( \bar{n}_i \) is the average photon number of the state before subtracting one photon, then the average photon number after subtraction, \( \bar{n}_{\text{sub}} \), is given by \( \bar{n}_{\text{sub}} = \bar{n}_i - 1 + F_i \) where \( F_i \) is the Fano factor of the initial state: \( F_i = \langle (\Delta \hat{n})^2 \rangle_i / \bar{n} \). The Fano factor can be rewritten in terms of the Mandel \( Q \) parameter, a means for which one can characterize the photon statistics of a given single-mode state, as \( F_i = Q + 1 \). If the initial state is a coherent state, for which \( Q = 0 \) signifying it has a Poissonian photon number distribution, then \( F_i = 1 \) and thus \( \bar{n}_{\text{sub}} = \bar{n}_i \). Zavatta et. al [7] have experimentally investigated the effects of single-photon subtraction on coherent states and have shown the invariance of the states under this action. For a state with sub-Poissonian statistics, characterized by \( Q < 0 \), \( F_i < 1 \) one has \( \bar{n}_{\text{sub}} > \bar{n}_i \). Indeed for a number state \( |n\rangle \), the fluctuations in the photon number are zero, and thus \( \bar{n}_{\text{sub}} = n - 1 \), as expected. For states with super-Poissonian statistics, \( Q > 1 \), \( F_i > 1 \), one has \( \bar{n}_{\text{sub}} > \bar{n}_i \). Of course, the distribution for the single-mode squeezed vacuum state, as is well known,
is super-Poissonian.

The photon subtraction operation can be carried out by the use of beam splitters with low reflectivity and with photon-number resolved projective measurements. Specifically, the detection of $p$-photons in the reflected beam, which must be performed by photon counters with single-photon resolution, projects the transmitted beam into the desired $p$-photon subtracted SVS. The detection of $p$-photons heralds this projection. As was pointed out by Dakna et al. [97], such a scheme can be used for the generation of Schrödinger cat states given as superpositions of the approximate form $|\beta\rangle \pm |\beta\rangle$, the even and odd cat states, where $|\pm\beta\rangle$ are low amplitude coherent states. Because of the low amplitudes of the component coherent states, the superposition states are sometimes known as Schrödinger kitten states. Ourjoumtsev [64] performed an experiment that subtracted one photon from a squeezed vacuum to generate an odd cat state $\sim |\beta\rangle - |\beta\rangle$ with a mean photon number of $\sim |\beta|^2 = 0.8$. The detection of the reflected photon was accomplished with an avalanche photodiode (APN), which, though not strictly a number-resolving detector, could be used as such to detect one photon, assuming a beam splitter reflectivity small enough to render the probability of subtracting two photons negligible. More recently, Gerrits et al. [128] performed an experiment wherein up to three photons were subtracted from a squeezed vacuum state creating low amplitude even or odd cat states and reaching, in the case of three photons subtracted, a mean average photon number of about 2.75. The counting of the number of reflected photons in this experiment was performed using high-efficiency photon-number-resolving superconducting transition edge detectors [1] [37]. These experiments demonstrate the ability to subtract low numbers of photons from a squeezed state. For applications to interferometry, we do not require the produced states to be low amplitude cat-like states. In fact, the higher the amplitude of the photon-subtracted SVS the better as long as fluctuations in the pump beam do not degrade the purity of the SVS produced by the downconversion process [36].

As we have mentioned, the $p$-photons are subtracted from the squeezed vacuum state via a low-reflectance beam splitter placed prior to the MZI. As a consequence, the
Chapter 5. Interferometry using the Photon Subtracted Squeezed Vacuum

Figure 5.3: The input state to the photon-subtracting beam splitter is taken to be $|r, 0\rangle_b \otimes |0\rangle_{b'}$. The beam splitter reflectivity and transitivity are given in terms of the angle $\theta$. As the angle $\theta \to 0$ and a detection of $p$-photons is made, the resultant single-mode state in the $b$-mode is the $p$-photon subtracted squeezed vacuum state, $|r, p\rangle_b$.

The state will not be continuously generated, but will be generated in bursts, as the generation of the state depends on the detection of $p$ photons from the photon-number resolving detector. This is not a trivial detail, as one needs to consider choosing parameters that maximize the probability of generating the desired state. Likewise, as the reflectivity of the photon-subtracting beam splitter increases, it will affect the statistics of the resulting single mode state, even if the desired number of photons are detected. We will discuss these affects in Chapter 6 in greater detail, but it is worth noting here, as the generation of the $p$-photon subtracted squeezed vacuum hinges on a projective measurement of $p$ photons. In order to properly discuss the generation of the photon-subtracted squeezed vacuum, we must first properly model the photon-subtracting beam splitter.

Consider, as the input state to the photon-subtracting beam splitter (p.s.-BS), the state

$$|\text{in, p.s.- BS}\rangle = |r, 0\rangle_b \otimes |0\rangle_{b'},$$  \hspace{1cm} (5.8) \hspace{1cm} as shown in Fig. (5.3). The action of the beam splitter is such that the resulting two mode state is given by
5.1. Generating the Photon-Subtracted Squeezed Vacuum

\[ \bar{n} = 1.05 \]
\[ r = 1.2 \]
\[ \theta = \pi / 4 \]
\[ P_{\text{Detect}} = 78\% \]

\[ \bar{n} = 4.16 \]
\[ r = 1.2 \]
\[ \theta = \pi / 4 \]
\[ P_{\text{Detect}} = 14\% \]

\[ \bar{n} = 6.12 \]
\[ r = 1.2 \]
\[ \theta = \pi / 4 \]
\[ P_{\text{Detect}} = 5\% \]

\[ \bar{n} = 8.89 \]
\[ r = 1.2 \]
\[ \theta = \pi / 4 \]
\[ P_{\text{Detect}} = 2\% \]

Figure 5.4: The photon-number probability distribution for \( p = 0, 1, 2 \) and 3, with \( r = 1.2 \). Here the beam splitter reflectivity is given by \( R = \sin^2(\theta/2) \) and \( P_{\text{Detect}} \) represents the probability of successfully detecting \( p \) photons.
where we use the Schwinger realization of the su(2) algebra to describe the beam splitter (see Appendices A and B). The probability of detecting $p$ photons in the output $b'$-mode, $P_{\text{Detect}}$, is then given by the expectation value of the $p$-photon projection operator

\[ \hat{P}_{p} = |p\rangle_{b'} \langle p|, \]

or simply put

\[ P_{\text{Detect}} = \langle \text{out, p.s.-BS} | \hat{P}_{p} | \text{out, p.s.-BS} \rangle = \langle \text{in, p.s.-BS} | e^{i\theta \hat{J}_1} | p\rangle_{b'} \langle p| e^{-i\theta \hat{J}_1} | \text{in, p.s.-BS} \rangle. \] (5.10)

We once again plot the photon-number distribution in Fig. (5.4), only this time special attention is paid to the probability for which the detection is made. Notice for $r = 1.2$ and $\theta = \pi/4$, the probability of generating the $p$-photon subtracted squeezed vacuum drops off considerably for increasing $p$. In fact, the case of $p = 0 - 3$ makes up 99% of the possible outcomes. It is also worth pointing out that the probabilities for detecting $p = 1 - 3$ drop off considerably faster as $\theta \to 0$; the limit for which one can obtain the $p$-photon subtracted squeezed vacuum state. In general, however, the resulting single mode state found in the $b$-mode after photon-number detection will depend on the beam splitter angle $\theta$, as we will show when we explore this technique of quantum state engineering in greater depth in Chapter 6.

Before moving on to the next section, we point out that the one-photon-subtracted ($p = 1$) SVS is equivalent to a squeezed one-photon state, as has been shown by Biswas and Agarwal [20] and demonstrated in Eq. (4.42). However, there is no equivalence between the squeezed number states and the corresponding subtracted SVS for $p > 1$. We now turn our attention to the use of the $p$-photon subtracted squeezed vacuum state in quantum optical phase-shift measurements.
5.2 Sensitivity of Phase Shift Measurements

The photon-number subtracted SVS are to be mixed with coherent states at the first beam splitter of the MZI as pictured in Fig. (5.1). Both beam splitters of the device are assumed to be 50:50. As was shown by Yurke et al [14], we can describe the action of a beam splitter as a rotation using the well known Schwinger realization of the su(2) algebra, explained in Appendices A and B. In terms of the angular momentum basis, $|j, m\rangle$, our input state can be written as

$$|\text{in}\rangle = |\alpha\rangle_a \otimes |r, p\rangle_b = \sum_{j=0,1/2,..}^{\infty} G_{j,m} |j,m\rangle,$$

where $G_{j,m} = A_{j+m} B_{j-m}^{(p)}$ and where $A_n$ are the usual coherent state probability amplitudes given in Eq. (2.51). We assume that the beam splitters are constructed so that the input state $|\text{in}\rangle$ of the MZI is related to the output state $|\text{out}\rangle$ according to

$$|\text{out}\rangle = e^{i\pi \hat{J}_1} e^{-i\varphi \hat{J}_3} e^{-i\pi \hat{J}_1} |\text{in}\rangle,$$

where the factors $\exp[\pm i(\pi/2) \hat{J}_1]$ represent the actions of the 50:50 beam splitter and where the factor $\exp[-i\varphi \hat{J}_3]$ represents the relative phase shift between the two arms, the angle $\varphi$ being the phase shift to be estimated. This set of operators constitutes a particular choice of beam-splitter types defined by the phase shift picked up by the reflected beam. The arrangement we have chosen is that discussed by Yurke et al [14]. Equivalently we can write for the output state

$$|\text{out}\rangle = e^{-i\varphi \hat{J}_2} |\text{in}\rangle,$$

where we have used the Baker-Hausdorff identity (see Appendix B). The photon-number parity operator of the $b$-mode is $\Pi_b = (-1)^{b^* b} = \exp[i\pi (\hat{J}_0 - \hat{J}_3)]$, and it’s expectation value with respect to the output state is
\[
\langle \hat{\Pi}_b \rangle = \langle \text{in}|e^{i\varphi \hat{J}_2}e^{i\pi (\hat{J}_0 - \hat{J}_3)}e^{-i\varphi \hat{J}_2}|\text{in}\rangle \\
= \langle \text{in}|e^{i\varphi \hat{J}_2}e^{i\pi \hat{J}_3}e^{-i\varphi \hat{J}_2}e^{i\pi \hat{J}_0}|\text{in}\rangle .
\] (5.14)

Resolving unity twice in terms of the angular momentum states, that is,
\[
\hat{I} = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} |J, M\rangle \langle J, M|,
\]
we obtain
\[
\langle \hat{\Pi}_b \rangle = \sum_{j=0, 1/2}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \sum_{m''=-j}^{j} (-1)^j G_{j,m}'G_{j,m}G_{j,m''}d_{j,m'',m}^j(\varphi) d_{j,m',m'}^j(-\varphi),
\] (5.15)

where, as usual, \(d_{j,m',m}^j(\varphi) = \langle j, m'|e^{-i\varphi \hat{J}_2}|j, m\rangle\) are the Wigner-\(d\) matrix elements (see Appendix C).

To obtain the sensitivity of phase-shift measurements with our input states, we use the quantum Fisher information \(F_Q\) to find the maximum level of sensitivity by the Cramér-Rao bound as given by Eq. (2.20). The quantum Fisher information, as has been shown by Ben-Aryeh [17] for a pure state is demonstrated in Eq. (2.50), and given by
\[
F_Q[\hat{\rho}(\varphi)] = 4\langle (\Delta \hat{J}_2^2) \rangle_{\text{in}},
\] (5.16)

where \(\varphi(\varphi) = e^{-i\varphi \hat{J}_3}e^{-i(\pi/2)\hat{J}_1}|\text{in}\rangle\) is the state vector just before the second beam splitter of the MZI and where \(\psi '(\varphi) = \partial_{\varphi} |\psi (\varphi)\rangle\). In terms of the input state, the quantum Fisher information is given as \(F_Q = 4\langle (\Delta \hat{J}_2^2) \rangle_{\text{in}}\), in agreement with Eq. (2.50).

On the other hand, for the measurement of photon number parity, the phase uncertainty based in the error propagation calculus is given by
\[
\Delta \varphi = \frac{\Delta \hat{\Pi}_b}{|\partial \langle \hat{\Pi}_b \rangle|},
\] (5.17)
5.2. Sensitivity of Phase Shift Measurements

where once again $\Delta \bar{\Pi}_b = \sqrt{1 - \langle \bar{\Pi}_b \rangle^2}$. As noted earlier, the qCRB, determined by the quantum Fisher information, yields the optimum phase-shift measurement sensitivity for pure, path symmetric, input states. That is, photon-number parity measurements saturates the qCRB.

First, we study the expectation value of the parity operator as a function of the phase shift $\varphi$ as given by Eq. (5.15). Fixing $\alpha = 2$, in Fig. (5.5) we plot this expectation value against $\varphi$ for $p = 0, 1, 2$ and 3. The increasing narrowness of the maxima or minima at $\varphi = 0$ as more photons are subtracted indicates that, with respect to resolution, the mixing of coherent light with a photon-subtracted SVS outperforms the SVS mixed with coherent state for the same choice of field parameters. Clearly the curves narrow for increasing $p$. It’s also worth pointing out that the value at $\varphi = 0$, that is $\pm 1$, reflects parity of the initial $p$-photon subtracted squeezed vacuum state; that is, the value of 1 is obtained for $p \in \text{Even}$ and the value $-1$ is obtained for $p \in \text{Odd}$.

5.2.1 Phase Uncertainty plotted for Fixed Parameters

We investigate the effects of photon subtraction for a wide range of parameters $p, r$ and $\alpha$, where we set $\varphi = 10^{-4}$. In Fig. (5.6) we fix the squeeze parameter at $r = 0.9$ and plot

![Figure 5.5: Expectation value of the parity operator for $p = 0, 1, 2$ and 3 with $\alpha = 2$ and (a) $r = 0.3$ and (b) $r = 0.9$.](image-url)
Figure 5.6: Plots of the phase uncertainty $\Delta \varphi$ with $\varphi = 10^{-4}$ for (a) a fixed squeeze parameter $r = 0.9$, with a varying $\alpha$ and (b) a fixed coherent state amplitude $\alpha = 2$, with a varying $r$. 
the phase uncertainty obtained from Eq. (5.17) as a function of $|\alpha|$ for the photon subtractions $p = 0, 1, 2, 3$ and 4. In all cases, we find a reduced phase uncertainty over that obtained by mixing coherent and squeezed vacuum states ($p = 0$) for the entire range of $|\alpha|$. We notice that a big jump to a lower phase uncertainty occurs after subtracting just one photon. This is to be expected as the lowest photon number state within the one one-photon subtracted squeezed vacuum state is the one-photon state. Only odd photon numbers are populated, and we have shown in the previous chapter, Chapter 4, there is a dramatic effect on interferometry when mixing a single photon with coherent light at a beam splitter; that is, a significant reduction in noise below the SQL. After one-photon subtraction from the SVS, the one-photon state has a high probability of occupation as can be seen in Fig. (5.2). For higher odd numbers of photon subtractions, the one-photon state has a lower likelihood of being occupied as the average photon number migrates to a higher value. In Fig. (5.6b), we have plotted the phase uncertainty by setting $\alpha = 2$ and varying $r$ for the same set of $p$ values. Again we notice the dramatic effect of subtracting just one photon, and we note that the subtraction of an odd number of photons crosses below the adjacent cases of even photon number subtraction, reinforcing the effect of having a nonzero probability of occupancy of the one-photon state. In summary, for fixed resources with respect to the initial light beams, i.e., for fixed values of $\alpha$ and $r$, we find a reduction in noise as a result of photon subtractions.

5.2.2 Phase Uncertainty against Total Average Photon Number

In order to compare the reduction in noise obtained with our states, we need to compare the obtained phase uncertainties along with the corresponding standard quantum and Heisenberg limits by plotting these quantities against the total average photon number assuming a very small phase shift. Setting $\varphi = 10^{-4}$, we plot the phase uncertainties against the total average photon number passing through the interferometer for fixed $r = 0.9$ and increasing $\alpha$ for $p = 0, 1, 2,$ and 3 in Fig. (5.7). It is not possible to make a direct comparison for different values of $p$ in this context as the average photon number varies non-linearly with $p$ for a given choice of $r$ and $\alpha$. Nevertheless, we still notice
Figure 5.7: Plots of the phase uncertainty $\Delta \varphi$ against total average photon number, along with the corresponding curves for the SQL and HL for $r = 0.9$, and for (a) $p = 0$, (b) $p = 1$, (c) $p = 2$ and (d) $p = 3$. Only the parameter $\alpha$ is being changed, but as the relationship of this parameter with the total average photon number is not linear, we indicate various values of $\alpha$ along the curves. For Fig. (5.7a), the green line denotes the minimum phase uncertainty obtained via calculation of the qCRB using the quantum Fisher information; it is included to once again show equivalence between this method and parity-based detection methods.
Figure 5.8: Plots of the joint photon-number probability distribution after the mixing of coherent states of amplitude $\alpha = 2$ and a photon-subtracted squeezed vacuum state with $r = 0.9$ for (a) $p = 1$ and (b) $p = 3$. 
the dramatic change in performance upon subtraction of one photon from the squeezed vacuum state. For the lower value of $r$, we note that the subtraction of two photons actually increases the noise level over that of the one-photon subtracted case; this is likely owing to the fact that the vacuum state has a higher occupation probability than in the corresponding case of large value of $r$.

Last, we examine the joint photon number distribution just after the first beam splitter of the MZI, which is determined from

$$P(n_1, n_2|p) = \left| \langle n_1, n_2 | e^{-i \pi \hat{J}_1} | \alpha \rangle a | r, p \rangle b \right|^2$$

(5.18)

In Fig. (5.8) we plot this distribution for $\alpha = 2$, $r = 0.9$ with (5.8a) $p = 1$ and (5.8b) $p = 3$. In Fig. (5.9) we repeat these plots but now we set $\alpha = 6$. In both cases, we can see that the distributions are bimodal as expected; the states along the borders are highly populated. We note here that the distributions plotted in Fig. (5.8) and (5.9) correspond to particular points in the phase uncertainty plots of Fig. (5.7). For higher $p$ and $\alpha$, the distributions display greater populations in the interior of the $\{n_1, n_2\}$-plane, but the corresponding phase uncertainties are still very close to their respective Heisenberg limits. Evidently, as in the case of the input twin-Fock states, the 'interior' populations that occur as a result of beam splitting have little effect on the sensitivity of the measurements. In fact, they may actually be beneficial in the same manner as are the arcsine states in that the loss of a few photons does not reveal the location of all the other photons and thus may act as a hedge against losses [105]. In the case of N00N states and their continuous analogs, the loss of one photon (action of the lowering operator of one of the modes) does reveal the location of all the other photons, and thus entanglement is destroyed. However, Joo et. al [75] have shown that superpositions of the N00N states having the form of the entangled coherent states discussed in the introduction are more robust to losses that are the N00N states themselves.

In this paper, we have generalized the idea of mixing coherent and squeezed vacuum light upon a 50:50 beam splitter to that of mixing coherent and photon-subtracted squeezed vacuum states for quantum optical interferometry. We have also touched
5.2. Sensitivity of Phase Shift Measurements

Figure 5.9: Plots of the joint photon-number probability distribution after the mixing of coherent states of amplitude $\alpha = 6$ and a photon-subtracted squeezed vacuum state with $r = 0.9$ for (a) $p = 1$ and (b) $p = 3$. 
upon the role of probabilistic determination in the generation of the $p$-photon subtracted squeezed vacuum state. Indeed, this method of quantum state engineering can be generalized and implemented to generate a wide range of quantum states of light. This is the subject of our next chapter, where we will discuss photon catalysis as a means of generating non-classical (and non-Gaussian) states of light. Furthermore, we will also discuss some particular non-classical properties of the resultant states. We will also discuss a method by which a correlated two-mode state of light, namely the pair coherent state, can be engineered in a laboratory setting with currently-existing technology.
Chapter 6

Photon Catalysis and Quantum State Engineering

Some time ago, Lvovsky and Mlynek [96] reported on the preparation of coherent superposition states of the (unnormalized) form \( t |0\rangle + \alpha |1\rangle \) in a single mode of the quantized electromagnetic field. This was performed by mixing a weak-amplitude (continuous-variable) coherent state \( |\alpha\rangle \), \((\alpha \ll 1)\) with a one-photon Fock state \( |1\rangle \) at a beam splitter of small transmissivity \( |t| \) followed by the detection of a single photon in one of the output beams of the beam splitter. The detection projects the other output beam into the state \( (t |0\rangle + \alpha |1\rangle) / \sqrt{|t|^2 + |\alpha|^2} \). Because the method involves the injection and subsequent detection of the same number of photons (one in this case) it was labeled quantum-optical catalysis. Recently, Bartley et al. [132] have extended the catalysis method in two ways: They consider cases of higher-amplitude coherent states and they considered cases where an arbitrary number of photons, \( k \), are mixed with the coherent state and then detected at one of the beam splitter outputs. By varying the transmissivity of the beam splitter, the authors of [132] found that they could produce a variety of non-classical states of the single-mode field in the other output beam of the beam splitter. They found states possessing quadrature squeezing, antibunched and super-bunched photon statistics, and states with over 90% fidelity to displaced coherent superposition states. The experiment reported in [132] involved just one photon mixed with coherent states with amplitudes in the range \( \alpha = 0.9 \) to \( \alpha = 2.7 \). Recently, Hu et al. [93] investigated the case where the number of photons in the input \( b \)-mode, \( m \),
is greater than 1, but the input $a$-mode state remains a coherent state. They coined the name "Laguerre polynomial excited coherent state" (LPECS) as the resultant photon catalyzed state. The state is then investigated for quadrature squeezing and sub-Poissonian statistics. They also investigated the Wigner function as a means of quantifying non-Gaussianity as well as analyze the decoherence in a thermal environment.

In general, photon catalysis is a method of photonic quantum state engineering for generating multiphoton continuous-variable non-classical and non-Gaussian states of a quantized single-mode field. Because there are many adjustable parameters, the coherent state amplitude, the number of photons $k$ used for the catalysis, and the choice of beam splitters of different reflectivities and transmissivities, a great variety of multiphoton states can be produced that could be suitable for applications in quantum information processing. In essence, one can make designer non-Gaussian states by carefully choosing values of the relevant parameters mentioned above. There is clearly a connection between this scheme for quantum state engineering and the schemes involving either photon addition or photon subtraction [14]. However, photon catalysis involves both addition and subtraction, performed sequentially, to produce non-Gaussian states.

In the present paper we theoretically make further extensions of the photon catalysis method. To begin with, we first replace the input coherent state with another continuous variable single-mode state, namely the squeezed coherent state, this being the most general form of a single-mode, pure, Gaussian state. Such a state contains as limiting cases the coherent state, a classical-like state, and the squeezed vacuum state, a Gaussian non-classical field state. Extension to the squeezed coherent state is motivated by the expectation that photon catalysis performed on a non-classical Gaussian state produces a non-Gaussian state with an even greater degree of non-classicality. We also consider cases where the number of photons detected at one of the beam splitter outputs is different than the number of photons mixed with the continuous state at input. Strictly speaking, this is not photon catalysis, but it is a natural variation on it that allows for one more degree of freedom to engineer non-Gaussian continuous variable states.
We consider one further extension to the photon catalysis scheme, this one concerned with the manipulation of two-mode quantized fields. Specifically, we consider the case of photon catalysis performed on one mode of the two-mode squeezed vacuum state (TMSVS). The TMSVS state is a Gaussian state consisting of a superposition of the form \( \sum_n^\infty C_n |n\rangle_a |n\rangle_b \) wherein the photon number states of each of the modes are tightly paired. The reduced density operator for each of the modes has the photon statistics of a thermal field [57]. When one of the modes of this field state is subjected to photon catalysis, the tight pairing of the photon states is preserved, such that we obtain a different superposition \( \sum_n^\infty D_n |n\rangle_a |n\rangle_b \) where the photon statistics of the total field is altered in such a way to render the output state non-Gaussian. States of this form are desirable since non-Gaussian states of this type could be used to perform loophole free homodyne tests of Bell’s inequality using quadrature-phase measurements. The TMSVS is unsuitable because its Wigner function, being Gaussian, is never negative, and a Bell inequality violation obtained with tests using quadrature-phase measurements requires a Wigner function that takes negative values in some regions of phase space [38].

The plan of this chapter is as follows: First, we describe the photon catalysis technique for an arbitrary single-mode field using the language of the SU(2) description of a beam splitter [87] and apply this method to the single-mode squeezed coherent state. Next, we will consider the case where one mode of the TMSVS is subjected to photon catalysis. We will then go into detail about the generation of a two-mode correlated state, the pair coherent state, via a state projective method. We will close the chapter with some brief remarks.

### 6.1 Photon Catalysis using Squeezed Coherent Light

A schematic for the photon catalysis procedure is given in Fig. (6.1). We consider mixing a single-mode squeezed coherent state and a number state of a given photon number \( q \) at a beam splitter with variable reflectivity given in terms of the angle \( \theta \), where the transmittance \( t \) and reflectance \( r \) of the beam splitter is given in terms of the beam splitter angle as \( t = \cos(\theta/2) \) and \( r = i \sin(\theta/2) \), respectively. In the limiting cases, \( \theta = 0 \)
corresponds to a fully transmitting beam splitter while $\theta = \pi$ corresponds to a fully reflecting beam splitter.

The input squeezed coherent state in the $a$-mode is given by

$$|\alpha, r\rangle_a = \hat{S}(\xi) |\alpha\rangle_a = \hat{S}(\xi) \hat{D}(\alpha) |0\rangle_a$$ (6.1)

where

$$|\alpha\rangle_a = \hat{D}(\alpha) |0\rangle_a = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_a$$ (6.2)

is the usual coherent state, with $\hat{D}(\alpha) = \exp \left[ \alpha \hat{a}^\dagger - \alpha^* \hat{a} \right]$ being the displacement operator and $\hat{S}(\xi) = \exp \left[ \frac{i}{2} (\xi^* \hat{a}^2 - \xi \hat{a}^\dagger^2) \right]$ being the single mode squeeze operator where $\xi = re^{i\phi}$. Expanded in terms of the number states, the squeezed coherent state reads

$$|\alpha, r\rangle_a = \sum_{n}^{\infty} C_n |n\rangle_a$$ (6.3)

where

\[\text{FIGURE 6.1: Schematic of the photon catalysis procedure for an input squeezed coherent state.}\]
6.1. Photon Catalysis using Squeezed Coherent Light

\[ C_n = \frac{1}{\sqrt{n!}} e^{-\frac{1}{2}(|\alpha|^2 - \alpha^2 \nu^2)} \frac{1}{\sqrt{2\mu}} \left( \sqrt{\frac{\nu}{2\mu}} \right)^n H_n \left( \frac{\alpha}{\sqrt{2\mu} \nu} \right), \]  

(6.4)

with \( \mu = \cosh r \) and \( \nu = e^{i\phi} \sinh r \). The input \( b \)-mode is occupied by the \( q \)-photon number state, \( |q\rangle_b \). The technique of photon catalysis involves performing a projective measurement at the output \( b \)-mode of the same number of photons initially occupying the \( b \)-mode prior to beamsplitting; this is achieved through the use of single-photon resolving detectors. It should be noted that such detectors, while not perfectly efficient, do exist [39]. The advantage of this method lies in the fact that, in general, the action of the beam splitter on an input state of the form \( |in\rangle = |\psi\rangle_a \otimes |q\rangle_b \) will result in entanglement between the two modes. As a result, a subsequent projective measurement made on the output \( b \)-mode will affect the state projected into the output \( a \)-mode.

In order to describe the beam splitter transformation, we use the Schwinger realization of the \( \text{su}(2) \) algebra with two sets of boson operators, as described in Appendix B [14]. The input state can be written in the number state basis as

\[ |in\rangle = |\alpha, r\rangle_a \otimes |q\rangle_b = \sum_{n=0}^{\infty} C_n |n\rangle_a |q\rangle_b. \]  

(6.5)

For convenience, we transform our input state from the number state basis to angular momentum basis, according to \( |n\rangle_a \otimes |q\rangle_b \rightarrow |j, m\rangle \) with \( j = (n + q) / 2 \) and \( m = (n - q) / 2 \), yielding

\[ |in\rangle = \sum_{n=0}^{\infty} C_n |n\rangle_a |q\rangle_b = \sum_{j=q/2,..}^{\infty} \sum_{m=-j}^{j} C_{2j-q} |j, m\rangle \langle j, m| e^{-i\theta \hat{J}_1} |j, j-q\rangle. \]  

(6.6)

Working in the Schrödinger picture, we write the state after the action of the beam splitter as

\[ |out\rangle = e^{-i\theta \hat{J}_1} |in\rangle \]

\[ = \sum_{j=q/2,..}^{\infty} \sum_{m=-j}^{j} C_{2j-q} |j, m\rangle \langle j, m| e^{-i\theta \hat{J}_1} |j, j-q\rangle \]  

(6.7)
where we have inserted a complete set of states, \( \hat{I}_j = \sum_{m=-j}^{j} |j, m \rangle \langle j, m| \). We can simplify the matrix elements of Eqn. (6.10) using the Baker-Hausdorf identity, see Appendix B, yielding

\[
\langle j, m | e^{-i\theta \hat{J}_1} | j, j - q \rangle = \langle j, m | e^{i\frac{\theta}{2} \hat{J}_3} e^{-i\theta \hat{J}_2} e^{-i\frac{\theta}{2} \hat{J}_3} | j, j - q \rangle = i^{m-j+q} d_{m,j-q}^j (\theta) \tag{6.8}
\]

where \( d_{m,j-q}^j (\theta) \) are the Wigner-\( d \) matrix elements, given formally by \( d_{m,m'}^j (\theta) = \langle j, m | e^{-i\theta \hat{J}_2} | j, m' \rangle \), see Appendix C. Reverting back to number state basis the final output state after beam splitting is given by

\[
|\text{out}\rangle = \sum_{j=q/2,..}^{\infty} \sum_{m=-j}^{j} \Gamma_{j,m} (\theta) | j + m \rangle_a | j - m \rangle_b \tag{6.9}
\]

where the coefficients \( \Gamma_{j,m} (\theta) \) are

\[
\Gamma_{j,m} (\theta) = C_{2j-q} i^{m-j+q} d_{m,j-q}^j (\theta) . \tag{6.10}
\]

Next we determine the probability of successful catalysis, that is, the probability of detecting \( l \) photons in the output \( b \)-mode. This is just the expectation value of the \( b \)-mode projection operator for \( l \) photons,

\[
P_{\text{cat}} = \langle \text{out} | l \rangle_b \langle l | \text{out} \rangle = \sum_{j=\max\{q,l\}/2,..}^{\infty} | \Gamma_{j,m} (\theta) |^2 . \tag{6.11}
\]

The probability of detecting \( l \) photons in the output \( b \)-mode is plotted in Fig. (6.2) for several different cases. The normalized single-mode state after catalysis is given by

\[
|\psi\rangle_a = \frac{b \langle l | \text{out} \rangle}{\sqrt{P_{\text{cat}}}} = \sum_{j=\max\{q,l\}/2,..}^{\infty} \Gamma_{j,j-l} (\theta) | 2j - l \rangle_a . \tag{6.12}
\]

The probability of generating this state will depend on the average number of photons incident on, as well as the reflectivity of, the beam splitter.
6.1. Photon Catalysis using Squeezed Coherent Light

Figure 6.2: Probability of detecting $l$ photons in the output $b$-mode for (a) an input coherent state, (b) an input squeezed coherent state, and (c) an input squeezed coherent state.
6.1.1 Photon Statistics — The Mandel $Q$ Parameter

Now we examine the non-classical properties of the produced state. We first consider the nature of the photon number distributions as characterized by the Mandel $Q$ Parameter given by

$$Q = \frac{\langle (\Delta \hat{n}_a)^2 \rangle}{\langle \hat{n}_a \rangle} - 1$$  \hspace{1cm} (6.13)

where the operator $\hat{n}_a$ is the photon number operator for the $a$-mode. Recall for a Poisson distribution, such as the case for a coherent state, $Q = 0$. If $Q > 0$ for some state, we have super-Poissonian statistics and if $Q < 0$ we have sub-Poissonian statistics. Only the latter case indicates non-classicality. For photon number states, the photon number variance $\langle (\Delta \hat{n}_a)^2 \rangle$ is equal to zero, thus giving us the lower bound $Q = -1$. Any state falling within the range $-1 \leq Q < 0$ is said to have sub-Poissonian statistics and is therefore non-classical [57]. The Mandel $Q$ parameter is plotted in Fig. (6.3) for different cases. The Mandel $Q$ parameter is related to the normalized second order correlation function, at zero-time delay $g^{(2)}(0)$ according to

$$g^{(2)}(0) = \frac{Q}{\langle \hat{n}_a \rangle} + 1.$$  \hspace{1cm} (6.14)

In terms of this quantity, sub-Poissonian statistics occurs when $g^{(2)}(0) < 1$. It should be noted that this relation is often taken to be a signal of photon anti-bunching. Photon anti-bunching is also a non-classical effect but is not identical to sub-Poissonian statistics. Photon anti-bunching occurs whenever the second-order correlation function at delay time $\tau$ satisfies the relation $g^{(2)}(\tau) > g^{(2)}(0)$ [125] [103], that is, the function $g^{(2)}(\tau)$ must have a positive slope for times $\tau$ after $\tau = 0$. To properly calculate $g^{(2)}(\tau)$, one requires a multi-mode representation of the fields. For single mode fields, the correlation function is independent of time delay so that the condition $g^{(2)}(\tau) > g^{(2)}(0)$ cannot be satisfied [57]. Bartley et. al [132] have claimed to show photon anti-bunching, but, in light of the above remarks, and because they described single-mode fields, their results only indicate sub-Poissonian statistics. See Zou and Mandel [138] for more on
6.1. Photon Catalysis using Squeezed Coherent Light

**Figure 6.3**: Mandel Q parameter for (a) an input coherent state, (b) an input squeezed coherent state, and (c) an input squeezed coherent state.
the distinction between sub-Poissonian statistics and photon anti-bunching.

In Fig. (6.3), we plot the Mandel $Q$ parameter versus beam splitter reflectivity for several different initial input states. For reference, it is worth comparing particular points on the plot with their corresponding probabilities of detection in Fig. (6.2). For the case where we initially have a coherent state, that is we set $r = 0$ so there is no squeezing, in the $a$-mode (Fig. (6.3a)), we find sub-Poissonian statistics across a broad range of reflectivity. For example, with one photon in and two photons detected, we see sub-Poissonian statistics at all values of the reflectivity beyond $R = 0.37$. Looking at the probability of successful catalysis, we see that for any reflectivity beyond that point, a detection of a single photon will be made $> 15\%$ of the times. Once again, it is important to take note of the success rate when studying a good result. For the case where we initially have a squeezed vacuum state, that is we set $\alpha \rightarrow 0$ (Fig. (6.3b)), consider the case where one has a single photon initially occupying the $b$-mode while a detection is made in the output $b$-mode for two photons. For the value of $R = 0.68$, the state exhibits sub-Poissonian statistics, since the Mandel $Q$ parameter falls below zero. However, a cursory glance at the probability of making a detection of two photons, Fig. (6.2a), tells us that that result has a success rate of around $1\%$. Lastly we consider the case where we have a squeezed coherent state as our initial input state (Fig. (6.3c)) and find sub-Poissonian statistics for both small and moderately larger values of beam splitter reflectivity.

6.1.2 Nonclassical Effects — Quadrature Squeezing

Next we examine the single mode photon-catalyzed state for another possible non-classical property: quadrature squeezing. One can define the quadrature operators for a single-mode field as

\[
\hat{X}_1 = \frac{1}{2} \left( \hat{a} + \hat{a}^\dagger \right), \quad \hat{X}_2 = \frac{1}{2i} \left( \hat{a} - \hat{a}^\dagger \right)
\]

satisfying the commutation relation $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$. For the vacuum, these operators satisfy, and indeed minimize, the uncertainty relation $\Delta \hat{X}_1 \Delta \hat{X}_2 \geq 1/4$, where each
quadrature variance is given individually by $(\Delta \hat{X}_{1,2}^{\text{vac}})^2 = 1/4$. For quadrature squeezed light the individual quadrature variances may fall below that of the vacuum, that is $(\Delta \hat{X}_{1,2})^2 < 1/4$, while the product of the quadrature variances still satisfies the uncertainty relation. In Fig. (6.4) we plot the quadrature variances as well as the product of uncertainty in each quadrature as a function of beam splitter reflectivity $R$. It has been pointed out [132] that in the case of a coherent state initially occupying the $a$-mode and balanced catalysis with one photon, squeezing is seen in one of the quadratures for low values of the beam splitter reflectivity. Extending this example to the unbalanced case where we initially have two photons in the input $b$-mode yet still project on one photon at the output, we obtain squeezing over a larger range of beam splitter reflectivity and maximized at a larger value of beam splitter reflectivity. Increasing the coherent state amplitude, the squeezing can be deepened, but becomes much more localized over a smaller span of beam splitter angle.

One needs to be a bit more careful when evaluating the merit in these results as the probability of successful detection needs to be considered. For the case of Fig. (6.4a) we find a probability of successful catalysis of around 32% when the squeezing is at a maximum. Likewise, for Fig. (6.4b), we find a probability of 21%. Finally, for the case in Fig. (6.4c) we find a probability of 16%. Interesting effects can also be seen in the limiting case of $\alpha \to 0$, wherein our initial state in the input $a$-mode is a squeezed vacuum state. In Fig. (6.4d) we can actually see slight squeezing effects in the opposite quadrature for low values of the squeeze parameter. This squeezing is maximized, however slightly, for a given beam splitter reflectivity of $R \simeq 0.58$. The probability of successful catalysis at this point is 17%, which is still fairly high. In the last figure, we plot the case of a squeezed coherent state in the initial input $a$-mode. We find a deeper degree of squeezing within a certain range of beam splitter reflectivity than one would find if the input state were a single mode squeezed vacuum state.
Figure 6.4: Quadrature variances and uncertainty relation versus squeezing [dB] for several different cases: (a) An input coherent state with \( \alpha = 1, q = 1, l = 1 \), (b) A coherent state with \( \alpha = 1, q = 1, l = 2 \), (c) A coherent state with \( \alpha = 2, q = 1, l = 2 \), (d) A squeezed vacuum state with \( r = 0.7, q = 2, l = 2 \), and (e) A squeezed coherent state with \( \alpha = 1, r = 0.7, q = 1, l = 1 \).
6.1.3 Non-Gaussianity via the Wigner Distribution

We analyze the Wigner function, a phase-space quasi-probability distribution, which serves as an important tool in determining non-classicality. This is due to the fact that non-classicality of a state can be determined by a Wigner distribution that takes on a negative value somewhere in phase space. We also utilize the Wigner function in determining the non-Gaussianity of the photon catalyzed state, as negativity of the distribution implies non-Gaussianity. The Wigner function can be written in terms of the displaced parity operator for pure states as \( W(\beta) = \frac{2}{\pi} \langle \hat{D}(\beta) \hat{\Pi} \hat{D}^\dagger(\beta) \rangle \).

We plot the Wigner Distribution in Fig. (6.5) for several different cases, with the designation \( \beta \rightarrow x + iy \). We see in Fig. (6.5a), where \( r \rightarrow 0 \) so the initial state is a coherent state, the distribution peaks negative close to the origin. Meanwhile, for the case where \( \alpha \rightarrow 0 \), Fig. (6.5b), such that the initial state is a squeezed vacuum state with a positive peak in the Wigner distribution centered at the origin, the distribution is now minimized and negative at the origin. Lastly, we consider the squeezed coherent state, Fig. (6.5c), which also peaks negative close to the origin. The figures discussed demonstrate both the non-classicality and consequent non-Gaussianity of the photon catalyzed state for just some of the possible choices of parameters, evident by the negativity of the Wigner function.

6.1.4 Lower Bound on the Wigner Function

It is mentioned in [132] that a pure state, such as coherent light, in which the product of quadrature variances does not minimize the associated uncertainty relation is necessarily non-Gaussian. From this argument, they go on to claim non-Gaussianity of the single-photon catalyzed coherent state for a particular range of beam splitter reflectivity. The goal of this section is the prove this claim to be demonstrably false. To this end, we analyze a single mode Gaussian state: the squeeze vacuum state. This state
Figure 6.5: Wigner Function for the following cases: (a) A coherent state input with $\alpha = 2, q = 2, l = 2, R = 0.8$, (b) A squeezed vacuum input with $r = 0.7, q = 1, l = 2, R = 0.2$, and (c) A squeezed coherent state with $\alpha = 2, r = 0.7, q = 2, l = 2, R = 0.7$.

Figure 6.6: Quadrature Variances and Uncertainty Product for the single-mode squeezed vacuum state for a given phase value (a) $\phi = 0$ and (b) $\phi = \pi/4$. 
was introduced earlier as the squeeze operator, given by
\[ \hat{S} (\xi) = \exp \left[ \frac{1}{2} (\xi^* \hat{a}^2 - \xi \hat{a}^\dagger^2) \right], \]
acting on the vacuum state |0⟩, where ξ = re^{iφ}. We reanalyze the properties of this state while considering values of the phase φ that are non-zero. We begin by analyzing the quadrature variances for the general case. Using the usual operator transformations we arrive at
\[
\begin{align*}
\langle (\Delta \hat{X}_1)^2 \rangle_{SVS} &= \frac{1}{4} \left[ \cosh^2 r + \sinh^2 r - 2 \sinh r \cosh r \cos \phi \right], \\
\langle (\Delta \hat{X}_2)^2 \rangle_{SVS} &= \frac{1}{4} \left[ \cosh^2 r + \sinh^2 r + 2 \sinh r \cosh r \cos \phi \right].
\end{align*}
\]
(6.17)

For the often-considered case of φ = 0, this result simplifies to the result:
\[
\begin{align*}
\langle (\Delta \hat{X}_1)^2 \rangle_{SVS, \phi \to 0} &= \frac{1}{4} e^{-2r}, \\
\langle (\Delta \hat{X}_2)^2 \rangle_{SVS, \phi \to 0} &= \frac{1}{4} e^{2r},
\end{align*}
\]
(6.18)

where the quadrature variance product will clearly always minimize the uncertainty relation, \(\Delta \hat{X}_1 \Delta \hat{X}_2 = 1/4\). However, for the choice of \(\phi = \pi/2\), the quadrature variances take on the values
\[
\begin{align*}
\langle (\Delta \hat{X}_1)^2 \rangle_{SVS} &= \frac{1}{4} [1 + 2 \sinh^2 r] \\
&= \langle (\Delta \hat{X}_2)^2 \rangle_{SVS},
\end{align*}
\]
(6.19)

which clearly do not minimize the quadrature uncertainty relation. We plot the individual quadrature variances as well as the uncertainty product in Fig. (6.6). We see that for a phase value of \(\phi = \pi/4\) the uncertainty product is not minimized over a large range of the squeeze parameter \(r\). For a given value of the squeeze parameter, \(r = 0.8\), we plot the corresponding Wigner function for two values of the phase, \(\phi = \{0, \pi/4\}\) to see the phase effects on the Wigner distribution. We see in Fig. (6.7) that the effect of the phase is to rotate the Wigner function in phase space by \(\pi/4\). With this in mind, we see that the state described by a \(\phi = \pi/2\) phase shift, Eq. (6.19), will remain Gaussian despite not minimizing the quadrature uncertainty relation.
Figure 6.7: Wigner function for the single-mode squeezed vacuum state for a given phase value (a) $\phi = 0$ and (b) $\phi = \pi/4$. The effect of the phase is such that the Wigner distribution simply rotates in phase space. Note that for both figures the squeeze parameter is taken to be $r = 0.8$, as indicated by the green line in Fig. 6.6.

To verify Gaussianity of the Wigner function using a more quantifiable metric, we turn to the lower bound on Gaussianity for the Wigner function [104] given by

$$ W [ |\psi_G\rangle \langle \psi_G | ] (0) \geq \frac{2}{\pi} e^{-2\bar{n}(1+\bar{n})}, \quad (6.20) $$

where $\bar{n}$ is the average photon number given by $\bar{n} = \langle \psi_G | \hat{a}^\dagger \hat{a} | \psi_G \rangle$. For the single mode squeezed vacuum, the average photon number is given in terms of the squeeze parameter $r$ as $\bar{n} = \sinh^2 r$. With this in mind, we can verify Gaussianity of the Wigner function for any value of the phase parameter. As the Wigner function simply rotates in phase space when varying the phase, the value at the origin remains the same, $W (x, y \to 0) \simeq 0.64$, while the right side of the inequality in Eq. (6.20) is simply $\simeq 0.1$. This proves by demonstration that a single mode pure state can remain Gaussian while not necessarily minimizing the quadrature uncertainty relation.

Next, we move on to discuss the effects of performing photon catalysis on one mode of a two-mode squeezed vacuum state. We will show that the resulting two-mode state is non-Gaussian for some of parameters proven by showing negativity of the Wigner distribution for some value in space. As one can not plot the Wigner distribution for a
two-mode state, we instead minimize the distribution for a given value of beam splitter angle $\theta$. As we have stated, such a two-mode state may find use in loop-free violations of Bell’s inequalities. Further more, this method of generation is advantageous over the case where one performs photon catalysis on both modes simultaneously as one need not have to worry about timing the detections.

6.2 Generating Non-Gaussian Two-mode Correlated States of Light

We start with a two-mode squeezed vacuum where one output beam, say the signal beam, is mixed with $q$ photons at a beam splitter while a detection of $l$ photons takes place at the output $b$-mode in Fig. 6.8. In the case of pure photon catalysis, when $q$ and $l$ are the same, the photon number correlations between the two modes remain unchanged, that is, the photon correlations of the two-mode squeezed vacuum state are preserved. In this sense, the photon catalysis process serves as a means of degaussifying the two-mode squeezed vacuum. As we have previously stated, this process may prove to be useful in tests of Bell’s Inequalities, as the resultant two-mode correlated state will be non-Gaussian for some values of the beam splitter reflectivity, thus closing the detection loophole. Our initial input state is

$$|\text{in}\rangle = |\text{TMSV}\rangle_{a,a'} \otimes |q\rangle_b = \sum_{j=q}^{\infty} \Lambda_{2j-q} |j,j-q\rangle |2j-q\rangle_{a'},$$  \hspace{1cm} (6.21)

where the coefficients for the two-mode squeezed vacuum are given by [104]

$$\Lambda_n = \frac{1}{\cosh r} \left( -1 \right)^n (\tanh r)^n, \hspace{1cm} (6.22)$$

which are obtained by solving the eigenvalue equation associated with acting on the vacuum with the two mode squeeze operator $\hat{S}_2 (\xi) = \exp \left[ \xi^* \hat{a} \hat{b} - \xi \hat{a}^\dagger \hat{b}^\dagger \right]$. In the last step of Eqn. (6.21) we used the su(2) algebra to write the joint $a,b$-modes as an angular
momentum state $|j, m = j - q\rangle$. Following the same procedure as in the single-mode case, we can write the output two-mode state as

$$|\text{out}\rangle = e^{-i\theta \hat{J}_1} |\text{in}\rangle = \sum_{j=\frac{q}{2}}^{\infty} \sum_{m=-j}^{j} \Omega_{j,m} |j, m\rangle |2j - q\rangle_{a'}$$

$$= \sum_{j=\frac{q}{2}}^{\infty} \sum_{m=-j}^{j} \Omega_{j,m} |j + m\rangle_{a} |j - m\rangle_{b} |2j - q\rangle_{a'}, \quad (6.23)$$

where the new probability amplitudes, $\Omega_{j,m}$ are given by $\Omega_{j,m} = \Lambda_{2j-q}\Omega_{m,j-q}(\theta)$. The probability of successfully detecting $l$ photons in the output b-mode is given by the expectation of the projection operator

$$P_{\text{cat}} = \langle \text{out}|l\rangle_{b} \langle l|\text{out}\rangle = \sum_{j=\max\{q,l\}/2}^{\infty} |\Omega_{j-j-l}|^2, \quad (6.24)$$

and the resulting normalized two-mode state occupying the $a - a'$ modes is given by
6.2. Generating Non-Gaussian Two-mode Correlated States of Light

\[ |\Psi\rangle_{a,a'} = \sqrt{\frac{P_{\text{cat}}}{\Omega_j j}} \sum_{j=\max\{q,l\}/2}^{\infty} \Omega_{j-1} \langle 2j-1 \rangle_{a} |2j-q\rangle_{a'} . \]  

(6.25)

It should be noted that in the case of balanced catalysis, i.e. the same number of photons incident on the beam splitter is the same number of photons being detected, the correlations of the two-mode squeezed vacuum remain. However, for unbalanced catalysis, the two modes differ in photon number by \(|q-l|\). Since each two-mode state in the resulting superposition differs by the same amount, the difference in average photon number between the two beams will as well, that is, \(|\langle n_a \rangle - \langle n_{a'} \rangle| = |q-l|\). The probability of successful catalysis is plotted for reference in Fig. (6.9).

### 6.2.1 Determining Non-Gaussianity in Two-Mode Systems

Extending Eqn. (6.16) to the two mode case, we can write the Wigner function as

\[ W(\alpha, \beta) = \frac{4}{\pi^2} \langle \hat{D}_a^\dagger(\alpha) \hat{\Pi}_a \hat{D}_a(\alpha) \otimes \hat{D}_b^\dagger(\beta) \hat{\Pi}_b \hat{D}_b(\beta) \rangle . \]  

(6.26)
Table 6.1: A table of the minimum values of the two-mode Wigner Function, $S(\theta)$, for various values of the beam splitter reflectivity as well as $q$ and $l$. Note that negative values for $S(\theta)$ show non-Gaussianity as a consequence of non-classicality of the state. Note: $r, R, q, l$ are the squeeze parameter, beam splitter reflectivity, input $b$-mode photons, and output $b$-mode photons detected, respectively.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$R$</th>
<th>$q$</th>
<th>$l$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x'$</th>
<th>$y'$</th>
<th>$S(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9</td>
<td>0.7</td>
<td>1</td>
<td>1</td>
<td>-0.37</td>
<td>0.91</td>
<td>-0.2</td>
<td>-0.49</td>
<td>-0.141</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7</td>
<td>2</td>
<td>2</td>
<td>0.489</td>
<td>0.566</td>
<td>0.25</td>
<td>-0.29</td>
<td>-0.143</td>
</tr>
<tr>
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<td>0.4</td>
<td>1</td>
<td>2</td>
<td>-4.2E-8</td>
<td>-1.4E-8</td>
<td>-3E-8</td>
<td>-2.2E-9</td>
<td>-0.405</td>
</tr>
<tr>
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<td>2</td>
<td>-0.35</td>
<td>0.92</td>
<td>-0.25</td>
<td>-0.65</td>
<td>-0.102</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>-0.475</td>
<td>0.91</td>
<td>-0.26</td>
<td>-0.49</td>
<td>-0.18</td>
</tr>
<tr>
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<td>1</td>
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<td>1.05</td>
<td>-0.52</td>
<td>-0.77</td>
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</tr>
<tr>
<td>1.9</td>
<td>0.4</td>
<td>2</td>
<td>2</td>
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</tr>
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<td>0.7</td>
<td>1</td>
<td>2</td>
<td>-0.13</td>
<td>0.81</td>
<td>-0.23</td>
<td>-1.49</td>
<td>-0.142</td>
</tr>
</tbody>
</table>

Without photon catalysis, that is, in the case of a two mode squeezed vacuum, the Wigner function is Gaussian and therefore does not take on negative values in any region of phase space. Our objective is to show that in the case of single-, two-photon, or unbalanced catalysis, the Wigner function will take on negative values at some point in phase space, thus proving non-Gaussianity of the Wigner function as a consequence of non-classicality. Simply put, we minimize the Wigner function with respect to the phase space coordinates defined by $\alpha \rightarrow x + iy, \beta \rightarrow x' + iy'$ for given values of the beam splitter angle. To this end, we define the quantity

$$S(\theta) = \min_{\theta}\{W(x, y|x', y')\}$$ (6.27)

Where values of $S(\theta) < 0$ signify non-Gaussianity. It should be noted that the opposite is not necessarily true; that is, values of $S(\theta) > 0$ do not necessitate Gaussianity of the Wigner function. A table of $S(\theta)$ can be found in Table 6.1 for various cases.

We also consider the single-mode Wigner functions. We note that for the case of pure catalysis, in which the same number of photons are taken out of the auxiliary mode as put in, the single-mode Wigner functions for each mode will be identical as the photon number correlations of the two mode squeezed vacuum are preserved. We calculate the single-mode Wigner function by
\[ W_i(\lambda) = \frac{2}{\pi} Tr\{\hat{\rho}_i \hat{D}_i^\dagger(\lambda) \hat{\Pi}_i \hat{D}_i(\lambda)\}, \]

where \( \hat{\rho}_i \) is the reduced density operator for one of the mode, obtained by tracing out one of the modes. For the two mode squeezed vacuum, the single-mode Wigner functions will be Gaussian. However, in the case of single- and two-photon catalysis, we find Wigner functions displaying non-Gaussian profiles.

### 6.2.2 The Mandel Q Parameter for the Two-Mode Case

Similar to the single-mode case, we can analyze photon statistics of the two-mode photon catalyzed squeezed vacuum state. For the case of a two-mode squeezed vacuum, each mode will have super-Poissonian statistics, that is, values of the Mandel Q parameter that is greater than 0. We plot the Mandel Q parameter in Fig. (6.10). In the case of the two-mode photon catalyzed state, we find sub-Poissonian statistics for large values of the beam splitter reflectivity, around 80\%, for all cases studied. In addition to that, for small values of the squeeze parameter, we find sub-Poissonian statistics for beam splitters with relatively smaller reflectivity, around 30\%. It should be noted from Fig. (6.9), that the probability of successful catalysis for all cases discussed is around 10 – 20\%. More specifically for the case of unbalanced catalysis, in Fig. (6.10c), we get sub-Poissonian statistics for low values of the beam splitter reflectivity, while still maintaining a detection probability of around 15\%, seen in Fig. (6.9a). For other cases, such as in Fig. (6.10d), we see sub-Poissonian statistics for a beam splitter reflectivity of around 75\%, with a detection probability of roughly 10\%, as revealed by Fig. (6.9b).

### 6.3 Generating the Pair Coherent State

Pair coherent states (PCS) were introduced in quantum optics by Agarwal [10] more than 25 years ago. In analogy to the usual coherent states \(|\alpha\rangle\) for a single-mode field, which are right eigenstates of the annihilation operator, \(\hat{a} |\alpha\rangle = \alpha |\alpha\rangle\), the PCS \(|\zeta, q\rangle\) are defined as right eigenstates of the double-beam annihilation operator \(\hat{a}\hat{b} |\zeta, q\rangle = \zeta |\zeta, q\rangle\).
Figure 6.10: Mandel $Q$ parameter as a function of beam splitter reflectivity for the $a$-mode for (a) a squeeze parameter $r = 0.9$ and (b) $r = 1.9$, and for the $a'$-mode for (c) a squeeze parameter of $r = 0.9$ and (d) $r = 1.9$. 
where $\zeta$ is a complex number, but with the restriction that they also be eigenstates of the number difference operator with eigenvalue $q$, i.e., $(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) |\zeta, q\rangle = q |\zeta, q\rangle$, where $q$ is an integer. For the special case of $q = 0$, the primary case of interest discussed in this chapter, the PCS are a superposition of twin-Fock states $|n\rangle_a |n\rangle_b$, given as

$$|\zeta, 0\rangle \rightarrow |\zeta\rangle = \left[I_0(2|\zeta|)\right]^{-1/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} |n\rangle_a |n\rangle_b,$$

(6.29)

where $I_0(x)$ is the modified Bessel function of order zero. The parameter $\zeta$ is related to the average photon number of the two modes according to the relation

$$\bar{n} = \langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle = 2|\zeta| \frac{I_1(2|\zeta|)}{I_0(2|\zeta|)},$$

(6.30)

where $I_1(x)$ is the modified Bessel function of order 1. The PCS are also known as 'circle states' as they can be written as

$$|\zeta\rangle \propto \int_{0}^{2\pi} |\mu e^{i\omega}\rangle_a |\nu e^{-i\omega}\rangle_b \ d\omega,$$

(6.31)

where the two single-mode states of modes $a$ and $b$ are coherent states and where $\zeta = \mu \nu$.

The PCS have very different quantum mechanical properties than do the more familiar squeezed vacuum states, which are also the superposition of twin-Fock states. The latter are Gaussian states, whereas the PCS are non-Gaussian and therefore generally have stronger nonclassical properties [76]. Because of their strong nonclassical properties and the fact that they are highly entangled two-mode field states, they extend the candidates of states that could provide continuous variable realizations [11] of the Einstein-Podolsky-Rosen (EPR) [2] paradox and for violations of local realism through violations of Bell-type inequalities [3]. In addition, there has recently been much interest in applications of the PCS to quantum information processing. For example, Gábris and Agarwal [50] have discussed quantum teleportation with the PCS, while Usenko and Paris [133] have discussed their applications to quantum communication. Recently,
Chapter 6. Photon Catalysis and Quantum State Engineering

we [58] have discussed the application of the PCS to the problem of quantum optical interferometry and have found them to yield, using parity measurements on one of the output beams, phase-shift detections with Heisenberg-limited sensitivity and super-resolved interference fringes. However, the optical PCS have yet to be generated in the laboratory. In his original work, Agarwal [10] showed that the states could be generated via steady-state optical balance where a two-photon parametric process competes with incoherent two-photon losses. Gilchrist and Munro [61] have examined the problem of the nondegenerate parametric oscillator with losses included and have shown that PCS appear in the transient regime if the pump field mode is adiabatically eliminated. Some experimental work [78] in that direction is encouraging.

In this section we take a different approach to generating the PCS that amounts to a projective measurement on a pair of ordinary (Glauber) coherent states. We start with the following observation: Suppose we have available in modes $a$ and $b$ the coherent states $|\alpha\rangle_a$ and $|\beta\rangle_b$, where

$$|\alpha\rangle_a \otimes |\beta\rangle_b = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n! m!}} |n\rangle_a |m\rangle_b.$$  \hspace{1cm} (6.32)

The projection operator $\hat{P}_0 = \sum_{N=0}^{\infty} |N\rangle_a \langle N| \otimes |N\rangle_b \langle N|$ acts on any product state of two modes of a quantized field, i.e., states of the form $|\psi\rangle_a |\xi\rangle_b$, to project out a superposition of the form $\sum_{N=0}^{\infty} C_N |N\rangle_a |N\rangle_b$ such that the difference in the photon numbers of the modes vanishes. If we now apply this projection operator to the product of coherent states above, we find

$$\hat{P}_0 |\alpha\rangle_a |\beta\rangle_b \propto \sum_{N=0}^{\infty} \frac{(\alpha \beta)^N}{N!} |N\rangle_a |N\rangle_b.$$  \hspace{1cm} (6.33)

where the right hand side, up to the normalization factor, is the same as the PCS of Eq. (6.29) with $\zeta = \alpha \beta$. In what follows we provide a method to perform this projection to obtain the PCS with currently available technology or technology that is on the horizon.
6.3. Generating the Pair Coherent State

6.3.1 A State Projective Scheme

We consider the setup pictured in Fig. (6.11) where modes $a$ and $b$ are each coupled in sequence to different cross-Kerr media but where both media are coupled to a third mode denoted as the $c$-mode. This mode is assumed to be prepared in a coherent state $|\gamma\rangle_c$. The interaction Hamiltonian for the first cross-Kerr interaction, the interaction coupling the $a - c$ modes, is $\hat{H}_{CK1} = \hbar \chi_1 \hat{a}^\dagger \hat{a} \hat{c}^\dagger \hat{c}$ while the interaction coupling the $b - c$ modes is $\hat{H}_{CK2} = -\hbar \chi_2 \hat{b}^\dagger \hat{b} \hat{c}^\dagger \hat{c}$. The sign change for the second interaction can be engineered by changing the sign of the detuning in the interaction creating the Kerr nonlinearity. The parameters $\chi_1$ and $\chi_2$ are proportional to the third-order nonlinear susceptibility $\chi^{(3)}$. We have allowed for the possibility that the magnitudes of the two nonlinearities might also be slightly different, hence the choices $\chi_1$ and $-\chi_2$. Working in the interaction picture, the first cross-Kerr interaction produces

$$e^{-i\hat{H}_{CK1}t_1/\hbar} |\alpha\rangle_a |\beta\rangle_b |\gamma\rangle_c = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2)} \sum_{n,m,l=0}^{\infty} \frac{\alpha^n \beta^m \gamma^l}{\sqrt{n!m!l!}} \times e^{-i\chi_1 t_1 n} |n_a\rangle |m_b\rangle |l_c\rangle,$$

(6.34)

where $t_1$ is the interaction time. The second interaction produces the state
\[ |\psi\rangle = e^{-i\hat{H}_{CK}\tau_2/\hbar} e^{-i\hat{H}_{CK}\tau_1/\hbar} |\alpha\rangle_a |\beta\rangle_b |\gamma\rangle_c \]
\[ = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\gamma|^2)} \sum_{n,m,l=0}^{\infty} \frac{\alpha^n \beta^m \gamma^l}{\sqrt{n!m!l!}} \times e^{-i(\chi_1 t_1 n - \chi_2 t_2 m)l} |n\rangle_a |m\rangle_b |l\rangle_c, \tag{6.35} \]

or equivalently,
\[ |\psi\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!m!}} |n\rangle_a |m\rangle_b |\gamma e^{i\Theta_{nm}}\rangle, \tag{6.36} \]

where \( \Theta_{nm} = \tau_1 n - \tau_2 m \) and where we have set \( \tau_i = \chi_i t_i \). If we now displace the \( c \)-mode by \( -\gamma \), we obtain
\[ |\Psi\rangle = \hat{D}_c(-\gamma) |\psi\rangle \simeq e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!m!}} \times \]
\[ \times e^{i|\gamma|^2 \sin \Theta_{nm}} |n\rangle_a |m\rangle_b |\gamma (e^{-i\Theta_{nm}} - 1)\rangle \tag{6.37} \]

where \( \hat{D}_c(-\gamma) \) is the usual displacement operator introduced in Chapter 2. We have used the result
\[ \hat{D}_c(-\gamma) |\gamma e^{-i\Theta_{nm}}\rangle_c = e^{-i|\gamma|^2 \sin \Theta_{nm}} |\gamma (e^{-i\Theta_{nm}} - 1)\rangle_c. \tag{6.38} \]

The displacement operation is realized as a beam splitter with a high transmittance \( T \to 1 \), for the \( c \)-mode, with one port fed by a strong coherent state \( |\mu\rangle \) of large amplitude \( |\mu| \to \infty \) such that \( \mu \sqrt{1-T} = -\gamma \) is finite [135]. This displacement operation has been realized experimentally [95]. However, as far as we are aware, this has so far only been implemented for small displacements. In [95], the displacement operation \( \hat{D}(\eta) \) has been implemented for \( \eta \) up to \( \eta = 2.4 \). On the other hand, there does not seem to be any fundamental obstacle to implementing larger-amplitude displacements.

Upon detection of the vacuum state in the \( c \)-mode, \( |0\rangle_c \), the \( a-b \)-modes are projected into the state
6.3. Generating the Pair Coherent State

\[ |\psi_{ab}\rangle = \Gamma_{ab} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!m!}} e^{-i|\gamma|^2 \sin \Theta_{nm}} e^{-|\gamma|^2(1-\cos \Theta_{nm})} |n\rangle_a |m\rangle_b, \quad (6.39) \]

where

\[ \Gamma_{ab} = \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^2n!|\beta|^2m!}{n!m!} e^{-2|\gamma|^2(1-\cos \Theta_{nm})} \right]^{-1/2}. \quad (6.40) \]

These states will actually be generated in bursts conditioned on the detection of the vacuum state in the \( c \)-mode. Next we move on to discuss this states’ fidelity with the PCS, that is, the faithfulness of this state to the PCS. A fidelity of \( F = 1 \) signifies that the two states in question are exactly the same while a fidelity of \( F = 0 \) denotes two states that are entirely dissimilar.

6.3.2 Calculating the Fidelity

The fidelity for obtaining the PCS is given by

\[ \mathcal{F} = |\langle \psi_{ab} | \zeta \rangle|^2 = \left| \Gamma_{ab} \left[ I_0(2|\zeta|) \right] \right|^{-1/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(|\alpha^*|^2|\beta|^2)^n}{(n!)^2} e^{i|\gamma|^2 \sin \Theta_{nm} e^{-|\gamma|^2(1-\cos \Theta_{nm})}} \right|^2. \quad (6.41) \]

To proceed, we assume that \( \tau_1 \) and \( \tau_2 \) differ by a small amount \( \Delta \tau \). We set \( \tau_1 = \tau \) and \( \tau_2 = \tau + \Delta \tau \) such that \( \Theta_{nm} = \tau(n - m) + m \Delta \tau \), and we shall assume that \( \tau \) is small so that \( \Delta \tau \) is even smaller: \( |\Delta \tau| << \tau << 1 \). Under these conditions we can approximate the state of Eq. (6.39) by

\[ |\psi_{ab}\rangle \approx \Gamma_{ab} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha e^{-i|\gamma|^2 \tau})^n (\beta e^{i|\gamma|^2(\tau+\Delta \tau)})^m}{\sqrt{n!m!}} \times e^{-|\gamma|^2 \tau^2(n-m)^2/2} \times e^{-|\gamma|^2 \Delta \tau^2/2} e^{i|\gamma|^2 \Delta \tau \tau (n-m)} |n\rangle_a |m\rangle_b, \quad (6.42) \]

where now the normalization factor is approximated by
\[ \Gamma_{ab} \simeq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{2n} |\beta|^{2m} n!m!}{e^{-2|\gamma|^2 r^2 (n-m)^2} e^{\gamma^2 r^2 \Delta \tau m(n-m)} e^{-|\gamma|^2 m^2 \Delta \tau^2 / 2}} \]  

In this approximation, the Fidelity becomes

\[ \mathcal{F} = \left| \Gamma_{ab} \left[ I_0(2|\zeta|) \right] \right|^{-1/2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{\alpha^* \beta^* \zeta e^{-i|\gamma|^2 \Delta \tau}}{(n!)^2} \right)^n \times e^{-|\gamma|^2 \Delta \tau^2 n^2} \right|^2. \]  

We start with the special case where \( \Delta \tau = 0 \), that is, the cross-Kerr media are of identical length. In this case our state becomes

\[ |\psi_{ab}\rangle = \Gamma_{ab} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \frac{\alpha e^{-i|\gamma|^2 \tau} \beta e^{i|\gamma|^2 \tau}}{\sqrt{n!m!}} \right)^n \times e^{-|\gamma|^2 \tau^2 (n-m)^2 / 2} |n\rangle_a |m\rangle_b, \]  

now with

\[ \Gamma_{ab} \simeq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{2n} |\beta|^{2m} n!m!}{e^{-2|\gamma|^2 r^2 (n-m)^2} e^{\gamma^2 r^2 \Delta \tau m(n-m)} e^{-|\gamma|^2 m^2 \Delta \tau^2 / 2}} \]  

If \( |\gamma \tau| \) is sufficiently large, we can make the approximation \( e^{-|\gamma|^2 r^2 (n-m)^2 / 2} \simeq \delta_{n,m} \) and \( |\psi_{ab}\rangle \) approaches the pair coherent state \( |\zeta\rangle \), where \( \zeta = \alpha \beta \). The fidelity in this case, with \( \Delta \tau = 0 \), becomes \( \mathcal{F} = \Gamma_{ab}^2 / \left[ I_0(2|\zeta|) \right] \) for the choice \( \zeta = \alpha \beta \).

It is useful at this point to consider physically realistic values of \( \tau \) and to determine the values of \( \bar{n}_c = |\gamma|^2 \) for which the fidelity approaches unity. The parameter \( \tau \) characterizes the strength of the cross-Kerr interaction at the single-photon level. In what follows, we shall assume that the cross-Kerr nonlinearity, even as enhanced by the techniques of electromagnetically induced transparency (EIT) [84], is still relatively weak, and specifically, we shall assume that the parameter \( \tau \) does not approach the value \( \tau = \pi \) rad, the value that would allow for the implementation of a universal gate set for all-optical quantum communication [34], for generating superpositions of coherent...
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Figure 6.12: Plot of the fidelity $F$ versus $|\gamma|$ for $|\zeta|^2 = 1, 10, \text{ and } 20$.

states [53], and for generating maximally path entangled N00N states and entangled coherent states [56] which have been shown to be useful for interferometric purposes [56] [43]. In addition, such large nonlinearities could be used for quantum nondemolition measurements of photon number [86] and of photon number parity [26]. As we have shown earlier, photon-number parity measurements are of importance in quantum optical metrology [60]. However, recent work [126] has shown that the casual, non-instantaneous behavior of any $\chi^{(3)}$ nonlinearity is enough to preclude the operation of a cross-Kerr interaction with high fidelity. Readily available cross-Kerr nonlinearities, such as in optical fibers, can produce only small phase shifts on the order of $10^{-20}\text{ rad}$ [22] at the single-photon level, which would require a $c$-mode laser field intensity of $\bar{n}_c = |\gamma|^2 > 8 \times 10^{40}$. But from an EIT-enhanced nonlinearity able to produce a phase shift of $\tau \sim 10^{-5}\text{ rad}$ at the level of a single photon, which is about the best that can be accomplished at present, we instead require $\bar{n}_c = |\gamma|^2 > 8 \times 10^{10}$, a considerable reduction in the required laser power.

With these considerations in mind, we set $\tau = 10^{-5}\text{ rad}$, and for various values of $\zeta$,
we plot the fidelity $\mathcal{F}$ against $|\gamma|$ in Fig. (6.12) and we see that for all cases, $\mathcal{F} \to 1$ as long as $|\gamma| \geq 2.5 \times 10^5$. We have confirmed this with both the exact and approximated forms of the fidelity, Eq. (6.41) and (6.44), respectively.

In the case where $\Delta \tau \neq 0$ but is small, we can see from Eq. (6.41) and (6.44) that we approach the PCS $|\zeta\rangle$ but now with $\zeta = \alpha \beta e^{i|\gamma|^2 \Delta \tau}$, provided $e^{-|\gamma|^2 \Delta \tau n^2} \simeq 1$, which will hold over the relevant values of $n$, determined by the values of $\bar{n}_a = |\alpha|^2$ and $\bar{n}_b = |\beta|^2$, assumed not to be too large, for small enough $\Delta \tau$. If we take, for example, $\Delta \tau = 10^{-8}$ rad, with $|\gamma|^2 \sim 10^{10}$, then $e^{-|\gamma|^2 \Delta \tau^2 n^2} = e^{-10^{-6} n^2} \simeq 1$ for $n$ not too large. In Fig. (6.13) we plot the corresponding fidelities as a function of $|\gamma|$ for various values of $\Delta \tau$. We see that the fidelity will be relatively low for all values of $|\gamma|$ unless $\Delta \tau$ is on the order of $\sim 10^{-7}$ or lower, in which case a fidelity of unity is obtained for $|\gamma| \geq 2.5 \times 10^5$. Thus our scheme can tolerate small discrepancies in the lengths of the Kerr media or the interaction times. On the other hand, it should be possible to adjust the interaction strengths through tuning in the same manner as suggested for creating cross-Kerr
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Finally, we examine the issue of projecting out the vacuum state of the displaced coherent state of the \( c \)-mode. The probability of detecting the vacuum state in the \( c \)-mode, \(|0\rangle_c\), is the expectation value of the projection operator \( \hat{P}_0 = |0\rangle_c \langle 0| \), which yields

\[
P_0 = \langle \psi' | \hat{P}_0 | \psi' \rangle = e^{-\left( |\alpha|^2 + |\beta|^2 \right)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|\alpha|^{2n} |\beta|^{2m}}{n!m!} \times e^{-2|\gamma|^2(1-\cos \Theta_{nm})}.
\]

(6.47)

in the regime of interest where the fidelity is unity for the case where \( \Delta \tau = 0 \), and with \( \beta = \alpha \) and \( \zeta = \alpha^2 \), this can be approximated by
\[
\bar{P}_0 = e^{-2|\zeta|^2} \sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{(n!)^2},
\]

(6.48)

We have numerically shown that these two expressions are in agreement under the above conditions. In Fig. (6.14) we plot \(\bar{P}_0\) versus \(|\zeta|\). We see that over the range of \(|\zeta|\) considered, the probability of detecting the vacuum state remains finite. As to the experimental prospects of detecting a ‘dark state’, such as a coherent state \(|\eta\rangle\) with \(n \approx 0\), one could use direct detection with single-photon-resolving detectors, such as superconducting transition edge sensors (TES) [1], or with silicon avalanche photo-diodes [1]. Detectors of this type can resolve photon numbers 0, 1, 2, up to about 8. However, our proposed scheme only requires the ability to discriminate between zero photons and any other number, which should be possible with such detectors. Lita et. al [1] have obtained detection efficiencies of 95\% ± 2\% in the near-infrared.
Chapter 7

Conclusion

In our opening chapter, we have reviewed the basic elements of phase estimation, with particular emphasis placed on the use of the quantum Fisher information and the quantum Cramér-Rao bound as it pertains to quantum optical interferometry. We have demonstrated how the path symmetric observable, the parity operator, saturates the quantum Cramér-Rao bound and is in agreement with results obtained via error propagation calculus when the parity operator is used as a detection observable. We have also discussed the prospect of using non-classical states of light in obtaining Heisenberg-limited phase uncertainty as well as some methods for which new and known non-classical states of light can be generated experimentally using currently available technology. We close the opening section by discussing the fundamental limits on phase sensitivity when using both classical (standard quantum limit) and non-classical (Heisenberg limit) states of light.

In Chapter 3, we have re-examined the two-mode squeezed coherent states (TMSCS) as obtained by the action of the two-mode squeeze operator representing the time evolution operator of the parametric down-conversion process acting on input coherent light fields. We have assessed the role of the various phases that enter the state, that is, the phase of the classical pump field and the two phases associated with the input coherent states of the parametric down-conversion process, and as to how these phases appear depending on the definition of the state, whether as two-mode squeezed coherent states as in Eq. (3.10) or as displaced TMSVS as given in Eq. (3.16). We have investigated the effect the phases have on the joint photon-number probability distributions of the
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TMSCS and on other statistical properties such as the average photon number and the linear entropy. Furthermore, we have studied the results of mixing the two beams of the TMSCS with a 50:50 beam splitter and have examined the means of controlling that output by making certain choices of the various phases. Finally, we have examined the prospect of utilizing the TMSCS for photon-number parity-based interferometry, and have found, for certain choices of parameters and phases, noise reductions approaching the level of the Heisenberg limit. In this work we have initially considered the pump field of the down-converter to be a classically prescribed field, which means we have ignored the effects of photon depletion of the pump field altogether. In the closing sections, we studied the states produced in the case where the pump field is quantized and assumed to be initially in a coherent state. We explored the effects of the phases on the evolution of the fully quantized model, especially their effects on the average photon numbers in the pump and signal/idler modes. For the case where all fields are initially in coherent states and with phase choices such that $\Phi = \pi$, we demonstrate a more rapid decrease in the average photon number of the pump field as the average photon numbers of the signal and idler modes increase compared to the case when $\Phi = 0$.

We have also studied, in Chapter 4, the multi-photon interference obtained by mixing coherent states of light $|\alpha\rangle$ with photon-number states $|N\rangle$ for low photon number $N = 0, 1, 2, \text{ and } 3$ at a 50:50 beam splitter and have investigated the prospects of performing quantum optical interferometric measurements with them. When coherent light is mixed with a photon-number state, the resulting multi-photon interference creates joint photon-number probability distributions that are multi-modal and symmetric about the diagonal line $n_a = n_b$. The structure of these distributions, the fact that they split into distributions that lead to uncertainty as to the location of the bulk of the photons, is key to the effectiveness of such states for approaching Heisenberg-limited sensitivity in the measurements of phase shifts. The distributions obtained from the mixing of coherent states with number states resembles the kinds of distributions obtained from the mixing of twin-number states at a 50 : 50 beam splitter where the output states are the arcsine states. With regard to phase-shift measurements, which are
performed with the use of photon-number parity detections on one of the output beams, we have noticed a significant improvement over the standard quantum limit obtained with coherent-state mixing with a vacuum state by mixing the coherent light with a single photon. This happens because of the dramatic effect that occurs in this case where the joint Poisson distribution obtained from coherent light alone is bifurcated along the line $m_a = m_b$ as the result of quantum interference with just one photon. To us, this suggested the possibility that the squeezed one-photon state, because the one-photon state itself is the lowest number state therein and has a relatively high probability of occurrence, should be more effective in obtaining substandard quantum limit noise reductions than the squeezed vacuum. This expectation was confirmed. We pointed out that it was not even necessary to squeeze a one-photon state (a difficult task) as one can obtain it identically by subtracting a single photon from a squeezed vacuum state as shown by Biswas and Agarawl [21]. The possibilities and benefits for using multiple photon-subtracted squeezed states in interferometry is discussed in further detail in Chapter 5.

We have generalized the idea of mixing at a 50:50 beam splitter coherent and squeezed vacuum light to that of mixing coherent and photon-subtracted squeezed vacuum states for quantum optical interferometry. We found that for given state parameters, $\alpha$ and $r$, improvement in the sensitivity (reduction in the noise of the phase-shift measurement) occurs generally with the increasing numbers of photons subtracted. We examined the effects of subtracting up to three or four photons. The greatest degree of improvement occurs upon the subtraction of just one photon, with more gradual noise reduction with further photon subtractions. At the same time, photon-subtraction from the squeezed vacuum state improves resolution. The photon number parity measurement approach to interferometry is known to result in super-resolution even with coherent states of light used alone 5.13, and here we have found that further narrowing of the signal indicating increasing resolution with increasing numbers of photons subtracted. As for the experimental prospects, we point out that parity-based interferometry with the proposed photon-subtracted squeezed vacuum states is within reach of currently available
technology. As mentioned, in the experiment of Gerrits et. al [128] up to three photons have been subtracted from squeezed vacuum states. In the experiment of Afek et. al [73], detectors with resolutions at the level of a single photon were used to perform interferometry with the mixing of coherent and squeezed vacuum states of light by projecting onto N00N states through coincidence counting. However, with their technology they could easily perform a parity-based interferometry experiment. More recently, Cohen et. al [91] have announced a laboratory realization of a photon-number parity-based interferometry experiment. They used only coherent light in their experiment and achieved, via parity measurements, super-resolved phase-shift measurements at the shot noise limit as predicted by Gao et. al [137]. Finally, we point out that very recently Lang and Caves [92] have considered the question: given that one input of an interferometer is powered by laser light, what is the optimal state for achieving high-sensitivity phase-shift measurements given a constraint on the average number of photons that the state of the input beam can carry? The answer, they found, is the squeezed vacuum state. Of course, it is the squeeze parameter \( r \) that determines the average photon number, and thus a constraint on that number is a constraint on \( r \). But as we have seen, the subtraction of photons from the squeezed vacuum state not only increases the average photon number for the fixed squeezed parameter, it also increases the corresponding sensitivity in the phase-shift measurement, see Fig. (5.4). Thus even within the constraint on the average photon of a squeezed vacuum state, it is still possible to attain higher sensitivity via photon subtraction. One could, as well, perform photon addition to the squeezed vacuum state, but photon subtraction appears to be easier to implement using the scheme discussed in [97].

Lastly, in Chapter 6, we have extended the method of photon catalysis for generating non-Gaussian single mode states of light to the more general case where a squeezed coherent state is considered as our input state as opposed to the previously studied case of coherent light as the input state. We have also relaxed the condition of photon catalysis by considering state projective measurements for \( l \neq q \) photons. The photon catalyzed state was then shown to exhibit non-classical properties such as quadrature...
squeezing and sub-Poissonian statistics for a wide range of state parameters. We note that the generation of such a state, in general, occurs with a fairly high probability for the state parameters we have studied. We have also considered applying photon catalysis to one mode of a two-mode squeezed vacuum. The resulting correlated two-mode state, which will be non-Gaussian for a range of beam splitter reflectivity, may find use in loophole free tests of Bell’s inequalities as well as quantum optical interferometry. Such two-mode states are advantageous over the case where one performs photon catalysis simultaneously on both modes due to their relative ease of generation since they only require a projective measurement be made on a single mode so one need not worry about the timing of simultaneous measurements. It was also noted that both modes of the two-mode photon catalyzed state will exhibit sub-Poissonian statistics for some range of the beam splitter reflectivity, regardless of the beam on which photon catalysis is performed.

In the closing sections, we have presented a scheme for generating pair coherent states through the implementation of the projection operator, \( \hat{P}_0 = \sum_{N=0}^{\infty} |N\rangle \langle N| \otimes |N\rangle \langle N| \), acting on a product of coherent states \( |\alpha\rangle \otimes |\beta\rangle \). The scheme, under certain conditions and to a good approximation, projects out the pair coherent state \( |\zeta\rangle \), where \( \zeta = \alpha \beta \). Furthermore, the scheme relies only on the enhanced but still weak cross-Kerr nonlinearities made possible by recent developments in electromagnetically induced transparency. The pair coherent states have yet to be generated in the laboratory by any of the means previously proposed, with those schemes usually involving competitive nonlinear processes, but the scheme here offers a possible method for generating the states in bursts conditioned on the detection of the vacuum state in an auxiliary field mode.
Appendix A

Beam Splitters

This appendix aims to very briefly address how one properly conceptualizes the quantum mechanical beam splitter. We will see that, when dealing with classical field states such as coherent or thermal light, one may take a classical approach when discussing beam splitting. However, things break down when reaching the level of single photons. Before discussing the fully quantum mechanical beam splitter, it is worth noting how one handles a classical beam splitter and why this treatment breaks down for most quantum states.

A.1 A Classical Treatment

Consider a classical light field of complex amplitude $\xi_1$ incident on a lossless beam splitter as shown in Fig. (A.1) [57]. The action of the beam splitter is to produce two orthogonal beams denoted $\xi_2$ and $\xi_3$, where $\xi_2$ is the reflected beam and $\xi_3$ is the transmitted beam.
beams. If \( r \) and \( t \) are the reflectance and transmittance of the beam, respectively, then the intensities of the two beams are simply

\[
\begin{align*}
\xi_2 &= r \xi_1, \\
\xi_3 &= t \xi_1.
\end{align*}
\]

(A.1)

For the simplest case, a 50:50 beam splitter, we have \(|r| = |t| = 1/\sqrt{2}\). However, regardless of the choice of beam splitter, the condition imposed on the field intensities,

\[
|\xi_1|^2 = |\xi_2|^2 + |\xi_3|^2,
\]

(A.2)

must hold true for a lossless beam splitter. This requires \(|r|^2 + |t|^2 = 1\) hold true. When dealing with the quantum mechanical beam splitter, it is tempting to replace the classical field amplitudes with by a set of boson annihilation operators \( \hat{a}_i \), \( \{i = 1, 2, 3\} \). Following through with the same reasoning we employed for the classical case, we could write

\[
\begin{align*}
\hat{a}_2 &= r \hat{a}_1, \\
\hat{a}_3 &= t \hat{a}_1,
\end{align*}
\]

(A.3)

however, this formalization quickly breaks down upon trying to reconcile the boson operator commutation relations \( [\hat{a}_i, \hat{a}_i^\dagger] = 1, \ [\hat{a}_i, \hat{a}_j] = \delta_{i,j} \). For example, we see that \( [\hat{a}_2, \hat{a}_3^\dagger] = t^* r \) and \( [\hat{a}_2, \hat{a}_2^\dagger] = |r|^2 \), not in agreement with the boson commutation relations. In order to rectify this, we must consider an alternative setup.

### A.2 The Quantum Mechanical Beam Splitter

Instead consider a new scheme where the unused port of the beam splitter is taken into account, as in Fig. (A.2). In this case the unused port is still occupied by a quantized field mode, the vacuum, and therefore has a set of boson operators \( \{\hat{a}_0, \hat{a}_0^\dagger\} \). We can now write for the beam splitter transformation

\[
\begin{align*}
\hat{a}_2 &= r \hat{a}_1 + t' \hat{a}_0, \\
\hat{a}_3 &= t \hat{a}_1 + r' \hat{a}_0,
\end{align*}
\]

(A.4)
Appendix A. Beam Splitters

Figure A.2: The correct quantum mechanical depiction of a beam splitter where the unused port is still characterized by a set of boson operators \{\hat{a}_0, \hat{a}_0^\dagger\}.

or in the more compact form

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}. \quad (A.5)$$

It is fairly straightforward to prove that the boson operator commutation relations are satisfied provided

$$|r'| = |r|, \quad |t'| = |t|, \quad |r|^2 + |t|^2 = 1, \quad \text{and} \quad r^*t' + r't^* = r^*t + r't^* = 0. \quad (A.6)$$

These equations are collectively known as the reciprocity relations [57]. Henceforth we will assume to have a 50:50 beam splitter and that the reflected beam picks up a $\pi/2$-phase shift, making the beam splitter scattering matrix

$$U_{\text{scatt}} = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad (A.7)$$

resulting in a beam splitter transformation given by
A.2. The Quantum Mechanical Beam Splitter

Using this transformation, let us briefly demonstrate the quantum mechanical beam splitter using a couple of simple examples.

A.2.1 Example: Number States

Consider, as the input to a 50:50 beam splitter, the state $|\text{in}\rangle = |\psi_1\rangle_a \otimes |\psi_2\rangle_b = |2\rangle_a |0\rangle_b$, that is we have two photon initially occupying the a-mode and a vacuum state occupying the b-mode, as in Fig. (A.3). The input state can be written as

$$|\text{in}\rangle = |2\rangle_a |0\rangle_b = \frac{1}{\sqrt{2}} \hat{a}_0^\dagger |0\rangle_a |0\rangle_b,$$  \hspace{1cm} (A.9)

where we have used the relation $|n\rangle = (n!)^{-1/2} \hat{a}^\dagger n |0\rangle$. Using the beam splitter transformation given in Eq. (A.8), we can plug in for $\hat{a}_0^\dagger$ in terms of the boson operators $\hat{a}_2^\dagger$ and $\hat{a}_3^\dagger$ to find
Appendix A. Beam Splitters

\[
\frac{1}{\sqrt{2}} \hat{a}_0^\dagger |0\rangle_a |0\rangle_b \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} (\hat{a}_2^\dagger + i\hat{a}_3^\dagger) \right)^2 |0\rangle_a |0\rangle_b
= \frac{1}{2} |2\rangle_a |0\rangle_b - \frac{1}{2} |0\rangle_a |2\rangle_b + \frac{i}{\sqrt{2}} |1\rangle_a |1\rangle_b, \tag{A.10}
\]
which makes perfect sense. If the two photons initially in the \(a\)-mode both transmit through the beam splitter, the output state will be \(|2\rangle_a |0\rangle_b\). If they both reflect off the beam splitter, each photon picks up a phase of \(\pi/2\), resulting in the output state \(-|0\rangle_a |2\rangle_b\) and if one photon transmits and the other reflects, the state picks up a single factor of a \(\pi/2\)-phase shift, resulting in the output state \(i |1\rangle_a |1\rangle_b\).

Another example would be to consider as the initial state \(|\text{in}\rangle = |\psi_1\rangle_a \otimes |\psi_2\rangle_b = |1\rangle_a |1\rangle_b\); that is, one photon initially prepared in the \(a\)-mode and one in the \(b\)-mode. We can rewrite this state in much the same way we did in the previous example

\[
|\text{in}\rangle = |1\rangle_a |1\rangle_b = \hat{a}_0^\dagger \hat{a}_1^\dagger |0\rangle_a |0\rangle_b, \tag{A.11}
\]
which yields, after beam splitting, the state

\[
\hat{a}_0^\dagger \hat{a}_1^\dagger |0\rangle_a |0\rangle_b \xrightarrow{\text{BS}} \left( \frac{1}{\sqrt{2}} (\hat{a}_2^\dagger + i\hat{a}_3^\dagger) \right) \left( \frac{1}{\sqrt{2}} (\hat{a}_3^\dagger + i\hat{a}_2^\dagger) \right) |0\rangle_a |0\rangle_b
= \frac{i}{\sqrt{2}} \left( |2\rangle_a |0\rangle_b + |0\rangle_a |2\rangle_b \right). \tag{A.12}
\]
Once again, this result should not be too shocking. In order to obtain the output state \(|2\rangle_a |0\rangle_b\) or \(|0\rangle_a |2\rangle_b\), one of the photons in either mode would have to be reflected into the opposite mode, thus obtaining a \(\pi/2\)-phase shift. Interestingly enough, the output \(|1\rangle_a |1\rangle_b\) is not possible if the beam splitter is 50:50. This is because these results corresponds to either both photons transmitted, \(|1\rangle_a |1\rangle_b\), or both photons reflecting, \(i^2 |1\rangle_a |1\rangle_b = - |1\rangle_a |1\rangle_b\). As a result, these probability amplitudes destructively interfere and cancel out. This is known as the Hong-Ou-Mandel effect and was first experimentally verified in 1987 [29]. Next we move on to the case of an input coherent state.
A.2. Example: Coherent State

As we have discussed in Chapter 2, the coherent state, \( |\alpha\rangle \), is the most classical of pure single-mode field states. It can be written as a superposition in the number state basis as

\[
|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .
\]  

(A.13)

More formally, the coherent state can be written as the displacement operator acting upon the vacuum

\[
|\alpha\rangle = \hat{D}(\alpha) |0\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle .
\]  

(A.14)

For this reason, the coherent state is also known as the displaced vacuum state. Assume our initial state incident on the beam splitter is given as

\[
|\text{in}\rangle = |\psi_1\rangle_a \otimes |\psi_2\rangle_b = |\alpha\rangle_a |0\rangle_b .
\]

We can follow the same prescription as we did with number states to write

\[
|\text{in}\rangle = |\alpha\rangle_a |0\rangle_b = \hat{D}_a(\alpha) |0\rangle_a |0\rangle_b = e^{\alpha \hat{a}_0^\dagger - \alpha^* \hat{a}_0} |0\rangle_a |0\rangle_b ,
\]  

(A.15)

which, upon beam splitting, becomes

\[
e^{\alpha \hat{a}_0^\dagger - \alpha^* \hat{a}_0} |0\rangle_a |0\rangle_b \xrightarrow{\text{BS}} e^{\alpha \left( \frac{\hat{a}_1^\dagger + i\hat{a}_3^\dagger}{\sqrt{2}} \right) - \alpha^* \left( \frac{\hat{a}_2 - i\hat{a}_3}{\sqrt{2}} \right)} |0\rangle_a |0\rangle_b
\]

\[
= e^{\frac{\alpha}{\sqrt{2}}} \left( \frac{\alpha}{\sqrt{2}} \right)^* \hat{a}_2 e^{\frac{i\alpha}{\sqrt{2}}} \hat{a}_1^\dagger - \left( \frac{\alpha}{\sqrt{2}} \right)^* \hat{a}_3
\]

\[
= \hat{D}_a\left( \frac{\alpha}{\sqrt{2}} \right) \hat{D}_b\left( \frac{i\alpha}{\sqrt{2}} \right) |0\rangle_a |0\rangle_b
\]

\[
= \left| \frac{\alpha}{\sqrt{2}} \right|_a |\frac{i\alpha}{\sqrt{2}} \rangle_b .
\]  

(A.16)

Note that a slightly different beam splitter scattering matrix was used when deriving a similar result in Eq. (2.52), which yielded a \( \pi/2 \)-phase shift in the output \( b \)-mode. The result in Eq. (A.16) is noteworthy as the action of the beam splitter results in a disentangled, entirely separable, state consisting of two coherent states of equal intensities, albeit...
differing by a phase factor. In this sense, a quantized field state behaves identically as one would expect a classical field to behave.
Appendix B

Brief Overview of the SU(2) Group

In a standard Mach-Zehnder interferometer (MZI), light initially prepared in two separate modes are incident on a beam splitter. One mode incurs a phase shift due to a relative path length difference between the two modes, before being incident upon a second beam splitter. The light is then analyzed with the intent of extracting some information related to the incurred phase shift. In the previous appendix, we have discussed a relatively straightforward method for performing beam splitter transformations, and indeed one can account for all steps within the MZI and arrive at the resulting output state. This was done for an input coherent-vacuum state, \(|\alpha\rangle_a |0\rangle_b\) in Chapter 2, where the output state obtained after passing through the MZI is given in Eq. (2.54). In that chapter we discussed the use of the su(2) algebra [115] [92] in deriving the final output state. In this appendix we plan to discuss the mechanics for which one can characterize a MZI in terms of the SU(2) group [14].

B.1 The su(2) Lie Algebra

Assume two separate modes incident upon a beam splitter. Each mode is characterized by a set of boson operators, \(\{\hat{a}, \hat{a}^\dagger\}\) for the a-mode and \(\{\hat{b}, \hat{b}^\dagger\}\) for the b-mode. These operators satisfy the usual boson operator commutation relations \([\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1\) and \([\hat{a}, \hat{b}] = 0\). One can introduce the Hermitian operators
Appendix B. Brief Overview of the SU(2) Group

\[ J_x = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \]
\[ J_y = -\frac{i}{2}(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}), \]
\[ J_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \]

and

\[ J_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \]

satisfying the commutation relations for the Lie algebra of the SU(2) group,

\[ [\hat{J}_i, \hat{J}_j] = i\hat{J}_k \epsilon_{ijk}, \]

where \( \epsilon_{ijk} \) is the Levi-Civita symbol. The Casimir invariant of the group, which commutes with all elements of the group, \( \hat{J}^2 = \frac{1}{2}(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2) = \hat{J}_0(\hat{J}_0 + 1) \). As this is precisely the angular momentum algebra, it is natural to write a two-mode quantized field state \( |n\rangle_a \otimes |n'\rangle_b \) in terms of the SU(2) multiplet states \( |j, m\rangle \). It is fairly straightforward to see

\[ \hat{J}_0 |n\rangle_a |n'\rangle_b = \frac{n + n'}{2}, \quad \hat{J}^2 |j, m\rangle = \hat{J}_0(\hat{J}_0 + 1) |j, m\rangle \quad \rightarrow \quad \hat{J}_0 |j, m\rangle = j |j, m\rangle. \]

and thus \( j = \frac{n + n'}{2} \). We can employ the same strategy to arrive at an expression for \( m \). Consider

\[ \hat{J}_z |n\rangle_a |n'\rangle_b = \frac{1}{2}(n - n'), \quad \hat{J}_z |j, m\rangle = m |j, m\rangle \]

making \( m = \frac{n - n'}{2} \). Combined, we can write out multiplet state as

\[ |n\rangle_a |n'\rangle_b \rightarrow |j, m\rangle = \left| \frac{n + n'}{2}, \frac{n - n'}{2} \right\rangle. \]
B.2 Beam Splitters Revisited using $su(2)$ Lie Algebra

Next we will discuss how one describes a lossless passive device, such as a beam splitter, in terms of the $su(2)$ algebra. Let $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$ represent the boson annihilation operators representing light entering the input ports and $\hat{a}_{\text{out}}$, $\hat{b}_{\text{out}}$ represent the boson operators representing the output ports. The scattering matrix linking them is given by (see Appendix A)

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad (B.7)$$

such that

$$\begin{pmatrix} \hat{a}_{\text{out}} \\ \hat{b}_{\text{out}} \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_{\text{in}} \\ \hat{b}_{\text{in}} \end{pmatrix}. \quad (B.8)$$

The scattering matrix in Eq. (B.7) must be unitary in order to conserve the boson operator commutation relations. In general, this transformation leads to a transformation of the group elements accordingly. Following the work of Yurke et. al [14], we take for example the scattering matrix

$$U = \begin{pmatrix} \cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\ -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}. \quad (B.9)$$

This scattering matrix will cause the SU(2) group elements to transform to

$$\begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{in}}$$

$$= e^{i\theta \hat{J}_x} \begin{pmatrix} \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{in}} e^{-i\theta \hat{J}_x}, \quad (B.10)$$
that is, our input state is rotated by an angle $\theta$ about the $x$-axis. It should be noted that the analogy between the number state basis and the angular momentum states of the SU(2) group is purely formal [94]; it arises because the Lie algebra of operators generating unitary transformations in two-dimensional space happen to be the same algebra of the operators generating rotations in a three-dimensional space. For this reason, these are often called ‘quasi-spins’, but have no direct physical interpretation in terms of any rotation in a real three-dimensional space [94]. Working in the Schrödinger picture, we can write the output state in terms of the input state as

$$|\text{out}\rangle = e^{-i\theta \hat{J}_x} |\text{in}\rangle .$$  \hspace{1cm} (B.11)

We can consider rotations about the $y$-axis by transforming the boson operator using a different scattering matrix

$$U = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} .$$  \hspace{1cm} (B.12)

This yields the transformed group elements

$$\begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{in}}$$

$$= e^{i\theta \hat{J}_y} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{\text{in}} e^{-i\theta \hat{J}_y} ,$$  \hspace{1cm} (B.13)

making the output state

$$|\text{out}\rangle = e^{-i\theta \hat{J}_y} |\text{in}\rangle .$$  \hspace{1cm} (B.14)

Lastly, let us examine how the group elements transform under an incurred phase shift.
Let the $a$-mode incur a phase shift of $\phi_a$ and the $b$-mode a phase shift $\phi_b$. We can denote the relative path length difference as $\varphi = \phi_b - \phi_a$. This corresponds to a scattering matrix given by

$$
U = \begin{pmatrix}
e^{i\phi_a} & 0 \\
0 & e^{i\phi_b}
\end{pmatrix}.
$$

(B.15)

Under this scattering matrix, the group elements will transform according to

$$
\begin{pmatrix}
\hat{J}_x \\
\hat{J}_y \\
\hat{J}_z
\end{pmatrix}_{\text{out}} = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{J}_x \\
\hat{J}_y \\
\hat{J}_z
\end{pmatrix}_{\text{in}}
= e^{i\varphi \hat{J}_z}
\begin{pmatrix}
\hat{J}_x \\
\hat{J}_y \\
\hat{J}_z
\end{pmatrix}_{\text{in}},
$$

(B.16)

hence, in the Schrödinger picture, this represents a phase shift

$$
|\text{out}\rangle = e^{-i\varphi \hat{J}_z} |\text{in}\rangle.
$$

(B.17)

Thus we can represent each component of a MZI in terms of a fictitious rotation in three dimensional space.

### B.3 Calculations in Interferometry using the SU(2) Group

Assuming our beam splitters are chosen such that they correspond to a rotation about the $x$-axis, $\hat{J}_x$ (or sometimes written in the text as $\hat{J}_1$), we can write the output state of the MZI in terms of the input state as

$$
|\text{out}\rangle = e^{i\varphi \hat{J}_x} e^{-i\varphi \hat{J}_z} e^{-i\varphi \hat{J}_x} |\text{in}\rangle = e^{-i\varphi \hat{J}_y} |\text{in}\rangle,
$$

(B.18)
where in the last step we used the Baker-Hausdorf relation to simplify

\[ e^{i\frac{\pi}{2} \hat{J}_x} e^{-i\varphi \hat{J}_z} e^{-i\frac{\pi}{2} \hat{J}_x} = \exp \left[ -i\varphi \ e^{i\frac{\pi}{2} \hat{J}_x} \ e^{-i\frac{\pi}{2} \hat{J}_x} \right] = e^{-i\varphi \hat{J}_y}, \tag{B.19} \]

where it is clear to see from Eq. (B.10) that \( e^{i\frac{\pi}{2} \hat{J}_x} \ e^{-i\frac{\pi}{2} \hat{J}_x} = \hat{J}_y \). Assume know that we have two arbitrarily written pure field states in our input ports, in the Schrödinger picture, we write this as

\[ |\text{in}\rangle = |\Lambda \rangle_a \otimes |\Gamma \rangle_b = \left( \sum_{n=0}^{\infty} \lambda_n |n\rangle_a \right) \otimes \left( \sum_{n'=0}^{\infty} \gamma_{n'} |n'\rangle_b \right) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \lambda_n \gamma_{n'} |n\rangle_a |n'\rangle_b \]

\[ = \sum_{j=0,1/2,..., m=-j}^{\infty} \sum_{m'=j}^{j} \lambda_{j+m} \gamma_{j-m} |j, m\rangle, \tag{B.20} \]

where in the last step, we used Eq. (B.6) to change from number state basis to angular momentum basis. We can write our output state as

\[ |\text{out}\rangle = e^{-i\varphi \hat{J}_y} |\text{in}\rangle = \sum_{j=0,1/2,...}^{\infty} \lambda_{j+m} \gamma_{j-m} e^{-i\varphi \hat{J}_y} |j, m\rangle \]

\[ = \sum_{j=0,1/2,..., m=-j}^{\infty} \sum_{m'=j}^{j} \lambda_{j+m} \gamma_{j-m} \langle j, m' | e^{-i\varphi \hat{J}_y} |j, m\rangle |j, m'\rangle \]

\[ = \sum_{j=0,1/2,..., m=-j}^{\infty} \sum_{m'=j}^{j} \lambda_{j+m} \gamma_{j-m} \ d_{m', m}^{j}(\varphi) |j, m'\rangle \]

\[ = \sum_{j=0,1/2,..., m=-j}^{\infty} \sum_{m'=j}^{j} \lambda_{j+m} \gamma_{j-m} \ d_{m', m}^{j}(\varphi) |j + m'\rangle_a |j - m'\rangle_b, \tag{B.21} \]

where \( d_{m', m}^{j}(\beta) \) are the Wigner-\( d \) matrix elements more thoroughly discussed in Appendix C, and we have inserted a complete set of states \( \hat{I}_j = \sum_{m'=-j}^{j} |j, m'\rangle \langle j, m'| \) to complete the relation. In the last step of Eq. (B.21), we revert back to number state basis.
B.3. Calculations in Interferometry using the SU(2) Group

for completeness. Throughout this paper, we often calculate the expectation value of a detection observable in the Heisenberg picture. For parity-based detection performed on the output $b$-mode, this becomes

$$
\langle \hat{\Pi}_b \rangle_{\text{out}} = \langle e^{i\varphi J_y} \hat{\Pi}_b e^{-i\varphi J_y} \rangle_{\text{in}}
$$

$$
= \sum_{j=0,1/2,...}^{\infty} \sum_{j'=0,1/2,...}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \lambda^*_{j+m,m'} \gamma^*_{j'-m',j'+m} \langle j', m' | e^{i\varphi J_y} \hat{\Pi}_b e^{-i\varphi J_y} | j, m \rangle
$$

$$
= \sum_{j=0,1/2,...}^{\infty} \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \lambda^*_{j+m',m-m'} \gamma_{j-m,m'-j} e^{i\pi (j-m')} d^j_{m',m'}(\varphi) d^j_{m',m}(\varphi). \tag{B.22}
$$

Such expressions, while complicated, can be evaluated numerically in a fairly straightforward fashion. It is now worth discussing the Wigner $d$ matrix elements in some depth, as they are a persistent occurrence in quantum optical interferometric calculations.
Appendix C

The Wigner-\(d\) Matrix Elements

Before delving into the specifics of the Wigner-\(d\) matrix elements, it may be prudent to briefly discuss where they formally come from. In order to do that, we must discuss what are known as Euler rotations. Simply put, Euler’s rotation theorem states that an arbitrary rotation of a rigid body can be accomplished in three steps, known as Euler rotations. In linear algebra terms, this tells us that in three dimensional space, any two Cartesian coordinate systems with a common origin are related by a rotation about some fixed axis; this implies that the product of two rotation matrices is also a rotation matrix. This is advantageous as the Euler rotation language, defined by three Euler rotations, provides a transparent and concise way of characterizing general rotations in three dimensional space. [123]

C.1 Derivation — Euler Angles

Imagine we have two separate coordinate systems: one that is fixed in space, which we will refer to as the space-fixed coordinate system, and another that is ’embedded’ in our rigid body, a body coordinate system. We denote these as \(\{x_f, y_f, z_f\}\) and \(\{x_b, y_b, z_b\}\), respectively. This notation is crude, as the body coordinate system will vary as we perform rotations, but is still sufficient to gain a functional understanding of the topic. Assuming before any rotations are performed that the two coordinate systems align such that the axes of the two coordinate systems are equivalent, the three steps of Euler rotations are as follows [123]. First, a rotation of the rigid body is performed about the
$z_b$-axis by an angle $\alpha$, counterclockwise from the positive $z$-side. As a result of this first rotation, note, that the $y_f$- and $y_b$-axes no longer align. The second rotation is performed about the $y_b$-axis by an angle $\beta$. As a result, the $z_f$- and $z_b$-axes no longer point in the same direction. The third and final rotation is about the $z_b$-axis by angle $\gamma$. Note that as a result of this final rotation, the $y_b$-axis has changed orientation. These rotations can be summarized as

$$R(\alpha, \beta, \gamma) \equiv R_{z_b}(\gamma) R_{y_b}(\beta) R_{z_f}(\alpha).$$  \hspace{1cm} (C.1)$$

Note that our expression for the product of these rotations depend on rotations about the body coordinate system axes. This is not ideal as we have already worked out the expressions for rotations about space-fixed coordinates. Luckily there are a couple of identities that will help us rewrite Eq. (C.1) strictly in terms of rotations about space-fixed axes, the first being

$$R_{y_b}(\beta) = R_{z_f}(\alpha) R_{y_f}(\beta) R_{z_f}^{-1}(\alpha)$$  \hspace{1cm} (C.2)$$

and the second is

$$R_{z_b}(\gamma) = R_{y_b}(\beta) R_{z_f}(\gamma) R_{y_b}^{-1}(\beta).$$  \hspace{1cm} (C.3)$$

Combining these two equations, we can rewrite Eq. (C.1) as

$$R(\alpha, \beta, \gamma) \equiv R_{z_b}(\gamma) R_{y_b}(\beta) R_{z_f}(\alpha)$$

$$= R_{z_f}(\alpha) R_{y_f}(\beta) R_{z_f}(\gamma),$$  \hspace{1cm} (C.4)$$

where we have used the fact that rotations about the same axes commute. Note that the final line of Eq. (C.4) depends solely on rotations about fixed spacial axes. To state more succinctly: this shows that a general arbitrary rotation can be characterized by three rotations, known as Euler rotations, about spatially fixed axes.
Appendix C. The Wigner-d Matrix Elements

We are now ready to discuss the matrix elements of an arbitrary rotation specified by an axis of rotation \( \hat{n} \) and angle of rotation \( \phi \). The matrix elements, with \( \hbar \to 1 \) for convenience, are

\[ D_{jm}^{\prime \prime} (R) = \langle j, m' | e^{-i\phi \hat{n} \cdot \hat{J}} | j, m \rangle. \]  

(C.5)

Since the rotation operator commutes with the \( \hat{J}^2 \) operator, a rotation cannot change the \( j \) value of a state. The \((2j + 1) \times (2j + 1)\) matrix formed by \( D_{jm}^{\prime \prime} (R) \) is referred to as the \((2j + 1)\)-dimensional irreducible representation of the rotation operator \( D(R) \) [123].

We now consider the matrix realization of the Euler Rotation,

\[ D_{m',m}^{\alpha \beta \gamma} (\alpha, \beta, \gamma) = \langle j, m' | e^{-i\alpha \hat{J}_x} e^{-i\beta \hat{J}_y} e^{-i\gamma \hat{J}_z} | j, m \rangle \]

(C.6)

These matrix elements are referred to as the Wigner-D matrix elements. Notice that the first and last rotation only add a phase factor to the expression, thus making only the rotation about the fixed \( y \)-axis the only non-trivial part of the matrix. For this reason, the Wigner-D matrix elements are written in terms of a new matrix

\[ D_{m',m}^{\alpha \beta \gamma} (\alpha, \beta, \gamma) = e^{-i(m'+m \gamma)} \langle j, m' | e^{-i\beta \hat{J}_y} | j, m \rangle = e^{-i(m'+m \gamma)} d_{m',m}^{\beta} (\beta), \]  

(C.7)

where the matrix elements \( d_{m',m}^{\beta} (\beta) = \langle j, m' | e^{-i\beta \hat{J}_y} | j, m \rangle \) are known as the Wigner-d matrix elements and are given by
\[ d_{m',m}^j (\beta) = \left( \frac{(j - m)!(j + m')!}{(j + m)!(j - m')!} \right)^{1/2} \frac{(-1)^{m' - m} \cos^{2j + m - m'} (\beta)}{(m' - m)!} \times \times _2 F_1 \left( m' - j, -m - j; m' - m + 1; -\tan^2 \left( \frac{\beta}{2} \right) \right), \]  
\[ \text{with the property} \]
\[ \begin{cases} 
  d_{m',m}^j (\beta) & m' \geq m \\
  d_{m,m'}^j (-\beta) & m > m' 
\end{cases} \]  
\[ \text{and where} \ _2 F_1 (a, b; c; z) \text{is the Hypergeometric function.} \]

\[ \text{It is worth noting that in our interferometric calculations such as in Eq. (B.21), we naturally end up with an expression that depends on the Wigner-}d \text{ matrix elements. However, when simply dealing with a single \( \hat{J}_x \) beam splitter of angle} \ \theta, \ \text{one encounters the matrix elements} \]
\[ \langle j, m' | e^{-i \theta \hat{J}_x} | j, m \rangle. \]
\[ \text{This can be simplified using the relations in Eq. (B.16) to} \]
\[ \langle j, m' | e^{-i \theta \hat{J}_x} | j, m \rangle = \langle j, m' | e^{i \pi/2} \hat{J}_z e^{-i \theta \hat{J}_x} e^{-i \pi/2} \hat{J}_z | j, m \rangle \]
\[ = D_{j, m'} (\alpha, \frac{\pi}{2}, \theta) \]
\[ = i^{m' - m} d_{m',m}^j (\theta). \]  
\[ \text{In our closing section we state, without proof, a handful of useful identities used throughout this paper pertaining to the Wigner-}d \text{ matrix elements.} \]

\section*{C.2 Useful Identities}

\textbf{Unitarity of Rotation} [129]:
\[ d_{m,k}^j (\beta) = d_{k,m}^j (-\theta). \]
Composition of two rotations \[129\]:

\[ d^j_{m,k} (\theta_1 + \theta_2) = \sum_{\nu=-j}^j d^j_{m,\nu} (\theta_1) d^j_{\nu,k} (\theta_2). \]  \hspace{1cm} (C.12)

Consequence of Eq. (C.9) \[129\]:

\[ d^j_{m,k} (\theta) = (-1)^{m-k} d^j_{m,k} (-\theta). \]  \hspace{1cm} (C.13)

\[ d^j_{m,k} (\theta) = d^j_{-m,-k} (\theta). \]  \hspace{1cm} (C.14)

\[ d^j_{m,k} (\pi) = (-1)^{j+m} d^j_{m,-k} (\pi). \]  \hspace{1cm} (C.15)

From Eq. (C.11) and (C.13) \[129\]:

\[ d^j_{m,k} (\theta) = (-1)^{m-k} d^j_{k,m} (\theta). \]  \hspace{1cm} (C.16)

From Eq. (C.14) and (C.16) \[129\]:

\[ d^j_{m,k} (\theta) = (-1)^{m-k} d^j_{-m,-k} (\theta). \]  \hspace{1cm} (C.17)

From Eq. (C.12), (C.15), and (C.16) \[129\]:

\[ d^j_{m,k} (\pi + \theta) = (-1)^{j+m} d^j_{-m,k} (\theta). \]  \hspace{1cm} (C.18)

\[ d^j_{m,k} (\pi - \theta) = (-1)^{j+m} d^j_{m,-k} (\theta). \]  \hspace{1cm} (C.19)

Miscellaneous:

\[ \sum_{m'=-j}^j (-1)^{j-m'} d^j_{-n,m'} (-\beta) d^j_{m',j-n} (\beta) = (-1)^n d^j_{-n,j-n} (2\beta). \]  \hspace{1cm} (C.20)
\[ \sum_{m'=-j}^{j} (-1)^{j-m'} d_{m',m}^j (\beta) d_{m',k}^j (\beta) = (-1)^{2j} d_{k,-m}^j (\pi - 2\beta) = (-1)^{2j} d_{-m,k}^j (2\beta - \pi). \] (C.21)

\[ \sum_{m'=-j}^{j} m' d_{m',m}^j (\beta) d_{m',k}^j (\beta) = \left( \sum_{m'=-j}^{j} m' d_{m',m}^j (\beta) d_{m',m+1}^j (\beta) \delta_{k,m+1} + m' d_{m',k+1}^j (\beta) \times d_{m',k}^j (\beta) \delta_{k+1,m} + m' d_{m',k}^j (\beta) d_{m',k}^j (\beta) \delta_{m,k} \right) \] (C.22)

\[ \sum_{m'=-j}^{j} m'^2 d_{m',m}^j (\beta) d_{m',k}^j (\beta) = \left( \sum_{m'=-j}^{j} m'^2 d_{m',m}^j (\beta) d_{m',m+2}^j (\beta) \delta_{k,m+2} + m'^2 d_{m',k+2}^j (\beta) \times d_{m',k}^j (\beta) \delta_{k+2,m} + m'^2 d_{m',k}^j (\beta) d_{m',k}^j (\beta) \delta_{m,k} \right) \] (C.23)
Bibliography


[44] Work done by Dr. Paul Alsing. In: ().


