Intercusp Geodesics and Cusp Shapes of Fully Augmented Links

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Intercusp Geodesics and Cusp Shapes Of Fully Augmented Links

by

Rochy Flint

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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Abstract

Intercusp Geodesics and Cusp Shapes Of Fully Augmented Links

by

Rochy Flint

Advisor: Abhijit Champanerkar

We study the geometry of fully augmented link complements in $S^3$ by looking at their link diagrams. We extend the method introduced by Thistlethwaite and Tsvietkova [24] to fully augmented links and define a system of algebraic equations in terms of parameters coming from edges and crossings of the link diagrams. Combining it with the work of Purcell [21], we show that the solutions to these algebraic equations are related to the cusp shapes of fully augmented link complements. As an application we use the cusp shapes to study the commensurability classes of fully augmented links.
Acknowledgements

B”H

“Each individual has the capacity to build communities and endow communities with life so that each community member becomes a source of inspiration.”

—The Rebbe, Rabbi Menachem Mendel Schneerson

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Acharon Acharon Chaviv: the last is most beloved. Thank you to my beloved soul mate. Yitzie, you have always believed in me and encouraged me to reach higher. I cannot wait to tackle this next phase of our lives together.
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Chapter 1

Overview

The field of *Geometric Topology* uses geometry to study and understand the topology of surfaces and 3-manifolds. This dissertation studies the interactions between geometry, topology and combinatorics. A useful way to understand 3-manifolds is by studying links and their complements. An active area of research is relating the combinatorics of a link diagram to the geometry and topology of its complement.

Thurston studied the interactions between geometry and combinatorics using ideal triangulations and gluing equations for hyperbolic links and 3-manifolds. The solutions to the gluing equations often allow us to construct the discrete faithful representation of the fundamental group of the link complement to $Isom^+ (\mathbb{H}^3)$ and help us to compute many geometric invariants. Although an ideal triangulation can be obtained from a link diagram, the solutions to the gluing equations, and
the geometric invariants obtained from them are difficult to relate to diagrammatic invariants obtained from the link diagram.

In [24] Thistlethwaite and Tsvietkova used link diagrams to study the geometry of hyperbolic alternating link complements by implementing a method to construct a system of algebraic equations directly from the link diagram. We refer to their method as the T-T method. The idea of the T-T method is as follows: by looking at the faces of the link diagram, and assigning parameters to crossings and edges in every face, they find relations on the parameters which determine algebraic equations. The solutions to these equations allow them to construct the discrete faithful representation of the link group into $\text{Isom}^+(\mathbb{H}^3)$.

In this dissertation we study the combinatorics and geometry of a class of links called fully augmented links, which are links that are obtained by augmenting every twist region of a given link with a circle component and removing all twists, see Figure 1.1. We show the following:
CHAPTER 1. OVERVIEW

(1) A way to extend the T-T method to fully augmented links. This is the first application of the T-T method to an infinite class of non-alternating links. This is done in Proposition 4.1.1, Theorem 4.2.1;

(2) A new method to determine the cusp shapes of fully augmented link complements using the solutions of the system of equations obtained from the T-T method. This is stated in Theorem 5.1.1 and Theorem 5.1.3;

(3) A way to study commensurability classes for different classes of fully augmented links. This is done in Theorems 6.1.6, 6.1.7, and 6.2.4;

(4) A way to choose the geometric solutions i.e. the solution which enables us to construct the discrete, faithful representation, from the solutions of the system of equations obtained from the T-T method. We demonstrate this in Theorem 6.3.1.

This dissertation is divided into 6 chapters. In §2 we give necessary background about knots, links, hyperbolic 3-manifolds, fully augmented links and the geometry of their complements. In §3 we introduce the T-T method for alternating links and give examples illustrating the T-T method on alternating links. In §4 we show that the T-T method can be extended to fully augmented links, and illustrate with examples. In §5 we state our main theorem relating the cusp shapes to the intercusp-geodesics, and give explicit examples. §6 discusses applications of our main theorem by studying
CHAPTER 1. OVERVIEW

the invariant trace fields of fully augmented links, studying commensurability of fully augmented links, and finding geometric solutions to systems of equations in the T-T method for FALs.
Chapter 2

Hyperbolic Knots and Links

This chapter states the necessary background, definitions and theorems that set up the framework for our theorems.

2.1 Knots and Links

Definition 2.1.1. A knot $K$ is a (smooth) embedding of the circle $S^1$ in $S^3$. A knot has one component. Similarly, a link $L$ of $m$ components is a (smooth) embedding of a disjoint union of $m$ circles in $S^3$.

We can think of a knot as a 1-component link. Thus we will primarily use the term links throughout. Two links are equivalent if there is an ambient isotopy of $S^3$ taking one to the other. Let $L$ be a link, a link diagram $D(L)$ is a projection of $L$ on
a plane which is a 4-valent graph with over-and under-crossing information e.g. see Figure 3.5. Even though there are infinitely many different diagrams for $L$, we can study link diagrams to find information about $L$. Let $S^3 \setminus L$ denote the complement of the link in $S^3$. A combinatorial way to study links is to study link diagrams. A topological way to study links is to study link complements. Links are very useful in the study of 3-manifolds.

**Definition 2.1.2.** A *link invariant* is a quantity whose value depends only on the equivalence class of the link.

Link invariants are used to prove that two links are distinct, and to measure the complexity of the link in various ways. An example of a combinatorial link invariant is the *crossing number* of a link $L$, which is the minimal crossing number taken over all diagrams of $L$. An example of a topological link invariant is the *fundamental group* of $S^3 \setminus L$, denoted $\pi_1(L)$.

We give some commonly studied classes of links, see [7] for more.

1. **Alternating links** are links that admit a diagram where crossings alternate between underpasses and overpasses.

2. **Torus links** $T_{p,q}$ are links which wrap around the standard solid torus $p$ times in the longitudinal direction, and $q$ times in the meridional direction. The $(2, n)$-torus links are alternating.
(3) *Satellite knots* are knots that contain an incompressible, non-boundary parallel torus in their complements. Composite links are of this kind.

A way to connect links with 3-manifolds $M$ is to use the operation of Dehn surgery.

**Definition 2.1.3.** [14] Let $L \subset M$ be a link with $k$ components in an orientable 3-manifold $M$. A *Dehn surgery* on $L$ is a Dehn filling of the complement of $L$ in $M$.

This is a two-step operation that consists of:

1. **drilling**– the removal of small open tubular neighborhoods of $L$, that creates new boundary tori $T_1, ..., T_k$;

2. **filling**– Dehn filling is a way of gluing solid tori $D^2 \times S^1$ to some of $T_1, ..., T_k$, along the boundary tori such that the $(p_i, q_i)$-curves on the $T_i$ is glued to the meridians of the solid tori.

We get a new manifold denoted as $M(q_1/p_1, ..., q_k/p_k)$. For more on Dehn filling see [4], [14], [22].

**Theorem 2.1.4.** *(Lickorish-Wallace)* Every closed, orientable, connected 3-manifold $M$ can be obtained from Dehn surgery on a link in $S^3$.

For more on knots and links see [22], [23].
2.2 Hyperbolic 3-Manifolds

We will be working with the upper half-space model of $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 | t > 0\}$$

with metric

$$ds^2 = \frac{(dx^2 + dy^2 + dt^2)}{t^2} \quad \text{and} \quad \partial \mathbb{H}^3 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$ 

One can visualize $\mathbb{H}^3$ as a space where triangles are skinny and the sum of their angles are less than $\pi$. Geodesics, which are shortest paths in $\mathbb{H}^3$ are vertical half-lines or semicircles perpendicular to the $xy$-plane. Geodesic planes (copies of $\mathbb{H}^2$) are vertical half-planes, or hemispheres with centers on the $xy$-plane. A complete geodesic has endpoints on $\partial \mathbb{H}^3$. An ideal vertex is a vertex that lies on $\partial \mathbb{H}^3$. Angles between geodesics that are incident to ideal vertices are 0. An important object in $\mathbb{H}^3$ which we will use is a horosphere.

**Definition 2.2.1.** A horosphere $H$ in $\mathbb{H}^3$ centered at $z \in \hat{\mathbb{C}}$ is a Euclidean sphere $S$ tangent to the $xy$-plane at $z$ and is orthogonal to all geodesics with endpoint $z$. A horosphere centered at $\infty$ is parallel to the $xy$-plane. The induced metric on the
horosphere is a scaled Euclidean metric. A horosphere divides $\mathbb{H}^3$ into two connected regions one of which is homeomorphic to 3-balls. The 3-ball meeting $z$ is called a \textit{horoball} centered at $z$.

$\text{Isom}^+(\mathbb{H}^3)$ is the group of orientation preserving isometries of $\mathbb{H}^3$, which is isomorphic to $PSL(2, \mathbb{C})$, the group of Möbius transformations which acts on $\hat{\mathbb{C}}$. This action on $\hat{\mathbb{C}}$ extends uniquely to an isometry on $\mathbb{H}^3$. We look at $\mathbb{H}^3/\Gamma$ where $\Gamma < PSL(2, \mathbb{C})$, is a subgroup of hyperbolic isometries acting freely and properly discontinuously on $\mathbb{H}^3$.

Möbius transformations are obtained by compositions of translations, rotations, dilations and inversions, where angles are preserved, lines and circles are mapped to lines and circles, and horospheres are mapped to horospheres.

There are 4 types of Möbius transformations, which can be detected from the trace $a + d$ of the element $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. $\alpha$ is parabolic if $tr\alpha = \pm 2$.

2. $\alpha$ is elliptic if $tr\alpha \in (-2, 2)$.

3. $\alpha$ is loxodromic if $tr\alpha \in \mathbb{C} \setminus [-2, 2]$.

4. $\alpha$ is hyperbolic if it’s loxodromic with $tr\alpha \in \mathbb{R}$.
Definition 2.2.2. A hyperbolic 3-manifold $M$ is a 3-manifold equipped with a complete Riemannian metric of constant sectional curvature -1, i.e. the universal cover of $M$ is $\mathbb{H}^3$ with covering translations acting as isometries. Equivalently, $M = \mathbb{H}^3/\Gamma$ where $\Gamma$ is a torsion free Kleinian group i.e. discrete subgroups of $PSL(2, \mathbb{C})$.

In this dissertation hyperbolic 3-manifold will mean an orientable, complete, finite volume hyperbolic 3-manifold. Hyperbolic 3-manifolds have a thick-thin decomposition that allows us to understand the topology of non-compact hyperbolic 3-manifolds. This decomposition consists of a thin part with tubular neighborhoods of closed geodesics and ends which are homeomorphic to $T^2 \times [0,1)$. The manifold is of finite volume if and only if its thick part is compact. When $M = \mathbb{H}^3/\Gamma$, then $\Gamma$ contains parabolic elements, which correspond to the cusps of $M$. For distinct cusps $\mathbb{H}^3/\Gamma$ there is a distinct conjugacy class of maximal parabolic subgroups in $\Gamma$. The manifolds we will be working will be of this kind.

Definition 2.2.3. [4] A cusp of a hyperbolic 3-manifold is the thin end isometric to $T^2 \times [0, \infty)$ with metric $ds^2 = e^{-2t}(dx^2 + dy^2) + dt^2$.

Note that the cross sectional tori $T^2 \times \{t\}$ are scaled Euclidean tori.

Definition 2.2.4. A link $L$ is hyperbolic if the complement $S^3 \setminus L$ is a hyperbolic 3-manifold.

In a link complement, the cusps are the tubular neighborhoods of every compo-
ent of the link where the actual components are removed.

The cusps lift to a set of horoballs with disjoint interiors in the universal cover \( \mathbb{H}^3 \). For each cusp, the set of horoballs are identified by the covering transformations.

**Theorem 2.2.5. (Thurston's hyperbolic Dehn surgery theorem) [4]** Let \( M \) be a cusped hyperbolic 3-manifold, then “almost all” manifolds obtained by Dehn surgery from \( M \) are hyperbolic.

**Theorem 2.2.6. (Thurston’s geometrization theorem for knot complements) [5]** Every knot is precisely one of the following: a torus knot, a satellite knot, or a hyperbolic knot.

Since torus links lie on an unknotted torus in \( \mathbb{R}^3 \), they have an annulus in their complements. A satellite link has an incompressible non-boundary parallel torus in its complement. A hyperbolic link has neither.

**Theorem 2.2.7. (Menasco) [15]** If \( L \) has a non-split prime alternating diagram which is not a \((2,n)\) torus link diagram, then \( L \) is hyperbolic.

In view of the Lickorish-Wallace theorem and Thurston’s hyperbolic Dehn surgery theorem, “almost” all 3-manifolds are hyperbolic and are related to link complements. Hence studying hyperbolic links and their invariants is an active area of research.

Thurston’s famous example of the figure-eight knot complement decomposing into two ideal tetrahedra [25] is the first example of finding the hyperbolic structure from
a link diagram. Jeff Weeks implemented this idea in the computer program SnapPea which finds the geometric structure of link complements from link diagrams [26]. This was extended by Marc Culler and Nathan Dunfield recently to the program SnapPy [10]. Hyperbolic structures are very useful to study knots and links. For example, the geometric complexity for a hyperbolic knot is the minimal number of ideal tetrahedra needed to triangulate its complement. Knots have been enumerated with geometric complexity instead of the diagrammatic complexity which is the crossing number. Hyperbolic knots with geometric complexity up to six tetrahedra were found by Callahan-Dean-Weeks [6], extended to seven tetrahedra by Champanerkar-Kofman-Paterson [9], and eight tetrahedra by Champanerkar-Kofman-Mullen [9]. The census of simplest hyperbolic knots is included in SnapPy.

**Theorem 2.2.8.** *(Mostow-Prasad Rigidity) [4]* Let $M_1$ and $M_2$ be complete, orientable, finite-volume hyperbolic 3-manifolds. If $f : M_1 \rightarrow M_2$ is a homotopy equivalence, then $\exists$ an isometry $\phi : M_1 \rightarrow M_2$ homotopic to $f$.

**Corollary 2.2.9.** *(Mostow-Prasad Rigidity)* Let $M_1$ and $M_2$ be complete, orientable, finite-volume hyperbolic 3-manifolds, if $\pi_1(M_1) \cong \pi_1(M_2)$, then $M_1$ and $M_2$ are isometric.

The Mostow-Prasad Rigidity Theorem implies that the hyperbolic structure on a 3-manifold is unique and hence geometric invariants, e.g. hyperbolic volume are
topological invariants. A useful way of studying hyperbolic 3-manifolds is by computing their geometric invariants. One such invariant we will study is the cusp shape.

**Definition 2.2.10.** [17] A horospherical section of a cusp of a hyperbolic 3-manifold \( M \) is a flat torus. This torus is isometric to \( \mathbb{C}/\Lambda \), for some lattice \( \Lambda \subset \mathbb{C} \), and the ratio of two generators of \( \Lambda \) is the conformal parameter of the flat torus, which we call the cusp shape of the cusp of \( M \). Choosing generators \([m]\) and \([\ell]\) of \( \pi_1(T^2) \), the Euclidean structure on the torus is obtained by mapping \([m]\) and \([\ell]\) to Euclidean translations \( T_1(z) = z + \mu \) and \( T_2(z) = z + \lambda \) respectively, where \( \mu \) and \( \lambda \in \mathbb{C} \). Then \( \mu \) and \( \lambda \) generate the lattice \( \Lambda \) and the cusp shape is obtained as \( \lambda/\mu \).

**Remark 2.2.11.** The cusp shape depends on the choice of generators of \( \Lambda \), but a different choice changes it by an integral Möbius transformation, so the field it generates is independent of choices.

The cusp shape gives us very important information about links. There are several tools developed to determine the cusp shape for links. For example, one can enter a link diagram into SnapPy, which computes the hyperbolic structure on its complement and many geometric invariants including the cusp shape.

For example, SnapPy evaluates the cusp shape of the alternating knot 6_2 as \( 6.744313618 + 3.498585842i \). Below we will see another way to compute the cusp shape for a special class of links directly from the diagram of the links using what we call the T-T polynomial.
Definition 2.2.12. Two hyperbolic 3-manifolds are *commensurable* if they have a common finite-sheeted cover.

The cusp shape gives us key information that will enable us to analyze whether certain links are commensurable. The cusp shapes generate a number field called the *cusp field* which is a commensurability invariant [16]. Hence the cusp shapes often determine if two links are commensurable or not. For example, we will see how the Borromean rings is commensurable with the 3-pretzel fully augmented link in §6.

### 2.3 Fully Augmented Links

The class of links that we will be studying is called fully augmented links.

**Definition 2.3.1.** A link diagram is *prime* if for any simple closed curve in the plane that intersects a component transversely in two points the simple closed curve bounds a subdiagram containing no crossings. See Figure 2.1(a).

![Diagram](image)

(a) (b)

Figure 2.1: (a) Prime diagram (b) Twist Reduced diagram
Definition 2.3.2. In a link diagram, a string of bigons, or a single crossing is called a *twist region*. A link diagram is *twist reduced* if for any simple closed curve in the plane that intersects the link transversely in four points, with two points adjacent to one crossing and the other two points adjacent to another crossing, the simple closed curve bounds a subdiagram consisting of a (possibly empty) collection of bigons strung end to end between these crossings. See Figure 2.1(b).

Definition 2.3.3. A *fully augmented link* (FAL) is a link that is obtained from a diagram of a link $K$ as follows: 1) augment every twist region with a circle component (called a *crossing circle*), 2) get rid of all full twists, and 3) remove all remaining half-twists. See Figure 2.2. A diagram obtained above will be referred to as a FAL diagram. The diagram obtained after step 2) is called a FAL diagram with half-twists.

Thus the FAL diagram consists of link components in the projection plane and
crossing circle components that are orthogonal to the projection plane and bound twice punctured discs. In [21] Purcell studied the geometry of FALs using a decomposition of the FAL complement into a pair of totally geodesic hyperbolic right-angled ideal polyhedra. We will describe how the geodesic faces of these polyhedra can be seen on the FAL diagrams.

FAL, while interesting in their own right, enable us to study the geometry of the original knot or link it’s built from.

**Theorem 2.3.4.** [2, 21] A fully augmented link is hyperbolic if and only if the associated knot or link diagram is non-splittable, prime, twist reduced, with at least two twist regions.

We will only consider hyperbolic FAL in this dissertation.

**The Cut-Slice-Flatten Method and Polyhedron \( P_L \)**

Given a FAL diagram \( L \), we can obtain the polyhedra decomposition by using a construction given by Agol and D. Thurston in [13] called the *cut-slice-flatten method*. Assume that the twice punctured discs are perpendicular to the plane. First, cut the link complement in half along the projection plane, which cuts the twice punctured disc bounded by the crossing circle into half. This creates a pair of polyhedra, see Figure 2.3(a). For each half, slice open the half disc like a pita bread and flatten it down on the projection plane, see Figure 2.3(b). Lastly, shrink the link components
to ideal vertices, see Figure 2.3(c). This gives us two copies of a polyhedron which we denote as $P_L$. For each crossing circle we get a bowtie on each copy of $P_L$, which consists of two triangular faces that share the ideal vertex corresponding to the crossing circle component. The cut-slice-flatten method is part of the proof of Proposition 2.2 in [21], which we state below:

**Proposition 2.3.5.** [21] Let $L$ be a hyperbolic FAL diagram. There is a decomposition of $S^3 \setminus L$ into two copies of geodesic, ideal, hyperbolic polyhedron $P_L$ with the following properties.

1. Faces of $P_L$ can be checkerboard colored, with shaded faces corresponding to bowties, and white faces corresponding to the regions of the FAL components in the projection plane.

2. Ideal vertices of $P_L$ are all 4-valent.
(3) The dihedral angle at each edge of $P_L$ is $\frac{\pi}{2}$.

**Gluing the Polyhedra**

For FAL with or without half-twists the polyhedron $P_L$ is the same. The difference is in how they glue up. For FAL without half-twist the shaded faces glue up such that the bowties on each polyhedron glue to each other, see Figure 2.4, and then the white faces on each polyhedron get glued to their respective copies. Whereas in the case a half-twist occurs, the shaded faces get glued to the opposite shaded face on the other polyhedron, see Figure 2.5, and then the white faces on each polyhedron get glued to their respective copies. Right handed and left handed twists produce the same link complement due to the presence of the crossing circle as one can add/delete full twists without changing the link complement. In §5 we use this gluing to study the fundamental domain of a cusp.
Circle Packings and Cusp Shapes

Definition 2.3.6. A circle packing is a finite collection of circles inside a given boundary such that no two overlap and some (or all) of them are mutually tangent.

The geometry of FAL complements is studied using the hyperbolic structure on $P_L$. Since all faces of $P_L$ are geodesic, for each face, the hyperbolic plane it lies on determines a circle or line in $\mathbb{C} \cup \{\infty\}$. Purcell showed that there is a corresponding circle packing for the white geodesic faces of $P_L$, and a dual circle packing for the geodesic shaded faces of $P_L$. We can visualize $P_L$ if you place the two circle packings on top of one another, and intersect it with half-spaces in $\mathbb{H}^3$.

In [21] Purcell described a technique to compute cusp shapes of FALs by examining the circle packings and the gluing of polyhedra. The main result of this paper is that we can extend the T-T method to fully augmented links, and determine the
cusp shapes of FALs by solving an algebraic system of equations, see §5. Since the equations are obtained directly from the FAL diagram, we can directly relate the combinatorics of FAL diagrams and the geometry of FAL complements.
Chapter 3

T-T Method

3.1 The T-T Method

We will work in the upper half-space model of hyperbolic 3-space $\mathbb{H}^3$.

**Definition 3.1.1.** [24] A diagram of a hyperbolic link is *taut* if each associated checkerboard surface is incompressible and boundary incompressible in the link complement, and moreover does not contain any simple closed curve representing an accidental parabolic.

The taut condition implies that the faces in the diagram correspond to ideal polygons in $\mathbb{H}^3$ with distinct vertices. Let $L$ be a taut, oriented link diagram of a hyperbolic link. A *crossing* arc is an arc which runs from the overcrossing to the
undercrossing. Let \( R \) be a face in \( L \) with \( n \) crossings. Then \( R \) corresponds to an ideal polygon \( F_R \) in \( \mathbb{H}^3 \) as follows:

1. Distinct edges of \( L \) around the boundary of \( R \) lift to distinct ideal vertices in \( \mathbb{H}^3 \) because of the no accidental parabolic condition in Definition 3.1.1.

2. The lifts of the crossing arcs can be straightened out in \( \mathbb{H}^3 \) to geodesic edges giving the ideal polygon \( F_R \). See Figure 3.1.

Remark 3.1.2. Although the vertices of \( F_R \) are ideal and the edges are geodesic, the face of \( F_R \) need not be geodesic, i.e. \( F_R \) needs not lie on a hyperbolic plane in \( \mathbb{H}^3 \).

There are two types of parameters we will focus on in \( R \). The first type of parameter is assigned to each crossing in \( R \) and is known as the crossing label, also
referred to in the literature as *crossing geodesic parameter*, or *intercusp geodesic parameter*, denoted by $\omega_i$. The second parameter we will focus on is assigned to the edges of $L$ in $R$, and is known as an *edge label*, also referred to in the literature as *translational geodesic parameter*, or *edge parameter* and denoted by $u_j$. See Figure 3.1.

**Remark 3.1.3.** When we are in the diagram we refer to $\omega_i$ and $u_j$ as crossing and edge labels respectively. When we are in $\mathbb{H}^3$ we refer to them as intercusp and translational parameters, respectively.

We will choose a set of horospheres in $\mathbb{H}^3$ such that for every cusp the meridian curve on the cross-sectional torus has length one. Furthermore, we will choose one horosphere to be the Euclidean plane $z = 1$. It follows from results of Adams on waist size of hyperbolic 3-manifolds [3] that the horoballs are at most tangent and have disjoint interiors.

The lift of the crossing arc is a geodesic $\gamma$ in $\mathbb{H}^3$ which is an edge of the ideal polygon $F_R$ and which travels from the center of one horoball to the center of an adjacent horoball. For each horosphere the meridional direction along with geodesic $\gamma$ defines a hyperbolic half-plane. The intercusp parameter $\omega_\gamma$, is defined as $|\omega_\gamma| = e^{-d}$ where $d$ is the hyperbolic distance between the horoballs along the geodesic $\gamma$, and the argument of $\omega_\gamma$ is the dihedral angle between these two half-planes, both of which contain $\gamma$. $\omega_\gamma$ encodes information about the intercusp translation taking into
account distance and angles formed by parallel transport. The isometry that maps one horoball to another is represented up to conjugation by the $2 \times 2$ matrix

$$
\begin{pmatrix}
0 & \omega \\
\gamma & 1
\end{pmatrix}
$$

in $GL(2, \mathbb{C})$, which maps horosphere $H_2$ to horosphere $H_1$ in Figure 3.2(a).

For each edge inside a region $F$ we assign edge labels. The edges lift to ideal vertices of the polygon $F_R$ in $\mathbb{H}^3$. The edge label $u_j$ represent the translation parameter along the horosphere centered at that ideal vertex that travels from one intercusp geodesic to another. From $u_j$ we can find the distance traveled along a horoball, and the direction of travel. Since the edge label is a translation, up to conjugation it is represented by a $2 \times 2$ matrix:

$$
\begin{pmatrix}
1 & \epsilon_j u_j \\
0 & 1
\end{pmatrix}
$$

where $\epsilon_j$ is positive if the direction of the edge in the diagram is the same as the direction of travel along the region, and negative otherwise, see Figure 3.1(b), the matrix is an isometry translating one endpoint of the $u_i$ curve to the other end along the horosphere, i.e. it maps $p_i$ to $q_i$. 

Figure 3.2: (a) The isometry for the intercusp geodesic maps $H_2$ to $H_1$. (b) The isometry that maps along the translational geodesic points $p_i$ to $q_i$ or the reverse.
or \( q_i \) to \( p_i \) along horosphere \( H_i \), see Figure 3.2(b).

We use the following conventions.

1. The basis of peripheral subgroups is the canonical meridian and longitude.
   The meridian is oriented using the right hand screw rule with respect to the orientation of the link.

2. The length of meridians along the horospherical cross section on a cusp are 1 [3]. Consequently, there is a natural relationship between the two faces incident to the diagram that share an edge in the diagram: let \( R \) and \( S \) be adjacent regions that share an edge \( u \), then the edge labels \( u_R \) and \( u_S \) satisfy \( u_R - u_S = \pm 1 \) or 0 depending on whether the edge is going from overpass to underpass, underpass to overpass, or staying leveled respectively from within region \( R \). See Figure 3.3.

3. The edge labels inside a bigon are zero.

**Remark 3.1.4.** For convention 2 above, this relationship holds if the actual region in the diagram corresponds to the ideal polygonal face in \( \mathbb{H}^3 \) as described above. In this case, the translation on either side of the region will start and end with the same intercusp geodesics as the other side. However, below we will see that for fully augmented links, the faces coming from the crossing circles will not be the faces from the diagram directly, but will require the polyhedral decomposition of the
complement first. In this case the edges will not share the same intercusp geodesics, thus the above relationship will not necessarily hold, and will require modification.

**Definition 3.1.5.** Let the ideal vertices of the \( n \)-sided ideal polygon \( F_R \) corresponding to the face \( R \) be \( z_1,\ldots,z_n \). We will assign a shape parameter to each edge of the polygon as follows: Let \( \gamma_i \) be a geodesic edge between ideal vertices \( z_i \) and \( z_{i+1} \) then its **shape parameter** \( \xi_i \) is defined as

\[
\xi_i = \frac{(z_{i-1} - z_i)(z_{i+1} - z_{i+2})}{(z_{i-1} - z_{i+1})(z_i - z_{i+2})},
\]

which is the cross-ratio of four consecutive vertices of \( F_R \).

Thistlethwaite and Tsvietkova show that the above shape parameter can be written in terms of crossing and edge labels in Proposition 4.1 in [24]. For our purposes...
all the faces in our class of links will have total geodesic faces as we shall see below.

**Proposition 3.1.6.** [24] Up to complex conjugation, \( \xi_i = \frac{\pm \omega_i}{u_i u_{i+1}} \) where the sign is positive if both edges are directed away or both are directed toward the crossing, negative if one edge is directed into the crossing and one is directed out.

**Proof.** Let \( z_0, z_1, z_2, z_3 \) be four consecutive ideal vertices in \( \mathbb{H}^3 \) that correspond to the edges with edge labels \( u_0, u_1, u_2, u_3 \) respectively in the link diagram \( L \), See Figure 3.4. We can always perform an isometry and let \( z_0 \) be placed at \( |u_1| \), \( z_1 \) at \( \infty \) where the horoball \( H_x \) is at Euclidean height 1, \( z_2 \) at \( (0,0,0) \). Let \( \gamma_0 \) connect \( z_0 \) to \( z_1 \), correspond to \( \omega_0 \) in \( L \), \( \gamma_1 \) be the geodesic connecting \( z_1 \) to \( z_2 \) correspond to \( \omega_1 \) in \( L \) and \( \gamma_2 \) connect \( z_2 \) to \( z_3 \) correspond to \( \omega_2 \) in \( L \). The horoball \( H_2 \) has diameter \( |\omega_1| \) since the hyperbolic distance between \( H_x \) and \( H_2 \) is \( \log \frac{1}{|\omega_1|} \) and \( |\omega| = e^{-d} \). In [24]
T-T showed that \( u_2 = \frac{|\omega_1|}{|z_3|} \). Thus the shape parameter \( \xi_1 \) is

\[
\xi_1 = \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_2)(z_1 - z_3)} = \frac{z_3}{z_0} = \frac{|\omega_1|}{u_2} = \frac{|\omega_1|}{u_1 u_2}.
\]

If either \( u_1 \) or \( u_2 \) exclusively were going in the opposite direction then it will cause the shape parameter to be of different sign.

Let \( R_i \) be a face in \( L \), which corresponds to \( F_{R_i} \) in the ideal polygon. Fix \( F_{R_i} \), we can perform an isometry sending ideal vertices \( z_{i-1}, z_i, \) and \( z_{i+1} \) to \( 1, \infty, \) and \( 0 \) respectively, then \( z_{i+2} \) will be placed at \( \xi_i \). Since the region closes up, the collection of shape parameters for each region determines the isometry class of the associated ideal polygon. The shape parameters \( \xi_i \) satisfy algebraic equations amongst themselves. For example, for a 3-sided region the shape parameters are equal to each other and are equal to 1, while in a 4-sided region the sum of consecutive shape parameters is equal to 1. For regions with \( n \geq 5 \) we use Proposition 4.2 in [24] to determine the algebraic equations in terms of crossing and edge labels.

For each region in the diagram there is an alternating sequence of edges and crossings until the region closes up. The product of the corresponding matrices is a scalar multiple of the identity. Consequently, we have a system of equations whose solution allows us to construct a discrete faithful representation of the complement. We state Proposition 4.2 in [24].
Proposition 3.1.7. Let $R$ be a region of an oriented link diagram with $n \geq 3$ sides, and, starting from some crossing of $R$, let

$$u_1, \omega_1, u_2, \omega_2, \ldots, u_n, \omega_n$$

be the alternating sequence of edge and crossing labels for $R$ encountered as one travels around the boundary of the region. Also, for $1 \leq i \leq n$ let $\epsilon_i = 1$ (resp. $\epsilon_i = -1$) if the direction of the edge corresponding to $u_i$ is with (resp. against) the direction of travel. Then the equation for $R$ is written as

$$\prod_{i=1}^{n} \begin{pmatrix} 0 & \omega_i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \epsilon_i u_i \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

This can be done for each region of $L$, thus we have algebraic equations for each face $R$ of $L$ in terms of crossing and edge labels, the solutions to the algebraic equations allow us to construct the discrete faithful representation of the link group into $PSL_2(\mathbb{C})$.

It is proved in [24] that the solutions to the above system of equations is discrete. Thus we can eliminate the variables and reduce this system of equations to a 1-variable polynomial, referred to as the $T-T$ polynomial. Neumann-Tsvietkova [18] related the solutions to the invariant trace field of the link complement.
3.2 Examples

3.2.1 Example 62

In Figure 3.5 we have a taut alternating diagram of the knot 62. Region 8: This is a three-sided region with shape parameters:

$$\xi_1 = \frac{-\omega_2}{(u_2 + 1)(u_1 + 1)} = 1, \quad \xi_2 = \frac{\omega_1}{(u_3 + 1)(u_1 + 1)} = 1, \quad \xi_3 = \frac{\omega_3}{(u_2 + 1)(u_3 + 1)} = 1.$$
**Region ॥**: This is a four-sided region with shape parameters:

\[ \xi_1 = \frac{\omega_1}{u_3}, \quad \xi_2 = -\omega_1, \quad \xi_3 = \frac{\omega_1}{u_4}, \quad \xi_4 = \frac{-\omega_3}{u_3u_4}. \]

Thus we have equations:

\[ \frac{\omega_1}{u_3} - \omega_1 = 1, \quad -\omega_1 + \frac{\omega_1}{u_4} = 1, \quad \frac{\omega_1}{u_4} - \frac{\omega_3}{u_3u_4} = 1, \quad \frac{-\omega_3}{u_3u_4} - \frac{\omega_1}{u_3} = 1 \]

**Region ॥**: This is a three-sided region with shape parameters:

\[ \xi_1 = \frac{-\omega_2}{(u_4 + 1)(u_6 + 1)} = 1, \quad \xi_2 = \frac{-\omega_2}{(u_5 + 1)(u_4 + 1)} = 1, \quad \xi_3 = \frac{-\omega_3}{(u_5 + 1)(u_4 + 1)} = 1. \]

**Region ॥**: This is a three-sided region with shape parameters:

\[ \xi_1 = \frac{-\omega_2}{u_2} = 1, \quad \xi_2 = \frac{-\omega_3}{u_2u_5} = 1, \quad \xi_3 = \frac{-\omega_2}{u_5} = 1. \]

Solving gives us the relations

\[ u_2 = u_5 = -\omega_2, \quad -\omega_3 = u_2^2. \]
Using these relations in the other regions we find that

\[ u_1 = u_5, \quad u_3 = u_4 \]

We can now simplify and get all the crossing geodesic parameters in terms of \( u_2 \)

Using the equation \( \frac{\omega_1}{u_4} - \frac{\omega_3}{u_3u_4} = 1 \) and substituting all the parameters in terms of \( u_2 \) we get the equation

\[
\frac{1}{\frac{1}{u_2^2} - 1} + \frac{u_2^2}{\frac{-u_2^2}{u_2^2 + 1} - 1} = \frac{-u_2^2}{u_2 + 1} - 1,
\]

which gives us the polynomial

\[
u_2^5 + u_2^4 - u_2^3 - 4u_2^2 - 3u_2 - 1 = 0.
\]

Solving the above equation gives us one real root and 2 pairs of complex roots.

\[
u_{2_1} \approx 1.70062, \quad u_{2_2} \approx -0.896438 \pm 0.890762i, \quad u_{2_3} \approx -0.453870 \pm 0.402731i.
\]

Neumann-Tsvietkova proved in [18] that one of the roots of the polynomial should give us the invariant trace field. Using Snap [12] we find the invariant trace field is given by polynomial

\[
x^5 - x^4 + x^3 - 2x^2 + x - 1
\]
with root $x \approx 0.276511 - 0.728237i$.

We can check the linear dependence using mathematica or pari-gp. Checking dependence we find that $u_{23} = x^2$. Hence $u_{23}$ generates the invariant trace field. We checked that this is not the case for the other roots $u_{21}$ and $u_{22}$.

It is suggested in [24] that the geometric solution will be the one that produces the highest volume, but finding the volume from the solutions can be difficult. In §6 we will show how to find the geometric solution for the class of fully augmented links.

Figure 3.6: Hamantash Link
3.2.2 Hamantash Link

Region $\mathbb{N}$: This is a four-sided region with shape parameters:

$$
\xi_1 = \frac{\omega_1}{u_1}, \quad \xi_2 = \frac{\omega_1}{u_2}, \quad \xi_3 = \frac{\omega_3}{u_2}, \quad \xi_4 = \frac{\omega_3}{u_1}.
$$

Thus the equations are:

$$
\frac{\omega_1}{u_1} + \frac{\omega_1}{u_2} = 1, \quad \frac{\omega_1}{u_2} + \frac{\omega_3}{u_2} = 1, \quad \frac{\omega_3}{u_2} + \frac{\omega_3}{u_1} = 1, \quad \frac{\omega_3}{u_1} + \frac{\omega_1}{u_1} = 1
$$

solving gives us the relations

$$
u_1 = u_2 = 2\omega_1 = 2\omega_3, \quad \text{and} \quad \omega_1 = \omega_3.
$$

Region $\mathbb{D}$: This is a four-sided region with shape parameters:

$$
\xi_1 = \frac{\omega_1}{u_4}, \quad \xi_2 = \frac{\omega_1}{u_3}, \quad \xi_3 = \frac{\omega_2}{u_3}, \quad \xi_4 = \frac{\omega_2}{u_4}.
$$

Thus the equations are:

$$
\frac{\omega_1}{u_4} + \frac{\omega_1}{u_3} = 1, \quad \frac{\omega_1}{u_3} + \frac{\omega_2}{u_3} = 1, \quad \frac{\omega_2}{u_3} + \frac{\omega_2}{u_4} = 1, \quad \frac{\omega_2}{u_4} + \frac{\omega_1}{u_4} = 1
$$
CHAPTER 3. T-T METHOD

solving gives us the relations

\[ u_3 = u_4 = 2\omega_1 = 2\omega_2, \quad \text{and} \quad \omega_1 = \omega_2. \]

Region 3: This is a four-sided region with shape parameters:

\[ \xi_1 = \frac{\omega_3}{u_6}, \quad \xi_2 = \frac{\omega_2}{u_6}, \quad \xi_3 = \frac{\omega_2}{u_5}, \quad \xi_4 = \frac{\omega_3}{u_5}. \]

Thus the equations are:

\[ \frac{\omega_3}{u_6} + \frac{\omega_2}{u_6} = 1, \quad \frac{\omega_2}{u_6} + \frac{\omega_2}{u_5} = 1, \quad \frac{\omega_2}{u_5} + \frac{\omega_3}{u_5} = 1, \quad \frac{\omega_3}{u_5} + \frac{\omega_3}{u_6} = 1 \]

solving gives us the relations

\[ u_5 = u_6 = 2\omega_2 = 2\omega_3, \quad \text{and} \quad \omega_2 = \omega_3. \]

Region 7: This is a three-sided region with edge labels \(-1 + u_2, -1 + u_4, -1 + u_6\).

\[ \xi_1 = \frac{-\omega_1}{(-1 + u_2)(-1 + u_4)} = 1, \quad \xi_2 = \frac{-\omega_2}{(-1 + u_4)(-1 + u_6)} = 1, \quad \xi_3 = \frac{-\omega_3}{(-1 + u_6)(-1 + u_2)} = 1 \]
solving these equations we get the T-T polynomial as

\[ 4\omega_1^2 - 3\omega_1 + 1 = 0 \]

thus

\[ \omega_i = \frac{3}{8} \pm \frac{\sqrt{7}}{8}i, \quad u_i = \frac{3}{4} \pm \frac{\sqrt{7}}{4}i. \]

Using Snap the invariant trace field for the Hamantash link is

\[ x^2 - x + 2, \quad x = \frac{1}{2} + \frac{\sqrt{7}}{2}i \quad \text{and} \quad \omega_1 = \frac{1 + x}{4}. \]
Chapter 4

T-T Method for FAL

4.1 T-T Method and FAL

In this chapter we show that the T-T method can be effectively extended to the class of fully augmented links. Our extension of the T-T method will be on a trivalent graph which is the intermediate step between the FAL diagram and the polyhedron $P_L$. We denote the planar trivalent graph $T_L$, for example see Figure 2.3(b). $T_L$ is in fact the ideal polyhedron $P_L$, truncated at the ideal vertices, along with the orientations on the links of the vertices. Since the vertices are all 4-valent, the link of the vertices are all rectangles which tessellate the cusp torus. Thinking of the long thin rectangular pieces as thick edges in Figure 4.1 ($T_L$ for Borromean FAL), one gets the trivalent graph $T_L$. The components of the link diagram and the crossing
geodesics are both visible on $T_L$. The crossing geodesics are on the boundary of the hexagonal regions corresponding to the bowties. We will assign the edge labels and the crossing labels on this type of diagram for the T-T method to work.

In order to use the T-T method we need to ensure that it can be applied to the class of FALs. The tautness condition on the diagram is to ensure that the faces in the link diagram correspond to ideal polygons in $\mathbb{H}^3$ with distinct vertices.

**Proposition 4.1.1.** Let $L$ be a FAL diagram, then the planar trivalent graph $T_L$ is taut.

**Proof.** Let $L$ be a hyperbolic FAL. By Lemma 2.1 in [21], the following surfaces are embedded totally geodesic surfaces in the link complement:
(1) twice punctured discs coming from the regions bounded by the crossing circles and punctured by two strands in the projection plane, and

(2) the surfaces in the projection plane.

Embedded totally geodesic surfaces are Fuchsian and Thurston’s trichotomy for surfaces in 3-manifolds implies a surface can either be quasifuchsian, accidental, or semi-fibered [11]. This implies they do not contain any accidental parabolics. We have a checkerboard coloring by shading the discs coming from the regions bounded by the crossing circles, and leaving the surfaces in the projection plane white. Thus by definition, the checkerboard surfaces of $T_L$ are incompressible and boundary incompressible. Hence $T_L$ is taut.

Since the regions of $T_L$ correspond to geodesic faces in $\mathbb{H}^3$, the definitions for crossing and edge labels in the T-T method, the corresponding matrices, the shape parameters and equations of Propositions 3.1.6 and 3.1.7 hold for $T_L$. The fundamental difference is in the relationship between parameters for edges incident to adjacent faces as will be discussed below.

### 4.2 Thrice Punctured Sphere

The twice punctured disc bounded by the crossing circle is geodesic and has the hyperbolic structure of the thrice punctured sphere formed by gluing two ideal tri-
angles. So we will refer to the twice punctured discs as thrice punctured spheres from now on. Let $L$ be a FAL, then $L$ contains at least two crossing circles. This implies $S^3 \setminus L$ contains at least two thrice punctured spheres. We will first study how the T-T method defines parameters on the thrice punctured sphere and use this as a basic building block for FALs.

The thrice punctured sphere has three components, two strands that lie in the projection plane, and another circle component know as a crossing circle that encircles the two other strands. The thrice punctured sphere is known to be totally geodesic constructed by gluing two ideal triangles together along their edges [1]. There are two cases based on the orientation: one where the strands in the projection plane are parallel and the other when they are anti-parallel. On $T_L$ the thrice punctured sphere corresponds to a hexagon, and on $P_L$ it corresponds to a bowtie. We will study the part of $T_L$ corresponding to the hexagon.

**Theorem 4.2.1.**  
(1) The crossing labels on opposite sides of the augmented circles with parallel strands in the link diagram will be equal and the intercusp geodesic along the projection plane equals $-1/4$.

(2) The crossing labels on opposite sides of the augmented circles with anti-parallel strands in the link diagram will differ by sign and the intercusp geodesic along the projection plane equals $1/4$. 
Proof. For each crossing circle there are four crossing labels $\omega_i$. The two labels that share a bigon are equivalent since the region collapses and has the same geodesic arc going from horoball to horoball [24]. For the relationship between the two crossing labels that don’t share a bigon we have two cases:

Case 1: Parallel strands in the link diagram

As the cusp torus for the crossing circle is cut in half, the translation parameters coming from the longitudinal strands in the projection plane will also be cut in half and are $1/2$ keeping with the convention that the meridional curve along the cross sectional torus has length 1 and keeping with the right hand screw rule. For region $\mathcal{N}_A$ in Figure 4.2(b) we have shape parameters:
\[ \xi_1 = \frac{\omega_1}{\frac{1}{2}u_1} = 1, \quad \xi_2 = \frac{-\omega_3}{\frac{1}{2} \times \frac{1}{2}} = 1, \quad \xi_3 = \frac{\omega_2}{\frac{1}{2}u_1} = 1, \]

solving these equations gives us the relations

\[ \omega_3 = -\frac{1}{4}, \quad \omega_1 = \omega_2, \quad \text{and} \quad u_1 = 2\omega_1. \]

Using Proposition 3.1.7 we can check that these parameters are correct. Starting from the edge \( \omega_1 \) in the left side of region \( \mathcal{A} \) and traveling counterclockwise we have:

\[
\begin{bmatrix}
0 & \omega_1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1/2 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & -1/4 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & \frac{1}{2} \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & \omega_2 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & -u_1 \\
0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
-\frac{\omega_1}{2} & 0 \\
0 & -\frac{\omega_1}{2} \\
\end{bmatrix}.
\]

Case 2: Anti-parallel strands in the link diagram.

Notice here that the translation parameters coming from the longitudinal strands will be \( \frac{1}{2} \) but their directions differ each going according to the right hand screw rule and the orientation on the strands, see Figure 4.3. For Region \( \mathcal{B} \) we have shape parameters

\[ \xi_1 = \frac{\omega_1}{\frac{1}{2}u_1} = 1, \quad \xi_2 = \frac{-\omega_2}{\frac{1}{2}u_1} = 1, \quad \xi_3 = \frac{\omega_3}{\frac{1}{2} \times \frac{1}{2}} = 1. \]
Figure 4.3: (a) Thrice punctured sphere with anti-parallel strands with the intercusp geodesics penciled in. (b) For each half we slice open and flatten. Here we can see the intercusp and translational parameters in the regions.

solving these equations gives us the relations

\[ \omega_3 = \frac{1}{4}, \quad \omega_1 = -\omega_2, \quad \text{and} \quad u_1 = 2\omega_1. \]

Using Proposition 3.1.7, starting from the red edge \( \omega_1 \) in the left side of region \( \aleph \) and traveling counterclockwise we have:

\[
\begin{pmatrix}
0 & \omega_1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1/2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1/4 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & -\frac{1}{2} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & \omega_2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & -u_1 \\
0 & 1
\end{pmatrix}
\]

substituting in the above relations =

\[
\begin{pmatrix}
-\frac{\omega_3}{2} & 0 \\
0 & -\frac{\omega_3}{2}
\end{pmatrix}
\].
4.3 Adaptation of T-T Method for FAL Diagram

All the shaded faces on a FAL diagram come from thrice punctured spheres, and have parameters as determined in Theorem 4.2.1. Hence to set up the T-T equations, we only need to understand the edge and crossing parameters on the regions in the projection plane. These regions are the white faces which have boundary alternating between the crossing geodesics and strands of the link diagram, except when it intersects the crossing circle. At the intersection of the region and the crossing circle, the boundary goes across a meridian of the crossing circle, see Figure 4.1. Thus with an adjustment, we can write down the equations directly from the FAL diagram, without using $T_L$.

**Lemma 4.3.1.** For each crossing circle, the two translational geodesics that correspond to the parts of the crossing circle component that bound the bigons, one going from $\omega_1$ to $\omega_1$ and the other going from $\omega_2$ to $\omega_2$ correspond to the meridional curve for that component, thus both are oriented the same way and are equal to 1.

**Proof.** The part of the cusp torus corresponding to a crossing circle lies on the hexagon in $T_L$, such that the meridians lie flat in the projection plane. The orientations on the meridians are obtained by the right hand screw rule. The meridians on opposite sides of the crossing circle are homotopic and are oriented as in Figure 4.4(b).
Consequently, starting from a FAL diagram, we can reorient parts of the crossing circle on the FAL diagram to agree with the orientations on the meridians. See Figure 4.4(c). With this adjustment we can now write the T-T equations directly from the FAL diagrams without using the trivalent graph $T_L$.

Now we need to analyze the relationship between edge labels coming from adjacent regions of an edge.

**Lemma 4.3.2.** The edge labels on opposite sides of an edge coming from the longitudinal strands without a half-twist in the FAL diagram are equal.

**Proof.** Purcell showed how the cusps for FAL are tiled by rectangles [21]. Since the white faces on the two polyhedra are glued by identity, these rectangles can be seen in between the white faces on $T_L$, see Figure 4.1. The parts of the longitude corresponding to the adjacent regions are homotopic across the sliced torus. Hence
they are equal.

In [21], Purcell showed that the complements of a FAL with and without a half-twist have the same polyhedral decomposition, but with different gluing on shaded faces. Thus the faces of the regions of a FAL diagram with half-twist do not represent white faces of \( P_L \). So for FAL with half-twists we will take \( T_L \) the same as the one for the corresponding FAL without half-twists. We look at faces of \( T_L \), we can find the intercusp geodesic parameters and translational parameters from analyzing the shear that is caused by the half-twist gluing. This will be done below when we look at the cusps in \( \S 5 \).

4.4 Examples

4.4.1 Borromean Ring FAL

See Figure 4.5. Recall, for a 3-sided region all \( \xi_i = 1 \). Region \( \aleph \):

\[
\xi_1 = \frac{-\omega_2}{u_1} = 1, \quad \xi_2 = \frac{-\omega_2}{u_2} = 1, \quad \xi_3 = \frac{-1}{u_1 u_2} = 1,
\]

\[\implies u_1 = u_2 = -\omega_2 \quad \text{and} \quad u_1^2 = -\frac{1}{4} \implies u_1 = \pm \frac{i}{2}.
\]

Region \( \daleth \):

\[
\xi_1 = \frac{-\omega_2}{u_3} = 1, \quad \xi_2 = \frac{-\omega_2}{u_4} = 1, \quad \xi_3 = \frac{-1}{u_3 u_4} = 1,
\]
Figure 4.5: Borromean ring FAL with crossing and edge parameters.

\[
\begin{align*}
\text{⇒ } u_3 &= u_4 = -\omega_2 = \pm \frac{i}{2}. \\
\text{Region } \mathbb{J}: \\
\xi_1 &= \frac{-\omega_1}{u_3}, \quad \xi_2 = \frac{-\omega_1}{u_2} = 1, \quad \xi_3 = \frac{-\frac{1}{3}}{u_2 u_3} = 1, \\
\text{⇒ } u_2 &= u_3 = -\omega_1 = \pm \frac{i}{2}.
\end{align*}
\]

4.4.2 4CC$_1$

We denote the FAL shown in Figure 4.6 as 4CC$_1$. 
Figure 4.6: $4CC_1$
Region \( \aleph \):

This is a four-sided region with shape parameters:

\[
\xi_1 = \frac{-\omega_1}{u_2}, \quad \xi_2 = \frac{-\omega_2}{u_2}, \quad \xi_3 = \frac{-\omega_2}{u_1}, \quad \xi_4 = \frac{-\omega_1}{u_1}.
\]

The sum of consecutive shape parameters are 1.

\[
\frac{-\omega_1}{u_2} - \frac{\omega_2}{u_2} = 1, \quad \frac{-\omega_2}{u_2} - \frac{\omega_2}{u_1} = 1, \quad \frac{-\omega_2}{u_1} - \frac{\omega_1}{u_1} = 1, \quad \frac{-\omega_1}{u_1} - \frac{\omega_1}{u_2} = 1
\]

solving gives us the relations

\[
u_1 = u_2 = -2\omega_1 = -2\omega_2, \quad \text{and} \quad \omega_1 = \omega_2.
\]

Region \( \beth \):

This is a four-sided region with shape parameters:

\[
\xi_1 = \frac{-\frac{1}{4}}{u_3u_5}, \quad \xi_2 = \frac{-\omega_1}{u_3}, \quad \xi_3 = \frac{-\omega_1}{u_4}, \quad \xi_4 = \frac{-\frac{1}{4}}{u_4u_5}.
\]

Thus we have equations:

\[
\frac{-\frac{1}{4}}{u_3u_5} - \frac{\omega_1}{u_3} = 1, \quad \frac{-\omega_1}{u_3} - \frac{\omega_1}{u_4} = 1, \quad \frac{-\omega_1}{u_4} - \frac{1}{4} = 1, \quad \frac{-\frac{1}{4}}{u_4u_5} - \frac{1}{4} = 1
\]
solving gives us the relations \( u_4 = u_3 = -2\omega_1 \), and \( u_5 = \frac{1}{4\omega_1} \).

Region \( \mathfrak{I} \):

\[
\xi_1 = -\frac{1}{u_8 u_6}, \quad \xi_2 = -\frac{1}{u_8 u_7}, \quad \xi_3 = \frac{-\omega_2}{u_7}, \quad \xi_4 = -\frac{\omega_2}{u_6}
\]

This is a four-sided region with equations:

\[
\begin{align*}
\frac{-1}{u_8 u_6} - \frac{1}{u_8 u_7} &= 1, \quad \frac{-1}{u_8 u_7} - \frac{\omega_2}{u_7} &= 1, \quad \frac{-\omega_2}{u_7} - \frac{\omega_2}{u_6} &= 1, \quad \frac{-\omega_2}{u_6} - \frac{1}{u_8 u_7} &= 1
\end{align*}
\]

solving gives us the relations

\[
u_6 = u_7 = -2\omega_2, \quad \text{and} \quad u_8 = \frac{1}{4\omega_2}.
\]

Region \( \mathfrak{I} \):

\[
\xi_1 = -\frac{1}{u_4 u_1}, \quad \xi_2 = -\frac{1}{u_4 u_7}, \quad \xi_3 = \frac{\omega_3}{u_7}, \quad \xi_4 = \frac{\omega_3}{u_4}
\]

This is a four-sided region with equations:

\[
\begin{align*}
\frac{-1}{u_4 u_1} - \frac{1}{u_4 u_7} &= 1, \quad \frac{-1}{u_4 u_7} + \frac{\omega_3}{u_7} &= 1, \quad \frac{\omega_3}{u_7} + \frac{\omega_3}{u_4} &= 1, \quad \frac{\omega_3}{u_4} - \frac{1}{u_4 u_1} &= 1
\end{align*}
\]

solving gives us the relations

\[
u_4 = u_7 = 2\omega_3, \quad \text{and} \quad u_1 = -\frac{1}{4\omega_3}.
\]
Region $E$: This is a four-sided region with shape parameters:

$$\xi_1 = \frac{-\omega_4}{u_5}, \quad \xi_2 = \frac{\omega_3}{u_5}, \quad \xi_3 = \frac{\omega_3}{u_8}, \quad \xi_4 = \frac{-\omega_4}{u_8}.$$ 

Thus the equations are:

$$\frac{-\omega_4}{u_5} + \frac{\omega_3}{u_5} = 1, \quad \frac{\omega_3}{u_5} + \frac{\omega_3}{u_8} = 1, \quad \frac{\omega_4}{u_8} - \frac{\omega_4}{u_8} = 1, \quad \frac{-\omega_4}{u_8} - \frac{\omega_4}{u_5} = 1$$

solving gives us the relations

$$u_5 = u_8 = 2\omega_3 = -2\omega_4, \quad \text{and} \quad \omega_3 = -\omega_4.$$ 

Using the fact that opposite sides of an edge are equal, we get

$$\omega_1 = \pm \frac{\sqrt{2}}{4} i.$$
Chapter 5

Cusp Shapes of FAL

5.1 FAL Cusp Shapes

In [21] Purcell described a method to compute the cusp shapes for each cusp of a FAL using the polyhedral decomposition, by lifting the ideal vertex corresponding to a crossing circle to $\infty$, constructing a circle packing and computing the radii of each circle.

In Theorem 5.1.1 below we prove that the extension of the T-T method to FALs in Section 4 enable us to compute cusp shapes in a simpler way, by solving algebraic equations derived directly from the FAL diagram, without constructing the polyhedral decomposition, and circle packings.
Theorem 5.1.1. Let \( L \) be a FAL diagram and let \( \omega \) be the parameter of the crossing geodesic for a crossing circle \( C \) of \( L \).

1. If \( L \) has no half-twist at \( C \), then the cusp shape of \( C \) is \( 4\omega \).

2. If \( L \) has a RH half-twist at \( C \), then the cusp shape of \( C \) is \( \frac{4\omega}{1 + 2\omega} \).

3. If \( L \) has a LH half-twist at \( C \), then the cusp shape of \( C \) is \( \frac{4\omega}{1 - 2\omega} \).

Remark 5.1.2. The FAL complements with RH half-twist and LH half-twist are isometric as a RH half-twist can be changed to a LH half-twist in presence of a crossing circle by adding a full twist, which is a homeomorphism. However the
canonical longitude for the crossing circle is different in each case, thus we get a different cusp shape.

Proof. We will determine the longitude and meridian curves in the fundamental domain for the given crossing circle. Let \( L \) be a FAL and \( C \) be a crossing circle. Let \( S^3 - L = P_1 \cup P_2 \), where \( P_1 \) and \( P_2 \) are isometric to the right angled polyhedron \( P_L \) described in Proposition 4.1.1. The twice punctured disc bounded by \( C \) becomes a bowtie on \( P_L \) and the ideal point corresponding to \( C \) is the center of the bowtie. Let \( p \) denote the ideal point corresponding to \( C \), since the faces of \( P_L \) are geodesic, they lie on hyperbolic planes, which are determined by circles or lines on \( \mathbb{C} \cup \infty \). The four faces incident to \( p \) are two white faces and two shaded faces. Correspondingly we have two tangent circles in the white circle packing, and two tangent circles in the dual shaded circle packing, see Figure 5.2(b).
Superimposing the two circle packings, and taking the point \( p \) to \( \infty \), the four circles tangent to \( p \) become lines that form the rectangle of the cusp on each polyhedron \( P_1 \) and \( P_2 \). Let \( H_\infty \) denote the horizontal plane corresponding to the horosphere centered at \( p \). See Figure 5.2 (c). All other circles lie inside this rectangle, since the circles are at most tangent to one another and do not overlap. To find the cusp shape we need to study the fundamental domain of the cusp.

Case 1: Purcell showed that for FAL without a half-twist present, the fundamental domain for the cusp torus for \( C \) is formed by two rectangles attached along a white edge (representing a white face).

Let’s describe the longitude and meridian curves along the crossing circle component of the FAL. Let \( s', r', t', q' \) denote the points on \( H_\infty \) that are directly above \( s, r, t, q \) respectively, translated along respective crossing geodesics \( \omega_i \). See Figure 5.2(b). The fundamental domain is formed by taking two copies of the rectangle \( s'r'q't' \) glued along the edge \( s'r' \). The lift of the meridian is \( s'r' \), and the longitude is double the curve \( s't' \).

From the computations of the thrice punctured sphere in Section 3.1 edge parameter \( u_1 \) is isometric to geodesic \( s't' \) which is the translation parameter along the horoball at hand and is \( 2\omega \). Since it is double in the actual cusp, the longitude parameter is \( 4\omega \). The translation parameter \( s'r' \) is isometric to the meridian and is \( 1 \). Therefore, the cusp shape \( \frac{\lambda}{\mu} = \frac{4\omega}{1} \), see Figure 5.3.
Case 2: for FAL with half-twists present, i.e. the crossing circle cusps that bound a half-twist will be tiled by rectangles but the fundamental domain will be a parallelogram due to a shear in the universal cover, its longitude curve will run along the shaded face (same as the case without a half-twist present) which is $4\omega$. The meridian curve will run diagonally across since it takes one step along a white face and one step along a shaded face. This is due to a twist in the gluing of the shaded faces. $s'_2$ will be identified with $q'_1$, and $t'_2$ will not be identified with $q'_2$.

There are two cases: The twist goes with a RH half-twist, see Figure 5.5(a), where the meridian goes diagonal increasing from left to right, thus it’s $1 + 2\omega$ so the cusp shape is $\frac{4\omega}{1 + 2\omega}$. When the twist goes with the LH half-twist, see Figure 5.6(b), the diagonal is decreasing from left to right, it goes one step down which is $-2\omega$ and one step across which is 1, thus it’s $1 - 2\omega$ and the cusp shape is $\frac{4\omega}{1 - 2\omega}$.
CHAPTER 5. CUSP SHAPES OF FAL

Figure 5.4: Gluing in Half-twist

Figure 5.5: (a) TPS with RH half-twist. (b) The corresponding fundamental domain of the cusp due to the RH half-twist.

Figure 5.6: (a) TPS with LH half-twist. (b) The corresponding fundamental domain of the cusp due to the LH half-twist.
Note that the imaginary part of the cusp shape will always be positive.

The cusp shapes for the strands in the projection plane can also be determined from the labels in the diagram.

**Theorem 5.1.3.** For a FAL without a half-twist, the cusp shape for a component in the projection plane will be the sum of the edge labels as one goes around the strand.

**Proof.** In [19] Purcell showed that the component in the projection plane is tiled by a sequence of rectangles two for each segment of a component which is then glued along the shaded edge as one goes around the strand. See Figure 5.7. For each component in the projection plane there is a cusp that is tiled by rectangles coming from each portion along the component. Two identical rectangles are glued along a white edge for the upper and lower polyhedra. Then along the shaded edge the portion of the component adjacent will be glued along the shaded edge. The meridian
is by convention 1, where we have a $\frac{1}{2}$ for the meridional segment along each shaded triangle, See Figure 4.2(b). The longitude will consist of $u_j$ for each edge. The sum for each portion along the strand will be the longitude of the cusp.

\[ \square \]

**Remark 5.1.4.** For a link component in the projection plane that has half-twists, tracking the longitude is a bit trickier. A FAL with a half-twist and more than one component in the projection plane, the component in the projection plane which goes through an odd number of half-twists will still have meridian of length 1. The longitude will travel parallel to the projection plane except when it will pass a half-twist where it will then travel down/up to the other polyhedron thus it will increase its length by $\pm k\mu/2$ where $k$ takes into account the direction of the half-twist and how many times it passes a half-twist. The cusp shape for the component will be the sum of the edge labels plus half an integer, $\sum u_i \pm k \times \frac{1}{2}$, where the sign will depend on the direction of the half-twists. This is due to a shear, thus the cusp will not necessarily be rectangular. However, if the component goes through an even number of half-twists, then it will have a rectangular cusp, just the longitude won’t be perpendicular to the real meridian–the cusp shape will be as if no half-twists are present. In addition, if a FAL has only one component in the projection plane then the cusp will be rectangular regardless if there is a half-twist present [20].

Experimentally, FAL with odd number of half-twists such that the presence of the half-twist reduces the number of components seem to be the ones impacted by
the shear. Moreover, when there is one half-twist present the cusp shape will be \( \sum u_i \pm 2 \).

## 5.2 Examples

### 5.2.1 Borromean Ring FAL with a half-twist

See Figure 5.8. Using the results from the Borromean Ring FAL without half-twist, we get

\[
u_2 = u_3 = -\omega_1 = -\omega_2 = \pm \frac{i}{2}.
\]

Thus there are three cusps: Cusp \( p \) with cusp shape

\[
4\omega_2 = 4 \times \frac{i}{2} = 2i.
\]
Cusp $q$ with cusp shape

$$\frac{4\omega_1}{1 - 2\omega_1} = \frac{4 \times \frac{i}{2}}{1 - 2 \times \frac{i}{2}} = -1 + i.$$  

The cusp shape for the single component in the projection plane, has rectangular cusp with longitude

$$u_1 + u_2 + u_3 + u_4 = 4 \times \frac{i}{2} = 2i.$$
5.2.2 3 Pretzel FAL without half-twist

Region $\aleph$: This is a three-sided region with shape parameters:

\[
\xi_1 = \frac{-1}{u_1u_3} = 1, \quad \xi_2 = \frac{-1}{u_1u_2} = 1, \quad \xi_3 = \frac{-1}{u_2u_3} = 1
\]

solving gives us the relations

\[
u_1 = u_2 = u_3 \quad \text{and} \quad u_1 = \pm \frac{i}{2}
\]

Region $\beth$: This is a three-sided region with shape parameters:

\[
\xi_1 = \frac{-1}{u_4u_6} = 1, \quad \xi_2 = \frac{-1}{u_4u_5} = 1, \quad \xi_3 = \frac{-1}{u_5u_6} = 1
\]

solving gives us the relations

\[
u_4 = u_5 = u_6 \quad \text{and} \quad u_4 = \pm \frac{i}{2}
\]

Region $\gimel$:

\[
\xi_1 = \frac{\omega_1}{u_4}, \quad \xi_2 = \frac{-\omega_2}{u_4}, \quad \xi_3 = \frac{-\omega_2}{u_2}, \quad \xi_4 = \frac{\omega_1}{u_2}
\]
This is a four-sided region with equations:

\[
\frac{\omega_1}{u_4} - \frac{\omega_2}{u_4} = 1, \quad \frac{-\omega_2}{u_4} - \frac{\omega_2}{u_2} = 1, \quad \frac{-\omega_2}{u_2} + \frac{\omega_1}{u_2} = 1, \quad \frac{\omega_1}{u_2} + \frac{\omega_1}{u_4} = 1
\]

solving gives us the relations

\[
u_2 = u_4, \quad \omega_1 = -\omega_2, \quad \text{and} \quad u_2 = 2\omega_1.
\]

Region 7:

\[
\xi_1 = \frac{-\omega_2}{u_5}, \quad \xi_2 = \frac{-\omega_3}{u_5}, \quad \xi_3 = \frac{-\omega_3}{u_3}, \quad \xi_4 = \frac{-\omega_2}{u_3}
\]

solving gives us the relations

\[
u_3 = u_5, \quad \omega_2 = \omega_3, \quad u_3 = -2\omega_2
\]

all

\[
u_i = \pm \frac{i}{2} \quad \text{and} \quad \omega_i = \pm \frac{i}{4}.
\]

The cusp shapes for all 6 components are equal to $i$. The three crossing circles have cusp shape

\[
4\omega_i = 4 \times \frac{i}{4} = i.
\]
The component $s + y$ has cusp shape

\[ u_1 + u_6 = 2 \times \frac{i}{2} = i. \]

The component $t + v$ has cusp shape

\[ u_2 + u_4 = 2 \times \frac{i}{2} = i. \]

The component $u + x$ has cusp shape

\[ u_3 + u_5 = 2 \times \frac{i}{2} = i. \]

Figure 5.10: (a) $FALP_3$ with crossing geodesics colored. (b) $T_{FALP_3}$ (c) $P_{FALP_3}$
5.2.3 3 Pretzel FAL with half-twist

In the diagram the five-sided region does not correspond to a five-sided ideal polygon, rather the polyhedral decomposition is the same as in the 3-pretzel without any half-twist, but the gluing of the faces change. Thus to find the cusp shapes we first obtain the parameters from the 3-pretzel without any half-twist and then calculate the cusps off those parameters and the above theorems. Using the information from the 3-pretzel FAL without a half-twist, we have $u_i = \pm \frac{i}{2}$ and $\omega_i = \pm \frac{i}{4}$. The cusp shape for the red crossing circle $p$ is

$$\frac{4\omega_1}{1 - 2\omega_1} = \frac{4 \times \frac{i}{4}}{1 - 2 \times \frac{i}{4}} = \frac{-2}{5} + \frac{4i}{5}.$$
The cusp shapes for the blue and green crossing circles \( n \) and \( m \) respectively are

\[
4\omega_2 = 4 \times \frac{i}{4} = i.
\]

The cusp shape for the light blue component in the projection plane \( r + s + t + q \) is

\[
u_2 + u_6 + u_1 + u_4 - 4\frac{1}{2} = 4 \times \frac{i}{2} - 2 = -2 + 2i.
\]

The cusp shape for the pink component in the projection plane \( u + v \) is

\[
u_3 + u_5 = 2\frac{i}{2} = i.
\]
Chapter 6

Applications

6.1 Invariant Trace Fields of FAL Complements.

Let $M$ be a complete orientable finite volume hyperbolic 3-manifold, then $M = \mathbb{H}^3/\Gamma$ where $\Gamma = \pi_1(M)$ (a Kleinian group) is a discrete subgroup of $PSL(2, \mathbb{C}) = Isom^+(\mathbb{H}^3)$. Let $\rho : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$ be quotient map and let $\Gamma = \rho^{-1}(\Gamma)$.

**Definition 6.1.1.** The trace field $KM = K\Gamma$ is the field over $\mathbb{Q}$ generated by all the traces of $\Gamma$, i.e. $K\Gamma := \mathbb{Q}\langle \{tr(\gamma) | \gamma \in \Gamma \} \rangle$. The invariant trace field of $\Gamma$ is $kM = k\Gamma := k\Gamma^{(2)}$ where $\Gamma^{(2)} := < \gamma^2 | \gamma \in \Gamma >$.

It follows from Mostow-Prasad rigidity that $KM$ and $kM$ are number fields, i.e. finite extensions of $\mathbb{Q}$ and are invariants of $M$. 

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Definition 6.1.2. \( M_1 \) and \( M_2 \) are commensurable if they have common finite-sheeted covers.

Theorem 6.1.3. \[17\] The invariant trace field \( kM \) is an invariant of the commensurability class of \( M \).

Definition 6.1.4. Let \( M \) be a cusped hyperbolic 3-manifold. The field generated by the cusp shapes of all the cusps of \( M \) is called the cusp field \( cM \) of \( M \).

It follows from results of Neumann-Reid in \[17\] that \( cM \) is contained in \( kM \) and is a commensurability invariant. It is often the case that for a link complement \( M \), \( cM = kM \).

The polynomials we derive from the T-T method in terms of intercusp and translational parameters play a central role in studying the invariant trace fields of FAL complements. If the images of the intercusp geodesics and translational geodesics are embedded in \( M \), in which case we call them intercusp arcs and cusp arcs (respectively) then the following theorem holds.

Theorem 6.1.5. \[18\] Suppose \( X \subset M \) is a union of cusp arcs and pairwise disjoint intercusp arcs, where any intercusp arcs which are not disjoint have been bent slightly near intersection points to make them disjoint, and suppose \( \pi_1(X) \to \pi_1(M) \) is surjective. Then the intercusp and translation parameters corresponding to these arcs generate \( kM \).
**Theorem 6.1.6.** Let $M$ be a FAL complement then $cM = kM$.

*Proof.* $kM$ is generated by all the meridian curves of the overstrands of the link diagram. Let $X$ be the union of cusp arcs and pairwise disjoint intercusp arcs, see Figure 6.1. The meridians are all realized by the translation parameters, while the intercusp geodesics ensure that $X$ is connected. For FALs the intercusp parameters $\omega_i$ and the translational parameters $u_j$ are parameters of the intercusp arcs and cusp arcs, respectively, since FALs decompose into totally geodesic polyhedra. To show that $\pi_1(X) \to \pi_1(M)$ is surjective, we need to see that all the meridians are included. This is quite explicit, see Figure 6.1. The meridians for the crossing circles, the meridians for the components in the projection plane, and the meridian that runs around the crossing circle in the projection plane are all combinations of the intercusp and translation parameters. By Theorem 6.1.5, $\omega_i$s and the $u_j$s generate $kM$. Moreover by Theorems 5.1.1 and 5.1.3 $\omega_i$s and $u_j$s generate the cusp field, thus $cM = kM$. \qed

**Theorem 6.1.7.** Let $L_1$ and $L_2$ be FAL that differ in half-twists and let $M_i = S^3 - L_i$, then $kM_1 = kM_2$.

*Proof.* This follows from Theorem 5.1.1, the cusp shapes for FAL complements differing in half-twists have cusp shapes $4\omega$ and $\frac{4\omega}{1+2\omega}$ respectively, generating the same field. \qed
FAL complements that differ in half-twists have the same volume and the same invariant trace fields, but are not isometric.

**Corollary 6.1.8.** There exists an arbitrarily large set of links with complements having the same volume and same invariant trace fields yet are not isometric.

*Proof.* The class of fully augmented pretzel links called $FALP_n$ have number of components ranging from $n + 1$ to $2n$ depending on half-twists, see Figure 6.3(a). $FALP_n$ without half-twists have $2n$ components, for each half-twist added the number of components decrease by 1. $FALP_n$ with $n - 1$ half-twists will have $n + 1$
components, see Figure 6.2. These links will all have the same volume and the same invariant trace field, yet they are non-isometric since they have different number of cusps which is an invariant of the link complement.

\[ \square \]

**Remark 6.1.9.** A very interesting question to study is the commensurability of these links. What happens to the commensurability of FAL when we add half-twists?

### 6.2 Commensurability of Pretzel FALs

![Figure 6.3: (a) $FALP_n$ (b) $FALR_n$](image)

We denote the fully augmented link for the $n$-pretzel link $FALP_n$, see Figure 6.3.

We explore the effects of a $\frac{\pi}{2}$ rotation on the left most crossing circle in a $FALP_n$ for $n \geq 3$. Let $FALR_n$ denote the link we obtain from $FALP_n$ by rotating the left most crossing circle by $\pi/2$, see Figure 6.3(b).
6.2.1 \textit{FALP}_3 and \textit{FALR}_3

We have studied \textit{FALP}_3 in detail in \S 5. We now compute the T-T equations for \textit{FALR}_3.

Region \( \aleph \): We have shape parameters:

\[
\xi_1 = \frac{-\omega_1}{u_1}, \quad \xi_2 = \frac{-\frac{1}{4}}{u_1u_3}, \quad \xi_3 = \frac{-\frac{1}{4}}{u_3u_2}, \quad \xi_4 = \frac{-\omega_1}{u_2}
\]

This is a four-sided region with equations:

\[
\frac{-\omega_1}{u_1} - \frac{1}{4} = 1, \quad \frac{-\frac{1}{4}}{u_1u_3} - \frac{1}{4} = 1, \quad \frac{-\frac{1}{4}}{u_3u_2} - \frac{\omega_1}{u_2} = 1, \quad \frac{-\omega_1}{u_2} - \frac{\omega_1}{u_1} = 1
\]
solving gives us the relations

\[ u_2 = u_1, \quad -2\omega_1 = u_2, \quad \text{and} \quad u_3 = -\frac{1}{2u_2}. \]

Region \( \Box \): We have a three-sided region with shape parameters

\[ \xi_1 = \frac{\omega_2}{u_2} = 1, \quad \xi_2 = \frac{\omega_2}{u_4} = 1, \quad \xi_3 = \frac{-1}{u_2u_4} = 1 \quad \implies \quad u_2 = u_4 = \omega_2 \]

and \( u_2^2 = -\frac{1}{4} \quad \implies \quad u_2 = \pm \frac{i}{2} \).

\( FALR_3 \) has invariant trace field \( x^2 + \frac{1}{4} \). It can be checked using snap [12] that both \( FALP_3 \) and \( FALR_3 \) are arithmetic. Since they have the same invariant trace field, they are commensurable. Note that they have the same volume yet they are not isometric links as they don’t have the same number of components.

Figure 6.5: (a) \( FALP_n \) with labels (b) Symmetric diagram of \( FALP_n \)
6.2.2 T-T Polynomial for $FALP_n$ and $FALR_n$

In this section we find a recurrence relation for the T-T polynomial for $FALP_n$ and $FALR_n$.

**Theorem 6.2.1.** Let $\mathcal{R}$ be the region in $FALP_n$ denoted in Figure 6.5(a). Let $C_n(x)$ be the $(2,1)$ entry of the matrix equation in Proposition 3.1.7, where $x = u_2$ is the edge parameter as shown in Figure 6.3(a). The T-T polynomial for $FALP_n$ is $C_n(x)$, which satisfies the recurrence relation

$$C_n(x) = \frac{C_{n-2}(x)}{4} + xC_{n-1}(x)$$

for $n \geq 5$ where $C_3(x) = x^2 + 1/4$, $C_4(x) = \frac{x(2x^2+1)}{2}$.

**Proof.** From the symmetries in these links (see Figure 6.5(b)) and the shape parameter equations for the four-sided regions we have

1. $-\omega_1 = \omega_2 = \ldots = \omega_n$

2. $u_2 = u_3 = \ldots = u_n$

3. $-2\omega_i = u_j$ where $i, j \neq 1$ or $2n + 2$.

For simplicity let $u_2 = x$ and $u_1 = z$. 

The smallest $FALP_n$ is when $n = 3$. Let $n = 3$ the matrix equation for Region $\aleph$ is:

\[
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -z
\end{bmatrix}
= \begin{bmatrix}
\frac{-x}{4} & \frac{1}{16} + \frac{xz}{4} \\
x^2 + \frac{1}{4} & -xz - \frac{x}{4} - \frac{z}{4}
\end{bmatrix}
\]

thus

\[C_3(x) = x^2 + \frac{1}{4}.\]

For $n = 4$ $FALP_4$ the matrix equation for Region $\aleph$ is:

\[
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -z
\end{bmatrix}
= \begin{bmatrix}
-\left(\frac{x^2}{4} + \frac{1}{16}\right) & \frac{x^2}{4} + \frac{x}{16} + \frac{z}{16} \\
\frac{x(2x^2+1)}{2} & -x^3z - \frac{x^2}{4} - \frac{xz}{2} - \frac{1}{16}
\end{bmatrix}
\]

thus

\[C_4(x) = \frac{x(2x^2+1)}{2}\]

For $FALP_{n-2}$ Region $\aleph$ is a $(n - 2)$-sided region with matrix equation

\[
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -z
\end{bmatrix}
= \begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -z
\end{bmatrix}
\]
Now for $FALP_{n-1}$ Region $\aleph$ is an $(n - 1)$-sided region with matrix equation

\[
\begin{bmatrix}
0 & -\frac{1}{4} & 1 & x \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
-4 & 0 & C_{n-2} & D_{n-2}
\end{bmatrix}
\begin{bmatrix}
A_{n-2} & B_{n-2} \\
C_{n-2} & D_{n-2}
\end{bmatrix}
= \begin{bmatrix}
0 & -\frac{1}{4} \\
-1 & x
\end{bmatrix}
\begin{bmatrix}
A_{n-2} & B_{n-2} \\
C_{n-2} & D_{n-2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{n-1} & B_{n-1} \\
C_{n-1} & D_{n-1}
\end{bmatrix}
= \begin{bmatrix}
-\frac{C_{n-2}}{4} & -\frac{D_{n-2}}{4} \\
-A_{n-2} + xC_{n-2} & -B_{n-2} + xD_{n-2}
\end{bmatrix}
\]

Now for $FALP_n$ Region $\aleph$ is an $n$-sided region with matrix equation

\[
\begin{bmatrix}
0 & -\frac{1}{4} & 1 & x \\
-1 & x & C_{n-1} & D_{n-1}
\end{bmatrix}
\begin{bmatrix}
A_{n-1} & B_{n-1} \\
C_{n-1} & D_{n-1}
\end{bmatrix}
= \begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\frac{C_{n-1}}{4} & -\frac{D_{n-1}}{4} \\
-A_{n-1} + xC_{n-1} & -B_{n-1} + xD_{n-1}
\end{bmatrix}
\]

Where $A_{n-1} = -\frac{C_{n-2}}{4}$, thus $C_n(x) = \frac{C_{n-2}}{4} + xC_{n-1}$.

In Table 6.1 below we compute $C_n(x)$ for some values of $n$. We list the factors of $C_n(x)$. The factor in bold corresponds to the invariant trace field, which is listed in
the last column. We checked using pari-gp that every root of this factor lies in the invariant trace field
Table 6.1: T-T polynomial and invariant trace field for $FALP_n$

<table>
<thead>
<tr>
<th>C</th>
<th>T-T Polynomial</th>
<th>Invariant trace field</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$(4x^2 + 1)/4$</td>
<td>$x^2 + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$x(2x^2 + 1)/2$</td>
<td>$x^2 + 2$</td>
</tr>
<tr>
<td>5</td>
<td>$(16x^4 + 12x^2 + 1)/16$</td>
<td>$x^4 + 3x^2 + 1$</td>
</tr>
<tr>
<td>6</td>
<td>$x(4x^2 + 1)(4x^2 + 3)/16$</td>
<td>$x^2 - x + 1$</td>
</tr>
<tr>
<td>7</td>
<td>$(64x^6 + 80x^4 + 24x^2 + 1)/64$</td>
<td>$x^6 + 5x^4 + 6x^2 + 1$</td>
</tr>
<tr>
<td>8</td>
<td>$x(2x^2 + 1)(8x^4 + 8x^2 + 1)/16$</td>
<td>$x^4 + 4x^2 + 2$</td>
</tr>
<tr>
<td>9</td>
<td>$(4x^2 + 1)(64x^6 + 96x^4 + 36x^2 + 1)/256$</td>
<td>$x^6 + 6x^4 + 9x^2 + 1$</td>
</tr>
<tr>
<td>10</td>
<td>$x(16x^4 + 12x^2 + 1)(16x^4 + 20x^2 + 5)/256$</td>
<td>$x^4 - x^3 + x^2 - x + 1$</td>
</tr>
<tr>
<td>11</td>
<td>$(1024x^{10} + 2304x^8 + 1792x^6 + 560x^4 + 60x^2 + 1)/1024$</td>
<td>$x^{10} + 9x^8 + 28x^6 + 35x^4 + 15x^2 + 1$</td>
</tr>
<tr>
<td>12</td>
<td>$x(2x^2 + 1)(4x^2 + 1)(4x^2 + 3)/512$</td>
<td>$x^4 + 4x^2 + 1$</td>
</tr>
<tr>
<td>13</td>
<td>$(4096x^{12} + 11264x^{10} + 11520x^8 + 5376x^6 + 1120x^4 + 84x^2 + 1)/4096$</td>
<td>$x^{12} + 11x^{10} + 45x^8 + 84x^6 + 70x^4 + 21x^2 + 1$</td>
</tr>
<tr>
<td>14</td>
<td>$x(64x^6 + 80x^4 + 24x^2 + 1)(64x^6 + 112x^4 + 56x^2 + 7)/4096$</td>
<td>$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$</td>
</tr>
</tbody>
</table>
Proposition 6.2.2. It follows from the recurrence that the degree of $C_n(x) = n - 1$.

Conjecture 6.2.3. $C_n(x)$ satisfies the following conditions:

1. If $n$ is prime then $C_n(x)$ is irreducible.
2. $C_m(x)|C_n(x)$ if and only if $m|n$.

See Table 6.1.

![Figure 6.6: FALR$_n$ with labels](image)

Computations for FALR$_n$:

For Region  \( \nabla \) we have a 3-sided region

\[
\xi_1 = \frac{\omega_2}{u_2} = 1, \quad \xi_2 = \frac{\omega_2}{u_{n+1}} = 1, \quad \xi_3 = \frac{-1}{4 u_2 u_{n+1}} = 1,
\]

\[
\implies u_2 = u_{n+1} = \omega_2 = \pm \frac{i}{2}.
\]
From the similarities in the 4-sided regions we get the following equations

\[ u_3 = ... = u_n, \quad \omega_2 = ... = \omega_n, \quad u_3 = 2\omega_2. \]

Without loss of generality let

\[ \omega_2 = \frac{i}{2}, \quad \omega_1 = x, \quad u_1 = z. \]

Then Region \( \aleph \) is a \((n + 1)\)-sided region with matrices equation

\[
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & i \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1}{4} \\
1 & 0
\end{bmatrix}^{n-3}
\begin{bmatrix}
0 & -\frac{1}{4} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -\frac{i}{2} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -x \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -x \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -z \\
0 & 1
\end{bmatrix} = \alpha
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The (2,1)-entry will give you a solution in \( \mathbb{Q}(i) \) for all \( n \). All the cusps other than the crossing circle that is rotated have equal cusp shapes of \( 2i \). To find the cusp shape for the cusp of the rotated crossing circle, solve the above equation and multiply the solution by 4.

**Theorem 6.2.4.** Let \( m, n \geq 3 \), then \( FALR_m \) and \( FALR_n \) are commensurable.
Proof. From Figure 6.8 we can see that $FALR_n$ is a $(n-1)$-sheeted cover of the Borromean rings. Hence $FALR_n$ is commensurable with the Borromean rings $FALR_3$. Since commensurability is an equivalence relation, $FALR_m$, which is a $(m-1)$-sheeted cover of the Borromean rings is commensurable with $FALR_n$. \qed

![Figure 6.7: (a) Symmetric diagram of $FALP_6$ (b) $FALR_6$ with rotated crossing circle green](image)

![Figure 6.8: $FALR_6$ where green dot is vertical axis viewed from $\infty$, Borromean Ring with green crossing circle viewed from $\infty$.](image)

**Corollary 6.2.5.** For $n \geq 3$, the invariant trace field for $FALR_n$ is $\mathbb{Q}(i)$. 

Conjecture 6.2.6. (1) \(\text{FALP}_n\) and \(\text{FALP}_m\) are incommensurable for \(n \neq m\).

(2) For \(n \geq 4\), \(\text{FALP}_n\) and \(\text{FALR}_n\) are incommensurable for all \(n\).

Remark 6.2.7. For \(n \neq m\), \(\text{FALP}_n\) has different invariant trace field than \(\text{FALP}_m\) by Conjecture 6.2.3. For the second part, by Conjecture 6.2.3 \(\text{FALP}_n\) for \(n > 3\) have invariant trace fields \(\not\subseteq \mathbb{Q}(i)\), while the invariant trace field for all \(\text{FALR}_n\) is \(\mathbb{Q}(i)\), thus they are incommensurable.

### 6.3 Geometric Solutions to T-T Equations

The T-T method gives us a way to construct algebraic equations in variables, which then gives us a representation \(\rho\) in \(PSL(2, \mathbb{C})\) with matrix entries in terms of root \(x_i\) of the T-T polynomial. Moreover, there exists a root \(x_0\) of the T-T polynomial such that \(\rho_{x_0}\) is discrete and faithful, this solution will be the geometric solution.

For FALs all the non-real complex solutions lie in \(k\Gamma\) thus to find the geometric solution we use Theorem 5.1.1, which states that \(4\omega\) will give us the cusp shape. We can therefore work backwards, use SnapPy to compute the cusp shape and see which solutions give us the cusp shape, thereby giving us the geometric solution.

**Theorem 6.3.1.** Let \(L\) be a FAL, then the solution of the T-T polynomials which corresponds to the cusp shape is the geometric solution.
Proof. The T-T polynomial for FALs can be written in a variable which is an edge parameter for a crossing circle, which is related to the cusp shape of that crossing circle. Due to Mostow-Prasad Rigidity the geometric structure of the manifold is unique. For fully augmented links, Theorems 5.1.1 and 5.1.3 show how the T-T polynomial gives us the cusp shape, which is a geometric invariant. Different solutions to the T-T polynomial would imply different choices of cusp shapes, which would contradict the uniqueness. Thus the solution to the discrete faithful representation must be the one that results in the correct cusp shape.

6.4 Future Research

By extending the T-T method to the class of Fully Augmented Links and relating their cusp shapes immediately lends to exciting exploration. Few immediate projects to explore are:

(1) Study commensurability of FALs: Our goal is to completely classify commensurability classes of FAL complements using combinatorial data of two types.

- The dimer on a trivalent graph. When we change the dimer the position of the augmented circles change but not the trivalent graph. It will be interesting to see how and if commensurability changes as the dimer is switched.
• $\pi/2$ rotations as was done in $FALR_n$. It will be interesting to see how
and if commensurability continues to change as we continue to increase
the number of crossing circles that we apply a $\pi/2$ rotation.

(2) FALs in thickened torus $T^2 \times I$ - Study geometry of FALs in thickened torus
by extending methods developed by Purcell to this family [8]. These links are studied by Champanerkar, Kofman, and Purcell.

(3) Alternating links in thickened torus $T^2 \times I$ - Extend the T-T method to alternating links in thickened torus $T^2 \times I$. 
Bibliography


