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JOINT LAVER DIAMONDS AND GROUNDED FORCING AXIOMS

by

MIHA E. HABIČ

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2017

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Joint Laver diamonds and grounded forcing axioms

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Miha E. Habič

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

JOINT LAVER DIAMONDS AND GROUNDED FORCING AXIOMS

by

MIHA E. HABIČ

Advisor: Joel David Hamkins

In chapter 1 a notion of independence for diamonds and Laver diamonds is investigated. A sequence of Laver diamonds for κ is *joint* if for any sequence of targets there is a single elementary embedding j with critical point κ such that each Laver diamond guesses its respective target via j . In the case of measurable cardinals (with similar results holding for (partially) supercompact cardinals) I show that a single Laver diamond for κ yields a joint sequence of length κ , and I give strict separation results for all larger lengths of joint sequences. Even though the principles get strictly stronger in terms of direct implication, I show that they are all equiconsistent. This is contrasted with the case of θ -strong cardinals where, for certain θ , the existence of even the shortest joint Laver sequences carries nontrivial consistency strength. I also formulate a notion of jointness for ordinary \diamond_κ -sequences on any regular cardinal κ . The main result concerning these shows that there is no separation according to length and a single \diamond_κ -sequence yields joint families of all possible lengths.

In chapter 2 the notion of a *grounded* forcing axiom is introduced and explored in the case of Martin's axiom. This grounded Martin's axiom, a weakening of the usual axiom, states that the universe is a ccc forcing extension of some inner model and the restriction of Martin's axiom to the posets coming from that ground model holds. I place the new axiom in the hierarchy of fragments of Martin's axiom and examine its effects on the cardinal characteristics of the continuum. I also show that the grounded version is quite a bit more robust under mild forcing than Martin's axiom itself.

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Lastly, thank you to my parents and other family members who have stood by me and supported me through my long journey. I hope I have made you proud. Rad vas imam.

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Introduction

This dissertation consists of two largely independent chapters in which I explore some questions in set theory related to guessing principles, and variants of forcing axioms.

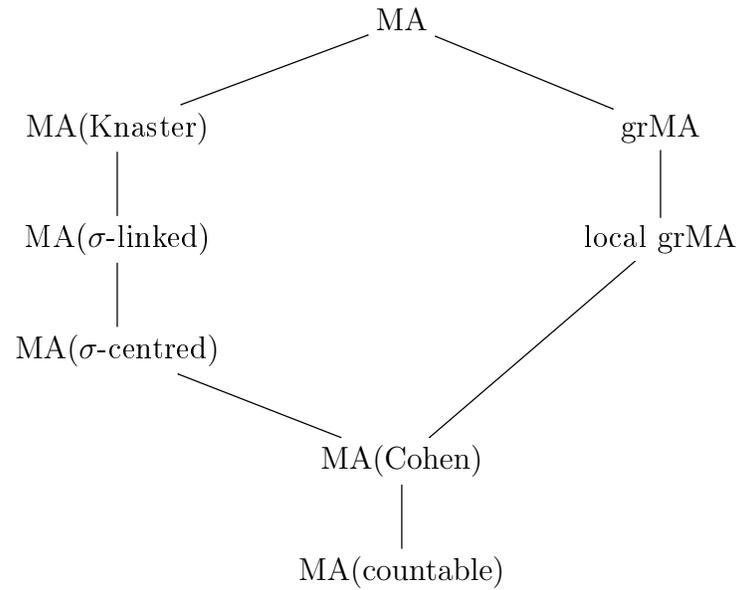
Set-theoretic guessing principles are combinatorial gems, exemplifying the compact nature of certain infinities. They find use in many inductive or diagonalization arguments, where they allow us to anticipate a wide variety of objects. For example, we might attempt to construct a branchless tree by using a guessing principle to anticipate what a potential branch might look like *while still building the tree* and avoiding it. A version of this idea was essentially the primordial application of the guessing principle \diamond by Jensen, and since then a number of other principles have emerged: some allow us to anticipate more complicated sets, others guarantee that the provided guesses are correct quite often, etc. An important variant are the Laver diamond principles which arise alongside many members of the large-cardinal hierarchy.

Chapter 1 is concerned with the question of how many guessing objects of a certain kind can a cardinal carry. For example, how many different \diamond_κ -sequences or Laver functions can there be for a given κ ? To avoid trivial perturbations of a single sequence, the question should rather ask for the number of independent such sequences, for some notion of independence. We suggest and investigate *jointness* as a suitable candidate; a family of guessing sequences is joint if all of its members can guess their targets simultaneously and independently of one another. We investigate jointness for Laver diamonds on (partially) supercompact, and

strong cardinals. In the case of partially supercompact cardinals we show that, while having joint Laver families of various different lengths requires no additional consistency strength, there are no nontrivial implications between these principles: a cardinal may carry λ many joint Laver functions but not λ^+ many, for example. The situation is markedly different in the case of partially strong cardinals. There, in certain cases, having any (infinite) number of joint Laver diamonds requires consistency strength beyond just the large cardinal itself. The chapter is concluded with a discussion of jointness for \diamond_κ -sequences. The definition needs to be modified to make sense in the small-cardinal context, but a natural coherence condition suggests itself, in terms of the existence of normal uniform filters or, equivalently, generic elementary embeddings.

Chapter 2 investigates *grounded* versions of forcing axioms, with a particular focus on Martin's axiom. Forcing axioms are combinatorial statements, generalizing the Baire category theorem, which state that certain “generic” objects exist (such as a generic point in a typical application of the mentioned theorem). They also serve to bridge the gap between set theory and other fields (particularly topology, analysis, and algebra) by presenting a ready-to-use account of the powerful technique of forcing. The strength of these forcing axioms can be calibrated by changing the meaning of the word “generic” in interesting ways. One might restrict or expand the field of structures (posets) that we find these generics in and arrive at Martin's axiom, or the proper forcing axiom, or Martin's maximum or many others.

Grounding a forcing axiom is exactly this kind of restriction. Specifically, we connect the axiom even more closely to its forcing nature and require that the posets under consideration come from a ground model of the universe of a suitable kind: a ccc ground model for Martin's axiom, a proper ground model for the proper forcing axiom etc. Our investigation centres on the grounded Martin's axiom (grMA) and its relationship to the more well-known fragments of Martin's axiom (MA). We show that grMA is largely independent of the other fragments, obtaining the following diagram of implications.



The grounded axiom is also more permissive regarding the combinatorics of the real line. For example, it is consistent with the cardinality of the continuum being singular and that the real line can be covered by \aleph_1 many Lebesgue-null sets. We also explore the behaviour of grMA under forcing adding a single real. We show, using a variant of term forcing, that, in contrast with MA, the grounded axiom is preserved when adding a single Cohen or random real.

Chapter 1

Joint diamonds and Laver diamonds

The concept of a Laver function, introduced for supercompact cardinals in [16], is a powerful strengthening of the usual \diamond -principle to the large cardinal setting. It is based on the observation that a large variety of large cardinal properties give rise to different notions of “large” set, intermediate between stationary and club, and these are then used to provide different guessing principles, where we require that the sequence guesses correctly on ever larger sets. This is usually recast in terms of elementary embeddings or extensions (if the large cardinal in question admits such a characterization), using various ultrapower constructions. For example, in the case of a supercompact cardinal κ , the usual definition states that a *Laver function* for κ is a function $\ell: \kappa \rightarrow V_\kappa$ such that for any θ and any $x \in H_{\theta^+}$ there is a θ -supercompactness embedding $j: V \rightarrow M$ with critical point κ such that $j(\ell)(\kappa) = x$ (this ostensibly second order definition can be rendered in first order language by replacing the quantification over arbitrary embeddings with quantification over ultrapowers by measures on $\mathcal{P}_\kappa(\theta)$, as in Laver’s original account).

Laver functions for other large cardinals were later defined by Gitik and Shelah (see [10]), Corazza (see [6]), Hamkins (see [13]) and others. The term *Laver diamonds* has been suggested to more strongly underline the connection between the large and small cardinal ver-

sions.

In this chapter we examine the notion of *jointness* for both ordinary and Laver diamonds. We shall give a simple example in section 1.1; for now let us just say that a family of Laver diamonds is joint if they can guess their targets simultaneously and independently of one another. Section 1.1 also introduces some terminology that will ease later discussion. Sections 1.2 and 1.3 deal with the outright existence or at least the consistency of joint Laver sequences for supercompact and strong cardinals, respectively. Our results will show that in almost all cases the existence of a joint Laver sequence of maximal possible length is simply equiconsistent with the particular large cardinal. The exception are the θ -strong cardinals where θ is a limit of small cofinality, for which we prove that additional strength is required for even the shortest joint sequences to exist. We also show that there are no nontrivial implications between the existence of joint Laver sequences of different lengths. Section 1.4 considers joint \diamond_κ -sequences and their relation to other known principles. Our main result there shows that \diamond_κ is simply equivalent to the existence of a joint \diamond_κ -sequence of any possible length.

1.1 Jointness: a motivating example

All of the large cardinals we will be dealing with in this chapter are characterized by the existence of elementary embeddings of the universe into certain transitive inner models which have that cardinal as their critical point. We can thus speak of embeddings associated to a measurable, a θ -strong, or a 17-huge cardinal κ etc. Since the definitions of (joint) Laver diamonds for these various large cardinals are quite similar, we give the following general definition as a framework to aid future exposition.

Definition 1.1. Let j be an elementary embedding of the universe witnessing the largeness of its critical point κ (e.g. a measurable, or a hugeness embedding) and let ℓ be a function

defined on κ . We say that a set a , the target, is *guessed by ℓ via j* if $j(\ell)(\kappa) = a$.

If A is a set or a definable class, say that ℓ is an *A -guessing Laver function* if for any $a \in A$ there is an embedding j , witnessing the largeness of κ , such that ℓ guesses a via j . If there is an A -guessing Laver function for κ , we shall say that $\Delta_\kappa(A)$ holds.¹

To simplify the notation even more, we shall, in the coming sections, associate to each type of large cardinal considered, a default set of targets A (for example, when talking about measurable cardinals κ we will mostly be interested in targets from $A = H_{\kappa^+}$). In view of this, whenever we neglect the mention of a particular class of targets, this default A will be intended.

We will often specify the type of large cardinal embeddings we have in mind explicitly, by writing $\Delta_\kappa^{\text{meas}}$, or $\Delta_\kappa^{\theta\text{-sc}}$ etc. This is to avoid ambiguity; for example, we could conceivably start with a supercompact cardinal κ but only be interested in its measurable Laver functions. Even so, to keep the notation as unburdened as possible, we may sometimes omit the specific large cardinal property under consideration when it is clear from context.

As a further complication, the stated definition of an A -guessing Laver function is second-order, since we are quantifying over all possible embeddings j . This seems unavoidable for arbitrary A . However, the default sets of targets we shall be working with are chosen in such a way that standard factoring arguments allow us to restrict our attention to ultrapower or extender embeddings. The most relevant definitions of Laver functions can therefore be recast in first-order language.

Given the concept of a Laver diamond for a large cardinal κ , one might ask how many different Laver diamonds can there be on a given cardinal κ . Immediately after posing the question we realize that it is not that interesting. Since the behaviour of Laver diamonds is determined by their restrictions to large (in the sense of measure) subsets of κ , we can

¹Different notation has been used by different authors to denote the existence of a Laver function. We chose here to follow Hamkins [13].

simply take a fixed Laver function and perturb it on an unbounded but small (nonstationary, say) subset of its domain and preserve its guessing property. There are 2^κ many such perturbations and thus trivially 2^κ many distinct Laver functions.

This answer is very much unsatisfactory. We are not interested in insignificantly modified versions of a single Laver function. We definitely do not want to count the functions above as distinct: they cannot even guess distinct targets! So perhaps we are interested in collections of Laver functions whose targets needn't all be equal, given a fixed embedding j . But even that seems insufficient. For example, given a Laver function ℓ we can define a new one ℓ' by letting $\ell'(\xi)$ be the unique element of $\ell(\xi)$, if one exists. Both of these are Laver functions, their targets under a fixed j are never equal, and yet the targets cannot be independently chosen at all. Again, one can always find large families of Laver functions like this, providing another unsatisfactory answer to our original question.

Finally we come to the right idea of when to Laver functions should be deemed to be distinct, or independent. The key property should be that their targets, under a single embedding j , can be chosen completely independently. Let us illustrate this situation with a simple example.

Suppose $\ell: \kappa \rightarrow V_\kappa$ is a supercompactness Laver function. We can then define two functions ℓ_0, ℓ_1 by letting $\ell_0(\xi)$ and $\ell_1(\xi)$ be the first and second components, respectively, of $\ell(\xi)$, if this happens to be an ordered pair. These two are then easily seen to be Laver functions themselves, but have the additional property that, given any pair of targets a_0, a_1 , there is a *single* supercompactness embedding j such that $j(\ell_0)(\kappa) = a_0$ and $j(\ell_1)(\kappa) = a_1$. This additional trait, where two Laver functions are, in a sense, enmeshed, we call jointness.

Definition 1.2. Let A be a set or a definable class and let κ be a cardinal with a notion of A -guessing Laver function. A sequence $\vec{\ell} = \langle \ell_\alpha; \alpha < \lambda \rangle$ of A -guessing Laver functions is an *A -guessing joint Laver sequence* if for any sequence $\vec{a} = \langle a_\alpha; \alpha < \lambda \rangle$ of targets from A there is a single embedding j , witnessing the largeness of κ , such that each ℓ_α guesses a_α via j .

If there is an A -guessing joint Laver sequence of length λ for κ , we shall say that $\triangleleft_{\kappa,\lambda}(A)$ holds.

In other words, a sequence of Laver diamonds is joint if, given any sequence of targets, these targets can be guessed simultaneously by their respective Laver diamonds. The guessing property of joint Laver sequences is illustrated in the following diagram:

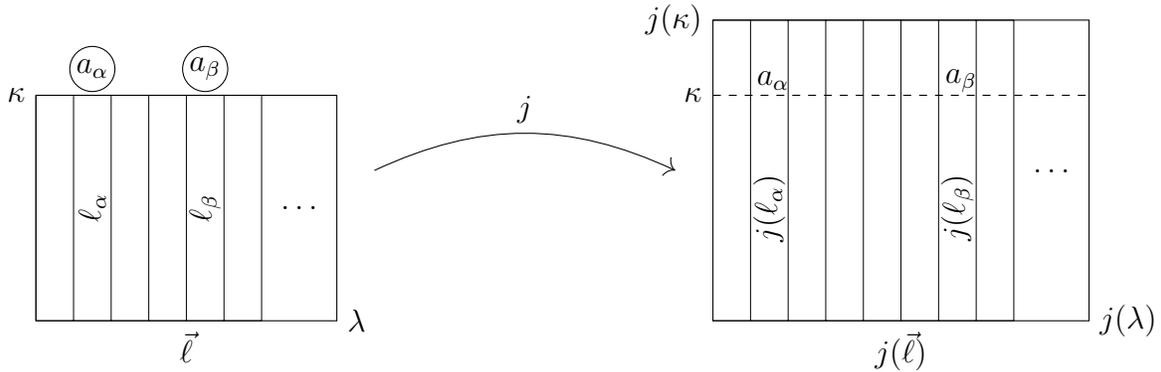


Figure 1.1: The guessing property of joint Laver sequences

We must be careful to distinguish between an entire sequence being jointly Laver or its members being pairwise jointly Laver. It is not difficult to find examples of three (or four or even infinitely many) Laver functions that are pairwise joint but not fully so. For example, given two joint Laver functions ℓ_0 and ℓ_1 we might define $\ell_2(\xi)$ to be the symmetric difference of ℓ_0 and ℓ_1 . It is easy to check that any two of these three functions can have their targets freely chosen, but the third one is uniquely determined by the other two.

Jointness also makes sense for ordinary diamond sequences, but needs to be formulated differently, since elementary embeddings do not (obviously) appear in that setting. Rather, we distil jointness for Laver diamonds into a property of certain ultrafilters and then apply this to more general filters and diamond sequences. We explore this further in section 1.4.

1.2 Joint Laver diamonds for supercompact cardinals

Definition 1.3. A function $\ell: \kappa \rightarrow V_\kappa$ is a θ -supercompactness Laver function for κ if it guesses elements of H_{θ^+} via θ -supercompactness embeddings with critical point κ . This also includes the case of κ being measurable (as this is equivalent to it being κ -supercompact).

If κ is fully supercompact then a function $\ell: \kappa \rightarrow V_\kappa$ is a Laver function for κ if it is a θ -supercompactness Laver function for κ for all θ .

Observe that there are at most 2^κ many θ -supercompactness Laver functions for κ , since there are only 2^κ many functions $\kappa \rightarrow V_\kappa$. Since a joint Laver sequence cannot have the same function appear on two different coordinates (as they could never guess two different targets), this implies that $\lambda = 2^\kappa$ is the largest cardinal for which there could possibly be a joint Laver sequence of length λ . Bounding from the other side, a single θ -supercompactness Laver function already yields a joint Laver sequence of length θ .

Proposition 1.4. *If $\Delta_\kappa^{\theta\text{-sc}}$ holds then there is a θ -supercompactness joint Laver sequence for κ of length $\lambda = \min\{\theta, 2^\kappa\}$.*

Proof. Let ℓ be a Laver function for κ . Fix a subset I of $\mathcal{P}(\kappa)$ of size λ and an enumeration $f: \lambda \rightarrow I$. For $\alpha < \lambda$ define $\ell_\alpha: \kappa \rightarrow V_\kappa$ by $\ell_\alpha(\xi) = \ell(\xi)(f(\alpha) \cap \xi)$ if this makes sense and $\ell_\alpha(\xi) = \emptyset$ otherwise. We claim that $\langle \ell_\alpha; \alpha < \lambda \rangle$ is a joint Laver sequence for κ .

To verify this let $\vec{a} = \langle a_\alpha; \alpha < \lambda \rangle$ be a sequence of elements of H_{θ^+} . Then $\vec{a} \circ f^{-1} \in H_{\theta^+}$, so by assumption there is a θ -supercompactness embedding $j: V \rightarrow M$ such that $j(\ell)(\kappa) = \vec{a} \circ f^{-1}$. But now observe that, by elementarity, for any $\alpha < \lambda$

$$j(\ell_\alpha)(\kappa) = j(\ell)(\kappa)(j(f(\alpha)) \cap \kappa) = j(\ell)(\kappa)(f(\alpha)) = a_\alpha \quad \square$$

Of course, if a given Laver function works for many degrees of supercompactness then the joint Laver sequence derived above will work for those same degrees. In particular, if

κ is fully supercompact then this observation, combined with Laver's original construction, gives us a supercompactness joint Laver sequence of length 2^κ .

Corollary 1.5. *If κ is supercompact then $\mathbb{L}_{\kappa, 2^\kappa}^{\text{sc}}$ holds.*

Proposition 1.4 essentially shows that joint Laver sequences of maximal length exist automatically for cardinals with a high degree of supercompactness. Since we will be interested in comparing the strengths of the principles $\mathbb{L}_{\kappa, \lambda}$ for various λ , we will in the remainder of this section be mostly concerned with cardinals κ which are not 2^κ -supercompact (but are at least measurable).

1.2.1 Creating long joint Laver diamonds

We now show that the existence of θ -supercompactness joint Laver sequences of maximal length does not require strength beyond θ -supercompactness itself. The following lemma is well-known.

Lemma 1.6. *If κ is θ -supercompact then there is a function $f: \kappa \rightarrow \kappa$ with the Menas property, i.e. we have $j(f)(\kappa) > \theta$ for some θ -supercompactness embedding $j: V \rightarrow M$ with critical point κ .*

Theorem 1.7. *If κ is θ -supercompact then there is a forcing extension in which $\mathbb{L}_{\kappa, 2^\kappa}^{\theta\text{-sc}}$ holds.*

Proof. Since θ -supercompactness of κ implies its $\theta^{<\kappa}$ -supercompactness, we may assume that $\theta^{<\kappa} = \theta$. Furthermore, we assume that $2^\theta = \theta^+$, since this may be forced without adding subsets to $\mathcal{P}_\kappa \theta$. Fix a Menas function f as in lemma 1.6. Let \mathbb{P}_κ be the length κ Easton support iteration which forces with $\mathbb{Q}_\gamma = \text{Add}(\gamma, 2^\gamma)$ at inaccessible closure points of f , i.e. those inaccessible γ for which $f[\gamma] \subseteq \gamma$. Finally, let $\mathbb{P} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa$. It is useful to note that forcing with \mathbb{P} does not change the value of 2^κ . Let $G * g \subseteq \mathbb{P}$ be generic; we will extract a joint Laver sequence from g .

If $g(\alpha)$ is the α -th subset added by g , we view it as a sequence of bits. Given any $\xi < \kappa$ we may then view the segment of $g(\alpha)$ between the ξ th bit and the next marker as the Mostowski code of an element of V_κ (employing bit doubling or some other coding scheme which admits end-of-code markers). We then define $\ell_\alpha: \kappa \rightarrow V_\kappa$ as follows: given an inaccessible ξ , let $\ell_\alpha(\xi)$ be the set coded by $g(\alpha)$ at ξ ; otherwise let $\ell_\alpha(\xi) = \emptyset$. We claim that $\langle \ell_\alpha; \alpha < 2^\kappa \rangle$ is a joint Laver sequence.

Let $\vec{a} = \langle a_\alpha; \alpha < 2^\kappa \rangle$ be a sequence of targets in $H_{\theta^+}^{V[G][g]}$. Let $j: V \rightarrow M$ be the ultrapower embedding by a normal measure on $\mathcal{P}_\kappa\theta$ which corresponds to f , i.e. such that $j(f)(\kappa) > \theta$. We will lift this embedding through the forcing \mathbb{P} in $V[G][g]$.

The argument splits into two cases, depending on the size of θ . We deal first with the easier case when $\theta \geq 2^\kappa$. In this case the poset $j(\mathbb{P}_\kappa)$ factors as $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_\kappa * \mathbb{P}_{\text{tail}}$. Since $j(f)(\kappa) > \theta$, the next stage of forcing in $j(\mathbb{P}_\kappa)$ above κ occurs after θ , so \mathbb{P}_{tail} is $\leq \theta$ -closed in $M[G][g]$ and has size $j(\kappa)$ there. Since M was an ultrapower, $M[G][g]$ has at most $|2^{j(\kappa)}| \leq (2^\kappa)^\theta = \theta^+$ many subsets of \mathbb{P}_{tail} , and so we can diagonalize against all of them to produce in $V[G][g]$ an $M[G][g]$ -generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ and lift j to $j: V[G] \rightarrow M[j(G)]$, where $j(G) = G * g * G_{\text{tail}}$.

Since $M[j(G)]$ is still an ultrapower and thus closed under θ -sequences in $V[G][g]$, we get $j[g] \in M[j(G)]$. Since $j(\mathbb{Q}_\kappa)$ is $\leq \theta$ -directed closed in $M[j(G)]$ it has $q = \bigcup j[g]$ as a condition. Since $M[j(G)]$ has both the sequence of targets \vec{a} and $j \upharpoonright 2^\kappa$, we can further extend q to q^* by coding a_α at κ in $q(j(\alpha))$ for each $\alpha < 2^\kappa$. We again diagonalize against the $|2^{2^{j(\kappa)}}| \leq \theta^+$ many dense subsets of $j(\mathbb{Q}_\kappa)$ in $M[j(G)]$ below the master condition q^* to get a $M[j(G)]$ -generic $g^* \subseteq j(\mathbb{Q}_\kappa)$ and lift j to $j: V[G][g] \rightarrow M[j(G)][g^*]$. Finally, observe that we have arranged the construction of g^* in such a way that $g^*(j(\alpha))$ codes a_α at κ for all $\alpha < 2^\kappa$ and, by definition, this implies that $j(\ell_\alpha)(\kappa) = a_\alpha$ for all $\alpha < 2^\kappa$. Thus we indeed have a joint Laver sequence for κ of length 2^κ in $V[G][g]$.

It remains for us to consider the second case, when $\kappa \leq \theta < 2^\kappa$. In this situation our

assumptions on θ imply that $2^\kappa = \theta^+$. The poset $j(\mathbb{P}_\kappa)$ factors as $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \tilde{\mathbb{Q}}_\kappa * \mathbb{P}_{\text{tail}}$, where $\tilde{\mathbb{Q}}_\kappa = \text{Add}(\kappa, (2^\kappa)^{M[G]})$ is isomorphic but not necessarily equal to \mathbb{Q}_κ . Nevertheless, the same argument as before allows us to lift j to $j: V[G] \rightarrow M[j(G)]$ where $j(G) = G * \tilde{g} * G_{\text{tail}}$ and \tilde{g} is the isomorphic image of g .

We seem to hit a snag with the final lift through the forcing \mathbb{Q}_κ , which has size 2^κ and thus resists the usual approach of lifting via a master condition, since this condition would simply be too big for the amount of closure we have. We salvage the argument by using a technique, originally due to Magidor [19], sometimes known as the “master filter argument”.

The forcing $j(\mathbb{Q}_\kappa) = \text{Add}(j(\kappa), 2^{j(\kappa)})$ has size $2^{j(\kappa)}$ and is $\leq \theta$ -directed closed and $j(\kappa)^+$ -cc in $M[j(G)]$. Since $M[j(G)]$ is still an ultrapower, $|2^{j(\kappa)}| \leq \theta^+ = 2^\kappa$ and so $M[j(G)]$ has at most 2^κ many maximal antichains of $j(\mathbb{Q}_\kappa)$. Let these be given in the sequence $\langle Z_\alpha; \alpha < 2^\kappa \rangle$. Since each Z_α has size at most $j(\kappa)$, it is in fact contained in some bounded part of the poset $j(\mathbb{Q}_\kappa)$. Furthermore (and crucially), since j is an ultrapower by a measure on $\mathcal{P}_\kappa \theta$, it is continuous at $2^\kappa = \theta^+$ and so there is for each α a $\beta_\alpha < 2^\kappa$ such that $Z_\alpha \subseteq \text{Add}(j(\kappa), 2^{j(\kappa)}) \upharpoonright j(\beta_\alpha)$. In particular, each Z_α is a maximal antichain in $\text{Add}(j(\kappa), 2^{j(\kappa)}) \upharpoonright j(\beta_\alpha)$. We will now construct in $V[G][g]$ a descending sequence of conditions, deciding more and more of the antichains Z_α , which will generate a filter, the “master filter”, that will allow us to lift j to $V[G][g]$ and also (lest we forget) witness the joint guessing property. We begin by defining the first condition q_0 . Consider the generic g up to β_0 . This piece has size θ and so $\bigcup j[g \upharpoonright \beta_0]$ is a condition in $j(\mathbb{Q}_\kappa) \upharpoonright j(\beta_0)$. Let q'_0 be the extension of $\bigcup j[g \upharpoonright \beta_0]$ which codes the target a_α at κ in $q'_0(j(\alpha))$ for each $\alpha < \beta_0$. This is still a condition in $j(\mathbb{Q}_\kappa) \upharpoonright j(\beta_0)$ and we can finally let q_0 be any extension of q'_0 in this poset which decides the maximal antichain Z_0 . Note that q_0 is compatible with every condition in $j[g]$, since we extended the partial master condition $\bigcup j[g \upharpoonright \beta_0]$ and made no commitments outside $j(\mathbb{Q}_\kappa) \upharpoonright j(\beta_0)$. We continue in this way inductively, constructing a descending sequence of conditions q_α for $\alpha < \theta^+$, using the θ -closure of $j(\mathbb{Q}_\kappa)$ and $M[j(G)]$ to pass through limit stages. Now consider the filter g^*

generated by the conditions q_α . It is $M[j(G)]$ -generic by construction and also extends (or can easily be made to extend) $j[g]$. We can thus lift j to $j: V[G][g] \rightarrow M[j(G)][g^*]$ and, since \mathbb{P} is θ^+ -cc and both G_{tail} and g^* were constructed in $V[G][g]$, the model $M[j(G)][g^*]$ is closed under θ -sequences which shows that κ remains θ -supercompact in $V[G][g]$. Finally, as in the previous case, g^* was constructed in such a way that $j(\ell_\alpha)(\kappa) = a_\alpha$ for all $\alpha < 2^\kappa$, verifying that these functions really do form a joint Laver sequence for κ . \square

As a special case of theorem 1.7 we can deduce the corresponding result for measurable cardinals.

Corollary 1.8. *If κ is measurable then there is a forcing extension in which there is a measurable joint Laver sequence for κ of length 2^κ .*

It follows from the results of Hamkins [14] that the forcing \mathbb{P} from theorem 1.7 does not create any measurable or (partially) supercompact cardinals below κ , since it admits a very low gap. We could therefore have started with the least large cardinal κ of interest and preserved its minimality through the construction.

Corollary 1.9. *If κ is the least θ -supercompact cardinal then there is a forcing extension where κ remains the least θ -supercompact cardinal and $\mathbb{D}_{\kappa, 2^\kappa}^{\theta\text{-sc}}$ holds.*

It is perhaps interesting to observe the peculiar arrangement of cardinal arithmetic in the model produced in the above proof. We have $2^\theta = \theta^+$ and, if $\theta \leq 2^\kappa$, also $2^\kappa = \theta^+$. In particular, we never produced a θ -supercompactness joint Laver sequence of length greater than θ^+ (assuming here, of course, that $\theta = \theta^{<\kappa}$ is the optimal degree of supercompactness). One has to wonder whether this is significant. Certainly the existence of long joint Laver sequences does not imply much about cardinal arithmetic, since, for example, if κ is indestructibly supercompact, we can manipulate the value of 2^κ freely, while maintaining the existence of a supercompactness joint Laver sequence of length 2^κ . On the other hand, even

in the case of measurable κ , the consistency strength of $2^\kappa > \kappa^+$ is known to exceed that of κ being measurable. The following question is therefore natural:

Question 1.10 (open). *If κ is θ -supercompact and $2^\kappa > \theta^+$, is there a forcing extension preserving these facts in which there is a joint Laver sequence for κ of length 2^κ ?*

We next show that the existence of joint Laver sequences is preserved under mild forcing. This will be useful later when we separate different lengths of these sequences.

Lemma 1.11. *Let κ be θ -supercompact and suppose that $\triangleleft_{\kappa,\lambda}^{\theta\text{-sc}}$ holds. Let \mathbb{P} be a poset such that either*

1. $\lambda \geq \theta^{<\kappa}$ and \mathbb{P} is $\leq \lambda$ -distributive, or
2. $|\mathbb{P}| \leq \kappa$ and many θ -supercompactness embeddings with critical point κ lift through \mathbb{P} .

Then forcing with \mathbb{P} preserves $\triangleleft_{\kappa,\lambda}^{\theta\text{-sc}}$.

We were intentionally vague in item (2) of the lemma. The hypotheses will definitely be satisfied if every θ -supercompactness embedding should lift through \mathbb{P} , as is very often the case with forcing iterations up to the large cardinal κ . The proof, however, will show that it is in fact sufficient that at least one ground model embedding associated to each target of the existing joint Laver sequence lift through \mathbb{P} . Furthermore, while the restriction $|\mathbb{P}| \leq \kappa$ in (2) is necessary for full generality, it can in fact be relaxed to $|\mathbb{P}| \leq \theta$ for a large class of forcings.

Proof. Under the hypotheses of (1) every ultrapower embedding by a measure on $\mathcal{P}_\kappa\theta$ lifts to the extension by \mathbb{P} and no elements of H_{θ^+} or λ -sequences of such are added, so any ground model joint Laver sequence of length λ retains this property in the extension.

Now suppose that the hypotheses of (2) hold, let $\langle \ell_\alpha; \alpha < \lambda \rangle$ be a joint Laver sequence for κ and let $G \subseteq \mathbb{P}$ be generic. We may also assume that the universe of \mathbb{P} is a subset of

κ . Define functions $\ell'_\alpha: \kappa \rightarrow V_\kappa[G]$ by $\ell'_\alpha(\xi) = \ell_\alpha(\xi)^{G \cap \xi}$ if this makes sense and $\ell'_\alpha(\xi) = \emptyset$ otherwise. We claim that $\langle \ell'_\alpha; \alpha < \lambda \rangle$ is a joint Laver sequence in $V[G]$.

Let $\vec{a} \in V[G]$ be a λ -sequence of targets in H_{θ^+} . Since \mathbb{P} is small we can find names \dot{a}_α for these targets in H_{θ^+} and the sequence of these names is in V . Let $j: V \rightarrow M$ be a θ -supercompactness embedding with critical point κ which lifts through \mathbb{P} and which satisfies $j(\ell_\alpha)(\kappa) = \dot{a}_\alpha$ for each α . It then follows that $j(\ell'_\alpha)(\kappa) = \dot{a}_\alpha$ in $V[G]$, verifying the joint Laver diamond property there. \square

1.2.2 Separating joint Laver diamonds by length

We next aim to show that it is consistent that there is a measurable Laver function for κ but no joint Laver sequences of length κ^+ . The following proposition expresses the key observation for our solution, connecting the question to the number of normal measures problem.

Proposition 1.12. *If there is a θ -supercompactness joint Laver sequence for κ of length λ then there are at least $2^{\theta \cdot \lambda}$ many normal measures on $\mathcal{P}_\kappa \theta$.*

Proof. The point is that any normal measure on $\mathcal{P}_\kappa \theta$ realizes a single λ -sequence of elements of H_{θ^+} via the joint Laver sequence and there are $2^{\theta \cdot \lambda}$ many such sequences of targets. \square

Theorem 1.13. *If κ is measurable then there is a forcing extension in which there is a measurable Laver function for κ but no measurable joint Laver sequences of length κ^+ .*

Proof. After forcing as in the proof of theorem 1.7, if necessary, we may assume that κ has a Laver function. A result of Apter–Cummings–Hamkins [1] then shows that κ still carries a Laver function in the extension by $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\kappa^+, 2^{2^\kappa})$ but only carries κ^+ many normal measures there. Proposition 1.12 now implies that there cannot be a joint Laver sequence of length κ^+ in the extension. \square

We can push this result a bit further to get a separation between any two desired lengths of joint Laver sequences. To state the sharpest result we need to introduce a new notion.

Definition 1.14. Let κ be a large cardinal supporting a notion of Laver diamond and λ a cardinal. We say that a sequence $\vec{\ell} = \langle \ell_\alpha; \alpha < \lambda \rangle$ is an *almost-joint Laver sequence* if $\vec{\ell} \upharpoonright \gamma$ is a joint Laver sequence for any $\gamma < \lambda$. We say that $\mathbb{A}_{\kappa, < \lambda}$ holds if there is an almost-joint Laver sequence of length λ .

Theorem 1.15. *Suppose κ is measurable and let λ be a regular cardinal satisfying $\kappa < \lambda \leq 2^\kappa$. If $\mathbb{A}_{\kappa, < \lambda}^{\text{meas}}$ holds then there is a forcing extension preserving this in which $\mathbb{A}_{\kappa, \lambda}^{\text{meas}}$ fails.*

Proof. We imitate the proof of theorem 1.13 but force instead with $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\lambda, 2^{2^\kappa})$. The analysis based on [1] now shows that the final extension has at most λ many normal measures on κ and thus there can be no joint Laver sequences of length λ there by proposition 1.12. That $\mathbb{A}_{\kappa, < \lambda}^{\text{meas}}$ still holds follows from (the proof of) lemma 1.11: part (2) implies that, by guessing names, the $\mathbb{A}_{\kappa, < \lambda}^{\text{meas}}$ -sequence from the ground model gives rise to one in the intermediate Cohen extension. Part (1) then shows that each of the initial segments of this sequence remains a joint Laver sequence in the final extension. \square

We can also extend these results to θ -supercompact cardinals without too much effort.

Theorem 1.16. *If κ is θ -supercompact, θ is regular, and $\theta^{< \kappa} = \theta$ then there is a forcing extension in which $\mathbb{A}_{\kappa}^{\theta\text{-sc}}$ holds but $\mathbb{A}_{\kappa, \theta^+}^{\theta\text{-sc}}$ fails.*

Of course, the theorem is only interesting when $\kappa \leq \theta < 2^\kappa$, in which case the given separation is best possible in view of proposition 1.4.

Proof. We may assume by prior forcing, as in theorem 1.7, that we have a Laver function for κ . We now force with $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\theta^+, 2^{2^\theta})$ to get an extension $V[g][G]$. By the results of [1] the extension $V[g][G]$ has at most θ^+ many normal measures on $\mathcal{P}_\kappa \theta$ and

therefore there are no joint Laver sequences for κ of length θ^+ there by proposition 1.12. It remains to see that there is a Laver function in $V[g][G]$. Let ℓ be a Laver function in V and define $\ell' \in V[g][G]$ by $\ell'(\xi) = \ell(\xi)^g$ if $\ell(\xi)$ is an $\text{Add}(\omega, 1)$ -name and $\ell'(\xi) = \emptyset$ otherwise. For a given $a \in H_{\theta^+}^{V[g][G]} = H_{\theta^+}^{V[g]}$ we can select an $\text{Add}(\omega, 1)$ -name $\dot{a} \in H_{\kappa^+}^V$ and find a θ -supercompactness embedding $j: V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{a}$. The embedding j lifts to $j: V[g][G] \rightarrow M[g][j(G)]$ since the Cohen forcing was small and the collapse forcing was $\leq \theta$ -closed. But then clearly $j(\ell')(\kappa) = \dot{a}^g = a$, so ℓ' is a Laver function. \square

Theorem 1.17. *Suppose κ is θ -supercompact and let λ be a regular cardinal satisfying $\theta^{<\kappa} < \lambda \leq 2^\kappa$. If $\mathbb{A}_{\kappa, < \lambda}^{\theta\text{-sc}}$ holds then there is a forcing extension preserving this in which $\mathbb{A}_{\kappa, \lambda}^{\theta\text{-sc}}$ fails.*

Proof. The relevant forcing is $\text{Add}(\omega, 1) * \text{Coll}(\lambda, 2^{2^{\theta^{<\kappa}}})$. Essentially the argument from theorem 1.15 then finishes the proof. \square

A question remains about the principles $\mathbb{A}_{\kappa, < \lambda}$, whether they are genuinely new or whether they reduce to other principles.

Question 1.18 (open). *Let κ be θ -supercompact and $\lambda \leq 2^\kappa$. Is $\mathbb{A}_{\kappa, < \lambda}^{\theta\text{-sc}}$ equivalent to $\mathbb{A}_{\kappa, \gamma}^{\theta\text{-sc}}$ holding for all $\gamma < \lambda$?*

An almost-joint Laver sequence definitely gives instances of joint Laver diamonds at each particular γ . The reverse implication is particularly interesting in the case when $\lambda = \mu^+$ is a successor cardinal. This is because simply rearranging the functions in a joint Laver sequence of length μ gives joint Laver sequences of any length shorter than μ^+ . The question is thus asking whether $\mathbb{A}_{\kappa, \mu}$ suffices for $\mathbb{A}_{\kappa, < \mu^+}$. The restriction to $\lambda \leq 2^\kappa$ is necessary to avoid the following triviality.

Proposition 1.19. $\mathbb{A}_{\kappa, < (2^\kappa)^+}^{\theta\text{-sc}}$ fails for every cardinal κ

Proof. Any potential $\mathbb{A}_{\kappa, < (2^\kappa)^+}^{\theta\text{-sc}}$ -sequence must necessarily have the same function appear on

at least two coordinates. But then any initial segment of this sequence cannot be joint, since it cannot guess distinct targets on those two coordinates. \square

An annoying feature of the models produced in the preceding theorems is that in all of them the least λ for which there is no joint Laver sequence of length λ is $\lambda = 2^\kappa$. One has to wonder whether this is significant.

Question 1.20. *Is it relatively consistent that there is a θ -supercompact cardinal κ , for some θ , such that $\mathbb{L}_\kappa^{\theta\text{-sc}}$ holds and $\mathbb{L}_{\kappa,\lambda}^{\theta\text{-sc}}$ fails for some λ satisfying $\lambda < 2^\kappa$?*

To satisfy the listed conditions, GCH must fail at κ (since we must have $\kappa < \lambda < 2^\kappa$ by proposition 1.4). We can therefore expect that achieving the situation described in the question will require some additional consistency strength.

In the case of a measurable κ the answer to the question is positive: we will show in theorem 1.22 that, starting from sufficient large cardinal hypotheses, we can produce a model where κ is measurable and has a measurable Laver function but no joint Laver sequences of length $\kappa^+ < 2^\kappa$. The proof relies on an argument due to Friedman and Magidor [8] which facilitates the simultaneous control of the number of measures at κ and the value of the continuum function at κ and κ^+ .

Let us briefly give a general setup for the argument of [8] that will allow us to carry out our intended modifications without repeating too much of the work done there.

Fix a cardinal κ and suppose GCH holds up to and including κ . Furthermore suppose that κ is the critical point of an elementary embedding $j: V \rightarrow M$ satisfying the following properties:

- j is an extender embedding, i.e. every element of M has the form $j(f)(\alpha)$ for some function f defined on κ and some $\alpha < j(\kappa)$;
- $(\kappa^{++})^M = \kappa^{++}$;

- there is a function $f: \kappa \rightarrow V$, such that $j(f)(\kappa)$ is, in V (and therefore also in M), a sequence of κ^{++} many disjoint stationary subsets of $\kappa^{++} \cap \text{Cof}_{\kappa^+}$.

Given this arrangement, Friedman and Magidor define a forcing iteration \mathbb{P} of length $\kappa + 1$ (with carefully chosen support) which forces at each inaccessible stage $\gamma \leq \kappa$ with $\text{Sacks}^*(\gamma, \gamma^{++}) * \text{Sacks}_{\text{id}^{++}}(\gamma) * \text{Code}(\gamma)$. Here the conditions in $\text{Sacks}_{\text{id}^{++}}$ are perfect trees of height γ , splitting on a club, and in which every splitting node of height δ has δ^{++} many successors; furthermore $\text{Sacks}^*(\gamma, \gamma^{++})$ is a large product of versions of $\text{Sacks}(\gamma)$ where splitting is restricted to a club of singular cardinals and $\text{Code}(\gamma)$ is a certain $\leq \gamma$ -distributive notion of forcing coding information about the stage γ generics into the stationary sets given by $f(\gamma)$.

Let $G \subseteq \mathbb{P}$ be generic. In the interest of avoiding a full analysis of the forcing notion \mathbb{P} , we list some of the properties of the extension $V[G]$ that we will use axiomatically to derive the subsequent results (the interested reader is encouraged to see [8] for proofs):

- (i) \mathbb{P} preserves cardinals and cofinalities, and increases the values of the continuum function by at most two cardinal steps. In particular, the inaccessible cardinals of V remain inaccessible in $V[G]$;
- (ii) We have $2^\kappa = \kappa^{++}$ in $V[G]$;
- (iii) \mathbb{P} has the κ -Sacks property: for any function $f: \kappa \rightarrow \text{Ord}$ in $V[G]$ there is a function $h \in V$ such that $f(\alpha) \in h(\alpha)$ for all α and $|h(\alpha)| \leq \alpha^{++}$;
- (iv) The generic G is self-encoding in a strong way: in $V[G]$ there is a unique M -generic for $j(\mathbb{P})_{< j(\kappa)}$ extending $j[G_{< \kappa}]$;
- (v) If S_κ is the generic added by $\text{Sacks}_{\text{id}^{++}}(\kappa)$ within \mathbb{P} , then $\bigcap j[S_\kappa]$ is a *tuning fork*: the union of κ^{++} many branches, all of which split off exactly at level κ and all of which are generic over $j(G_{< \kappa})$;

- (vi) In $V[G]$ there are exactly κ^{++} many M -generics for $j(\mathbb{P})$ extending $j[G]$, corresponding to the κ^{++} many branches in $\bigcap j[S_\kappa]$. In particular, there are exactly κ^{++} many lifts j_α of j in $V[G]$, distinguished by $j_\alpha(S_\kappa)(\kappa) = \alpha$ for $\alpha < \kappa^{++}$.

Proposition 1.21. *In the above setup, the iteration \mathbb{P} adds a measurable Laver function for κ .*

Proof. Let $G \subseteq \mathbb{P}$ be generic. As we stated in item (i), for any $\alpha < \kappa^{++}$ there is a lift $j_\alpha: V[G] \rightarrow M[j_\alpha(G)]$ of j such that $j_\alpha(S_\kappa)(\kappa) = \alpha$, where S_κ is the Sacks subset of κ added by the κ th stage of G . This shows that $\bar{\ell}(\gamma) = S_\kappa(\gamma)$ is a κ^{++} -guessing measurable Laver function for κ .

Note that all of the subsets of κ in $M[j_\alpha(G)]$ (and $V[G]$) appear already in $M[G]$. Let $\vec{e} = \langle e_\alpha; \alpha < \kappa^{++} \rangle$ be an enumeration of $H_{\kappa^+}^{M[G]}$ in $M[G]$ and let $\dot{e} \in M$ be a name for \vec{e} . We can write $\dot{e} = j(F)(\kappa)$ for some function F , defined on κ . Now define a function $\ell: \kappa \rightarrow V_\kappa$ in $V[G]$ by $\ell(\gamma) = (F(\gamma)^G)(\bar{\ell}(\gamma))$. This is, in fact, our desired Laver function; given an arbitrary element of $H_{\kappa^+}^{V[G]}$, we can find it in the enumeration \vec{e} . If α is its index, then

$$j_\alpha(\ell)(\kappa) = (j_\alpha(F)(\kappa)^{j_\alpha(G)})(j_\alpha(\bar{\ell})(\kappa)) = (j(F)(\kappa)^{j_\alpha(G)})(\alpha) = \vec{e}(\alpha) = e_\alpha \quad \square$$

Theorem 1.22. *Suppose κ is $(\kappa+2)$ -strong and assume that $V = L[\vec{E}]$ is the minimal extender model witnessing this. Then there is a forcing extension in which $2^\kappa = \kappa^{++}$, the cardinal κ remains measurable, κ carries a measurable Laver function but there are no measurable joint Laver sequences for κ of length κ^+ .*

This finally answers question 1.20 in the positive.

Proof. Let $j: V \rightarrow M$ be the ultrapower embedding by the top extender of \vec{E} , the unique extender witnessing the $(\kappa+2)$ -strongness of κ . In particular, every element of M has the form $j(f)(\alpha)$ for some $\alpha < j(\kappa)$, and M computes κ^{++} correctly. Furthermore, V has a

canonical $\diamond_{\kappa^{++}}(\text{Cof}_{\kappa^+})$ -sequence, which is definable over $H_{\kappa^{++}}$. Since $H_{\kappa^{++}} \in M$, this same sequence is also in M and, by definability, is of the form $j(\bar{f})(\kappa)$ for some function \bar{f} . By having this diamond sequence guess the singletons $\{\xi\}$ for $\xi < \kappa^{++}$, we obtain a sequence of κ^{++} many disjoint stationary subsets of $\kappa^{++} \cap \text{Cof}_{\kappa^+}$, and this sequence itself has the form $j(f)(\kappa)$ for some function f . We are therefore in a situation where the definition of the Friedman–Magidor iteration we described above makes sense. But first, we need to carry out some preliminary forcing.

Let $g \subseteq \text{Add}(\kappa^+, \kappa^{+3})$ be generic. Since this Cohen poset is $\leq \kappa$ -distributive, the embedding j lifts (uniquely) to an embedding $j: V[g] \rightarrow M[j(g)]$.² Let us examine the lifted embedding j . It is still an extender embedding. Additionally, since GCH holds in V , the forcing $\text{Add}(\kappa^+, \kappa^{+3})$ preserves cardinals, cofinalities, and stationary subsets of κ^{++} . Together this means that $M[j(g)]$ computes κ^{++} correctly and the stationary sets given by the sequence $j(f)(\kappa)$ above remain stationary. Therefore we may still define the Friedman–Magidor iteration \mathbb{P} over $V[g]$.

Let $G \subseteq \mathbb{P}$ be generic over $V[g]$. We claim that $V[g][G]$ is the model we want. We have $2^\kappa = \kappa^{++}$ in the extension, by item (ii) of our list, and proposition 1.21 implies that κ is measurable in $V[g][G]$ and $\mathfrak{N}_{\kappa}^{\text{meas}}$ holds there. So it remains for us to see that $\mathfrak{N}_{\kappa, \kappa^+}^{\text{meas}}$ fails. By proposition 1.12 it suffices to show that κ does not carry $2^{\kappa^+} = \kappa^{+3}$ many normal measures in $V[g][G]$.

Let $U^* \in V[g][G]$ be a normal measure on κ and let $j^*: V[g][G] \rightarrow N[j^*(g)][j^*(G)]$ be its associated ultrapower embedding. This embedding restricts to $j^*: V \rightarrow N$. Since V is the core model from the point of view of $V[g][G]$, the embedding j^* arises as the ultrapower map associated to a normal iteration of extenders on the sequence \vec{E} .

We first claim that the first extender applied in this iteration is the top extender of \vec{E} .

²The lifted embedding will not be a $(\kappa+2)$ -strongness embedding and, in fact, κ is no longer $(\kappa+2)$ -strong in $V[g]$. Nevertheless, the residue of strongness will suffice for our argument.

Let us write $j^* = j_1 \circ j_0$, where $j_0: V \rightarrow N_0$ results from the first applied extender. Clearly j_0 has critical point κ . Now suppose first that $j_0(\kappa) < \kappa^{++}$. Of course, $j_0(\kappa)$ is inaccessible in N_0 and, since N is an inner model of N_0 , also in N . But $j_0(\kappa)$ is not inaccessible in $N[j^*(g)][j^*(G)]$, since $2^\kappa = \kappa^{++}$ there. This is a contradiction, since passing from N to $N[j^*(g)][j^*(G)]$ preserves inaccessibility, by item (i) of our list.

It follows that we must have $j_0(\kappa) \geq \kappa^{++}$. We will argue that the extender E applied to get j_0 witnesses the $(\kappa + 2)$ -strongness of κ , so it must be the top extender of \vec{E} and $j_0 = j$. Using a suitable indexing of \vec{E} , the extender E has index $(j_0(\kappa)^+)^{N_0} > \kappa^{++}$, and the coherence of the extender sequence implies that the sequences in V and in N_0 agree up to κ^{++} . By the acceptability of these extender models it now follows that $H_{\kappa^{++}}^V = H_{\kappa^{++}}^{N_0}$ or, equivalently, $V_{\kappa+2} \in N_0$.

Finally, we claim that the iteration giving rise to j^* ends after one step, meaning that $j^* = j$. Suppose to the contrary that j_1 is nontrivial. By the normality of the iteration, the critical point of j_1 must be some $\lambda > \kappa$. We can find a function $h \in V[g][G]$, defined on κ , such that $j^*(h)(\kappa) = \lambda$, since j^* is given by a measure ultrapower of $V[g][G]$. By the κ -Sacks property of \mathbb{P} (see item (iii)) we can cover the function h by a function $\bar{h} \in V[g]$; in fact, since the forcing to add g was $\leq \kappa$ -closed, we have $\bar{h} \in V$. Now

$$\lambda = j^*(h)(\kappa) \in j^*(\bar{h})(\kappa) = j_1(j(\bar{h}))(j_1(\kappa)) = j_1(j(\bar{h})(\kappa))$$

and $A = j(\bar{h})(\kappa)$ has cardinality at most κ^{++} in M . In particular, since $\kappa^{++} < \lambda$, we have $\lambda \in j_1(A) = j_1[A]$, which is a contradiction, since λ was the critical point of j_1 .

We can conclude that any embedding j^* arising from a normal measure on κ in $V[g][G]$ is a lift of the ground model $(\kappa + 2)$ -strongness embedding j . But there are exactly κ^{++} many such lifts: the lift to $V[g]$ is unique, and there are κ^{++} many possibilities for the final lift to $V[g][G]$, according to item (vi). Therefore there are only κ^{++} many normal measures

on κ in $V[g][G]$. □

Ben Neria and Gitik have recently announced that the consistency strength required to achieve the failure of GCH at a measurable cardinal carrying a unique normal measure is exactly that of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ (see [3]). Their method is flexible enough to allow us to incorporate it into our proof of theorem 1.22, reducing the consistency strength hypothesis required there from a $(\kappa + 2)$ -strong cardinal κ to just $o(\kappa) = \kappa^{++}$. We have chosen to present the proof based on the original Friedman–Magidor argument since it avoids some complications arising from using the optimal hypothesis.

1.2.3 (Joint) Laver diamonds and the number of normal measures

The only method of controlling the existence of (joint) Laver diamonds we have seen is by controlling the number of large cardinal measures, relying on the rough bound given by proposition 1.12. One has to wonder whether merely the existence of sufficiently many measures guarantees the existence of (joint) Laver diamonds. We focus on the simplest form of the question, concerning measurable cardinals.

Question 1.23 (open). *Suppose κ is measurable and there are at least 2^λ many normal measures on κ for some $\lambda \geq \kappa$. Does there exist a measurable joint Laver sequence for κ of length λ ?*

As the special case when $\lambda = \kappa$, the question includes the possibility that having 2^κ many normal measures, the minimum required, suffices to give the existence of a measurable Laver function for κ . Even in this very special case it seems implausible that simply having enough measures would automatically yield a Laver function. Nevertheless, in all of the examples of models obtained by forcing and in which we have control over the number of measures that we have seen, Laver functions have existed. On the other hand, Laver functions and joint Laver sequences also exist in canonical inner models that have sufficiently many measures.

These models carry long Mitchell-increasing sequences of normal measures that we can use to obtain ordinal-guessing Laver functions. We can then turn these into actual Laver functions by exploiting the coherence and absoluteness of these models.

Definition 1.24. Let A be a set (or class) of ordinals and let $\bar{\ell}$ be an A -guessing Laver function for some large cardinal κ . Let \triangleleft be some well-order (one arising from an L -like inner model, for example). We say that \triangleleft is *suitable* for $\bar{\ell}$ if, for any $\alpha \in A$, there is an elementary embedding j , witnessing the largeness of κ , such that $\bar{\ell}$ guesses α via j and $j(\triangleleft) \upharpoonright (\alpha + 1) = \triangleleft \upharpoonright (\alpha + 1)$; that is, the well-orders $j(\triangleleft)$ and \triangleleft agree on their first $\alpha + 1$ many elements.

If \mathcal{J} is a class of elementary embeddings witnessing the largeness of κ , we say that \triangleleft is *supersuitable* for \mathcal{J} if $j(\triangleleft) \upharpoonright j(\kappa) = \triangleleft \upharpoonright j(\kappa)$ for any $j \in \mathcal{J}$.

We could, for example, take the class \mathcal{J} to consist of all ultrapower embeddings by normal measures on κ or, more to the point, all ultrapower embeddings arising from a fixed family of extenders. We should also note that, for the notion to make sense, the order type of \triangleleft must be quite high: at least $\sup A$ in the case of well-orders suitable for an A -guessing Laver function and at least $\sup_{j \in \mathcal{J}} j(\kappa)$ for supersuitable well-orders (the latter would also make sense if the order type of \triangleleft were smaller than κ , but that case is not of much interest).

Clearly any supersuitable well-order is suitable for any ordinal-guessing Laver function $\bar{\ell}$, provided that the class \mathcal{J} includes the embeddings via which $\bar{\ell}$ guesses its targets. The following obvious lemma describes the way in which suitable well-orders will be used to turn ordinal-guessing Laver functions into set-guessing ones.

Lemma 1.25. *Let A be a set (or class) of ordinals and let $\bar{\ell}$ be an A -guessing Laver function for some large cardinal κ . Let \triangleleft be a well-order such that $\text{otp}(\triangleleft) \subseteq A$. If \triangleleft is suitable for $\bar{\ell}$, then there is a B -guessing Laver function for κ , where B is the field of \triangleleft .*

Proof. We can define a B -guessing Laver function by simply letting $\ell(\xi)$ be the $\bar{\ell}(\xi)$ th element

of \triangleleft . Then, given a target $b \in B$, we can find its index α in the well-order \triangleleft and an embedding j such that $j(\bar{\ell})(\kappa) = \alpha$. Since \triangleleft is suitable for $\bar{\ell}$, the orders \triangleleft and $j(\triangleleft)$ agree on their α th element and so ℓ guesses b via j . \square

It follows from the above lemma that in any model with a sufficiently supersuitable well-order, being able to guess ordinals suffices to be able to guess arbitrary sets.

Lemma 1.26. *Let X be a set (or class) of ordinals and let \mathcal{J} be a class of elementary embeddings of $L[X]$ with critical point κ such that $j(X) \cap j(\kappa) = X \cap j(\kappa)$ for any $j \in \mathcal{J}$. Then \leq_X , the canonical order of $L[X]$, is supersuitable for \mathcal{J} .*

Proof. This is obvious; the order $\leq_X \upharpoonright j(\kappa)$ is definable in $L_{j(\kappa)}[X]$, but by our coherence hypothesis this structure is just the same as $L_{j(\kappa)}[j(X)]$. \square

We are mostly interested in this lemma in the case when $X = \vec{E}$ is an extender sequence and $L[\vec{E}]$ is an extender model in the sense of [25]. In particular, we want \vec{E} to be *acceptable* (a technical condition which implies enough condensation properties in $L[\vec{E}]$ to conclude $H_\lambda^{L[\vec{E}]} = L_\lambda[\vec{E}]$), *coherent* (meaning that if $j: L[\vec{E}] \rightarrow L[\vec{F}]$ is an ultrapower by the α th extender of \vec{E} then $\vec{F} \upharpoonright (\alpha + 1) = \vec{E} \upharpoonright \alpha$), and to use Jensen indexing (meaning that the index of an extender E on \vec{E} with critical point κ is $j_E(\kappa)^+$, as computed in the ultrapower).

Corollary 1.27. *Let $V = L[\vec{E}]$ be an extender model. Then the canonical well-order is supersuitable for the class of ultrapower embeddings by the extenders on the sequence \vec{E} .*

Proof. This is immediate from the preceding lemma and the fact that our extender sequences are coherent and use Jensen indexing. \square

Theorem 1.28. *Let $V = L[\vec{E}]$ be an extender model. Let κ be a cardinal such that every normal measure on κ appears on the sequence \vec{E} . If $o(\kappa) \geq \kappa^+$ then $\triangleleft_\kappa^{\text{meas}}$ holds. Moreover, if $o(\kappa) = \kappa^{++}$ then $\triangleleft_{\kappa, \kappa^+}^{\text{meas}}$, and even $\triangleleft_\kappa^{\text{meas}}(H_{\kappa^{++}})$, holds.*

In particular, the above theorem implies $\mathbb{N}_\kappa^{\text{meas}}$ holds in the least inner model with the required number of measures and the same holds for $\mathbb{N}_{\kappa, \kappa^+}^{\text{meas}}$. This provides further evidence that the answer to question 1.23, which remains open, might turn out to be positive

Proof. We can argue for the two cases more or less uniformly: let $\lambda \in \{\kappa^+, \kappa^{++}\}$ such that $\lambda \leq o(\kappa)$. The function $\bar{\ell}(\xi) = o(\xi)$ is a λ -guessing measurable Laver function for κ . By the acceptability of \vec{E} we have that $H_\lambda = L_\lambda[\vec{E}]$. The canonical well-order $\leq_{\vec{E}} \cap L_\lambda[\vec{E}]$ has order type λ and, by corollary 1.27, is supersuitable for the class of ultrapower embeddings by normal measures on κ . It follows that $\leq_{\vec{E}}$ is suitable for $\bar{\ell}$, so, by lemma 1.25, there is an H_λ -guessing measurable Laver function for κ .

To finish the proof we still need to produce a joint measurable Laver sequence for κ , in the case that $o(\kappa) = \kappa^{++}$. This is done in exactly the same way as in proposition 1.4; one simply uses the $H_{\kappa^{++}}$ -guessing Laver function to guess the whole sequence of targets for a joint Laver sequence. \square

We should also mention that, if we restrict to a smaller set of targets, having enough normal measures *does* give us Laver functions.

Lemma 1.29. *Let κ be a regular cardinal and $\gamma \leq \kappa$ and suppose that $\langle \mu_\alpha; \alpha < \gamma \rangle$ is a sequence of distinct normal measures on κ . Then there is a sequence $\langle X_\alpha; \alpha < \gamma \rangle$ of pairwise disjoint subsets of κ such that $X_\alpha \in \mu_\beta$ if and only if $\alpha = \beta$.*

Proof. We prove the lemma by induction on γ . In the base step, $\gamma = 1$, we simply observe that, since $\mu_0 \neq \mu_1$, we must have a set $X_0 \subseteq \kappa$ such that $X_0 \in \mu_0$ and $\kappa \setminus X_0 \in \mu_1$.

The successor step proceeds similarly. Suppose that the lemma holds for sequences of length γ and fix a sequence of measures $\langle \mu_\alpha; \alpha < \gamma + 1 \rangle$. By the induction hypothesis we can find pairwise disjoint sets $\langle Y_\alpha; \alpha < \gamma \rangle$ such that each Y_α picks out a unique measure among those with indices below γ . Again, since μ_γ is distinct from all of the other measures, we

can find sets $Z_\alpha \in \mu_\gamma \setminus \mu_\alpha$ for each $\alpha < \gamma$. Then the sets $X_\alpha = Y_\alpha \setminus Z_\alpha$ for $\alpha < \gamma$ and $X_\gamma = \bigcap_\alpha Z_\alpha$ are as required.

In the limit step suppose that the lemma holds for all $\delta < \gamma$. We can then fix sequences $\langle X_\alpha^\delta; \alpha < \delta \rangle$ for each $\delta < \gamma$ as above. The argument proceeds slightly differently depending on whether $\gamma = \kappa$ or not. If $\gamma < \kappa$ we can simply let $X_\alpha = \bigcap_{\alpha < \delta < \gamma} X_\alpha^\delta \in \mu_\alpha$. If, on the other hand, we have $\gamma = \kappa$ then first let $Y_\alpha = \Delta_{\alpha < \delta < \kappa} X_\alpha^\delta \in \mu_\alpha$. Observe that the Y_α are almost disjoint: $Y_\alpha \cap Y_\beta$ is bounded in κ for any $\alpha, \beta < \kappa$. Now consider

$$X_\alpha = Y_\alpha \setminus \bigcup_{\beta < \alpha} (Y_\alpha \cap Y_\beta)$$

for $\alpha < \kappa$. Since $Y_\alpha \cap Y_\beta$ is bounded for all $\beta < \alpha$, we still have $X_\alpha \in \mu_\alpha$. Furthermore, we obviously have $X_\alpha \cap X_\beta = \emptyset$ for $\beta < \alpha$ and this implies that the X_α are pairwise disjoint. \square

Theorem 1.30. *Let κ be a measurable cardinal and $\gamma < \kappa^+$ an ordinal. There is a γ -guessing measurable Laver function for κ if and only if there are at least $|\gamma|$ many normal measures on κ .*

Proof. First suppose that $\Delta_\kappa^{\text{meas}}(\gamma)$ holds. Then, just as in proposition 1.12, each target $\alpha < \gamma$ requires its own embedding j via which it is guessed and this gives us $|\gamma|$ many distinct normal measures.

Conversely, suppose that we have at least $|\gamma|$ many normal measures on κ . We can apply lemma 1.29 to find a sequence of pairwise disjoint subsets distinguishing these measures. By reorganizing the measures and the distinguishing sets we may assume that they are given in sequences of length γ . We now have normal measures $\langle \mu_\alpha; \alpha < \gamma \rangle$ and sets $\langle X_\alpha; \alpha < \gamma \rangle$ such that μ_α is the unique measure concentrating on X_α ; we may even assume that the X_α partition κ . Let f_α for $\alpha < \gamma$ be the representing functions for α , that is, $j(f_\alpha)(\kappa) = \alpha$ for any ultrapower embedding j by a normal measure on κ . We can now define a γ -guessing Laver function ℓ by letting $\ell(\xi) = f_\alpha(\xi)$ where α is the unique index such that $\xi \in X_\alpha$. This

function indeed guesses any target $\alpha < \gamma$: simply let $j: V \rightarrow M$ be the ultrapower by μ_α . Since μ_α concentrates on X_α we have $j(\ell)(\kappa) = j(f_\alpha)(\kappa) = \alpha$. \square

Corollary 1.31. *Let κ be a measurable cardinal and fix a subset $A \subseteq H_{\kappa^+}$ of size at most κ . Then there is an A -guessing measurable Laver function for κ if and only if there are at least $|A|$ many normal measures on κ .*

Proof. The forward direction follows just as before: each target in A gives its own normal measure on κ . Conversely, if there are at least $|A|$ many normal measures on κ then, by theorem 1.30, there is an $|A|$ -guessing measurable Laver function $\bar{\ell}$. Fix a bijection $f: |A| \rightarrow A$. We may assume, moreover, that $A \subseteq \mathcal{P}(\kappa)$. Then we can define an A -guessing Laver function ℓ by letting $\ell(\xi) = f(\bar{\ell}(\xi)) \cap \xi$. This definition works: to guess $f(\alpha)$ we let $\bar{\ell}$ guess α via some j . Then $j(\ell)(\kappa) = j(f(\alpha)) \cap \kappa = f(\alpha)$. \square

Lemma 1.29 can be recast in somewhat different language, giving it, and the subsequent results, a more topological flavour.

Given a cardinal κ let $\mathcal{M}(\kappa)$ be the set of normal measures on κ . We can topologize $\mathcal{M}(\kappa)$ by having, for each $X \subseteq \kappa$, a basic neighbourhood $[X] = \{\mu \in \mathcal{M}(\kappa); X \in \mu\}$ (this is just the topology induced on $\mathcal{M}(\kappa)$ by the Stone topology on the space of ultrafilters on κ). Lemma 1.29 can now be restated to say that any subspace of $\mathcal{M}(\kappa)$ of size at most κ is discrete and, moreover, the *discretizing family* of subsets of κ witnessing this can be taken to be pairwise disjoint. One might thus hope to show the existence of Laver functions by exhibiting even larger discrete subspaces of $\mathcal{M}(\kappa)$. In pursuit of that goal we obtain the following generalization of corollary 1.31.

Theorem 1.32. *Let κ be a measurable cardinal and $A \subseteq \mathcal{P}(\kappa)$. Then $\Delta_\kappa^{\text{meas}}(A)$ holds if and only if there are for each $a \in A$ a set $S_a \subseteq \kappa$ and a normal measure μ_a on κ such that $\{\mu_a; a \in A\}$ is discrete in $\mathcal{M}(\kappa)$, as witnessed by $\{S_a; a \in A\}$, and $S_a \cap S_b \subseteq \{\xi; a \cap \xi = b \cap \xi\}$.*

Note that we could have relaxed our hypothesis to $A \subseteq H_{\kappa^+}$ by working with Mostowski codes.

Proof. Assume first that ℓ is a measurable A -guessing Laver function for κ . Then we can let $S_a = \{\xi; \ell(\xi) = a \cap \xi\}$. Obviously we have $j(\ell)(\kappa) = a$ if and only if the measure derived from j concentrates on S_a . It follows that the measures μ_a derived this way form a discrete subspace of $\mathcal{M}(\kappa)$ and we obviously have $S_a \cap S_b \subseteq \{\xi; a \cap \xi = b \cap \xi\}$.

Conversely, assume we have such a discrete family of measures μ_a and a discretizing family of sets S_a . We can define an A -guessing measurable Laver function ℓ by letting $\ell(\xi) = a \cap \xi$ where a is such that $\xi \in S_a$. This is well defined by the coherence condition imposed upon the S_a and it is easy to see that ℓ satisfies the guessing property. \square

This topological viewpoint presents a number of questions which might suggest an approach to question 1.23. For example, it is unclear whether, given a discrete family of normal measures one can find an almost disjoint discretizing family as in the above theorem. Even more pressingly, we do not know whether it is possible for $\mathcal{M}(\kappa)$ to have no discrete subspaces of size κ^+ (while itself having size at least κ^+).

1.2.4 Laver trees

Thus far we have thought of joint Laver diamonds as simply matrices or sequences of Laver diamonds. To better facilitate the reflection properties required for the usual forcing iterations using prediction, we would now like a different representation. A reasonable attempt seems to be trying to align the joint Laver sequence with the full binary tree of height κ .

Definition 1.33. Let κ be a large cardinal supporting a notion of Laver diamond. A *Laver tree*³ for κ is a labelling of the binary tree such that the labels along the branches form a joint Laver sequence. More precisely, a Laver tree is a function $D: {}^{<\kappa}2 \rightarrow V$ such that

³Not to be confused with particularly bushy trees used as conditions in Laver forcing.

for any sequence of targets $\langle a_s; s \in {}^\kappa 2 \rangle$ there is an elementary embedding j , witnessing the largeness κ , such that $j(D)(s) = a_s$ for all $s \in {}^\kappa 2$.

Given an $I \subseteq {}^\kappa 2$, an I -Laver tree for κ is a function D as above, satisfying the same guessing property but only for sequences of targets indexed by I .

The guessing property of a Laver tree is illustrated in the following diagram:

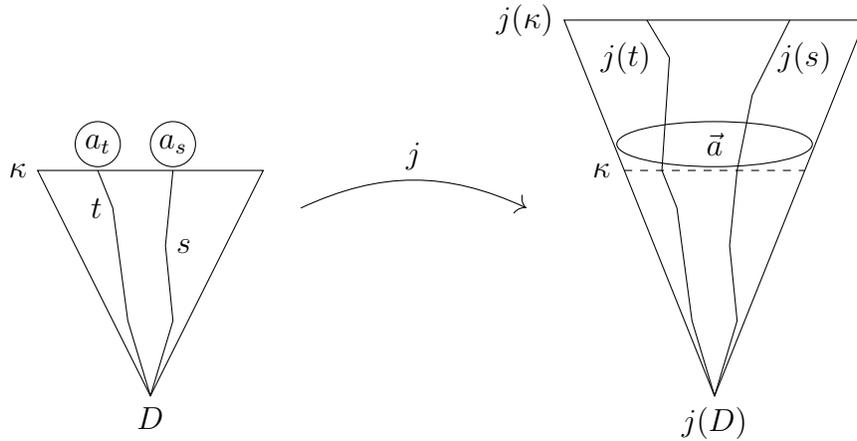


Figure 1.2: The guessing property of Laver trees

If the degree of supercompactness of κ is sufficiently large then Laver trees are nothing new.

Proposition 1.34. *Suppose κ is θ -supercompact and $\theta \geq 2^\kappa$. Then a θ -supercompactness Laver tree for κ exists if and only if a θ -supercompactness Laver function for κ does (if and only if $\triangleleft_{\kappa, 2^\kappa}^{\theta\text{-sc}}$ holds).*

Proof. The forward implication is trivial, so we focus on the reverse implication. Let ℓ be a Laver function. For any $t \in {}^{<\kappa} 2$ define $D(t) = \ell(|t|)(t)$ if this makes sense and $D(t) = \emptyset$ otherwise. We claim this defines a Laver tree for κ . Indeed, let $\vec{a} = \langle a_s; s \in {}^\kappa 2 \rangle$ be a sequence of targets. Since $\theta \geq 2^\kappa$ we get $\vec{a} \in H_{\theta^+}$, so there is a θ -supercompactness embedding j such that $j(\ell)(\kappa) = \vec{a}$. Therefore, given any $s \in {}^\kappa 2$, we have $j(D)(s) = j(\ell)(\kappa)(s) = a_s$ \square

In other situations, however, the existence of a Laver tree can have strictly higher consistency strength than merely a θ -supercompact cardinal.

Definition 1.35. Let X be a set and θ a cardinal. A cardinal κ is X -strong with closure θ if there is an elementary embedding $j: V \rightarrow M$ with critical point κ such that ${}^\theta M \subseteq M$ and $X \in M$.

Proposition 1.36. *Suppose κ is θ -supercompact and there is a θ -supercompactness Laver tree for κ . Then κ is X -strong with closure θ for any $X \subseteq H_{\theta^+}$ of size at most 2^κ .*

Proof. Suppose $D: {}^{<\kappa}2 \rightarrow V_\kappa$ is a Laver tree and fix an $X \subseteq H_{\theta^+}$ of size at most 2^κ . Let $f: {}^\kappa 2 \rightarrow X$ enumerate X . We can then find a θ -supercompactness embedding $j: V \rightarrow M$ with critical point κ such that $j(D)(s) = f(s)$ for all $s \in {}^\kappa 2$. In particular, $X = j(D)[{}^\kappa 2]$ is an element of M , as required. \square

If $2^\kappa \leq \theta$ then X -strongness with closure θ for all $X \subseteq H_{\theta^+}$ of size 2^κ amounts to just θ -supercompactness and proposition 1.34 gives the full equivalence of Laver functions and Laver trees. But if $\theta < 2^\kappa$ then X -strongness with closure θ can have additional consistency strength. For example, we might choose X to be a normal measure on κ to see that κ must have nontrivial Mitchell rank (by iterating this idea we can even deduce that $o(\kappa) = (2^\kappa)^+$). In the typical scenario where $2^\kappa = 2^\theta = \theta^+$, we can also reach higher and choose X to be a normal measure on $\mathcal{P}_\kappa \theta$ and see that κ must also have nontrivial θ -supercompactness Mitchell rank. We can use this observation to show that there might not be any Laver trees, even in the presence of very long joint Laver sequences.

Theorem 1.37. *Suppose GCH holds and let κ be θ -supercompact where either $\theta = \kappa$ or $\text{cf}(\theta) > \kappa$. Then there is a cardinal-preserving forcing extension in which κ remains θ -supercompact, has a θ -supercompactness joint Laver sequence of length 2^κ , but is also the least measurable cardinal. In particular, $\theta < 2^\kappa$ and κ has no θ -supercompactness Laver trees in the extension.*

Proof. We may assume by prior forcing as in the proof of theorem 1.7, that κ has a Laver function. Additionally, by performing either Magidor's iteration of Prikry forcing (see [18]) or applying an argument due to Apter and Shelah (see [2]), depending on whether $\theta = \kappa$ or not, we may assume that, in addition to being θ -supercompact, κ is also the least measurable cardinal.⁴

We now apply corollary 1.9 and arrive at a model where κ carries a θ -supercompactness joint Laver sequence of length 2^κ , but is also the least measurable. It follows that there can be no (θ -supercompactness or even measurable) Laver trees for κ , since, by the discussion above, their existence would imply that κ has nontrivial Mitchell rank, implying that there are many measurables below κ . \square

Proposition 1.36 can be improved slightly to give a jump in consistency strength even for I -Laver trees where I is not the whole set of branches. A simple modification of the proof given there yields the following result, together with the corresponding version of theorem 1.37.

Theorem 1.38. *Suppose κ is θ -supercompact and there is a θ -supercompactness I -Laver tree for κ for some $I \subseteq {}^\kappa 2$ of size 2^κ . If I is definable (with parameters) over H_{κ^+} then κ is X -strong with closure θ for any $X \subseteq H_{\theta^+}$ of size at most 2^κ .*

The above theorem notwithstanding, we shall give a construction which shows that the existence of an I -Laver tree does not yield additional consistency strength, provided that we allow I to be sufficiently foreign to H_{κ^+} . The argument will rely on being able to surgically alter a Cohen subset of κ^+ in a variety of ways. To this end we fix some notation beforehand.

Definition 1.39. Let f and g be functions. The *graft* of f onto g is the function $g \wr f$,

⁴Of course, if $\theta > \kappa$, this arrangement requires a strong failure of GCH at κ . In fact, $2^\kappa = \theta^+$ in the Apter–Shelah model.

defined on $\text{dom}(g)$ by

$$(g \wr f)(x) = \begin{cases} f(x); & x \in \text{dom}(g) \cap \text{dom}(f) \\ g(x); & x \in \text{dom}(g) \setminus \text{dom}(f) \end{cases}$$

Essentially, the graft replaces the values of g with those of f on their common domain.

Theorem 1.40. *Let λ be a regular cardinal and assume \diamond_λ holds. Suppose M is a transitive model of ZFC (either set- or class-sized) such that $\lambda \in M$ and $M^{<\lambda} \subseteq M$ and $|\mathcal{P}(\lambda)^M| = \lambda$. Then there are an unbounded set $I \subseteq \lambda$ and a function $g: \lambda \rightarrow H_\lambda$ such that, given any $f: I \rightarrow H_\lambda$, the graft $g \wr f$ is generic for $\text{Add}(\lambda, 1)$ over M .⁵*

The hypothesis of \diamond_λ is often automatically satisfied. Specifically, our assumptions about M imply that $2^{<\lambda} = \lambda$. If $\lambda = \kappa^+$ is a successor, this gives $2^\kappa = \kappa^+$ which already implies \diamond_λ by a result of Shelah [24].

Proof. Let $\langle f_\alpha; \alpha < \lambda \rangle$, with $f_\alpha: \alpha \rightarrow H_{|\alpha|}$, be a \diamond_λ -sequence and fix an enumeration $\langle D_\alpha; \alpha < \lambda \rangle$ of the open dense subsets of $\text{Add}(\lambda, 1)$ in M . We shall construct by recursion a descending sequence of conditions $p_\alpha \in \text{Add}(\lambda, 1)$ and an increasing sequence of sets I_α as approximations to g and I . Specifically, we shall use \diamond_λ to guess pieces of any potential function f and ensure along the way that the modified conditions $p_\alpha \wr f$ meet all of the listed dense sets.

Suppose we have built the sequences $\langle p_\alpha; \alpha < \gamma \rangle$ and $\langle I_\alpha; \alpha < \gamma \rangle$ for some $\gamma < \lambda$. Let $I_\gamma^* = \bigcup_{\alpha < \gamma} I_\alpha$. Let $p_\gamma^* \in M$ be an extension of $\bigcup_{\alpha < \gamma} p_\alpha$ such that $I_\gamma^* \subseteq \text{dom}(p_\gamma^*) \in \lambda$; such an extension exists in M by our assumption on the closure of M .

Let us briefly summarize the construction. We shall surgically modify the condition p_γ^* by grafting the function given by \diamond_λ onto it. We shall then extend this modified condition

⁵Here we take the forcing-equivalent version of $\text{Add}(\lambda, 1)$ which adds a function $g: \lambda \rightarrow H_\lambda$ by initial segments.

to meet one of our dense sets, after which we will undo the surgery. We will be left with a condition p_γ which is one step closer to ensuring that the result of one particular grafting $g \restriction f$ is generic. At the same time we also extend I_γ^* by adding a point beyond the domains of all the conditions constructed so far.

More precisely, let $\tilde{p}_\gamma^* = p_\gamma^* \restriction (f_\gamma \restriction I_\gamma^*)$. This is still a condition in M . Let \tilde{p}_γ be any extension of this condition inside D_{η_γ} , where η_γ is the least such that $\tilde{p}_\gamma^* \notin D_{\eta_\gamma}$, and satisfying $\text{dom}(\tilde{p}_\gamma) \in \lambda$. Finally, we undo the initial graft and set $p_\gamma = \tilde{p}_\gamma \restriction (p_\gamma^* \restriction I_\gamma^*)$. Note that we have $p_\gamma \leq p_\gamma^*$. We also extend our approximation to I with the first available point, letting $I_\gamma = I_\gamma^* \cup \{\min(\lambda \setminus \text{dom}(p_\gamma))\}$.

Once we have completed this recursive construction we can set $I = \bigcup_{\gamma < \lambda} I_\gamma$ and $g = \bigcup_{\gamma < \lambda} p_\gamma$. Let us check that these do in fact have the desired properties.

Let $f: I \rightarrow H_\lambda$ be a function. We need to show that $g \restriction f$ is generic over M . Using \diamond_λ , we find that there are stationarily many γ such that $f_\gamma = f \restriction \gamma$. Note also that there are club many γ such that $I_\gamma^* \subseteq \gamma$ is unbounded, and together this means that $S = \{\gamma; f_\gamma \restriction I_\gamma^* = f \restriction (I \cap \gamma)\}$ is stationary. The conditions \tilde{p}_γ for $\gamma \in S$ extend each other and we have $\bigcup_{\gamma \in S} \tilde{p}_\gamma = g \restriction f$. Furthermore, since the sets D_α are open, the construction of \tilde{p}_γ ensures that eventually these conditions will meet every such dense set, showing that $g \restriction f$ really is generic. \square

The construction in the above proof is quite flexible and can be modified to make the set I generic in various ways as well (for example, we can arrange for I to be Cohen or dominating over M etc.).

Theorem 1.41. *If κ is θ -supercompact then there is a forcing extension in which there is a θ -supercompactness I -Laver tree for κ for some $I \subseteq {}^\kappa 2$ of size 2^κ .*

Proof. If $\theta \geq 2^\kappa$ then even a single Laver function for κ gives rise to a full Laver tree, by proposition 1.34, and we can force the existence of a Laver function by theorem 1.7. We thus

focus on the remaining case when $\kappa \leq \theta < 2^\kappa$.

We make similar simplifying assumptions as in theorem 1.7. Just as there we assume that $\theta = \theta^{<\kappa}$. Furthermore, we may assume that $2^\theta = \theta^+$, since this can be forced without adding subsets to $\mathcal{P}_\kappa\theta$ and affecting the θ -supercompactness of κ . Note that these cardinal arithmetic hypotheses imply that $2^\kappa = \theta^+$.

Let \mathbb{P} be the length κ Easton support iteration which adds, in a recursive fashion, a labelling of the tree ${}^{<\kappa}2$ of the extension. Specifically, let \mathbb{P} force with $\mathbb{Q}_\gamma = \text{Add}(2^\gamma, 1)$ at each inaccessible $\gamma < \kappa$ stage γ . Let $G \subseteq \mathbb{P}$ be generic and let G_γ be the piece added at stage γ . Using suitable coding, we can see each G_γ , in $V[G]$, as a function $G_\gamma: {}^\gamma 2 \rightarrow H_{\gamma^+}$; in particular, we should note that the iteration \mathbb{P} does not add any nodes to the tree $(\leq^\gamma 2)^{V[G]}$ at stage γ or later, so that G_γ really does label the whole level ${}^\gamma 2$. Thus G induces a map $D: {}^{<\kappa}2 \rightarrow V_\kappa[G]$, by extending the G_γ in any way we like to the entire tree. We shall show that D is an I -Laver tree for some specifically chosen I .

Fix a θ -supercompactness embedding $j: V \rightarrow M$ in V . Note that $M[G]^\theta \subseteq M[G]$ in $V[G]$ as well, since the forcing \mathbb{P} is θ^+ -cc. Furthermore, in $V[G]$ we still have $2^\theta = \theta^+$, which implies \diamond_{θ^+} by a result of Shelah [24]. Now apply theorem 1.40 to $M[G]$ and $\lambda = \theta^+$ to obtain an $I \subseteq {}^\kappa 2$ of size θ^+ and a function $g: {}^\kappa 2 \rightarrow H_{\theta^+}$ such that for any $f: I \rightarrow H_{\theta^+}$ in $V[G]$, the graft $g \wr f$ is generic over $M[G]$. We claim that D is an I -Laver tree.

To check the guessing property, fix a sequence of targets $\vec{a} = \langle a_s; s \in I \rangle$ in $V[G]$. We shall lift the embedding j to $V[G]$. Let us write $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q}_\kappa * \mathbb{P}_{\text{tail}}$. We know that $g \wr \vec{a}$ is $M[G]$ -generic for \mathbb{Q}_κ , so we only need to find the further generic for \mathbb{P}_{tail} . We easily see that $M[G][g \wr \vec{a}]^\theta \subseteq M[G][g \wr \vec{a}]$ in $V[G]$, that \mathbb{P}_{tail} is $\leq \theta$ -closed in that model, and that $M[G][g \wr \vec{a}]$ only has θ^+ -many subsets of \mathbb{P}_{tail} . We can thus diagonalize against these dense sets in θ^+ -many steps and produce a generic G_{tail} for \mathbb{P}_{tail} . Putting all of this together, we can lift j to $j: V[G] \rightarrow M[j(G)]$ in $V[G]$, where $j(G) = G * (g \wr \vec{a}) * G_{\text{tail}}$. Now consider $j(D)$. This is exactly the labelling of the tree ${}^{<j(\kappa)}2$ in $M[j(G)]$ given by $j(G)$. Furthermore, for

any $s \in I$, we have $j(D)(s) = (g \upharpoonright \vec{a})(s) = a_s$, verifying the guessing property. \square

Given a Laver tree for κ , it is easy to produce a joint Laver sequence of length 2^κ from it by just reading the labels along each branch of the Laver tree. The resulting sequence then exhibits a large degree of coherence. We might wonder about the possibility of reversing this process, starting with a joint Laver sequence and attempting to fit it into a tree. A sequence for which this can be done might be called treeable. But, taken literally, this notion is not very robust. For example, all functions in a treeable joint Laver sequence must have the same value at 0. This means that we could take a treeable sequence and modify it in an inessential way to destroy its treeability. To avoid such trivialities, we relax the definition to only ask for coherence modulo bounded perturbations.

Definition 1.42. Let κ be a regular cardinal and $\vec{f} = \langle f_\alpha; \alpha < \lambda \rangle$ a sequence of functions defined on κ . The sequence \vec{f} is *treeable* if there are a bijection $e: \lambda \rightarrow {}^\kappa 2$ and a tree labelling $D: {}^{<\kappa} 2 \rightarrow V$ such that, for all $\alpha < \lambda$, we have $D(e(\alpha) \upharpoonright \xi) = f_\alpha(\xi)$ for all but boundedly many $\xi < \kappa$.

Given an $I \subseteq {}^\kappa 2$, the sequence \vec{f} is *I-treeable* if the above holds for a bijection $e: \lambda \rightarrow I$.

Lemma 1.43. *Let κ be a regular cardinal and assume that $(2^{<\kappa})^+ \leq 2^\kappa$. Let $G \subseteq \text{Add}(\kappa, 2^\kappa)$ be generic. Then G is not treeable.*

Proof. Let us write $G = \langle g_\alpha; \alpha < 2^\kappa \rangle$ as a sequence of its slices. Now suppose that this sequence were treeable and let \dot{e} and \dot{D} be names for the indexing function and the labelling of ${}^{<\kappa} 2$, respectively. Our cardinal arithmetic assumption implies that the name \dot{D} only involves conditions from a bounded part of the poset $\text{Add}(\kappa, 2^\kappa)$, so we may assume that the labelling D exists already in the ground model. Let p be an arbitrary condition and $\alpha < \kappa$. Since we assumed that G was forced to be treeable, there is a name $\dot{\gamma}$ for an ordinal, beyond which G_α agrees with $D \upharpoonright f(\alpha)$. By strengthening p if necessary, we may assume that the value of $\dot{\gamma}$ has been decided. We now inductively construct a countable descending sequence

of conditions below p , deciding longer and longer initial segments of $\dot{e}(\alpha)$, in such a way that, for some $\delta > \gamma$, their union $p^* \leq p$ decides $\dot{e}(\alpha) \upharpoonright \delta$ but does not decide $G_\alpha(\delta)$. Then p^* can be further extended to a condition forcing $G_\alpha(\delta) \neq D(\dot{e}(\alpha) \upharpoonright \delta)$, which contradicts the fact that p forces that G is treeable. \square

Corollary 1.44. *If κ is θ -supercompact then there is a forcing extension in which there is a nontreeable θ -supercompactness joint Laver sequence for κ of length 2^κ .*

Proof. The joint Laver sequence constructed in theorem 1.7 was added by forcing with $\text{Add}(\kappa, 2^\kappa)$, so it is not treeable by lemma 1.43. \square

1.3 Joint Laver diamonds for strong cardinals

Definition 1.45. A function $\ell: \kappa \rightarrow V_\kappa$ is a θ -strongness Laver function if it guesses elements of V_θ via θ -strongness embeddings with critical point κ .

If κ is fully strong then a function $\ell: \kappa \rightarrow V_\kappa$ is a Laver function for κ if it is a θ -strongness Laver function for κ for all θ .

As in the supercompact case, 2^κ is the largest possible cardinal length of a θ -strongness joint Laver sequence for κ , just because there are only 2^κ many functions $\ell: \kappa \rightarrow V_\kappa$.

The set of targets V_θ is a bit unwieldy and lacks some basic closure properties, particularly in the case when θ is a successor ordinal. The following lemma shows that, modulo some coding, we can recover a good deal of closure under sequences.

Lemma 1.46. *Let θ be an infinite ordinal and let $I \in V_\theta$ be a set. If θ is successor ordinal or $\text{cf}(\theta) > |I|$ then V_θ is closed under a coding scheme for sequences indexed by I . Moreover, this coding is Δ_0 -definable.*

Proof. If $\theta = \omega$ then the I under consideration are finite. Since V_ω is already closed under finite sequences we need only deal with $\theta > \omega$.

Fix in advance a simply definable flat pairing function $[\cdot, \cdot]$ (flat in the sense that any infinite V_α is closed under it; the Quine–Rosser pairing function will do).

Let $\vec{a} = \langle a_i; i \in I \rangle$ be a sequence of elements of V_θ . For each $i \in I$ we can find an (infinite) ordinal $\theta_i < \theta$ such that $a_i \cup \{i\} \subseteq V_{\theta_i}$. Now let $\tilde{a}_i = \{[i, b]; b \in a_i\} \subseteq V_{\theta_i}$ and finally define $\tilde{a} = \bigcup_{i \in I} \tilde{a}_i$. We see that $\tilde{a} \subseteq V_{\sup_i \theta_i}$ and, under our hypotheses, $\sup_i \theta_i < \theta$. It follows that $\tilde{a} \in V_{(\sup_i \theta_i)+1} \subseteq V_\theta$ as required. \square

Proposition 1.47. *Let κ be θ -strong with $\kappa + 2 \leq \theta$ and let $\lambda \leq 2^\kappa$ be a cardinal. If there is a θ -strongness Laver function for κ and θ is either a successor ordinal or $\lambda < \text{cf}(\theta)$ then there is a θ -strongness joint Laver sequence of length λ for κ .*

In particular, if θ is a successor then a single θ -strongness Laver function already yields a joint Laver sequence of length 2^κ , the maximal possible.

Proof. We aim to imitate the proof of proposition 1.4. To that end, fix an $I \subseteq \mathcal{P}(\kappa)$ of size λ and a bijection $f: \lambda \rightarrow I$. If ℓ is a Laver function for κ , we define a joint Laver sequence by letting $\ell_\alpha(\xi)$, for each $\alpha < \lambda$, be the element of $\ell(\xi)$ with index $f(\alpha) \cap \xi$ in the coding scheme described in lemma 1.46.

It is now easy to verify that the functions ℓ_α form a joint Laver sequence: given a sequence of targets \vec{a} , we can replace it, by using f and lemma 1.46, with a coded version $\tilde{a} \in V_\theta$. We can then use ℓ to guess \tilde{a} and the θ -strongness embedding obtained this way will witness the joint guessing property of the ℓ_α . \square

Again, as in the supercompact case, if the Laver diamond we started with works for several different θ then the joint Laver sequence derived above will also work for those same θ . In particular, if κ is strong then combining the argument from proposition 1.47 with the Gitik–Shelah construction of a strongness Laver function in [10] gives an analogue of corollary 1.5.

Corollary 1.48. *If κ is strong then there is a strongness joint Laver sequence for κ of length 2^κ .*

Proposition 1.47 implies that in most cases (that is, for most θ) we do not need to do any additional work beyond ensuring that there is a θ -strongness Laver function for κ to automatically also find the longest possible joint Laver sequence. For example, if θ is a successor or if $\text{cf}(\theta) > 2^\kappa$ then a single θ -strongness Laver function yields a joint Laver sequence of length 2^κ . To gauge the consistency strength of the existence of θ -strongness joint Laver sequences for κ we should therefore only focus on the consistency strength required for a single Laver diamond, and, separately, on θ of low cofinality.

The following lemma is implicit in many arguments involving the preservation of strongness; we include a proof for completeness.

Lemma 1.49. *Let κ be a cardinal and suppose $\mathbb{P} \subseteq V_\kappa$ is a poset. Let $G \subseteq \mathbb{P}$ be generic and assume that κ is a \beth -fixed point in $V[G]$. If $\alpha \geq \kappa + 1$ is an ordinal then V_α contains (codes for) names for all elements of $V[G]_\alpha = V_\alpha^{V[G]}$.*

Proof. We argue by induction on α ; the limit case is clear. For the base case $\alpha = \kappa + 1$ notice that, since κ is a \beth -fixed point in $V[G]$, names for elements of $V[G]_{\kappa+1}$ can be recovered from names for subsets of κ . But nice names for subsets of κ are essentially just subsets of $\kappa \times \mathbb{P}$ and are thus elements of $V_{\kappa+1}$.

For the successor case, observe that nice names for subsets of $V[G]_\alpha$ can be represented by subsets of $V_\alpha \times \mathbb{P}$ since, by the induction hypothesis, V_α has names for elements of $V[G]_\alpha$. Therefore $V_{\alpha+1}$ has all of these nice names for elements of $V[G]_{\alpha+1}$. \square

In the following theorem we give a forcing construction of a θ -strongness Laver function. Combined with proposition 1.47 this will give us better bounds on the consistency strength of the existence of θ -strongness joint Laver sequences of various lengths. For this we need a preliminary lemma, analogous to lemma 1.6.

Lemma 1.50. *If κ is θ -strong then there is a function $f: \kappa \rightarrow \kappa$ with the Menas property; i.e. we have $j(f)(\kappa) = \theta$ for some θ -strongness embedding $j: V \rightarrow M$ with critical point κ .*

Theorem 1.51. *Let κ be θ -strong with $\kappa + 2 \leq \theta$. Then there is a forcing extension in which there is a θ -strongness Laver function for κ .*

Proof. We may assume by prior forcing that GCH holds. Fix a Menas function f as in lemma 1.50 and let \mathbb{P} be the length κ Easton support iteration which forces with $\mathbb{Q}_\gamma = \text{Add}(\beth_{f(\gamma)}^+, 1)$ at inaccessible γ . If $G \subseteq \mathbb{P}$ is generic we can extract a Laver function ℓ from it by letting $\ell(\xi)$ be the set whose Mostowski code appears as the length $\beth_{f(\xi)}$ initial segment of G_ξ if ξ is inaccessible and $\ell(\xi) = \emptyset$ otherwise.

To see that this definition works let $A \in V_\theta^{V[G]}$. Since $\mathbb{P} \subseteq V_\kappa$, lemma 1.49 implies that A has a name $\dot{A} \in V_\theta$.

Now fix a θ -strongness embedding $j: V \rightarrow M$ such that $j(f)(\kappa) = \theta$. We may additionally assume that this embedding is an ultrapower given by a (κ, V_θ) -extender, i.e. that $M = \{j(h)(a); a \in V_\theta, h: V_\kappa \rightarrow V\}$. We will lift this embedding through the forcing \mathbb{P} .

The poset $j(\mathbb{P})$ factors as $j(\mathbb{P}) = \mathbb{P} * \mathbb{P}_{\text{tail}}$, where $M[G] \models \text{“}\mathbb{P}_{\text{tail}} \text{ is } \leq \beth_\theta\text{-closed”}$. Let $X = \{j(h)(\kappa, \dot{A}); h: V_\kappa \rightarrow V\}$ be the Skolem hull of $\{\kappa, \dot{A}\} \cup \text{ran}(j)$ in M . Then $X^\kappa \subseteq X$ in V , $\mathbb{P}_\kappa \in X$ and $\mathbb{P}_{\text{tail}} \in X[G]$. Furthermore, since $\theta = j(f)(\kappa) \in X$, we also have $V_\theta \in X$. Now observe that, since $X[G]$ is essentially an ultrapower by a measure on κ , \mathbb{P}_{tail} has size $j(\kappa)$ in $X[G]$, and GCH holds, $X[G]$ has at most κ^+ many maximal antichains for \mathbb{P}_{tail} , counted in $V[G]$. In addition, as $X[G]^\kappa \subseteq X[G]$ in $V[G]$ and \mathbb{P}_{tail} is $\leq \kappa$ -closed in $X[G]$, we may diagonalize against these maximal antichains and build a $X[G]$ -generic $G_{\text{tail}}^* \subseteq \mathbb{P}_{\text{tail}} \cap X[G]$. Since we put $A, V_\theta \in X[G]$, we may also ensure in this construction that the length \beth_θ initial segment of G_κ codes A .

To finish the proof we show that G_{tail}^* generates an $M[G]$ -generic filter G_{tail} . To see this let $D \in M[G]$ be an open dense subset of \mathbb{P}_{tail} . Choosing a name $\dot{D} \in M$ for D , we can find

a function \vec{D} and an $a \in V_\theta$ such that $j(\vec{D})(a) = \dot{D}$. We now let $\bar{D} = \bigcap_b j(\vec{D}(b))^G$ where the intersection runs over those $b \in V_\theta$ for which $j(\vec{D}(b))^G$ is a dense open subset of \mathbb{P}_{tail} . Clearly $\bar{D} \subseteq D$ and \bar{D} is a dense subset of \mathbb{P}_{tail} , as this poset is $\leq \beth_\theta$ -closed in $M[G]$. But now notice that \bar{D} was defined using only parameters in $X[G]$ (recall that we had $V_\theta \in X$). Thus, since $X[G] \prec M[G]$, we get $\bar{D} \in X[G]$. The filter G_{tail}^* being $X[G]$ -generic, the intersection $G_{\text{tail}}^* \cap \bar{D}$ is nonempty, which shows that G_{tail}^* meets D and generates an $M[G]$ -generic filter.

We may thus lift the embedding j to $j: V[G] \rightarrow M[j(G)]$, where $j(G) = G * G_{\text{tail}}$. It now follows from lemma 1.49 that the lift of j witnesses the θ -strongness of κ in $V[G]$. Finally, we have arranged matters to ensure that $j(\ell)(\kappa) = A$, witnessing the guessing property. \square

Corollary 1.52. *Let κ be θ -strong with $\kappa + 2 \leq \theta$. If θ is either a successor ordinal or $\text{cf}(\theta) \geq \kappa^+$ then there is a forcing extension in which $\mathbb{N}_{\kappa, 2^\kappa}^{\theta\text{-str}}$ holds.*

Of course, the question of the necessity of the hypotheses of proposition 1.47 remains very attractive.

Question 1.53. *Suppose there is a θ -strongness Laver function for κ (with θ possibly being a limit of low cofinality). Is there a θ -strongness joint Laver sequence of length κ ? Or even of length ω ?*

We give a partial answer to this question. In contrast to the supercompact case, some restrictions are in fact necessary to allow for the existence of joint Laver sequences for θ -strong cardinals. The existence of even the shortest of such sequences can surpass the existence of a θ -strong cardinal in consistency strength.

To give a better lower bound on the consistency strength required, we introduce a notion of Mitchell rank for θ -strong cardinals, inspired by Carmody [5].

Definition 1.54. Let κ be a cardinal and θ an ordinal. The θ -strongness Mitchell order \triangleleft for κ is defined on the set of (κ, V_θ) -extenders, by letting $E \triangleleft F$ if E is an element of the (transitive collapse of the) ultrapower of V by F .

Unsurprisingly, this Mitchell order has properties analogous to those of the usual Mitchell order on normal measures on κ or the θ -supercompactness Mitchell order on normal fine measures on $\mathcal{P}_\kappa\theta$, as studied by Carmody. In particular, the θ -strongness Mitchell order is well-founded and gives rise to a notion of a θ -strongness Mitchell rank. Having θ -strongness Mitchell rank at least 2 implies that many cardinals below κ have reflected versions of θ -strongness; for example, if κ has $(\kappa + \omega)$ -strongness Mitchell rank at least 2, then there are stationarily many cardinals $\lambda < \kappa$ which are $(\lambda + \omega)$ -strong (and much more is true).

We should mention a bound on the θ -strongness Mitchell rank of a cardinal κ . If $j: V \rightarrow M$ is the ultrapower by a (κ, V_θ) -extender then any (κ, V_θ) -extenders in M appear in $V_{j(\kappa)}^M$. It follows that these extenders are represented by a function $f: V_\kappa \rightarrow V_\kappa$ and a seed $a \in V_\theta$. In particular, there are at most \beth_θ many such extenders, counted in V . Any given extender therefore has at most \beth_θ many predecessors in the Mitchell order, so the highest possible θ -strongness Mitchell rank of κ is \beth_θ^+ .

Theorem 1.55. *Let κ be a θ -strong cardinal, where θ is a limit ordinal, and $\text{cf}(\theta) \leq \kappa < \theta$. If there is a θ -strongness joint Laver sequence for κ of length $\text{cf}(\theta)$ then κ has maximal θ -strongness Mitchell rank.*

Proof. We first show that the existence of a short θ -strongness joint Laver sequence implies a certain degree of hypermeasurability for κ . Let $\vec{\ell} = \langle \ell_\alpha; \alpha < \text{cf}(\theta) \rangle$ be the joint Laver sequence. If $\vec{a} = \langle a_\alpha; \alpha < \text{cf}(\theta) \rangle$ is any sequence of targets in V_θ there is, by definition, a θ -strongness embedding $j: V \rightarrow M$ with critical point κ such that $j(\ell_\alpha)(\kappa) = a_\alpha$. But we can recover $\vec{\ell}$ from $j(\vec{\ell})$ as an initial segment, since $\vec{\ell}$ is so short. Therefore we actually get the whole sequence $\vec{a} \in M$, just by evaluating that initial segment at κ . Now consider any $a \subseteq V_\theta$. We can resolve a into a $\text{cf}(\theta)$ -sequence of elements a_α of V_θ such that $a = \bigcup_\alpha a_\alpha$. Our argument then implies that a is an element of M .

Now let E be an arbitrary (κ, V_θ) -extender. Since E can be represented as a family of

measures on κ indexed by V_θ , it is coded by a subset of V_θ (using the coding scheme from lemma 1.46, for example). Applying the argument above, there is a θ -strongness embedding $j: V \rightarrow M$ with critical point κ such that $E \in M$. It follows that κ is θ -strong in M , giving κ nontrivial θ -strongness Mitchell rank in V .

The argument in fact yields more: given any collection of at most \beth_θ many (κ, V_θ) -extenders, we can code the whole family by a subset of V_θ and, again, obtain an extender whose ultrapower contains the entire collection we started with. It follows that, given any family of at most \beth_θ many extenders, we can find an extender which is above all of them in the θ -strongness Mitchell order. Applying this fact inductively now shows that κ must have maximal θ -strongness Mitchell rank. \square

Just as in the case of θ -supercompactness we can also consider θ -strongness Laver trees. In view of propositions 1.47 and 1.34 it is not surprising that again, for most θ , a single θ -strongness Laver diamond yields a full Laver tree.

Proposition 1.56. *Suppose κ is θ -strong where $\kappa + 2 \leq \theta$ and θ is either a successor ordinal or $\text{cf}(\theta) > 2^\kappa$. Then a θ -strongness Laver tree for κ exists if and only if a θ -strongness Laver function for κ does (if and only if $\mathbb{D}_{\kappa, 2^\kappa}^{\theta\text{-sc}}$ holds).*

Proof. We follow the proof of proposition 1.34. Note that, since $\theta \geq \kappa + 2$, we have ${}^\kappa 2 \in V_\theta$, so V_θ is closed under sequences indexed by ${}^\kappa 2$ via the coding scheme given by lemma 1.46. If ℓ is a θ -strongness Laver function for κ we define a Laver tree by letting $D(t)$ be the element with index t in the sequence coded by $\ell(|t|)$, if this makes sense. It is now easy to check that this truly is a Laver tree: given any sequence of targets \vec{a} we simply use the Laver function ℓ to guess it (or, rather, its code), and the embedding j obtained this way will witness the guessing property for D . \square

1.4 Joint diamonds

Motivated by the joint Laver sequences of the previous sections we now apply the jointness concept to smaller cardinals. Of course, since we do not have any elementary embeddings of the universe with critical point ω_1 , say, we need a reformulation that will make sense in this setting as well.

As motivation, consider a measurable Laver function ℓ and let $a \subseteq \kappa$. By definition there is an elementary embedding $j: V \rightarrow M$ such that $j(\ell)(\kappa) = a$. Let U be the normal measure on κ derived from this embedding. Observe that a is represented in the ultrapower by the function $f_a(\xi) = a \cap \xi$ and thus, by Łoś's theorem, we conclude that $\ell(\xi) = a \cap \xi$ for U -almost all ξ . Therefore ℓ is (essentially) a special kind of \diamond_κ -sequence, guessing not just on stationary sets but on sets of measure one. Similarly, if we are dealing with a joint Laver sequence $\langle \ell_\alpha; \alpha < \lambda \rangle$ there is for every sequence $\langle a_\alpha; \alpha < \lambda \rangle$ of subsets of κ a normal measure on κ with respect to which each of the sets $\{\xi < \kappa; \ell_\alpha(\xi) = a_\alpha \cap \xi\}$ has measure one.

This understanding of jointness seems amenable to transfer to smaller cardinals. There are still no normal measures on ω_1 , but perhaps we can weaken that requirement slightly.

Recall that a filter on a regular cardinal κ is *normal* if it is closed under diagonal intersections, and *uniform* if it extends the cobounded filter. It is a standard result that the club filter on κ is the least normal uniform filter on κ (in fact, a normal filter is uniform if and only if it extends the club filter). It follows that any subset of κ contained in a (proper) normal uniform filter is stationary. Conversely, if $S \subseteq \kappa$ is stationary, then it is easy to check that S , together with the club filter, generates a normal uniform filter on κ . Altogether, we see that a set is stationary if and only if it is contained in a normal uniform filter. This observation suggest an analogy between Laver functions and \diamond_κ -sequences: in the same way that Laver functions guess their targets on positive sets with respect to some large cardinal measure, \diamond_κ -sequences guess their targets on positive sets with respect to some normal uniform filter.

Extending the analogy, in the same way that a joint Laver sequence is a collection of Laver functions that guess sequences of targets on positive sets with respect to a *common* large cardinal measure (corresponding to the single embedding j), a collection of \diamond_κ -sequences will be joint if they guess sequences of targets on positive sets with respect to a common normal uniform filter.

We will adopt the following terminology: if κ is a cardinal then a κ -list is a function $d: \kappa \rightarrow \mathcal{P}(\kappa)$ with $d(\alpha) \subseteq \alpha$.

Definition 1.57. Let κ be an uncountable regular cardinal. A $\diamond_{\kappa,\lambda}$ -sequence is a sequence $\vec{d} = \langle d_\alpha; \alpha < \lambda \rangle$ of κ -lists such that for every sequence $\langle a_\alpha; \alpha < \lambda \rangle$ of subsets of κ there is a (proper) normal uniform filter \mathcal{F} on κ such that for every α the *guessing set* $S_\alpha = S(d_\alpha, a_\alpha) = \{\xi < \kappa; d_\alpha(\xi) = a_\alpha \cap \xi\}$ is in \mathcal{F} .

An alternative, apparently simpler attempt at defining jointness would be to require that all the κ -lists in the sequence guess their respective targets on the same stationary set. However, a straightforward diagonalization argument shows that, with this definition, there are no $\diamond_{\kappa,\kappa}$ -sequences at all. Upon reflection, the definition we gave above corresponds more closely to the one in the case of Laver diamonds and, hopefully, is not vacuous.

We will not use the following proposition going forward, but it serves to give another parallel between \diamond_κ -sequences and Laver diamonds. It turns out that \diamond_κ -sequences can be seen as Laver functions, except that they work with generic elementary embeddings.⁶

Proposition 1.58. *Let κ be an uncountable regular cardinal and let d be a κ -list. Then d is a \diamond_κ -sequence if and only if there is, for any $a \subset \kappa$, a generic elementary embedding $j: V \rightarrow M$ with critical point κ and M wellfounded up to $\kappa + 1$ such that $j(d)(\kappa) = a$.*

Proof. Suppose d is a \diamond_κ -sequence and fix a target $a \subseteq \kappa$. Let $S(d, a) = \{\xi < \kappa; d(\xi) = a \cap \xi\}$ be the guessing set. By our discussion above there is a normal uniform filter \mathcal{F} on κ with

⁶In the terminology introduced by Gitman and Schindler in [11], we could say that \diamond_κ -sequences are *virtual* Laver functions.

$S \in \mathcal{F}$. Let G be a generic ultrafilter extending \mathcal{F} and $j: V \rightarrow M$ the generic ultrapower by G . Then M is wellfounded up to κ^+ and $\kappa = [\text{id}]_G$. Since $S \in G$, Łoś's theorem now implies that $j(d)(\kappa) = a$.

Conversely, fix a target $a \subseteq \kappa$ and suppose that there is a generic embedding j with the above properties. We can replace j with the induced normal ultrapower embedding and let U be the derived ultrafilter in the extension. Since $j(d)(\kappa) = a$ it follows that $S(d, a) \in U$. But since U extends the club filter, $S(d, a)$ must be stationary. \square

Similarly to the above proposition, a sequence of κ -lists is joint if they can guess any sequence of targets via a single generic elementary embedding.

We would now like to find a “bottom up” criterion deciding when a collection of subsets of κ (namely, some guessing sets) is contained in a normal uniform filter. The following key lemma gives such a criterion, which is completely analogous to the finite intersection property characterizing containment in a filter.

Definition 1.59. Let κ be an uncountable regular cardinal. A family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ has the *diagonal intersection property* if for any function $f: \kappa \rightarrow \mathcal{A}$ the diagonal intersection $\Delta_{\alpha < \kappa} f(\alpha)$ is stationary.

Lemma 1.60. *Let κ be uncountable and regular and let $\mathcal{A} \subseteq \mathcal{P}(\kappa)$. The family \mathcal{A} is contained in a normal uniform filter on κ if and only if \mathcal{A} satisfies the diagonal intersection property.*

Proof. The forward direction is clear, so let us focus on the converse. Consider the family of sets

$$E = \left\{ C \cap \Delta_{\alpha < \kappa} f(\alpha); f \in {}^\kappa \mathcal{A}, C \subseteq \kappa \text{ club} \right\}$$

We claim that E is directed under diagonal intersections: any diagonal intersection of κ many elements of E includes another element of E . To see this take $C_\alpha \cap \Delta_{\beta < \kappa} f_\alpha(\beta) \in E$ for $\alpha < \kappa$. Let $\langle \cdot, \cdot \rangle$ be a pairing function and define $F: \kappa \rightarrow \lambda$ by $F(\langle \alpha, \beta \rangle) = f_\alpha(\beta)$. A

calculation then shows that

$$\Delta_{\alpha < \kappa} (C_\alpha \cap \Delta_{\beta < \kappa} f_\alpha(\beta)) = \Delta_\alpha C_\alpha \cap \Delta_\alpha \Delta_\beta f_\alpha(\beta) \supseteq \Delta_\alpha C_\alpha \cap D \cap \Delta_\alpha F(\alpha)$$

where D is the club of closure points of the pairing function.

It follows that closing E under supersets yields a normal uniform filter on κ . By considering constant functions f we also see that every $a \in \mathcal{A}$ is in this filter. \square

Lemma 1.60 will be the crucial tool for verifying $\diamond_{\kappa, \lambda}$. More specifically, we shall often apply the following corollary.

Corollary 1.61. *A sequence $\vec{d} = \langle d_\alpha; \alpha < \lambda \rangle$ is a $\diamond_{\kappa, \lambda}$ -sequence if and only if every subsequence of length κ is a $\diamond_{\kappa, \kappa}$ -sequence.*

Proof. The forward implication is obvious; let us check the converse. Let $\vec{a} = \langle a_\alpha; \alpha < \lambda \rangle$ be a sequence of targets and let S_α be the corresponding guessing sets. By lemma 1.60 we need to check that the family $\mathcal{S} = \{S_\alpha; \alpha < \lambda\}$ satisfies the diagonal intersection property. So fix a function $f: \kappa \rightarrow \mathcal{S}$ and let $r = \{\alpha; S_\alpha \in f[\kappa]\}$. By our assumption $\vec{a} \upharpoonright r$ is a $\diamond_{\kappa, |r|}$ -sequence, so $f[\kappa]$ is contained in a normal uniform filter and, in particular, $\Delta_\alpha f(\alpha)$ is stationary. \square

This characterization leads to fundamental differences between joint diamonds and joint Laver diamonds. While the definition of joint diamonds was inspired by large cardinal phenomena, the absence of a suitable analogue of the diagonal intersection property in the large cardinal setting provides for some very surprising results.

Definition 1.62. Let κ be an uncountable regular cardinal. A \diamond_κ -tree is a function $D: {}^{<\kappa}2 \rightarrow \mathcal{P}(\kappa)$ such that for any sequence $\langle a_s; s \in {}^\kappa 2 \rangle$ of subsets of κ there is a (proper) normal uniform filter on κ containing all the guessing sets $S_s = S(D, a_s) = \{\xi < \kappa; D(s \upharpoonright \xi) = a_s \cap \xi\}$.

This definition clearly imitates the definition of Laver trees. We also have a correspondence in the style of proposition 1.58: a \diamond_κ -tree acts like a Laver tree for κ using generic elementary embeddings.

The following theorem, the main result of this section, shows that, in complete contrast to our experience with joint Laver diamonds, \diamond_κ already implies all of its stronger, joint versions.

Theorem 1.63. *Let κ be an uncountable regular cardinal. The following are equivalent:*

1. \diamond_κ
2. $\diamond_{\kappa,\kappa}$
3. $\diamond_{\kappa,2^\kappa}$
4. *there exists a \diamond_κ -tree.*

Proof. For the implication (1) \implies (2), let $d: \kappa \rightarrow \mathcal{P}(\kappa)$ be a \diamond_κ -sequence and fix a bijection $f: \kappa \rightarrow \kappa \times \kappa$. Define

$$d_\alpha(\xi) = (f[d(\xi)])_\alpha \cap \alpha = \{\eta < \alpha; (\alpha, \eta) \in f[d(\xi)]\}$$

We claim that $\langle d_\alpha; \alpha < \kappa \rangle$ is a $\diamond_{\kappa,\kappa}$ -sequence.

To see this take a sequence of targets $\langle a_\alpha; \alpha < \lambda \rangle$ and let $S_\alpha = \{\xi < \kappa; d_\alpha(\xi) = a_\alpha \cap \xi\}$ be the guessing sets. The set

$$T = \left\{ \xi < \kappa; f^{-1} \left[\bigcup_{\alpha < \kappa} \{\alpha\} \times a_\alpha \right] \cap \xi = d(\xi) \right\}$$

is stationary in κ . Let \mathcal{F} be the filter generated by the club filter on κ together with T . This is clearly a proper filter and, since a set Y is \mathcal{F} -positive if and only if $Y \cap T$ is stationary,

Fodor's lemma immediately implies that it is also normal. We claim that we have $S_\alpha \in \mathcal{F}$ for all $\alpha < \kappa$, so that \mathcal{F} witnesses the defining property of a $\diamond_{\kappa, \kappa}$ -sequence. Since $f[\xi] = \xi \times \xi$ for club many $\xi < \kappa$, the set

$$T' = \left\{ \xi < \kappa; d(\xi) = f^{-1} \left[\bigcup_{\alpha < \kappa} \{\alpha\} \times a_\alpha \right] \cap f^{-1}[\xi \times \xi] \right\}$$

is just the intersection of T with some club and is therefore in \mathcal{F} . But now observe that

$$\begin{aligned} T' &= \left\{ \xi < \kappa; d(\xi) = f^{-1} \left[\bigcup_{\alpha < \xi} \{\alpha\} \times (a_\alpha \cap \xi) \right] \right\} \\ &= \{ \xi < \kappa; \forall \alpha < \xi: a_\alpha \cap \xi = (f[d(\xi)])_\alpha = d_\alpha(\xi) \} = \Delta_{\alpha < \kappa} S_\alpha \end{aligned}$$

We see that $T' \in \mathcal{F}$ is, modulo a bounded set, contained in each S_α and can thus conclude, since \mathcal{F} is uniform, that $S_\alpha \in \mathcal{F}$ for all $\alpha < \kappa$.

Instead of proving (2) \implies (3) it will be easier to show (2) \implies (4) directly. Since the implications (4) \implies (3) \implies (1) are obvious, this will finish the proof.

Fix a $\diamond_{\kappa, \kappa}$ -sequence \vec{d} . We proceed to construct the \diamond_κ -tree D level by level; in fact the only meaningful work will take place at limit levels. At a limit stage $\gamma < \kappa$ we shall let the first γ many \diamond_κ -sequences anticipate the labels and their positions. Concretely, consider the sets $d_\alpha(\gamma)$ for $\alpha < \gamma$. For each α we interpret $d_{2\alpha}(\gamma)$ as a node on the γ -th level of ${}^{<\kappa}2$ and let $D(d_{2\alpha}(\gamma)) = d_{2\alpha+1}(\gamma)$, provided that there is no interference between the different \diamond_κ -sequences. If it should happen that for some $\alpha \neq \beta$ we get $d_{2\alpha}(\gamma) = d_{2\beta}(\gamma)$ but $d_{2\alpha+1}(\gamma) \neq d_{2\beta+1}(\gamma)$ we scrap the whole level and move on with the construction higher in the tree. At the end we extend D to be defined on the nodes of ${}^{<\kappa}2$ that were skipped along the way in any way we like.

We claim that the function D thus constructed is a \diamond_κ -tree. To check this let us fix a sequence of targets $\vec{a} = \langle a_s; s \in {}^\kappa 2 \rangle$ and let S_s be the guessing sets. By lemma 1.60 it now

suffices to check that $\Delta_{\alpha < \kappa} S_{s_\alpha}$ is stationary for any sequence of branches $\langle s_\alpha; \alpha < \kappa \rangle$.

For $\alpha < \kappa$ let $T_{2\alpha} = \{\xi; s_\alpha^{-1}[\{1\}] \cap \xi = d_{2\alpha}(\xi)\}$ and $T_{2\alpha+1} = \{\xi; A^{s_\alpha} \cap \xi = d_{2\alpha+1}(\xi)\}$. Since our construction was guided by a $\diamond_{\kappa, \kappa}$ -sequence, there is a normal uniform filter on κ which contains every T_α . In particular, $T = \Delta_{\alpha < \kappa} T_\alpha$ is stationary. By a simple bootstrapping argument there is a club C of limit ordinals γ such that all $s_\alpha \upharpoonright \gamma$ for $\alpha < \gamma$ are distinct. Let $\gamma \in C \cap T$. We now have $s_\alpha^{-1}[\{1\}] \cap \gamma = d_{2\alpha}(\gamma)$ and $a_{s_\alpha} \cap \gamma = d_{2\alpha+1}(\gamma)$ for all $\alpha < \gamma$. But this means precisely that the construction of D goes through at level γ and that $\gamma \in \bigcap_{\alpha < \gamma} S_{s_\alpha}$ and it follows that $\Delta_{\alpha < \kappa} S_{s_\alpha}$ is stationary. \square

We can again consider the treeability of joint diamond sequences, as we did in definition 1.42. We get the following analogue of corollary 1.44.

Theorem 1.64. *If κ is an uncountable regular cardinal and GCH holds then after forcing with $\text{Add}(\kappa, 2^\kappa)$ there is a nontreeable $\diamond_{\kappa, 2^\kappa}$ -sequence.*

Proof. Let $\mathbb{P} = \text{Add}(\kappa, 2^\kappa)$ and $G \subseteq \mathbb{P}$ generic; we refer to the α -th subset added by G as G_α . We will show that the generic G , seen as a sequence of 2^κ many \diamond_κ -sequences in the usual way, is a nontreeable $\diamond_{\kappa, 2^\kappa}$ -sequence.

Showing that G is a $\diamond_{\kappa, 2^\kappa}$ -sequence requires only minor modifications to the usual proof that a Cohen subset of κ codes a \diamond_κ -sequence. Thus, we view each G_α as a sequence, defined on κ , with $G_\alpha(\xi) \subseteq \xi$. Fix a sequence $\langle \dot{a}_\alpha; \alpha < 2^\kappa \rangle$ of names for subsets of κ , a name \dot{f} for a function from κ to 2^κ and a name \dot{C} for a club in κ as well as a condition $p \in \mathbb{P}$. We will find a condition $q \leq p$ forcing that $\dot{C} \cap \Delta_{\alpha < \kappa} S_{\dot{f}(\alpha)}$ is nonempty, where $S_{\dot{f}(\alpha)}$ names the set $\{\xi < \kappa; \dot{a}_{\dot{f}(\alpha)} \cap \xi = G_\alpha(\xi)\}$; this will show that G codes a $\diamond_{\kappa, 2^\kappa}$ -sequence by lemma 1.60.

We build the condition q in ω many steps. To start with, let $p_0 = p$ and let γ_0 be an ordinal such that $\text{dom}(p_0) \subseteq 2^\kappa \times \gamma_0$. We now inductively find ordinals γ_n , sets $B_n^\alpha \subseteq \gamma_n$, functions f_n and a descending sequence of conditions p_n satisfying $\text{dom}(p_n) \subseteq 2^\kappa \times \gamma_n$ and $p_{n+1} \Vdash \gamma_n \in \dot{C}$ as well as $p_{n+1} \Vdash \dot{f} \upharpoonright \gamma_n = f_n$ and $p_{n+1} \Vdash \dot{a}_{f_n(\alpha)} = B_n^\alpha$ for $\alpha < \gamma_n$. Let

$\gamma = \sup_n \gamma_n$ and $p_\omega = \bigcup_n p_n$ and $f_\omega = \bigcup_n f_n$ and $B_\omega^\alpha = \bigcup_n B_n^\alpha$. The construction of these ensures that $\text{dom}(p_\omega) \subseteq 2^\kappa \times \gamma$ and p_ω forces that $\dot{f} \upharpoonright \gamma = f_\omega$ and $\dot{a}_{f(\alpha)} \cap \gamma = B_\omega^\alpha$ for $\alpha < \gamma$ as well as $\gamma \in \dot{C}$. To obtain the final condition q we now simply extend p_ω by placing the code of B_ω^α on top of the $f(\alpha)$ -th column for all $\alpha < \gamma$. It now follows immediately that $q \Vdash \gamma \in \dot{C} \cap \Delta_{\alpha < \kappa} S_{f(\alpha)}$.

It remains to show that the generic $\diamond_{\kappa, 2^\kappa}$ -sequence is not treeable. This follows from lemma 1.43. □

In the case of Laver diamonds we were able to produce models with quite long joint Laver sequences but no Laver trees simply on consistency strength grounds (see theorem 1.37). In other words, we have models where there are long joint Laver sequences, but none of them are treeable. The situation seems different for ordinary diamonds, as theorem 1.63 tell us that treeable joint diamond sequences exist as soon as a single diamond sequence exists. While theorem 1.64 shows that it is at least consistent that there are nontreeable such sequences, we should ask whether this is simply always the case.

Question 1.65 (open). *Is it consistent for a fixed κ that every $\diamond_{\kappa, 2^\kappa}$ -sequence is treeable? Is it consistent that all $\diamond_{\kappa, 2^\kappa}$ -sequences are treeable for all κ ?*

Chapter 2

The grounded Martin's axiom

The standard Solovay-Tennenbaum proof of the consistency of Martin's axiom with a large continuum starts by choosing a suitable cardinal κ and then proceeds in an iteration of length κ by forcing with ccc posets of size less than κ , and not just those in the ground model but also those arising in the intermediate extensions. To ensure that all of the potential ccc posets are considered, some bookkeeping device is usually employed.

Consider now the following reorganization of the argument. Instead of iterating for κ many steps we build a length κ^2 iteration by first dealing with the κ many small posets in the ground model, then the small posets in that extension, and so on. The full length κ^2 iteration can be seen as a length κ iteration, all of whose iterands are themselves length κ iterations.

In view of this reformulation, we can ask what happens if we halt this construction after forcing with the first length κ iteration, when we have, in effect, ensured that Martin's axiom holds for posets from the ground model. What combinatorial consequences of Martin's axiom follow already from this weaker principle? We aim in this chapter to answer these questions (at least partially).

2.1 Forcing the grounded Martin's axiom

Definition 2.1. The *grounded Martin's axiom* (grMA) asserts that V is a ccc forcing extension of some ground model W and V satisfies the conclusion of Martin's axiom for posets $\mathbb{Q} \in W$ which are still ccc in V .

To be clear, in the definition we require that V have \mathcal{D} -generic filters for any family $\mathcal{D} \in V$ of fewer than \mathfrak{c}^V dense subsets of \mathbb{Q} . We should also note that, while the given definition is second-order, grMA is in fact first-order expressible, using the result of Reitz [20] that the ground models of the universe are uniformly definable.

If Martin's axiom holds, we may simply take $W = V$ in the definition, which shows that the grounded Martin's axiom is implied by Martin's axiom. In particular, by simply performing the usual Martin's axiom iteration, $\text{grMA} + \mathfrak{c} = \kappa$ can be forced from any model where κ is regular and satisfies $2^{<\kappa} = \kappa$. It will be shown in theorem 2.6, however, that the grounded Martin's axiom is strictly weaker than Martin's axiom. Ultimately we shall see that grMA retains some of the interesting combinatorial consequences of Martin's axiom (corollary 2.11), while also being more robust with respect to mild forcing (theorems 2.23 and 2.29).

As in the case of Martin's axiom, a key property of the grounded Martin's axiom is that it is equivalent to its restriction to posets of small size.

Lemma 2.2. *The grounded Martin's axiom is equivalent to its restriction to posets of size less than continuum, i.e. to the following principle:*

The universe V is a ccc forcing extension of some ground model W and V satisfies the conclusion of Martin's axiom for posets $\mathbb{Q} \in W$ of size less than \mathfrak{c}^V which are still ccc in V . ()*

Proof. Assume V satisfies $(*)$ and let $\mathbb{Q} \in W$ be a poset which is ccc in V and $\mathcal{D} = \{D_\alpha; \alpha < \kappa\} \in V$ a family of $\kappa < \mathfrak{c}^V$ many dense subsets of \mathbb{Q} . Let $V = W[G]$ for some W -generic $G \subseteq \mathbb{P}$ and let $\dot{D}_\alpha \in W$ be \mathbb{P} -names for the D_α . Choose θ large enough so that \mathbb{P}, \mathbb{Q} and all of the \dot{D}_α are in $H_{\theta^+}^W$. We can then find an $X \in W$ of size at most κ such that $X \prec H_{\theta^+}^W$ and X contains \mathbb{P}, \mathbb{Q} and the \dot{D}_α .

Now let

$$X[G] = \{ \tau^G; \tau \in X \text{ is a } \mathbb{P}\text{-name} \} \in W[G]$$

We can verify the Tarski-Vaught criterion to show that $X[G]$ is an elementary substructure of $H_{\theta^+}^W[G] = H_{\theta^+}^{W[G]}$. Specifically, suppose that $H_{\theta^+}^W[G] \models \exists x: \varphi(x, \tau^G)$ for some $\tau \in X$. Let S be the set of conditions $p \in \mathbb{P}$ which force $\exists x: \varphi(x, \tau)$. Since S is definable from the parameters \mathbb{P} and τ we get $S \in X$. Let $A \in X$ be an antichain, maximal among those contained in S . By mixing over A we can obtain a name $\sigma \in X$ such that $p \Vdash \varphi(\sigma, \tau)$ for any $p \in A$ and it follows that $H_{\theta^+}^W[G] \models \varphi(\sigma^G, \tau^G)$, which completes the verification.

Now let $\mathbb{Q}^* = \mathbb{Q} \cap X[G]$ and $D_\alpha^* = D_\alpha \cap X[G]$. Then $\mathbb{Q}^* \prec \mathbb{Q}$ and it follows that \mathbb{Q}^* is ccc (in $W[G]$) and that D_α^* is dense in \mathbb{Q}^* for any $\alpha < \kappa$. Furthermore, \mathbb{Q}^* has size at most κ . Finally, since \mathbb{P} is ccc, the filter G is X -generic and so $\mathbb{Q}^* = \mathbb{Q} \cap X[G] = \mathbb{Q} \cap X$ is an element of W . If we now apply $(*)$ to \mathbb{Q}^* and $\mathcal{D}^* = \{D_\alpha^*; \alpha < \kappa\}$, we find in $W[G]$ a filter $H \subseteq \mathbb{Q}^*$ intersecting every D_α^* , and thus every D_α . Thus H generates a \mathcal{D} -generic filter on \mathbb{Q} . □

The reader has likely noticed that the proof of lemma 2.2 is somewhat more involved than the proof of the analogous result for Martin's axiom. The argument there hinges on the straightforward observation that elementary subposets of ccc posets are themselves ccc. While that remains true in our setting, of course, matters are made more difficult since we require that all of our posets come from a ground model that may not contain the dense sets under consideration. It is therefore not at all clear that taking appropriate elementary

subposets will land us in the ground model and a slightly more elaborate argument is needed.

Let us point out a deficiency in the definition of grMA. As we have described it, the principle posits the existence of a ground model for the universe, a considerable global assumption. On the other hand, lemma 2.2 suggests that the operative part of the axiom is, much like Martin's axiom, a statement about H_c . This discrepancy allows for some undesirable phenomena. For example, Reitz [20] shows that it is possible to perform arbitrarily closed class forcing over a given model and obtain a model of the ground axiom, the assertion that the universe is not a nontrivial set forcing extension over any ground model at all. This implies that there are models which have the same H_c as a model of the grounded Martin's axiom but which fail to satisfy it simply because they have no nontrivial ground models at all. To avoid this situation we can weaken the definition of grMA in a technical way.

Definition 2.3. The *local grounded Martin's axiom* asserts that there are a cardinal $\kappa \geq \mathfrak{c}$ and a transitive ZFC^- model $M \subseteq H_{\kappa^+}$ such that H_{κ^+} is a ccc forcing extension of M and V satisfies the conclusion of Martin's axiom for posets $\mathbb{Q} \in M$ which are still ccc in V .

Of course, if the grounded Martin's axiom holds, over the ground model W via the forcing notion \mathbb{P} , then its local version holds as well. We can simply take κ to be large enough so that $M = H_{\kappa^+}^W$ contains \mathbb{P} and that $M[G] = H_{\kappa^+}^V$. One should view the local version of the axiom as capturing all of the relevant combinatorial effects of grMA (which, as we have seen, only involve H_c), while disentangling it from the structure of the universe higher up.

We now aim to give a model where the Martin's axiom fails but the grounded version holds. The idea is to imitate the Solovay-Tennenbaum argument, but to only use ground model posets in the iteration. While it is then relatively clear that grMA will hold in the extension, a further argument is needed to see that MA itself fails. The key will be a kind of product analysis, given in the next few lemmas. We will show that the iteration of ground model posets, while not exactly a product, is close enough to a product to prevent Martin's

axiom from holding in the final extension by a result of Roitman. An extended version of this argument will also yield the consistency of grMA with a singular continuum.

Lemma 2.4. *Let \mathbb{Q}_0 and \mathbb{R} be posets and τ a \mathbb{Q}_0 -name for a poset such that $\mathbb{Q}_0 * \tau$ is ccc and $\mathbb{Q}_0 \Vdash$ “if $\check{\mathbb{R}}$ is ccc then $\tau = \check{\mathbb{R}}$ ”. Furthermore, suppose that $\mathbb{Q}_0 * \tau \Vdash$ “ $\check{\mathbb{Q}}_0$ is ccc”. Then either $\mathbb{Q}_0 \Vdash$ “ $\check{\mathbb{R}}$ is ccc” or $\mathbb{Q}_0 \Vdash$ “ $\check{\mathbb{R}}$ is not ccc”.*

Proof. Suppose the conclusion fails, so that there are conditions $q_0, q_1 \in \mathbb{Q}_0$ which force $\check{\mathbb{R}}$ to either be or not be ccc, respectively. It follows that $\mathbb{Q}_0 \restriction q_1 \times \mathbb{R}$ is not ccc. Switching the factors, there must be a condition $r \in \mathbb{R}$ forcing that $\check{\mathbb{Q}}_0 \restriction \check{q}_1$ is not ccc. Now let $G * H$ be generic for $\mathbb{Q}_0 * \tau$ with $(q_0, \check{r}) \in G * H$ (note that (q_0, \check{r}) is really a condition since $q_0 \Vdash \tau = \check{\mathbb{R}}$). Consider the extension $V[G * H]$. On the one hand \mathbb{Q}_0 must be ccc there, since this was one of the hypotheses of our statement, but on the other hand $\mathbb{Q}_0 \restriction q_1$ is not ccc there since $r \in H$ forces this. \square

Lemma 2.5. *Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \tau_\alpha; \alpha < \gamma \rangle$, with $\gamma > 0$, be a finite-support ccc iteration such that for each α there is some poset \mathbb{Q}_α for which*

$$\mathbb{P}_\alpha \Vdash \text{“if } \check{\mathbb{Q}}_\alpha \text{ is ccc then } \tau_\alpha = \check{\mathbb{Q}}_\alpha \text{ and } \tau_\alpha \text{ is trivial otherwise”}$$

Furthermore assume that $\mathbb{P} \Vdash$ “ $\check{\mathbb{Q}}_0$ is ccc”. Then \mathbb{P} is forcing equivalent to the product $\mathbb{Q}_0 \times \overline{\mathbb{P}}$ for some poset $\overline{\mathbb{P}}$.

Before we give the (technical) proof, let us provide some intuition for this lemma. We can define the iteration $\overline{\mathbb{P}}$ in the same way as \mathbb{P} (i.e. using the same \mathbb{Q}_α) but skipping the first step of forcing. The idea is that, by lemma 2.4, the posets which appear in the iteration \mathbb{P} do not depend on the first stage of forcing \mathbb{Q}_0 . We thus expect that generics $G \subseteq \mathbb{P}$ will correspond exactly to generics $H \times \overline{G} \subseteq \mathbb{Q}_0 \times \overline{\mathbb{P}}$, since the first stage G_0 of the generic G does not affect the choice of posets in the rest of the iteration.

Proof. We show the lemma by induction on γ , the length of the iteration \mathbb{P} . In fact, we shall work with a stronger induction hypothesis. Specifically, we shall show that for each $\alpha < \gamma$ there is a poset $\bar{\mathbb{P}}_\alpha$ such that $\mathbb{Q}_0 \times \bar{\mathbb{P}}_\alpha$ embeds densely into \mathbb{P}_α , that the $\bar{\mathbb{P}}_\alpha$ form the initial segments of a finite-support iteration and that the dense embeddings extend one another. For the purposes of this proof we shall take all two-step iterations to be in the style of Kunen, i.e. the conditions in $\mathbb{P} * \tau$ are pairs (p, σ) such that $p \in \mathbb{P}$ and $\sigma \in \text{dom}(\tau)$ and $p \Vdash \sigma \in \tau$. Furthermore we shall assume that the τ_α are full names. These assumptions make no difference for the statement of the lemma, but ensure that certain embeddings will in fact be dense.

Let us start with the base case $\gamma = 2$, when $\mathbb{P} = \mathbb{Q}_0 * \tau_1$. By lemma 2.4 whether or not $\check{\mathbb{Q}}_1$ is ccc is decided by every condition in \mathbb{Q}_0 . But then, by our assumption on τ_1 , if \mathbb{Q}_0 forces $\check{\mathbb{Q}}_1$ to be ccc then τ_1 is forced to be equal to $\check{\mathbb{Q}}_1$ and $\mathbb{Q}_0 \times \mathbb{Q}_1$ embeds densely into \mathbb{P} , and otherwise τ_1 is forced to be trivial and \mathbb{Q}_0 embeds densely into \mathbb{P} . Depending on which is the case, we can thus take either $\bar{\mathbb{P}}_2 = \mathbb{Q}_1$ or $\bar{\mathbb{P}}_2 = 1$.

For the induction step let us assume that the stronger induction hypothesis holds for iterations of length γ and show that it holds for iterations of length $\gamma + 1$. Let us write $\mathbb{P} = \mathbb{P}_\gamma * \tau_\gamma$. By the induction hypothesis there is a $\bar{\mathbb{P}}_\gamma$ such that $\mathbb{Q}_0 \times \bar{\mathbb{P}}_\gamma$ embeds densely into \mathbb{P}_γ .

Before we give the details, let us sketch the string of equivalences that will yield the desired conclusion. We have

$$\mathbb{P} \equiv (\mathbb{Q}_0 \times \bar{\mathbb{P}}_\gamma) * \bar{\tau}_\gamma \equiv (\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma \equiv \bar{\mathbb{P}}_\gamma * (\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma)$$

Here $\bar{\tau}_\gamma$ is the $\mathbb{Q}_0 \times \bar{\mathbb{P}}_\gamma$ -name (or $\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0$ -name) resulting from pulling back the \mathbb{P}_γ -name τ_γ along the dense embedding provided by the induction hypothesis. We will specify what exactly we mean by $\check{\mathbb{Q}}_0 * \bar{\tau}_\gamma$ later.

We can apply the base step of the induction to the iteration $\check{Q}_0 * \bar{\tau}_\gamma$ in $V^{\bar{\mathbb{P}}_\gamma}$ and obtain a $\bar{\mathbb{P}}_\gamma$ -name $\dot{\mathbb{R}}$ such that $\check{Q}_0 \times \dot{\mathbb{R}}$ is forced to densely embed into $\check{Q}_0 * \bar{\tau}_\gamma$. We can then continue the chain above with

$$\bar{\mathbb{P}}_\gamma * (\check{Q}_0 * \bar{\tau}_\gamma) \equiv \bar{\mathbb{P}}_\gamma * (\check{Q}_0 \times \dot{\mathbb{R}}) \equiv \bar{\mathbb{P}}_\gamma * (\dot{\mathbb{R}} \times \check{Q}_0) \equiv \bar{\mathbb{P}}_{\gamma+1} \times Q_0$$

where $\bar{\mathbb{P}}_{\gamma+1} = \bar{\mathbb{P}}_\gamma * \dot{\mathbb{R}}$. While this is apparently enough to finish the successor step for the bare statement of the lemma, we wish to preserve the stronger induction hypothesis, and this requires a bit more work.

We first pick a specific $\bar{\mathbb{P}}_\gamma$ -name for $\check{Q}_0 * \bar{\tau}_\gamma$. Let

$$\tau = \{ ((\check{q}_0, \rho), \bar{p}) ; q_0 \in Q_0, \rho \in \text{dom}(\bar{\tau}_\gamma), \bar{p} \in \bar{\mathbb{P}}_\gamma, (\bar{p}, q_0) \Vdash \rho \in \bar{\tau}_\gamma \}$$

Then $\bar{\mathbb{P}}_\gamma \Vdash \tau = \check{Q}_0 * \bar{\tau}_\gamma$. Next we pin down $\dot{\mathbb{R}}$. Note that $\bar{\mathbb{P}}_\gamma$ forces that \check{Q}_0 decides whether \check{Q}_γ is ccc or not by lemma 2.4. Let $A = A_0 \cup A_1 \subseteq \bar{\mathbb{P}}_\gamma$ be a maximal antichain such that each $\bar{p} \in A_0$ forces \check{Q}_γ to not be ccc and each $\bar{p} \in A_1$ forces it to be ccc. Now let

$$\dot{\mathbb{R}} = \{ (\dot{\mathbb{I}}, \bar{p}) ; \bar{p} \in A_1 \} \cup \{ (\check{q}, \bar{p}) ; q \in Q_\gamma, \bar{p} \in A_0 \}$$

Observe that $\dot{\mathbb{R}}$ has the properties we require of it: it is forced by $\bar{\mathbb{P}}_\gamma$ that $\dot{\mathbb{R}} = \check{Q}_\gamma$ if \check{Q}_0 forces that \check{Q}_γ is ccc, and $\dot{\mathbb{R}}$ is trivial otherwise, and that $\check{Q}_0 \times \dot{\mathbb{R}}$ embeds densely into τ .

Finally, let us define $\bar{\mathbb{P}}_{\gamma+1} = \bar{\mathbb{P}}_\gamma * \dot{\mathbb{R}}$. We can now augment the equivalences given above with dense embeddings:

- The embedding $\bar{\mathbb{P}}_{\gamma+1} \times Q_0 \hookrightarrow \bar{\mathbb{P}}_\gamma * (\check{Q}_0 \times \dot{\mathbb{R}})$ is clear.
- To embed $\bar{\mathbb{P}}_\gamma * (\check{Q}_0 \times \dot{\mathbb{R}})$ into $\bar{\mathbb{P}}_\gamma * \tau$ we can send $(\bar{p}, (\check{q}_0, \rho))$ to $(\bar{p}, (\check{q}_0, \rho'))$ where ρ' is some element of $\text{dom}(\bar{\tau}_\gamma)$ for which $(\bar{p}, q_0) \Vdash \rho = \rho'$ (this is where the fullness of τ_γ is

needed).

- With our specific choice of the name τ we in fact get an isomorphism between $\bar{\mathbb{P}}_\gamma * \tau$ and $(\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma$, given by sending $(\bar{p}, (\check{q}_0, \rho))$ to $((\bar{p}, q_0), \rho)$.
- The final embedding from $(\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0) * \bar{\tau}_\gamma$ into \mathbb{P} is given by the induction hypothesis.

After composing these embeddings, we notice that the first three steps essentially fixed the $\bar{\mathbb{P}}_\gamma$ part of the condition and the last step fixed the τ_γ part. It follows that the embedding $\bar{\mathbb{P}}_{\gamma+1} \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_{\gamma+1}$ we constructed extends the embedding $\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_\gamma$ given by the induction hypothesis. This completes the successor step of the induction.

We now look at the limit step of the induction. The induction hypothesis gives us for each $\alpha < \gamma$ a poset $\bar{\mathbb{P}}_\alpha$ and a dense embedding $\bar{\mathbb{P}}_\alpha \times \mathbb{Q}_0 \hookrightarrow \mathbb{P}_\alpha$ and we also know that the $\bar{\mathbb{P}}_\alpha$ are the initial segments of a finite-support iteration and that the dense embeddings extend each other. If we now let $\bar{\mathbb{P}}_\gamma$ be the direct limit of the $\bar{\mathbb{P}}_\alpha$, we can easily find a dense embedding of $\bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0$ into \mathbb{P}_γ . Specifically, given a condition $(\bar{p}, q) \in \bar{\mathbb{P}}_\gamma \times \mathbb{Q}_0$, we can find an $\alpha < \gamma$ such that \bar{p} is essentially a condition in $\bar{\mathbb{P}}_\alpha$, since $\bar{\mathbb{P}}_\gamma$ is the direct limit of these. Now we can map (\bar{p}, q) using the stage α dense embedding, landing in \mathbb{P}_α and interpreting this as an element of \mathbb{P}_γ . This map is independent of the particular choice of α since all the dense embeddings extend one another and it is itself a dense embedding since all the previous stages were. \square

Theorem 2.6. *Let $\kappa > \omega_1$ be a cardinal of uncountable cofinality satisfying $2^{<\kappa} = \kappa$. Then there is a ccc forcing extension that satisfies $\text{grMA} + \neg\text{MA} + \mathfrak{c} = \kappa$.*

Proof. Fix a well-order \triangleleft of H_κ of length κ ; this can be done since $2^{<\kappa} = \kappa$. We can assume, without loss of generality, that the least element of this order is the poset $\text{Add}(\omega, 1)$. We define a length κ finite-support iteration \mathbb{P} recursively: at stage α we shall force with the next poset with respect to the order \triangleleft if that is ccc at that stage and with trivial forcing otherwise. Let G be \mathbb{P} -generic.

Notice that any poset $\mathbb{Q} \in H_\kappa$ occurs, up to isomorphism, unboundedly often in the well-order \triangleleft . Specifically, we can first find an isomorphic copy whose universe is a set of ordinals bounded in κ and then simply move this universe higher and higher up. In particular, isomorphic copies of Cohen forcing $\text{Add}(\omega, 1)$ appear unboundedly often. Since these are ccc in a highly robust way (being countable), they will definitely be forced with in the iteration \mathbb{P} . Therefore we have at least κ many reals in the extension $V[G]$. Since the forcing is ccc of size κ and κ has uncountable cofinality, a nice-name argument shows that the continuum equals κ in the extension.

To see that MA fails in $V[G]$, notice that \mathbb{P} is exactly the type of iteration considered in lemma 2.5. The lemma then implies that \mathbb{P} is equivalent to $\bar{\mathbb{P}} \times \text{Add}(\omega, 1)$ for some $\bar{\mathbb{P}}$. Therefore the extension $V[G]$ is obtained by adding a Cohen real to some intermediate model. But, as CH fails in the final extension, Roitman has shown in [21] that Martin's axiom must also fail there.

Finally, we show that the grounded Martin's axiom holds in $V[G]$ with V as the ground model. Before we consider the general case let us look at the easier situation when κ is regular. Thus, let $\mathbb{Q} \in V$ be a poset of size less than κ which is ccc in $V[G]$ and $\mathcal{D} \in V[G]$ a family of fewer than κ many dense subsets of \mathbb{Q} . We can assume without loss of generality that $\mathbb{Q} \in H_\kappa$. Code the elements of \mathcal{D} into a single set $D \subseteq \kappa$ of size $\lambda < \kappa$. Since \mathbb{P} is ccc, the set D has a nice \mathbb{P} -name \dot{D} of size λ . Since the iteration \mathbb{P} has finite support, the set D appears before the end of the iteration. As we have argued before, up to isomorphism, the poset \mathbb{Q} appears κ many times in the well-order \triangleleft . Since \mathbb{Q} is ccc in $V[G]$ and \mathbb{P} is ccc, posets isomorphic to \mathbb{Q} will be forced with unboundedly often in the iteration \mathbb{P} and, therefore, eventually a \mathcal{D} -generic will be added for \mathbb{Q} .

If we now allow κ to be singular we run into the problem that the dense sets in \mathcal{D} might not appear at any initial stage of the iteration \mathbb{P} . We solve this issue by using lemma 2.5 to factor a suitable copy of \mathbb{Q} out of the iteration \mathbb{P} and see it as coming after the forcing that

added \mathcal{D} .

Let $\mathbb{Q}, \mathcal{D}, D, \lambda$ and \dot{D} be as before. As mentioned, it may no longer be true that \mathcal{D} appears at some initial stage of the iteration. Instead, note that, since \dot{D} has size λ , there are at most λ many indices α such that some condition appearing in \dot{D} has a nontrivial α -th coordinate. It follows that there is a $\delta < \kappa$ such that no condition appearing in \dot{D} has a nontrivial δ -th coordinate and the poset considered at stage δ is isomorphic to \mathbb{Q} . Additionally, if we fix a condition $p \in G$ forcing that \mathbb{Q} is ccc in $V[G]$, we can find such a δ beyond the support of p . Now argue for a moment in $V[G_\delta]$. In this intermediate extension the quotient iteration $\mathbb{P}^\delta = \mathbb{P} \upharpoonright [\delta, \kappa)$ is of the type considered in lemma 2.5 and, since we chose δ beyond the support of p , we also get that \mathbb{P}^δ forces that $\check{\mathbb{Q}}$ is ccc. The lemma now implies that \mathbb{P}^δ is equivalent to $\bar{\mathbb{P}} \times \mathbb{Q}$ for some $\bar{\mathbb{P}}$. Moving back to V , we can conclude that $\mathbb{P} \upharpoonright p$ factors as $\mathbb{P}_\delta \upharpoonright p * (\bar{\mathbb{P}} \times \check{\mathbb{Q}}) \equiv (\mathbb{P}_\delta \upharpoonright p * \bar{\mathbb{P}}) \times \mathbb{Q}$ and obtain the corresponding generic $(G_\delta * \bar{G}) \times H$. Furthermore, the name \dot{D} is essentially a $\mathbb{P}_\delta * \bar{\mathbb{P}}$ -name, since no condition in \dot{D} has a nontrivial δ -th coordinate. It follows that the set D appears already in $V[G_\delta * \bar{G}]$ and that the final generic $H \in V[G_\delta * \bar{G}][H] = V[G]$ is \mathcal{D} -generic for \mathbb{Q} . We have thus shown that (*) from lemma 2.2 holds in $V[G]$, which implies that the grounded Martin's axiom also holds. \square

We should reflect briefly on the preceding proof. If we would have been satisfied with obtaining a model with a regular continuum, the usual techniques would apply. Specifically, if κ were regular then all the dense sets in \mathcal{D} would have appeared by some stage of the iteration, after which we would have forced with (a poset isomorphic to) \mathbb{Q} , yielding the desired \mathcal{D} -generic. This approach, however, fails if κ is singular, as small sets might not appear before the end of the iteration. Lemma 2.5 was key in resolving this issue, allowing us to factor the iteration \mathbb{P} as a product and seeing the forcing \mathbb{Q} as happening at the last stage, after the dense sets had already appeared. The lemma implies that the iteration

factors at any stage where we considered an absolutely ccc poset (for example, we can factor out any Knaster poset from the ground model). However, somewhat fortuitously, we can also factor out any poset to which we might apply the grounded Martin's axiom, at least below some condition. There is no surrogate for lemma 2.5 for the usual Solovay-Tennenbaum iteration and indeed, Martin's axiom implies that the continuum is regular.

Corollary 2.7. *The grounded Martin's axiom is consistent with the existence of a Suslin tree.*

Proof. We saw that the model of grMA constructed in the above proof was obtained by adding a Cohen real to an intermediate extension. Adding that Cohen real also adds a Suslin tree by a result of Shelah [22]. \square

A further observation we can make is that the cofinality of κ plays no role in the proof of theorem 2.6 beyond the obvious König's inequality requirement on the value of the continuum. This allows us to obtain models of the grounded Martin's axiom with a singular continuum and violate cardinal arithmetic properties which must hold in the presence of Martin's axiom.

Corollary 2.8. *The grounded Martin's axiom is consistent with $2^{<\mathfrak{c}} > \mathfrak{c}$.*

Proof. Starting from some model, perform the construction of theorem 2.6 with κ singular. In the extension the continuum equals κ . But the desired inequality is true in any model where the continuum is singular: of course $\mathfrak{c} = 2^\omega \leq 2^{\text{cf}(\mathfrak{c})} \leq 2^{<\mathfrak{c}}$ is true but equalities cannot hold since the middle two cardinals have different cofinalities by König's inequality. \square

On the other hand, assuming we start with a model satisfying GCH, the model of theorem 2.6 will satisfy the best possible alternative to $2^{<\mathfrak{c}} = \mathfrak{c}$, namely $2^{<\text{cf}(\mathfrak{c})} = \mathfrak{c}$. Whether this always happens remains open.

Question 2.9 (open). *Does the grounded Martin's axiom imply that $2^{<\text{cf}(\mathfrak{c})} = \mathfrak{c}$?*

2.2 The axiom's relation to other fragments of Martin's axiom

Let us now compare some of the combinatorial consequences of the grounded Martin's axiom with those of the usual Martin's axiom. We first make an easy observation.

Proposition 2.10. *The local grounded Martin's axiom implies $\text{MA}(\text{countable})$.*

Proof. Fix the cardinal $\kappa \geq \mathfrak{c}$ and the ZFC^- ground model $M \subseteq H_{\kappa^+}$ witnessing local grMA. Observe that the the model M contains the poset $\text{Add}(\omega, 1)$, since its elements are effectively coded by the natural numbers. This poset is therefore always a valid target for local grMA. \square

It follows from the above proposition that the (local) grounded Martin's axiom will have some nontrivial effects on the cardinal characteristics of the continuum. In particular, we obtain the following.

Corollary 2.11. *The local grounded Martin's axiom implies that the cardinals on the right side of Cichoń's diagram equal the continuum. In particular, this holds for both the covering number for category $\text{cov}(\mathcal{B})$ and the reaping number \mathfrak{r} .*

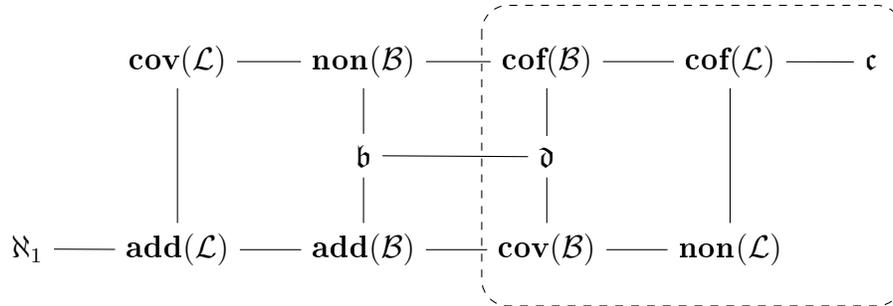


Figure 2.1: Cichoń's diagram; the indicated region is pushed up to \mathfrak{c} under grMA.

Proof. All of the given equalities follow already from MA(countable); we briefly summarize the arguments from [4].

The complement of any nowhere dense subsets of the real line is dense. It follows that, given fewer than continuum many nowhere dense sets, we can apply MA(countable) to obtain a real number not contained in any of them. Therefore the real line cannot be covered by fewer than continuum many nowhere dense sets and, consequently, also not by fewer than continuum many meagre sets.

To see that the reaping number must be large, observe that, given any infinite $x \subseteq \omega$, there are densely many conditions in $\text{Add}(\omega, 1)$ having arbitrarily large intersection with both x and $\omega \setminus x$. It follows that a Cohen real will split x . Starting with fewer than continuum many reals and applying MA(countable), we can therefore find a real splitting all of them, which means that the original family was not a reaping family. \square

But where Martin's axiom strictly prescribes the size of all cardinal characteristics of the continuum, the grounded Martin's axiom allows for more leeway in some cases. Observe that, since $\kappa > \omega_1$, the iteration \mathbb{P} of theorem 2.6 contains $\text{Add}(\omega, \omega_1)$ as an iterand. Thus, by lemma 2.5, there is a poset $\bar{\mathbb{P}}$ such that \mathbb{P} is equivalent to $\bar{\mathbb{P}} \times \text{Add}(\omega, \omega_1)$.

Theorem 2.12. *It is consistent that the grounded Martin's axiom holds, CH fails and the cardinal characteristics on the left side of Cichoń's diagram, as well as the splitting number \mathfrak{s} are equal to \aleph_1 .*

Proof. Consider a model $V[G]$ of grMA satisfying $\mathfrak{c} > \aleph_1$ which was obtained by forcing with the iteration \mathbb{P} from theorem 2.6 over a model of GCH. We have argued that this model is obtained by adding \aleph_1 many Cohen reals to some intermediate extension. We again briefly summarize the standard arguments for the smallness of the indicated cardinal characteristics in such an extension (see [4] for details).

Let X be the set of ω_1 many Cohen reals added by the final stage of forcing. We claim

it is both nonmeager and splitting. Note that any real in $V[G]$ appears before all of the Cohen reals in X have appeared. It follows that every real in $V[G]$ is split by some real in X . Furthermore, if X were meager, it would be contained in a meager Borel set, whose Borel code also appears before all of the reals in X do. But this leads to contradiction, since any Cohen real will avoid any meager set coded in the ground model. \square

To summarize, while the grounded Martin's axiom implies that the right side of Cichoń's diagram is pushed up to \mathfrak{c} , it is consistent with the left side dropping to \aleph_1 (while CH fails, of course). This is the most extreme way in which the effect of the grounded Martin's axiom on Cichoń's diagram can differ from that of Martin's axiom. The precise relationships under grMA between the cardinal characteristics on the left warrant further exploration in the future.

We can consider further the position of the grounded Martin's axiom within the hierarchy of the more well-known fragments of Martin's axiom. As we have already mentioned, (local) grMA implies MA(countable). We can strengthen this slightly. Let MA(Cohen) denote Martin's axiom restricted to posets of the form $\text{Add}(\omega, \lambda)$ for some λ . It will turn out that local grMA also implies MA(Cohen).

Lemma 2.13. *The axiom MA(Cohen) is equivalent to its restriction to posets of the form $\text{Add}(\omega, \lambda)$ for $\lambda < \mathfrak{c}$.*

*Proof*¹. Let $\mathbb{P} = \text{Add}(\omega, \kappa)$ and fix a collection \mathcal{D} of $\lambda < \mathfrak{c}$ many dense subsets of \mathbb{P} . As usual, let \mathbb{Q} be an elementary substructure of $\langle \mathbb{P}, D \rangle_{D \in \mathcal{D}}$ of size λ . We shall show that \mathbb{Q} is isomorphic to $\text{Add}(\omega, \lambda)$. The lemma then follows easily.

To demonstrate the desired isomorphism we shall show that \mathbb{Q} is determined by the single-bit conditions it contains. More precisely, \mathbb{Q} contains precisely those conditions which are meets of finitely many single-bit conditions in \mathbb{Q} .

¹The proof of the key claim was suggested by Noah Schweber.

First note that being a single-bit conditions is definable in \mathbb{P} : these are precisely the coatoms of the order. Furthermore, given a coatom p , its complementary coatom \bar{p} with the single bit flipped is definable from p as the unique coatom such that any condition is compatible with either p or \bar{p} . It follows by elementarity that the coatoms of \mathbb{Q} are precisely the single-bit conditions contained in \mathbb{Q} and that \mathbb{Q} is closed under the operation $p \mapsto \bar{p}$. In \mathbb{P} any finite collection of pairwise compatible coatoms has a meet, therefore the same holds in \mathbb{Q} and the meets agree. Conversely, any given condition in \mathbb{P} uniquely determines the finitely many coatoms it strengthens and therefore all the coatoms determined by conditions in \mathbb{Q} are also in \mathbb{Q} . Taken together, this proves the claim.

It follows immediately from the claim that \mathbb{Q} is isomorphic to $\text{Add}(\omega, |X|)$ where X is the set of coatoms of \mathbb{Q} , and also that $|X| = |\mathbb{Q}|$. \square

Proposition 2.14. *The local grounded Martin's axiom implies MA(Cohen).*

Proof. Suppose the local grounded Martin's axiom holds, witnessed by $\kappa \geq \mathfrak{c}$ and a ZFC⁻ model $M \subseteq H_{\kappa^+}$. In particular, the height of M is κ^+ and M contains all of the posets $\text{Add}(\omega, \lambda)$ for $\lambda < \kappa^+$. But this means that Martin's axiom holds for all the posets $\text{Add}(\omega, \lambda)$ where $\lambda < \mathfrak{c}$ and lemma 2.13 now implies that MA(Cohen) holds. \square

As we have seen, the local grounded Martin's axiom implies some of the weakest fragments of Martin's axiom. The following corollary of theorem 2.12 tells us, however, that this behaviour stops quite quickly.

Corollary 2.15. *The grounded Martin's axiom does not imply MA(σ -centred).*

Proof. By theorem 2.12 there is a model of the grounded Martin's axiom where the bounding number is strictly smaller than the continuum. But this is impossible under MA(σ -centred), since applying the axiom to Hechler forcing yields, for any family of fewer than continuum many reals, a real dominating them all. \square

As mentioned earlier, Reitz has shown that we can perform class forcing over any model in such a way that the resulting extension has the same H_c and is also not a set-forcing extension of any ground model. Performing this construction over a model of $\text{MA}(\sigma\text{-centred})$ (or really any of the standard fragments of Martin's axiom) shows that $\text{MA}(\sigma\text{-centred})$ does not imply grMA , for the disappointing reason that the final model is not a ccc forcing extension of anything. However, it turns out that already local grMA is independent of $\text{MA}(\sigma\text{-centred})$, and even of $\text{MA}(\text{Knaster})$. This places the grounded Martin's axiom, as well as its local version, outside the usual hierarchy of fragments of Martin's axiom.

Theorem 2.16. *Assume $V = L$ and let $\kappa > \omega_1$ be a regular cardinal. Then there is a ccc forcing extension which satisfies $\text{MA}(\text{Knaster}) + \mathfrak{c} = \kappa$ and in which the local grounded Martin's axiom fails.*

Proof. Let \mathbb{P} be the usual finite-support iteration forcing $\text{MA}(\text{Knaster}) + \mathfrak{c} = \kappa$. More precisely, we consider the names for posets in H_κ using appropriate bookkeeping and append them to the iteration if they, at that stage, name a Knaster poset. Let $G \subseteq \mathbb{P}$ be generic. We claim that the local grounded Martin's axiom fails in the extension $L[G]$.

Notice first that \mathbb{P} , being a finite-support iteration of Knaster posets, is Knaster. It follows that the product of \mathbb{P} with any ccc poset is still ccc. In particular, forcing with \mathbb{P} preserves the Suslin trees of L .

Now fix a $\lambda \geq \kappa$ and let $M \in L[G]$ be a transitive ZFC^- model of height λ^+ . It is straightforward to see that M builds its constructible hierarchy correctly so that, in particular, $L_{\omega_2} \subseteq M$. This implies that M has all of the Suslin trees of L . Since these trees are still Suslin in $L[G]$, partially generic filters do not exist for them and the model M does not witness local grMA in $L[G]$. As λ and M were completely arbitrary, local grMA must fail in $L[G]$. \square

We summarize the relationships between grMA and the other fragments of Martin's

axiom in the following diagram.

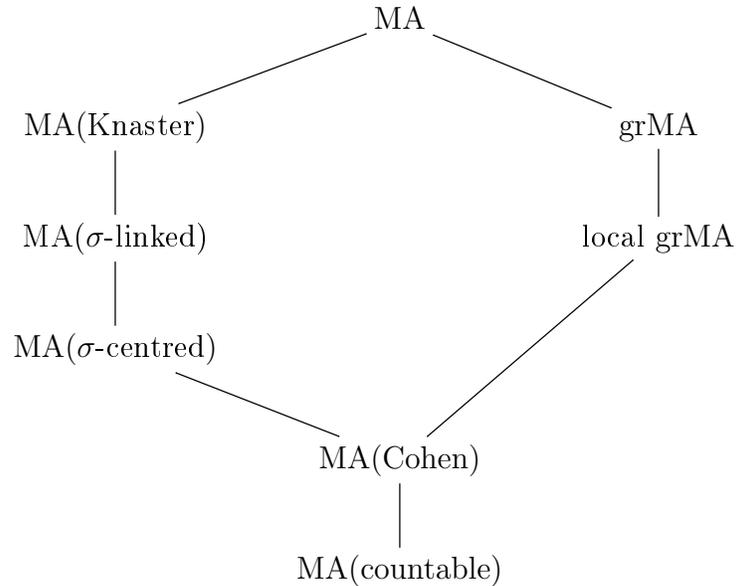


Figure 2.2: Diagram of implications between fragments of Martin's axiom

Let us mention that it is quite easy to perform ccc forcing over any model and have grMA fail in the extension.

Corollary 2.17. *Given any model V there is a ccc forcing extension $V[G]$ in which the local grounded Martin's axiom fails.*

Proof. We may assume that CH fails in V . If \mathbb{P} is the length ω_1 finite-support iteration of Hechler forcing and $G \subseteq \mathbb{P}$ is generic then it is easily seen that G is a dominating family in $V[G]$ and therefore the dominating number of $V[G]$ equals \aleph_1 . It now follows from corollary 2.11 that the local grMA fails in $V[G]$. \square

In the following two sections we shall explore the other side of the coin: grMA is preserved by certain kinds of ccc forcing.

2.3 Adding a Cohen real to a model of the grounded Martin's axiom

An interesting question when studying fragments of Martin's axiom is what effect adding various kinds of generic reals has on it. It was shown by Roitman [21] that MA_{\aleph_1} is destroyed after adding a Cohen or a random real. At the same time, it was shown that adding a Cohen real preserves a certain fragment, $\text{MA}(\sigma\text{-centred})$. In this section we follow the spirit of Roitman's arguments to show that the grounded Martin's axiom is preserved, even with respect to the same ground model, after adding a Cohen real.

It is well known that $\text{MA} + \neg\text{CH}$ implies that any ccc poset is Knaster (recall that a poset \mathbb{P} is Knaster if any uncountable subset of \mathbb{P} has in turn an uncountable subset of pairwise compatible elements). We start this section by transposing this fact to the grMA setting.

Lemma 2.18. *Let V satisfy the local grounded Martin's axiom over the ground model $M \subseteq H_{\kappa^+}$ and suppose CH fails in V . Then any poset $\mathbb{P} \in M$ which is ccc in V is Knaster in V .*

Proof. Let \mathbb{P} be as in the statement of the lemma and let $A = \{p_\alpha; \alpha < \omega_1\} \in V$ be an uncountable subset of \mathbb{P} . We first claim that there is a $p^* \in \mathbb{P}$ such that any $q \leq p^*$ is compatible with uncountably many elements of A . For suppose not. Then there would be for any $\alpha < \omega_1$ some $q_\alpha \leq p_\alpha$ which was compatible with only countably many elements of A . We could thus choose $\beta(\alpha) < \omega_1$ in such a way that q_α would be incompatible with any p_β for $\beta(\alpha) \leq \beta$. Setting $\beta_\alpha = \beta^\alpha(0)$ (meaning the α -th iterate of β), this would mean that $\{q_{\beta_\alpha}; \alpha < \omega_1\}$ is an uncountable antichain in \mathbb{P} , contradicting the fact that \mathbb{P} was ccc in V .

By replacing \mathbb{P} with the cone below p^* and modifying A appropriately, we may assume that in fact every element of \mathbb{P} is compatible with uncountably many elements of A . We now let $D_\alpha = \bigcup_{\beta \leq \alpha} \mathbb{P} \upharpoonright p_\beta$ for $\alpha < \omega_1$. The sets D_α are dense in \mathbb{P} and by $\text{grMA} + \neg\text{CH}$, we can find, in V , a filter $H \subseteq \mathbb{P}$ which intersects every D_α . But then $H \cap A$ is an uncountable set

of pairwise compatible elements. \square

We now introduce the main technical device we will use in showing that the grounded Martin's axiom is preserved when adding a Cohen real. In the proof we will be dealing with a two step extension $W \subseteq W[G] \subseteq W[G][c]$ where the first step is some ccc extension, the second adds a Cohen real and $W[G]$ satisfies the grounded Martin's axiom over W . To utilize the forcing axiom in $W[G]$ in verifying it in $W[G][c]$, we need to find a way of dealing with (names for) dense sets from $W[G][c]$ in $W[G]$. The termspace forcing construction (due to Laver and possibly independently also Abraham, Baumgartner, Woodin and others) comes to mind (for more information on this construction we point the reader to [7]), however the posets arising from this construction are usually quite far from being ccc and are thus unsuitable for our context. We attempt to rectify the situation by radically thinning out the full termspace poset and keeping only the simplest conditions.

Definition 2.19. Given a poset \mathbb{P} , a \mathbb{P} -name τ will be called a *finite \mathbb{P} -mixture* if there exists a finite maximal antichain $A \subseteq \mathbb{P}$ such that for every $p \in A$ there is some x satisfying $p \Vdash_{\mathbb{P}} \tau = \check{x}$. The antichain A is called a *resolving antichain* for τ and we denote the value x of τ at p by τ^p .

Definition 2.20. Let \mathbb{P} and \mathbb{Q} be posets. The *finite-mixture termspace poset* for \mathbb{Q} over \mathbb{P} is

$$\text{Term}_{\text{fin}}(\mathbb{P}, \mathbb{Q}) = \{ \tau ; \tau \text{ is a finite } \mathbb{P}\text{-mixture and } 1 \Vdash_{\mathbb{P}} \tau \in \check{\mathbb{Q}} \}$$

ordered by letting $\tau \leq \sigma$ iff $1 \Vdash_{\mathbb{P}} \tau \leq_{\mathbb{Q}} \sigma$.

As a side remark, let us point out that in all interesting cases the finite-mixture termspace poset is not a regular suborder of the full termspace poset and we can expect genuinely different properties. In fact, this occurs as soon as \mathbb{P} and \mathbb{Q} are nontrivial. To see this, suppose $\{p_n ; n < \omega\}$ and $\{q_n ; n < \omega\}$ are infinite antichain in \mathbb{P} and \mathbb{Q} , respectively. By

mixing we can find a $\tau \in \text{Term}(\mathbb{P}, \mathbb{Q})$ such that $p_n \Vdash \tau = q_n$; we claim that this τ does not have a reduction to $\text{Term}_{\text{fin}}(\mathbb{P}, \mathbb{Q})$. Suppose $\sigma \in \text{Term}_{\text{fin}}(\mathbb{P}, \mathbb{Q})$ were such a reduction. Then there is a condition p in its resolving antichain that is compatible with at least two conditions p_i , say p_0 and p_1 . Since σ is a reduction of τ , it and all stronger conditions are compatible with τ , and this means that σ^p is compatible with q_0 and q_1 . Let $q' \leq \sigma^p, q_1$ and define a strengthening $\sigma' \leq \sigma$ by setting $\sigma'^p = q'$ and keeping the rest of the mixture the same as in σ . But now σ' and τ are clearly incompatible.

In what follows, let us write $\mathbb{C} = {}^{<\omega}2$. We should mention two key issues with the finite-mixture termspace poset construction. Firstly, the construction is very sensitive to the concrete posets being used. For example, the forthcoming lemma 2.21 will show that $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{C})$ is Knaster, but it is not difficult to see that $\text{Term}_{\text{fin}}(\text{ro}(\mathbb{C}), \mathbb{C})$ already has antichains of size continuum. Therefore we cannot freely substitute forcing equivalent posets in the construction. In fact, if \mathbb{B} is a complete Boolean algebra then one easily sees that $\text{Term}_{\text{fin}}(\mathbb{B}, \mathbb{Q})$ consists of exactly those names that have only finitely many interpretations and, if \mathbb{Q} is nontrivial and \mathbb{B} has no atoms, this poset will have antichains of size continuum. The second issue is that it is quite rare for a poset to have a large variety of finite maximal antichains. Some, such as ${}^{<\omega}\omega$ or various collapsing posets, have none at all except the trivial one-element maximal antichain, while others, such as ${}^{<\omega_1}2$, have a few, but they do not capture the structure of the poset very well. In all of these cases we do not expect the finite-mixture termspace poset to be of much help. Nevertheless, in the case of Cohen forcing \mathbb{C} it turns out to be a useful tool.

Lemma 2.21. *If \mathbb{Q} is a Knaster poset then $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$ is also Knaster.*

Proof. Let $T = \{\tau_\alpha; \alpha < \omega_1\}$ be an uncountable subset of $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$ and choose resolving antichains A_α for τ_α . By refining the A_α we may assume that each of them is a level of the tree \mathbb{C} and, by thinning out T if necessary, that they are all in fact the same level A .

Let us enumerate $A = \{s_0, \dots, s_k\}$ and write τ_α^i instead of $\tau_\alpha^{s_i}$.

Since \mathbb{Q} is Knaster, an uncountable subset Z_0 of ω_1 such that the set $\{\tau_\alpha^0; \alpha \in Z_0\} \subseteq \mathbb{Q}$ consists of pairwise compatible elements. Proceeding recursively, we can find an uncountable $Z \subseteq \omega_1$ such that for every $i \leq k$ the set $\{\tau_\alpha^i; \alpha \in Z\} \subseteq \mathbb{Q}$ consists of pairwise compatible elements. We can mix the lower bounds of τ_α^i and τ_β^i over the antichain A to produce a name $\sigma_{\alpha\beta} \in \text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$ such that $\sigma_{\alpha\beta}$ is a lower bound for τ_α and τ_β . Thus $\{\tau_\alpha; \alpha \in Z\}$ is an uncountable subset of T consisting of pairwise compatible elements, which proves that $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$ is Knaster. \square

The following lemma is somewhat awkward, but it serves to give us a way of transforming a name for a dense subset of \mathbb{Q} into a closely related actual dense subset of $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$. With the usual termspace forcing construction simply taking $E = \{\tau; \Vdash \tau \in \dot{D}\}$ would have sufficed, but this set is not dense in $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$, so modifications are necessary.

Lemma 2.22. *Let \mathbb{Q} be poset and \dot{D} a \mathbb{C} -name for a dense subset of \mathbb{Q} . Then for any $n < \omega$ the set*

$$E_n = \{ \tau \in \text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q}); \exists A \text{ a resolving antichain for } \tau \forall s \in A: \\ n \leq |s| \wedge \exists s' \leq s: s' \Vdash \tau \in \dot{D} \}$$

is a dense subset of $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$.

One can think of the set E_n as the set of those τ that have a sufficiently deep resolving antichain, none of whose elements force τ to not be in \dot{D} .

Proof. Let $\sigma \in \text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$ and let A be a resolving antichain for it. Any finite refinement of a resolving antichain is, of course, another resolving antichain, so we may assume that we already have $n \leq |s|$ for all $s \in A$. By fullness we can find a name ρ for an element of \mathbb{Q} such that $1 \Vdash_{\mathbb{C}} (\rho \leq \sigma \wedge \rho \in \dot{D})$. For each $s \in A$ we can find an $s' \leq s$ such that $s' \Vdash_{\mathbb{C}} \rho = \check{q}_s$

for some $q_s \in \mathbb{Q}$. By mixing the q_s over the antichain A , we get a name $\tau \in E_n$ such that $\tau \leq \sigma$, which shows that E is dense in $\text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q})$. \square

Theorem 2.23. *Assume the local grounded Martin's axiom holds in V over the ground model $M \subseteq H_{\kappa^+}$ and let $V[c]$ be obtained by adding a Cohen real to V . Then $V[c]$ also satisfies the local grounded Martin's axiom over the ground model $M \subseteq H_{\kappa^+}^{V[c]} = H_{\kappa^+}[c]$.*

Proof. By assumption there is a ccc poset $\mathbb{P} \in M$ such that $H_{\kappa^+}^V = M[G]$ for an M -generic $G \subseteq \mathbb{P}$. We may assume that CH fails in V , for otherwise it would also hold in the final extension $V[c]$, which would then satisfy the full Martin's axiom. Consider a poset $\mathbb{Q} \in M$ which is ccc in $V[c]$. Since \mathbb{C} is ccc, \mathbb{Q} must also be ccc in V and by lemma 2.18 is in fact Knaster in V .

Let $\lambda < \mathfrak{c}^{V[c]}$ be a cardinal and let $\mathcal{D} = \{D_\alpha; \alpha < \lambda\} \in V[c]$ be a collection of dense subsets of \mathbb{Q} . Pick names $\dot{D}_\alpha \in V$ for these such that $1 \Vdash_{\mathbb{C}} \text{``}\dot{D}_\alpha \subseteq \check{\mathbb{Q}} \text{ is dense''}$.

Consider $\mathbb{R} = \text{Term}_{\text{fin}}(\mathbb{C}, \mathbb{Q}) \in M$. Note that \mathbb{R} is computed the same in M and in V . It now follows from lemma 2.21 that \mathbb{R} is Knaster in V (although not necessarily in M).

Let $E_{\alpha,n} \subseteq \mathbb{R}$ be the dense sets associated to the \dot{D}_α as in lemma 2.22. Write $\mathcal{E} = \{E_{\alpha,n}; \alpha < \lambda, n < \omega\}$. Applying the grounded Martin's axiom in V , we get an \mathcal{E} -generic filter $H \subseteq \mathbb{R}$. We will show that the filter generated by the set $H^c = \{\tau^c; \tau \in H\}$ is \mathcal{D} -generic. Pick a \dot{D}_α and consider the set

$$B_\alpha = \{s' \in \mathbb{C}; \exists \tau \in H \exists A \text{ a resolving antichain for } \tau \exists s \in A: \\ s' \leq s \wedge s' \Vdash_{\mathbb{C}} \tau \in \dot{D}_\alpha\}$$

We will show that B_α is dense in \mathbb{C} . To that end, pick a $t \in \mathbb{C}$. Since H is \mathcal{E} -generic, there is some $\tau \in H \cap E_{\alpha,|t|}$. Let A be a resolving antichain for τ . Since A is maximal, t must be compatible with some $s \in A$, and, since $|t| \leq |s|$, we must in fact have $s \leq t$. But then, by the definition of $E_{\alpha,|t|}$, there exists a $s' \leq s$ such that $s' \Vdash_{\mathbb{C}} \tau \in \dot{D}_\alpha$. This exactly says that

$s' \in B_\alpha$ and also $s' \leq s \leq t$. Thus B_α really is dense in \mathbb{C} .

By genericity we can find an $s' \in B_\alpha \cap c$. If $\tau \in H$ is the corresponding name, the definition of B_α now implies that $\tau^c \in D_\alpha \cap H^c$. Thus H^c really does generate a \mathcal{D} -generic filter. \square

The proof is easily adapted to show that, starting from the full grounded Martin's axiom in V over a ground model W , we obtain the full grounded Martin's axiom in $V[c]$ over the same ground model W .

2.4 Adding a random real to a model of the grounded Martin's axiom

Our next goal is to prove a preservation theorem for adding a random real. The machinery of the proof in the Cohen case will be slightly modified to take advantage of the measure theoretic structure in this context.

Recall that a measure algebra is a pair (\mathbb{B}, m) where \mathbb{B} is a complete Boolean algebra and $m: \mathbb{B} \rightarrow [0, 1]$ is a countably additive map such that $m(b) = 1$ iff $b = 1$.

Definition 2.24. Let (\mathbb{B}, m) be a measure algebra and $0 < \varepsilon < 1$. A \mathbb{B} -name τ will be called an ε -deficient finite \mathbb{B} -mixture if there is a finite antichain $A \subseteq \mathbb{B}$ such that $m(\sup A) > 1 - \varepsilon$ and for every $w \in A$ there exists some x such that $w \Vdash_{\mathbb{B}} \tau = \check{x}$. The antichain A is called a *resolving antichain* and we denote the value x of τ at w by τ^w .

Definition 2.25. Let (\mathbb{B}, m) be a measure algebra, \mathbb{Q} a poset and $0 < \varepsilon < 1$. The ε -deficient finite mixture termspace poset for \mathbb{Q} over (\mathbb{B}, m) is

$$\text{Term}_{\text{fn}}^\varepsilon(\mathbb{B}, \mathbb{Q}) = \{ \tau ; \tau \text{ is an } \varepsilon\text{-deficient finite } \mathbb{B}\text{-mixture and } 1 \Vdash_{\mathbb{B}} \tau \in \check{\mathbb{Q}} \}$$

ordered by letting $\tau \leq \sigma$ iff there are resolving antichains A_τ and A_σ such that A_τ refines A_σ and $\sup A_\tau \Vdash_{\mathbb{B}} \tau \leq \sigma$.

The following lemma is the analogue of lemma 2.22 for ε -deficient finite mixtures.

Lemma 2.26. *Let (\mathbb{B}, m) be a measure algebra, \mathbb{Q} a poset and $0 < \varepsilon < 1$. If \dot{D} is a \mathbb{B} -name for a dense subset of \mathbb{Q} then*

$$E = \{ \tau \in \text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}, \mathbb{Q}) ; \exists A \text{ a resolving antichain for } \tau : \sup A \Vdash_{\mathbb{B}} \tau \in \dot{D} \}$$

is dense in $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}, \mathbb{Q})$.

Proof. Let $\sigma \in \text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}, \mathbb{Q})$ and pick a resolving antichain $A = \{w_0, \dots, w_n\}$ for it. Let $\delta = m(\sup A) - (1 - \varepsilon)$. By fullness there are \mathbb{B} -names ρ_i for elements of \mathbb{Q} such that $w_i \Vdash \rho_i \leq \sigma \wedge \rho \in \dot{D}$. There are maximal antichains A_i below w_i such that each element of A_i decides the value of ρ_i . We now choose finite subsets $A'_i \subseteq A_i$ such that $m(\sup A_i) - m(\sup A'_i) < \frac{\delta}{n}$. Write $A' = \bigcup_i A'_i$. We then have $m(\sup A') > 1 - \varepsilon$. By mixing we can find a \mathbb{B} -name τ for an element of \mathbb{Q} which is forced by each element of A' to be equal to the appropriate ρ_i . Thus A' is a resolving antichain for τ and we have ensured that τ is in E and $\sigma \leq \tau$. \square

In what follows we let \mathbb{B}_{null} be the random Boolean algebra with the induced Lebesgue measure μ . The next lemma is the analogue of lemma 2.21.

Lemma 2.27. *Let \mathbb{Q} be a Knaster poset and $0 < \varepsilon < 1$. Then $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$ is Knaster as well.*

Proof. Let $\{ \tau_\alpha ; \alpha < \omega_1 \}$ be an uncountable subset of $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$. Choose resolving antichains A_α for the τ_α . We may assume that there is a fixed δ such that $1 - \varepsilon < \delta < \mu(\sup A_\alpha)$ for all α . We may also assume that all of the A_α have the same size n and enumerate them as $A_\alpha = \{w_\alpha^0, \dots, w_\alpha^{n-1}\}$; we shall write τ_α^i instead of $\tau_\alpha^{w_\alpha^i}$. By inner regularity of the measure

we may assume further that the elements of each A_α are compact. Using this and the outer regularity of the measure we can find open neighbourhoods $w_\alpha^i \subseteq U_\alpha^i$ such that U_α^i and U_α^j are disjoint for all α and distinct i and j and additionally satisfy

$$\mu(U_\alpha^i \setminus w_\alpha^i) < \frac{\delta - (1 - \varepsilon)}{n}$$

Fix a countable basis for the topology. Since the w_α^i are compact, we may take the U_α^i to be finite unions of basic opens. Since there are only countably many such finite unions, we can assume that there are fixed U^i such that $U_\alpha^i = U^i$ for all α .

We now obtain

$$\begin{aligned} \mu(w_\alpha^i \cap w_\beta^i) &= \mu(w_\alpha^i) - \mu(w_\alpha^i \cap (U^i \setminus w_\beta^i)) \\ &\geq \mu(w_\alpha^i) - \mu(U^i \setminus w_\beta^i) > \mu(w_\alpha^i) - \frac{\delta - (1 - \varepsilon)}{n} \end{aligned}$$

In particular, this gives that $\sum_i \mu(w_\alpha^i \cap w_\beta^i) > 1 - \varepsilon$.

Since \mathbb{Q} is Knaster we may assume that the elements of $\{\tau_\alpha^i; \alpha < \omega_1\}$ are pairwise compatible and that this holds for any i . Pick lower bounds $q_{\alpha\beta}^i$ for the τ_α^i and τ_β^i . By mixing we can construct \mathbb{B}_{null} -names $\sigma_{\alpha\beta}$ for elements of \mathbb{Q} such that $w_\alpha^i \cap w_\beta^i \Vdash \sigma_{\alpha\beta} = q_{\alpha\beta}^i$ for all i . By construction the $\sigma_{\alpha\beta}$ are ε -deficient finite \mathbb{B}_{null} -mixtures and are lower bounds for τ_α and τ_β . \square

While the concept of ε -deficient finite mixtures makes sense for any measure algebra, finding a good analogue of the preceding proposition for algebras of uncountable weight has proven difficult.

Lemma 2.28. *Let (\mathbb{B}, m) be a measure algebra and $0 < \varepsilon < 1$. Suppose \mathcal{A} is a family of finite antichains in \mathbb{B} , downward directed under refinement, such that $m(\sup A) > 1 - \varepsilon$ for any $A \in \mathcal{A}$. If we let $d_{\mathcal{A}} = \inf\{\sup A; A \in \mathcal{A}\}$ then $m(d_{\mathcal{A}}) \geq 1 - \varepsilon$.*

Proof. By passing to complements it suffices to prove the following statement: if I is an upward directed subset of \mathbb{B} all of whose elements have measure less than ε then $\sup I$ has measure at most ε .

Using the fact that \mathbb{B} is complete, we can refine I to an antichain Z that satisfies $\sup I = \sup Z$. Since \mathbb{B} is ccc, Z must be countable. Applying the upward directedness of I and the countable additivity of the measure, we can conclude that $m(\sup Z) \leq \varepsilon$. \square

We are finally ready to state and prove the preservation theorem we have been building towards.

Theorem 2.29. *Assume Martin's axiom holds in V and let $V[r]$ be obtained by adding a random real to V . Then $V[r]$ satisfies the grounded Martin's axiom over the ground model V .*

Proof. If CH holds in V then it holds in $V[r]$ as well, implying that $V[r]$ satisfies the full Martin's axiom. We may therefore assume without loss of generality that V satisfies $\text{MA} + \neg\text{CH}$.

Assume toward a contradiction that $V[r]$ does not satisfy the grounded Martin's axiom over V . Then there exist a poset $\mathbb{Q} \in V$ which is ccc in $V[r]$, a cardinal $\kappa < \mathfrak{c}$ and a collection $\mathcal{D} = \{D_\alpha; \alpha < \kappa\} \in V[r]$ of dense subsets of \mathbb{Q} such that $V[r]$ has no \mathcal{D} -generic filters on \mathbb{Q} . There must be a condition $b_0 \in \mathbb{B}_{\text{null}}$ forcing this. Let $\varepsilon < \mu(b_0)$.

Since \mathbb{B}_{null} is ccc, \mathbb{Q} must be ccc in V and, since $\text{MA} + \neg\text{CH}$ holds there, is also Knaster there. Thus $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q}) \in V$ is Knaster by lemma 2.27. We now choose names \dot{D}_α for the dense sets D_α such that \mathbb{B}_{null} forces that the \dot{D}_α are dense subsets of \mathbb{Q} . Then, by lemma 2.26, the E_α are dense in $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$, where E_α is defined from \dot{D}_α as in that lemma. We can thus obtain, using Martin's axiom in V , a filter H on $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$ which meets all of the E_α .

Pick a resolving antichain A_τ for each $\tau \in H$ and consider $\mathcal{A} = \{A_\tau; \tau \in H\}$. This

family satisfies the hypotheses of lemma 2.28, whence we can conclude that $\mu(d_{\mathcal{A}}) \geq 1 - \varepsilon$, where $d_{\mathcal{A}}$ is defined as in that lemma. Interpreting H as a \mathbb{B}_{null} -name for a subset of \mathbb{Q} , we now observe that

$$d_{\mathcal{A}} \Vdash_{\mathbb{B}_{\text{null}}} \text{“} H \text{ generates a } \dot{\mathcal{D}}\text{-generic filter on } \mathbb{Q}\text{”}$$

Now, crucially, since we have chosen $\varepsilon < \mu(b_0)$, the conditions b_0 and d_H must be compatible in \mathbb{B}_{null} . But this is a contradiction, since they force opposing statements. Therefore $V[r]$ really does satisfy the grounded Martin's axiom over V . \square

Corollary 2.30. *The grounded Martin's axiom is consistent with there being no Suslin trees.*

Proof. If Martin's axiom holds in V and r is random over V then $V[r]$ satisfies the grounded Martin's axiom by the above theorem and also has no Suslin trees by a theorem of Laver [17]. \square

Unfortunately, the employed techniques do not seem to yield the full preservation result as in theorem 2.23. If V satisfied merely the grounded Martin's axiom over a ground model W we would have to argue that the poset $\text{Term}_{\text{fin}}^{\varepsilon}(\mathbb{B}_{\text{null}}, \mathbb{Q})$ as computed in V was actually an element of W , so that we could apply grMA to it. But we cannot expect this to be true if passing from W to V added reals; not only will the termspace posets be computed differently in W and in V , even the random Boolean algebras of these two models will be different. Still, these considerations lead us to the following improvement to the theorem above.

Theorem 2.31. *Assume the grounded Martin's axiom holds in V over the ground model W via a forcing which is countably distributive (or, equivalently, does not add reals), and let $V[r]$ be obtained by adding a random real to V . Then $V[r]$ also satisfies the grounded Martin's axiom over the ground model W .*

Proof. By assumption there is a ccc countably distributive poset $\mathbb{P} \in W$ such that $V = W[G]$ for some W -generic $G \subseteq \mathbb{P}$. Since W and V thus have the same reals, they must also have the same Borel sets. Furthermore, since the measure is inner regular, a Borel set having positive measure is witnessed by a positive measure compact (i.e. closed) subset, which means that W and V agree on which Borel sets are null. It follows that the random Boolean algebras as computed in W and in V are the same.

Now let $0 < \varepsilon < 1$ and let $\mathbb{Q} \in W$ be a poset which is ccc in V . We claim that V and W compute the poset $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$ the same. Clearly any ε -deficient finite mixture in W is also such in V , so we really only need to see that V has no new such elements. But \mathbb{B}_{null} is ccc, which means that elements of \mathbb{Q} have countable nice names and these could not have been added by G . So V and W in fact agree on the whole termspace poset $\text{Term}(\mathbb{B}_{\text{null}}, \mathbb{Q})$, and therefore also on the ε -deficient finite mixtures.

The rest of the proof proceeds as in theorem 2.29. The key step there, where we apply Martin's axiom to the poset $\text{Term}_{\text{fin}}^\varepsilon(\mathbb{B}_{\text{null}}, \mathbb{Q})$, goes through, since we have shown that this poset is in W and we may therefore apply grMA to it. \square

Just as in theorem 2.23 we may replace the grounded Martin's axiom in the above theorem with its local version.

It is not immediately obvious that the hypothesis of the above theorem is ever satisfied in a nontrivial way, that is, whether grMA can ever hold via a *nontrivial* countably distributive extension. The following theorem, due to Larson, shows that this does happen and gives yet another construction of a model of grMA. For the purposes of this theorem we shall call a Suslin tree T *homogeneous* if for any two nodes $p, q \in T$ of the same height, the cones below them are isomorphic. Note that homogeneous Suslin trees may be constructed from \diamond .

Theorem 2.32 (Larson [15]). *Let $\kappa > \omega_1$ be a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let T be a homogeneous Suslin tree. Then there is a ccc poset \mathbb{P} such that, given a V -generic*

$G \subseteq \mathbb{P}$, the tree T remains Suslin in $V[G]$ and, if b is a generic branch through T , the extension $V[G][b]$ satisfies $\mathfrak{c} = \kappa$ and the grounded Martin's axiom over the ground model $V[G]$.

Proof. The idea is to attempt to force $\text{MA} + \mathfrak{c} = \kappa$, but only using posets that preserve the Suslin tree T . More precisely, fix a well-order \triangleleft of H_κ of length κ and define \mathbb{P} as the length κ finite support iteration which forces at stage α with the next \mathbb{P}_α -name for a poset $\dot{\mathbb{Q}}_\alpha$ such that \mathbb{P}_α forces that $T \times \dot{\mathbb{Q}}_\alpha$ is ccc.

Let $G \subseteq \mathbb{P}$ be V -generic. It is easy to see by induction that $\mathbb{P}_\alpha \times T$ is ccc for all $\alpha \leq \kappa$; the successor case is clear from the definition of the iteration \mathbb{P} and the limit case follows by a Δ -system argument. We can thus conclude that T remains a Suslin tree in $V[G]$. Furthermore, standard arguments show that there are exactly κ many reals in $V[G]$ and that this extension satisfies Martin's axiom for small posets which preserve T , i.e. those \mathbb{Q} such that $\mathbb{Q} \in H_\kappa^{V[G]}$ and $\mathbb{Q} \times T$ is ccc.

Finally, let us see that adding a branch b through T over $V[G]$ yields a model of the grounded Martin's axiom over $V[G]$. Thus let $\mathbb{Q} \in V[G]$ be a poset which is ccc in $V[G][b]$ and has size less than κ there. There is a condition in T forcing that \mathbb{Q} is ccc, so by our homogeneity assumption T forces this, meaning that $\mathbb{Q} \times T$ is ccc in $V[G]$. The key point now is that, since T is countably distributive, all of the maximal antichains (and open dense subsets) of \mathbb{Q} in $V[G][b]$ are already in $V[G]$. Furthermore, any collection \mathcal{D} of less than κ many of these in $V[G][b]$ can be covered by some $\tilde{\mathcal{D}}$ in $V[G]$ of the same size. Our observation from the previous paragraph then yields a $\tilde{\mathcal{D}}$ -generic filter for \mathbb{Q} in $V[G]$ and therefore $V[G][b]$ satisfies grMA over $V[G]$ by lemma 2.2. \square

Starting from a Suslin tree with a stronger homogeneity property, Larson also shows that there are no Suslin trees in the extension $V[G][b]$ above. This gives an alternative proof of corollary 2.30.

From the argument of theorem 2.32 we can actually extract another preservation result for grMA.

Theorem 2.33. *Assume the grounded Martin's axiom holds in V over the ground model W and let $T \in V$ be a Suslin tree. If $b \subseteq T$ is a generic branch then $V[b]$ also satisfies the grounded Martin's axiom over the ground model W .*

Proof. The point is that, just as in the proof of theorem 2.32, forcing with T does not add any new maximal antichains to posets from W that remain ccc in $V[b]$ and any collection of these antichains in $V[b]$ can be covered by a collection of the same size in V . \square

If grMA holds over a ground model that reals have been added to, it seems harder to say anything about preservation after adding a further random real. Nevertheless, we fully expect the answers to the following question to be positive.

Question 2.34 (open). *Does adding a random real to a model of grMA preserve grMA? Does it preserve it with the same witnessing ground model?*

Generalizations of theorems 2.23 and 2.29 to larger numbers of reals added seem the natural next step in the exploration of the preservation phenomena of the grounded Martin's axiom. Such preservation results would also help in determining the compatibility of grMA with various configurations of the cardinal characteristics on the left side of Cichoń's diagram. The constructions Term_{fin} and $\text{Term}_{\text{fin}}^{\varepsilon}$ seem promising, but obtaining a good chain condition in any case at all, except those shown, has proven difficult.

2.5 The grounded proper forcing axiom

We can, of course, also consider grounded versions of other forcing axioms. We define one and note that similar definitions can be made for grSPFA, grMM and so on.

Definition 2.35. The *grounded proper forcing axiom* (grPFA) asserts that V is a forcing extension of some ground model W by a proper poset and V satisfies the conclusion of the proper forcing axiom for posets $\mathbb{Q} \in W$ which are still proper in V .

Theorem 2.36. *Let κ be supercompact. Then there is a proper forcing extension that satisfies grPFA + $\mathfrak{c} = \kappa = \omega_2$ and in which PFA, and even MA, fails.*

The idea of the proof is similar to the one used in the proof of theorem 2.6. Specifically, we modify the Baumgartner PFA iteration, starting from a supercompact cardinal, to only use posets coming from the ground model. Since the iteration uses countable support, a product analysis like the one given in lemma 2.5 is not possible, but this turns out not to matter for this particular result.

Proof. Start with a Laver function ℓ for κ and build a countable-support forcing iteration \mathbb{P} of length κ which forces at stage α with $\dot{\mathbb{Q}}$, some full name for the poset $\mathbb{Q} = \ell(\alpha)$ if it is proper at that stage and with trivial forcing otherwise. Note that \mathbb{P} is proper. Now let $V[G][H]$ be a forcing extension by $\mathbb{R} = \text{Add}(\omega, 1) \times \mathbb{P}$. We claim that $V[G][H]$ is the required model.

Since the Laver function ℓ will quite often output the poset $\text{Add}(\omega, 1)$ and this will always be proper, the iteration \mathbb{P} will add reals unboundedly often. Furthermore, since \mathbb{R} is κ -cc, we will obtain $\mathfrak{c} = \kappa$ in $V[G][H]$.

Next we wish to see that $\kappa = \omega_2^{V[G][H]}$. For this it suffices to see that any $\omega_1 < \lambda < \kappa$ is collapsed at some point during the iteration. Recall the well-known fact that any countable-support iteration of nontrivial posets adds a Cohen subset of ω_1 at stages of cofinality ω_1 and therefore collapses the continuum to ω_1 at those stages. Now fix some $\omega_1 < \lambda < \kappa$. Since \mathbb{P} ultimately adds κ many reals and is κ -cc, there is some stage α of the iteration such that \mathbb{P}_α has already added λ many reals and therefore $\mathfrak{c}^{V[H_\alpha]} \geq \lambda$. Since \mathbb{P}_α is proper, the rest of the iteration $\mathbb{P} \upharpoonright [\alpha, \kappa)$ is a countable-support iteration in $V[H_\alpha]$ and the fact mentioned above

implies that λ is collapsed to ω_1 by this tail of the iteration.

Note that \mathbb{R} is proper, since $\mathbb{P} * \text{Add}(\omega, 1)$ is proper and has a dense subset isomorphic to \mathbb{R} . To verify that grPFA holds in $V[G][H]$ let $\mathbb{Q} \in V$ be a poset that is proper in $V[G][H]$ and let $\mathcal{D} = \{ D_\alpha; \alpha < \omega_1 \} \in V[G][H]$ be a family of dense subsets of \mathbb{Q} . In V we can fix (for some large enough θ) a θ -supercompactness embedding $j: V \rightarrow M$ such that $j(\ell)(\kappa) = \mathbb{Q}$. Since the Cohen real forcing is small, the embedding j lifts to a θ -supercompactness embedding $j: V[G] \rightarrow M[G]$. We can factor $j(\mathbb{P})$ in $M[G]$ as $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$. Let $h * H_{\text{tail}} \subseteq \mathbb{Q} * \mathbb{P}_{\text{tail}}$ be $V[G][H]$ -generic. As usual, we can now lift the embedding j in $V[G][H * h * H_{\text{tail}}]$ to $j: V[G][H] \rightarrow \overline{M} = M[G][H * h * H_{\text{tail}}]$. Note that the closure of this embedding implies that $j[h] \in \overline{M}$. But $j[h]$ is a $j(\mathcal{D})$ -generic filter on $j(\mathbb{Q})$ in \overline{M} and so, by elementarity, there is a \mathcal{D} -generic filter on \mathbb{Q} in $V[G][H]$.

Finally, since we can see $V[G][H]$ as obtained by adding a Cohen real to an intermediate extension and since CH fails there, PFA and even Martin's axiom must fail there by Roitman's [21]. \square

With regard to the above proof, we should mention that one usually argues that κ becomes ω_2 after an iteration similar to ours because at many stages the poset forced with was explicitly a collapse poset $\text{Coll}(\omega_1, \lambda)$. In our case, however, the situation is different. It turns out that a significant number of proper posets from V (the collapse posets among them) cease to be proper as soon as we add the initial Cohen real. Therefore the possibility of choosing $\text{Coll}(\omega_1, \lambda)$ never arises in the construction of the iteration \mathbb{P} and a different argument is needed. We recount a proof of this fact below. The argument is essentially due to Shelah, as communicated by Goldstern in [12].

Theorem 2.37 (Shelah). *Let \mathbb{P} be a ccc poset and let \mathbb{Q} be a countably distributive poset which collapses ω_2 . Let $G \subseteq \mathbb{P}$ be V -generic. If $V[G]$ has a new real then \mathbb{Q} is not proper in $V[G]$.*

Proof. Fix at the beginning a \mathbb{Q} -name \dot{f} , forced to be a bijection between ω_1 and ω_2^V . Let θ be a sufficiently large regular cardinal. By claim XV.2.12 of [23] we can label the nodes $s \in {}^{<\omega}2$ with countable models $M_s \prec H_\theta$ such that:

- the M_s are increasing along each branch of the tree ${}^{<\omega}2$;
- $\mathbb{P}, \mathbb{Q}, \dot{f} \in M_\emptyset$;
- there is an ordinal δ such that $M_s \cap \omega_1 = \delta$ for all s ;
- for any s there are ordinals $\alpha_s < \beta_s < \omega_2$ such that $\alpha_s \in M_{s \smallfrown 0}$ and $\alpha_s \notin M_{s \smallfrown 1}$ and $\beta_s \in M_{s \smallfrown 1}$.

Now consider, in $V[G]$, the tree of models $M_s[G]$. By the argument given in the proof of lemma 2.2, the models $M_s[G]$ are elementary in $H_\theta^{V[G]}$ and, since \mathbb{P} is ccc, we still have $M_s[G] \cap \omega_1 = \delta$. Let $M = M_r[G]$ be the branch model determined by the new real $r \in V[G]$. We shall show that there are no M -generic conditions in \mathbb{Q} .

Suppose that q were such a generic condition. We claim that q forces that $\dot{f} \upharpoonright \delta$ maps onto $M \cap \omega_2^V$ (note that \dot{f} still names a bijection $\omega_1 \rightarrow \omega_2^V$ over $V[G]$). First, suppose that q does not force that $\dot{f}[\delta] \subseteq M$. Then we can find $q' \leq q$ and an $\alpha < \delta$ such that $q' \Vdash \dot{f}(\alpha) \notin M$. But if $q' \in H \subseteq \mathbb{Q}$ is generic then $M[H]$ is an elementary substructure of $H_\theta^{V[G][H]}$ and, of course, $f, \alpha \in M[H]$, leading to a contradiction.

Conversely, suppose that q does not force that $M \cap \omega_2^V \subseteq \dot{f}[\delta]$. We can again find $q' \leq q$ and an $\alpha \in M \cap \omega_2^V$ such that $q' \Vdash \alpha \notin \dot{f}[\delta]$. Let $q' \in H \subseteq \mathbb{Q}$ be generic. As before, $M[H]$ is an elementary substructure of $H_\theta^{V[G][H]}$ and $f, \alpha \in M[H]$. Since $f: \omega_1 \rightarrow \omega_2^V$ is a bijection, we must have $f^{-1}(\alpha) \in M[H]$. But by construction $f^{-1}(\alpha)$ is an ordinal greater than δ while simultaneously $M[H] \cap \omega_1 = \delta$ by the M -genericity of q , giving a contradiction.

Fixing our putative generic condition q , we can use the countable distributivity of \mathbb{Q} in V to see that $\dot{f} \upharpoonright \delta$, and consequently $M \cap \omega_2^V$, exist already in V . But we can extract r from

$M \cap \omega_2$.

Notice that, given a model $M_{s \frown 1}$ in our original tree, no elementary extension $M_{s \frown 1} \prec X$ satisfying $X \cap \omega_1 = \delta$ can contain α_s . This is because $M_{s \frown 1}$ contains a bijection $g: \omega_1 \rightarrow \beta_s$ and, by elementarity, g must restrict to a bijection between δ and $M_{s \frown 1} \cap \beta_s$. But seen from the viewpoint of X , that same function g must restrict to a bijection between δ and $X \cap \beta_s$ and so $X \cap \beta_s = M_{s \frown 1} \cap \beta_s$.

Using this fact, we can now extract r from $M \cap \omega_2$ in V . Specifically, we can decide at each stage whether the branch determined by r went left or right depending on whether $\alpha_s \in M \cap \omega_2$ or not. We conclude that r appears already in V , contradicting our original assumption. Therefore there is no generic condition q as above and \mathbb{Q} is not proper in $V[G]$. \square

Ultimately, one hopes that by grounding the forcing axiom we lower its consistency strength while still being able to carry out some of the usual arguments and obtain some of the standard consequences. However, theorem 2.37 severely limits the kind of arguments we can carry out under grPFA. Many arguments involving PFA use, among other things, collapsing posets such as $\text{Coll}(\omega_1, 2^\omega)$. In contrast, if the poset witnessing grPFA in a model is any kind of iteration that at some stage added, say, a Cohen real, the theorem prevents us from applying the forcing axiom to any of these collapsing posets. It is thus unclear exactly how much strength of PFA can be recovered from grPFA. In particular, while grPFA implies that CH fails, the following key question remains open:

Question 2.38 (open). *Does grPFA imply that the continuum equals ω_2 ?*

Regarding the relation of grPFA to other forcing axioms, a lot remains unknown. Theorem 2.36 shows that grPFA does not imply MA. Beyond this a few more things can be said.

Proposition 2.39. *Martin's axiom does not imply grPFA.*

Proof. Starting over some model, force with the Solovay-Tennenbaum iteration to produce a model $V[G]$ satisfying MA and $\mathfrak{c} = \omega_3$. Now perform Reitz's ground axiom forcing (cf. [20]) above ω_3 to produce a model $V[G][H]$ still satisfying MA and $\mathfrak{c} = \omega_3$ but which is not a set-forcing extension of any model (note that H is added by class-sized forcing). Therefore the only way $V[G][H]$ could satisfy grPFA is if it actually satisfied PFA in full. But that cannot be the case since PFA implies that the continuum equals ω_2 . \square

Proposition 2.40. *The grounded proper forcing axiom does not imply MA(σ -centred) (and not even MA_{\aleph_1} (σ -centred)).*

Proof. We could have replaced the forcing $\text{Add}(\omega, 1) \times \mathbb{P}$ in the proof of theorem 2.36 with $\text{Add}(\omega, \omega_1) \times \mathbb{P}$ without issue. As in the proof of theorem 2.12 we get a model whose bounding number equals \aleph_1 , but this contradicts MA_{\aleph_1} (σ -centred) as in the proof of corollary 2.15. \square

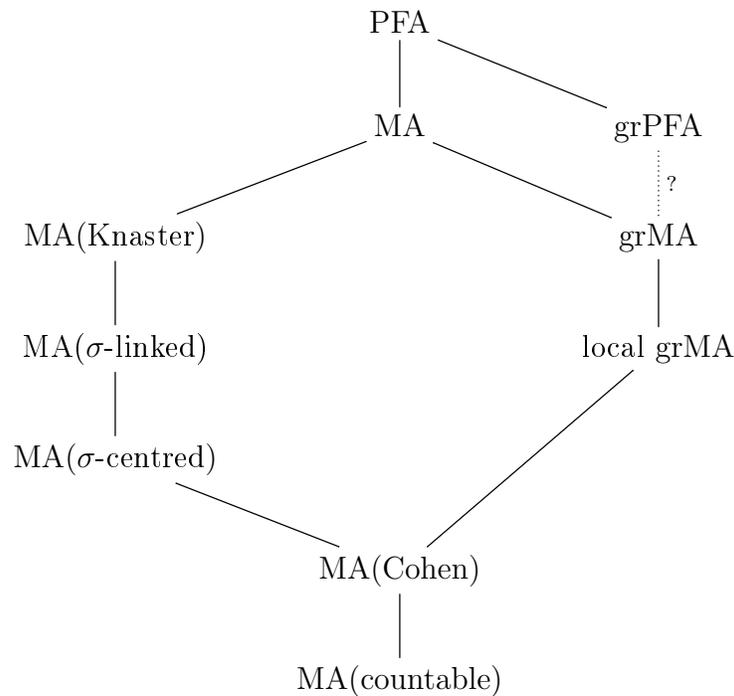


Figure 2.3: Expanded diagram of forcing axioms

While it is easy to see that grPFA implies MA_{\aleph_1} (Cohen), whether or not it even implies $\text{MA}(\text{countable})$ is unclear (a large part of the problem being that we do not have an answer to question 2.38). But an even more pressing question concerns the relationship between grPFA and grMA :

Question 2.41 (open). *Does grPFA imply grMA ?*

Even if the answer to question 2.38 turns out to be positive, we conjecture that the answer to this last question is negative. We expect that it is possible to use the methods of [20] or, more generally, [9] in combination with the forcing construction of theorem 2.36 to produce a model of grPFA which has no ccc ground models and in which MA (and consequently also grMA) fails.

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