Some Results in Combinatorial Number Theory

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Recommended Citation
https://academicworks.cuny.edu/gc_etds/2182
Some Results in Combinatorial Number Theory

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2017
This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

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THE CITY UNIVERSITY OF NEW YORK
The first chapter establishes results concerning equidistributed sequences of numbers. For a given $d \in \mathbb{N}$, $s(d)$ is the largest $N \in \mathbb{N}$ for which there is an $N$-regular sequence with $d$ irregularities. We compute lower bounds for $s(d)$ for $d \leq 10000$ and then demonstrate lower and upper bounds $\lfloor \sqrt{4d + 895} + 1 \rfloor \leq s(d) < 24801d^3 + 942d^2 + 3$ for all $d \geq 1$.

In the second chapter we ask if $Q(x) \in \mathbb{R}[x]$ is a degree $d$ polynomial such that for $x \in [x_k] = \{x_1, \ldots, x_k\}$ we have $|Q(x)| \leq 1$, then how big can its lead coefficient be? We prove that there is a unique polynomial, which we call $L_{d,[x_k]}(x)$, with maximum lead coefficient under these constraints and construct an algorithm that generates $L_{d,[x_k]}(x)$. 
Acknowledgements

I thank my advisor, Professor Kevin O’Bryant, for his continued patience, expert help and wise guidance. Professor O’Bryant, yours is an enviably fast and incisive mind. You are one of the smartest people I have ever met. I hope that one day I am worthy of being called your student. Since patience has been mentioned I must now thank my wife Adriane Levy. Adriane, your patient love and support are gifts I hope one day to deserve. I also would like to thank Professor Melvyn Nathanson for running the number theory courses, seminars, and conferences from which I and so many others have learned so much. Professor Nathanson, you are the pillar of the community. I would also like to thank Professor Cormac O’Sullivan for taking the time to read my thesis and being on the committee for its defense. And to all my other friends and family members who have helped me along the way, thank you!
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Chapter 1

Lower and Upper Bounds for Irregularities of Distribution

1.1 Introduction and History

Definition 1.1.1. A sequence

$$X = (x_1, x_2, \ldots, x_N)$$

of $N$ distinct real numbers $x_i \in [0, 1)$ is an $N$-regular sequence if

$$\{\lfloor nx_1 \rfloor, \lfloor nx_2 \rfloor, \ldots, \lfloor nx_n \rfloor \} = \{0, 1, 2, \ldots, n-1\}$$

for all natural numbers $n \leq N$. The geometric sense here is that for all $n \leq N$ each one of the $n$ intervals

$$[0, 1), [1, 2), \ldots, [n-1, n)$$
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contains one of the \( n \)-dilated elements from the set

\[ \{nx_1, nx_2, \cdots, nx_n\}. \]

Steinhaus [St64] introduced the following question

**Question** (Steinhaus). Is there a longest \( N \)-regular sequence?

Warmus [Wa76] proved that an \( N \)-regular sequences could have at most 17 elements and Graham [Gr13] gave one such sequence

\[
\left( \frac{4}{7}, \frac{2}{7}, \frac{16}{17}, \frac{1}{14}, \frac{8}{11}, \frac{5}{13}, \frac{1}{14}, \frac{3}{11}, \frac{15}{12}, \frac{1}{17}, \frac{13}{16}, \frac{5}{10} \right).
\]

Berlekamp and Graham [BG70] asked a more general question, relaxing the regularity of Steinhaus. We give two more definitions before stating their question.

**Definition 1.1.2.** A sequence

\[ X = (x_1, x_2, \cdots, x_{N+d}) \]

of \( N + d \) distinct real numbers \( x_i \in [0, 1) \) is an *\( N \)-regular sequence with at most \( d \) irregularities* if

\[ \{[nx_1], [nx_2], \cdots, [nx_{n+d}]\} = \{0, 1, 2, \cdots, n-1\} \]

for all natural numbers \( n \leq N \). The geometric sense here is that for all \( n \leq N \) each
one of the $n$ intervals

$$[0, 1), [1, 2), \cdots, [n - 1, n)$$

contains at least one of the $n$-dilated elements from the set

$$\{nx_1, nx_2, \cdots, nx_{n+d}\}.$$

**Definition 1.1.3.** For a given $d \in \mathbb{N}$, $s(d)$ is equal to the largest $N \in \mathbb{N}$ for which there is an $N$-regular sequence with at most $d$ irregularities.

**Question** (Berlekamp and Graham [BG70]). What is the biggest $N$ for which there is an $N$-regular sequence with $d$ irregularities, i.e., what is $s(d)$?

Though Warmus proved that $s(0) = 17$ [Wa76] in 1976, $s(d)$ remains unknown for $d \geq 1$.

**Theorem 1.1.4** (Graham [Gr13]). There exists a cubic polynomial $p(d)$ such that

$$s(d) < p(d)$$

for all $d \geq 1$.

A briefly outline of our main results: in Section 1.2 we compute lower bounds for $s(d)$ for $d \leq 10000$, using a computer to construct sequences. In Section 1.3 we demonstrate a lower bound $s(d) \geq \lfloor \sqrt{4d + 895} + 1 \rfloor$ for all $d \geq 1$. In Section 1.4 we fill-in and make explicit the arguments outlined by pin [Gr13] and show by way of
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an algebraic construction that Theorem 1.1.4 holds for \( p(x) = 24801d^3 + 942d^2 + 3 \).

1.2 Computational Lower Bounds for \( s(d) \) for \( d \leq 1000 \)

Recall that \( s(d) \) (Definition 1.1.1) is the maximum \( N \) for which there is an \( N \)-regular sequence with at most \( d \) irregularities (Definition 1.1.2). In this section we describe the Pick-the-Middle algorithm we use to find lower bounds for \( s(d) \) for \( d \leq 10000 \) and then view some results from the algorithm.

1.2.1 Description of the Pick-the-Middle algorithm

For a set \( d \) we attempt to construct the longest possible sequence with at most \( d \) irregularities as follows.

We begin with a set of first terms, \( x_1 \)'s \( \in [0, 1) \), one for each of the test sequences we will construct.

Next we choose an \( x_2 \) for each \( x_1 \) in the following manner. Call this step \( n = 2 \). If \( 2x_1 \) is in the interval \([0, 1)\) then we choose \( x_2 = \frac{x_1 + 1}{2} \) and thus \( 2x_2 \) is in the empty interval \([1, 2)\). If however \( 2x_1 \) is in the interval \([1, 2)\) then we choose \( x_2 = \frac{0 + x_1}{2} \) so that \( 2x_2 \) is in the empty interval \([0, 1)\). So far we have constructed 2-regular sequences that each have 2 terms and no irregularities.

\footnote{The shape and size of a set of \( x_i \)'s depending on various considerations including running time, processor power and computer memory. The implementations of the algorithm, whose results we reported later in this section, used sets of \( x_i \)'s that are equidistributed in the unit interval.}
Next we more or less repeat step 2 but now we can add more than one term to a test sequence as long as doing so doesn’t make the test sequence contain more than $n + d$ terms. For each of the $n$ intervals

$[0, 1), [1, 2), \ldots, [n - 1, n)$

empty of an $n$-dilated $nx_i$ from a sequence’s existing terms, we pick the maximum $x_L$ from that sequence so that $nx_L$ is to the left of the empty interval and then we pick the minimum $x_R$ from that sequence so that $nx_R$ is to the right of the empty interval\(^2\). We then append the average, $\frac{x_L + x_R}{2}$, for each such empty interval to our test sequence (again, as long as doing so doesn’t make our sequence contain more than $\leq n + d$ terms).

At any step $n$, if a test sequence would have more than $n + d$ terms then we stop extending it. That test sequence remains an $(n - 1)$-regular sequence with at most $d$ irregularities. The algorithm continues until there is no test sequence left to extend and the test sequence with highest $n$-regularity gives us our lower bound for $s(d)$ for our particular $d$.

Two versions of our Sagemath code for the Pick-the-Middle algorithm appear in Sections 1.6.1 and 1.6.2.

We now prove two facts we implicitly assumed in the description of our algorithm.

---

\(^2\)If the empty interval is $[0, 1)$ or $[n - 1, n)$ we set $x_L = 0$ or $x_R = 1$, respectively.
First, our manner of choosing $x_L$ and $x_R$ assumes that there were never two consecutive empty unit-length intervals. If there were two consecutive empty unit-length intervals then our algorithm would fill only one of them and thus produce a sequence that is not $n$-regular\footnote{Theorem 1.2.1 will also get used in Section 1.3 to establish lower bounds for $s(d)$ for all $d \geq 1$}.

**Lemma 1.2.1.** Given $n - 1 \geq 1$, $d \geq 0$ and

$$X = (x_1, x_2, \ldots, x_{n-1+d}),$$

if $X$ is an $(n-1)$-regular sequence with $d$ irregularities then there does not exist a natural number $l \leq n - 1$ such that

$$[l-1, l+1) \cap \{nx_1, nx_2, \ldots, nx_{n-1+d}\} = \emptyset.$$

**Proof.** For $n-1 = 1$ this is true because the only $[l-1, l+1)$ is $[0, 2)$, which contains all of the 2-dilated terms $\{2x_1, 2x_2, \ldots, 2x_{2+d}\}$, so $[0, 2)$ can not be empty.

For $n-1 \geq 2$ we assume that there are two consecutive intervals $[l-1, l)$ and $[l, l+1)$ such that

$$[l-1, l+1) \cap \{nx_1, nx_2, \ldots, nx_{n-1+d}\} = \emptyset.$$
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This is equivalent to assuming that for each \( x \in X \) either

\[ nx < l - 1 \]

or

\[ l + 1 \leq nx. \]

This implies, since \( 0 \leq x < 1 \), that for each \( x \in X \) either

\[ (n - 1)x < l - 1 \]

or

\[ l \leq (n - 1)x. \]

This is equivalent to

\[ [l - 1, l) \cap \{(n - 1)x_1, (n - 1)x_2, \ldots, (n - 1)x_{n-1+d}\} = \emptyset, \]

which contradicts our assumption that \( X \) is \((n - 1)\)-regular. \( \square \)

The second fact we implicitly assume for our algorithm is that

\[ (n + 1)\frac{x_L + x_R}{2} \]

is contained in the empty interval of which \((n + 1)x_L\) and \((n + 1)x_R\) are picked to the left and right.
Lemma 1.2.2. Given two real numbers $l, n$ and two real numbers $x_L, x_R \in [0, 1)$, if

$$(n + 1)x_L \in [l - 1, l)$$

and

$$(n + 1)x_R \in [l + 1, l + 2)$$

then

$$(n + 1)\frac{x_L + x_R}{2} \in [l, l + 1).$$

Proof. Adding the assumptions gives us

$$(l - 1) + (l + 1) \leq (n + 1)x_L + (n + 1)x_R < (l) + (l + 2)$$

and thus

$$l \leq (n + 1)\frac{x_L + x_R}{2} < l + 1.$$

We now state results from some implementations of our Pick-the-Middle algorithm.
1.2.2 Results for $d \leq 100$

For each $d \leq 100$ we constructed 10000 test sequences, each with a different element from the set $\{0, \frac{1}{10000}, \frac{2}{10000}, \cdots, \frac{9999}{10000}\}$ as its first term. We then ran our Pick-the-Middle algorithm as described above in Section 1.2.1. For each $d$, the largest $N$ for which one of our 10000 test sequences was $N$-regular with at most $d$-irregularities gave us our computational lower bound for $s(d)$. The results are displayed below in Figure 1.1 and Table 1.1. The line $y = 3.1d$ is included in Figure 1.1 for comparison.

![Figure 1.1: Computational lower bound for $s(d)$ for $d \leq 100$](image-url)
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Table 1.1: Computational lower bound for $s(d)$ for $d \leq 100$
1.2.3 Results for $d \leq 1000$

For each $d \leq 1000$ we constructed 100 test sequences, each with a different element from the set $\{0, \frac{1}{100}, \frac{2}{100}, \cdots, \frac{99}{100}\}$ as its first term. We then ran our Pick-the-Middle algorithm as described above in Section 1.2.1. For each $d$, the largest $N$ for which one of our 100 test sequences was $N$-regular with at most $d$-irregularities gave us our computational lower bound for $s(d)$. These results are displayed below in Figure 1.2, with the line $y = 2.3d$ included for comparison.

![Figure 1.2: Computational lower bound for $s(d)$ for $d \leq 1000$](image-url)
1.2.4 Results for $d \leq 10000$

For each $d \leq 10000$ we constructed 10 test sequences, each with a different element from the set $\{0, \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}\}$ as its first term. We then ran our Pick-the-Middle algorithm as described above in Section 1.2.1. For each $d$, the largest $N$ for which one of our 10 test sequences was $N$-regular with at most $d$-irregularities gave us our computational lower bound for $s(d)$. These results are displayed below in Figure 1.3 with the line $y = 1.8d$ included for comparison.

![Figure 1.3: Computational lower bound for $s(d)$ for $d \leq 10000$](image-url)
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1.2.5 Further computational investigation into lower bounds for $s(d)$

So far we have taken the initial terms in our test sequences, the set of $x_1$’s, to be uniformly distributed between 0 and 1. Taking the initial terms from the set of Farey fractions has not yielded any noticeable improvements to our search for longer $N$-regular sequences with at most $d$ irregularities. Further restructuring of the algorithm is being considered.

1.2.6 Increasing the lower bounds for $s(d)$ for a specific $d$

From [Wa76] we have that $s(0) = 17$ and from [Ol17] we have that $s(1) \geq 31$. For a few specific small values of $d$ we continue to search for longer $N$-regular sequences with at most $d$ irregularities. The Sagemath source code of our search algorithm is in Section 1.6.3. As of this writing no sequences improving our current lower bounds have been found. More running time or further improvement of our algorithm is necessary.

1.3 Lower Bounds for $s(d)$ for All $d \geq 1$

Recall that $s(d)$ (Definition 1.1.1) is the maximum $N$ for which there is an $N$-regular sequence with $d$ irregularities (Definition 1.1.2). In this section we construct a lower bound for $s(d)$ for all $d \geq 1$ by first proving that $N$-regular sequences with $d$ irregularities have certain geometric properties. We then use these geometric properties to
extend \( N \)-regular sequences with at most \( d \) irregularities to \( N + 1 \) regular sequences with at most \( d \) plus some fixed number of irregularities.

**Lemma 1.3.1.** If a sequence \( X = (x_1, x_2, \ldots, x_{N+d}) \) is \( N \)-regular with \( d \) irregularities then there is a sequence

\[
X' = (x'_1, x'_2, \ldots, x'_N)
\]

such that the sequence

\[
XX' = (x_1, \ldots, x_{N+d}, x'_1, \ldots, x'_N)
\]

is \((N + 1)\)-regular with \((d + N - 1)\) irregularities.

**Proof.** The worst case is that

\[
\{(\lfloor (N + 1)x_1 \rfloor, \lfloor (N + 1)x_2 \rfloor, \ldots, \lfloor (N + 1)x_{N+d} \rfloor) = \{i\}
\]

for some \( i \in \{0, 1, 2, \ldots, N\} \) so we set \( i = \lfloor (N + 1)x_1 \rfloor \) and pick \( X' \) such that

\[
\{(\lfloor (N + 1)x'_1 \rfloor, \lfloor (N + 1)x'_2 \rfloor, \ldots, \lfloor (N + 1)x'_{N+d} \rfloor) = \{0, 1, 2, \ldots, i - 1, i + 1, \ldots, N\}.
\]

This guarantees that

\[
XX' = (x_1, \ldots, x_{N+d}, x'_1, \ldots, x'_N)
\]

is an \((N + 1)\)-regular sequence with \((d + N - 1)\) irregularities. \(\Box\)
We now use Lemma 1.3.1 to construct our first lower bound for \( s(d) \).

**Theorem 1.3.2.** Given two natural numbers \( d' \) and \( N \), if

\[
s(d') \geq N
\]

then

\[
s(d) \geq \left\lfloor \sqrt{2d - 2d'} + \frac{(2N - 3)^2}{4} + \frac{3}{2} \right\rfloor
\]

for all \( d \geq d' \).

**Proof.** We first observe, by our definition of \( s \) (Definition 1.1.3), that \( s(d') \geq N \) is equivalent to the existence of an \( N \)-regular sequence with at most \( d' \) irregularities. Combining this with the results of Lemma 1.3.1 implies the existence of an \( (N + 1) \)-regular sequence with at most \( (d' + N - 1) \) irregularities. This, combined with our first observation, implies that

\[
s(d' + (N - 1)) \geq N + 1.
\]

By a recursive application we have that

\[
s(d' + \sum_{i=1}^{k} (N + i - 2)) \geq N + k
\]

for all \( k \in \mathbb{N} \). With a change of variables we have for \( d \in D = \{d' + \sum_{i=1}^{k} (N + i - 2) : k \in \mathbb{N} \} \), that

\[
s(d) \geq \sqrt{2d - 2d'} + \frac{(2N - 3)^2}{4} + \frac{3}{2}.
\]
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By definition $s(d)$ is a nondecreasing arithmetic function. Since its lower bound given in Inequality (1.3.3) takes on consecutive integer values for consecutive $d \in D$, we can extend Inequality (1.3.3) to all the natural numbers skipped in $D$ by taking the floor of

$$\sqrt{2d - 2d' + \frac{(2N - 3)^2}{4} + \frac{3}{2}}.$$ 

Thus

$$s(d) \geq \left\lfloor \sqrt{2d - 2d' + \frac{(2N - 3)^2}{4} + \frac{3}{2}} \right\rfloor$$

for all $d \geq d'$.

Corollary 1.3.3. For all $d \geq 1$,

$$s(d) \geq \left\lfloor \sqrt{2d + \frac{3473}{4} + \frac{3}{2}} \right\rfloor.$$ 

Proof. Definition 1.1.3 together with the 31-regular sequence with 1 irregularity demonstrated in [Ol17] implies that $s(1) \geq 31$. Setting $d' = 1$ and $N = 31$ in Theorem 1.3.2 gives us our result.

Lemma 1.3.1 assumes a worst case wherein all the $(N + 1)$-dilates of a sequence $X$ are clumped together into just one unit-length interval. This seems impossible in light of the fact that we also assume that the sequence $X$ is $N$-regular with $d$ irregularities and thus has at least one of its $N$-dilates in each unit-length interval from 0 to $N$. We can in fact show that this is impossible by demonstrating a better
worst-case of how our terms clump up. To do this we use Theorem 1.2.1, which shows that if a sequence \( X \) is \( N \)-regular with \( d \) irregularities then the \((N + 1)\)-dilates of \( X \) can not miss two consecutive unit-length intervals. This leads to an improvement over Corollary 1.3.3.

**Lemma 1.3.4.** If a sequence \( X = (x_1, x_2, \cdots, x_{N+d}) \) is \( N \)-regular with \( d \) irregularities then there is a sequence

\[
X' = (x'_1, x'_2, \cdots, x'_{\lceil \frac{N+1}{2} \rceil})
\]

such that the series

\[
XX' = (x_1, \cdots, x_{N+d}, x'_1, \cdots, x'_{\lceil \frac{N+1}{2} \rceil})
\]

is \((N + 1)\)-regular with at most \((d + \lceil \frac{N+1}{2} \rceil) - 1\) irregularities.

**Proof.** Theorem 1.2.1 shows that if a sequence is \( N \)-regular then the \((N + 1)\)-dilates of its terms will not miss any two consecutive unit-length intervals between 0 and \( N + 1 \). This implies that the worst possible case is that

\[
|\{(N + 1)x_1, (N + 1)x_2, \cdots, (N + 1)x_{N+d}\}| = (N + 1) - \left\lfloor \frac{N + 1}{2} \right\rfloor.
\]

So we pick the terms in our sequence

\[
X' = (x'_1, x'_2, \cdots, x'_{\lceil \frac{N+1}{2} \rceil})
\]
so that each of the at most $\left\lceil \frac{N+1}{2} \right\rceil$ unit-length intervals missed by all of the $(N+1)x_i$ contains at least one of the $(N+1)x'_i$. This guarantees that

$$XX' = (x_1, \cdots, x_{N+d}, x'_1, \cdots, x'_{\left\lceil \frac{N+1}{2} \right\rceil})$$

is an $(N+1)$-regular sequence with at most $(d + \left\lceil \frac{N+1}{2} \right\rceil - 1)$ irregularities. \qed

We now use Lemma 1.3.4 to construct a second and better lower bound for $s(d)$.

**Theorem 1.3.5.** Given two natural numbers $d'$ and $N$, if

$$s(d') \geq N$$

then

$$s(d) \geq \left\lfloor \sqrt{4d - 4d' - 1 + (N - 1)^2} + 1 \right\rfloor$$

for all $d \geq d'$

**Proof.** We first observe, by our definition of $s$ (Definition 1.1.3), that $s(d') \geq N$ is equivalent to the existence of an $N$-regular sequence with at most $d'$ irregularities. Combining this with the results of Lemma 1.3.4 implies the existence of an $(N+1)$-regular sequence with at most $(d' + \left\lceil \frac{N+1}{2} \right\rceil - 1)$ irregularities. This, combined with our first observation, implies that

$$s(d' + \left\lceil \frac{N+1}{2} \right\rceil - 1) \geq N + 1.$$
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By a recursive application we have that

\[ s(d' + \sum_{i=1}^{k} \left( \left\lfloor \frac{N + i}{2} \right\rfloor - 1 \right)) \geq N + k \]

for all \( k \in \mathbb{N} \). By a change of variables we have for

\[ d \in D = \{ d' + \sum_{i=1}^{k} \left( \left\lfloor \frac{N + i}{2} \right\rfloor - 1 \right) : k \in \mathbb{N} \} \tag{1.3.2} \]

that when \( k \) is even

\[ s(d) \geq \sqrt{4d - 4d' + (N - 1)^2} + 1 \]

and that when \( k \) is odd

\[ s(d) \geq \begin{cases} \sqrt{4d - 4d' - 1 + (N - 1)^2} + 1, & \text{when } N \text{ is even} \\ \sqrt{4d - 4d' + 1 + (N - 1)^2} + 1, & \text{when } N \text{ is odd} \end{cases} \]

We simplify this lower bound for \( s(d) \) by using the smallest right-side of the last three inequalities and so have that

\[ s(d) \geq \sqrt{4d - 4d' - 1 + (N - 1)^2} + 1. \tag{1.3.3} \]

By definition \( s(d) \) is an increasing arithmetic function. Since its lower bound given in Inequality (1.3.3) takes on consecutive integer values for consecutive \( d \in D \), we can extend Inequality (1.3.3) to all the natural numbers skipped in \( D \) by taking the floor of

\[ \sqrt{4d - 4d' - 1 + (N - 1)^2} + 1. \]
Thus

\[ s(d) \geq \left\lfloor \sqrt{4d - 4d'} - 1 + (N - 1)^2 + 1 \right\rfloor \]

for all \( d \geq d' \).

**Corollary 1.3.6.** For all \( d \geq 1 \),

\[ s(d) \geq \left\lfloor \sqrt{4d + 895} + 1 \right\rfloor. \]

**Proof.** Definition 1.1.3 together with the 31-regular sequence with 1 irregularity demonstrated in [Ol17] implies that \( s(1) \geq 31 \). Setting \( d' = 1 \) and \( N = 31 \) in Theorem 1.3.5 gives us our result.

\[ \] 

### 1.3.1 Improving the lower bound for \( s(d) \) for all \( d \geq 1 \)

The theorems in this section used lemmas that put an upper bound for the number of unit-length intervals in \([0, N+1)\) that are left empty by the \((N+1)\)-dilates of the terms from an \( N \)-regularity sequence but this upper bound seems rather loose in light of the sequence’s \( N \)-regularity. Perhaps our current upper bound for the highest number of empty unit-length intervals, \( \left\lceil \frac{N+1}{2} \right\rceil \), could be made smaller by examining the geometric structure of the \( n \)-regular subsequences of \( N \)-regular sequences? Perhaps there is a specific set of \( N \)-regular sequences whose dilated subsequences leave very few empty unit-length intervals?
1.4 Upper bounds for $s(d)$ for All $d \geq 1$

Recall that $s(d)$ (Definition 1.1.1) is the maximum $N$ for which there is an $N$-regular sequence with $d$ irregularities (Definition 1.1.2). In this section we show that a sequence $X$, which we assume to be $N$-regular with at most $d$ irregularities, contains a certain set of terms we call $P'$. The set $P'$ is used to show that if $N$ is allowed to be greater than some $d$-dependent value then $X$ is forced to have more than $d$ irregularities, which contradicts the assumption that $X$ has at most $d$ irregularities. The process of establishing this contradiction yields an upper bound for $s(d)$ for all $d \geq 1$.

1.4.1 Subsequence $P \subset X$ and subsequence $P' \subset P \subset X$

First we show that there is a subsequence

$$P \subset X$$

such that when we dilate the terms in $P$ by some natural number $n_0$ each dilated term is contained in a unit-length interval separated from the next dilated term by an empty unit-length interval. Next we show that there is a certain subsequence

$$P' \subset P \subset X$$
such that all paired terms in this subsequence $P'$ are separated by almost the same distance. Controlling the distances between the paired terms in $P'$ is key to forcing the contradiction that yields our upper bound for $s(d)$.

**Lemma 1.4.1.** Let $X$ be an $N$-regular sequence with at most $d$ irregularities. If $l$, $m$ and $n_0$ are natural numbers such that

$$l + 8md + 3 \leq n_0 \leq N$$

then there is a (not necessarily order preserving) subsequence of the first $n_0 + d$ terms of $X$,

$$P = (v_1 < w_1 < v_2 < w_2 < \cdots < v_{2md+1} < w_{2md+1}) \subset (x_1, x_2, \cdots, x_{n_0+d}) \subset X$$

such that for all $v_i, w_i \in P$

$$n_0v_i \in [l + 4i - 4, l + 4i - 3) \text{ and } n_0w_i \in [l + 4i - 2, l + 4i - 1).$$

**Proof.** The existence of such a subsequence follows directly from the definition of an $N$-regular sequence with $d$ irregularities (Definition 1.1.2). For all $n_0 \leq N$ the definition guarantees at least one of the $n_0$-dilated terms from $(n_0x_1, n_0x_2, \cdots, n_0x_{n_0+d}) \subset n_0X$ is in each of the half-open unit-length intervals from 0 to $n_0$. Since $n_0 \geq l + 8md + 3$, neither $n_0w_k$ nor any other $n_0$-dilated term from $P$ will fall into a unit-length interval beyond $[n_0 - 1, n_0)$. \qed
Corollary 1.4.2. Lemma 1.4.1 leads directly to the following bounds for the distance between and values of $P$’s paired terms $v_i, w_i$:

$$1 < n_0(w_i - v_i) < 3 \quad (1.4.1)$$

and

$$\frac{l}{n_0} \leq v_i < w_i < \frac{l + 8md + 3}{n_0}. \quad (1.4.2)$$

Proof. Inequalities (1.4.1) follow from $n_0v_i$ and $n_0w_i$ each being contained in one of the intervals $[l + 4i - 4, l + 4i - 3)$ and $[l + 4i - 2, l + 4i - 1)$, respectively. Inequalities (1.4.2) follow from the fact that $n_0v_1 \in [l, l + 1)$, $n_0w_k \in [l + 2, l + 3)$ and that all other $n_0v_i$ and $n_0w_i$ must be between $n_0v_1$ and $n_0w_k$. \qed

Lemma 1.4.3. Let $X$ be an $N$-regular sequence with at most $d$ irregularities. If $l, m$ and $n_0$ are natural numbers such that

$$l + 8md + 3 \leq n_0 \leq N$$

then there is a (not necessarily order preserving) subsequence of the first $n_0 + d$ terms of $X$,

$$P' = (y_1 < z_1 < y_2 < z_2 < \cdots < y_{d+1} < z_{d+1}) \subset (x_1, x_2, \cdots, x_{n_0+d}) \subset X,$$

such that for some positive natural number $r \leq 2m$,

$$1 + \frac{r - 1}{m} < n_0(z_i - y_i) \leq 1 + \frac{r}{m}$$
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holds for all paired terms $y_i, z_i \in P'$.

Proof. By Lemma 1.4.1 there is the (not necessarily order preserving) subsequence of the first $n_0 + d$ terms of $X$,

$$P = (v_1 < w_1 < v_2 < w_2 < \cdots < v_{2md+1} < w_{2md+1}) \subset (x_1, x_2, \cdots, x_{n_0+d}) \subset X.$$ 

From Corollary 1.4.2 we know that the interval $(1, 3)$ contains the value

$$n_0(w_i - v_i)$$

for each of the $2md + 1$ pairs of $w_i, v_i$ in $P$. By the pigeon-hole principle we have that there is a (not necessarily order preserving) subsequence

$$P' = (y_1 < z_1 < y_2 < z_2 < \cdots < y_{d+1} < z_{d+1}) \subset P$$

made up of $d + 1$ of the $2md + 1$ pairs of $v_i, w_h$ from $P$ such that for some positive natural number $r \leq 2m$,

$$1 + \frac{r - 1}{m} < n_0(z_i - y_i) \leq 1 + \frac{r}{m}$$

for $y_i, z_i \in P'$.

\[ \square \]

1.4.2 Dilated distances between pairs $y_i, z_i \in P'$

Next we find an $n_1$ greater than the $n_0$ so that when pairs of terms $z_i, y_i$ from the subsequence $P'$ described in Lemma 1.4.3 are dilated by $n_1$ their differences
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$n_1(z_i - y_i)$ are all just slightly greater than 3. This is crucial to controlling the number of unit-length intervals between dilated pairs of $z_i, y_i$ and controlling this number of unit-length intervals is what gives us our upper bound for $s(d)$.

**Lemma 1.4.4.** Let $X$ be an $N$-regular sequence with at most $d$ irregularities. If

1. $l, m$ and $n_0$ are natural numbers such that $l + 8md + 3 \leq n_0 \leq N$; and

2. the $d + 1$ paired terms $y_i, z_i \in P' \subset X$ are as given by Lemma 1.4.3,

then there is some natural number $n_1$, where $n_0 < n_1 \leq 3n_0 + (l + 8md + 3)$, such that

$$3 + \frac{l + 8md + 3}{n_0} \leq n_1(z_i - y_i) \leq \left(1 + \frac{1}{m}\right) \left(3 + \frac{l + 8md + 3}{n_0}\right) + \frac{3}{n_0}$$

holds for all $y_i, z_i \in P'$.

**Proof.** By Lemma 1.4.3 there is a (not necessarily order preserving) subsequence of the first $n_0 + d$ terms of $X$,

$$P' = (y_1 < z_1 < y_2 < z_2 < \cdots < y_{d+1} < z_{d+1}) \subset (x_1, x_2, \cdots, x_{n_0+d}) \subset X,$$

such that for positive some natural number $r \leq 2m$

$$1 + \frac{r - 1}{m} < n_0(z_i - y_i) \leq 1 + \frac{r}{m} \quad (1.4.3)$$
holds for all \( y_i, z_i \in P' \). Multiplying Inequalities (1.4.3) by

\[
\frac{m}{m + r - 1} \left( 3 + \frac{l + 8md + 3}{n_0} \right)
\]

gives us

\[
3 + \frac{l + 8md + 3}{n_0} < \frac{m}{m + r - 1} (3n_0 + l + 8md + 3) (z_i - y_i) \leq \left( 1 + \frac{1}{m + r - 1} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right)
\]

for all \( y_i, z_i \in P' \). The the right-side inequality in (1.4.3) implies that \( z_i - y_i \leq \frac{1}{n_0} (1 + \frac{r}{m}) \). This in turn implies that when the ceiling function is applied to the middle of the proceeding string of inequalities we have that

\[
3 + \frac{l + 8md + 3}{n_0} < \left[ \frac{m}{m + r - 1} (3n_0 + l + 8md + 3) \right] (z_i - y_i) \leq \left( 1 + \frac{1}{m + r - 1} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{1}{n_0} \left( 1 + \frac{r}{m} \right)
\]

(1.4.4)

for all \( y_i, z_i \in P' \). Now set

\[
n_1 = \left[ \frac{m}{m + r - 1} (3n_0 + l + 8md + 3) \right]
\]

(1.4.5)

and since \( 1 \leq r \leq 2m \) it follows that

\[
n_0 \leq n_1 \leq 3n_0 + (l + 8md + 3),
\]

\[
1 + \frac{1}{m + r - 1} \leq 1 + \frac{1}{m}
\]
and

\[ \frac{1}{n_0} \left( 1 + \frac{r}{m} \right) \leq \frac{3}{n_0}. \]

Combining the last two inequalities with Inequalities (1.4.4) and Equation (1.4.5) gives us

\[ 3 + \frac{l + 8md + 3}{n_0} < n_1(z_i - y_i) \leq \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \]

for all \( y_i, z_i \in P' \).

\[ \square \]

1.4.3 Forcing \( d + 1 \) irregularities

In this final subsection of our construction of an upper bound for \( s(d) \) we find an \( n_2 \geq n_1 \) so that for each of the \( d + 1 \) pairs \( y_i, z_i \) in \( P' \) there is a subset

\[ N_{y_i,z_i} \subset \{ n_1, n_1 + 1, \ldots, n_2 - 1, n_2 \} \]

with density greater than \( \frac{d}{d+1} \) such that for each \( n \in N_{y_i,z_i} \) one of the unit-length intervals

\[ [\lceil ny_i \rceil, \lceil ny_i \rceil + 1), [\lceil ny_i \rceil + 1, \lceil ny_i \rceil + 2), \ldots, [\lceil nz_i \rceil, \lceil nz_i \rceil + 1) \]

contains, for a sufficiently long sequence \( X \), two terms from the first \( n + d \) terms of the \( n \)-dilated sequence \( nX \).

We then show how this forces there to be a natural number \( n' \) between \( n_1 \) and
n_2 so that for each of the \( d + 1 \) pairs \( y_i, z_i \), one of the unit-length intervals 
\[ ([n'y_i], [n'y_i] + 1), ([n'y_i] + 1, [n'y_i] + 2), \ldots, ([n'z_i], [n'z_i] + 1). \]
contains two terms from the first \( n' + d \) terms of the sequence \( n'X \).

Having two such terms in each of the \( d + 1 \) unit-length intervals means, by Definition 1.1.2, that the subsequence 
\[ (x_1, \ldots, x_{n'+d}) \]
of \( X \) has \( d + 1 \) irregularities. Having \( d + 1 \) irregularities contradicts the assumption that \( X \) is an \( N \)-regular sequence with at most \( d \) irregularities.

Thus \( n' \) gives us our upper bound for \( s(d) \).

**Theorem 1.4.5.** Let \( X \) be an \( N \)-regular sequence with at most \( d \) irregularities. If

1. \( l, m \) and \( n_0 \) are natural numbers such that \( l + 8md + 3 \leq n_0 \leq N \); and

2. \( l = 351d^2, m = 35d \) and \( n_0 = 8267d^3 \); and

3. \( n_1 \) and the \( d + 1 \) paired terms \( y_i, z_i \in P' \subset X \) are as given by Lemma 1.4.4;

and

4. \( n_2 = n_1 + 311d^2 \),

then for each of the \( d + 1 \) paired terms \( y_i, z_i \) some set of more than \( \frac{d}{d+1}(n_2 - n_1) \) of the natural numbers \( n \) between \( n_1 \) and \( n_2 \) force at least two points from the first \( n + d \)
terms of the \( n \)-dilated sequence \( nX \) to be contained in one of the 4 intervals

\[
([ny_i], [ny_i] + 1), ([ny_i] + 1, [ny_i] + 2), ([ny_i] + 2, [nz_i]), ([nz_i], [nz_i] + 1).
\]

Proof. We begin by dilating each of the \( y_i \) from our subsequence

\[
P' = (y_1 < z_1 < y_2 < z_2 < \cdots < y_{d+1} < z_{d+1})
\]

by the smallest natural number \( n^*_i \geq n_1 \) so that \( n^*_i \cdot y_i \) is immediately to the left of a natural number or, more rigorously put, the smallest natural number \( n^*_i \geq n_1 \) so that

\[
[n^*_i \cdot y_i + y_i] = [n^*_i \cdot y_i] + 1,
\]

which implies that

\[
n^*_i \leq n_1 + \left\lceil \frac{1}{y_i} \right\rceil.
\]

Next we show that if a pair \( y_i, z_i \in P' \) is dilated by the natural number \( n^*_i \) then the two intervals containing \( n^*_i \cdot y_i \) and \( n^*_i \cdot z_i \),

\[
([n^*_i \cdot y_i], [n^*_i \cdot y_i] + 1) \text{ and } ([n^*_i \cdot z_i], [n^*_i \cdot z_i] + 1),
\]

are separated by 3 unit-length intervals or, more concisely put, that

\[
[n^*_i \cdot z_i] - [n^*_i \cdot y_i] \geq 4.
\]
In Lemma 1.4.4 we proved that

\[ n_1(z_i - y_i) \geq 3 + \frac{l + 8md + 3}{n_0} \]

for all \( y_i, z_i \in P' \) and, since \( n_i^* \geq n_1 \),

\[ n_i^*(z_i - y_i) \geq 3 + \frac{l + 8md + 3}{n_0} . \]

Rearranging this last inequality and then taking the floor of both sides gives us that

\[ \lfloor n_i^* \cdot z_i \rfloor - 3 \geq \lfloor n_i^* \cdot y_i + \frac{l + 8md + 3}{n_0} \rfloor . \] (1.4.8)

Lemma 1.4.3 picked the terms of \( P' \) from Lemma 1.4.1’s \( P \), so by Corollary 1.4.2 we have that

\[ y_i < z_i < \frac{l + 8md + 3}{n_0} \] (1.4.9)

for all \( y_i, z_i \in P' \subset P \). This combined with Inequality (1.4.8) implies the looser inequality

\[ \lfloor n_i^* \cdot z_i \rfloor - 3 \geq \lfloor n_i^* \cdot y_i + y_i \rfloor . \]

Combining this looser inequality with Equation (1.4.6) gives us

\[ \lfloor n_i^* \cdot z_i \rfloor - \lfloor n_i^* \cdot y_i \rfloor \geq 4 \]

as desired.
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Next we show that for each of the $d + 1$ pairs $y_i, z_i \in P'$ more than

$$\frac{d}{d+1}(n_2 - (n_i^* + 1))$$

of the natural numbers between $n_i^* + 1$ and $n_2$ dilate the pair so that the two intervals containing $ny_i$ and $nz_i$,

$$\left[\lfloor ny_i \rfloor, \lfloor ny_i \rfloor + 1 \right) \text{ and } \left[\lfloor nz_i \rfloor, \lfloor nz_i \rfloor + 1 \right),$$

are themselves separated by 2 unit-length intervals or, equivalently, so that

$$\lfloor (n_i^* + 1)z_i \rfloor - \lfloor (n_i^* + 1)y_i \rfloor = 3.$$ 

Put concisely, we show that

$$\min_{1 \leq i \leq d+1} \frac{\# \{ n : (n_i^* + 1 \leq n \leq n_2) \wedge ([nz_i] - [ny_i] = 3) \}}{n_2 - n_1} > \frac{d}{d+1}.$$ 

We do this by finding a large natural number $k$ such that whenever an $ny_i$ is immediately to the left of an integer (but before $ny_i$ is immediately to the left of the next integer for some larger value of $n$) the next $k$ dilated values of the pair $y_i$ and $z_i$ are separated by two unit-length intervals. Put more concisely, if for some $n \geq n_i^*$ we have that

$$\lfloor ny_i \rfloor = \lfloor (n + 1)y_i \rfloor - 1,$$
then how large can a natural number $k$ be so that

$$\lfloor (n+1)z_i \rfloor - \lfloor (n+1)y_i \rfloor = \lfloor (n+2)z_i \rfloor - \lfloor (n+2)y_i \rfloor = \cdots$$

$$\cdots = \lfloor (n+k)z_i \rfloor - \lfloor (n+k)y_i \rfloor = 3? \quad (1.4.10)$$

This will hold for $k$’s obeying the slightly stricter condition that

$$ky_i + (n+k)(z_i - y_i) < 4. \quad (1.4.11)$$

Assuming that $n + k \leq n_2$, which means we will have to throw away some of the $n$’s between $n_i^* + 1$ and $n_2$ below, this last inequality follows from the stricter inequality

$$ky_i + \frac{n_2}{n_1} [n_1(z_i - y_i)] < 4.$$

Which, using the conclusion of Theorem 1.4.4 to replace $n_1(z_i - y_i)$, follows from the stricter inequality

$$ky_i + \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] < 4.$$

Which is equivalent to

$$k < \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\}.$$

So, since we were looking for a large natural number $k$ that satisfies all the previous
inequalities, we set
\[
 k = \left\lceil \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} \right\rceil - 1. \quad (1.4.12)
\]

There are at most \( \left\lceil \frac{1}{y_i} \right\rceil \) values of \( n \) for which \( ny_i \) is between two integers. We will divide \( k \) by this amount. Also (to insure, as we assume above, that \( n + k \leq n_2 \) for all \( n \) for which \( ny_i \) is immediately to the left of an integer) we throw away \( \left\lceil \frac{1}{y_i} \right\rceil \) of the \( n \)'s immediately less than \( n_2 \). Thus we have for at least
\[
 \frac{k}{\left\lceil \frac{1}{y_i} \right\rceil} (n_2 - \left\lfloor \frac{1}{y_i} \right\rfloor) - (n_i^* + 1)
\]
of the natural numbers \( n \) between \( n_i^* + 1 \) and \( n_2 - \left\lfloor \frac{1}{y_i} \right\rfloor \) that all of the pairs \( ny_i, nz_i \) are separated by 2 unit-length intervals. Put concisely, we have that
\[
 \min_{1 \leq i \leq d+1} \frac{\#\{ n : (n_i^* + 1 \leq n \leq n_2) \land (\lfloor nz_i \rfloor - \lfloor ny_i \rfloor = 3) \}}{n_2 - n_1} \geq \min_{1 \leq i \leq d+1} \frac{k}{\left\lceil \frac{1}{y_i} \right\rceil} \frac{n_2 - \left\lfloor \frac{1}{y_i} \right\rfloor - n_i^*}{n_2 - n_1}. \quad (1.4.13)
\]

By the upper bound for \( n_i^* \) from Inequality (1.4.7) and the lower bound for \( y_i \) from Corollary 1.4.2 we have that
\[
 \min_{1 \leq i \leq d+1} \frac{k}{\left\lceil \frac{1}{y_i} \right\rceil} \frac{n_2 - \left\lfloor \frac{1}{y_i} \right\rfloor - n_i^*}{n_2 - n_1} \geq \min_{1 \leq i \leq d+1} \frac{k}{\left\lceil \frac{1}{y_i} \right\rceil} \frac{n_2 - \left\lfloor \frac{n_0}{T} \right\rfloor - (n_1 + \left\lfloor \frac{n_0}{T} \right\rfloor)}{n_2 - n_1}.
\]

Using Equation (1.4.12) to substitute for \( k \) together with the fact that \( \left\lfloor \frac{n_0}{T} \right\rfloor \leq \frac{n_0}{T} + 1 \)
gives us that

\[
\min_{1 \leq i \leq d+1} \left\{ \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} - 1 \right\} \cdot \left( 1 - \frac{2(n_0 + l)}{l(n_2 - n_1)} \right).
\]

Confining our focus to the left two factors directly above together with the fact that \(\left\lceil \frac{1}{y_i} \right\rceil \leq \frac{1}{y_i} + 1\) gives us that

\[
\min_{1 \leq i \leq d+1} \left\{ \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} - 1 \right\} \cdot \frac{1}{\left\lceil \frac{1}{y_i} \right\rceil} \geq
\]

\[
\min_{1 \leq i \leq d+1} \left\{ \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} - \frac{y_i}{y_i + 1} \right\} \cdot \frac{1}{\left\lceil \frac{1}{y_i} \right\rceil} \geq
\]

Since \(y_i \in P\) we have by Corollary 1.4.2 that \(y_i \leq \frac{l + 8md + 3}{n_0}\) which gives us that

\[
\min_{1 \leq i \leq d+1} \left\{ \frac{1}{y_i} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} - \frac{y_i}{y_i + 1} \right\} \geq
\]

\[
\frac{n_0}{l + 8md + 3 + n_0} \left\{ 4 - \frac{n_2}{n_1} \left[ \left( 1 + \frac{1}{m} \right) \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] \right\} - \frac{l + 8md + 3}{l + 8md + 3 + n_0}.
\]
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Using first that \( n_2 = n_1 + 190d^2 \) from Premise (4) and then The lower bound \( n_1 \geq n_0 \) from Lemma 1.4.4 give us that

\[
\frac{n_2}{n_1} \geq 1 + \frac{190d^2}{n_0}
\]

So now, combining the chain of inequalities all the way back to (1.4.13) gives us that

\[
\min_{1 \leq i \leq d+1} \#\{ n : (n_i^* + 1 \leq n \leq n_2) \land ([nz_i] - [ny_i] = 3) \} \geq \frac{n_2 - n_1}{n_0} \left\{ 4 - \left( 1 + \frac{190d^2}{n_0} \right) \left[ \left( 3 + \frac{l + 8md + 3}{n_0} \right) + \frac{3}{n_0} \right] - \frac{l + 8md + 3}{l + 8md + 3 + n_0} \left( 1 - \frac{n_0 + l}{l(n_2 - n_1)} \right) \right\}.
\]

By substituting in the values from Premise (2) we have that

\[
\min_{1 \leq i \leq d+1} \#\{ n : (n_i^* + 1 \leq n \leq n_2) \land ([nz_i] - [ny_i] = 3) \} \geq \frac{n_2 - n_1}{1724293890(1914d^3 + 150d^2 + 1)} \cdot \frac{(20710d^2 - 2871d - 109)}{d^4} \cdot \frac{(159357726d^5 - 57282192d^4 - 1389390d^3 - 264027d^2 - 8381d - 95)}{1724293890(1914d^3 + 150d^2 + 1)}
\]

and since

\[
\frac{(159357726d^5 - 57282192d^4 - 1389390d^3 - 264027d^2 - 8381d - 95)}{1724293890(1914d^3 + 150d^2 + 1)} \cdot \frac{(20710d^2 - 2871d - 109)}{d^4} > \frac{d}{d+1}.
\]
for all $d \geq 1$ we can combine the last two inequalities and have that

$$\min_{1 \leq i \leq d+1} \frac{\# \{ n : (n_i^* + 1 \leq n \leq n_2) \land ([nz_i] - [ny_i] = 3) \}}{n_2 - n_1} > \frac{d}{d+1}.$$ 

Now, since we assume $X$ to be $N$-regular (with at most $d$ irregularities) we have that the 5 intervals

$$[\lfloor n_i^* \cdot y_i \rfloor, \lfloor n_i^* \cdot y_i \rfloor + 1), [\lfloor n_i^* \cdot y_i \rfloor + 1, \lfloor n_i^* \cdot y_i \rfloor + 2), [\lfloor n_i^* \cdot y_i \rfloor + 2, \lfloor n_i^* \cdot y_i \rfloor + 3), [\lfloor n_i^* \cdot y_i \rfloor + 3, \lfloor n_i^* \cdot z_i \rfloor), [\lfloor n_i^* \cdot z_i \rfloor, \lfloor n_i^* \cdot z_i \rfloor + 1).$$

must each contain an $n_i^*$-dilated term from the first $n_i^* + d$ terms of $n_i^*X$, let’s call them

$$n_i^* \cdot y_i < n_i^* \cdot x_i^{(a)} < n_i^* \cdot x_i^{(b)} < n_i^* \cdot x_i^{(c)} < n_i^* \cdot z_i.$$

But then Inequality (1.4.3) implies that for more than $\frac{d}{d+1}(n_2 - n_1)$ of the natural numbers $n$ between $n_1$ and $n_2$ the 4 intervals

$$[\lfloor ny_i \rfloor, \lfloor ny_i \rfloor + 1), [\lfloor ny_i \rfloor + 1, \lfloor ny_i \rfloor + 2), [\lfloor ny_i \rfloor + 2, \lfloor nz_i \rfloor), [\lfloor nz_i \rfloor, \lfloor nz_i \rfloor + 1).$$

contain the 5 $n$-dilated terms

$$ny_i < nx_i^{(a)} < nx_i^{(b)} < nx_i^{(c)} < nz_i.$$

This, by the pigeonhole principle, implies that for more than $\frac{d}{d+1}(n_2 - n_1)$ of the natural numbers $n$ between $n_1$ and $n_2$ two of these 5 terms (which are all from the
first \( n + d \) terms of the \( n \)-dilated sequence \( nX \) are contained in one of the 4 intervals
\[
\left[ \lfloor ny_i \rfloor, \lfloor ny_i \rfloor + 1 \right), \left[ \lfloor ny_i \rfloor + 1, \lfloor ny_i \rfloor + 2 \right), \left[ \lfloor ny_i \rfloor + 2, \lfloor nz_i \rfloor \right), \left[ \lfloor nz_i \rfloor, \lfloor nz_i \rfloor + 1 \right).
\]

\[\Box\]

**Corollary 1.4.6.** There is an \( n' \) between \( n_1 \) and \( n_2 \) such that for all the \( d + 1 \) of pairs \( y_i, z_i \) one of the 4 intervals
\[
\left[ \lfloor n'y_i \rfloor, \lfloor n'y_i \rfloor + 1 \right), \left[ \lfloor n'y_i \rfloor + 1, \lfloor n'y_i \rfloor + 2 \right), \left[ \lfloor n'y_i \rfloor + 2, \lfloor n'z_i \rfloor \right), \left[ \lfloor n'z_i \rfloor, \lfloor n'z_i \rfloor + 1 \right).
\]
contains two of the first \( n' + d \) terms from the \( n' \)-dilated sequence \( n'X \).

**Proof.** This corollary’s proof is by contradiction. We assume that no such \( n' \) exists. This implies that for any single \( n \) between \( n_1 \) and \( n_2 \) there are at most \( d \) pairs \( y_i, z_i \) such that one of the 4 intervals
\[
\left[ \lfloor ny_i \rfloor, \lfloor ny_i \rfloor + 1 \right), \left[ \lfloor ny_i \rfloor + 1, \lfloor ny_i \rfloor + 2 \right), \left[ \lfloor ny_i \rfloor + 2, \lfloor nz_i \rfloor \right), \left[ \lfloor nz_i \rfloor, \lfloor nz_i \rfloor + 1 \right).
\]
contains two of the first \( n + d \) terms from the \( n \)-dilated sequence \( nX \). But this implies that if we sum over all \( n \) between \( n_1, n_2 \) then the total amount of times that, for a pair \( y_i, z_i \), one of the 4 intervals
\[
\left[ \lfloor ny_i \rfloor, \lfloor ny_i \rfloor + 1 \right), \left[ \lfloor ny_i \rfloor + 1, \lfloor ny_i \rfloor + 2 \right), \left[ \lfloor ny_i \rfloor + 2, \lfloor nz_i \rfloor \right), \left[ \lfloor nz_i \rfloor, \lfloor nz_i \rfloor + 1 \right).
\]
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contains two of the first \( n + d \) terms from the \( n \)-dilated sequence \( nX \) is at most

\[ d(n_2 - n_1). \]

But then this implies that for any single pair \( y_i, z_i \) the average amount of \( n \) between \( n_1 \) and \( n_2 \) for which one of the 4 intervals

\[ ([ny_i], [ny_i] + 1), ([ny_i] + 1, [ny_i] + 2), ([ny_i] + 2, [nz_i]), ([nz_i], [nz_i] + 1). \]

contains two of the first \( n + d \) terms from the \( n \)-dilated sequence \( nX \) is at most

\[ \frac{d}{d + 1}(n_2 - n_1) \]

which, since an average can not be less than all the values being average, contradicts Theorem 1.4.5’s result that for strictly more than \( \frac{d}{d + 1}(n_2 - n_1) \) of the \( (n_2 - n_1) \) natural numbers \( n \) between \( n_1 \) and \( n_2 \) one of the 4 intervals

\[ ([ny_i], [ny_i] + 1), ([ny_i] + 1, [ny_i] + 2), ([ny_i] + 2, [nz_i]), ([nz_i], [nz_i] + 1). \]

contains two of the first \( n + d \) terms from the \( n \)-dilated sequence \( nX \). \( \square \)

Finally we put the pieces together to form our upper bound on \( s(d) \).

Corollary 1.4.7.

\[ s(d) < 24801d^3 + 942d^2 + 3 \]

Proof. Given a sequence \( X \) if there is an \( n' \) such that for some \( d + 1 \) of pairs \( y_i, z_i \subset X \)
one of the 4 intervals
\[
[[n'y_i, [n'y_i + 1), [[n'y_i + 1, [n'y_i + 2), [[n'y_i + 2, [n'z_i), [[n'z_i, [n'z_i + 1).
\]
contains two of the first \(n' + d\) terms from the \(n'\)-dilated sequence \(n'X\) then, using
the terminology of Definition (1.1.2), the sequence \(X\) has at least \(d + 1\) irregularities.

Now for a given \(d\) Theorem 1.4.5 along with Corollary 1.4.6 tell us that if a
sequence \(X\) has
\[
24801d^3 + 942d^2 + d + 3
\]
terms\(^4\) then there must be an \(n'\) guaranteeing that the sequence \(X\) has at least \(d + 1\)
irregularities. But this contradicts Theorem 1.4.5’s assumption that the \(X\) has as
most \(d\) irregularities. Thus, again using the terminology of Definition (1.1.2), if \(X\) is
any \(N\)-regular sequence with at most \(d\) irregularities then
\[
N + d < 24801d^3 + 942d^2 + d + 3.
\]
This, using the notation from Definition 1.1.3, is equivalent to
\[
s(d) < 24801d^3 + 942d^2 + 3.
\]
\(^4\)24801d^3 + 942d^2 + d + 3 comes from adding \(d\) to the largest possible value of Theorem 1.4.5’s \(n_2\)
1.5 A Combinatorial Game Based on Regular Sequences

We define the following game based on regular sequence distribution: Two players, $a$ and $b$, alternate picking points from the interval $[0,1)$. A player picks the $N^{th}$ point in the game so that for $0 \leq n \leq N$ each of the intervals

$$\left[ \frac{n \cdot n + 1}{N}, \frac{n + 1}{N} \right)$$

contains either this $N^{th}$ point or one of the $N - 1$ points already chosen by both players. If a player can not pick such a point then they lose.

The notation

$$a_N : [x, y)$$

stands for player $a$, on the $N^{th}$ turn, picking any one point from the interval $[x, y)$. *Mutatis mutandis*, the same goes for player $b$.

The player $a$ can force the player $b$ to lose on $b$'s second move. We now demonstrate the three possible $b_2$ responses to a winning $a_1$ opening and see how they each result in the loss of the game when $b$ tries to pick the $4^{th}$ point:

1. $a_1 : \left[ \frac{1}{4}, \frac{1}{3} \right) \rightarrow b_2 : \left[ \frac{1}{2}, \frac{2}{3} \right) \rightarrow a_3 : \left[ \frac{2}{3}, \frac{3}{4} \right) \rightarrow b_4 :！$

Player $b$ finds intervals $[0, \frac{1}{4})$ and $[\frac{3}{4}, 0)$ both empty when trying to pick the
4th point and thus loses!

2.

\[ a_1 : \left[ \frac{1}{4}, \frac{1}{3} \right] \rightarrow b_2 : \left[ \frac{2}{3}, \frac{3}{4} \right] \rightarrow a_3 : \left[ \frac{1}{3}, \frac{1}{2} \right] \rightarrow b_4 :! \]

Again player b finds intervals \([0, \frac{1}{4})\) and \([\frac{3}{4}, 0)\) both empty when trying to pick the 4th point and thus loses!

3.

\[ a_1 : \left[ \frac{1}{4}, \frac{1}{3} \right] \rightarrow b_2 : \left[ \frac{3}{4}, 1 \right] \rightarrow a_3 : \left[ \frac{1}{3}, \frac{1}{2} \right] \rightarrow b_4 :! \]

In this case player b finds intervals \([0, \frac{1}{4})\) and \([\frac{1}{2}, \frac{3}{4})\) both empty when trying to pick the 4th point and thus loses!

We leave the reader with two questions.

What is the Grundy value of this game?

Can the rules of this game be altered (for example, in the manner we altered Definition 1.1.1 to have Definition 1.1.2) so that the first player cannot force a win on his or her second move?

1.6 Sagemath Code for Algorithms

1.6.1 The Pick-the-Middle algorithm without sequence storage

This version takes up less memory but more computations than the version from Section 1.6.2.
1.6.2 The Pick-the-Middle algorithm with sequence storage

This version takes fewer computations but more memory that the version from Section 1.6.1.
sage: Max_d= 
sage: dMaxSeqN=[] 
sage: seqStore=[[1,0,[((i-1)/p)]] for i in [1..p]] 
sage: for d in [0..Max_d, step=1]: 
... dMaxSeq=[] 
... s_d=0 
... for y in [1..p]: 
...   redistributed=true 
...   while redistributed==True: 
...     missingIntervals=list(set([0..(seqStore[y-1][0])])
- set([floor((seqStore[y-1][0]+1)*x)
for x in seqStore[y-1][2]])) 
...     if (len(missingIntervals)-1)+seqStore[y-1][1]<=d: 
...       for j in [0..(len(missingIntervals)-1)]: 
...         leftOfHole=max([x for x in seqStore[y-1][2]
if (seqStore[y-1][0]+1)*x <\ 
missingIntervals[j]]+[0]) 
...         rightOfHole=min([x for x in seqStore[y-1][2]
if (seqStore[y-1][0]+1)*x >=\ 
(missingIntervals[j]+1)]+[1]) 
...         seqStore[y-1][2]=\ 
seqStore[y-1][2]+[((leftOfHole+rightOfHole)/2)] 
...       seqStore[y-1][1]=\ 
seqStore[y-1][1]+(len(missingIntervals)-1) 
...     seqStore[y-1][0]=seqStore[y-1][0]+1 
...   else: 
...     if seqStore[y-1][0]>s_d: 
...       dMaxSeq=seqStore[y-1][2] 
...       s_d=seqStore[y-1][0] 
...       redistributed=False 
...     dMaxSeqN=dMaxSeqN+[(d,s_d)] 
sage: print dMaxSeqN

1.6.3  The "Is There an \( N \)-Regular Sequence with at Most \( d \) Irregularities?" algorithm
sage: import itertools
sage: d=

sage: def fareySet(m,n):
  ... F=Set([])
  ... for i in [m..n]:
  ...   F=F.union(Set([0..((i-1)/i), step=(1/i)]))
  ... return F
sage: def nRegular(n,Xn):
  ... if len(set([floor(n*x1) for x1 in Xn]))==n:
  ...   return True
  ... else:
  ...   return False
sage: def findSequence(n,Xn):
  ... if nRegular(n,Xn):
  ...   if n==N:
  ...     print Xn,'is',N,'-Regular with at most'
  ...     ',d,'irregularities found with m=',m
  ...     exit()
  ... else:
  ...   for x in F.difference(Set(Xn)):
  ...     findSequence(n+1,Xn+[x])

sage: F=fareySet(1,N)

sage: for Xd in itertools.combinations(F,(d+1)):
  ...   findSequence(1,list(Xd))

sage: print 'There is no',N,'-Regular sequence with at most'
  ... ',d,'irregularities with m=',m
Chapter 2

Discrete Chebyshev Type Polynomials

2.1 Introduction and History

What is the maximum lead coefficient of a degree $d$ polynomial $Q(x) \in \mathbb{R}[x]$ if we insist that $|Q(1)|, |Q(2)|, \ldots, |Q(k)|$ are all less than or equal to 1? A polynomial with such a lead coefficient can be thought of as a discrete analog to a Chebyshev $T$-polynomial. The Chebyshev $T$-polynomial $T_d(x)$ is the degree $d$ polynomial with maximum lead coefficient when bound between $-1$ and 1 for all $x$ in the interval $[-1, 1]$.

The reciprocal of such a maximum lead coefficient turns out to be the minimum, over all sets of $d$ points, of the maximum, over all $x \in \{1, 2, \ldots, k\}$, of the product of distances between $d$ points and $x$. This combinatorial geometric view of the problem is discussed in Section 2.3.

This question of finding such a maximum lead coefficient first came to our at-
tention in connection with Kevin O’Bryant’s work on the long standing problem of finding the maximum density of sets that avoid arithmetic progressions [Ro, Sz].

O’Bryant’s paper [OB] develops a technique for constructing a lower bound on this density. His technique involves building sets that avoid having $k$-term arithmetic progressions map into a certain interval contained in the image of a degree $2d$ polynomial. The problem of finding the maximum lead coefficient of such a polynomial was suggested by O’Bryant.

**Definition 2.1.1.** As notation we use 

$$[k] = \{1, 2, 3, \ldots, k\}.$$ 

More generally we use 

$$[x_k] = \{x_1, x_2, x_3, \ldots, x_k\}$$


to denote an increasing set of $k$ real numbers.

In general we call polynomials $L$-polynomials if they have maximum possible lead coefficient when their absolute value is bounded on some finite set. More specifically

**Definition 2.1.2.** Given $k > d \geq 1$ and $[x_k] = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}$ we let 

$$L_{d, [x_k]}(x)$$
denote the unique degree $d$ polynomial with maximum lead coefficient such that

$$|L_{d,[x_k]}(x)| \leq 1$$

for all $x \in [x_k]$.

We prove that this polynomial is unique in Theorem 2.4.3.

**Theorem 2.1.3.** Given a $k$-term arithmetic progression

$$[x_k] = \{x_1, x_1 + \Delta, ..., x_1 + (k-1)\Delta\},$$

for every $d < k$ there is a unique polynomial

$$|L_{d,[x_k]}(x)| = |a_dx^d + \cdots + a_1x + a_0|$$

with maximum lead coefficient $a_d$ such that for all $x \in [x_k]$ we have that

$$|L_{d,[x_k]}(x)| \leq 1.$$

Moreover, we have for
$d = 1$, $a_1 = \frac{M}{\Delta} \cdot \frac{2}{k-1}$

$d = 2$, $a_2 = \frac{M}{\Delta^2} \cdot 8 \cdot \begin{cases} 
\frac{1}{(k-1)^2} & \text{if } k \equiv 1 \mod 2 \\
\frac{1}{k(k-2)} & \text{if } k \equiv 0 \mod 2
\end{cases}$

$d = 3$, $a_3 = \frac{M}{\Delta^3} \cdot \frac{32}{k-1} \cdot \begin{cases} 
\frac{1}{(k-1)^2} & \text{if } k \equiv 1 \mod 4 \\
\frac{1}{k(k-2)} & \text{if } k \equiv 2 \mod 4 \\
\frac{1}{(k+1)(k-3)} & \text{if } k \equiv 3 \mod 4 \\
\frac{1}{k(k-2)} & \text{if } k \equiv 0 \mod 4
\end{cases}$

$d = 4$, $a_4 = \frac{M}{\Delta^4} \cdot \begin{cases} 
\min_{x \in I} \left\{ \frac{8}{x^2(k-1)^2 - 4x^4} \right\} & \text{if } k \equiv 1 \mod 2 \\
\min_{x \in H} \left\{ -32 \right\} & \text{if } k \equiv 0 \mod 2
\end{cases}$

with $I = \left\{ \left\lfloor \frac{k-1}{2\sqrt{2}} \right\rfloor, \left\lceil \frac{k-1}{2\sqrt{2}} \right\rceil \right\}$ and $H = \left\{ \left\lfloor \frac{k-1}{2\sqrt{2}} + \frac{1}{2} \right\rfloor - \frac{1}{2}, \left\lfloor \frac{k-1}{2\sqrt{2}} + \frac{1}{2} \right\rfloor - \frac{1}{2} \right\}$. For $k > d \geq 5$ we can use the algorithm described in Section 2.5.5 to find $a_d$.

2.1.1 A brief summary

In Section 2.2 we discuss the connection between $L$-polynomials and Chebyshev $T$-polynomials (our $L$-polynomials being a discrete analog of Chebyshev $T$-polynomials).

In Section 2.3 we recast the problem from the perspective of combinatorial geometry.

From this perspective we rule out the existence of $L$-polynomials when $k \leq d$ and
prove theorems that we later use to: prove the uniqueness of \( L_{d,[x_k]} \) for a specific \( d \) and \([x_k]\), compute the lead coefficients of some of the \( L_{d,[k]} \), and describe an algorithm that generates all of the \( L_{d,[x_k]} \). In Section 2.4 we prove more theorems (no longer using the combinatorial geometry perspective) and end by proving the uniqueness of \( L_{d,[x_k]} \). In Section 2.5 lead coefficients for some of the \( L_{d,[k]} \) (for \( d \leq 4 \) and \( k > d \)) are calculated and an algorithm that generates all of the \( L_{d,[x_k]} \) is described. In Section 2.6 we write some of the \( L_{d,[k]} \) in terms of the corresponding degree \( d \) Chebyshev \( T \)-polynomials.

![Figure 2.1: \( L_{5,[6]}(x) \)](image-url)
2.2 Chebyshev Polynomials

The Chebyshev $T$-polynomials are defined by the following recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x \quad \text{and} \quad T_{d+1}(x) = 2xT_d(x) - T_{d-1}(x).$$

Alternatively, each $T_d(x)$ is the unique polynomial of degree $d$ satisfying the relationship

$$T_d(x) = \cos(d \arccos(x))$$

which, by substituting $\cos(x)$ for $x$, becomes

$$T_d(\cos(x)) = \cos(dx).$$
The first few Chebyshev \( T \)-polynomials are

\[
\begin{align*}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{align*}
\]

An elementary proof \([Ri]\) shows that

\[
f(x) = \frac{T_d(x)}{2^{d-1}}
\]

is the unique monic polynomial of degree \( d \) with minimum infinity norm on the interval \([-1, 1]\). To wit \( \|f(x)\|_\infty = \frac{1}{2^{d-1}} \). It follows that \( T_d \) is the polynomial of degree \( d \) that, while bounded between \(-1\) and 1 on the interval \([-1, 1]\), has the maximum possible lead coefficient equal to \( 2^{d-1} \). Thus our \( L \)-polynomials, which are bounded between \(-1\) and 1 on a set of \( k \) values, can be thought of as discrete analogs of \( T \)-polynomials.

If we stretch \( T_d(x) \) by composing it with

\[
s(x) = \frac{2x - (x_k + x_1)}{x_k - x_1},
\]
instead of being bounded between $-1$ and $1$ on the interval $[-1, 1]$ it is now bounded between $-1$ and $1$ on the interval $[x_1, x_k]$. The lead coefficient of $T_d(s(x))$ is

$$2^{d-1} \left( \frac{2}{x_k - x_1} \right)^d = \frac{2^{2d-1}}{(x_k - x_1)^d}.$$ 

Since $|T(s(x))| \leq 1$ for $x \in [x_1, x_k]$ is a stronger constraint than $|L_{d,[k]}(x)| \leq 1$ for $x \in \{x_1, x_2, \cdots, x_k\} = [x_k]$ the lead coefficient of $T_d(s(x))$ is a lower bound for the lead coefficient of $L_{d,[k]}(x)$.

### 2.3 A Combinatorial Geometry Perspective on $L$-Polynomials

The problem of finding our maximum coefficient starts looking combinatorial when we remember that polynomials can be factored. Consider a polynomial written in terms of its factors

$$Q(x) = a_d \prod_{i \in [d]} (x - r_i)$$

where $d = \deg(Q(x))$ and $r_i \in \mathbb{C}$.

Our question: how big can $a_d$ be if, for all $x \in [x_k]$, the following inequality holds

$$|Q(x)| = |a_d| \prod_{i \in [d]} |(x - r_i)| \leq 1?$$

Thus finding the maximum value of $a_d$ is equivalent to finding the minimum over all
multisets \{r_1, \ldots, r_d\} \subset \mathbb{C} of the following maximum product of distances,

\[
\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} \left| (x - r_i) \right| \right\}.
\]

Put concisely:

**Question 2.3.1.** For a given \([x_k] = \{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}\) what is

\[
\min_{|A|=d} \left\{ \max_{x \in [x_k]} \left\{ \prod_{r_i \in A} |x - r_i| \right\} \right\}
\]

if \(A\) is taken over all multisets \(\{r_1, \ldots, r_d\} \subset \mathbb{C}\)?

This minimum is equal to \(1/a_d\) where \(a_d\) is the lead coefficient of \(L_{d,[x_k]}\). We do not necessarily have to find a multiset \(A\) that gives us this minimum in order to find this minimum but it will be useful to consider some properties that such a multiset \(A\) must necessarily have. Before doing so we rule out the existence of our maximum lead coefficient for some \(k\) and \(d\).

**2.3.1 When \(k \leq d\)**

We pick a multiset of roots \(A\) such that \([x_k] \subset A\). For example, we define

\[
\frac{|Q(x)|}{a_d} = |x - x_1|^{1+(d-k)}|x - x_2||x - x_3| \cdots |x - x_k|.
\]

then it follows that

\[
|Q(x)| = 0 \leq 1
\]
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for all $x \in [x_k]$ no matter the value of $a_d$. Therefore when $k \leq d$ our constraints leave $a_d$ unbounded. From here on out we only concerned ourselves with cases where $k > d$.

2.3.2 Some combinatorial theorems

We prove a few lemmas using this combinatorial geometry view of things.

**Theorem 2.3.2.** If $k > d \geq 2$ and

$$Q(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial such that for $[x_k] = \{x_1 < x_2 < \cdots < x_k\}$

$$\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\}$$

(2.3.1)

is minimal then the multiset of the $d$ roots of $Q(x)$ is real and contains no duplicates for $k > d \geq 2$.

**Proof.** First we prove the roots are all real. Assume that $r_i = a_i + (\sqrt{-1})b_i$ with $b_i \neq 0$ for some $i$. This means that for all $x \in [x_k]$

$$|x - r_i| = \sqrt{(x - a_i)^2 + (b_i)^2} > \sqrt{(x - a_i)^2} = |x - a_i|.$$ 

But then

$$\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\} > \max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - a_i| \right\},$$
which contradicts the assumed minimality of the left-side of this inequality. It follows
then that

$$\{r_1, \cdots, r_d\} = \{a_1, \cdots, a_d\} \subset \mathbb{R}.$$  

Next we prove that the roots are all distinct. Assume $Q(x)$ has a root of multi-

plicity greater than one. Since we have made no claims based on the ordering of the

$r_i$ we can, without loss of generality, set

$$r = r_1 = r_2$$

and rewrite the product of distances, i.e., the factors of our polynomial, as

$$|x - r|^2 |x - r_3| \cdots |x - r_d|.$$  

We will show that replacing the two identical $r$’s with two distinct roots, $r + \varepsilon$ and

$r - \varepsilon$, gives us a smaller maximum product than (2.3.1), a contradiction. This $\varepsilon$

is the minimum of a subset of $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, all of which we now define:

**Case I:** We define

$$\varepsilon_1 = \min_{x \in [x_k], x > r} \left\{ \frac{x - r}{2} \right\}.$$  

If $x > r$ and $x \in [x_k]$ then it follows that

$$|x - (r + \varepsilon_1)||x - (r - \varepsilon_1)| = (x - r)^2 - \varepsilon_1^2 < |x - r|^2$$

and by multiplying both sides of the inequality by the other roots’ distances to
$x$, i.e., the absolute values of $Q(x)$’s factors, we have that

$$|x - (r + \varepsilon_1)||x - (r - \varepsilon_1)||x - r_3| \ldots |x - r_d| < |x - r|^2|x - r_3| \ldots |x - r_d|.$$  

**Case II:** We define

$$\varepsilon_2 = \min_{x \in [x_k]} \left\{ \frac{r - x}{2} \right\}.$$  

If $x < r$ and $x \in [x_k]$ then it follows that

$$|x - (r + \varepsilon_2)||x - (r - \varepsilon_2)| = (r - x)^2 - \varepsilon_2^2 < |x - r|^2$$  

and by multiplying both sides of the inequality by the other roots’ distances to $x$, i.e., the absolute values of $Q(x)$’s factors, we have that

$$|x - (r + \varepsilon_2)||x - (r - \varepsilon_2)||x - r_3| \ldots |x - r_d| < |x - r|^2|x - r_3| \ldots |x - r_d|.$$  

**Case III:** If $r \in [x_k]$ and $r$ is a root of multiplicity two we define

$$\varepsilon_3 = \max_{x \in [x_k]} \left\{ \sqrt{|x - r|^2|x - r_3| \ldots |x - r_d|} \right\} \cdot \frac{1}{\sqrt{2|r - r_3| \ldots |r - r_d|}}.$$
If $x = r$ and $x \in [x_k]$ then it follows that

$$|x - (r + \varepsilon_3)||x - (r - \varepsilon_3)| = \varepsilon_3^2$$

$$= \max_{x \in [x_k]} \left\{ \frac{|x - r|^2 |x - r_3| \cdots |x - r_d|}{2|r - r_3| \cdots |r - r_d|} \right\}$$

$$< \max_{x \in [x_k]} \left\{ \frac{|x - r|^2 |x - r_3| \cdots |x - r_d|}{|r - r_3| \cdots |r - r_d|} \right\}.$$ 

and by multiplying both sides of the inequality by the other roots’ distances to $x = r$, i.e., the absolute values of $Q(x)$’s factors, we have that

$$|x - (r + \varepsilon_3)||x - (r - \varepsilon_3)||x - r_3| \cdots |x - r_d| <$$

$$\max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \cdots |x - r_d| \right\}.$$ 

Now we define $\varepsilon$. If $r \in [x_k]$ and $r$ is a root of multiplicity two we define $\varepsilon = \min \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, otherwise we define $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Replacing $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ in the three cases above with this value of $\varepsilon$ and then combining all three cases gives us that

$$\max_{x \in [x_k]} \left\{ |x - (r + \varepsilon)||x - (r - \varepsilon)||x - r_3| \cdots |x - r_d| \right\} <$$

$$\max_{x \in [x_k]} \left\{ |x - r|^2 |x - r_3| \cdots |x - r_d| \right\}$$

which contradicts the minimality of (2.3.1).
This proves that the roots of our \( L \)-polynomials must be real and distinct.

**Lemma 2.3.3.** If \( k > d \geq 2 \) and

\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]

is a degree \( d \) polynomial such that for \([x_k] = \{x_1 < x_2 < \cdots < x_k\}\)

\[
\max_{x \in [x_k]} \left\{ \prod_{i \in [d]} |x - r_i| \right\}
\]

(2.3.2)

is minimal then the set \( \{r_1 < r_2 < \cdots < r_d\} \) of \( d \) distinct (Theorem 2.3.2) roots of \( Q(x) \) is contained in the open interval \((x_1, x_k)\).

**Proof.** Suppose that one of the roots is outside of the open interval \((x_1, x_k)\), that is for some \( \delta \geq 0 \) we have \( r_d = x_k + \delta \). We will show that replacing the root \( r_d \) with \( x_k - \varepsilon \) for some soon to be defined \( \varepsilon > 0 \) yields a smaller maximum product (2.3.2), a contradiction. This \( \varepsilon \) is the minimum of \( \varepsilon_1 \) and \( \varepsilon_2 \) which we now define:

**Case I:** We define

\[
\varepsilon_1 = \frac{x_k - x_{k-1}}{2}.
\]

If \( \delta \geq 0 \) and \( x \in \{x_1, \cdots, x_{k-1}\} = [x_{k-1}] \) then it follows that

\[
|x - r_d| = |x - (x_k + \delta)| = (x_k + \delta) - x > \frac{x_k + x_{k-1}}{2} - x = \left| x - \left( x_k - \frac{x_k - x_{k-1}}{2} \right) \right| = |x - (x_k - \varepsilon_1)|
\]
and by multiplying both sides of the inequality by the other roots’ distances to $x$, i.e., the absolute values of $Q(x)$’s factors, we have that

$$|x - r_1| \cdots |x - r_{d-1}| |x - r_d| > |x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon_1)|.$$ 

**Case II:** We define

$$\varepsilon_2 = \max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_d|\} \cdot \frac{1}{2|x_k - r_1| \cdots |x_k - r_{d-1}|}.$$ 

If $x = x_k$ then it follows that

$$|x - (x_k - \varepsilon_2)| = \varepsilon_2$$

$$= \max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_d|\} \cdot \frac{1}{2|x_k - r_1| \cdots |x_k - r_{d-1}|}$$

and by multiplying both sides of the inequality by the other roots’ distances to $x = x_k$, i.e., the absolute values of $Q(x)$’s factors, we have that

$$|x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon_2)| < \max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_d|\}.$$ 

Now we define $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Replacing both $\varepsilon_1$ and $\varepsilon_2$ in the two cases above with this value of $\varepsilon$ and then combining these two cases gives us that

$$\max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_{d-1}| |x - (x_k - \varepsilon)|\} < \max_{x \in [x_k]} \{|x - r_1| \cdots |x - r_d|\}.$$
which contradicts the minimality of (2.3.2). We note that the similar arguments work if we attempt to place a root to the left of the interval \((x_1, x_k)\), i.e., if for some \(\delta \geq 0\) we have \(r_1 = x_1 - \delta\).

This proves that the roots of our \(L\)-polynomials must be in the interval \((x_1, x_k)\). By putting Lemma 2.3.2 and Lemma 2.3.3 together we can now say that the roots of our \(L\)-polynomials are all distinct real numbers in the interval \((x_1, x_k)\).

### 2.4 Some Non-combinatorial Theorems

First we show that a degree \(d\) \(L\)-polynomial bounded between \(-1\) and \(1\) for \(x \in [x_k]\) must pass through both of the points \((x_1, (-1)^d)\) and \((x_k, 1)\).

**Lemma 2.4.1.** If \(k > d \geq 2\) and

\[
Q(x) = a_d \prod_{i \in [d]} (x - r_i)
\]

is a degree \(d\) polynomial with maximum lead coefficient \(a_d\) when bound between \(-1\) and \(1\) for

\[
x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}
\]

then

\[
Q(x_1) = (-1)^d \text{ and } Q(x_k) = 1.
\]

**Proof.** We assume that \(Q(x_k) < 1\). Theorem 2.3.2 and Lemma 2.3.3 proved that \(Q(x)\) has \(d\) distinct roots and that these roots are all contained in the interval \((x_1, x_k)\). In
other words, for some

\[ x_1 < r_1 < r_2 < \cdots < r_{d-1} < r_d < x_k, \]

we have

\[ Q(x) = a_d(x - r_1) \cdots (x - r_d). \]

With some soon to be determined \( \varepsilon \) we define

\[ \hat{Q}(x) = Q(x) + \varepsilon(x - r_1) \cdots (x - r_{d-1}). \]

For \( x \neq r_d \) we can write

\[ \hat{Q} = \left(1 + \frac{\varepsilon}{a_d(x - r_d)}\right)Q(x). \]

We use this \( \hat{Q}(x) \) to contradict the maximality of \( Q(x) \)'s lead coefficient. We choose \( \varepsilon \) to be the minimum of a subset of \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \), which we define for three cases of \( x \):

**Case I:** We define

\[ \varepsilon_1 = \frac{a_d}{2} \cdot \min_{x \in [x_k]} \{r_d - x\}. \]

If \( x < r_d \) and \( x \in [x_k] \) then it follows that

\[ \left|1 + \frac{\varepsilon_1}{a_d(x - r_d)}\right| \cdot |Q(x)| = \left|1 - \min_{x \in [x_k]} \{r_d - x\} \cdot \frac{1}{2(r_d - x)}\right| \cdot |Q(x)| < |Q(x)| \leq 1. \]
Case II: Since Lemma 2.3.3 gives us that $Q(x_k) > 0$, we can define

$$
\varepsilon_2 = \left(\frac{1}{Q(x_k)} - 1\right) \cdot \frac{a_d}{2} \cdot \min_{x \in [x_k]} \{x - r_d\}.
$$

Recall that we assumed $Q(x_k) < 1$. So if $x > r_d$ and $x \in [x_k]$ then it follows that

$$
\left|1 + \frac{\varepsilon_2}{a_d(x - r_d)}\right| \cdot |Q(x)| = 
\left[1 + \left(\frac{1}{Q(x_k)} - 1\right) \cdot \min_{x \in [x_k]} \{x - r_d\} \cdot \frac{1}{2(x - r_d)}\right] \cdot |Q(x)| \leq
\left[1 + \left(\frac{1}{Q(x_k)} - 1\right) \cdot \frac{1}{2}\right] \cdot |Q(x)|.
$$

Since $Q(x)$ is a polynomial with positive lead coefficient it increases to the right of its largest root $r_d$. We use this fact to continue the above string of inequalities and have that

$$
\left[1 + \left(\frac{1}{Q(x_k)} - 1\right) \cdot \frac{1}{2}\right] \cdot |Q(x)| \leq \left[1 + \left(\frac{1}{Q(x_k)} - 1\right) \cdot \frac{1}{2}\right] \cdot Q(x_k) =
\frac{Q(x_k) + 1}{2} < 1.
$$

Case III: If $r_d \in [x_k]$ we define

$$
\varepsilon_3 = \frac{1}{(r_d - r_1)(r_d - r_2) \cdots (r_d - r_{d-1})}.
$$
If \( x = r_d \) and \( x \in [x_k] \) then it follows that

\[
|Q(x) + \varepsilon_3(x - r_1)(r_d - r_2) \cdots (x - r_{d-1})| =
\]

\[
|Q(r_d) + \varepsilon_3(r_d - r_1)(r_d - r_2) \cdots (r_d - r_{d-1})| = |Q(r_d) + \frac{1}{2}| = \frac{1}{2} < 1.
\]

Otherwise, if \( r_d \notin [x_k] \), we define

\[
\varepsilon_3 = 1.
\]

Now we define \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \). Replacing \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \) in the three cases above with the value of this \( \epsilon \) and then combining all three cases gives us that

\[
|\hat{Q}(x)| < 1
\]

when \( x \in [x_k] \). Clearly the lead coefficients of \( \hat{Q}(x) \) and \( Q(x) \) are equal since we defined \( \hat{Q}(x) \) as \( Q(x) \) plus a degree \( d - 1 \) polynomial. But since \( |\hat{Q}(x)| \) is strictly less than one we can have some \( \lambda > 1 \) such that

\[
|\lambda \hat{Q}(x)| \leq 1
\]

for \( x \in [x_k] \). But then \( \lambda a_d \), the lead coefficient of \( \lambda \hat{Q} \) is greater than \( a_d \). This contradicts the maximality of the lead coefficient \( a_d \) of \( Q(x) \). Similar arguments work to show that \( Q(x_1) = (-1)^d \).

Next we prove that \( L \)-polynomial must pass through some point \((x_i, \pm 1)\) between
any two of its consecutive roots.

**Lemma 2.4.2.** If $k > d \geq 2$ and

$$Q(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial with maximum lead coefficient $a_d$ when bound between $-1$ and $1$ for

$$x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

then for any two of $Q(x)$’s consecutive roots $r_i$ and $r_{i+1}$ there exists an $x' \in [x_k]$ such that

$$r_i < x' < r_{i+1}$$

and

$$|Q(x')| = 1.$$

**Proof.** We assume that whenever $x \in [x_k]$ and $r_i < x < r_{i+1}$ we have that $|Q(x)| < 1$. From Theorem 2.3.2 and Lemma 2.3.3 we know that $Q(x)$ has $d$ distinct roots and that these roots are contained in the interval $(x_1, x_k)$. In other words, for some

$$x_1 < r_1 < r_2 < \cdots < r_{d-1} < r_d < x_k,$$

we have that

$$Q(x) = a_d(x - r_1) \cdots (x - r_d).$$
Now, with some soon to be determined $\varepsilon > 0$, we define

$$\tilde{Q}(x) = Q(x) - \varepsilon(x - r_1) \cdots (x - r_{i-1})(x - r_{i+2}) \cdots (x - r_d).$$

For $x \notin \{r_i, r_{i+1}\}$ and $x \in [x_k]$ we can write

$$\tilde{Q}(x) = \left(1 - \frac{\varepsilon}{a_d(x - r_i)(x - r_{i+1})}\right)Q(x).$$

We will use this $\tilde{Q}(x)$ to contradict the maximality of $Q(x)$’s lead coefficient. We define the $\varepsilon$ which we used above to be the minimum of $\varepsilon_1, \varepsilon_2, \varepsilon_3,$ and $\varepsilon_4$ which we define below:

**Case I:** We define

$$\varepsilon_1 = \frac{a_d}{2} \cdot \left(\min_{x \in [x_k], x \notin [r_i, r_{i+1}]} \{(x - r_i)(x - r_{i+1})\}\right).$$

If $(x < r_1$ or $r_{i+1} < x)$ and $x \in [x_k]$ then it follows that

$$\left|\frac{\varepsilon_1}{a_d(x - r_i)(x - r_{i+1})}\right| \cdot |Q(x)| =$$

$$\left|1 - \left(\min_{x \in [x_k], x \notin [r_i, r_{i+1}]} \{(x - r_i)(x - r_{i+1})\}\right) \cdot \frac{1}{2(x - r_i)(x - r_{i+1})}\right| \cdot |Q(x)| <$$

$$|Q(x)| \leq 1.$$

**Case II:** If there exists an $x \in [x_k]$ such that $r_i < x < r_{i+1}$ then we define

$$Q_{\text{max}} = \max \{|Q(x)| : x \in [x_k] \text{ and } r_i < x < r_{i+1}\}.$$
Recall that we assumed that $|Q(x)| < 1$ for any and all $x \in [x_k]$ where $r_i < x < r_{i+1}$. This means $0 < Q_{max} < 1$. Next we define

$$
\varepsilon_2 = \frac{a_d}{2} \cdot \left( \frac{1}{Q_{max}} - 1 \right) \cdot \left( \min_{x \in [x_k]} \{ -(x - r_i)(x - r_{i+1}) \} \right).
$$

If $x \in [x_k]$ and $r_i < x < r_{i+1}$ then it follows that

$$
\left| 1 - \frac{\varepsilon_2}{a_d(x - r_i)(x - r_{i+1})} \right| \cdot |Q(x)| =
\left[ 1 - \left( \frac{1}{Q_{max}} - 1 \right) \cdot \left( \min_{x \in [x_k]} \{ (r_i - x)(x - r_{i+1}) \} \right) \cdot \frac{1}{2(x - r_i)(x - r_{i+1})} \right] \cdot |Q(x)| \leq
\left[ 1 + \frac{1}{2} \cdot \left( \frac{1}{Q_{max}} - 1 \right) \right] \cdot |Q(x)| \leq
\left[ 1 + \frac{1}{2} \cdot \left( \frac{1}{Q_{max}} - 1 \right) \right] \cdot Q_{max} = \frac{Q_{max} + 1}{2} < 1.
$$

If there does not exist an $x \in [x_k]$ such that $r_i < x < r_{i+1}$ then we define $\varepsilon_2 = 1$.

**Case III:** If $r_i \in [x_k]$ we define

$$
\varepsilon_3 = \frac{1}{2(r_i - r_1) \cdots (r_i - r_{i-1})(r_i - r_{i+2}) \cdots (r_i - r_d)}.
$$

If $x = r_i$ and $x \in [x_k]$ then it follows that

$$
|Q(x) + \varepsilon_3(x - r_1) \cdots (x - r_{i-1})(x - r_{i+2}) \cdots (x - r_d)| =
|Q(r_i) + \varepsilon_3(r_i - r_1) \cdots (r_i - r_{i-1})(r_i - r_{i+2}) \cdots (r_i - r_d)| =
\left| Q(r_i) + \frac{1}{2} \right| = 0 + \frac{1}{2} < 1.
$$
Otherwise, if \( r_i \notin [x_k] \), we define
\[
\varepsilon_3 = 1.
\]

**Case IV:** If \( r_{i+1} \in [x_k] \) we define
\[
\varepsilon_4 = \frac{1}{2(r_{i+1} - r_1) \cdots (r_{i+1} - r_{i-1})(r_{i+1} - r_{i+2}) \cdots (r_{i+1} - r_d)}.
\]

If \( x = r_{i+1} \) and \( x \in [x_k] \) then it follows that
\[
|Q(x) + \varepsilon_4 (x - r_1) \cdots (x - r_{i-1})(x - r_{i+2}) \cdots (x - r_d)| = |Q(r_{i+1}) + \varepsilon_4 (r_{i+1} - r_1) \cdots (r_{i+1} - r_{i-1})(r_{i+1} - r_{i+2}) \cdots (r_{i+1} - r_d)| = \left| Q(r_{i+1}) + \frac{1}{2} \right| = 0 + \frac{1}{2} < 1.
\]

Otherwise, if \( r_{i+1} \notin [x_k] \), we define
\[
\varepsilon_4 = 1.
\]

Now we define \( \varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \). Replacing \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \) and \( \varepsilon_4 \) in the four cases above with the value of this \( \varepsilon \) and then combining all four cases gives us that
\[
|\tilde{Q}(x)| < 1
\]
for \( x \in [x_k] \). Clearly the lead coefficients of \( \tilde{Q}(x) \) and \( Q(x) \) are equal since we have defined \( \tilde{Q}(x) \) as \( Q(x) \) plus a degree \( d - 2 \) polynomial. But since \( |\tilde{Q}(x)| \) is strictly less
than one we can have some $\lambda > 1$ such that

$$|\lambda \hat{Q}(x)| \leq 1$$

for $x \in [x_k]$. But then $\lambda a_d$, the lead coefficient of $\lambda \hat{Q}$, is greater than $a_d$. This contradicts the maximality of the lead coefficient $a_d$ of $Q(x)$.

Finally we prove that there is a unique $L_{d,[x_k]}(x)$ for every $[x_k]$ and $d$ whenever $k > d \geq 1$.

**Theorem 2.4.3.** If $k > d \geq 2$ and

$$L_{d,[x_k]}(x) = a_d \prod_{i \in [d]} (x - r_i)$$

is a degree $d$ polynomial with maximum lead coefficient $a_d$ when bound between $-1$ and $1$ for

$$x \in [x_k] = \{x_1 < x_2 < \cdots < x_k\}$$

then $L_{d,[x_k]}(x)$ is the unique degree $d$ polynomial with maximum lead coefficient $a_d$ when bound between $-1$ and $1$ for $x \in [x_k]$.

**Proof.** We assume that $\hat{Q}(x)$ and $\hat{Q}(x)$ are two degree $d$ polynomials with the same lead coefficient $a_d > 0$ satisfying the condition that $a_d$ be the maximum lead coefficient possible if both

$$|\hat{Q}(x)| \leq 1$$
and

$$|\dot{Q}(x)| \leq 1$$

for \(x \in [x_k]\). Next we form the average

$$\bar{Q}(x) = \frac{\dot{Q}(x) + \dot{Q}(x)}{2}.$$  

Clearly this average also has lead coefficient \(a_d\) and also satisfies the condition that

$$|\bar{Q}(x)| = \frac{|\dot{Q}(x) + \dot{Q}(x)|}{2} \leq \frac{|\dot{Q}(x)| + |\dot{Q}(x)|}{2} \leq 1$$

for \(x \in [x_k]\). From Theorem 2.3.2, Lemma 2.4.1, and Lemma 2.4.2 we know that there must be some

$$A = \{b_1 < b_2 < \cdots < b_{d-1}\} \subset \{x_2, x_3, \cdots, x_{k-1}\} \subset [x_k],$$

and further

$$\{c_1, c_2, \cdots, c_{d+1}\} = \{x_1, b_1, b_2, \cdots, b_{d-1}, x_k\},$$

such that the polynomial \(\bar{Q}(x)\) passes through the \(d + 1\) points

$$(c_i, (-1)^{(d+1)-i}) \text{ for } i \in [d + 1]$$

or, stated differently, that

$$\bar{Q}(c_i) = (-1)^{(d+1)-i}$$

for \(i \in [d + 1]\). But since \(\dot{Q}(x)\) and \(\dot{Q}(x)\) are bounded between \(-1\) and \(1\) for \(x \in [x_k]\)
and since $\bar{Q}(x)$ is an average of these two polynomials, we must also have that

$$\bar{Q}(c_i) = \hat{Q}(c_i) = (-1)^{(d+1)-i}$$

for $i \in [d+1]$. But if two degree $d$ polynomials intersect at $d + 1$ points then the two polynomials are equal. Thus

$$L_{d,[x_k]}(x) = \hat{Q}(x) = \bar{Q}(x) = \hat{Q}(x).$$

2.5 Calculating the Lead Coefficients of $L_{d,[k]}$ for $d \leq 4$ and an Algorithm that Finds All the $L_{d,[x_k]}$

We begin this section by constructing algebraic formulas for the lead coefficients of $L_{d,[x_k]}$ for $d \leq 4$ and $[x_k] = [k] = \{1, \cdots, k\}$. We recall from Definition 2.1.2 that $L_{d,[x_k]}$ is the unique degree $d$ polynomial with maximum lead coefficient bounded between $-1$ and $1$ on a set $[x_k]$. We end this section with a description of an algorithm that generates all $L_{d,[x_k]}$ for $k > d$. 
2.5.1 Lead coefficients of $L_{1,[k]}$

This case is quite simple. Lemma 2.4.1 forces $L_{1,[k]}(1) = -1$ and $L_{1,[k]}(k) = 1$. Since we know two points that the line $L_{1,[k]}(x)$ passes through we write

$$L_{1,[k]}(x) = a_1x + a_0 = \frac{2}{k-1}x - \frac{k+1}{k-1},$$

which has lead coefficient

$$a_1 = \frac{2}{k-1}.$$

2.5.2 Lead coefficients of $L_{2,[k]}$

We recall that $L_{2,[k]}(x)$ is the unique (by Theorem 2.4.3) degree 2 polynomial with the maximum lead coefficient when bounded between $-1$ and 1 for $x \in [k] = \{1, \cdots, k\}$. We shift $L_{2,[k]}(x)$ so that the $k$ consecutive $x$ values it is bounded on are centered at zero (that is, instead of being bounded for $x \in \{1, \cdots, k\} = [k]$ it is bounded for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$). This alters neither the shape of its graph nor its lead coefficient. We define

$$k' = \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}.$$

Since $\deg(L_{2,k'}(x)) = 2$ is even it follows that the average

$$\frac{L_{2,k'}(x) + L_{2,k'}(-x)}{2} = a_2x^2 + a_0$$
is an even-function with the same lead coefficient as $L_{2,k'}(x)$. Since $|L_{2,k'}(x)| \leq 1$ for $x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2}\}$ it follows that $|L_{2,k'}(x) + L_{2,k'}(-x)|/2 \leq 1$ for $x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2}\}$. But by the uniqueness of $L_{2,k'}(x)$ (from Theorem 2.4.3) we must have

$$L_{2,k'}(x) = \frac{L_{2,k'}(x) + L_{2,k'}(-x)}{2},$$

so

$$L_{2,k'}(x) = a_2 x^2 + a_0.$$

Two unknown coefficients are better than three. From Lemma 2.4.1 we know that $L_{2,k'}$ passes through $(\frac{k-1}{2}, 1)$. Thus

$$L_{2,k'}\left(\frac{k - 1}{2}\right) = a_2 \left(\frac{k - 1}{2}\right)^2 + a_0 = 1.$$

Solving for $a_0$ gives us that

$$a_0 = 1 - \frac{a_2 (k - 1)^2}{4}.$$

Plugging this into

$$a_2 x^2 + a_0 \geq -1,$$

the constraining inequalities on $L_{2,k'}(x)$ for $x \in \{-\frac{k-1}{2} - 1, \ldots, \frac{k-1}{2} - 1\}$, gives us that

$$a_2 \leq \frac{8}{(k - 1)^2 - 4x^2} \quad (2.5.1)$$
for \( x \in \{-\left(\frac{k-1}{2}-1\right), \cdots, \frac{k-1}{2}-1\}\).

The right-side of inequality (2.5.1) is minimized on \( x \in \{-\left(\frac{k-1}{2}-1\right), \cdots, \frac{k-1}{2}-1\}\) when \( x \) is closest or equal to zero. For odd \( k \) it is minimized when \( x = 0 \). For even \( k \) it is minimized when \( x = \pm \frac{1}{2} \). Plugging these minimizing values into Inequality (2.5.1) gives us

\[
a_2 = 8 \cdot \begin{cases} 
\frac{1}{(k-1)^2} & \text{for } k \equiv 1 \mod 2 \\
1 & \text{for } k \equiv 0 \mod 2 \\
\frac{1}{k(k-2)} & \text{for } k \equiv 0 \mod 2
\end{cases}
\]

as the lead coefficient of \( L_{2,k'}(x) \) and thus also of \( L_{2,[k]}(x) \).

### 2.5.3 Lead coefficients of \( L_{3,[k]} \)

We recall that \( L_{3,[k]}(x) \) is the unique (by Theorem 2.4.3) degree 3 polynomial with the maximum lead coefficient when bounded between \(-1\) and \(1\) for \( x \in [k] = \{1, \cdots, k\} \). We shift \( L_{3,[k]}(x) \) so that, as with \( L_{2,[k]}(x) \) above, the \( k \) consecutive \( x \) values it is bounded on are centered at zero (that is, instead of being bounded for \( x \in \{1, \cdots, k\} \) it is bounded for \( x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\} \)). This alters neither the shape of its graph nor its lead coefficient. We define

\[
k' = \left\{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\right\}.
\]

Since \( \deg(L_{3,k'}(x)) = 3 \) is odd it follows that the difference

\[
\frac{L_{3,k'}(x) - L_{3,k'}(-x)}{2} = a_3x^3 + a_1x
\]
is an odd-function with the same lead coefficient as $L_{3,k'}(x)$. Since $|L_{3,k'}(x)| \leq 1$ for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$ it follows that $|L_{3,k'}(x) - L_{3,k'}(-x)|/2 \leq 1$ for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$. But then by uniqueness (from Theorem 2.4.3) we must have that

$$L_{3,k'}(x) = \frac{L_{3,k'}(x) - L_{3,k'}(-x)}{2}$$

which implies that

$$L_{3,k'}(x) = a_3x^3 + a_1x.$$  

Two unknown coefficients are better than four.

From Lemma 2.4.1, $L_{3,k'}$ passes through $(\frac{k-1}{2}, 1)$. Thus

$$L_{3,k'}(\frac{k-1}{2}) = a_3(\frac{k-1}{2})^3 + a_1(\frac{k-1}{2}) = 1.$$  

Solving for $a_1$ gives us that

$$a_1 = \frac{8 - a_3(k-1)^3}{4(k-1)}.$$  

Plugging this into

$$a_3x^3 + a_1x \geq -1,$$  

the constraining inequalities on $L_{3,k'}(x)$ for $x \in \{-\frac{k-1}{2} - 1, \cdots, \frac{k-1}{2} - 1\}$, gives us
CHAPTER 2. DISCRETE CHEBYSHEV TYPE POLYNOMIALS

that

\[ a_3 \leq \frac{-4}{2(k - 1)x^2 - (k - 1)^2x} \]  

(2.5.2)

for \( x \in \{-(k-1)/2, \ldots, k-1/2 - 1\} \).

It is clear from looking at the graph of the right-side of the inequality (2.5.2) that it is minimized by the \( x \in \{-(k-1)/2, \ldots, k-1/2 - 1\} \) closest or equal to \( k - 1/4 \).

Plugging these minimizing values into Inequality (2.5.2) gives us

\[
a_3 = \frac{32}{k - 1} \begin{cases} 
\frac{1}{(k - 1)^2} & \text{when } k \equiv 1 \mod 4, \text{ by setting } x = \frac{k - 1}{4} \\
\frac{1}{k(k - 2)} & \text{when } k \equiv 2 \mod 4, \text{ by setting } x = \frac{k}{4} \\
\frac{1}{(k + 1)(k - 3)} & \text{when } k \equiv 3 \mod 4, \text{ by setting } x = \frac{k - 3}{4} \text{ or } x = \frac{k + 1}{4} \\
\frac{1}{k(k - 2)} & \text{when } k \equiv 0 \mod 4, \text{ by setting } x = \frac{k - 2}{4}
\end{cases}
\]

as the lead coefficient of \( L_{3,k'}(x) \) and thus also of \( L_{3,[k]}(x) \).

2.5.4 Lead coefficients of \( L_{4,[k]} \)

By the same argument we used for \( L_{2,[k]} \) above we can say that the maximum lead coefficient of the polynomial

\[ L_{4,k'}(x) = a_4 x^4 + a_2 x^2 + a_0, \]

(for \( k' = \{-(k-1)/2, \ldots, k-1/2 \} \) and \( k > 4 \)) is the same as the lead coefficient of \( L_{4,[k]} \).

Now we split the \( d = 4 \) case into two subcases, one for odd \( k \) and one for even \( k \):
**Case I:** For odd $k$. From Theorem 2.3.2 and Lemma 2.4.2 we know that $L_{4,k'}(0) = 1$ which implies that $a_0 = 1$. From Lemma 2.4.1 we know that

\[ L_{4,k'} \left( \frac{k - 1}{2} \right) = 1 \]

which implies that

\[ a_4 \left( \frac{k - 1}{2} \right)^4 + a_2 \left( \frac{k - 1}{2} \right)^2 + a_0 = 1. \]

Rewriting this in terms of $a_2$ after substituting 1 in for $a_0$ gives us that

\[ a_2 = -a_4 \left( \frac{k - 1}{2} \right)^2. \]

We recall our lower bound

\[ L_{4,k'}(x) \geq -1 \]

when $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$. Eliminating $a_2$ and $a_0$ from this lower bound gives us that

\[ a_4x^4 - a_4 \left( \frac{k - 1}{2} \right)^2 x^2 + 1 \geq -1 \]

which implies that

\[ a_4 \leq \frac{8}{x^2(k - 1)^2 - 4x^4} \]

for $x \in \{-\frac{k-1}{2}, \cdots, \frac{k-1}{2}\}$.

Optimization of the right-side of this last inequality when $x$ is in the interval
(0, \frac{k-1}{2}) gives us that

\[ a_4 = \min_{x \in I} \left\{ \frac{8}{x^2(k-1)^2 - 4x^4} \right\}, \]

for \( I = \left\{ \left\lfloor \frac{k-1}{2\sqrt{2}} \right\rfloor, \left\lceil \frac{k-1}{2\sqrt{2}} \right\rceil \right\}. \) This is the lead coefficient of \( L_{4,k'}(x) \) and thus also of \( L_{4,[k]}(x) \) when \( k' \) and \( k \) are odd.

**Case II:** For even \( k \). From Theorem 2.3.2 and Lemma 2.4.2 we know that

\[ L_{4,k'} \left( \frac{1}{2} \right) = a_4 \left( \frac{1}{2} \right)^4 + a_2 \left( \frac{1}{2} \right)^2 + a_0 = 1. \]

Writing this in terms of \( a_0 \) gives us that

\[ a_0 = 1 - \frac{a_2}{4} - \frac{a_4}{16} \] (2.5.3)

From Lemma 2.4.1 we have a second equation

\[ L_{4,k'} \left( \frac{k-1}{2} \right) = 1 \]

which implies that

\[ a_4 \left( \frac{k-1}{2} \right)^4 + a_2 \left( \frac{k-1}{2} \right)^2 + a_0 = 1. \] (2.5.4)

We combine equations (2.5.3) and (2.5.4) and have that

\[ a_0 = a_4 \frac{(k-1)^2}{16} + 1 \]
and

\[ a_2 = -a_4 \frac{(k - 1)^2 + 1}{4}. \]

We recall our lower bound

\[ L_{4,\lfloor k \rfloor}(x) \geq -1 \]

when \( x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2}\} \). Eliminating \( a_2 \) and \( a_0 \) from this lower bound gives us that

\[ a_4 \leq \frac{-32}{16x^4 - 4((k - 1)^2 + 1)x^2 + (k - 1)^2} \]

for \( x \in \{-\frac{k-1}{2}, \ldots, \frac{k-1}{2}\} \).

Optimization of the right-side of this last inequality when \( x \) is in the interval \((\frac{1}{2}, \frac{k-1}{2})\) gives us that

\[ a_4 = \min_{x \in H} \left\{ \frac{-32}{16x^4 - 4((k - 1)^2 + 1)x^2 + (k - 1)^2} \right\} \]

for \( H = \left\{ \left[ \frac{k-1}{2\sqrt{2}} + \frac{1}{2} \right] - \frac{1}{2}, \frac{k-1}{2\sqrt{2}} + \frac{1}{2}, \frac{k-1}{2\sqrt{2}} - \frac{1}{2}, \frac{k-1}{2\sqrt{2}} - \frac{1}{2} \right\} \). This is the lead coefficient of \( L_{4,k'}(x) \) and thus also of \( L_{4,\lfloor k \rfloor}(x) \) when \( k' \) and \( k \) are even.

### 2.5.5 An algorithm for finding all the \( L_{d,\lfloor x_k \rfloor} \)

When \( d = 5 \) only one of the three coefficients \( \{a_5, a_3, a_1\} \) can be eliminated using the algebraic techniques we used for \( d \leq 4 \). When \( d = 6 \) only two of the four coefficients \( \{a_6, a_4, a_2, a_0\} \) can be eliminated. The situation continues to get worse as \( d \) increases.
Fortunately, putting together some of our theorems and lemmas yields an algorithm that finds \( L_{d,[x_k]}(x) \) for all \( d \geq 1 \) and \( k > d \).

We recall that \( L_{d,[x_k]}(x) \) is the unique degree \( d \) polynomial with maximum lead coefficient \( a_d \) such that

\[
|L_{d,[x_k]}(x)| \leq 1
\]

for \( x \in [x_k] \). From Theorem 2.3.2, Lemma 2.4.1 and Lemma 2.4.2 we know that there must be some

\[
B = \{b_1 < b_2 < \cdots < b_{d-1}\} \subset \{x_2, x_3, \cdots , x_{k-1}\} \subset [x_k],
\]

and further some

\[
\{c_1, c_2, \cdots , c_{d+1}\} = \{x_1, b_1, b_2, \cdots , b_{d-1}, x_k\},
\]

so that the polynomial \( L_{d,[x_k]}(x) \) passes through the \( d + 1 \) points

\[
(c_i, (-1)^{(d+1)-i}) \text{ for } i \in [d+1].
\]

Since \( L_{d,[x_k]}(x) \) is a degree \( d \) polynomial we can solve for it explicitly if we know the \( d \) points in the set \( B \). So, for each of the \( \binom{k-2}{d-1} \) possible \( d - 1 \) element sets \( B \subset \{x_2, x_3, \cdots , x_{k-1}\} \), we find the polynomial passing through the corresponding points

\[
(c_i, (-1)^{(d+1)-i}) \text{ for } i \in [d+1].
\]
From the set of all \( \binom{k-2}{d-1} \) such polynomials we choose the subset of polynomials bounded between \(-1\) and \(1\) for \(x \in [x_k]\). From this subset the polynomial with maximum lead coefficient must be our unique \( L_{d,[x_k]}(x) \). In particular, setting

\[
\{x_1, x_2, \cdots, x_k\} = \{1, 2, \cdots, k\}
\]

will give us \( L_{d,[k]}(x) \).

Here is a table of the lead coefficients of \( L_{d,[k]} \) for some values of \(d\) and \(k\):

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2/3</td>
<td>1/2</td>
<td>2/5</td>
<td>1/3</td>
<td>2/7</td>
<td>1/4</td>
<td>2/9</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
<td>1/3</td>
<td>2/9</td>
<td>1/6</td>
<td>1/8</td>
<td>1/10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4/3</td>
<td>1/2</td>
<td>4/15</td>
<td>1/6</td>
<td>2/21</td>
<td>1/16</td>
<td>2/45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2/3</td>
<td>1/4</td>
<td>1/10</td>
<td>1/18</td>
<td>2/63</td>
<td>1/48</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4/15</td>
<td>1/12</td>
<td>4/105</td>
<td>1/60</td>
<td>5/567</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4/45</td>
<td>1/36</td>
<td>11/120</td>
<td>7/1440</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8/315</td>
<td>1/144</td>
<td>1/360</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We recall from Section 2.3 that the reciprocal of the lead coefficient of \( L_{d,[k]} \) is the smallest possible maximum product of distances between each point in the set \([k]\) and any set \(A\) of \(d\)-points. Here is a table of the reciprocals of these lead coefficients, these smallest maximum products of distances

\[
\min_{|A|=d} \left\{ \max_{x \in [k]} \left\{ \prod_{r_i \in A} |x - r_i| \right\} \right\}
\]

taken over all multisets \(A = \{r_1, \cdots, r_d\} \subset \mathbb{C}\) for some values of \(d\) and \(k\):
2.6 \( L_{d, [k]} \) in Terms of Chebyshev Polynomials

To more clearly demonstrate how our \( L \)-polynomials relate to the Chebyshev \( T \)-polynomials—to which they are, with respect to constraints, discrete analogs—we write some of the \( L \)-polynomials in terms of the corresponding Chebyshev \( T \)-polynomials.

We compose \( L_{d, [k]}(x) \) with

\[
t(x) = \frac{k - 1}{2} x + \frac{k + 1}{2}
\]

in order to squeeze the points \( [k] = \{1, \ldots, k\} \) (on which our \( L_{d, [k]}(x) \) are bounded between \(-1\) and \(1\)) into the interval \((-1, 1)\) (on which Chebyshev’s \( T_d(x) \) are bounded between \(-1\) and \(1\)).

### 2.6.1 \( L_{1, [k]} \) in terms of \( T_1 \)

\[
L_{1, [k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = x = T_1(x)
\]
2.6.2 \( L_{2,[k]} \) in terms of \( T_2 \)

\[
L_{2,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_2(x) \quad \text{for } k \equiv 1 \text{ mod } 2
\]

\[
L_{2,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_2(x) + \frac{2}{k(k - 2)}(x^2 - 1) \quad \text{for } k \equiv 0 \text{ mod } 2
\]

2.6.3 \( L_{3,[k]} \) in terms of \( T_3 \)

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) \quad \text{for } k \equiv 1 \text{ mod } 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{4}{k(k - 2)}(x^3 - x) \quad \text{for } k \equiv 2 \text{ mod } 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{16}{(k + 1)(k - 3)}(x^3 - x) \quad \text{for } k \equiv 3 \text{ mod } 4
\]

\[
L_{3,[k]} \left( \frac{k - 1}{2} x + \frac{k + 1}{2} \right) = T_3(x) + \frac{4}{k(k - 2)}(x^3 - x) \quad \text{for } k \equiv 0 \text{ mod } 4
\]
2.6.4 \( L_{4,[k]} \) in terms of \( T_4 \)

Unfortunately, for \( d = 4 \) there is no nice pattern in terms of congruency classes of \( k \) mod some-number like there is for \( d = 1, 2, \) and 3. For \( 5 \leq k \leq 21 \) we have that

\[
\begin{align*}
L_{4,[5]} \left( \frac{4}{2} x + \frac{6}{2} \right) &= T_4(x) + \frac{8}{3} (x^4 - x^2) \\
L_{4,[6]} \left( \frac{5}{2} x + \frac{7}{2} \right) &= T_4(x) + \frac{1}{64} (x^2 - 1)(113x^2 - 25) \\
L_{4,[7]} \left( \frac{6}{2} x + \frac{8}{2} \right) &= T_4(x) + \frac{1}{10} (x^4 - x^2) \\
L_{4,[8]} \left( \frac{7}{2} x + \frac{9}{2} \right) &= T_4(x) + \frac{1}{288} (x^2 - 1)(97x^2 - 49) \\
L_{4,[9]} \left( \frac{8}{2} x + \frac{10}{2} \right) &= T_4(x) + \frac{8}{63} (x^4 - x^2) \\
L_{4,[10]} \left( \frac{9}{2} x + \frac{11}{2} \right) &= T_4(x) + \frac{1}{256} (x^2 - 1)(139x^2 - 27) \\
L_{4,[11]} \left( \frac{10}{2} x + \frac{12}{2} \right) &= T_4(x) + \frac{49}{72} (x^4 - x^2) \\
L_{4,[12]} \left( \frac{11}{2} x + \frac{13}{2} \right) &= T_4(x) + \frac{1}{1728} (x^2 - 1)(817x^2 - 121) \\
L_{4,[13]} \left( \frac{12}{2} x + \frac{14}{2} \right) &= T_4(x) + \frac{1}{10} (x^4 - x^2) \\
L_{4,[14]} \left( \frac{13}{2} x + \frac{15}{2} \right) &= T_4(x) + \frac{1}{3520} (x^2 - 1)(401x^2 - 169) \\
L_{4,[15]} \left( \frac{14}{2} x + \frac{16}{2} \right) &= T_4(x) + \frac{1}{300} (x^4 - x^2) \\
L_{4,[16]} \left( \frac{15}{2} x + \frac{17}{2} \right) &= T_4(x) + \frac{1}{416} (x^2 - 1)(47x^2 - 15) \\
L_{4,[17]} \left( \frac{16}{2} x + \frac{18}{2} \right) &= T_4(x) + \frac{8}{63} (x^4 - x^2) \\
L_{4,[18]} \left( \frac{17}{2} x + \frac{19}{2} \right) &= T_4(x) + \frac{1}{10080} (x^2 - 1)(2881x^2 - 289) \\
L_{4,[19]} \left( \frac{18}{2} x + \frac{20}{2} \right) &= T_4(x) + \frac{1}{10} (x^4 - x^2) \\
L_{4,[20]} \left( \frac{19}{2} x + \frac{21}{2} \right) &= T_4(x) + \frac{1}{16128} (x^2 - 1)(1297x^2 - 361) \\
L_{4,[21]} \left( \frac{20}{2} x + \frac{22}{2} \right) &= T_4(x) + \frac{8}{2499} (x^4 - x^2).
\end{align*}
\]
Bibliography


