Detecting Majorana Fermion Induced Crossed Andreev Reflection

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Detecting Majorana Fermion Induced Crossed Andreev Reflection

by

Lei Fang

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2017
This manuscript has been read and accepted by the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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The City University of New York
Abstract

Detecting Majorana Fermion Induced Crossed Andreev Reflection

by

Lei Fang

Adviser: Professor David Schmeltzer

This dissertation is devoted to a study of detecting the Majorana fermion induced crossed Andreev reflection.

Majorana fermions are particles that constitute their own antiparticles. In condensed matter physics, Majorana fermions are zero energy modes that reside at edges or around vortices of topological superconductors. The special properties of Majorana fermions result in their potential to conduct topological quantum computation, which has been attracting a lot of current research. One of the most important issues in the field of the Majorana fermion physics now is to detect their existence in realistic systems. Among many classes of detecting methods, a transport experiment is the most direct one. The existence of Majorana fermions induces uncommon Andreev reflections that can happen at interfaces between normal regions and superconducting regions. There are two types of Andreev reflections: local Andreev reflection and crossed Andreev reflection. The Majorana fermion induced local Andreev reflection yields zero-bias conductance peaks that have been observed by many experimentalists. Nonetheless, the zero-bias peak alone cannot be taken as a conclusive evidence for the existence of Majorana fermions, because several other mechanisms can lead to analogous results. On the other hand, the Majorana fermion induced crossed Andreev reflection yields equal probabilities of electron tunneling and hole tunneling. It is thus usually believed that a direct measurement of the tunneling current is impossible and a measure-
ment of the shot noise is necessary for the detection of the Majorana fermion induced crossed Andreev reflection.

My main contribution is a new experimental proposal that is aimed at changing this opinion. In my proposal, a metallic ring structure is employed to separate the electron tunneling signals and the hole tunneling signals, and as a result the tunneling current is measurable. The key idea behind this proposal is that the constructive interference condition for the tunneling electrons in the metallic ring is different from that for the tunneling holes, utilizing the fact that their wave-vectors are different. The signature of the tunneling current in my intended set-up is that it changes sign from the electron tunneling dominated regime to the hole tunneling dominated regime, where the control parameter is the magnetic flux threading the ring.
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Chapter 1

Introduction

Majorana fermions are fermions that are their own antiparticles. They were first proposed by Ettore Majorana in [Majorana (1937)]. The Majorana fermions are in contrast to the Dirac fermions [Dirac (1928)], which like electrons and quarks constitute the main part of our universe. In high energy physics, it is hypothesized that neutrinos are Majorana fermions, but this has never been verified [Avignone et al. (2008)].

In condensed matter physics, due to the particle-hole symmetry in the formalism to describe superconductor [Elliott and Franz (2015)], the Bogoliubov quasi-particles can be taken as Majorana fermions. The particle-hole symmetry here is actually a redundancy, revealed in the fact that to create a positive energy state is the same as to annihilate a negative energy state: $\gamma^+_E = \gamma_{-E}$. If the quasi-particle state has zero energy, then we have $\gamma^+ = \gamma$, which is very special. These zero-energy Bogoliubov quasi-particle states with the property of $\gamma^+ = \gamma$ are called the Majorana zero modes. For certain unclear reasons, however, it is a convention in condensed matter physics community to call Majorana zero modes just Majorana fermions. So usually when people mention Majorana fermions they mean Majorana zero modes. In this dissertation, I take this convention.

The Majorana fermions’ characterization identity $\gamma^+ = \gamma$ makes them neither fermions nor
bosons. They belong to a general class of anyons [Wilczek (1982)]. Majorana fermions can be employed to store quantum informations in a non-local manner, resulting in a promising way to implement topological quantum computations [Ivanov (2001)] [Nayak et al. (2008)] [Alicea et al. (2011)] [Mong et al. (2014)].

It has been proposed theoretically that Majorana fermions can appear as edge states and vortex states in p-wave superconductors [Read and Green (2000)], on the surface of a topological insulator in proximity to an s-wave superconductor [Fu and Kane (2008)], at the ends of a spin-orbit coupled nano-wire that is in proximity to an s-wave superconductor and experiencing a Zeeman field [Sau et al. (2010)] [Lutchyn et al. (2010)] [Oreg et al. (2010)].

On the experimental side, there have been now many groups claiming to have observed Majorana fermions. The works of [Mourik et al. (2012)] [Deng et al. (2012)] employed a strong spin-orbit coupled InSb nanowire in proximity to the superconducting Nb. The work of [Das et al. (2012)] used a spin-orbit coupled InAs nanowire in proximity to the superconducting Al. The work of [Nadj-Perge et al. (2014)] fabricated ferromagnetic iron atomic chains on the surface of superconducting lead (Pb). The works of [Xu et al. (2015)] [Sun et al. (2016)] grew topological insulator Bi$_2$Te$_3$ films on a superconductor NbSe$_2$.

Majorana fermions are charge neutral due to the particle-hole symmetry, which makes the detection work non-trivial. There are three classes of detection methods [Alicea (2012)]: tunneling experiments [Law et al. (2009)] [Fidkowski et al. (2012)] [Nilsson et al. (2008)], fractional Josephson effect [Kitaev (2001)] and interferometry [Fu and Kane (2009)] [Akhmerov et al. (2009)]. Within them, the tunneling experiments are the most direct and have been performed most often.

The tunneling experiments are based on the Andreev reflection phenomenon [Andreev (1964)]. There are two types of Andreev reflections: local and crossed. The Majorana fermion induced local Andreev reflections lead to zero-bias conductance peaks [Fidkowski et al. (2012)], which have been observed by many experimental groups [Mourik et al. (2012)] [Das et al.
CHAPTER 1. INTRODUCTION

Figure 1.1: The Majorana fermion induced Andreev reflection. Here $\gamma_1$ and $\gamma_2$ are two Majorana fermions. “NM” denotes normal metal, “SC” denotes superconductor, “AR” denotes Andreev reflection. Nonetheless, it is widely thought that the observation of zero-bias conductance peaks alone cannot be taken as the conclusive evidence for the existence of Majorana fermions. Many other mechanisms may give analogous results. Thus, we are required to find more evidences.

On the other hand, the Majorana fermion induced crossed Andreev reflection leads to equal tunneling probabilities of electron and hole. So a direct tunneling current measurement gives nothing and it is usually believed that the measurement of shot noise is necessary.

My work is devoted to change this opinion. We note that tunneling electrons and tunneling holes actually do not constitute particles and anti-particles. They are distinguished by wave-vectors. Electrons are above the Fermi level, having wave-vectors
larger than the Fermi wave-vector, while holes are below the Fermi level, having wave-vectors smaller than the Fermi wave-vector. The point of our method is to employ this difference to separate electron and hole tunneling signals, and our key set-ups are metallic rings.

From the very beginning of the quantum mechanics we know that the quantization condition of an electron staying in the ring is \( kL = 2n\pi \) (\( k \) is its wave-vector, \( L \) is the circumference of the ring and \( n \) is an integer). If we thread a magnetic flux through the ring, the electron experiences the AharonovBohm effect and acquires a magnetic phase [Aharonov and Bohm (1959)]. The quantization condition is modified to \( kL + \phi_M = 2n\pi \) (\( \phi_M = \frac{2\pi}{\Phi_0} \Phi \)) denotes the magnetic phase an electron acquires when circling the ring).

We can adjust the magnetic fluxes to let tunneling electrons or tunneling holes be at resonance in the ring. Since tunneling electrons and tunneling holes have different wave-vectors, they do not need to be at resonance simultaneously. If electrons are at resonance but holes are not, we are able to see an electron tunneling signal; vice versa.

This dissertation is dedicated to a detail study of this mechanism. After the introduction of some physical background in Chapter 2, the properties of the metallic rings are investigated in Chapter 3. Then the coupled systems of metallic rings and Majorana fermions are studied and their transport properties are investigated in Chapter 4. Finally, a conclusion is given in Chapter 5.
Chapter 2

Physical Background

2.1 Single Particle Theories

The modern solid state physics originated from the discovery of quantum mechanics, which was immediately used to explain the behaviour of a single electron in periodic lattices. The Bloch theorem states that the existence of lattice atoms does not prevent the electron’s free propagation, but only modulates it in a form of Bloch waves. The importance of the Bloch theorem is shown in the explanation of the fact that free electrons in metals have very long mean free path, compared to the distance between two adjacent atoms [Ashcroft and Mermin (1976)].

However, electrons are charged particles, with very strong Coulomb interactions between each other. Normally, a many-body system with interactions is extremely difficult, if not impossible, to deal with. A great insight came from Landau, who showed in his Fermi liquid theory [Landau (1958)] that at low temperature, the low energy physics is dominated by quasi-particles, which are almost free of interactions and have very long life-time [Abrikosov et al. (1975)] [Lifschitz and Pitaevskii (1980)]. The concept of quasi-particle plays an important role in the traditional condensed matter physics, because it makes a lot of single particle
models, which are easy to deal with, relevant to our realistic world.

The band theory (based on the Bloch theorem) belongs to the single particle models. It forms the foundation to our knowledge, besides many other things, of the classification of solids into metals, insulators and semi-conductors \cite{Ashcroft, Mermin}. Although the band theory had a very long history, it turned out that we had not got its full picture until very recently. The missing part was its topological structure, which was brought to us gradually only after the discovery of the integer quantum Hall effect \cite{Klitzing, et al.}. The work of Thouless, Kohmoto, Nightingale and Nijs \cite{Thouless, et al.} (abbr. TKNN) was the first to reveal the topological structure of the band. They expressed the Hall conductance as an integral of the Berry curvature over the whole Brillouin zone. The integral was later identified as the first Chern number \cite{Chern, Bott, Chern}, which is a well-known topological character in mathematics. The theory of TKNN was put in a more general framework by \cite{Haldane}, but since then had been put away for many years, and people had been thinking that the Chern number characterized the band topology in a complete way. It was not until 2005 that Kane and Mele, in their famous work \cite{Kane, Mele}, proposed a new topological band structure under the constraint of the time-reversal symmetry. This work started an era of the topological insulators \cite{Hasan, Kane}, \cite{Qi, Zhang}.

The superconductivity was initially discovered by Onnes in 1911 \cite{Onnes}, and has been a fascinating phenomenon for physicists since then. The microscopic theory of superconductivity came from the classical work of Bardeen, Cooper and Schrieffer \cite{Bardeen, et al.}. Their theory, now named as the BCS theory, is based on the key idea that electrons at Fermi surface are unstable under attractive interactions and tend to form electron pairs called Cooper pairs \cite{Cooper}. A Cooper pair is a bound state of two electrons, of which the statistical property is like that of the boson. The superconducting ground-state is consisted of a condensate of Cooper pairs, which is analogous to the Bose-Einstein condensate.
Bose (1924)] [Einstein (1925)]. Obviously, it is very different from the normal Fermi liquid. However, interestingly enough, the elementary excitations upon the superconducting ground-state are also quasi-particles, called the Bogoliubov quasi-particles [Bogoliubov (1958)]. They are also almost interaction-free and can be described by a single particle equation — the Bogoliubov-de Gennes equation [Gennes (1999)] (abbr. BdG equation). It turns out that the idea of the band topology can be applied to the BdG equation [Hasan and Kane (2010)] as well, and it results in the concept of the topological superconductor [Hasan and Kane (2010)] [Qi and Zhang (2011)].

Although the band topology is defined as a bulk property of the material, its physical effect is manifested most evidently on the boundary between two materials with different topologies. This is because the bulk topology is protected by the single particle spectrum gap and cannot change smoothly from one material to another without closing the gap. Thus, there must exist states locating at the boundary which have energies inside the bulk gap. These boundary states are also protected by the bulk topology. At the boundary of a topological superconductor, due to the particle-hole symmetry of the BdG equations, the kind of topological protected boundary states are Majorana fermions [Alicea (2012)], which are their own antiparticles. A Majorana fermion is only half of a normal fermion. It belongs to actually the class of neither fermion nor boson but anyon [Wilczek (1982)]. Majorana fermions have been attracting a lot of interests among condensed matter physicists in recent years because of its potential application in topological quantum computations [Navak et al. (2008)], which serves a promising way to realize quantum computers.

2.2 Topological Band Theory

In this section, I would like to explain the topological band theory [Hasan and Kane (2010)]. After a brief introduction to the traditional band theory, I will show its topological structure
through examples of the integer quantum Hall effect and topological insulators. At the end, the topologically protected edge states are illustrated.

2.2.1 Band theory

The band theory is based on the Bloch theorem [Bloch (1929)]. In this section, we do not pursue the most general form but use a simple model to illustrate the theory.

Suppose we have a single spin-less electron inside a one-dimensional (1D) lattice. The Hamiltonian is

\[ H = \frac{p^2}{2m} + V(x) , \]

where \( m \) is the electron’s mass and \( V(x) \) is the lattice potential that fulfil \( V(x+a) = V(x) \) (\( a \) is the lattice constant). This Hamiltonian has a lattice-translation symmetry, which means if \( \psi(x) \) is an eigenfunction of \( H \) with energy \( E \)

\[ H \psi(x) = E \psi(x) , \]

then \( \psi(x + a) \) is also an eigenfunction with energy \( E \). We can define a lattice-translational operator \( T_a \):

\[ T_a \psi(x) = \psi(x + a) , \]

and it is easy to check

\[ [T_a, H] = 0 . \]

This means we can form a complete set of eigenfunctions of \( H \) that are eigenfunctions of \( T_a \) at the same time. \( T_a \) is a unitary operator with eigenvalues being a phase factor (a complex number of unit modules). It is convenient for us to denote this phase factor as \( e^{ika} \), where \( k \) is a real number that we call the lattice wave-vector. \( k \) is a good quantum number, which we can use to characterize the eigenstate. Noting the periodic property of the phase factor,
$k$ and $k + \frac{2\pi}{a}$ actually denote the same state. In this sense, we can take $k$ to be defined on a circle with a period $\frac{2\pi}{a}$. This circle is called the irreducible Brillouin zone [Brillouin (1930)][Ashcroft and Mermin (1976)].

Let $\psi_k(x)$ be the eigenfunction with

$$T_a \psi_k(x) = e^{ika} \psi_k(x) . \tag{2.5}$$

If we define

$$\psi_k(x) \equiv e^{ikx} u_k(x) , \tag{2.6}$$

then $u_k(x)$ is a periodic function with the period to be the lattice constant $a$:

$$u_k(x + a) = u_k(x) . \tag{2.7}$$

This is just the content of the Bloch theorem, and $\psi_k(x)$ is called the Bloch wave.

We can go one step further here. Now we take $u_k(x)$ to be defined in the unit cell $[0, a]$ with the periodic boundary condition. It’s not difficult to see that $u_k(x)$ satisfies

$$H_k u_k(x) = E(k) u_k(x) , \tag{2.8}$$

where

$$H_k = \frac{p^2}{2m} + V(x) + \frac{\hbar^2 k^2}{2m} + \frac{\hbar}{m} k \cdot p . \tag{2.9}$$

Solving this Schrödinger equation for $u_k(x)$, we can get a discrete set of eigenvalues $E_n(k)$, where the index $n$ is another good quantum number charactering the energy.

Since $H_k$ is a group of Hamiltonians parametrized by the lattice wave-vector $k$, for each $n$ we have a group of eigenstates $u_{nk}$ with eigenvalues $E_n(k)$. Now $E_n(k)$ is a continuous function of $k$ and we say it denotes the $n$-th energy band. It is obvious there may exist an
energy gap between two adjacent energy bands.

Although we have only shown the band theory in its simplest form, namely a single spinless electron in a 1D periodic lattice, all its key characteristics for our later use are included. We make a summary here: 1) The Hamiltonian has a lattice-translational symmetry; 2) The lattice wave-vector $k$ and the band index $n$ are two good quantum numbers; 3) The lattice vector $k$ is defined on the irreducible Brillouin zone, which has a periodic structure (circle in 1D); 4) The eigenfunction has the form $\psi_{nk}(x) = e^{ikx}u_{nk}(x)$, where $u_{nk}(x)$ is a periodic function with the period to be the lattice constant and can be described by a $k$-parametrized Hamiltonian $H_k$ defined on the unit cell.

\subsection*{2.2.2 Integer Quantum Hall Effect}

Historically, the Hall effect was discovered by Hall \cite{Hall1879} even before the discovery of the electron \cite{Thomson1897}. Hall considered the effect of an external magnetic field to the electric current inside a conductor, and found out that when the external magnetic field is perpendicular to the direction of the current, there appears a voltage in the transverse direction of the conductor (perpendicular to both the direction of the current and the direction of the magnetic field). This induced voltage is now called the Hall voltage $V_H$. The ratio between the Hall voltage and the current is called the Hall resistance $R_{xy}$. The inverse of $R_{xy}$ is called the Hall conductance $G_{xy}$. $G_{xy}$ is proportional to the cross sectional area $S$ of the conductor. The Hall conductivity is defined by $\sigma_{xy} = \frac{G_{xy}}{S}$. From the classical physics of Lorentz force, it’s a simple exercise to find $\sigma_{xy} = \frac{ne}{B}$, where $B$ is the magnitude of the external magnetic field, $n$ is the electron number density and $e$ represents the electron charge. Thus we get to know $\sigma_{xy}$ is a continuous function with respect to $B$ from the classical physics.

However, about one century later, a rather surprising result of the Hall effect came out \cite{Klitzing1980}. It was found that at extremely low temperature and in a strong
perpendicular magnetic field, the Hall conductivity at a two-dimensional interface between a semiconductor and an oxide insulator is quantized. The reason of course cannot be explained in the framework of the classical physics.

The motion of an electron in a uniform magnetic field was first studied by Landau using quantum mechanics [Landau (1930)] [Landau and Lifschitz (1981)]. Landau found that if we restrict the electron in a two-dimensional plane perpendicular to the magnetic field, the electron’s energy levels are highly degenerate. Each energy level, now called a Landau level, holds a number of $\frac{eB}{h}$ electron states per unit area. The energy gap between two adjacent Landau levels is $\frac{eB}{m}$. The Landau levels are actually very like flat energy bands.

Now suppose electrons occupy an integer number $\nu$ of Landau levels. The electron number density is $\nu\frac{eB}{h}$. If we tentatively plug this into the classical expression of $\sigma_{xy}$, we obtain $\sigma_{xy} = \nu\frac{e^2}{h}$ ($\nu \in \mathbb{Z}$). These are just the quantized Hall conductivities that have been observed in experiment. Since $\nu$ is an integer here, it is called the integer quantum Hall effect (abbr. IQHE). If we change the magnetic field $B$ a little, and thus change the number of states in

Figure 2.1: The Hall effect.
each Landau level a little, then it may happen a non-integer number of Landau levels are occupied. But in this case the Hall conductivity can still be quantized, because the excess electrons can occupy localized states (due to the disorder in the material) that do not affect the transport properties.

The above explanation of IQHE is more or less naive. After the experimental discovery there appeared many theoretical works, of which the notable ones are [Laughlin (1981)] [Halperin (1982)] [Thouless et al. (1982)] [Haldane (1988)]. The TKNN paper [Thouless et al. (1982)] was the first to reveal that the quantized Hall conductivity is related to the topology of the band structure. They used the Kubo formula of the linear response theory [Kubo (1957)] to express the Hall conductivity as an integral involving all the eigenfunctions in the occupied bands, and found out that the final result was a winding number, which was later identified as the Chern number [Simon (1983)] [Kohmoto (1985)].

In the following I would like to show this topological structure of band while keeping the theory in a minimum level. Suppose we have a two-dimensional square lattice with the lattice constants $a$ the same in both directions. Using the Bloch theorem, we can write the eigenfunction in the Bloch wave form \( \psi_{k_x k_y}(x, y) = e^{i(k_x x + k_y y)} u_{k_x k_y}(x, y) \), where \( u_{k_x k_y}(x, y) \) can be described by a parametrized Hamiltonian \( H(k_x, k_y) \). When there is a perpendicular magnetic field in the system, \( u_{k_x k_y}(x, y) \) fulfils a general boundary condition in the unit cell (or more precisely the magnetic unit cell).

Let’s restrict to one band and suppose it is well separated with all other bands in energy. We know that the global phase of \( u_{k_x k_y}(x, y) \) is indeterminate in the Schrödinger equation

\[
H(k_x, k_y) u_{k_x k_y} = E(k_x, k_y) u_{k_x k_y} ,
\]

which means we have a freedom to choose the eigenfunction. Let \( u_{k_x k_y}(x, y) \) be one choice,
then

\[ u'_{k_x,k_y}(x,y) = e^{i\phi(k_x,k_y)} u_{k_x,k_y}(x,y) \]  (2.11)

is another choice. They differ by a gauge transformation in the momentum space (the irreducible Brillouin zone). For the parametrized Hamiltonian, we can define a Berry connection in the parameter space [Berry (1984)]

\[ A(k_x,k_y) \equiv i \langle u_{k_x,k_y} | \nabla_k | u_{k_x,k_y} \rangle = i \int_{\text{unit cell}} d^2r \ u^*_{k_x,k_y}(x,y) \nabla_k u_{k_x,k_y}(x,y). \]  (2.12)

After the gauge transformation, \( A \) becomes

\[ A'(k_x,k_y) = A(k_x,k_y) + \nabla_k \phi(k_x,k_y). \]  (2.13)

Next, we can construct a Berry curvature

\[ B(k_x,k_y) \equiv \nabla_k \times A(k_x,k_y) = \partial_{k_x} A_{k_y} - \partial_{k_y} A_{k_x}. \]  (2.14)

The Berry curvature is gauge-invariant, since \( \nabla_k \times \nabla_k \phi(k_x,k_y) = 0 \). Now if we integrate \( B(k_x,k_y) \) over the whole irreducible Brillouin zone, we obtain the Chern number

\[ C \equiv \frac{1}{2\pi} \int d^2k \ B(k_x,k_y). \]  (2.15)

The Chern number is obviously gauge-invariant and thus an inherent property of the band. To see why it characterizes the topology, first we try to use the Stocks theorem to convert the two-dimensional integral into a line integral around the Brillouin zone’s boundary. Since the Brillouin zone is actually a torus, it has no boundary and the result of the line integral is zero. It seems not very interesting. The catch is that the global phase of \( u_{k_x,k_y} \) may be impossible to be smoothly defined all over the Brillouin zone. In the case that it cannot,
A\((k_x, k_y)\) is not single-valued and thus the use of the Stocks theorem is not proper. Let’s consider a situation where we can divide the Brillouin zone into two regions, on each of which we have a smooth global phase of \(u_{k_x,k_y}\), and thus we can define a single-valued \(A^{(i)}(k_x, k_y)\) \((i = 1, 2)\). The boundary between these two regions is a circle. We can now use the Stocks theorem in each region and transform the curvature integral into the line integral of \(A^{(i)}\) on the circle. The Chern number equals to

\[
\frac{1}{2\pi} \oint d\mathbf{r} \cdot (A^{(1)} - A^{(2)}) = \frac{1}{2\pi} \oint d\mathbf{r} \cdot \nabla \phi .
\]  

(2.16)

Since \(\phi\) is the phase difference between \(u^{(1)}_{k_x,k_y}\) and \(u^{(2)}_{k_x,k_y}\), its change on the circle should equal to \(2n\pi\) \((n \in \mathbb{Z})\). Thus the Chern number is an integer. It is easy to understand why it’s also called the winding number (meaning the winding of the phase). The argument here is similar to that of the Dirac monopole [Dirac (1931)] [Wu and Yang (1969)], except we have a gauge field in the momentum space. In a more mathematical way, the space of the band can be taken as a line-bundle on the irreducible Brillouin zone, and the Chern number just characterizes the topology of this line-bundle [Nakahara (2003)].

We have defined the Chern number for a single band. The work of TKNN [Thouless et al. (1982)] showed the value \(\nu\) in the IQHE equals to the sum of the Chern numbers for all the occupied bands. Thus, the existence of the IQHE reveals the band can have non-trivial topology. The material that has non-zero Chern number is now called the Chern insulator.

Shortly after the discovery of the IQHE, experimentalists found that under an even stronger magnetic field and in a more clean sample, the Hall conductivity could be fractionally quantized [Tsui et al. (1982)]. The theoretical explanation of the fractional quantum Hall effect (abbr. FQHE) involves the interaction between electrons and shows that it is really a many-body problem [Laughlin (1983b)] [Laughlin (1983a)]. The FQHE stands beyond the Landau Fermi liquid theory, and so the band theory (which is a single particle theory)
CHAPTER 2. PHYSICAL BACKGROUND

is not able to describe it alone. Interestingly, the FQHE also involves topology, and leads to the concept of topological order [Wen (1990)] [Wen (1995)]. Since FQHE is not directly relevant to my work in this thesis, which is almost restricted to the single particle theory, we do not go further any more.

There was a period of time when physicists thought that the external perpendicular magnetic field was necessary for the appearance of the IQHE. It was until Haldane [Haldane (1988)] proposed a model in which the local magnetic field changes sign and the net magnetic flux in a unit cell is zero. Haldane showed that the Chern number in this model could be non-zero and thus it led to the IQHE. Haldane’s model was a toy model, which did not correspond to any realistic system at the time he proposed it. However, Haldane’s idea had a long influence and actually spurred the work to the topological insulator [Kane and Mele (2005a)] [Kane and Mele (2005b)]. Moreover, the phenomenon of the IQHE without a global external magnetic field is now called the quantum anomalous Hall effect (abbr. QAHE) [Liu et al. (2016)] and has been experimentally observed already [Chang et al. (2013)].

Haldane’s work showed the global external magnetic field is not a necessary condition for the appearance of the IQHE, but the broken of the time-reversal symmetry is. If we do not consider the electron spin (or there is a spin up-down symmetry), it can be proved that the Berry curvature at Brillouin zone points that are connected by the time-reversal symmetry is opposite to each other, and thus the Chern number, which is an integral of the Berry curvature, is definitely zero. Based on this reasoning, it was believed that the topology of a time-reversal symmetric band was always trivial.

The IQHE appears in a 2D system, while not in 1D or 3D. This is because the Berry curvature, defined through the derivative of the Berry connection, is a 2-form [Nakahara (2003)], which can only be integrated on a 2D Brillouin zone. There was an effort trying to generalize the IQHE to 4D, in which the 2nd Chern number is utilized to characterize the topology [Zhang and Hu (2001)]. But since the realistic world is only of 3D, it seems that
the models in 4D have only conceptual values.

### 2.2.3 Topological Insulator

The breakthrough in a new topological band structure came from a deeper investigation to the time-reversal symmetry. The Kramers theorem [Kramers (1930)] states that the eigenstates of a time-reversal symmetric system with half-integer total spin are doubly degenerate. Now since we are restricted to the single particle models and the electron is spin-half, the energy band should be doubly degenerate when the time-reversal symmetry exists.

If the system has also a spin up-down symmetry, then a band and its time-reversal partner are in coincidence with each other. They actually can be taken as two independent same bands and so nothing is very interesting in this case.

However, if there is spin-orbit coupling in the system, which breaks the spin up-down symmetry, then the band and its time-reversal partner are not in coincidence, as shown in Fig. 2.2. Suppose now we have two bands which are time-reversal partner to each other, and we denote them by the superscripts (1) and (2). Let $\mathcal{T}$ be the time-reversal operator, which is anti-unitary and commutes with the Hamiltonian $[\mathcal{T}, H] = 0$. It transforms states between the two bands in a way that

\begin{align*}
\mathcal{T} u_k^{(1)} &= e^{i\chi_k} u_{-k}^{(2)}, \\
\mathcal{T} u_k^{(2)} &= -e^{i\chi_{-k}} u_{-k}^{(1)}
\end{align*}

where $u_k^{(1)}$ and $u_{-k}^{(2)}$ constitute a time-reversal pair, and $\chi_k$ is a global phase of the wavefunction. It is simple to check that $\mathcal{T}^2 = -1$ is fulfilled.

It is tempted to redefine the wave-function $u_k$ in order to let the global phase $\chi_k$ in the expression of the time-reversal transform be 0. However, it happens that this is not always possible [Fu and Kane (2006)]. The obstruction to do this in certain band structure serves
Figure 2.2: Time-reversal symmetric bands in 1D.

as a new type of topology for the time-reversal symmetric system.

We note that, due to the spin-orbit coupling, in general \( E^{(1)}(k) \neq E^{(2)}(k) \). But there exist certain high-symmetry points \( k^* \) in the Brillouin zone that fulfill \( k^* = -k^* \) (two \( k \) differing by \( \frac{2\pi}{a} \) represent the same point in the Brillouin zone and are taken to be equal also). Combining to the time-reversal symmetry, we have \( E^{(1)}(k^*) = E^{(2)}(k^*) \), which means the pairs of time-reversal symmetric bands have intersections at these points. We call these \( k^* \) time-reversal invariant momenta (abbr. TRIM). Fig. (2.2) shows an example of time-reversal symmetric bands in 1D. It’s obvious from the graph there are two TRIM \( k^* = 0 \) and \( k^* = \pm \pi/a \), where the pairs of the time-reversal symmetric bands intersect. Similarly, in 2D there are four TRIM, and in 3D there are eight.

The TRIM stand at a very special position in the band structure of a time-reversal invariant system, due to its symmetry protected irremovable degeneracy. The obstruction we mentioned above is intimately related to the properties of the degenerate eigenstates at
these points. We will not go further to the details here but just want to point out that this new topology can be characterized by a $Z_2$ index, which has two possible values 0 and 1. “0” represents trivial, meaning the obstruction does not exist; “1” represents non-trivial, meaning the obstruction exists. The $Z_2$ index is very different from the Chern number, which can take any integer value.

Historically, the first topological non-trivial time-reversal band model was proposed by [Kane and Mele (2005a), Kane and Mele (2005b)] in a 2D graphene system. It is then realized that the spin-orbit coupling in graphene is too weak to have observable effect. The first realistic theoretical prediction was made by [Bernevig et al. (2006)] in HgTe-CdTe quantum well system, which belongs to a family of semiconductors with strong spin-orbit coupling. This proposal was soon realized in experiment by [König et al. (2007)] and constitutes the first topological insulator (abbr. TI, which was first named in [Moore and Balents (2007)]). During the same time, the theory was generalized to 3D system [Fu et al. (2007)] with some specific materials including BiSb being predicted [Fu and Kane (2007)]. The first experimental realization of 3D TI BiSb was then reported by [Hsieh et al. (2008)]. Later, a second generation of TI BiSe was experimentally identified in [Xia et al. (2009)]. For a summary of TI materials, see the review [Ando (2013)].

### 2.2.4 Edge States

The above discussions of the topological band structure focused on the bulk of the system. In this section, we would like to show the most fascinating properties of the topological non-trivial materials are actually shown in their edge states.

Suppose that two materials with different Chern numbers or different $Z_2$ indexes are in touch. Now the bulk band topology is protected by the bulk band gap (in the TI case, also by the time-reversal symmetry), which means we cannot change the topology smoothly from one material to the other without closing the gap. There should be states inside the
bulk band gap. These states are localized at the boundary and are called edge states. Fairly speaking, there may also exist edge states at the boundary of the topological trivial materials (the vacuum outside the material can be taken to be topological trivial). However, these states are not protected by the bulk band topology and may merge into the bulk band under perturbations.

Let’s consider a 2D system, of which the boundary conditions we choose to be periodic in \( x \)-direction but open in \( y \)-direction. Since the translational symmetry is still preserved in \( x \)-direction, the wave-vector \( k_x \) continues to serve as a good quantum number that can be used to denote the eigenstates. Now focus on the electron motion in \( y \)-direction. There are two kinds of eigen-modes in \( y \)-direction. The first kind is the bulk state, which looks like a standing wave, formed by reflections of the traveling waves on the boundaries. We will see an example in Sec. 3.3.2. When the lattice is of finite size, the exact eigenvalues are different from those in the situation where a periodic boundary condition is taken. However, the spectrum becomes almost continuous as the size of the system is large, and they tends to be essentially the same. Therefore, the boundary condition has no essential effect to the bulk states for a large system. The second kind is the edge state. It corresponds to the situation that the wave-vector \( k_y \) becomes a complex number. The wave-function of this kind of mode localizes at the boundary, and exponentially decays when going deep into the bulk. Its energy level stays inside the energy gap, which is thus forbidden to the bulk states.

Fig. 2.3 is a sketch of the spectrum for the system we have just described. The good quantum number \( k_x \) is put on the horizontal axis. The lower green color represents the bulk states for the valence band; the upper yellow color represents the bulk states for the conduction band. Note that for a specific \( k_x \), the spectrum for the bulk states is continuous. The energy levels of the edge states are inside the band gap.

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Depending on the topology, the edge states have different properties. Fig. 2.3(a) shows the topological trivial situation. In this case the bulk band gap is not closed, and the edge
Figure 2.3: A sketch of the dispersion relation for a quantum Hall system with the periodic boundary condition in $x$-direction while the open boundary condition in $y$-direction. The green color below represents the valence band; the yellow color above represents the conduction band. The Fermi level is inside the band gap. (a) Topological trivial situation. The bulk gap is not closed by the edge states. (b) Topological non-trivial situation. The bulk gap is closed by the edge states.

states are possible to merge into the bulk spectrum and disappear under perturbations. Fig. 2.3(b) shows the topological non-trivial situation for an integer quantum Hall state. It is seen that the dispersion of the edge states links the valence band and the conduction band, fulfilling the requirement that the edge states must close the bulk gap. Now as long as the gap exists, the perturbation cannot change this type of link, so the edge states can never disappear. In other words, the robustness of the edge states is protected by the bulk band gap. Furthermore, the edge states shown in Fig. 2.3(b) have a single direction of the group velocity, which means the electrons can move in one way along the edge [Halperin (1982)]. This type of edge states is called chiral.

There are many more examples that showing the interplay between the non-trivial bulk topology and robust edge states. The famous ones in 1D include [Jackiw and Rebbi (1976)] and [Su et al. (1979)].
CHAPTER 2. PHYSICAL BACKGROUND

2.3 Superconductors

The microscopic description of the superconductivity, namely the Bogoliubov-de Gennes (BdG) formalism, bears almost the same structure as the band theory. The superconductor has a superconducting gap that plays the same role as the band gap to the normal material. The bulk band topology has a counterpart in superconductors.

2.3.1 Bogoliubov-de Gennes Formalism of superconductivity

In the BCS theory, the microscopic reason for the superconductivity is that electrons near the Fermi surface form Cooper pairs, which then phase-coherently condensate. The original work of BCS [Bardeen et al. (1957)] gave a ground-state wave-function ansatz and investigated the excitations upon it. It was later found that it is more convenient to use an equivalent mean-field theory [Gennes (1999)], which is named the BdG formalism.

Generally, the mean-field Hamiltonian for a superconductor can be written as

\[ H = \sum_{mn} \left\{ c_m^\dagger h_{mn} c_n + \frac{1}{2} \Delta_{mn} c_m^\dagger c_n^\dagger + \frac{1}{2} \Delta_{mn}^* c_n c_m \right\} . \] (2.19)

Here \( c \) and \( c^\dagger \) are electron annihilation and creation operators, \( h \) is the single particle Hamiltonian, \( \Delta_{mn} = \langle c_m c_n \rangle \) represents the superconductor paring potential, and the sum is taken over all the degrees of freedom (including orbitals and spins). We have \( h_{mn} = h_{nm}^* \) due to the hermiticity of \( h \) and \( \Delta_{mn} = -\Delta_{nm} \) due to the anti-commutation of the fermion operators.

Do a transformation

\[ \gamma = \sum_n (u_n c_n + v_n c_n^\dagger) , \] (2.20)

and suppose the operator \( \gamma \) can diagonalize the Hamiltonian in the sense that

\[ [\gamma, H] = EH. \] (2.21)
Substituting Eq. (2.20) into Eq. (2.21) and employing the commutation relations \(\{c_m, c_n^\dagger\} = \delta_m n\) and \(\{c_m, c_n\} = \{c_m^\dagger, c_n^\dagger\} = 0\), we obtain an equation for all the operators of \(c_m\) and \(c_m^\dagger\).

By requiring their coefficients to be zero, we have

\[
\sum_n \begin{pmatrix} h_{mn} & \Delta_{mn} \\ \Delta_{nm}^* & -h_{nm}^* \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = E \begin{pmatrix} u_m \\ v_m \end{pmatrix}.
\]

(2.22)

Now the transformation Eq. (2.20) is called the BdG transformation and the equation Eq. (2.22) is called the BdG equation. The BdG equation can be taken as an eigenvalue equation. \((u_n, v_n)\) is the state vector. It can be checked that the matrix on the left side of Eq. (2.22) is hermitian. So the BdG equation is the same as a Schrödinger equation.

There is a particle-hole symmetry in the BdG equation (2.22). If \((u, v)^T\) is an eigenfunction of (2.22) with an eigenvalue \(E\), then \((v^*, u^*)^T\) is an eigenfunction with an eigenvalue \(-E\). This relation is also reflected in the commutator Eq. (2.21). If \([\gamma E, H] = EH\), then \([\gamma E^\dagger, H] = -EH\). So we have \(\gamma_{-E} = \gamma_{E}^\dagger\). This is just the property of the Majorana fermion in the general sense that we have mentioned in the introduction.

When we find out all the eigenvalues and eigenvectors of the BdG equation, the original Hamiltonian can be diagonalized in the form

\[
H = E_g + \sum_{E_n \geq 0} E_n \gamma_n^\dagger \gamma_n,
\]

(2.23)

where \(E_g\) is the ground-state energy. The sum is taken over only the positive energy states due to the particle-hole redundancy \(\gamma_{-E} = \gamma_{E}^\dagger\).

### 2.3.2 Topological Superconductors and Majorana Fermions

We have seen that the BdG equation bears the same structure as the usual Schrödinger equation. So the topological band structure applies to superconductors as well. In the case of the
topological insulators, we have the time-reversal symmetry as a constraint to the eigenstates in the Brillouin zone. Now we have the particle-hole symmetry as the constraint. Actually, there is a complete classification of possible topological band structures based on the discrete symmetry of the system \cite{Schnyder2008,Ryu2010}. Superconductors that have topological non-trivial band structures are called topological superconductors.

A superconductor usually has a superconducting gap in the bulk, while the edge states can stay inside the bulk gap. The existence of edge states in topological superconductors is protected by the particle-hole symmetry. The zero energy edge states are the Majorana zero modes, which in this dissertation we follow the convention to call Majorana fermions.

A concrete model of Majorana fermions is given in Appendix A.

2.4 Quantum Transport and Landauer-Büttiker Formalism

Traditional transport problems are studied in the framework of the Boltzmann transport equation \cite{LifschitzPitaevskii1981}, of which the theory is usually based on the assumptions: 1. the scattering process is local in space and instantaneous in time; 2. the scattering is weak and the fields are low; 3. the time scale is much longer than the mean free time between two adjacent collisions. The Boltzmann transport theory is a classical or semi-classical one that works well in the macroscopic world.

Starting from 1980s, physicists become capable to manufacture structures and devices that are smaller than the mean free path of electrons in materials. The Boltzmann transport equation does not apply to such situations because the quantum mechanics becomes essential. Transport processes in such small scale are called quantum transport.

The method to describe the quantum transport bears the name of the Landauer-Büttiker formalism, for their original contributions to this field \cite{Landauer1957,Landauer1970}.
Figure 2.4: The configuration of a two-terminal quantum transport experiment. “S” stands for a source reservoir. “D” stands for a drain reservoir. The box in the middle represents the core structure that we are interested in.

There are now many good review articles and books that illustrate the quantum transport theory. We do not go deep into this topic here but are satisfied to illustrate it in a form that we will use later.

Fig. (2.4) shows a set-up for a typical two-terminal quantum transport experiment. First we have two reservoirs: source and drain. Reservoirs are very large contacts that can hold many electrons and are assumed to be in equilibrium even in the process of the transport. Then leads, which are metallic wires, are used to connect the reservoirs to the core of a mesoscopic structure that we would like to detect. Electrons are able to leak out from a reservoir, flow along the lead into the core structure, after some quantum coherent processes jump out of the core structure and flow along leads to a different reservoir or back to the original reservoir. When there is no voltage bias and temperature difference between the two reservoirs, such processes are balanced and no net current exists. However if we exert a voltage bias between the reservoirs, a transport current appear.

The Landauer-Büttiker formalism is devoted to calculate the transport current. It is of
a two-step to use. First, we take the leads to be infinite long and solve a scattering problem assuming an electron is incident from the far end of one lead. We find out the transmission amplitude \( t \) as a function of the incident energy \( E \). Second, when a voltage bias \( V \) is exerted between the source and the drain, the transmission current in the zero temperature is given by

\[
I = \frac{e}{h} \int_0^{eV} dE \, |t|^2,
\]

where \( e \) is the electron charge and \( h \) is the Planck constant. It is equivalent to an expression of the differential conductance

\[
G(E) \equiv \frac{dI}{dV} = \frac{e^2}{h} |t|^2.
\]

When the system involves a superconductor in the core structure, holes can be generated from the Andreev reflection process. Since holes carry opposite charges compared to electrons, the expression for the transmission current need to be adjusted.
Chapter 3

Metallic Rings

In this chapter we study the properties of metallic rings.

We first focus on a single metallic ring, through the center of which a magnetic flux can be threaded. In this set-up, the persistent current inside the ring as a function of the magnetic flux is of basic interest. We would like also to study the transport problem by letting two external leads separately connect the ring to a source reservoir and a drain reservoir. For our purpose, we limit to the phase coherent transport and assume no disorder in the ring.

Next, we place two metallic rings side by side and let them couple through the tunneling effect. Magnetic fluxes are threaded through the two rings. We first investigate the eigenstates for this coupled system and then study the phase coherent transport.

3.1 A Single Metallic Ring

In this section we study a single metallic ring threaded by a magnetic flux.
3.1.1 Persistent Current

Let’s consider a single isolated metallic ring, with a magnetic flux \( \Phi \) threaded through its center. The system can be described by a Hamiltonian

\[
H = \int_{-L/2}^{+L/2} dx \, \Psi\!\dagger(x) \left[ \frac{1}{2m} \left( p - eA \right)^2 - \mu \right] \Psi(x),
\]

where \( L/2 \) and \( -L/2 \) denote the same point in the ring, \( m \) is the electron effective mass, \( e \) is the electron charge, \( A \cdot L \) equals to the magnetic flux \( \Phi \), \( \Psi\!\dagger(x) \) and \( \Psi(x) \) are electron creation and annihilation operators, and \( \mu \) is the chemical potential. We do not explicitly consider the spin degree of freedom here. In realistic situations, the spin may be polarized under a magnetic field. Alternatively, when there is no spin-orbit coupling, the spin-up and spin-down states are degenerate and has no essential difference. In this case only the density of states doubles.

Now suppose that an operator

\[
\Gamma = \int_{-L/2}^{+L/2} dx \, \phi(x) \Psi(x)
\]

diagonalize the Hamiltonian. We have

\[
[\Gamma, H] = E\Gamma,
\]

where \( E \) is the energy of this eigenstate. Explicitly calculating the commutator and comparing the coefficient in front of \( \Psi(x) \) leads to the Schrödinger equation for the wave-function \( \phi(x) \):

\[
\hbar \phi(x) = E\phi(x),
\]
CHAPTER 3. METALLIC RINGS

Figure 3.1: A single metallic ring with a magnetic flux $\Phi$ threaded through its center.

where

$$h = \frac{1}{2m} (p - eA)^2 - \mu$$

is the Hamiltonian in the first quantization form.

It is easy to see that the solution to this Schrödinger equation is a plane wave

$$\phi(x) = \frac{1}{\sqrt{L}} e^{iKx}.$$  \hfill (3.6)

The boundary condition $\phi(+L/2) = \phi(-L/2)$ requires

$$K = \frac{2n\pi}{L} \quad (n \in \mathbb{Z}) .$$  \hfill (3.7)

The eigenenergy has the form

$$E = \frac{1}{2m} (\hbar K - eA)^2 - \mu .$$  \hfill (3.8)

In the ground-state of the system, all the single particle eigenstates with negative energies are occupied. It is convenient to set $k = K - \frac{e}{\hbar} A$, then all the states with $\frac{(hk)^2}{2m} < \mu$ are
occupied, as shown in Fig. (3.2).

The persistent current that a single eigenstate carries is

\[ j_n = -\frac{\partial E_n}{\partial \Phi} = \frac{e}{L} \frac{\hbar k_n}{m}, \]  

where the subscript \( n \) denotes the same as that appears in Eq. (3.7). Note that \( \hbar k_n/m \) is the velocity of the electron and \( L \) is the circumference of the ring.

The total persistent current in the ground-state of the metallic ring is the sum of that carried by all the single particle eigenstates

\[ j = \sum_{E_n < \mu} j_n. \]
CHAPTER 3. METALLIC RINGS

Then a simple algebra leads to

\[ j = \frac{e^2 \Phi_0}{mL^2} \left[ \left( \sum_{E_n < \mu} n \right) - N \frac{\Phi}{\Phi_0} \right], \tag{3.11} \]

where \( N \) equals to the number of electrons in the ground states, and \( \Phi_0 = \frac{\hbar}{e} \) is the flux quantum for a single electron. From this expression, we can see the relationship between the persistent current and the magnetic flux \( \Phi \).

If we assume the number of electrons in the ring is a constant, then \( N \) is a constant for whatever \( \Phi \). But the sum \( \left( \sum_{E_n < \mu} n \right) \) is a jump function of \( \Phi \), of which the jump happens when an energy-level enters into the Fermi sea from one side of \( k \) and at the same time an energy-level leaves out of the Fermi sea from the other side.

If we vary the magnetic flux \( \Phi \), normally when there is no jump happening, the persistent current \( j \) is a linear function of \( \Phi \). When the jump happens, the persistent current has a sudden decrease of \( 2j_0 \) where \( j_0 = \frac{e^2 \Phi_0}{mL^2} \).

The exact relationship between \( j \) and \( \Phi \) is related to the number of electrons in the ring. There is a qualitatively difference between the situations of odd number of electrons and even number of electrons, as can be seen in Fig. (3.3).

Note that there is a period of \( \Phi_0 \) for the function \( j(\Phi) \) in both situations, which is due to the spectrum of the system keeps the same when the magnetic flux changes by \( \Phi_0 \). This is a special case of a more general theorem [Byers and Yang (1961)], which states that all the physics are the same when the magnetic fluxes differ by an integer number of the flux quantum \( \Phi_0 \).

3.1.2 Phase Coherent Transport

Now we connect the ring through two external leads to a source reservoir and a drain reservoir separately. Using the Landauer-Büttiker formalism, we can study its transport property.
(a) when the number of electrons in the ring is odd

(b) when the number of electrons in the ring is even

Figure 3.3: The persistent current inside the metallic ring.

The details of this study was first given in Büttiker et al. (1984). For our purpose, we restrict the discussion to the weak coupling case and only show the main calculation method and the final results.

When we exert a negative voltage bias to the source reservoir and ground the drain
reservoir, electrons emit from the source and flow along the first lead until encountering the ring. At the contact between the lead and the ring, a scattering process happens. After the scattering, electrons have some probability hopping on to the ring. They travel along the ring arms until encountering the contact between the ring and the second lead. Then a second scattering process happens, after which there is some probability that electrons can hop into the second lead and flow into the drain.

The above is only a rough narrative of what happens during the transport. In reality, there exist processes like electrons reflecting from the second contact and scattering at the first contact again, and so on. We need to focus on a steady process, for what the electron amplitudes do not change with time. As shown in Fig. 3.5, we can set the electron amplitudes to be $1, r, t$, corresponding to the electron incoming from the first lead, the electron reflection in the first lead, the electron outgoing in the second lead, and $A_1, A_2, A_3, A_4$, corresponding to the electron clockwise circling the ring in the corresponding regions, and $B_1, B_2, B_3, B_4$, corresponding to the electron counter-clockwisely circling the ring in the corresponding regions.
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Figure 3.5: An analysis of the single ring transport process. An electron is incident from the left lead, with amplitude 1. It has an amplitude $r$ to reflect and an amplitude $t$ to travel through the ring to the second lead.

At the contacts, the scattering process can be described by a tri-junction scattering matrix

$$S_t = \begin{pmatrix} -(a + b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & a & b \\ \sqrt{\epsilon} & b & a \end{pmatrix},$$

(3.12)

where $a = \frac{1}{2}(\sqrt{1 - 2\epsilon} - 1)$ and $b = \frac{1}{2}(\sqrt{1 - 2\epsilon} + 1)$. $\epsilon$ is a parameter denoting the coupling strength: $\epsilon = \frac{1}{2}$ is the strong coupling limit and $\epsilon = 0$ is the weak or zero coupling limit. The three channels of this matrix represent the lead and the two arms of the ring. We note that there is a symmetry in this matrix between the two ring arm channels. In general the value of $\epsilon$ depends on the energy of the incident electron. In our case, we restrict to the weak coupling regime and the detail value of $\epsilon$ does not affect the physics qualitatively. Thus, we would like to set it as as constant, i.e. it does not depend on the energy.

Now suppose an electron with a wave-vector $k$ is incident from the source. Its energy equals to $\frac{(hk)^2}{2m}$. When it encounters the ring, there happens a scattering process at the
lead-ring tri-junction. This scattering process can be described using Eq. (3.12) as

\[
\begin{pmatrix}
-(a + b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
1 \\
A_2 \\
B_1
\end{pmatrix}
= 
\begin{pmatrix}
r \\
B_2 \\
A_1
\end{pmatrix}.
\] (3.13)

An electron acquires two different phases when traveling along the ring arms: 1) a dynamical phase \( kl \) (\( l \) is the arm’s length); 2) a magnetic phase \( \phi_M \), which depends on the magnetic flux \( \Phi \). The electron’s motion in the ring can be described by

\[
A_4 = A_1 e^{i[l^{(1)} - \phi_M^{(1)}]},
\]

(3.14)

\[
B_4 = B_1 e^{i[-l^{(1)} - \phi_M^{(1)}]},
\]

(3.15)

\[
A_3 = A_2 e^{i[-l^{(2)} + \phi_M^{(2)}]},
\]

(3.16)

\[
B_3 = B_2 e^{i[l^{(2)} + \phi_M^{(2)}]},
\]

(3.17)

where \( l^{(1)} \) and \( l^{(2)} \) are the lengths of the upper arm and the lower arm (\( l^{(1)} + l^{(2)} = L \) is the circumference of the ring), \( \phi_M^{(1)} \) and \( \phi_M^{(2)} \) are the magnetic phases an electron acquires when traveling along the upper arm and the lower arm counter-clockwisely (\( \phi_M^{(1)} + \phi_M^{(2)} = \phi_M = 2\pi \frac{\Phi}{\Phi_0} \)).

Next, at the right lead-ring tri-junction, the scattering process can be described by

\[
\begin{pmatrix}
-(a + b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
0 \\
B_3 \\
A_4
\end{pmatrix}
= 
\begin{pmatrix}
t \\
A_3 \\
B_4
\end{pmatrix}.
\] (3.18)

Combining these equations, we can solve out all the amplitudes. Within them, \( t \) is what we care most, since \( |t|^2 \) determines the current in the second lead. In principle, this group
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Figure 3.6: The transmission probability $|t|^2$ as a function of the dynamical phase $k \cdot L$. The magnetic flux has been chosen such that the magnetic phase $\phi_M = 0.33 \times 2\pi$ in this simulation.

of equations can be solved analytically, since it is linear. But the analytical result may not help to give much physical insight, because it is very complicated. So we are satisfied here to show the numerical result and explain it using physical arguments.

Fig. 3.6 shows the result of a numerical calculation in which we set the magnetic phase $\phi_M = 0.33 \times 2\pi$. It is clear that the transmission probability has peaks around the dynamical phases that fulfill the resonant condition

$$\phi_D \pm \phi_M = 2n\pi \quad (n \in \mathbb{Z}) \, ,$$

(3.19)

where we define $\phi_D = k \cdot L$ to be the dynamical phase. The width of the peak depends on the scattering parameter $\epsilon$, which characterizes the coupling between the lead and the ring. It is obvious that when $\epsilon$ is small, the coupling is weak and the peak is more sharp; when $\epsilon$ is relatively large, the coupling is relatively strong and the peak is more wide.
Figure 3.7: The system of two coupled rings. The left ring is threaded by a magnetic flux $\Phi_1$; the right ring is threaded by a magnetic flux $\Phi_2$.

Since we restrict to the weak coupling regime, the appearance of the transmission peaks indicates the existence of eigenstates of the isolated ring. From the point of view of the scattering theory, transmission peaks represent resonant scattering.

3.2 Two Coupled Metallic Rings

In this section, we study the system of two coupled metallic rings.

3.2.1 Eigenstates

Fig. 3.7 shows the configuration of two coupled rings, in which magnetic fluxes $\Phi_1$ and $\Phi_2$ thread through the center of the left ring and the center of the right ring separately. In this section, we solve the eigenstates of this system by employing a general approach.

Suppose the coupling between the two rings could be described by a 4-channel scattering
Figure 3.8: An analysis of the eigenstate of two coupled rings.

matrix

$$S_c = \begin{pmatrix}
  r^c & t_1^c & t_2^c & t_2^c \\
  t_1^c & r^c & t_2^c & t_2^c \\
  t_2^c & t_2^c & r^c & t_1^c \\
  t_2^c & t_2^c & t_1^c & r^c \\
\end{pmatrix}$$  \hspace{1cm} (3.20)

where \(r^c = -\sqrt{\gamma}\), \(t_1^c = 1 - \sqrt{\gamma}\) and \(t_2^c = i\sqrt{\sqrt{\gamma} - \gamma}\). \(\gamma\) is the scattering parameter, which has three different regimes: 1) \(\gamma = 0\) is the weak or zero coupling limit; 2) \(\gamma = \frac{1}{4}\) the maximal tunnelling limit; 3) \(\gamma = 1\) is the maximal reflection limit.

The scattering process between the two rings can be described by

$$\begin{pmatrix}
  r^c & t_1^c & t_2^c & t_2^c \\
  t_1^c & r^c & t_2^c & t_2^c \\
  t_2^c & t_2^c & r^c & t_1^c \\
  t_2^c & t_2^c & t_1^c & r^c \\
\end{pmatrix} \begin{pmatrix}
  A_1 \\
  B_2 \\
  B_3 \\
  A_4 \\
\end{pmatrix} = \begin{pmatrix}
  B_1 \\
  A_2 \\
  A_3 \\
  B_4 \\
\end{pmatrix}.$$  \hspace{1cm} (3.21)
The electron propagation around the ring can be described by

\[ A_1 = A_2 e^{i[kL^{(1)} - \phi_M^{(1)}]} , \]
\[ B_2 = B_1 e^{i[kL^{(1)} + \phi_M^{(1)}]} , \]
\[ B_3 = B_4 e^{i[kL^{(2)} + \phi_M^{(2)}]} , \]
\[ A_4 = A_3 e^{i[kL^{(2)} - \phi_M^{(2)}]} , \]

where \( L^{(1)} \) and \( L^{(2)} \) are the circumferences of the left ring and the right ring, \( \phi_M^{(1)} \) and \( \phi_M^{(2)} \) are the magnetic phases an electron acquires when counter-clockwisely circling the left ring and the right ring.

Substituting Eq.(3.22) - (3.25) into Eq.(3.21), we have

\[
\begin{pmatrix}
A_1 \\
B_2 \\
B_3 \\
A_4
\end{pmatrix}
= 0 ,
\]

where the coefficient matrix

\[
M = \begin{pmatrix}
r_c & t_1^c - e^{-i[kL^{(1)} + \phi_M^{(1)}]} & t_2^c & t_2^c \\
t_1^c - e^{-i[kL^{(1)} - \phi_M^{(1)}]} & r_c & t_2^c & t_2^c \\
t_2^c & t_2^c & r_c & t_1^c - e^{-i[kL^{(2)} + \phi_M^{(2)}]} \\
t_2^c & t_2^c & t_1^c - e^{-i[kL^{(2)} - \phi_M^{(2)}]} & r_c
\end{pmatrix} .
\]

In order for a non-zero eigenstate, there should be

\[ \det(M) = 0 , \]
which, after some manipulation, can be written as

\[
[\cos \phi_D^{(1)} - \cos \phi_M^{(1)}][\cos \phi_D^{(2)} - \cos \phi_M^{(2)}] = \frac{\sqrt{7}}{\gamma_1} \sin \phi_D^{(1)} \sin \phi_D^{(2)}.
\]  

(3.29)

Here we have set \(\phi_D^{(i)} = kL^{(i)}\) (\(i = 1, 2\)) to be the dynamical phase that an electron acquires when traveling one complete circle around the \(i\)-th ring.

If there is no coupling between the rings, then \(\gamma = 0\) and Eq. (3.29) regresses to the situation of two independent rings.

Generically, the scattering parameter \(\gamma\) depends on \(k\) (the detail dependence relies on the coupling mechanism between the two rings), and the secular equation Eq. (3.29) that determines the spectrum can be very complicated. We will give an example of a specific lattice model in the next section.

### 3.2.2 Phase Coherent Transport

Now we study the transport properties of the system of two coupled rings. The configuration of the set-up is shown in Fig. 3.9. We let a lead connect one ring to the source reservoir and another lead connect another ring to the drain reservoir. When we exert a voltage bias between the source and the drain, there will be a current flowing from the source to the drain. We would like to see how the current depends on the magnetic fluxes that thread through the rings.

Just like what we did for the single ring transport process, we write down the eigenstate amplitudes on each part of the system, as shown in Fig. 3.10.
At the first lead-ring tri-junction, we have

\[
\begin{pmatrix}
-(a_1 + b_1) & \sqrt{\epsilon_1} & \sqrt{\epsilon_1} \\
\sqrt{\epsilon_1} & a_1 & b_1 \\
\sqrt{\epsilon_1} & b_1 & a_1
\end{pmatrix}
\begin{pmatrix}
1 \\
A_2 \\
B_1
\end{pmatrix}
=
\begin{pmatrix}
r \\
B_2 \\
A_1
\end{pmatrix},
\tag{3.30}
\]

where \( \epsilon_1 \) characterizes the scattering strength of the first lead-ring tri-junction.

Then, the electron’s motion in the first ring can be described by

\[
A_4 = A_1 e^{i[kl(a) - \phi_M^{(a)}]},
\tag{3.31}
\]

\[
B_4 = B_1 e^{i[-kl(a) - \phi_M^{(a)}]},
\tag{3.32}
\]
Figure 3.10: Analysis of two coupled rings transport process

\[
A_3 = A_2 e^{i[-kl^{(b)} + \phi^{(b)}_M]}, \quad (3.33)
\]
\[
B_3 = B_2 e^{i[kl^{(b)} + \phi^{(b)}_M]}, \quad (3.34)
\]

where \(l^{(a)}\) and \(l^{(b)}\) are the length of the two arms of the first ring, \(\phi^{(a)}_M\) and \(\phi^{(b)}_M\) are the magnetic phases an electron acquires when travelling counter-clockwise around the corresponding arms.

Next, the scattering at the junction between the two rings is

\[
\begin{pmatrix}
  r^c & t^c_1 & t^c_2 & t^c_2 \\
  t^c_1 & r^c & t^c_2 & t^c_2 \\
  t^c_2 & t^c_2 & r^c & t^c_1 \\
  t^c_2 & t^c_2 & t^c_1 & r^c
\end{pmatrix}
\begin{pmatrix}
  A_3 \\
  B_4 \\
  B_5 \\
  A_6
\end{pmatrix}
= \begin{pmatrix}
  B_3 \\
  A_4 \\
  A_5 \\
  B_6
\end{pmatrix}.
\]

(3.35)

The electron’s motion in the second ring can be described by

\[
A_8 = A_5 e^{i[kl^{(c)} - \phi^{(c)}_M]}, \quad (3.36)
\]
where $l^{(c)}$ and $l^{(d)}$ are the length of the two arms of the second ring, $\phi^{(c)}_M$ and $\phi^{(d)}_M$ are the magnetic phases an electron acquires when travelling counter-clockwise around the corresponding arms.

Finally, at the second lead-ring tri-junction, we have

$$
\begin{pmatrix}
-(a_2 + b_2) & \sqrt{\epsilon_2} & \sqrt{\epsilon_2} \\
\sqrt{\epsilon_2} & a_2 & b_2 \\
\sqrt{\epsilon_2} & b_2 & a_2 
\end{pmatrix}
\begin{pmatrix}
0 \\
B_7 \\
A_8 
\end{pmatrix}
= 
\begin{pmatrix}
t \\
A_7 \\
B_8 
\end{pmatrix},
$$

(3.40)

where $\epsilon_2$ characterizes the scattering strength of the second lead-ring tri-junction.

Combining all these equations, we can solve them and obtain all the amplitudes. We expect that the transmission peak appears when the energy of the incident electron matches one of the eigenstate of the closed system of two coupled rings, if the coupling between the leads and the system is weak. In an opposite way, we can infer from the transmission peaks the eigenstates of the system.

The result of a numerical simulation, under the regime of weak coupling between the leads and the rings and weak coupling between the two rings, is shown in Fig. 3.11. In this simulation, we fix the electron wave-vector and the coupling between the leads and the rings, and adjust the coupling between the two rings. In the weak limit, the transmission peaks correspond to the magnetic fluxes that one of them fulfills the resonant condition $kL \pm \phi_M = 2n\pi$ ($n \in \mathbb{Z}$). At the point where both the resonant conditions are fulfilled, it is seen that the peak lines have a small repelling with each other and no crossing happens.
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(a) $\gamma = 0.00001$

(b) $\gamma = 0.0001$

(c) $\gamma = 0.001$

(d) $\gamma = 0.01$

Figure 3.11: Transmission as a function of two magnetic phases in the transport process of the system of two coupled rings, where we set $\epsilon = 0.01$, $k = 0.15 \text{ nm}^{-1}$, $l^{(a)} = 1700 \text{ nm}$, $l^{(b)} = 990 \text{ nm}$, $l^{(c)} = 1280 \text{ nm}$ and $l^{(d)} = 1500 \text{ nm}$ in this simulation. The dynamical phase an electron acquires when travelling one complete circle around the left ring is $64.219 \times 2\pi$; around the right ring is $66.368 \times 2\pi$. The dashed lines in the graphs correspond to the magnetic fluxes that satisfy $kL^{(i)} \pm \phi_M^{(i)} = 2n\pi$ (where $i = 1, 2$ and $n \in \mathbb{Z}$), which are $\phi_M^{(1)} = 0.219 \times 2\pi$, $0.781 \times 2\pi$, and $\phi_M^{(2)} = 0.368 \times 2\pi$, $0.632 \times 2\pi$.

This result could be understood in terms of the perturbation theory of quantum mechanics. Suppose there is no coupling between the two rings, then an eigenstate of the system is the sum of the independent eigenstate of each ring. The degeneracy happens at the point where the resonant conditions are fulfilled for both the rings. Now when there is a small coupling
between the rings, it will lift this degeneracy and results in no crossing. This is just what we have seen in Fig. 3.11. As we gradually intensify the coupling, we see the repelling between the eigenstates becomes large and large, which also fulfill the prediction of the perturbation theory.

Finally, we would like to compare the transmission peaks directly with the positions of the eigenstates. To do this, we calculate $|\text{det}(M)|$ and see if the trajectories of its zero
points, which correspond to the eigenstates, match the transmission peak lines. The result of $|\det(M)|$ is shown in Fig. 3.12 where we have chosen the circumferences of the rings $L^{(1)} = l^{(a)} + l^{(b)}$ and $L^{(2)} = l^{(c)} + l^{(d)}$. It is seen that they indeed match.

### 3.3 Lattice Models

In this section, we investigate the lattice model of metallic rings, which serves us a concrete example to the solutions of the two coupled rings. We stress that the physics of the lattice model has no essential difference from the continuous one. The discussion below mainly follows my work in [Fang and Schmeltzer (2016)].

Since the system of two coupled rings is too complicated to solve directly, we would like to start from relatively simple lattice, and go forward step by step to find the solution of the two coupled rings. In the following, we first develop a method of transfer matrix by studying a semi-infinite wire.

#### 3.3.1 Semi-Infinite Wire

Our starting point of a lattice is a semi-infinite wire, of which the configuration is shown in Fig. 3.13. The semi-infinite wire has only one end. The Hamiltonian in the second-quantization formalism is

$$H = -t \sum_{n=0}^{\infty} c^\dagger_{n+1} c_n + \text{h.c.} , \tag{3.41}$$

where $t$ is the hopping constant, $c^\dagger_n$ and $c_n$ are creation and annihilation operators for electrons on the lattice site $n$.

Let $A = (c_0, c_1, ...)^T$. We can rewrite the Hamiltonian in a matrix form

$$H = -t A^\dagger h A , \tag{3.42}$$
Figure 3.13: Lattice model for a semi-infinite open wire. The red spots on the line represent lattice sites. They are linked by allowing electrons to hop between nearest neighbours.

where

\[
    h = \begin{pmatrix}
    0 & 1 & & & \\
    1 & 0 & 1 & & \\
    & 1 & 0 & 1 & \\
    & & 1 & 0 & \\
    & & & & \ddots
    \end{pmatrix} \tag{3.43}
\]

is equivalent to the Hamiltonian matrix in the first-quantization formalism. Let \( \psi = (x_0, x_1, ...)^T \) be an eigenvector of \( h \) with eigenvalue \( \lambda \). It’s easy to check that \( (x_0c_0^\dagger + x_1c_1^\dagger + x_2c_2^\dagger + \cdots) |\Omega\rangle \) (\(|\Omega\rangle \) represents the vacuum state) is a single particle eigenstate of \( H \) with energy \( E = -\lambda t \).

In the following, we want to diagonalize \( h \), which is to find all the eigenvalues and corresponding eigenvectors of \( h \).

The eigenvalue equation of \( h \) reads

\[
    \begin{pmatrix}
    0 & 1 & & & \\
    1 & 0 & 1 & & \\
    & 1 & 0 & 1 & \\
    & & 1 & 0 & \\
    & & & & \ddots
    \end{pmatrix}
    \begin{pmatrix}
    x_0 \\
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots
    \end{pmatrix}
    = \lambda
    \begin{pmatrix}
    x_0 \\
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots
    \end{pmatrix}, \tag{3.44}
\]
which, in an explicit form, is

\[ x_1 = \lambda x_0, \]

\[ x_0 + x_2 = \lambda x_1, \]

\[ x_1 + x_3 = \lambda x_2, \]

\[ x_2 + x_4 = \lambda x_3, \]

\[ \ldots \]

Let \( x_0 = 1 \), then \( x_1 = \lambda \). For \( n > 1 \), \( x_n = \lambda x_{n-1} - x_{n-2} \). Combining an obvious identity \( x_{n-1} = x_{n-1} \), we have

\[
\begin{pmatrix}
  x_n \\
  x_{n-1}
\end{pmatrix} = L \begin{pmatrix}
  x_{n-1} \\
  x_{n-2}
\end{pmatrix},
\]

where

\[
L = \begin{pmatrix}
  \lambda & -1 \\
  1 & 0
\end{pmatrix}
\]

is the transfer matrix.

Since \( L \) is independent of \( n \), we are able to iterate Eq. (3.45) and express \( (x_n, x_{n-1})^T \) in terms of \( (x_1, x_0)^T \) as

\[
\begin{pmatrix}
  x_n \\
  x_{n-1}
\end{pmatrix} = L^n \begin{pmatrix}
  x_1 \\
  x_0
\end{pmatrix}.
\]

Now if we know \( L^n \), we can calculate out the entire wave-function. The expression of \( L^n \) depends on the value of \( \lambda \). We list the results here. For a detail derivation, consult Fang and Schmeltzer (2016).
1. When $\lambda \in (-2, 2)$, let $\lambda = 2 \cos \theta \ (0 < \theta < \pi)$, then

$$L^n = \begin{pmatrix} \frac{\sin(n+1)\theta}{\sin \theta} & -\frac{\sin n\theta}{\sin \theta} \\ \frac{\sin n\theta}{\sin \theta} & -\frac{\sin(n-1)\theta}{\sin \theta} \end{pmatrix}.$$ \hspace{1cm} (3.48)

2. i) When $\lambda = 2$,

$$L^n = \begin{pmatrix} n + 1 & -n \\ n & -(n - 1) \end{pmatrix},$$ \hspace{1cm} (3.49)

which corresponds to $\theta \to 0$ in (3.48).

ii) When $\lambda = -2$,

$$L^n = (-1)^n \begin{pmatrix} n + 1 & n \\ -n & -(n - 1) \end{pmatrix},$$ \hspace{1cm} (3.50)

which corresponds to $\theta \to \pi$ in (3.48).

3. i) When $\lambda > 2$, let $\lambda = 2 \cosh \eta \ (\eta > 0)$, then

$$L^n = \begin{pmatrix} \frac{\sinh(n+1)\eta}{\sinh \eta} & -\frac{\sinh n\eta}{\sinh \eta} \\ \frac{\sinh n\eta}{\sinh \eta} & -\frac{\sinh(n-1)\eta}{\sinh \eta} \end{pmatrix},$$ \hspace{1cm} (3.51)

which corresponds to $\theta \to i\eta$ in (3.48).

ii) When $\lambda < -2$, let $\lambda = -2 \cosh \eta \ (\eta > 0)$, then

$$L^n = (-1)^n \begin{pmatrix} \frac{\sinh(n+1)\eta}{\sinh \eta} & -\frac{\sinh n\eta}{\sinh \eta} \\ -\frac{\sinh n\eta}{\sinh \eta} & -\frac{\sinh(n-1)\eta}{\sinh \eta} \end{pmatrix},$$ \hspace{1cm} (3.52)

which corresponds to $\theta \to \pi + i\eta$ (3.48).

Now substituting $L^n$ and the initial value $(x_0, x_1) = (1, \lambda)$ into Eq. (3.47), we obtain the eigenfunction:
1. When $\lambda = 2 \cos \theta$ ($0 < \theta < \pi$),

\[
\psi = (1, 2 \cos \theta, \frac{\sin 3\theta}{\sin \theta}, \frac{\sin 4\theta}{\sin \theta}, \ldots)^T .
\] (3.53)

Note here that $U_n(\cos \theta) \equiv \frac{\sin(n+1)\theta}{\sin \theta}$ is the second kind of Chebyshev polynomial [Mason and Handscomb (2002)].

2. i) When $\lambda = 2$,

\[
\psi = (1, 2, 3, 4, \ldots)^T .
\] (3.54)

ii) When $\lambda = -2$,

\[
\psi = (1, -2, 3, -4, \ldots)^T .
\] (3.55)

3. i) When $\lambda = 2 \cosh \eta$ ($\eta > 0$),

\[
\psi = (1, 2 \cosh \eta, \frac{\sinh 3\eta}{\sinh \eta}, \frac{\sinh 4\eta}{\sinh \eta}, \ldots)^T .
\] (3.56)

ii) When $\lambda = -2 \cosh \eta$ ($\eta > 0$),

\[
\psi = (1, -2 \cosh \eta, \frac{\sinh 3\eta}{\sinh \eta}, -\frac{\sinh 4\eta}{\sinh \eta}, \ldots)^T .
\] (3.57)

In quantum mechanics the wave-function is required to be normalizable, implying it cannot blow up at infinity. Thus the solutions 2 and 3 are abandoned. The spectrum for a semi-infinite wire is in the range of $(-2, 2)$. Actually, any value in $(-2, 2)$ belongs to the set of the spectrum, for there is no other constraint now.

In the solution 2 if setting $x_0 = \sin \theta$, we have a simple-looking wave-function

\[
\psi = (\sin \theta, \sin 2\theta, \sin 3\theta, \sin 4\theta, \ldots)^T .
\] (3.58)
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Figure 3.14: Lattice model for a finite open wire. The red spots in the middle of the line represent realistic sites, while the blue spots at the opposite ends represent imaginary sites.

The form of this wave-function represents a standing wave with a fixed end, which can be interpreted as a superposition of the incident wave and the reflected wave as an electron travels along the wire from infinity to the end and reflects.

Now we have finished the discussion of the semi-infinite wire. The key to the solution is that we can express the entire wave-function in terms of \((x_0, x_1)^T\) through the transfer matrix \((3.47)\). We note that the simple geometry of the semi-infinite wire gives us a great advantage: the unique end of the wire serves as a natural starting point for iteration.

### 3.3.2 Finite Open Wire

We next discuss a finite open wire, of which the configuration is shown in Fig. 3.14. Compared to the semi-infinite wire, it has two open ends and only a finite number of lattice sites. Let the number of sites be \(N\), and denote the wave-function \(\psi = (x_1, ..., x_N)^T\).

For the reason of the open wire’s geometry, it is appropriate to employ a trick at this point. Let’s assume an imaginary site 0 at the left end and another imaginary site \((N + 1)\) at the right end. It is required that \(x_0 = x_{N+1} = 0\), which serves as the new boundary condition. The original boundary conditions \(x_2 = \lambda x_1\) and \(x_{N-1} = \lambda x_N\) can be tailored as \(x_2 = \lambda x_1 - x_0\) and \(x_{N+1} = \lambda x_N - x_{N-1}\), which fits the general form of the iteration relation \(x_{n+1} = \lambda x_n - x_{n-1}\).

Now using the transfer matrix, we can express \(x_n\) in terms of \(x_0\) and \(x_1\). If we let \(x_1 = 1\), then \(x_2 = \lambda\), and the same expression as that of the semi-infinite wire follows for \(n > 2\).
For the boundary condition requires \( x_{N+1} = 0 \), we have to abandon the solutions 2 and 3. Thus the eigenvalue \( \lambda \) of a finite open wire also stays in the range of \((-2, 2)\).

Let \( \lambda = 2 \cos \theta \) \((0 < \theta < \pi)\) and \( x_1 = \sin \theta \) (instead of 1 for convenience). Applying the transfer matrix \( (3.48) \), we obtain \( x_n = \sin n\theta \). Then from \( x_{N+1} = 0 \), we have

\[
\sin(N + 1)\theta = 0.
\] (3.59)

This equation has \( N \) distinct roots

\[
\lambda_m = 2 \cos \theta_m, \quad \theta_m = \frac{m}{N + 1} \pi, \quad m = 1, 2, ..., N
\] (3.60)

These eigenvalues correspond to energy levels \( E_m = -\lambda_m t \) of the finite open wire, and the corresponding eigenfunction (unnormalized) is

\[
\psi_m \sim (0, \sin \theta_m, \sin 2\theta_m, \cdots, \sin N\theta_m, 0)^T,
\] (3.61)

representing a standing wave with two fixed ends.

### 3.3.3 Single Closed Ring

If we link the two opposite ends of the finite open wire, a single closed ring is formed. The configuration, as shown in Fig. 3.15 is the same as a 1D lattice with the Born-von Karman periodic boundary condition. Due to the translational symmetry, it is usually solved by the Fourier transformation. Here we use the transfer matrix method.

Let the number of sites in the ring be \( N \), and the wave-function be \( \psi = (x_0, x_1, ..., x_{N-1})^T \). Like before, using the transfer matrix, we can express \( x_n \) in terms of \( x_0, x_1 \). Now since we
Figure 3.15: Lattice model for a single closed ring. The red spots on the circle represent lattice sites.

have a periodic boundary condition, we expect \( x_n = x_{n-N} \). So \( x_N = x_0 \), \( x_{N+1} = x_1 \). For

\[
\begin{pmatrix}
    x_{N+1} \\
    x_N
\end{pmatrix} = L^N \begin{pmatrix}
    x_1 \\
    x_0
\end{pmatrix},
\]

we have

\[
\begin{pmatrix}
    x_1 \\
    x_0
\end{pmatrix} = L^N \begin{pmatrix}
    x_1 \\
    x_0
\end{pmatrix}.
\]

At this moment, it is convenient to unify the three different situations of \( \lambda \), by enlarging the domain of \( \theta \) in \( \mathbb{I} \). As noted when we wrote down the expression of \( L^n \), we can extend the domain of \( \theta \) into the complex plane. The illustration is shown in Fig. 3.16. In the following we will interpret \( \theta \) in its generalized complex form. The first benefit of doing this is that \( L^n \) can now be simply expressed as (3.48) with no exceptions.

Insert (3.48) into (3.63), we have

\[
\begin{pmatrix}
    \frac{\sin(N+1)\theta}{\sin \theta} - 1 \\
    \frac{\sin N\theta}{\sin \theta} - \frac{\sin(N-1)\theta}{\sin \theta} - 1
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_0
\end{pmatrix} = 0.
\]

(3.64)
Figure 3.16: The domain of $\theta$ in the complex plane. The blue U-shape line in the graph represents the generalized domain of $\theta$, which unifies the three different situations.

In order of a non-zero eigenfunction, we have

\[
\begin{vmatrix}
\frac{\sin(N+1)\theta}{\sin \theta} - 1 & -\frac{\sin N\theta}{\sin \theta} \\
-\frac{\sin(N-1)\theta}{\sin \theta} - 1 & \frac{\sin N\theta}{\sin \theta}
\end{vmatrix} = 0 .
\] (3.65)

The roots of this equation constitute the spectrum of the single closed ring.

Next we intend to generalize the model by letting a magnetic flux thread the ring.

Assume a magnetic flux $\Phi$ is threaded through the ring. The well-known Peierls substitution [Peierls (1933)] can be used to include the effect of the magnetic flux into our lattice model. Simply speaking, the Peierls substitution varies the hopping constant $t$ by a phase $\phi$, such that $t$ is substituted by $te^{i\phi}$. The phase $\phi$ is the magnetic phase an electron acquires when travelling a distance between two nearest-neighbour sites in the ring.

With a magnetic flux, the Hamiltonian for the single closed ring becomes

\[
H = - \sum_{n=0}^{N-1} te^{i\phi} a_{n+1}^\dagger a_n + \text{h.c.} ,
\] (3.66)

where $a_N = a_0$ is assumed to respect the periodic boundary condition. We can turn this
Hamiltonian into a matrix form as (3.42), with \( h_\phi \) being the Hamiltonian matrix in the first-quantization formalism.

Now we want to diagonalize \( h_\phi \). To emphasize the difference from the situation without magnetic flux, we let the eigenfunction of \( h_\phi \) be \( (y_0, y_1, \cdots, y_{N-1})^T \).

The eigenvalue equation of \( h_\phi \) is

\[
y_{n-1}e^{-i\phi} + y_{n+1}e^{i\phi} = \lambda y_n ,
\]

which differs from our familiar one \( x_{n-1} + x_{n+1} = \lambda x_n \) only by a gauge transformation. If we let \( y_n = e^{-in\phi}x_n \), the two equations are the same. Thus, we have

\[
y_n = \frac{\sin n\theta}{\sin \theta} e^{-i(n-1)\phi} y_1 - \frac{\sin(n-1)\theta}{\sin \theta} e^{-in\phi} y_0
\]

(3.68)

Next, the boundary condition becomes

\[
y_0 = \frac{\sin N\theta}{\sin \theta} e^{-i(N-1)\phi} y_1 - \frac{\sin(N-1)\theta}{\sin \theta} e^{-iN\phi} y_0
\]

(3.69)

\[
y_1 = \frac{\sin(N+1)\theta}{\sin \theta} e^{-iN\phi} y_1 - \frac{\sin N\theta}{\sin \theta} e^{-i(N+1)\phi} y_0
\]

(3.70)

Although Eq. (3.69) and Eq. (3.70) are of equal importance here, they are not in the system of two coupled rings. We will see that, due to the coupling, Eq. (3.70) becomes different. From Eq. (3.69), we can express \( y_1 \) in terms of \( y_0 \)

\[
y_1 = \frac{e^{i(N/2)\phi}\sin \theta + e^{-i(N/2)\phi}\sin(N-1)\theta}{\sin N\theta} e^{i(N-1)\phi} y_0
\]

(3.71)

Then substituting Eq. (3.71) into Eq. (3.68), we have

\[
y_n = \frac{e^{i(N-n)\phi}\sin n\theta + e^{-in\phi}\sin(N-n)\theta}{\sin N\theta} y_0
\]

(3.72)
A notable property of this wave-function is $y_{N-n} = y_n^*$, resulting from that the system remains unchanged if we reverse both the direction of the magnetic flux and the direction of the ring.

By setting $n = 0$ in Eq. (3.67), we have

$$y_{N-1} e^{-i\phi} + y_1 e^{i\phi} = 2 \cos \theta y_0 .$$

(3.73)

Then, combining Eq. (3.72) with Eq. (3.73), we get

$$\cos N\theta = \cos N\phi .$$

(3.74)

Note $N\phi = \phi_M$ is just the magnetic phase an electron acquires when traveling one complete circle around the ring. Due to $\cos N\theta \in [-1, 1]$ here, $\theta$ could only be in the range of $[0, \pi]$. We can set $\theta = ka$ ($k$ is the electron wave-vector, $a$ is the crystal constant) to be the dynamical phase an electron acquires when traveling a distance between two nearest-neighbour sites in the ring, and thus $N\theta = \phi_D$ is the dynamical phase an electron acquires when traveling one complete circle around the ring. It is obvious that the solution to Eq. (3.74) is

$$\phi_D \pm \phi_M = 2n\pi \ (n \in \mathbb{Z}) .$$

(3.75)

Therefore, we have recovered our familiar result using the transfer matrix method.

### 3.3.4 Two Coupled Rings

In this part, we come to study the system of two coupled rings. The configuration is shown in Fig. 3.17. The rings are coupled by allowing electrons to hop from the site $y_0$ in the left ring to the site $z_0$ in the right ring. A magnetic flux $\Phi_1$ threads the left ring and another magnetic flux $\Phi_2$ threads the right ring. The Hamiltonian of the entire system can be cast
CHAPTER 3. METALLIC RINGS

into

\[ H = H_1 + H_2 + H_{\text{coup}}, \] (3.76)

where

\begin{align*}
H_1 &= -\sum_{m=0}^{N_1-1} [t e^{i\phi_1} c_m^\dagger c_{m+1} + \text{h.c.}] , \\
H_2 &= -\sum_{n=0}^{N_2-1} [t e^{i\phi_2} d_n^\dagger d_{n+1} + \text{h.c.}] , \\
H_{\text{coup}} &= -V_0(c_0^\dagger d_0 + d_0^\dagger c_0) .
\end{align*}

Here \( H_1 \) and \( H_2 \) are the Hamiltonians for the left ring and the right ring. \( H_{\text{coup}} \) represents the coupling between the rings. Like before, \( c_{N_1} = c_0 \) and \( d_{N_2} = d_0 \) are assumed to respect the periodic boundary condition of the rings.

It is noted that the model in Schmeltzer [2008] and Avishai and Luck [2009] is different from our considering one. In those two works, \( y_0 \) and \( z_0 \) are assumed to be the same single site, which connects to \( y_1, y_{N_1-1}, z_1 \) and \( z_{N_2-1} \).

Let the wave-function of the entire system be \( \psi = (y_1, y_2, \ldots, y_{N_1-1}, y_0, z_0, z_1, \ldots, z_{N_2-1})^T \). Just like Eq. (3.72), we can express the wave-function of the left ring part in terms of \( y_0 \) and the wave-function of the right ring part in terms of \( z_0 \):

\begin{align*}
y_n &= \frac{e^{i(N_1-n)\phi_1} \sin n\theta + e^{-in\phi_1} \sin(N_1 - n)\theta}{\sin N_1 \theta} y_0 \\
z_n &= \frac{e^{i(N_2-n)\phi_2} \sin n\theta + e^{-in\phi_2} \sin(N_2 - n)\theta}{\sin N_2 \theta} z_0.
\end{align*}

Next, we express the Hamiltonian Eq. (3.76) in a matrix form

\[ H = -t A^\dagger h A \] (3.82)
Figure 3.17: Lattice model for the two coupled rings. The red spots in the rings represent lattice sites. The blue link between the rings means that the electron can hop between $y_0$ and $z_0$.

where $A = (c_1, c_2, ..., c_{N_1-1}, c_0, d_0, d_1, ..., d_{N_2-1})^T$, and

$$
\begin{bmatrix}
0 & e^{i\phi_1} & e^{-i\phi_1} \\
e^{-i\phi_1} & ... & ... \\
... & ... & e^{i\phi_1} \\
e^{i\phi_1} & 0 & e^{i\phi_1} \\
e^{-i\phi_1} & e^{-i\phi_1} & 0
\end{bmatrix}
\begin{bmatrix}
V_0/t \\
V_0/t \\
0 & e^{i\phi_2} & e^{-i\phi_2} \\
e^{-i\phi_2} & 0 & e^{i\phi_2} \\
e^{-i\phi_2} & ... & ... \\
e^{i\phi_2} & 0 & e^{-i\phi_2}
\end{bmatrix}
$$

Then, the rows for $y_0$ and $z_0$ in this matrix give us the coupling equations

$$
y_{N_1-1}e^{-i\phi_1} + y_1e^{i\phi_1} + (V_0/t)z_0 = \lambda y_0
$$
Substituting Eq. (3.80) and Eq. (3.81) into Eq. (3.84) and Eq. (3.85), we obtain

\[
\begin{pmatrix}
2 \sin \theta (\cos N_1 \theta - \cos N_1 \phi_1) & -(V_0/t) \sin N_2 \theta \\
-(V_0/t) \sin N_1 \theta & 2 \sin \theta (\cos N_2 \theta - \cos N_2 \phi_2)
\end{pmatrix}
\begin{pmatrix}
y_0 \\
z_0
\end{pmatrix}
= 0.
\] (3.86)

In order for a non-zero solution, we have

\[
\begin{vmatrix}
2 \sin \theta (\cos N_1 \theta - \cos N_1 \phi_1) & -(V_0/t) \sin N_2 \theta \\
-(V_0/t) \sin N_1 \theta & 2 \sin \theta (\cos N_2 \theta - \cos N_2 \phi_2)
\end{vmatrix}
= 0,
\] (3.87)

which is equivalent to

\[
\left( \cos N_1 \theta - \cos N_1 \phi_1 \right) \left( \cos N_2 \theta - \cos N_2 \phi_2 \right) = \frac{(V_0/t)^2}{4 \sin^2 \theta} \sin N_1 \theta \sin N_2 \theta.
\] (3.88)

If \( \theta \in [0, \pi] \), we can cast Eq. (3.88) into

\[
\left( \cos \phi_D^{(1)} - \cos \phi_M^{(1)} \right) \left( \cos \phi_D^{(2)} - \cos \phi_M^{(2)} \right) = \frac{(V_0/t)^2}{4 \sin^2 k a} \sin \phi_D^{(1)} \sin \phi_D^{(2)},
\] (3.89)

which fulfills the general form of Eq. (3.29). Comparing these two equations, we have for our lattice model

\[
\frac{\sqrt{\gamma}}{1 - \sqrt{\gamma}} = \frac{(V_0/t)^2}{4 \sin^2 k a}.
\] (3.90)

Note that the scattering parameter \( \gamma \) indeed depends on the wave-vector \( k \) in this case.

An interesting thing is that now \( \theta \) may be outside the range \([0, \pi]\), when the coupling between the rings is strong \((V_0/t)\) large). In this case our general interpretation of \( \theta \) applies.

Fig. 3.18 shows a sketch of the solutions of Eq. (3.88) under the setting of \( N_1 = N_2 = 8 \), \( V_0/t = 1 \) and some random values of \( \phi_1 \) and \( \phi_2 \).
Figure 3.18: Solutions of Eq. (3.88) when \( N_1 = N_2 = 8, V_0/t = 1 \). (a) the situation of the eigenvalue \( \lambda \) in the range \([-2, 2]\) \( (\lambda = 2 \cos \theta) \); (b) the situation of \( \lambda > 2 \) \( (\lambda = 2 \cosh \eta) \).

In Fig. 3.18(a), we show the solutions of Eq. (3.88) when \( \theta \) is in the range of \([0, \pi]\) \((-2 \leq \lambda \leq 2)\). Since there are a total of 16 sites in our set-up, we expect 16 eigenstates in all. We find 14 intersections in Fig. 3.18(a), which correspond to only 14 eigenstates. There must be 2 additional ones, for which \( \theta \) is outside the range \([0, \pi]\) \((\lambda > 2 \text{ or } \lambda < -2)\).

In the range of \( \lambda > 2 \), we let \( \theta \) be \( i\eta \). Fig. 3.18(b) shows there is a solution in such case. Likewise, there exists another eigenstate in the range of \( \lambda < -2 \) \( (\theta = \pi + i\eta) \). The eigenstates for which \( \theta \notin [-2, 2] \) are localized states (bound states). In this situation the wave-function is concentrated around the junction and decays exponentially when being away.
Chapter 4

Detecting Majorana Fermion Induced Crossed Andreev Reflection

In this chapter, we focus on a scheme to detect the Majorana fermion induced crossed Andreev reflection. We would like to utilize the properties of metallic rings to separate the electron tunneling signals and the hole tunneling signals. The key idea is that electrons and holes have different wave-vectors and so they can be constructively coherent in the ring under different conditions.

4.1 Coupled Systems of Majorana Fermions and Metallic Rings

In this section we study properties of the Majorana fermions and metallic rings coupled systems.
4.1.1 A Metallic Ring Coupled to An Isolated Majorana Fermion

We start from the simplest case of a metallic ring coupled with an isolated Majorana fermion. The configuration is shown in Fig. 4.1.

The Hamiltonian of the system can be written as

$$H = \int_{-L/2}^{+L/2} dx \left\{ \Psi^\dagger(x) \left[ \frac{1}{2m} (p - eA)^2 - \mu \right] \Psi(x) + \frac{t}{\sqrt{2}} \gamma_1 [\psi^\dagger(x) - \Psi(x)] \delta(x) \right\} , \quad (4.1)$$

where $\gamma_1$ represents the isolated Majorana fermion: $\gamma_1 = \gamma_1^\dagger$ and $\gamma_1^2 = 1$. The last term denotes the coupling between the Majorana fermion and the metallic ring, resulted from the tunneling effect. $t$ represents the tunneling amplitude.

Employing the BdG transformation we write an operator

$$\Gamma = \int dx [D(x)\Psi(x) + B(x)\Psi^\dagger(x)] + \frac{C}{\sqrt{2}} \gamma_1 \quad (4.2)$$

and assume it diagonalizes the Hamiltonian in the sense that $[\Gamma, H] = E \Gamma$. Expanding this commutator and comparing the coefficients in front of $\Psi(x)$, $\Psi^\dagger(x)$ and $\gamma_1$, we obtain the BdG equations

$$\left[ \frac{1}{2m} (p - eA)^2 - \mu \right] D(x) - t \delta(x) C = ED(x) , \quad (4.3)$$

$$- \left[ \frac{1}{2m} (p + eA)^2 - \mu \right] B(x) + t \delta(x) C = EB(x) , \quad (4.4)$$

$$t [- D(0) + B(0)] = EC . \quad \quad (4.5)$$

This group of equations can be solved by making an ansatz

$$D(x) = \begin{cases} 
D_1^+ e^{iK_1^D x} + D_2^+ e^{-iK_2^D x}, & 0 < x < \frac{L}{2} , \\
D_1^- e^{iK_1^P x} + D_2^- e^{-iK_2^P x}, & -\frac{L}{2} < x < 0 , 
\end{cases} \quad (4.6)$$
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Figure 4.1: A metallic ring coupled with an isolated Majorana fermion.

\[ B(x) = \begin{cases} 
  B_1^+ e^{-iK_D^1 x} + B_2^+ e^{iK_D^2 x}, & 0 < x < \frac{L}{2}, \\
  B_1^- e^{-iK_D^1 x} + B_2^- e^{iK_D^2 x}, & -\frac{L}{2} < x < 0 ,
\end{cases} \tag{4.7} \]

where

\[ k_e = K_D^1 - \frac{e}{\hbar} A = K_D^2 + \frac{e}{\hbar} A , \tag{4.8} \]
\[ k_h = K_D^1 - \frac{e}{\hbar} A = K_D^2 + \frac{e}{\hbar} A \tag{4.9} \]

are the electron wave-vector and the hole wave-vector. The energy of the state is

\[ E = \frac{(\hbar k_e)^2}{2m} - \mu = \mu - \frac{(\hbar k_h)^2}{2m} . \tag{4.10} \]

Next, we have:

1. The boundary conditions at \( x = \pm L/2 \) yield

\[ D_1^+ e^{+iK_D^1 L/2} = D_1^- e^{-iK_D^1 L/2} , \tag{4.11} \]
\[ D_2^+ e^{-iK_D^2 L/2} = D_2^- e^{+iK_D^2 L/2} , \tag{4.12} \]
\[ B_1^+ e^{-iK_D^1 L/2} = B_1^- e^{+iK_D^1 L/2} , \tag{4.13} \]
\[ B_2^+ e^{iK_2^B L/2} = B_2^- e^{-iK_2^B L/2}. \] (4.14)

2. The boundary conditions at \( x = 0 \) yield

\[ D_1^+ + D_2^+ = D_1^- + D_2^-, \] (4.15)
\[ B_1^+ + B_2^+ = B_1^- + B_2^-, \] (4.16)
\[ \frac{i\hbar K_1^D}{2m}(D_1^+ - D_1^-) - \frac{i\hbar K_2^D}{2m}(D_2^+ - D_2^-) = -\frac{t}{\hbar} C, \] (4.17)
\[ \frac{i\hbar K_1^B}{2m}(B_1^+ - B_1^-) - \frac{i\hbar K_2^B}{2m}(B_2^+ - B_2^-) = +\frac{t}{\hbar} C. \] (4.18)

3. The coupling with the Majorana fermion yields

\[ C = \frac{t}{E}[-D(0) + B(0)] \]
\[ = \frac{t}{E}[-D_1^+ - D_2^+ + B_1^+ + B_2^+] \] (4.19)
\[ = \frac{t}{E}[-D_1^- - D_2^- + B_1^- + B_2^-]. \]

Solving all these equations, we can obtain the spectrum and the eigenstate wave-function.

Now the resonant condition in the ring is one of \( K_1^D L, K_2^D L, K_1^B L, K_2^B L \) equals to \( 2n\pi \) \((n \in \mathbb{Z})\), or

\[ \phi_{D}^{e/h} \pm \phi_{M} = 2n\pi, \] (4.20)

where \( \phi_{D}^{e/h} = k_{e/h} L \) is the dynamical phase and \( \phi_{M} = \frac{e}{\hbar}AL \) is the magnetic phase an electron/hole acquires when traveling one complete circle round the ring. Depending on the system being in resonance or not, we divide the solutions into:
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Non-Resonant Situation

When the system is not in resonance, a straightforward calculation leads to

\[ D^+_1 = N \left( \frac{e^{-i(\phi^e_D + \phi_M)/2}}{2k_e \sin[(\phi^e_D + \phi_M)/2]} \right), \]  
(4.21)
\[ D^+_2 = N \left( \frac{e^{i(\phi^e_D - \phi_M)/2}}{2k_e \sin[(\phi^e_D - \phi_M)/2]} \right), \]  
(4.22)
\[ D^-_1 = N \left( \frac{e^{i(\phi^h_D + \phi_M)/2}}{2k_e \sin[(\phi^h_D + \phi_M)/2]} \right), \]  
(4.23)
\[ D^-_2 = N \left( \frac{e^{-i(\phi^h_D - \phi_M)/2}}{2k_e \sin[(\phi^h_D - \phi_M)/2]} \right), \]  
(4.24)
\[ B^+_1 = N \left( \frac{e^{i(\phi^h_D + \phi_M)/2}}{2k_h \sin[(\phi^h_D + \phi_M)/2]} \right), \]  
(4.25)
\[ B^+_2 = N \left( \frac{e^{-i(\phi^h_D - \phi_M)/2}}{2k_h \sin[(\phi^h_D - \phi_M)/2]} \right), \]  
(4.26)
\[ B^-_1 = N \left( \frac{e^{i(\phi^h_D + \phi_M)/2}}{2k_h \sin[(\phi^h_D + \phi_M)/2]} \right), \]  
(4.27)
\[ B^-_2 = N \left( \frac{e^{-i(\phi^h_D - \phi_M)/2}}{2k_h \sin[(\phi^h_D - \phi_M)/2]} \right), \]  
(4.28)
\[ C = -N \frac{\hbar^2}{mt}, \]  
(4.29)

where \( N \) is a normalization constant to make

\[ \{\Gamma, \Gamma^\dagger\} = \int [||D(x)||^2 + ||B(x)||^2] dx + |C|^2 = 1 . \]  
(4.30)

The energy of the eigenstate is determined by

\[ \frac{2\hbar^2}{mt} E = \frac{1}{k_e} \left( \cot \frac{\phi^e_D + \phi_M}{2} + \cot \frac{\phi^e_D - \phi_M}{2} \right) - \frac{1}{k_h} \left( \cot \frac{\phi^h_D + \phi_M}{2} + \cot \frac{\phi^h_D - \phi_M}{2} \right), \]  
(4.31)

Now we would like to investigate if a zero mode can exist in this situation.
For the zero mode, we have \( k_e = k_h = k_F \), where the Fermi wave-vector \( k_F \) is defined by \( \mu = \frac{\hbar k_F^2}{2m} \). The resonant condition becomes \( \phi_D \pm \phi_M = 2n\pi \) \( (\phi_D = k_F L) \). This condition is fulfilled or not depends on the value of the magnetic flux. It’s not difficult to obtain the form of the zero mode

\[
\Gamma_0 = \frac{N_0}{2\sqrt{m\mu}} \int_0^{L/2} dx \left\{ \frac{e^{-i(\phi_D + \phi_M)/2}}{\sin[(\phi_D + \phi_M)/2]} e^{iK_1 x} + \frac{e^{i(\phi_D - \phi_M)/2}}{\sin[(\phi_D - \phi_M)/2]} e^{-iK_2 x} \right\} \Psi(x) \]

\[
+ \frac{N_0}{2\sqrt{m\mu}} \int_{-L/2}^{0} dx \left\{ \frac{e^{i(\phi_D + \phi_M)/2}}{\sin[(\phi_D + \phi_M)/2]} e^{iK_1 x} + \frac{e^{-i(\phi_D - \phi_M)/2}}{\sin[(\phi_D - \phi_M)/2]} e^{-iK_2 x} \right\} \Psi(x) 

- \frac{N_0}{m\hbar} \right\} \Psi(x) \] (4.32)

where \( K_1 = k_F + \frac{e}{\hbar} A \), \( K_2 = k_F - \frac{e}{\hbar} A \). The normalization constant

\[
N_0 = \left\{ \frac{L}{2m\mu} \left( \frac{1}{\sin^2[(\phi_D + \phi_M)/2]} + \frac{1}{\sin^2[(\phi_D - \phi_M)/2]} \right) + \frac{1}{k_F L} \left( \frac{\cot \phi_D + \phi_M}{2} + \cot \frac{\phi_D - \phi_M}{2} \right) + \frac{\hbar^2}{m^2 t^2} \right\} \] (4.33)

is chosen to let \( \Gamma_0^2 = 1 \) \( (\{\Gamma_0, \Gamma_0\} = 2) \). From this result we see the zero mode lies mainly at the Majorana fermion \( \gamma_1 \) when the coupling is weak \( (t \text{ small}) \). As the coupling becomes strong \( (t \text{ large}) \), the zero mode gradually transfers to the ring.

**Resonant Situation**

If one of the conditions \( \phi_D^{e/h} \pm \phi_M = 2n\pi \) \( (n \in \mathbb{Z}) \) is fulfilled, then the resonance happens and the solution Eq. (4.21)-(4.29) is not valid. For the resonant eigenstate, we have
(a) $D_1^+ = D_1^- \equiv D_1$. If $D_1 \neq 0$, the resonant condition $\phi^e_D + \phi_M = 2n_1\pi \ (n_1 \in \mathbb{Z})$ should be fulfilled.

(b) $D_2^+ = D_2^- \equiv D_2$. If $D_2 \neq 0$, the resonant condition $\phi^e_D - \phi_M = 2n_2\pi \ (n_2 \in \mathbb{Z})$ should be fulfilled.

(c) $B_1^+ = B_1^- \equiv B_1$. If $B_1 \neq 0$, the resonant condition $\phi^h_D + \phi_M = 2m_1\pi \ (m_1 \in \mathbb{Z})$ should be fulfilled.

(d) $B_2^+ = B_2^- \equiv B_2$. If $B_2 \neq 0$, the resonant condition $\phi^h_D - \phi_M = 2m_2\pi \ (m_2 \in \mathbb{Z})$ should be fulfilled.

(e) $C = 0$ and $D_1 + D_2 = B_1 + B_2$.

The property (e) shows the wave-functions of the resonant eigenstates are concentrated in the ring, and we have at least two of $D_1$, $D_2$, $B_1$, $B_2$ being non-zero. There exist six different possibilities and we divide them into three classes:

(i) $D_1$ and $D_2$ are non-zero or $B_1$ and $B_2$ are non-zero.

If $D_1 = -D_2 \neq 0$, then $\phi^e_D + \phi_M = 2n_1\pi$ and $\phi^e_D - \phi_M = 2n_2\pi$. These conditions are also equivalent to $\phi_M = n\pi$ and $\phi^e_D = (2\bar{n} - n)\pi \ (n, \bar{n} \in \mathbb{Z})$. In this situation, the counter-clockwise electron mode ($D_1$) and clockwise electron mode ($D_2$) are paired to be in resonance.

If $B_1 = -B_2 \neq 0$, then $\phi^h_D + \phi_M = 2m_1\pi$ and $\phi^e_D - \phi_M = 2m_2\pi$. These conditions are also equivalent to $\phi_M = n\pi$ and $\phi^h_D = (2\bar{n} - n)\pi \ (n, \bar{n} \in \mathbb{Z})$. In this situation, the counter-clockwise hole mode ($B_1$) and clockwise hole mode ($B_2$) are paired to be in resonance.

(ii) $D_1$ and $B_2$ are non-zero or $D_2$ and $B_1$ are non-zero.
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In the following discussion of (ii) and (iii) we assume the energy $E$ is small compared to the Fermi energy $E_F = \mu$, and the dispersion relation can be linearised near the Fermi surface, such that we can set $k_e = k_F + q$ and $k_h = k_F - q$.

If $D_1 = B_2 \neq 0$, then $\phi^e_D + \phi_M = 2n_1 \pi$ and $\phi^h_D - \phi_M = 2m_2 \pi$. These conditions are also equivalent to $k_F L = n \pi$ and $qL + \phi_M = (2\bar{n} - n) \pi$ $(n, \bar{n} \in \mathbb{Z})$. In this situation, the counter-clockwise electron mode ($D_1$) and the clockwise hole mode ($B_2$) are paired to be in resonance.

If $D_2 = B_1 \neq 0$, then $\phi^e_D - \phi_M = 2n_2 \pi$ and $\phi^h_D + \phi_M = 2m_1 \pi$. These conditions are also equivalent to $k_F L = n \pi$ and $qL - \phi_M = (2\bar{n} - n) \pi$ $(n, \bar{n} \in \mathbb{Z})$. In this situation, the clockwise electron mode ($D_2$) and the counter-clockwise hole mode ($B_1$) are paired to be in resonance.

Note that for this class of resonance, since the electron and the hole have opposite charges, the eigenstates carry the maximal persistent current.

(iii) $D_1$ and $B_1$ are non-zero or $D_2$ and $B_2$ are non-zero.

If $D_1 = B_1 \neq 0$, then $\phi^e_D + \phi_M = 2n_1 \pi$ and $\phi^h_D - \phi_M = 2m_1 \pi$. These conditions are also equivalent to $k_F L + \phi_M = n \pi$ and $qL = (2\bar{n} - n) \pi$ $(n, \bar{n} \in \mathbb{Z})$. In this situation, the counter-clockwise electron mode ($D_1$) and the clockwsl hole mode ($B_1$) are paired to be in resonance.

If $D_2 = B_2 \neq 0$, then $\phi^e_D - \phi_M = 2n_2 \pi$ and $\phi^h_D - \phi_M = 2m_2 \pi$. These conditions are also equivalent to $k_F L - \phi_M = n \pi$ and $qL = (2\bar{n} - n) \pi$ $(n, \bar{n} \in \mathbb{Z})$. In this situation, the clockwise electron mode ($D_2$) and the clockwise hole mode ($B_2$) are paired to be in resonance.

Finally, we would like to investigate the zero mode under the resonant conditions. Since for the zero mode we have $q = 0$ and $\phi^e_D = \phi^h_D = k_F L$, the class (i) resonant condition and
the class (ii) resonant condition are equivalent and they both become special situations of the class (iii) resonant condition. So there is actually only one class of resonant zero mode, of which the necessary condition is \(k_F L \pm \phi_M = 2n\pi\ (n \in \mathbb{Z})\). The zero mode operator has the form
\[
\Gamma_0 = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \left[ e^{\pm i2n\pi x/L} \Psi(x) + e^{\mp i2n\pi x/L} \Psi^\dagger(x) \right].
\] (4.34)

### 4.1.2 A Metallic Ring Coupled to A Pair of Majorana Fermions

Now assume the Majorana fermion \(\gamma_1\) is not isolated but coupled through tunnelling effect to another Majorana fermion \(\gamma_2\). We add a term \(H_M = i\frac{E_M}{2} \gamma_1 \gamma_2\) to the Hamiltonian Eq.(4.1) and rewrite the operator
\[
\Gamma = \int dx \left[ D(x)\psi(x) + B(x)\psi^\dagger(x) \right] + \frac{C^{(1)}}{\sqrt{2}} \gamma_1 + \frac{C^{(2)}}{\sqrt{2}} \gamma_2.
\] (4.35)

The BdG equations for \(D(x)\) and \(B(x)\) are the same as Eq. (4.3) and (4.4), while the BdG equations for \(C^{(1)}\) and \(C^{(2)}\) are
\[
t \left[ -D(0) + B(0) \right] - iE_M C^{(2)} = EC^{(1)},
\] (4.36)
\[
iE_M C^{(1)} = EC^{(2)}.
\] (4.37)

So we have \(C^{(2)} = (iE_M/E)C^{(1)}\) and
\[
t \left[ -D(0) + B(0) \right] = (E - \frac{E_M^2}{E})C^{(1)},
\] (4.38)

which can be taken as to replace the Eq. (4.5).
Non-Resonant Situation

When the system is not in resonance, the secular equation is changed from Eq. (4.31) to

\[
\frac{2\hbar^2}{ml^2}(E - \frac{E_M^2}{E}) = \frac{1}{k_c}\left(\cot \frac{\phi_D + \phi_M}{2} + \cot \frac{\phi_D - \phi_M}{2}\right) - \frac{1}{k_h}\left(\cot \frac{\phi_D + \phi_M}{2} + \cot \frac{\phi_D - \phi_M}{2}\right).\]

(4.39)

It’s seen there is no zero energy solution to this equation when \(E_M \neq 0\). So we conclude that there is no non-resonant zero mode when the pair of Majorana fermions are coupled.

Resonant Situation

When the system is in resonance, the properties (a) - (d) in Sec. 4.1.1 are still fulfilled. For the property (e), we still have \(C^{(1)} = 0\). But the rest depends on the resonant mode to be zero energy or not.

For the non-zero energy resonant mode, we have \(C^{(2)} = 0\) and \(D_1 + D_2 = B_1 + B_2\). Thus, the solutions are the same as that in Sec. 4.1.1.

For the zero energy resonant mode, we have \(C^{(2)}\) not necessarily being zero and

\[
D_1 + D_2 - B_1 - B_2 = -\frac{iE_M}{t}C^{(2)}.\]

(4.40)
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So unlike the situation of Sec. 4.1.1 where at least two of $D_1, D_2, B_1, B_2$ should be non-zero, now we can have only one of them being non-zero. The resonant condition is $\phi_D \pm \phi_M = 2n\pi \ (n \in \mathbb{Z})$. We have two zero modes:

$$
\Gamma_{0,D} = \frac{E_M}{\sqrt{LE_M^2 + t^2}} \int_{-L/2}^{L/2} dx \ e^{i2n\pi x/L} \Psi(x) + \frac{it}{\sqrt{LE_M^2 + t^2}} \gamma_2 , \quad (4.41)
$$

$$
\Gamma_{0,B} = \frac{E_M}{\sqrt{LE_M^2 + t^2}} \int_{-L/2}^{L/2} dx \ e^{i2n\pi x/L} \Psi^\dagger(x) - \frac{it}{\sqrt{LE_M^2 + t^2}} \gamma_2 . \quad (4.42)
$$

It’s obvious that $\Gamma_{0,D}^\dagger = \Gamma_{0,B}$. We can also form two Majorana zero modes:

$$
\Gamma_{0,I} = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \left[ e^{i2n\pi x/L} \Psi(x) + e^{i2n\pi x/L} \Psi^\dagger(x) \right] , \quad (4.43)
$$

$$
\Gamma_{0,II} = \frac{-iE_M}{\sqrt{LE_M^2 + t^2}} \int_{-L/2}^{L/2} dx \left[ e^{i2n\pi x/L} \Psi(x) - e^{i2n\pi x/L} \Psi^\dagger(x) \right] + \frac{t}{\sqrt{LE_M^2 + t^2}} \gamma_2 . \quad (4.44)
$$

4.1.3 Two Metallic Rings Coupled through A Pair of Majorana Fermions

In this section we study the system of two metallic rings coupled by a pair of Majorana fermions. The configuration is shown in Fig. 4.3. The entire system can actually be taken as two copies of the single ring - single Majorana systems, coupled through $H_M = i\frac{E_M}{2} \gamma_1 \gamma_2$.

In the following we focus on the effect of the coupling to the separate systems.

Now we have two copies of BdG Eq. (4.3) and (4.4), and a pair of coupling equations:

$$
t_1 \left[ - D^{(1)}(0) + B^{(1)}(0) \right] - iE_M C^{(2)} = EC^{(1)} , \quad (4.45)
$$

$$
t_2 \left[ - D^{(2)}(0) + B^{(2)}(0) \right] + iE_M C^{(1)} = EC^{(2)} . \quad (4.46)
$$
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Figure 4.3: Two metallic rings coupled by a pair of Majorana fermions.

Non-Resonant Situation

We first study the situation when neither ring is in resonance. Denote $X^{(i)} \equiv \left[ D^{(i)}(0) - B^{(i)}(0) \right] / N^{(i)}$, where $N^{(i)}$ is the normalization constant for the wave-function in the $i$-th ring.

A straightforward calculation leads to

$$
X^{(i)} = \frac{1}{k^{(i)}_e} \left( \cot \phi_D^{(i)} + \frac{\phi_M^{(i)}}{2} + \cot \phi_D^{(i)} - \frac{\phi_M^{(i)}}{2} \right) - \frac{1}{k_h} \left( \cot \phi_D^{(i)} + \frac{\phi_M^{(i)}}{2} + \cot \phi_D^{(i)} - \frac{\phi_M^{(i)}}{2} \right).
$$

(4.47)

Using the notation of $X^{(i)}$, Eq. (4.45) and (4.46) can be written as

$$
\begin{align*}
\left[ \frac{\hbar^2 E}{mt_1^2} - X^{(1)} \right] N^{(1)} + i\hbar^2 E_M \frac{N^{(2)}}{mt_1 t_2} &= 0, \\
-\frac{i\hbar^2 E_M}{mt_1 t_2} N^{(1)} + \left[ \frac{\hbar^2 E}{mt_2^2} - X^{(2)} \right] N^{(2)} &= 0.
\end{align*}
$$

(4.48)

(4.49)

Thus the necessary condition for the existence of a non-zero wave-function is

$$
\left[ \frac{\hbar^2 E}{mt_1^2} - X^{(1)} \right] \left[ \frac{\hbar^2 E}{mt_2^2} - X^{(2)} \right] = \left( \frac{\hbar^2 E_M}{mt_1 t_2} \right)^2.
$$

(4.50)

This is the secular equation for the non-resonant situation. It can be seen that there is no zero energy solution when the pair of Majorana fermions are coupled ($E_M \neq 0$).
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The Situation of Only One Ring Being in Resonance

Assuming the 1st ring is and the 2nd ring is not in resonance, we have $C^{(1)} = 0$. From Eq. (4.45) and (4.46), the entire system can be taken as a combination of two subsystems:

1. The 1st ring and the pair of Majorana fermions form a resonant single ring - paired Majorana system.

2. The 2nd ring and the Majorana fermion $\gamma_2$ form a non-resonant single ring - single Majorana system.

It is worth to mention that the Majorana fermion $\gamma_2$ involves in both subsystems.

Now the 1st system has 4 classes of resonant modes, corresponding to the resonant conditions $\phi^{e/h(1)}_D + \phi^{(1)}_M = 2n\pi$. If the energy of one resonant mode matches that of a non-resonant mode of the 2nd system, then an eigenstate exists, consisted of the resonant mode of the 1st system and the non-resonant mode of the 2nd system. The weight of the wave-function on each part is determined by Eq. (4.45) and (4.46). If there are two resonant conditions that are satisfied simultaneously in the 1st ring, then an eigenstate can exist by allowing the wave-function in the 2nd ring to be zero. Such an eigenstate is consisted of the pair of resonant modes in the 1st ring only.

Note that if one of the resonant conditions is satisfied at zero energy for the 1st ring, then a zero energy eigenstate of the entire system is guaranteed to exist, for the zero energy is always a solution to the 2nd system.

The Situation of Both Rings Being in Resonance

When both rings are in resonance, we have $C^{(1)} = C^{(2)} = 0$. The coupling equations Eq. (4.45) and Eq. (4.46) simply decouple and we obtain two independent resonant single ring - single Majorana systems.
There are 4 classes of resonant modes in each ring. Now from our analysis, there should be at least two resonant conditions that are satisfied in the same ring, in order for a resonant eigenstate to exist. What happens if two resonant conditions are satisfied for two separate rings? If we solve the BdG equations directly we can only obtain the entire wave-function being zero. The situation is actually analogous to the system of two directly coupled metallic rings, which we have discussed in Sec. 3.2. There is a level repelling effect at the point when both the resonant conditions are satisfied. There does not exists such kind of eigenstate indeed.

However, when the coupling between the two Majorana fermions is weak, we expect that the level repelling effect is weak and the actual eigenstates are close the resonant point, although strictly speaking they belong to the non-resonant situation.

For example, there exist $\phi^{(1)}_M$ and $\phi^{(2)}_M$ for which the resonant conditions $\phi^{(1)}_D + \phi^{(1)}_M = 2n\pi$ and $\phi^{h(2)}_D + \phi^{(2)}_M = 2m\pi$ are almost satisfied, and a corresponding eigenstate exists. This eigenstate is an almost resonant one, dominated by a counter-clockwise electron mode in the 1st ring and a counter-clockwise hole mode in the 2nd ring. This kind of eigenstate is very useful in a transport set-up for us to detect the existence of Majorana fermions.

### 4.2 Majorana Fermion Induced Andreev Reflection

In this section we discuss the Majorana fermion induced Andreev reflection in metallic wires.

#### 4.2.1 Scattering by An Isolated Majorana Fermion in A Metallic Wire

Consider a configuration of one metallic wire placed near an isolated Majorana fermion. The wire and the Majorana fermion is coupled through the tunneling effect. The set-up is shown in Fig. 4.4.
The Hamiltonian of the system can be expressed as
\[
H = \int dx \left\{ \Psi^\dagger(x) \left[ \frac{p^2}{2m} + V(x) - \mu \right] \Psi(x) + \frac{t}{\sqrt{2}} \gamma_1 [\Psi^\dagger(x) - \Psi(x)] \delta(x) \right\} .
\] (4.51)

Here we have added an impurity term \( V\delta(x) \) for the convenience to compare the normal scattering process and the Andreev reflection process.

Making the BdG transformation as Eq.(4.2), we are able to obtain the BdG equation
\[
\left[ \frac{1}{2m} p^2 + V(x) - \mu \right] D(x) - t\delta(x)C = ED(x) ,
\] (4.52)
\[
- \left[ \frac{1}{2m} p^2 + V(x) - \mu \right] B(x) + t\delta(x)C = EB(x) ,
\] (4.53)
\[
t[ - D(0) + B(0) ] = EC .
\] (4.54)

We first suppose that an electron is incident from the left side of the wire. The wave-function representing this situation can be written as
\[
D(x) = \begin{cases} e^{ik_e x} + r_e e^{-ik_e x} , & x < 0 , \\ te^{ik_e x} , & x > 0 , \end{cases}
\] (4.55)
\[
B(x) = \begin{cases} r_h e^{ik_h x} , & x < 0 , \\ t_h e^{-ik_h x} , & x > 0 , \end{cases}
\] (4.56)

By matching the boundary conditions, we can solve out all the scattering amplitudes
\[
r_e = \frac{-\hbar v_e + i(E/t^2) V(i\hbar v_e + V)}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)} ,
\] (4.57)
\[
t_e = \frac{\hbar v_e - (E/t^2)(i\hbar v_h + V) h v_e}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)} ,
\] (4.58)
\[
r_h = \frac{\hbar v_e}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)} ,
\] (4.59)
and the wave-function for the Majorana fermion

\[ C = \frac{\hbar v_e (i \hbar v_h + V)/t}{\hbar (v_e + v_h) + i (E/t^2)(i \hbar v_e - V)(i \hbar v_h + V)}. \]  

(4.61)

Here \( v_e = k_e/m \) represents the velocity of the electron and \( v_h = k_h/m \) represents the velocity of the hole.

When the incident energy is small compared to the Fermi energy, we can linearize the dispersion relation to be \( E = v_F q \) (note that \( k_{e/h} = k_F \pm q \)). Expanding to the first order of \( v_q/v_F \), the scattering amplitudes become

\[ r_e = -\frac{1}{2} + \frac{v_q}{v_F} \left[ \left( \frac{1}{2} - \frac{mv_F^2}{2t^2} V \right) + i \frac{mv_F}{4\hbar t^2} (V^2 - \hbar^2 v_F^2) \right], \]  

(4.62)

\[ t_e = \frac{1}{2} + \frac{v_q}{v_F} \left[ \left( \frac{1}{2} - \frac{mv_F^2}{2t^2} V \right) + i \frac{mv_F}{4\hbar t^2} (V^2 - \hbar^2 v_F^2) \right], \]  

(4.63)

\[ r_h = t_h = \frac{1}{2} + \frac{v_q}{v_F} \left[ \frac{1}{2} + i \frac{mv_F}{4\hbar t^2} (V^2 + \hbar^2 v_F^2) \right], \]  

(4.64)

It’s seen that for our set-up, the scattering probabilities of the electron and of the hole are equal in the zero energy limit. This is very different from the usual local Andreev reflection where the scattering probability of the hole dominates over that of the electron, but is much
like the crossed Andreev reflection.

Note that the quasi-particle probability is conserved in the sense that $j_P^{\text{(in)}} = j_P^{\text{(out)}}$, where the incoming probability current is

$$j_P^{\text{(in)}} = (v_F + v_q) \cdot 1 \quad (4.65)$$

and the outgoing probability current is

$$j_P^{\text{(out)}} = (v_F + v_q)(|t_e|^2 - |r_e|^2) + (v_F - v_q)(|t_h|^2 - |r_h|^2). \quad (4.66)$$

Meanwhile, the transmitted charge current is

$$j_Q^{\text{(trans)}} = (v_F + v_q)t_e^2 - (v_F - v_q)t_h^2 = v_q \left( \frac{1}{2} - \frac{mv_F^2}{2t^2} V \right). \quad (4.67)$$

It’s interesting to see that when $V > t^2/mv_F^2$, the transmitted charge current is negative. This can happen because, restricted to the first order, the electron transmitted probability $|t_e|^2$ depends on the scattering potential $V$ while the hole transmitted probability $|t_h|^2$ is independent of $V$ (it is generated entirely by the Andreev reflection).

Next, we consider the situation of a hole incoming from the left. The wave-function has the form

$$\tilde{B}(x) = \begin{cases} e^{-ik_hx} + \tilde{r}_he^{ik_hx}, & x < 0, \\ \tilde{t}_he^{-ik_hx}, & x > 0, \end{cases} \quad (4.68)$$

$$\tilde{D}(x) = \begin{cases} \tilde{r}_e^{-ik_e}x, & x < 0, \\ \tilde{t}_ee^{ik_e}x, & x > 0. \end{cases} \quad (4.69)$$
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The solution is

\[ \tilde{r}_h = \frac{-\hbar v_e - i(E/t^2)V(i\hbar v_e - V)}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)}, \quad (4.70) \]

\[ \tilde{t}_h = \frac{\hbar v_h - (E/t^2)(i\hbar v_e - V)\hbar v_h}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)}, \quad (4.71) \]

\[ \tilde{r}_e = \frac{\hbar v_h}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_B + V)}, \quad (4.72) \]

\[ \tilde{t}_e = \tilde{r}_e, \quad (4.73) \]

and

\[ \tilde{C} = \frac{-\hbar v_h(i\hbar v_e - V)/t}{\hbar(v_e + v_h) + i(E/t^2)(i\hbar v_e - V)(i\hbar v_h + V)}. \quad (4.74) \]

In the low energy limit, we can expand them to the first order of \( v_q/v_F \)

\[ \tilde{r}_h = -\frac{1}{2} - \frac{v_q}{v_F} \left[ \frac{1}{2} - \frac{mv_F^2}{2t^2}V - i\frac{mv_F}{4\hbar t^2}(V^2 - \hbar^2 v_F^2) \right], \quad (4.75) \]

\[ \tilde{t}_h = \frac{1}{2} - \frac{v_q}{v_F} \left[ \frac{1}{2} - \frac{mv_F^2}{2t^2}V - i\frac{mv_F}{4\hbar t^2}(V^2 - v_F^2) \right], \quad (4.76) \]

\[ \tilde{r}_e = \tilde{t}_e = \frac{1}{2} - \frac{v_q}{v_F} \left[ \frac{1}{2} - i\frac{mv_F}{4\hbar t^2}(V^2 + \hbar^2 v_F^2) \right]. \quad (4.77) \]

Now using the symmetry between the left and the right, we can form a scattering matrix under the basis of left electron, right electron, left hole and right hole:

\[
S = \begin{pmatrix}
  r_e & t_e & \tilde{r}_e & \tilde{t}_e \\
  t_e & r_e & \tilde{t}_e & \tilde{r}_e \\
  r_h & t_h & \tilde{r}_h & \tilde{t}_h \\
  t_h & r_h & \tilde{t}_h & \tilde{r}_h
\end{pmatrix}.
\quad (4.78)
\]
4.2.2 Scattering by A Pair of Majorana Fermions between Two Metallic Wires

Now we consider the scattering between two metallic wires that are coupled by a pair of Majorana fermions. The configuration is shown in Fig. 4.5 and the Hamiltonian can be written as

\[ H = \int dx \, \Psi_1^\dagger(x) \left[ \frac{p_x^2}{2m} - \mu \right] \Psi_1(x) + \int dy \, \Psi_2^\dagger(y) \left[ \frac{p_y^2}{2m} - \mu \right] \psi_2(y) + i \frac{E_M}{2} \gamma_1 \gamma_2 \\
+ \int dx \, \frac{t_1}{\sqrt{2}} \gamma_1 [\Psi_1^\dagger(x) - \Psi_1(x)] \delta(x) + \int dy \, \frac{t_2}{\sqrt{2}} \gamma_2 [\Psi_2^\dagger(y) - \Psi_2(y)] \delta(y). \tag{4.79} \]

Assuming the operator

\[ \Gamma = \int dx [D^{(1)}(x)\Psi_1(x) + B^{(1)}(x)\Psi_1^\dagger(x)] + \int dy [D^{(2)}(y)\Psi_2(y) + B^{(2)}(y)\Psi_2^\dagger(y)] + C^{(1)} + \frac{C^{(2)}}{\sqrt{2}} \gamma_1 + \frac{C^{(2)}}{\sqrt{2}} \gamma_2 \tag{4.80} \]

diagonalize the Hamiltonian, we obtain the BdG equations:

\[ \frac{1}{2m} p_x^2 - \mu \right] D^{(1)}(x) - t_1 \delta(x) C^{(1)} = E D^{(1)}(x), \tag{4.81} \]

\[ \frac{1}{2m} p_y^2 - \mu \right] B^{(1)}(x) + t_1 \delta(x) C^{(1)} = E B^{(1)}(x), \tag{4.82} \]

\[ t_1 [ - D^{(1)}(0) + B^{(1)}(0)] - iE_M C^{(2)} = EC^{(1)}, \tag{4.83} \]

\[ \frac{1}{2m} p_y^2 - \mu \right] D^{(2)}(y) - t_2 \delta(y) C^{(2)} = E D^{(2)}(y), \tag{4.84} \]

\[ \frac{1}{2m} p_y^2 - \mu \right] B^{(2)}(y) + t_2 \delta(y) C^{(2)} = E B^{(2)}(y), \tag{4.85} \]

\[ t_2 [ - D^{(2)}(0) + B^{(2)}(0)] + iE_M C^{(1)} = EC^{(2)}. \tag{4.86} \]

First, suppose an electron is incident from the left lower side, as shown in Fig. 4.5. The
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Figure 4.5: Scattering process in two metallic wires that are coupled through a pair of Majorana fermions.

The eigenstate wave-function can be written as

\[
D^{(1)}(x) = \begin{cases} 
  e^{ik_x} + r^{(1)}_e e^{-ik_x}, & x < 0 , \\
  t^{(1)}_e e^{ik_x}, & x > 0 ,
\end{cases}
\]  
\tag{4.87}

\[
B^{(1)}(x) = \begin{cases} 
  r^{(1)}_h e^{ik_h x}, & x < 0 , \\
  t^{(1)}_h e^{-ik_h x}, & x > 0 ,
\end{cases}
\]  
\tag{4.88}

\[
D^{(2)}(y) = \begin{cases} 
  w^{(1)}_e e^{-ik_y}, & y < 0 , \\
  w^{(1)}_e e^{ik_y}, & y > 0 ,
\end{cases}
\]  
\tag{4.89}

\[
B^{(2)}(y) = \begin{cases} 
  w^{(1)}_h e^{ik_y}, & y < 0 , \\
  w^{(1)}_h e^{-ik_y}, & y > 0 .
\end{cases}
\]  
\tag{4.90}

Substituting this wave-function ansatz into the BdG equations, we can solve out all the
unknown amplitudes. The result is

\[ r_e^{(1)} = i \frac{t_2^1}{\hbar v_e} \left[ E + i \left( \frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z , \]

(4.91)

\[ t_e^{(1)} = 1 + r_e^{(1)} , \]

(4.92)

\[ r_h^{(1)} = -i \frac{t_1^2}{\hbar v_h} \left[ E + i \left( \frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z , \]

(4.93)

\[ t_h^{(1)} = r_h^{(1)} , \]

(4.94)

\[ w_e^{(1)} = -E_M \left( \frac{t_1 t_2}{\hbar v_e} \right) / Z , \]

(4.95)

\[ w_h^{(1)} = E_M \left( \frac{t_1 t_2}{\hbar v_h} \right) / Z , \]

(4.96)

and

\[ C^{(1)} = t_1 \left[ E + i \left( \frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z , \]

(4.97)

\[ C^{(2)} = i E_M t_1 / Z , \]

(4.98)

where

\[ Z = E_M^2 - \left[ E + i \left( \frac{t_1^2}{\hbar v_e} + \frac{t_1^2}{\hbar v_h} \right) \right] \left[ E + i \left( \frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h} \right) \right] . \]

(4.99)

In the low energy regime, there is \( v_e \simeq v_h \simeq v_F \). The electron tunneling probability and the hole tunneling probability in the process of the crossed Andreev reflection are equal: \( |w_e|^2 \simeq |w_h|^2 \). Furthermore, in the regime where \( E_M \gg E \) and \( E_M \gg t^2/\hbar v \), we can see the crossed Andreev reflection dominate over the local Andreev reflection.

Next, we consider the situation of a hole incoming from the left lower side. The wave-
function can be written as

\[
\tilde{B}^{(1)}(x) = \begin{cases} 
  e^{-ik_h x} + \tilde{r}_h^{(1)} e^{ik_h x}, & x < 0, \\
  \tilde{t}_h^{(1)} e^{-ik_h x}, & x > 0,
\end{cases} \quad (4.100)
\]

\[
\tilde{D}^{(1)}(x) = \begin{cases} 
  \tilde{r}_c^{(1)} e^{-ik_e x}, & x < 0, \\
  \tilde{t}_c^{(1)} e^{ik_e x}, & x > 0,
\end{cases} \quad (4.101)
\]

\[
\tilde{B}^{(2)}(y) = \begin{cases} 
  \tilde{w}_h^{(1)} e^{ik_h y}, & y < 0, \\
  \tilde{w}_h^{(1)} e^{-ik_h y}, & y > 0,
\end{cases} \quad (4.102)
\]

\[
\tilde{D}^{(2)}(y) = \begin{cases} 
  \tilde{w}_e^{(1)} e^{-ik_e y}, & y < 0, \\
  \tilde{w}_e^{(1)} e^{ik_e y}, & y > 0.
\end{cases} \quad (4.103)
\]

Substituting into the BdG equations and solving them, we obtain

\[
\tilde{r}_h^{(1)} = \frac{i \frac{t_1^2}{\hbar v_h} [E + i(\frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h})]}{Z}, \quad (4.104)
\]

\[
\tilde{t}_h^{(1)} = 1 + \tilde{r}_1^{(h)}, \quad (4.105)
\]

\[
\tilde{r}_c^{(1)} = \frac{-i \frac{t_1^2}{\hbar v_e} [E + i(\frac{t_2^2}{\hbar v_e} + \frac{t_2^2}{\hbar v_h})]}{Z}, \quad (4.106)
\]

\[
\tilde{t}_c^{(1)} = \tilde{r}_1^{(c)}, \quad (4.107)
\]

\[
\tilde{w}_h^{(1)} = -E_M \left( \frac{t_1 t_2}{\hbar v_h} \right) / Z, \quad (4.108)
\]

\[
\tilde{w}_e^{(1)} = E_M \left( \frac{t_1 t_2}{\hbar v_e} \right) / Z. \quad (4.109)
\]
Now using the symmetry, we can write down the scattering matrix for the set-up:

\[
S_M = \begin{pmatrix}
    r_e^{(1)} & t_e^{(1)} & w_e^{(1)} & w_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{t}_e^{(1)} & \tilde{w}_e^{(1)} & \tilde{w}_e^{(2)} \\
    t_e^{(1)} & r_e^{(1)} & w_e^{(1)} & w_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{t}_e^{(1)} & \tilde{w}_e^{(1)} & \tilde{w}_e^{(2)} \\
    w_e^{(1)} & w_e^{(1)} & r_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{w}_e^{(1)} & \tilde{w}_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{r}_e^{(2)} \\
    w_e^{(1)} & w_e^{(1)} & r_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{w}_e^{(1)} & \tilde{w}_e^{(2)} & \tilde{r}_e^{(1)} & \tilde{r}_e^{(2)} \\
    r_h^{(1)} & t_h^{(1)} & w_h^{(2)} & w_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{t}_h^{(1)} & \tilde{w}_h^{(1)} & \tilde{w}_h^{(2)} \\
    t_h^{(1)} & r_h^{(1)} & w_h^{(2)} & w_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{t}_h^{(1)} & \tilde{w}_h^{(1)} & \tilde{w}_h^{(2)} \\
    w_h^{(1)} & w_h^{(1)} & r_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{w}_h^{(1)} & \tilde{w}_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{r}_h^{(2)} \\
    w_h^{(1)} & w_h^{(1)} & r_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{w}_h^{(1)} & \tilde{w}_h^{(2)} & \tilde{r}_h^{(1)} & \tilde{r}_h^{(2)}
\end{pmatrix}, \quad (4.110)
\]

of which the eight channels are left lower electron, right lower electron, left upper electron, right upper electron, left lower hole, right lower hole, left upper hole and right upper hole.

The not yet given matrix elements can be determined by symmetry:

\[
r_e^{(2)} = i \frac{t_2^2}{hv_e} \left[ E + i \left( \frac{t_1^2}{hv_e} + \frac{t_1^2}{hv_h} \right) \right] / Z, \quad (4.111)
\]

\[
t_e^{(2)} = 1 + r_2^{(e)}, \quad (4.112)
\]

\[
r_h^{(2)} = -i \frac{t_2^2}{hv_h} \left[ E + i \left( \frac{t_1^2}{hv_e} + \frac{t_1^2}{hv_h} \right) \right] / Z, \quad (4.113)
\]

\[
t_h^{(2)} = r_2^{(h)}, \quad (4.114)
\]

\[
w_e^{(2)} = E_M \left( \frac{t_1 t_2}{hv_e} \right) / Z, \quad (4.115)
\]

\[
w_h^{(2)} = -E_M \left( \frac{t_1 t_2}{hv_h} \right) / Z, \quad (4.116)
\]

\[
\tilde{r}_h^{(2)} = i \frac{t_2^2}{hv_h} \left[ E + i \left( \frac{t_1^2}{hv_e} + \frac{t_1^2}{hv_h} \right) \right] / Z, \quad (4.117)
\]

\[
\tilde{t}_h^{(2)} = 1 + \tilde{r}_2^{(h)}, \quad (4.118)
\]
\[ \tilde{r}_{e}^{(2)} = -i t_2^2 \left[ E + i \left( \frac{t_1^2}{hv_e} + \frac{t_1^2}{hv_h} \right) \right] / Z , \]  
\[ \tilde{t}_{e}^{(2)} = \tilde{r}_{e}^{(c)} , \]  
\[ \tilde{w}_{h}^{(2)} = E_M \left( \frac{t_1 t_2}{hv_h} \right) / Z , \]  
\[ \tilde{w}_{e}^{(2)} = -E_M \left( \frac{t_1 t_2}{hv_e} \right) / Z . \]

### 4.3 Detecting Majorana Fermion Induced Crossed Andreev Scattering

From previous results, we know that in a direct crossed Andreev reflection process, the tunneling probability of the electron equals to that of the hole. So a direct tunneling current measurement is impossible to detect the crossed Andreev reflection. It was proposed that a measurement of the shot noise was necessary [Nillson et al. (2008)] [Law et al. (2009)]. In this section, we will see that by using metallic rings, we can distinguish the electron tunneling signals and the hole tunneling signals.

Our proposed tunneling current measurement set-up is shown in Fig. (4.6). It is consisted of two metallic rings coupled by a pair of Majorana fermions. An external lead connects the source reservoir to the 1st ring, and another external lead connects the drain reservoir to the 2nd ring. When exerting a negative voltage bias to the source reservoir, electrons are supposed to flow out from the source, and tunneling current that flows into the drain is measured.

#### 4.3.1 An Intuition

We first give an intuition of why our set-up can distinguish the electron tunneling signal and the hole tunneling signal.
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Figure 4.6: The set-up of tunneling current measurement to detect the Majorana fermion induced crossed Andreev reflection. Two metallic rings, threaded by magnetic fluxes $\Phi_1$ and $\Phi_2$, are coupled with two Majorana fermions and two external leads through tunneling effect. The superconductor hosting the Majorana fermions is grounded. Electrons are incident from the lead contacting the source (S). Tunneling electrons/holes outflow along the second lead to the drain (D).

In order for a current to flow from the source to the drain, electrons or holes need to tunnel through the two metallic rings. For electrons or holes to stay in the ring, the resonant condition $\phi_D + \phi_M = 2n\pi \ (n \in \mathbb{Z})$ should be satisfied. Now we know that the wave-vector of the electron is slightly different from that of the hole. Since $\phi_D = kL$, the dynamical phase of the electron is different from that of the hole. So for a specific magnetic flux the resonant condition for electrons and the resonant condition for holes are not satisfied simultaneously. Thus, we can distinguish the electron tunneling signals and the hole tunneling signals by adjusting the magnetic flux. When the resonant condition for the electron is satisfied, it is the electron tunneling dominated regime. When the resonant condition for the hole
satisfied, it is the hole tunneling dominated regime.

4.3.2 A Detail Analysis

We can use the techniques that were employed in Chapter 3 to calculate the tunneling probabilities.

There are a total of 3 tunneling processes: 1. electrons incident from the 1st lead tunnels to the 1st ring; 2. electrons or holes in the 1st ring tunnel through the Majorana fermions to the 2nd ring; 3. electrons or holes in the 2nd ring tunnels to the 2nd lead.

The first and the third tunneling processes can be described by the scattering matrix $S_t$, given in Eq. (3.12). The second tunneling process can be described by the scattering matrix $S_M$, given in Eq. (4.110).

Now we start to set equations to describe the steady transport process. Let $D$ denote the electron amplitude, $B$ denote the hole amplitude. Let the subscript 1 denote counterclockwise electron/hole motion on the ring, and the subscript 2 denote clockwise motion. Let the superscript $(i)$ denote the corresponding part as shown in Fig. 4.7. For the convenience, we let the tunneling parameters be the same at the two tri-junctions.

Suppose an electron is incident from the first lead. The tunneling process at the tri-junction between the 1st lead and the 1st ring can be described as

$$
\begin{pmatrix}
-(a + b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
1 \\
D_1^{(2)} \\
D_2^{(3)}
\end{pmatrix}
= 
\begin{pmatrix}
D_1^{(1)} \\
D_2^{(2)} \\
D_1^{(3)}
\end{pmatrix},
$$

(4.123)

$$
\begin{pmatrix}
-(a + b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
0 \\
B_1^{(2)} \\
B_2^{(3)}
\end{pmatrix}
= 
\begin{pmatrix}
B_1^{(1)} \\
B_2^{(2)} \\
B_1^{(3)}
\end{pmatrix}.
$$

(4.124)
Figure 4.7: A detail analysis to the transport process.

Here the electron incoming amplitude in the first lead is 1 while the hole incoming amplitude is 0. $D^{(1)}$ and $B^{(1)}$ denote the electron outgoing amplitude and the hole outgoing amplitude separately.

The electron’s propagation in the 1st ring can be described by

\[
D_1^{(5)} = D_1^{(2)} \exp[i(-k_e L_1^a - \phi_1^a)] , \quad (4.125)
\]
\[
D_2^{(5)} = D_2^{(2)} \exp[i(+k_e L_1^a - \phi_1^a)] , \quad (4.126)
\]
\[
D_1^{(4)} = D_1^{(3)} \exp[i(+k_e L_1^b + \phi_1^b)] , \quad (4.127)
\]
\[
D_2^{(4)} = D_2^{(3)} \exp[i(-k_e L_1^b + \phi_1^b)] . \quad (4.128)
\]
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The hole’s propagation in the 1st ring can be described by

\[
B_1^{(5)} = B_1^{(2)} \exp[i(+k_h L_1^a + \phi_1^a)] , \\
B_2^{(5)} = B_2^{(2)} \exp[i(-k_h L_1^a + \phi_1^a)] , \\
B_1^{(4)} = B_1^{(3)} \exp[i(-k_h L_1^b - \phi_1^b)] , \\
B_2^{(4)} = B_2^{(3)} \exp[i(+k_h L_1^b - \phi_1^b)] .
\]

Here \( L_1^a \) (\( L_1^b \)) denotes the length of the arm \( a \) (\( b \)) of the left ring, and \( \phi_1^a \) (\( \phi_1^b \)) denotes the magnetic phase an electron/hole acquires when circulating counter-clockwise/clockwise along the arm \( a \) (\( b \)).

Next, the tunneling process between the two metallic rings can be described as

\[
\begin{pmatrix}
    r_e^{(1)} & t_e^{(1)} & w_e^{(2)} & \bar{w}_e^{(2)} & \bar{t}_e^{(1)} & \bar{r}_e^{(1)} & \bar{w}_e^{(2)} & \bar{w}_e^{(2)} \\
    t_e^{(1)} & r_e^{(1)} & w_e^{(2)} & \bar{w}_e^{(2)} & \bar{r}_e^{(1)} & \bar{t}_e^{(1)} & \bar{w}_e^{(2)} & \bar{w}_e^{(2)} \\
    w_e^{(1)} & w_e^{(1)} & r_e^{(2)} & \bar{t}_e^{(2)} & \bar{r}_e^{(1)} & \bar{w}_e^{(1)} & \bar{t}_e^{(2)} & \bar{t}_e^{(2)} \\
    w_e^{(1)} & w_e^{(1)} & t_e^{(2)} & r_e^{(2)} & \bar{r}_e^{(1)} & \bar{w}_e^{(1)} & \bar{t}_e^{(2)} & \bar{t}_e^{(2)} \\
    r_h^{(1)} & t_h^{(1)} & w_h^{(2)} & \bar{w}_h^{(2)} & \bar{t}_h^{(1)} & \bar{r}_h^{(1)} & \bar{w}_h^{(2)} & \bar{w}_h^{(2)} \\
    t_h^{(1)} & r_h^{(1)} & w_h^{(2)} & \bar{w}_h^{(2)} & \bar{r}_h^{(1)} & \bar{t}_h^{(1)} & \bar{w}_h^{(2)} & \bar{w}_h^{(2)} \\
    w_h^{(1)} & w_h^{(1)} & r_h^{(2)} & \bar{t}_h^{(2)} & \bar{t}_h^{(1)} & \bar{w}_h^{(1)} & \bar{t}_h^{(2)} & \bar{t}_h^{(2)} \\
    w_h^{(1)} & w_h^{(1)} & t_h^{(2)} & r_h^{(2)} & \bar{r}_h^{(1)} & \bar{w}_h^{(1)} & \bar{t}_h^{(2)} & \bar{t}_h^{(2)} \\
\end{pmatrix}
\begin{pmatrix}
    D_2^{(5)} \\
    D_1^{(4)} \\
    D_1^{(7)} \\
    D_2^{(6)} \\
    B_2^{(5)} \\
    B_1^{(4)} \\
    B_1^{(7)} \\
    B_2^{(6)} \\
\end{pmatrix}
= 
\begin{pmatrix}
    D_1^{(5)} \\
    D_2^{(4)} \\
    D_2^{(7)} \\
    D_1^{(6)} \\
    B_1^{(5)} \\
    B_2^{(4)} \\
    B_1^{(7)} \\
    B_2^{(6)} \\
\end{pmatrix}
\]

The electron’s propagation in the 2nd ring can be described by

\[
D_1^{(9)} = D_1^{(6)} \exp[i(+k_e L_2^a + \phi_2^a)] , \\
D_2^{(9)} = D_2^{(6)} \exp[i(-k_e L_2^a + \phi_2^a)] , \\
D_1^{(8)} = D_1^{(7)} \exp[i(-k_e L_2^b - \phi_2^b)] , \\
\]
\[ D_2^{(8)} = D_2^{(7)} \exp[i(+k_e L^b_2 - \phi^b_2)]. \] (4.137)

The hole’s propagation in the 2nd ring can be described by

\[ B_1^{(9)} = B_1^{(6)} \exp[i(-k_e L^a_2 - \phi^a_2)], \] (4.138)
\[ B_2^{(9)} = B_2^{(6)} \exp[i(+k_e L^a_2 - \phi^a_2)], \] (4.139)
\[ B_1^{(8)} = B_1^{(7)} \exp[i(+k_e L^b_2 + \phi^b_2)], \] (4.140)
\[ B_2^{(8)} = B_2^{(7)} \exp[i(-k_e L^b_2 + \phi^b_2)]. \] (4.141)

Here \( L^a_2 \) (\( L^b_2 \)) denotes the length of the arm \( a \) (\( b \)) of the right ring, and \( \phi^a_2 \) (\( \phi^b_2 \)) denotes the magnetic phase an electron/hole acquires when circulating counter-clockwise/clockwise along the arm \( a \) (\( b \)).

The tunneling process at the tri-junction between the 2nd ring and the 2nd lead can be described as

\[
\begin{pmatrix}
-(a+b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
0 \\
D_2^{(8)} \\
D_1^{(9)}
\end{pmatrix}
= 
\begin{pmatrix}
D^{(10)} \\
D_1^{(8)} \\
D_2^{(9)}
\end{pmatrix},
\] (4.142)

\[
\begin{pmatrix}
-(a+b) & \sqrt{\epsilon} & \sqrt{\epsilon} \\
\sqrt{\epsilon} & a & b \\
\sqrt{\epsilon} & b & a
\end{pmatrix}
\begin{pmatrix}
0 \\
B_2^{(8)} \\
B_1^{(9)}
\end{pmatrix}
= 
\begin{pmatrix}
B^{(10)} \\
B_1^{(8)} \\
B_2^{(9)}
\end{pmatrix}.\] (4.143)

Here the electron incoming amplitude and the hole incoming amplitude are both 0 in the second lead. \( D^{(10)} \) and \( B^{(10)} \) denote the electron outgoing amplitude and the hole outgoing amplitude separately.

Combining all these equations, we can solve out all the amplitudes. Since the equations are linear, it is easy to find numerical solutions. What we care about most are the electron tunneling probability \( P_e = |D^{(10)}|^2 \) and the hole tunneling probability \( P_h = |B^{(10)}|^2 \).
4.3.3 Numerical Results

We now set a group of parameters to show the results of a numerical calculation, leaving the explanations of choosing such values later in the next section.

Let the Fermi energy and the Fermi vector in the rings be $E_F = 12.835 \text{ meV}$ and $k_F = 0.15 \text{ nm}^{-1}$. The circumferences of the two rings are $L_1 = 2.69 \mu \text{m}$ and $L_2 = 2.78 \mu \text{m}$. The MZM coupling energy is $E_M = 0.01 \text{ meV}$. Let the coupling strength between the rings and the Majorana fermions be equal $t_1 = t_2$. The width of the Majorana state is $\Gamma = t^2/\hbar v_F = 0.001 \text{ meV}$. The scattering parameter at the lead-ring tri-junction is $\epsilon = 0.05$.

Now let the incident electron has an energy $E = 0.003 \text{ meV}$. Then the dynamical phase the electron/hole acquires when circling the 1st ring is $\phi_{e/h}^{(1)} = (64.219 \pm 0.008) \times 2\pi$, circling the 2nd ring is $\phi_{e/h}^{(2)} = (66.368 \pm 0.008) \times 2\pi$.

Under this setting, the tunneling probabilities for electrons and for holes as functions of the two magnetic fluxes are shown in Fig. 4.8. The graph is a heat map, of which the color is set in a log-scale. We can see that there are four tunneling peaks in each graph, corresponding to the values of the magnetic phases satisfying the resonant conditions. The dashed lines in the graph correspond to the magnetic phases that the resonant conditions for the Fermi wave-vector are satisfied: $k_F L_1 \pm \phi_M^{(1)} = 2n\pi$ and $k_F L_2 \pm \phi_M^{(2)} = 2n'\pi$, which can be calculated out being $\phi_M^{(1)} = 0.219 \times 2\pi$, $0.781 \times 2\pi$ and $\phi_M^{(2)} = 0.368 \times 2\pi$, $0.632 \times 2\pi$. Obviously the electron tunneling peaks and the hole tunneling peaks locate at different positions in the graph, because they have different dynamical phases resulting in different resonant conditions.

The arrows in the graphs indicate the tunneling peaks (dark red color) close to the magnetic phases $\phi_M^{(1)} = 0.219 \times 2\pi$ and $\phi_M^{(2)} = 0.368 \times 2\pi$. Both the electron tunneling peak and the hole tunneling peak are above the dashed line $\phi_M^{(1)} = 0.219 \times 2\pi$, for it is the electron (not hole) incident from the 1st lead, dominating in the 1st ring. The electron peak is to
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Figure 4.8: A heat map showing the electron and hole tunneling probabilities as functions of magnetic phases.

The right while the hole peak is to the left of the dashed line $\phi^{(2)}_M = 0.368 \times 2\pi$, for the dynamical phase of the tunneling electron/hole in the right ring is slightly larger/smaller than $k_F L_2 = 66.368 \times 2\pi$.

Next, we calculate the tunneling current

$$j = j_{(in)}[P_e - (v_h/v_e)P_h],$$

(4.144)

where $j_{(in)}$ represents the current carried by the incident electron, and $P_{e/h}$ is the electron/hole tunneling probability. In our case since the incident energy $E$ is small (compared to $E_F$), $v_e \simeq v_h \simeq v_F$, and we have $j \simeq j_{(in)}(P_e - P_h)$.

The result of the tunneling current is shown in Fig. 4.9. From Fig. 4.9(a) we see there are positive current regions and negative current regions in the heat map. The former denotes the electron tunneling dominated regime, while the latter denotes the hole tunneling dominated regime. These two regions are separated by the boundary lines that correspond
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Figure 4.9: The tunneling current $j/j_{(in)}$ resulted from the incident electron. (a) The general picture in terms of heat map. (b) The situation when $\phi_M^{(1)}/2\pi = 0.195, 0.205, 0.215, 0.225, 0.235, 0.245$ is fixed.

to the resonant conditions $k_F L_2 \pm \phi_M^{(2)} = 2n\pi$, for the Fermi wave-vector is just the boundary between electron states and hole hole states. Fig. 4.9(b) shows if we fix the magnetic flux threading the 1st ring near the resonant condition and scan the magnetic flux threading the 2nd ring, we can observe a negative current peak followed by a positive current peak, or vice versa. The adjacent positive and negative peaks have almost the same heights, reflecting the fact that the tunneling probabilities of electrons and holes are equal in the direct process of the Majorana fermion induced crossed Andreev reflection. This kind of tunneling current signal can be taken as a typical feature for the existence of the coupled Majorana fermions. It is the core of our scheme for the detection of the Majorana fermion induced crossed Andreev reflection.
4.3.4 Experimental Considerations

For metallic rings, we would like them to be as large as possible. First, we need to thread magnetic fluxes through the center of the rings. If the rings are too small, it is difficult in experiment to finely adjust the magnetic fluxes. Second, the tunneling experiment is performed under a small bias, resulting in that the electron wave-vector and the hole wave-vector are very close. In order to have enough resolution between the adjacent electron and hole tunneling peaks, the difference of the electron’s and the hole’s dynamical phases in the ring need to be big. Since the dynamical phase difference is proportional to the circumference of the ring, it is better for the ring to be large. However, in another aspect, the size of the ring should be much smaller than the mean free path of the underlying material, in order to make the entire transport process phase coherent.

Based on these considerations, we have chosen the semiconductor GaAs-GaAlAs heterostructure as our intended material to manufacture metallic rings. The 2D electron gas confined at the interface between high quality GaAs and GaAlAs samples is famous for its high mobility and long mean free path \cite{Manfra2014}. In the work of \cite{Mailly1993}, GaAs-GaAlAs mesoscopic rings of diameters about 2.5\,\mu m have been manufactured to study the persistent current. The mean free path of the sample at that time was about 11\,\mu m. Now it has been improved to be about 300\,\mu m \cite{Umansky2009}. In our numerical simulations, we set the circumference of the rings to be no more than 3\,\mu m, much smaller than the available mean free path.

The effective mass of conduction band electrons in GaAs is $m^* = 0.067m_e$ (here $m_e$ is the electron mass). When the electron density at the heterointerface is $3.6 \times 10^{11}\text{cm}^{-2}$, the Fermi energy is $E_F = 12.835\text{meV}$ and the Fermi wave-vector is $k_F = 0.15\text{nm}^{-1}$.

Next, to realize Majorana fermions, the mechanism proposed in \cite{Sau2010}, \cite{Lutchyn2010}, \cite{Oreg2010} is the most convenient one, which is based on the structure of a
strong spin-orbit coupled semiconductor nanowire in proximity to an s-wave superconductor. We consult the work of [Mourik et al. (2012)] [Deng et al. (2012)] to consider the InSb nanowire in proximity to the superconducting Nb. The superconducting gap of Nb is about 1.55\,meV. The Nb can induce a superconducting gap of 0.25\,meV in the InSb nanowire through the proximity effect. When the spin-orbit energy of InSb is 0.3\,meV, the superconducting coherence length $l$ in the wire is estimated to be 185\,nm. Then, if we take the length of the InSb nanowire to be about 595\,nm, the Majorana fermion coupling energy $E_M \approx \Delta e^{-l/\xi}$ is about 0.01\,meV.

In order to let the crossed Andreev reflection dominate over the local Andreev reflection, the electron incident energy should be much smaller than the Majorana coupling energy. Thus, in our simulation we set the incident energy $E = 0.003\,\text{meV}$. This energy corresponds to a temperature of about 30\,mK. If we can make the experimental temperature to be about 1\,mK, then the Landauer-Büttiker formalism in the zero temperature limit works.
Chapter 5

Conclusion

In this dissertation, we have investigated properties of metallic rings and the coupled systems of Majorana fermions and metallic rings. Through theoretical analysis and numerical simulations, we have found that the resonant condition is important to the transmission through a metallic ring. The resonant condition can be controlled by adjusting the magnetic flux threaded through the ring. In a transport set-up, we have used this property of metallic rings to separate the electron tunneling signal and the hole tunneling signal in the Majorana fermion induced crossed Andreev process. The wave-vector difference between electrons and holes plays an important role. It results in that electrons and holes have different dynamical phases and thus different resonant conditions. The tunneling current is dominated by the tunneling electrons or the tunneling holes depending on whose resonant condition is satisfied.
Appendices
Appendix A

Kitaev Chain

The Kitaev chain is a 1D toy model, in which the Majorana fermions appear as edge states [Kitaev (2001)].

The Hamiltonian for a Kitaev chain is

\[ H = \sum_{n=1}^{N-1} \left\{ (\epsilon - \mu)c_n^\dagger c_n - tc_{n+1}^\dagger c_n c_n^\dagger c_{n+1} + \Delta c_{n+1}^\dagger c_n^\dagger + \Delta^* c_n c_{n+1} \right\}, \] \hspace{1cm} (A.1)

where \( \epsilon \) is the on-site energy that we set to be zero from now, \( \mu \) is the chemical potential, \( t \) is the hopping constant (taken to be real), \( \Delta \) is the pairing potential. The chain is assumed to have a total of \( N \) sites.

In the following we follow the method that was used in [Semenoff and Sodano (2006a)] [Semenoff and Sodano (2006b)] to discuss the Kitaev chain in its most general form.

Recall that the key of the BdG formalism is to diagonalize the BdG matrix (the left-hand
side of Eq. (2.22). Now for the Kitaev chain, from Eq. (A.1) the BdG matrix is of the form

\[
\begin{pmatrix}
-\mu & 0 & -t & \Delta \\
0 & \mu & -\Delta^* & t \\
-t & -\Delta & -\mu & 0 & -t & \Delta \\
\Delta^* & t & 0 & \mu & -\Delta^* & t \\
\Delta^* & t & 0 & \mu & -\Delta^* & t \\
\Delta^* & t & 0 & \mu & -\Delta^* & t \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.
\] (A.2)

We see that there is a recursion relation in the bulk

\[
\begin{pmatrix}
-t & -\Delta \\
\Delta^* & t \\
\end{pmatrix}
\begin{pmatrix}
u_{n-1} \\
v_{n-1} \\
\end{pmatrix} + \begin{pmatrix}
-\mu & 0 \\
0 & \mu \\
\end{pmatrix}
\begin{pmatrix}
u_n \\
v_n \\
\end{pmatrix} + \begin{pmatrix}
-t & \Delta \\
-\Delta^* & t \\
\end{pmatrix}
\begin{pmatrix}
u_{n+1} \\
v_{n+1} \\
\end{pmatrix} = E
\begin{pmatrix}
u_n \\
v_n \\
\end{pmatrix}.
\] (A.3)

This difference equation is linear, homogeneous and with constant coefficients. To find its solutions, we can make an ansartz

\[
\begin{pmatrix}
u_n \\
v_n \\
\end{pmatrix} = \xi^n
\begin{pmatrix}
U \\
V \\
\end{pmatrix},
\] (A.4)

where \(U\) and \(V\) are undetermined constants. Substituting Eq. (A.4) into Eq. (A.3), we have

\[
M
\begin{pmatrix}
U \\
V \\
\end{pmatrix} = 0,
\] (A.5)
where
\[ M = \xi^{-1} \begin{pmatrix} -t & -\Delta \\ \Delta^* & t \end{pmatrix} + \begin{pmatrix} -\mu - E & 0 \\ 0 & \mu - E \end{pmatrix} + \xi \begin{pmatrix} -t & \Delta \\ -\Delta^* & t \end{pmatrix} . \] (A.6)

The requirement for the existence of non-zero solutions to the eigen-spinor \( \chi \equiv (U,V)^T \) enforces
\[ \det(M) = 0 , \]
which then leads to a relation between \( E \) and \( \xi \)
\[ E = \pm \sqrt{[(\xi + \xi^{-1})t + \mu]^2 - (\xi - \xi^{-1})^2|\Delta|^2} . \] (A.8)

The corresponding eigen-spinor has the form (unnormalized)
\[ \chi = \begin{pmatrix} (\xi - \xi^{-1})\Delta \\ (\xi + \xi^{-1})t + \mu + E \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (\xi + \xi^{-1})t + \mu - E \\ (\xi - \xi^{-1})\Delta^* \end{pmatrix} . \] (A.9)

Let \( \lambda \equiv \xi + \xi^{-1} \), then \( E = \pm \sqrt{(\lambda t + \mu)^2 + (4 - \lambda^2)|\Delta|^2} \) (the particle-hole symmetry is obvious). Now a specific \( E \) gives two possible values of \( \lambda \), which then give four possible values of \( \xi \). This means that the energy level of the Kitaev chain is four-fold degenerate if no other constraint exists (when the chain is infinite long). It will be shown that for the bulk states the four corresponding eigen-modes represent the electron-like and hole-like left movers and right movers. A linear combination of the four eigen-modes constitutes the most general form of the solution to Eq. (A.3). To find the solutions for the configuration of a finite open chain, we can choose the coefficients of the linear combination to fulfil the open boundary conditions.

Next we want to stay in the topological non-trivial phase and show the existence of edge states. For the convenience we set \( \mu = 0. \)
According to the earlier analysis, the general form of the eigenfunction is
\[ \psi(n) = A_1 \xi_1^n \chi_1 + A_2 \xi_2^n \chi_2 + A_3 \xi_3^n \chi_3 + A_4 \xi_4^n \chi_4, \] (A.10)
where \( A_i \) (\( i = 1, 2, 3, 4 \)) are undermined coefficients, and \( \chi_i \) are eigen-spinors with corresponding \( \xi_i \).

The boundary condition can be taken as \( \psi(0) = \psi(N + 1) = 0 \), which is the same as the open normal metallic wire.

Now we divide the discussion into the bulk states and the edge states:

1. The bulk states correspond to \( \xi = e^{ika} \), where \( k \in [-\pi/a, \pi/a] \) is the wave-vector and \( a \) is the crystal constant. In this case \( \lambda = 2 \cos ka \in [-2, 2] \) and
   \[ E = \pm 2 \sqrt{t^2 \cos^2 ka + |\Delta|^2 \sin^2 ka}. \] (A.11)

   For the four eigen-modes we have
   \[ k = k_F + q, -(k_F + q), k_F - q, -(k_F - q), \] (A.12)
   where \( k_F = \frac{\pi}{2a} \) is the Fermi wave-vector and \( q \in [0, \frac{\pi}{2a}] \). The corresponding eigen-spinors are
   \[ \chi = \begin{pmatrix} 2i\Delta \cos qa \\ -2t \sin qa + E \end{pmatrix}, \begin{pmatrix} -2i\Delta \cos qa \\ -2t \sin qa + E \end{pmatrix}, \begin{pmatrix} 2i\Delta \cos qa \\ -2t \sin qa + E \end{pmatrix}, \begin{pmatrix} -2i\Delta \cos qa \\ -2t \sin qa + E \end{pmatrix}. \] (A.13)

   Now we have \( E = \pm 2 \sqrt{t^2 \sin^2 qa + |\Delta|^2 \cos^2 qa} \). It’s obvious that these eigen-modes correspond to the electron-like right mover, the electron-like left mover, the hole-like left mover and the hole-like right mover separately.
Next, by matching the boundary conditions $\psi(0) = \psi(N + 1) = 0$ we can obtain all the possible values of $q$ and therefore the corresponding $E$. As for the case of the normal metallic wire, when $N$ is large the spectrum of the bulk states is almost continuous.

2. The edge states correspond to $|\xi| \neq 1$.

i) If $|\Delta| > t$, let $\kappa > 0$. The four eigen-spinors corresponding to $\xi = e^{\kappa a}, -e^{\kappa a}, -e^{-\kappa a}, e^{-\kappa a}$ are

$$
\chi = \begin{pmatrix}
2\Delta \sinh \kappa a \\
E + 2t \cosh \kappa a
\end{pmatrix},
\begin{pmatrix}
-2\Delta \sinh \kappa a \\
E - 2t \cosh \kappa a
\end{pmatrix},
\begin{pmatrix}
-2\Delta \sinh \kappa a \\
E + 2t \cosh \kappa a
\end{pmatrix},
\begin{pmatrix}
2\Delta \sinh \kappa a \\
E - 2t \cosh \kappa a
\end{pmatrix}.
$$

Next we write the general solution in a little bit different way

$$
\psi(n) = A_1 e^{-\kappa (N+1-n)a} \chi_1 + (-1)^n A_2 e^{-\kappa (N+1-n)a} \chi_2 + A_3 e^{-\kappa na} \chi_3 + (-1)^n A_4 e^{-\kappa na} \chi_4.
$$

(A.15)

The first two terms represent the part of the wave-function that is localized near the site-$N$ end of chain and the last two terms represent the part that is localized near the site-$1$ end.

Now employing the boundary conditions $\psi(0) = \psi(N + 1) = 0$, we can fix the values of $\kappa$ and $E$:

(a) When $N$ is odd,

$$
E = 0,
$$

$$
\tanh \kappa a = \frac{t}{|\Delta|}.
$$

(A.16) (A.17)
The two Majorana fermion solutions are

\[ \psi_{\gamma_1}(n) = A \frac{1 - (-1)^n}{2} e^{-\kappa(N+1-n)a} \left( e^{i\theta_2} e^{-i\theta_2} \right), \] (A.18)

and

\[ \psi_{\gamma_2}(n) = -iA \frac{1 - (-1)^n}{2} e^{-\kappa na} \left( e^{i\theta_2} e^{-i\theta_2} \right), \] (A.19)

where \( A \) is a real normalization constant, \( \theta \) is the phase of \( \Delta \) defined by \( \Delta = |\Delta| e^{i\theta} \).

(b) When \( N \) even,

\[ E = \pm \frac{2t \cosh \kappa a}{\cosh \kappa(N+1)a} , \] (A.20)

\[ \tanh \kappa a = \frac{t}{|\Delta|} \tanh \kappa(N+1)a . \] (A.21)

The two Majorana fermion solutions are

\[ \psi_{\gamma_1}(n) = A \frac{1 + (-1)^n}{2} \left[ w e^{-\kappa(N+1-n)a} - w^{-1} e^{-\kappa na} \right] \left( e^{i\theta_2} e^{-i\theta_2} \right), \] (A.22)

and

\[ \psi_{\gamma_2}(n) = -iA \frac{1 - (-1)^n}{2} \left[ w e^{-\kappa na} - w^{-1} e^{-\kappa(N+1-n)a} \right] \left( e^{i\theta_2} e^{-i\theta_2} \right), \] (A.23)

where \( w = e^{\kappa(N+1)a}/2 \) is a very large constant. \( \gamma_1 \) is the Majorana fermion that locates near the site-\( N \) end of chain, and \( \gamma_2 \) is the Majorana fermion
that locates near the site-1 end. The eigenstate corresponding to $E = E_+$ is

$$
\psi_+ = \psi_{\gamma_1} + i\psi_{\gamma_2},
$$

(A.24)

and the eigenstate corresponding to $E = E_-$ is

$$
\psi_- = \psi_{\gamma_1} - i\psi_{\gamma_2}.
$$

(A.25)

ii) If $|\Delta| < t$, we have all the results the same as if we replace $\kappa$ by $i\frac{\pi}{2} + \kappa$. 


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