2-2018

Infinitely Many Solutions to Asymmetric, Polyharmonic Dirichlet Problems

Edger Sterjo

The Graduate Center, City University of New York

How does access to this work benefit you? Let us know!
Follow this and additional works at: https://academicworks.cuny.edu/gc_etds

Part of the Analysis Commons

Recommended Citation
https://academicworks.cuny.edu/gc_etds/2443

This Dissertation is brought to you by CUNY Academic Works. It has been accepted for inclusion in All Dissertations, Theses, and Capstone Projects by an authorized administrator of CUNY Academic Works. For more information, please contact deposit@gc.cuny.edu.
Infinitely many solutions to asymmetric, polyharmonic Dirichlet problems

by

Edger Sterjo

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2018
Infinitely many solutions to asymmetric, polyharmonic Dirichlet problems
by
Edger Sterjo

This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Date

Marcello Lucia
Chair of Examining Committee

Date

Ara Basmajian
Executive Officer

Supervisory Committee
Zheng Huang
Leon Karp
Marcello Lucia

THE CITY UNIVERSITY OF NEW YORK
ABSTRACT

INFINITELY MANY SOLUTIONS TO ASYMMETRIC, POLYHARMONIC DIRICHLET PROBLEMS

by

EDGER STERJO

Adviser: Professor Marcello Lucia

This dissertation consists of four chapters. In Chapter 1 we introduce the problem of proving the existence of infinitely many solutions to nonlinear Partial Differential Equations that are perturbed from symmetry. We also introduce the methods that have been developed to attack this problem, and give a more detailed outline of the dissertation. In Chapter 2 we prove the variational principle of Bolle [15] on the behavior of critical values under perturbation, and the variational principle of Tanaka [51] on the existence of critical points of large augmented Morse index. In Chapter 3 we use the framework originating in Birman and Solomyak [11] for deriving eigenvalue estimates to find alternatives of the CLR inequality specifically designed for our nonlinear PDE applications. Chapters 2 and 3 comprise the tools of the perturbation argument. In Chapter 4 we bring everything together and prove our main new results. Our primary concern is three new theorems on equations with exponential nonlinearities. These are theorems 12, 13, and 14. They are now published in Sterjo [46]. Theorems 15, 16, and 17 attack cases such as non-homogenous boundary values, and unbounded domains. Theorems 12, 15, 16, and 17 are generalizations of known results to higher order equations. The other two theorems seem to be new.
Acknowledgements

Foremost I would like to thank my advisor, Professor Marcello Lucia, for his patience, guidance, consistent words of encouragement, and for introducing me to the beautiful field of nonlinear analysis. If it were not for him I would not be the mathematician I am today.

Secondly, I would like to thank my advisory committee for taking the time out of their very busy days to preside over the defense of my dissertation. I would also like to thank all of the professors that I have come to know at the Graduate Center over the years. They have made it a very fulfilling experience. I consider myself privileged to have had the chance to study Mathematics at the Graduate Center.

Lastly I must thank my family for the undying support they gave me over the years, through times both easy and tough. I love you with all my heart.
Contents

1 Introduction 1

2 The Variational Principles of Bolle and Tanaka 7
   2.1 Bolle’s theorem on the preservation of minimax critical levels along a path of functionals 7
   2.2 Tanaka’s theorem on critical points with large augmented Morse Index 21
   2.3 Proof of Tanaka’s theorem in the finite dimensional case 24
   2.4 Proof of Tanaka’s theorem in the infinite dimensional case 32

3 Spectral estimates for Schrödinger operators 35
   3.1 Solomyak’s theorem on piecewise-polynomial approximation 38
   3.2 An eigenvalue estimate in the Orlicz setting 48
   3.3 An estimate in the radially symmetric case on an annulus 55
   3.4 An estimate in the radially symmetric case when $d > 2l$ 70
   Appendix A: Proof of Lemma 11 87
   Appendix B: The Birman-Schwinger Principle and its proof 91

4 Applications to polyharmonic Dirichlet problems 101
   4.1 Problem (P), the case of a general, bounded domain 104
   4.2 Problem (R), the radial problem on an annulus 124
CONTENTS

4.3 Problem (H), the radial problem with Hardy potential . . . . . . . . . . . . . . . 129

4.4 A radial problem with a power-type nonlinearity in the case \( d > 2l \) . . . . . 132

4.5 A problem on \( \mathbb{R}^d \), in the case \( d > 2l \) . . . . . . . . . . . . . . . . . 149

Bibliography . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 162
Chapter 1

Introduction

Many papers have been written on the existence and multiplicity of solutions for second order, nonlinear, elliptic problems, primarily by means of variational methods. The archetype has been the Dirichlet problem

\[
\begin{cases}
-\Delta u = u|u|^{p-2} + \varphi(x) & \text{in } \Omega \\
u|_{\partial \Omega} = 0.
\end{cases}
\]

In its simplest form $\Omega \subset \mathbb{R}^d$ is an open, bounded domain with a smooth boundary, and the perturbation $\varphi(x) \in L^2(\Omega)$. In the simplest case, the exponent of the nonlinearity is such that if $d \geq 3$, then $2 < p < \frac{2d}{d-2}$, while if $d = 2$, then $2 < p < \infty$. These restrictions on $p$ come when one makes use of the Sobolev embeddings $W^{1,2}_0(\Omega) \hookrightarrow L^p(\Omega)$ and their compactness. If, in addition, $\varphi \equiv 0$ then the above problem possesses a $\mathbb{Z}_2$-symmetry with respect to the group of reflections in Sobolev space. That is, $u$ is a (weak) solution if and only if $-u$ is a (weak) solution. In this case the Symmetric Mountain Pass theorem of Ambrosetti and Rabinowitz guarantees the existence of an unbounded sequence of critical values for the functional associated with the variational formulation of the problem. In other
words, infinitely many solutions exist. Such equivariant variational methods can be applied to a great variety of nonlinear problems invariant under compact groups of symmetries (see [3], [39], [49] and references therein). However, this brings up the question of what exactly happens to this multitude of critical values when the symmetry of the problem is broken by some non-equivariant perturbation. Although an great deal of effort has gone into answering this question, there are no satisfactory general answers as of yet. Methods to deal with this problem in certain cases appeared first in the papers [4], [5], and [48]. The general variational principle employed in these works was later formulated by Rabinowitz in [38] (see also [39] or section 2.7 of [49]). Roughly speaking, the idea is to estimate the spacing between consecutive symmetric mountain pass levels of the unperturbed functional, and then compare this spacing to the effect of the perturbation. Whenever the perturbation is not sufficient to eliminate this spacing, then the variational principle formulated by Rabinowitz guarantees the existence of a critical value to the perturbed functional. The first methods for estimating the spacing (or more practically the growth rate) of the symmetric mountain pass levels was based on the Weyl asymptotics for the Dirichlet eigenvalues of the Laplacian. A more refined approach, that could be more tailored for a specific problem, came in the papers of Bahri-Lions [6], and Tanaka [51]. Based on Morse theory, these works make use of an estimate for the number of non-positive eigenvalues of Schrödinger operators further described below. However, even for linear perturbations of the functional, (like the one coming from \( \varphi \)), the value of \( p \) needs to be further restricted to \( 2 < p < \frac{2d-2}{d-2} \). To improve the range of \( p \) with these methods one must weaken the perturbation. It is still a central open question of exactly how necessary is this trade-off.

Many sorts of perturbations other than a non-homogenous term \( \varphi \) are of interest. A natural one to consider is the problem of an unperturbed equation, itself formally equivariant, but with a non-homogenous boundary condition \( u|_{\partial \Omega} = u_0 \neq 0 \), which destroys the \( \mathbb{Z}_2 \)-symmetry of the problem. This time however, the perturbation is of much higher order.
CHAPTER 1. INTRODUCTION

When one uses a change of variable to reduce the problem to one with homogenous boundary conditions, the perturbation directly enters into the nonlinearity. So if the nonlinearity is large, the perturbation will be as well. For example, in the problem

\[
\begin{cases}
-\Delta u = u|u|^{p-2} & \text{in } \Omega \\
u|_{\partial \Omega} = u_0
\end{cases}
\]

one would use the change of variable \( u(x) = v(x) + \xi(x) \), where \( \xi \) is a harmonic function equal to \( u_0 \) on \( \partial \Omega \), so that the new unknown \( v \) may be taken to be an element of the Hilbert Space \( H^1_0(\Omega) \). The function \( \xi \) is to be thought of as the perturbation (if \( \xi \equiv 0 \) then the problem would be symmetric). But now the problem becomes

\[
\begin{cases}
-\Delta v = (v + \xi)|v + \xi|^{p-2} & \text{in } \Omega \\
v|_{\partial \Omega} = 0,
\end{cases}
\]

with \( \xi \) entering directly into the nonlinearity.

To deal with such complications Bolle in his paper [15] developed a new approach to the perturbation theory of minimax levels. Similar in spirit to the earlier approach, but considerably more streamlined, the new approach considers the perturbed functional \( I \) as the endpoint \( I_1 \) of a continuous path of functionals \( I_\theta, \theta \in [0, 1] \) which starts at the unperturbed functional, denoted \( I_0 \). Bolle’s general theorem explains quantitatively how far apart two consecutive mountain pass levels of the unperturbed functional need to be for a critical level to persist to \( \theta = 1 \). Roughly speaking, it’s not the size of the perturbation at general points that determines this, but the size of \( \frac{\partial}{\partial \theta} I_\theta(u) \) at the critical points \( u \) of \( I_\theta \). This can certainly be helpful because these \( u \) satisfy the corresponding Euler-Lagrange equation. Furthermore, it becomes clearer how the size of the perturbation, as a functional in \( u \), enters into the problem. This makes it easier to consider perturbations other than simple non-homogenous
terms like $\varphi(x)$. This approach is further developed and applied to a number of problems in [16].

Other very important open questions concern the type of nonlinearities considered, in particular ones of exponential growth as opposed to power-type growth. In the two dimensional case the proper Sobolev embedding is into an Orlicz space given by an exponential $N$-function, see [2]. The maximal growth rate of the nonlinearity for which a variational treatment of the problem is possible is like $e^{Ku^2}$. This is related to the optimality of the Moser-Trudinger inequality [37], [53], [33]. A typical problem now is

$$
\begin{aligned}
-\Delta u &= g(x,u) + \varphi(x) \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0
\end{aligned}
$$

where $g(x,-t) = -g(x,t)$ is of exponential growth in $t$. To guarantee the convergence of general Palais-Smale sequences in $H^1_0(\Omega)$, $g$ is taken subcritical, which in this case means that $g(x,t)$ is of order of growth strictly below any positive power of $e^{\beta t^2}$. That is

$$
\lim_{t \to \infty} \frac{|g(x,t)|}{e^{\beta t^2}} = 0 \quad \text{for all } \beta > 0 \text{ and a.e } x \in \Omega.
$$

In [50] Sugimura proved that the perturbed symmetric problem above has an infinite number of solutions if the nonlinear term $g(x,t)$ has growth like $e^{\beta |t|\alpha}$, where $0 < \alpha < 1/2$. One of the key points in this paper comes when applying the Morse index approach of [6] and [51]. At that stage one typically applies estimates for the number of non-positive eigenvalues of Schrödinger operators. Previous results for the problem involving the power-type nonlinearity $u|u|^{p-2}$ had made use of a famous such estimate from mathematical physics known as the CLR inequality, discovered independently by Rozenblum, Lieb, and Cwikel [41],[42], [29], [19]. To get his result Sugimura proved a 2-dimensional version of this estimate. (Later,
using the foundation laid by Sugimura, Tarsi [52] streamlined these results by using Bolle’s approach.)

One advantage of Rozenblum’s proof of the CLR estimate is that it automatically applies to higher order Schrödinger operators (which is the original form in which Rozenblum stated his result), where the Laplacian is replaced by the poly-Laplacian. Using this fact, it was proved in [26] that the problem

\[
\begin{cases}
(-\Delta)^l u = u|u|^{p-2} + \varphi(x) & \text{in } \Omega \\
\left( \frac{\partial}{\partial \nu} \right)^j u \bigg|_{\partial\Omega} = \phi_j, & j = 0, \ldots, l - 1
\end{cases}
\]

where \( \Omega \subset \subset \mathbb{R}^d, d > 2l \) has infinitely many solutions for \( p \) suitably restricted.

This thesis is concerned with applying the perturbative approach described above to new semi-linear, poly-harmonic Dirichlet problems. We do this by systematizing the derivation of appropriate eigenvalue estimates for the necessary Schrödinger operators. These are purpose-built alternatives of the CLR inequality. The method behind Rozenblum’s proof of the CLR inequality seems to have been originally developed in the paper [11] by Birman and Solomyak. This method, originally created for problems of mathematical quantum mechanics and far removed from nonlinear PDEs, provides a general framework with which to design alternative versions of the CLR inequality. A brief outline of the thesis is as follows:

(i) In chapter 2 we present the variational principle of Bolle on the behavior of critical values under perturbation, and the variational principle of Tanaka on the existence of critical points of large augmented Morse index. These are the most basic tools of the method.

(ii) In chapter 3 we use the framework created by Birman and Solomyak for deriving
eigenvalue estimates to find alternatives of the CLR inequality specifically designed for our nonlinear PDE applications. This is the tool needed to allow a comparison between the augmented Morse index and the symmetric mountain pass values.

(iii) Chapters 2 and 3 comprise the tools of the perturbation argument. In chapter 4 we bring everything together and prove our main new results. Our primary concern is three new theorems on equations with exponential nonlinearities. These are theorems 12, 13, and 14. They are now published in Sterjo [46]. Theorems 15, 16, and 17 attack cases such as non-homogenous boundary values, and unbounded domains. Theorems 12, 15, 16, and 17 are generalizations of known results to higher order equations. The other two theorems seem to be new.

Each of these steps requires a great deal of machinery, and so we will not describe them further in this introduction, leaving the details for the corresponding chapters. An effort has been made to include at least as much detail as is present in the existing literature, and almost always more detail. The thesis has been written in the order in which one would carry out the whole perturbation argument. What is new here is not just theorems 12-17. It is also the paradigm of chapter 3 in which eigenvalue estimates can be cooked up for any problem to which one might like to apply the perturbation argument.
Chapter 2

The Variational Principles of Bolle and Tanaka

2.1 Bolle’s theorem on the preservation of minimax critical levels along a path of functionals

In this section we will prove Bolle’s main variational principle. The reference is [15], Theorem 3.

Let $E$ be a Hilbert space, and consider a $C^2$ family of functionals $I : [0, 1] \times E \to \mathbb{R}$. For simplicity, as in [15], we will prove the theorem in the $C^2$ case. However the theorem holds as stated for the more general case of a $C^1$ functional on a Banach Space. As is common, the only difference in the proof is that instead of using the gradient of $I(\theta, \cdot)$ to create the desired flow, one uses a locally-Lipschitz pseudo-gradient in the sense of Palais. The construction of such a vector field is classical, see [36] Theorem 4.4, or [49] Lemma 3.2. See also [18] Lemma
2.4.

We denote by $\langle \cdot, \cdot \rangle$ the inner product on $E$ and $\| \cdot \|$ the associated norm. Depending on how much emphasis we want to put on the dependence on $\theta$, we will use the symbols $I_{\theta}(u)$ and $I(\theta, u)$ interchangeably. The symbol $I_{\theta}'(u)$ denotes the gradient of $I$ with respect to $u$, with the parameter $\theta$ held constant. We also need the following definition. Let $A$ and $B$ be two closed subsets of $E$ with $A \subset B$. Then for, some $R > 0$ fixed,

$$S_{B,A} := \{ g \in C(B, E) : g(x) = x \text{ if } x \in A \text{ or } \|x\| \geq R \}$$

(2.1)

We also define

$$c_A := \sup_{A} I_0 \quad \text{and} \quad c_{B,A} := \inf_{g \in S_{B,A}} \sup_{g(B)} I_0.$$  

(2.2)

The functional will satisfy the following hypotheses:

(H1) The family $I$ satisfies an analogue of the Palais-Smale Condition: For a sequence $\{(\theta_n, u_n)\}_{n \in \mathbb{N}}$ in $[0, 1] \times E$ for which $\|I_{\theta_n}'(u_n)\|_{E^*} \to 0$ and $|I_{\theta_n}(u_n)| \leq C$ there is a subsequence of it converging strongly in $[0, 1] \times E$.

(H2) For all $b > 0$ there exists a constant $C(b)$ such that

$$|I_{\theta}(u)| \leq b \quad \text{implies} \quad \left| \frac{\partial}{\partial \theta} I_{\theta}(u) \right| \leq C(b)(|I_{\theta}'(u)| + 1)(\|u\| + 1).$$

(H3) There exist two continuous functions $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$, with $f_1 \leq f_2$, which are Lipschitz continuous relative to the second variable, and such that for all critical points $u$ of $I_{\theta}$

$$f_1(\theta, I_{\theta}(u)) \leq \frac{\partial}{\partial \theta} I_{\theta}(u) \leq f_2(\theta, I_{\theta}(u)).$$

These are referred to as estimator functions.
There are two closed subsets of $E$, $B$ and $A \subset B$, such that:

(i) $\int_0^1 \sup_{u \in B} I_\theta(u) = -\infty$

(ii) $c_{B,A} > c_A$.

Consider two continuous functions $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous relative to the second variable, and $f_1 \leq f_2$, as in (H3). Define the associated flows

$$
\begin{align*}
\psi_i(0, s) &= s \\
\frac{\partial}{\partial \theta} \psi_i(\theta, s) &= f_i(\theta, \psi_i(\theta, s))
\end{align*}
$$

$i = 1, 2$.

The flows $\psi_1$ and $\psi_2$ are continuous in both variables. Moreover, by the comparison theorem for ODEs, since $f_1 \leq f_2$, we have $\psi_1 \leq \psi_2$. See [10]. Lastly, each of the $\psi_i(\theta, \cdot)$ is non-decreasing in $s$, for $\theta$ fixed. (To see this fix an $i = 1, 2$ and let $s_1 \leq s_2$ be two real numbers. Let $h_1(\theta) = \psi_1(\theta, s_1)$ and $h_2(\theta) = \psi_1(\theta, s_2)$. Clearly $h_1(0) \leq h_2(0)$ and they each satisfy the same ODE. Assume there is a $\theta_0 \in [0, 1]$ at which $h_1(\theta_0) > h_2(\theta_0)$. Then because each of the $h$’s is continuous there is a $\tau \in [0, \theta_0)$ at which $h_1(\tau) = h_2(\tau)$ and $h_1(\theta) > h_2(\theta)$ for $\theta \in (\tau, \theta_0]$. Therefore, for $\theta \in [\tau, \theta_0]$,

$$
(h_1(\theta) - h_2(\theta))' = f_i(\theta, h_1(\theta)) - f_i(\theta, h_2(\theta)) \leq L_i|h_1(\theta) - h_2(\theta)|
$$

$$
= L_i(h_1(\theta) - h_2(\theta))
$$

where $L_i$ is the Lipschitz constant of $f_i$. Hence upon integrating this differential inequality, we get that $h_1(\theta_0) - h_2(\theta_0) \leq (h_1(\tau) - h_2(\tau))e^{L_i(\theta_0 - \tau)} = 0$. This is a contradiction.
CHAPTER 2. THE VARIATIONAL PRINCIPLES OF BOLLE AND TANAKA

$h_1(\theta_0) > h_2(\theta_0).$) We can now state the main variational principle.

**Theorem 1** (Bolle). Assume that $I : [0, 1] \times E \to \mathbb{R}$ is $C^2$ and satisfies (H1)-(H4). Then if $\psi_1(1, c_{B,A}) > \psi_2(1, c_A)$, $I_1$ has a critical point of critical value greater than or equal to $\psi_1(1, c_{B,A})$.

Before we proceed with the proof we note the general idea of the theorem. It starts with a separation between the minimax values $c_{B,A}$ and $c_A$. We also have flows $\psi_1$, $\psi_2$ defined using the estimator functions $f_1$, $f_2$. Briefly, the theorem says that if the initial separation between $c_{B,A}$ and $c_A$ is sufficient to survive the flow by $\psi_1$ and $\psi_2$, then a critical level will persist for the family $I_\theta$, as $\theta$ goes from 0 to 1. We see that it is the spacing between $c_{B,A}$ and $c_A$, and the estimator functions $\{f_1, f_2\}$ that determine the “stability” of the critical points at the level $c_{B,A}$. From now on we will assume the hypotheses of the theorem, i.e. conditions (H1)-(H4) and that $\psi_1(1, c_{B,A}) > \psi_2(1, c_A)$. We will need a few lemmas. First fix $\eta > 0$ small enough so that $\psi_2(1, c_A) < \psi_1(1, c_{B,A} - \eta)$. Also, let $D := \{x \in E : I_0(x) \geq c_{B,A} - \eta\}$.

**Lemma 1.** Let $H \in C([0, 1] \times E, E)$ satisfy

1. $H(0, \cdot) = Id$
2. There is an $R > 0$ such that $\forall \theta \in [0, 1]$ ($x \in B$ and $\|x\| > R$) $\implies H(\theta, x) = x$.
3. $\forall \theta \in [0, 1]$ $H(\theta, A) \cap D = \emptyset$. Note this requires $c_A < c_{B,A} - \eta$.

Then $H(1, B) \cap D \neq \emptyset$.

**Proof:** Note that $A$ and $D$ are closed subsets of $E$, and $H$ is continuous. So since $H([0, 1], A) \cap D = \emptyset$ there is an open neighborhood $U$ of $A$ such that $H([0, 1], U) \cap D = \emptyset$\(^1\). Let $V \subset E$ be open such that $A \subset \overline{V} \subset U$. Let $l \in C(E, [0, 1])$ be such that $l|_V = 0$ and

---

\(^1\)To see why note that for every $t \in [0, 1]$ and $x \in A$ we have $H(t, x) \notin D$. So since $D$ is closed there exists an interval $I_t$ centered at $t$ and a neighborhood $U_x$ of $x$ such that $H(I_t, U_x) \cap D = \emptyset$. Since such sets cover $[0, 1] \times A$ we get $H([0, 1], U) \cap D = \emptyset$, where $U = \bigcup_x U_x$.\]
Lemma 2. For any $l|_{E \setminus U} = 1$. Let $g(x) = H(l(x), x)$. We have $g \in S_{B,A}$. By definition of $c_{B,A}$ there is an $x \in B$ with $g(x) \in D$. This $x$ cannot be an element of $U$ because $g(U) \subset H([0,1], U)$ does not intersect $D$. Thus $l(x) = 1$ and $H(1, x) \in D$. Since $x \in B$, this proves that $H(1, B) \cap D \neq \emptyset$.

\[ \blacksquare \]

**Corollary 1.** Let $H \in C([0,1] \times E, E)$ and $G \in C([0,1] \times E, E)$ satisfy (i) and (ii) of Lemma 1. Moreover we assume that

(iv) for all $\theta \in [0,1]$ $G(\theta, \cdot)$ is a homeomorphism and $G_{-1} : [0,1] \times E \to E$ defined by $G_{-1}(\theta, x) := (G(\theta, \cdot))^{-1}(x)$ is continuous on $[0,1] \times E$.

(v) for all $\theta \in [0,1]$ $H(\theta, A) \cap G(\theta, D) = \emptyset$.

Then $H(1, B) \cap G(1, D) \neq \emptyset$.

**Proof:** Setting $\tilde{H}(\theta, x) = G_{-1}(\theta, H(\theta, x)) = (G(\theta, \cdot))^{-1}(H(\theta, x))$, we see that $\tilde{H}(\theta, x)$ satisfies the hypotheses of Lemma 1. $\blacksquare$

**Remark 1:** Let $H$ and $G$ be as in Corollary 1. Denote by $A' := H(1,A)$, $B' := H(1,B)$, and $D' := G(1,D)$, and consider $g \in S_{B',A'}$. Then $g(B') \cap D' \neq \emptyset$. To see this let $\tilde{H}(\theta, x) = H(2\theta, x)$ for $0 \leq \theta \leq 1/2$; $\tilde{H}(\theta, x) = (2 - 2\theta)H(1,x) + (2\theta - 1)g(H(1,x))$ for $1/2 \leq \theta \leq 1$. Let $\tilde{G}(\theta, x) = G(2\theta, x)$ for $0 \leq \theta \leq 1/2$; $\tilde{G}(\theta, x) = G(1,x)$ for $1/2 \leq \theta \leq 1$.

For $0 \leq \theta \leq 1/2$ $\tilde{H}(\theta, A) \cap \tilde{G}(\theta, D) = H(2\theta, A) \cap G(2\theta, D) = \emptyset$. Now if $x \in A$ then $H(1,x) \in A'$ and so since $g \in S_{B',A'}$, $g(H(1,x)) = H(1,x)$. Thus $\tilde{H}(\theta, x) = H(1,x)$ for $x \in A$ and $1/2 \leq \theta \leq 1$. Hence for $1/2 \leq \theta \leq 1$, $\tilde{H}(\theta, A) \cap \tilde{G}(\theta, D) = H(1,A) \cap G(1,D) = \emptyset$. So corollary 1 applies and we have $\tilde{H}(1,B) \cap \tilde{G}(1,D) \neq \emptyset$. That is, $g(B') \cap D' \neq \emptyset$.

**Lemma 2.** For any $\delta > 0$ and $b > 0$ there exists $\zeta > 0$ such that, $\forall \theta \in [0,1]$, $\forall x \in E$,

\[
(|I_\theta(x)| \leq b \text{ and } ||I_\theta'(x)|| < \zeta) \quad \Longrightarrow \quad f_1(\theta, I_\theta(x)) - \delta < \frac{\partial}{\partial \theta} I(\theta, x) < f_2(\theta, I_\theta(x)) + \delta \tag{2.3}
\]
Proof: This is a direct consequence of conditions (H3) and (H1). ■

We will abbreviate \((\partial/\partial\theta)I(\theta,x)\) as \(J_\theta(x)\). We proceed with the proof of Theorem 1:

For \(\delta > 0\) fixed, let \(\bar{\psi}_1\) and \(\bar{\psi}_2\) be flows defined by

\[
\begin{aligned}
\bar{\psi}_1(0,s) &= s \\
\frac{\partial}{\partial \theta} \bar{\psi}_1(\theta,s) &= f_1(\theta, \bar{\psi}_1(\theta,s)) - \delta
\end{aligned}
\]

and

\[
\begin{aligned}
\bar{\psi}_2(0,s) &= s \\
\frac{\partial}{\partial \theta} \bar{\psi}_2(\theta,s) &= f_2(\theta, \bar{\psi}_2(\theta,s)) + \delta.
\end{aligned}
\]

\(\bar{\psi}_i\) is continuous with respect to \(\delta\). Thus since \(\psi_2(1,c_A) < \psi_1(1,c_B,A - \eta)\) we can assume, by choosing \(\delta\) sufficiently small, that

\[
\bar{\psi}_2(1,c_A) < \bar{\psi}_1(1,c_B,A - \eta). \tag{2.4}
\]

Define \(\varphi_1(\theta) := \bar{\psi}_1(\theta,c_{B,A} - \eta)\) and \(\varphi_2(\theta) := \bar{\psi}_2(\theta,c_A)\).

Since \(-f_1 + \delta > -f_2 - \delta\), then by the comparison theorem for ODEs and inequality (2.4) we have\(^1\)

\[
\varphi_2(\theta) < \varphi_1(\theta). \tag{2.5}
\]

Let \(\alpha := \inf\{\varphi_2(\theta) : \theta \in [0,1]\}\) and \(\beta := \sup\{\varphi_1(\theta) : \theta \in [0,1]\}\). Let \(u \in C^\infty(\mathbb{R},[0,1])\) be

\(^1\)To be more explicit let \(\tau = 1 - \theta\). Then \(\partial_{\tau}\varphi_1 = -\partial_\theta\varphi_1 = -f_1(1-\tau,\varphi_1) + \delta\). Similarly \(\partial_\tau\varphi_2 = -f_2(1-\tau,\varphi_2) - \delta\). Since \(\varphi_1|_{\tau=0} > \varphi_2|_{\tau=0}\) the comparison theorem implies that \(\varphi_1 > \varphi_2\) for all \(\tau \in [0,1]\).
such that \( u(t) = 0 \) if \( t \in (-\infty, \alpha - 2] \cup [\beta + 2, \infty) \) and \( u(t) = 1 \) if \( t \in [\alpha - 1, \beta + 1] \).

By applying Lemma 2, there is a \( \zeta \in (0, \delta) \) such that

\[
(\alpha - 2 < I_\theta(x) < \beta + 2 \text{ and } ||I_\theta'(x)|| < \zeta) \implies f_1(\theta, I_\theta(x)) - \delta < \frac{\partial}{\partial \theta} I(\theta, x) < f_2(\theta, I_\theta(x)) + \delta.
\]

Let \( v \in C^\infty(\mathbb{R}, [0, 1]) \) be such that \( v(t) = 0 \) if \( |t| \leq \zeta/2 \) and \( v(t) = 1 \) if \( |t| \geq \zeta \). We define the following vector fields

\[
X_1(\theta, x) := (J_\theta^-(x) + 1 + f_1^+(\theta, \varphi_1(\theta))u(I_\theta(x))v(||I_\theta'(x)||_E)\frac{I_\theta'(x)}{||I_\theta'(x)||_E^2})
\]

and

\[
X_2(\theta, x) := (-J_\theta^+(x) - 1 - f_2^-(\theta, \varphi_2(\theta))u(I_\theta(x))v(||I_\theta'(x)||_E)\frac{I_\theta'(x)}{||I_\theta'(x)||_E^2})
\]

where \( a^+ \) denotes \( \sup\{a, 0\} \) and \( a^- \) denotes \( \sup\{-a, 0\} \). The only difference in a more general Banach space setting is that instead of using \( I_\theta' \) above one would use a locally Lipschitz continuous pseudo-gradient vector field. See Lemma 2.2 of [56] or the references given earlier. Next we define the flows associated to these fields: For \( x \in E \) let

\[
\begin{cases}
\phi_i(0, x) = x \\
\frac{\partial}{\partial \theta} \phi_i(\theta, x) = X_i(\theta, \phi_i(\theta, x))
\end{cases}
\]

Since \( X_i(\theta, x) = 0 \) if \( I_\theta(x) \notin [\alpha - 2, \beta + 2] \) or \( ||I_\theta'(x)|| \leq \zeta/2 \) we get by condition (H2) that

\[
||X_i(\theta, x)||_E \leq \bar{C}(||x||_E + 1) \quad \forall x \in E
\]

where \( \bar{C} \) depends on \( \zeta \) (but not on \( \theta \) since \([0, 1]\) is compact so \( f_i^\pm(\theta, \varphi_i(\theta)) \)), being continuous,
is bounded). Inequality 2.6 and the fact that $X_i$ is continuous in $\theta$ and locally Lipschitz continuous in $x$ imply that $\phi_i$ is well-defined and continuous on $[0, 1] \times E$. Flows of time dependent vector fields are homeomorphisms. More precisely, for $\theta$ fixed, $\phi(\theta, \cdot)$ is a homeomorphism and that the map $(\theta, x) \mapsto \phi_i(\theta, \cdot)^{-1}(x)$ is continuous on $[0, 1] \times E$. Using condition (H4) and the definition of the function $u(t)$ there exists an $R > 0$ such that

$$x \in B \text{ and } ||x|| > R \implies \forall \theta \in [0, 1] \text{ we have } X_i(\theta, x) = 0.$$ 

Thus for all $\theta \in [0, 1]$ and all $x \in B$ for which $||x|| > R$, we have $\phi_i(\theta, x) = x$. We wish to apply corollary 1 with $H = \phi_2$ and $G = \phi_1$. In order to do so we must show that

$$\phi_2(\theta, A) \cap \phi_1(\theta, D) = \emptyset \quad \forall \theta \in [0, 1]. \quad (2.7)$$

To prove (2.7) we proceed as follows. First we show that

if $I_0(x) \leq c_A$ then $\forall \theta \in [0, 1], \ I_\theta(\phi_2(\theta, x)) \leq \varphi_2(\theta). \quad (2.8)$

Let $x \in E$ be such that $I_0(x) \leq c_A$. Holding $x$ fixed denote by $Q(\theta) := I_\theta(\phi_2(\theta, x))$. Since $Q(0) \leq c_A = \varphi_2(0)$, it suffices to show that

if $Q(\theta) = \varphi_2(\theta)$ then $Q'(\theta) < \varphi'_2(\theta). \quad (2.9)$

So assuming $Q(\theta) = \varphi_2(\theta)$, we compute $Q'$ by using the definition of $\phi_2$:

$$Q'(\theta) = J_\theta(\phi_2(\theta, x)) + \langle I'_\theta(\phi_2(\theta, x)), X_2(\theta, \phi_2(\theta, x)) \rangle$$
So

\[
Q'(\theta) = J_\theta(\phi_2(\theta, x)) - (J_\theta^+(\phi_2(\theta, x)) + 1 + f_2^- (\theta, \varphi_2(\theta))) \\
\times u(I_\theta(\phi_2(\theta, x))) v(\|I_\theta'(\phi_2(\theta, x))\|).
\]

Since \(\alpha = \inf_{t \in [0, 1]} \varphi_2(t) \leq \varphi_2(\theta) \leq \varphi_1(\theta) \leq \sup_{t \in [0, 1]} \varphi_1(t) = \beta\) we have that \(u(Q(\theta)) = u(\varphi_2(\theta)) = 1\).

We consider two cases. First when \(\|I_\theta'(\phi_2(\theta, x))\| < \zeta\). In this case we recall that since \(I_\theta(\phi_2(\theta, x)) = Q(\theta) = \varphi_2(\theta) \in [\alpha, \beta]\) we can apply Lemma 2. By Lemma 2, \(J_\theta(\phi_2(\theta, x)) < f_2(\theta, Q(\theta)) + \delta\). Thus \(Q'(\theta) = J_\theta(\phi_2(\theta, x)) - (\text{positive terms}) < f_2(\theta, Q(\theta)) + \delta = f_2(\theta, \varphi_2(\theta)) + \delta = \varphi_2'(\theta)\). So (2.9) holds in this case.

In the case that \(\|I_\theta'(\phi_2(\theta, x))\| \geq \zeta\) we have \(v(\|I_\theta'(\phi_2(\theta, x))\|) = 1\). Thus, keeping in mind that \(Q(\theta) = \varphi_2(\theta)\), we compute

\[
Q'(\theta) = J_\theta(\phi_2(\theta, x)) - J_\theta^+(\phi_2(\theta, x)) - 1 - f_2^- (\theta, \varphi_2(\theta)) \\
< f_2(\theta, \varphi_2(\theta)) \\
= \varphi_2'(\theta) - \delta.
\]

So (2.9) holds in all cases. Hence (2.8) holds. In a completely analogous way we can show that

\[
\text{if } I_0(x) \geq c_{B,A} - \eta \text{ then } \forall \theta \in [0, 1], \ I_\theta(\phi_1(\theta, x)) \geq \varphi_1(\theta)
\] (2.10)
We now recall (2.5). Assume \( x \in A \) and \( y \in D \). Then from the definitions

\[
I_0(x) \leq c_A \text{ and } I_0(y) \geq c_{B,A} - \eta
\]

So by (2.8) and (2.10) for all \( \theta \in [0,1] \)

\[
I_\theta(\phi_2(\theta, x)) \leq \varphi_2(\theta) < \varphi_1(\theta) \leq I_\theta(\phi_1(\theta, y)).
\]

Therefore (2.7) is proven. We can apply corollary (1) to get

\[
\phi_2(1, B) \cap \phi_1(1, D) \neq \emptyset.
\]

We define the sets \( A' := \phi_2(1, A) \), \( B' := \phi_2(1, B) \), and \( D' := \phi_1(1, D) \). That is, we have used \( \phi_2 \) to flow the initial geometric data \( B \) and \( A \) to \( B' \) and \( A' \). By the remark following corollary (1) we have that

\[
\text{for all } g \in S_{B',A'} \ g(B') \cap D' \neq \emptyset.
\]

Therefore

\[
\inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 \geq \inf_{D'} I_1.
\]

But by (2.10), (2.5), and (2.8)

\[
\inf_{D'} I_1 \geq \varphi_1(1) > \varphi_2(1) \geq \sup_{A'} I_1.
\]

Hence \( \inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 > \sup_{A'} I_1 \). Since \( I_1 \) satisfies the (PS) condition the classical deformation lemma applies at the level \( \inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 \) showing that it is a critical value of \( I_1 \). Also, \( \inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 \geq \varphi_1(1) = \bar{\psi}_1(1, c_{B,A} - \eta) \). If we let \( \delta \) and \( \eta \) tend to
zero, $\bar{\psi}_1(1, c_{B,A} - \eta)$ tends to $\psi_1(1, c_{B,A})$. Hence given $\epsilon > 0$ we can choose $\delta$ and $\eta$ sufficiently small so that $\inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 \geq \psi_1(1, c_{B,A}) - \epsilon$. If in fact for some $\delta$ and $\eta$ we have that $\inf_{g \in S_{B',A'}} \sup_{g(B')} I_1 \geq \psi_1(1, c_{B,A})$ we are done, and the theorem is proven.

Otherwise all these critical levels would lie between $\psi_1(1, c_{B,A}) - \epsilon$ and $\psi_1(1, c_{B,A})$, for differing $\delta$ and $\eta$. So we can apply the (PS) condition to their corresponding critical points, showing they converge up to a subsequence as $\delta$, $\eta$, and $\epsilon$ tend to zero. The limiting critical point would have a critical value of exactly $\psi_1(1, c_{B,A})$. Thus in either case $I_1$ has at least one critical point with a critical value $\geq \psi_1(1, c_{B,A})$, which is the statement of the Theorem (1). ■

**Remark 2:** It can also be shown that the critical level obtained above actually belongs to the interval $[\psi_1(1, c_{B,A}), \psi_2(1, c_{B,A})]$. However we will not need this.

The problem we are interested in here is the effect of the perturbation on the symmetric mountain pass levels. Specifically, when the perturbation destroys the evenness of a functional. To address this case, we specialize Theorem 1 as was done in [16]. Let $E$ be a separable Hilbert Space and let $e_k$ be a basis for $E$. Decompose $E$ as

$$E = \bigcup_{k=0}^{\infty} E_k,$$

where $E_0 = \{0\}$. For a given increasing sequence of real numbers $R_k > 0$ set

$$\Gamma_k := \{g \in C(E, E) : g \text{ is odd and } g(u) = u \text{ for } \|u\| \geq R_k\}$$

For an even functional $I_0(u)$ on $E$ set

$$c_k := \inf_{g \in \Gamma_k} \sup_{u \in g(E_k)} I_0(u)$$
Under typical assumptions on $I_0(u)$, the Symmetric Mountain Pass Theorem of Ambrosetti and Rabinowitz shows that (for appropriate values of $R_k$) the minimax values $c_k$ form an unbounded sequence of critical values of $I_0(u)$.

Let $f_i$ and $\psi_i$ be as in Theorem 1, and denote by

$$\bar{f}_i(s) := \sup_{\theta \in [0,1]} |f_i(\theta, s)|.$$ 

Then

**Theorem 2** (Bolle-Ghoussoub-Tehrani). Let $E$ be a Hilbert space and $I : [0,1] \times E \to \mathbb{R}$ be a $C^2$ functional satisfying (H1), (H2), and (H3) from the above. Also, assume $I$ satisfies

$(H4')$ $I_0$ is even and for any finite dimensional subspace $W \subseteq E$ we have

$$\sup_{\theta \in [0,1]} I_{\theta}(y) \to -\infty \text{ as } ||y|| \to \infty \text{ for } y \in W$$

Then there is a $K > 0$ such that for every $k \in \mathbb{N}$ only one of the two possibilities below holds:

1. Either $I_1$ has a critical level $\bar{c}_k$ with

$$\psi_2(1, c_k) < \psi_1(1, c_k + 1) \leq \bar{c}_k$$

2. Or $c_{k+1} - c_k \leq K(\bar{f}_1(c_{k+1}) + \bar{f}_2(c_k) + 1)$

**Proof:** Case 1) $\psi_2(1, c_k) < \psi_1(1, c_k + 1)$.

The idea is of course to see that Theorem 1 can be applied. To see this, let $\epsilon > 0$ be chosen such that $\psi_2(1, c_k + \epsilon) < \psi_1(1, c_{k+1})$. Fix $g \in \Gamma_k$ such that $\sup_{g(E_k)} I_0 < c_k + \epsilon$. Denote by $E_{k+1}^+ := E_k \oplus \mathbb{R}^+ e_{k+1}$, and set $A_k := g(E_k)$ and $B_k := g(E_{k+1}^+)$. These $A_k \subseteq B_k$. 

will play the role of the $A \subseteq B$ in Theorem 1. To apply that theorem we need to show that

$$\psi_1(1, c_{B_k,A_k}) > \psi_2(1, \sup_{A_k} I_0)$$

where $c_{B_k,A_k} := \inf_{h \in S_{B_k,A_k}} \sup_{h(B_k)} I_0$ and $S_{B_k,A_k}$ is given by equation (2.1) with $R = R_{k+1}$.

Let $h \in S_{B_k,A_k}$. The map $m = h \circ g|_{E_{k+1}}$ is odd on $E_k$ since $h$ is the identity on $A_k = g(E_k)$ and $g$ is odd. Therefore $m$ naturally extends to an odd map $\bar{m}$ on $E_{k+1}$. Hence to an odd map on the whole space satisfying $\bar{m}(x) = x$ for $||x|| > R_{k+1}$, by first extending using the Tietze Extension theorem, and then taking the odd part of the resulting extension. So we have $\bar{m} \in \Gamma_{k+1}$. Now since $I_0$ is even and $\bar{m}$ is odd we have

$$\sup_{h(B_k)} I_0 = \sup_{m(E_{k+1}^+)} I_0 = \sup_{\bar{m}(E_{k+1}^+)} I_0 = \sup_{\bar{m}(E_{k+1})} I_0 \geq c_{k+1}.$$  \hspace{1cm} (2.12)

Taking the infimum over all $h$ we have $c_{B_k,A_k} \geq c_{k+1}$. By the monotonicity of $\psi_i(1, \cdot)$ we obtain

$$\psi_1(1, c_{B_k,A_k}) \geq \psi_1(1, c_{k+1}) > \psi_2(1, c_k + \epsilon) \geq \psi_2(1, \sup_{A_k} I_0) \hspace{1cm} (2.13)$$

where the last inequality comes from $A_k = g(E_k)$ and $\sup_{g(E_k)} I_0 < c_k + \epsilon$. Therefore Theorem 1 applies. So $I_1$ has a critical level $\bar{c}_k$ with $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \leq \bar{c}_k$. This proves the theorem in the first case.

Case 2) $\psi_2(1, c_k) \geq \psi_1(1, c_{k+1})$

We will show that $|\psi_i(1, s) - s| \leq K_i \bar{f}_i(s) + M_i$ for $s \geq 0$ and $i = 1, 2$ where $K_i$ and
$M_i$ are positive constants. If this is true then we would have

$$c_{k+1} - c_k \leq \psi_1(1, c_{k+1}) + K_1 \bar{f}_1(c_{k+1}) + M_1 - \psi_2(1, c_k) + K_2 \bar{f}_2(c_k) + M_2$$

which proves the theorem in the second case. In order to prove the stated inequality fix $s$ and let $\sigma(\theta) := |\psi_i(\theta, s) - s|^2$ which is in $C^1([0, 1], \mathbb{R})$. Let $\theta_0 \in [0, 1]$ be such that $\sigma(\theta_0) = \sup_{[0, 1]} \sigma(\theta)$. If $\sigma(\theta_0) = 0$ then the inequality is trivially satisfied. So we may assume that $\sigma(\theta_0) > 0$. Then choose $0 < \epsilon < 1$ so that $\sigma(\theta_0) > \epsilon$. Since $\sigma(\theta)$ is continuous, the set where $\sigma(\theta) > \epsilon$ is open. Let $U$ be the interval in this set that contains $\theta_0$ and let $a$ be the left endpoint of $U$. Then $\sigma(a) = \epsilon$. For $\theta \in U$

$$\sigma'(\theta) = 2(\psi_i(\theta, s) - s) \cdot \frac{\partial \psi_i}{\partial \theta} = 2(\psi_i(\theta, s) - s) \cdot f_i(\theta, \psi_i(\theta, s))$$

$$\leq 2 \bar{f}_i(s) |\psi_i(\theta, s) - s| + 2L_i |\psi_i(\theta, s) - s|^2$$

where $L_i$ is the Lipschitz constant of $f_i$. For brevity let $u = |\psi_i(\theta, s) - s|$, so that $u^2 = \sigma$, $2uu' = \sigma'$, and by the above

$$2uu' \leq 2 \bar{f}_i(s)u + 2L_i u^2.$$ 

For $\theta \in U$, $u > \sqrt{\epsilon}$ is positive. Hence dividing the above inequality by $2u$ gives

$$u' \leq \frac{\bar{f}_i(s)}{L_i} + L_i u$$

Therefore $u' e^{-L_i \theta} - L_i e^{-L_i \theta} u \leq \bar{f}_i(s) e^{-L_i \theta}$, i.e. $(ue^{-L_i \theta})' \leq \bar{f}_i(s) e^{-L_i \theta}$. Integrating this from $a$ to $\theta_0$ gives

$$u(\theta_0) e^{-L_i \theta_0} \leq u(a) e^{-L_i a} + \frac{(e^{-L_i a} - e^{-L_i \theta_0})}{L_i} \bar{f}_i(s)$$
Dividing and using $u(a) = \sqrt{\epsilon}$ we have

\[
u(\theta_0) \leq \sqrt{\epsilon} e^{L_i(\theta_0-a)} + \frac{(e^{L_i(\theta_0-a)} - 1)}{L_i} \bar{f}_i(s) \\
\quad \leq e^{L_i} + \frac{(e^{L_i} - 1)}{L_i} \bar{f}_i(s)
\]

And so

\[
|\psi_i(1, s) - s| \leq K_i \bar{f}_i(s) + M_i.
\]

2.2 Tanaka’s theorem on critical points with large augmented Morse Index

Before presenting the precise framework, we begin with some general remarks. For an even functional $J$ satisfying the Palais-Smale condition, the typical theorem used to show the existence of infinitely many critical values is the Symmetric Mountain Pass Theorem. There are many versions of this theorem, with one of the simplest being Theorem 6.5 in [49]. Here the candidate critical values are the symmetric mountain pass levels

\[
b_k := \inf_{g \in \Gamma_k} \sup_{u \in g(E_k)} J(u)
\]

whose precise definition is given below (we’ve already seen them in Section 2.1 following equation (2.11)). An essential hypothesis in the original formulation of the theorem is that the functional $J$ satisfies the property that there exist $\alpha > 0$ and $\beta > 0$ for which

\[
||u|| = \beta \implies J(u) \geq \alpha.
\] (2.14)
This property forces the values $b_k$ to be unbounded. Therefore infinitely many of them have to be distinct. In one respect, Tanaka’s result is yet another version of the Symmetric Mountain Pass Theorem. It can be seen as seeking to distinguish between critical points (related to the minimax levels $b_k$) by their Morse indices, instead of their levels. It does not assume property (2.14). Particularly important for our purposes however, it also makes it possible to estimate the $b_k$. This of course makes it possible to estimate the spacing between the $b_k$’s, which is what we need in order to apply Theorem 2.

Let $E$ be a separable Hilbert Space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $e_k$ be a basis for $E$. Decompose $E$ as

$$E = \bigcup_{k=1}^{\infty} E_k, \quad \text{where} \quad E_k = E_{k-1} \oplus \mathbb{R} e_k$$

(2.15)

and $E_0 = \{0\}$. Let $J \in C^2(E)$ be a functional satisfying the following

(J1) $J(0) = 0$

(J2) $J(-u) = J(u)$ for all $u \in E$

(J3) For each finite dimensional subspace $W \subseteq E$, there is an $R = R(W)$ such that

$$J(u) < 0 \text{ for all } u \in W \text{ with } \|u\| \geq R(W).$$

(J4) $J'$ is a compact perturbation of the Riesz representation map. That is for $u \in E$

$$J'(u)(\cdot) = \langle u, \cdot \rangle + K(u)(\cdot)$$

where $K : E \to E^*$ is compact.
(PS) If for some $M > 0$, a sequence $\{u_k\}$ satisfies
\[ u_k \in E, \quad J(u_k) \leq M \quad \text{for all } k, \quad \|J'(u_k)\|_{E^*} \to 0 \text{ as } k \to \infty \]
then $\{u_k\}$ is precompact.

(PS)$_m$ If for some $M > 0$, a sequence $\{u_k\}$ satisfies
\[ u_k \in E_m, \quad J(u_k) \leq M \quad \text{for all } k, \quad \|(J|_{E_m})'(u_k)\|_{E_m^*} \to 0 \text{ as } k \to \infty \]
then $\{u_k\}$ is precompact.

(PS)$_*$ If for some $M > 0$, $\{u_k\}$ satisfies
\[ u_k \in E_k, \quad J(u_k) \leq M \quad \text{for all } k, \quad \|(J|_{E_k})'(u_k)\|_{E_k^*} \to 0 \text{ as } k \to \infty \]
then $\{u_k\}$ is precompact.

For $R_k = R(E_k)$ set
\[ \Gamma_k := \{g \in C(E, E) : g \text{ is odd and } g(u) = u \text{ for } \|u\| \geq R_k\}. \]

Finally set
\[ b_k := \inf_{g \in \Gamma_k} \sup_{u \in g(E_k)} J(u). \quad (2.16) \]

Tanaka [51] proves two theorems on the Morse indices of critical points associated to the $b_k$’s. One is an estimate from above on the Morse index. The other one, which we are more interested in, is an estimate from below. More precisely, using an idea of Ambrosetti-Rabinowitz [3], Tanaka proves
Theorem 3. Assume that $J$ satisfies (J1)-(J4), (PS), (PS)$_m$, and (PS)$_*$. Then for each $k \in \mathbb{N}^+$ there exists a $u_k \in E$ such that

$$J(u_k) \leq b_k$$  \hspace{1cm} (2.17)

$$J'(u_k) = 0$$ \hspace{1cm} (2.18)

$$\text{index}_0 J''(u_k) \geq k$$ \hspace{1cm} (2.19)

where the augmented Morse index, \(\text{index}_0 J''(u)\), is defined as the Morse index + nullity of \(J''(u)\). That is,

$$\text{index}_0 J''(u) := \sup \{ \dim(W) \}$$

where the supremum is taken over all $W \subseteq E$ for which \(J''(u)(w, w) \leq 0\) for all $w \in W$.

Remark 3: Note that the theorem gives an estimate for the augmented Morse index of a critical point at a level $\leq b_k$. This is good enough for our purposes. Tanaka’s theorem will be used only to obtain a bound from below on the $b_k$’s.

2.3 Proof of Tanaka’s theorem in the finite dimensional case

We will first prove Theorem 3 in the finite dimensional case. The infinite dimensional case will be treated as a limit of the finite dimensional one. Here we will let $\dim E = m < \infty$. Since later we will be making $m \to \infty$, we will emphasize the dependence on $m$ by writing $E$ as $E_m$. To this end we introduce a new family of mappings, and their associated maximin values as follows:
For $k \leq m$ let $S^{m-k}$ denote the unit sphere in $\mathbb{R}^{m-k+1}$. Set

$$
\Sigma_k := \{ \sigma \in C(S^{m-k}, E_m); \sigma(-v) = -\sigma(v) \text{ for all } v \in S^{m-k} \} \tag{2.20}
$$

and

$$
\beta_k := \sup_{\sigma \in \Sigma_k} \inf_{u \in \sigma(S^{m-k})} J(u). \tag{2.21}
$$

Before we proceed to relate $\beta_k$ to $b_k$ we need a lemma.

**Lemma 3.** Let $g \in \Gamma_k$ and $\sigma \in \Sigma_k$ be arbitrary. Then

$$
g(E_k) \cap \sigma(S^{m-k}) \neq \emptyset.
$$

**Proof:** Choose a number $R > 0$ such that $R > \max\{R_k, \sup_{v \in S^{m-k}} ||\sigma(v)||_{E_m} \}$. We write

$$
B^{m-k+1} = \{ tv \in \mathbb{R}^{m-k+1}; t \in [0,1], v \in S^{m-k} \}
$$

and

$$
B_k = B_k(R) := \{ u \in E_k; ||u||_{E_m} \leq R \}.
$$

Now define a map $\Phi \in C(\partial(B^{m-k+1} \times B_k), E_m)$ by $\Phi(tv, u) := t\sigma(v) - g(u)$. This is a well-defined and odd map. Also note that $\partial(B^{m-k+1} \times B_k)$ is homeomorphic to $S^m$ by an odd homeomorphism.

Therefore, by the Borsuk-Ulam theorem, there is a $(t_0 v_0, u_0) \in \partial(B^{m-k+1} \times B_k)$ at which $\Phi(t_0 v_0, u_0) = 0$. That is, $t_0 \sigma(v_0) = g(u_0)$. We claim that $t_0 = 1$. Otherwise $t_0 < 1$. In this case then, since $(t_0 v_0, u_0) \in \partial(B^{m-k+1} \times B_k)$, we must have that $u_0 \in \partial B_k$. So $||u_0|| = R$, and thus $g(u_0) = u_0$ by the choice of $R$. But again by the choice of $R$, $||u_0|| = ||g(u_0)|| = ||t_0 \sigma(v_0)|| < tR < R$, which is a contradiction to assuming $t_0 < 1$. Hence $t_0 = 1$ and $\sigma(v_0) = g(u_0)$. That is, $(v_0, u_0) \in S^{m-k} \times B_k$ verifies $\sigma(v_0) = g(u_0)$, which
proves the lemma. ■

The following proposition allows us to compare $\beta_k$ with $b_k$.

**Proposition 1.** The values $\beta_k$ satisfy

i) $\beta_k$ is a critical value of $J(u)$.

ii) $0 \leq \beta_k \leq b_k$.

**Proof:** Property i) is well known and follows from the usual proof of the criticality of a maximin value by the classical deformation lemma. See, for example, the classic text [39]. Since $\sigma \equiv 0$ is in $\Sigma_k$ we see that $\beta_k \geq 0$. Given $g \in \Gamma_k$ and $\sigma \in \Sigma_k$, by Lemma 3 we have that

$$\sup_{u \in g(B_k)} J(u) \geq \inf_{u \in \sigma(S^{m-k})} J(u).$$

Therefore, taking the infimum over $g \in \Gamma_k$ and the supremum over $\sigma \in \Sigma_k$ gives $\beta_k \leq b_k$. ■

We first approximate our functional $J \in C^2(E_m)$ by a Morse function. By lemma 4.9 of Wasserman [54] we have

**Proposition 2.** Assume $J$ satisfies (J1)-(J3) and (PS)$_m$. For any $\epsilon > 0$ and $M > 0$, there exists a $F_\epsilon \in C^2(E_m)$ such that

1. $F_\epsilon(0) = J(0) = 0$.
2. $F_\epsilon(-u) = F_\epsilon(u)$ for all $u \in E_m$.
3. $F_\epsilon$ is $C^2$ within $\epsilon$ of $J$. That is, $|F_\epsilon(u) - J(u)|$, $||F'_\epsilon(u) - J'(u)||_{E_m}$, $||F''_\epsilon(u) - J''(u)||_{B(E_m)} \leq \epsilon$ for all $u \in E_m$. Here $B(E_m)$ is the space of bounded bilinear forms on $E_m$ with its usual norm.
(ε4) The critical points of $F_\epsilon$ in $\{u \in E_m; |F_\epsilon(u)| \leq M\}$ are finite in number and nondegenerate.

(ε5) $F_\epsilon$ satisfies $(PS)_m$.

We fix $k \leq m$ and choose $M > \beta_k + 1$. Then by (ε3) above there is an $\epsilon_0 \in (0, 1)$ such that

$$F_\epsilon(u) < 0 \text{ for all } u \in E_k \text{ with } ||u|| \geq R_k$$

(2.22) for $\epsilon \in (0, \epsilon_0]$. For such an $\epsilon$ define

$$\beta_k(\epsilon) := \sup_{\sigma \in \Sigma_k} \inf_{u \in \sigma(S^{m-k})} F_\epsilon(u)$$

(2.23)

By (ε1), (ε2), (ε5) and (2.22) $\beta_k(\epsilon)$ is a critical value of $F_\epsilon(u)$. By (ε3) $|\beta_k(\epsilon) - \beta_k| \leq \epsilon$. We proceed with the following proposition concerning the $\beta_k(\epsilon)$'s:

**Proposition 3.** Suppose $\beta_k > 0$. Let $\epsilon_1 \in (0, \epsilon_0]$ be small enough so that $\beta_k(\epsilon) > 0$ for $\epsilon \in (0, \epsilon_1]$. Then for $\epsilon \in (0, \epsilon_1]$ there is a $u_k(\epsilon) \in E_m$ such that

$$F_\epsilon(u_k(\epsilon)) = \beta_k(\epsilon)$$

(2.24)

$$F'_\epsilon(u_k(\epsilon)) = 0$$

(2.25)

$$\text{index} F''_\epsilon(u_k(\epsilon)) \geq k$$

(2.26)

where $\text{index} F''_\epsilon(u)$ is the Morse index of $F_\epsilon$ at a critical point $u$.

**Proof:** We first denote

$$[F_\epsilon \geq a] := \{u \in E_m; F_\epsilon(u) \geq a\}$$
and

\[ B^d = \{ x \in \mathbb{R}^d; |x| \leq 1 \} \text{ for } d \in \mathbb{N} \]

\[ S^d = \{ x \in \mathbb{R}^{d+1}; |x| = 1 \} \text{ for } d \in \mathbb{N} \]

By (\( \epsilon 4 \)) and the hypothesis we can choose \( \delta \in (0, \beta_k(\epsilon)) \) such that \( \beta_k(\epsilon) \) is the only critical value of \( F_\epsilon(u) \) in \( (\beta_k(\epsilon) - \delta, \beta_k(\epsilon) + \delta) \). We denote by \( \{ \pm v_j \}_{j=1}^l \) the critical points on the level \( \beta_k(\epsilon) \) and \( I_j = \text{index} F'_\epsilon(\pm v_j) \) their respective Morse indices.

We apply the Critical Neck Theorem (See [35], or Theorem 4.87 of [43]). We have that

\[ [F_\epsilon \geq \beta_k(\epsilon) - \delta] \text{ is diffeomorphic to } [F_\epsilon \geq \beta_k(\epsilon) + \delta] \cup H^{+1}_1 \cup H^{-1}_1 \cup \cdots \cup H^{-l}_1 \text{ by an odd diffeomorphism } \Phi, \]

and there are mappings \( h_j^\pm : B^{m-I_j} \times B^{I_j} \to H^j_\pm \) such that for \( 1 \leq j \leq l \)

1. \( h_j^- = -h_j^+ \).
2. \( h_j^\pm \) is a homeomorphism.
3. The restriction \( h_j^\pm |_{\text{int}(B^{m-I_j}) \times B^{I_j}} \) is a diffeomorphism of \( \text{int}(B^{m-I_j}) \times B^{I_j} \) onto \( H^j_\pm \setminus [F_\epsilon \geq \beta_k(\epsilon) + \delta] \).
4. The restriction \( h_j^\pm |_{\partial B^{m-I_j} \times B^{I_j}} \) is an embedding of \( \partial B^{m-I_j} \times B^{I_j} \) into \( [F_\epsilon = \beta_k(\epsilon) + \delta] \).

Here \( \Phi \) satisfies \( \Phi(u) = u \) if \( u \in [F_\epsilon \leq 0] \) or \( u \in [F_\epsilon \geq \beta_k(\epsilon) + 2\delta] \) (using the fact that \( \beta_k(\epsilon) - \delta > 0 \)). The oddness of \( \Phi \), which is not usually part of the statement of the Critical Neck Theorem, comes from the fact that \( \Phi \) is constructed from the gradient flow of \( F_\epsilon \). Since \( F_\epsilon \) is even, its gradient is odd, and hence so is \( \Phi \).

The argument for the proof is by contradiction. Suppose that \( I_j < k \) for all \( j \). So that \( m - I_j > m - k \) for all \( j \). By definition of the maximin values \( \beta_k(\epsilon) \) there is a smooth \( \sigma \in \Sigma_k \) such that \( \sigma(S^{m-k}) \subseteq [F_\epsilon \geq \beta_k(\epsilon) - \delta] \). Set \( V_j^\pm = (\Phi \circ \sigma)^{-1}(H_j^\pm) \). Note that
\[ V_{j}^{\pm} \subseteq S^{m-k}. \] Since \( \Phi \) and \( \sigma \) are odd, \( V_{j}^{-} = -V_{j}^{+} \). Also \( \{ V_{j}^{\pm} \}_{j=1}^{l} \) are pairwise disjoint, for if \( u \in V_{j}^{\pm} = (\Phi \circ \sigma)^{-1}(H_{j}^{\pm}) \) then \( (\Phi \circ \sigma)(u) \in H_{j}^{\pm} \), and the \( H_{j}^{\pm} \) are all pairwise disjoint.

Consider the projection \( P : B^{m-I_{j}} \times B^{I_{j}} \to B^{m-I_{j}} \), which is a smooth, odd map. Since \( S^{m-k} \) is \((m-k)\)-dimensional and \( m - I_{j} > m - k \) by assumption, we claim that there must exist a \( \pm y_{j} \in \text{int}(B^{m-I_{j}}) \) such that

\[ \pm y_{j} \notin P \circ (h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma(S^{m-k}). \]

Indeed, \((h_{j}^{\pm})^{-1} \) is smooth near any point in \( H_{j}^{\pm} \) that is mapped into \( \text{int}(B^{m-I_{j}}) \times B^{I_{j}} \) by \( h_{j}^{\pm} \), by property 3 above. The maps \( P, \Phi, \) and \( \sigma \) are smooth everywhere they are defined. Hence in the open set of points of \( S^{m-k} \) that are mapped into \( \text{int}(B^{m-I_{j}}) \) by \( \Pi := P \circ (h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma \), the map \( \Pi \) is smooth. Since \( \dim S^{m-k} = m - k < m - I_{j} = \dim B^{m-I_{j}} \) every point in \( \Pi^{-1}(\text{int}(B^{m-I_{j}})) \) is a critical point of \( \Pi \). Therefore by Sard’s Theorem, \( \Pi \circ (\Pi^{-1}(\text{int}(B^{m-I_{j}}))) \) has Lebesgue measure 0, hence is not all of \( \text{int}(B^{m-I_{j}}) \). This proves the claim.

Hence

\[ \pm y_{j} \notin P \circ (h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma(V_{j}^{\pm}). \]

and so

\[ \{ \pm y_{j} \} \times B^{I_{j}} \cap [(h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma(V_{j}^{\pm})] = \emptyset. \]

That is,

\[ (h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma(V_{j}^{\pm}) \subseteq (B^{m-I_{j}} \setminus \{ \pm y_{j} \}) \times B^{I_{j}}. \]

Let \( \rho_{j} : (B^{m-I_{j}} \setminus \{ \pm y_{j} \}) \times B^{I_{j}} \to \partial B^{m-I_{j}} \times B^{I_{j}} \) be a continuous mapping such that \( \rho_{j} \) is the identity on \( \partial B^{m-I_{j}} \times B^{I_{j}} \). Let us define

\[ \tilde{\sigma}(v) = \begin{cases} (h_{j}^{\pm} \circ \rho_{j} \circ (h_{j}^{\pm})^{-1} \circ \Phi \circ \sigma)(v) & \text{if } v \in V_{j}^{\pm} \\ (\Phi \circ \sigma)(v) & \text{if } v \in S^{m-k} \setminus \bigcup_{j=1}^{l}(V_{j}^{+} \cup V_{j}^{-}). \end{cases} \]
Then $\tilde{\sigma} \in \Sigma_k$ and $\tilde{\sigma}(S^{m-k}) \subseteq [F_\epsilon \geq \beta_k(\epsilon) + \delta]$. Indeed, the image of $h_j^+ \circ \rho_j$ is in $[F_\epsilon = \beta_k(\epsilon) + \delta]$, and $S^{m-k} \setminus \bigcup_{j=1}^l (V_j^+ \cup V_j^-)$ is the part of $S^{m-k}$ which is mapped to $[F_\epsilon > \beta_k(\epsilon) + \delta]$ by $\Phi \circ \sigma$.

The existence of such a $\tilde{\sigma} \in \Sigma_k$ contradicts the definition of $\beta_k(\epsilon)$. ■

**Proposition 4.** There is a $u_k \in E_m$ such that

1. $J(u_k) = \beta_k$
2. $J'(u_k) = 0$
3. $\text{index}_0 J''(u_k) \geq k$.

**Proof:** We first consider the case where $\beta_k > 0$. Let $\epsilon \in (0, \epsilon_1]$ be as in proposition 3 and $F_\epsilon$ be the associated approximating Morse function satisfying (ε1) – (ε5). By proposition 3 there exists a $u_k(\epsilon) \in E_m$ such that

$$F_\epsilon(u_k(\epsilon)) = \beta_k(\epsilon) > 0$$

$$F_\epsilon'(u_k(\epsilon)) = 0$$

$$\text{index}_0 F_\epsilon''(u_k(\epsilon)) \geq k$$

By proposition 2

$$\beta_k(\epsilon) = F_\epsilon(u_k(\epsilon)) \to \beta_k \quad \text{as } \epsilon \to 0$$

$$J'(u_k(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0$$

$$||F_\epsilon''(u_k(\epsilon)) - J''(u_k(\epsilon))|| \to 0 \quad \text{as } \epsilon \to 0$$

By (PS)$_m$ we take a convergent subsequence $u_k(\epsilon_i) \to u_k$ as $\epsilon_i \to 0$. We have that $J(u_k) = \beta_k$.
and \( J'(u_k) = 0 \). Let \( \text{Sym}(m) \) be the space of \( m \times m \) symmetric matrices. Clearly the function \( \text{Sym}(m) \to \mathbb{N} : M \mapsto \#\{\text{non-positive eigenvalues of } M\} \) is upper semicontinuous. Since \( F_{\epsilon_i}'(u_k(\epsilon_i)) \to J''(u_k) \) strongly in \( \text{Sym}(m) \) we deduce

\[
\text{index}_0 J''(u_k) \geq \limsup_{\epsilon_i \to 0} \text{index}_0 F''_{\epsilon_i}(u_k(\epsilon_i)) \geq k.
\]

We now treat the case where \( \beta_k = 0 \). By evenness \( 0 \in E_m \) is a critical point with critical value \( 0 \in \mathbb{R} \). Therefore it suffices to prove that

\[
\text{index}_0 J''(0) \geq k
\]

For the sake of contradiction suppose that \( \text{index}_0 J''(0) < k \). Then in this case \( J''(0) \) possesses at least \( m - k + 1 \) positive eigenvalues. Let \( \epsilon > 0 \) and consider the map

\[
\sigma_\epsilon : S^{m-k} = \{(v_1, \ldots, v_{m-k+1}); \sum v_j^2 = 1\} \to E_m
\]

defined by

\[
\sigma_\epsilon(v_1, \ldots, v_{m-k+1}) := \epsilon \sum_{j=1}^{m-k+1} v_j e_j
\]

where \( e_1, \ldots, e_{m-k+1} \) are eigenvectors of \( J''(0) \) with positive eigenvalues. Clearly \( \sigma_\epsilon \) is odd. Also, for \( \epsilon > 0 \) taken sufficiently small, we have that \( \inf_{u \in \sigma_\epsilon(S^{m-k})} J(u) > 0 \), using \( J(0) = 0 \) and \( J'(0) = 0 \). But this shows \( \beta_k > 0 \), a contradiction. Therefore the proposition is proved in this case as well. ■

**Proof of Tanaka’s theorem in finite dimensions:** Clearly since \( \beta_k \leq b_k \) the above proposition proves Tanaka’s theorem in the finite dimensional case. ■
2.4 Proof of Tanaka’s theorem in the infinite dimensional case

As was mentioned earlier, the infinite dimensional case of Tanaka’s theorem is proved using the finite dimensional case by way of a limit argument. Fix \( k \in \mathbb{N} \) and for \( m > k \) denote

\[
\Gamma_k^m := \{ g \in C(E_k, E_m); g(-u) = -g(u), \text{ and } g(u) = u \text{ if } ||u|| \geq R_k \}
\]

and

\[
b_k^m := \inf_{g \in \Gamma_k^m} \sup_{u \in g(E_k)} J(u)
\]

These values satisfy the properties

(i) \( b_k^m \) is a critical value of \( J|_{E_m} \in C^2(E_m) \),

(ii) For \( k \) fixed, \( b_k^m \searrow b_k \) as \( m \to \infty \).

The first assertion is clear and comes from the fact that \( b_k^m \) is one of the usual symmetric mountain pass levels of \( J|_{E_m} \). To prove the second property note that if \( n \leq m \) then \( E_n \subseteq E_m \) and so \( \Gamma_n^k \subseteq \Gamma_k^m \). Thus \( b_n^m \leq b_k^m \). Let \( P_m : E \to E_m \) denote the orthogonal projection onto \( E_m \). Given a \( g \in \Gamma_k, P_m \circ g \in \Gamma_k^m \) with \( k < m \). Then \( P_m g(u) \to g(u) \) uniformly for \( u \in E_k \). Indeed if \( ||u|| \geq R_k \) then \( P_m g(u) = P_m(u) = u \). Next, \( g(\{u \in E_k; ||u|| \leq R_k\}) \) is compact and so we can take a finite \( \epsilon \)-net around this set. Then we can choose \( m_0 \) large enough so that \( P_m \) is the identity on the elements of the net itself, and within \( \epsilon \) of the identity on \( g(\{u \in E_k; ||u|| \leq R_k\}) \), for \( m \geq m_0 \). So \( P_m g(u) \to g(u) \) uniformly. This proves (ii). Now a lemma:

**Lemma 4.** Let \( m \in \mathbb{N} \) and \( u \in E \) be a critical point of \( J \). Then

\[
(i) \ text{index}_E J''(u) \leq \text{index}_{E_m} J''(u)
\]
Where
\[
\text{index}_{E_m} J''(u) := \sup \{ \dim W; W \subseteq E_m, J''(u)(w, w) \leq 0 \text{ for } w \in W \}
\]

(ii) There is an \( \epsilon = \epsilon(u) > 0 \) such that
\[
\text{index}_{E_m} J''(u) = \text{index}_{E_m} (J''(u) - \epsilon \langle \cdot, \cdot \rangle_E)
\]

Proof: The first assertion follows immediately from the definitions of \( \text{index}_{E_m} \) and \( \text{index}_{E_m} \).

To prove the second part we use property (J4) and Proposition 8.2 from [20]. That is, by (J4) \( K \) is a compact map. Hence it’s Frechet derivative at any given point \( u \in E \) is a compact linear operator. Therefore there is a compact, self-adjoint operator \( \kappa_u : E \to E \) such that
\[
J''(u)(w, w) = \langle w, w \rangle_E + \langle \kappa_u w, w \rangle_E
\]

where \( \langle \cdot, \cdot \rangle_E \) denotes the inner product on \( E \). Hence \( \text{index}_{E_m} J''(u) \) is equal to the number of non-positive eigenvalues of \( \text{id} + \kappa_u \). Since \( \kappa_u \) is a compact, self-adjoint linear operator, any eigenvalue of \( \text{id} + \kappa_u \) not equal to 1 is of finite multiplicity with the only accumulation point being 1. This also shows that \( \text{index}_{E_m} J''(u) < \infty \). Now choose \( \epsilon \in (0, \min \{ \lambda^+_1, 1 \}) \) where \( \lambda^+_1 \) is the first positive eigenvalue of \( \text{id} + \kappa_u \). Then
\[
\text{index}_{E_m} J''(u) = \text{index}_{E_m} (J''(u) - \epsilon \langle \cdot, \cdot \rangle_E).
\]

Proof of Tanaka’s theorem in the infinite dimensional case: By Tanaka’s theorem in the finite dimensional, there exist \( u^m_k \in E_m \) such that

(i) \( J(u^m_k) \in [0, b^m_k] \),

(ii) \( (J|_{E_m})'(u^m_k) = 0 \),

(iii) \( \text{index}_{E_m} J''(u^m_k) \geq k \).

By (i), (ii) above, and assumption \((PS)_*\), there is a subsequence \( u^m_{k_i} \to u_k \) strongly in \( E \).
Since \( J \in C^2 \) we get (2.17) and (2.18). For \( k \) fixed, by Lemma 4 (ii), there is an \( \epsilon > 0 \) for which

\[
\text{index}_0 J''(u_k) = \text{index}_0 (J''(u_k) - \epsilon \langle \cdot, \cdot \rangle_E) \tag{2.27}
\]

Since \( J \in C^2 \) and \( u_k^{m_i} \to u_k \), there is a \( i_0 \) such that

\[
\| J''(u_k^{m_i}) - J''(u_k) \|_{B(E)} < \epsilon \quad \text{for all } i \geq i_0. \tag{2.28}
\]

This implies that

\[
\text{index}_0 (J''(u_k) - \epsilon \langle \cdot, \cdot \rangle_E) \geq \text{index}_0 J''(u_k^{m_i}) \quad \text{for all } i \geq i_0. \tag{2.29}
\]

Therefore,

\[
\text{index}_0 J''(u_k) = \text{index}_0 (J''(u_k) - \epsilon \langle \cdot, \cdot \rangle_E) \\
\geq \text{index}_0 J''(u_k^{m_i}) \\
\geq \text{index}_0 J''(u_k^{m_i}) \quad \text{using Lemma 4 (i)} \\
\geq k. \tag{2.30}
\]

\[\blacksquare\]

\[\text{To be more explicit for (2.18) we would have}\]

\[
\| J'(u_k) \| = \sup_{v \in E, \| v \| \leq 1} | J'(u_k)(v) | \leq \sup_{v \in E, \| v \| \leq 1} | J'(u_k)(P_m v) | + \| (I - P_m) \circ J'(u_k)(v) | \\
= \sup_{v \in E, \| v \| \leq 1} \| (I - P_m) \circ J'(u_k)(v) \| \leq \| (I - P_m) \circ J'(u_k) \| \to 0 \text{ as } m \to \infty
\]
Chapter 3

Spectral estimates for Schrödinger operators

Our main application of Tanaka’s theorem will be to give a lower bound for the minimax values $c_k$ defined in Section 2.1. The precise details will be given in the next chapter. In order to apply Tanaka’s result in such a way we will find a mechanism by which we can compare the augmented Morse index of $J$ to the value of $J$. More precisely, in the notation of Theorem 3 of section 2.2, we need a way to compare $\text{index}_0 J''(u_k)$ to $J(u_k)$. Since $k \leq \text{index}_0 J''(u_k)$ and $J(u_k) \leq b_k$ this will give us a comparison between $k$ and $b_k$. In other words, a growth rate for $b_k$. So what is needed is an upper bound of $\text{index}_0 J''(u_k)$ by some monotone function of $J(u_k)$.

To motivate the results of this chapter, let us first consider a typical application of the calculus of variations to PDEs. For boundary value problems of elliptic partial differential equations, the space we are concerned with is usually a Sobolev space, for example $H^1_0(\Omega)$. From here on $\Omega$ will denote a smooth, bounded region of $\mathbb{R}^d$ and $d$ will denote the dimension.
of the domain. The functional associated with the variational formulation of the problem is an energy integral, a simple example of which is

\[ J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} H(u) dx \]

for \( u \in H^1_0(\Omega) \). Assuming proper growth restrictions on \( H \), critical points of such a functional are generalized solutions of the boundary value problem

\[ -\Delta u = h(u) \quad \text{for } x \in \Omega \]

\[ u = 0 \quad \text{for } x \in \partial \Omega \]

where \( H(t) = \int_0^t h(s)ds \). For this example we see that \( J''(u) \) has the form

\[ J''(u)(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx - \int_{\Omega} h'(u) vw \, dx \]

Therefore \( \text{index}_0 J''(u) \) represents the number of non-positive eigenvalues of the Schrödinger operator

\[ A := -\Delta - h'(u). \]

on \( L^2(\Omega) \). The operator \( A \) is defined as the Friedrichs operator of the quadratic form \( J''(u) \). We will be more precise later. For now, we see from the above example that an upper bound to \( \text{index}_0 J''(u) \) is really an upper bound on the number of non-positive eigenvalues of Schrödinger operators. These are sometimes referred to as the “bound states” in quantum mechanics. As a more general example consider the unbounded linear operator on \( L^2(\Omega) \) given by

\[ A_{V,l,b} := (-\Delta)^l + b|x|^{-2l} - V(x), \quad l \in \mathbb{N}, \ b \in \mathbb{R} \]  

(3.1)
where $V \geq 0$ is a locally integrable potential function on $\Omega$. For $\lambda \leq 0$ let us denote by $N(\lambda, A_{V,l,b})$ the number of eigenvalues of $A_{V,l,b}$ that are less than or equal to $\lambda$, counted according to multiplicity. If $l = 1$, $b = 0$, and $d > 2$ the CLR inequality, (discovered independently by Rozenblum, Lieb, and Cwikel [41],[42], [29], [19]) states that if $V \in L^{d/2}(\Omega)$ then

$$N(0, A_{V,1,0}) \leq C_d \int_{\Omega} V^{d/2} \, dx.$$  \hfill (3.2)

Actually Rozenblum’s result in [42] is that if $d > 2l$ and $V \in L^{d/2l}(\Omega)$ then

$$N(0, A_{V,l,0}) \leq C_{d,l} \int_{\Omega} V^{d/2l} \, dx.$$  \hfill (3.3)

The proof is based on an idea of Birman and Solomyak [11], that we elaborate later. By taking into account Hardy type inequalities, essentially the same proof shows that if $d > 2l$, $V \in L^{d/2l}(\Omega)$, and $b > -(\ldots(d - 2l))^{2}/2l$ then

$$N(0, A_{V,l,b}) \leq C_{d,l,b} \int_{\Omega} V^{d/2l} \, dx,$$  \hfill (3.4)

see [12] and [14]. In the above estimates $\Omega$ may also be taken to be unbounded. For the case $d < 2l$ the corresponding spectral inequality is substantially different, but the same general machinery applies to prove it. See [12] and [14], also. Inequalities (3.2) and (3.3) were used in [6], [51], [26], [16], and many other papers to prove multiplicity of solutions to non-symmetric boundary value problems with power type nonlinearities. The inequality (3.4) can also be used in such a way, to prove infinitely many solutions to a non-symmetric problem involving a subcritical Hardy potential.

However, the first case that interests us is the borderline case $d = 2l$. This case was first attacked using the machinery alluded to above by Solomyak in [45]. The proofs of the previous inequalities are derived from the Sobolev Embedding inequalities. The case $d = 2l$
requires an analogous embedding inequality, but one involving exponential Orlicz norms, which are more subtle than the $L^p$ norms. Adapting the method to this case was one of the achievements of [45].

Finally in this introduction, let us summarize the idea of Birman and Solomyak that we’ve been alluding to. A general outline is as follows. The first step is to begin with a group of related embedding inequalities. These are then used to prove a theorem on piecewise-polynomial approximation. This approximation theorem then leads to an eigenvalue estimate for a compact operator on an appropriate Hilbert space. Finally one relates the eigenvalues of this operator to non-positive eigenvalues of the Schrödinger operator of interest using the Birman-Schwinger principle. For the reader who might want to refresh the basic concepts of operators and quadratic forms on Hilbert spaces, it might be helpful to read Appendix B on the Birman-Schwinger principle first.

3.1 Solomyak’s theorem on piecewise-polynomial approximation

We will first consider the operator (3.1), with $b = 0$ and where $\Omega \subset \subset \mathbb{R}^d$, is an arbitrary smooth domain, with $d = 2l$. We essentially follow the same argument as [45]. First let us set some notation. In the Hilbert space $H^l(\Omega) = W^{l,2}(\Omega)$ we consider the subspace

$$H^l_0(\Omega) := \text{the completion of } C_0^\infty(\Omega) \text{ in the following norm}$$

$$\|u\|_{H^l_0(\Omega)} := \begin{cases} \|\Delta^k u\|_{L^2(\Omega)} & \text{if } l = 2k \\ \|\nabla(\Delta^k u)\|_{L^2(\Omega)} & \text{if } l = 2k + 1 \end{cases} \quad (3.5)$$

and where $d = 2l$. 

CHAPTER 3. SPECTRAL ESTIMATES FOR SCHRÖDINGER OPERATORS
Typically on the space $H^0_0(\Omega)$ one might use the norm $\|D^l u\|_{L^2(\Omega)}$, after taking into account Poincare’s Inequality for the lower order terms. However, this norm is equivalent to $\|u\|_{H^0_0(\Omega)}$ on $C^\infty_0(\Omega)$ by integration by parts. When convenient, we shall denote the norm $\|u\|_{H^l_0(\Omega)}$ by $\|u\|$. As shorthand on $H^0_0(\Omega)$ we also define the $l$th power of the gradient as

$$\nabla^l u = \begin{cases} 
\Delta^k u & \text{if } l = 2k \\
\nabla(\Delta^k u) & \text{if } l = 2k + 1.
\end{cases} \quad (3.6)$$

As in [45], the approximation theorem will be proven in a cube. Let $Q = (0,1)^d$ be the unit cube in $\mathbb{R}^d$ and let $u \in H^l(Q) = W^{l,2}(Q)$. Let $\Delta \subset \mathbb{R}^d$ be a parallelepiped with edges parallel to those of $Q$, and denote

$$\mathcal{P}(l,d) = \text{vector space of all polynomials of degree } < l \text{ in } \Delta,$$

$$m(l,d) := \dim \mathcal{P}(l,d).$$

That is, we regard $\mathcal{P}(l,d)$ as that subspace of of $L^2(\mathbb{R}^d)$ consisting of functions supported in $\Delta$, and which in $\Delta$ are polynomials of degree less than $l$. We let $\mathcal{P}_{l,\Delta}$ be the corresponding orthonormal projection onto $\mathcal{P}(l,d)$. That is, $\mathcal{P}_{l,\Delta}$ is the $L^2$-orthogonal projection of $L^2(\mathbb{R}^d)$ onto $\mathcal{P}(l,d)$.

Furthermore, let $\Xi$ be a finite covering of $Q$ by parallelepipeds $\Delta$. To any such covering and any $l > 0$ we associate an operator of piecewise-polynomial approximation in $L^2(\mathbb{R}^d)$: For $\Xi = \{\Delta_j\}, 1 \leq j \leq \text{card}(\Xi)$, and with $\chi_j$ the characteristic function of the set $\Delta_j \setminus \cup_{i<j} \Delta_i$, we denote

$$K_{\Xi,l} = \sum_j \chi_j \mathcal{P}_{\Delta_j,l}. \quad (3.7)$$

Note that $\text{rank}(K_{\Xi,l}) \leq m(l,d) \cdot \text{card}(\Xi)$.

In this section we will need to recall the theory of Orlicz spaces (see [2],[25],[40]). Let $\mathcal{B}, \mathcal{A}$
be a pair of mutually complementary \(N\)-functions, and \(L_B(\omega), L_A(\omega)\) be the corresponding Orlicz spaces on a set \(\omega \subset \mathbb{R}^d\) of finite Lebesgue measure. We are primarily interested in the pair 

\[
A(t) = e^{|t|} - 1 - |t|, \quad \text{and} \quad B(t) = (|t| + 1) \ln(|t| + 1) - |t|
\]

Then Solomyak’s main theorem is

**Theorem 4** (Solomyak [45]). Let \(Q = (0,1)^d, V \in L_B(Q), V \geq 0\). Then for any \(n \in \mathbb{N}\) there exists a covering \(\Xi = \Xi(V,n)\) of \(Q\) by parallelepipeds \(\Delta \subset Q\) such that

\[
\text{card}(\Xi) \leq C_1 n \quad (3.8)
\]

and for any \(u \in H^l(Q), 2l = d\) we have

\[
\int_Q V|u - K_{\Xi,l}u|^2 dx \leq C_2 n^{-1} ||V||_{B,Q} \int_Q |\nabla^l u|^2 dx \quad (3.9)
\]

where \(C_1, C_2\) depend only on \(d\).

The proof will use a few lemmas. Let \(\Omega\) be a set of finite Lebesgue measure. Let \(A, B\) be mutually complementary \(N\)-functions, and let \(L_A(\Omega), L_B(\Omega)\) denote the corresponding Orlicz spaces. One of the typical norms defined on the space \(L_B(\Omega)\) is

\[
||u||_{B,\Omega} := \sup \left\{ \left| \int_\Omega vu \ dx \right| ; \text{over all } v \text{ such that } \int_\Omega A(v(x)) \ dx \leq 1 \right\} \quad (3.10)
\]

Unfortunately this norm doesn’t necessarily possess a (super)additivity property common to the \(L^p\) norms, which is crucial to the proof of Theorem 4. Following Solomyak[45], we introduce an “averaged Orlicz norm”

\[
|||u|||_{B,\Omega} := \sup \left\{ \left| \int_\Omega vu \ dx \right| ; \text{over all } v \text{ such that } \int_\Omega A(v(x)) \ dx \leq |\Omega| \right\} \quad (3.11)
\]
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. These norms are in fact equivalent (see [25]).

Let $\xi$ be an affine transformation of $\mathbb{R}^d$, $v \in L_B(\xi(\Omega))$, and $u(x) = v(\xi(x)) \in L_B(\Omega)$. Then

\[
\|||u|||_{B,\Omega} = \sup \left\{ \left| \int_{\Omega} f(x)v(\xi(x)) \, dx \right|; \int_{\Omega} A(f(x)) \, dx \leq |\Omega| \right\}
\]

\[
= \sup \left\{ \left| \int_{\xi(\Omega)} f(\xi^{-1}(y))v(y)|D\xi^{-1}| \, dy \right|; \int_{\xi(\Omega)} A(f(\xi^{-1}(y)))|D\xi^{-1}| \, dy \leq |\Omega| \right\}
\]

\[
= |D\xi|^{-1} \sup \left\{ \left| \int_{\xi(\Omega)} g(y)v(y) \, dy \right|; \int_{\Omega} A(g(y)) \, dy \leq |\xi(\Omega)| \right\}
\]

\[
= |D\xi|^{-1} \|||v|||_{B,\xi(\Omega)}
\]

So since $|D\xi| = \frac{|\xi(\Omega)|}{|\Omega|}$ we have

**Lemma 5.** Let $\xi$ be an affine transformation of $\mathbb{R}^d$, $v \in L_B(\xi(\Omega))$, and $u(x) = v(\xi(x)) \in L_B(\Omega)$. Then

\[
|\Omega|^{-1} \|||v \circ \xi|||_{B,\Omega} = |\xi(\Omega)|^{-1} \|||v|||_{B,\xi(\Omega)}.
\] (3.12)

Therefore the norm behaves nicely with respect to affine rescaling. As we mentioned earlier in this chapter, a proper embedding inequality is needed. Theorem 8.25 of [2] gives the Sobolev embedding for the critical exponent

**Proposition 5.** There exists a constant $C$ such that for every $u \in H^l(\Omega)$, $2l = d$

\[
\|||u^2|||_{A,\Omega} \leq C \|||u|||_{H^l(\Omega)}^2
\] (3.13)

where $A(t) = e^{|t|} - 1 - |t|$.

Here $\||\cdot||_{H^l(\Omega)}$ is the usual norm in $W^{l,2}(\Omega)$. Consider the subspace $\ker \mathcal{P}_{\Omega,l} \subset H^l(\Omega)$. On this subspace the norms $\||u||_{H^l}$ and $\||u||_{H^l_0}$ are equivalent. The proof of this fact is exactly the same as the usual Poincaré-Wirtinger inequality. Therefore (3.13) becomes

\[
\|||u^2|||_{A,\Omega} \leq C_{14} \int_{\Omega} |\nabla^l u|^2 \, dx.
\] (3.14)
for \( u \in \ker \mathcal{P}_{\Omega,l} \). To prove Theorem 4 we first need a few lemmas.

**Lemma 6.** Let \( \Delta \subset \mathbb{R}^d \) be a parallelepiped with edges of lengths \( l_1, \ldots, l_d \) parallel to the coordinate axes. Let \( u \in H^l(\Delta), 2l = d, \) such that \( \mathcal{P}_{\Delta,l}u = 0, \) and let \( V \in L_B(\Delta) \) be non-negative. Then

\[
\int_{\Delta} V(x)|u|^2 \, dx \leq C_{14} \max_{i,j} \left( \frac{l_i}{l_j} \right)^d ||V||_{B,\Delta} \int_{\Delta} |\nabla^l u|^2 \, dx
\]

**(3.15)**

**Proof:** First take \( \Delta = Q \). By the Hölder inequality for Orlicz spaces we have

\[
\int_{Q} V(x)|u|^2 \, dx \leq ||V||_{B,Q} \cdot ||u^2||_{A,Q} \leq C_{14} ||V||_{B,Q} \cdot \int_{Q} |\nabla^l u|^2 \, dx
\]

\[
= C_{14} ||V||_{B,Q} \cdot \int_{Q} |\nabla^l u|^2 \, dx.
\]

For the general case, let \( \Delta \subset \mathbb{R}^d \) be an arbitrary parallelepiped as in the hypothesis. Let \( \xi \) be an affine transformation of \( \mathbb{R}^d \) with \( \xi(Q) = \Delta \), and let \( u \in H^l(\Delta) \). Then \( u \circ \xi \in H^l(Q) \). By a simple change of variable we have

\[
\int_{Q} |\nabla^l (u \circ \xi)|^2 \, dx \leq \max_{i,j} \left( \frac{l_i}{l_j} \right)^d \int_{\Delta} |\nabla^l u|^2 \, dx
\]

where one uses the hypothesis that \( 2l = d \). Before we proceed, notice also that \( \mathcal{P}_{\Delta,l}u = 0 \) implies that \( \mathcal{P}_{Q,l}(u \circ \xi) = 0 \). This is easy to see from the usual integral expressions for the coefficients of these polynomials (for example, the constant term of \( \mathcal{P}_{\Delta,l}u \) would just be the
average value of \( u \) in \( \Delta \)). Therefore we compute

\[
\int_{\Delta} V(x)|u|^2 \, dx = |\Delta| \int_{Q} V \circ \xi(x)|u \circ \xi|^2 \, dx
\]

\[
\leq C_{14} \max_{i,j} \left( \frac{l_i}{l_j} \right)^d |\Delta| \||V \circ \xi||_{B,Q} \int_{\Delta} |\nabla^t u|^2 \, dx
\]

\[
= C_{14} \max_{i,j} \left( \frac{l_i}{l_j} \right)^d \||V||_{B,\Delta} \int_{\Delta} |\nabla^t u|^2 \, dx
\]

(3.16)

where we made use of (3.12) in the last line.

A key step in the proof is a super-additivity property of the Orlicz norm involved. More precisely, for \( u \in L_B(\Omega) \) we consider the set-function

\[
\mathcal{J}(\omega) = \mathcal{J}_{B,u}(\omega) := \||u||_{B,\omega}
\]

(3.17)

defined for \( \omega \subseteq \Omega \). Then

**Lemma 7.** For any \( \omega \subseteq \Omega \) and any finite collection of pairwise disjoint subsets \( \omega_j \subseteq \omega \)

\[
\sum_{j} \mathcal{J}(\omega_j) \leq \mathcal{J}(\omega).
\]

**Proof:** Let \( f_j \) be a function on \( \omega_j \) such that

\[
\int_{\omega_j} A(f_j(x)) \, dx \leq |\omega_j|.
\]

Extend \( f_j \) by letting \( \tilde{f}_j \) equal \( f_j \) on \( \omega_j \), and 0 on \( \omega \setminus \omega_j \). Then form the function

\[
F := \sum_{j} \tilde{f}_j.
\]

(3.18)
Notice that
\[
\int_\omega A(F(x)) \, dx = \sum_j \int_{\omega_j} A(f_j(x)) \, dx \leq \sum_j |\omega_j| \leq |\omega|.
\] (3.19)
Therefore
\[
|||u|||_{B,\omega} = \sup \left\{ \left| \int_\omega Fu \, dx \right| : \int_\omega A(F(x)) \, dx \leq |\omega| \right\}
\]
\[
\geq \sup \left\{ \left| \int_\omega Fu \, dx \right| : F \text{ of the form (3.18)} \right\}
\]
\[
= \sum_j \sup \left\{ \left| \int_{\omega_j} f_j u \, dx \right| : \int_{\omega_j} A(f_j(x)) \, dx \leq |\omega_j| \right\}
\]
\[
= \sum_j |||u|||_{B,\omega_j}.
\]
Which is the required statement. \(\blacksquare\)

We take \(\Omega = Q\), and for a fixed \(u \in L_B(Q)\) we consider \(J = J_{B,u}\). For a given \(x \in \bar{Q}\) and \(t > 0\) we denote by \(\Delta_x(t)\) the cube in \(\mathbb{R}^d\) centered at \(x\), with edges of length \(t\). Let \(\tilde{\Delta}_x(t) := \Delta_x(t) \cap Q\), and consider the function \(j(t) := J_{B,u}(\tilde{\Delta}_x(t))\), \(t > 0\). It is clear from the definition (or even more so from lemma 7) that for every \(x \in \bar{Q}\), \(j(t)\) is a non-decreasing function and \(j(t) = J_{B,u}(Q)\) for \(t \geq 2\).

**Lemma 8.** \(j(t)\) satisfies the following

\(i\) \(j(0+) = 0\)

\(ii\) \(j(t)\) is continuous for \(t > 0\).

**Remark:** As will be clear from the proof these properties hold for any \(N\)-function \(B\) satisfying the \(\Delta_2\)-condition (see [25], section 4).

**Proof:** \(i\) This follows from the fact that since \(B\) satisfies the \(\Delta_2\)-condition, any element
$u \in L_B(Q)$ has an absolutely continuous norm (see [25] Theorem 10.3).

ii) Notice that for given $t_0, t > 0$ the sets $\tilde{\Delta}_x(t_0)$ and $\tilde{\Delta}_x(t)$ are both parallelepipeds. Hence there is an affine transformation $\xi_{t_0, t}$ of $\mathbb{R}^d$ such that $\xi_{t_0, t}(\tilde{\Delta}_x(t_0)) = \tilde{\Delta}_x(t)$. Then by (3.12)

$$j(t) = \frac{|\tilde{\Delta}_x(t)|}{|\Delta_x(t_0)|} \cdot |||u \circ \xi_{t_0, t}|||_{B, \Delta_x(t_0)}.$$  (3.20)

We first assume that $u \in C(\bar{Q})$. In this case $u \circ \xi_{t_0, t} \to u$ uniformly on $\tilde{\Delta}_x(t_0)$ as $t \to t_0$. Therefore we have “convergence in the mean”:

$$\int_{\tilde{\Delta}_x(t_0)} B(u(\xi_{t_0, t}(z)) - u(z)) \, dz \to 0.$$  

Since $B$ satisfies the $\Delta_2$-condition, mean convergence implies convergence in norm:

$$|||u \circ \xi_{t_0, t} - u|||_{B, \Delta_x(t_0)} \to 0$$  (3.21)

(see [25], Theorem 9.4). However, again since $B$ satisfies the $\Delta_2$-condition, $C(\bar{Q})$ is dense in $L_B(Q)$ (see [2], Theorem 8.20). Therefore by an approximation argument (3.21) remains valid for all $u \in L_B(Q)$. It follows by the triangle inequality that $|||u \circ \xi_{t_0, t}|||_{B, \Delta_x(t_0)} \to |||u|||_{B, \Delta_x(t_0)}$ as $t \to t_0$. This together with (3.20) show that $j(t) \to j(t_0)$ as $t \to t_0$. ■

In order to apply these lemmas in the proof of Theorem 4 we will use the Besicovitch covering lemma. We first define the notion of the “linkage” of a cover $\Xi$ of $\bar{Q}$ by cubes $\Delta \subseteq \mathbb{R}^d$. Suppose that $\Xi$ can be partitioned into $\tau$ subsets $\Xi_1, \ldots, \Xi_\tau$ such that for each $k = 1, \ldots, \tau$ the set $\Xi_k$ consists of pairwise-disjoint cubes. The linkage of $\Xi$ is the smallest $\tau$ for which such a partition of $\Xi$ is possible. It is denoted as $\text{link}(\Xi)$.

**Proposition 6** (Besicovitch Covering Lemma). Assume that for each $x \in \bar{Q} = [0,1]^d$ we
are given a (nontrivial) closed cube $\Delta_x \subseteq \mathbb{R}^d$ centered at $x$. Then a finite subset $\Xi = \{\Delta_{x_j}\}$ can be chosen in such a way that

i) $\bar{Q} \subseteq \bigcup_j \Delta_{x_j}$,

ii) $\text{link}(\Xi) \leq \tau_d$, where $\tau_d$ is a constant depending only on the dimension $d$.

For the proof see [23], Theorem 1.1 or [21], Theorem 18.1c.

**Proof of Theorem 4**: The proof makes use of the set function $J_{B,V}(\omega) = \|||V|||_{B,\omega}$. After normalizing $V$ we can assume that $J(Q) = \|||V|||_{B,Q} = ||V||_{B,Q} = 1$. We may also suppose that $V(x) \geq \delta > 0$ a.e. on $Q$ instead of simply $V(x) \geq 0$ by replacing $V(x)$ by $\tilde{V}(x) = V(x) + \delta$, proving the theorem first for $\tilde{V}$ and then letting $\delta \to 0$. Then $\tilde{V} \to V$ in “mean convergence” on $Q$, and so by the $\Delta_2$-condition of $B$, $\tilde{V} \to V$ in $L_{B}(Q)$.

Therefore, by the previous lemma, we may fix an $n \in \mathbb{N}$ and for each $x \in \bar{Q}$ we may find a cube $\Delta_x$ centered around $x$ such that $\|||V|||_{B,\Delta_x \cap Q} = n^{-1}$. By the Besicovitch covering lemma we may select a covering $\Xi$ of $\bar{Q}$ consisting of some of these cubes where $\Xi = \bigcup_k \Xi_k$, $1 \leq k \leq \text{link}(\Xi) \leq \tau_d$ is a partition of $\Xi$ as in the definition of linkage. Then by lemma 7

$$ n^{-1} \text{card}\{\Xi_k\} = \sum_{\Delta \in \Xi_k} J(\Delta \cap Q) \leq J(Q) = 1. $$

So

$$ \text{card}(\Xi) \leq \text{link}(\Xi) \max_k \{\text{card}(\Xi_k)\} \leq \text{link}(\Xi) \cdot n \leq \tau_d \cdot n. \quad (3.22) $$
For each cube $\Delta \in \Xi$, let $\tilde{\Delta} = \Delta \cap Q$. The parallelepipeds $\tilde{\Delta}$ are elements of a covering $\tilde{\Xi}$ of $Q$. If $l_1, \ldots, l_d$ are the edge lengths of $\Delta \in \tilde{\Xi}$ then since $\Delta$ is centered at $x \in \bar{Q}$, we have

$$\max_{i,j} \frac{l_i}{l_j} \leq 2.$$ 

We will show that $\tilde{\Xi}$ is the required covering. Notice first that $\text{card}(\tilde{\Xi}) \leq \tau d n$, which proves the first part of the theorem. Let $K_{\tilde{\Xi},l}$ be the operator of piecewise-polynomial approximation given in (3.7). Then

$$\text{rank}(K_{\tilde{\Xi},l}) \leq n \cdot m(l, d) \cdot \tau_d. \tag{3.23}$$

Now for any $u \in H^1(Q)$ we have

$$\int_Q V|u - K_{\tilde{\Xi},l}u|^2 \, dx \leq \sum_{\Delta_j \in \tilde{\Xi}} \int_{\tilde{\Delta}_j \cup \Delta_k < \Delta_i} V|u - P_{\Delta_k,l}u|^2 \, dx \leq \sum_{\Delta_j \in \tilde{\Xi}} \int_{\tilde{\Delta}_j} V|u - P_{\Delta_j,l}u|^2 \, dx \tag{3.24}$$

Since $P_{\Delta_j,l}(u - P_{\Delta_j,l}u) = 0$ we can apply (3.15) at this point to get

$$\int_Q V|u - K_{\tilde{\Xi},l}u|^2 \, dx \leq 2^d C_{14} \sum_{\Delta \in \tilde{\Xi}} J(\tilde{\Delta}) \int_{\tilde{\Delta}} |\nabla^l u|^2 \, dx \leq 2^d C_{14} n^{-1} \sum_{\Delta \in \Xi} \int_{\tilde{\Delta}} |\nabla^l u|^2 \, dx = 2^d C_{14} n^{-1} \sum_k \sum_{\Delta \in \Xi_k} \int_{\Delta \cap Q} |\nabla^l u|^2 \, dx \leq 2^d C_{14} \tau d \cdot n^{-1} \int_Q |\nabla^l u|^2 \, dx.$$ 

This proves the second part of the theorem and $\tilde{\Xi}$ is the promised covering. ■
3.2 An eigenvalue estimate in the Orlicz setting

Defined on $H_0^1(\Omega)$, we consider the quadratic form

$$a(u, v) := \int_{\Omega} \nabla^l u \cdot \nabla^l v dx,$$

with

$$a(u) := a(u, u) = \int_{\Omega} |\nabla^l u|^2 dx.$$

As an unbounded quadratic form on $L^2(\Omega)$, $a(u)$ is symmetric and positive. It is also closed in $L^2(\Omega)$. To see this note that if $u_n \to u$ in $L^2(\Omega)$ and $a(u_m - u_n) \to 0$ then $\{u_n\}$ is Cauchy in $H_0^1(\Omega)$, and hence converges to some $\bar{u}$ in that space. By the generalized Poincaré Inequality $u_n \to \bar{u}$ in $L^2(\Omega)$. Thus $u = \bar{u}$ a.e. and so $u$ is (representable by) an element of $H_0^1(\Omega)$ and $\lim_{n \to \infty} a(u_n - u) = 0$. On the space $L^2(\Omega)$ the unbounded operator $(-\Delta)^l$ is defined as the self-adjoint Friedrichs operator associated to the form $a(u)$. That is,

$$D((-\Delta)^l) := \{ u \in H_0^1(\Omega) : \text{the linear functional} \quad v \mapsto \int_{\Omega} \nabla^l u \cdot \nabla^l v dx \quad \text{is } L^2\text{-continuous,} \quad \text{where} \quad v \in H_0^1(\Omega) \}$$

and

$$\langle (-\Delta)^l f, g \rangle_{L^2} = a(f, g)$$

for $f \in D((-\Delta)^l)$ and $g \in H_0^1(\Omega)$. We can do this since $a$ is closed. See for example section 5.5 in [55].
Suppose $A$ is a self-adjoint operator on a Hilbert space and that the spectrum of $A$ less than or equal to $\lambda \in \mathbb{R}$ is discrete. Then define $N(\lambda, A)$ to be the number of eigenvalues of $A$ less than or equal to $\lambda$, counted according to multiplicity. For a compact, non-negative, symmetric operator $T$ denote by

$$n(\lambda, T) = N(-\lambda, -T)$$

the number of eigenvalues of $T$ greater than or equal to $\lambda$. We now consider the quadratic functional

$$b_V(u) := \int_{\Omega} V(x)|u(x)|^2 dx$$

where $V \in L_B$. If $b_V$ is bounded on $(H_0^1(\Omega), ||\cdot||_{H_0^1(\Omega)})$, then it generates a bounded, self-adjoint, non-negative operator on $H_0^1(\Omega)$ - say $T_V$. By definition

$$u = T_V f \iff u \in H_0^1(\Omega);$$

$$\int_{\Omega} \nabla^i u \cdot \nabla^j w dx = \int_{\Omega} V f w dx, \quad \forall w \in H_0^1(\Omega)$$

**Theorem 5.** Let $\Omega \subset \mathbb{R}^d$ be a bounded region with smooth boundary, and $V \in L_B(\Omega)$. Then the operator $T_V$ is well-defined and compact on $H_0^1(\Omega)$, and there exists a constant $C_3 = C_3(\Omega)$ such that for any $\lambda > 0$

$$n(\lambda; T_V) \leq C_3 ||V||_{L_B, \Omega} \lambda^{-1}$$  \hspace{1cm} (3.25)

**Proof:** Let $Q \subset \mathbb{R}^d$ be a cube such that $\bar{\Omega} \subseteq Q$. We can regard $Q$ as a unit cube, after rescaling. Let $W$ be the function on $Q$ equal to $V$ on $\Omega$ and $W = 0$ on $Q \setminus \Omega$. Note that by
the Hölder inequality for Orlicz spaces we write

\[ \int_\Omega V \cdot |u|^2 \, dx \leq ||V||_{B,\Omega} \cdot ||u^2||_{A,\Omega} \]  \hspace{1cm} (3.26)

where the \( N \)-function \( A(t) := e^{|t|} - 1 - |t| \) is the Young function conjugate to \( B(t) \). Now apply Proposition 5. Since \( u \in H^l_0(\Omega) \) we can use the norm of \( H^l_0(\Omega) \) instead of \( H^l(\Omega) \). We get

\[ \int_\Omega V \cdot |u|^2 \, dx \leq C_d ||V||_{B,\Omega} \cdot ||\nabla^l u||^2_{L^2} \]  \hspace{1cm} (3.27)

Thus \( b_V \) is bounded as a quadratic form on \( H^l_0(\Omega) \) (as well as on \( \Pi \circ H^l_0(\Omega) \subseteq H^l_0(Q) \), where \( \Pi \) is the natural “extension by 0” operator). So \( T_V \) is bounded on \( H^l_0(\Omega) \) and

\[ n(\lambda; T_V) = 0 \quad \text{for } \lambda > C_d ||V||_{B,\Omega} \] \hspace{1cm} (3.28)

Now fix \( \lambda \in (0, \lambda_0] \), where \( \lambda_0 = C_2 C_l ||V||_{B,\Omega} \), with \( C_2 \) coming from Theorem 4, and \( C_l \) coming from

\[ ||u||^2_{W^{l,2}} \leq C_l ||u||^2_{H^l_0} \]

for \( u \in H^l_0(\Omega) \). Let \( n \) be the minimal integer such that \( n\lambda > \lambda_0 \). We apply Theorem 4 for this particular \( n \) and the weight function \( W \). Let \( \Xi \) be the covering of \( Q \) constructed in Theorem 4 and \( K := K_{\Xi,l} \) the corresponding operator (3.7).

For the subspace \( F := \ker(K \circ \Pi) \) of \( H^l_0(\Omega) \)

\[ \text{codim } F \leq \text{rank } K \leq m(l, d) C_1 \cdot n \]

For \( u \in H^l_0(\Omega) \) denote by \( U := \Pi(u) \). Let \( u \in F \). Then by Theorem 4 the following inequality
holds

\[
\int_{\Omega} V|u|^2 \, dx = \int_{Q} W|U - K(U)|^2 \, dx \\
\leq C_2 n^{-1} ||W||_{B,Q} \int_{Q} |\nabla l u|^2 \, dx \\
\leq C_l C_2 n^{-1} ||W||_{B,Q} ||u||_{H_0^1}^2 \\
< \lambda ||u||_{H_0^1}^2
\]

by the choice of \( n \). This is enough to show that \( T_V \) is compact. For, \( T_V = T_V|_{F} + T_V|_{F^\perp} \) and \( F^\perp \) is finite dimensional. The above shows that \( ||T_V|_{F}|| \leq \lambda \). So taking \( \lambda \to 0 \), we see that \( T_V \) is the norm limit of \( T_V|_{F^\perp} \) (defined as \( T_V \) on \( F^\perp \) and 0 on \( F \)). So \( T_V \) is compact and its spectrum consists of eigenvalues. If \( \{u_j\} \) is an eigenvector with eigenvalue \( \lambda_j \geq \lambda \) then

\[
\lambda \leq \lambda_j = \frac{\langle T_V u_j, u_j \rangle_{H_0^1}}{\langle u_j, u_j \rangle_{H_0^1}} = \frac{\int_{\Omega} V|u_j|^2 \, dx}{\int_{\Omega} |\nabla l u_j|^2 \, dx}.
\]

So \( u_j \notin F \) by (3.29). Since eigenvectors are orthogonal

\[
n(\lambda; T_V) \leq codim F \leq m(l, d) C_1 n \\
< m(l, d) C_1 \left( \frac{\lambda_0}{\lambda} + 1 \right) \\
\leq 2m(l, d) C_1 \frac{\lambda_0}{\lambda}, \quad \lambda \leq \lambda_0
\]

The required estimate (3.25), with \( C_3 = 2m(l, d) C_1 \max\{C_l C_2, C_d\} \), where \( C_d \) is given in (3.28) and \( C_l \) is given in the inequality after (3.29), is a consequence of (3.29) and (3.31):

i) If \( C_l C_2 \geq C_d \) then (3.31) gives the result, since by (3.28) we do not need to look for eigenvalues \( \lambda \) greater than \( C_d ||V||_{B,\Omega} \leq C_l C_2 ||V||_{B,\Omega} = \lambda_0 \).

ii) If \( \lambda \leq \lambda_0 = C_l C_2 ||V||_{B,\Omega} \), again (3.31) gives the result.
iii) If $C_lC_2\|V\|_{B,\Omega} < \lambda \leq C_d\|V\|_{B,\Omega}$ then (3.31) is applied to $\tilde{\lambda} = \lambda \frac{C_lC_2}{C_d}$ and that gives the result.

\[\blacksquare\]

**Remark:** Let us examine the result above a little more thoroughly using the language of quadratic forms on Hilbert spaces. On the Hilbert space $(L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)})$ we consider the quadratic form $a(u) = \int_{\Omega} \nabla^l u|^2 \, dx$, with the form domain defined as $H^l_0(\Omega)$. Since $\Omega$ is bounded, $a(u)$ is a positive definite form on $L^2(\Omega)$ by the Poincaré inequality (i.e. $\exists \gamma > 0$ such that $a(u) \geq \gamma \|u\|^2_{L^2(\Omega)}$ for all $u \in H^l_0(\Omega)$). Also, $a(u)$ is a closed quadratic form. Therefore $(H^l_0(\Omega), a(\cdot, \cdot))$ is a Hilbert space, and continuously embeds into $(L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)})$. Now let us consider the operator $T_V$ defined above. As an operator on $L^2(\Omega)$ it is typically unbounded. However as an operator on $(H^l_0(\Omega), a(\cdot, \cdot))$ it is compact according to the above theorem. This implies that the form $b_V(u) = \int_{\Omega} V|u|^2 \, dx$ has a zero-bound relative to $a(u)$ in $L^2(\Omega)$. That is, for every $\epsilon > 0$ there exists a $C(\epsilon)$ such that

$$b_V(u) \leq \epsilon a(u) + C(\epsilon) \|u\|^2_{L^2(\Omega)}.$$ 

We will prove this fact in Appendix B.

Also, on $L^2(\Omega)$ we consider the form

$$a_V(u) := a(u) - \int_{\Omega} V(x)|u|^2 \, dx$$

(3.32)

where $V \in L_B, V \geq 0$. Here the form domain is $H^l_0(\Omega) \cap L^2(\Omega, V \, dx)$. As we saw in the proof of Theorem 5

$$\|u\|^2_{L^2(\Omega, V \, dx)} \leq C_{\Omega} \cdot a(u)$$

for $u \in H^l_0(\Omega)$. So the domain of $a_V$ is really just $D(a_V) = H^l_0(\Omega)$ and $H^l_0(\Omega)$ embeds into
$L^2(\Omega, Vdx)$. As a matter of fact by the remark following theorem 5, for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$||u||^2_{L^2(\Omega, Vdx)} \leq \epsilon a(u) + C(\epsilon)||u||^2_{L^2(\Omega)}.$$  \hspace{1cm} (3.33)

That is, the quadratic form

$$\int_{\Omega} V(x)|u|^2dx$$  \hspace{1cm} (3.34)

has “zero bound” relative to the form $a(u)$ in $L^2(\Omega)$. This implies that $a_V(u)$ is semi-bounded in $L^2(\Omega)$, i.e. there is a constant $C > 0$ such that $a_V(u) \geq -C||u||^2_{L^2(\Omega)}$. This is easy to see, but the details are also given in Appendix B. It also implies that $a_V(u)$ is closed in $L^2(\Omega)$. To see this let $\{u_n\} \subset H^1_0(\Omega)$ be such that $u_n \to u$ in $L^2(\Omega)$ and $a_V(u_n - u_m) \to 0$. Then $a(u_n - u_m) \to 0$. So $\{u_n\}$ is Cauchy in $H^1_0(\Omega)$, and hence in $L^2(\Omega)$. Therefore $u \in H^1_0(\Omega)$.

So as before we can define the associated self-adjoint Friedrichs operator on $L^2(\Omega)$: 

$$A_V(u) := (-\Delta)^l u - V(x)u.$$ 

whose domain is a subset of $D((-\Delta)^l)$.

The Briman-Schwinger Principle

The reference here is section 1 in [14], and the details are found in Appendix B. Let $a(u)$ be a positive, symmetric, and closed quadratic form in a Hilbert space $\mathcal{H}$ with domain $D(a) \subseteq \mathcal{H}$. Let $b(u)$ be another non-negative, symmetric quadratic form such that

$$b(u) \leq C \cdot a(u), \hspace{1cm} u \in D(a)$$  \hspace{1cm} (3.35)
Consider the space $\tilde{D}(a)$ - the completion of $D(a)$ in the inner product given by $a(\cdot, \cdot)$. By (3.35) $b$ can be extended to all of $\tilde{D}(a)$. The extended form defines on $\tilde{D}(a)$ a bounded, self-adjoint, non-negative operator, which we denote by $B : (\tilde{D}(a), a(\cdot, \cdot)) \to (\tilde{D}(a), a(\cdot, \cdot))$.

**Proposition 7** (Birman-Schwinger Principle). Suppose (3.35) is satisfied and the operator $B$ is compact as an operator from $(\tilde{D}(a), a(\cdot, \cdot))$ to itself. Then for any $\alpha > 0$ the quadratic form

$$a_\alpha(u) := a(u) - \alpha b(u), \quad u \in D(a)$$

is semi-bounded from below and closed in $\mathcal{H}$. As usual, this implies that there is a corresponding self-adjoint Friedrichs operator $A_{ab}$ associated with this form. For $A_{ab}$ the non-positive spectrum is finite and satisfies

$$N(0; A_{ab}) = n(\alpha^{-1}; B). \quad (3.36)$$

For us $\mathcal{H} = L^2(\Omega)$, $D(a) = \tilde{D}(a) = H_0^1(\Omega)$, $a(u) = \int_\Omega |\nabla u|^2 dx$, $b(u) = \int_\Omega V|u|^2 dx$, $\alpha = 1$, and $B = T_V$. From the proof of Theorem 5, (3.35) is satisfied. The form $a(u)$ is closed in $L^2(\Omega)$ as was noted at the beginning of the section. The precise definition of the operator $(-\Delta)^f - V(x)$ is that it is the operator $A_{ab}$ with the above data. We thus obtain

$$N(0, (-\Delta)^f - V(x)) = n(1, T_V).$$

So by Theorem 5

$$N(0, (-\Delta)^f - V(x)) \leq C_3||V||_{\mathcal{B}, \Omega}.$$ 

That is, we obtain the main result of this section:

**Theorem 6.** Let $\mathcal{B}(t) := (|t| + 1) \ln(|t| + 1) - |t|$ be an $N$-function, and $L_{\mathcal{B}}(\Omega)$ be the corresponding Orlicz space. Let $V(x) \in L_{\mathcal{B}}(\Omega)$, and on $L^2(\Omega)$ consider the unbounded lin-
ear operator \((-\Delta)^l - V(x)\). Denote by \(N(0; (-\Delta)^l - V(x))\) the number of its non-positive eigenvalues. Then there exists a constant \(C = C(l, \Omega)\) such that

\[
N(0; (-\Delta)^l - V(x)) \leq C \|V\|_{B, \Omega}
\]

We will not use Theorem 6 but rather a corollary of it. As is well-known in the theory of Orlicz spaces \(\|V\|_{B, \Omega} \leq \int_\Omega B(V(x))dx + 1\). See (9.12) in [25].

**Corollary 2.** \(N(0; (-\Delta)^l - V(x)) \leq C\int_\Omega B(V(x))dx + C\).

This generalizes the eigenvalue estimate originally obtained by Sugimura [50].

### 3.3 An estimate in the radially symmetric case on an annulus

Here \(\Omega = A^R_{R_0} := \{x \in \mathbb{R}^2 : R_0 < |x| < R\}\) will denote an annulus, with \(R_0 \geq 0\) and \(R < +\infty\). Sometimes we will denote \(A^R_{R_0}\) as simply \(A\). We prove the following:

**Proposition 8.** Let \(\Omega\) be an annulus of outer radius \(R < +\infty\) and inner radius \(R_0 > 0\). Let \(V(x) = V(|x|)\). On the space \(L^2_r(\Omega)\), of radially symmetric, square integrable functions, consider the unbounded linear operator \((-\Delta)^l - V(x)\). Denote by \(N(0; (-\Delta)^l - V(x))\) the number of its non-positive eigenvalues. There exists a constant \(C = C(l, R_0, R)\) such that

\[
[N(0; (-\Delta)^l - V(x))]^{2l} \leq C \int_\Omega V^+(x) \left[1 + \log \left(\frac{R}{|x|}\right)\right]^{2i} dx
\]

where \(i = 1/2\) when \(l = 1\), and \(i = 1\) when \(l > 2\).

As mentioned earlier, we first need some appropriate inequalities in the radial case to take the place of the Orlicz-Sobolev inequality of Proposition 3. An important space for this
section is
\[ H_r := \{ u \in H^1_0(\Omega) : u(x) = u(|x|) \text{ a.e. in } \Omega \}. \]

Lemma 9 below is a generalization of an inequality by Ni, see [34].

**Lemma 9.** Let \( u \in H_r \).

a) If \( d = 2l = 2 \) then
\[
|u(x)| \leq C_d ||\partial_r u||_{L^2} \cdot \sqrt{\log \left( \frac{R}{|x|} \right)}
\]
for \( x \in \Omega = A^R_{R_0} \).

b) If \( d = 2l > 2 \) then
\[
|u(x)| \leq C_d ||\partial^l_r u||_{L^2} \cdot \log \left( \frac{R}{|x|} \right)
\]
for \( x \in \Omega = A^R_{R_0} \).

**Proof:**

a) For simplicity we write \( u = u(r) \) as a function of the radial variable. By a density argument we may assume that \( u \in C^\infty_0(\Omega) \). For \( r \in [R_0, R] \)
\[
-u(r) = u(R) - u(r) = \int_r^R u'(\rho) d\rho
\]
so
\[
|u(r)| \leq \int_r^R |u'(\rho)| d\rho \\
\leq \left( \int_r^R |u'(\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_r^R \frac{1}{\rho} d\rho \right)^{1/2} \\
\leq C_d ||\partial_r u||_{L^2} \cdot \left[ \log \left( \frac{R}{r} \right) \right]^{1/2}
\]
which is the required result.

b) Again we take $u \in C_0^\infty$.

$$ u(r) = u(r) - u(R) = - \int_r^R u'(\rho_1) d\rho_1 $$
$$ = \int_r^R u'(R) - u'(\rho_1) d\rho_1 = \int_r^R \int_{\rho_1}^R u''(\rho_2) d\rho_2 d\rho_1 $$
$$ \vdots $$
$$ = (-1)^l \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-1}}^R u^{(l)}(\rho_l) d\rho_l \cdots d\rho_1 $$

So

$$ |u(r)| \leq \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)| d\rho_l \cdots d\rho_1 $$
$$ = \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)| \rho_l^{d-1} \rho_{l-1}^{1-d} d\rho_l \cdots d\rho_1 $$
$$ \leq \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-2}}^R \left( \int_{\rho_{l-1}}^R |u^{(l)}(\rho_l)|^2 \rho_{l-1}^{d-1} d\rho_l \right)^{1/2} \left( \int_{\rho_{l-1}}^R \rho_{l-1}^{1-d} d\rho_l \right)^{1/2} d\rho_{l-1} \cdots d\rho_1 $$
$$ \leq C_d \| \partial_r^l u \|_{L^2} \cdot \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-1}}^R \rho_{l-1}^{1-d} d\rho_{l-1} \cdots d\rho_1 $$
$$ \leq C_d \| \partial_r^l u \|_{L^2} \cdot \int_r^R \frac{1}{\rho_1} d\rho_1 $$
$$ = C_d \| \partial_r^l u \|_{L^2} \cdot \log \left( \frac{R}{r} \right) $$

which is the required estimate. ■
When proving a radial version of Theorem 4 it is necessary to have at one’s disposal inequalities of the above type, but without the zero boundary conditions. The key is to find the appropriate \((l - 1)\)th-degree polynomial to subtract from \(u\), so that the remainder can be controlled by the \(l\)th-order derivative of \(u\). It is no surprise that this is the same polynomial approximation which appears in the radial version of Theorem 4.

**Lemma 10.** Let \(u \in H^l(A)\), where \(A = A_{R_0}^R\) is an annulus centered at the origin in \(\mathbb{R}^2\), and \(u(x) = u(|x|)\) a.e in \(A\).

a) If \(d = 2l = 2\) then

\[
|u(x) - \bar{u}_A| \leq C_d \|\partial_r u\|_{L^2(A)} \cdot \left[ 1 + \sqrt{\log \left( \frac{R}{|x|} \right)} \right]
\]

for \(x \in A\), where \(\bar{u}_A\) is the average value of \(u\) in \(A\).

b) When \(d = 2l > 2\). First define

\[
\tau_l(u)(r, s) := \sum_{n=0}^{l-1} \frac{u^{(n)}(s)}{n!} (r - s)^n
\]

and

\[
P_{l,A}(u)(r) := \frac{1}{|A|} \int_A \tau_l(u)(r, |x|)dx
\]

which is a polynomial in \(r\) of degree \(\leq l - 1\) and linear in \(u\). Then

\[
|u(x) - P_{l,A}(u)(|x|)| \leq C_d \|\partial_r^l u\|_{L^2(A)} \cdot \left[ 1 + \log \left( \frac{R}{|x|} \right) \right]
\]
for $x \in A = A_{R_0}^R$.

**Proof:** We assume $u \in C^\infty(A)$ and radially symmetric. Then

$$|u(r) - \bar{u}_A| = \left| \frac{1}{|A|} \int_A u(r) - u(x) dx \right| \leq \frac{1}{|A|} \int_A |u(r) - u(x)| dx$$

(3.37)

Now

$$u(r) - u(x) = \int_{|x|}^{r} u'(\rho)d\rho$$

So

$$|u(r) - u(x)| \leq \int_{|x|,r} |u'(\rho)| \rho^{1/2} \rho^{-1/2} d\rho$$

where the notation $\int_{a,b}$ denotes unoriented integration over the interval with endpoints $a$ and $b$. So

$$|u(r) - u(x)| \leq \left( \int_{|x|,r} |u'(\rho)|^2 \rho d\rho \right)^{1/2} \left( \int_{|x|,r} \rho^{-1} \rho \right)^{1/2} \leq C_d \|\partial_r u\|_{L^2(A)} \cdot \left| \log \left( \frac{r}{|x|} \right) \right|^{1/2}$$

Plugging this into the earlier inequality gives

$$|u(r) - \bar{u}_A| \leq \frac{1}{|A|} \int_A C_d \|\partial_r u\|_{L^2(A)} \cdot \left| \log \left( \frac{r}{|x|} \right) \right|^{1/2} dx$$

$$= \frac{C_d}{|A|} \|\partial_r u\|_{L^2(A)} \cdot \int_{R_0}^{R_0} \left| \log \left( \frac{r}{\rho} \right) \right|^{1/2} \rho d\rho$$

It’s easy to check that the value of the integral is bounded above by a constant times

$$R(R - R_0) \left[ 1 + \log \left( \frac{R}{r} \right) \right]^{1/2}.$$
To see this we evaluate the integral in two parts:

\[ I_1 = \int_{R_0}^{r} \log \left( \frac{r}{\rho} \right)^{1/2} \rho d\rho \]

and

\[ I_2 = \int_{r}^{R} \log \left( \frac{\rho}{r} \right)^{1/2} \rho d\rho. \]

In \( I_1 \) we let \( t = \log(r/\rho) \) and so

\[ I_1 = r^2 \int_{0}^{\log(r/R_0)} t^{1/2} e^{-2t} dt \leq C_0 r^2 \int_{0}^{\log(r/R_0)} e^{-t} dt = C_0 r^2 (r - R_0) \leq C_0 R(R - R_0). \]

For \( I_2 \) we simply notice

\[ I_2 = \int_{r}^{R} \log \left( \frac{\rho}{r} \right)^{1/2} \rho d\rho \leq \log \left( \frac{R}{r} \right)^{1/2} R(R - R_0). \]

Since \( A \) is a \( d = 2 \) dimensional annulus we have that

\[ \frac{R(R - R_0)}{|A|} \leq C_d \]

Thus

\[ |u(r) - \bar{u}_A| \leq C_d \| \partial_r u \|_{L^2(A)} \left[ 1 + \log \left( \frac{R}{r} \right)^{1/2} \right] \]

which is the required result.
b) As before, we assume $u$ is smooth. Then

$$|u(r) - P_{\tau,A}(u)(r)| = \left| \frac{1}{|A|} \int_A u(r) - \tau_l(u)(r, |x|) dx \right|$$

$$\leq \frac{1}{|A|} \int_A |u(r) - \tau_l(u)(r, |x|)| dx$$

(3.38)

Set $v(r) := u(r) - \tau_l(u)(r, |x|)$. Note that, when keeping $|x|$ fixed, we have

$$v^{(n)}(r)|_{r=|x|} = 0 \quad \text{for } 0 \leq n \leq l - 1$$

where the differentiation is partial differentiation w.r.t. $r$. We seek to estimate $v(r)$ by repeatedly applying this property.

$$v(r) = v(r) - v(|x|) = \int_r^{|x|} v'(\rho_1) d\rho_1$$

$$= \int_r^{|x|} v'(\rho_1) - v'(|x|) d\rho_1 = \int_r^{|x|} \int_{|x|}^{\rho_1} v''(\rho_2) d\rho_2 d\rho_1$$

$$= \int_r^{|x|} \int_{|x|}^{\rho_1} v''(\rho_2) - v''(|x|) d\rho_2 d\rho_1$$

$$\vdots$$

$$= \int_r^{|x|} \int_{|x|}^{\rho_1} \cdots \int_{|x|}^{\rho_{l-1}} v^{(l)}(\rho_l) d\rho_l \cdots d\rho_1$$

thus

$$|v(r)| \leq \int_{|x|,r} \int_{|x|,\rho_1} \cdots \int_{|x|,\rho_{l-1}} |v^{(l)}(\rho_l)| d\rho_l \cdots d\rho_1$$
We apply Hölder’s inequality to the inner most integral

\[
|v(r)| \leq \int_{|x|,r} \cdots \int_{|x|,\rho_{l-2}} \left( \int_{|x|,\rho_{l-1}} \left| v^{(l)}(\rho_{l}) \right|^2 \rho_{l}^{d-1} d\rho_{l} \right)^{1/2} \times \left( \int_{|x|,\rho_{l-1}} \rho_{l}^{1-d} d\rho_{l} \right)^{1/2} d\rho_{l-1} \cdots d\rho_{1}
\]

\[
\leq C_d \|\partial_{\rho_{l}} u\|_{L^2(A)} \cdot \int_{|x|,r} \cdots \int_{|x|,\rho_{l-2}} |\rho_{l-1}^{2-d} - |x|^{2-d}|^{1/2} d\rho_{l-1} \cdots d\rho_{1} dx
\]

after using \( v^{(l)}(\rho_{l}) = \partial_{\rho_{l}} u(\rho_{l}) \). Returning to 3.38 we get

\[
|u(r) - P_{l,A}(u)(r)| \leq \frac{C_d}{|A|} \|\partial_{\rho_{l}} u\|_{L^2(A)} \times \int_{|x|,r} \cdots \int_{|x|,\rho_{l-2}} |\rho_{l-1}^{2-d} - |x|^{2-d}|^{1/2} \ d\rho_{l-1} \cdots d\rho_{1} dx
\]  

\tag{3.39}

We seek to estimate the above integral. First, by converting to polar coordinates the integral becomes (after factoring out a constant depending only on the dimension)

\[
\int_{R_0}^{R} \int_{\rho_0}^{r} \cdots \int_{\rho_0}^{\rho_{l-2}} |\rho_{l-1}^{2-d} - \rho_0^{2-d}|^{1/2} d\rho_{l-1} \cdots d\rho_{1} \rho_0^{d-1} d\rho_0
\]

We divide the integration by \( d\rho_0 \) into two pieces. One where \( R_0 \leq \rho_0 \leq r \) and one where \( r \leq \rho_0 \leq R \). This allows us to properly orient the endpoints. The first integral is

\[
\int_{R_0}^{r} \int_{\rho_0}^{r} \cdots \int_{\rho_0}^{\rho_{l-2}} (\rho_0^{2-d} - \rho_{l-1}^{2-d})^{1/2} d\rho_{l-1} \cdots d\rho_{1} \rho_0^{d-1} d\rho_0
\]

\[
\leq \int_{R_0}^{r} \int_{\rho_0}^{r} \cdots \int_{\rho_0}^{\rho_{l-2}} \rho_0^{1-d/2} d\rho_{l-1} \cdots d\rho_{1} \rho_0^{d-1} d\rho_0
\]

\[
= \int_{R_0}^{r} \int_{\rho_0}^{r} \cdots \int_{\rho_0}^{\rho_{l-3}} \rho_0^{1-d/2}(\rho_{l-2} - \rho_0) d\rho_{l-2} \cdots d\rho_{1} \rho_0^{d-1} d\rho_0
\]

\[
= \int_{R_0}^{r} \int_{\rho_0}^{r} \cdots \int_{\rho_0}^{\rho_{l-3}} \rho_0^{1-d/2}(\rho_{l-2} - \rho_0) d\rho_{l-2} \cdots d\rho_{1} \rho_0^{d-1} d\rho_0
\]
\[ \leq \int_{R_0}^{r} \int_{\rho_0}^{\rho_1} \cdots \int_{\rho_0}^{\rho_{l-3}} \rho_0^{1-d/2} \rho_{l-2} \rho_{l-2} \cdots \rho_1 \rho_0^{d-1} d\rho_0 \]

\[ \vdots \]

\[ \leq C_l \int_{R_0}^{r} \int_{\rho_0}^{\rho_1} \rho_0^{1-d/2} \rho_1^{l-2} \rho_1 \rho_0^{d-1} d\rho_0 \]

\[ \leq C \int_{R_0}^{r} \rho_0^{1-d/2} r^{l-1} \rho_0^{d-1} d\rho_0 \]

\[ = C \int_{R_0}^{r} \rho_0^{d/2} r^{l-1} d\rho_0 \]

\[ \leq C r^{d/2+l-1} (r - R_0) \]

\[ \leq C R^{d/2+l-1} (R - R_0) = CR^{d-1} (R - R_0) \quad (3.40) \]

where we have used \( d = 2l \) in the final equality.

The second integral is

\[ \int_{r}^{R} \int_{\rho_0}^{\rho_0} \cdots \int_{\rho_0}^{\rho_0} (\rho_{l-1}^{2-d} - \rho_0^{2-d})^{1/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \]

\[ \leq \int_{r}^{R} \int_{r}^{\rho_0} \cdots \int_{r}^{\rho_0} \rho_{l-1}^{1-d/2} d\rho_{l-1} \cdots d\rho_1 \rho_0^{d-1} d\rho_0 \]

\[ \vdots \]

\[ \leq C_d \int_{r}^{R} \int_{r}^{\rho_0} \rho_1^{l-1-d/2} d\rho_1 \rho_0^{d-1} d\rho_0 \]

Now using \( l - 1 - d/2 = -1 \),

\[ = C_d \int_{r}^{R} \log \left( \frac{\rho_0}{r} \right) \rho_0^{d-1} d\rho_0 \leq C_d R^{d-1} (R - R_0) \log \left( \frac{R}{r} \right) \quad (3.41) \]
Combining 3.39, 3.40, and 3.41 gives

$$|u(r) - P_l,A(u)(r)| \leq C_d \frac{R^{d-1}(R - R_0)}{|A|} ||\partial_r u||_{L^2(A)} \left[ 1 + \log \left( \frac{R}{r} \right) \right]$$  \hspace{1cm} (3.42)

Since \( A \) is an annulus, \( \frac{R^{d-1}(R - R_0)}{|A|} \leq C_d \). So

$$|u(r) - P_l,A(u)(r)| \leq C_d ||\partial_r u||_{L^2(A)} \left[ 1 + \log \left( \frac{R}{r} \right) \right]$$  \hspace{1cm} (3.43)

Which is the required estimate. \( \blacksquare \)

We will apply Lemma 10 to our main region \( \Omega = A_{R_0}^R \), where \( R < \infty \) and \( R_0 > 0 \) are the fixed outer and inner radii of \( \Omega \), respectively. Let \( \tilde{u}(x), x \in \Omega \), be as in that lemma. We have that for \( \tilde{r} \in [R_0, R] \)

$$|\tilde{u}(\tilde{r}) - P_{l,\Omega}(\tilde{u})(\tilde{r})|^2 \leq c_d \left[ 1 + \log \left( \frac{R}{R_0} \right) \right]^{2i} \int_{R_0}^{R} |\partial_{\tilde{r}} \tilde{u}|^2 \tilde{\rho}^{d-1} d\tilde{\rho}$$

So

$$|\tilde{u}(\tilde{r}) - P_{l,\Omega}(\tilde{u})(\tilde{r})|^2 \leq \kappa_0 \int_{R_0}^{R} |\partial_{\tilde{r}} \tilde{u}|^2 \tilde{\rho}^{d-1} d\tilde{\rho}$$  \hspace{1cm} (3.44)

where

$$\kappa_0 = \kappa_0(d, R_0, R) = c_d \left[ 1 + \log \left( \frac{R}{R_0} \right) \right]^{2i}$$

Now consider a change of variable, replacing the domain \( \Omega = A_{R_0}^R \) with a smaller annulus contained in it, \( A \). It is centered at the origin, with inner radius \( R_A \), and outer radius \( R_A' \), \( R_0 \leq R_A < R_A' \leq R \):

Let \( r = \mu \tilde{r} + \beta \), where \( \mu := \frac{R_A - R}{R - R_0} \) and \( \beta := \frac{R_0 R_A^A - R R_A}{R A^A - R_A} \). For a radially symmetric \( \tilde{u} \in H^l(\Omega) \), let \( u(r) := \tilde{u}(\tilde{r}) \). Define
Inequality (3.44) becomes

$$|u(r) - P_{l,A}u(r)|^2 \leq \kappa_0 \mu^{2l-1} \int_{R_A^A} |\partial_r^l u|^2 \tilde{\rho}^{d-1} d\rho$$

(3.46)

where $\rho = \mu \tilde{\rho} + \beta$. Since $R_0 > 0$

$$\frac{R_0}{R} \leq \frac{\tilde{\rho}}{\rho} \leq \frac{R}{R_0}$$

This and inequality (3.46) give

$$|u(r) - P_{l,A}u(r)|^2 \leq \kappa' \mu^{2l-1} \int_A |\partial_r^l u|^2 dx, \quad \kappa' := \kappa_0 (R/R_0)^{d-1}$$

(3.47)

But clearly

$$\mu^{2l-1} \leq \left( \frac{c_d}{R_0^{d-1}(R - R_0)} \right)^2 |A|^{2l-1}$$

So finally, inequality (3.47) gives

$$|u(r) - P_{l,A}u(r)|^2 \leq \kappa'' |A|^{2l-1} \int_A |\partial_r^l u|^2 dx, \quad \kappa'' := \kappa' \left( \frac{c_d}{R_0^{d-1}(R - R_0)} \right)^{2l-1}$$

(3.48)

for all radially symmetric $u \in H^l(A)$. We will use this result as the basis for establishing a radial analogue of Theorem (4). Before we proceed with that, we need a lemma on functions of sets.

Let $\mathcal{J}$ be a nonnegative function of half-open annuli $A \subseteq \Omega$ (always taken to be centered at the origin), which is super-additive. That is, if an annulus $A$ is partitioned into finitely
many annuli \( \{ A_j \} \), then \( \sum_j J(A_j) \leq J(A) \). The \( J \) that we are interested in is

\[
J(A) := \int_A V(x) \left[ 1 + \log \left( \frac{R}{|x|} \right) \right]^{2l} dx
\]

For a partition \( \Xi = \{ A_j \} \) of \( \Omega \) into (half-open) annuli define

\[
G(J, \Xi) := \max_{A \in \Xi} |A|^{2l-1} J(A)
\]

Then by Theorem 1.5 in [12] we have

**Lemma 11.** For any natural number \( n \) there exists a partition \( \Xi \) of \( \Omega \) into (half-open) annuli such that

\[
\text{card}(\Xi) \leq n \quad \text{and} \quad G(J, \Xi) \leq C(l, d)n^{-2l} J(\Omega).
\]

The proof of this lemma is given in Appendix A. Let \( \Xi \) be the partition guaranteed by the lemma. For a set \( A \) let \( \chi_A \) denote its characteristic function. Define

\[
K_{\Xi, l} := \sum_{A \in \Xi} \chi_A P_{l, A}
\]

the operator of piecewise-polynomial approximation, where \( P_{l, A} \) is given in equation (3.45).

Note that \( \text{rank}(K_{\Xi, l}) \leq l \cdot \text{card}(\Xi) \leq ln \).

**Theorem 7.** With the above notation we have

\[
\int_{\Omega} V(x)|u - K_{\Xi, l} u|^2 dx \leq \kappa n^{-2l} J(\Omega) \int_{\Omega} |\partial_x^l u|^2 dx \tag{3.49}
\]

where \( \kappa = \kappa(d, l, R_0, R) \).
Proof: First
\[
\int_{\Omega} V(x) |u - K_{l,\Xi} u|^2 dx = \sum_{A \in \Xi} \int_A V(x) |u - P_{l,A} u|^2 dx
\]
\[
\leq \sum_{A \in \Xi} \sup_{x \in A} \frac{|u(x) - P_{l,A} u(|x|)|^2}{[1 + \log(R/|x|)]^{2i}} \cdot \int_A V(x) [1 + \log(R/|x|)]^{2i} dx
\]
\[
\leq \sum_{A \in \Xi} \sup_{x \in A} |u(x) - P_{l,A} u(|x|)|^2 \cdot J(A)
\]
where we have used \([1 + \log(R/|x|)]^{2i} \geq 1\) in the last inequality. So by (3.48)
\[
\int_{\Omega} V(x) |u - K_{l,\Xi} u|^2 dx \leq \kappa'' \sum_{A \in \Xi} |\partial_{r}^l u|^2 dx \cdot |A|^{2l-1} \cdot J(A)
\]
\[
\leq G(J, \Xi) \cdot \kappa'' \sum_{A \in \Xi} |\partial_{r}^l u|^2 dx
\]
\[
\leq \kappa n^{-2l} J(\Omega) \int_{\Omega} |\partial_{r}^l u|^2 dx
\]
where we used lemma (11) in the last line. 

On the space \(H_r = H_r(\Omega)\) endowed with the norm \((\int_{\Omega} |\nabla^l u|^2 dx)^{1/2}\) we consider the quadratic form
\[
b_V(u) := \int_{\Omega} V(x) |u(x)|^2 dx
\]
where \(V(x)\) is radial and integrable when weighted with the weight \([1 + \log(R/|x|)]^{2i}\). If \(b_V\) is bounded on \(H_r\), then it generates a bounded, self-adjoint, non-negative operator on \(H_r\), which we will denote by \(T_V\). By definition, for \(f \in H_r\)
\[
u = T_V f \iff u \in H_r;
\]
\[
\int_{\Omega} \nabla^l u \cdot \nabla^l w dx = \int_{\Omega} V f w dx, \forall w \in H_r
\]
As before we let

\[ n(\lambda; T_V) = N(-\lambda; -T_V) = \#\{\text{eigenvalues of } T_V \text{ that are } \geq \lambda\} \]

**Theorem 8.** Let \( V(x) \) and \( \Omega \) be as above. Then the operator \( T_V \) is well-defined and compact on \( H_r \), and there exists a constant \( C_4 = C_4(\Omega) \) such that for any \( \lambda > 0 \)

\[ n(\lambda; T_V)^2 \leq C_4 \lambda^{-1} \int_{\Omega} V(x) \left[ 1 + \log \left( R/|x| \right) \right]^{2i} dx \]

**Proof:** First note that \( b_V \) is bounded by Lemma 9. We will actually prove an upper bound for the eigenvalues of \( T_V \) and use this to derive the estimate for \( n(\lambda; T_V) \). Lemma 9 shows that \( b_V \) is bounded on \( H_r \), and hence \( T_V \) is bounded as well. Fix a natural number \( n \).

By Theorem (7) there exists a partition \( \Xi \) of \( \Omega \) into smaller annuli such that \( \text{card}(\Xi) \leq n \) and for any radially symmetric \( u \in H^l(\Omega) \) the estimate (3.49) holds. Let \( \mathcal{F} := \text{ker}(K_{\Xi,l}) \).

We have

\[ \text{codim}(\mathcal{F}) = \text{rank}(K_{\Xi,l}) \leq \ln n \]

For \( u \in \mathcal{F} \) we compute

\[ (T_V u, u)_{H^l_0(\Omega)} = b_V(u) = \int_{\Omega} V(x)|u|^2 dx \]

\[ = \int_{\Omega} V(x)|u - K_{\Xi,l} u|^2 dx \]

\[ \leq \kappa n^{-2l} J(\Omega) \int_{\Omega} |\nabla^l u|^2 dx \]

Where Theorem (7) was applied in the last line. This gives a bound of \( b_V \) on \( \mathcal{F} \), which gives a bound for \( T_V \) on \( \mathcal{F} \). Thus by taking \( n \to \infty \) we see that \( T_V \) is the norm-limit of finite rank
operators, hence compact. Also

\[
\max_{u \in F} \frac{(T_V u, u)_{H_0^1(\Omega)}}{(u, u)_{H_0^1(\Omega)}} = \max_{u \in F} \frac{\int_{\Omega} V(x)|u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx} \leq \kappa n^{-2l} \cdot J(\Omega) \tag{3.50}
\]

So by Courant’s Minimax Theorem for general symmetric compact operators we have

\[
\lambda_{nl+1} \leq \kappa n^{-2l} \cdot J(\Omega)
\]

where \( \lambda_j \) denotes the \( j \)-th eigenvalue of the positive operator \( T_V \) in \( H_r \). Hence there is a constant \( C = C(\Omega, l) \) such that

\[
\lambda_n \leq C(\Omega)n^{-2l} \cdot J(\Omega), \quad n = 1, 2, \ldots \tag{3.51}
\]

To prove the required estimate we proceed as follows

\[
n(\lambda; T_V) = \# \{ n : \lambda_n \geq \lambda \} \\
\leq \# \{ n : C(\Omega)J(\Omega)n^{-2l} \geq \lambda \} \\
= \# \{ n : n \leq C(\Omega)^{1/2l} J(\Omega)^{1/2l} \cdot \lambda^{-1/2l} \} \\
\leq C(\Omega)^{1/2l} J(\Omega)^{1/2l} \cdot \lambda^{-1/2l}
\]

raising both sides to the power \( 2l \) gives the result. ■

**Proof of Proposition 8:** To finish the proof of Proposition 8 we again use the Birman-Schwinger Principle with \( \mathcal{H} = L^2_{\text{rad}}(\Omega) \), \( D(a) = \tilde{D}(a) = H_r \), \( a(u) = \int_{\Omega} |\nabla u|^2 dx \), \( b(u) = \int_{\Omega} V|u|^2 dx \), and \( B = T_V \) to obtain

\[
N(0; (-\Delta)^l - V)^{2l} = n(1; T_V)^{2l} \leq C_4 J(\Omega).
\]
3.4 An estimate in the radially symmetric case when $d > 2l$

In this section we generalize a result in [9] by using the methods of this chapter. Namely, for $d > 2l$, let $\Omega = B_R(0)$ be the ball in $\mathbb{R}^d$ of radius $R$. With essentially no modification to the argument, we may consider an annulus with nonzero inner radius centered at the origin, as well. Denote by $L^2_r(\Omega)$ the set of radially symmetric, square integrable functions on $\Omega$. Let

$$H_r = H_r(\Omega) := \{ u \in H^1_0(\Omega) : u(x) = u(|x|) \text{ a.e. in } \Omega \}$$  \hspace{1cm} (3.52)

As usual, this space will be normed by

$$||u|| = ||u||_{H^1_0} = \sqrt{a(u)} = \left( \int_\Omega |\nabla^l u|^2 dx \right)^{1/2}.$$  

On the space $L^2_r(\Omega)$ we consider the unbounded linear operator $(-\Delta)^l - V(x)$. The precise definition is that, as before, it is the Friedrichs operator associated to the quadratic form

$$a_V(u) = a(u) - \int_\Omega V(x)|u|^2 dx = \int_\Omega |\nabla^l u|^2 dx - \int_\Omega V(x)|u|^2 dx$$

just as in (3.32), but acting on $L^2_r(\Omega)$ instead of $L^2(\Omega)$. Indeed, assuming appropriate conditions on the potential $V(x)$, we will see below that for any $\epsilon > 0$ inequality (3.33) holds. Therefore, as in section (3.2), this shows that $a_V$ is semi-bounded and closed on $L^2_r$. Then $(-\Delta)^l - V(x)$ is defined as the associated self-adjoint Friedrichs operator on $L^2_r(\Omega)$. The semi-boundedness and closedness of $a_V$ is also needed when we apply the Birman-Schwinger principle later on.
Finally let us denote by $N(0; (−Δ)^l − V(x))$ the number of non-positive eigenvalues, counted according to multiplicity, of $−Δ^l − V(x)$ in the space of radial functions $H_r(Ω)$. Then the main result of this section is

**Theorem 9.** Let $d > 2l$, and let $V(x) ≥ 0$ be a potential such that $V(x) = V(|x|)$ a.e., and $V(x)|x|^{2l−d} ∈ L^1(B_R(0))$. Then there exists a constant $C_{l,d}$ such that

$$N(0; (−Δ)^l − V(x)) ≤ C_{l,d} ∫_{B_R(0)} V(x)|x|^{2l−d} dx.$$  \(3.53\)

In order to prove this theorem we need a number of lemmas.

**Lemma 12.** Let $A ⊆ R^d$ be an annulus or a ball centered at the origin. Let $u ∈ H_r(A)$. If $d > 2l ≥ 2$ then

$$|u(x)| ≤ C_{l,d}||∂^l_r u||_{L^2(A)} · |x|^{l−d/2}$$

for $x ∈ A$.

**Proof:** Let $R_0$ denote the inner radius of $A$ and $R$ denote the outer radius of $A$. First, let’s consider the case $d > 2l = 2$. For simplicity we write $u = u(r)$ as a function of the radial variable. By a density argument we may assume $u ∈ C^∞_0(A)$. For $r ∈ [R_0, R]$

$$−u(r) = u(R) − u(r) = ∫_r^R u'(ρ)dρ$$
\[ |u(r)| \leq \int_r^R |u'(\rho)|d\rho \leq \left( \int_r^R |u'(\rho)|^2 \rho^{2-d}d\rho \right)^{1/2} \left( \int_r^R \frac{1}{\rho^{d-1}}d\rho \right)^{1/2} \leq C_d ||\partial_r u||_{L^2(A)} \cdot \left( \frac{R^{2-d}}{2-d} \right)^{1/2} \]

\[ = C_d ||\partial_r u||_{L^2(A)} \cdot \left( R^{2-d} - R^{2-d} \right)^{1/2} \leq C_d ||\partial_r u||_{L^2(A)} \cdot r^{1-d/2} \]

which proves the lemma in the case \( l = 1 \). Now assume \( d > 2l > 2 \) and again take \( u \in C_0^\infty \).

Then

\[ u(r) = u(r) - u(R) = - \int_r^R u'(\rho_1)d\rho_1 = \int_r^R u'(R) - u'(\rho_1)d\rho_1 = \int_r^R \int_{\rho_1}^R u''(\rho_2)d\rho_2d\rho_1 = \ldots = (-1)^l \int_r^R \int_{\rho_1}^R \ldots \int_{\rho_{l-1}}^R u^{(l)}(\rho_1)d\rho_1 \ldots d\rho_1 \]

So

\[ |u(r)| \leq \int_r^R \int_{\rho_1}^R \ldots \int_{\rho_{l-1}}^R |u^{(l)}(\rho_1)|d\rho_1 \ldots d\rho_1 \leq \int_r^R \int_{\rho_1}^R \ldots \int_{\rho_{l-2}}^R \left( \int_{\rho_{l-1}}^R |u^{(l)}(\rho_1)|^2 \rho_1^{d-1}d\rho_1 \right)^{1/2} \left( \int_{\rho_{l-1}}^R \rho_1^{1-d}d\rho_1 \right)^{1/2} \int_{\rho_{l-1}}^R \ldots d\rho_1 \leq C_d ||\partial_{\rho_1} u||_{L^2(A)} \int_r^R \int_{\rho_1}^R \ldots \int_{\rho_{l-2}}^R \left( \frac{\rho_1^{2-d}}{2-d} \right)^{R_{\rho_1-1}} \int_{\rho_{l-1}}^R \ldots d\rho_1 \]
\[ \leq C_d \| \partial_r^l u \|_{L^2(A)} \int_r^R \int_{\rho_1}^R \cdots \int_{\rho_{l-2}}^R \rho_{l-1}^{1-d/2} d\rho_{l-1} \cdots d\rho_1 \]
\[ \leq C_{l,d} \| \partial_r^l u \|_{L^2(A)} \int_r^R \rho_1^{1-d/2} d\rho_1 \]
\[ \leq C_{l,d} \| \partial_r^l u \|_{L^2(A)} \cdot r^{1-d/2} \]

which is the required result. \[\blacksquare\]

**Lemma 13.** Let \( u \in H^l(A) \), where \( A \) is an annulus or a ball centered at the origin in \( \mathbb{R}^d \), and \( u(x) = u(|x|) \) a.e in \( A \).

a) If \( d > 2l = 2 \) then
\[ |u(x) - \bar{u}_A| \leq C_d \| \partial_r u \|_{L^2(A)} \cdot |x|^{1-d/2} \]
for \( x \in A \), where \( \bar{u}_A \) is the average value of \( u \) in \( A \).

b) More generally, let \( d > 2l \geq 2 \). First define
\[ \tau_l(u)(r, s) := \sum_{n=0}^{l-1} \frac{u^{(n)}(s)}{n!} (r - s)^n \]
and
\[ P_{l,A}(u)(r) := \frac{1}{|A|} \int_A \tau_l(u)(r, |x|)dx \]
which is a polynomial in \( r \) of degree \( \leq l - 1 \) and linear in \( u \). Then
\[ |u(x) - P_{l,A}(u)(|x|)| \leq C_d \| \partial_r^l u \|_{L^2(A)} \cdot |x|^{l-d/2} \]
for \( x \in A \).
Proof: a) As before we assume \( u \in C^\infty(A) \). Then

\[
|u(r) - \bar{u}_A| = \left| \frac{1}{|A|} \int_A u(r) - u(x) dx \right| \\
\leq \frac{1}{|A|} \int_A |u(r) - u(x)| dx
\]

Now

\[
|u(r) - u(x)| \leq \int_{r,|x|} |u'(\rho)| \rho^{\frac{d-1}{2}} \rho^{\frac{1-d}{2}} d\rho
\]

where the notation \( \int_{a,b} \) denotes the positively oriented integration over the interval with endpoints \( a \) and \( b \). So

\[
|u(r) - u(x)| \leq \left( \int_{r,|x|} |u'(\rho)|^2 \rho^{d-1} d\rho \right)^{1/2} \left( \int_{r,|x|} \rho^{1-d} d\rho \right)^{1/2}
\]

\[
\leq C_d \| \partial_r u \|_{L^2(A)} |r^{2-d} - |x|^{2-d}|^{1/2}.
\]

Plugging this into the earlier inequality gives

\[
|u(r) - \bar{u}_A| \leq \frac{1}{|A|} \int_A C_d \| \partial_r u \|_{L^2(A)} |r^{2-d} - |x|^{2-d}|^{1/2} dx
\]

\[
= \frac{C_d}{|A|} \| \partial_r u \|_{L^2(A)} \int_{R_0} |r^{2-d} - \rho^{2-d}|^{1/2} \rho^{d-1} d\rho
\]
where $R_0$ and $R$ are the inner and outer radii of $A$. We estimate the value of the integral as

\[
\int_{R_0}^{R} |r^{2-d} - \rho^{2-d}|^{1/2} \rho^{d-1} d\rho = \int_{R_0}^{R} (\rho^{2-d} - r^{2-d})^{1/2} \rho^{d-1} d\rho + \int_{r}^{R} (r^{2-d} - \rho^{2-d})^{1/2} \rho^{d-1} d\rho
\]

\[
\leq \int_{R_0}^{r} \rho^{1-d/2} \rho^{d-1} d\rho + \int_{r}^{R} r^{1-d/2} \rho^{d-1} d\rho
\]

\[
\leq \int_{R_0}^{r} \rho^{d/2} d\rho + C_d' r^{1-d/2} |A|
\]

\[
\leq r^{d/2} (r - R_0) + C_d' r^{1-d/2} |A|
\]

\[
\leq C_d r^{1-d/2} |A|
\]

So

\[
|u(r) - \bar{u}_A| \leq C_d \|\partial_r u\|_{L^2(A)} r^{1-d/2}
\]

which is the required result.

b) For the general case we proceed similarly

\[
|u(r) - P_{l,A}(u)(r)| \leq \frac{1}{|A|} \int_A |u(r) - \tau_l(u)(r, |x|)| dx.
\]

Set $v(r) := u(r) - \tau_l(u)(r, |x|)$. Note that for $x$ fixed we have

\[
v^{(n)}(r)|_{r=|x|} = 0 \quad \text{for } 0 \leq n \leq l - 1
\]

where the differentiation is partial differentiation w.r.t. $r$. We seek to estimate $v(r)$ by
repeatedly applying this property.

\[ v(r) = v(r) - v(|x|) = \int_{|x|}^{r} v'(\rho_1)d\rho_1 \]
\[ = \int_{|x|}^{r} v'(\rho_1) - v'(|x|)d\rho_1 = \int_{|x|}^{r} \int_{|x|}^{\rho_1} v''(\rho_2)d\rho_2d\rho_1 \]
\[ = \int_{|x|}^{r} \int_{|x|}^{\rho_1} v''(\rho_2) - v''(|x|)d\rho_2d\rho_1 \]
\[ \vdots \]
\[ = \int_{|x|}^{r} \int_{|x|}^{\rho_1} \cdots \int_{|x|}^{\rho_l} v^{(l)}(\rho_l)d\rho_l \cdots d\rho_1 \]

thus

\[ |v(r)| \leq \int_{|x|,r} \int_{|x|,\rho_1} \cdots \int_{|x|,\rho_l} |v^{(l)}(\rho_l)|d\rho_l \cdots d\rho_1 \]

We apply Hölder’s inequality to the inner most integral

\[ |v(r)| \leq \int_{|x|,r} \int_{|x|,\rho_1} \cdots \int_{|x|,\rho_{l-2}} \left( \int_{|x|,\rho_{l-1}} |v^{(l)}(\rho_l)|^2\rho_l^{l-1}d\rho_l \right)^{1/2} \times \]
\[ \left( \int_{|x|,\rho_{l-1}} \rho_l^{1-d}d\rho_l \right)^{1/2} d\rho_{l-1} \cdots d\rho_1 \]
\[ \leq C_d \|\partial^l r u\|_{L^2(A)} \cdot \int_{|x|,r} \int_{|x|,\rho_1} \cdots \int_{|x|,\rho_{l-2}} |\rho_l^{2-d} - |x|^{2-d}|^{1/2}d\rho_{l-1} \cdots d\rho_1 \]

after using \( v^{(l)}(\rho_l) = \partial^l r u(\rho_l) \). Returning to the earlier estimate we get

\[ |u(r) - P_{l,A}(u)(r)| \leq \frac{C_d}{|A|} \|\partial^l r u\|_{L^2(A)} \times \]
\[ \int_{A} \int_{|x|,r} \int_{|x|,\rho_1} \cdots \int_{|x|,\rho_{l-2}} |\rho_l^{2-d} - |x|^{2-d}|^{1/2} d\rho_{l-1} \cdots d\rho_1 dx \]
As before the integral on the right hand side is bounded from above by $C_{l,d}|A|$. Hence

$$|u(r) - P_{l,A}(u)(r)| \leq C_{l,d}||\partial^l u||_{L^2(A)}r^{l-d/2},$$

which is the required result. ■

Let $A$ be an annulus centered at the origin (or a ball in the case that the inner radius is 0). Let $V(x)$ be as in the statement of Theorem 9. By lemma 13, for $u$ as in that lemma, we have

$$\sup_{x \in A} |u(x) - P_{l,A}(u)(|x|)|^2 \cdot |x|^{d-2l} \leq C_{l,d} \int_A |\partial^l u|^2 dx.$$ 

So

$$\int_A V(x)|u(x) - P_{l,A}(u)(|x|)|^2 dx \leq J(A) \cdot C_{l,d}||\partial^l u||_{L^2(A)}^2$$

where

$$J(A) := \int_A V(x)|x|^{2l-d} dx,$$

a notation we will keep for the rest of this section.

Let $\overline{\Omega} = \{R_0 \leq |x| \leq R\}$ be the closure of an annulus (or of a ball if $R_0 = 0$). We will apply the Besicovitch Covering lemma to coverings of $\overline{\Omega}$ by concentric annuli (or balls) centered at the origin. We note that such coverings are equivalent to coverings of the interval $[R_0, R]$ by subintervals of $\mathbb{R}$. We proceed as follows:

Given a finite interval $[R_0, R]$, let $\Xi$ be a covering of it by finite nontrivial (i.e. non-empty interior) intervals $I \subseteq \mathbb{R}$. Suppose that $\Xi$ can be split into $\tau$ subsets $\Xi_1, \ldots, \Xi_\tau$ in such a way that for each $k = 1, \ldots, \tau$ the set $\Xi_k$ contains only pairwise disjoint intervals. The smallest number $\tau$ for which such a partition of $\Xi$ is possible is called the linkage of $\Xi$, and is denoted
by $\text{link}(\Xi)$.

**Proposition 9** (One-Dimensional Besicovitch Covering Lemma). *Suppose that for any $r \in [R_0, R]$ we are given a nontrivial closed interval $I_r \subseteq \mathbb{R}$ centered at $r$. Then a finite subset $\Xi = \{I_r\}$ can be chosen so that $[R_0, R] \subseteq \bigcup I_r$ and $\text{link}(\Xi) \leq \tau_1$, where $\tau_1$ is an absolute constant.*

As before, we also need to define an operator of piecewise-polynomial approximation. Let $\Xi = \{I_j\}$ be a finite covering of the interval $[R_0, R]$ by nontrivial intervals. Let $A_j = \{x \in \mathbb{R}^d : |x| \in I_j\}$. Let $\chi_j$ denote the characteristic function of $A_j \setminus \bigcup_{i<j} A_i$. Let $P_{l,A_j}(u(|x|))$ be as in lemma 13. Then define

$$K_{\Xi,l}(u) := \sum_j \chi_j P_{l,A_j}(u).$$

Note that $\text{rank}(K_{\Xi,l}) \leq l \cdot \text{card}(\Xi)$. Then the following theorem on piecewise-polynomial approximation holds:

**Theorem 10.** *Let $d > 2l$, and let $\Omega = \{x \in \mathbb{R}^d : R_0 < |x| < R\}$ (or in case $R_0 = 0$, $\Omega = B_R(0)$). Let $V \geq 0$ be a potential such that $V(x) = V(|x|)$ a.e. and $V(x)|x|^{2l-d} \in L^1(\Omega)$. Let $u \in H^l(\Omega)$ with $u(x) = u(|x|)$ a.e. Then for any $n \in \mathbb{N}^+$ there exists a covering $\tilde{\Xi} = \tilde{\Xi}(V,n)$ of $[R_0, R]$ by subintervals and a constant $C_{l,d}$ such that

$$\text{card}(\tilde{\Xi}) \leq \tau_1 n$$

and

$$\int_{\Omega} V(x)|u - K_{\tilde{\Xi},l}u|^2 dx \leq C_{l,d} \frac{\mathcal{J}(\Omega)}{n} \int_{\Omega} |\partial^l_x u|^2 dx$$

(3.56)

where $\mathcal{J}(\Omega) := \int_{\Omega} V(x)|x|^{2l-d} dx$. 
Proof: Again, for a given annulus/ball $A$ we denote

$$J(A) = \int_A V(x)|x|^{2d-d}dx.$$ 

By replacing $V$ by $V + \epsilon$, and letting $\epsilon \to 0$ later in the proof, we may suppose for now that $V(x) \geq \epsilon$ instead of simply $V(x) \geq 0$ on $\Omega$. Fix an $n \in \mathbb{N}^+$. For each $r \in [R_0, R]$ we can find an interval $I_r \subseteq \mathbb{R}$ centered at $r$ such that

$$J(A_r \cap \Omega) = \frac{J(\Omega)}{n}$$

where $A_r = \{x \in \mathbb{R}^d : |x| \in I_r\}$. This possible is since $V(x)|x|^{2d-d} \in L^1(\Omega)$. Let $\Xi$ be the covering of $[R_0, R]$ by such intervals as selected according to Besicovitch’s lemma. Let $\Xi = \bigcup_k \Xi_k$, $1 \leq k \leq link(\Xi)$, be any partition as in that lemma. Then

$$\frac{J(\Omega)}{n} card(\Xi_k) = \sum_{I_r \in \Xi_k} J(A_r \cap \Omega) \leq J(\Omega).$$

Hence

$$card(\Xi_k) \leq n,$$

and so

$$card(\Xi) \leq n \cdot link(\Xi) \leq \tau_1 n. \tag{3.57}$$

The set $\tilde{\Xi} := \{I \cap [R_0, R] : I \in \Xi\}$ constitutes a covering of $[R_0, R]$. The corresponding set of annuli or balls $\tilde{A} = \{x \in \mathbb{R}^d : |x| \in \tilde{I} : \tilde{I} \in \tilde{\Xi}\}$, constitute a covering of $\Omega$.

Let $K_{\Xi, I}$ be the operator of piecewise-polynomial approximation, described earlier, corre-
CHAPTER 3. SPECTRAL ESTIMATES FOR SCHRÖDINGER OPERATORS

For any \( u \in H^l(\Omega) \), \( u(x) = u(|x|) \) a.e. we have

\[
\int_{\Omega} V(x)|u - K_{\tilde{\Xi},l}u|^2 dx \leq \sum_{\tilde{A}} \int_{\tilde{A}} V(x)|u - P_{l,\tilde{A}}u|^2 dx
\]

where the summation is over all \( \tilde{A} = \{x \in \mathbb{R}^d : |x| \in \tilde{I} \} \) for \( \tilde{I} \in \tilde{\Xi} \). We apply (3.55) to get

\[
\int_{\Omega} V(x)|u - K_{\tilde{\Xi},l}u|^2 dx \leq C_{l,d} \sum_{\tilde{A}} J(\tilde{\Omega}) \cdot \int_{\tilde{A}} |\partial_{l,r} u|^2 dx
\]

This calculation and (3.57) prove the theorem.

In order to prove the theorem we return to the case \( \Omega = B_R(0) \). On the space \( H_r(B_R(0)) \) consider the quadratic form

\[
b_V(u) := \int_{B_R(0)} V(x)|u(x)|^2 dx.
\]

By Lemma 12 \( b_V \) is bounded on \( H_r \). Thus it generates a bounded, self-adjoint, non-negative
operator on $H_r$, which we will denote by $T_V$. By definition, for $f \in H_r$,

$$ u = T_V f \iff u \in H_r; \quad \int_{\Omega} \nabla^l u \cdot \nabla^l w dx = \int_{\Omega} V f w dx, \quad \forall w \in H_r $$

As before we let

$$ n(\lambda; T_V) = N(-\lambda; -T_V) = \#\{\text{eigenvalues of } T_V \text{ that are } \geq \lambda\} $$

**Theorem 11.** Let $V$ be as above. Then the operator $T_V$ is compact on $H_r$, and there exists a constant $C_{l,d}$ such that for any $\lambda > 0$

$$ n(\lambda; T_V) \leq C_{l,d} \cdot \lambda^{-1} \int_{B_R(0)} V(x)|x|^{2l-d} dx $$

**Proof:** We will prove an upper bound on the eigenvalues of $T_V$ and use this to derive the required estimate for $n(\lambda; T_V)$. Fix a natural number $n$. By theorem 10 there exists a partition $\Xi$ of $[0, R]$ into subintervals such that $card(\Xi) \leq \tau_1 n$, and for radially symmetric $u \in H^l(B_R(0))$ the estimate (3.56) holds. Let $F := ker(K_{\Xi,l})$. We have

$$ codim(F) = rank(K_{\Xi,l}) \leq \tau_1 ln. $$

For $u \in F$ we compute

$$ \langle T_V u, u \rangle_{H^l_0} = \int_{B_R(0)} V|u|^2 dx = \int_{B_R(0)} V|u - K_{\Xi,l} u|^2 dx $$

$$ \leq C_{l,d} \cdot n^{-1} J(B_R(0)) \int_{B_R(0)} |\nabla^l u|^2 dx. $$

Note that by taking $n \to \infty$ we see that $T_V$ is the norm limit of finite rank operators, hence
is compact. Also

$$\max_{u \in F} \langle Tu, u \rangle_{H^1_0(\Omega)} = \max_{u \in F} \frac{\int_{\Omega} V(x)|u|^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} \leq C_{l,d} \cdot n^{-1} J(B_R(0)) \quad (3.58)$$

We apply Courant’s Minimax Principle, for general positive, symmetric, compact operators. Since \text{codim}(F) \leq \tau \ln n and since the eigenvalues of $T_V$ are decreasing we have

$$\lambda_{\tau \ln n + 1} \leq C_{l,d} \cdot n^{-1} J(B_R(0)) \quad \forall n \in \mathbb{N}^+,$$

where $\lambda_j$ denotes the $j$-th eigenvalue of the positive operator $T_V$ in $H_r$. Again since the eigenvalues of $T_V$ are decreasing there is another constant $C_{l,d}$ such that

$$\lambda_n \leq C_{l,d} \cdot n^{-1} J(B_R(0)) \quad n \in \mathbb{N}^+ \quad (3.59)$$

To prove the required estimate we proceed as follows

$$n(\lambda; T_V) = \# \{ n : \lambda_n \geq \lambda \}$$

$$\leq \# \{ n : C_{l,d} J(B_R(0)) n^{-1} \geq \lambda \}$$

$$= \# \{ n : n \leq C_{l,d} J(B_R(0)) \cdot \lambda^{-1} \}$$

$$\leq C_{l,d} J(B_R(0)) \lambda^{-1}$$

$$= C_{l,d} \lambda^{-1} \int_{B_R(0)} V(x)|x|^{2l-d} \, dx.$$  

\[\blacksquare\]

**Proof of Theorem 9:** To finish the proof we again use the Birman-Schwinger Principle with $\mathcal{H} = L^2_{\text{rad}}(B_R(0))$, $D(a) = \tilde{D}(a) = H_r(B_R(0))$, $a(u) = \int_{\Omega} |\nabla u|^2 \, dx$, $b(u) = \int_{\Omega} V|u|^2 \, dx$, \ldots
CHAPTER 3. SPECTRAL ESTIMATES FOR SCHRÖDINGER OPERATORS

\[ B = T_V, \text{ and } \alpha = 1 \text{ to obtain} \]

\[ N(0; (-\Delta)^l - V) = n(1; T_V) \leq C_{l,d} \mathcal{J}(B_R(0)). \]

■

**A final, extended remark:** As we remarked at the beginning of this section, theorem 9 holds for any finite ball \( B_R(0) \subset \mathbb{R}^d \) or any finite annulus \( A \subset \mathbb{R}^d \) centered at the origin. In particular, the constant \( C_{l,d} \) in Theorem 9 is independent of the inner and outer radii of the annulus or ball. In this remark we show that, with a modification, the result of Theorem 9 holds if the domain is taken to be all of \( \mathbb{R}^d \). The setup is as follows:

Let \( L_r^2(\mathbb{R}^d) \) denote the Hilbert space of radially symmetric, square integrable functions on \( \mathbb{R}^d \) and

\[ H_r = H_r(\mathbb{R}^d) := \{ u \in H^1(\mathbb{R}^d) : u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^d \}. \]

Note that since \( H^1(\mathbb{R}^d) = H^1_0(\mathbb{R}^d) \) this notation is consistent with that of (3.52). Let \( m > 0 \) be a number to be held fixed throughout. Define on \( H_r \) the quadratic form

\[ a(u) := \int_{\mathbb{R}^d} |\nabla^l u|^2 dx + m \int_{\mathbb{R}^d} |u|^2 dx. \]

This quadratic form gives a typical inner product and norm on \( H_r \), making it a Hilbert space. Let \( V(x) \geq 0 \) be a potential such that \( V(x) = V(|x|) \) a.e. and \( V(x)|x|^{2l-d} \in L^1(\mathbb{R}^d) \). Then the quadratic form

\[ b(u) := \int_{\mathbb{R}^d} V(x)|u|^2 dx \]

is well-defined and bounded on \((H_r, a(\cdot, \cdot))\). This follows from lemma 12, keeping in mind that the set of smooth, compactly supported, radially symmetric functions is dense in \( H_r(\mathbb{R}^d) \).
Thus $b(u)$ defines a bounded, non-negative, symmetric operator on $(H_r, a(\cdot, \cdot))$ which we denote by $T_V$. Let us also define the quadratic form

$$a_V(u) := a(u) - b(u).$$

(3.63)

In order to prove that $a_V$ is lower semibounded on $L^2_r(\mathbb{R}^d)$ it suffices to show that $b(u)$ has a zero-bound relative to $a(u)$. That is,

$$b(u) \leq \epsilon a(u) + C(\epsilon) \|u\|_{L^2(\mathbb{R}^d)}^2, \quad u \in H_r$$

for each $\epsilon > 0$. To prove this it suffices to prove that $T_V$ is a compact operator on $(H_r, a(\cdot, \cdot))$, analogously to what was done in theorem 11. The only special ingredient in the proof of theorem 11 was the use of theorem 10. Therefore we simply need to extend theorem 10 to the case where $\Omega = \mathbb{R}^d$, which we proceed to do now.

As before, for a given annulus/ball $A$ we denote

$$J(A) = \int_A V(x) |x|^{2l-d} dx.$$

Since $V(x)|x|^{2l-d} \in L^1(\mathbb{R}^d)$ then for a given $n \in \mathbb{N}^+$ there exists an $R = R(V, n) > 0$ such that if $A_0 := \{x \in \mathbb{R}^d : |x| > R\}$ then

$$J(A_0) \leq \frac{J(\mathbb{R}^d)}{n}.$$

As before, we need to define an operator of piecewise-polynomial approximation. The idea is that on $A_0$ the approximation is going to be identically 0, and inside the ball of radius $R$ it will be the same as it was in theorem 10. Let $\Xi = \{I_j\}_{j=1}^{\text{card}(\Xi)}$ be a finite covering of the interval $[0, R]$ by nontrivial intervals. Let $A_j = \{x \in \mathbb{R}^d : |x| \in I_j\}$, for $j \geq 1$. Let $\chi_j$ denote
the characteristic function of \( A_j \setminus \cup_{i<j} A_i \). Let \( P_{l,A_j}(u)(|x|) \) be as in lemma 13. Then define

\[
K_{\Xi,l} u(x) := \begin{cases} 
\sum_j \chi_j P_{l,A_j} u(x) & \text{if } x \in \mathbb{R}^d \setminus A_0, \\
0 & \text{if } x \in A_0.
\end{cases}
\]

Note that \( \text{rank}(K_{\Xi,l}) \leq l \cdot \text{card}(\Xi) \).

**Theorem 10'.** Let \( V(x) \geq 0 \) be a potential such that \( V(x) = V(|x|) \) a.e. and \( V(x)|x|^{2l-d} \in L^1(\mathbb{R}^d) \). Let \( R \) and \( A_0 \) be as above. Let \( u \in H^r(\mathbb{R}^d) \). Then for any \( n \in \mathbb{N}^+ \) there exists a covering \( \tilde{\Xi} = \tilde{\Xi}(V,n) \) of \([0,R]\) by subintervals and a constant \( C_{l,d} \) such that

\[
\text{card}(\tilde{\Xi}) \leq \tau_1 n
\]

and

\[
\int_{\mathbb{R}^d} V(x)|u - K_{\Xi,l} u|^2 dx \leq C_{l,d} \frac{\mathcal{J}(\mathbb{R}^d)}{n} \int_{\mathbb{R}^d} |\partial^j_x u|^2 dx.
\]

**Proof:** The idea is to split the integral as

\[
\int_{\mathbb{R}^d} V(x)|u - K_{\Xi,l} u|^2 dx = \int_{A_0} V(x)|u|^2 dx + \int_{\mathbb{R}^d \setminus A_0} V(x)|u - K_{\Xi,l} u|^2 dx.
\]

For the integral over the finite ball \( \mathbb{R}^d \setminus A_0 \) we choose the covering \( \tilde{\Xi} = \tilde{\Xi}(V,n) \) exactly as we did in theorem 10. Hence the estimate is given by (3.56) with \( \Omega = \mathbb{R}^d \setminus A_0 \). For the other integral we simply apply lemma 12 and the definition of \( A_0 \). Notice that in the proof of lemma 12 we only needed that \( u \) and its radial derivatives be 0 on the outer boundary of the annulus (which in the case of \( A_0 \) is at infinity), and not necessarily on the inner boundary.
Hence we have

\[
\int_{A_0} V(x)|u|^2 dx \leq C_{l,d} \mathcal{J}(A_0) \int_{A_0} |\partial^l u|^2 dx \\
\leq C_{l,d} \frac{\mathcal{J}(\mathbb{R}^d)}{n} \int_{A_0} |\partial^l u|^2 dx.
\]

This then allows us to carry out the same exact proof of theorem 11 when the domain is all of \( \mathbb{R}^d \) instead of a finite ball. Therefore \( T_V \) is a compact operator on \((H_r, a(\cdot,\cdot))\) and

\[
n(\lambda; T_V) \leq C_{l,d} \cdot \lambda^{-1} \int_{\mathbb{R}^d} V(x)|x|^{2l-d} dx.
\]

To reiterate, \( T_V \) being compact implies that \( b(u) \) has a zero-bound relative to \( a(u) \), which implies that \( a_V \) is closed and lower semibounded on \( L^2_r(\mathbb{R}^d) \). Therefore \( a_V \) defines an associated self-adjoint operator:

\[ (-\Delta)^l + m - V. \]

To finish we apply the Birman-Schinger Principle to the following data: \( \mathcal{H} = L^2_r(\mathbb{R}^d), D(a) = H_r, a(u) \) is as in (3.61), \( b(u) \) is as in (3.62), \( B = T_V \), and \( \alpha = 1. \) We get

\[ N(0; (-\Delta)^l + m - V) = n(1; T_V). \]

So

\[ N(0; (-\Delta)^l + m - V) \leq C_{l,d} \int_{\mathbb{R}^d} V(x)|x|^{2l-d} dx, \tag{3.64} \]

which is the analogue to theorem 9 that we were after. With a little more work we can actually prove the above for \( m = 0 \), but this will not be needed.
Appendices to Chapter 3

Appendix A: Proof of Lemma 11

We will follow the argument and notation of [12]. We will prove a result that has lemma 11 as a special case. To see why this is the case, we note that a partition of an annulus into smaller concentric annuli equates to a partition of an interval into subintervals. Instead of merely proving a theorem on the partitions of intervals into subintervals, we will be concerned with the higher dimensional analogue of partitions of cubes into subcubes with faces parallel to those of the original cube.

Let \( Q := [0, 1]^N \) denote the unit cube in \( \mathbb{R}^N \). Let \( J \) be a nonnegative function of half-open cubes \( \Delta \subseteq Q \) which satisfies the following super additivity property: if a cube \( \Delta \) is partitioned into finitely many disjoint cubes \( \Delta_j \), then

\[
\sum_{j} J(\Delta_j) \leq J(\Delta).
\]

We will denote the set of such functions \( J \) as \( \mathcal{J}^+ \). For a given \( a > 0 \), a given \( J \in \mathcal{J}^+ \), and a partition \( \Xi \) of \( Q \) into cubes we define the expression

\[
G_a(J, \Xi) := \max_{\Delta \in \Xi} |\Delta|^a J(\Delta).
\]

**Theorem A 1.** Suppose that \( J \in \mathcal{J}^+ \) and a natural number \( n \) are given. Then there exists
a partition $\Xi$ of $Q$ into cubes such that $\text{card}(\Xi) \leq n$ and

$$G_a(J, \Xi) \leq C_0 n^{-(a+1)} J(Q) \quad (a > 0)$$

where the constant $C_0 = C_0(N, a)$ does not depend on $J$.

The proof will require a preliminary lemma. First we may normalize so that $J(Q) = 1$. Suppose $\Xi$ is a partition of $Q$ into cubes with faces parallel to those of $Q$. We define a new partition as follows. Each $\Delta \in \Xi$ such that

$$|\Delta|^a J(\Delta) > 2^{-N} G_a(J, \Xi)$$  \hspace{1cm} (3.65)

is divided into $2^N$ equal cubes. This new partition is called an *elementary refinement* of $\Xi$.

Fix an initial partition $\Xi_0$ of $Q$. We shall construct a sequence of partitions $\{\Xi_i\}$ by induction, choosing each $\Xi_{i+1}$ to be an elementary refinement of $\Xi_i$. We denote by $n_i = \text{card}(\Xi_i)$ and $\delta_i = G_a(J, \Xi_i)$, $i = 0, 1, \ldots$. The number of cubes $\Delta \in \Xi_i$ which are divided when constructing $\Xi_{i+1}$ will be denoted by $\sigma_i$. This implies the following identity about the number of new cubes produced after a refinement:

$$n_{i+1} - n_i = (2^N - 1)\sigma_i$$  \hspace{1cm} (3.66)

for $i = 0, 1, \ldots$.

**Lemma A 1.** For any $J \in \mathcal{J}^+$ and any initial partition $\Xi_0$ the quantities $n_i$ and $\delta_i$ satisfy the inequality

$$\delta_i \leq \kappa(n_i - n_0)^{-(a+1)}, \quad i = 1, 2, \ldots,$$  \hspace{1cm} (3.67)

where

$$\kappa = (2^N - 1)^{a+1}(1 - 2^{-Na(a+1)^{-1}})^{-(a+1)}.$$
For the proof of this lemma we need the following two results:

**Lemma A 2.** Suppose a cube $\Delta$ is partitioned into $2^N$ equal cubes $\Delta_j$, $j = 1, 2, \ldots, 2^N$. Then
\[
\max_j |\Delta_j|^a J(\Delta_j) \leq 2^{-Na} |\Delta|^a J(\Delta).
\]

To see this simply note that
\[
|\Delta_j|^a J(\Delta_j) = 2^{-Na} |\Delta|^a J(\Delta_j) \leq 2^{-Na} |\Delta|^a J(\Delta).
\]

**Lemma A 3.** Let $\sigma$ be a positive natural number. Suppose that the numbers $x_j, y_j > 0$, $j = 1, \ldots, \sigma$ are given and that they satisfy
\[
\sum_j x_j \leq 1, \quad \sum_j y_j \leq 1, \quad \min_j x_j y_j^a \geq \eta
\]
for some $a > 0$. Then $\eta \leq \sigma^{-(a+1)}$.

This is not proven in [12], so we give an independent proof. In order to see how this lemma is true, let us note that it suffices to prove that
\[
\min_j \{ \log(x_j) + a \log(y_j) \} \leq -(a + 1) \log(\sigma).
\]

To prove this first we see that by the AM-GM inequality we have
\[
\frac{1}{\sigma} \sum_j \log(x_j) \leq \log \left( \frac{1}{\sigma} \sum_j x_j \right) \leq \log \left( \frac{1}{\sigma} \right).
\]

Similarly
\[
\frac{1}{\sigma} \sum_j a \log(y_j) \leq a \log \left( \frac{1}{\sigma} \right).
\]
Therefore

\[ \frac{1}{\sigma} \sum_j \log(x_j) + a \log(y_j) \leq -(a + 1) \log(\sigma). \]

Since the minimum is smaller than the average this implies

\[ \min_j \{\log(x_j) + a \log(y_j)\} \leq -(a + 1) \log(\sigma), \]

and hence the proof of the lemma.

**Proof of Lemma A 1:** From (3.65) and lemma A 2 it follows that

\[ \delta_i \leq 2^{-Na} \delta_{i-1}, \quad i = 1, 2, \ldots. \quad (3.68) \]

Referring to lemma A 3, we set \( \sigma = \sigma_i, x_j = J(\Delta_j) \) and \( y_j = |\Delta_j| \). Using \( |Q| = 1, J(Q) = 1 \), and (3.65) we see that the conditions in Lemma A 3 are satisfied for \( \eta = 2^{-Na} \delta_i \). Hence

\[ \delta_i \leq 2^{Na} \sigma_i^{-(a+1)}. \quad (3.69) \]

Let \( 0 \leq p < i \). From (3.68) and (3.69) it follows that

\[ \delta_i \leq 2^{-(i-p-1)Na} \sigma_p^{-(a+1)}, \]

or, equivalently,

\[ \sigma_p \leq 2^{-(i-p-1)Na(a+1)^{-1}} \sigma_i^{-(a+1)^{-1}}. \]
If we sum with respect to $p$ and use 3.66 we find that

$$n_i - n_0 \leq (2^N - 1)\delta_i^{-(a+1)^{-1}} \sum_{p=0}^{i-1} 2^{-(i-p-1)Na(a+1)^{-1}} < (2^N - 1)\left(1 - 2^{-Na(a+1)^{-1}}\right)^{-1} \cdot \delta_i^{-(a+1)^{-1}}.$$  

This inequality is equivalent to (3.67). ■

**Proof of Theorem A 1**: Let $\Xi_0$ be the trivial partition which contains just the cube $\Delta = Q$. Then $n_0 = 1$ and $\delta_0 = \mathcal{J}(Q) = 1$. Hence inequality 3.67 takes the form

$$\delta_i \leq \kappa(n_i - 1)^{-(a+1)}\mathcal{J}(Q).$$

Then, it follows,

$$\delta_i \leq 2^{a+1}\kappa n_i^{-(a+1)}\mathcal{J}(Q), \quad i = 0, 1, \ldots.$$  \hspace{1cm} (3.70)

Now let $n$ be an arbitrary natural number. Choose the integer $i$ for which $n_i \leq n < n_{i+1}$. For the partition $\Xi$ corresponding to $n$ choose $\Xi_i$. Then $\text{card}(\Xi) \leq n$. From (3.70) and the inequality $n \leq 2^N n_i$ we obtain

$$G_a(\mathcal{J}, \Xi) = \delta_i \leq 2^{(N+1)(a+1)}\kappa n_i^{-(a+1)}\mathcal{J}(Q).$$

Thus the theorem holds for $C_0 = 2^{(N+1)(a+1)}\kappa$. ■
Appendix B: The Birman-Schwinger Principle and its proof

Let $\mathcal{H}$ be a Hilbert space. Let $(\cdot, \cdot)$ and $||\cdot||$ be the inner product and norm on $\mathcal{H}$. Sometimes we add a subscript $\mathcal{H}$ to these two expressions to emphasize that they are the inner product or norm on $\mathcal{H}$. For us typically $\mathcal{H}$ is $L^2(\Omega)$ with its usual inner product.

Let $a(u, v)$ be a symmetric sesquilinear form on $\mathcal{H}$, and $a(u) := a(u, u)$ be the associated quadratic form. We denote by $D(a)$ the form domain of $a$, that is the set of $u \in \mathcal{H}$ for which $a(u)$ is finite. We will always assume that $a(u)$ is lower semibounded on $\mathcal{H}$. That is,

$$a(u) \geq m_a ||u||^2, \quad \forall u \in D(a),$$

(3.71)

for some $m_a \in \mathbb{R}$. We will also always assume that $a(u)$ is closed. That is, if $\{u_n\} \subseteq D(a)$ is such that $u_n \rightarrow u$ in $\mathcal{H}$ and $a(u_n - u_m) \rightarrow 0$ as $n, m \rightarrow \infty$ then $u \in D(a)$ and $\lim_{n \rightarrow \infty} a(u_n - u) = 0$.

In this case it is well-known that $a$ has an associated self-adjoint Friedrichs operator $A$ which is defined as follows

$$D(A) = \{u \in D(a) : \text{the linear functional } v \mapsto a(u, v) \text{ is } \mathcal{H}-\text{continuous}\}$$

and

$$(Af, g)_{\mathcal{H}} = a(f, g)$$

(3.72)

for all $f \in D(A)$ and $g \in D(a)$. For the details see, for example, section 5.5 of [55].

Since $a(u)$ is closed and lower semibounded, the space $D(a)$ along with the inner product $a(\cdot, \cdot) + (m_a + 1)(\cdot, \cdot)_{\mathcal{H}}$, form a Hilbert Space. Also, $D(A)$ is dense in $(D(a), a(\cdot, \cdot) + (m_a + 1)(\cdot, \cdot)_{\mathcal{H}})$. Later, when we take $a(\cdot, \cdot)$ to be positive definite in $\mathcal{H}$ we will not need the extra
term \((m_a + 1)(\cdot, \cdot)_H\), which was only included to make the inner product positive definite.

Let \(E_A\) denote the spectral measure of \(A\). Then it is well-known that

\[
a(u, v) = \int_{[m_a, \infty)} td(E_A(t)u, v)_H \quad \forall u, v \in D(a).
\]  

(3.73)

For example, see chapter 10, section 2, paragraph 1 of the book [13]. Also let us define

\[
N(\lambda; A) := \dim (E_A[m_a, \lambda]H), \quad \lambda \in \mathbb{R}.
\]  

(3.74)

Then we have the following result known as Glazman’s lemma

**Lemma B 1** (Glazman’s Lemma). Suppose \(a\) is a symmetric, lower semibounded, closed quadratic form on \(H\), with lower semibound \(m_a\). Let \(A\) be the associated self-adjoint Friedrichs operator. Then

\[
N(\lambda; A) = \dim \{ F \subseteq D(a), \text{ and } a(u) \leq \lambda \|u\|^2_H, \forall u \in F \}.
\]

**Proof:** Let \(H_\lambda := E_A[m_a, \lambda]H\). Since \(m_a\) and \(\lambda\) are finite

\[
H_\lambda \subseteq D(a).
\]

For \(u \in H_\lambda\) we have

\[
a(u) = \int_{[m_a, \lambda]} td(E_A(t)u, u) \leq \lambda \|u\|^2_H.
\]

Thus \(H_\lambda\) is one of the admissible linear subspaces \(F\). Consequently

\[
N(\lambda; A) = \dim H_\lambda \leq \sup \{ \dim F : F \subseteq D(a), \text{ and } a(u) \leq \lambda \|u\|^2_H, \forall u \in F \}.
\]

This proves the proposition in the case \(\dim H_\lambda = \infty\). Assume \(\dim H_\lambda < \infty\) and let \(F \subseteq D(a)\)
be a linear subset on which \( a(u) \leq \lambda ||u||^2 \), for all \( u \in F \).

If \( \dim F > \dim H_\lambda \) then there exists an element \( u_0 \in F, u_0 \neq 0 \), such that \( u_0 \perp H_\lambda \). Therefore we have

\[
a(u_0) = \int_{(\lambda, \infty)} td(E_A(t)u_0, u_0) > \lambda(u_0, u_0)
\]

which is a contradiction. Therefore \( \dim F \leq \dim H_\lambda \). ■

Suppose that \( a(u) > 0 \) for \( u \neq 0 \in D(a) \). So that in particular \( m_a \geq 0 \). Let \( b(u) \) be a non-negative symmetric quadratic form defined on \( D(a) \) such that

\[
b(u) \leq C a(u), \quad u \in D(a).
\]

Consider the Hilbert space \( \tilde{D}(a) \) which is defined as the completion of \( D(a) \) in the metric \( a(u) \).

If the form \( a(u) \) is positive definite (i.e. \( m_a > 0 \)) then since it is also closed we would have \( D(a) = \tilde{D}(a) \). In this case \( (D(a), a(\cdot, \cdot)) \) is a Hilbert space which embeds continuously into \( \mathcal{H} \).

**Examples:** In the following examples we always assume that \( \Omega \) is sufficiently smooth.

1) For us typically \( a \) is given by the Dirichlet energy

\[
a(u) = \int_{\Omega} |\nabla u|^2 dx
\]

and we may chose, for example, \( D(a) = H^1_0(\Omega) \). If \( \Omega \) is bounded then the Poincaré inequality implies that \( a(u) \) is positive definite on \( \mathcal{H} = L^2(\Omega) \).

2) In the above example take \( l = 1 \) and assume \( \Omega \) is bounded. Instead of \( H^1_0(\Omega) \) we can take \( D(a) = H^1(\Omega) := \{ u \in H^1(\Omega) : \int_{\Omega} u dx = 0 \} \). Then by the Poincaré-Wirtinger inequality
$a(u)$ is positive definite on $L^2(\Omega)$. The higher order version of this statement is analogous.

3) Let $\Omega = \mathbb{R}^d$. For some $m > 0$ consider the form

$$a(u) = \int_{\Omega} |\nabla^l u|^2 dx + m \int_{\Omega} |u|^2 dx$$

with $D(a) = H^l_0(\Omega) = H^l(\Omega)$. This form is clearly positive definite in $L^2(\Omega)$.

4) Let $\Omega = \mathbb{R}^d$ and $a(u) = \int_{\Omega} |\nabla^l u|^2 dx$ with $D(a) = H^l_0(\Omega) = H^l(\Omega)$. In this case $a(u)$ is not positive definite in $L^2(\Omega)$. Here $\tilde{D}(a) = \dot{H}^l(\mathbb{R}^d)$, the homogeneous Sobolev space. This is not a space of functions, and does not embed into $L^2(\Omega)$.

For simplicity we will stick to the case where $a(u)$ is positive definite in $\mathcal{H}$. So that $D(a) = \tilde{D}(a)$ is a Hilbert space with the inner product $a(\cdot, \cdot)$, and embeds continuously into $\mathcal{H}$. By (3.75) the form $b(u)$ is bounded on $(D(a), a(\cdot, \cdot))$, and hence defines a bounded, self-adjoint Friedrichs operator $B : (D(a), a(\cdot, \cdot)) \to (D(a), a(\cdot, \cdot))$. To be precise $B$ is defined as follows: for a fixed $u \in D(a)$ let

$$f_u(v) := b(u, v).$$

Then $f_u$ is a linear functional on $D(a)$ and

$$||f_u||_{D(a)^*} = \sup_{v \in D(a); a(v) \leq 1} |b(u, v)| \leq \sup_{v \in D(a); a(v) \leq 1} \sqrt{b(u)} \sqrt{b(v)} \leq C \sqrt{a(u)}$$

where we used (3.75). So $f_u$ is bounded in $(D(a), a(\cdot, \cdot))$ and hence by the Riesz representation theorem there is an element $\beta_u \in D(a)$ such that

$$f_u(\cdot) = a(\beta_u, \cdot)$$

satisfying $a(\beta_u) \leq C^2 a(u)$. Thus we defined $B : u \mapsto \beta_u$, which is evidently bounded.
Lemma B 2. If the operator $B$ defined above is a compact operator from $(D(a), a(\cdot, \cdot))$ into itself, then the form $b$ has a zero bound with respect to $a$ in $\mathcal{H}$. That is, for all $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that

$$b(u) \leq \epsilon a(u) + C(\epsilon)\|u\|_{\mathcal{H}}^2, \quad u \in D(a). \quad (3.76)$$

Proof: Let us denote by $\|\cdot\|_a := \sqrt{a(\cdot, \cdot)}$ the norm in $(D(a), a(\cdot, \cdot))$. Assume that $B$ is compact. Let $B := \{u \in D(a) : a(u) \leq 1\}$ be the unit ball in $D(a)$. Also, let $K := \overline{B(B)}$, where the closure is in the topology of $D(a)$. Since $B$ is a compact operator, $K$ is a compact subset of $D(a)$.

Therefore, for each $\epsilon > 0$ there exists a finite set $\{f_i\}_{i=1}^N \subset D(a)$ such that

$$K \subseteq \bigcup_{i=1}^N \{u \in D(a) : \|f_i - u\|_a \leq \epsilon/2\}. \quad (3.77)$$

Let $A$ denote the Friedrichs operator of $a$ in $\mathcal{H}$. Since $D(A)$ is dense in $(D(a), a(\cdot, \cdot))$ we can assume that $f_i \in D(A)$ for each $i$. Let $V := \langle \{f_j\}_{j=1}^N \rangle$ and $P_V$ be the orthogonal projection (in $a(\cdot, \cdot)$) onto $V$.

Now let $u \in B$ be given. Since $I - P_V$ is also a projection we have

$$\|u\|_a \leq 1 \implies \|(I - P_V)u\|_a \leq 1,$$

so that $(I - P_V)u \in B$. By (3.77) there exists an $f_{j(u)}$ such that

$$\|B(I - P_V)u - f_{j(u)}\|_a \leq \epsilon/2.$$
So

\[ b(u) = a(Bu, u) = a(BP_V u, u) + a(f_{j(u)}, u) + a(B(I - P_V)u - f_{j(u)}, u). \]

Hence

\[
\begin{align*}
    b(u) & \leq |a(BP_V u, u)| + |a(f_{j(u)}, u)| + \|B(I - P_V)u - f_{j(u)}\|_a \cdot \|u\|_a \\
        & \leq |a(BP_V u, u)| + |a(f_{j(u)}, u)| + \epsilon/2 \\
        & \leq \|BP_V u\|_a \cdot \|u\|_a + \|(Af_{j(u)}, u)_{\mathcal{H}}\| + \epsilon/2 \\
        & \leq \|BP_V u\|_a + \sup_j \|Af_j\|_{\mathcal{H}} \cdot \|u\|_{\mathcal{H}} + \epsilon/2. 
\end{align*}
\]

(3.78)

Now

\[ P_V u = \sum_{j=1}^N a(f_j, u)f_j. \]

So

\[ BP_V u = \sum_{j=1}^N a(f_j, u)Bf_j. \]

and thus

\[
\begin{align*}
    \|BP_V u\|_a & \leq N \sup_j |a(f_j, u)| \cdot \|Bf_j\|_a \\
                  & = N \sup_j \|(Af_j, u)_{\mathcal{H}}\| \cdot \|Bf_j\|_a \\
                  & \leq \left( N \sup_j \|Af_j\|_{\mathcal{H}} \cdot \|Bf_j\|_a \right) \|u\|_{\mathcal{H}}. 
\end{align*}
\]

Combining this with (3.78) gives

\[ b(u) \leq \epsilon/2 + \left( N \sup_j \|Af_j\|_{\mathcal{H}} \cdot \|Bf_j\|_a + \sup_j \|Af_j\|_{\mathcal{H}} \right) \|u\|_{\mathcal{H}}. \]
For a general $u$ we apply the above inequality to $\frac{u}{||u||_a}$. This gives

$$b(u) \leq \frac{\epsilon}{2} a(u) + \left( N \sup_j ||Af_j||_\mathcal{H} \cdot ||Bf_j||_a + \sup_j ||Af_j||_\mathcal{H} \right) ||u||_\mathcal{H} ||u||_a.$$ 

To finish the proof we apply the AM-GM inequality: $rs \leq \frac{r^2}{2} + \frac{1}{2\epsilon} s^2$, to the second term on the RHS above. We get

$$b(u) \leq \epsilon a(u) + C(\epsilon) ||u||^2_\mathcal{H},$$

where

$$C(\epsilon) = \frac{1}{2\epsilon} \left( N \sup_j ||Af_j||_\mathcal{H} \cdot ||Bf_j||_a + \sup_j ||Af_j||_\mathcal{H} \right)^2.$$

From now on we will assume that the form $b(u)$ is such that the inequality (3.75) is satisfied and that the operator $B$ is compact. Let us define on $D(a)$ the form

$$a_\alpha(u) := a(u) - \alpha b(u), \quad u \in D(a), \quad \alpha > 0. \quad (3.79)$$

Crucially this form is lower semibounded on $\mathcal{H}$. To see this we apply Lemma 2. Given $\epsilon > 0$ that lemma gives

$$a_\alpha(u) = a(u) - \alpha b(u) \geq (1 - \epsilon \alpha) a(u) - C(\epsilon) ||u||^2_\mathcal{H}.$$ 

Choosing $\epsilon > 0$ sufficiently small gives the required bound

$$a_\alpha(u) \geq -C_\alpha ||u||^2_\mathcal{H}.$$ 

This form is also closed in $\mathcal{H}$. Indeed suppose $\{u_n\}_{n=1}^\infty$ is a sequence in $D(a)$ such that
CHAPTER 3. SPECTRAL ESTIMATES FOR SCHRÖDINGER OPERATORS

$u_n \to \bar{u}$ in $\mathcal{H}$, and $a_\alpha(u_n - u_m) \to 0$. Then

$$a(u_n - u_m) = a_\alpha(u_n - u_m) + \alpha b(u_n - u_m)$$

$$\leq a_\alpha(u_n - u_m) + \epsilon a(u_n - u_m) + C(\epsilon)||u_n - u_m||^2_H$$

Choosing $\epsilon = 1/2$ gives

$$\frac{1}{2}a(u_n - u_m) \leq a_\alpha(u_n - u_m) + C||u_n - u_m||^2_H.$$

Therefore $a(u_n - u_m) \to 0$. Since we always assume that $a$ is a closed form we have that $\bar{u} \in D(a)$ and $\lim_{n \to \infty} a(u_n - \bar{u}) = 0$. Therefore, $\lim_{n \to \infty} a_\alpha(u_n - \bar{u}) = 0$, showing that $a_\alpha$ is closed in $\mathcal{H}$. Given these two facts we can define the Friedrichs operator $A_{\alpha b}$ associated to the form $a_\alpha$. We will apply Glazman’s Lemma to $A_{\alpha b}$. First, we can define the number

$$n(\lambda; B) := \#\{\text{eigenvalues of } B \text{ which are } \geq \lambda\}. \quad (3.80)$$

This number is finite since $B$ is compact. It corresponds to the dimension of the subspace of $(D(a), a(\cdot, \cdot))$ spanned by those eigenvectors of $B$ whose eigenvalues are $\geq \lambda$. Hence

$$n(\lambda; B) = \sup\{\dim F : F \subseteq D(a), b(u) \geq \lambda a(u), \forall u \in F\}. \quad (3.81)$$

In particular, for $\alpha > 0$

$$n(\alpha^{-1}; B) = \sup\{\dim F : F \subseteq D(a), b(u) \geq \alpha^{-1} a(u), \forall u \in F\}. \quad (3.82)$$

However, by applying Glazman’s Lemma to the operator $A_{\alpha b}$, this number is also equal to

$$N(0; A_{\alpha b}) = \sup\{\dim F : F \subseteq D(a), a(u) - \alpha b(u) \leq 0, \forall u \in F\}. \quad (3.83)$$
Therefore we arrive at

**Theorem B 1 (Birman-Schwinger Principle).** Suppose the form $a(u)$ is a symmetric, positive definite, closed quadratic form on the Hilbert space $\mathcal{H}$. Suppose the form $b(u)$ is a non-negative symmetric quadratic form satisfying (3.75), and is such that the corresponding operator $B : (D(a), a(\cdot, \cdot)) \to (D(a), a(\cdot, \cdot))$ is compact. Then the quadratic form

$$a_\alpha(u) := a(u) - \alpha b(u), \quad u \in D(a), \quad \alpha > 0$$

is lower semibounded and closed in $\mathcal{H}$. Its associated operator $A_{ab}$ satisfies

$$N(0; A_{ab}) = n(\alpha^{-1}; B).$$ (3.84)
Chapter 4

Applications to polyharmonic Dirichlet problems

In this chapter we will apply the machinery of previous chapters to the problem of finding infinitely many solutions to polyharmonic, semilinear Dirichlet problems. We will consider six such problems.

Common to the first three is that the order the equations also equals the dimension of the domain. In the notation below, \( d = 2l \) where \( d \) is the dimension, and \( l \) is the power of the Laplacian. Related to this fact, they all concern odd nonlinearities of exponential growth. Like all our applications, they also include perturbations which aren’t odd. First we seek weak solutions to

\[
\begin{cases}
(-\Delta)^l u = g(x, u) + \varphi(x) & \text{in } \Omega \\
\left(\frac{\partial}{\partial \nu}\right)^j u \bigg|_{\partial \Omega} = 0, & j = 0, \ldots, l - 1
\end{cases}
\]  

(P)
where the domain $\Omega \subset \mathbb{R}^{2l}$ has a smooth boundary. Here $\varphi \in L^2(\Omega)$, and $g(x,u)$ is an odd-in-$u$, exponential nonlinearity satisfying conditions (g1)-(g5) given below. A typical example to keep in mind would be $g(x,u) = u e^{\left|u\right|^\alpha}$, with $0 < \alpha < 1$. So if $\varphi = 0$ the problem would possess odd symmetry (i.e. $u$ is a solution if and only if $-u$ is a solution). Next in the case that $\Omega = A_{R_0}^R := \{x \in \mathbb{R}^{2l} : R_0 < |x| < R\}$ is an annulus, with $R_0 > 0$ and $R < +\infty$ we seek weak, radial solutions to

$$
\begin{cases}
(-\Delta)^l u = 2ue^{u^2} + \varphi(x,u) & \text{in } \Omega \\
\left(\frac{\partial}{\partial \nu}\right)^j u \bigg|_{\partial \Omega} = 0, \quad j = 0, \ldots, l - 1
\end{cases}
$$

where $\varphi(x,u) = \varphi(|x|,u)$ is not odd in $u$. For the third problem we seek weak, radial solutions to

$$
\begin{cases}
(-\Delta)^l u + \bar{b}|x|^{-2l}u = g(x,u) + \varphi(x) & \text{in } B_R(0) \\
\left(\frac{\partial}{\partial \nu}\right)^j u \bigg|_{\partial B_R(0)} = 0, \quad j = 0, \ldots, l - 1
\end{cases}
$$

where $B_R(0) \subset \mathbb{R}^{2l}$ denotes the ball of radius $R$ centered at the origin, and $\bar{b} > 0$ is a constant. Here $\varphi(x) = \varphi(|x|)$, $g(x,u) = g(|x|,u)$, and $g$ satisfies conditions (g1)-(g5) given below.

(g1) $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$

(g2) Given any constant $\sigma > 0$, $\exists$ constant $A_\sigma > 0$, such that

$$
|g(x,t)| \leq A_\sigma e^{\sigma |t|^2} \quad \forall (x,t) \in \bar{\Omega} \times \mathbb{R}
$$
(g3) Let $G(x, t) := \int_0^t g(x, s)ds$. There are constants $\mu > 0$ and $r_0 \geq 0$ such that

$$0 < G(x, t) \ln G(x, t) \leq \mu t g(x, t)$$

for $x \in \bar{\Omega}$, $|t| \geq r_0$.

(g4) $g(x, -t) = -g(x, t)$ for $(x, t) \in \bar{\Omega} \times \mathbb{R}$

(g5) There exists $0 < \alpha_1 \leq \alpha_2 < 1$, and $A_1, A_2, B_1$ such that

$$A_1 e^{|t|\alpha_1} - B_1 \leq G(x, t) \leq A_2 e^{|t|\alpha_2}$$

for $(x, t) \in \bar{\Omega} \times \mathbb{R}$.

Our first result is

**Theorem 12.** Suppose that $g$ satisfies conditions (g1)-(g5). Then if $2/\alpha_2 - 2 > 1/\alpha_1$, problem (P) has an unbounded sequence of solutions.

As a prototypical example we may take $g(x, u) := ue^{|u|\alpha}$. Then the above theorem asserts that for $0 < \alpha < 1/2$ problem (P) has an unbounded sequence of solutions. If we impose radial assumptions on the problem we can improve the result. In problem (R), let $\Phi(x, u) := \int_0^u \varphi(x, t)dt$

**Theorem 13.** Suppose that there exists a $\beta \in (0, 1)$ and a $C > 0$ such that $|\Phi(x, t)| + |\varphi(x, t)t| \leq C(t^2 e^{t^2})^\beta$ for sufficiently large values of $|t|$. If $2l > \frac{1}{1-\beta}$ then problem (R) has an unbounded sequence of radial solutions.

**Theorem 14.** Let $\Omega = B_R(0)$, be the open ball centered at the origin with a finite radius $R$. Suppose that, in addition to conditions (g1)-(g5), we have $g(x, u) = g(|x|, u)$ and $\varphi(x) = \varphi(|x|)$. Then if $2/\alpha_2 - 1 > 1/\alpha_1$, problem (H) has an unbounded sequence of radial solutions.
In the above problems the perturbation which destroys the symmetry of the problem takes the form of a function $\varphi$ entering into the equation. This function prevents the equation from being invariant under the transformation $u \mapsto -u$. However, there are many other perturbations breaking the symmetry, this being one of the simplest and most natural. Another very natural perturbation would be to perturb the boundary values, leaving open the case where

$$
\left. \left( \frac{\partial}{\partial \nu} \right)^j u \right|_{\partial \Omega}, \quad j = 0, \ldots, l - 1
$$

be non-zero. To approach this problem one makes the change of variable $u = v + \xi$. Here $\xi$ is a specially chosen function, and $v$ is the new unknown, which now has zero boundary values and hence is a member of the usual Sobolev space. One thinks of $\xi$ as the perturbation from zero boundary values. This is a much higher level perturbation than the one considered in the previous problems. It takes place everywhere $v$ is perturbed to $v + \xi$, particularly in the nonlinearity of the equation. The limits of the overall method described in this dissertation become even more apparent, and one must restrict the growth behavior of the nonlinearity in the equation even further. In section 4.4, we consider this problem with a power-type nonlinearity. Finally, in section 4.5, we consider a problem on $\mathbb{R}^d$, instead of on a bounded region $\Omega$.

### 4.1 Problem (P), the case of a general, bounded domain

#### 4.1.1 The variational setup

Let $\Omega \subset \subset \mathbb{R}^d$ be a smooth domain, where $d = 2l$. In the Hilbert space $L^2(\Omega)$ we consider
the dense subspace

\[ H_0^l(\Omega) := \text{the completion of } C_0^\infty(\Omega) \text{ in the following norm} \]

\[
||u||_{H_0^l(\Omega)} := \begin{cases} 
||\Delta^k u||_{L^2(\Omega)} & \text{if } l = 2k \\
||\nabla(\Delta^k u)||_{L^2(\Omega)} & \text{if } l = 2k + 1.
\end{cases}
\] (4.1)

Typically on the space \( H_0^l(\Omega) \) one might use the norm \( ||D^l u||_{L^2(\Omega)} \), after taking into account Poincare’s Inequality for the lower order terms. However, this norm is equivalent to \( ||u||_{H_0^l(\Omega)} \) on \( C_0^\infty(\Omega) \) by integration by parts. When convenient, we shall denote the norm \( ||u||_{H_0^l(\Omega)} \) by \( ||u|| \). As a shorthand for members of \( H_0^l(\Omega) \) define the \( l \)th power of the gradient as

\[
\nabla^l u = \begin{cases} 
\Delta^k u & \text{if } l = 2k \\
\nabla(\Delta^k u) & \text{if } l = 2k + 1.
\end{cases}
\] (4.2)

The variational setup for our problem is as follows. On the space \( H_0^l(\Omega) \) we consider the functional

\[
I_1(u) := \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx - \int_\Omega G(x, u) dx - \int_\Omega \varphi u dx.
\] (4.3)

This is the functional whose critical points correspond to generalized solutions of the boundary value problem

\[
\begin{cases} 
(-\Delta)^l u = g(x, u) + \varphi(x), & x \in \Omega \\
\left. \left( \frac{\partial}{\partial \nu} \right)^j u \right|_{\partial \Omega} = 0, & j = 0, \ldots, l - 1
\end{cases}
\] (P)

where \( G(x, u) = \int_0^u g(x, s) ds \), in the space \( H_0^l(\Omega) \). See [22].
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

For $I_1(u)$ to be well-defined on all of $H^l_0(\Omega)$, and for such a variational treatment to be viable, we must restrict the growth rate with respect to $u$ of the nonlinearity $G(x, u)$. The maximal growth rate for which such a variational treatment is viable is related to Adam's generalization of the Moser-Trudinger inequality. Namely, on the space $W^{l, \frac{d}{l}}(\Omega)$, $1 \leq l < d$, Adams showed

$$
\sup_{u \in W^{l, \frac{d}{l}}_0(\Omega), \|\nabla^l u\|_{\frac{d}{l}} \leq 1} \int_{\Omega} e^{\beta|u|^d} \, dx \begin{cases} 
\leq C|\Omega|, & \text{if } \beta \leq \beta(d, l) \\
= +\infty, & \text{if } \beta > \beta(d, l)
\end{cases}
$$

where $\beta(d, l)$ is given explicitly. For this classic result see [1]. In our case $d = 2l$, so the exponent $\frac{d}{d-1}$ equals 2. In this case, $\beta(2l, l) = l!(4\pi)^l$. It follows from the proof of (4.4) that if $u \in H^l_0(\Omega)$ then, for any $K > 0$,

$$
\int_{\Omega} e^{Ku^2} \, dx
$$

exists, and is a $C^1$ functional of $u$. This is not true if $e^{Ku^2}$ is replaced by a function of $u$ which grows substantially faster (more precisely, by one whose logarithm is super-quadratic). So for a variational treatment to be possible in the space $H^l_0(\Omega)$, $G(x, u)$ can’t grow faster than $e^{Ku^2}$ for all $K$.

Conditions (g1)-(g5) imply that $I_1(u)$ is a $C^1$ functional on $H^l_0(\Omega)$. As for the corresponding path of functionals to which we apply Bolle’s theorem, we simply consider

$$
I_\theta(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 \, dx - \int_{\Omega} G(x, u) \, dx - \theta \int_{\Omega} \varphi u \, dx
$$

where $\theta \in [0, 1]$, and for which $I_0$ is even.
4.1.2 Bolle’s Requirements

(H1) \( I \) satisfies the following analogue of the Palais-Smale Condition: For a sequence \( \{ (\theta_n, u_n) \}_{n \in \mathbb{N}} \) in \([0, 1] \times E\) such that \( ||I'_{\theta_n}(u_n)||_{E^*} \to 0 \) and \( |I_{\theta_n}(u_n)| \leq C \) there is a subsequence converging strongly in \([0, 1] \times E\).

**Proof**: Let \( \{ (\theta_n, u_n) \}_{n \in \mathbb{N}} \) be such a sequence. Then after taking a subsequence we may assume

\[
I_{\theta_n}(u_n) = \frac{1}{2} \int_\Omega |\nabla l u_n|^2 dx - \int_\Omega G(x, u_n) dx - \theta_n \int_\Omega \varphi u_n dx \to C \quad (4.6)
\]

where \( \theta_n \to \theta_0 \), and

\[
\left| \int_\Omega \nabla^l u_n \cdot \nabla^l v - g(x, u_n)v - \theta_n \varphi(x)v \right| \leq \epsilon_n ||v||_{H^l_0(\Omega)} \quad (4.7)
\]

for all \( v \in H^l_0(\Omega) \), where \( \epsilon_n \to 0 \) as \( n \to \infty \). Choosing \( v = u_n \) in (4.7) and rewriting (4.6) we get, for \( \bar{\mu} > 2 \)

\[
\frac{\bar{\mu}}{2} \int_\Omega |\nabla^l u_n|^2 dx - \bar{\mu} \int_\Omega G(x, u_n) dx - \bar{\mu} \theta_n \int_\Omega \varphi(x) u_n dx \leq C \quad (4.6')
\]

and

\[
-||u_n||^2_{H^l_0(\Omega)} + \int_\Omega g(x, u_n) u_n dx + \theta_n \int_\Omega \varphi(x) u_n dx \leq \epsilon_n ||u_n||_{H^l_0(\Omega)} \quad (4.8)
\]

Adding (4.6') and (4.8) gives

\[
\left( \frac{\bar{\mu}}{2} - 1 \right) ||u_n||^2 + \int_\Omega g(x, u_n) u_n dx - (1 - \bar{\mu}) \theta_n \int_\Omega \varphi(x) u_n dx \leq C' + \epsilon_n ||u_n|| \quad (4.9)
\]

By assumption (g3)
\[
\int_{\Omega} g(x, u_n) u_n - \bar{\mu} G(x, u_n) \, dx \geq -C
\]

Hence (4.9) gives

\[
\left( \frac{\bar{\mu}}{2} - 1 \right) ||u_n||^2 \leq C' + \epsilon_n ||u_n|| + C \int |\varphi(x) u_n| \, dx \leq C' + \epsilon_n ||u_n|| + C||u_n||_{L^2(\Omega)}.
\]

By the generalized Poincaré Inequality, if \( u \in H^1_0(\Omega) \) then there exists a constant \( C_0 > 0 \) such that

\[
||u||_{L^2(\Omega)} \leq C_0 ||u||_{H^1_0(\Omega)}.
\]

So we get

\[
\left( \frac{\bar{\mu}}{2} - 1 \right) ||u_n||^2 \leq C' + C||u_n||.
\]

Thus

\[
||u_n|| \leq K. \tag{4.10}
\]

Having proven the boundedness of Palais-Smale sequences, we will show that they are pre-compact by proving that \( I'_\theta(u)(\cdot) \) has the form \( L(u)(\cdot) + K(u)(\cdot) \) where \( L : H^1_0(\Omega) \to H^{-l}(\Omega) \) is an isomorphism, and \( K : H^1_0(\Omega) \to H^{-l}(\Omega) \) is compact. Although this isn’t entirely necessary, and a shorter proof which doesn’t rely explicitly on this fact is possible. However, the fact that \( I'_\theta(u) \) has this form will be needed later to apply Tanaka’s Theorem, hence we prove it now. Now

\[
I'_\theta_0(u)(\cdot) = \langle u, \cdot \rangle_{H^1_0(\Omega)} - \langle g(x, u), \cdot \rangle_{L^2(\Omega)} - \theta_0 \langle \varphi(x), \cdot \rangle_{L^2(\Omega)} \tag{4.11}
\]

Clearly \( L : H^1_0(\Omega) \to H^{-l}(\Omega) : u \mapsto \langle u, \cdot \rangle_{H^1_0(\Omega)} \) is the Riesz map, hence a Hilbert space iso-
morphism. Clearly the map $K_1 : H^l_0(\Omega) \to H^{-l}(\Omega) : u \mapsto \langle \varphi(x), \cdot \rangle_{L^2(\Omega)}$ is compact because it's a constant map. To show that $K_2 : H^l_0(\Omega) \to H^{-l}(\Omega) : u \mapsto \langle g(x, u), \cdot \rangle_{L^2(\Omega)}$ is compact it suffices to show that if $\{u_n\} \subset H^l_0(\Omega)$ is bounded then, up to a subsequence, $g(x, u_n)$ converges in $L^2(\Omega)$. Without loss of generality we may assume, after taking a subsequence, that

$$||u_n|| \leq K$$

$$u_n \rightharpoonup u \text{ weakly in } H^l_0(\Omega)$$

$$u_n \to u \text{ strongly in } L^p(\Omega), \ p \geq 1$$

$$u_n(x) \to u(x) \text{ a.e in } \Omega$$

Now since $g$ has subcritical growth in $u$ by (g2), we can find $C_K > 0$ such that

$$|g(x, t)| \leq C_K \exp \left( \frac{\beta(2l, l)}{2K^2} t^2 \right)$$

(4.12)

where $\beta(2l, l) = l!/(4\pi)^l$ is the optimal constant in Adam’s inequality. We apply Adam’s inequality:

$$||g(x, u_n)||^2_{L^2} \leq C_K \int_\Omega \exp \left( \frac{\beta(2l, l)}{K^2} |u_n|^2 \right) dx$$

$$\leq C_K \int_\Omega \exp \left( \frac{\beta(2l, l)}{||u_n||^2} |u_n|^2 \right) dx$$

$$\leq C'_K$$

Similarly we have

$$\int_\Omega |g(x, u_n)|^2 |u_n| dx \leq C''_K$$
To obtain the required result we use the following lemma

**Lemma 14.** Let \( \{u_n\} \) be a convergent sequence of functions in \( L^2(\Omega) \), with \( u_n(x) \to u(x) \) a.e. Assume that \( g(x,u_n) \) and \( g(x,u) \) are also in \( L^2(\Omega) \) with \( g(x,t) \) continuous in \( t \) uniformly in \( x \). If

\[
\int_{\Omega} |g(x,u_n)|^2 |u_n| dx \leq C_1
\]

then \( g(x,u_n) \) converges in \( L^2(\Omega) \) to \( g(x,u) \).

**Proof:** Note that since \( g(x,t) \) is continuous in \( t \) and \( u_n(x) \to u(x) \) a.e. then \( g(x,u_n(x)) \to g(x,u(x)) \) a.e. We have that

\[
|g(x,u_n(x)) - g(x,u(x))|^2 \leq \left[ |g(x,u_n(x))| + |g(x,u(x))| \right]^2 \\
\leq 2|g(x,u_n(x))|^2 + 2|g(x,u(x))|^2
\]

If we assume \( ||g(x,u_n)||_{L^2} \to ||g(x,u)||_{L^2} \) then

\[
\int_{\Omega} 2|g(x,u_n(x))|^2 + 2|g(x,u(x))|^2 dx \to 4 \int_{\Omega} |g(x,u(x))|^2 dx
\]

Also \( |g(x,u_n(x)) - g(x,u(x))| \to 0 \) a.e. So we can apply the Generalized Lebesgue Dominated Convergence Theorem and get that \( \int |g(x,u_n(x)) - g(x,u(x))|^2 dx \to 0 \), which is the required result. So it suffices to prove that \( \int |g(x,u_n)|^2 \to \int |g(x,u)|^2 dx \). Let \( f(x,t) := g(x,t)^2 \)

Since \( f(x,u(x)) \in L^1(\Omega) \) it follows that for a given \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
\int_A f(x,u(x)) dx \leq \epsilon \quad \text{if} \quad |A| \leq \delta \tag{4.13}
\]

for all measurable subsets \( A \subseteq \Omega \). Next using the fact that \( u \in L^1(\Omega) \) we find \( M_1 > 0 \) such that

\[
|\{x \in \Omega : |u(x)| \geq M_1\}| \leq \delta \tag{4.14}
\]
Let \( M := \max\{M_1, C_1/\epsilon\} \). We write
\[
\left| \int f(x, u_n(x))dx - \int f(x, u(x))dx \right| \leq I_1 + I_2 + I_3
\]
(4.15)
and estimate each integral separately:
\[
I_1 \equiv \int_{|u_n(x)| \geq M} f(x, u_n(x))dx = \int_{|u_n(x)| \geq M} \frac{|g(x, u_n(x))|^2}{|u_n(x)|} |u_n(x)|dx 
\leq \frac{C_1}{M} \leq \epsilon
\]
By the choices we have made above
\[
I_3 \equiv \int_{|u(x)| \geq M} f(x, u(x))dx \leq \epsilon
\]
Next we claim that
\[
I_2 \equiv \left| \int_{|u_n(x)| < M} f(x, u_n(x))dx - \int_{|u(x)| < M} f(x, u(x))dx \right| \to 0
\]
as \( n \to \infty \). Indeed, \( h_n(x) := f(x, u_n(x))\chi_{|u_n|<M} - f(x, u(x))\chi_{|u|<M} \) tends to 0 a.e. in \( \Omega \).
Moreover \( |h_n(x)| \leq |f(x, u(x))| \) if \( |u_n(x)| \geq M \) and \( |h_n(x)| \leq C + f(x, u(x)) \) if \( |u_n(x)| < M \).
So \( I_2 \to 0 \) as \( n \to \infty \) by the Lebesgue Dominated Convergence Theorem. 

Thus \( I' \) has the stated form and (H1) is satisfied. Indeed
\[
L(u_n) + K(u_n) = I'_\theta(u_n) = I'_\theta(u_n) + (\theta_n - \theta_0) \langle \varphi(x), \cdot \rangle_{L^2}
\]
So
\[
L(u_n) = -K(u_n) + O(1),
\]
and
\[ u_n = L^{-1}(-K(u_n) + O(1)) \]
which converges strongly up to a subsequence.

(H2) Here \( \frac{\partial}{\partial \theta} I_\theta(u) = -\int_\Omega \varphi(x) u(x) \, dx \) is bounded in absolute value by \( ||\varphi||_{L^2(\Omega)} ||u||_{L^2(\Omega)} \). By the Generalized Poincaré inequality this is bounded by \( C_\varphi ||u|| \).

(H3) Determining \( f_1, f_2 \)

Lemma 15. There exists a constant \( C > 0 \) such that if \( u \in H^1_0(\Omega) \) is a critical point of \( I_\theta \) then
\[
\left| \frac{\partial}{\partial \theta} I_\theta(u) \right| \leq C \left[ \ln(|I_\theta(u)| + 1) \right]^{1/\alpha_1} + C \tag{4.16}
\]

Proof: From (H2) above we have that
\[
\left| \frac{\partial}{\partial \theta} I_\theta(u) \right| \leq C ||u||_{L^2(\Omega)}. \]
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

So it suffices to estimate $\|u\|_{L^2(\Omega)}$. Assume $I'_{\theta}(u) = 0$. Then

\[
I_{\theta}(u) = I_{\theta}(u) - \frac{1}{2}\langle I'_{\theta}(u), u \rangle \\
= \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} G(x, u) dx - \theta \int_{\Omega} \varphi u dx \\
- \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx + \int_{\Omega} \frac{1}{2} g(x, u) u dx + \frac{\theta}{2} \int_{\theta} \varphi u dx \\
= \int_{\Omega} \frac{1}{2} g(x, u) u - G(x, u) dx - \frac{\theta}{2} \int_{\theta} \varphi u dx
\]  

(4.17)

We now apply condition (g3). When $|u(x)| > r_0$ we bound

\[
\frac{1}{2} g(x, u) u - G(x, u) \geq \frac{1}{2\mu} G(x, u) \ln[G(x, u)] - G(x, u) \\
\geq \left( \frac{1}{2\mu} - \epsilon \right) G(x, u) \ln[G(x, u)] - C
\]

When $|u(x)| \leq r_0$ the expression $\frac{1}{2} g(x, u) u - G(x, u)$ is bounded by a constant since $g$ and $G$ are continuous. Since $\Omega$ is of finite measure we get from the above and equation (4.17)

\[
I_{\theta}(u) \geq \left( \frac{1}{2\mu} - \epsilon \right) \int_{|u(x)| \geq r_0} G(x, u) \ln[G(x, u)] dx - C \|u\|_{L^2(\Omega)} - C
\]

Now applying the growth condition (g5) and again the fact that $\Omega$ is of finite measure

\[
I_{\theta}(u) \geq C \int_{\Omega} |u|^\alpha e^{\beta |u|} dx - C \|u\|_{L^2(\Omega)} - C
\]

(4.18)

Observe that for $\alpha, \beta > 0$ there exists a constant $t_0 = t_0(\alpha, \beta)$ such that the function $t^\beta e^{\alpha t}$ is convex for $t \geq t_0$. We take $\alpha = \beta = \alpha_1/2$, and apply Jensen’s inequality

\[
C \int_{|u| \geq t_0^{1/2}} |u|^{\alpha_1} e^{\beta |u|} dx \geq \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \exp \left[ \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right]
\]

(4.19)
Also note that
\[ ||u||_{L^2(\Omega)}^{\alpha_1} \leq C + \left\{ \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \] (4.20)

Hence we proceed as follows

\[ ||u||_2^{\alpha_1} \exp \left[ \left( \frac{1}{|\Omega|} \right)^{\alpha_1/2} ||u||_2^{\alpha_1} \right] \leq C + \left\{ \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \exp \left[ C + \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right] \]

\[ \leq C + C \left\{ \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \exp \left[ \left\{ \frac{1}{|\Omega|} \int_{|u| \geq t_0^{1/2}} |u|^2 dx \right\}^{\alpha_1/2} \right] \]

Here apply (4.19)

\[ \leq C + C \int_{|u| \geq t_0^{1/2}} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx \]
\[ \leq C + C \int_{\Omega} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx \] (4.21)

Inequality (4.21) now implies

\[ ||u||_{L^2} \leq C \left\{ \ln \left( \int_{\Omega} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx + 1 \right) \right\}^{1/\alpha_1} + C \] (4.22)

So (4.22) and (4.18) give

\[ I_\theta(u) \geq C \int_{\Omega} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx - \left\{ \ln \left( \int_{\Omega} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx + 1 \right) \right\}^{1/\alpha_1} - C \]
\[ \geq C \int_{\Omega} |u|^{\alpha_1} e^{||u||_{L^2}^{\alpha_1}} dx - C \] (4.23)
So by (4.21) and (4.23)

\[ I_\theta(u) \geq C \|u\|_2^{\alpha_1} \exp \left( \left( \frac{1}{|\Omega|} \right)^{\alpha_1/2} \|u\|_2^{\alpha_1} \right) - C \]

that is,

\[ \|u\|_2 \leq C \left[ \ln(|I_\theta(u)| + 1)^{1/\alpha_1} + C \right] \]

(4.24) which proves the lemma. ■

Thus we may take

\[ f_i(\theta, t) = f_i(t) = (-1)^i C \left\{ \ln(|t| + 1)^{1/\alpha_1} + 1 \right\} \]

(4.25)

\((\text{H4})\) This condition is easily satisfied by assumption (g5), which shows that \(G(x,u)\) is super-quadratic (uniformly in \(x\)) and tends to \(+\infty\) as \(|u| \to \infty\).

4.1.3 The Alleged Upper Bound

As noted earlier, we will operate under the assumption that alternative 2) of Theorem 2 holds for sufficiently large \(n\). That is, for \(n > n_0\), it will be assumed that

\[ c_{n+1} - c_n \leq K \left[ (\ln(c_{n+1}))^{1/\alpha_1} + (\ln(c_n))^{1/\alpha_1} + 1 \right]. \]

(4.26)

We will show this implies that \(c_n \leq An[\ln(n)]^{1/\alpha_1}\) for sufficiently large \(n\), and for some con-
stant $A$ to be chosen appropriately. Later, we shall derive a contradiction to this estimate, proving that alternative 1) of Theorem 2 must in fact hold for infinitely many $n$.

Let $\gamma := 1/\alpha_1$ and let $b_n := A[n\ln(n)]^\gamma$. First we can choose $A > 0$ so large that $c_{n_0} < b_{n_0}$ where $n_0$ is large and fixed. For $n > n_0$

\[ b_{n+1} - b_n = A(n+1)[\ln(n+1)]^\gamma - A[n\ln(n)]^\gamma \
= A[\ln(n+\theta)^\gamma + \gamma \ln(n+\theta)^{\gamma-1}] \]

for some $\theta \in [0, 1]$ by the Mean Value Theorem. Hence

\[ b_{n+1} - b_n \geq \frac{A}{2} [\ln(n)]^\gamma. \]  

(4.27)

Now from the definition of $b_n$ we compute

\[ K[\ln(b_{n+1})^\gamma + \ln(b_n)^\gamma + 1] \leq C_\gamma K \ln(n)^\gamma + C_\gamma K \ln(A) \]  

(4.28)

for $n > n_0$ sufficiently large. So we take $A >> 2C_\gamma K$. Then (4.27) and (4.28) combine to give

\[ b_{n+1} - b_n > K[\ln(b_{n+1})^\gamma + \ln(b_n)^\gamma + 1]. \]

This is the reverse of the inequality satisfied by $c_n$. We already have $b_{n_0} \geq c_{n_0}$. Assume that
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

\( b_i > c_i \) for \( i = n_0, \ldots, n \). We will show that \( b_{n+1} \geq c_{n+1} \):

\[
\begin{aligned}
 b_{n+1} - c_{n+1} &= b_{n+1} - b_n - (c_{n+1} - c_n) + (b_n - c_n) \\
 &\geq b_{n+1} - b_n - (c_{n+1} - c_n) \\
 &\geq K\left[ \ln(b_{n+1})^\gamma + \ln(b_n)^\gamma + 1 \right] - K\left[ \ln(c_{n+1})^\gamma + \ln(c_n)^\gamma + 1 \right] \\
 &= K \ln(b_{n+1})^\gamma - K \ln(c_{n+1})^\gamma + [K \ln(b_n)^\gamma - K \ln(c_n)^\gamma] \\
 &\geq K \ln(b_{n+1})^\gamma - K \ln(c_{n+1})^\gamma. 
\end{aligned}
\]  

(4.29)

Assume that \( b_{n+1} < c_{n+1} \). Then

\[
K \ln(b_{n+1})^\gamma - K \ln(c_{n+1})^\gamma = -K \gamma \int_{b_{n+1}}^{c_{n+1}} \frac{\ln(t)^{\gamma-1}}{t} dt \\
> b_{n+1} - c_{n+1},
\]

where we have used the fact that \( -K \gamma \frac{\ln(t)^{\gamma-1}}{t} > -1 \) for \( t > b_{n+1} \geq b_{n_0} \), when \( n_0 \) is taken to be sufficiently large. This contradicts (4.29), and so \( c_{n+1} \leq b_{n+1} \). Thus by induction \( c_n \leq b_n \) for all \( n > n_0 \), i.e.

\[
c_n \leq A n \left[ \ln(n) \right]^{1/\alpha_1} \quad \text{for } n > n_0,
\]

(4.31)

when assuming alternative 2) of Theorem 2.

4.1.4 The requirements for applying Tanaka’s theorem

We need to show that we can apply Tanaka’s theorem to our problem. For the moment,
we are concerned with \( I_0(u) \)

\[
I_0(u) = \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} G(x, u) dx
\]

We already know from the proof of the Palais-Smale condition that \( I_0' \) has the form of a compact perturbation of a Hilbert Space isomorphism. Actually we will apply Tanaka’s Theorem to a slightly smoother functional:

\[
J(u) := \frac{1}{2} \int_{\Omega} |\nabla^l u|^2 dx - \int_{\Omega} H(u) dx + C_H
\]  \hspace{1cm} (4.32)

where

\[
H(t) = a \exp[(t^2 + 1)^b],
\]

\( b = \alpha_2/2 \), and where \( C_H := H(0)|\Omega| \) is a constant chosen so that \( J(0) = 0 \). By assumption (g5) we can choose \( a > 0 \) so that

\[
G(x, t) \leq H(t) \quad \text{for} \quad (x, t) \in \bar{\Omega} \times \mathbb{R}
\]

Thus \( I_0(u) \geq J(u) - C_H \).

\( J(u) \) has a nonlinearity of subcritical and super-quadratic growth, and so all compactness properties of \( I_0(u) \) also hold for \( J(u) \). In particular \( J' \) has the form \( L + K \) where \( L : H^l_0(\Omega) \to H^{-l}(\Omega) \) is an isomorphism, and \( K : H^l_0(\Omega) \to H^{-l}(\Omega) \) is compact. In addition we have the following compactness conditions needed in the application of Tanaka’s theorem: Let \( \{E_j\} \) be the decomposition in equation (2.15).

\((PS)_m\) If for some \( M > 0 \), \( \{u_j\} \) satisfies

\[
u_j \in E_m, \quad J(u_j) \leq M \quad \forall j, \quad \|(J|_{E_m})'(u_j)\|_{E_m} \to 0 \quad \text{as} \quad j \to \infty
\]
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

then \( \{ u_j \} \) is precompact.

\((PS)_*\) If for some \( M > 0 \), \( \{ u_j \} \) satisfies

\[
  u_j \in E_j, \quad J(u_j) \leq M \quad \forall j, \quad ||(J|_{E_j})'(u_j)||_{E_j'} \to 0 \quad \text{as } j \to \infty
\]

then \( \{ u_j \} \) is precompact.

We will verify \((PS)_*\). The condition \((PS)_m\) follows from the boundedness of such a sequence, and the Heine-Borel Theorem. (Recall that in the proof of the Palais-Smale condition we did not need \( |I_0(u_n)| \leq C \), but only the weaker statement that \( I_0(u_n) \leq C \) to prove that the sequence was bounded. See equation (4.6'). This proves \((PS)\) and \((PS)_m\).) Let \( \{ u_j \} \) be a sequence such that \( u_j \in E_j, \ J(u_j) \leq M, \) and \( ||(J|_{E_j})'(u_j)||_{E_j'} \to 0 \). Now with \( u \) and \( v \) restricted to \( E_j \)

\[
  (J|_{E_j})'(u)v = J'(u)v. \tag{4.33}
\]

As in the verification of (H1) earlier, by taking \( u = v = u_j \) we see that \( \{ u_j \} \) is bounded in \( E = H_0^1(\Omega) \). Now

\[
  (J|_{E_j})'(u)v = J'(u)v = L(u)v + K(u)v, \quad u, v \in E_j. \tag{4.34}
\]

Let \( L_j : E_j \to E_j' : L_j(u)v := L(u)v \) for \( u, v \in E_j \). Similarly let \( K_j : E_j \to E_j' : K_j(u)v := K(u)v \) for \( u, v \in E_j \).

Now

\[
  \xi_j := (J|_{E_j})'(u_j) \to 0,
\]
and we have
\[ L_j(u_j) = \xi_j - K_j(u_j). \] (4.35)

Extend \( K_j(u_j) \) and \( \xi_j \) from elements of \( E_j^\prime \) to elements of \( H^{-l} \) by the following formulas and linearity
\[
\bar{K}_j(u_j)v := \begin{cases} 
K_j(u_j)v & \text{if } v \in E_j \\
0 & \text{if } v \in E_j^\perp
\end{cases}
\]
and
\[
\bar{\xi}_j v := \begin{cases} 
\xi_j v & \text{if } v \in E_j \\
0 & \text{if } v \in E_j^\perp
\end{cases}
\]
We claim
\[ L(u_j)v = \bar{\xi}_j v - \bar{K}_j(u_j)v, \quad \forall v \in E \] (4.36)

If \( v \in E_j \) then equation (4.36) is just equation (4.35). If \( v \in E_j^\perp \) both sides are 0 since, using the fact that \( u_j \in E_j \), we have
\[ L(u_j)v = \langle u_j, v \rangle_{H_0^l(\Omega)} = 0 \]
So we get
\[ L(u_j) = \bar{\xi}_j - \bar{K}_j(u_j) \] (4.37)
as an equation in \( H^{-l} \). So
\[ u_j = L^{-1}[\bar{\xi}_j - \bar{K}_j(u_j)] \] (4.38)
Clearly \( ||\bar{\xi}_j|| \rightarrow 0 \). Let \( P_j : H_0^l(\Omega) \rightarrow H_0^l(\Omega) \) be the orthogonal projection onto \( E_j \) (which is finite dimensional). Then
\[ \bar{K}_j(u_j) = K(u_j) \circ P_j \]
Since \( \{u_j\} \) is bounded and \( K \) is a compact operator, we can assume that \( K(u_j) \to \eta \in H^{-l} \).

So

\[
||K(u_j)P_j - \eta P_j|| \to 0
\]

Hence to show that \( \bar{K}_j(u_j) = K(u_j)P_j \) converges it’s sufficient to show that \( \eta P_j \) converges.

By the Riesz representation theorem we can find an \( e \in H_0^l(\Omega) \) such that \( \eta(v) = \langle e, v \rangle_{H_0^l} \) for all \( v \in H_0^l(\Omega) \). Then

\[
||\eta - \eta P_j|| = \sup_{v \in H_0^l \atop ||v|| \leq 1} |\eta(I - P_j)v| \\
= \sup_{v \in H_0^l \atop ||v|| \leq 1} |\langle e, (I - P_j)v \rangle_{H_0^l}| \\
= \sup_{v \in H_0^l \atop ||v|| \leq 1} |\langle (I - P_j)e, v \rangle_{H_0^l}| \\
\leq ||(I - P_j)e||_{H_0^l(\Omega)} \to 0
\]

(4.39)

as \( j \to \infty \). Hence \( \bar{K}_j(u_j) \to \eta \) as \( j \to \infty \). Thus we have that \( u_j = L^{-1}[\bar{\xi}_j - \bar{K}_j(u_j)] \) converge up to a subsequence. and so \((PS)_*\) is satisfied.

### 4.1.5 Applying Tanaka’s Theorem: The lower bound

The goal is to obtain a lower bound for \( c_n \) that will contradict (4.31). For \( J(u) \) define the symmetric minimax levels

\[
b_n := \inf_{g \in \Gamma_n} \sup_{u \in g(E_n)} J(u)
\]

(4.40)

Since \( J(u) - C_H \leq I_0(u) \) by construction, we have \( b_n - C_H \leq c_n \). (Recall that \( C_H = H(0)|\Omega| \)).

So it will suffice to obtain a good lower bound on \( b_n \). By Tanaka’s theorem, there exists a
sequence $u_n$ such that

i) $J(u_n) \leq b_n$

ii) $J'(u_n) = 0$

iii) $n \leq index_0 J''(u_n)$

Where the extended Morse index $index_0 J''(u)$ is the dimension of the maximal, negative semidefinite subspace corresponding to the form $J''(u)$. For simplicity we simply denote $u_n$ as $u$, holding $n$ fixed for the time being. Now

$$J''(u)(v, w) = \langle v, w \rangle_{\mathcal{H}_0^2(\Omega)} - \int_{\Omega} H''(u) v w \, dx$$

One basis for the maximal negative semidefinite subspace of this bilinear form is the set of eigenfunctions of $(-\Delta)^l - H''(u)$ with non-positive eigenvalues. So

$$index_0 J''(u) = \text{number of non-positive eigenvalues of } (-\Delta)^l - H''(u) \text{ on } L^2(\Omega),$$

where eigenvalues are counted according to multiplicity. By applying the corollary to Theorem 6 we get

$$index_0 J''(u) \leq C \int_{\Omega} \mathcal{B}(H''(u(x))) \, dx + C \quad (4.41)$$

So by Tanaka’s theorem

$$n \leq C \int_{\Omega} \mathcal{B}(H''(u_n(x))) \, dx + C$$

where we take $n$ sufficiently large. Since $\Omega$ is of finite measure, the exact form of $H''(u)$ isn’t important, only that it behaves like $(|u| + 1)^{2\alpha - 2} e^{(u^2 + 1)^{\frac{h}{2}}} - 2e^{(u^2 + 1)^{\frac{h}{2}}}$ for $|u|$ large. So that for some
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

$C > 0$

$$
\mathcal{B}(H''(u)) \leq C(|u| + 1)^{3\alpha_2 - 2}e^{(u^2+1)b}
$$

So

$$
\frac{n}{C} \leq \int_\Omega (|u| + 1)^{3\alpha_2 - 2}e^{(u^2+1)b}dx
$$

Since $u = u_n$ is a critical point of $J$

$$
J(u) = A\int_\Omega [bu^2(u^2 + 1)^b - 1]e^{(u^2+1)b}dx
\geq C\int_\Omega (|u| + 1)^{\alpha_2}e^{(u^2+1)b}dx - C \tag{4.42}
$$

Let $\tau = (|u| + 1)^{\alpha_2}e^{(u^2+1)b}$, and for some $\gamma$ consider

$$
\kappa(\tau) := \frac{\tau}{[\ln(\tau)]^\gamma}
\geq C(|u| + 1)^{\alpha_2 - \alpha_2\gamma}e^{(u^2+1)b} \tag{4.43}
$$

We let $\alpha_2 - \alpha_2\gamma = 3\alpha_2 - 2$, so $\gamma = 2/\alpha_2 - 2$. Note that $\gamma > 0$ since $\alpha_2 < 1$. From (4.43) and the last bound on $n$ we have

$$
n \leq C\int_\Omega \kappa\left[(|u| + 1)^{\alpha_2}e^{(u^2+1)b}\right]dx + C \tag{4.44}
$$

For large values of $\tau$, $\kappa''(\tau) < 0$. So by the eventual concavity of $\kappa$, (4.44) and Jensen’s inequality give

$$
n \leq C\kappa\left[\int_\Omega (|u| + 1)^{\alpha_2}e^{(u^2+1)b}dx\right] + C \tag{4.45}
$$
Now apply inequality (4.42) by using the sublinearity of $\kappa$, inequality (4.45) gives

$$n \leq C \cdot \kappa\left[ J(u_n) \right] + C$$

where we have included the subscript on $u$. Using the fact that $J(u_n) \leq b_n$, and that $\kappa$ is eventually increasing we have

$$n \leq C \cdot \kappa[b_n] + C \quad (4.46)$$

Let $\theta(\tau) := \tau[\ln(\tau)]^\gamma$, which is increasing and subexponential. Apply $\theta(\cdot)$ to both sides of (4.46)

$$\theta(n) \leq C \cdot \theta \circ \kappa[b_n] + C \quad (4.47)$$

Now for large $\tau$

$$\theta(\kappa(\tau)) = \tau \left[ 1 - \gamma \frac{\ln \ln(\tau)}{\ln(\tau)} \right]^\gamma \leq \tau$$

So that for large $n$

$$C\theta(n) \leq b_n$$

i.e.

$$Cn[\ln(n)]^\gamma - H(0)|\Omega| \leq b_n - H(0)|\Omega| \leq c_n, \quad \gamma = \frac{2}{\alpha_2} - 2$$

If $2/\alpha_2 - 2 > 1/\alpha_1$, as in the hypothesis of Theorem 12, this contradicts (4.31) and proves that theorem. ■

4.2 Problem (R), the radial problem on an annulus

4.2.1 The problem and its variational setup
Here \( \Omega = A_{R_0}^R := \{ x \in \mathbb{R}^2 : R_0 < |x| < R \} \) will denote an annulus, with \( R_0 > 0 \) and \( R < +\infty \). We seek multiple radial solutions to the problem

\[
\begin{aligned}
(-\Delta)^lu &= 2ue^{u^2} + \varphi(x,u) & \text{in } \Omega \\
\left( \frac{\partial}{\partial \nu} \right)^j u &= 0, & j = 0, \ldots, l-1
\end{aligned}
\]  

where \( \varphi(x,u) = \varphi(|x|,u) \). The proper space for this problem is

\[
H_r := \{ u \in H^l_0(\Omega) : u(x) = u(|x|) \text{ a.e. in } \Omega \}
\]

and the corresponding functional is

\[
I_1(u) := \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx - \int_\Omega (e^{u^2} - 1) dx - \int_\Omega \Phi(x,u) dx
\]

where \( \Phi(x,u) = \int_0^u \varphi(x,t)dt \) is not even in \( u \). Since \( \varphi(x,u) \) is radial in its explicit dependence on \( x \), critical points of \( I_1 \), even when restricted to \( H_r \), still correspond to generalized solutions of \( (R) \). This can be seen from a simple direct calculation using spherical coordinates\(^1\).

Concerning the size of the perturbation, we assume there exists \( \beta < 1 \) and \( C > 0 \) such that

\[
|\Phi(x,t)| + |\varphi(x,t)t| \leq Ct^2 e^{\beta t^2}
\]  

(4.48)

for all \( x \in \Omega \), and \( t \in \mathbb{R} \) with \( |t| \) large. As before, the path of functionals to which we will apply the method is

\[
I_\theta(u) := \frac{1}{2} \int_\Omega |\nabla^l u|^2 dx - \int_\Omega (e^{u^2} - 1) dx - \theta \int_\Omega \Phi(x,u) dx
\]

\(^1\)The general principle behind this is called the principle of symmetric criticality, but we do not need this.
where $\theta \in [0, 1]$. More complex paths (dependence on $\theta$) can be considered but they don’t seem necessary here.

### 4.2.2 Bolle’s and Tanaka’s requirements

The compactness of Palais-Smale sequences in the space $H_r$ as required by Bolle’s condition (H1) follows as it did for the general case in section 4.1. The proof is exactly the same with the exception that instead of Adam’s inequality we use Lemma 9. Thus the radial setting allows us to consider a nonlinearity which was of critical growth in the unrestricted setting earlier. For the condition (H2) assume $|I_{\theta}(u)| \leq b$. In fact, we only need to assume that $I_{\theta}(u) \leq b$:

$$
||I'_{\theta}(u)||_{H^{-1}}||u||_{H^r} \geq -\langle I'_{\theta}(u), u \rangle = -\int |\nabla^l u|^2 dx + \int 2u^2 e^{-u^2} + \theta \varphi(x, u) u dx \\
= \int |\nabla^l u|^2 dx + \int (2u^2 - 4)e^{-u^2} + \theta \varphi(x, u) u - 4\Phi(x, u)dx - 4I_{\theta}(u) - C \\
\geq ||u||^2_{H^r(\Omega)} + \frac{1}{2} \int u^2 e^{-u^2} dx - C - b \\
\geq c \int \Phi(x, u)dx - C = c \left| \frac{\partial}{\partial \theta} I_{\theta}(u) \right| - C
$$

where $c$ is a small positive constant. We used (4.48) for the second to last inequality. This verifies condition (H2). Next for condition (H3), let $u$ be a critical point of $I_{\theta}$. Then

$$
I_{\theta}(u) = I_{\theta}(u) - \frac{1}{2}(I'_{\theta}(u), u) \\
= \int (u^2 - 1)e^{-u^2} + \frac{\theta}{2} \varphi(x, u) u - \theta \Phi(x, u)dx - C \\
\geq \frac{1}{2} \int u^2 e^{-u^2} dx - C
$$

where we used (4.48) for the last inequality. Continuing, we next apply Jensen’s inequality
and (4.48) again. We can find sufficiently large constants $C$ and $C_0$ such that

$$C[|I_\theta(u)| + 1]^\beta \geq \int \left( u^2 e^{u^2} \right)^\beta \, dx + C_0 \quad \text{by Jensen}$$

$$\geq \int |\Phi(x, u)| \, dx \quad \text{by (4.48)}$$

$$\geq \left| \int \Phi(x, u) \, dx \right|$$

$$= \left| \frac{\partial}{\partial \theta} I_\theta(u) \right|$$

(4.49)

So condition (H3) holds, and the estimator functions are $f_i(\theta, t) = (-1)^i C[|t| + 1]^\beta$. So $\bar{f}_i(t) = C[|t| + 1]^\beta$. The condition (H4) follows as before. If we assume that only the second possibility of Theorem 2 holds for sufficiently large $n \in \mathbb{N}$ then there is a $K > 0$ such that

$$c_{n+1} - c_n \leq K(\bar{f}_1(c_{n+1}) + \bar{f}_2(c_n) + 1)$$

for sufficiently large $n$. More concisely, by enlarging $K$ if necessary, this means

$$c_{n+1} - c_n \leq K((c_{n+1})^\beta + (c_n)^\beta + 1)$$

for sufficiently large $n$. Finally, using the fact that $\beta < 1$, this implies that for some $A > 0$

$$c_n \leq An^{1-\beta} \quad \text{for} \quad n > n_0$$

with $n_0$ sufficiently large. The argument is the same as that used for (4.31).

4.2.3 The lower bound
As before, the goal is to obtain a lower bound for \( c_n \) that will contradict the alleged upper bound. The requirements in Tanaka’s Theorem are all verified as before, with no new phenomena appearing. By Tanaka’s theorem there exists a sequence \( u_n \) in the Hilbert space \( H_r \) such that

i) \( I_0(u_n) \leq c_n \)

ii) \( I'_0(u_n) = 0 \)

iii) \( n \leq index_0 I''_0(u_n) \)

For simplicity we denote \( u_n \) as \( u \), holding \( n \) fixed for the moment. As before,

\[
index_0 I''_0(u) = \text{number of non-positive eigenvalues of } (-\Delta)^l - 2(u^2 + 2)e^{u^2}
\]

By applying Proposition 8 we get

\[
n^{2l} \leq [index_0 I''_0(u)]^{2l} \leq C \int_{A_{R_0}^R} 2(u^2 + 2)e^{u^2} \log \left( \frac{R}{|x|} \right)^2 \, dx \leq C_{R_0} \int_{A_{R_0}^R} 2(u^2 + 2)e^{u^2} \, dx
\]

Since \( u = u_n \) is a critical point of \( I_0 \), as before we have that \( c_n \geq I_0(u) = I_0(u) - \frac{1}{2} \langle I'_0(u), u \rangle \geq \frac{1}{2} \int u^2 e^{u^2} \, dx - C \). Combining this with the previous inequality we obtain

\[
c_n \geq C_0 n^{2l}
\]

for sufficiently large \( n \). Therefore if \( 2l > \frac{1}{1-\beta} \) this contradicts the alleged upper bound

\[
c_n \leq A n^{\frac{1}{1-\beta}} \text{ for } n > n_0
\]
and proves Theorem 13.

4.3 Problem (H), the radial problem with Hardy potential

Here we consider problem (H). That is we seek weak radial solutions to

\[
\begin{cases}
(-\Delta)^l u + \frac{\bar{b}}{|x|^2} u = g(x, u) + \varphi(x) & \text{in } B_R(0) \\
\left(\frac{\partial}{\partial \nu} \right)^j u \bigg|_{\partial B_R(0)} = 0, & j = 0, \ldots, l - 1
\end{cases}
\]  

(H)

where $\bar{b} > 0$, and $g$ and $\varphi$ are radially symmetric in their explicit dependence on $x$. Notice that the Hardy exponent is the critical one. That is, the corresponding Hardy inequality in $H^1_0(B_R)$ doesn’t hold in general. Define the space $\mathcal{C}_r$ as the set of $u \in C_0^\infty(B_R \setminus \{0\})$ with $u(x) = u(|x|)$ a.e. Let

$\mathcal{H}_r := \text{the closure of } \mathcal{C}_r \text{ in the following norm}$

\[
||u||_{\mathcal{H}_r}^2 := \int_{B_R} \left|\nabla^l u\right|^2 dx + \bar{b} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx
\]  

(4.50)

Note that $\mathcal{H}_r \hookrightarrow H^1_0(B_R)$ continuously, an important fact to keep in mind. We will prove the existence of solutions by looking for critical points of the functional

\[
I_1(u) := \frac{1}{2} \int_{B_R} \left|\nabla^l u\right|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^{2l}} dx - \int_{B_R} G(x, u) dx - \int_{B_R} \varphi u dx
\]
on \( H_r \), where \( G(x,u) = \int_0^u g(x,t)dt \). Because \( g \) and \( \varphi \) are radially symmetric in their explicit dependence on \( x \), the critical points of this functional on \( H_r \) are also critical points on \( H^1_0(B_r) \). Hence they are generalized solutions for (H). Since \( G(x,u) \) has the same growth restrictions in this problem as it did in (P) all conditions in Theorem 2 and in Tanaka’s theorem follow exactly the same reasoning as they did earlier in section 4.1. Since \( \bar{b} > 0 \) Palais-Smale sequences are bounded in the norm of \( H_r \), instead of merely in the norm of \( H^1_0 \). In any case the argument proceeds as it did in section 4.1 and we need to contradict the possibility that the minimax values \( c_n \) of the unperturbed functional satisfy inequality (4.31)

\[
c_n \leq An \left[ \ln(n) \right]^{1/\alpha_1} \quad \text{for } n > n_0
\]

with \( n_0 \) large. In section 4.1 we defined the smoother functional \( J(u) \) by equation (4.32), and its minimax levels

\[
b_n := \inf_{g \in \Gamma_n} \sup_{u \in g(E_n)} J(u).
\]

We do exactly the same thing here, except \( J(u) \) is now defined as

\[
J(u) = \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^2} dx - \int_{B_R} H(u) dx + C_H,
\]

where \( C_H := H(0)|B_R(0)| \) is a constant chosen so that \( J(0) = 0 \). That is, the form \( \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx \) is replaced by the form \( \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx + \frac{\bar{b}}{2} \int_{B_R} \frac{|u|^2}{|x|^2} dx \). Since \( J(u) - C_H \leq I_0(u) \) by construction, we have \( b_n - C_H \leq c_n \). So it will suffice to obtain a good lower bound on \( b_n \). By Tanaka’s theorem, there exists a sequence \( u_n \) such that

i) \( J(u_n) \leq b_n \)

ii) \( J'(u_n) = 0 \)

iii) \( n \leq \text{index}_0 J''(u_n) \)
Therefore
\[ n \leq \text{index}_0 J''(u) = N(0; (-\Delta)^l + b|x|^{-2l} - H''(u_n(x))) \]

The eigenvalue estimate we apply is the following result from [27]

**Lemma 16** (Laptev-Netrusov). *Consider the unbounded linear operator \( H := (-\Delta)^l + b|x|^{-2l} - V(x) \) acting on \( L^2(B_R) \), \( B_R \) the ball of radius \( R \) in \( \mathbb{R}^{2l} \). Suppose \( V(x) = V(|x|) \geq 0 \) and \( V(x) \in L^1(B_R) \). Then*

\[ N(0; (-\Delta)^l + b|x|^{-2l} - V(x)) \leq C(l, b) \int_{B_R} V(x) dx \]

So
\[ n \leq C \int_{B_R} H''(u_n(x)) dx \leq C \int_{B_R} (|u_n| + 1)^{2\alpha_2 - 2} e^{(u_n^2 + 1)^b} dx \]

Since \( u_n \) is a critical point of \( J \) the last term in the line above can be controlled by \( J(u_n) \).

Like in the end of section 4.1, this gives

\[ Cn[\log(n)]^\gamma \leq J(u_n) \leq b_n \leq c_n + C_H \]

for sufficiently large \( n \), where this time \( \gamma = 2/\alpha_2 - 1 \). Note that \( C_H = H(0)|B_R(0)| \) does not depend on \( n \). Thus if \( 1/\alpha_1 < 2/\alpha_2 - 1 \) this contradicts the alleged upper bound, and proves that \( I_1 \) an unbounded sequence of radial critical points which are generalized solutions of problem (H).

**Remarks:** 1) As a side, we note that \( \mathcal{H}_r \) embeds continuously into \( C_0(B_R) \), and so is a Hilbert space of continuous functions. Indeed let \( u \in C_0^\infty(B_R(0) \setminus \{0\}) \) be radially symmet-
ric, and let \( r \) denote the radial variable. Then

\[
|u(r)|^2 = -2 \int_r^R u'(\rho)u(\rho) \, d\rho
\]

\[
\leq 2 \left( \int_r^R |u'(\rho)| \rho \, d\rho \right)^{1/2} \left( \int_r^R \frac{|u(\rho)|^2}{\rho} \, d\rho \right)^{1/2}
\]

\[
\leq c_d \left( \int_{B_R} \frac{|\partial_r u|^2}{|x|^{d-2}} \, dx \right)^{1/2} \left( \int_{B_R} \frac{|u(x)|^2}{|x|^d} \, dx \right)^{1/2}
\]

When \( d = 2 \) we are done after applying the AM-GM inequality. When \( d > 2 \) we apply the generalized Hardy-Rellich inequality: If \( d > p \) and \( f \in C^\infty_0(B_R) \) then

\[
\int_{B_R} \frac{|f(x)|^p}{|x|^{pk}} \, dx \leq c_{d,p,k} \int_{B_R} |\nabla^k f|^p \, dx
\]

See for example [31]. Letting \( pk = d - 2 \), \( p = 2 \), so that \( k = d/2 - 1 = l - 1 \), and \( f = \partial_r u \) gives

\[
|u(r)|^2 \leq c_d \left( \int_{B_R} |\nabla^l u|^2 \, dx \right)^{1/2} \left( \int_{B_R} \frac{|u(x)|^2}{|x|^{2l}} \, dx \right)^{1/2}
\]

\[
\leq C_d \int_{B_R} |\nabla^l u|^2 \, dx + C_d \int_{B_R} \frac{|u(x)|^2}{|x|^{2l}} \, dx.
\]

### 4.4 A radial problem with a power-type nonlinearity in the case \( d > 2l \)

In this section we generalize some of the results of [17] to higher order equations. We use the same argument as that paper, except we make use of the eigenvalue estimates we developed in section 3.4. Namely we consider the following problem with non-homogeneous boundary
data
\[\begin{cases}
(-\Delta)^lu = g(x,u) + \varphi(x), & x \in B_R \\
\left( \frac{\partial}{\partial \nu} \right)^ju \bigg|_{\partial B_R} = \xi_j, & j = 0, \ldots, l - 1.
\end{cases}
\] (P_\xi)

Here \(B_R := B_R(0) = \{x \in \mathbb{R}^d : |x| < R\}\), where \(d > 2l\). The boundary values \(\{\xi_j\}_{j=0}^{l-1}\) are constants. The term \(\varphi(x) = \varphi(|x|)\) is a member of

\[L^2_r(B_R) = \{g \in L^2(B_R) : g(x) = g(|x|) \text{ a.e.}\},\]

the space of radially symmetric, square integrable functions. The precise conditions on the nonlinearity \(g(x,u)\) are given below, but briefly it will be odd in \(u\) and of power type growth, such as for example \(|u|^{p-2}u\). And so if \(\varphi \equiv 0\) and \(\xi_j = 0\) for \(j = 0, \ldots, l - 1\) the problem would possess odd symmetry (i.e. \(u\) is a solution iff \(-u\) is a solution). The function \(g\) is required to satisfy

\((g_0)\) \(g : \bar{B}_R \times \mathbb{R} \to \mathbb{R}\) is continuous;

\((g_1)\) There exist constants \(A > 0\) and \(2 < p < \frac{2d}{d-2l}\) such that

\[|g(x,t)| \leq A(|t|^{p-1} + 1) \quad \forall (x,t) \in B_R \times \mathbb{R};\]

\((g_2)\) There exist constants \(\mu > 2\) and \(t_0 \geq 0\) such that if \((x,t) \in \bar{B}_R \times \mathbb{R}\) and \(|t| \geq t_0\) then

\[0 < \mu G(x,t) \leq tg(x,t),\]

where \(G(x,t) = \int_0^t g(x,s)ds;\)

\((g_3)\) \(g(x,-t) = -g(x,t)\) for all \((x,t) \in B_R \times \mathbb{R};\)
(g₄) \( g(x, t) = g(|x|, t) \) for all \( (x, t) \in B_R \times \mathbb{R} \).

The main results of this section are

**Theorem 15** (Non-homogeneous boundary conditions). Let \( g : \bar{B}_R \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy conditions \((g₀)-(g₄)\). In addition, suppose that for some positive constants \( Cᵢ \) it satisfies

\[
C₁|t|^{p−1} − C₂ \leq |g(x, t)| \leq C₃|t|^{p−1} + C₄,
\]

for all \( (x, t) \in B_R \times \mathbb{R} \). Then for any radially symmetric \( ϕ \in L²(B_R) \) and \( ξᵢ \in \mathbb{R} \), \( i = 0, \ldots, l − 1 \), the problem \((P_ξ)\) possesses an unbounded sequence of radial solutions provided that

\[
2 < p < \min \left\{ 3, \frac{2d}{d − 2l} \right\}.
\]

In particular, if \( d \geq 6l \) there are infinitely many radial solutions for all exponents in the subcritical range \( p \in (2, \frac{2d}{d − 2l}) \).

**Theorem 16** (Homogeneous boundary conditions). Let \( g : \bar{B}_R \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy conditions \((g₀)-(g₄)\). If \( ξᵢ = 0 \) for \( i = 0, \ldots, l − 1 \) then for any radially symmetric \( ϕ \in L²(B_R) \) the problem \((P_ξ)\) possesses an unbounded sequence of radial solutions provided that

\[
2 < p < \min \left\{ 2µ, \frac{2d}{d − 2l} \right\}.
\]

In particular if

\[
d \geq \frac{2µl}{µ − 1},
\]

there are infinitely many radial solutions for all exponents in the subcritical range \( p \in (2, \frac{2d}{d − 2l}) \).

Several steps are required for the proofs. Conditions \((g₁)\) and \((g₂)\) imply that there exist
constants $C_i$ such that
\[ C_5|t|^\mu \leq G(x,t) + C_6 \leq \frac{1}{\mu} t g(x,t) + C_7 \leq C_8(|t|^p + 1) \]  
(4.51)

for all $(x,t) \in \overline{B}_R \times \mathbb{R}$. In order to set up the variational structure we make some reductions. First we reduce the problem to one with homogeneous boundary conditions as follows: Let
\[ P_l(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{l-1} t^{l-1} \]
be a polynomial of degree at most $l - 1$, and let $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ be the radial variable in $\mathbb{R}^d$. Consider the function
\[ \xi(x) := P_l(r^2) = a_0 + a_1 r^2 + a_2 r^4 + \cdots + a_{l-1} r^{2l-2}, \]
on $B_R$, which satisfies $(-\Delta)^l \xi = 0$. By choosing the coefficients $a_i$ appropriately, we may solve the system of linear equations
\[ \left. \frac{\partial}{\partial \nu} \right|_{\partial B_R}^j \xi = \xi_j, \quad j = 0, \ldots, l - 1. \]

Then letting $u = v + \xi$, we transform the problem $P_\xi$ into
\[
\begin{cases}
(-\Delta)^l v = g(x,v + \xi) + \varphi(x), & x \in B_R \\
\left. \frac{\partial}{\partial \nu} \right|_{\partial B_R}^j v = 0, & j = 0, \ldots, l - 1.
\end{cases}
\]

We will find radial solutions of the above problem by looking for critical points of the
functional

\[ I_1(u) = \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx - \int_{B_R} G(x, u + \xi) dx - \int_{B_R} \varphi u dx, \quad (4.52) \]

defined on the space

\[ H_r := \{ u \in H^l_0(B_R) : u(x) = u(|x|) \text{ a.e. in } B_R \}, \quad (4.53) \]

normed by

\[ ||u||^2_{H^l_0(B_R)} = \int_{B_R} |\nabla^l u|^2 dx. \]

At first sight it may seem that for such critical points to correspond to weak solutions of our PDE problem the function space ought to be \( H^l_0(B_R) \). However, because \( g \) and \( \varphi \) are radially symmetric in their dependence on \( x \), critical points of \( I_1 \) on \( H_r \) are also critical points of the same functional on \( H^l_0(B_R) \). So we only need to work on \( H_r \), a closed subspace of \( H^l_0(B_R) \).

By the usual theorems \( I_1 \) is a \( C^1 \)-functional on the Hilbert space \( H_r \). To carry out the Bolle perturbation argument we consider the path of functionals

\[ I_\theta(u) := \frac{1}{2} \int_{B_R} |\nabla^l u|^2 dx - \int_{B_R} G(x, u + \theta \xi) dx - \theta \int_{B_R} \varphi u dx, \quad \theta \in [0, 1]. \quad (4.54) \]

Note that \( I_0 \) is even and \( I_1 \) is the functional corresponding to our variational problem. As before, standard theorems imply that \( I \) is \( C^1 \) on \( [0, 1] \times H_r \). In order to verify the conditions in theorem 2 we first need a technical lemma:

**Lemma 17.** For all \( \delta \in (1/\mu, 1/2) \) there exist constants \( \gamma_1(\delta), \gamma_2(\delta) > 0 \) such that for all \( (\theta, u) \in [0, 1] \times H_r \)

\[ ||u||^2_{H^l_0} + \int_{B_R} |u + \theta \xi|^\mu dx \leq \gamma_1(\delta) \left[ I_\theta(u) - \delta I'_\theta(u)(u) \right] + \gamma_2(\delta) \quad (4.55) \]
and

$$\|u\|_{H_0^2}^2 + \int_{B_u} G(x, u + \theta \xi)dx + \int_{B_u} g(x, u + \theta \xi)(u + \theta \xi)dx \leq \gamma_1(\delta) [I_{\theta}(u) - \delta I_{\theta}'(u)] + \gamma_2(\delta)$$

(4.56)

where $B_u := \{x \in B_R : |u(x) + \theta \xi(x)| \geq t_0\}$, with $t_0$ as in (g2).

**Proof:** Fix $(\theta, u) \in [0, 1] \times H_r$ and $\delta \in (1/\mu, 1/2)$. Then

$$I_{\theta}(u) - \delta I_{\theta}'(u)(u) = \left(\frac{1}{2} - \delta\right)\|u\|_{H_0^2}^2 + \int_{B_R} \delta g(x, u + \theta \xi)u - G(x, u + \theta \xi)dx$$

$$- (1 - \delta)\theta \int_{B_R} \varphi u dx$$

Hoping to apply estimates (4.51), we proceed as follows: Fix an $s \in (1/\mu, \delta)$, then

$$\int_{B_R} \delta g(x, u + \theta \xi)u - G(x, u + \theta \xi)dx$$

$$= \int_{B_R} \delta g(x, u + \theta \xi)(u + \theta \xi) - G(x, u + \theta \xi)dx$$

$$- \int_{B_R} \delta g(x, u + \theta \xi)\theta \delta \xi dx$$

$$= \int_{B_R} sg(x, u + \theta \xi)(u + \theta \xi) - G(x, u + \theta \xi)dx$$

$$+ \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi)(u + \theta \xi)dx$$

$$+ \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi) \left( u + \theta \xi - \frac{2\theta \delta}{\delta - s} \xi \right) dx$$
Therefore by the second inequality in (4.51)

\[
\int_{B_R} \delta g(x, u + \theta \xi) u - G(x, u + \theta \xi) \, dx \\
\geq (\mu s - 1) \int_{B_R} G(x, u + \theta \xi) \, dx + \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi)(u + \theta \xi) \, dx \\
+ \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi) \left(u + \theta \xi - \frac{2\theta \delta}{\delta - s} \xi\right) \, dx - C_5.
\] (4.57)

In order to estimate the term

\[
\frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi) \left(u + \theta \xi - \frac{2\theta \delta}{\delta - s} \xi\right) \, dx
\]

we consider the set

\[
\tilde{B}_u := \left\{ x \in B_R : |u(x) + \theta \xi(x)| \geq \max \left\{ t_0, \frac{2\theta \delta}{\delta - s} ||\xi||_{L^\infty} \right\} \right\}.
\]

Clearly since \( g \) is continuous and \( B_R \) is bounded

\[
\left| \int_{B_R \setminus \tilde{B}_u} g(x, u + \theta \xi) \left(u + \theta \xi - \frac{2\theta \delta}{\delta - s} \xi\right) \, dx \right| \leq C_6.
\] (4.58)
Moreover,

\[
\int_{\tilde{B}_u} g(x, u + \theta \xi) \left( u + \theta \xi - \frac{2\theta \delta \xi}{\delta - s} \right) dx \\
= \int_{\tilde{B}_u} g(x, u + \theta \xi)(u + \theta \xi) \left[ 1 - \frac{2\theta \delta \xi}{(\delta - s)(u + \theta \xi)} \right] dx
\]

\[
= \int_{\tilde{B}_u} \left[ g(x, u + \theta \xi)(u + \theta \xi) + C_3 \right] \left[ 1 - \frac{2\theta \delta \xi}{(\delta - s)(u + \theta \xi)} \right] dx - C_3 \int_{\tilde{B}_u} \left[ 1 - \frac{2\theta \delta \xi}{(\delta - s)(u + \theta \xi)} \right] dx
\]

\[
\geq -C_7 \quad (4.59)
\]

where in the last inequality we used the fact that for \( x \in \tilde{B}_u \)
\[ g(x, u + \theta \xi)(u + \theta \xi) + C_3 \geq 0, \] as well as \( 1 - \frac{2\theta \delta \xi}{(\delta - s)(u + \theta \xi)} \geq 0. \) Therefore (4.57), (4.58), and (4.59) combine to give

\[
(\mu s - 1) \int_{B_R} G(x, u + \theta \xi) dx + \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi)(u + \theta \xi) dx \\
\leq \int_{B_R} \delta g(x, u + \theta \xi) u - G(x, u + \theta \xi) dx + C_8. \quad (4.60)
\]

By Young's inequality

\[
(1 - \delta)\theta \int_{B_R} \varphi u dx \leq \frac{1 - 2\delta}{4} ||u||_{H^1_0}^2 + C_9.
\]
Therefore

\[
\frac{1 - 2\delta}{4} ||u||_{H_0^s}^2 + (\mu s - 1) \int_{B_R} G(x, u + \theta \xi) dx + \frac{\delta - s}{2} \int_{B_R} g(x, u + \theta \xi)(u + \theta \xi) dx
\]

\[
\leq \frac{1 - 2\delta}{4} ||u||_{H_0^s}^2 + \int_{B_R} \delta g(x, u + \theta \xi) dx + C_8
\]

\[
= \left( \frac{1}{2} - \delta \right) ||u||_{H_0^s}^2 + \int_{B_R} \delta g(x, u + \theta \xi) dx - \frac{1 - 2\delta}{4} ||u||_{H_0^s}^2 + C_8
\]

\[
\leq \left( \frac{1}{2} - \delta \right) ||u||_{H_0^s}^2 + \int_{B_R} \delta g(x, u + \theta \xi) dx - (1 - \delta) \theta \int_{B_R} \varphi udx + C_9
\]

\[
= I_\theta(u) - \delta I'_\theta(u)(u) + C_9. \quad (4.61)
\]

This inequality yields (4.55) when taking into account (4.51). It also proves (4.56) after taking into account the fact that

\[
\left| (\mu s - 1) \int_{B_R \setminus B_u} G(x, u + \theta \xi) dx + \frac{\delta - s}{2} \int_{B_R \setminus B_u} g(x, u + \theta \xi)(u + \theta \xi) dx \right| \leq C_{10}.
\]

We can now begin to verify the requirements of Theorem 2.

**Lemma 18.** If \( \{ (\theta_n, u_n) \}_{n=1}^\infty \subset [0, 1] \times H_r \) is a sequence such that \( I_{\theta_n}(u_n) \leq C \) and \( ||I'_{\theta_n}(u_n)||_{H^{-1}} \to 0 \) then \( \{ (\theta_n, u_n) \}_{n=1}^\infty \) is precompact.

**Proof:** By the Heine-Borel theorem we may assume that \( \theta_n \to \theta_0 \) after taking a subsequence. Since \( H_r \) is a closed subspace of \( H^1_0(B_R) \), we need only show that the sequence \( \{ u_n \} \) is precompact in \( H^1_0(B_R) \). Inequality (4.55) implies that \( \{ u_n \} \) is bounded in \( H^1_0(B_R) \).
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS 141

Consider the sequence of maps

\[ [0, 1] \times H_0^l(B_R) \xrightarrow{i} [0, 1] \times L^p(B_R) \xrightarrow{\Gamma} L^{\frac{p}{p-1}}(B_R) \xrightarrow{j} H^{-l}(B_R) \quad (4.62) \]

where \( \Gamma(\theta, u)(x) = g(x, u(x) + \theta \xi(x)) \). Since \( p \leq \frac{2d}{d-2l} \), the injection

\[ i : [0, 1] \times H_0^l(B_R) \hookrightarrow [0, 1] \times L^p(B_R) \]

is compact by the Rellich-Kondrachov theorem (and Heine-Borel). In particular, since \( \{u_n\} \) is bounded in \( H_0^l(B_R) \), we may assume that it converges to some \( u \) in \( L^p(B_R) \) after taking a subsequence. The injection

\[ j : L^{\frac{p}{p-1}}(B_R) \hookrightarrow H^{-l}(B_R) : v \mapsto \langle v, \cdot \rangle_{H^{-l} \times H_0^l} \]

is continuous by the Sobolev inequality. The proof that \( \Gamma \) is continuous follows very similar lines to the proof of Lemma 14, and is a classical result. See [24]. It is a consequence of the fact that \( g(\cdot, \cdot) \) is continuous and of sub-critical growth. Therefore the map \( u \mapsto \langle g(x, u + \theta \xi), \cdot \rangle_{H^{-l} \times H_0^l} \) is compact and continuous. So

\[
I'_\theta(u)(\cdot) = \langle u, \cdot \rangle_{H_0^l} - \langle g(x, u + \theta \xi), \cdot \rangle_{H^{-l} \times H_0^l} - \theta \langle \varphi, \cdot \rangle_{L^2}
\]

is a compact perturbation of the Riez representation map on \( H_r \). For the Palais-Smale sequence \( u_n \) we write this as

\[
\langle u_n, \cdot \rangle_{H_0^l} = I'_{\theta_n}(u_n) + \langle g(x, u_n + \theta_n \xi), \cdot \rangle_{H^{-l} \times H_0^l} + \theta_n \langle \varphi, \cdot \rangle_{L^2}
\]

\[
= I'_{\theta_n}(u_n) + \langle g(x, u_n + \theta_0 \xi), \cdot \rangle_{H^{-l} \times H_0^l} + \theta_0 \langle \varphi, \cdot \rangle_{L^2} + o(1)
\]
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

Everything on the right hand side converges strongly in $H^{-l}$ as $n \to \infty$. So $\langle u_n, \cdot \rangle_{H^l_0}$ converges in $H^{-l}$, and thus $u_n$ converges strongly in $H^l_0(B_R)$. ■

Lemma 19. For any $b > 0$ there exists a constant $C_b > 0$ such that

$$|I_\theta(u) \leq b| \implies \left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_b(||I_\theta'(u)||_{H^{-l}} + 1)(||u||_{H^l_0} + 1).$$

Proof: We have

$$\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| = \left| -\int_{B_R} \xi(x)g(x, u + \theta \xi)dx - \int_{B_R} \varphi udx \right| \leq ||\xi||_{L^\infty} \int_{B_R} |g(x, u + \theta \xi)|dx + C_1||u||_{H^l_0}.$$

We may assume, without loss of generality, that $t_0 > 1$. Then, with $B_u$ as in Lemma 17

$$\int_{B_R} |g(x, u + \theta \xi)|dx \leq \int_{B_u} |g(x, u + \theta \xi)|dx + C_2 \leq \int_{B_u} |g(x, u + \theta \xi)| \cdot |(u + \theta \xi)|dx + C_2 = \int_{B_u} |g(x, u + \theta \xi)(u + \theta \xi)|dx + C_2 \leq \int_{B_u} |g(x, u + \theta \xi)(u + \theta \xi)|dx + C_3|dx + C_4 = \int_{B_u} g(x, u + \theta \xi)(u + \theta \xi)dx + C_5$$

where the last equality is by (4.51) if $C_3$ is chosen large enough. The proof now follows by (4.56). ■

The next two lemmas are to determine the control functions $f_i$ in Bolle’s theorem.
Lemma 20. Suppose there exist constants $C_i$ such that

$$C_1 |t|^{p-1} - C_2 \leq |g(x, t)| \leq C_3 |t|^{p-1} + C_4,$$

for all $(x, t) \in B_R \times \mathbb{R}$. Then if $(\theta, u) \in [0, 1] \times H_r$ is such that $I_\theta'(u) = 0$, then there is a constant $C > 0$ such that

$$\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C (I_\theta(u)^2 + 1)^{\frac{p-1}{2p}}$$

(4.63)

Remark: Therefore in this case we may take $f_i(\theta, t) = (-1)^i C (t^2 + 1)^{\frac{p-1}{2p}}$ for $i = 1, 2$ as the control functions appearing in condition (H3) of theorem 2.

Proof: We compute

$$\frac{\partial}{\partial \theta} I(\theta, u) = - \int_{B_R} g(x, u + \theta \xi) \xi dx - \int_{B_R} \varphi u dx.$$

Hence

$$\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_0 \int_{B_R} |g(x, u + \theta \xi)| \xi dx + \int_{B_R} |\varphi| |u| dx$$

$$\leq C_1 \int_{B_R} |u + \theta \xi|^{p-1} dx + \int_{B_R} |\varphi| |u| dx + C_2.$$

Therefore by the Hölder and Young inequalities we have

$$\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_4 ||u + \theta \xi||_{L^p}^{p-1} + C_5.$$

However, in the present case $\mu = p$ in estimate (4.51) and hence also in estimate (4.55). Therefore, by (4.55)

$$||u + \theta \xi||_{L^p} \leq C (I_\theta(u)^2 + 1)^{1/(2p)},$$
since \( I_\theta'(u)(u) = 0 \). So
\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_6 (I_\theta(u)^2 + 1)^{\frac{p-1}{2p}}.
\]
\[\blacksquare\]

**Lemma 21.** Suppose \( \xi_i = 0 \) for \( i = 0, \ldots, l - 1 \), i.e. the problem has homogeneous boundary conditions. Then if \( (\theta, u) \in [0, 1] \times H_r \) is such that \( I_\theta'(u) = 0 \), then there is a constant \( C > 0 \) such that
\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C (I_\theta(u)^2 + 1)^{\frac{1}{2p}}. \tag{4.64}
\]

**Remark:** Therefore in this case we may take \( f_i(\theta, t) = (-1)^i C (t^2 + 1)^{\frac{1}{2p}} \) for \( i = 1, 2 \) as the control functions appearing in condition \( (H3) \) of theorem 2.

**Proof:** This time we may take \( \xi \equiv 0 \), and so
\[
\frac{\partial}{\partial \theta} I(\theta, u) = -\int_{B_R} \varphi u dx = -\int_{B_R} \varphi (u + \theta \xi) dx \pm C_0.
\]

Therefore by Hölder’s inequality
\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_1 ||u + \theta \xi||_\mu + C_2.
\]

This along with estimate (4.55) give
\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C (I_\theta(u)^2 + 1)^{\frac{1}{2p}},
\]

since \( I_\theta'(u)(u) = 0 \). \[\blacksquare\]

The last requirement of Bolle’s theorem comes from the super-linear nature of the non-linearity \( g \). More precisely
**Lemma 22.** Let \( W \subset H_r \) be a finite dimensional subspace. Then

\[
\lim_{\|u\| \to \infty} \sup_{\theta \in [0,1]} I_\theta(u) = -\infty.
\]

**Proof:** This follows because by (4.51) there exist positive constants \( C_i \) for which

\[
I_\theta(u) \leq C_1 \|u\|^2_{H^1_0} - C_2 \|u\|_{\mu} + C_3.
\]

Since \( \mu > 2 \), and on the finite dimensional subspace \( W \) all norms are equivalent, the above inequality implies the result.  

**Proof of Theorem 15:** We have shown that the conditions required to apply theorem 2 hold for \( I_\theta(u) \). In particular, by lemma 20, we have determined the control functions

\[
f_i(\theta, t) = C_1 (t^2 + 1)^{\frac{p-1}{2p}}
\]

under the hypothesis of theorem 15. Therefore

\[
\bar{f}_i(t) = C_1 (t^2 + 1)^{\frac{p-1}{2p}} \leq C_2 (|t|^{\frac{p-1}{p}} + 1).
\]

Assume that \( I_1(u) \) does not have a sequence of critical values tending to \(+\infty\). By theorem 2 this implies a bound on the symmetric mountain pass levels \( \{c_k\} \) of the unperturbed functional \( I_0(u) \). More precisely, for \( k \) sufficiently large we have

\[
c_{k+1} - c_k \leq C_3 \left( c_{k+1}^{\frac{p-1}{p}} + c_k^{\frac{p-1}{p}} + 1 \right).
\]  

(4.65)
This implies that \( \exists C_4, k_0 > 0 \) such that for all \( k \geq k_0 \)
\[
c_k \leq C_4k^p. \tag{4.66}
\]
By (4.51) there are constants \( C_5, C_6 > 0 \) such that
\[
I_0(u) \geq \frac{1}{2} \int_{B_R} |\nabla^4 u|^2 dx - C_5 \int_{B_R} |u|^p dx - C_6.
\]
Define
\[
J(u) := \frac{1}{2} \int_{B_R} |\nabla^4 u|^2 dx - C_5 \int_{B_R} |u|^p dx
\]
and let
\[
b_k := \inf_{g \in \Gamma_k} \sup_{u \in g(E_k)} J(u)
\]
be the symmetric mountain pass levels of \( J(u) \). Then
\[
c_k \geq b_k - C_6.
\]
So it suffices to get a good lower bound on the \( b_k \)'s in hopes of contradicting (4.66). In order to apply Tanaka’s theorem (theorem 3) we note that conditions (J1), (J2), and (J3) of that theorem are clearly satisfied by the functional \( J(u) \). Condition (J4), that \( J'(u) \) is a compact perturbation of the Riesz representation map, follows exactly as it did for \( I_\theta(u) \). The fact that \( J \) satisfies the compactness condition (PS) follows exactly as in did for \( I_\theta(u) \) in lemma 18. For similar reasons \( J \) satisfies conditions (PS)_m and (PS)_*. Therefore we can apply theorem 3 to find, for each \( k \in \mathbb{N} \), a \( u_k \in H_r \) such that
\begin{enumerate}
  \item \( J(u_k) \leq b_k \),
  \item \( J'(u_k) = 0 \),
\end{enumerate}
iii) $\text{index}_0 J''(u_k) \geq k$.

Therefore

$$k \leq N_0 ((-\Delta)^l - C_5 p(p - 1)|u_k|^{p - 2}).$$

We apply theorem 9 to get

$$k \leq C_7 \int_{B_R} |u_k|^{p - 2}|x|^{2l - d} dx$$

$$= C_8 \int_0^R |u_k|^{p - 2} r^{2l - 1} dr$$

$$= C_8 \int_0^R |u_k|^{p - 2} r^\alpha r^{2l - 1 - \alpha} dr$$

$$\leq C_8 \left\{ \int_0^R |u_k|^p r^{\alpha p} dr \right\}^{p - 2 \over p} \left\{ \int_0^R r^{p(2l - 1 - \alpha)} dr \right\}^{2 \over p}.$$

Let

$$\frac{\alpha p}{p - 2} = d - 1,$$

and so

$$\alpha = \frac{(d - 1)(p - 2)}{p}.$$

This also implies

$$\frac{p(2l - 1 - \alpha)}{2} = d - 1 - \frac{p(d - 2l)}{2},$$

which is $> -1$ since $p < \frac{2l}{d - 2l}$. Therefore

$$k \leq C_9 \left\{ \int_{B_R} |u_k|^p dx \right\}^{p - 2 \over p}. \quad (4.67)$$
Since $J'(u_k)(u_k) = 0$ this implies

$$k \leq C_{10} \left\{ J(u_k) \right\}^{\frac{p-2}{p}} \leq C_{10} \left\{ b_k \right\}^{\frac{p-2}{p}}.$$

Therefore

$$b_k \geq C_{11} k^{\frac{p}{p-2}}$$

and so

$$c_k \geq C_{11} k^{\frac{p}{p-2}} \quad (4.68)$$

for $k$ sufficiently large. This contradicts (4.66) if

$$\frac{p}{p-2} > p,$$

i.e. if $2 < p < 3$. This proves the theorem. ■

Proof of Theorem 16: In this case we apply lemma 21 instead of lemma 20. Inequality (4.65) is replaced by

$$c_{k+1} - c_k \leq C_1 \left( c_{k+1}^{1/\mu} + c_k^{1/\mu} + 1 \right). \quad (4.69)$$

Which implies that

$$c_k \leq C_2 k^{\frac{\mu}{\mu-1}} \quad (4.70)$$

for $k$ sufficiently large. However, exactly as before estimate (4.68) holds. Therefore there is a contradiction if

$$\frac{p}{p-2} > \frac{\mu}{\mu-1},$$

i.e. if $2 < p < 2\mu$. ■
4.5 A problem on $\mathbb{R}^d$, in the case $d > 2l$

So far we have dealt only with problems on bounded domains. In this section we will use our eigenvalue estimates to generalize the result of [7] to higher order equations. We are concerned with finding radial solutions to the problem

$$
\begin{cases}
(-\Delta)^l u + mu = |u|^{p-2}u + \varphi(x), & x \in \mathbb{R}^d \\
\left(\frac{\partial}{\partial \nu}\right)^j u \to 0, & \text{as } |x| \to \infty, \quad j = 0, \ldots, l - 1.
\end{cases}
$$

(P$_{un}$)

where $d > 2l$, $m > 0$, $2 < p < \frac{2d}{d - 2l}$, and $\varphi$ is radially symmetric. Here we need to take $\varphi \in L^{p'}(\mathbb{R}^d)$ where $p'$ is the Young’s conjugate of $p$.

**Theorem 17.** Let $d > 2l$, $m > 0$. Let $\varphi$ be radially symmetric with $\varphi \in L^{p'}(\mathbb{R}^d)$, where $p' = \frac{p}{p-1}$. Then problem (P$_{un}$) has infinitely many weak, radially symmetric solutions provided that

$$
\frac{d}{d - 2l} < p < \frac{2d}{d - 2l}.
$$

In particular, if $d \geq 4l$ then (P$_{un}$) has infinitely many weak, radially symmetric solutions for all $2 < p < \frac{2d}{d - 2l}$ in the subcritical range.

As in the previous section we will search for radial solutions by looking for critical points of the functional

$$
I_1(u) := \frac{1}{2} \int (|\nabla^l u|^2 + m|u|^2) dx - \frac{1}{p} \int |u|^p dx - \int \varphi u dx,
$$

(4.71)

defined on the space

$$
H_r := \{u \in H^l(\mathbb{R}^d) : u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^d\},
$$

(4.72)
CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

150

normed by

\[ ||u||_{H^l(R^d)}^2 = \int \left( |\nabla^l u|^2 + m|u|^2 \right) dx. \]

Here all integrals will be over \( R^d \) unless otherwise indicated. By the well-known theorems
\( I_1 \) is a \( C^1 \) functional on \( H_r \). The fact that \( \varphi \) is radially symmetric with respect to \( x \) means that critical points of \( I_1 \) in \( H_r \) are also critical points of the same functional defined on all
of \( H^l(R^d) \), and so they still correspond to generalized solutions of the PDE. One problem
with working on unbounded domains is that the Rellich-Kondrachov compactness theorem
does not hold. However, as the following lemma shows, the restriction of radial symmetry
corrects this.

**Lemma 23.** The injection

\[ H_r \hookrightarrow L^p(R^d), \]

for \( 2 < p < \frac{2d}{d-2l} \), is compact.

This lemma is classical. It is essentially due to Strauss [47], and P. L. Lions [8].

For the path of functionals we simply take

\[ I_\theta(u) := \frac{1}{2} \int \left( |\nabla^l u|^2 + m|u|^2 \right) dx - \frac{1}{p} \int |u|^p dx - \theta \int \varphi u dx. \quad (4.73) \]

We first verify the requirements of Theorem 2.

**Lemma 24.** If \( \{(\theta_n, u_n)\}_{n=1}^\infty \subset [0, 1] \times H_r \) is a sequence such that \( I_{\theta_n}(u_n) \leq C \) and \( ||I'_{\theta_n}(u_n)||_{H^{-l}} \to 0 \) then \( \{(\theta_n, u_n)\}_{n=1}^\infty \) is precompact.

**Proof:** Let \( \{(\theta_n, u_n)\}_{n=1}^\infty \) be such a sequence. Take \( C \) such that \( I_{\theta_n}(u_n) \leq C \) and \( \epsilon_n \to 0 \)
such that $|I'_{\theta_n}(u_n)(u_n)| \leq \epsilon_n ||u_n||_{H^l}$. Then

$$C + \frac{\epsilon_n}{p} ||u_n||_{H^l} \geq I_{\theta_n}(u_n) - \frac{1}{p} I'_{\theta_n}(u_n)(u_n) = \left( \frac{1}{2} - \frac{1}{p} \right) ||u_n||_{H^l}^2 - \left( 1 - \frac{1}{p} \right) \theta_n \int \varphi u_n dx$$

$$\geq c_1 ||u_n||_{H^l}^2 - c_2 ||u_n||_{H^l}$$

and so $\{u_n\}$ is bounded in $H_r$. By lemma 23, after taking a subsequence, it converges in $L^p(\mathbb{R}^d)$. So the sequence $|u_n|^{p-2}u_n$ converges in $L^{\frac{p}{p-\tau}}(\mathbb{R}^d)$. In particular, by considering the injection of $L^{\frac{p}{p-\tau}}(\mathbb{R}^d)$ into $H^{-\tau}(\mathbb{R}^d)$, the sequence $|u_n|^{p-2}u_n$ converges in $H^{-\tau}(\mathbb{R}^d)$. Finally, we may also take $\{\theta_n\}$ to be converging. Now

$$I'_{\theta_n}(u_n) = \langle u_n, \cdot \rangle_{H^l} - \langle |u_n|^{p-2}u_n, \cdot \rangle_{H^{-\tau} \times H^l} - \theta_n \langle \varphi, \cdot \rangle_{L^{p'} \times L^p},$$

which we may write as

$$\langle u_n, \cdot \rangle_{H^l} = I'_{\theta_n}(u_n) + \langle |u_n|^{p-2}u_n, \cdot \rangle_{H^{-\tau} \times H^l} + \theta_n \langle \varphi, \cdot \rangle_{L^2}.$$ 

The right hand side converges strongly in $H^{-\tau}(\mathbb{R}^d)$. Therefore $u_n$ converges strongly in $H^l(\mathbb{R}^d)$. This proof also shows that for each $\theta$, $I'_{\theta}$ is a compact perturbation of the Riesz representation map. ■

**Lemma 25.** For any $b > 0$ there exists a constant $C_b > 0$ such that

$$|I_\theta(u) \leq b| \implies \left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C_b \left( ||I_\theta'(u)||_{H^{-l}} + 1 \right) \left( ||u||_{H^l} + 1 \right).$$
Proof: We have

\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| = \left| \int \varphi u dx \right| \leq \|\varphi\|_{L^{p'}} \|u\|_{L^p} \leq c_1 \|u\|_{H^r}.
\]

Lemma 26. If \((\theta, u) \in [0, 1] \times H_r \) is such that \(I_\theta'(u) = 0\), then there is a constant \(C > 0\) such that

\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C (I_\theta(u)^2 + 1)^{\frac{1}{2p}}. \tag{4.74}
\]

Proof: We have that

\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| = \left| \int \varphi u dx \right| \leq \|\varphi\|_{L^{p'}} \|u\|_{L^p}
\]

However, since \(I_\theta'(u)(u) = 0\) we also have that

\[
I_\theta(u) = I_\theta(u) - \frac{1}{2} I_\theta'(u)(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \|u\|_{L^p}^p - \left( 1 - \frac{1}{2} \right) \theta \int \varphi u dx 
\]

Therefore

\[
\left| \frac{\partial}{\partial \theta} I(\theta, u) \right| \leq C (I_\theta(u)^2 + 1)^{\frac{1}{2p}}.
\]
Lemma 27. Let $W \subset H_r$ be a finite dimensional subspace. Then

$$\lim_{||u|| \to \infty} \sup_{\theta \in [0,1]} I_\theta(u) = -\infty.$$ 

Proof: By Young’s inequality

$$I_\theta(u) \leq C_1 ||u||^2_{H^1} - C_2 ||u||^p_{L^p} + C_3$$

and so the result follows since all the norms are equivalent on $W$, and $2 < p$. ■

We have shown that the conditions required to apply theorem 2 hold for $I_\theta(u)$. In particular, we have determined the control functions

$$f_i(\theta, t) = (-1)^i C_1 \left( t^2 + 1 \right)^{\frac{1}{2p}}$$

under the hypothesis of theorem 15. Therefore

$$\tilde{f}_i(t) = C_1 \left( t^2 + 1 \right)^{\frac{1}{2p}} \leq C_2 \left( |t|^{\frac{1}{p}} + 1 \right).$$

Assume that $I_1(u)$ does not have a sequence of critical values tending to $+\infty$. By theorem 2 this implies a bound on the symmetric mountain pass levels $\{c_k\}$ of the unperturbed functional $I_0(u)$. More precisely, for $k$ sufficiently large we have

$$c_{k+1} - c_k \leq C_3 \left( c_{k+1}^{\frac{1}{p}} + c_k^{\frac{1}{p}} + 1 \right).$$ (4.75)
This implies that \( \exists C_4, k_0 > 0 \) such that for all \( k \geq k_0 \)

\[ c_k \leq C_4 k^{\frac{p}{p-1}}. \]  

As before, we will contradict this upper bound by using Tanaka’s theorem (theorem 3) to derive an adequate lower bound on the \( c_k \)'s. As before \( I_0 \) satisfies all of the requirements of theorem 3. Therefore we can apply theorem 3 to find, for each \( k \in \mathbb{N} \), a \( u_k \in H_r \) such that

i) \( I_0(u_k) \leq c_k \),

ii) \( I_0'(u_k) = 0 \),

iii) \( \text{index}_0 I_0''(u_k) \geq k \).

Therefore

\[ k \leq N_0((-\Delta)^l + mI - C_5 p(p-1)|u_k|^{p-2}) \]

We apply the remark at the end of section 3.4 (see estimate (3.64)). Therefore

\[ k \leq C_5 \int_{\mathbb{R}^d} |u_k|^{p-2} |x|^{2l-d} \, dx \]

\[ = C_6 \int_0^{\infty} |u_k|^{p-2} r^{2l-1} \, dr. \]

So we need to bound

\[ \int_0^{\infty} |u_k|^{p-2} r^{2l-1} \, dr \]

by \( c_k \). Using the fact that \( u_k \) is a critical point, we first note that

\[ c_k \geq I_0(u_k) = \left( \frac{1}{2} - \frac{1}{p} \right) ||u_k||_{H^l}^2 = \left( \frac{1}{2} - \frac{1}{p} \right) ||u_k||_{L^p}^p. \]  

(4.77)
We will follow the rest of the argument in [7]. First we have by Hölder’s inequality
\[
\int_0^1 |u_k|^{p-2} r^{2l-1} dr \leq \left\{ \int_0^1 r^\alpha dr \right\}^{\frac{2}{p}} \left\{ \int_0^1 |u_k|^{p} r^{d-1} dr \right\}^{\frac{p-2}{p}}
\]
where \( \alpha \) is such that
\[
\frac{(d - 1)(p - 2)}{p} + \frac{2\alpha}{p} = 2l - 1.
\]
So that
\[
\alpha = d - 1 + \frac{(2l - d)p}{2}.
\]
Now since \( p < \frac{2d}{d-2l} \)
\[
\alpha = d - 1 + \frac{(2l - d)p}{2} > -1.
\]
Hence (4.77) gives
\[
\int_0^1 |u_k|^{p-2} r^{2l-1} dr \leq C_7 \left\{ \int_0^1 |u_k|^{p} r^{d-1} dr \right\}^{\frac{p-2}{p}}
\]
\[
\leq C_8 |u_k|_{L^p_r}^{p-2}
\]
\[
\leq C_9 c_k^{p-2}.
\]
For the second part of the integral we proceed as follows. First take \( \beta > 0 \) and \( q > 2 \), which will be further restricted later. Then
\[
\int_1^\infty |u_k|^{p-2} r^{2l-1} dr = \int_1^\infty |u_k|^{p-2} r^{2l-1+\beta} r^{-\beta} dr
\]
\[
\leq \left\{ \int_1^\infty r^{-\beta \frac{q}{2}} dr \right\}^{\frac{2}{q}} \left\{ \int_1^\infty |u_k|^{\frac{(p-2)q}{q-2}} r^{(2l-1+\beta) \frac{q}{q-2}} dr \right\}^{\frac{q-2}{q}}.
\]
(4.79)
We need to have $-\frac{\beta}{2} < -1$ and

$$(2l - 1 + \beta) \frac{q}{q - 2} \leq d - 1.$$ 

Before we proceed let us note that these conditions require that $q > \frac{2d}{d - 2l}$. Indeed, since $\beta > \frac{2}{q}$ we have

$$\left(2l - 1 + \frac{2}{q}\right) \frac{q}{q - 2} < (2l - 1 + \beta) \frac{q}{q - 2} \leq d - 1$$

$$\implies (2l - 1)q + 2 < (d - 1)(q - 2)$$

$$\implies (2l - 1)q < (d - 1)q - 2d$$

$$\implies (2l - d)q < -2d$$

$$\implies q > \frac{2d}{d - 2l}.$$ 

By the reverse manipulations for any $q > \frac{2d}{d - 2l}$ there is a $\beta > \frac{2}{q}$ for which

$$(2l - 1 + \beta) \frac{q}{q - 2} \leq d - 1$$

holds. Now we continue from (4.79)

$$\int_1^{\infty} |u_k|^{p-2} r^{2l-1} dr \leq \bar{C}_1 \left\{ \int_1^{\infty} |u_k|^{(\frac{p-2)g}{q-2}} r^{d-1} dr \right\}^{\frac{2-q}{q}}$$

$$\leq \bar{C}_2 ||u_k||_{L^{(\frac{p-2)g}{q-2}}} (\mathbb{R}^d)$$

where the constant $\bar{C}_1$ depends on $\beta$ and $q$. Therefore it suffices to estimate $||u_k||_{L^{(\frac{p-2)g}{q-2}}} (\mathbb{R}^d)$.
from above by $c_k$, or rather by $\|u_k\|_{L^p(\mathbb{R}^d)}$. To do this we will require that

$$2 \leq \frac{(p-2)q}{q-2} < p.$$  \hfill (4.80)

Since $q > \frac{2d}{d-2l}$ the first inequality in (4.80) implies that we need to restrict $p$ so that

$$2 + \frac{4l}{d} < p < \frac{2d}{d-2l},$$ \hfill (4.81)

For any such $p$ we may choose $q > \frac{2d}{d-2l}$ but sufficiently close to $\frac{2d}{d-2l}$ so that (4.80) is satisfied. With this in mind, we proceed by using the interpolation inequality

$$\|u_k\|^p_{L^q(\mathbb{R}^d)} \leq \left\{ \|u_k\|^q_{L^2(\mathbb{R}^d)} \left\| u_k \right\|_{L^p(\mathbb{R}^d)}^{1-a} \right\}^{p-2}$$

$$\leq C_{10} \left\{ c_k^{\frac{a}{q}} c_k^{\frac{1-a}{p}} \right\}^{p-2}$$

$$= C_{10} c_k^{\frac{q-2}{q}}$$ \hfill (4.82)

where $0 \leq a \leq 1$ is such that $\frac{q}{2} + \frac{1-a}{p} = \frac{q-2}{(p-2)q}$. Therefore by (4.78) and (4.82) we obtain

$$k \leq \tilde{C}_3 \left( c_k^{\frac{q-2}{p}} + c_k^{\frac{q-2}{q}} \right) \leq \tilde{C}_4 c_k^{\frac{q-2}{q}}$$

using $q > p$, in the special case $2 + \frac{4l}{d} < p < \frac{2d}{d-2l}$. Here $\tilde{C}_4$ depends on $\beta$ and $q$. This gives

$$c_k \geq C_q k^{\frac{q}{q-2}}$$

for $k$ sufficiently large, in the case that $2 + \frac{4l}{d} < p < \frac{2d}{d-2l}$. 

CHAPTER 4. APPLICATIONS TO POLYHARMONIC DIRICHLET PROBLEMS

Given $\delta > 0$ we can choose $q$ sufficiently close to $\frac{2d}{d-2l}$ so that

$$\frac{q}{q-2} \geq \frac{d}{2l} - \delta.$$ 

Then, given $\delta > 0$, there is a constant $C_\delta$ such that

$$c_k \geq C_\delta k^{\frac{d}{\p} - \delta}$$

(4.83)

for $k$ sufficiently large, so long as $2 + \frac{4l}{d} < p < \frac{2d}{d-2l}$. However, the estimate (4.83) holds for any $2 < p < \frac{2d}{d-2l}$. To prove this we need a lemma.

**Lemma 28.** Let $2 < p \leq 2 + \frac{4l}{d}$. Then there is a $\bar{p}$ such that

$$2 + \frac{4l}{d} < \bar{p} < \frac{2d}{d-2l}$$

and such that for any $\epsilon > 0$ a positive constant $C_\epsilon$ exists for which

$$\int |u|^p dx \leq \epsilon \int |u|^2 dx + C_\epsilon \int |u|^\bar{p} dx$$

for all $u \in H_r$.

**Proof:** Let $s > 1$ and $\alpha > 0$ be constants to be chosen appropriately later. By Young’s inequality we have

$$|u|^p = (\epsilon s)^{\frac{1}{s}} |u|^{\alpha} \frac{1}{(\epsilon s)^{\frac{1}{s}}} |u|^{p-\alpha} \leq \epsilon |u|^\alpha s + \frac{1}{s'(\epsilon s)^{\frac{s'}{s}}} |u|^{(p-\alpha)s'}$$

where $s' = \frac{s}{s-1}$ is the Young’s conjugate of $s$. The result would follow if we can show that
\( \alpha > 0 \) and \( s > 1 \) can be chosen so that

\[ \alpha s = 2 \]  
(4.84)

\[ 2 + \frac{4d}{d} < (p - \alpha) \frac{s}{s-1} < \frac{2d}{d - 2l}. \]  
(4.85)

To simplify the manipulations let \( M := \frac{d}{l} \) and denote by \( 2M := \frac{2d}{d - 2l} \). Then (4.84) and (4.85) take the form

\[ \alpha = \frac{2}{s} \]  
(4.86)

\[ 2 + \frac{4}{M} < (p - \alpha) \frac{s}{s-1} < 2M. \]  
(4.87)

We proceed

\[ 2 + \frac{4}{M} < \left( p - \frac{2}{s} \right) \frac{s}{s-1} < 2M, \]

\[ \iff 2 + \frac{4}{M} < p \left( 1 + \frac{1}{s-1} \right) - \frac{2}{s-1} < 2M, \]

\[ \iff 2 + \frac{4}{M} - p < \frac{p - 2}{s-1} < 2M - p. \]

At this point we note that the left most term in this inequality is \( \geq 0 \) since \( 2 + \frac{4}{M} - p = 2 + \frac{4}{d} - p \geq 0 \) by assumption. Therefore

\[ \frac{1}{2 + \frac{4}{M} - p} > \frac{s-1}{p-2} > \frac{1}{2M - p}, \]

remembering to take \( +\infty \) instead of \( \frac{1}{2 + \frac{4}{M} - p} \) in the case \( p = 2 + \frac{4}{M} \). Thus

\[ \frac{p - 2}{2 + \frac{4}{M} - p} > s - 1 > \frac{p - 2}{2M - p}, \]

\[ \iff \frac{4}{2 + \frac{4}{M} - p} > s > \frac{2M - 2}{2M - p}. \]
Therefore it suffices to prove that

\[ 1 < \frac{2M - 2}{2M - p} < \frac{4}{M} \frac{M}{2 + \frac{4}{M} - p}. \]

The first inequality is clear since \( p > 2 \). The second inequality follows because

\[ \frac{2M - 2}{2M - p} = \frac{4}{M} \frac{M}{2 + \frac{2p}{M} - p} \]

and \( 2p > 4 \). Therefore \( \alpha > 0 \) and \( s > 1 \) can be found for which (4.84) and (4.85) hold. Thus we may take

\[ \bar{p} = (p - \alpha) \frac{s}{s - 1}. \]

\[ \blacksquare \]

**Proof of Theorem 17:** We have already proven the estimate (4.83) in the case

\[ 2 + \frac{4l}{d} < p < \frac{2d}{d - 2l}. \]

Now assume

\[ 2 < p \leq 2 + \frac{4l}{d}. \]

By the previous lemma there is a \( \bar{p} \in (2 + \frac{4l}{d}, \frac{2d}{d - 2l}) \) such that for any \( \epsilon > 0 \)

\[ I_0(u) \geq J(u) \]

where

\[ J(u) := \frac{1}{2} \int (|\nabla^l u|^2 + A_\epsilon |u|^2) \, dx - B_\epsilon \int |u|^\bar{p} \, dx \]

for some positive constants \( A_\epsilon \), and \( B_\epsilon \). So if we denote by \( \bar{c}_k \) the symmetric mountain-pass
levels of $J$ then
\[ c_k \geq \bar{c}_k. \]

By applying the earlier proof to $J$ instead of $I_0$ we have that
\[ c_k \geq \bar{c}_k \geq C \delta k^{\frac{d}{2l}-\delta} \]
for all $\delta > 0$. So (4.83) holds for all $2 < p < \frac{2d}{d-2l}$. Finally we see that there is a contradiction when
\[ \frac{p}{p-1} < \frac{d}{2l}, \]
i.e. when
\[ p > \frac{d}{d-2l}, \]
which proves the theorem. ■
Bibliography


