

5-2018


On Some Geometry of Graphs

Zachary S. McGuirk

The Graduate Center, City University of New York

How does access to this work benefit you? Let us know!

Follow this and additional works at: https://academicworks.cuny.edu/gc_etds

 Part of the [Analysis Commons](#), [Discrete Mathematics and Combinatorics Commons](#), and the [Geometry and Topology Commons](#)

Recommended Citation

McGuirk, Zachary S., "On Some Geometry of Graphs" (2018). *CUNY Academic Works*.
https://academicworks.cuny.edu/gc_etds/2681

This Dissertation is brought to you by CUNY Academic Works. It has been accepted for inclusion in All Dissertations, Theses, and Capstone Projects by an authorized administrator of CUNY Academic Works. For more information, please contact deposit@gc.cuny.edu.

ON SOME GEOMETRY OF GRAPHS

by

ZACHARY S. MCGUIRK

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2018

© 2018

ZACHARY S. MCGUIRK

All Rights Reserved

This manuscript has been read and accepted by the Graduate Faculty in
Mathematics in satisfaction of the dissertation requirement for the degree of
Doctor of Philosophy.

Professor Melvyn Nathanson

Date

Chair of Examining Committee

Professor Ara Basmajian

Date

Executive Officer

Professor Melvyn Nathanson

Professor Józef Dodziuk

Professor Radosław Wojciechowski

Supervisory Committee

Abstract
ON SOME GEOMETRY OF GRAPHS
by
ZACHARY S. MCGUIRK

Adviser: Professor Dr. Melvyn Nathanson

In this thesis we study the intrinsic geometry of graphs via the constants that appear in discretized partial differential equations associated to those graphs. By studying the behavior of a discretized version of Bochner's inequality for smooth manifolds at the cone point for a cone over the set of vertices of a graph, a lower bound for the internal energy of the underlying graph is obtained. This gives a new lower bound for the size of the first non-trivial eigenvalue of the graph Laplacian in terms of the curvature constant that appears at the cone point and the size of the vertex set for the underlying graph. For the sake of completeness, the main analysis for cones is actually done for cones over subsets of the vertex set. We follow this analysis up by studying which types of functions can achieve equality in the discrete Bochner inequality, in particular functions which yield the largest possible curvature bound at the cone point come with a dynamical definition. We are then able to classify the space of all such functions via spectral graph theory and recast the regularity of a graph in terms of the dimension of this space of functions.

Acknowledgments

*“There are no atheists in foxholes’ isn’t an argument
against atheism, it’s an argument against foxholes.”*

-James K. Morrow

I acknowledge everyone who played a role in my graduate school experience. There were many special individuals with whom I interacted. However, in particular I owe an immense debt of gratitude to Dr. Burton Randol, Dr. Józef Dodziuk, Dr. Melvyn Nathanson, Dr. Byungdo Park, Dr. Sajjad Lakzian, Dr. Enrique Pujals, and Dr. Radosław Wojciechowski.

Under usual circumstances the above should suffice, however nothing about the last six years has been usual and something really must be said about the tireless contributions of Dr. Burton Randol to my academics and my life. During a particularly sad and difficult stretch of my graduate school experience, Dr. Randol took to meeting with me nearly every day just to keep me sane, thinking about math, and moving forward. His wisdom, cool demeanor, and guidance is likely the only reason I survived my time at the CUNY Graduate Center. Even after things started to calm down, he still met with me once or twice a week. For a long while, he was the only person who would talk to me about mathematics. I spent years toiling in obscurity, chasing in the footsteps of Field’s medalist, trying to prove deep conjectures on my own. If not for the encouragement and sustained support of Dr. Randol, I wouldn’t be here today. Truly, if not for his tireless efforts over the years

I would have never graduated. Dr. Randol, I'm interminably grateful for everything.

I am also very thankful for the help and support provided to me by Dr. Joseph Hirsch and Dr. Linda Keen.

Furthermore, I am thankful for the camaraderie provided by Dr. Jorge Basilio, Dr. Martin Bendersky, Dr. David Edward Bruschi, Bora Ferlengez, Dr. Aron Fischer, Dr. Jorge Florez, Dr. Cihan Karabulut, Dr. Asaf Katz, Dr. Seungwon Kim, Alice Kwon, Dr. Ana Turchetti-Maia, and Johannes Niediek.

In conclusion, I should also like to thank my friends, family, girlfriend, and my undergraduate school's mathematics department. Thank you all.

THIS THESIS IS DEDICATED TO MY FATHER, JACK P. MCGUIRK.

Contents

1	Introduction	1
2	Preliminaries	8
3	Cone Construction	15
3.1	Γ -Calculus for Cones	19
3.2	Γ_2^c for $C(G)$	30
4	Graph Energy	32
5	Generalized Harmonic Functions on Finite Graphs	40
5.1	Unit Spherical Averaging Operator	42
5.2	Unit Ball Averaging Operator	46
5.3	Explicit Examples	49
	Bibliography	58

List of Figures

3.1	Example of a partial cone $C(X, G)$	16
3.2	Example of a complete cone $C(G)$	17
3.3	Metric spheres centered at p	18

Chapter 1

Introduction

The relation between Ricci curvature bounds and the analytic and geometric properties of a smooth Riemannian manifold is a well studied subject in geometric analysis (see, for example, [14]). Lower bounds on the Ricci curvature of a manifold can provide eigenvalue estimates and functional inequalities. Thanks to the work of Sturm [27], [28], [26] and Lott-Villani [22], a notion of lower Ricci curvature bounds has been generalized to the setting of metric-measure spaces via a $CD(K, N)$ curvature-dimension condition due to Bakry and Émery [3]. Bakry and Émery's original work provided a lower Ricci curvature bound for a measure space via a generalized heat semigroup with some strong regularity conditions for the generator of that semigroup and a curvature-dimension condition realized by what has come to be known as the Γ -Calculus of Bakry and Émery. This curvature-dimension condition is essentially a discretization of Bochner's inequality for smooth manifolds and the Γ -Calculus of Bakry and Émery is a method of iterated gradients used to study semigroups of linear operators on functions spaces. Refinements of these discretization methods to simply metric spaces, or rather graphs, have yielded a notion of lower Ricci curvature bounds on graphs (see, for example, [4], [9], [8], [15], [16], [20], [21]).

In this thesis, we make use of Bakry and Émery's Γ -Calculus to study lower Ricci curvature bounds for cones over subsets of vertices. These cones are then used

to study local and global properties of the underlying graph and establish a global Poincaré inequality. In addition, studying the space of functions which achieved the maximum curvature at the cone point lead to an interesting family of functions with a dynamical nature, which we were able to classify for finite graphs via the spectral properties of the graph Laplacian. So, in chapter 3 we cover the Γ -Calculus for cones over subsets of vertices, in chapter 4 we apply those results to the cone over the full vertex set and derive a Poincaré inequality, and lastly we define generalized harmonic functions on graphs and study the space of such functions.

Let's begin filling in the above with definitions. First, the curvature-dimension condition:

Definition 1.0.1 (Bakry-Émery Curvature-Dimension Condition). *Suppose $K \in \mathbb{R}$ and $N \in (1, \infty]$. We say that a graph $G = (V, E)$ satisfies the curvature-dimension conditions, $CD(K, N)$, if for every $x \in V$ and every $f \in \ell^2(V)$,*

$$\Gamma_2(f)(x) \geq \frac{(\Delta f)^2(x)}{N} + K\Gamma_1(f)(x).$$

Where, for a linear operator \mathcal{L} acting on a Hilbert space,

$$\Gamma_1(f, g) = \frac{1}{2} [\mathcal{L}(fg) - \mathcal{L}f \cdot g - f \cdot \mathcal{L}g],$$

and

$$\Gamma_2(f, g) = \frac{1}{2} [\mathcal{L}\Gamma_1(f, g) - \Gamma_1(\mathcal{L}f, g) - \Gamma_1(f, \mathcal{L}g)]$$

and in this setting we have taken \mathcal{L} to be the unnormalized graph Laplacian Δ (For more detail, see the beginning of chapter 2).

Also, note that when $N = \infty$, the second term in the inequality above is understood to be 0.

An object of interest in this thesis is the complete cone, $C(G)$, over a finite graph

$G = (V, E)$. In the smooth and metric-measure settings, there is a close relation between the lower Ricci bounds of the space, X , and the lower Ricci curvature bounds of the cones over X . For example, a result due to Bacher and Sturm says that the Ricci curvature of a complete n -dimensional Riemannian manifold M is bounded below by $n - 1$ if and only if the cone over M (as a metric-measure space) satisfies the $CD(0, n + 1)$ condition [2]. In the setting of $CD(K, N)$ metric-measure spaces the relation between the weak Ricci curvature bound of X and that of the cone(s) over X has been explored in [17]. We pursue a similar vein in this thesis by studying cones over subsets of the vertices of a graph and looking for relations between the curvature bounds for the cone versus the curvature bounds for the underlying graph. Formally, $C(G)$ is constructed by taking the graph Cartesian product of G and the complete graph with two vertices K_2 , i.e. $G \square K_2$, where $K_2 = (\{q, p\}, \{(q, p)\})$, and then identifying all the vertices whose second component is p . This complete cone is often referred to as the cone over the vertices of a finite graph. Furthermore, $C(X, G)$ is the subgraph of $C(G)$ restricted to the subset $X \subset V(G)$. Note, G needs to be finite so that $C(G)$ is locally-finite and summability concerns can be avoided. Lastly, in this paper the point p will always refer to the cone point of whatever cone is currently under consideration.

When studying the properties of an underlying graph G that can be extracted when the cone over G it is convenient to restrict the $CD(K, N)$ curvature-dimension conditions to just the cone point (a property which will be called the conical curvature-dimension, or $CCD(K, N)$ condition). This interest in the curvature at just the cone point stems from the fact that the $CD(K, N)$ inequality depends on vertices which are at most two steps away, with respect to the graph metric, on the left hand side, however the right hand side is made up of operators that only depend on vertices that are at most one away. So when considering a function whose value at the cone point is zero, the right hand side will only depend on a function's values on the underlying graph and it bounds from below the left hand side which will take into account the

effect of adding on the cone point. Furthermore, when taking $f(p) = 0$, the right hand side of the $CD(K, N)$ inequality begs for a Cauchy-Schwarz application. Thus, we have:

Definition 1.0.2 (*CCD(K, N) Conical Curvature-Dimension Conditions [19]*). *Let $G = (V, E)$ be a finite, connected, undirected, loop-edge free graph and consider the cone over the vertex set of G . G is said to satisfy the conical curvature-dimension condition, $CCD(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty]$, if the cone over G satisfies the $CD(K, N)$ curvature-dimension conditions at the vertex p , namely if*

$$\Gamma_2^c(f)(p) \geq \frac{(\Delta^c f)^2(p)}{N} + K\Gamma_1^c(f)(p),$$

holds for any function f defined on the cone and Δ^c , Γ_1^c and Γ_2^c are the usual Δ , Γ_1 and Γ_2 operators (see (2.4), (2.7) and (2.8)) except on the cone $C(G)$ over G . We note that the second term in (2.1) is understood to be zero when $N = \infty$.

Furthermore, we refer to the curvature at the cone point K as the conical curvature of the finite graph G .

While studying the functions that result in the largest possible curvature at the cone point over a set of vertices, it became natural to consider functions that evenly spread their $\ell^2(V, \mu)$ mass over V in the limit of iterated applications of a unit spherical averaging operator on G . Thus we define a unit spherical averaging operator and a unit ball averaging operator.

Definition 1.0.3. *Let $G = (V, E)$ be a simple, finite, connected, unweighted, loop-edge free graph and let $f : V \rightarrow \mathbb{R}$. The unit spherical averaging operator is then given by $g : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|V|}$, such that:*

$$g[f](v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(u)$$

for all $v \in V$.

The unit ball averaging operator is then:

Definition 1.0.4. Let $G = (V, E)$ be a simple, finite, connected, unweighted, loop-edge free graph and let $f : V \rightarrow \mathbb{R}$. The unit ball averaging operator is then given by $h : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|V|}$, such that:

$$h[f](v) = \frac{1}{\deg_G(v) + 1} \left(f(v) + \sum_{u \sim v} f(u) \right)$$

for all $v \in V$.

For completeness, we note here that by the average of $f : V \rightarrow \mathbb{R}$, $\text{avg}(f)$, we mean $\frac{1}{|V|} \sum_{v \in V} f(v)$. Now, We have made these definitions in order to say the following:

Definition 1.0.5. Let $G = (V, E)$ be a simple, finite, connected, unweighted, loop-edge free graph and let $f : V \rightarrow \mathbb{R}$. We call f generalized harmonic of type I if and only if,

$$\lim_{i \rightarrow \infty} g_i[f](v) = \text{avg}(f)$$

for all $v \in V$ and we call f generalized harmonic of type II if and only if,

$$\lim_{i \rightarrow \infty} h_i[f](v) = \text{avg}(f)$$

for all $v \in V$.

With this definition, $g_i[f](v) = g[g_{i-1}[f]](v)$, $g_0[f](v) = f(v)$ and $g_1[f](v) = g[f](v)$.

And, similarly for h , $h_i[f](v) = h[h_{i-1}[f]](v)$, $h_0[f](v) = f(v)$, and $h_1[f](v) = h[f](v)$.

Now we can state our main theorems and corollaries:

Theorem 1.0.6. *If a finite graph G satisfies $CCD(K, N)$ curvature-dimension condition, then for any function f on $V(G)$ one has*

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y).$$

For functions f with $\text{avg}(f) = 0$, this reduces to the following global Poincaré inequality,

$$\|f\|_2 \leq \sqrt{\frac{4}{2K + |V| - 3}} \|\nabla f\|_2,$$

where $\|\nabla f\|_2$ is understood in the graph setting to be $2 \cdot \sum_{y \in V} \Gamma_1(f)(y)$.

Theorem 1.0.7. *For any finite graph G and a given $N > 1$ the conical curvature cannot exceed the following number:*

$$K_{max}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N}.$$

Theorem 1.0.8 (Curvature Maximizers). *Suppose a finite graph G satisfies $CCD(K_{max}^c, N)$.*

Then any function f realizes K_{max}^c if and only if f is either constant or $f - \text{avg}(f)$ is an eigenfunction corresponding to $\lambda_1(G) = \frac{N-2}{4N}|V|$.

Furthermore, when G is a complete graph, f must be constant (harmonic).

Corollary 1.0.9 (Ricci Curvature of Complete Graphs). *Suppose G is the complete graph on n vertices, then the $CD(K, N)$ property coincides with the $CCD(K^c, N)$ condition on the complete subgraph with $n - 1$ vertices and the curvature of G is $\frac{n}{2} + 1 - 2\frac{(n-1)}{N}$.*

Furthermore any function that realizes this curvature bound is constant (harmonic).

Remark 1.0.10. When $N = \infty$, our bound $K_{max}^c = 1 + \frac{n}{2}$ coincides with the maximum Ricci curvature of complete graphs as found in [18].

The following theorem illustrates an applications of our Γ -calculus on cones:

Theorem 1.0.11. Suppose a finite graph G satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$ then the subgraph $G \subset C(G)$ satisfies $CD(K + \frac{1}{2}, \infty)$.

Theorem 1.0.12. The set of generalized harmonic functions on a finite graph forms a subspace of $\ell^2(V, \mu)$ that contains harmonic functions.

Theorem 1.0.13. A function on the vertices of a connected, finite, undirected, non-bipartite graph is generalized harmonic of type I if and only if its weighted average is equal to its average, i.e. if and only if

$$\frac{\sum_{v \in V} \deg(v) f(v)}{\sum_{v \in V} \deg(v)} = \frac{1}{|V|} \sum_{v \in V} f(v).$$

Theorem 1.0.14. Given a connected, undirected, non-bipartite, finite graph $G = (V, E)$ and $f : V(G) \rightarrow \mathbb{R}$, f is generalized harmonic of type II if and only if for $H = G \square K_2$, with $V(H) = \{(v, t) \in V(G) \times \{0, 1\}\}$ and $F : V(H) \rightarrow \mathbb{R}$ such that $F(v, t) = f(v)$, $\lim_{i \rightarrow \infty} g_i[F](x) = \text{avg}(F)$.

The next result has to do with the behavior of δ_v (for $v \in V$) functions on the complete graph with m vertices. It provides an explicit formula for the n -th level iteration of $g[\delta_v]$ on G .

Theorem 1.0.15. For any $y \sim x \in K_m$, and any n ,

$$g_n[\delta_x](y) = \begin{cases} \frac{1}{(m-1)^n} \sum_{j=0}^{\frac{n}{2}-1} \binom{\frac{n}{2}+j}{2j+1} (m-1)^{\frac{n}{2}-1-j} (m-2)^{2j+1}; & n \equiv 0 \pmod{2} \\ \frac{1}{(m-1)^n} \sum_{j=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}+j}{2j} (m-1)^{\frac{n-1}{2}-j} (m-2)^{2j}; & n \equiv 1 \pmod{2} \end{cases}.$$

Chapter 2

Preliminaries

The fundamental objects in this thesis are graphs, denoted $G = (V, E)$. Unless otherwise stated these graphs will be connected, simple, unweighted, finite, and loop-edge free.

Let \mathcal{L} be a linear operator on the Hilbert space of square-summable functions from the set of vertices, V , to the real numbers with measure μ and inner product $\langle f, g \rangle := \sum_{v \in V} \mu(v) f(v) g(v)$; i.e. \mathcal{L} acts on

$$\ell^2(V, \mu) = \{f : V \rightarrow \mathbb{R} : \sum_{v \in V} \mu(v) |f(v)|^2 < \infty\}. \quad (2.1)$$

The Γ -Calculus of Bakry and Émery associated to \mathcal{L} is then defined (see [3]) as:

$$\Gamma_1(f, g) = \frac{1}{2} [\mathcal{L}(fg) - \mathcal{L}f \cdot g - f \cdot \mathcal{L}g], \quad (2.2)$$

and

$$\Gamma_2(f, g) = \frac{1}{2} [\mathcal{L}\Gamma_1(f, g) - \Gamma_1(\mathcal{L}f, g) - \Gamma_1(f, \mathcal{L}g)]. \quad (2.3)$$

Given any graph, G , there is an associated adjacency operator, $A(G)$, for which the $A_{i,j}$ -entry is 1 whenever there is an edge from vertex i to vertex j and 0 otherwise.

Since we have taken G to be finite, $A(G)$ will be a finite dimensional, square matrix. This allows us to avoid summability issues. It should be noted that when G is undirected, $A(G)$ is symmetric.

In this thesis the linear operator that we will be studying is the combinatorial graph Laplacian (see [11]), both unnormalized (denoted Δ) and normalized (denoted $\bar{\Delta}$) as given by:

$$\Delta f(v) = \sum_{u \in V \sim v} (f(u) - f(v)) \quad (2.4)$$

and

$$\bar{\Delta} f(v) = \frac{1}{\deg(v)} \sum_{u \in V \sim v} (f(u) - f(v)), \quad (2.5)$$

where $u \in V \sim v$ (if the that contains u is unambiguous or the entire vertex set then we simply write $u \sim v$) means that $u \in V$ and there exists an edge $(u, v) \in E$, and

$$\deg(v) = \#\{u : u \sim v\}.$$

Note: When using the unnormalized operator, Δ , the measure μ in $\ell^2(V, \mu)$ will be the counting measure, i.e. $\mu(v) = 1$, for all $v \in V$. However, a standard modification to the Hilbert space for the normalized operator, $\bar{\Delta}$, results in $\mu(v) = \deg(v)$ for all $v \in V$ (see [12]). Let $D(G)$ be the diagonal degree operator for a graph G . Thus,

$$D(G) = (d_{i,j}) = \begin{cases} \deg(i) & , \text{when } i = j \\ 0 & , \text{otherwise} \end{cases}.$$

The combinatorial graph Laplacians can then be realized as the difference between the operators $D(G)$ and $A(G)$, i.e. $-\Delta = (D(G) - A(G))$ and $-\bar{\Delta} =$

$(I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}})$ (see [6]). A graph is said to be k -regular if the degree of every vertex is k . When G is k -regular, $-\Delta = (k \cdot I - A)$ or $-\bar{\Delta} = (I - \frac{1}{k} \cdot A)$ (see [10]). When thinking of the graph Laplacians (both normalized and unnormalized) as operators on the function space of a finite graph, they are realized to be matrices with real entries. For undirected graphs these matrices will be symmetric and therefore self-adjoint. It should be noted that after the work of Wojciechowski (see [29]), these self-adjoint matrices can be extended uniquely to a self-adjoint linear operator acting on the Hilbert space of functions on an infinite graph's set of vertices. However, for this thesis we won't need the essential self-adjointness of the Laplacian on graphs. Furthermore, the first non-zero eigenvalue of the graph Laplacian, on a locally-finite graph, may be bounded below via the Rayleigh-Ritz quotient. However in the finite setting, which is the case for this thesis, the Rayleigh-Ritz quotient actually yields the first non-zero eigenvalue.

$$\lambda_1 = \inf \left\{ \frac{\frac{1}{2} \sum_{v \in V} \sum_{u \sim v} (f(u) - f(v))^2}{\sum_{v \in V} f^2(v)} : f \in \ell^2(V) \text{ and } \text{avg}(f) = 0 \right\}.$$

Applying the definitions for the Γ -Calculus of Bakry-Émery to the combinatorial graph Laplacian, one can see that

$$\Gamma_1(f, g)(v) = \frac{1}{2} [\Delta(fg)(v) - \Delta f(v)g(v) - f(v)\Delta g(v)] \quad (2.6)$$

$$= \frac{1}{2} \sum_{u \in V \sim v} (f(u) - f(v))(g(u) - g(v)), \quad (2.7)$$

and so in this discrete setting we think of $\Gamma_1(f, g)$ as $\langle \nabla f, \nabla g \rangle$.

Furthermore,

$$\Gamma_2(f, g)(v) = \frac{1}{2} [\Delta \Gamma_1(f, g)(v) - \Gamma_1(\Delta f, g)(v) - \Gamma_1(f, \Delta g)(v)]. \quad (2.8)$$

Remark 2.0.1. *It should be noted that both Δf and $\bar{\Delta} f$ are invariant under the addition of a constant to f , i.e. $\Delta f = \Delta(f + c)$ and $\bar{\Delta} f = \bar{\Delta}(f + c)$. As a result*

$\Gamma_1(f, g)$ and $\Gamma_2(f, g)$ are invariant under translations by constant functions, in both f and g . Furthermore, any scalar $c \in \mathbb{R}$ applied to f , or in the case of Γ_1 and Γ_2 g as well, may be factored out from the operator's argument.

Now in the case of the normalized graph Laplacian there is by definition a factor of $\frac{1}{\deg(v)}$ that comes up. Hence,

$$\begin{aligned}\bar{\Gamma}_1(f, g)(v) &= \frac{1}{2\deg(v)} \sum_{u \in V \sim v} (f(u) - f(v))(g(u) - g(v)) \\ &= \frac{1}{\deg(v)} \Gamma_1(f, g)(v),\end{aligned}$$

and

$$\bar{\Gamma}_2(f, g) = \frac{1}{\deg^2(v)} \Gamma_2(f, g)(v).$$

Let $\Gamma_1(f, f) := \Gamma_1(f)$. Thus, $\Gamma_1(f)(v)$ is thought of as $|\nabla f(v)|^2$ in the unnormalized setting and one can verify (provided that Δ is bounded, which is true for finite graphs) the following divergence identity,

$$\|\nabla f\|_2^2 = \sum_{v \in V} \Gamma_1(f)(v) = - \sum_{v \in V} f(v) \Delta f(v).$$

In thinking of $\Gamma_1(f)(v)$ as $|\nabla f(v)|^2$ one is lead to a discrete notion of energy for a graph G via its Dirichlet sum [6].

Definition 2.0.2. *The energy of a finite, simple, undirected, unweighted graph is given by:*

$$\begin{aligned}\|\nabla f\|_2^2 &= \frac{1}{2} \sum_{v \in V} \sum_{u \sim v} (f(u) - f(v))^2 \\ &= \sum_{v \in V} \Gamma_1(f)(v).\end{aligned}$$

We can now unambiguously define the Curvature-Dimension condition of Bakry-

Émery.

Definition 2.0.3. *If $K \in \mathbb{R}$ and $N \in (1, \infty]$ are such that for every $v \in V$ and every $f \in \ell^2(V, \mu)$,*

$$\Gamma_2(f)(v) \geq \frac{(\Delta f)^2(v)}{N} + K\Gamma_1(f)(v),$$

then we say that $G = (V, E)$ satisfies the $CD(K, N)$ curvature-dimension condition.

Note, when $N = \infty$ the middle term in the inequality above is understood to be 0. The largest possible value of K such that the above inequality holds for all $v \in V$ and all $f \in \ell^2(V, \mu)$ is thought of as the lower ‘‘Ricci curvature’’ bound, in analogy (see [13]) to the lower Ricci curvature bound, k , that appears in Bochner’s famous inequality for manifolds:

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq \frac{1}{n}|\Delta f|^2 + k|\nabla f|^2,$$

where f is a smooth function on a manifold M , n is an upper bound for its dimension, and k is a lower bound for the Ricci curvature. Furthermore, the $CD(K, N)$ condition,

$$\Gamma_2(f)(v) \geq \frac{(\Delta f)^2(v)}{N} + K\Gamma_1(f)(v),$$

is invariant under translations and scalar multiplication of f . i.e. let $c \in \mathbb{R}$

$$\begin{aligned} \Gamma_2(c \cdot f)(v) &\geq \frac{(\Delta c \cdot f)^2(v)}{N} + K\Gamma_1(c \cdot f)(v) \\ \Leftrightarrow c^2\Gamma_2(f)(v) &\geq \frac{c^2(\Delta f)^2(v)}{N} + c^2K\Gamma_1(f)(v) \\ \Leftrightarrow \Gamma_2(f)(v) &\geq \frac{(\Delta f)^2(v)}{N} + K\Gamma_1(f)(v) \end{aligned}$$

and,

$$\begin{aligned}\Gamma_2(f+c)(v) &\geq \frac{(\Delta(f+c))^2(v)}{N} + K\Gamma_1(f+c)(v) \\ \Leftrightarrow \Gamma_2(f)(v) &\geq \frac{(\Delta f)^2(v)}{N} + K\Gamma_1(f)(v).\end{aligned}$$

This curvature-dimension condition provides an avenue for defining a notion of Ricci curvature for graphs and we make the following definitions.

Definition 2.0.4 (Uniform and Pointwise Ricci Curvature Bounds). *We define the pointwise curvatures by*

$$\text{Ric}_N(y) := \sup_f \{K : \Gamma_2(f)(y) \geq \frac{1}{N}(\Delta f)^2(y) + K\Gamma_1(f)(y)\}$$

and

$$\text{Ric}_\infty(y) := \sup_f \{K : \Gamma_2(f)(y) \geq K\Gamma_1(f)(y)\},$$

and similarly we define the dimensional (respectively, dimensionless) Ricci curvature of the graph G , $\text{Ric}_N(G)$ (respectively, $\text{Ric}_\infty(G)$) by,

$$\text{Ric}_N(G) := \sup \{K : G \text{ satisfies } CD(K, N)\}$$

and

$$\text{Ric}_\infty(G) := \sup \{K : G \text{ satisfies } CD(K, \infty)\}.$$

Definition 2.0.5 (Conical Ricci Curvatures). *We define the conical Ricci curvature by*

$$CRic_N(G) := \sup \{K : G \text{ satisfies } CCD(K, N)\},$$

where $CCD(K, N)$ is defined in 2.1 and

$$CRic_\infty(G) := \sup\{K : G \text{ satisfies } CCD(K, \infty)\}.$$

Graphs come with a natural metric, which is simply the sum of the least number of edge one must cross to get from vertex u to vertex v in the graph.

$$d_G(u, v) = \inf \left\{ \sum_{i=0}^{m-1} 1 : v = x_0 \sim x_1 \sim x_2 \sim \dots \sim x_m = u \right\}.$$

Given $F \subset V$, then

$$\partial F := \{(u, v) \in E : u \in F \text{ and } v \in V \setminus F\}.$$

From which we can define the finite graph isoperimetric constant,

$$h(G) := \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \setminus F|\}} : 0 < |F| < \infty \right\}.$$

For convenience, let $m = \sup\{\deg(v) : v \in V\}$ and recall a well-known result of Dodziuk and Alon-Milman from the 1980's (see [11], [1], [7]):

Theorem 2.0.6. *Let $G = (V, E)$ be a finite, simple, connected, loop-edge free graph and let λ_1 be the first non-trivial eigenvalue of Δ and let $m = \max\{\deg(v) : v \in V\}$. Then, the following inequalities hold:*

$$\frac{\lambda_1}{2} \leq h(G) \leq \sqrt{2m\lambda_1}.$$

Definition 2.0.7. *Let $\{G_i\}_{i \geq 0}$ be a countably infinite family of graphs with $\sup\{\deg(v) : v \in V(G_i)\} = m_i \leq M$ for some $M \in \mathbb{N}$, ordered so that $|V_i| \leq |V_{i+1}|$ and $\lim_{i \rightarrow \infty} |V_i| \rightarrow \infty$. If there exists a constant, $C > 0$, such that $h(G_i) \geq C > 0$, for all $i \geq 0$, then $\{G_i\}_{i \geq 0}$ is an expanding family of graphs (see, [24] and [10]).*

Chapter 3

Cone Construction

The complete cone, $C(G)$, over a finite graph G is constructed by taking the graph Cartesian product of G and H , $G \square K_2$, where $K_2 = (\{q, p\}, \{(q, p)\})$ is the complete graph on two vertices q and p , and then identifying all the vertices whose second component is p . In this thesis, as mentioned before, p always refers to the cone point of $C(G)$. More generally, for a subset $X \subset V(G)$, the partial cone, $C(X, G)$, is a subgraph of $C(G)$ containing X , all edges (x, y) , for $x, y \in X$ and all edges (x, p) , for $x \in X$. For brevity, we will use a superscript c to denote any operation that is taking place on a cone over G or on a subset of the vertices $X \subset V(G)$. Notice that any vertex $v \in V(G)$ can be thought of as the cone point of the 1-sphere based at v , i.e. $S_v^1 := \{y \in V \mid y \sim v\} = X$ in the above construction. In this way partial cones can be useful in studying cliques. The Figures 3.1 and 3.2 on the next page demonstrate examples of a partial cone and, when $X = V$, the complete cone over the vertices of a finite graph respectively.

First we will prove a few lemmas that calculate the Δ and Γ operators of a partial cone in terms of the analogous operators on the base graph plus some error terms. In this way the equations and lemmas that follow are pointwise equivalences between operators acting on a cone and operators acting on an underlying graph. The last subsection is then devoted to an immediate result. In the lemmas that

follow a helpful visual is that of a tower stratifying the metric spheres in $C(G)$ that are centered at the cone point p (see Figure 3.3). Underlying the usefulness of this tower is the fact that the operator $\Gamma_2(f)(v)$ depends on vertices that are at most two steps away from v , while for $\Gamma_1(f)(v)$ and $\Delta f(v)$ only depend on vertices that are one step away.

Figure 3.1: Example of a partial cone $C(X, G)$

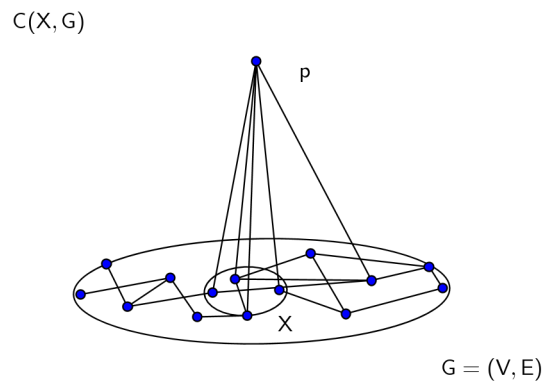


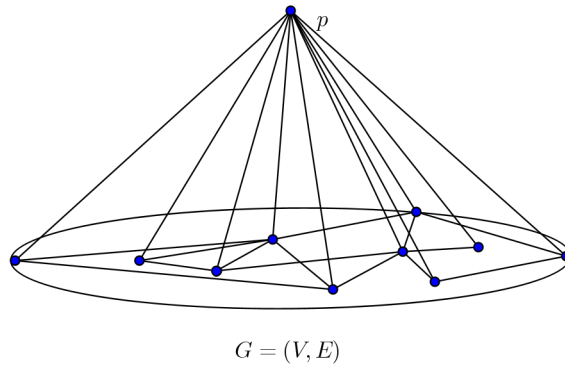
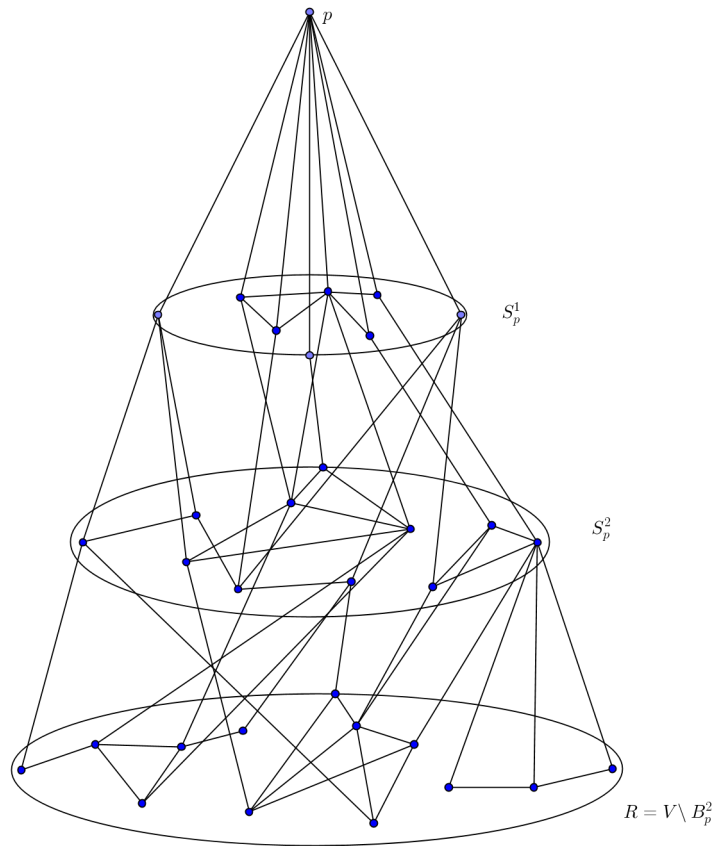
Figure 3.2: Example of a complete cone $C(G)$ 

Figure 3.3: Metric spheres centered at p 

3.1 Γ -Calculus for Cones

In this section we compute the operators Δ^c , Γ_1^c , and Γ_2^c in terms of Δ , Γ_1 , and Γ_2 , plus some error terms that result from the addition of the point p to G , for a cone $C(X, G)$ over a finite graph $G = (V, E)$ with $X \subset V$ some arbitrary subset of G 's vertices. For convenience we take $C = V \cup \{p\}$ to be the vertex set of $C(X, G)$. Since the operators Δ and Γ are invariant under translation by a constant, we may assume, without loss of generality, that $f(p) = 0$.

Denote by S_p^n and B_p^n the metric spheres and balls (resp.) with radius n and center p in the cone. For any subset $B \subset V$, the notation $v \in B \sim x$ means $v \in B$ and $v \sim x$. Thus, if $x \in X \subset V$, then $x \in C \sim p$ and in particular we often say $x \in S_p^1 \subset V$.

Remark 3.1.1. *Note that Δ and Γ_1 only depend on vertices that are at most one step away. Thus, $\Delta^c f(x) = \Delta f(x)$ and $\Gamma_1^c(f)(x) = \Gamma_1(f)(x)$ when $x \not\sim p$.*

Lemma 3.1.2. *Let f be a function on the cone $C(X, G)$ with $f(p) = 0$ then,*

$$\Delta^c f(x) = \begin{cases} \Delta f(x) - f(x), & x \sim p \\ \sum_{y \in S_p^1} f(y), & x = p \end{cases}.$$

Proof. 1. If $x \sim p$, then

$$\begin{aligned} \Delta^c f(x) &= \sum_{y \in C \sim x} (f(y) - f(x)) \\ &= \sum_{y \in V \sim x} (f(y) - f(x)) + (f(p) - f(x)) \\ &= \Delta f(x) - f(x). \end{aligned}$$

2. If $x = p$, then

$$\begin{aligned}\Delta^c f(p) &= \sum_{y \in C \sim p} (f(y) - f(p)) \\ &= \sum_{y \in S_p^1} f(y).\end{aligned}$$

□

Lemma 3.1.3. *Let f be a function on the cone $C(X, G)$ with $f(p) = 0$ then,*

$$\Gamma_1^c(f)(x) = \begin{cases} \Gamma_1(f)(x) + \frac{1}{2}f^2(x), & x \sim p \\ \frac{1}{2} \sum_{y \in S_p^1} f^2(y), & x = p \end{cases}.$$

Proof. 1. If $x \sim p$, then using (2.7)

$$\begin{aligned}\Gamma_1^c(f)(x) &= \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x))^2 \\ &= \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x))^2 + \frac{1}{2} (f(p) - f(x))^2 \\ &= \Gamma_1(f)(x) + \frac{1}{2}f^2(x).\end{aligned}$$

2. If $x = p$, then using (2.7)

$$\begin{aligned}\Gamma_1^c(f)(p) &= \frac{1}{2} \sum_{y \in V} (f(y) - f(p))^2 \\ &= \frac{1}{2} \sum_{y \in S_p^1} f^2(y).\end{aligned}$$

□

In the next few lemmas we calculate the constituent parts that appear in the definition of Γ_2^c .

Remark 3.1.4. *Note that Γ_2^c depends on vertices at most two steps away. Thus*

$\Gamma_2^c(f)(x)$ coincides with $\Gamma_2(f)(x)$ pointwise when $x \in V \setminus B_p^2$.

Lemma 3.1.5. *Let f be a function defined on the cone $C(X, G)$, and suppose $f(p) = 0$, then whenever*

1. $x \in S_p^2$,

$$\Gamma_1^c(f, \Delta^c f)(x) = \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)).$$

2. $x \in S_p^1$,

$$\begin{aligned} \Gamma_1^c(f, \Delta^c f)(x) &= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\ &\quad - \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) + \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\ &\quad + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x). \end{aligned}$$

3. $x = p$,

$$\Gamma_1^c(f, \Delta^c f)(x) = \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2.$$

Proof. 1. If $x \in S_p^2$, then using (2.7)

$$\begin{aligned}
\Gamma_1^c(f, \Delta^c f)(x) &= \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x)) (\Delta^c f(y) - \Delta^c f(x)) \\
&= \frac{1}{2} \sum_{y \in V \setminus S_p^1 \sim x} (f(y) - f(x)) (\Delta^c f(y) - \Delta^c f(x)) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x)) (\Delta^c f(y) - \Delta^c f(x)) \\
&= \frac{1}{2} \sum_{y \in V \setminus S_p^1 \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x)) (\Delta f(y) - f(y) - \Delta f(x)) \\
&= \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \\
&\quad - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y) (f(y) - f(x)) \\
&= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y) (f(y) - f(x)).
\end{aligned}$$

2. If $x \in S_p^1$, then using (2.7)

$$\begin{aligned}
\Gamma_1^c(f, \Delta^c f)(x) &= \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x)) (\Delta^c f(y) - \Delta^c f(x)) \\
&= \frac{1}{2} \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x) + f(x)) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x)) (\Delta f(y) - f(y) - \Delta f(x) + f(x)) \\
&\quad + \frac{1}{2} (f(p) - f(x)) \left(\sum_{y \in S_p^1} f(y) - \Delta f(x) + f(x) \right) \\
&= \frac{1}{2} \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \\
&\quad + \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \\
&\quad - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) \\
&\quad + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x) \\
&= \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x)) (\Delta f(y) - \Delta f(x)) \\
&\quad - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) \\
&\quad + \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x) \\
&= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) \\
&\quad + \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x).
\end{aligned}$$

3. If $x = p$, then using (2.7)

$$\begin{aligned}
\Gamma_1^c(f, \Delta^c f)(p) &= \frac{1}{2} \sum_{y \in C \sim p} (f(y) - f(p)) (\Delta^c f(y) - \Delta^c f(p)) \\
&= \frac{1}{2} \sum_{y \in S_p^1} (f(y) - f(p)) (\Delta f(y) - f(y) - \sum_{z \in S_p^1} f(z)) \\
&= \frac{1}{2} \sum_{y \in S_p^1} \left[f(y) \Delta f(y) - f^2(y) - f(y) \sum_{z \in S_p^1} f(z) \right] \\
&= \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2.
\end{aligned}$$

□

Lemma 3.1.6. *Let f be a function defined on the cone $C(X, G)$, and suppose $f(p) = 0$, then whenever*

1. $x \in S_p^2$,

$$\Delta^c \Gamma_1^c(f)(x) = \Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y).$$

2. $x \in S_p^1$,

$$\begin{aligned}
\Delta^c \Gamma_1^c(f)(x) &= \Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) \\
&\quad + \frac{1}{2} \left[\sum_{y \in S_p^1 \sim x} f^2(y) + \sum_{y \in S_p^1} f^2(y) \right] - \frac{\deg_G(x) + 1}{2} f^2(x).
\end{aligned}$$

3. $x = p$,

$$\Delta^c \Gamma_1^c(f)(x) = \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{2} \sum_{y \in S_p^1} f^2(y).$$

Proof. 1. If $x \in S_p^2$, then

$$\begin{aligned}
\Delta^c \Gamma_1^c(f)(x) &= \sum_{y \in C \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
&= \sum_{y \in V \setminus S_p^1 \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
&= \sum_{y \in V \setminus S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] \\
&\quad + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) + \frac{1}{2} f^2(y) - \Gamma_1(f)(x) \right] \\
&= \sum_{y \in V \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) \\
&= \Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y).
\end{aligned}$$

2. If $x \in S_p^1$, then

$$\begin{aligned}
\Delta^c \Gamma_1^c(f)(x) &= \sum_{y \in C \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
&= \sum_{y \in S_p^2 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) - \frac{1}{2} f^2(x) \right] \\
&\quad + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) + \frac{1}{2} (f^2(y) - f^2(x)) \right] \\
&\quad + \Gamma_1^c(f)(p) - \Gamma_1(f)(x) - \frac{1}{2} f^2(x) \\
&= \sum_{y \in S_p^2 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \frac{1}{2} \deg_C(x) f^2(x) \\
&\quad + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \Gamma_1(f)(x) \\
&= \sum_{y \in V \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \Gamma_1(f)(x) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} (\deg_G(x) + 1) f^2(x) \\
&= \Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) + \frac{1}{2} \left[\sum_{y \in S_p^1 \sim x} f^2(y) + \sum_{y \in S_p^1} f^2(y) \right] \\
&\quad - \frac{\deg_G(x) + 1}{2} f^2(x).
\end{aligned}$$

3. If $x = p$, then

$$\begin{aligned}
\Delta^c(\Gamma_1^c(f))(p) &= \sum_{y \in S_p^1} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(p) \right] \\
&= \sum_{y \in S_p^1} \left[\Gamma_1(f)(y) + \frac{1}{2}f^2(y) - \frac{1}{2} \sum_{z \in S_p^1} f^2(z) \right] \\
&= \sum_{y \in S_p^1} \Gamma_1(f)(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{|S_p^1|}{2} \sum_{z \in S_p^1} f^2(z) \\
&= \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{(|S_p^1| - 1)}{2} \sum_{y \in S_p^1} f^2(y).
\end{aligned}$$

□

Lemma 3.1.7. *Let f be a function defined on the cone $C(X, G)$, and suppose $f(p) = 0$, then whenever*

1. $x \in S_p^2$

$$\Gamma_2^c(f)(x) = \Gamma_2(f)(x) + \frac{3}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2}f(x) \sum_{y \in S_p^1 \sim x} f(y).$$

2. $x \in S_p^1$

$$\begin{aligned}
\Gamma_2^c(f)(x) &= \Gamma_2(f)(x) - \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\
&\quad + \frac{1}{4} \left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x)f^2(x) \right] - \frac{1}{2}f(x)\Delta f(x) \\
&\quad + \frac{1}{4} \left[\sum_{y \in S_p^1} f^2(y) + f^2(x) \right] - \frac{1}{2}f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)).
\end{aligned}$$

3. $x = p$

$$\begin{aligned}\Gamma_2^c(f)(x) &= \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) \\ &\quad - \frac{|S_p^1| - 3}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2.\end{aligned}$$

Proof. 1. If $x \in S_p^2$, then

$$\begin{aligned}\Gamma_2^c(f)(x) &= \frac{1}{2} \Delta^c \Gamma_1^c(f, f)(x) - \Gamma_1^c(f, \Delta^c f)(x) \\ &= \frac{1}{2} \Delta \Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \Gamma_1(f, \Delta f)(x) \\ &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)) \\ &= \Gamma_2(f)(x) + \frac{3}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} f(y).\end{aligned}$$

2. If $x \in S_p^1$, then

$$\begin{aligned}
\Gamma_2^c(f)(x) &= \frac{1}{2} \Delta^c \Gamma_1^c(f)(x) - \Gamma_1^c(f, \Delta^c f)(x) \\
&= \frac{1}{2} \Delta \Gamma_1(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) \\
&\quad - \frac{\deg_G(x) + 1}{4} f^2(x) - \Gamma_1(f, \Delta f)(x) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\
&\quad + \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) - \frac{1}{2} f(x) \Delta f(x) + \frac{1}{2} f^2(x) \\
&= \left[\frac{1}{2} \Delta \Gamma_1(f)(x) - \Gamma_1(f, \Delta f)(x) \right] - \frac{1}{2} \Gamma_1(f)(x) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 + \frac{1}{4} \left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg_G(x) f^2(x) \right] \\
&\quad + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) - \frac{1}{2} f(x) \Delta f(x) \\
&\quad + \frac{1}{4} f^2(x) \\
&= \Gamma_2(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\
&\quad + \frac{1}{4} \left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg_G(x) f^2(x) \right] - \frac{1}{2} f(x) \Delta f(x) \\
&\quad + \frac{1}{4} \left[\sum_{y \in S_p^1} f^2(y) + f^2(x) \right] - \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)).
\end{aligned}$$

3. If $x = p$, then

$$\begin{aligned}
\Gamma_2^c(f)(p) &= \frac{1}{2} \Delta^c \Gamma_1^c(f)(p) - \Gamma_1^c(f, \Delta^c f)(p) \\
&= \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{4} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) \\
&\quad + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2 \\
&= \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{|S_p^1| - 3}{4} \sum_{y \in S_p^1} f^2(y) \\
&\quad + \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2.
\end{aligned}$$

□

3.2 Γ_2^c for $C(G)$

When $C(X, G) = C(G)$ is the full cone over $V(G)$, then $S_p^1 = V$ and $S_p^2 = \emptyset$. Thus, Lemma 3.1.7 reduces to the following Lemma.

Lemma 3.2.1.

$$\Gamma_2^c(f)(x) = \begin{cases} \Gamma_2(f)(x) + \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in V} f^2(y) + \frac{1}{4} f^2(x), & x \sim p \\ \sum_{y \in V} \Gamma_1(f)(y) - \frac{|V| - 3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left(\sum_{y \in V} f(y) \right)^2, & x = p \end{cases}$$

Proof. Since $S_p^1 = V$ and $S_p^2 = \emptyset$ the first case in Lemma 3.1.7 disappears. In case 2 notice that when $S_p^1 = V$, then $\frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 = \Gamma_1(f)(x)$ and $\sum_{y \in S_p^1 \sim x} (f^2(y) - f^2(x)) = (\Delta f^2)(x)$. Since $\frac{1}{2} \Gamma_1(f)(x) = \frac{1}{4} (\Delta f^2)(x) - \frac{1}{2} f(x) \Delta f(x)$ the case when $x \sim p$ follows. When $x = p$, applying the identity (2) gives the desired result. □

This leads to the following result regarding the curvature of the cone,

Theorem 1.0.11. *Suppose G satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$ then the subgraph $G \subset C(G)$ satisfies $CD(K + \frac{1}{2}, \infty)$.*

Proof of Theorem 1.0.11. Suppose G satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$. Since G satisfies $CD(K, \infty)$ then by Lemma 3.2.1 for $x \sim p$,

$$\Gamma_2^c(f)(x) = \Gamma_2(f)(x) + \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in V} f^2(y) + \frac{1}{4}f^2(x).$$

Therefore, given that,

$$\Gamma_2(f)(x) \geq \frac{1}{2}\Gamma_1(f)(x).$$

one has,

$$\begin{aligned} & \Gamma_2(f)(x) + \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in V} f^2(y) + \frac{1}{4}f^2(x) \\ & \geq K\Gamma_1^c(f)(x) - \frac{K}{2}f^2(x) + \frac{1}{2}\Gamma_1(f)(x) - \frac{1}{4}f^2(x) + \frac{1}{4}\sum_{y \in V} f^2(y) + \frac{1}{4}f^2(x) \end{aligned}$$

and so,

$$\Gamma_2^c(f)(x) \geq (K + \frac{1}{2})\Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in V} f^2(y) - \frac{K}{2}f^2(x).$$

Since $K \leq \frac{1}{2}$, then $\frac{1}{4}\sum_{y \in V} f^2(y) - \frac{K}{2}f^2(x) \geq 0$. Hence we may drop both terms from the inequality and $C(G)$ satisfies $CD(K + \frac{1}{2}, \infty)$ for $x \sim p$. \square

Chapter 4

Graph Energy

Let us recall here some pertinent definitions for readability. First, the Conical Curvature-Dimension condition:

Definition 4.0.1 (*CCD(K, N) Conical Curvature-Dimension Conditions [19]*). *1.0.2*

Let $G = (V, E)$ be a finite, connected, undirected, loop-edge free graph and consider the cone over the vertex set of G . G is said to satisfy the conical curvature-dimension condition, $CCD(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty]$, if the cone over G satisfies the $CD(K, N)$ curvature-dimension conditions at the vertex p , namely if

$$\Gamma_2^c(f)(p) \geq \frac{(\Delta^c f)^2(p)}{N} + K\Gamma_1^c(f)(p),$$

holds for any function f defined on the cone and Δ^c , Γ_1^c and Γ_2^c are the usual Δ , Γ_1 and Γ_2 operators (see (2.4), (2.7) and (2.8)) except on the cone $C(G)$ over G . We note that the second term in (2.1) is understood to be zero when $N = \infty$.

Furthermore, we refer to the curvature at the cone point K as the conical curvature of the finite graph G .

Definition 4.0.2 (Conical Ricci Curvatures). *We define the conical Ricci curvature*

by

$$CRic_N(G) := \sup\{K : G \text{ satisfies } CCD(K, N)\},$$

where $CCD(K, N)$ is defined in 2.1 and

$$CRic_\infty(G) := \sup\{K : G \text{ satisfies } CCD(K, \infty)\}.$$

Lastly, we recall the definition for the energy of a graph.

Definition 4.0.3. *The energy of a finite, simple, undirected, unweighted graph is given by:*

$$\begin{aligned} \|\nabla f\|_2^2 &= \frac{1}{2} \sum_{v \in V} \sum_{u \sim v} (f(u) - f(v))^2 \\ &= \sum_{v \in V} \Gamma_1(f)(v). \end{aligned}$$

If the cone $C(G)$, over a finite graph G , satisfies the $CD(K, N)$ inequality at the vertex, p , then by Lemmas 3.1.3 and 3.1.2, we get

$$\Gamma_2^c(f)(p) \geq \frac{1}{N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y) \quad (4.1)$$

$$= \frac{1}{N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \|f\|_2^2 \quad (4.2)$$

This leads to the following,

Theorem 1.0.6. *If a finite graph, G , satisfies the $CCD(K, N)$ curvature-dimension condition, then for any function $f \in \ell^2(V)$ on G one has*

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \|f\|_2^2. \quad (4.3)$$

For functions $f \in \ell^2(V)$ with $\text{avg}(f) = 0$, this reduces to the following global

Poincaré inequality,

$$\|f\|_2 \leq \sqrt{\frac{4}{2K + |V| - 3}} \|\nabla f\|_2,$$

where $\|\nabla f\|_2$ is the energy of the graph G that the cone is based on.

Proof of Theorem 1.0.6. Suppose a finite graph G satisfies $CCD(K, N)$ condition.

By Lemma 3.2.1,

$$\Gamma_2^c(f)(p) = \sum_{y \in V} \Gamma_1(f)(y) - \frac{|V| - 3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left(\sum_{y \in V} f(y) \right)^2, \quad (4.4)$$

Upon combining (4.4) and (4.2), we arrive at

$$\sum_{y \in V} \Gamma_1(f)(y) - \frac{|V| - 3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left(\sum_{y \in V} f(y) \right)^2 \geq \frac{1}{N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y),$$

which simplifies to

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \|f\|_2^2.$$

For $f \in \ell^2(V)$ with $\text{avg}(f) = 0$, the above reduces to

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2K + |V| - 3}{4} \|f\|_2^2.$$

Thinking of $\Gamma_1(f)$ as a discrete analogue for $|\nabla f|^2$ and recalling 4.1 this yields the

Poincaré inequality,

$$\|f\|_2 \leq \sqrt{\frac{4}{2K + |V| - 3}} \|\nabla f\|_2, \text{ when } f \in \ell^2(V), \text{ and } \text{avg}(f) = 0.$$

□

Corollary 4.0.4.

$$\lambda_1 \geq \frac{2K + |V| - 3}{4}$$

Proof. By the inequalities in the previous proof one has that for all f such that $\text{avg}(f) = 0$,

$$\lambda_1 = \inf \left\{ \frac{\frac{1}{2} \sum_{v \in V} \sum_{u \sim v} (f(u) - f(v))^2}{\sum_{v \in V} f^2(v)} : f \in \ell^2(V) \text{ and } \text{avg}(f) = 0 \right\}.$$

$$\begin{aligned} \sum_{y \in V} \Gamma_1(f)(y) &\geq \frac{2K + |V| - 3}{4} \|f\|_2^2 \\ \frac{\frac{1}{2} \sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))^2}{\sum_{y \in V} f^2(y)} &\geq \frac{2K + |V| - 3}{4}. \end{aligned}$$

Thus, by the Rayleigh-Ritz quotient, the result follows. \square

This result allows one to bound λ_1 from below in terms of the curvature just at the cone point. So rather than calculating the largest K such that the $CD(K, N)$ inequality holds at every vertex, the $CD(K, N)$ inequality only needs to be checked at a single point. Several years ago, Yann Ollivier [25], asked whether there exists a family of expanding graphs with non-negative Ricci curvature. Since λ_1 is bounded from above by $\frac{|V|}{|V|-1}$ [6] and the size of the vertex set for an expanding family must go to infinity, then the curvature at the cone point must be going to negative infinity, but it isn't the curvature of the underlying graph that is going to negative infinity, it's the curvature at the cone point. Note: expanders must have bounded degree.

Corollary 4.0.5.

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{(4 - N)|V| + (2K^c - 3)N}{4N|V|} \left(\sum_{v \in V} f(v) \right)^2$$

Proof. By Cauchy-Schwarz, $\sum_{v \in V} f^2(v) \geq \frac{1}{|V|} (\sum_{v \in V} f(v))^2$. Thus,

$$\begin{aligned} \sum_{y \in V} \Gamma_1(f)(y) &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K+|V|-3}{4} \sum_{y \in V} f^2(y) \\ &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K+|V|-3}{4|V|} \left(\sum_{v \in V} f(v) \right)^2 \\ &= \frac{4|V|-2|V|N+2K^cN+N|V|-3N}{4N|V|} \left(\sum_{v \in V} f(v) \right)^2 \\ &= \frac{(4-N)|V|+(2K^c-3)N}{4N|V|} \left(\sum_{v \in V} f(v) \right)^2 \end{aligned}$$

□

Theorem 1.0.7. *For any finite graph, G , and a given $N > 1$, the conical curvature, K^c , cannot exceed the following number:*

$$K_{max}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N}.$$

Proof of Theorem 1.0.7. Suppose a finite graph G satisfies the $CCD(K, N)$ and f is a non-zero harmonic function, then one has $\sum_{y \in V} \Gamma_1(f)(y) = 0$. Thus,

$$\frac{2K+|V|-3}{4} \sum_{y \in V} f^2(y) \leq \frac{N-2}{2N} \left(\sum_{y \in V} f(y) \right)^2.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{y \in V} f(y) \right)^2 \leq |V| \cdot \sum_{y \in V} f^2(y),$$

which implies

$$\frac{2K+|V|-3}{4} \sum_{y \in V} f^2(y) \leq \frac{N-2}{2N} |V| \cdot \sum_{y \in V} f^2(y).$$

Since, f is not constant zero,

$$K \leq \frac{|V|}{2} - \frac{2|V|}{N} + \frac{3}{2}.$$

□

Having established relevant bounds for the curvature at the cone point over the vertex set of a graph, and the first non-trivial eigenvalue of Δ , we now turn to an investigation of when the maximum curvature value is achieved.

Lemma 4.0.6. *For any finite graph, G , the Ricci curvatures $\text{Ric}_\infty(G)$, $\text{Ric}_N(G)$, $CRic_\infty(G)$ and $CRic_N(G)$ are realized by some functions, i.e. there are functions that achieve the equality in the (corresponding) defining Bakry-Émery curvature-dimension inequalities.*

Proof. We will prove the result for $\text{Ric}_N(G)$. The proofs for the other Ricci curvatures are basically the same. Since, $\text{Ric}_N(G)$ is the supremum of all possible lower curvature bounds, one can find a sequence, f_i such that for all $v \in V$

$$\frac{1}{N}(\Delta f_i)^2(v) + \text{Ric}_N(G)\Gamma_1(f_i)(v) \leq \Gamma_2(f_i)(v),$$

and

$$\Gamma_2(f_i)(v) < \frac{1}{N}(\Delta f_i)^2(v) + \left(\text{Ric}_N(G) + \frac{1}{i}\right)\Gamma_1(f_i)(v).$$

Recall that the inequalities above are invariant under rescaling of the f_i 's. Hence, without loss of generality, we may assume that $\text{Range}(f_i) \subset [-1, 1]$, for all i . Since $V(G)$ is finite, then there exists subsequence f_j of the f_i 's that converges to a function f . Taking the limit of the inequalities above as $j \rightarrow \infty$ shows that f achieves $\text{Ric}_N(G)$. □

Theorem 1.0.8. *Suppose G satisfies $CCD(K_{max}^c, N)$. Then $f : V \rightarrow \mathbb{R}$ realizes*

K_{max}^c if and only if f is either constant or $f - \text{avg}(f)$ is an eigenfunction corresponding to $\lambda_1(G) = \frac{N-2}{4N}|V|$. Furthermore, when G is a complete graph, f must be harmonic.

Proof. Suppose G satisfies $CCD(K_{max}^c, N)$, then for any f

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K_{max}^c + |V| - 3}{4} \sum_{y \in V} f^2(y).$$

Since $K_{max}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N} = \frac{N \cdot |V| + 3N - 4|V|}{2N}$, this simplifies to

$$\begin{aligned} \sum_{y \in V} \Gamma_1(f)(y) &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N \cdot |V| + 3N - 4|V| + N \cdot |V| - 3N}{4N} \sum_{y \in V} f^2(y) \\ &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N \cdot |V| - 2|V|}{2N} \sum_{y \in V} f^2(y) \\ &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N-2}{2N} \cdot |V| \sum_{y \in V} f^2(y) \\ &\geq \frac{N-2}{2N} \left[|V| \sum_{y \in V} f^2(y) - \left(\sum_{y \in V} f(y) \right)^2 \right]. \end{aligned}$$

Take $\phi : V \rightarrow \mathbb{R}$ to be any variational function on the vertex set of G and let $t \in \mathbb{R}$, then

$$\sum_{y \in V} \Gamma_1(f + t\phi)(y) \geq \frac{N-2}{2N} \left[|V| \sum_{y \in V} (f + t\phi)^2(y) - \left(\sum_{y \in V} (f + t\phi)(y) \right)^2 \right].$$

Suppose now that f achieves K_{max}^c , then for any ϕ the above inequality becomes an equality (i.e. $\frac{d}{dt}|_{t=0}$ of both sides must be equal for any variation ϕ). Hence a straightforward calculation yields the linearized equation,

$$\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))(\phi(z) - \phi(y)) = \frac{N-2}{2N} \left[|V| \sum_{y \in V} f(y)\phi(y) - \sum_{y \in V} f(y) \sum_{y \in V} \phi(y) \right]. \quad (4.5)$$

Now fix $r \in V$ and let $\phi(y) = \delta_r(y)$. Notice that $\sum_{z \sim y} (f(z) - f(y))(\delta_r(z) - \delta_r(y))$ is

zero except when $y = r$ or $y \sim r$. If $y = r$ the result is $-\sum_{z \sim r} (f(z) - f(r))$. When $y \sim r$ there is exactly one non-zero term in the sum, $(f(r) - f(y))$ and summing over all $y \sim r$ we get $\sum_{y \sim r} (f(r) - f(y))$. Thus

$$\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y)) (\delta_r(z) - \delta_r(y)) = -2\Delta f(r), \quad (4.6)$$

and (4.5) reduces to

$$-2\Delta f(r) = \frac{N-2}{2N} \left[|V|f(r) - \sum_{y \in V} f(y) \right], \quad (4.7)$$

which is equivalent to

$$\Delta f(r) = \frac{N-2}{4N} \widehat{\Delta} f(r), \quad (4.8)$$

where $\widehat{\Delta}$ denotes the Laplacian for the graph completion \widehat{G} of G to a complete graph on $|V(G)|$ vertices. Thus, when G is a complete graph, f must be harmonic on G . Since otherwise, this would imply that $3N = -2$, which is impossible.

Now, suppose G is arbitrary. By equation (4.7)

$$\Delta(f - \text{avg}(f))(r) = -\frac{N-2}{4N} |V| (f - \text{avg}(f))(r).$$

Now if f is not constant then by the Rayleigh quotient, (2.6), we see that $\lambda_1(G) = \frac{N-2}{4N} |V|$ and $f - \text{avg}(f)$ is an eigenfunction for λ_1 .

For the "if" direction suppose for some non-constant function, f , that $f - \text{avg}(f)$ is an eigenfunction for $\lambda_1 = \frac{N-2}{4N} |V|$. Tracing back the above computations one has (4.5) holds for $\phi = \delta_y$'s. Then since (4.5) is linear in ϕ , one can use $f = \sum_{y \in V} f(y) \delta_y$ instead of ϕ which will translate to f realizing K_{max}^c . \square

Chapter 5

Generalized Harmonic Functions on Finite Graphs

We begin this chapter with a motivating lemma. This lemma came as an outgrowth of the curvature maximizer result above (see 1.0.8).

Remark 5.0.1. *The spectrum of the adjacency matrix for a finite, k -regular graph G is contained inside the interval $[-k, k]$. After diagonalizing the adjacency matrix and subtracting it from the diagonal degree matrix $k \cdot I$ one finds that the eigenvalues from the unnormalized graph Laplacian on a k -regular graph lie in the interval $[0, 2k]$. [10]*

Lemma 5.0.2. *Let G be a k -regular graph and f an eigenfunction of the unnormalized graph Laplacian corresponding to the first non-trivial eigenvalue λ_1 such that $\text{avg}(f) = 0$. If $0 < \lambda_1 < 2k$, then f is generalized harmonic of both types I and II.*

Proof. By our hypothesis, $\Delta f = -\lambda_1 f$, hence for any $v \in V$,

$$\sum_{u \sim v} f(u) - \deg(v)f(v) = -\lambda_1 f(v),$$

and so,

$$(k - \lambda_1)f(v) = \sum_{u \sim v} f(u) = kg_1[f](v).$$

Therefore, since $g_1[f]$ is a multiple of f , $g_1[f]$ is again an eigenfunction corresponding to λ_1 and the same argument can be applied, yielding

$$(k - \lambda_1)^2 f(v) = k(k - \lambda_1)g_1[f](v) = k^2 g_2[f](v),$$

where $g_2[f] = g_1[g_1[f]]$. Iterating this construction, we will get

$$(k - \lambda_1)^n f(v) = k^n g_n[f](v),$$

or

$$g_n[f](v) = \left(\frac{k - \lambda_1}{k} \right)^n f(v).$$

Since we assumed $0 < \lambda_1 < 2k$ then $\left| \frac{k - \lambda_1}{k} \right| < 1$. Taking the limit as $n \rightarrow \infty$ the result follows.

To prove that f is generalized harmonic of type II we notice that

$$(k + 1 - \lambda_1)f(v) = (k + 1)h_1(v).$$

A similar iteration argument as above will lead to

$$h_n(v) = \left(\frac{k + 1 + \lambda_1}{k + 1} \right)^n f(v),$$

and again this is converging to 0 (which is $\text{avg}(f)$ after normalization). \square

Remark 5.0.3. *The condition that $0 < \lambda_1 < 2k$ holds whenever G is connected and not bipartite (see Proposition 0.5 in [10]).*

5.1 Unit Spherical Averaging Operator

This leads one to consider functions with the property that $\lim_{n \rightarrow \infty} g_n[f](v) = \text{avg}(f)$. Hence, the preceding definitions 1.0.3 and 1.0.4.

Recall from the introduction, that for $G = (V, E)$ a simple, finite, connected, unweighted, loop-edge free graph and $f : V \rightarrow \mathbb{R}$, the unit spherical averaging operator is $g : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|V|}$, such that:

$$g[f](v) = \frac{1}{\text{deg}(v)} \sum_{u \sim v} f(u)$$

for all $v \in V$. Furthermore, define $g_i[f](v) := g[g_{i-1}[f]](v)$, $g_0[f](v) = f(v)$ and $g_1[f](v) = g[f](v)$. Then $\{g_i[f](v)\}_{0 \leq i < \infty}$ is the orbit of $v \in V$ via the transformation $T^i := g_i[f]$. We say f is generalized harmonic of type I if and only if,

$$\lim_{i \rightarrow \infty} g_i[f](v) = \text{avg}(f)$$

for all $v \in V$. Note that, by definition, $g_1[f]$ is there normalized adjacency matrix for the graph G . Thus, $g_1[f](x) = (\bar{\Delta} + I)f(v)$ and therefore $g_i[f](x) = (I + \bar{\Delta})^i f(x)$.

Let \mathfrak{G} be the set of functions, f , such that $\lim_{n \rightarrow \infty} g_n[f](v) = \text{avg}(f)$ i.e. the set of generalized harmonic functions of type I on a graph G . By the maximum principle for harmonic functions on graphs, any harmonic function on a finite graph must be constant on connected components. Furthermore, it is clear that constant functions satisfy the limit definition given above (in fact, for a constant function f , $g_n[f]$ is constant for all n) and therefore they are also generalized harmonic. Hence, \mathfrak{G} is non-empty. Thus, we conclude:

Lemma 5.1.1. *The set \mathfrak{G} of generalized harmonic functions of type I on a finite graph is non-empty and contains the set of harmonic (constant) functions.*

Furthermore,

Theorem 1.0.12. *The set of generalized harmonic functions of type I on a finite graph forms a subspace of $\ell^2(V, \deg)$ that contains harmonic functions.*

Proof. Let f and h be generalized harmonic functions and $\alpha, \beta \in \mathbb{R}$, and consider $\alpha f + \beta h$.

$$\lim_{n \rightarrow \infty} g_n[\alpha f + \beta h](v) = \lim_{n \rightarrow \infty} \frac{1}{\deg(v)} \sum_{u \sim v} g_{n-1}[\alpha f + \beta h](u)$$

When $n = 1$,

$$\begin{aligned} g_1[\alpha f + \beta h](v) &= \frac{\alpha}{\deg(v)} \sum_{u \sim v} f(u) + \frac{\beta}{\deg(v)} \sum_{u \sim v} h(u) \\ &= \alpha g_1[f](v) + \beta g_1[h](v) \end{aligned}$$

Suppose that for $n < m$, $g_n[\alpha f + \beta h](v) = \alpha g_n[f](v) + \beta g_n[h](v)$. Then if $n = m$,

$$g_m[\alpha f + \beta h](v) = \frac{1}{\deg(v)} \sum_{u \sim v} g_{m-1}[\alpha f + \beta h](v) \quad (5.1)$$

$$= \frac{1}{\deg(v)} \sum_{u \sim v} (\alpha g_{m-1}[f](v) + \beta g_{m-1}[h](v)) \quad (5.2)$$

$$= \alpha g_m[f](v) + \beta g_m[h](v). \quad (5.3)$$

Thus the space of generalized harmonic functions of type I is a subspace of $\ell^2(V, \deg)$, that contains the harmonic functions. \square

Remark 5.1.2. *From a dynamical point of view, the unit spherical averaging operator defined above is a Markov operator on G , and the requirement that its iterations stabilize means that $\text{avg}(f)$ is a fixed point for $g_n[f]$ on G .*

Since, $g_n[f](x) = (I + \bar{\Delta})^n f(x)$. The spectrum of $\bar{\Delta}$ is a multiset of eigenvalues $0 \leq \lambda_j \leq 2$ with $0 \leq j \leq |V| - 1$ (for convenience let $|V| = m$), such that for some eigenfunction, $f_j : V \rightarrow \mathbb{R}$, $\bar{\Delta} f_j = -\lambda_j \cdot f_j$. Standard analysis on graphs yields that $\lambda_0 = 0$, $\lambda_0 < \lambda_1$ if and only if G is connected, and $\lambda_{m-1} = 2$ if and only if

G is bipartite (see [6]). Furthermore, these eigenfunctions f_j are orthogonal to one another with respect to the inner product,

$$\langle f, g \rangle = \sum_{v \in V} \deg(v) f(v) g(v),$$

and therefore given any $f : V \rightarrow \mathbb{R}$ one may write f as a sum of weighted projections onto eigenfunctions, $\{f_j\}_{0 \leq j \leq m-1}$, associated to $\bar{\Delta}$'s eigenvalues. i.e.

$$f = c_0 f_0 + c_1 f_1 + \dots + c_{m-1} f_{m-1},$$

where $c_j = \frac{\langle f, f_j \rangle}{\langle f_j, f_j \rangle}$ and,

$$\bar{\Delta} f = -c_1 \lambda_1 f_1 - c_2 \lambda_2 f_2 - \dots - c_{m-1} \lambda_{m-1} f_{m-1}$$

Thus,

$$g_n[f] = c_0 f_0 + c_1 (1 - \lambda_1)^n f_1 + \dots + c_{m-1} (1 - \lambda_{m-1})^n f_{m-1}.$$

Recall that, if G is a connected, non-bipartite graph, then $0 < \lambda_j < 2$, for $0 < j < m$. This implies that $|1 - \lambda_j| < 1$ and $\lim_{n \rightarrow \infty} |1 - \lambda_j|^n = 0$. So, the condition that $\lim_{n \rightarrow \infty} g_n[f](x) = \text{avg}(f)$ for all $x \in V$ is equivalent to $c_0 f_0 = \text{avg}(f)$. Since, λ_0 is the eigenvalue associated to the constant functions and we've taken $\langle f, g \rangle = \sum_{v \in V} \deg(v) f(v) g(v)$, then without loss of generality, take $f_0(v) = 1, \forall v \in V$ and thus,

$$c_0 f_0 = \frac{\sum_{v \in V} \deg(v) f(v)}{\sum_{v \in V} \deg(v)}.$$

Definition 5.1.3. Let the weighted average of a function, f , on a graph with respect to the measure μ be denote $w.\text{avg}_\mu(f) := \frac{\sum_{v \in V} \mu(v) f(v)}{\sum_{v \in V} \mu(v)}$.

Therefore, taking $\mu(v) = \deg(v)$ we see that:

Theorem 1.0.13. A function on a non-bipartite graph is generalized harmonic of

type I if and only if its weighted average is equal to its average, i.e. if and only if:

$$\frac{\sum_{v \in V} \deg(v)f(v)}{\sum_{v \in V} \deg(v)} = \frac{1}{|V|} \sum_{v \in V} f(v).$$

This immediately implies:

Corollary 5.1.4. *Every function on a finite k -regular graph is generalized harmonic of type I.*

In particular, every function on a complete graph is generalized harmonic of type I.

Remark 5.1.5. *So, if there exists a function on your graph G such that $w.\text{avg}_{\deg(f)} \neq \text{avg}(f)$, then G can't be regular for any k .*

Remark 5.1.6. *Furthermore, there isn't a maximum principle for generalized harmonic functions of type I on finite graphs (like there is for harmonic functions on a graph). Since a delta function, $\delta_v(x) = \begin{cases} 1, & \text{when } x = v \\ 0, & \text{otherwise} \end{cases}$, would be generalized harmonic on any k -regular graph.*

If G is connected and bipartite then $\lambda_{m-1} = -1$. Since,

$$g_i[f] = c_0 f_0 + c_1 (1 - \lambda_1)^i f_1 + \dots + c_{m-1} (1 - \lambda_{m-1})^i f_{m-1}$$

and in the limit everything except for the terms $c_0 f_0$ and $c_{m-1} (-1)^i f_{m-1}$ collapse to zero, there is no unique value for $\lim_{i \rightarrow \infty} g_i[f]$ for any f with a non-zero c_{m-1} coefficient. Thus, the limit in the definition for generalized harmonic functions of type I does not exist for bipartite graphs. However, one should note that the Cesàro mean of the $g_i[f]$'s for a connected bipartite graph G will converge. For convenience, let $g_i^\perp[f](x)$ represent the component in the Fourier decomposition of $g_i[f](x)$ that

is orthogonal to the λ_{n-1} eigenspace. Then,

$$\lim_{i \rightarrow \infty} g_i^\perp[f](x) = c_0.$$

Since Cesáro means preserve limits, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N g_i^\perp[f](x) = c_0 f_0$ and since

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N (-1)^i = 0,$$

then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N g_i[f](x) = c_0 f_0.$$

So, if f is generalized harmonic of type I then the Cesáro mean of $\{g_i[f]\}_{i \geq 0}$ converges to $w.\text{avg}(f)$, for all $x \in V$. Since

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N (1 - \lambda_{n-1})^i = 0,$$

then if the Cesáro mean of $\{g_i[f]\}_{i \geq 0}$ converges to $w.\text{avg}(f)$, for all $x \in V$, f is generalized harmonic of type I. Thus, f is generalized harmonic of type I if and only if the Cesáro mean of $\{g_i[f]\}_{i \geq 0}$ converges to $w.\text{avg}(f)$.

5.2 Unit Ball Averaging Operator

We now turn our attention to the unit ball averaging operator, given by $h : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|V|}$, such that:

$$h[f](v) = \frac{1}{\deg_G(v) + 1} \left(f(v) + \sum_{u \sim v} f(u) \right)$$

for all $v \in V$. Furthermore, define $h_i[f](v) := h[h_{i-1}[f]](v)$, $h_0[f](v) = f(v)$, and $h_1[f](v) = h[f](v)$. We say f is generalized harmonic of type II if and only if,

$$\lim_{i \rightarrow \infty} h_i[f](v) = \text{avg}(f)$$

for all $v \in V$ Notice that from the definition though that:

$$\begin{aligned}
 h_1[f](v) &= \frac{1}{\deg(v) + 1} \left[f(v) + \sum_{u \sim v} f(u) \right] \\
 &= \frac{1}{\deg(v) + 1} \left[f(v) + \sum_{u \sim v} f(u) - \deg(v)f(v) + \deg(v)f(v) \right] \\
 &= \frac{1}{\deg(v) + 1} [(\deg(v) + 1)f(v) + \deg(v)\bar{\Delta}f(v)] \\
 &= (I + \frac{\deg(v)}{\deg(v) + 1}\bar{\Delta})f(v).
 \end{aligned}$$

If we consider the graph Cartesian (sometimes called box) product of G with the complete graph on two vertices. The result is a connected graph H made up of two copies of G , call them G_0 and G_1 in which each vertex in G_0 , $(v, 0)$, is connected to its copy in G_1 , $(v, 1)$, via an edge. This is a sort of double cover of G and each $x \in H$ has degree one greater than it did in G , i.e. $\deg_H(x) = \deg_G(x) + 1$. Take $F : V(H) \rightarrow \mathbb{R}$ such that $F(x, 0) = F(x, 1) = f(x)$ for all $x \in G$, then the unit spherical average flow on H is exactly the unit ball average flow on G and the results from the previous section yield:

Theorem 1.0.14. *Given a complete, undirected, non-bipartite, finite graph, $G = (V, E)$ and $f : V(G) \rightarrow \mathbb{R}$, $\lim_{i \rightarrow \infty} h_i[f](x) = \text{avg}(f)$ if and only if for $H = G \square K_2$, with $V(H) = \{(v, t) \in V(G) \times \{0, 1\}\}$ and $F : V(H) \rightarrow \mathbb{R}$ such that $F(v, t) = f(v)$, $\lim_{i \rightarrow \infty} g_i[F](x) = \text{avg}(F)$.*

Proof. This follows from a direct computation for the unit spherical averaging flow on $H = G \square K_2$. Let $(v, t) \in V(H)$ for some $t \in \{0, 1\}$ and $F : V(H) \rightarrow \mathbb{R}$ such that $F(v, t) = f(v)$, then without loss of generality we can take $t = 0$ and then split the

sum into a sum across the edges in G and the edges in K_2 , i.e.

$$\begin{aligned} g_1[F](v, t) &= \frac{1}{\deg_G(x) + 1} \left[\sum_{(u,0) \sim (v,0)} F(u, 0) + \sum_{(v,1) \sim (v,0)} F(v, 1) \right] \\ &= \frac{1}{\deg_G(x) + 1} \left[\sum_{u \sim v} f(u) + f(v) \right] \\ &= h_1[f](v). \end{aligned}$$

□

Corollary 5.2.1. *Since $\lim_{i \rightarrow \infty} g_i[F](x) = \text{avg}(F)$ if and only if $\text{avg}(F) = \text{w. avg}(F)$, then $f : V(G) \rightarrow \mathbb{R}$ is generalized harmonic of type II, if and only if*

$$\frac{1}{|V(G)|} \sum_{v \in V(G)} f(v) = \frac{\sum_{v \in V(G)} \deg_G v f(v) + \sum_{v \in V(G)} f(v)}{\sum_{v \in V(G)} \deg_G(v) + \sum_{v \in V(G)} 1}.$$

Proof.

$$\begin{aligned} \text{avg}(F) &= \text{w. avg}(F) \\ \frac{\sum_{(v,t) \in V(H)} F(v, t)}{|V(H)|} &= \frac{\sum_{(v,t) \in V(H)} \deg_H(v, t) F(v, t)}{\sum_{(v,t) \in V(H)} \deg_H(v, t)} \\ \frac{2 \sum_{v \in V(G)} f(v)}{2|V(G)|} &= \frac{2 \sum_{v \in V(G)} (\deg_G(v) + 1) f(v)}{2 \sum_{v \in V(G)} (\deg_G(v) + 1)} \\ \frac{\sum_{v \in V(G)} f(v)}{|V(G)|} &= \frac{\sum_{v \in V(G)} \deg(v) f(v) + \sum_{v \in V(G)} f(v)}{\sum_{v \in V(G)} \deg(v) + \sum_{v \in V(G)} 1}. \end{aligned}$$

□

5.3 Explicit Examples

Let $G = K_n$, the complete graph on n -vertices. Take

$$f(y) = \delta_x(y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{else} \end{cases}.$$

Fix $x \in K_n$ and let $y_1, y_2 \sim x$ in K_n . Then,

$$\begin{aligned} g_1[\delta_x](y_1) &= \frac{1}{\deg(y_1)} \sum_{z \sim y_1} \delta_x(z) \\ &= \frac{1}{n-1} \end{aligned}$$

and,

$$\begin{aligned} g_1[\delta_x](y_2) &= \frac{1}{\deg(y_1)} \sum_{z \sim y_1} \delta_x(z) \\ &= \frac{1}{n-1}. \end{aligned}$$

Suppose that for all $0 \leq i < m$, $g_i[\delta_x](y_1) = g_i[\delta_x](y_2)$. Since, $g_{m-1}[\delta_x](y_1) = g_{m-1}[\delta_x](y_2)$ then

$$\begin{aligned} g_m[\delta_x](y_1) &= \frac{1}{n-1} \sum_{z \sim y_1} g_{m-1}[\delta_x](z) \\ &= \frac{1}{n-1} \left(\sum_{z \sim y_1 \setminus y_2} g_{m-1}[\delta_x](z) + g_{m-1}[\delta_x](y_2) \right) \\ &= \frac{1}{n-1} \left(\sum_{z \sim y_1 \setminus y_2} g_{m-1}[\delta_x](z) + g_{m-1}[\delta_x](y_1) \right) \\ &= g_m[\delta_x](y_2). \end{aligned}$$

Thus, for all $y_1, y_2 \sim x$, $g_i[\delta_x](y_1) = g_i[\delta_x](y_2)$. Furthermore, since $g_{i-1}[\delta_x](y_1) =$

$g_{i-1}[\delta_x](y_2)$ for all $y_1, y_2 \sim x \in K_n$,

$$\begin{aligned} g_i[\delta_x](x) &= \frac{1}{n-1} \sum_{y \sim x} g_{i-1}[\delta_x](y) \\ &= \frac{1}{n-1} ((n-1)g_{i-1}[\delta_x](y_1)) \\ &= g_{i-1}[\delta_x](y_1). \end{aligned}$$

Since $g_i[\delta_x](x) = g_{i-1}[\delta_x](y)$, then for $i > 1$

$$\begin{aligned} g_i[\delta_x](y) &= \frac{1}{n-1} \sum_{z \sim y} g_{i-1}[\delta_x](z) \\ &= \frac{1}{n-1} ((n-2)g_{i-1}[\delta_x](y) + g_{i-1}[\delta_x](x)) \\ &= \frac{1}{n-1} ((n-2)g_{i-1}[\delta_x](y) + g_{i-2}[\delta_x](y)). \end{aligned}$$

Let $\partial g_i[f](x)$ denote $g_i[f](x) - g_{i-1}[f](x)$. Since

$$g_i[\delta_x](y) = \frac{1}{n-1} ((n-2)g_{i-1}[\delta_x](y) + g_{i-2}[\delta_x](y)),$$

then,

$$\partial g_i[\delta_x](y) = g_i[\delta_x](y) - g_{i-1}[\delta_x](y) = \frac{1}{n-1} ((n-2)\partial g_{i-1}[f](x) + \partial g_{i-2}[f](x)).$$

$$\begin{aligned}
\partial g_0[\delta_x](y) &= 0 \\
\partial g_1[\delta_x](y) &= \frac{1}{n-1} \\
\partial g_2[\delta_x](y) &= \frac{(n-2)}{(n-1)^2} - \frac{1}{n-1} \\
&= \frac{-1}{(n-1)^2} \\
\partial g_3[\delta_x](y) &= \frac{(n-2)^2 + (n-1)}{(n-1)^3} - \frac{(n-2)}{(n-1)^2} \\
&= \frac{1}{(n-1)^3}.
\end{aligned}$$

Assuming that for $0 < i < m$, $\partial g_i[\delta_x](y) = \frac{(-1)^{i-1}}{(n-1)^i}$, then the above relation implies that

$$\begin{aligned}
\partial g_m[\delta_x](y) &= \frac{1}{n-1} ((n-2)\partial g_{m-1}[f](x) + \partial g_{m-2}[f](x)) \\
&= \frac{1}{n-1} ((n-2)\frac{(-1)^{m-1}}{(n-1)^m} + \frac{(-1)^{m-2}}{(n-1)^{m-1}}) \\
&= \frac{(-1)^{m-1}}{(n-1)^m} ((n-1) - (n-2)) \\
&= \frac{(-1)^{m-1}}{(n-1)^m}.
\end{aligned}$$

Combining all of these smaller results, we get the following theorem:

Theorem 5.3.1. *Let $G = K_n$ be the complete graph on n -vertices. Fix an $x \in K_n$, then the limit,*

$$\lim_{i \rightarrow \infty} g_i[\delta_x](z) = \frac{1}{n}$$

for all $z \in K_n$ (i.e. δ_x is generalized harmonic on K_n).

Proof. Since $\partial g_i[\delta_x](y) = \frac{(-1)^{i-1}}{(n-1)^i}$ for $i > 0$, then

$$\begin{aligned}
\lim_{i \rightarrow \infty} g_i[\delta_x](y) &= \lim_{i \rightarrow \infty} \sum_{j=1}^i (g_j[\delta_x](y) - g_{j-1}[\delta_x](y)) + g_0[\delta_x](y) \\
&= \sum_{j=1}^{\infty} \partial g_j[\delta_x](y) \\
&= \frac{1}{n-1} \sum_{j=0}^{\infty} \left(\frac{-1}{n-1} \right)^j \\
&= \frac{1}{n-1} \left(\frac{1}{1 - \left(\frac{-1}{n-1} \right)} \right) \\
&= \frac{1}{n}.
\end{aligned}$$

□

Corollary 5.3.2. *Given any $f : V \rightarrow \mathbb{R}$ for $G = K_n$, f is generalized harmonic of type I.*

Proof. Since $g_i[f](y)$ is a linear operator and $f(y) = \sum_{x \in V} c_x \delta_x(y)$, where $c_x = f(x)$. The result follows.

□

Perturb the complete graph on n -vertices ($n \geq 4$) by removing a random edge from it. The resulting graph looks like a suspension of a complete graph on $(n-2)$ -vertices. We will denote this graph by SK_{n-2} . In this graph there are two vertices, x_1 and x_2 , that have degree $n-2$ and $n-2$ vertices with degree $n-1$. This graph is symmetric with respect to x_1 and x_2 , so let $x \in \{x_1, x_2\}$ and again let's look at the flow of $g_i[\delta_x](x)$.

Let $y_1, y_2 \sim x \in SK_{n-2}$ again and notice that

$$g_i[\delta_x](y_1) = g_i[\delta_x](y_2) \text{ for } i = 0 \text{ and } i = 1.$$

When $i > 1$, $g_i[\delta_x](y_1)$ differs from $g_i[\delta_x](y_2)$ by $g_{i-1}[\delta_x](y_1) - g_{i-2}[\delta_x](y_2)$. So by induction $g_i[\delta_x](y_1) = g_i[\delta_x](y_2)$.

This then implies that for $i > 0$, $g_i[\delta_x](x_1) = g_{i-1}[\delta_x](y) = g_i[\delta_x](x_2)$.

Substituting these statements back into the definition for $g_i[\delta_x](y)$, one finds

$$g_i[\delta_x](y) = \frac{1}{deg(y)} ((n-3)g_{i-1}[\delta_x](y) + 2g_{i-2}[\delta_x](y)).$$

Consider now, $\partial g_i[\delta_x](y) = g_i[\delta_x](y) - g_{i-1}[\delta_x](y)$, for $i > 0$. Then we have the following,

$$\begin{aligned} \partial g_1[\delta_x](y) &= \frac{1}{n-1} \\ \partial g_2[\delta_x](y) &= \frac{-2}{(n-1)^2}. \end{aligned}$$

Suppose then that $\partial g_i[\delta_x](y) = \frac{(-2)^{i-1}}{(n-1)^i}$, whenever $0 < i < m$, for some $m \in \mathbb{N}$.

Notice that

$$\begin{aligned}
g_m[\delta_x](y) - g_{m-1}[\delta_x](y) &= \frac{1}{deg(y)} ((n-3)\partial g_{m-1}[\delta_x](y) + 2\partial g_{m-2}[\delta_x](y)) \\
&= \frac{1}{n-1} \left((n-3) \frac{(-2)^{m-2}}{(n-1)^{m-1}} + 2 \frac{(-2)^{m-3}}{(n-1)^{m-2}} \right) \\
&= \frac{(-2)^{m-3}}{(n-1)^m} (-2(n-3) + 2(n-1)) \\
&= \frac{(-2)^{m-3}}{(n-1)^m} (4) \\
&= \frac{(-2)^{m-3}}{(n-1)^m} (-2)^2 \\
&= \frac{(-2)^{m-1}}{(n-1)^m}.
\end{aligned}$$

Therefore, by induction $\partial g_i[\delta_x](y) = \frac{(-2)^{i-1}}{(n-1)^i}$, for all $i > 0$.

Following up on this we get that

$$\begin{aligned}
\lim_{i \rightarrow \infty} g_i[\delta_x](y) &= g_0[\delta_x](y) + \sum_{i=1}^{\infty} \partial g_i[\delta_x](y) \\
&= 0 + \sum_{i=1}^{\infty} \frac{(-2)^{i-1}}{(n-1)^i} \\
&= \frac{1}{(n-1)} \sum_{j=0}^{\infty} \frac{(-2)^j}{(n-1)^j} \\
&= \frac{1}{(n-1)} \frac{1}{1 - \frac{-2}{(n-1)}} \\
&= \frac{1}{n+1} < \frac{1}{n}.
\end{aligned}$$

So, the functions δ_x are not generalized harmonic on SK_{n-2} .

Recall the analysis of δ_x functions on complete graphs under the unit spherical

average flow. In particular recall that,

$$\partial g_i[\delta_x](y) = \frac{(-1)^{i-1}}{(n-1)^i},$$

for all $i > 0$.

Theorem 5.3.3. *For any $y \sim x \in K_n$,*

$$g_i[\delta_x](y) = \begin{cases} \frac{1}{(n-1)^i} \sum_{j=0}^{\frac{i}{2}-1} \binom{\frac{i}{2}+j}{2j+1} (n-1)^{\frac{i}{2}-1-j} (n-2)^{2j+1}; & i \equiv 0 \pmod{2} \\ \frac{1}{(n-1)^i} \sum_{j=0}^{\frac{i-1}{2}} \binom{\frac{i-1}{2}+j}{2j} (n-1)^{\frac{i-1}{2}-j} (n-2)^{2j}; & i \equiv 1 \pmod{2} \end{cases}$$

for all $i > 0$.

Proof. The proof is by brute force induction. First, let $i = 1$ so then $g_1[\delta_x](y) = \frac{1}{(n-1)}$ and $\frac{1}{(n-1)} \sum_{j=0}^0 \binom{j}{2j} (n-1)^{-j} (n-2)^{2j} = \frac{1}{(n-1)} \binom{0}{0} = \frac{1}{(n-1)}$. Also, we must check when $i = 2$. From before we know that $g_2[\delta_x](y) = \frac{(n-2)}{(n-1)^2}$ and $\frac{1}{(n-1)^2} \sum_{j=0}^0 \binom{1+j}{2j+1} (n-1)^{-j} (n-2)^{2j+1} = \frac{1}{(n-1)^2} \binom{1}{1} (n-2)$. Suppose that this relationship holds all $i < m$.

If $m \equiv 0 \pmod{2}$, then $m - 1$ is odd and

$$\begin{aligned}
g_m[\delta_x](y) &= \frac{1}{n-1} \left((n-2)g_{m-1}[\delta_x](y) + g_{m-2}[\delta_x](y) \right) \\
&= \frac{(n-2)}{(n-1)^m} \sum_{j=0}^{\frac{m-2}{2}} \binom{\frac{m-2}{2} + j}{2j} (n-1)^{\frac{m-2}{2}-j} (n-2)^{2j} \\
&\quad + \frac{1}{(n-1)^{m-1}} \sum_{j=0}^{\frac{m-2}{2}-1} \binom{\frac{m-2}{2} + j}{2j+1} (n-1)^{\frac{m-2}{2}-1-j} (n-2)^{2j+1} \\
&= \frac{1}{(n-1)^m} \left[\sum_{j=0}^{\frac{m-2}{2}} \binom{\frac{m-2}{2} + j}{2j} (n-1)^{\frac{m-2}{2}-j} (n-2)^{2j+1} \right. \\
&\quad \left. + \sum_{j=0}^{\frac{m-2}{2}-1} \binom{\frac{m-2}{2} + j}{2j+1} (n-1)^{\frac{m-2}{2}-j} (n-2)^{2j+1} \right] \\
&= \frac{1}{(n-1)^m} \left[(n-2)^{m-1} \right. \\
&\quad \left. + \sum_{j=0}^{\frac{m-2}{2}-1} \left[\binom{\frac{m-2}{2} + j}{2j} + \binom{\frac{m-2}{2} + j}{2j+1} \right] (n-1)^{\frac{m-2}{2}-j} (n-2)^{2j+1} \right].
\end{aligned}$$

Note here that direct calculation gives us:

$$\binom{\frac{m-2}{2} + j}{2j} + \binom{\frac{m-2}{2} + j}{2j+1} = \binom{\frac{m}{2} + j}{2j+1},$$

since,

$$\begin{aligned}
\binom{\frac{m-2}{2} + j}{2j} + \binom{\frac{m-2}{2} + j}{2j+1} &= \frac{(\frac{m-2}{2} + j)!}{(2j)! (\frac{m-2}{2} + j - 2j)!} + \frac{(\frac{m-2}{2} + j)!}{(2j+1)! (\frac{m-2}{2} + j - 2j - 1)!} \\
&= \left((2j+1) + \left(\frac{m-2}{2} - j \right) \right) \frac{(\frac{m-2}{2} + j)!}{(2j+1)! (\frac{m-2}{2} - j)!} \\
&= \binom{\frac{m-2}{2} + j + 1}{2j+1}.
\end{aligned}$$

Thus,

$$g_m[\delta_x](y) = \frac{1}{(n-1)^m} \left[(n-2)^{m-1} + \sum_{j=0}^{\frac{m-2}{2}-1} \binom{\frac{m}{2} + j}{2j+1} (n-1)^{\frac{m-2}{2}-j} (n-2)^{2j+1} \right].$$

Further note that when $j = \frac{m-2}{2}$ in the sum above, the $\frac{m-2}{2}$ -th term is:

$$\binom{m-1}{m-1} (n-1)^0 (n-2)^{m-1} = (n-2)^{m-1}.$$

And so we have at last,

$$g_m[\delta_x](y) = \frac{1}{(n-1)^m} \sum_{j=0}^{\frac{m}{2}-1} \binom{\frac{m}{2}+j}{2j+1} (n-1)^{\frac{m}{2}-1-j} (n-2)^{2j+1}.$$

The proof for when $i \equiv 1 \pmod{2}$ is basically the same and will be omitted. □

Bibliography

- [1] N. Alon and V. D. Milman, λ_1 , *isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory Ser. B **38** (1985), no. 1, 73–88, DOI 10.1016/0095-8956(85)90092-9. MR782626
- [2] Kathrin Bacher and Karl-Theodor Sturm, *Ricci bounds for Euclidean and spherical cones*, Singular phenomena and scaling in mathematical models, Springer, Cham, 2014, pp. 3–23, DOI 10.1007/978-3-319-00786-1₁. MR3205034
- [3] Dominique Bakry and Michel Émery, *Hypercontractivité de semi-groupes de diffusion*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 15, 775–778 (French, with English summary). MR772092
- [4] F. Bauer, F. Chung, Y. Lin, and Y. Liu, *Curvature aspects of graphs*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 2033–2042. MR3611318
- [5] Tullio Ceccherini-Silberstein, Michel Coornaert, and Józef Dodziuk, *The surjectivity of the combinatorial Laplacian on infinite graphs*, Enseign. Math. (2) **58** (2012), no. 1-2, 125–130, DOI 10.4171/LEM/58-1-5. MR2985012
- [6] Fan R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, vol. 92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR1421568
- [7] Fan Chung, *Four proofs for the Cheeger inequality and graph partition algorithms*, Fourth International Congress of Chinese Mathematicians, AMS/IP Stud. Adv. Math., vol. 48, Amer. Math. Soc., Providence, RI, 2010, pp. 331–349. MR2744229
- [8] F. R. K. Chung and S.-T. Yau, *Logarithmic Harnack inequalities*, Math. Res. Lett. **3** (1996), no. 6, 793–812, DOI 10.4310/MRL.1996.v3.n6.a8. MR1426537
- [9] Fan Chung, Yong Lin, and S.-T. Yau, *Harnack inequalities for graphs with non-negative Ricci curvature*, J. Math. Anal. Appl. **415** (2014), no. 1, 25–32, DOI 10.1016/j.jmaa.2014.01.044. MR3173151
- [10] Giuliana Davidoff, Peter Sarnak, and Alain Valette, *Elementary number theory, group theory, and Ramanujan graphs*, London Mathematical Society Student Texts, vol. 55, Cambridge University Press, Cambridge, 2003. MR1989434
- [11] Jozef Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984), no. 2, 787–794, DOI 10.2307/1999107. MR743744
- [12] J. Dodziuk and W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From local times to global geometry, control and physics (Coventry, 1984/85), Pitman Res. Notes Math. Ser., vol. 150, Longman Sci. Tech., Harlow, 1986, pp. 68–74. MR894523
- [13] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm, *On the Equivalence of the Entropic Curvature-Dimension Condition and Bochner’s Inequality on Metric Measure Spaces*, ArXiv e-Print (2013). arXiv:1303.4382v2 [math.DG].
- [14] Jürgen Jost, *Riemannian geometry and geometric analysis*, 6th ed., Universitext, Springer, Heidelberg, 2011. MR2829653

- [15] Jürgen Jost and Shiping Liu, *Ollivier's Ricci curvature, local clustering and curvature-dimension inequalities on graphs*, *Discrete Comput. Geom.* **51** (2014), no. 2, 300–322. MR3164168
- [16] Mark Kempton, Florentin Munch, and Shing-Tung Yau, *Relationships between cycles spaces, gain graphs, graph coverings, path homology, and graph curvature*, arXiv:1710.01264 (10/2017).
- [17] Christian Ketterer, *Cones over metric measure spaces and the maximal diameter theorem*, *J. Math. Pures Appl.* (9) **103** (2015), no. 5, 1228–1275, DOI 10.1016/j.matpur.2014.10.011 (English, with English and French summaries). MR3333056
- [18] Bo'az Klartag, Gady Kozma, Peter Ralli, and Prasad Tetali, *Discrete curvature and abelian groups*, *Canad. J. Math.* **68** (2016), no. 3, 655–674, DOI 10.4153/CJM-2015-046-8. MR3492631
- [19] Sajjad Lakzian and Zachary Mcguirk, *A Global Poincaré inequality on Graphs via a Conical Curvature-Dimension Condition*, *Anal. Geom. Metr. Spaces* **6** (2018), 32–47, DOI 10.1515/agms-2018-0002. MR3767387
- [20] Yong Lin, Linyuan Lu, and Shing-Tung Yau, *Ricci curvature of graphs*, *Tohoku Math. J.* (2) **63** (2011), no. 4, 605–627, DOI 10.2748/tmj/1325886283. MR2872958
- [21] Yong Lin and Shing-Tung Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, *Math. Res. Lett.* **17** (2010), no. 2, 343–356, DOI 10.4310/MRL.2010.v17.n2.a13. MR2644381
- [22] John Lott and Cédric Villani, *Ricci curvature for metric-measure spaces via optimal transport*, *Ann. of Math.* (2) **169** (2009), no. 3, 903–991, DOI 10.4007/annals.2009.169.903. MR2480619
- [23] ———, *Weak curvature conditions and functional inequalities*, *J. Funct. Anal.* **245** (2007), no. 1, 311–333, DOI 10.1016/j.jfa.2006.10.018. MR2311627
- [24] Alexander Lubotzky, *Discrete groups, expanding graphs and invariant measures*, *Progress in Mathematics*, vol. 125, Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski. MR1308046
- [25] Yann Ollivier, *A survey of Ricci curvature for metric spaces and Markov chains*, *Probabilistic approach to geometry*, *Adv. Stud. Pure Math.*, vol. 57, Math. Soc. Japan, Tokyo, 2010, pp. 343–381. MR2648269
- [26] Karl-Theodor Sturm, *A curvature-dimension condition for metric measure spaces*, *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 3, 197–200 (English, with English and French summaries). MR2198193
- [27] ———, *On the geometry of metric measure spaces. I*, *Acta Math.* **196** (2006), no. 1, 65–131, DOI 10.1007/s11511-006-0002-8. MR2237206
- [28] ———, *On the geometry of metric measure spaces. II*, *Acta Math.* **196** (2006), no. 1, 133–177, DOI 10.1007/s11511-006-0002-8. MR2237206
- [29] Radosław Krzysztof Wojciechowski, *Stochastic completeness of graphs*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—City University of New York. MR2711706