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Geometry and Analysis of some Euler-Arnold Equations

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Abstract

GEOMETRY AND ANALYSIS OF SOME EULER-ARNOLD EQUATIONS

by

JAE MIN LEE

Adviser: Professor Stephen Preston

In 1966, Arnold showed that the Euler equation for an ideal fluid can arise as the geodesic flow on the group of volume preserving diffeomorphisms with respect to the right invariant kinetic energy metric. This geometric interpretation was rigorously established by Ebin and Marsden in 1970 using infinite dimensional Riemannian geometry and Sobolev space techniques. Many other nonlinear evolution PDEs in mathematical physics turned out to fit in this universal approach, and this opened a vast research on the geometry and analysis of the Euler-Arnold equations, i.e., geodesic equations on a Lie group endowed with one-sided invariant metrics. In this thesis, we investigate two Euler-Arnold equations; the Camassa-Holm equation from the shallow water equation theory and quasi-geostrophic equation from geophysical fluid dynamics.

First, we will prove the local-wellposedness of the Camassa-Holm equation on the real line in the space of continuously differentiable diffeomorphisms, satisfying certain asymptotic conditions at infinity. Motivated by the work of Misiołek, we will re-express the equation in Lagrangian variables, by which the PDE becomes an ODE on a Banach manifold with a locally Lipschitz right-side. Consequently, we obtain the existence and uniquenss of the solution, and the topological group property of the diffeomorphism group ensures the continuous dependence on the initial data.

Second, we will construct global weak conservative solutions of the Camassa-Holm equation on the periodic domain. We will use a simple Lagrangian change of variables, which

removes the wave breaking singularity of the original equation and allows the weak continuation. Furthermore, we obtain the global spatial smoothness of the Lagrangian trajectories via this construction. This work was motivated by Lenells who proved similar results for the Hunter-Saxton equation using the geometric interpretation.

Lastly, we will study some geometric aspects for the quasi-geostrophic equation, which is the geodesic on the quantomorphism group, a subgroup of the contactomorphism group. We will derive an explicit formula for the sectional curvature and discuss the nonpositive curvature criterion, which extends the work of Preston on two dimensional incompressible fluid flows.

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In memory of my uncle

Cheol Seob Lee

1961–2011

Contents

Chapter 1

Introduction

1.1 Motivation and history

The story of this research originates from the rediscovery of the Euler equations by Arnold [2] in his seminal paper of 1966. In 1765, two centuries before Arnold, Euler [23] showed that the motion of a rigid body in three dimensional Euclidean space is described as geodesics in the group of rotations endowed with a left-invariant metric. Arnold rederived this equation in the modern theory of Lie groups endowed with a one-sided invariant metric, and realized that this Euler's equation holds true for an arbitrary Lie group. In the year 1776, Euler proposed another important equation that is named after him, which is the Euler equation for ideal fluid in hydrodynamics. It is the following evolution equation

$$u_t + u \cdot \nabla u = -\nabla p,$$

where u is the velocity of the fluid particle and p is the pressure, and it was derived from the principles of physics such as the Newton's law, continuity equation, etc. One can view the fluid motion from how the velocity field is evolving over time, but one can also equivalently describe the fluid motion by how the particles move in the fluid vessel. In the general fluid mechanics perspective, the former viewpoint is observing the fluid with a fixed observer and the observer is moving along with the fluid in the latter case. Arnold's idea was that since the motion of inertial system is governed by the principle of least action, the fact that the Euler equation for rigid body motion is described as the geodesic on the group of rotations was a natural consequence. Furthermore, fluid motion, in the particle trajectory perspective, should be described in the similar fashion, if we find the correct configuration space (Lie group) and the invariant metric from the physical system. Arnold showed that the Euler equation for an ideal fluid can arise as the geodesic flow on the group of volume preserving diffeomorphisms with respect to the right invariant kinetic energy metric. This is the reason why we call equations that can be realized as a geodesic on a Lie group endowed with one-sided invariant metrics as the *Euler-Arnold equations*.

However, Arnold's argument assumed that the group of volume preserving diffeomorphisms, which is an infinite dimensional Lie group, would behave just like a finite dimensional Lie group or possesses all the necessary properties of it. However, the group of smooth diffeomorphisms is a Frechét space which is a complete vector space generated by a family of seminorms. This causes many complications to use Banach space techniques such as Inverse Function Theorem, Existence and Uniqueness Theorem for ODEs, due to the lack of local compactness. In fact, the existence of Riemannian connection, which allows one to define a geodesic, is not guaranteed. This is where Ebin and Marsden [21] contributed in their celebrated paper of 1970 by furnishing Arnold's geometric interpretation with rigorous mathematical foundation. Their main idea was to *enlarge* the diffeomorphism group by requiring less regularity, namely Sobolev class H^s. Then $\mathcal{D}^s$, the group of H^s-diffeomorphisms, is a Banach manifold where we can employ the general theory of Banach manifold to construct the infinite dimensional Riemannian geometry from scratch. Then, by taking the Sobolev index s arbitrarily large, we obtain the group of C^∞-diffeomorphisms and one can show that the geometric objects constructed in the space $\mathcal{D}^s$ extend to $\mathcal{D}$.

This opened a vast research on the analysis of other nonlinear evolution PDEs in the Arnold's geometric framework. Many other nonlinear PDEs in fluid dynamics turned out to be Euler-Arnold equations on appropriate Lie groups with respect to one-sided invariant metrics. For example, Burgers equation on the diffeomorphism group with respect to the L^2 metric [15], Korteweg-de Vries equation on the Virasoro-Bott group with respect to the L^2 metric [71], the Camassa-Holm equation with the H^1 metric [68]. Also, on the homogemeous difeomorphism group, we have the Hunter-Saxton equation with respect to the homogeneous $\dot{H}^1$ metric [46], Wunsch equation with $\dot{H}^{1/2}$ metric [4], surface quasi-geostrophic equation with $\dot{H}^{-1/2}$ metric [80, 82], and the list goes on. One can answer many questions on the PDEs, such as local/global aspects of the solutions, or provide a different description of their mathematical properties in this geometric approach. For more information, see [45, 44].

The geometric approach on PDEs as geodesics naturally led to the study of the infinite dimensional manifold where the geodesics lie in, e.g., the Riemannian geometry of the diffeomorphism groups and their subgroups. In general, this could be seen as an attempt to better understand the nature of the infinite dimensional Riemannian geometry compared to its finite dimensional counterpart. For example, the Hopf-Rinow theorem, which states that a connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space, does not hold true when the dimension of a manifold is infinite; this is due to the loss of local compactness.

In addition, one can compute local quantities like curvature to obtain information about the stability of flow. Arnold first computed the sectional curvature of the diffeomorphism group on the flat 2-torus to demonstrate the unpredictibality of the weather due to the Lagrangian instability. That is, positive curvature in all sections implies that geodesics with close initial data locally converge, while negative sectional curvature implies that the geodesics spread apart, which can be measured by studying the Jacobi field.

1.2 Summary of the main results

In this thesis, we will study two Euler-Arnold equations; the Camassa-Holm equation from the shallow water equation theory and quasi-geostrophic equation from geophysical fluid dynamics. First, we will prove the local well-posedness in the sense of Hadamard of the Camassa-Holm(CH) equation on the real line, that is, existence, uniqueness and continuous dependence on the initial data to the solution for the initial value problem. The CH equation

$$u_t - u_{txx} + 3uu_x - uu_{xxx} - 2u_x u_{xx} = 0, \quad t, x \in \mathbb{R} \tag{1.1}$$

was originally proposed as a model for shallow water waves and has remarkable properties like complete integrability, soliton-like solutions, bi-hamiltonian structures, and finite time blowup from wave breaking. Also, the solution can be interpreted as the geodesics on $\mathcal{D}(\mathbb{S}^1)$, the group of circle diffeomorphisms, endowed with the right-invariant Sobolev H^1 metric

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}} uv + u_x v_x \, dx. \tag{1.2}$$

The main theorem is following:

Theorem 1.2.1. *The Cauchy problem for the Camassa-Holm equation is equivalent to the system*

$$\begin{cases} \frac{d\eta}{dt} = U \\ \frac{dU}{dt} = -\mathcal{L}_\eta\left(U^2 + \frac{U_x^2}{2\eta_x^2}\right) \end{cases} \tag{1.3}$$

with initial conditions $\eta(0,\cdot) = \mathrm{Id}$*,* $U(0,\cdot) = u_0$*. Here,* $\mathcal{L}_\eta$ *is defined by*

$$\mathcal{L}_\eta(\phi) = \mathcal{L}(\phi \circ \eta^{-1}) \circ \eta, \text{ and } \mathcal{L} = \partial_x(1 - \partial_x^2)^{-1} \text{ for any function } \phi.$$

This system describes the flow of a C^1 *vector field on* $T\mathcal{D}(\mathbb{R})$*, the tangent bundle of the* *diffeomorphism group, and the solution curve* (η, U) *exists for some time* $T > 0$. *Defining* $u = U \circ \eta^{-1}$, *we obtain a* C^1 *vector field* $u : [0, T) \times \mathbb{R} \to \mathbb{R}$ *satisfying the Camassa-Holm equation which depends continuously on* u_0.

We have closely followed the approach by Misiołek [69] who proved the local well-posedness of the CH equation in the space of C^1 functions on the periodic domain $\mathbb{S}^1$. In the nonperiodic domain case, however, we need to require diffeomorphisms to be not only C^1 but also H^1 with extra decaying conditions at infinity for the H^1 energy to be well-defined. To prove the theorem, we first show that the right-side of the ODE (4.11) is continuously differentiable to obtain the existence and uniqueness of the solution via Contraction Mapping Principle. From the Lagrangian solution (η, η_t), we can construct the solution of the CH equation (1.1) by defining $u := \eta_t \circ \eta^{-1}$. This gives the solution mapping $u_0 \mapsto u(t, x)$ for the initial value problem of the CH equation, and this mapping is continuous from the topological group properties of $\mathcal{D}(\mathbb{R})$.

Next, we will construct global weak conservative solutions of the CH equation on the circle by using a simple change of variables on the Lagrangian variable η:

$$\rho = \sqrt{\eta_x}. \tag{1.4}$$

This idea is motivated by Lenells [58] who constructed global weak conservative solutions of the Hnter-Saxton(HS) equation

$$u_{txx} + 2u_x u_{xx} + 3u u_{xxx} = 0, \quad t \in \mathbb{R},\ x \in \mathbb{S}^1, \tag{1.5}$$

which is another Euler-Arnold equation with a degenerate right invariant $\dot{H}^1$ metric

$$\langle u, v \rangle_{\dot{H}^1} = \int_{\mathbb{S}^1} u_x v_x \, dx. \tag{1.6}$$

on the homogeneous space $\mathcal{D}(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$, where $\mathrm{Rot}(\mathbb{S}^1)$ is the subgroup of rotations. Lenells used the geometric interpretation that the HS geodesic remains on the unit sphere for all time, which makes it possible to continue the geodesic in the weak sense after the blowup time. Using this geometric interpretation, we will show that the transformation (4.2) removes the wave breaking singularities in the CH equation case as well, even though we don't have exactly the same geometric picture and explicit formula as in the HS equation case. We can show that the CH equation, written in ρ variables, has a global solution from which we can reconstruct the flow η defined by

$$\eta(t,x) := \int_0^x \rho^2(t,y)dy + c(t), \text{ where } c(t) \text{ is some function of time,} \tag{1.7}$$

in the space of absolutely continuous functions. This flow is a weak geodesic for almost all time since the spatial derivative of η vanishes on a set of measure zero for the most time. The idea is that the new variable ρ can assume both signs and ρ passes through the x-axis with a finite speed whenever it reaches the axis. Consequently, the flow η remains a homeomorphism for almost all time and this is precisely how the singularity of the CH equation is removed.

In addition, the construction of this global weak solution shows that the spatial smoothness of the Lagrangian trajectories η in (1.7) depends on the smoothness of ρ, which is determined by the smoothness of the initial condition u_0. This is an interesting phenomenon of the CH equation observed by McKean [65]; the solution of the CH equation experiences the jump discontinuity of its slope even for the smooth initial data, but the Lagrangian trajectory η is spatially smooth for all time. We show that for each $k \geq 1$ the Lagrangian flow η is in C^k for all time whenever the initial condition u_0 is in C^k. This improves the result of McKean by showing the exact correspondence between the smoothness of the initial condition and the smoothness of the Lagrangian flow. Also, this approach does not use the complete integrability and explicit formula for the solutions of the CH equation. Hence we prove the following theorems:

Theorem 1.2.2. *The Cauchy problem for the periodic Camassa-Holm equation*

$$\begin{cases} u_t + uu_x = -(1-\partial_x^2)^{-1}\partial_x \left(u^2 + \frac{u_x^2}{2}\right) \\ u(0,x) = u_0(x) \end{cases} \tag{1.8}$$

has a global solution $u \in AC([0,\infty), H^1(S^1)) \cap L^\infty([0,\infty), L^2(S^1))$. The solution is weak in the sense that the equality in the equation (1.8) *is satisfied in the distributional sense. Also, the solution u is conservative; $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ for $t \in [0,\infty)$ almost everywhere.*

Theorem 1.2.3. *Let $\eta(t,x)$ be the Lagrangian flow of the weak solution u of the CH equation* (1.8)*. Then η is spatially absolutely continuous for all time. Furthermore, for each integer $k \geq 1$, if η is initially C^k, then η remains C^k for all time.*

Finally, we compute the sectional curvature of the quantomorphism group. A quantomorphism is a diffeomorphism that preserves the contact form on the base manifold exactly. Quantomorphism group can be viewed geometrically as a central extension of the Lie algebra of exact divergence free vector fields via a Lie algebra 2-cocycle [22, 81]. We consider base manifolds $M = \mathbb{R}^3$, $N = \mathbb{S}^1 \times [0, 2\pi]$(i.e., the flat cylinder), and the quantomorphism group $\mathcal{D}_q(M)$ endowed with the right invariant metric

$$\langle X, Y\rangle_{\mathcal{D}_q} = \int_M \left(\langle \nabla f, \nabla g\rangle + \alpha^2 fg\right)\, dA + \beta\gamma,$$

where $X, Y \in T_{\mathrm{Id}}\mathcal{D}_q(M) \simeq C^\infty(N) \times \mathbb{R}$ are identified by $X = (f,\beta)$ and $Y = (g,\gamma)$, and α^2 is the Froude number. Then the geodesic equation on $\mathcal{D}_q(M)$ is

$$(\Delta - \alpha^2)f_t + \{f, \Delta f\} + \beta\{f, \psi\} = 0, \text{ where } \psi(x,y) = y,$$

and this is the quasi-geostrophic(QG) equation in geophysics with fixed constants α and β [22, 72]. We compute the Riemann curvature tensor R explicitly and then try to find the condition that the curvature operator $R_X : Y \mapsto R(Y,X)X$ is nonpositive in all directions when X is a vector field tangent along the geodesic. This work extends the result of Preston [75] who computed the sectional curvature of the area preserving diffeomorphism group and derived the nonpositivity criterion. The main theorem is following:

Theorem 1.2.4. *Let $X, Y \in T_{\mathrm{Id}}\mathcal{D}_q(M) \simeq C^\infty(N) \times \mathbb{R}$ be tangent vectors identified by $X = (f, \beta)$ and $Y = (g, \gamma)$ as above, and assume that $f = f(y)$ is a function of only y variable. Then the curvature of the quantomorphism group $\mathcal{D}_q(M)$ with section spanned by X and Y has the following explicit formula $K(X,Y) = \sum_{n\in\mathbb{Z}} n^2 K_n$, where*

$$K_n := -\int_0^{2\pi} \frac{Q' - Q^2 + \lambda^2}{(\lambda - Q)^2} e^{2\lambda y} |J_n|^2 dy - \frac{|J_n(0)|^2}{\lambda - Q(0)} - \frac{e^{-2\pi\lambda}\,|H_n(2\pi) - J_n(0)|^2}{2\lambda(e^{2\pi\lambda} - e^{-2\pi\lambda})}.$$

Here, $Q = \frac{\alpha f' + \beta}{2f''}$ and $\lambda = \sqrt{\alpha^2 + n^2}$, and H_n and J_n are auxiliary functions defined in Chapter 5 from the equation (5.18)*.*

From this explicit formula, we hope to derive the criterion for the nonpositive curvature as in [75].

1.3 Outline of the thesis

In Chapter 2, we will begin with preliminaries. The main goal is to concisely present Arnold's original idea of the geometric approach on fluid motion and the subsequent work of Ebin and Marsden. We will provide some backgrounds on Sobolev space, diffeomorphism groups, functional analysis/differential calculus on Banach manifold, and Lie group theory. These are the main tools and language that Arnold, Ebin, and Marsden used and so in this current research. At the end of Chapter 2, we will introduce the Camassa-Holm equation and motivate the main problems in this thesis.

In Chapter 3, we will prove the local well-posedness of the Camassa-Holm equation on the real line in the space of $C^1 \cap H^1$ diffeomorphisms with a decaying condition. We express the CH equation in Lagrangian variables and show that the PDE becomes an infinite dimensional ODE with a C^1 right-side, which proves the existence and uniqueness of the solution. Also, we will prove the continuity of the initial data to solution mapping by using the topological group properties of the diffeomorphism group.

In Chapter 4, we will investigate the weak continuation of the solution of the Camassa-Holm equation on the circle by introducing a new variable ρ given by (1.4). We will reformulate the Cauchy problem for the CH equation in ρ variable and first show that the CH equation written in ρ variable has a global solution. Then we will construct global weak conservative solution of the original CH equation by reversing the change of variables (1.4). Finally, this construction will show that the persistence of the spatial smoothness of the Lagrangian trajectories holds in the class C^k for every $k \geq 1$.

In Chapter 5, we will study the quantomorphism group and compute its sectional curvature. We will derive an explicit formula for the sectional curvature and discuss the nonpositive curvature criterion.

1.4 Publications

Some results from this thesis have been disseminated in mathematical publications. The local-wellposedness of the Camassa-Holm equation on the real line (Chapter 3) was a joint work with my advisor Stephen Preston and was published in [52]. Also, global conservative weak continuation of the Camassa-Holm equation (Chapter 4) is published in arXiv [51] and under the revision process for publication in the Journal of Differential Equations.

Chapter 2

Preliminaries

2.1 Differential Calculus on Banach Spaces

In this section, we establish some differential calculus that will be used throughout this thesis. The references are [20, 21, 42, 49].

Let $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ denote Banach spaces. Let $U \subset \mathbf{E}$ and $V \subset \mathbf{G}$ be open sets and $x \in U$. Let $f : U \to \mathbf{F}$ be a map.

Definition 2.1.1. *differentiable*
We say that f is differentiable at x if there exists a continuous linear map $A : \mathbf{E} \to \mathbf{F}$ and a map $R : \mathbf{E} \to \mathbf{F}$ that is $o(h)$ as $h \to 0$ such that

$$f(x+h) = f(x) + A(h) + R(h).$$

In this case, A is called the derivative of f and is denoted by $D_x f$.

This notion of derivative is also known as the *Frechét derivative*. If $f : U \to \mathbf{F}$ is differentiable at every point $x \in U$, then we say that f is differentiable on U and we can define the derivative function $Df : U \to L(\mathbf{E}, \mathbf{F})$ by $Df(x) := D_x f$. Here, $L(\mathbf{E}, \mathbf{F})$ is the space of continuous linear maps from $\mathbf{E}$ to $\mathbf{F}$, which itself is a Banach space under the operator norm topology.

Definition 2.1.2. *continuously differentiable*
If the derivative $Df : U \to L(\mathbf{E}, \mathbf{F})$ is continuous, we say that f is continuously differentiable, or f is of class C^1. Then we can define that f is of class C^k if Df is of class C^{k-1} and we define $D^k f = D\left(D^{k-1} f\right)$, inductively.

The typical application of continuous differentiability is when the function f is a vector field. Let $f : U \to \mathbf{E}$ be a vector field on U, which we interpret as assigning a vector to each point of U, for now. We say that f satisfies a *Lipschitz condition* on U if there exists a number $K > 0$ such that

$$|f(y) - f(x)| \leq K\,|x - y|\,,$$

for all $x, y \in U$. Here, $|\cdot|$ denotes an appropriate norm. If f is of class C^1, then it follows from the Mean Value Theorem that f is locally Lipschitz. That is, f is Lipschitz in the neighborhood of every point, and that it is bounded on such a neighborhood. For a vector field satisfying the Lipschitz condition, we have the following existence and uniqueness theorem for differential equations on a Banach space.

Theorem 2.1.3. *Let J be an open interval of $\mathbb{R}$ containing 0. Suppose that $f : J \times U \to \mathbf{E}$ be continuous, and satisfy a Lipschitz condition on U uniformly with respect to J. Let x_0 be a point of U. Then there exists an open subinterval J_0 of J containing 0, and an open subset U_0 of U containing x_0 such that f has a unique flow*

$$\alpha : J_0 \times U_0 \to U$$

satisfying

$$\frac{d\alpha}{dt}(t, x) = f(t, \alpha(t, x)), \quad \alpha(0, x) = x.$$

We can select J_0 and U_0 such that α is continuous and satisfies a Lipschitz condition on $J_0 \times U_0$.

If f is C^k, then we have $D^k f : U \to L(\mathbf{E}, L(\mathbf{E}, \ldots, L(\mathbf{E}, \mathbf{F}), \ldots))$, where there are k-L's on the right-side. Let $L^k(\mathbf{E}, \mathbf{F})$ be the set of continuous multilinear maps from $\mathbf{E} \times \mathbf{E} \times \cdots \times \mathbf{E}$($k$-$\mathbf{E}$'s) to $\mathbf{F}$. Then $L^k(\mathbf{E}, \mathbf{F})$ is again a Banach space and we have the canonical identification,

$$L^k(\mathbf{E}, \mathbf{F}) = L(\mathbf{E}, L(\mathbf{E}, \ldots, L(\mathbf{E}, \mathbf{F}), \ldots)).$$

For the map $f : U \times V \to \mathbf{G}$ where $U \times V \subset \mathbf{E} \times \mathbf{F}$ is open, we can define partial derivative of f as the derivative with respect to each component. We write $D_1 f : U \times V \to L(\mathbf{E}, \mathbf{G})$ for the partial derivative of f with respect to the first variable. As in multivariable calculus, we have the following criterion for the differentiability of f in terms of the partial derivatives.

Proposition 2.1.4. *The map $f : U \times V \to \mathbf{G}$ is of class C^k if and only if both partial derivatives $D_1 f$ and $D_2 f$ are of class C^{k-1}. If this is the case, then*

$$Df(x)v = D_1 f(x)(v_1) + D_2 f(x)(v_2), \quad x \in U \times V, \ v = (v_1, v_2) \in \mathbf{E} \times \mathbf{F}.$$

There is another notion of derivative called *Gâteaux derivative*, which is the generalization of the directional derivative.

Definition 2.1.5. *Gâteaux derivative*

We say that the map $f : U \to \mathbf{F}$ has Gâteaux derivative at a point u in the direction v if the limit

$$\left.\frac{d}{dt}\right|_{t=0} f(u + tv) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

exists in $\mathbf{F}$. We denote this limit as $df(u, v)$. If the limit exists for all $u \in \mathbf{E}$, we say that f is Gâteaux differentiable at u.

If f is Frechét differentiable, then f is Gâteaux differentiable, but the converse is not true. However, we have the following result:

Proposition 2.1.6. *If $f : U \to \mathbf{F}$ has Gâteaux derivative $df(u,v)$ for all $u \in U$ and all $v \in \mathbf{E}$ and the map $df : U \to L(\mathbf{E}, \mathbf{F})$ given by $df(u)(v) := df(u,v)$ is continuous, then f is also Frechét differentiable.*

We can call the function f that satisfies the hypothesis in the above proposition as Gâteaux-C^1 and define the notion of Gâteaux-C^k, for $k \geq 1$, inductively. Then we have the following proposition

Proposition 2.1.7. *If $f : U \to \mathbf{F}$ is Gateaux-C^{k+1} for some $k \geq 0$, then f is of class C^k in the sense of Frechét. In particular, for smooth maps between Banach spaces, the two definitions coincide.*

We will also need the following inverse mapping theorem.

Proposition 2.1.8. *Let $f : U \to \mathbf{F}$ be a C^k map. Assume that $p \in U$ and $D_x f : \mathbf{E} \to \mathbf{F}$ is a C^1-diffeomorphism. Then f is a local C^k-diffeomorphism at x.*

2.2 Sobolev Spaces

Definition 2.2.1. *Sobolev space $W^{k,p}$*
Let $U \subset \mathbb{R}^n$ be an open subset with C^∞ boundary and $\overline{U}$ compact. We define the Sobolev space $W^{k,p}(U)$ as the space of all locally summable functions $f : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha f$ exists in the weak sense and belongs to $L^p(U)$.

In the case when $p = 2$, we write $H^k(U) = W^{k,2}(U)$ for $k = 0, 1, \ldots$. In particular, $H^0(U) = L^2(U)$.

Definition 2.2.2. *Norm in $W^{k,p}(U)$*

If $f \in W^{k,p}(U)$, we define its norm to be

$$\|f\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha f|^p \, dx\right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha f| & (p = \infty) \end{cases}$$

The above definitions make the Sobolev space $W^{k,p}(U)$ a Banach space for each $k = 1, \ldots$ and $1 \le p \le \infty$. In particular, $H^k(U)$ is the Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{|\alpha|\le k} D^\alpha f \cdot D^\alpha g. \tag{2.1}$$

We can also define the Sobolev space $H^k(U)$ as the completion of $C^\infty(U)$ with respect to the H^k norm

$$\|f\|_{H^k}^2 = \sum_{|\alpha|\le k} \int_U |D^\alpha f|^2 \, dx.$$

We have the following important facts about the Sobolev spaces, whose proof can be found in [1].

Theorem 2.2.3. *(Sobolev Lemma) If $s > k + \frac{n}{2}$, we have the continuous embedding $H^s(U) \hookrightarrow C^k(U)$.*

Theorem 2.2.4. *(Algebra Property) If $s > \frac{n}{2}$, then $H^s(U, \mathbb{R}^m)$ is an algebra under pointwise multiplication, i.e.,*

$$\|f \cdot g\|_{H^s} \le \|f\|_{H^s} \|g\|_{H^s}.$$

Theorem 2.2.5. *(Composition Lemma) If $s > \frac{n}{2} + 1$, $f \in H^s(U, \mathbb{R}^m)$, and $g \in H^s(U, \mathbb{R}^n)$ is a C^1 diffeomorphism from $\overline{U}$ onto itself,*

$$\|f \circ g\|_{H^s} \le C \|f\|_{H^s} (\|g\|_{H^s}^s + 1),$$

for some constant C and the map $g \mapsto f \circ g$ is continuous onto $H^s(U, \mathbb{R}^m)$.

Theorem 2.2.6. *(Rellich Lemma) If $s > t$ and $\overline{U} \subset \mathbb{R}^n$ is compact, then $H^s(U, \mathbb{R}^m) \subset H^t(U, \mathbb{R}^m)$ continuously, i.e., the inclusion map is continuous.*

2.3 Diffeomorphism Groups

2.3.1 Diffeomorphism group as Banach(Hilbert) Manifolds

Let M be a Riemannian manifold with $\dim M = n$, and $\langle \cdot, \cdot \rangle$ and μ be the Riemannian metric and the corresponding volume form on M, respectively. For the simplicity, we will consider the case when M is compact without boundary. Most of the results continue to work when M is not compact or $\partial M \neq \emptyset$ without much modifications. For more detail, we refer to [21, 62].

Let s be an integer that is large enough, which will be specified later. We define the space of C^1 diffeomorphisms as

$$C^1\mathcal{D} := \{f : M \to M \mid f \text{ bijection, and } f, f^{-1} \in C^1\},$$

and the space of H^s mappings on M as

$$H^s(M, M) := \{f : M \to M \mid \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(\varphi(U)) \text{ is in Sobolev class } H^s\},$$

for every chart (U, φ) and (V, ψ) of M. Then we define the group of H^s diffeomorphisms of M as the intersection of the above two spaces:

$$\mathcal{D}^s(M) := C^1\mathcal{D} \cap H^s(M, M).$$

Note that

- $C^1\mathcal{D} \subset C^1(M, M)$ is open, and
- $H^s(M, M) \subset C^1(M, M)$ continuously by Sobolev embedding, if $s > \frac{n}{2} + 1$.

Then we have the following classical result from Eells [20].

Theorem 2.3.1. *If $s > \frac{n}{2}$, then $H^s(M, M)$ is a C^∞ Hilbert manifold.*

Remark 1. *Eells' construction works for any set of maps $Map(M_1, M_2)$, where M_1 is compact and M_2 is complete with $\partial M_2 = \emptyset$, which can be topologized by a topology that is at least C^0. For example,*

- *(C^k mappings) $C^k(M, M)$ with $k = 0, 1, 2, \ldots$*
- *(Hölder mappings) $C^{k,\alpha}(M, M)$ with $k \geq 0$ and $0 < \alpha < 1$*
- *(H^s mappings) $H^s(M, M)$ with $s > \frac{n}{2}$*

2.3.2 $H^s(M, M)$ with $s > \frac{n}{2}$

In this section, we will sketch the construction of Eells. That is, we will show that the space $H^s(M, M)$ is modeled on a Hilbert manifold and admits an atlas of class C^∞. We will construct the tangent space by taking a derivative of a curve through a point in $H^s(M, M)$. For the chart, we will use the Riemannian exponential map of the base manifold M.

Let $f \in H^s(M, M)$ and consider a curve $c : (-\epsilon, \epsilon) \to H^s(M, M)$ with $c(0) = f$. For a fixed point $p \in M$, the map $t \mapsto c_p(t)$ is a curve in M with $c_p(0) = f(p)$. Hence, the derivative $\frac{d}{dt}\Big|_{t=0} c_p(t)$ is an element of $T_{f(p)}M$, and we can see that the map $p \mapsto \frac{d}{dt}\Big|_{t=0} c_p(t)$ is a section of TM through the fiber $T_{f(\cdot)}M$. By identifying

$$\frac{d}{dt}\Big|_{t=0} c(t)(p) = \frac{d}{dt}\Big|_{t=0} c_p(t),$$

we can determine the tangent space at f as

$$T_f H^s(M,M) = \{X : M \to TM \mid X \in H^s(M,TM) \text{ and } \pi \circ X = f\},$$

where $H^s(M,TM)$ is the space of all sections from M to TM which is of class H^s and $\pi : TM \to M$ is the canonical projection. Note that all sections are continuous by the Sobolev lemma since $s > \frac{n}{2}$. Since the tangent space is a linear space, we can make it into a Hilbert space with inner product defined by

$$\langle\langle X, Y \rangle\rangle := \sum_{|k| \leq s} \int_M \langle \nabla^k X, \nabla^k Y \rangle \mu,$$

where ∇^k is the k-th order covariant derivative on the Riemannian manifold M.

Next, we construct a chart at $f \in H^s(M,M)$. Since M is closed, it is geodesically complete and for each $x \in M$, the exponential map $\exp_x : T_xM \to M$ is defined on the whole tangent space. Hence, we can extend the $\exp_x$ to the map $\text{Exp} : TM \to M$, where for $u \in T_xM$, $\text{Exp}(u) = \exp_x(u)$. Since $f(M)$ is compact, there is a number $\lambda_f > 0$ such that any point of M whose distance from $f(x)$ is less than λ_f can be joined by a unique geodesic arc of length less than λ_f. That is, for any point p of M with $d(p, f(x)) < \lambda_f$, there is $X(x) \in T_{f(x)}M$ which lies in the disk of radius λ_f centered at 0 so that $\exp_{f(x)} X(x) = p$. Since the map

$$x \mapsto \exp_{f(x)} X(x)$$

is a map from M to M, the map

$$\begin{aligned} \Phi : T_f H^s(M,M) &\to H^s(M,M) \\ X &\mapsto \text{Exp} \circ X \end{aligned}$$

gives a bijective correspondence between the disk of radius λ_f in $T_f H^s(M,M)$ centered at 0 and the disk of radius λ_f in $H^s(M,M)$ centered at f.

Since Exp : $TM \to M$ is a local diffeomorphism, the transition maps are smooth since they are compositions of smooth maps. Finally, the compactness of M ensures that the H^s topology in terms of the chart is independent of the choice of covers and charts of M. Thus, $H^s(M,M)$ has a smooth manifold structure. See [20] for more detail.

Let $e \in \mathcal{D}^s(M)$ be the identity. We can identify the tangent spaces of the diffeomorphism groups as following:

Theorem 2.3.2. *(tangent structures of $\mathcal{D}^s$ and $\mathcal{D}^s_\mu$)*

1. $T_e\mathcal{D}^s(M) = \{X : M \to TM \mid X$ *is a* H^s *vector field*$\} = H^s(TM)$.

2. $T_\eta\mathcal{D}^s(M) = H^s(\eta^{-1}TM) = \{X \in H^s(TM) \mid \pi \circ X = \eta\}$.

3. $T_e\mathcal{D}^s_\mu(M) = \{v \in T_e\mathcal{D}^s(M) \mid \mathrm{div}v = 0\}$, *where*

$$\mathcal{D}^s_\mu(M) = \{\eta \in \mathcal{D}^s(M) \mid \eta^*\mu = \mu\}$$

is the group of volume preserving H^s *diffeomorphisms.*

Note that $D^s_\mu(M)$ is a C^∞ submanifold of $\mathcal{D}^s(M)$ from the Implicit Function Theorem for Banach manifold (see [50]). The identification of the tangent space of the volume preserving diffeomorphism groups comes from the following simple computations. Consider the map $t \mapsto \eta(t) \in \mathcal{D}^s_\mu(M)$ with $\eta(0) = e$ and $\frac{d}{dt}\Big|_{t=0}\eta(t) = v \in T_e\mathcal{D}^s_\mu(M)$. Since η_t^* is the identity mapping, we have

$$0 = \frac{d}{dt}\Big|_{t=0}(\eta_t^*\mu) = \eta_0^*(\mathcal{L}_v\mu) = \mathrm{div}v,$$

where $\mathcal{L}$ is the Lie derivative. Here, $\mathcal{L}_v\mu = \mathrm{div}v$ by the definition of divergence. Hence, $\mathrm{div}v = 0$.

2.3.3 $\mathcal{D}^s(M)$ and $\mathcal{D}^s_\mu(M)$ as Lie groups

When $s > \frac{n}{2}+1$, both $\mathcal{D}^s(M)$ and $\mathcal{D}^s_\mu(M)$ are topological groups under the composition of diffeomorphisms. However, both $\mathcal{D}^s$ and $\mathcal{D}^s_\mu$ are *not* Lie groups, unless $s=\infty$ in which case we have Frechét Lie groups. We can check the loss of smoothness in the group operations. For $\xi, \eta \in \mathcal{D}^s$ define $L_\eta(\xi) = \eta \circ \xi$ and $R_\eta(\xi) = \xi \circ \eta$, i.e., left and right translations by η. Note that L_η and R_η are maps from $\mathcal{D}^s$ to itself and we can compute their derivatives as following; first, for the right translations we have

$$\begin{aligned} d_\xi R_\eta : T_\xi \mathcal{D}^s &\to T_{R_\eta(\xi)}\mathcal{D}^s \\ X &\mapsto \tfrac{d}{dt}\Big|_{t=0} (R_\eta(\xi_t)) \end{aligned}$$

where

$$\frac{d}{dt}\Big|_{t=0} (R_\eta(\xi_t)) = \frac{d}{dt}\Big|_{t=0} (\xi_t \circ \eta) = X \circ \eta.$$

So $d_\xi R_\eta(X) = X \circ \eta \in H^s$ since X is H^s, η is $\mathcal{D}^s$ and by using the composition lemma (2.2.5). In fact, the mapping R_η is of class C^∞ with respect to the Sobolev H^s topology. However, for the left translations we have

$$d_\xi L_\eta(X) = \frac{d}{dt}\Big|_{t=0} (\eta \circ \xi_t) = D\eta_\xi \cdot X,$$

where $D\eta_\xi$ is H^{s-1} since $\xi \in \mathcal{D}^s$ and X is H^s. So we are facing a loss of derivative, which gives a reason why $\mathcal{D}^s$ fails to be a Lie group. In fact, L_η is continuous, but not even Lipschitz continuous. Similarly, we can show that the inversion map $\mathrm{Inv} : \mathcal{D}^s \to \mathcal{D}^s$ is continuous, but not even Lipschitz continuous. We will see in the Chapter 3(see Lemma 3.2.4) that this is a generic phenomenon of the diffeomorphism group. For more discussion on the diffeomorphism group as a Lie group, see [15, 21].

2.4 Riemannian Structures of $\mathcal{D}^s(M)$ and $\mathcal{D}^s_\mu(M)$

2.4.1 L^2 inner product on $T_e\mathcal{D}^s_\mu(M)$

Let $\langle\cdot,\cdot\rangle$ be the Riemannian metric on M and μ be the corresponding volume form. For $v, w \in T_e\mathcal{D}^s(M)$ we define the L^2 inner product on $T_e\mathcal{D}^s(M)$ as follows:

$$\langle\!\langle v, w\rangle\!\rangle_e := \int_M \langle v(x), w(x)\rangle_{T_xM}\mu(x). \tag{2.2}$$

Then we can use the right translation $R_\eta : \mathcal{D}^s \to \mathcal{D}^s$ to extend this inner product on the entire group $\mathcal{D}^s(M)$; for $V, W \in T_\eta\mathcal{D}^s(M)$ and $\eta \in \mathcal{D}^s(M)$,

$$\langle\!\langle V, W\rangle\!\rangle_\eta := \langle\!\langle d_\eta R_{\eta^{-1}}V, d_\eta R_{\eta^{-1}}W\rangle\!\rangle_e = \langle\!\langle V\circ\eta^{-1}, W\circ\eta^{-1}\rangle\!\rangle_e.$$

On the group of volume preserving diffeomorphisms $\mathcal{D}^s_\mu(M)$, this metric is right invariant which enables us to simplify a number of formulas.

Note that this Riemannian metric is *weak* since it generates the L^2 topology which is weaker than the H^s topology of the manifold $\mathcal{D}^s(M)$. This, the gap between the topology and geometry, is one of the main issues when working with the infinite dimensional Riemannian manifold. For example, unlike the finite dimensional case, the smoothness of the metric and Riemannian exponential map is not guaranteed and one needs to prove such facts for the given weak Riemannian metric. See [21, 57] for more discussion on the weak Riemannian metric.

2.4.2 Hodge Decomposition

For $\eta \in \mathcal{D}^s$, we have the following orthogonal decomposition from the Hodge theory [21, 70]:

$$T_\eta \mathcal{D}^s(M) = T_\eta \mathcal{D}^s_\mu(M) \oplus_{L^2} \nabla H^{s+1}(M) \circ \eta. \tag{2.3}$$

In the case when $\eta = e$, the above decomposition reads as

$$H^s(TM) = T_e \mathcal{D}^s(M) = T_e \mathcal{D}^s_\mu(M) \oplus_{L^2} \nabla H^{s+1}(M).$$

That is, if $v \in H^s(TM)$ is a vector field on M, we can write

$$v = w + \nabla f,$$

where $\mathrm{div} w = 0$, ∇f is the gradient of some function f, and $w \perp \nabla f$. By taking a divergence on both sides of this equation, we obtain an elliptic equation $\Delta f = \mathrm{div} v$. By solving this equation, we get

$$f = \Delta^{-1}(\mathrm{div} v) \in H^{s+1},$$

so that $v = (v - \nabla f) + \nabla f =: w + \nabla f$. In the case when $\partial M \neq \emptyset$, we impose the Neumann boundary condition $\langle \nabla f, \nu \rangle = \langle v, \nu \rangle$, where ν is the outward unit normal to ∂M. The Neumann problem for f has a unique solution up to an arbitrary constant, which we can specify. Note that w is L^2 orthogonal to ∇f since

$$\begin{aligned}\langle w, \nabla f\rangle_{L^2} &= \int_M \langle w(x), \nabla f(x)\rangle \, d\mu \\ &= \int_M \mathrm{div}(f \cdot w) - f \mathrm{div} w \, d\mu \\ &= \int_M \mathrm{div}(f \cdot w) \, d\mu \; (\because \mathrm{div} w = 0) \\ &= 0 \; (\because \text{Stokes Theorem})\end{aligned}$$

For an alternative approach on the Hodge decomposition in terms of differential forms, see [21, 79].

Let P_η and Q_η denote the corresponding L^2 orthogonal projections onto $T_\eta \mathcal{D}^s_\mu(M)$ and $\nabla H^{s+1}(M) \circ \eta$ in (2.3), respectively. We have $P_\eta = 1 - Q_\eta$ and the maps

$$\begin{aligned}\mathcal{D}^s(M) \to L(T_\eta \mathcal{D}^s, T_\eta \mathcal{D}^s_\mu) : \eta \mapsto P_\eta, \text{ and} \\ \mathcal{D}^s(M) \to L(T_\eta \mathcal{D}^s, \nabla H^{s+1} \circ \eta) : \eta \mapsto Q_\eta\end{aligned}$$

are smooth. In the dimension one case, we can check the smoothness of the mapping $\eta \mapsto Q_\eta = \nabla_\eta \Delta_\eta^{-1} \mathrm{div}_\eta$ by a direct computation. Note that

$$\eta \mapsto Q_\eta = \big(\partial_x (f \circ \eta^{-1})\big) \circ \eta = \frac{f'}{\eta'},$$

which is smooth mapping in η. For the general case, see [21].

2.4.3 Euler equation

Finally, we prove that the Euler equation for ideal fluid is the geodesic equation on $\mathcal{D}^s_\mu$ with respect to the right invariant L^2 metric (2.2). The following theorem is due to Ebin and Marsden [21]. For the simplicity, we consider the case when $M = \mathbb{T}^n$, the n-dimensional flat torus.

Theorem 2.4.1. *The map $t \mapsto \eta(t)$ is a geodesic of the L^2-metric on $\mathcal{D}_\mu(M)$ with $\eta(0) = e$ and $\dot{\eta}(0) = u_0$ if and only if the map*

$$t \mapsto u(t) := \dot{\eta}(t) \circ \eta^{-1}(t) = d_{\eta_t} R_{\eta^{-1}(t)} \left(\dot{\eta}(t)\right)$$

is a 1-parameter family of vector fields on M that solves the Euler equation.

Proof. Recall that geodesics are critical points of the energy functional of the Riemannian metric

$$\mathcal{E}(\eta) = \frac{1}{2} \int_0^1 \|\dot{\eta}(t)\|_{L^2}^2 \, dt.$$

So it suffices to compute the First Variation Formula for $\mathcal{E}$.

Let $\epsilon > 0$ and $(l,t) \in (-\epsilon, \epsilon) \times [0,1]$. Let $(l,t) \mapsto \eta_s(t)$ be a 1 parameter variation of $\eta(t) = \eta_0(t)$ with fixed end points $\eta_l(0) = e$ and $\eta_l(1) = \eta(1)$. Let $X(t) := \frac{\partial}{\partial l}\Big|_{l=0} \eta_l(t)$ be the associated variational vector field. Note that $X(0) = X(1) = 0$ since endpoints are fixed. Then we compute

$$\begin{aligned}
0 =& \frac{\partial}{\partial s}\Big|_{l=0} \mathcal{E}(\eta_l) \\
=& \frac{\partial}{\partial l}\Big|_{l=0} \frac{1}{2} \int_0^1 \int_{\mathbb{T}^n} \langle \partial_t \eta_l(t,x), \partial_t \eta_l(t,x) \rangle \, dxdt \\
=& \int_0^1 \int_{\mathbb{T}^n} \langle \partial_l\Big|_{l=0} \partial_t \eta_l(t,x), \partial_l\Big|_{l=0} \partial_t \eta_l(t,x) \rangle \, dxdt \\
=& \int_0^1 \int_{\mathbb{T}^n} \langle \partial_t \partial_l \eta_l(t,x), \partial_l \partial_t \eta_l(t,x) \rangle \, dxdt \\
=& \langle X(t), \dot{\eta}(t) \rangle_{L^2} \Big|_{t=0}^{t=1} - \int_0^1 \int_{\mathbb{T}^n} \langle X(t,x), \ddot{\eta}(t,x) \rangle \, dxdt \; (\because \text{ integration by parts}) \\
=& 0 - \int_0^1 \langle X(t) \circ \eta^{-1}(t), \ddot{\eta}(t) \circ \eta^{-1}(t) \rangle_{L^2} \, dt (\because \text{ right invariance of metric}).
\end{aligned}$$

Since the variational vector field X is an arbitrary divergence free vector field, the inner product inside the integral must vanish. Since

$$\operatorname{div}(X(t) \circ \eta^{-1}(t)) = 0,$$

by using the Hodge decomposition, we have for each $t \in (0,1)$,

$$\ddot{\eta}(t) \circ \eta^{-1}(t) = \nabla(-p(t)),$$

for some function $p(t)$. Finally, write

$$\begin{aligned} -(\nabla p) \circ \eta = \ddot{\eta}_t = \partial_t(\dot{\eta}_t) = \partial_t(u_t \circ \eta_t) &= \partial_t u \circ \eta + Du \circ \eta \cdot \dot{\eta} \\ &= \partial_t u \circ \eta + Du \circ \eta \cdot (u \circ \eta) \\ &= (\partial_t u + \nabla_u u) \circ \eta, \end{aligned}$$

which is the Euler equation. □

2.5 Lagrangian perspectives on hydrodynamics

2.5.1 Arnold's Lie group approach on fluid

Let G be a (possibly infinite dimensional) Lie group and $\mathfrak{g} = T_e G$ be the Lie algebra where $e \in G$ is the identity element with the Lie bracket $[\cdot,\cdot]$. Let $\langle \cdot,\cdot \rangle$ be the inner product on $\mathfrak{g}$. In the seminal paper of 1966, Arnold [2] observed that the Euler's equation for the motions of a rigid body is the geodesic equation on the group of rotations of three dimensional Euclidean space endowed with a left-invariant metric. Furthermore, he noticed that this Lagrangian formalism(which dates back to Hamilton [30] and Lagrange [48]) could be a universal approach to describe physical motions of continuum mechanics. That is, if we find an appropriate configuration space G with a symmetry for the given physical system, the motions are described by a path on G. This idea goes back to Poincaré [73]. Then if there is no potential energy, this path is necessarily a geodesic by the least action principle. The general procedure for computing the geodesic equation is following; we first construct a right(or left) invariant Riemannian metric on G induced by the inner product $\langle \cdot, \cdot \rangle$. Then we can obtain a Levi-Civita connection ∇ from this metric and define the geodesic as the curve whose acceleration is identically zero.

Unlike a finite dimensional Riemannian manifold, the existence of a covariant derivative ∇ that is compatible with the metric is not guaranteed on an infinite dimensional manifold. See Constantin-Kolev [15] for more discussion on this. Hence, deriving the geodesic equation on the Lie group G using the connection ∇ is highly technical and not efficient. Instead, we can use the Lie group theory to define the geidesic equation on the Lie algebra $\mathfrak{g}$. See Arnold-Khesin [3] for more details.

2.5.2 Hamiltonian formulation

On the Lie group $G = \mathcal{D}^s$, the group adjoint operator ad is a representation of the group on its Lie algebra $\mathfrak{g} = \mathrm{Vect}(M)$, the space of all vector fields of class H^s. It is given by the negative of the usual Lie bracket of vector fields, that is,

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ u, v &\mapsto \mathrm{ad}_u v := -[u, v] \end{aligned}$$

Let $\Lambda : \mathfrak{g} \to \mathfrak{g}^*$ be the inertia operator that induces the right-invariant metric $\langle \cdot, \cdot \rangle$. That is, for $u, v \in \mathfrak{g}$, the operator A satisfies the following relation

$$(\Lambda u, v) = \langle u, v \rangle,$$

where $(\cdot,\cdot)$ denotes the pairing between the Lie algebra and its dual. We also have the coadjoint representation ad^* of the group in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$. That is, $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ is defined by the following relation:

$$(\mathrm{ad}^*{}_w\, m, v) = (m, \mathrm{ad}_w\, v) = -(m, [w,v]) = -\langle u, [w,v]\rangle,$$

where $m = \Lambda u \in \mathfrak{g}^*$ and $u, v, w \in \mathfrak{g}$. Then the Euler-Arnold equation can be written as the following equation on $\mathfrak{g}^*$:

$$\begin{cases} m_t = -\,\mathrm{ad}^*{}_u\, m \\ m(0) = m_0 \end{cases} \tag{2.4}$$

where $m = \Lambda u$. This equation reads as the following on the Lie algebra $\mathfrak{g}$:

$$\begin{cases} (\Lambda u)_t = -\,\mathrm{ad}^*{}_u(\Lambda u) \\ u(0) = u_0 \end{cases}$$

Then we have the following theorem from Arnold [2].

Theorem 2.5.1. *Let G be a Lie group equipped with a right invariant metric $\langle\cdot,\cdot\rangle$. Then $t \mapsto \eta(t)$ is a geodesic in G with $\eta(0) = e$ and $\dot\eta(0) = u_0$ if and only if $t \mapsto u(t) = dR_{\eta^{-1}(t)}\dot\eta(t)$ satisfies the Euler-Arnold equation in T_eG where $\partial_t u = -\,\mathrm{ad}^*_u\, u$ and $u(0) = u_0$.*

Proof. Let $(l,t) \mapsto \gamma_l(t)$ be a 1 parameter variation with $\gamma_0(t) = \eta(t)$ and fixed ends at $t = a$ and $t = b$. Using the right invariance of the metric, we can write the energy functional of $\langle\cdot,\cdot\rangle$ in the form

$$\mathcal{E}(\gamma_l) = \frac{1}{2}\int_0^1 \|\partial_t\gamma_l(t)\|^2\, dt = \frac{1}{2}\int_a^b \langle dR_{\gamma_l^{-1}(t)}\partial_t\gamma_l(t), dR_{\gamma_l^{-1}(t)}\partial_t\gamma_l(t)\rangle\, dt.$$

Let

$$w(t) := \Big(dR_{\gamma_l^{-1}(t)}\partial_l\gamma_l(t)\Big)\Big|_{l=0} \in T_eG$$

be the right translation of the associated variational field $X(t) = \partial_l\Big|_{l=0}\gamma_l(t)$ along η. Note that

$$\begin{aligned}\partial_l\Big|_{l=0}\left(dR_{\gamma_l^{-1}(t)}\partial_t\gamma_l(t)\right) =&\partial_l\Big|_{l=0}\left(\partial_t\gamma_l(t)\circ\gamma_l^{-1}(t)\right)\\ =&\partial_t\left(\partial_l\Big|_{l=0}\gamma_l(t)\right)\circ\eta^{-1}(t)+D\left(\partial_t\eta(t)\right)\circ\eta^{-1}(t)\cdot\partial_l\Big|_{l=0}\gamma_l^{-1}(t)\\ =&\partial_t\left(w(t)\circ\eta(t)\right)\circ\eta^{-1}(t)-D\left(\partial_t\eta(t)\right)\circ\eta^{-1}(t)\cdot D\eta^{-1}(t)\cdot w(t)\\ =&\partial_t w(t)+Dw(t)\cdot\partial_t\eta(t)\circ\eta^{-1}(t)-Du(t)\cdot w(t)\\ =&\partial_t w(t)+Dw(t)u(t)-Du(t)\cdot w(t)\\ =&\partial_t w(t)-\operatorname{ad}_{u(t)}w(t).\end{aligned}$$

Now, differentiating the energy functional and integrating by parts gives

$$0=\partial_l\Big|_{l=0}\mathcal{E}(\gamma_l)=\int_a^b\langle\partial_t w(t)-\operatorname{ad}_{u(t)}w(t),u(t)\rangle\,dt=-\int_a^b\langle w(t),\partial_t u(t)+\operatorname{ad}^*_{u(t)}u(t)\rangle\,dt.$$

Since $w(t)$ can be chosen arbitrary, we must have $\partial_t u(t)+\operatorname{ad}^*_{u(t)}u(t)=0$ in order to have the integral vanish. □

2.6 Camassa-Holm equation

The Camassa-Holm equation

$$u_t-u_{txx}+3uu_x-uu_{xxx}-2u_xu_{xx}=0 \tag{2.5}$$

was originally proposed by Camassa and Holm [10] as a model for shallow water waves, which was obtained by using an asymptotic expansion in the Hamiltonian of the Euler equation in the shallow water regime. This equation was also discovered by Fuchssteiner and Fokas independently [24] as an abstract bi-Hamiltonian equation that generalizes the Korteweg-de Vries(KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.6}$$

which is another model for shallow water wave. The two equations share remarkable properties such as traveling wave solutions, i.e., solutions of the form $u(t,x) = \phi(x - ct)$ which travel with fixed speed c maintaining its shape and vanish at infinity. Furthermore, these traveling wave solutions are solitions, which means that two traveling waves reconstitute their shape and size after having an interaction. Also, CH and KdV equations are completely integrable. That is, the PDEs can be turned into an infinite system of linear ODEs, which can be integrated to give an explicit solution. They have bi-hamiltonian structures which produce an infinite hierarchy of conserved quantities.

The solutions of KdV and CH equations have different global behavior. As soon as the initial condition $u_0 \in H^1(\mathbb{R})$, the solutions of (2.6) are global [43]. However, the solution of the CH equation admits both global solution and finite time blowup via wave breaking. This means that the solution u remains bounded but its slope u_x becomes unbounded at the breakdown time. This is what makes the CH equation very interesting and distinct from other shallow water wave equations. For more discussion on KdV, refer to [6, 41, 43, 63]. For more background on the CH equation, see the following papers and references therein: for symmetries, complete integrability, and bi-hamiltonian structures, see [10, 24]; for peakon and soliton solutions, see [5, 16, 17, 18, 53, 54, 55]; for wave breaking, see [13, 14, 16, 64, 66].

As in the Euler equation for ideal fluid, the solutions to the CH equation can be interpreted as geodesics of the right invariant Sobolev H^1 metric

$$\langle u, v \rangle_{H^1} = \int uv + u_x v_x \, dx. \tag{2.7}$$

on the diffeomorphism group on $\mathbb{R}$ or also on $\mathbb{S}^1$.

Proposition 2.6.1. *The geodesic equation for a curve $\eta(t) \in \mathcal{D}$ in the metric* (2.7) *is given by the following system:*

$$\begin{cases} \eta_t = u \circ \eta, \\ u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \end{cases} \tag{2.8}$$

with initial data $\eta(0) = \mathrm{Id}$, $u(0,x) = u_0(x)$.

Proof. We will compute ad^* and plug it into the geodesic equation (2.4). Note that

$$\begin{aligned} \langle \mathrm{ad}^*{}_u\, v, w \rangle =& \langle v, \mathrm{ad}_u\, w \rangle = \int v\Lambda(wu_x - uw_x)dx \\ =& \int w\left[u_x \Lambda v + \partial_x(u\Lambda v)\right] dx \\ =& \langle w, \Lambda^{-1}\left[u_x \Lambda v + \partial_x\left(u\Lambda v\right)\right]\rangle, \end{aligned}$$

where $\Lambda = 1 - \partial_x^2$ is the inertia operator. Hence, we conclude that

$$\mathrm{ad}^*{}_u\, v = \Lambda^{-1}\left(2u_x\Lambda v + u\Lambda v_x\right).$$

By substituting in to the Euler-Arnold equation (2.4) and expanding, we get the CH equation (3.1). □

The geometric interpretation of the CH equation was first discovered by Misiołek [68] on the Bott-Virasoro group and Kouranbaeva [47] on the diffeomorphism group. See also Constantin-McKean [16].

The local well-posedness of the corresponding Cauchy problem of the CH equation in both periodic and nonperiodic cases has been studied extensively. In 1997, Constantin [11] showed the local well-posedness in the Sobolev spaces $H^s(\mathbb{S}^1)$ for $s \geq 4$ where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then Constantin and Escher [14] improved the result with $s \geq 3$ in 1998. Another approach was taken by Danchin [19] in 2001 using the Besov spaces $B^s_{p,r}(\mathbb{S}^1)$ with $s > \max\{1 + 1/p, 3/2\}$, $1 \leq p \leq \infty$, and $1 \leq r < \infty$. In 2002, Misiołek [69] proved the result in the space $C^1(\mathbb{S}^1)$ with the Arnold's geometric framework.

For the nonperiodic case, local well-posedness was proved for initial data in $H^s(\mathbb{R})$ with $s > 3/2$ by Li and Olver [59] in 2000 and Rodriguez-Blanco [78] in 2001. In 2017, the author and Preston [52] proved the result in $H^1(\mathbb{R}) \cap C^1(\mathbb{R})$ space with a decaying condition using geometric approach, which is nicely compared with the result of Linares-Ponce-Sideris [60] who proved the existence and uniqueness of the solutions, but not the continuous dependence of on the initial data, in the space $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Also in 2017, Holmes and Thompson [39] proved the local well-posedness in $C^1(\mathbb{R})$ by using the technique of Bressan-Constantin [7].

As mentioned earlier, the solution of the CH equation forms a wave breaking singularity and this can be nicely illustrated in terms of peakon-antipeakon interaction. Peakons and antipeakons are peaked solution solutions whose first derivative has a jump discontinuity, which have the general form $u(t, x) = ce^{-|x-ct|}$. When these two waves collide at some time, the combined wave forms an infinite slope and there are two possible scenarios after the collision; either two waves pass through each other with total energy preserved, or annihilate each other with a lose of energy. The solutions in the former case is called conservative and the latter case is called dissipative.

The continuation of the solutions after wave breaking has been studied extensively, as well. There are different situations where such solutions are constructed; energy preservation after the breakdown(conservative/dissipative), type of spatial domain(periodic/non-periodic), vanishing/non-vanishing asymptotics in the case of non-periodic domain, etc. Here, we list some previous known results on the global weak solutions by their distinct approaches. Bressan-Constantin [7, 8] and Holden-Raynaud [34, 33, 35, 36, 37] obtained global weak solutions by reformulating the CH equation into a semilinear system of ODEs after introducing a new set of independent and dependent variables. Another approach was taken by Xin-Zhang [83] using the limit of viscous approximation. Bressan-Fonte [9] defined a Lipschitz distance functional to extract global weak solutions as the uniform limit of multi-peakon solutions. Grunert-Holden-Raynaud [26, 29] defined the new Lipschitz metric that is consistent with the construction of the solutions as in [34] and [36]. They also studied the aspects of global conservative solutions of the CH equation with nonvanishing asymptotics [27] and as a limit of vanishing density in the two-component CH system [28].

In the following Chapter 3, we will discuss the first main result of the thesis. In Chapter 3, we will prove the local well-posedness of the CH equation on the real line. This chapter is reprinted from the paper [52] published in the peer-reviewed journal Discrete and Continuous Dynamical Systems - Series A, and it is a joint work with my advisor Stephen Preston.

Chapter 3

Local Well-posedness of the Camassa-Holm equation on the real line

3.1 Introduction

We study the local well-posedness of the nonperiodic Camassa-Holm [CH] equation

$$u_t - u_{txx} + 3uu_x - uu_{xxx} - 2u_x u_{xx} = 0, \quad x, t \in \mathbb{R}. \tag{3.1}$$

In 2002, Misiołek [69] proved the local well-posedness of the periodic CH equation in the space of continuously differentiable functions on S^1, by viewing the equation as an ODE in a Banach space using the geometric interpretation. We want to establish an analogous C^1 result for the non-periodic problem using a similar technique. The main difference when we consider the non-periodic case is that the domain is not compact, so we must consider C^1 diffeomorphisms with an appropriate decaying condition. We study this diffeomorphism group and show that it is a topological group, i.e., operations of composition and inversion are continuous.

It is interesting to compare the present work with the recent paper by Linares-Ponce-Sideris [60]. There, the authors assume slightly weaker hypotheses: their initial data u_0 is in $H^1 \cap W^{1,\infty}$ rather than $H^1 \cap C^1$, which allows them to include peakon solutions of the form $u(t,x) = e^{-|x-ct|}$. Using different methods, they obtain local existence and uniqueness in this space, but significantly they do not obtain continuous dependence on the initial data (and in fact an explicit example shows it is *false* in that context). This demonstrates that the group of piecewise C^1 diffeomorphisms on $\mathbb{R}$ does *not* form a topological group, since the continuity properties of the inversion and composition are the primary tool to make our proofs work.

Note that even when the solution operator is continuous, it is not uniformly continuous even in very strong Sobolev topologies; see for example Himonas-Kenig-Misiołek [32]. This is a consequence of the failure of the group operations to be uniformly continuous, as we will see.

In section 3.3, we use the Lagrangian approach for the local well-posedness of the CH equation. That is, we will write the equation entirely in terms of the flow η of particle trajectories, which turns the equation into an ODE on the open subset of a Banach space. We will show that the resulting vector field is locally Lipschitz, so that we can apply the Picard Theorem for ODEs in Banach space (see Lang [50]). Topological group properties will then ensure that the solution depends continuously on the initial data, which completes the proof of the local well-posedness.

3.2 The group of $C^1 \cap H^1$ diffeomorphisms

Definition 3.2.1. *We denote by $\mathcal{D}(\mathbb{R})$ the set of maps $\eta : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions*

1. *$\eta(x)$ is a C^1 function with a bound $a \le \eta'(x) \le b$ on $\mathbb{R}$ for some $0 < a < b$,*
2. *$\int_{\mathbb{R}} |\eta(x) - x|^2 \, dx < \infty$ and $\int_{\mathbb{R}} |\eta'(x) - 1|^2 \, dx < \infty$,*
3. *$\lim_{|x|\to\infty} \eta'(x) = 1$.*

Similarly, denote by $\mathcal{V}_1(\mathbb{R})$ the set of maps $u : \mathbb{R} \to \mathbb{R}$ satisfying the conditions

1. *$u(x)$ is a C^1 function with a bound $|u'(x)| \le M$ on $\mathbb{R}$ for some $M > 0$,*
2. *$\int_{\mathbb{R}} |u(x)|^2 \, dx < \infty$ and $\int_{\mathbb{R}} |u'(x)|^2 \, dx < \infty$,*
3. *$\lim_{|x|\to\infty} u'(x) = 0$.*

The topology of $\mathcal{V}_1(\mathbb{R})$ is generated by the following norm:

$$\|u\|_{1,1} = \|u\|_{C^1} + \|u\|_{H^1} = \sup_{x\in\mathbb{R}} |u(x)| + \sup_{x\in\mathbb{R}} |u'(x)| + \sqrt{\int_{\mathbb{R}} u(x)^2 dx + \int_{\mathbb{R}} u'(x)^2 dx}, \tag{3.2}$$

and the corresponding distance on the space $\mathcal{D}(\mathbb{R})$ is given by

$$D(\eta, \xi) = \|\eta - \xi\|_{1,1} . \tag{3.3}$$

The conditions in the above definition have useful consequences, which we list here; the proof is straightforward.

Lemma 3.2.2. *Suppose that f is a C^1 function with $|f'(x)| \le M$ for all x and $\int_{\mathbb{R}} f(x)^2 dx < \infty$. Then $\lim_{|x|\to\infty} f(x) = 0$. If, in addition, we have $\lim_{|x|\to\infty} f'(x) = 0$, then f' is uniformly continuous.*

From this lemma, we conclude another decaying property: $\lim_{|x|\to\infty} |\eta(x) - x| = 0$ and $\lim_{|x|\to\infty} u(x) = 0$. Furthermore we know that η' and u' are uniformly continuous. This uniform continuity of the derivatives is a very useful tool for many estimates in this paper, and in fact it is not hard to check that uniform continuity of the derivative together with an H^1 bound implies the decay properties above.

Remark 2. *We can easily check that $\mathcal{D}(\mathbb{R})$ is a topological manifold: given $\eta \in \mathcal{D}(\mathbb{R})$, let $v(x) := \eta(x) - x$; then v and v' are bounded continuous functions approaching zero asymptotically, satisfying $v'(x) > -1$ for all x and $\|v\|_{H^1} < \infty$. Conversely, any such v gives $\eta = \mathrm{Id} + v$ in $\mathcal{D}(\mathbb{R})$, so that the map $\eta \mapsto v$ is a bijection. The image of this bijection is the set $\{v \in \mathcal{V}_1(\mathbb{R}) \,:\, \Phi(v) > -1\}$ where $\Phi(v) := \min_{x\in\mathbb{R}} v'(x)$, which is obviously a continuous function in the topology* (3.3)*; hence, we have a map to an open convex subset of a Banach space $\mathcal{V}_1(\mathbb{R})$. This makes $\mathcal{D}(\mathbb{R})$ a manifold with just one chart.*

Next, we show that $\mathcal{D}(\mathbb{R})$ is a group. We know that C^1 diffeomorphisms themselves form a group, but we need to check that it is closed under the additional conditions we have imposed. After that, we will verify that it is a topological group as well, which is a bit more involved.

Proposition 3.2.3. *The set $\mathcal{D}(\mathbb{R})$ is a group.*

Proof. First we show that $\mathcal{D}(\mathbb{R})$ contains all inverses. If $\eta \in \mathcal{D}(\mathbb{R})$, then $\xi := \eta^{-1}$ exists for all x by the inverse function theorem on $\mathbb{R}$, and it is in C^1. Furthermore, since $\xi'(x) = \eta'(\xi(x))^{-1}$, we see that bounds $a \le \eta'(x) \le b$ imply the bounds $b^{-1} \le \xi'(x) \le a^{-1}$. Also, we can easily see that $\xi'(x) \to 1$ as $x \to \infty$, since $\eta'(x) \to 1$. The fact that ξ has finite L^2 distance from the identity comes from

$$\int_{\mathbb{R}} |\xi(x) - x|^2 \, dx = \int_{\mathbb{R}} |\xi(\eta(y)) - \eta(y)|^2 \, \eta'(y) \, dy \le b \int_{\mathbb{R}} |\eta(y) - y|^2 \, dy < \infty,$$

using the change of variables formula with $x = \eta(y)$. Similarly, we can check

$$\int_{\mathbb{R}} |\xi'(x) - 1|^2 \, dx \leq \int_{\mathbb{R}} \left| \frac{1}{\eta'(y)} - 1 \right|^2 \eta'(y) \, dy \leq \frac{1}{a} \int_{\mathbb{R}} |\eta'(y) - 1|^2 \, dy < \infty.$$

Next, we show that $\mathcal{D}(\mathbb{R})$ contains compositions. Let $\eta, \phi \in \mathcal{D}(\mathbb{R})$ with the bounds $0 < a \leq \eta'(x) \leq b$ and $0 < c \leq \phi'(x) \leq d$. Since $(\phi \circ \eta)'(x) = \phi'(\eta(x))\eta'(x)$, the limit $(\phi \circ \eta)'(x) \to 1$ is obvious, and we also have the obvious bounds $ac \leq (\phi \circ \eta)'(x) \leq bd$. Lastly, we can check that

$$\begin{aligned}\int_{\mathbb{R}} |\phi(\eta(y)) - y|^2 \, dy &= \int_{\mathbb{R}} |\phi(x) - \xi(x)|^2 \, \xi'(x) \, dx \\ &\leq 2a^{-1} \int_{\mathbb{R}} |\phi(x) - x|^2 + 2a^{-1} \int_{\mathbb{R}} |\xi(x) - x|^2 \, dx < \infty,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} |\phi'(\eta(y))\eta'(y) - 1|^2 \, dy &\leq 2 \int_{\mathbb{R}} |\phi'(\eta(y))\eta'(y) - \eta'(y)|^2 \, dy + 2 \int_{\mathbb{R}} |\eta'(y) - 1|^2 \, dy \\ &\leq 2b \int_{\mathbb{R}} |\phi'(x) - 1|^2 \, dx + 2 \int_{\mathbb{R}} |\eta'(y) - 1|^2 \, dy < \infty.\end{aligned}$$

□

We now show that $\mathcal{D}(\mathbb{R})$ is a topological group, which means that the inversion and composition maps are continuous in the distance (3.3). We prove this in the following lemmas.

Lemma 3.2.4. *The map* $\mathrm{Inv} : \eta \mapsto \eta^{-1}$ *is continuous.*

Proof. Let $\eta_1 \in \mathcal{D}(\mathbb{R})$ with $\mathrm{Inv}(\eta_1) = \xi_1$. We want to show that the map Inv is continuous at η_1. Let $\epsilon > 0$ be given. We have to bound $\|\xi_1 - \xi_2\|_{1,1}$ in terms of $\rho := \|\eta_1 - \eta_2\|_{1,1}$ for all $\eta_2 \in \mathcal{D}(\mathbb{R})$ with the inverse ξ_2. We will estimate L^∞ and L^2 norms of $\xi_1 - \xi_2$ and $\xi_1' - \xi_2'$ separately.

From the definition, we have $a_i \leq \eta_i'(x) \leq b_i$ for $i = 1, 2$. Note that $a_2 \geq a_1 - \rho$ and $b_2 \leq b_1 + \rho$ which we will use later. Our goal is to get the bound of $\|\xi_1 - \xi_2\|_{1,1}$ solely in terms of quantities depending on η_1. From Lemma 3.2.2, η_1' is uniformly continuous and so there is a modulus of continuity $\omega_1 : \mathbb{R} \to \mathbb{R}$ satisfying $\omega_1(0) = \lim_{\rho\to 0} \omega_1(\rho) = 0$ with ω_1 increasing such that

$$|\eta_1'(x_1) - \eta_1'(x_2)| \leq \omega_1(|x_1 - x_2|)$$

for all $x_1, x_2 \in \mathbb{R}$. First, note that

$$|\xi_1(x) - \xi_2(x)| = |\xi_1(\eta_2(y)) - y| = |\xi_1(\eta_2(y)) - \xi_1(\eta_1(y))| \leq a_1^{-1} |\eta_1(y) - \eta_2(y)| \leq \rho/a_1$$

for all x, where $x = \eta_2(y)$. So

$$\|\xi_1 - \xi_2\|_{L^\infty} \leq \frac{\rho}{a_1}.$$

Next, using the same trick as above with $x = \eta_2(y)$, we write

$$\begin{aligned}
\int_{\mathbb{R}} |\xi_1(x) - \xi_2(x)|^2 \, dx =& \int_{\mathbb{R}} |\xi_1(\eta_2(y)) - \xi_1(\eta_1(y))|^2 \, \eta_2'(y) \, dy \\
\leq& \frac{b_2}{a_1^2} \int_{\mathbb{R}} |\eta_2(y) - \eta_1(y)|^2 \, dy \\
\leq& \frac{b_1 + \rho}{a_1^2} \rho^2,
\end{aligned}$$

since $b_2 \leq b_1 + \rho$. We conclude that

$$\|\xi_1 - \xi_2\|_{L^2} \leq \frac{\sqrt{b_1 + \rho}}{a_1} \rho.$$

Now we estimate $\|\xi_1' - \xi_2'\|_{L^\infty}$. We have

$$\begin{aligned}
|\xi_1'(x) - \xi_2'(x)| &= \left|\eta_1'(\xi_1(x))^{-1} - \eta_2'(\xi_2(x))^{-1}\right| \\
&= \frac{1}{|\eta_1'(\xi_1(x))|\,|\eta_2'(\xi_2(x))|} |\eta_1'(\xi_1(x)) - \eta_2'(\xi_2(x))| \\
&\leq \frac{1}{a_1a_2} \left(|\eta_1'(\xi_1(x) - \eta_1'(\xi_2(x)))| + |\eta_1'(\xi_2(x)) - \eta_2'(\xi_2(x))|\right) \\
&\leq \frac{1}{a_1(a_1-\rho)} \left(\omega_1\left(|\xi_1(x) - \xi_2(x)|\right) + \rho\right) \\
&\leq \frac{1}{a_1(a_1-\rho)} \left(\omega_1(\rho/a_1) + \rho\right).
\end{aligned}$$

Hence

$$\|\xi_1' - \xi_2'\|_{L^\infty} \leq \frac{1}{a_1(a_1-\rho)} \left(\omega_1(\rho/a_1) + \rho\right).$$

The last estimate $\|\xi_1' - \xi_2'\|_{L^2}$ is a little bit more complicated than the previous ones. We use heavily the uniform continuity and work directly in terms of the $\epsilon > 0$ given above. First, note that we can write

$$\begin{aligned}
\|\xi_1' - \xi_2'\|_{L^2}^2 &= \int_{\mathbb{R}} \left(\xi_1'(x) - \xi_2'(x)\right)^2 \, dx \\
&= \int_{\mathbb{R}} \left(\frac{1}{\eta_1'(\xi_1(x))} - \frac{1}{\eta_2'(\xi_2(x))}\right)^2 \, dx \\
&= \int_{\mathbb{R}} \frac{1}{\eta_1'(\xi_1(x))^2\eta_2'(\xi_2(x))^2} \left(\eta_1'(\xi_1(x)) - \eta_2'(\xi_2(x))\right)^2 \, dx \\
&\leq \frac{1}{(a_1a_2)^2} \int_{\mathbb{R}} \left(\eta_1'(\xi_1(x)) - \eta_2'(\xi_2(x))\right)^2 \, dx \\
&\leq \frac{2}{a_1^2(a_1-\rho)^2} \Big(\int_{\mathbb{R}} \left(\eta_1'(\xi_2(x)) - \eta_2'(\xi_2(x))\right)^2 \, dx \\
&\qquad\qquad + \int_{\mathbb{R}} \left(\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))\right)^2 \, dx \Big)
\end{aligned}$$

Then for the first integral, we have

$$\int_{\mathbb{R}} \left(\eta_1'(\xi_2(x)) - \eta_2'(\xi_2(x))\right)^2 \, dx = \int_{\mathbb{R}} \left(\eta_1'(y) - \eta_2'(y)\right)^2 \eta_2'(y) \, dy \le b_2\rho^2 \le (b_1+\rho)\rho^2.$$

For the second integral, we use the fact that the function $\eta_1'(x) - 1$ is in L^2 from the assumption on $\mathcal{D}(\mathbb{R})$. So there exists $M > 0$ such that

$$\int_{|x|>M} (\eta_1'(z) - 1)^2 \, dz < \epsilon^2 \frac{a_1^2(a_1-\rho)^2}{32(2b_1+\rho)}.$$

Now choose $M' = M + 2\left\|\xi_1 - \mathrm{Id}\right\|_{L^\infty}$; then $|x| > M'$ implies $|\xi_1(x)| > M$. In addition if $\rho < \left\|\xi_1 - \mathrm{Id}\right\|_{L^\infty}$ then $|x| > M'$ also implies $|\xi_2(x)| > M$.

Then

$$\begin{aligned}\int_{\mathbb{R}} \left(\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))\right)^2 \, dx &= \int_{|x|\le M'} \left(\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))\right)^2 \, dx \\ &\quad + \int_{|x|>M'} \left(\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))\right)^2 \, dx \\ &=: (I) + (II).\end{aligned}$$

Then

$$\begin{aligned}(II) &\le 2\int_{|x|>M'} (\eta_1'(\xi_1(x)) - 1)^2 \, dx + 2\int_{|x|>M'} (\eta_1'(\xi_2(x)) - 1)^2 \, dx \\ &\le 2\int_{|z|>M} (\eta_1'(z) - 1)^2 \eta_1'(z) \, dz + 2\int_{|z|>M} (\eta_1'(z) - 1)^2 \eta_2'(z) \, dz \\ &< \frac{\epsilon^2 a_1^2(a_1-\rho)^2}{16}.\end{aligned}$$

For (I), we use the uniform continuity of η_1' directly. That is we can find $\delta > 0$ such that if

$|\xi_1(x) - \xi_2(x)| < \delta$, then

$$|\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))| < \frac{\epsilon a_1(a_1 - \rho)}{\sqrt{32M'}},$$

for all $x \in \mathbb{R}$. Requiring ρ to be smaller than δ implies that $(I) \leq \frac{\epsilon^2 a_1^2 (a_1-\rho)^2}{16}$.

Combining the bounds for (I) and (II), we get $\int_{\mathbb{R}} \left(\eta_1'(\xi_1(x)) - \eta_1'(\xi_2(x))\right)^2 \, dx < \frac{\epsilon^2 a_1^2 (a_1-\rho)^2}{8}$, and then

$$\|\xi_1' - \xi_2'\|_{L^2} < \frac{\sqrt{2(b_1+\rho)}}{a_1(a_1-\rho)}\rho + \frac{\epsilon}{2}.$$

Combining the four inequalities above, we see that

$$\|\xi_1 - \xi_2\|_{1,1} \leq \left[\frac{1}{a_1} + \frac{1}{a_1(a_1-\rho)} + \frac{\sqrt{b_1+\rho}}{a_1} + \frac{\sqrt{2(b_1+\rho)}}{a_1(a_1-\rho)}\right]\rho + \frac{\omega_1(\rho/a_1)}{a_1(a_1-\rho)} + \frac{\epsilon}{2}. \tag{3.4}$$

Choosing ρ sufficiently small, the right hand side can be made less than ϵ. This proves the continuity of the inversion map. □

Note that in the periodic case the same proof applies, except we do not need to estimate the L^2 tail. On the other hand the estimate (3.4) still requires us to use the modulus of continuity ω_1 of the fixed diffeomorphism η_1, and thus we do not get *uniform* continuity of the inversion map. This is responsible for the failure of the Camassa-Holm solution map to be uniformly continuous in the data even in spaces with higher smoothness, as mentioned in the Introduction.

It remains to prove the continuity of the composition mapping. We will prove the following lemma which is a more general statement than the continuity of composition of diffeomorphisms. That is, we will prove that the composition mapping of a vector $\phi \in \mathcal{V}_1(\mathbb{R})$ and a diffeomorphism $\eta \in \mathcal{D}(\mathbb{R})$ is continuous. Then the composition mapping of two diffeomorphisms will be continuous as a consequence. Furthermore, this type of composition will appear later in the argument of local well-posedness.

Lemma 3.2.5. *The map* $\mathrm{Comp}^1 : \mathcal{V}_1(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \to \mathcal{V}_1(\mathbb{R})$ *given by* $(\phi, \eta) \mapsto \phi \circ \eta$ *is continuous.*

Proof. Let $(\phi_i, \eta_i) \in \mathcal{V}_1 \times \mathcal{D}$ for $i = 1, 2$. Then $a_i \leq \eta_i'(x) \leq b_i$ with inverses ξ_i for η_i and we have the bounds $|\phi_i'(x)| \leq C_i$ on $\mathbb{R}$. In addition, there is a modulus of continuity ω_1 since ϕ_1' is uniformly continuous as before. We claim that we can control the norm $\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_2\|_{1,1}$ in terms of $\rho := \|\eta_1 - \eta_2\|_{1,1}$ and $\sigma := \|\phi_1 - \phi_2\|_{1,1}$. First, note that $|\phi_1(\eta_1(x)) - \phi_2(\eta_2(x))| \leq |\phi_1(\eta_1(x)) - \phi_1(\eta_2(x))| + |\phi_1(\eta_2(x)) - \phi_2(\eta_2(x))|$. By the Mean Value Theorem and the bound for ϕ_1',

$$|\phi_1(\eta_1(x)) - \phi_1(\eta_2(x))| \leq C_1 |\eta_1(x) - \eta_2(x)| \leq C_1 \|\eta_1 - \eta_2\|_{L^\infty}.$$

Also, $|\phi_1(\eta_2(x)) - \phi_2(\eta_2(x))| \leq \|\phi_1 - \phi_2\|_{L^\infty}$. Hence,

$$\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^\infty} \leq C_1 \|\eta_1 - \eta_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty} \leq C_1\rho + \sigma.$$

Next,

$$\begin{aligned}
|(\phi_1 \circ \eta_1)' - (\phi_2 \circ \eta_2)'| &= |\phi_1'(\eta_1(x))\eta_1'(x) - \phi_2'(\eta_2(x))\eta_2'(x)| \\
&\leq |\phi_1'(\eta_1(x))\eta_1'(x) - \phi_1'(\eta_1(x))\eta_2'(x)| \\
&\quad + |\phi_1'(\eta_1(x))\eta_2'(x) - \phi_1'(\eta_2(x))\eta_2'(x)| \\
&\quad + |\phi_1'(\eta_2(x))\eta_2'(x) - \phi_2'(\eta_2(x))\eta_2'(x)| \\
&=: (I) + (II) + (III).
\end{aligned}$$

Then

$$\begin{aligned}(I) &\leq C_1 \left|\eta_1'(x) - \eta_2'(x)\right| \leq C_1 \left\|\eta_1' - \eta_2'\right\|_{L^\infty}, \\ (II) &\leq b_2 \left|\phi_1'(\eta_1(x)) - \phi_1'(\eta_2(x))\right| \leq b_2\omega_1(|\eta_1(x) - \eta_2(x)|) \leq b_2\omega_1(\|\eta_1 - \eta_2\|_{L^\infty}), \\ (III) &\leq b_2 \left|\phi_1'(\eta_2(x)) - \phi_2'(\eta_2(x))\right| \leq b_2 \left\|\phi_1' - \phi_2'\right\|_{L^\infty}.\end{aligned}$$

Hence,

$$\|(\phi_1 \circ \eta_1)' - (\phi_2 \circ \eta_2)'\|_{L^\infty} \leq C_1\rho + b_2\sigma + b_2\omega_1(\rho) \leq C_1\rho + (b_1 + \rho)(\sigma + \omega_1(\rho)).$$

For $\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^2}$, we have

$$\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^2} \leq \|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_1\|_{L^2} + \|\phi_2 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^2},$$

by the triangle inequality. Then

$$\begin{aligned}\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_1\|_{L^2}^2 &= \int_{\mathbb{R}} \left(\phi_1(\eta_1(x)) - \phi_2(\eta_1(x))\right)^2 \, dx \\ &= \int_{\mathbb{R}} \left(\phi_1(y) - \phi_2(y)\right)^2 \xi_1'(y) \, dy \\ &\leq \frac{1}{a_1} \|\phi_1 - \phi_2\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\|\phi_2 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^2}^2 &= \int_{\mathbb{R}} \left(\phi_2(\eta_1(x)) - \phi_2(\eta_2(x))\right)^2 \, dx \\ &\leq C_2^2 \int_{\mathbb{R}} (\eta_1(x) - \eta_2(x))^2 \, dx \\ &= C_2^2 \|\eta_1 - \eta_2\|_{L^2}^2.\end{aligned}$$

Hence,

$$\|\phi_1 \circ \eta_1 - \phi_2 \circ \eta_2\|_{L^2} \leq \frac{1}{\sqrt{a_1}}\sigma + C_2\rho \leq \frac{1}{\sqrt{a_1}}\sigma + (C_1 + \sigma)\rho.$$

Lastly, we estimate $\|(\phi_1 \circ \eta_1)' - (\phi_2 \circ \eta_2)'\|_{L^2}$. Note that

$$\begin{aligned}\|\phi_1'(\eta_1(x))\eta_1'(x) - \phi_2'(\eta_2(x))\eta_2'(x)\|_{L^2} &\leq \|\phi_1'(\eta_1(x))\eta_1'(x) - \phi_1'(\eta_1(x))\eta_2'(x)\|_{L^2} \\ &\quad + \|\phi_1'(\eta_1(x))\eta_2'(x) - \phi_1'(\eta_2(x))\eta_2'(x)\|_{L^2} \\ &\quad + \|\phi_1'(\eta_2(x))\eta_2'(x) - \phi_2'(\eta_2(x))\eta_2'(x)\|_{L^2} \\ &=: (I) + (II) + (III).\end{aligned}$$

By using a similar technique, we can find $(I) \leq C_1\rho$ and $(III) \leq \sqrt{b_2}\sigma$. For (II), we use the method of splitting the integral into two parts and then using the uniform continuity as in the proof of Lemma 6. Here, the situation is slightly simpler since ϕ_1' itself is in L^2. So for any $\epsilon > 0$ we have

$$\|\phi_1'(\eta_1(x))\eta_2'(x) - \phi_1'(\eta_2(x))\eta_2'(x)\|_{L^2} < \frac{\epsilon}{2}$$

if ρ and σ are sufficiently small, and

$$\|(\phi_1 \circ \eta_1)' - (\phi_2 \circ \eta_2)'\|_{L^2} \leq C_1\rho + \sqrt{b_2}\sigma + \frac{\epsilon}{2} \leq C_1\rho + \sqrt{b_1 + \rho}\sigma + \frac{\epsilon}{2}$$

Now combining the four inequalities, we obtain

$$\begin{aligned}\|\eta_1 \circ \phi_1 - \eta_2 \circ \phi_2\|_{1,1} \leq &\left(1 + \frac{1}{\sqrt{a_1}} + b_1 + 2\rho\right)\sigma + 4C_1\rho \\ &+ \sqrt{b_1 + \rho}\sigma + (b_1 + \rho)\omega_1(\rho) + \frac{\epsilon}{2}.\end{aligned}$$

Then we can make the right side as small as we want by choosing σ and ρ sufficiently small;

hence $\eta \circ \phi$ is continuous as a function of (η, ϕ) in the product metric. □

Again we note that this map is not uniformly continuous.

Corollary 3.2.6. *The map* $\mathrm{Comp}^2 : \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ *is continuous.*

Proof. Let $(\xi, \eta) \in \mathcal{D} \times \mathcal{D}$. Then $\phi := \xi - \mathrm{Id}$ is an element of $\mathcal{V}_1$. From Lemma 3.2.5, the mapping $(\phi, \eta) \mapsto \phi \circ \eta$ is continuous. Note that

$$\phi \circ \eta = (\xi - \mathrm{Id}) \circ \eta = \xi \circ \eta - \eta,$$

and so we can identify $\mathrm{Comp}^2 = \mathrm{Comp}^1 + \mathrm{Proj}_2$, where Proj_2 is the projection map onto the second component. Since Comp^2 is the sum of two continuous mappings, it is continuous. □

Therefore, $\mathcal{D}(\mathbb{R})$ is a topological group.

3.3 Local well-posedness

It is convenient to rewrite the Camassa-Holm equation in its equivalent form

$$u_t + uu_x = -\partial_x (1 - \partial_x^2)^{-1} \left(u^2 + \frac{1}{2} u_x^2 \right). \tag{3.5}$$

Note that the operator $(1 - \partial_x^2)$ is invertible since we can can check, for a bounded function g, that

$$f(x) - f''(x) = g(x) \text{ and } \lim_{|x| \to \infty} f(x) = 0 \implies f(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} g(y) dy. \tag{3.6}$$

Now, consider the Lagrangian flow equation

$$\frac{\partial \eta}{\partial t}(t, x) = u\left(t, \eta(t, x)\right). \tag{3.7}$$

Since u is in $\mathcal{V}_1(\mathbb{R})$, the function $\phi := u^2 + \frac{1}{2}u_x^2$ in the parentheses of (4.5) is continuous, integrable, and decaying to zero at infinity. Then this implies that the function ϕ is also square integrable. So we will define the collection of such functions ϕ as $\mathcal{V}_0$ below and proceed with the local existence in the space $\mathcal{V}_0(\mathbb{R})$. Then we will realize ϕ as the function $u^2 + \frac{1}{2}u_x^2$ to finish the proof of local well-posedness.

Definition 3.3.1. *We denote by $\mathcal{V}_0(\mathbb{R})$ the set of maps $u : \mathbb{R} \to \mathbb{R}$ satisfying the conditions*

1. *$u(x)$ is a continuous function with $\lim_{|x|\to\infty} u(x) = 0$, and*
2. *$\int_{\mathbb{R}} |u(x)|^2\, dx < \infty$.*

So $\mathcal{V}_0 = C_0^0 \cap L^2$, where C_0^0 denotes the space of continuous functions that decay to zero at infinity. Hence the norm in $\mathcal{V}_0$ is the sum of L^∞ and L^2 norms. Now we can write the equation (4.5) in terms of η.

Proposition 3.3.2. *The equation (4.5) can be rewritten in terms of the flow η as*

$$\eta_{tt} = -\mathcal{L}_\eta\left(\eta_t^2 + \frac{\eta_{tx}^2}{2\eta_x^2}\right), \tag{3.8}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}_\eta(\phi) = \mathcal{L}(\phi \circ \eta^{-1}) \circ \eta, \text{ and } \mathcal{L} = \partial_x(1-\partial_x^2)^{-1} \text{ for any function } \phi. \tag{3.9}$$

Proof. We differentiate the equation (3.7) with respect to t and use the chain rule. We obtain

$$\begin{aligned}\frac{\partial^2\eta}{\partial t^2}(t,x) =& \frac{\partial u}{\partial t}(t,\eta(t,x)) + \frac{\partial u}{\partial x}(t,\eta(t,x))\frac{\partial \eta}{\partial t}(t,x)\\ =& \frac{\partial u}{\partial t}(t,\eta(t,x)) + u(t,\eta(t,x))\frac{\partial u}{\partial x}(t,\eta(t,x)),\end{aligned}$$

which is the left hand side of equation (4.5) composed with η. Also,

$$\frac{\partial^2 \eta}{\partial t \partial x}(t,x) = \frac{\partial u}{\partial x}(t, \eta(t,x)) \frac{\partial \eta}{\partial x}(t,x),$$

so we get $u_x \circ \eta = \frac{\eta_{tx}}{\eta_x}$. Now, if $p - p_{xx} = u^2 + \frac{1}{2}u_x^2$, then the equation (4.5) becomes

$$\eta_{tt} = -p_x \circ \eta.$$

After taking the inverse flow η^{-1} to set the variables at the right place and writing everything in terms of operators, we get the equation (3.8). □

The first thing we notice is that as a function of $(\eta, V) = (\eta, \eta_t)$, the right side of equation (3.8) does not lose derivatives. If η and V are both C^1, then the term inside parentheses is continuous, while $\mathcal{L}$ gains a derivative so that if $\phi \in C^0$, then $\mathcal{L}_\eta(\phi) \in C^1$. Hence, the equation (3.8) becomes a first-order equation on an open subset of the Banach space $\mathcal{D}(\mathbb{R}) \times \mathcal{V}_1(\mathbb{R})$. We claim that the right side function is C^1 in the (η, V) variables. We begin with the following lemma.

Lemma 3.3.3. *For each $\eta \in \mathcal{D}(\mathbb{R})$, $\mathcal{L}_\eta : \mathcal{V}_0(\mathbb{R}) \to \mathcal{V}_1(\mathbb{R})$ is a bounded linear operator.*

Proof. By formula (3.6) we can write (3.8) as

$$\mathcal{L}(\phi)(x) = -\frac{1}{2}e^{-x} \int_{-\infty}^{x} e^z \phi(z) dz + \frac{1}{2} e^x \int_x^\infty e^{-z}\phi(z)\, dz \tag{3.10}$$

and thus

$$\mathcal{L}_\eta(\phi)(x) = -\frac{1}{2}e^{-\eta(x)} \int_{-\infty}^{x} e^{\eta(y)} \phi(y)\eta'(y)\, dy + \frac{1}{2} e^{\eta(x)} \int_x^\infty e^{-\eta(y)}\phi(y)\eta'(y)\, dy =: f(x).$$

We need to show that $\|f\|_{1,1}$ is bounded by $\|\phi\|_{L^\infty}$ and $\|\phi\|_{L^2}$. Since $\eta \in \mathcal{D}(\mathbb{R})$ and $\phi \in \mathcal{V}_0(\mathbb{R})$, we have $a \leq \eta' \leq b$ and $-M \leq \phi \leq M$ for some constants $a, b, M > 0$. Since η is strictly increasing,

$$\begin{cases} -\eta(x)+\eta(y) \leq -a(x-y) & \text{if } -\infty < y < x, \\ \eta(x)-\eta(y) \leq -a(y-x) & \text{if } x < y < \infty. \end{cases}$$

Hence,

$$|f(x)| \leq \frac{Mb}{2}\left(\int_{-\infty}^{x} e^{-a(x-y)}\, dy + \int_{x}^{\infty} e^{-a(y-x)}\, dy\right) = \frac{Mb}{a}.$$

So $\|f\|_{L^\infty} \leq \frac{b}{a}\|\phi\|_{L^\infty}$. Similarly, we can find $\|f'\|_{L^\infty} \leq \left(\frac{b^2}{a}+b\right)\|\phi\|_{L^\infty}$.

Note that by definition (3.9) we can write $f(x) = q(\eta(x))$ where q satisfies the differential equation $q(x) - q''(x) = \psi'(x)$, with $\psi = \phi \circ \xi$ and $\xi = \eta^{-1}$. Multiplying by $q(x)$ on both sides and integrating on $\mathbb{R}$, we get

$$\int_{-\infty}^{\infty} q(z)^2 + q'(z)^2\, dz = \int_{-\infty}^{\infty} \psi'(z)q(z)\, dz = -\int_{-\infty}^{\infty} \psi(z)q'(z)\, dz.$$

This implies that $\|q\|_{H^1}^2 \leq \|\psi\|_{L^2}\|q\|_{H^1}$, and thus that

$$\|q\|_{H^1} \leq \|\psi\|_{L^2}.$$

Since in addition we have

$$\|\psi\|_{L^2}^2 = \int_{\mathbb{R}} |\phi(\xi(z))|^2\, dz = \int_{\mathbb{R}} |\phi(y)|^2\, \eta'(y)\, dy \leq b\,\|\phi\|_{L^2}^2 < \infty$$

we obtain $\|q\|_{H^1} \leq \sqrt{b}\,\|\phi\|_{L^2}$. Note that

$$\|q\circ\eta\|_{L^2}^2 = \int_{\mathbb{R}} |q(\eta(y))|^2\, dy = \int_{\mathbb{R}} |q(z)|^2\, \xi'(z)\, dz \leq \frac{1}{a}\int_{\mathbb{R}} |q(z)|^2\, dz = \frac{1}{a}\|q\|_{L^2}^2.$$

Similarly, $\|(q\circ\eta)'\|_{L^2}^2 \le b\,\|q'\|_{L^2}^2$. Hence,

$$\|f\|_{H^1} = \|q\circ\eta\|_{H^1} \le \left(\frac{1}{\sqrt{a}}+\sqrt{b}\right)\|q\|_{H^1} \le \left(\frac{\sqrt{b}}{\sqrt{a}}+b\right)\|\phi\|_{L^2}.$$

By combining all inequalities, we get

$$\|f\|_{1,1} = \|f\|_{C^1} + \|f\|_{H^1} \le \left(\frac{b+b^2}{a}+b\right)\|\phi\|_{L^\infty} + \left(\frac{\sqrt{b}}{\sqrt{a}}+b\right)\|\phi\|_{L^2},$$

and this proves the continuity. □

Theorem 3.3.4. *The function $F(\phi,\eta)=\mathcal{L}_\eta(\phi)$, where the operator $\mathcal{L}_\eta$ is defined by formula* (3.9)*, is a continuously differentiable function from $\mathcal{V}_0(\mathbb{R})\times\mathcal{D}(\mathbb{R})$ to $\mathcal{V}_1(\mathbb{R})$.*

In the definition of $\mathcal{L}_\eta$, note that the operator $\partial_x(1-\partial_x^2)^{-1}$ is essentially ∂_x^{-1} up to lower-order terms. So heuristically, it suffices to show that $(\partial_x^{-1})_\eta$ is of class C^1. Let

$$S:(\eta,\phi)\mapsto\left(\eta,\left[R_\eta\circ\partial_x\circ R_{\eta^{-1}}\right](\phi)\right)$$

be the inverse. Then by a direct computation,

$$D_1S_1=\mathrm{Id},\quad D_2S_1=0,$$

and

$$\begin{aligned}\frac{d}{d\epsilon}S_2(\eta+\epsilon\psi,\phi)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\Big\{\partial_x[\phi\circ(\eta+\epsilon\psi)^{-1}]\circ(\eta+\epsilon\psi)\Big\}_{\epsilon=0}\\ &= \frac{d}{d\epsilon}\left\{\phi_x\frac{1}{\eta_x+\epsilon\psi_x}\right\}_{\epsilon=0}\\ &= \frac{\phi_x\psi_x}{(\eta_x)^2},\\ \frac{d}{d\epsilon}S_2(\eta,\phi+\epsilon\rho)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\Big\{(\phi_x+\epsilon\rho_x)\frac{1}{\eta_x}\Big\}_{\epsilon=0} = \frac{\rho_x}{\eta_x}.\end{aligned}$$

Hence,

$$DS\Big|_{(\mathrm{Id},0)} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \partial_x \end{pmatrix} \in \mathrm{Isom}\left(\mathcal{D}(\mathbb{R}) \times \mathcal{V}_1(\mathbb{R}), \mathcal{D}(\mathbb{R}) \times \mathcal{V}_0(\mathbb{R})\right),$$

which shows that $(\partial_x)_\eta$ is C^1. Thus, by the Implicit Function Theorem, we get that $(\partial_x^{-1})_\eta$ is C^1.

Now, we present the rigorous proof using the C^1 estimates.

Proof. Note that F is smooth as a function of ϕ since it is linear with respect to ϕ and bounded in ϕ from Lemma 3.3.3. So we only have to show the continuous differentiability of F with respect to η. We can compute

$$\begin{aligned} \partial_\eta F(\phi,\eta)(\rho) =& \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(\phi, \eta + \epsilon\rho) \\ =& \frac{1}{2}\int_{-\infty}^{x} e^{-\eta(x)+\eta(y)}\phi(y)\left(\rho(x)\eta'(y) - \rho(y)\eta'(y) - \rho'(y)\right)\, dy \\ &+ \frac{1}{2}\int_{x}^{\infty} e^{\eta(x)-\eta(y)}\phi(y)\left(\rho(x)\eta'(y) - \rho(y)\eta'(y) + \rho'(y)\right)\, dy. \end{aligned} \tag{3.11}$$

We first want to show that $G(x) := \partial_\eta F(\phi,\eta)(\rho)(x)$ is a function in the correct target space $\mathcal{V}_1(\mathbb{R})$. So we must check the following:

$$\begin{aligned} &\max_{x\in\mathbb{R}} |G(x)| < \infty, \quad \lim_{|x|\to\infty} |G(x)| = 0, \quad \int_{\mathbb{R}} |G(x)|^2\, dx < \infty, \\ &\max_{x\in\mathbb{R}} |G'(x)| < \infty, \quad \lim_{|x|\to\infty} |G'(x)| = 0. \quad \int_{\mathbb{R}} |G'(x)|^2\, dx < \infty, \end{aligned}$$

Observe that the two conditions in the first column follow from the two conditions in the second column. Note that $\rho \in T_\eta\mathcal{D}(\mathbb{R}) = \mathcal{V}_1(\mathbb{R})$, so we have $-c \le \rho \le c$ and $-N \le \rho' \le N$ for some constants $c, N > 0$.

1. $\lim_{x\to\infty} G(x) = 0$ and $\lim_{x\to\infty} G'(x) = 0$

Denote $G(x) = (I) + (II)$ in terms of the two integrals in (3.11). By using l'Hôpital's Rule, we have

$$\begin{aligned}
&\lim_{x\to\infty} (I) \\
&= \lim_{x\to\infty} \frac{\rho(x)\int_{-\infty}^{x} e^{\eta(y)}\phi(y)\eta'(y)\,dy - \int_{-\infty}^{x} e^{\eta(y)}\phi(y)\rho(y)\eta'(y)\,dy - \int_{-\infty}^{x} e^{\eta(y)}\phi(y)\rho'(y)\,dy}{2e^{\eta(x)}} \\
&= \lim_{x\to\infty} \frac{\rho'(x)}{\eta'(x)} \frac{\int_{-\infty}^{x} e^{\eta(y)}\phi(y)\eta'(y)\,dy}{2e^{\eta(x)}} - \frac{\phi(x)\rho'(x)}{2\eta'(x)} \\
&= \lim_{x\to\infty} \frac{\rho'(x)}{\eta'(x)} \frac{\int_{-\infty}^{x} e^{\eta(y)}\phi(y)\eta'(y)\,dy}{2e^{\eta(x)}}, \quad \left(\because \lim_{x\to\infty} \rho'(x) = 0\right)
\end{aligned}$$

Note that

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{x} e^{\eta(y)}\phi(y)\eta'(y)\,dy}{2e^{\eta(x)}} = \lim_{x\to\infty} \frac{e^{\eta(x)}\phi(x)\eta'(x)}{2e^{\eta(x)}\eta'(x)} = \lim_{x\to\infty} \frac{\phi(x)}{2} = 0.$$

Hence, $\lim_{x\to\infty}(I) = 0$. For (II), we get another limit of indeterminate form $\frac{0}{0}$ and the computation is essentially the same. So $\lim_{x\to\infty}(II) = 0$ and we get $\lim_{x\to\infty} G(x) = 0$. The proofs for $\lim_{x\to-\infty} G(x) = 0$ and $\lim_{|x|\to\infty} G'(x) = 0$ are similar.

2. $\displaystyle\int_{\mathbb{R}} |G(x)|^2 + |G'(x)|^2\,dx < \infty$

Note that

$$\begin{aligned}
G(x) &:= \frac{1}{2}\int_{-\infty}^{x} e^{-\eta(x)+\eta(y)}\phi(y)\left(\rho(x)\eta'(y) - \rho(y)\eta'(y) - \rho'(y)\right)\,dy \\
&\quad + \frac{1}{2}\int_{x}^{\infty} e^{\eta(x)-\eta(y)}\phi(y)\left(\rho(x)\eta'(y) - \rho(y)\eta'(y) + \rho'(y)\right)\,dy \\
&= \left(\frac{1}{2}\rho(x)\int_{-\infty}^{x} e^{-\eta(x)+\eta(y)}\phi(y)\eta'(y)\,dy + \frac{1}{2}\rho(x)\int_{x}^{\infty} e^{\eta(x)-\eta(y)}\phi(y)\eta'(y)\,dy\right) \\
&\quad + \left(-\frac{1}{2}\int_{-\infty}^{x} e^{-\eta(x)+\eta(y)}\phi(y)\rho(y)\eta'(y)\,dy - \frac{1}{2}\int_{x}^{\infty} e^{\eta(x)-\eta(y)}\phi(y)\rho(y)\eta'(y)\,dy\right) \\
&\quad + \left(-\frac{1}{2}\int_{-\infty}^{x} e^{-\eta(x)+\eta(y)}\phi(y)\rho'(y)\,dy + \frac{1}{2}\int_{x}^{\infty} e^{\eta(x)-\eta(y)}\phi(y)\rho'(y)\,dy\right) \\
&=: h_1(x) + h_2(x) + h_3(x).
\end{aligned}$$

Then we use the same method as in Lemma 3.3.3. We can identify

$$\begin{cases} h_1(x) = \rho(x) q_1(\eta(x)), & q_1(x) - q_1''(x) = \psi(x), \\ h_2(x) = q_2(\eta(x)), & q_2(x) - q_2''(x) = -\psi(x)\rho(\xi(x)), \\ h_3(x) = q_3(\eta(x)), & q_3(x) - q_3''(x) = [\psi(x)\rho'(\xi(x))\xi'(x)]', \end{cases}$$

where $\psi(x) = \phi(\xi(x))$ and $\xi = \eta^{-1}$ as before. Then we have

$$\begin{aligned} \|q_1\|_{H^1} &\leq \|\psi\|_{L^2}, \\ \|q_2\|_{H^1} &\leq \|\psi(\rho \circ \xi)\|_{L^2}, \\ \|q_3\|_{H^1} &\leq \|\psi(\rho' \circ \xi)\xi'\|_{L^2}, \end{aligned}$$

where we can check that each right side is finite.

Note that since $1 = s > \frac{n}{2} = \frac{1}{2}$, the Sobolev space H^1 is an algebra under pointwise multiplication (see Lemma 2.7 in [40].) Then,

$$\|h_1\|_{H^1} \leq K \|\rho\|_{H^1} \|q_1 \circ \eta\|_{H^1},$$

for some constant K. We have

$$\|q_i \circ \eta\|_{L^2}^2 = \int_{\mathbb{R}} q_i(\eta(y))^2 \, dy = \int_{\mathbb{R}} q_i(z)^2 \xi'(z) \, dz \leq \frac{1}{a} \int_{\mathbb{R}} q_i(z)^2 \, dz < \infty,$$

for $i = 1, 2, 3$. Similarly, $\|(q_i \circ \eta)'\|_{L^2} < \infty$ and we conclude that $\|h_1\|_{H^1} < \infty$. We can show $\|h_2\|_{H^1} < \infty$ by using the same method. The proof for $\|h_3\|_{H^1} < \infty$ is similar to that of Lemma 3.3.3. This completes the proof for $G \in \mathcal{V}_1(\mathbb{R})$.

It remains to show that G is continuous with respect to η to complete the proof of continuous differentiability of F with respect to η. We will show that the maps q_i in the proof of this Theorem, are continuous in η. The idea is to identify q_i as the composition of continuous operations. We prove this in the following Lemmas. We first check that the composition map that appears in q_i is continuous.

□

Lemma 3.3.5. *The composition mapping* $\mathrm{Comp}^3 : \mathcal{V}_0(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \to \mathcal{V}_0(\mathbb{R})$ *is continuous.*

Proof. The proof is analogous to that of Lemma 3.2.5. □

Lemma 3.3.6. *The three maps* $q_i : \mathcal{D}(\mathbb{R}) \to \mathcal{V}_1(\mathbb{R})$*, which are defined as above, are continuous with respect to* η*.*

Proof. First, we can identify q_i as following:

$$\begin{aligned}
q_1 &:= (1 - \partial_x^2)^{-1}(\phi \circ \eta^{-1}), \\
q_2 &:= -(1 - \partial_x^2)^{-1}[(\phi \circ \eta^{-1})(\rho \circ \eta^{-1})], \\
q_3 &:= -\partial_x(1 - \partial_x^2)^{-1}[(\phi \circ \eta^{-1})((\partial_x \rho) \circ \eta^{-1})\partial_x \eta^{-1}],
\end{aligned}$$

We will prove the continuity of q_3, since the proofs for q_1 and q_2 are much easier.

In the definition of q_3, let $r(\eta) := (\phi \circ \eta^{-1})((\partial_x \rho) \circ \eta^{-1})\partial_x \eta^{-1}$, the function inside the square bracket. Then the map $r : \mathcal{D} \to \mathcal{V}_0$ is continuous with respect to η. This is because the following maps are continuous:

- $\mathrm{Inv} : \eta \mapsto \eta^{-1}$ ($\because$ Lemma 3.2.4),
- $\mathrm{Comp}^3 : \eta \mapsto \phi \circ \eta^{-1}$ ($\because$ Lemma 3.3.5),
- $\partial_x : \rho \mapsto \partial_x \rho$, differentiation with respect to x,
- multiplication of all three continuous maps.

Note that $r(\eta) \in \mathcal{V}_0$ since each term in the product is a function in C_0^0 and

$$\|r(\eta)\|_{L^2}^2 \leq \int_{\mathbb{R}} \left|(\phi \circ \eta^{-1})((\partial_x \rho) \circ \eta^{-1})\partial_x \eta^{-1}\right|^2 \, dx$$
$$\leq \|\phi \circ \eta\|_{L^\infty}^2 \left\|\partial_x \eta^{-1}\right\|_{L^\infty}^2 \left\|(\partial_x \rho) \circ \eta^{-1}\right\|_{L^2}^2 < \infty.$$

Lastly, from the explicit formula (3.10) we have

$$q_3 = -\partial_x (1 - \partial_x^2)^{-1}(r) = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^z r(z) dz - \frac{1}{2} e^x \int_x^\infty e^{-z} r(z) \, dz.$$

Then we can check that

$$\|q_3\|_{1,1} = \|q_3\|_{C^1} + \|q_3\|_{H^1} \leq 2 \|r\|_{L^\infty} + \|r\|_{L^2},$$

by doing essentially the same estimates as in Lemma 3.3.3. Hence, q_3 is continuous. □

Corollary 3.3.7. *The map G is continuous with respect to η.*

Proof. Recall that $G = h_1 + h_2 + h_3$. The composition $q_i \circ \eta$ is continuous by Theorem 3.2.5, so h_1, h_2, and h_3 are continuous. Since the multiplication map is continuous, h_1 is continuous, and so the sum of three maps is continuous. □

Then by the following existence and uniqueness theorem for ODEs in Banach space, we get the local existence and uniqueness of the solution of η of the Lagrangian equation (3.8).

Theorem 3.3.8. *[50] Let $f : J \times U \to \mathbf{E}$ be continuous, and satisfy a Lipschitz condition on U uniformly with respect to J. Let x_0 be a point of U. Then there exists an open subinterval J_0 of J containing 0, and an open subset of U containing x_0 such that f has a unique flow*

$$\alpha : J_0 \times U_0 \to U$$

satisfying

$$\frac{d\alpha}{dt}(t,x) = f(t,\alpha(t,x)), \quad \alpha(0,x) = x.$$

We can select J_0 *and* U_0 *such that* α *is continuous and satisfies a Lipschitz condition on* $J_0 \times U_0$*.*

In our situation, $f = F(\phi,\eta)$, $U = \mathcal{D}(\mathbb{R})$, and $\mathbf{E} = T\mathcal{D}(\mathbb{R})$ which is the tangent bundle of $\mathcal{D}(\mathbb{R})$. The integral curve α corresponds to $\eta(t)$ which is a curve in $\mathcal{D}(\mathbb{R})$. Finally, we have the following theorem, which proves the local well-posedness of the Camassa-Holm equation (3.1).

Theorem 3.3.9. *The Cauchy problem for the Camassa-Holm equation is equivalent to the system*

$$\begin{cases} \frac{d\eta}{dt} = U \\ \frac{dU}{dt} = -\mathcal{L}_\eta\left(U^2 + \frac{U_x^2}{2\eta_x^2}\right) \end{cases} \tag{3.12}$$

with initial conditions $\eta(0,\cdot) = \mathrm{Id}$, $U(0,\cdot) = u_0$*. This system describes the flow of a* C^1 *vector field on* $T\mathcal{D}(\mathbb{R})$ *and the solution curve* (η, U) *exists for some time* $T > 0$*. Defining* $u = U \circ \eta^{-1}$*, we obtain a* C^1 *vector field* $u : [0,T) \times \mathbb{R} \to \mathbb{R}$ *satisfying the Camassa-Holm equation* (3.1) *which depends continuously on* u_0*.*

Proof. Clearly, the map $(\eta, U) \mapsto U^2 + \frac{U_x^2}{2\eta_x^2} : \mathcal{D}(\mathbb{R}) \times \mathcal{V}_1(\mathbb{R}) \to \mathcal{V}_0(\mathbb{R})$ is smooth. We have shown in the Theorem 3.3.4 that L_η is C^1. Hence, the composition of these two mappings is C^1. Then by Theorem 3.3.8, there exists a time $T > 0$ such that the solution curve (η, U) of (4.11) exists on the interval $[0,T)$. This means that there is a solution mapping

$$\begin{aligned} \Upsilon : T\mathcal{D} &\to T\mathcal{D} \\ (\eta_0, u_0) &\mapsto (\eta, u) \end{aligned}$$

Then we can construct the following composition of maps

$$\begin{array}{ccccccc} \mathcal{V}_1 & \overset{\iota}{\hookrightarrow} & T\mathcal{D} & \overset{\Upsilon}{\longrightarrow} & T\mathcal{D} & \longrightarrow & \mathcal{V}_1 \\ u_0 & \longmapsto & (\mathrm{Id}, u_0) & \longmapsto & (\eta, U) & \longmapsto & U \circ \eta^{-1}, \end{array}$$

where the first map is inclusion and the last map is the inversion followed by composition. Hence, we obtain the solution $u = U \circ \eta^{-1}$ of the original equation (3.1). Continuity of u follows from the fact that the above mapping is a composition of continuous maps. □

Although the map from u_0 to $u(t)$ is continuous, it is not even uniformly continuous, as mentioned earlier [32]. On the other hand the map from u_0 to $\eta(t)$ is not only continuous, it is C^1 in both variables, which follows from the fact that the vector field in (4.11) is C^1. With more work we could show that in fact the vector field and thus the solution map is C^∞, as happens for fluids [21]. Analogously we can show that Lagrangian trajectories are C^∞ functions of time, even though the data is only spatially C^1. The essential feature here is that the PDE can be written in Lagrangian form in a way that does not lose derivatives, and typically this is enough to make the vector field not merely continuous but in fact C^∞. Similar techniques should work for other Euler-Arnold equations in the C^1 topology, at least in one dimension. For specific types of initial data, this type of geometric approach can be used to prove global existence or finite time blowup as in [12].

3.4 Future research

From this research, we can investigate the Riemannian geometry of the $C^1 \cap H^1$ diffeomorphisms on $\mathbb{R}$ rigorously. First, we need to show that the H^1 right invariant metric on $\mathcal{V}_1$ defines a smooth Riemannian metric on $\mathcal{D}$, and explicitly construct a smooth compatible affine connection. Once we have the smooth Riemannian metric and Levi-Civita connection, then we can use the results from the general theory of affine connections on Banach manifolds. So we can study the exponential map, geodesic flow, parallel translation, curvature, Jacobi fields, etc.

Chapter 4

Global Conservative Weak Solutions of the Camassa-Holm equation on the circle

In the previous chapter, I showed the local well-posedness of the CH equation on the real line by using the geometric interpretation of the PDE as an ODE in Lagrangian variables on the Banach manifold. As mentioned in Chapter 2, some solutions of the CH equation blowup via wave breaking but it is possible to continue the solution in the weak sense after the blowup [7, 8]. This means that the solution retains less regularity than it is required to be a classical solution and the PDE is satisfied only in the distributional sense.

In this chapter, we will investigate the global weak continuation of the CH equation on the circle. Note that we are switching to a periodic domain. Again, we will take the geometric approach which involves a simple change of variable on the Lagrangian variable. We will show that this new variable removes the singularity of the CH equation, which makes it possible to construct global weak conservative solutions. Also, we can show the persistence of spatial smoothness of the Lagrangian trajectories in an easier way from our construction than that of [65] without using the complete integrability.

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4.1 Introduction

The periodic Camassa-Holm(CH) equation

$$u_t - u_{txx} + 3uu_x - uu_{xxx} - 2u_x u_{xx} = 0, \quad t \in \mathbb{R}, \; x \in \mathbb{S}^1. \tag{4.1}$$

develops a finite time singularity due to the wave breaking and one can continue the solution after the breakdown time in the weak sense. We construct global weak conservative solutions of the CH equation by using a simple change of variables on the Lagrangian flow variable η:

$$\rho = \sqrt{\eta_x}. \tag{4.2}$$

This idea is motivated by Lenells [58] who constructed global weak conservative solutions of the Hnter-Saxton(HS) equation

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0, \quad t \in \mathbb{R}, \; x \in \mathbb{S}^1.$$

The author used the geometric interpretation that the HS equation describes the geodesic flow on the L^2-sphere via (4.2). In particular, the geodesic remains on the L^2-sphere for all time and this makes it possible to continue the geodesic in the weak sense after the blowup time. Since both CH and HS equations form singularities via wave breaking, it is natural to expect that the same kind of weak continuation holds true in the CH equation. We will show that the transformation (4.2) removes the wave breaking singularities in the CH equation case as well, even though we don't have exactly the same geometric picture and explicit formula as in the HS equation.

From the geometric perspective, wave breaking singularity is described by the particle trajectory η in the group of diffeomorphisms forming a horizontal tangent as it evolves in time. In other words, the geodesic flow hits the boundary of the diffeomorphism group. We will show that the CH equation, written in ρ variables via (4.2), has a global solution from which we can reconstruct the flow η defined by

$$\eta(t,x) := \int_0^x \rho^2(t,y)dy + c(t), \text{ where } c(t) \text{ is some function of time,} \tag{4.3}$$

in the space of absolutely continuous functions. This flow is a weak geodesic for almost all time since the spatial derivative of η vanishes on a set of measure zero for the most time. The idea is that ρ can assume both signs and freely passes through the axis during its evolution. This sign change of ρ ensures that if ρ vanishes at a point x_0 at some time T, $\rho(t,x_0)$, as a function of a time, does not vanish on the punctured neighborhood of T. It is certainly possible that ρ vanishes at different places as it evolves, but the imporatant point is that the weak geodesic flow η constructed above remains a homeomorphism for almost all time. This is precisely how the singularity of the CH equation is removed by introducing the new Lagrangian variable ρ.

Our construction of the global weak solution shows that the spatial smoothness of the Lagrangian trajectories η in (4.3) is completely determined by the smoothness of ρ, which is dependent on the smoothness of the initial condition. This is an interesting phenomenon of the CH equation observed by McKean [65]; the solution of the CH equation experiences the jump discontinuity of its slope even for the smooth initial data, but the Lagrangian trajectory η is spatially smooth for all time. We will prove that the Lagrangian flow η is of class C^k for all time whenever the initial condition u_0 is in C^k. Our result improves the McKean's by showing the exact correspondence between the smoothness of the initial condition and the smoothness of the Lagrangian flow. Also, our approach does not use the complete integrability and explicit formulae for the solutions of the CH equation.

The outline of the chapter is following. In Section 4.2, we write the CH equation in ρ variables defined by (4.2). Then we will interpret the resulting equation as an abstract ODE independent of the derivation and obtain the global solution. Using this global solution in ρ variables, in Section 4.3 we will construct global weak conservative solutions of the original CH equation. In particular, we obtain the same global results for the CH equation, which is due to Bressan-Constantin [7], in much simpler way. Also, by using the estimates we already have, we improve the result of McKean [65] on the persistence of the smoothness of Lagrangian trajectories. Finally, Section 4.4 contains some conclusions and remarks. The main theorems of this chapter are the following:

Theorem 4.1.1. *The Cauchy problem for the periodic Camassa-Holm equation*

$$\begin{cases} u_t + uu_x = -(1-\partial_x^2)^{-1}\partial_x \left(u^2 + \frac{u_x^2}{2}\right) \\ u(0,x) = u_0(x) \end{cases} \tag{4.4}$$

has a global solution $u \in AC([0,\infty), H^1(\mathbb{S}^1)) \cap L^\infty([0,\infty), L^2(\mathbb{S}^1))$. *The solution is weak in the sense that the equality in the equation* (4.4) *is satisfied in the distributional sense. Also, the solution* u *is conservative;* $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ *for* $t \in [0,\infty)$ *almost everywhere.*

Theorem 4.1.2. *Let* $\eta(t,x)$ *be the Lagrangian flow of the weak solution* u *of the CH equation* (4.4). *Then* η *is spatially absolutely continuous for all time. Furthermore, for each integer* $k \geq 0$, *if* η *is initially* C^k, *then* η *remains* C^k *for all time.*

4.2 Reformulation of the Cauchy problem

4.2.1 The Camassa-Holm equation in ρ variables

In terms of the diffeomorphism η defined by the flow equation (3.7), the Cauchy problem (4.4) can be written in (η, η_t) variables as following:

$$\begin{cases} \eta_{tt} = -\left\{\Lambda^{-1}\partial_x\left[(\eta_t \circ \eta^{-1})^2 + \frac{1}{2}\left(\frac{\eta_{tx}}{\eta_x} \circ \eta^{-1}\right)^2\right]\right\} \circ \eta \\ \eta(0,x) = x, \quad \eta_t(0,x) = u_0(x) \end{cases} \tag{4.5}$$

Here, $\Lambda^{-1} = (1-\partial_x^2)^{-1}$ is the operator defined by

$$\Lambda^{-1}u(x) = \int_{\mathbb{S}^1} g(x-y)u(y)dy, \tag{4.6}$$

where $g(x) = \frac{\cosh(|x|-1/2)}{2\sinh(1/2)}$. We want to write the equation (4.5) in terms of the new variable ρ defined by (4.2). To this end, we will use the conserved quantities of the CH equation to redefine quantities that appear in the equation. We will first assume that $\|\rho\|_{L^2} = 1$ to derive the equation and prove the local existence and uniqueness of the solution without this norm constraint. This is because the weak geodesic flow (4.15) can be defined for any $\rho \in L^2(\mathbb{S}^1)$. However, we will need to 'restrict' the solution by the constraint $\|\rho\|_{L^2} = 1$ to extend the solution globally.(see Proposition 4)

First, the mean of the velocity u of the CH equation is conserved, so let $\mu := \int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx$ be the constant determined by the initial condition u_0. By changing variables, we have

$$\mu = \int_0^1 (u\circ\eta)\cdot\eta_x dx = \int_0^1 \eta_t\eta_x\, dx = \int_0^1 \eta_t\rho^2 dx. \tag{4.7}$$

So η_t can be determined by μ and ρ. By integrating $\eta_{tx} = 2\rho\rho_t$ in the spatial variable x, we get $\eta_t = \int_0^x 2\rho\rho_t dy + c(t)$ for some function of time $c(t)$. We can substitute this expression into the equation (4.7), and determine $c(t) = \mu - \int_0^1 \int_0^y 2\rho\rho_t dz \rho^2 dy$. Hence, we have determined η_t completely in terms of μ, ρ, and ρ_t. Denoting η_t in new variable G, we have

$$G(\mu, \rho, \rho_t)(t, x) := \int_0^x 2\rho\rho_t dy + \mu - \int_0^1 \int_0^y 2\rho\rho_t dz \rho^2 dy. \tag{4.8}$$

Next, we can write $\frac{\eta_{tx}^2}{2\eta_x} = 2\rho_t^2$ since $\rho_t = \frac{\eta_{tx}}{2\sqrt{\eta_x}}$.

We are now ready to rewrite the equation (4.5) in ρ variables. By differentiating the equation (4.5) with respect to x, we get

$$\eta_{xtt} = \left(\eta_t^2 \eta_x + \frac{\eta_{tx}^2}{2\eta_x}\right) - \Lambda^{-1}\left[\left(\eta_t \circ \eta^{-1}\right)^2 + \frac{1}{2}\left(\frac{\eta_{tx}}{\eta_x} \circ \eta^{-1}\right)^2\right] \circ \eta \cdot \eta_x. \tag{4.9}$$

Here, we used the identity $\Lambda^{-1}\left(-\partial_x^2\right) = 1 - \Lambda^{-1}$ and the chain rule. By using the explicit formula (4.6) for Λ^{-1} and changing variables in the integration, we have

$$\begin{aligned}
&\Lambda^{-1}\left[\left(\eta_t \circ \eta^{-1}\right)^2 + \frac{1}{2}\left(\frac{\eta_{tx}}{\eta_x} \circ \eta^{-1}\right)^2\right] \circ \eta \\
&= \int_0^1 \frac{\cosh\left(|\eta(x) - y| - \frac{1}{2}\right)}{2\sinh(1/2)} \left[\left(\eta_t \circ \eta^{-1}\right)^2 + \frac{1}{2}\left(\frac{\eta_{tx}}{\eta_x} \circ \eta^{-1}\right)^2\right] dy. \\
&= \int_0^1 \frac{\cosh\left(|\eta(x) - \eta(y)| - \frac{1}{2}\right)}{2\sinh(1/2)} \left(\eta_t^2 \eta_y + \frac{\eta_{ty}^2}{2\eta_y}\right) dy.
\end{aligned}$$

Since $\eta(x) - \eta(y) = \int_y^x \eta_z dz = \int_y^x \rho^2 dz$, we can write this integral completely in terms of ρ, ρ_t, and G. Denoting it in new variable F, we have

$$F(\mu, \rho, \rho_t)(t, x) := \int_0^1 \frac{\cosh\left(\left|\int_y^x \rho^2 dz\right| - \frac{1}{2}\right)}{2\sinh(1/2)} (\rho^2 G^2 + 2\rho_t^2) dy \tag{4.10}$$

Since $\eta_{xtt} - \frac{\eta_{tx}^2}{2\eta_x} = 2\rho\rho_{tt}$, the equation (4.9) can be written as

$$2\rho\rho_{tt} = G^2\rho^2 - \rho^2 F.$$

Dividing by 2ρ on both sides and writing initial conditions of (4.5) in ρ variables, we obtain the following Cauchy problem for the CH equation in ρ variables:

$$\begin{cases} \rho_{tt} = \frac{1}{2}\rho\left(G^2 - F\right) \\ \rho(0,x) = 1(\text{constant function}), \quad \rho_t(0,x) = \frac{1}{2}u_0'(x) \end{cases} \tag{4.11}$$

4.2.2 Global solution of the CH equation in ρ variables

Now, we want to solve this equation by viewing it as an abstract ODE in (ρ, ρ_t) variables independent of the above derivation. That is, we assume that ρ and ρ_t are just functions in $L^2(\mathbb{S}^1)$ satisfying the equation (4.11). Note that this second order equation describes the integral curve of the vector field

$$f(\rho,\rho_t) := \frac{1}{2}\rho\left(G^2(\rho,\rho_t) - F(\rho,\rho_t)\right). \tag{4.12}$$

We will show that f is a smooth vector field on the manifold $TL^2(\mathbb{S}^1)$, and this implies the local existence and uniqueness of the solution of (4.11). We prove the smoothness of mappings consisting of f in the following lemmas.

Lemma 4.2.1. *The mapping $G : L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1)$ which is defined as* (4.8) *is smooth.*

Proof. Let $G(\rho,\rho_t) =: \phi(x)$. Since ϕ is an absolutely continuous function defined on $\mathbb{S}^1$, it is continuous and bounded. Note that

$$|\phi| \le \int_0^1 2|\rho\rho_t|dx + C + \int_0^1\int_0^1 2|\rho\rho_t|dz\rho^2 dy \le C\left(\|\rho\|_{L^2} + \|\rho\|_{L^2}^3\right)\|\rho_t\|_{L^2},$$

where C is a constant. Hence, $\|\phi\|_{L^\infty} \le C\left(\|\rho\|_{L^2} + \|\rho\|_{L^2}^3\right)\|\rho_t\|_{L^2}$, which shows that G is a bounded linear operator in ρ_t variable. Thus, G is smooth with respect to ρ_t. Next, G is a continuous mapping in ρ variable and

$$\begin{aligned}\partial_\rho G(\rho,\rho_t)(\psi) :=& \frac{d}{d\epsilon}\Big|_{\epsilon=0} G(\rho+\epsilon\psi,\rho_t)\\ =& \int_0^x 2\psi\rho_t dy - \int_0^1\int_0^y 2\rho\rho_t dz 2\rho\psi dy - \int_0^1\int_0^y 2\psi\rho_t dz \rho^2 dy =: \varphi(x),\end{aligned}$$

where we have

$$|\varphi| \le C\left(\|\rho_t\|_{L^2} + \|\rho\|_{L^2}^2 \|\rho_t\|_{L^2}\right)\|\psi\|_{L^2}.$$

Hence, $\partial_\rho G$ is a bounded linear operator. Note that $\partial_\rho G$ is continuous with repect to ρ and we can continue differentiating G with respect to ρ. This shows that G is a smooth function in ρ variable. Thus, the map G is of class C^∞(see [49] for the differential Calculus on Banach spaces.) □

Next, we prove the smoothness of the mapping F. We can check that the following maps are smooth:

- $\phi \mapsto \int_y^x \phi^2 dz$, $L^2(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1)$,
- $\phi \mapsto \cosh(\phi \pm 1/2)$, $L^\infty(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1)$,
- $(\phi,\psi) \mapsto \phi^2 G(\phi,\psi)^2 + 2\psi^2$, $L^2(\mathbb{S}^1)\times L^2(\mathbb{S}^1) \to L^1(\mathbb{S}^1)$, and
- $(\phi,\psi) \mapsto \int_0^1 \phi\psi dx$, $L^\infty(\mathbb{S}^1)\times L^1(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1)$.

Then F is a smooth mapping since it is a composition of smooth maps. Hence, we have

Lemma 4.2.2. *The mapping* $F : L^2(\mathbb{S}^1)\times L^2(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1)$ *which is defined as* (4.10) *is smooth.*

Finally, the mapping $(\rho,\rho_t) \mapsto \frac{1}{2}\rho(G^2 - F)$ is smooth by the product rule for the derivatives. From the existence and uniqueness theorem for ODEs in Banach space(see [50]), we get the following local existence and uniqueness of the solution of the equation (4.11).

Proposition 4.2.3. *The system*

$$\begin{cases} \frac{d\rho}{dt} = \rho_t \\ \frac{d\rho_t}{dt} = f(\rho, \rho_t) \end{cases} \tag{4.13}$$

with initial conditions $\rho(0, \cdot) = 1$*(constant function) and* $\rho_t(0, \cdot) = \frac{1}{2}u_0'$ *describes the flow of a* C^∞ *vector field on* $TL^2(\mathbb{S}^1) = L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$ *and the curve* (ρ, ρ_t) *exists for some time* $T > 0$.

In order to show that the integral curve desribed by (4.13) exists for all time, we need to have $\|\rho\|_{L^2} = 1$, i.e., ρ is on the L^2-sphere $U := \{\rho \in L^2(\mathbb{S}^1) \ : \ \|\rho\|_{L^2} = 1\}$. Note that $U = F^{-1}(1)$, where $F : L^2(\mathbb{S}^1) \to \mathbb{R}$ where $\phi \mapsto \|\phi\|_{L^2}^2$ is a smooth functional. With this identification, U is a closed submanifold of $L^2(\mathbb{S}^1)$ endowed with the induced weak Riemannian metric $\langle \xi, \zeta \rangle_\phi = \langle \xi, \zeta \rangle_{L^2}$, where $\xi, \zeta \in T_\phi U$. See [56] for more detail.

There are couple of reasons why this norm constraint is necessary. First, from the definition of the change of variable (4.2), we must have

$$\int_0^1 \rho^2 dx = \int_0^1 \eta_x dx = \eta(1) - \eta(0) = 1,$$

since η is a diffeomorphism on $\mathbb{S}^1$. Hence, we must constrain $\|\rho\|_{L^2} = 1$ in order for the weak geodesic flow (4.15) for the CH equation to be defined spatially on $\mathbb{S}^1$. Also, the functions F and G are periodic on $\mathbb{S}^1$ only if $\int_0^1 \rho^2 dx = 1$.

We first prove that the vector field f on $TL^2(\mathbb{S}^1)$ induces a vector field on the submanifold TU. This 'restriction' of the vector field on the sphere makes the induced vector field uniformly bounded and this will enable the solution of the system (4.13) with initial data in TU to stay on TU for all time.

Proposition 4.2.4. *Let* $f(\rho, \rho_t)$ *be the vector field on* $TL^2(\mathbb{S}^1)$ *defined by the equation* (4.11).

Then f induces a smooth vector field $\tilde{f}$ on TU.

Proof. Let $F : L^2(\mathbb{S}^1) \to \mathbb{R}$ be the functional defined above and $\pi : TL^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ be the canonical projection. Let $f : TL^2(\mathbb{S}^1) \to TTL^2(\mathbb{S}^1)$ be the smooth vector field defined by equation (4.11) and $\pi_1 : TTL^2(\mathbb{S}^1) \to TL^2(\mathbb{S}^1)$ be the projection. Note that we can lift F on $TL^2(\mathbb{S}^1)$ by defining $F_1 := F \circ \pi$ and similarly on $TTL^2(\mathbb{S}^1)$ by defining $F_2 := F_1 \circ \pi_1$. We claim that the vector field f induces a vector field $\tilde{f} : F_1^{-1}(1) \to F_2^{-1}(1)$ defined by $\tilde{f}(\rho, \rho_t) := (\rho, \rho_t, \rho_t, f(\rho, \rho_t))$.

We first check that $\tilde{f}$ is well-defined. Let $(\rho, \rho_t) \in F_1^{-1}(1)$. Then $F_2(\tilde{f}(\rho, \rho_t)) = F_1(\rho, \rho_t) = F(\rho) = 1$ and so $\tilde{f}(\rho, \rho_t) \in F_2^{-1}(1)$. Next, we need to show that $(\rho_t, f(\rho, \rho_t))$ is tangent to $F_2^{-1}(1)$. It suffices to show that

$$\langle (\rho, \rho_t), (\rho_t, f(\rho, \rho_t)) \rangle_{L^2} = 0.$$

Since $\rho \perp \rho_t$ on $F_0^{-1}(1)$, we have $\langle \rho, \rho_t \rangle_{L^2} = 0$ for the first components. Also, $\langle \rho_t, f(\rho, \rho_t) \rangle_{L^2} = \langle \rho_t, \rho_{tt} \rangle_{L^2} = 0$ from the definition of f in (4.11). Hence, $\tilde{f}$ is a well-defined vector field on TU and it is smooth since f is smooth. □

Next, we want to show that the conservation of the H^1 energy in the original CH equation holds true in ρ variables as well. This energy conservation will ensure the uniform boundedness of the estimates that we will need later. We first prove the following lemma.

Lemma 4.2.5. *Let G be defined as above. Then $G_t = -H$, where*

$$\begin{aligned} H(t,x) = \int_0^x \frac{\sinh\left(\int_y^x \rho^2 dz - \frac{1}{2}\right)}{2\sinh(1/2)} (\rho^2 G^2 + 2\rho_t^2) dy \\ + \int_x^1 \frac{\sinh\left(-\int_y^x \rho^2 dz - \frac{1}{2}\right)}{2\sinh(1/2)} (\rho^2 G^2 + 2\rho_t^2) dy. \end{aligned}$$

Proof. Note that

$$\frac{\partial H}{\partial x} = -(\rho^2 G^2 + 2\rho_t^2) + \rho^2 F.$$

By differentiating G with respect to t, we have

$$G_t = \int_0^x 2\rho_t^2 + 2\rho\rho_{tt} dy + c'(t) = \int_0^x 2\rho_t^2 + \rho^2 G^2 - \rho^2 F dy + c'(t),$$

where $c(t) = \mu - \int_0^1 \int_0^x 2\rho\rho_t \, dy \rho^2 \, dx$. So

$$G_t = \int_0^x \frac{\partial H}{\partial y} dy + c'(t) = -H(t,x) + H(t,0) + c'(t).$$

So it remains to show that $H(t,0) + c'(t) = 0$. We can compute

$$\begin{aligned}
c'(t) &= -\int_0^1 \int_0^x 2\rho_t^2 + 2\rho\rho_{tt} dy \rho^2 dx \\
&= -\int_0^1 \int_0^x 2\rho_t^2 + \rho^2 G^2 - \rho^2 F dy \rho^2 dx \\
&= -\int_0^1 (2\rho_t^2 + \rho^2 G^2 - \rho^2 F) \int_y^1 \rho^2 dx dy \ (\because \text{changing order of integration}) \\
&= \int_0^1 \frac{\partial H}{\partial y} \int_y^1 \rho^2 dx dy \\
&= -H(t,0) + \int_0^1 H\rho^2 dy \\
&= -H(t,0),
\end{aligned}$$

since $H\rho^2 = F_x$ and F is periodic. Hence, $G_t = -H$ as desired. □

Proposition 4.2.6. *The 'H^1 energy' of the ρ equation is conserved:*

$$\frac{d}{dt} \int_0^1 \rho^2 G^2 + 4\rho_t^2 \, dx = 0. \tag{4.14}$$

Proof. Recall that we have

$$G_x = 2\rho\rho_t \text{ and } G_t\rho^2 = -F_x.$$

Then

$$\begin{aligned}\frac{d}{dt}\int_0^1 \rho^2 G^2 + 4\rho_t^2\,dx &= \int_0^1 2\rho\rho_t G^2 + 2\rho^2 G G_t + 8\rho_t\rho_{tt}\,dx \\ &= \int_0^1 2\rho\rho_t G^2 - 2GF_x + 4\rho\rho_t(G^2 - F)\,dx \\ &= \int_0^1 6\rho\rho_t G^2\,dx - 2G(1)F(1) + 2G(0)F(0) \\ &= \int_0^1 \frac{d}{dx}\left[G^3\right]dx = 0,\end{aligned}$$

since F and G are periodic. □

Now, we are ready to prove that the solution of the equation (4.13) is global. The idea is that all estimates are bounded in terms of $\|\rho\|_{L^2}$ and $\|\rho_t\|_{L^2}$, and the restriction of ρ on the unit sphere toegether with the energy conservation imply that the two norms are uniformly bounded.

Proposition 4.2.7. *Suppose that the system* (4.13) *is given by the induced vector field* $\tilde{f}$ *on* TU *as in the Proposition 6 with initial data in* TU*. Then the solution of the system* (4.13) *exists for all time.*

Proof. Note that we have

$$\begin{aligned}\left\|\tilde{f}(\rho,\rho_t)\right\|_{L^2} &\le \|\rho_t\|_{L^2} + \|f(\rho,\rho_t)\|_{L^2} \\ &\le \|\rho\|_{L^2} + \left\|\frac{1}{2}\rho(G^2 - F)\right\|_{L^2} \\ &\le \|\rho\|_{L^2} + \frac{1}{2}\|\rho\|_{L^2}\left(\|G\|_{L^\infty}^2 + \|F\|_{L^\infty}\right),\end{aligned}$$

where

$$\begin{aligned}\|G\|_{L^\infty} \leq & C\left(\|\rho\|_{L^2} + \|\rho\|_{L^2}^3\right)\|\rho_t\|_{L^2}, \\ \|F\|_{L^\infty} \leq & C\|\rho\|_{L^2}^2\left(\|G\|_{L^\infty}^2\|\rho\|_{L^2}^2 + 2\|\rho_t\|_{L^2}^2\right).\end{aligned}$$

Since $\|\rho\|_{L^2} = 1$ and $\|\rho_t\|_{L^2}$ is uniformly bounded from the energy conservation, $\left\|\tilde{f}(\rho, \rho_t)\right\|_{L^2}$ is uniformly bounded by a constant. Since the RHS of the equation (4.13) is smooth and uniformly bounded, the solution of the Cauchy problem (4.13) can be extended for all time by Wintner's Theorem from the ODE theory(see [31].) □

4.3 Global weak conservative solution of the CH equation

4.3.1 Global weak conservative solution in η variables

From the global solution of the CH equation in ρ variables that we obtained, we can readily construct global weak solutions for the original CH equation (4.5). We first introduce new Lagrangian variables. Define

$$K(t,x) := \int_0^x \rho^2 dy + tu_0(0) - \int_0^t \int_0^\tau H(s,0)\, ds d\tau, \tag{4.15}$$

$$G(t,x) := \int_0^x 2\rho\rho_t dy + \mu - \int_0^1 \int_0^x 2\rho\rho_t dy \rho^2 dx. \tag{4.16}$$

As we have seen in the Lemma 7, $-H(s,0) = c'(t)$ where $c(t)$ is the function of time introduced in the equation (4.8). Hence, K satisfies the first order equation $\frac{\partial K}{\partial t} = G$. Next, we claim that (K, G) solves the second order equation (4.5). As in Lenells's paper [58], we can decompose $\mathbb{S}^1$ by $\mathbb{S}^1 = N \cup A \cup Z$ where

$$\begin{aligned} N :=& \{x \in \mathbb{S}^1 \ : \ K_x \text{ exists and equals } 0, \text{ i.e., } \rho(t,x) = 0\}, \\ A :=& \{x \in \mathbb{S}^1 \ : \ K_x \text{ exists and } K_x(x) > 0, \text{ i.e., } \rho(t,x) > 0\}, \text{ and} \end{aligned}$$

Z is a set of measure zero. We first prove the following lemma.

Lemma 4.3.1. *For almost all time $t \in [0, \infty)$, we have $\int_N \rho_t^2 dy = 0$.*

Proof. It suffices to show that the set N has a measure zero for almost all time. As in [58], the Fubini theorem gives

$$\int_0^T m(N) dt = \int_{\mathbb{S}^1} \int_0^T \chi_{\{\rho^{-1}(0)\}} dt dx, \text{ for } T < \infty,$$

where $\chi_{\{\rho^{-1}(0)\}} : [0, \infty) \times \mathbb{S}^1 \to \{0, 1\}$ is the characteristic function. We claim that the RHS of the equation vanishes. Let $x_0 \in \mathbb{S}^1$.Then the following set

$$N'(x_0) = \{0 \le t \le T \ : \ \rho(t, x_0) = 0\}$$

has the Lebesgue measure zero. This is because $\rho_t(t, x_0)$, as a function of time t, cannot vanish on this set. If not, there is a time t_0 such that $\rho(t_0, x_0) = 0 = \rho_t(t_0, x_0)$. Note that we have the following differential inequality satisfied by the solution of (4.11):

$$\begin{aligned} \frac{d}{dt} \left[\rho^2(t, x_0) + \rho_t^2(t, x_0)\right] =& 2\rho\rho_t + 2\rho_t\rho_{tt} \\ =& 2\rho\rho_t + \rho\rho_t(G^2 - F) \\ =& 2\rho\rho_t \left(1 + \frac{G^2 - F}{2}\right) \\ \le& C(\rho^2 + \rho_t^2), \end{aligned}$$

where $C = \max\left\{1 + \left\|\frac{G^2-F}{2}\right\|_{L^\infty}\right\}$ is the uniform constant. Hence, by the Gronwall's lemma, we have

$$\rho^2(t,x_0) + \rho_t^2(t,x_0) \leq \left(\rho^2(t_0,x_0) + \rho_t^2(t_0,x_0)\right) e^{Ct}.$$

If $\rho(t_0,x_0) = 0 = \rho_t(t_0,x_0)$, the RHS vanishes and this implies that $\rho(t,x_0) = 0 = \rho_t(t,x_0)$ for all time t. In particular, $\rho(0,x_0) = 0$ and this contradicts the initial condition of ρ. Hence, $\rho_t(t,x_0)$ must be nonzero on $N'(x_0)$. Since $\rho(t,x_0)$ is continuous as a function of a time, every neighborhood of t_0 contains a point where ρ is nonzero, i.e., the set $N'(x_0)$ is isolated. Hence the set $N'(x_0)$ must be finite and it has a measure zero. □

Remark 3. *It is possible that ρ vanishes on a set of positive measure, e.g., on an interval, at some time. However, the proof of the lemma shows that these appearances are rare and ρ can vanish only on a set of measure zero for almost all time. Since $K_x = \rho^2$, the slope of the particle trajectories looks like a parabola. As a result, K is generically a homeomorphism whenever K_x vanishes on a set of measure zero. This implies that K^{-1} is well-defined for almost all time.*

Proposition 4.3.2. *For almost all time t, (K,G) satisfies the CH equation*

$$G_t = -\left\{\partial_x \Lambda^{-1}\left[(G\circ K^{-1})^2 + \frac{1}{2}\left(\frac{G_x}{K_x}\circ K^{-1}\right)^2\right]\right\}\circ K. \tag{4.17}$$

Note that this solution is weak since K is only absolutely continuous in the spatial variable x and the equation is satisfied for almost all time t.

Proof. We want to show that the equation $G_t = -H$ is equivalent to the equation (4.5) in the weak sense. Since K is a diffeomorphism on A, we can change variables via K and get

$$\begin{aligned} G_t &= -\int_{\mathbb{S}^1} \frac{\sinh\left(\left|\int_y^x \rho^2 dz\right| - \frac{1}{2}\right)}{2\sinh(1/2)}(\rho^2 G^2 + 2\rho_t^2)dy \\ &= -\int_A \frac{\sinh\left(|K(x) - K(y)| - \frac{1}{2}\right)}{2\sinh(1/2)} \left(K_x G^2 + \frac{G_x^2}{2K_x}\right) dy + \int_N \rho_t^2 dy \\ &= -\int_{K(A)} \frac{\sinh\left(|K(x) - y| - \frac{1}{2}\right)}{2\sinh(1/2)} \left(\left(G \circ K^{-1}\right)^2 + \frac{1}{2}\left(\frac{G_x}{K_x} \circ K^{-1}\right)^2\right) dy \\ &\qquad + \int_N \rho_t^2 dy \\ &= -\partial_x \left[\int_{K(A)} \frac{\cosh\left(|x - y| - \frac{1}{2}\right)}{2\sinh(1/2)} \left(\left(G \circ K^{-1}\right)^2 + \frac{1}{2}\left(\frac{G_x}{K_x} \circ K^{-1}\right)^2\right) dy\right] \circ K \\ &\qquad + \int_N \rho_t^2 dy. \end{aligned}$$

Since the Lebesgue measure of the set $K(A)$ is 1, the first integral on the RHS is equivalent to the RHS of the equation (4.5). Also, $\int_N \rho_t^2 dy = 0$ for almost all time from the previous lemma. Thus, for almost all time $t \in [0, \infty)$, (K, G) satisfies the equation (4.17) as desired. □

4.3.2 Global weak conservative solution in u variables

Now, we prove the main theorems of the paper. We first check that the velocity field u satifying the flow equation (3.7) is well-defined in H^1 for all time.

Proposition 4.3.3. *Let (K, G) be a weak solution of the CH equation (4.5). Then the velocity field $u \in AC([0, \infty), H^1(\mathbb{S}^1)) \cap L^\infty([0, \infty), L^2(\mathbb{S}^1))$ is well-defined by the formula*

$$u(t, K(t, x)) = G(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{S}^1.$$

Proof. First, note that the flow K is a bijection from $A \to K(A)$, so we can define u a.e. on $\mathbb{S}^1$ by

$$u(y) := G(K^{-1}(y)) \text{ for } y \in K(A).$$

It remains to show that u is well-defined by this formula on N, as well. The idea is that the Lagrangian velocity G is well-defined for all time, so we can define the velocity u from it.

Suppose that K_x vanishes at some time T. Since K is a nondecreasing absolutely continuous funciton, the set N must be the union of isolated points and intervals. In the case where $x \in N$ is an isolated point, K is still a bijection around a neighborhood of x so we can defined the velocity u in the same way as above. Suppose that K_x vanishes on an interval $I \subset N$, i.e., $K_x(T,x) = 0$ for $x \in I$. Then we have $K(T,x) = x^*$ for some $x^* \in \mathbb{S}^1$ on I. Also, $\rho(T,x) = 0$ on that interval and this implies that $G_x(T,x) = 0$ since $G_x = 2\rho\rho_t$. Hence, $G(T,x) = G(T,x^*)$ on I and so we can define $u(T,x) := G(T,x^*)$ for $x \in I$. This completes the proof for u being well-defined.

Once we define $u = G \circ K^{-1}$, we can show that its distributional derivative is $u_x = \frac{G_x}{K_x} \circ K^{-1}$. We omit the proofs for the regularity since it is exactaly the same argument as in [58]. □

Remark 4. *When $K(T,x) = x^*$ on an interval I, we can think of this as a set of particles concentrating at one point x^*. This is consistent with the physical interpretation of the CH equation since it describes a compressible fluid motion. In this case, a set of particles concentrate at a point and moves in the same velocity.*

4.3.3 Proof of Theorem 4.1.1

We claim that u is a weak solution of the equation (4.4). We want to show that

$$\int_{\mathbb{S}^1\times[0,\infty)} (u_t + uu_x)\phi \, dxdt = \int_{\mathbb{S}^1\times[0,\infty)} -p_x\phi \, dxdt \text{ for all } \phi \in C_c^\infty(\mathbb{S}^1 \times [0,\infty)),$$

where $p = \Lambda^{-1}(u^2 + \frac{1}{2}u_x^2)$. We have

$$\begin{aligned}\int_{\mathbb{S}^1\times[0,\infty)} (u_t + uu_x)\phi \, dxdt &= \int_{\mathbb{S}^1\times[0,\infty)} -u\phi_t + uu_x\phi \, dxdt \\ &= \int_{\mathbb{S}^1\times[0,\infty)} -UK_x\phi_t \circ K + UU_x\phi \circ K \, dxdt, \\ &\qquad\qquad \text{where } U = u \circ K \\ &= \int_{\mathbb{S}^1\times[0,\infty)} U_tK_x\phi \circ K \, dxdt,\end{aligned}$$

since

$$(UK_x\phi \circ K)_t - \left(U^2\phi \circ K\right)_x = U_tK_x\phi \circ K - UU_x\phi \circ K + UK_x\phi_t \circ K,$$

and ϕ has a compact support. Since $U_t = -\partial_x\Lambda^{-1}\left[u^2 + \frac{1}{2}u_x^2\right] \circ K$, we get

$$\int_{\mathbb{S}^1\times[0,\infty)} U_tK_x\phi \circ K \, dxdt = \int_{\mathbb{S}^1\times[0,\infty)} -p_x\phi \, dxdt,$$

by another change of variables. Finally, the conservation of H^1 energy of the weak solution comes from rewriting all quantities appearing in (4.14) in u variables. That is,

$$\begin{aligned}\int_{\mathbb{S}^1} u^2 + u_x^2 dx &= \int_{K(A)} \left(G \circ K^{-1}\right)^2 + \left(\frac{G_x}{K_x} \circ K^{-1}\right)^2 dx \\ &= \int_A G^2K_x + \frac{G_x^2}{K_x} dx \\ &= \int_{\mathbb{S}^1} G^2\rho^2 + 4\rho_t^2 dx\end{aligned}$$

is conserved for almost all time t. This completes the proof of the Theorem 1.

4.3.4 Proof of Theorem 4.1.2

We can observe that by providing extra smoothness on ρ and ρ_t variables, we can improve the smoothness of the G. In fact, the same estimates in the proof of the Proposition 9 will continue to work with L^2 norms replaced by C^k norms since the spatial domain is compact. Hence, the solutions for the Cauchy problem (4.11) in ρ variables will be global in C^k spaces. Consequently, the Lagrangian variables K and G will be C^{k+1} on $\mathbb{S}^1$ for all time whenever ρ and ρ_t is C^k. This completes the proof of the Theorem 2.

Remark 5. *Theorem 2 improves the result of McKean [65]. The difference between the current work and McKean's work is that we don't need the assumption for the momentum* $m = u - u_{xx}$ *to satisfy* $m, m_x \in H^1$ *initially. Our result shows that that for each* k*,* K *is exactly in* C^k *whenever the initial condition* u_0 *is in* C^k*.*

4.4 Future Research

The global weak continuation of the CH equation in this research suggests that there might be a general theory that explains why it works. We can first suspect that the metrics corresponding to the HS and CH equations are close. That is, in the space of all Riemannian metrics on the group of diffeomorphisms, the Sobolev H^1 metric can be regarded as a nonlinear perturbation of $\dot{H}^1$ metric, where the global weak continuation property is a consequence of the robustness of the perturbation. However, the geometry of two diffeomorphism groups are different since the sectional curvature of $\dot{H}^1$ metric is a positive constant, whereas the H^1 metric has sectional curvature positive in 'most directions' but also assumes negative sign(see [56, 68])

We can apply the change of variable method in this paper to other Euler-Arnold equations or generalize this Lagrangian change of variable technique. In particular, we can consider the Wunsch equation which is the Euler-Arnold equation with a right invariant $\dot{H}^{1/2}$ metric:

$$\begin{cases} \omega_t + u\omega_x + 2\omega u_x = 0, \\ \omega = Hu_x, \ H \text{ is the Hilbert transform,} \end{cases} \tag{4.18}$$

It was studied by Bauer-Kolev-Preston [4] as a one dimensional vorticity equation for the three dimensional Euler's equation. The solution of the Wunsch equation forms a finite time singularity along a particle trajectory due to wave breaking and the blow up result was further extended in the framework of Teichmüller theory by Preston-Washabaugh [76]. In this case, we can instroduce a new variable $Z = u_x + i\omega$ to get

$$Z_t + uZ_x + Z^2 = -F,$$

where F is some positive function. In this framework, we can study the geodesic equation for the Sobolev $\dot{H}^s$ metric for $\frac{1}{2} < s < 1$ in a uniform way. Since the sign change of the vorticity is the crucial assumption to get the blow up in all known cases, we can analyze the second-order ODE in this new Lagrangian variable Z to find a direct proof of the blow up.

Lastly, it would be interesting to interpret the global weak solution of the CH equation constructed in this paper in the context of optimal transport. Recently, Gallouët-Vialard [25] formulated a generalized CH equation as the Euler-Arnold equation which can be identified as a particular solution of the incompressible Euler's equation on the group of homeomorphisms on $\mathbb{R}^2 \setminus \{0\}$. We can investigate how the blow up and the global weak continuation of the CH equation can be described in the language of optimal transport, and understand the role of the change of variables (4.2).

Chapter 5

Nonpositive Curvature of the Quantomorphism Group and Quasi-Geostrophic Motion

In this chapter, we present the last result of the thesis which is about the geometry of the quantomorphism group, which is one of the subgroups of the group of diffeomorphisms. The Euler-Arnold equation of the quantomorphism group is the quasi-geostrophic equation in geophysics. We will compute the sectional curvature of the quantomorphism group explicitly and try to find the criterion for the nonpositivity, as in Preston [75].

This work is a joint project with my advisor Stephen Preston and it is in progress to be finished and submitted for publication.

5.1 Introduction

The quasi-geostrophic equation(QG) describes large scale flows in atmosphere and ocean which has large horizontal to vertical aspect ratio. Here, quasi-geostrophy means that Coriolis force and horizontal pressure gradient forces are nearly in balance, which allows the momentum equation for the flow to be prognostic and include nonlinear dynamics. In terms of the stream function $\psi(t, x, y)$ of the velocity u of the barotropic fluid, the QG equation in the β-plane approximation is given by

$$\partial_t \left(\Delta\psi - \alpha^2\psi\right) + \{\psi, \Delta\psi\} + \beta\psi_x = 0, \tag{5.1}$$

where α denotes the Froude number, $\{\cdot, \cdot\}$ is the Poisson bracket, and β is the gradient for the Coriolis parameter. So $\{g, h\} = h_y g_x - g_x h_y$ and the Coriolis parameter f is approximated in the β-plane by $f = f_0 + \beta y$ with constants f_0 and β. The case when $\beta = 0$ is the f-plane approximation. The Froude number α is a nondimensionalized parameter defined by

$$\alpha := \frac{U}{\sqrt{gL}},$$

where U is the velocity scale, g is the gravitational constant, and L is the horizontal length scale. So α measures the effect of gravity and $\alpha \ll 1$ in the mesoscale motions of the atmosphere and oceans in the midlatitudes. The equation (5.1) can also be written in terms of the potential vorticity as

$$\partial_t\omega + \{\psi, \omega\} = 0, \qquad \omega = \Delta\psi - \alpha^2\psi - \beta\psi, \tag{5.2}$$

which is similar to the vorticity-stream formulation of the 2-dimensional incompressible Euler equation. The QG equation can be derived as the inviscid limit of the rotating shallow water equations, as well. For more mathematical theory of atmospheric and oceanic fluid, see Majda [61]. For more comprehensive backgrounds on the geostrophical fluid dynamics, see Pedlosky [72].

From the geometric point of view, the QG equation is of interest since it is an example of the Euler-Arnold equation. In 1994, Zeitlin-Pasmanter [84] showed that the QG equation can arise as the geodesic equation in the infinite dimensional Lie algebra and its extension. They also computed the sectional curvature and showed that it's negative in the section spanned by the cosinusoidal stationary flows. Then in 1998, Holm-Zeitlin [38] showed that the QG equation in the f- and β-plane approximations are the geodesic equations on the group of symplectic diffeomorphisms by using variational principles for QG dynamics. Also, in 2008, Vizman [81] showed that the equation (5.1) is the Euler-Arnold equation on the central extension of the group of Hamiltonian diffeomorphisms in the case when $\alpha = 0$. Finally, Ebin-Preston [22] showed in 2015 that the QG equation is the geodesic equation on the quantomorphism group and this is the perspective we will follow.

On a contact manifold (M, θ), the quantomorphism group $\mathcal{D}_q(M)$ is defined as the space of diffeomorphisms on M that preserve the contact form θ exactly. So the quantomorphism group is a subgroup of the the contactomorphism group $\mathcal{D}_\theta(M)$, whose elements preserve the contact structure. If the contact form is regular, then $\mathcal{D}_\theta(M)$ is related to a symplectic manifold by a Boothby-Wang fibration and the tangent space of $\mathcal{D}_q(M)$ can be identified with the space of functions $f : M \to \mathbb{R}$ such that $E(f) = 0$, where E is the Reeb field. Furthermore, one can show that $\mathcal{D}_q(M) \subset \mathcal{D}_\theta(M)$ is the totally geodesic submanifold. For more Riemannian geometry of the quantomorphism group in general, see Ratiu-Schmid [77].

As in the finite dimensional Lie group case, the sectional curvature of the diffeomorphism group provides an information about the stability of geodesics, which we call the Lagrangian stability. For example, positive curvature in all sections implies that geodesics with close initial data locally converge(stable) while negative sectional curvature implies that the geodesics spreading apart(instable). However, the Lagrangian stability is different from the usual Eulerian stability; see Misiołek [67] and Preston [74] for more discussion on this. For the curvature of the Euler-Arnold equations in general, see Khesin et al. [44].

In this research, we compute the sectional curvature $K(X, Y)$ of the quantomorphism group $\mathcal{D}_q(M)$ by the plane spanned by $X, Y \in T_{\mathrm{Id}}\mathcal{D}_q(M)$ where $M = \mathbb{R}^3$. Then from the explicit formula of the curvature, we will find a necessary and sufficient condition for the curvature operator $R_X : Y \mapsto K(X, Y)$ to be nonpositive. The explicit computation of the curvature formula is inspired by the work of Preston [75] where the nonpositive curvature criterion for the area-preserving diffeomorphism group of a rotationally symmetric surface was presented.

The outline of the chapter is following. In Section 5.2, we will review the Riemannian geometry of the quantomorphism group and sectional curvature formula. We will observe that the curvature formula simplifies significantly when one of the tangent vector is chosen to be a function of only y variable. Then in Section 5.2.3, we will compute the sectional curvature formula explicitly by using the explict Green's function and writing the curvature formula in terms of the combination of first integrals of known quantities. Then we will derive the nonpositive curvature criterion and discuss the asymptotics of the curvature as the Froude number α becomes large and its physical interpretation.

5.2 Riemannian geometry of quantomorphism group

5.2.1 The space of quantomorphisms

Let $N = \mathbb{R}^2$ be a 2-dimensional manifold with symplectic form $\omega = dx \wedge dy$. On top of N, there is a 3-dimensional manifold $M = \mathbb{R}^3$ with a contact form $\theta = dz - y dx$ and a projection map $\pi : M \to N$ satifying $\pi^*\omega = d\theta$. Recall that for the contact form θ, there is a unique vector field E, called the Reeb field, satisfying two conditions $\theta(E) = 1$ and $\iota_E d\theta = 0$. In our case, the Reeb field is $E = \partial_z$. In our curvature computation, we will work with the case when N is the flat cylinder; $N = \mathbb{S}^1 \times [0, 2\pi]$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. This is a general construction and we are specializing it for simplicity.

The space of quantomorphisms $\mathcal{D}_q(M)$ consists of diffeomorphisms η on M that preserves the contact form exactly, i.e., $\eta^*\theta = \theta$. Its tangent space at the identity consists of vector fields X such that $\mathcal{L}_X\theta = 0$ and such a vector field X is uniquely determined by the function $f = \theta(X)$. For any vector field X on the contact manifold (M, θ) we have, there is a function $f : M \to \mathbb{R}$ such that $f_z = 0$ and

$$X = -f_y\partial_x + f_x\partial_y.$$

We write $X = S_\theta f$, and X determines f and vice versa. That is, we can identify elements $X \in T_{\mathrm{Id}}\mathcal{D}_q(M)$ with E-invariant functions on M, which are identified with all functions on N.

5.2.2 The Riemannian structure of $\mathcal{D}_q(M)$

With the identification mentioned above, on the space of quantomorphisms $\mathcal{D}_q(M)$, we put a right-invariant metric which at the identity is given by

$$\langle\langle X, Y\rangle\rangle = \int_N \alpha^2 fg + \langle\nabla f, \nabla g\rangle d\nu, \quad X, Y \in T_{\mathrm{Id}}\mathcal{D}_q(M), \tag{5.3}$$

where ν is the volume form on N. Here, α is the rescaled length of the Reeb field on M. The Lie algebra structure on $\mathcal{D}_q(M)$ is given by

$$\mathrm{ad}_X Y = -[S_\theta f, S_\theta g] = -S_\theta \{f, g\}, \tag{5.4}$$

where $X = S_\theta f$ and $Y = S_\theta g$. Then the Euler-Arnold equation on $\mathcal{D}_q(M)$ is

$$\left(\Delta - \alpha^2\right) f_t + \{\Delta f, f\} = 0, \tag{5.5}$$

which is the quasi-geostrophic equation in f-plane approximation on N.

Now, we consider the central extension of $T_{\mathrm{Id}}\mathcal{D}_q(M)$ by $\mathbb{R}$ which is given by the cocycle of the form

$$b(f,g) = -\int_N \psi\,\{f,g\}\,d\nu,$$

where ψ is a fixed function defined by $\psi(x,y)=y$. Then the new Lie algebra on $C^\infty(M)\times\mathbb{R}$ is given by

$$\mathrm{ad}_{\tilde{X}}\,\tilde{Y} = -\left[(f,\beta),(g,\gamma)\right] = \left(-S_\theta\{f,g\}, b(f,g)\right), \tag{5.6}$$

and the metric is given by

$$\langle\!\langle \tilde{X},\tilde{Y}\rangle\!\rangle = \langle\!\langle (f,\beta),(g,\gamma)\rangle\!\rangle = \int_N \alpha^2 fg + \langle\nabla f,\nabla g\rangle d\nu + \beta\gamma d\nu, \tag{5.7}$$

where $\tilde{X}=(S_\theta f,\beta)$ and $\tilde{Y}=(S_\theta g,\gamma)$. Then we can compute that

$$\mathrm{ad}^*{}_{\tilde{X}}\,\tilde{Y} = \mathrm{ad}^*{}_{(S_\theta f,\beta)}(S_\theta g,\gamma) = \Big(\left(\Delta-\alpha^2\right)^{-1} S_\theta\left(\{f,\Delta g\}-\alpha^2\{f,g\}+\gamma\{f,\psi\}\right),0\Big), \tag{5.8}$$

and the corresponding the geodesic equation is

$$\beta_t = 0, \quad \left(\Delta-\alpha^2\right) f_t + \{f,\Delta f\} + \beta\psi_x = 0, \tag{5.9}$$

which is the equation (5.1), the quasi-geostrophic equation in β-plane approximation. We can also write this equation in terms of the potential vorticity ξ as following:

$$\xi_t + \{f,\xi\} = 0, \quad \xi = f_{xx}+f_{yy}-\alpha^2 f+\beta y. \tag{5.10}$$

If we assume that f is a function of only y-variable, f is a steady solution since $\xi = f''(y) - \alpha^2 f(y) + \beta y$ and $\{f,\xi\}=0$.

5.2.3 Sectional curvature formula

Recall that the Arnold's sectional curvature formula is

$$K(X,Y) := \langle R(X,Y)Y, X\rangle \tag{5.11}$$

$$= \frac{1}{4}\Big(|\mathrm{ad}^*{}_X Y + \mathrm{ad}^*{}_Y X|^2 + 2\langle \mathrm{ad}_X Y, \mathrm{ad}^*{}_Y X - \mathrm{ad}^*{}_X Y\rangle \tag{5.12}$$

$$- 3|\mathrm{ad}_X Y|^2 - 4\langle \mathrm{ad}^*{}_X X, \mathrm{ad}^*{}_Y Y\rangle\Big), \tag{5.13}$$

where $X, Y \in T_{\mathrm{Id}}\mathcal{D}_q(M)$ which are identified by $X = (f, \beta)$ and $Y = (g, \gamma)$ for functions $f, g : N \to \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}$. From the assumption that f is a function of only y variable, we can simplify the formula (5.11) in a nice form so that we can use the explicit computation technique suggested by Preston [75].

Observe that

$$\mathrm{ad}^*{}_X Y = \big(-\Lambda^{-1} S_\theta(f'\Lambda g_x), 0\big)$$

where $\Lambda = \alpha^2 - \Delta$, and

$$-\,\mathrm{ad}_X Y = (-S_\theta(f' g_x), 0),$$

which is very close to $\mathrm{ad}^*{}_X Y$. In fact, these two are exactly the same when $f'' \equiv 0$. Define the following nonsymmetric commutator operator

$$D(X,Y) := \mathrm{ad}^*{}_X Y + \mathrm{ad}_X Y.$$

Note that the deformation tensor of X is given by

$$\mathrm{Def}\ X(Y, W) = \langle\!\langle \mathrm{ad}_X Y + \mathrm{ad}^*{}_X Y, W\rangle\!\rangle.$$

Hence, we can conclude that the operator $D(X,Y) := \mathrm{Def}\ X(Y)$ satisfies the condition that

$D(X,\cdot) = 0$ if and only if X is an isometry. For example, $f''(y) = 0$ implies that X is an isometry. So, in terms of this operator D, we can write

$$\operatorname{ad}^*{}_X Y = -\operatorname{ad}_X Y + D(X,Y), \tag{5.14}$$

and we have the following simplification of the Arnold curvature formula in the case when $D(X,Y)$ is simple.

Theorem 5.2.1. *The Arnold curvature formula can be written in terms of the operator D as following:*

$$K(X,Y) = \frac{1}{4}\left|\operatorname{ad}^*{}_Y X + D(X,Y)\right|^2 - \langle\!\langle \operatorname{ad}_X Y, D(X,Y)\rangle\!\rangle. \tag{5.15}$$

Proof. By substituting the equation (5.14) and expanding, we get

$$\begin{aligned}
\langle R(X,Y)Y,X\rangle =& \frac{1}{4}|\operatorname{ad}^*{}_Y X - \operatorname{ad}_X Y + D(X,Y)|^2 + \frac{1}{2}\langle \operatorname{ad}_X Y, \operatorname{ad}^*{}_Y X + \operatorname{ad}_X Y - D(X,Y)\rangle \\
& - \frac{3}{4}|\operatorname{ad}_X Y|^2 \\
=& \frac{1}{4}|\operatorname{ad}^*{}_Y X|^2 + \frac{1}{4}|\operatorname{ad}_X Y|^2 + \frac{1}{4}|D(X,Y)|^2 - \frac{1}{2}\langle \operatorname{ad}^*{}_Y X, \operatorname{ad}_X Y\rangle \\
& + \frac{1}{2}\langle \operatorname{ad}^*{}_Y X, D(X,Y)\rangle - \frac{1}{2}\langle \operatorname{ad}_X Y, D(X,Y)\rangle + \frac{1}{2}\langle \operatorname{ad}_X Y, \operatorname{ad}^*{}_Y X\rangle \\
& + \frac{1}{2}|\operatorname{ad}_X Y|^2 - \frac{1}{2}\langle \operatorname{ad}_X Y, D(X,Y)\rangle - \frac{3}{4}|\operatorname{ad}_X Y|^2 \\
=& \frac{1}{4}|\operatorname{ad}^*{}_Y X + D(X,Y)|^2 - \langle \operatorname{ad}_X Y, D(X,Y)\rangle.
\end{aligned}$$

□

Note that the above simplification of the Arnold curvature formula works on any Lie algebra.

As mentioned earlier, we will compute the sectional curvature in the case when $f = f(y)$.

In this case, we have

$$\operatorname{ad}^*{}_Y X + D(X,Y) = \left(\Lambda^{-1} S_\theta \left(-2\partial_y(f'' g_x) + (\alpha^2 f' + \beta) g_x\right), 0\right),$$
$$\langle\langle \operatorname{ad}_X Y, D(X,Y)\rangle\rangle = \int_N (f'' g_x)^2 d\nu,$$

and finally the curvature formula becomes

$$K(X,Y) = \int_N \left(\partial_y(f'' g_x) - \frac{1}{2}(\alpha f' + \beta) g_x\right) \Lambda^{-1} \left(\partial_y(f'' g_x) - \frac{1}{2}(\alpha f' + \beta) g_x\right) d\nu - \int_N (f'' g_x)^2 d\nu. \tag{5.16}$$

Remark 6. *We can see that if $\alpha = \beta = 0$, then the curvature formula reduces to the nonpositive sectional curvature of the 2-dimensional area preserving diffeomorphism group case.*

5.3 Explicit curvature formula and Nonpositive criterion

5.3.1 Green's function

To precede with the explicit computation of the formula (5.16), we compute the Greens function for Λ^{-1} explicitly. We will expand the function g in terms of the Fourier series in x variables of the form

$$g(x,y) = \sum_{n\in\mathbb{Z}} g_n(y) e^{inx}.$$

Then the boundary value problem associated with the Green's function for Λ^{-1} reduces to the ODE of the functions in y variables. From the boundary condition of the flat cylinder, we obtain the following BVP

$$\begin{cases} -u'' + \lambda^2 u = \delta \\ u(0) = 0 = u(2\pi) \end{cases}$$

whose explicit solution is given by

$$G(y,s) = \frac{1}{2\lambda(e^{2\pi\lambda} - e^{-2\pi\lambda})} \begin{cases} e^{2\pi\lambda}(e^{\lambda y} - e^{-\lambda y})e^{-\lambda s} - e^{-2\pi\lambda}(e^{\lambda y} - e^{-\lambda y})e^{\lambda s} & \text{if } 0 \leq y \leq s \\ (e^{-2\pi\lambda}e^{\lambda y} - e^{2\pi\lambda}e^{-\lambda y})(e^{-\lambda s} - e^{\lambda s}) & \text{if } s < y \leq 2\pi \end{cases}$$

where $\lambda^2 = \alpha^2 + n^2$.

5.3.2 Explicit computation of the curvature

Now, we want to compute the sectional curvature formula (5.16) explicitly using the Green's function. By first substituting the Fourier expansion of g, we have

$$K(X,Y) = \sum_{n\in\mathbb{Z}} K_n = \sum_{n\in\mathbb{Z}} n^2 \left[\int_0^{2\pi} (\overline{h}' - Q\overline{h})(\lambda^2 - \partial_y^2)^{-1} \left[h' - Qh\right] dy - \int_0^{2\pi} |h|^2 dy \right], \quad (5.17)$$

where $h = f''g$ and $Q = \frac{\alpha f' + \beta}{2f''}$. Here, we assume that f'' never vanishes. Furthermore, let

$$h'(y) - Q(y)h(y) = e^{P(y)}\partial_y \left[e^{-P(y)}h(y)\right] =: e^{P(y)}z'(y), \ \text{ where } P'(y) = Q(y).$$

We can compute $(\lambda^2 - \partial_y^2)^{-1}\left[e^P z'\right]$ using the auxiliary functions

$$H_n(y) := \int_0^y (\lambda + Q)e^{\lambda s} f'' g \, ds, \text{ and} \quad (5.18)$$

$$J_n(y) := \int_y^{2\pi} (-\lambda + Q)e^{-\lambda s} f'' g \, ds. \quad (5.19)$$

Then by integrating against $e^P \overline{z}$, we get

$$K_n := \frac{1}{2\lambda}\int_0^{2\pi} \overline{H_n}' J_n - H_n \overline{J_n}'\, dy - \frac{|J_n(0)|^2}{2\lambda} - \frac{e^{-2\pi\lambda}\left|H_n(2\pi) - J_n(0)\right|^2}{2\lambda},$$

Note that we can compute the integral term explicitly:

$$\begin{aligned}
\frac{1}{2\lambda}\int_0^{2\pi} \overline{H_n}' J_n - H_n \overline{J_n}'\, dy &= \frac{1}{\lambda}\mathrm{Re}\left[\int_0^{2\pi} H_n' \overline{J_n} dy\right] \\
&= \frac{1}{2\lambda}\mathrm{Re}\left[\int_0^{2\pi} \frac{H_n'}{J_n'} \frac{d}{dy}\left(|J_n|^2\right) dy\right] \\
&= \frac{1}{2\lambda}\int_0^{2\pi} \left(\frac{\lambda + Q}{\lambda - Q} e^{2\lambda y}\right) \frac{d}{dy}\left(|J_n|^2\right) dy \\
&= \frac{1}{2\lambda}\left[-\frac{\lambda + Q(0)}{\lambda - Q(0)}|J_n(0)|^2 - \int_0^{2\pi} \frac{2\lambda(Q' - Q^2 + \lambda^2)}{(\lambda - Q)^2} e^{2\lambda y}|J_n|^2 dy\right] \\
&= -\int_0^{2\pi} \frac{Q' - Q^2 + \lambda^2}{(\lambda - Q)^2} e^{2\lambda y}|J_n|^2 dy - \frac{\lambda + Q(0)}{2\lambda(\lambda - Q(0))}|J_n(0)|^2.
\end{aligned}$$

Hence,

$$K_n = -\int_0^{2\pi} \frac{Q' - Q^2 + \lambda^2}{(\lambda - Q)^2} e^{2\lambda y}|J_n|^2 dy - \frac{|J_n(0)|^2}{\lambda - Q(0)} - \frac{e^{-2\pi\lambda}\left|H_n(2\pi) - J_n(0)\right|^2}{2\lambda(e^{2\pi\lambda} - e^{-2\pi\lambda})}, \tag{5.20}$$

by combining the boundary terms. By reversing the role of H_n and J_n above, we get

$$K_n = -\int_0^{2\pi} \frac{Q' - Q^2 + \lambda^2}{(\lambda - Q)^2} e^{2\lambda y}|H_n|^2 dy - \frac{e^{-4\pi\lambda}\left|H_n(2\pi)\right|^2}{\lambda + Q(2\pi)} - \frac{e^{2\pi\lambda}\left|e^{-4\pi\lambda}H_n(2\pi) + J_n(0)\right|^2}{2\lambda(e^{2\pi\lambda} - e^{-2\pi\lambda})}. \tag{5.21}$$

5.3.3 Nonpositive curvature criterion

We first establish the properties of the differential inequality that appears in the integrand of the formula (5.20) and (5.21).

Proposition 5.3.1. *Suppose that Q is a C^1 function on $[0, 2\pi]$ satisfying the differential* *inequality*

$$Q' - Q^2 + \lambda^2 \geq 0 \quad \textit{for all } y \in (0, 2\pi). \tag{5.22}$$

Then for any $0 < a < b < 2\pi$*, if* $|Q(y)| < \lambda$ *on* $[a, b]$*, we have the inequality*

$$\operatorname{arctanh}\left(\frac{Q(b)}{\lambda}\right) - \operatorname{arctanh}\left(\frac{Q(a)}{\lambda}\right) \geq -\lambda(b - a). \tag{5.23}$$

In addition, we have the following.

- *If* $|Q(y)| > \lambda$*, then* Q *is strictly increasing at* y*.*
- *If* $Q(0) \leq -\lambda$ *and* $Q(2\pi) \geq -\lambda$*, then either* $Q(y) \equiv -\lambda$ *on* $[0, 2\pi]$ *or* $Q(y_0) = -\lambda$ *for exactly one* $y_0 \in (0, 2\pi)$*.*

Proof. Let $Z = -Q$. Then Z satisfies the differential inequality $Z' + Z^2 \leq \lambda^2$ for all $y \in (0, 2\pi)$. By integrating this on $[a, b]$ where $0 < a < b < 2\pi$, we get

$$\operatorname{arctanh}(Z(b)) - \operatorname{arctanh}(Z(a)) \leq \lambda(b - a),$$

which is the inequality (5.23) when written in terms of the function $Q(y)$. The first statement is obvious. Suppose that $Q(0) \leq -\lambda$ and $Q(2\pi) \geq -\lambda$. If $Q(0) < -\lambda$, then $Q(y) < -\lambda$ around the neighborhood of zero. Then Q is strictly increasing and since $Q(2\pi) \geq -\lambda$, there exists a point $y_0 \in (0, 2\pi]$ such that $Q(y_0) = -\lambda$. Such a point y_0 is unique because the inequality (5.23) confines that $Q(y) > -\lambda$ once $Q(y)$ becomes greater than $-\lambda$. Next, suppose that we have $Q(0) = -\lambda$ but $Q(y)$ is not identically equal to $-\lambda$. If $Q(y) < -\lambda$ for some $y \in (0, 2\pi)$, then we have the same result as in the previous case. If $Q(y) > -\lambda$, then the inequality (5.23) forces $Q(2\pi) > -\lambda$. This completes the proof of the proposition. □

Using this, we obtain the following statement about the nonpositive sectional curvature.

Theorem 5.3.2. *Let $X = (f, \beta)$ be a vector tangent to the identity of the quantomorphism group $\mathcal{D}_q(M)$ where M is the flat cylinder. Suppose that $f = f(y)$ is a function of y alone and at least C^2 on $[0, 2\pi]$. Then the curvature operator R_X is nonpositive if the function $Q = \frac{\alpha f' + \beta}{2f''}$ is defined for all $y \in [0, 2\pi]$ and satisfies the differential inequality (5.22) for all $y \in (0, 2\pi)$.*

Proof. Suppose that Q is a function satisfying the differential inequality (5.22). Then there are three cases for the boundary conditions: (i) $Q(0) < -\lambda$ and $Q(2\pi) < -\lambda$, (ii) $Q(0) > -\lambda$ and $Q(2\pi) > -\lambda$, or $Q(0) \leq -\lambda$ and $Q(2\pi) \geq -\lambda$. The case $Q(0) > -\lambda$ and $Q(2\pi) < -\lambda$ is impossible by the above proposition. In case (i) and (ii), we have $K_n \leq 0$ from the formula (5.20) and (5.21). Consider the case (iii). From the above proposition, we must have either $Q(y) \equiv -\lambda$ on $[0, 2\pi]$ or $Q(y_0) = -\lambda$ for exactly one $y_0 \in (0, 2\pi)$. If $Q(y)$ is identically equal to $-\lambda$, then $H_n(y) \equiv 0$ and the first two terms of the formula (5.21) vanishes, so we have $K_n \leq 0$. In the latter case, we need to make sure that the formula (5.21) remains finite even when $Q(2\pi) = -\lambda$. The idea is to use the fact that $Q(y_0) = 0$ for some $y_0 \in (0, 2\pi)$ to cancel out the zero in the denominator. By using the function $[H_n(y) - H_n(y_0)]$ insted of $H_n(y)$, we can write the formula (5.21) as

$$\begin{aligned} K_n = &-\int_0^{2\pi} \frac{Q' - Q^2 + \lambda^2}{(\lambda + Q)^2} e^{-2\lambda y} \left|H_n(y) - H_n(y_0)\right|^2 dy - \frac{e^{-4\pi\lambda}}{\lambda + Q(2\pi)} \left|H_n(2\pi) - H_n(y_0)\right|^2 \\ &+ \frac{\left|H_n(y_0)\right|^2}{\lambda + Q(0)} + \frac{e^{-4\pi\lambda}}{2\lambda} \left|H_n(2\pi) - H_n(y_0)\right|^2 - \frac{1}{2\lambda} \left|H_n(y_0) - J_n(0)\right|^2 \\ &- \frac{e^{-2\pi\lambda}}{2\lambda\left(e^{2\pi\lambda} - e^{-2\pi\lambda}\right)} \left|H_n(2\pi) - J_n(0)\right|^2 . \end{aligned}$$

Clearly, the the first two terms on the first line are nonpositive and the third term is also nonpositive since we are considering the case when $Q(0) < -\lambda$. Also, note that the two denominators $\lambda + Q(y)$ and $\lambda + Q(2\pi)$ vanish precisely when $y = y_0$ and $y_0 = 2\pi$, respectively.

Hence, the first line always remains finite. Finally, by using the triangle inequality, we have

$$|H_n(2\pi) - H_n(y_0)| \leq |H_n(2\pi) - J_n(0)| + |H_n(y_0) - J_n(0)|,$$

which implies that the second line above is also nonpositive. Hence, $K_n \leq 0$ in this case as well. □

5.4 Future research

The theorem (5.3.2) is currently in progress to be completed. We want to show that the condition on the differential inequality (5.22) on the function Q is also a necessary condition for the nonpositive sectional curvature. Once we have the nonpositive curvature criterion, we will be able to estimate the divergence of neighboring geodesics by using the explicit curvature calculation. Also, we can analyze the asymptotics of the sectional curvature as we manipulate the parameters α and β. It would be interesting to understand the physical meaning(i.e., geophysical fluid flow) of the separation(or convergence) of geodesics of the quantomorphism group.

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