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Linear Progress with Exponential Decay in Weakly Hyperbolic Groups

by

Matthew H. Sunderland

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

Linear Progress with Exponential Decay
in Weakly Hyperbolic Groups

by

Matthew H. Sunderland

Advisor: Joseph Maher

A random walk w_n on a separable, geodesic hyperbolic metric space X converges to the boundary ∂X with probability one when the step distribution supports two independent loxodromics. In particular, the random walk makes positive linear progress. Progress is known to be linear with exponential decay when (1) the step distribution has exponential tail and (2) the action on X is acylindrical. We extend exponential decay to the nonacylindrical case. We give an application to random Heegaard splittings.

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Chapter 1

Introduction

Suppose X_n is a sequence of independent, identically distributed random variables, each taking the values 1 and -1 with probabilities p and $1 - p$, respectively, for some fixed $p \in [\frac{1}{2}, 1]$. Since the X_n have finite expectation $2p - 1$, by the law of large numbers, $\frac{1}{n}(X_1 + \cdots + X_n) \rightarrow 2p - 1$ almost surely. Setting $Z_n := X_1 + \cdots + X_n$, one obtains the stochastic process that Woess calls Pólya's Walk (in one dimension) [Woe00, §I.1.A]. In this process, our walker starts at zero and then wanders randomly on the real line, each time taking either a unit-one step to the left with probability p , or a unit-one step to the right with probability $1 - p$.

Pólya's Walk with $p > \frac{1}{2}$ makes *positive linear progress*, meaning that

$$\liminf_{n \rightarrow \infty} \frac{1}{n}|Z_n| > 0 \text{ almost surely.}$$

Indeed, $\frac{1}{n}|Z_n| \geq \frac{1}{n}Z_n \rightarrow 2p - 1 > 0$ almost surely. Pólya's Walk with $p = \frac{1}{2}$,

however, has expected number of returns $\sum_n \mathbb{P}(Z_n = 0) = \infty$, since $\mathbb{P}(Z_{2n} = 0) = 2^{-2n} \binom{2n}{n} \sim cn^{-0.5}$ for some constant c . Note that in Pólya's Walk (with any p), the walker can only return to zero at even times. It follows that the almost sure number of returns is infinite as well, and so Pólya's Walk with $p = \frac{1}{2}$ does not make positive linear progress. See, for example, Woess [Woe00, §I.1.A] or Lawler [Law10, Theorem 1.1].

We say that a real-valued stochastic process Z_n makes *linear progress with exponential decay* (Definition 4) if we can find a sufficiently large constant $C > 0$ so that for all n ,

$$\mathbb{P}(Z_n \leq n/C) \leq Ce^{-n/C}. \quad (1)$$

Note that if Z_n satisfies (1), then so does $|Z_n|$. Pólya's Walk with $p > \frac{1}{2}$ can be shown to make linear progress with exponential decay using standard results in concentration of measure; see for example, [Mah12, Proposition A.1]. Exponential decay can also be shown using the following argument, which will generalize to the setting of Theorem 1.

If a (not necessarily Markov) stochastic process Z_n has constants $t, \epsilon > 0$ so that for all n ,

$$\mathbb{E}(e^{-t(Z_{n+1} - Z_n)} \mid Z_n) \leq 1 - \epsilon, \quad (2)$$

then Z_n makes linear progress with exponential decay (Proposition 15). We say

a process with such t, ϵ makes *uniformly positive progress* (Criterion 13). For any random variable Z , the moment generating function $f(t) = \mathbb{E}(e^{tZ})$ has the property that $f'(0) = \mathbb{E}(Z)$. Thus (2) captures a notion of positive progress independent of location.

Therefore, it remains only to show that Pólya's Walk with $p > \frac{1}{2}$ makes uniformly positive progress. Let $p \in (\frac{1}{2}, 1]$, and let X_n be the sequence of independent, identically distributed random variables taking the values 1 and -1 with probabilities p and $1 - p$, respectively. The claim is that $Z_n := X_1 + \dots + X_n$ satisfies (2) for some positive t, ϵ . Since X_{n+1}, Z_n are independent and $Z_{n+1} - Z_n = X_{n+1}$,

$$\mathbb{E}(e^{-t(Z_{n+1} - Z_n)} \mid Z_n) = \mathbb{E}(e^{-tX_{n+1}}) = pe^{-t} + (1 - p)e^t$$

for all n and t . The righthand side $f(t) := pe^{-t} + (1 - p)e^t$ has value $f(0) = 1$ and derivative $f'(0) = 1 - 2p$. Moreover, this derivative is negative, since $p > \frac{1}{2}$.

Hence $f(t) = 1 - \epsilon$ for some $t, \epsilon > 0$, as desired.

Weakly hyperbolic groups

Thinking of the steps X_n in Pólya's walk as elements of $\mathbb{Z}$ acting on the real line inspires a notion of random walks on non-abelian groups. Let G be a group acting by isometries on a metric space (X, d_X) , and let μ be a probability distribution on G . Following the notation from Tiozzo [Tio15, §1], we define

the (G, μ) -random walk w_n on X by independently drawing at each time n an element s_n from G with distribution μ , and then defining the random variable $w_n := s_1 \cdots s_n$. Fixing a basepoint x_0 in X , we obtain $(w_n x_0)_{n \in \mathbb{N}}$, a stochastic process taking values in X .

The notions of positive linear progress and linear progress with exponential decay also generalize to non-abelian groups by putting $Z_n := d_X(x_0, w_n x_0)$. We say that the (G, μ) -random walk on X is *weakly hyperbolic* if X is separable, geodesic, and δ -hyperbolic (Definition 7). Note that X need not be locally compact. We say that the random walk is *nonelementary* if the support of μ generates a subgroup containing two loxodromics with disjoint endpoints on ∂X (Definition 7). Such loxodromics are sometimes called independent [Osi16, Theorem 1.1; MT16, §1]. Notably, the definition of nonelementary excludes Pólya's Walk.

As shown by Maher and Tiozzo, every nonelementary, weakly hyperbolic random walk makes positive linear progress. Progress is linear with exponential decay, moreover, when the support of μ is bounded in X [MT16, Theorem 1.2].

We say that μ has *exponential tail* if there exists $\lambda > 0$ such that

$$\sum_{g \in G} e^{\lambda d_X(x_0, gx_0)} \mu(g) < \infty.$$

Mathieu and Sisto prove linear progress with exponential decay in the case of

geodesic, hyperbolic X and μ with exponential tail and with support generating a subgroup not virtually cyclic, acting acylindrically with unbounded orbits in X [MS15, Theorem 9.1]. If a group G is not virtually cyclic and acts acylindrically with unbounded orbits on a hyperbolic space, then G contains infinitely many loxodromics with disjoint endpoints [Osi16, Theorem 1.1]. In particular, G is non-elementary (Definition 7).

The goal of this paper is to show the following.

Theorem 1 (Linear progress with exponential decay). *Every nonelementary, weakly hyperbolic random walk with exponential tail makes linear progress with exponential decay.*

The result applies, for example, to the action of $\text{Out}(F_n)$ on the complex of free splittings $\mathcal{FS}(F_n)$. The action is shown to be nonelementary, weakly hyperbolic, and nonacylindrical by Handel and Mosher [HM16, Theorem 1.4]. Even when a group is acylindrically hyperbolic, we may care about a nonacylindrical action on another hyperbolic space because of the geometric insight garnered from the particular action and its loxodromics.

We may care about the rate of convergence, instead of just whether something tends to zero, depending on the technical details of the random methods at hand. For example, Lubotzky, Maher, and Wu use exponential decay in an essential way in their study of the Casson invariant of random Heegaard

splittings [LMW16].

The idea of the proof is essentially the same as in Pólya's Walk with $p > \frac{1}{2}$: prove that every nonelementary, weakly hyperbolic random walk with exponential tail makes uniformly positive progress (Criterion 13), which we know implies linear progress with exponential decay (Proposition 15).

Proof of Theorem 1. Working backwards, the goal is to find $C > 0$ so that for all n ,

$$\mathbb{P}(d_X(x_0, w_n x_0) \leq n/C) \leq C e^{-n/C}. \quad (3)$$

This proof is structured so that each equation (3)–(8) is implied by the next.

We show that (3) follows from a -iterated linear progress with exponential decay, i.e., the existence of $C > 0$ and integer $a > 0$ so that for all n ,

$$\mathbb{P}(d_X(x_0, w_{an} x_0) \leq n/C) \leq C e^{-n/C}, \quad (4)$$

and exponential tail in Corollary 17.

The proof of this corollary is based on the following intuition. If the random walk satisfies (3) at all times $n = ai$ ($i \in \mathbb{Z}$) and the random walk cannot wander too far during the intervening times (by exponential tail), then the random walk satisfies (3) at all times n . The corollary is a special case of the general fact that if Y_n has exponential tail and Z_n makes linear progress with exponential decay, then their sum also makes linear progress with exponential decay.

We show that linear progress with exponential decay follows from uniformly positive progress in Proposition 15. Hence, to show (4), it suffices to find constants $b, \epsilon > 0$ such that for all n ,

$$\mathbb{E}\left(e^{-b(d_X(x_0, w_{an+a}x_0) - d_X(x_0, w_{an}x_0))} \mid d_X(x_0, w_{an}x_0)\right) \leq 1 - \epsilon. \quad (5)$$

Crucially, (5) implies (4) even though the process $Z_n = d_X(x_0, w_{an}x_0)$ is not necessarily Markov. The proof of the proposition is purely probabilistic.

Given w, g in G , define the horofunction $\rho_g(w) = d_X(gx_0, wx_0) - d_X(x_0, gx_0)$.

We prove that (5) holds as long as there are constants $a, b > 0$ so that

$$\sup_{g \in G} \left(\mathbb{E}\left(e^{-b\rho_g(w_a)}\right) \right) < 1 \quad (6)$$

in Lemma 14. We say a random walk with such a, b has *uniformly positive horofunctions* at time a .

Given w, g in G , define the Gromov product to be $(w, g)_1 = \frac{1}{2}d_X(x_0, wx_0) + \frac{1}{2}d_X(x_0, gx_0) - \frac{1}{2}d_X(wx_0, gx_0)$. In the hyperbolic setting, one can think of $(w, g)_1$ as measuring the distance w and g (or rather a geodesic from x_0 to wx_0 and a geodesic from x_0 to gx_0) fellow-travel. We establish that (6) holds as long as there exist sufficiently large d, k so that

$$\sup_{g, h \in G} \mathbb{P}((w_k h, g)_1 \geq d) \leq 0.01 \quad (7)$$

in Lemma 12. We say a random walk with such d, k has *uniform shadow decay*.

Figure 1: Stepping from x_0 to wx_0 , the horofunction $\rho_g(w) = d_X(gx_0, wx_0) - d_X(x_0, gx_0)$ measures progress away from the basepoint gx_0 (regardless of hyperbolicity). In the proof of Theorem 1, working backwards we show that (4) linear progress for the a -iterated random walk follows from (5) uniformly positive progress, which follows from (6) uniformly positive horofunctions.

The proofs of Lemmas 14 and 12 are adapted from Mathieu and Sisto [MS15, Theorem 9.1]. The key to the proof of Lemma 14 is the observation that $\mathbb{E}(\rho_{w_{ai}^{-1}}(w_a))$ equals $\mathbb{E}(d_X(w_{ai}x_0, w_{ai+a}x_0) - d_X(x_0, w_{ai+a}x_0))$ and hence the horofunction $\rho_g(w_a)$ measures a notion of progress; see Figures 1 and 14. In the proof of Lemma 12, we essentially argue that if you are unlikely to be in the shadow of g , then you are unlikely to be in the horoball about g (regardless of hyperbolicity).

We use the hyperbolicity of X and positive linear progress of $d_X(x_0, w_n x_0)$ to show that uniform shadow decay (7) follows from *shadow decay*, i.e.,

$$\lim_{d \rightarrow \infty} \sup_{n, g} \mathbb{P}((w_n, g)_1 \geq d) = \lim_{d \rightarrow \infty} \sup_{n, g} \mathbb{P}((w_n^{-1}, g)_1 \geq d) = 0, \quad (8)$$

in Lemma 10. The proof uses only the Gromov four-point condition and does not require the action of G on X to be acylindrical. See Figure 2.

Figure 2: Uniform shadow decay (7) means it is unlikely that $(w_k h, g)_1 \geq d$ for large d, k . Shadow decay (8) means it is unlikely that $(w_n, g)_1 \geq d$ for large d . We derive (7) from (8) using positive linear progress and hyperbolicity.

Finally, both shadow decay (8) and positive linear progress are consequences of the convergence of nonelementary, weakly hyperbolic random walks (Theorem 8 [MT16, 1.1, 1.2, 5.3]). □

Chapter 2

Background

2.1 Metric hyperbolicity

Given a metric space (X, d_X) and three points a, b, c in X , the *Gromov product* of b and c with respect to a is defined to be

$$(b, c)_a := \frac{1}{2}d_X(a, b) + \frac{1}{2}d_X(a, c) - \frac{1}{2}d_X(b, c).$$

The metric space is said to be δ -*hyperbolic*, where δ is some fixed real number, if every quadruple of points a, b, c, d in X satisfies the *Gromov four-point condition*

$$(b, c)_a \geq \min\{(b, d)_a, (c, d)_a\} - \delta. \tag{9}$$

This δ cannot depend on the four points: the same fixed constant δ must work for all quadruples in X . This condition can be shown to be equivalent to others, such as the slim triangles condition [BH99, Propositions III.H 1.17 and 1.22]. When δ is obvious or not important, we may say that X is a (Gromov) hyperbolic

metric space, or colloquially, that X has slim triangles.

Each hyperbolic space X has an associated *Gromov boundary* ∂X . If X is locally compact, then both ∂X and $\overline{X} = X \cup \partial X$ are compact. For a construction of the Gromov boundary, see for example Bridson and Haefliger [BH99, §III.H.3].

In this paper we exclusively take the Gromov product of points in the orbit of a fixed basepoint x_0 in X under a countable group of isometries $G \rightarrow \text{Isom } X$. Therefore, to simplify the notation, for each w in G , we denote

$$|w| := d_X(x_0, wx_0)$$

(following Kaimanovich [Kai00, §7.2]), and given two more elements g, h in G we denote the Gromov product of wx_0 and gx_0 with respect to hx_0 ,

$$(w, g)_h := \frac{1}{2}d_X(wx_0, hx_0) + \frac{1}{2}d_X(gx_0, hx_0) - \frac{1}{2}d_X(wx_0, gx_0)$$

(following Mathieu and Sisto [MS15, §9, p. 29]).

In the slim triangles setting, the intuition is that the Gromov product measures distance fellow-travelled, in the sense of Figure 3. We sometimes say that w and g fellow-travel to mean that all geodesics $[x_0, wx_0]$ and $[x_0, gx_0]$ in X fellow-travel. We use the notion of fellow-travelling for intuition only; for proofs we appeal to the Gromov four-point condition directly.

As in Figure 5, the Gromov four-point condition (9) applied to the points

Figure 3: When triangles are slim, the Gromov product $(w, g)_1$ measures the distance that w, g fellow-travel. The shadow $S(g, d)$ contains w if and only if that distance $(w, g)_1 \geq d$.

Figure 4: Suppose w avoids $S(g, d)$. Then its shadow $S(w, d + \delta)$ avoids $S(g, d + \delta)$.

Figure 5: The Gromov four point condition (10) means that h cannot fellow-travel both w and g beyond distance $(w, g)_1 + \delta$.

x_0, wx_0, gx_0, hx_0 , implies that

$$\begin{aligned} &\text{either } (h, w)_1 \leq (w, g)_1 + \delta \\ &\text{or } (h, g)_1 \leq (w, g)_1 + \delta. \end{aligned} \tag{10}$$

We will also use the following version of the triangle inequality in terms of the Gromov product:

$$0 \leq (w, g)_h \leq \min\{d_X(wx_0, hx_0), d_X(gx_0, hx_0)\}. \tag{11}$$

This equation is equivalent to the triangle inequality, and always holds regardless of whether or not X is hyperbolic.

Loxodromics in X

A map between metric spaces $f : (X, d_X) \rightarrow (Y, d_Y)$ is a C -quasi-isometric embedding if

$$d_X(x, x')/C - C \leq d_Y(f(x), f(x')) \leq d_X(x, x')C + C \tag{12}$$

for all x, x' in X , where C is independent of the choice of x, x' . When the domain is $X = \mathbb{Z}$, we call such a map a *quasigeodesic*. A reparametrized quasigeodesic is a map $f \circ p$ where $p : \mathbb{Z} \rightarrow \mathbb{R}$ is strictly increasing and $f : \mathbb{R} \rightarrow X$ is a C -quasi-isometric embedding. The map $f \circ p$ need not have infinite diameter image in X . The *diameter* of a subset $A \subset X$ is defined to be $\sup\{d_X(a, a') : a, a' \in A\}$.

An isometry g of X is called a *loxodromic in X* , or more briefly, an X -loxodromic, if for some (equivalently, all) x_0 in X , the map $n \mapsto g^n x_0$ is a quasigeodesic. The *translation length* of an isometry g in X is defined

$$\tau(g) = \lim_{n \rightarrow \infty} \frac{1}{n} d_X(x_0, g^n x_0). \quad (13)$$

By Gromov's classification of isometries (below), an isometry of a hyperbolic space is loxodromic if and only if it has positive translation length.

A loxodromic isometry of a hyperbolic space $g : \overline{X} \rightarrow \overline{X}$ fixes exactly two points, which are on the boundary. These fixed points are $g^+ = \lim_{n \rightarrow \infty} g^n x_0$ and $g^- = \lim_{n \rightarrow -\infty} g^n x_0$ regardless of x_0 , and are sometimes called the *endpoints* of the loxodromic. See, for example, Kapovich and Benakli [KB02, Theorem 4.1]. Two loxodromics g, h in X with distinct endpoints

$$\#\{g^-, g^+, h^-, h^+\} = 4 \quad (14)$$

are called *independent loxodromics for X* .

Acyndricity

We do not use acylindricity in any of our proofs, but the definition is included for completeness. Suppose G acts on (X, d_X) . The stabilizer (or isotropy group) of a point x in X is defined to be $G_x := \{g \in G : gx = x\}$. The r -coarse stabilizer of x is

$$G_{x,r} := \{g \in G : d_X(x, gx) \leq r\}.$$

We say that the action of G is *acyndric* if every $r > 0$ has constants $d(r)$, $n(r)$ such that the r -coarse stabilizers of any two points of sufficient distance $d_X(x, y) \geq d$ have not too many group elements in common

$$\#(G_{x,r} \cap G_{y,r}) \leq n.$$

Classification of actions on hyperbolic spaces

Consider a (possibly not proper) action of G on the δ -hyperbolic space X by isometries. As shown by Gromov [Gro87, §8.1–8.2], an isometry g is either *elliptic* (fixes one point in X , no points in ∂X), *parabolic* (fixes one point in ∂X , no points in X), or *loxodromic* (12). Gromov shows in the same paper that the action of G is either

1. *elliptic* in X (G has bounded orbits in X),
2. *parabolic* in X (G has unbounded orbits, but no loxodromics),

3. *lineal* in X (G has loxodromics, all of which share the same two endpoints),
4. *quasiparabolic* in X (G has two loxodromics that share exactly one endpoint λ , and all loxodromics have an endpoint at λ), or
5. *nonelementary* in X (G has a pair of independent loxodromics; see 14)

If G has a pair of independent loxodromics for X , then it has infinitely many nonconjugate pairs of independent loxodromics for X .

Some sources define nonelementary to include quasiparabolics [Osi16, §3]. However, if the action is acylindrical, then G is neither parabolic nor quasiparabolic [Osi16, Theorem 1.1]. A group that admits a nonelementary, acylindrical action on a hyperbolic space is called *acylindrically hyperbolic*.

The *limit set* of G on ∂X is defined to be $\Lambda G = \overline{Gx_0} \cap \partial X$, and is independent of the choice of basepoint x_0 in X . The limit set is empty for elliptic G , has one point for parabolic G , has two points for lineal G , and has uncountably many points for quasiparabolic G and nonelementary G . We say that G is *complete in X* if the limit set of G equals (equivalently, is dense in) the Gromov boundary ∂X .

Horofunctions

Let (X, d_X) be a metric space with basepoint x_0 and $G \rightarrow \text{Isom } X$ a (not necessarily injective) group of isometries. For each g in G , the *horofunction*

Figure 6: The horofunction $\rho_g(w) := d_X(gx_0, wx_0) - d_X(x_0, gx_0)$. Fixing every horofunction to be zero at x_0 allows us to take the limit of horofunctions $\rho_{g_i}(w)$ as $g_i x_0$ approaches the boundary.

$\rho_g : G \rightarrow \mathbb{R}$ is defined to be

$$\rho_g(w) := -d_X(x_0, gx_0) + d_X(gx_0, wx_0). \quad (15)$$

Think of horofunctions as distance functions normalized to be zero at x_0 ; see

Figure 6. The normalization allows us to take limits of horofunctions. Consider, for example, $G = X = \mathbb{R}$. Then the horofunction $\rho_x(y) = -x + |y - x|$ approaches the identity function $y \mapsto y$, as x approaches $-\infty$.

The intuition in the slim triangles setting is that the horofunction measures minus how far you fellow-travel plus how far you then veer off:

$$\rho_g(w_n) \stackrel{\text{slim}\Delta\text{s}}{\approx} - \underset{\text{follows } g}{\text{how far } w_n} + \underset{\text{veers off}}{\text{how far } w_n}.$$

See Figure 7.

Figure 7: When triangles are slim, the horofunction $\rho_g(w_n)$ measures $\approx -(\text{how far } w_n \text{ follows } g) + (\text{how far it veers off})$.

2.2 (G, μ) -random walk w_n on (X, d_X, x_0)

A *stochastic process* in a group G is a sequence of G -valued random variables w_n . We say that w_n is *Markov* if $\mathbb{P}(w_n = g_n \mid w_1 = g_1, \dots, w_{n-1} = g_{n-1})$ equals $\mathbb{P}(w_n = g_n \mid w_{n-1} = g_{n-1})$. In the construction below, the stochastic process w_n in G will be Markov, but the stochastic process $w_n x_0$ in X will not necessarily be Markov.

Let X be a metric space with basepoint x_0 , and let $G \rightarrow \text{Isom } X$ be a (not necessarily injective) group of isometries with probability measure μ . We will call μ the *step distribution* and the product $(\Omega, \mathbb{P}) := (G, \mu)^{\mathbb{N}}$ the *step space* [Tio15, §2.1]. From there, the (G, μ) -random walk on (X, d_X, x_0) is the stochastic process $(w_n)_{n \in \mathbb{N}}$ with $w_n : \Omega \rightarrow G$ defined to be

$$(g_1, g_2, g_3, \dots) \mapsto g_1 g_2 g_3 \cdots g_n$$

for every positive integer n ; we additionally require the semigroup generated by

the support of μ

$$\langle \text{supp } \mu \rangle_+ := \{g \in G : \text{there exists } k \in \mathbb{N} \text{ with } \mu^k(g) > 0\} \quad (16)$$

to be a group. If called for, we define $w_0 \equiv 1_G$. Each w_n is referred to as the n th (random) *location*, and has distribution μ^n , the n -fold *convolution*:

$$\mathbb{P}(w_n = g) = \mu^n(g) = \sum_{g_1 \cdots g_n = g} \mu(g_1) \cdots \mu(g_n). \quad (17)$$

Define the n th *step* to be the random variable $s_n := w_{n-1}^{-1} w_n$, which by construction is the n th projection $s_n : (G, \mu)^{\mathbb{N}} \rightarrow G$. It follows that the steps are independent and identically distributed G -valued random variables with law μ . Each outcome $(w_n(\omega))_{n \in \mathbb{N}}$ is a sequence of elements of G , and is referred to as a *sample path*.

The map $G^{\mathbb{N}} \rightarrow G^{\mathbb{N}}$ defined $\omega \mapsto (w_n(\omega))_{n \in \mathbb{N}}$ induces a pushforward probability measure on the codomain (range) $G^{\mathbb{N}}$. The codomain endowed with this pushforward measure is called variously the path space, location space, or Kolmogorov representation space [Saw97, §2]. One can alternatively define the path space first and then take each w_n to be the n th projection from the path space [Mah12, §2.1].

Reflected random walk

Given a group G and a probability measure μ , we define the *reflected probability measure* $\check{\mu}(g) := \mu(g^{-1})$. The $(G, \check{\mu})$ -random walk is sometimes called the *reflected random walk* $(\check{w}_n)_{n \in \mathbb{N}}$.

The bi-infinite (G, μ) -random walk $(w_n)_{n \in \mathbb{Z}}$ is defined to be the unique bi-infinite sequence of G -valued random variables with $w_0 \equiv 1_G$ and steps $s_n := w_{n-1}^{-1} w_n$ that are independently and identically distributed according to μ . It follows that w_n equals $s_1 \cdots s_n$ and $s_0^{-1} \cdots s_{n+1}^{-1}$ for positive and negative integers n , respectively. In particular, the sequence $(w_{-n})_{n \in \mathbb{N}}$ is a random walk with the same distribution as the $(G, \check{\mu})$ -random walk.

Definition 2 (*a*-iterated random walk). Given a random walk $(w_n)_{n \in \mathbb{N}}$ and positive integer a , the *a-iterated random walk* is the sequence of G -valued random variables $(w_{ai})_{i \in \mathbb{N}}$ indexed by integers $i > 0$.

Definition 3 (Exponential tail). The (G, μ) -random walk w_n on the metric space (X, d_X) with basepoint x_0 is said to have *exponential tail* in X if there exists $\lambda > 0$ such that

$$\sum_{g \in G} \mu(g) e^{\lambda |g|} < \infty, \quad (18)$$

where $|g|$ denotes $d_X(x_0, gx_0)$. Some sources refer to exponential tail as finite exponential moment.

More generally, a real-valued stochastic process Y_n is said to have *uniformly exponential tails* in $\mathbb{R}$ if there exists a single $\lambda > 0$ such that for all n ,

$$\mathbb{E}(e^{\lambda|Y_n|}) < \infty.$$

Note that there need not be a uniform upper bound for all n .

If Y_n has uniformly exponential tails in $\mathbb{R}$, then so does $-Y_n$. If μ satisfies (18) for some $\lambda > 0$ then so does every convolution power μ^n for the same λ . Indeed, if $\sum_g \mu(g)e^{\lambda|g|} = \ell$ finite, then

$$\begin{aligned} \ell^2 &= \sum_a \mu(a)e^{\lambda|a|} \ell \\ &= \sum_{a,b} \mu(a)\mu(b)e^{\lambda|a|}e^{\lambda|b|} \end{aligned}$$

(recall that $|a|$ and $|b|$ denote $d_X(x_0, ax_0)$ and $d_X(x_0, bx_0)$, respectively)

$$\geq \sum_{a,b} \mu(a)\mu(b)e^{\lambda|ab|}$$

(since $d_X(a^{-1}x_0, x_0) + d_X(x_0, bx_0) \geq d_X(a^{-1}x_0, bx_0)$ by the triangle inequality)

$$\begin{aligned} &= \sum_{a,g} \mu(a)\mu(a^{-1}g)e^{\lambda|g|} \\ &= \sum_g \mu^2(g)e^{\lambda|g|}. \end{aligned}$$

Thus $Y_n = |w_n|$ has uniformly exponential tails if μ has exponential tail.

Definition 4 (Positive linear progress). We say that the (G, μ) -random walk

w_n on the metric space (X, d_X) with basepoint x_0 makes *linear progress with exponential decay* in X if there is $C > 0$ so that for all n ,

$$\mathbb{P}(|w_n| \leq n/C) \leq Ce^{-n/C}, \quad (19)$$

makes *positive linear progress* (or has *positive drift*) in X if

$$\liminf_{n \rightarrow \infty} |w_n|/n > 0 \text{ almost surely,} \quad (20)$$

is *asymptotically probability zero* on bounded sets in X if for every r ,

$$\mathbb{P}(|w_n| \leq r) \xrightarrow{n} 0 \quad (21)$$

[MT16, §5.2, p. 40], and is *not positive-recurrent* on bounded sets in X if

$$\mathbb{E}(|w_n|) \xrightarrow{n} \infty, \quad (22)$$

where $|w_n|$ denotes $d_X(x_0, w_n x_0)$. Recall that a Markov chain is either transient, positive-recurrent, or neither (null-recurrent) [Woe00, I.1.B]. For linear progress with exponential decay, it suffices to show that there exist $a, b, c > 0$ so that for all but finitely many n , the probability $\mathbb{P}(|w_n| \leq n/a) \leq$ a finite sum $\sum_i b_i e^{-n/c_i}$.

More generally, a (not necessarily Markov) sequence of real-valued random variables Z_n makes linear progress with exponential decay in $\mathbb{R}$ if there is $C > 0$ so that for all n , $\mathbb{P}(Z_n \leq n/C) \leq Ce^{-n/C}$; see equation (1). Properties (20)–(22)

generalize to Z_n in $\mathbb{R}$ by replacing $|w_n|$ with $|Z_n|$.

Remark 5 (Progress and recurrence). Equations (19)–(22) are in descending order of strength:

$$\begin{array}{ccccccc} \text{linear progress with} & \implies & \text{positive linear} & \implies & \text{asymptotically} & \implies & \text{not positive-} \\ \text{exponential decay} & & \text{progress} & & \text{probability zero} & & \text{recurrent.} \end{array}$$

The first claim is that linear progress with exponential decay is indeed a special case of positive linear progress. For any sequence of events A_n , the Borel-Cantelli Lemma states that $\sum_n \mathbb{P}(A_n) < \infty$ implies $\mathbb{P}(\limsup_n A_n) = 0$. Exponential decay implies $\sum_n \mathbb{P}(|Z_n| \leq n/C)$ is finite, and so by Borel-Cantelli, $|Z_n| \leq n/C$ for only finitely many n almost surely. Thus, $\liminf_n \frac{1}{n}|Z_n| \geq 1/C$ almost surely.

The second claim is that positive linear progress in turn implies bounded sets have zero asymptotic probability. To show the contrapositive, assume there exists an r such that the sequence of events $A_n := \{|Z_n| \leq r\}$ has probabilities uniformly bounded away from zero $\liminf_n \mathbb{P}(A_n) > 0$. Then there is an $\epsilon > 0$ and a subsequence n_k such that $\inf_k \mathbb{P}(A_{n_k}) \geq \epsilon$. It follows that $\mathbb{P}(\limsup_k A_{n_k}) \geq \epsilon$, i.e., the subsequence of events A_{n_k} occurs infinitely often with probability $\geq \epsilon$. But then so does the full sequence: $\mathbb{P}(\limsup A_n) \geq \epsilon$. The event $\limsup A_n$ implies (is contained in) the event $\liminf_n \frac{1}{n}|Z_n| = 0$, which thus also occurs with probability $\geq \epsilon$. In other words, it is not the case that $\liminf_n \frac{1}{n}|Z_n| > 0$

almost surely.

The last claim is that Z_n cannot be both positive-recurrent and asymptotically probability zero on bounded subsets in $\mathbb{R}$. Suppose Z_n is positive-recurrent: $\sup_n \mathbb{E}(|Z_n|) \leq L$ finite. By Markov's inequality, $\mathbb{P}(|Z_n| < 3L) \geq 2/3$. Thus $\mathbb{P}(|Z_n| < r)$ does not $\xrightarrow{n} 0$ for every r .

2.3 Shadows

Consider the metric space (X, d_X) and the group of isometries $G \rightarrow \text{Isom } X$. Fix a basepoint $x_0 \in X$ and put $|g| = d_X(x_0, gx_0)$. For each g in G and real number r , the shadow about gx_0 of radius r is defined to be the set

$$\text{Shad}(g, r) := \{w \in G : (w, g)_1 \geq |g| - r\}.$$

For each real number d , we define the *shadow about gx_0 of depth d* to be the set

$$S(g, d) := \{w \in G : (w, g)_1 \geq d\};$$

in other words, $w \in S(g, d)$ if and only if $(w, g)_1 \geq d$. Note that it is possible to define shadows with basepoints other than x_0 , but we will not need such shadows. Most sources define shadows in terms of radius $\text{Shad}(g, r)$. However, following Maher [Mah12, §2.3], we will always define shadows in terms of depth $S(g, d)$. The two definitions are equivalent $S(g, d) = \text{Shad}(g, |g| - d)$, and shadows in terms of depth will be more convenient for our purposes, including

Definition 6 below.

By the triangle inequality (11), if d is negative or zero, then $S(g, d) = G$, and if d exceeds $|g|$, then $S(g, d)$ is empty.

Definition 6 (Shadow decay). We say that *shadows decay in X* if both measures $\mu^n(S(g, d))$ and $\check{\mu}^n(S(g, d)) \xrightarrow{d} 0$ in the sense that

$$\lim_{d \rightarrow \infty} \sup_{n, g} \mathbb{P}(w_n \in S(g, d)) = \lim_{d \rightarrow \infty} \sup_{n, g} \mathbb{P}(w_n^{-1} \in S(g, d)) = 0,$$

where we take each supremum over all $n \in \mathbb{N}$ and $g \in G$. In other words there is a function f that decays $f(d) \rightarrow 0$ as $d \rightarrow \infty$ such that $\mu^n(S(g, d)) \leq f(d)$ for all n, g , and there is an analogous function f' for $\check{\mu}$. Note that if f, f' differ, we may replace them with $\max\{f, f'\}$. However, the optimal f for μ and f' for $\check{\mu}$ may differ. See Maher and Tiozzo [MT16, Corollary 5.3].

It is known [Mah12, Lemma 2.10; MT16, Equation 16] that when the support of μ is bounded in X , shadows decay exponentially in d , meaning that there is some constant $C > 0$ so that for every d and n , $\mu^n(S(g, d)) \leq Ce^{-d/C}$. Crucially, C does not depend on n (or d).¹ However, the proof in this paper will not use exponential decay of shadows.

¹ Exponential decay of shadows in d for fixed n follows immediately from exponential tail for μ . Indeed, take $\lambda > 0$ so that $\sum_g \mu(g)e^{\lambda|g|}$ is finite. Then

$$\mathbb{P}((w_n, g)_1 \geq d/\lambda) \leq \mathbb{P}(|w_n| \geq d/\lambda) = \mathbb{P}(|w_n| \geq d/\lambda) = \mathbb{P}(e^{\lambda|w_n|} \geq e^d) \leq \mathbb{E}(e^{\lambda|w_n|})e^{-d}$$

where the first inequality follows from $(w_n, g)_1 \leq |w_n|$ (from the triangle inequality) and the last inequality follows from Markov's Inequality.

Chapter 3

Proofs

The results needed for Theorem 1 are organized as follows. Section 3.1 cites the convergence theorem for nonelementary, weakly hyperbolic random walks and extracts positive linear progress and shadow decay.

Section 3.2 uses these two properties along with hyperbolicity to prove a stronger shadow decay result (uniform shadow decay). Section 3.3 uses this shadow decay result along with exponential tails to prove a property concerning horofunctions (uniformly positive horofunctions).

Section 3.4 uses that property to derive linear progress with exponential decay for the a -iterated random walk by way of a progress Criterion (uniformly positive progress). Lastly, Section 3.5 uses exponential tails again to pass from

the a -iterated random walk to the full random walk.

$$\begin{array}{ccccccc}
 \begin{array}{c} \text{non-el} \\ \text{wk-hyp} \\ (\text{Def 7}) \end{array} & \xrightarrow{[\text{MT16}]} & \begin{array}{c} \text{pos lin prog} \\ \& \text{shad decay} \\ (\text{Def 4 \& 6}) \end{array} & \xrightarrow[\& \text{hyp}]{\text{Lem 10}} & \begin{array}{c} \text{unif shad} \\ \text{decay} \\ (\text{Crit 9}) \end{array} & \xrightarrow[\& \text{exp tail}]{\text{Lem 12}} & \begin{array}{c} \text{unif pos} \\ \text{horofns} \\ (\text{Crit 11}) \end{array} \\
 & \xRightarrow{\text{Lem 14}} & \begin{array}{c} \text{unif pos} \\ \text{progress} \\ (\text{Crit 13}) \end{array} & \xRightarrow{\text{Prop 15}} & \begin{array}{c} a\text{-iter lin prog} \\ \text{exp decay} \\ (\text{Eq 4}) \end{array} & \xrightarrow[\& \text{exp tail}]{\text{Cor 17}} & \begin{array}{c} \text{lin prog} \\ \text{exp decay} \\ (\text{Def 4}) \end{array}
 \end{array}$$

Our use of metric hyperbolicity is confined to Sections 3.1 and 3.2.

3.1 Convergence

We cite the convergence of nonelementary, weakly-hyperbolic random walks to obtain positive linear progress and shadow decay.

Definition 7 (Nonelementary, weakly hyperbolic). Let X be a metric space, not necessarily locally compact. A countable group of isometries $G \rightarrow \text{Isom } X$ is called

- *weakly hyperbolic* if X is separable, geodesic, and δ -hyperbolic (9), and
- *nonelementary in X* if G contains two independent loxodromics for X ,

where loxodromics (12) are called independent (14) if they have disjoint fixed point sets on ∂X , the Gromov boundary [MT16, §1; Osi16, Theorem 1.1].

The (G, μ) -random walk is called *nonelementary in X* if there are $g, h \in G$ and $k > 0$ such that g^k, h^k are independent loxodromics in X . Our definition of nonelementary follows Maher and Tiozzo [MT16, §1] and excludes the quasiparabolic case (see Section 2.1, page 15).

Theorem 8 (Boundary convergence [MT16, 1.1, 1.2, 5.3]). *If the random walk w_n on X is nonelementary and weakly hyperbolic (Definition 7), then $w_n x_0$ converges to the Gromov boundary almost surely, and the associated hitting measure is non-atomic. In particular, w_n makes positive linear progress in X (Definition 4) and shadows in X decay in d (Definition 6).*

Note that we do not require any moment or tail conditions on μ , we do not require X to be locally compact, and we do not require the action of G to be acylindrical.

3.2 Shadows

In this section we use convergence to the boundary (specifically, positive linear progress and shadow decay) along with hyperbolicity to prove a stronger shadow decay result.

In general, $2(g, w)_1$ equals $d_X(gx_0, x_0) + d_X(x_0, wx_0) - d_X(gx_0, wx_0)$, the difference between distance and displacement when one travels along a concatenation of geodesics $[gx_0, x_0] \cup [x_0, wx_0]$. Thus the Gromov product $(g, w)_1$ measures the inefficiency in traveling gx_0 to x_0 to wx_0 , and so shadow decay controls the inefficiency in a random walk as it escapes to infinity, even when X is not hyperbolic.

If X is hyperbolic, then this inefficiency looks like a “backtrack.” In this

section, we use hyperbolicity to prove the two “single backtracks” in Figure 8 imply the one “double backtrack” in Figure 2.

After this section, we will not use hyperbolicity for any of the remaining proofs.

Criterion 9 (Uniform shadow decay). We say that *shadows in X decay uniformly* if $\mathbb{P}((w_k h, g)_1 \geq d) \xrightarrow{d, k} 0$ in the sense that for all $\epsilon > 0$, for every depth $d \geq \text{some } d'(\epsilon)$, and for every time $k \geq \text{some } k'(d)$,

$$\sup_{g, h \in G} \mathbb{P}(w_k h \in S(g, d)) < \epsilon. \quad (23)$$

Note that for $n \geq k$, the random variables w_k and $w_k^{-1}w_n$ are independent, and so $\mathbb{P}(w_k h \in S(g, d))$ is equal to $\mathbb{P}(w_n \in S(g, d) \mid w_k^{-1}w_n = h)$. Hence equation (23) means that shadows decay independently from the value of $w_k^{-1}w_n$, and in particular, independently from the distance $|w_k^{-1}w_n|$. For n much larger than k , this distance $|w_k^{-1}w_n|$ approximates progress $|w_n|$.

Lemma 10 (Uniform shadow decay). *Consider the (G, μ) -random walk w_n on the metric space X with basepoint x_0 . Suppose X is δ -hyperbolic, bounded subsets have asymptotic probability zero (Definition 4), and shadows decay (Definition 6). Then shadows decay uniformly (Criterion 9).*

Note that positive linear progress implies that bounded subsets have asymptotic probability zero (Remark 5). We do not assume the action of G to be

Figure 8: In the geometric step of the proof of Lemma 10, we assume shadow decay (25, 26) and positive linear progress (24). Figure 2 depicts the corresponding claim.

acylindrical.

Proof. Hyperbolic geometry step. Recall that we write $(w, g)_1$ to denote $\frac{1}{2}|w| + \frac{1}{2}|g| - \frac{1}{2}d_X(wx_0, gx_0)$. As depicted in Figure 8, suppose we are given g, w, h in G and $d > 0$ such that

$$|w| > 2d, \quad (24)$$

$$(w, g)_1 < d - \delta, \text{ and} \quad (25)$$

$$(w^{-1}, h)_1 < d. \quad (26)$$

As in Figure 2, we claim that $(wh, g)_1 < d$.

By G -equivariance of the Gromov product, $(w^{-1}, h)_1$ equals $(1, wh)_w$ and so (26) indeed controls the backtrack at the bottom in Figure 8. Note that equations (25) and (26) are equivalent to $w \notin S(g, d - \delta)$ and $w^{-1} \notin S(h, d)$, respectively, and that the claim is equivalent to $wh \notin S(g, d)$.

We want to bound how far wh and g can fellow-travel. By the Gromov four-

point condition for hyperbolicity (Figure 5), wh cannot fellow-travel both w and g beyond distance $(w, g)_1 + \delta$. That distance is in turn bounded $(w, g)_1 + \delta < d$ by assumption (25). Putting the two together, we have that

$$(wh, w)_1 \text{ or } (wh, g)_1 \stackrel{(10)}{\leq} (w, g)_1 + \delta \stackrel{(25)}{<} d. \quad (27)$$

Therefore if we can show $(wh, w)_1 > d$, then $(wh, g)_1$ must be $< d$. But by our other two assumptions (24, 26) and the definition of Gromov product,

$$\begin{aligned} d &\stackrel{(24, 26)}{<} |w| - (w^{-1}, h)_1 \\ &= |w| - \frac{1}{2}|w| - \frac{1}{2}|h| + \frac{1}{2}|wh| \\ &= \frac{1}{2}|w| - \frac{1}{2}|h| + \frac{1}{2}|wh| \\ &= (wh, w)_1. \end{aligned} \quad (28)$$

Therefore $(wh, g)_1 < d$, as claimed.

In words, the calculation in (28) shows that w is too long (24) for the subsequent short backtrack (26) to cancel out, and thus wh fellow-travels w instead of g . See also Figure 9.

Probabilistic step

Let $\epsilon > 0$. We claim that we can find sufficiently large real $d > 0$ and integer $k > 0$ so that all three equations (24)–(26) hold simultaneously for $w = w_k$ for all g, h in G with probability $> 1 - \epsilon$.

It turns out we must first choose d and then choose k depending on d (hence

Figure 9: Equation (27) means in words that the geodesics between four points in a hyperbolic space look approximately like a tree in one of three possible configurations, and so showing that wh branches off w rules out the case where wh branches off g .

the order of quantifiers in Criterion 9). By decay of shadows (Definition 6), we can find a large real $d' > 0$ (depending only on μ and ϵ) so that for all real $d \geq d'$, integer $k > 0$, and g, h in G , the two events

$$\{(w_k, g)_1 < d - \delta\} \quad (29)$$

$$\{(w_k^{-1}, h)_1 < d\} \quad (30)$$

each occur with probability greater than $1 - \epsilon/3$. These two events (29) and (30) each occur when w_k and w_k^{-1} avoid $S(g, d - \delta)$ and $S(h, d)$, respectively. Fix arbitrary $d \geq d'$. Now using our zero asymptotic probability hypothesis (Definition 4), we can find a large integer $k' > 0$ (depending on d) so that for all integers $k \geq k'$, the event

$$A := \{|w_k| > 2d\}$$

has $\mathbb{P}(A) > 1 - \epsilon/3$. Fix arbitrary $k \geq k'$. Since (29) and (30) hold for all $n \in \mathbb{N}$ and in particular for $n = k$, therefore for all g, h in G ,

$$B := \{(w_k, g)_1 < d - \delta\}$$

$$C := \{(w_k^{-1}, h)_1 < d\}$$

have $\mathbb{P}(B) > 1 - \epsilon/3$ and $\mathbb{P}(C) > 1 - \epsilon/3$. Note that these events are defined using our fixed constants r and k . It follows (regardless of dependence) that the intersection of these events occurs with probability $\mathbb{P}(A \cap B \cap C) > 1 - \epsilon$, as claimed. Crucially, d and k do not depend on g or h .

Final step

For all g, h in G , the Gromov product $(w_k h, g)_1 < d$ whenever the three events A, B, C all occur (Geometric step), which is more than $1 - \epsilon$ of the time (Probabilistic step). Therefore $w_k h \notin S(g, d)$ more than $1 - \epsilon$ of the time, and so $\mathbb{P}(w_k h \in S(g, d)) < \epsilon$, as desired. $\square$

3.3 Horofunctions

In this section, we use the result of the previous one (Criterion 9) to prove a result concerning horofunctions (Criterion 11). We do not use hyperbolicity in these remaining Sections 3.3–3.5.

Criterion 11 (Uniformly positive horofunctions). The (G, μ) -random walk w_n on (X, d_X, x_0) is said to have *uniformly positive horofunctions* by time n_0 in X

if for all $n \geq n_0$ and all positive $t \leq$ some $t_0(n)$,

$$\sup_{g \in G} (\mathbb{E}(e^{-t\rho_g(w_n)})) < 1.$$

In other words, $\mathbb{E}(e^{-t\rho_g(w_n)}) \leq 1 - \epsilon$ where $\epsilon > 0$ depends on n and t but not g .

Lemma 12 (Uniformly positive horofunctions [cf. MS15, 9.1]). *Suppose the (G, μ) -random walk w_n has exponential tail and is not positive-recurrent in X (Definitions 3 and 4). Then uniform shadow decay in X implies uniformly positive horofunctions in X (Criteria 9 and 11).*

We do not require the action of G to be acylindrical, nor do we require X to be a priori hyperbolic. However, uniform shadow decay is a consequence of hyperbolicity by Lemma 10.

The following proof is adapted from Mathieu and Sisto [MS15, Theorem 9.1].

Proof. Step Zero. It will help guide the proof to think of the goal as being to show

$$\text{“Goal”}: \quad \mathbb{E}(\rho_g(w_n)) > 0.$$

Indeed, if the horofunction $\rho_g(w_n)$ for a given g in G and n in $\mathbb{N}$ has positive expectation, then

$$\frac{d}{dt} \mathbb{E}(e^{-t\rho_g(w_n)})|_{t=0} = \mathbb{E}(-\rho_g(w_n)) < 0,$$

since $\frac{d}{dt}\mathbb{E}(e^{tY})|_{t=0} = \mathbb{E}(Y)$ in general.¹ It would then follow that $\mathbb{E}(e^{-t\rho_g(w_n)}) < 1$ for sufficiently small $t > 0$.

For arbitrary g and w_n in G ,

$$\rho_g(w_n) \geq -|w_n| \text{ always,} \quad (31)$$

since $|w_n| + \rho_g(w_n) = d_X(x_0, w_n x_0) + d_X(w_n x_0, g x_0) - d_X(g x_0, x_0)$, which is non-negative by the triangle inequality. Then the strategy for obtaining $\mathbb{E}(\rho_g(w_n))$ positive is to name an event that (1) is highly likely, and (2) implies $\rho_g(w_n)$ is highly positive. In the event w_n lies outside some shadow $S(g, d)$, from the definition of horofunction and shadow, $|w_n| - \rho_g(w_n) = 2(w_n, g)_1 < 2d$. So up to some constant d ,

$$\rho_g(w_n) \approx |w_n| \text{ when } w_n \notin S(g, d). \quad (32)$$

The shadow $S(g, d)$ is unlikely to contain w_n for large d under the assumption of shadow decay; however, we will need the stronger property of uniform shadow decay (obtained from Lemma 10) for the following reason.

¹ For each g in G and integer n , the real-valued random variable $\rho_g(w_n)$ has a lower bound (depending on g and n). It follows that $e^{-t\rho_g(w_n)}$ has finite expectation for all positive t , integer n , and g in G . In other words, we get a one-sided moment generating function for free. We also get a one-sided derivative for each moment generating function $\mathbb{E}(e^{-t\rho_g(w_n)})$.

However, since we want to bound the whole family of moment generating functions, it does not suffice to first differentiate and then bound the derivative $\mathbb{E}(\rho_g(w_n))$. Instead, we must reverse the order: in Step Three (Estimation) below, we first construct an estimate $f(K, N, t)$ whose expectation bounds $\mathbb{E}(e^{-t\rho_g(w_n)})$ from above. Since we assume μ has exponential tail, this $f(K, N, t)$ will be finite and differentiable with respect to t in an open neighborhood about zero.

We work with $e^{-t\rho_g(w_n)}$ instead of $e^{t\rho_g(w_n)}$ so that our estimates hold for $t \geq 0$.

Suppose we have a random *estimate* and *event* so that $\rho_g(w_n) \geq -(\text{estimate})$ in general, and $\rho(w_n) \geq (\text{estimate})$ in the case that the event occurs. Then

$$\begin{aligned}\rho_g(w_n) &= \rho_g(w_n)(1 - \mathbb{1}_{\text{event}}) + \rho_g(w_n)(\mathbb{1}_{\text{event}}) \\ &\geq -(\text{estimate})(1 - \mathbb{1}_{\text{event}}) + (\text{estimate})(\mathbb{1}_{\text{event}})\end{aligned}$$

where $\mathbb{1}_{\text{event}}$ denotes the characteristic function, i.e., the random variable which takes the value 1 for outcomes (sample paths) in the *event* and the value 0 otherwise. Then

$$\begin{aligned}\mathbb{E}(\rho_g(w_n)) &\geq \mathbb{E}((\text{estimate})(2\mathbb{1}_{\text{event}} - 1)) \\ &= \mathbb{E}((\text{estimate})\mathbb{E}(2\mathbb{1}_{\text{event}} - 1 \mid \text{estimate})),\end{aligned}$$

since $\mathbb{E}(YZ) = \mathbb{E}(\mathbb{E}(YZ \mid Y)) = \mathbb{E}(Y\mathbb{E}(Z \mid Y))$ for arbitrary random variables Y and Z . Thus we want to show not only that $\mathbb{P}(\text{event})$ is close to 1, but that $\mathbb{P}(\text{event} \mid \text{estimate} = h)$ is close to 1 for all h . The former does not suffice because the *event* and *estimate* may be correlated—we do not know a priori that the *event* is not somehow less likely when the *estimate* is large. The latter will give us just enough independence to establish the lemma.

We will not use $|w_n|$ as our *estimate*, since $\mathbb{P}(w_n \notin S(g, d) \mid w_n = h)$ equals either 0 or 1 depending on whether or not h is in $S(g, d)$. In other words, we cannot bound $\mathbb{P}(w_n \notin S(g, d) \mid w_n = h)$ near 1 for all $h \in G$ because the estimate $|w_n|$ contains too much information about the event $w_n \notin S(g, d)$. Hence we pass to an estimate with less information with the following adjustment.

By the assumption that the (G, μ) -random walk is not positive-recurrent, the distance $|w_n|$ has much greater expectation than $|w_k|$ for $n \gg k$, and so

$$|w_n| \approx |\bar{w}_k^{-1} w_n|. \quad (33)$$

Putting (31)–(33) together, the strategy becomes to find d, k so that for all n, g ,

$$\rho_g(w_n) \gtrsim -|\bar{w}_k^{-1} w_n| \text{ always, and}$$

$$\rho_g(w_n) \approx -|\bar{w}_k^{-1} w_n| \text{ when } w_n \notin S(g, d).$$

More precisely, let $K = |w_k|$, let $N = |\bar{w}_k^{-1} w_n|$, and let $A = \mathbb{1}\{w_n \notin S(g, d)\}$.

Step One (Metric Geometry) shows in (38) that

$$\text{claim: } \begin{cases} \rho_g(w_n) \geq -K - N \text{ always, and} \\ \rho_g(w_n) \geq -K + N - 2d \text{ when } w_n \notin S(g, d). \end{cases}$$

Then Step Two (Probability) uses uniform shadow decay to find a sufficiently large integer k and real d so that (40) for all g in G and $n \geq k$,

$$\text{claim: } \mathbb{E}(1 - A \mid N) < 0.01.$$

Lastly, having fixed such d and k , Step Three (Estimation) constructs in (43) a random variable $f(K, N, t)$ that is finite in a neighborhood of $t = 0$ and finds a constant $n_0 \geq k$ so that

$$\text{claim: } \begin{cases} \mathbb{E}(e^{-t\rho_g(w_n)}) \leq \mathbb{E}(f(K, N, t)) \text{ for all } t \geq 0 \text{ and } g \text{ and } n \geq k, \\ f(K, N, 0) \equiv 1 \text{ for all } n \geq k, \text{ and} \\ \frac{d}{dt} \mathbb{E}(f(K, N, t))|_{t=0} < 0 \text{ for all } n \geq n_0. \end{cases}$$

Crucially, $f(K, N, t)$ will not depend on g . As depicted in Figure 10, it will then

follow that each $n \geq n_0$ has some $t_0(n)$ such that for all positive $t \leq t_0$, the family

Figure 10: We bound for all g in G , positive t , and $n \geq k$, $\mathbb{E}(e^{-t\rho_g(w_n)}) \leq \mathbb{E}(f(K, N, t))$ in Step Three (Estimation). Moreover, $f(K, N, t)$ is defined without using g , is finite in a neighborhood of $t = 0$, evaluates to one at $t = 0$, and has derivative $\frac{d}{dt}\mathbb{E}(f(K, N, t))|_{t=0}$ that is negative for all $n \geq$ some constant n_0 . It then follows that $\sup_g(\mathbb{E}(e^{-t\rho_g(w_n)})) \leq \mathbb{E}(f(K, N, t)) < 1$ for all $n \geq n_0$ and all positive $t \leq$ some $t_0(n)$.

of moment generating functions is bounded $\sup_g \mathbb{E}(e^{-t\rho_g(w_n)}) \leq \mathbb{E}(f(K, N, t)) < 1$, as desired.

Step One (Metric geometry)

From the definitions of horofunction and shadow,

$$|w_n| - \rho_g(w_n) = 2(w_n, g)_1 < 2d \quad \text{when } w_n \notin S(g, d). \quad (34)$$

By the triangle inequality:

$$|w_n| + \rho_g(w_n) = d_X(x_0, w_n x_0) + d_X(w_n x_0, g x_0) - d_X(g x_0, x_0) \geq 0; \quad (35)$$

$$K + |w_n| - N = d_X(w_k x_0, x_0) + d_X(x_0, w_n x_0) - d_X(w_n x_0, w_k x_0) \geq 0; \quad (36)$$

$$K + N - |w_n| = d_X(x_0, w_k x_0) + d_X(w_k x_0, w_n x_0) - d_X(w_n x_0, x_0) \geq 0. \quad (37)$$

Figure 11: Define the random variables K, N as above and define A to be the event $w_n \in S(g, d)$, that is, $(w_n, g)_1 \geq d$.

Figure 12: The Gromov product is non-negative in every metric space regardless of hyperbolicity: $2(y, z)_x = b + c - a \geq 0$.

Combining equations (34)–(37) we get

$$\begin{aligned}
 \rho_g(w_n) &\stackrel{(35)}{\geq} -|w_n| && \stackrel{(37)}{\geq} -K - N && \text{always, and} \\
 \rho_g(w_n) &\stackrel{(34)}{>} |w_n| - 2d && \stackrel{(36)}{\geq} -K + N - 2d && \text{when } w_n \notin S(g, d).
 \end{aligned} \tag{38}$$

We will refer to these two bounds $-K - N$ and $-K + N - 2d$ as the “weak”

and “strong” bounds of equation (38), respectively.

Step Two (Probability)

Fix k and d large enough that

$$\mathbb{P}(w_k h \in S(g, d)) < 0.01 \quad \text{for all } g, h \text{ in } G$$

Figure 13: As an aside, these trees are configurations that minimize $\rho_g(w_n)$ in general and in the case $w_n \notin S(g, d)$, respectively, whence an alternate derivation of (38). For example, if the “worst case in general” geodesic tree depicted connects $x_0, w_k x_0, w_n x_0, g x_0$, then $\rho_g(w_n) = -K - N$ by Figure 7. If no such geodesic tree exists, then $\rho_g(w_n) > -K - N$. In particular, we do not assume such geodesic trees exist, and we do not assume X is δ -hyperbolic.

using uniform shadow decay, Criterion 9. This explicit 0.01 bound for a single pair k, d will be good enough for our estimate in (43). Since w_k and $\bar{w}_k^{-1} w_n$ are independent, thus $\mathbb{P}(w_k \bar{w}_k^{-1} w_n \in S(g, d) \mid \bar{w}_k^{-1} w_n = h) < 0.01$ for all g, h in G and $n \geq k$. We have defined $A = \mathbb{1}\{w_n \notin S(g, d)\}$, and it follows that

$$\mathbb{E}(1 - A \mid \bar{w}_k^{-1} w_n) < 0.01 \quad \text{for all } g \text{ in } G \text{ and } n \geq k. \quad (39)$$

This bound means that no matter what value h the random variable $\bar{w}_k^{-1} w_n$ takes, the conditional probability $\mathbb{P}(w_n \in S(g, d) \mid \bar{w}_k^{-1} w_n = h) < 0.01$. Note that the conditional expectation $\mathbb{E}(1 - A \mid \bar{w}_k^{-1} w_n)$ is a real-valued random variable taking values in the interval from 0 to 1 depending on the (random) value of $\bar{w}_k^{-1} w_n$. Equation (39) bounds the entire range of outcomes for this random variable.

We cannot make a similar bound on $\mathbb{E}(1 - A \mid w_n)$ because w_n contains too

much information, and so conditional expectation $\mathbb{E}(1 - A \mid w_n) = 1 - A = 0$ or 1 depending on whether or not $w_n \in S(g, d)$.

Lastly, since N contains strictly less information than $w_k^{-1}w_n$ in the sense that N completely depends on $w_k^{-1}w_n$, thus

$$\mathbb{E}(1 - A \mid N) = \mathbb{E}(\mathbb{E}(1 - A \mid w_k^{-1}w_n) \mid N) < 0.01 \quad (40)$$

for all g in G and $n \geq k$.

Step Three (Estimation)

To obtain the g -independent bound in Criterion 11, we must work directly with the moment generating function first before passing to the first moment.

Recall that $K = |w_k|$, $N = |w_k^{-1}w_n|$, and $A = \mathbb{1}\{w_n \notin S(g, d)\}$. We get to apply $-K + N - 2d < \rho_g(w_n)$ the “strong” bound in equation (38) for the sample paths where $A = 1$. We fall back on the “weak” bound $-K - N \leq \rho_g(w_n)$ for the sample paths where $A = 0$. Hence for all real $t \geq 0$, integer $n \geq k$, and g in G ,

$$\begin{aligned} e^{-t\rho_g(w_n)} &\stackrel{(38)}{\leq} e^{-t(-K+N-2d)}A + e^{-t(-K-N)}(1-A) \\ &= e^{t(K-N+2d)} + \left(e^{t(K+N)} - e^{t(K-N+2d)}\right)(1-A) \\ &= e^{t(K-N+2d)} + \left(e^{tN} - e^{t(-N+2d)}\right)e^{tK}(1-A). \end{aligned}$$

This upper bound on $e^{-t\rho_g(w_n)}$ means that the horofunction $\rho_g(w_n)$ is no

less than $-K + N - 2d$, except when w_n lies in $S(g, d)$, in which case we must weaken the bound from $-K + N - 2d$ to $-K - N$.

Taking expectation of both sides, it follows that for all real $t \geq 0$, integer $n \geq k$, and g in G ,

$$\begin{aligned} \mathbb{E}(e^{-t\rho_g(w_n)}) &\leq \mathbb{E}\left(e^{t(K-N+2d)} + (e^{tN} - e^{t(-N+2d)})e^{tK}(1-A)\right) \\ &= \mathbb{E}\left(e^{t(K-N+2d)} + (e^{tN} - e^{t(-N+2d)})\mathbb{E}(e^{tK}(1-A) \mid N)\right) \end{aligned} \quad (41)$$

since $\mathbb{E}(\phi(Y)Z) = \mathbb{E}(\mathbb{E}(\phi(Y)Z \mid Y)) = \mathbb{E}(\phi(Y)\mathbb{E}(Z \mid Y))$ for any arbitrary function ϕ and random variables Y, Z , regardless of dependence. By exponential tail (Definition 3), each term on the right is finite for t in an open neighborhood of zero depending on (i.e., not necessarily uniform in) n .

We want to apply our Step Two bound (40) to $\mathbb{E}(e^{tK}(1-A) \mid w_k^{-1}w_n)$. By Cauchy-Schwarz, the fact that characteristic functions equal their squares, and (40), it follows that for all real $t \geq 0$, integer $n \geq k$, and g in G ,

$$\begin{aligned} \mathbb{E}(e^{tK}(1-A) \mid N) &\stackrel{\text{C-S}}{\leq} \sqrt{\mathbb{E}(e^{2tK} \mid N)\mathbb{E}((1-A)^2 \mid N)} \\ &\stackrel{(40)}{\leq} \sqrt{\mathbb{E}(e^{2tK} \mid N)} 0.1 \end{aligned}$$

which, since K and N are independent,

$$= \sqrt{\mathbb{E}(e^{2tK})} 0.1$$

which, since t and K are non-negative,

$$\leq \mathbb{E}(e^{2tK})0.1. \quad (42)$$

We are looking to bound (41) from above using $\mathbb{E}(e^{tK}(1-A) \mid N) \leq \mathbb{E}(e^{2tK})0.1$, equation (42). Doing so requires the relevant term in (41) to be non-negative. Hence we make a technical adjustment, replacing $e^{tN} - e^{t(-N+2d)}$ with the strictly larger $e^{t(N+2d)} - e^{t(-N+2d)}$. Only the latter is non-negative (for $t \geq 0$) on all sample paths. Without this adjustment, the sign and inequality flip for those sample paths where N is small enough relative to d to make the “weak” bound $-K - N$ stronger than the “strong” bound $-K + N - 2d$.

Putting it all together, we find that for all real $t \geq 0$, integer $n \geq k$, and g in G ,

$$\mathbb{E}\left(e^{-t\rho_g(w_n)}\right) \leq \mathbb{E}\left(\underbrace{e^{t(K-N+2d)} + \left(e^{t(N+2d)} - e^{t(-N+2d)}\right)e^{2tK}}_{=:f(K,N,t)}0.1\right), \quad (43)$$

where we can combine in a single expectation all terms and factors by linearity and independence of K, N respectively. We define the righthand side of equation (43) to be our estimate $\mathbb{E}(f(K, N, t))$. Crucially, the random variable $f(K, N, t)$ depends only on n and t , and does not depend on g .

By construction, $\mathbb{E}(e^{-t\rho_g(w_n)}) \leq \mathbb{E}(f(K, N, t))$ for all g , positive t , and $n \geq k$. Also by construction, $f(K, N, 0) \equiv 1$ for all $n \geq k$. As argued in Step Zero

and depicted in Figure 10, all that remains is to find n_0 such that for all $n \geq n_0$,

$$\text{claim: } \frac{d}{dt} \mathbb{E}(f(K, N, t))|_{t=0} < 0.$$

By direct computation from (43), the derivative

$$\begin{aligned} \frac{d}{dt} \mathbb{E}(f(K, N, t))|_{t=0} &= \mathbb{E}(K - N + 2d + 0.2N) \\ &= \mathbb{E}(K - 0.8N + 2d). \end{aligned}$$

Note that for any random variable X , $\frac{d}{dt}(\mathbb{E}(e^{tX}))|_{t=0} = \mathbb{E}(X)$.

Recall that d, k are fixed constants. Since $\lim_{n \rightarrow \infty} \mathbb{E}(|w_n|)$ is infinite by assumption that the (G, μ) -random walk is not positive-recurrent (22), thus so is $\lim_{n \rightarrow \infty} \mathbb{E}(N)$. Therefore, $\frac{d}{dt} \mathbb{E}(f(K, N, t))|_{t=0}$ is negative for all $n \geq$ some sufficiently positive n_0 . The existence of such n_0 establishes the lemma. $\square$

3.4 Progress

In this section, we use the result of the previous one to prove linear progress with exponential decay for the a -iterated random walk.

We proceed in two steps. First we show that the a -iterated random walk satisfies a progress condition. Then we show that every (not necessarily Markov) process that satisfies this condition makes linear progress with exponential decay.

Criterion 13 (Uniformly positive progress). Let Z_n be a (not necessarily Mar-

kov) sequence of real-valued random-variables. We say that Z_n makes *uniformly positive progress in $\mathbb{R}$* to mean that there exist $b, \epsilon > 0$ so that $\mathbb{E}(e^{-bZ_0})$ is finite and for each integer $n \geq 0$,

$$\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid Z_n) \leq 1 - \epsilon \quad \text{almost surely,} \quad (44)$$

where ϵ is not allowed to depend on n . We say that the (G, μ) -random walk w_n makes *uniformly positive progress in X* if (44) holds for $Z_n = |w_n|$.

The lefthand side of (44) is an $\mathbb{R}_{\geq 0}$ -valued random variable taking values depending on the (random) value of Z_n . Specifically, the lefthand side is defined to be the random variable taking the real value $\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid Z_n = \ell)$ for the outcomes where $Z_n = \ell$. Note that since $-b(Z_{n+1} - Z_n)$ is not strictly positive, equation (44) is not an exponential tail statement.

Lemma 14 (Uniformly positive progress [cf. MS15, 9.1]). *Uniformly positive horofunctions by time a for w_n (Criterion 11) implies uniformly positive progress for $Z_n = |w_{an}|$ (Criterion 13), for the same constant a .*

As suggested by Figures 1 and 14, the horofunction measures progress. This lemma shows that uniformly positive horofunctions implies uniformly positive progress (for the a -iterated random walk).

Having established this lemma, linear progress with exponential decay in X for the a -iterated random walk will immediately follow, i.e., there exists $C > 0$

Figure 14: Uniformly positive horofunctions means that there exist $a, b, c > 0$ such that for all g in G , the expectation $\mathbb{E}(e^{-b\rho_g(w_a)}) \leq e^{-c}$. No matter where you are at time an , you expect to escape further from the basepoint a steps later.

so that for all n , $\mathbb{P}(|w_{an}| \leq n/C) \leq Ce^{-n/C}$, by Proposition 15 below.

Proof. Suppose the (G, μ) -random walk w_n on (X, d_X, x_0) has uniformly positive horofunctions by time a . Then we can fix $a, b, c > 0$ so that for all g in G ,

$$\mathbb{E}(e^{-b\rho_g(w_a)}) \leq e^{-c};$$

see Figure 14. Since w_a and $w_{an}^{-1}w_{an+a}$ are identically distributed,

$$\mathbb{E}(e^{-b\rho_g(w_{an}^{-1}w_{an+a})}) \leq e^{-c}$$

for all integers $n \geq 0$ and g in G . Since w_{an}^{-1} and $w_{an}^{-1}w_{an+a}$ are independent,

$$\mathbb{E}(e^{-b\rho_{w_{an}^{-1}}(w_{an}^{-1}w_{an+a})} \mid w_{an}^{-1}) \leq e^{-c}$$

for all $n \geq 0$. Using the definition of horofunction (15), we can then rewrite

$\rho_{w_{an}^{-1}}(w_{an}^{-1}w_{an+a})$ as $|w_{an+a}| - |w_{an}|$; see Figure 1. Letting $Z_n := |w_{an}|$, we find

$$\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid w_{an}^{-1}) \leq e^{-c}. \quad (45)$$

Since Z_n completely depends on w_{an}^{-1} , we can take conditional expectation on both sides and simplify to

$$\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid Z_n) = \mathbb{E}(\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid w_{an}^{-1}) \mid Z_n) \stackrel{(45)}{\leq} e^{-c}.$$

Thus the $Z_n := |w_{an}|$ makes uniformly positive progress in $\mathbb{R}$ (Criterion 13). $\square$

Proposition 15 (Criterion for exponential decay). *Uniformly positive progress in $\mathbb{R}$ (Criterion 13) implies linear progress with exponential decay in $\mathbb{R}$ (Equation 1).*

This proposition is a special case of stochastic dominance; see for example Lindvall [Lin02, Chapter III Theorem 5.8]. However, we provide a short direct proof of Proposition 15 below, for the convenience of the reader.

Recall that although w_n is a Markov process in G , the process $w_n x_0$ in X and the process $d_X(x_0, w_n x_0)$ in $\mathbb{R}$ do not necessarily have the Markov property. Accordingly, we do not assume the Markov property in this proposition.

Proof. Conditional expectation step. Suppose Z_n is a (not necessarily Markov) sequence of random-variables that makes uniformly positive progress in $\mathbb{R}$. Then

we can find $b, c > 0$ so that

$$\mathbb{E}(e^{-b(Z_{n+1}-Z_n)} \mid Z_n) \leq e^{-c}$$

for each integer $n \geq 0$. Since e^{bZ_n} is completely determined by Z_n , we can pull it out of the expectation, so that for all n ,

$$\mathbb{E}(e^{-bZ_{n+1}} \mid Z_n) \leq e^{-bZ_n-c}.$$

Taking expectation on both sides, we obtain that for all n ,

$$\mathbb{E}(e^{-bZ_{n+1}}) \leq \mathbb{E}(e^{-bZ_n})e^{-c}. \quad (46)$$

Inductive step

We claim that $\mathbb{E}(e^{-bZ_n}) \leq Le^{-cn}$ for all n , where $L = \mathbb{E}(e^{-bZ_0})$, which is finite by assumption (Criterion 13). By definition, the claim holds for the base case $n = 0$. Now assume for induction that the claim holds for some given integer $n \geq 0$. From (46) we have that $\mathbb{E}(e^{-bZ_{n+1}}) \leq \mathbb{E}(e^{-bZ_n})e^{-c}$, which by our inductive hypothesis is $\leq Le^{-c(n+1)}$. The claim follows by induction.

Rewriting, we have that for each integer $n \geq 0$,

$$\mathbb{E}(e^{-bZ_n+cn}) \leq L. \quad (47)$$

Markov inequality step

Recall that for every $\mathbb{R}_{\geq 0}$ -valued random variable Y and real number ℓ , Markov's inequality gives $\mathbb{P}(Y \geq \ell) \leq \mathbb{E}(Y)\ell^{-1}$. Hence for all n ,

$$\mathbb{P}\left(Z_n \leq \frac{cn}{2b}\right) = \mathbb{P}\left(e^{-bZ_n+cn} \geq e^{cn/2}\right) \stackrel{\text{Markov Inequality and (47)}}{\leq} Le^{-cn/2},$$

and we have shown that the real-valued stochastic process Z_n makes linear progress with exponential decay in $\mathbb{R}$ (Equation 1). $\square$

3.5 Remainders

In this section we prove that linear progress with exponential decay for the full random walk follows from exponential tail for μ and linear progress with exponential decay for the a -iterated random walk. The result is a special case of the following probabilistic fact.

Proposition 16 (Exponential decay + exponential tail). *Suppose Y_n has uniformly exponential tail in $\mathbb{R}$ (Definition 3), and suppose Z_n makes linear progress with exponential decay in $\mathbb{R}$ (Equation 1). Then the sum $Z_n + Y_n$ and the difference $Z_n - Y_n$ both make linear progress with exponential decay as well.*

Note that we do not make any assumptions of independence.

Proof. It suffices to show that $-|Y_n| + Z_n$ makes linear progress with exponential decay, since the sum $Y_n + Z_n$ and difference $-Y_n + Z_n$ are both $\geq -|Y_n| + Z_n$.

Fix sufficiently large $C > 0$ and sufficiently small $\lambda > 0$ so that

$$\mathbb{P}(Z_n \leq n/C) \leq Ce^{-n/C} \text{ and}$$

$$\mathbb{E}(e^{\lambda|Y_n|}) < \infty$$

for all n . By Markov's inequality,

$$\begin{aligned} \mathbb{P}(|Y_n| \geq \frac{n}{2C}) &= \mathbb{P}(e^{\lambda|Y_n|} \geq e^{\lambda n/(2C)}) \\ &\leq \mathbb{E}(e^{\lambda|Y_n|}) e^{-\lambda n/(2C)}. \end{aligned}$$

If one real number is $< \frac{n}{2C}$ and another is $> \frac{n}{C}$, then their difference is $> \frac{n}{2C}$. By the contrapositive,

$$\begin{aligned} \mathbb{P}(-|Y_n| + Z_n \leq \frac{n}{2C}) &\leq \mathbb{P}(|Y_n| \geq \frac{n}{2C} \text{ or } Z_n \leq \frac{n}{C}) \\ &\leq \mathbb{E}(e^{\lambda|Y_n|}) e^{-\lambda n/(2C)} + Ce^{-n/C}. \end{aligned}$$

(Note that for arbitrary events, $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$, regardless of dependence.) It follows that $-|Y_n| + Z_n$ makes linear progress with exponential decay. □

Corollary 17 (*a*-iterated random walk [cf. MS15, 9.1]). *Suppose the random walk w_n on X has exponential tail in X (Definition 3). If the *a*-iterated random walk w_{ai} (Definition 2) makes linear progress with exponential decay in X (Definition 4), then so does the full random walk w_n .*

Proof. Fix the (G, μ) -random walk w_n on (X, d_X, x_0) and $a > 0$ from the hy-

potheses. For clarity, we will index the a -iterated walk by i , $(w_{ai})_{i \in \mathbb{N}}$. For each integer $n > 0$, write

$$n := ai(n) + r(n)$$

where $i(n)$ and $r(n)$ are non-negative integers depending on n such that $r(n) \leq a - 1$. Using linear progress with exponential decay (indexed by i), fix $C > 0$ so that

$$\mathbb{P}(|w_{ai}| \leq i/C) \leq Ce^{-i/C}$$

for all i , where $|w_{ai}|$ denotes $d_X(x_0, w_{ai}x_0)$. For all $n \geq a$, we have that $n \leq 2ai(n)$, and thus $\frac{n}{2a} \leq i(n)$, and thus

$$\mathbb{P}(|w_{ai(n)}| \leq \frac{n}{2aC}) \leq Ce^{-n/(2aC)}.$$

This equation holds for all but finitely many n , and so $Z_n := |w_{ai(n)}|$ makes linear progress with exponential decay indexed n .

Since μ has exponential tail in X , thus $|w_n|$ has uniformly exponential tail in X ; see Definition 3. It follows that $Y_n := |w_{ai(n)}^{-1}w_n|$, which is distributed identically to $|w_{r(n)}|$, has uniformly exponential tails as well.

By Proposition 16, the difference $-Y_n + Z_n$ makes linear progress with exponential decay in $\mathbb{R}$. Since $|w_n| \geq -Y_n + Z_n$ by the triangle inequality, hence $|w_n|$ makes linear progress with exponential decay in $\mathbb{R}$. By definition, the (G, μ) -random walk makes linear progress with exponential decay in X . $\square$

Chapter 4

Application

The classification of surfaces tells us that every closed (compact, without boundary), orientable topological surface is homeomorphic to S_g , the connect sum of a sphere and g copies of the torus. We call g the genus of the surface S_g . Every surface S_g with $g \geq 2$ is a *hyperbolic manifold* in the sense that S_g admits a Riemannian metric of constant negative sectional curvature.¹ Thus, “most” closed, orientable surfaces are hyperbolic. Can a similar statement be made for 3-manifolds?

Dunfield and W. Thurston pioneered the following enumeration of 3-manifolds [DT06]. Each closed, orientable 3-manifold admits a standard construction known as a *Heegaard splitting* (defined in Section 4.1). Each Heegaard splitting is associated with an element of a group known as the mapping class group $G = \text{MCG}(S_g)$ for some g (defined in Section 4.2). Thus we have a sequence

¹In this chapter we always take hyperbolic to mean hyperbolic in the sense of Riemannian geometry. Note that a closed manifold is a finite diameter metric space and thus trivially Gromov hyperbolic.

of groups G whose elements taken together enumerate every closed, orientable 3-manifold at least once.

From here, we can use random walks to define a notion of “most” group elements. We say that the subset $E \subset G$ is *exponentially large* for the (G, μ) -random walk w_n if the probability that w_n is in E approaches one exponentially fast (defined in Section 4.3).

Let $G = \text{MCG}(S_g)$ with $g \geq 2$. Lubotzky, Maher, and Wu [LMW16, Theorem 1.2] show the set of hyperbolic and *minimal* (defined in Section 4.1) Heegaard splittings is exponentially large for finitely supported (G, μ) -random walks that are *complete* in $\mathcal{C}_1(S_g)$ the *curve graph* (defined in Section 4.4). Maher and Schleimer [MS18, Theorem 7.3] extend this result to the larger class of (G, μ) -random walks that are nonelementary and bounded in $\mathcal{CB}(S_g)$ the *compression body graph* (defined in Section 4.5). In the same paper, they characterize independent loxodromics for $\mathcal{CB}(S_g)$ and show they exist (Theorem 19).

Applying Theorem 1, we obtain exponential largeness from the natural hypothesis of exponential tail (Definition 3).

Theorem 18 (“Most” 3-manifolds are hyperbolic). *Let $G = \text{MCG}(S_g)$ with $g \geq 2$. The set of hyperbolic, minimal Heegaard splittings is exponentially large for (G, μ) -random walks that are nonelementary and have exponential tail in the compression body graph $X = \mathcal{CB}(S_g)$ defined in Section 4.5.*

Proof. Fix a basepoint x_0 in X . Suppose the (G, μ) -random walk is nonelementary on X (Definition 7) with exponential tail in X (Definition 3). The mapping class group G acts by simplicial isometries on the compression body graph X , which is separable, geodesic, and Gromov hyperbolic; see Section 4.5. Thus by Theorem 1, the random walk makes linear progress with exponential decay in X (Definition 4): there is $C' > 0$ large enough that

$$\mathbb{P}(Z_n \geq n/C') \geq 1 - C' e^{-n/C'} \quad \text{for all } n \in \mathbb{N},$$

where $Z_n = d_X(x_0, w_n x_0)$. If Z_n exceeds $2g + 3$, then the Heegaard splitting $M(w_n)$ is a minimal splitting of a hyperbolic 3-manifold by Theorem 20 [Kob88, Hem01, MT07, ST06]. (Note that since we have assumed $g \geq 2$, we could replace $2g + 3$ with $2g$.)

Recall that the genus g is a fixed constant. For all but finitely many n in $\mathbb{N}$, we have that $n/C' > 2g + 3$, in which case the event $Z_n \geq n/C'$ implies (is contained in) the event $Z_n > 2g + 3$. Thus for all but finitely many n in $\mathbb{N}$,

$$\begin{aligned} \mathbb{P}(M(w_n) \text{ is } \begin{smallmatrix} \text{hyperbolic} \\ \& \text{minimal} \end{smallmatrix}) &\geq \mathbb{P}(Z_n > 2g + 3) \\ &\geq \mathbb{P}(Z_n \geq n/C') \\ &\geq 1 - C' e^{-n/C'}. \end{aligned}$$

Since the lefthand side is $\geq 1 - C' e^{-n/C'}$ for all but finitely many n , thus there is

an even larger $C \geq C'$ such that the lefthand side is $\geq 1 - Ce^{-n/C}$ for all n . $\square$

Note that the orbit map $h \mapsto hx_0$ for any group acting by isometries is distance decreasing up to multiplication by a constant when passing from a word metric on G to the metric on X . We construct the compression body graph as a quotient of the curve graph in Section 4.5. Therefore, exponential tail in a word metric on $G = \text{MCG}(S_g)$ implies exponential tail in $\mathcal{C}_1(S_g)$, which in turn implies exponential tail in $X = \mathcal{CB}(S_g)$.

4.1 Heegaard splitting $M(h)$ over S_g

A *handlebody* V is a compact 3-manifold homeomorphic to a regular neighborhood in $\mathbb{R}^3$ of a finite graph, or equivalently, homeomorphic to $S_{0,b} \times I$ a 2-sphere with boundary cross a closed interval. The boundary of a handlebody is a closed surface $\partial V = S_g$, and the genus of a handlebody is defined to be the genus of its boundary. Any two handlebodies of the same genus are homeomorphic. Given $h : \partial V \rightarrow \partial W$ a homeomorphism between the boundaries of two genus g handlebodies, the *Heegaard splitting* $M(h)$ over S_g is the 3-manifold $V \cup_h W$ obtained by identifying the boundaries of V and W according to $x \sim h(x)$. As shown by Moise, every closed 3-manifold M admits at least one Heegaard splitting over some S_g [Moi52]. If g is the smallest genus for which M admits a Heegaard splitting $M(h)$ over S_g , then M is said to have *Heegaard genus* g , and the splitting

$M(h)$ is said to be *minimal*.

r -Handles for M A manifold that splits $M(h) = V \cup_h W$ over S_g can be shown to split over S_{g+1} by removing a 1-handle from V and gluing it to W . Recall that an r -handle for an n -manifold with boundary M is a copy of the n -ball $B^r \times B^{n-r}$ that is to be glued to ∂M along $(\partial B^r) \times B^{n-r}$; see for example, Lickorish [Lic97, Definition 12.9]. Thus, a 1-handle for a 3-manifold M is a cylinder to be glued to two circles on ∂M ; a 2-handle is a cylinder to be glued to an annulus on ∂M . Some sources (equivalently) define a handlebody to be a 3-ball with g 1-handles attached.

Low Heegaard genus Every Heegaard splitting $M(h)$ over the 2-sphere S_0 is homeomorphic to the 3-sphere. Heegaard splittings over the torus S_1 are known as lens spaces. Lens spaces are spherical 3-manifolds, also known as elliptic 3-manifolds.

4.2 Mapping class group $G = \text{MCG}(S_g)$

Heegaard splittings over isotopic gluing maps are homeomorphic, and so each isotopy class $[h]$ of self-homeomorphisms of the closed, orientable surface S_g determines a well-defined $M([h])$. For a compact, orientable surface S with or without boundary, the *mapping class group* $\text{MCG}(S) = \pi_0(\text{Homeo}^+(S))$ is the

group of isotopy classes of orientation-preserving homeomorphisms, sometimes called mapping classes. We may suppress the brackets and write $h \in \text{MCG}(S)$. A random walk with $G = \text{MCG}(S_g)$ thus becomes a random model of (all) closed, orientable 3-manifolds of Heegaard genus at most g . We will write $S_{g,b}$ for the compact, orientable surface of genus g with b boundary components. When $b = 0$ we may write S_g .

Dehn twists For example, the *Dehn twist* h_γ in $\text{MCG}(S_{g,b})$ about a simple closed curve γ in $S_{g,b}$ is the (isotopy class of the) homeomorphism defined as follows. Identify an annular neighborhood of γ with $(\mathbb{R}/\mathbb{Z}) \times I$. Then define $h_\gamma(\theta, t) = (\theta + t, t)$ on that annular neighborhood, and define h_γ to be the identity everywhere else.

The mapping class group is finitely generated by $2g + 1$ Dehn twists; see for example Birman [Bir74]. The mapping class group is not hyperbolic. Indeed, two Dehn twists h, h' about nonisotopic, disjoint, *essential* (does not bound a disk), *nonperipheral* (does not cobound an annulus with a boundary component) simple closed curves γ, γ' in $S_{g,b}$ commute, and so generate a $\mathbb{Z}^2$ subgroup inside the mapping class group.

4.3 Exponentially large for the random walk

The set of group elements $E \subset G$ with some given property P is said to be *exponentially large* for the (G, μ) -random walk if the probability that w_n has property P approaches one exponentially fast: there is $C > 0$ such that

$$\mathbb{P}(w_n \text{ has } P) \geq 1 - Ce^{-n/C} \text{ for all } n.$$

Using convolution measure (17), the lefthand side may be rewritten $\mu^n(E)$. For example, Maher and Tiozzo show that the set of loxodromics is exponentially large for any finitely supported, nonelementary, weakly hyperbolic random walk [MT16, Theorem 1.4]. Masai shows that the set of non-arithmetic mapping tori is exponentially large for an appropriate class of random walks [Mas16, Theorem 1.5].

4.4 Curve graph $X = \mathcal{C}_1(S_{g,b})$

The mapping class group $\text{MCG}(S_{g,b})$ admits an action by graph isometries on a countable, infinite diameter, hyperbolic (but not locally compact) graph known as the curve graph $\mathcal{C}_1(S_{g,b})$ [MM99]. In fact, the curve graph is the 1-skeleton of a $(3g - 3 + b)$ -dimensional simplicial complex known as the curve complex, on which the mapping class group acts by simplicial isometries, but we will not need this additional structure.

First defined by Harvey [Har81], the *curve graph* $\mathcal{C}_1(S_{g,b})$ has a vertex for each isotopy class of essential (does not bound a disk) nonperipheral (does not cobound an annulus with a boundary component) simple closed curves on $S_{g,b}$. Each pair of vertices that correspond to disjoint curves are adjacent. If the curve graph has edges, then it is *not* locally compact; in fact, all vertices have infinite degree.

Sporadic surfaces The *complexity* of a surface is defined $\xi(S_{g,b}) = 3g - 3 + b$. The seven surfaces of complexity $\xi(S_{g,b}) \leq 1$ are called *sporadic* and have no pairs of disjoint nonisotopic, essential, nonperipheral simple closed curves. For these seven surfaces, the curve graph as defined above has no edges, and so the definition is usually altered. However, we will not need the curve graph of sporadic surfaces. Indeed, for Heegaard splittings, we are interested in closed surfaces, and it is already known that splittings over the sporadic S_0 and S_1 yield elliptic 3-manifolds.

Nonelementary action on $\mathcal{C}_1(S_{g,b})$ For surfaces of complexity $\xi(S_{g,b}) \geq 2$, the action of the mapping class group $\text{MCG}(S_{g,b})$ on the curve graph $\mathcal{C}_1(S_{g,b})$ is nonelementary, weakly hyperbolic (Definition 7). Firstly, countable simplicial complexes are separable. Secondly, Masur and Minsky show that the curve graph $\mathcal{C}_1(S_{g,b})$ for $\xi(S_{g,b}) \geq 2$ is Gromov hyperbolic with infinite diameter

[MM99, Theorem 1]. Thirdly, the action of the mapping class group has (infinitely many) independent loxodromics. It turns out that the action is acylindrically hyperbolic, and even hierarchically hyperbolic, but we will not be using these stronger properties.

Pseudo-Anosov elements Recall that for a weakly hyperbolic group $G \curvearrowright X$, each loxodromic group element (12) fixes exactly two distinct *endpoints* on the Gromov boundary ∂X , the forward limit $h^+ = \lim_{n \rightarrow \infty} h^n x_0$ and the backward limit $h^- = \lim_{n \rightarrow -\infty} h^n x_0$, each of which is the same regardless of the choice of basepoint x_0 in X . Now consider the case of the mapping class group $G = \text{MCG}(S_{g,b})$ acting on the curve graph $X = \mathcal{C}_1(S_{g,b})$. The set of loxodromics for X is equal to the set of *pseudo-Anosovs*, which have some other definition predating the curve graph. The endpoints of a pseudo-Anosov h are conventionally known as the *stable lamination* h^+ and the *unstable lamination* h^- .

Complete subgroups The result of Lubotzky, Maher, and Wu [LMW16, Theorem 1.2] assumes the random walk is complete. Rather than requiring completeness, Theorem 18 requires the random walk to be nonelementary in $\mathcal{C}_1(S_{g,b})$. Complete subgroups are a special case of nonelementary subgroups, as we now show. A subgroup $H < \text{MCG}(S_{g,b})$ is called *complete* if the set of all the endpoints of the pseudo-Anosovs in H is dense in the Gromov boundary

$\partial\mathcal{C}_1(S_{g,b})$. Equivalently, H is complete in $\mathcal{C}_1(S_{g,b})$, as defined in Section 2.1, page 16.

The (G, μ) -random walk is called complete if the semigroup (16) generated by the support of μ is a complete subgroup of G . Since the mapping class group does not act quasiparabolically on the curve graph, thus by Gromov's classification of actions on Gromov hyperbolic spaces (Section 2.1, page 15), complete subgroups of $\text{MCG}(S_{g,b})$ are nonelementary, i.e., they contain independent pseudo-Anosovs.

4.5 Compression body graph $X = \mathcal{CB}(S_g)$

Identify the closed surface S_g with the boundary of the handlebody V . A properly (i.e., $\partial D \subset \partial V$ and $D^\circ \subset V^\circ$) embedded compact disk $D \subset V$ is called a *compressing disk* if its boundary ∂D equals an essential, nonperipheral curve on S_g . The *disk graph* $\mathcal{D}_1(V)$ has a vertex for each isotopy class of compressing disks $D \subset V$ and an edge between each pair of distinct classes $[D], [D']$ that have disjoint representatives D, D' in V , respectively. We may suppress the brackets and write $D \in \mathcal{D}_1(V)$. By identifying each compressing disk with its boundary, we can identify the disk graph $\mathcal{D}_1(V)$ with a subgraph of the curve graph $\mathcal{D} \subset \mathcal{C}_1(S_g)$. For each mapping class $h \in \text{MCG}(S_g)$, the translate $h\mathcal{D} \subset \mathcal{C}_1(S_g)$ is called a *disk set*. The *limit set* of the disk set $h\mathcal{D}$ is defined to

be the subset of the Gromov boundary $\overline{h\mathcal{D}} \cap \partial C_1(S_g)$.

Electrification Using this identification, we construct the following electrification of the curve graph. For each disk set $h_i\mathcal{D}$, add a vertex x_i to $\mathcal{C}_1(S_g)$, and declare every vertex in $h_i\mathcal{D}$ to be adjacent to x_i . For our purposes, it will suffice to define this electrified graph to be the *compression body graph* $\mathcal{CB}(S_g)$. This graph is quasi-isometric to the compression body graph as originally defined by Biringer and Vlamis, which is the graph with a vertex for each isomorphism class $[V]$ of so called *nontrivial marked compression bodies* and an edge for each pair $[V], [W]$ with representatives $V \subset W$ [BV17, §1].

Nonelementary action on $\mathcal{CB}(S_g)$ The mapping class group admits a weakly hyperbolic action (Definition 7) on the compression body graph. Indeed, by construction, the isometric action of the mapping class group on $\mathcal{C}_1(S_g)$ descends to an isometric action on $\mathcal{CB}(S_g)$ [BV17]. Moreover, the compression body graph is separable, as it is a countable graph. Lastly, $\mathcal{CB}(S_g)$ is Gromov hyperbolic by the following argument. As shown by Masur and Minsky, the curve graph is Gromov hyperbolic [MM99], and each disk set $h\mathcal{D}$ is quasiconvex [MM04, Theorem 1.1]. By Bowditch, the electrification X' of a Gromov hyperbolic space X along quasiconvex subsets is Gromov hyperbolic [Bow12, Proposition 7.12].

Loxodromics for $\mathcal{CB}(S_g)$ From Kapovich and Rafi [KR14, Corollary 2.4], it is known that after electrifying a Gromov hyperbolic space X along quasiconvex sets, each C -quasigeodesic in X projects to a reparameterized C' -quasigeodesic in X' (12). Using this result and the infinite diameter of $\mathcal{CB}(S_g)$ [MS18, Theorem 1.1], Maher and Schleimer classify the loxodromics for $\mathcal{CB}(S_g)$ as follows.

Theorem 19 (Classification of Loxodromics [MS18, 7.3, proof]). *Identify S_g with the boundary of a handlebody V and identify the vertices of the disk graph $\mathcal{D}_0(V)$ with a subset of the curve graph $\mathcal{C}_1(S_g)$.*

1. *The loxodromics for $\mathcal{CB}(S_g)$ are precisely the pseudo-Anosovs whose stable (equivalently, unstable) laminations do not lie in the limit set of any disk set, i.e., do not lie in $\bigcup_{h \in \text{MCG}(S_g)} \overline{h\mathcal{D}} \cap \partial \mathcal{C}_1(S_g)$.*
2. *Loxodromics h_1, h_2 for $\mathcal{CB}(S_g)$ are independent for $\mathcal{CB}(S_g)$ if h_1, h_2 are independent for $\mathcal{C}_1(S_g)$.*
3. *Every complete subgroup of $\text{MCG}(S_g)$ with $g \geq 2$ contains independent loxodromics for $\mathcal{CB}(S_g)$.*

The proof of Theorem 19 from Maher and Schleimer is included for the convenience of the reader.

Proof. Suppose the stable lamination h^+ of some pseudo-Anosov h lies in the compression body V . Then the unstable lamination h^- also lies in V [MS18,

Theorem 7.1]. Therefore, h^-, h^+ project to the same point on the Gromov boundary $\partial\mathcal{CB}(S_g)$ and h is not a loxodromic for the compression body graph $\mathcal{CB}(S_g)$.

Now suppose h^+ avoids every compression body. Then h^- also avoids every compression body. Therefore h^-, h^+ project to distinct points on $\partial\mathcal{CB}(S_g)$. Geodesics in $\mathcal{C}(S_g)$ project to reparameterized quasi-geodesics in $\mathcal{CB}(S_g)$, which has infinite diameter [Casson-Jungreis, Theorem 4.1]. Therefore the quasixis of h projects to the bi-infinite quasixis of h in $\mathcal{C}(S_g)_{\mathcal{D}}$. Hence h is loxodromic for $\mathcal{CB}(S_g)$.

Suppose h_1, h_2 are loxodromics for $\mathcal{CB}(S_g)$ that are independent in $\mathcal{C}(S_g)$, i.e., the four limit points $h_1^-, h_1^+, h_2^-, h_2^+$ are distinct in the Gromov boundary $\partial\mathcal{C}(S_g)$. The geodesic from h_1^+ to h_2^+ in $\mathcal{C}(S_g)$ projects to reparameterized quasigeodesic in $\mathcal{CB}(S_g)$. Since the compression body graph $\mathcal{CB}(S_g)$ has infinite diameter [MS18, Theorem 4.1], thus h_1^+, h_2^+ project to distinct boundary points of $\mathcal{CB}(S_g)$. Applying a similar argument to h_1^-, h_2^- , we obtain four distinct limit points $h_1^-, h_1^+, h_2^-, h_2^+$ in $\partial\mathcal{CB}(S_g)$, and so h_1, h_2 are independent for $\mathcal{CB}(S_g)$. Note that since $\mathcal{CB}(S_g)$ is quasi-isometric to a quotient of $\mathcal{C}(S_g)$, pseudo-Anosovs that are not independent in $\mathcal{C}(S_g)$ cannot be independent in $\mathcal{CB}(S_g)$. $\square$

Theorem 20 (Criterion for Hyperbolicity, Minimality [Kob88, Hem01, MT07, ST06]). *Take h in $\text{MCG}(S_g)$ and x_0 in $\mathcal{CB}(S_g)$, where $g \geq 2$.*

1. If $d_{\mathcal{CB}(S_g)}(x_0, hx_0) \geq 3$, then $M(h)$ is hyperbolic.
2. If $d_{\mathcal{CB}(S_g)}(x_0, hx_0) > 2g$, then $M(h)$ is minimal.

Splitting distance In the literature, the Theorem 20 criteria are usually stated in terms of the *splitting distance* as defined by Hempel [Hem01]

$$\text{Split}(h) := d_{\mathcal{C}_1(S_g)}(\mathcal{D}, h\mathcal{D}),$$

where a subset of the curve graph $\mathcal{D} \subset \mathcal{C}_1(S_g)$ is identified with the disk graph $\mathcal{D}_1(V)$ of a handlebody whose boundary we identify $\partial V = S_g$. Note that in general, we define the *distance between subsets* $A, B \subset X$ as

$$d_X(A, B) := \inf\{d_X(a, b) : a \in A, b \in B\}.$$

For $\text{Split}(h)$, the subsets A, B in question are subgraphs of a graph, and so the infimum is realized as a minimum. By construction, distance in the compression body graph bounds splitting distance from below $d_{\mathcal{CB}(S_g)}(x_0, hx_0) \leq \text{Split}(h)$.

The hyperbolicity criterion follows from the work of Kobayashi [Kob88], Hempel [Hem01], and Perelman [MT07]. The minimality criterion is shown by Scharlemann and Tomova [ST06].

Appendix A

Conditional Expectation and Moment Generation

In Lemma 12 we use the following properties.

1. $\mathbb{E}(f(Y) \mid Y) = f(Y)$.

2. $\mathbb{E}(YZ \mid Y) = Y\mathbb{E}(Z \mid Y)$ and $\mathbb{E}(\mathbb{E}(- \mid Y)) = \mathbb{E}(-)$.

So $\mathbb{E}(YZ) = \mathbb{E}(\mathbb{E}(YZ \mid Y)) = \mathbb{E}(Y\mathbb{E}(Z \mid Y))$.

3. $\frac{d}{dt}\mathbb{E}(e^{tY})|_{t=0} = \mathbb{E}(Y)$ if $\mathbb{E}(e^{tY})$ is defined in a neighborhood about $t = 0$.

In Lemma 14 we use the following properties.

4. If $Y \sim Z$ then $\mathbb{E}(f(Y)) = \mathbb{E}(f(Z))$.

5. If Y, Z are independent and $\mathbb{E}(f(Y, b)) \leq c$ for all b ,

then $\mathbb{E}(f(Y, Z) \mid Z) \leq c$.

6. $\mathbb{E}(\mathbb{E}(Y \mid Z) \mid f(Z)) = \mathbb{E}(Y \mid f(Z))$.

Appendix B

Approximate Tree Lemma

Theorem 21 (Approximate Tree Lemma [GdlH90, §2, 12]). *Consider a δ -hyperbolic metric space X with basepoint x_0 and subset $S = \{x_1, \dots, x_n\}$ where $n \geq 4$. Let S' be the union of n geodesic rays from x_0 through these x_i .*

1. *There is a finite pointed metric tree T and a map $f : S \rightarrow T$ such that for*

all a, b in S , we have $d_T(f(x_0), f(a)) = d_X(x_0, a)$ and

$$0 \leq d_T(f(a), f(b)) - d_X(a, b) \leq 2\delta \lceil \log_2(n-2) \rceil.$$

2. *There is also a pointed real tree T' and a map $f' : S' \rightarrow T'$ such that for*

all a, b in S' , we have $d_{T'}(f'(x_0), f'(a)) = d_X(x_0, a)$ and

$$0 \leq d_{T'}(f'(a), f'(b)) - d_X(a, b) \leq 2\delta \lceil \log_2 n \rceil + 2\delta + \max\{d_X(x_0, x_i)\}.$$

See also Masai [Mas16, Lemma 4.13] and Bowditch [Bow06, Proposition 6.7].

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