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Studying the Space of Almost Complex Structures on a Manifold Using de Rham Homotopy Theory

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Abstract

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Advisor: Dennis Sullivan

In his seminal paper *Infinitesimal Computations in Topology*, Sullivan constructs algebraic models for spaces and then computes various invariants using them. In this thesis, we use those ideas to obtain a finiteness result for such an invariant (the de Rham homotopy type) for each connected component of the space of cross-sections of certain fibrations. We then apply this result to differential geometry and prove a finiteness theorem of the de Rham homotopy type for each connected component of the space of almost complex structures on a manifold. As a special case, we discuss the space of almost complex structures on the six sphere and conclude a conjecture about the ordinary homotopy type of that space.

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I dedicate this work to my family: My dear and precious wife Zeynep, who never stopped believing in me even in the heights of my self-doubt, our sweet daughter Zehra Lara, my mature little sister Burcu, always supportive father Ahmet Kemal Ferlengez and virtuous mother Şenay Ferlengez. I am a very lucky person to have them, and this writing and whatever I will ever produce in the future should be considered as miniature memorabilia representing their unending love and aid.

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Contents

Chapter 1

Introduction

One can speak of the differential forms and de Rham theorem for general spaces (that are not necessarily manifolds) in various ways. For instance, Whitney defines differential forms on simplices of a triangulation in [Whi57] by associating to a simplex a differential form in the affine space generated by that simplex and by putting a coherence condition to ensure that the forms on the neighboring simplices agree on common faces.

In §7 of [Sul77], Sullivan uses this idea to define differentiable forms on various spaces so that de Rham theorem holds. Those forms form a commutative differentiable graded algebra (like the de Rham complex of forms on a smooth manifold), and using those forms one can compute further invariants beyond the de Rham cohomology (such as the de Rham homotopy).

Chapter 2

Fundamentals

2.1 Algebraic models in rational homotopy theory

A commutative differential graded algebra, shortly a *cdga*, is a non-negatively graded cochain complex with coefficients in the field $\mathbb{Q}$ (or $\mathbb{R}$) that has a graded commutative multiplication with a unit. In this manuscript, the cdga's contain only the coefficient field in degree zero (i.e. are connected).

A *free cdga* is a cdga whose multiplication yields no relation but graded commutativity. A *triangular cdga* is a cdga that has a partially-ordered set of generators so that the differential on a generator is a polynomial in earlier generators. A *free triangular model* of a space X is a free triangular cdga and a map into the forms on X which induces an isomorphism in cohomology. We say a free triangular model has *finite type*, if in each degree it has finite rank.

The following lifting theorem is needed for Sullivan's theory to work:

Proposition 1 (Lifting theorem)**.** *Given a map φ between two cdga's that induces an isomorphism in cohomology and a map from a free triangular cdga into the target of φ, one can construct a map from the free triangular cdga into the domain of φ uniquely up to homotopy*[1] *so the obvious diagram commutes up to homotopy.*

Remark 1. By the lifting theorem[2], any two free triangular models of a space are homotopy equivalent. In other words, free triangular models are well-defined.

For a free triangular cdga, there is a second notion of cohomology called *the linearized cohomology* where the vector spaces are the indecomposibles and the differential is the induced $\mathrm{d}_{\mathrm{linearized}}$ which ignores the decomposibles in the differential d of the original cohomology.

By remark [1], it is easy to check this linearized notion of cohomology is well-defined (i.e. model independent) and we will relate it to the *de Rham homotopy* of the free triangular cdga (resp. the *de Rham homotopy* of the space, if the free triangular cdga is a model for a space). We say a space X has finite rank, if its linearized cohomology has finite total rank.

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[1] The notion of homotopy is defined in §3 of [Sul77].

[2] The lifting theorem can be seen as the analogue of obstruction theory for cdga's.

Remark 2. The natural map in each degree from usual cohomology to linearized cohomology when dualized is the de Rham analogue of the Hurewicz homomorphism from homotopy to homology. Thus the dual of linearized cohomology is called the *de Rham homotopy groups* (cf. [Sul75] for the comparison to regular homotopy groups).

2.2 Almost complex structures on a manifold

An almost complex structure on $\mathbb{R}^{2n}$ is a linear endomorphism $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -\mathrm{id}$, and such a J endows $\mathbb{R}^{2n}$ with the structure of a complex vector space, where the complex scalar multiplication can be defined as $(x + iy)\vec{v} := x\vec{v} + yJ(\vec{v})$.

$GL(2n, \mathbb{R})$ acts on almost complex structures on $\mathbb{R}^{2n}$ by conjugation. This action is transitive and its stabilizer is $GL(n, \mathbb{C})$. Therefore, the space of almost complex structures on $\mathbb{R}^{2n}$ can be identified with ${GL(2n, \mathbb{R})}/{GL(2n, \mathbb{C})}$ (see page 116 of [KN96]).

An almost complex structure on a manifold is a cross-section of the bundle associated to the space of almost complex structures on each tangent tangent space, i.e. of the bundle ${GL(2n, \mathbb{R})}/{GL(n, \mathbb{C})} \to \mathfrak{J}(M) \to M$, where $\mathfrak{J}(M)$ denotes the bundle of linear frames over M modulo $GL(n, \mathbb{C})$ (see page 113 in [KN96]).

Chapter 3

The main theorems

Theorem 1. *Let M be a smooth, closed, connected manifold of dimension $2n$ with first real Betti number zero. Then the de Rham homotopy of each connected component of the space of almost complex structures on M has total finite rank.*[1]

Proof. Since M is a smooth manifold, its tangent bundle is classified up to homotopy by a well-defined map $\tau : M \to BGL(2n, \mathbb{R})$. Using τ, we obtain the bundle of ACS's on M as a pullback of a nilpotent fibration[2]:

[1] Under some known technical conditions this means the total rank of the geometric homotopy groups is finite (page 3 of [Sul75]).

[2] We call $E \xrightarrow{p} B$ a *nilpotent fibration*, if it can be decomposed (up to homotopy equivalence) into a possibly infinite sequence of principal $K(G, n)$ fibrations that converges (see pp. 437-444 in [Spa66]).

$$\begin{array}{ccc}
GL(2n,\mathbb{R})/GL(n,\mathbb{C}) & & GL(2n,\mathbb{R})/GL(n,\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathfrak{J}(M) & \longrightarrow & BGL(n,\mathbb{C}) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tau} & BGL(2n,\mathbb{R})
\end{array}$$

For that reason, $GL(2n,\mathbb{R})/GL(n,\mathbb{C}) \to \mathfrak{J}(M) \to M$ is itself a nilpotent fibration.

Note that $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$ is homotopy equivalent to $SO(2n)/U(n)$[3] and also that $SO(2n)$ and $U(n)$ have finite rank homotopy being compact Lie groups.[4] The finiteness of the rank of $SO(2n)/U(n)$ follows from the exact sequence of homotopy.

Then Theorem [1] follows from Corollary [3]. □

Theorem 2. *Let* $F \longrightarrow E \underset{s}{\overset{p}{\rightleftarrows}} B$ *be a nilpotent fibration, where the fiber* F *has finite rank and the base* B *has finite type. Then the* Γ*-construction yields a finite type model for the connected component of* s *in the space of sections of the fibration.*

Proof. The Γ-construction suggested by Sullivan (see page 314 of [Sul77]) and completed by Haefliger [Hae82] yields a free model for the connected

[3] $SO(2n)/U(n)$ is the space of almost complex structures on $\mathbb{R}^{2n}$ that preserve the orientation and the metric.

[4] by the Elie Cartan argument that the bi-invariant forms produce a finite dimensional free cdga on one dimensional generators mapping into forms on a compact Lie group.

component of the section s.

Then Theorem [2] follows from the main theorem on page 615 of [Hae82] generalized by the remark (i) on page 620 of the same paper, after observing that the free model of $\Gamma(s)$ has only finitely many generators by degree reasons:

An algebra generator of the free model of $\Gamma(s)$ obtained by Γ-construction has the form $b' \otimes f$, where b' is the dual vector of a linear basis vector b in a free model of the base and f is an algebra generator of a free model of the fiber such that difference $|f| - |b|$ is non-negative, where $|f|$ and $|b|$ are degrees of f and b, respectively.

If the base has a finite type model and the fiber has a finite rank model, then the free model of $\Gamma(s)$ given by the Γ-construction can have only finitely many generators. □

Corollary 3. *Let $F \to E \to M$ be a fibration, where the base is a smooth, closed, connected manifold with first real Betti number zero and the fiber has finite rank de Rham homotopy. Then each connected component of the space of sections has de Rham homotopy of finite rank.*

Proof. Such a manifold M admits a free model of finite type (see page 3 in [Sul75]). □

Chapter 4

Special case: S^6

We consider the special case

$$\begin{array}{ccc} GL(2n,\mathbb{R})/GL(n,\mathbb{C}) & \longrightarrow & \mathfrak{J}(S^6) \\ & & \downarrow \\ & & S^6 \end{array}$$

Firstly, from elementary obstruction theory, we know that the space of cross-sections is non-empty and has two components (given orientation, the standard almost complex structures given by octonion multiplication provides a family of such a cross-section).

Therefore, Theorem 1 applied to $M = S^6$ tells us that the space of almost complex structures on S^6 has finite rank de Rham homotopy. In this chapter, we will show that rank is 1 (and is concentrated in dimension 7).

We replace $GL(6,\mathbb{R})/GL(3,\mathbb{C})$ by $SO(6)/U(3)$, as they are homotopy equivalent. Since the space of almost complex structures on $\mathbb{R}^8$ fiber over

S^6 with fibers being the space of almost complex structures on $\mathbb{R}^6$, we can replace $\mathfrak{J}(S^6)$ by $SO(8)/U(4)$.

The Γ-construction (suggested by Sullivan on page 314 of [Sul77]) works as follows:

Given a fibration $F \to E \xrightarrow{p} B$, we consider the pairs $b^* \otimes f$, where b^* is a linear basis vector of the differential graded coalgebra that is the dual of the (finite type) model $\mathcal{B}$ of the base and f is a cdga generator of a (finite rank) model $\Lambda(V)$ of the fiber. The degree of the pair $b^* \otimes f$ is defined to be $|f| - |b|$. We mod out the free algebra generated by those $b^* \otimes f$ pairs by the ideal generated by pairs that have a negative degree and cocycles in degree zero. That way, we obtain a connected free cdga denoted by Γ.

Haefliger shows in [Hae82] that there is a unique differential we can put on this free cdga that makes the evaluation map $\mathrm{ev} : \mathcal{B} \otimes \Lambda(V) \to \mathcal{B} \otimes \Gamma$, a map of cdga's:

$$d_\Gamma(b^* \otimes f) := \pm \partial b^* \otimes f \pm b^* \otimes \mathrm{ev}(df),$$

where $\mathcal{B} \otimes \Lambda(V)$ models the total space E of the fibration, ∂b^* is the transpose of $d_{\mathcal{B}}$ and the $b^* \otimes \mathrm{ev}(df)$ term is obtained by taking the differential of f in the the total space and then using the coalgebra structure of $\mathcal{B}^*$ to evaluate them at B to reduce the term to a product of the generators of Γ.

The generators of Γ (with positive degrees) in the S^6 case are $1^*_{\mathcal{B}} \otimes x_2$

(with degree 2), $1^*_{\mathcal{B}} \otimes y_7$ (with degree 7) and $w_6 * \otimes y_7$ (with degree 1). (The indices indicate the degrees of the generators in their respective algebras.)

$\Lambda(1_{\mathcal{B}}, w_6, z_{11})$ is a model of S^6 of finite type with $\mathrm{d}_{\mathcal{B}}(1) = \mathrm{d}_{\mathcal{B}}(w_6) = 0$ and $\mathrm{d}(z_{11}) = w_6^2$.

$\Lambda(1_F, x_2, y_7)$ is a model of $\mathbb{C}P^3$, which is diffeomorphic to ${}^{SO(6)}\!/\!{}_{U(3)}$ and it has finite rank linearized cohomology.

The differential of $1^*_{\mathcal{B}} \otimes x_2 = 0$. To find the differential of $1^*_{\mathcal{B}} \otimes y_7$ and $w_6^* \otimes y_7$, we need to figure out what $\mathrm{d}y_7$ is. The crucial part of the computation is to show that the fibration

$$\begin{array}{ccc} \mathbb{C}P^3 & \longrightarrow & {}^{SO(8)}\!/\!{}_{U(4)} \\ & & \downarrow \\ & & S^6 \end{array}$$

is not a product fibration and that follows from the fact that ${}^{SO(8)}\!/\!{}_{U(4)}$ is Hermitian symmetric (see page 518 in [Hel76]) and hence a Kähler manifold.

Therefore, it should have a closed 2-form whose sixth power is nonzero. The only closed 2-form in our model is x_2. That means $x_2^4 \neq 0$ and therefore $0 \neq c \in \mathbb{R}$ in $\mathrm{d}y_7 = x_2^4 + cx_2w_6$. Then $\mathrm{d}_\Gamma(w_6^* \otimes y_7) = (1^*_{\mathcal{B}} \otimes x_2)$ and we end up with one cohomology class represented by $(1^*_{\mathcal{B}}, y_7)$ as claimed.

This shows that the space of almost complex structures has the de Rham homotopy type of S^7 (or $\mathbb{R}P^7$, which is $\mathbb{Q}$-equivalent to S^7).

The octonion family of J's is actually S^7 mod antipodal action, which is $\mathbb{R}P^7$. This leads to the following conjecture:

Conjecture. *The space of almost complex structures on* S^6 *has the octonion* J*'s as a deformation retract in ordinary homotopy theory.*

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