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Essays in New Keynesian Monetary Policy

Tzu-Hao Huang

The Graduate Center, City University of New York

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ESSAYS IN NEW KEYNESIAN MONETARY POLICY

by

Tzu-Hao Huang

A dissertation submitted to the Graduate Faculty in Economics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2018
Essays In New Keynesian Monetary Policy

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Tzu-Hao Huang

This manuscript has been read and accepted for the Graduate Faculty in Economics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

Date

Thom Thurston
Chair of Examining Committee

Date

Wim Vijverberg
Executive Officer

Supervisory Committee:

Temisan Agbeyegbe

Wim Vijverberg

THE CITY UNIVERSITY OF NEW YORK
ABSTRACT

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by

Tzu-Hao Huang

Advisor: Thom Thurston

The dissertation consists of three Chapters. I consider New Keynesian models which involve tradeoffs between output gap and inflation variances. Such policy strategy is often referred to as flexible inflation targeting rules (e.g., Lars Svensson 2011, pp.1238-95). Taylor rules, in general, have the symbolic expression $i_t = \phi_x x_t + \phi_\pi \pi_t + \phi_g g_t$, where $i_t$ is the nominal interest rate at period $t$, $x_t$ is the target variable output gap at period $t$, $\pi_t$ is the target variable inflation rate at period $t$, $g_t$ is realized shock to output gap at period $t$, and $\phi_x$, $\phi_\pi$ and $\phi_g$ are coefficients. This three-term Taylor rule is the most efficient Taylor rule in terms of the social welfare loss measurement (i.e., the minimized social welfare loss involved with the three-term Taylor rule is the smallest value when we compare it with the minimized social welfare loss involved with a one-term Taylor rule ($i_t = \phi_\pi \pi_t$) or a two-term Taylor rule ($i_t = \phi_x x_t + \phi_\pi \pi_t$). Thus, the three-term Taylor rule is used as the benchmark for comparing the performance of Taylor rules in the dissertation.

Chapter 1 argues that the dynamic interpretation most authors have put on the “stability and uniqueness” (determinacy) condition of the new Keynesian monetary policy model is inappropriate. Literatures authors maintain a belief when monetary policy is operating through a
Taylor rule, the model stability and uniqueness requires the real interest rate move in the same
direction as inflation (Taylor Principle). This chapter shows the determinacy condition does not
necessarily require the Taylor Principle to hold. The Taylor Principle and the determinacy
condition are two different kettles of fish.

Although the three-term Taylor rule is applied in Chapter 1, some people may object or
think that it is impractical or “unrealistic” to expect the central bank (“the Fed”) bases a rule on a
shock term \( \varphi_g g_t \). Thus, in Chapter 2 and Chapter 3, I examine two-term (“simple”) Taylor
rules which do not have \( \varphi_g g_t \) term—i.e., \( i_t = \varphi_x x_t + \varphi_\pi \pi_t \).

Chapter 2 is a study of the linear relationship of the coefficients \( \varphi_x \) and \( \varphi_\pi \) in Taylor rules,
which \( \varphi_x \) is the coefficient to the target variable output gap \( x_t \) and \( \varphi_\pi \) is the coefficient to the
target variable inflation rate \( \pi_t \). Furthermore, since I use only \( x_t \) and \( \pi_t \) in Taylor rules instead
of using \( x_t \) and \( p_t - p_{t-1} \) (i.e., the difference between price levels in two periods) in Taylor
rules, the Taylor rules do not cause optimal inertia. In other words, the Fed has once-and-for-all
response to the new development in either \( x_t \) or \( \pi_t \), or both. Such new developments are either
from realized output gap shocks or inflation rate shocks or both. The monetary policy objective
function is then treated as a period quadratic social welfare loss function for two target variables
and their coefficients because the solution expectation for all periods is the same as the solution
for period \( t \). The optimal policy implies that, especially, the coefficients \( \varphi_x \) and \( \varphi_\pi \) must
produce minimum social welfare loss to the economy when the Fed’s monetary policy target is
based on the tradeoffs between two target variables inflation rate \( \pi_t \) (not price levels) and output
gap \( x_t \). For those policy-rate paths (expressed by Taylor rules) which the minimum social
welfare losses are guaranteed, I use the term optimal Taylor rules, and for those coefficient
values satisfied this purpose, I called them optimal coefficients or optimal linear relationship among those coefficients. The natural optimum Taylor rule, as pointed out by Woodford (2001), would have the $\varphi_g$ term ($= \sigma$), but for the reason in the previous paragraph, I only examine the case of a simpler Taylor rule, $i_t = \varphi_x x_t + \varphi_\pi \pi_t$ (hereafter this Taylor rule is called the simple Taylor rule or the simple TR), when the rule is specified as the optimal interest rate rule for governing the optimal paths of output gap and inflation rate. The global-type solutions with “optimal inertia” will not be considered in all chapters.

The first part of Chapter 2 develops an approach to obtain the linear relationship of $\varphi_x$ and $\varphi_\pi$ which is the first order condition for minimum social welfare loss, $L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2]$, where $L$ denotes social welfare loss, $E$ is the expectational operator and $\Gamma$ is the weights on output gap.

The second part of Chapter 2 is the discussion of two properties of the linear relationship of $\varphi_x$ and $\varphi_\pi$ that are observed by comparing with the three-term Taylor: (a) the linear relationship is the same for governing the optimal paths of $x_t$ and $\pi_t$ whether $g$-shocks are nullified by containing $\varphi_g = \sigma$ in the baseline new Keynesian model or not; (b) the limit of the social welfare loss containing the simple Taylor rule ($i_t = \varphi_x x_t + \varphi_\pi \pi_t$) is at the minimum when the values of $\varphi_x$ and $\varphi_\pi$ are very big (or approaching infinity), and such minimum is the same as the social welfare loss containing the three-term Taylor rule. This implies the three-term Taylor rule with $\varphi_g (= \sigma)$ suggested by Woodford (2001), whose model has different setup but it works out with the same result, is more efficient than the simple (two-term) Taylor rule.

In Chapter 3, using the method developed from and the two properties discovered in Chapter 2, I propose a combination monetary policy rule when the Fed sets the interest rate before observing current variables of output gap ($x_t$) and inflation ($\pi_t$). The missing information
is $\varepsilon_t$ in $x_t$ equation—i.e., $g_t = \lambda g_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim iid \ N (0, \sigma^2_{\varepsilon})$, and $\eta_t$ in $\pi_t$ equation—i.e., $u_t = \rho u_{t-1} + \eta_t$ where $\eta_t \sim iid \ N (0, \sigma^2_{\eta})$. Thus, the Fed cannot adjust their interest rate for those shocks because the Fed cannot observe $\varepsilon_t$ and $\eta_t$. On the other hand, the information of money is immediately available to the Fed when I use a model as abstract representation of the Fed’s observation of money surprise, so the Fed can use signals about money to adjust their interest rate. My model of the Fed’s operation on how they observe money surprise is a simplified model for making a theoretical point, not for the purpose of improving what the Fed is actually doing. The combination policy of a Taylor rule and money signal can improve the social welfare loss when the Fed sets their monetary policy with unobservable shocks. Chapter 3 uses an inverted version of Poole’s (1970) combination policy analysis and shows that the social welfare loss is improved from the money signals.
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1 THE TAYLOR PRINCIPLE AND TAYLOR RULE: TWO DIFFERENT KETTLES OF FISH

with Thom Thurston
Abstract

In recent literature in monetary theory it is nearly universal to specify monetary policy as operating through a Taylor rule and to interpret the well-known “stability and uniqueness” (determinacy) condition as meaning that the Taylor rule makes the real interest rate move in the same direction as inflation (Taylor Principle).

We first argue that the dynamic interpretation most authors have put on the NKM determinacy condition is inappropriate. Second, we show the standard NKM determinacy condition does not necessarily require the Taylor Principle to hold. The Taylor Principle and the NKM determinacy condition are two different kettles of fish.
1.1 Introduction

In recent years a great deal of attention among monetary economists has been focused on the issue of uniqueness, stability and/or “determinacy” in macroeconomic models, particular in the New Keynesian Model (NKM). Modeling in NKM usually represents monetary policy as following a Taylor rule, and the parameters of the Taylor rule must meet certain “determinacy conditions” which may be necessary to rule out both “sunspots” and explosive solutions. At least since the widely-cited article by Woodford (2001) there has been a consensus that the determinacy condition in these models is essentially a restatement of the Taylor Principle – the policy rule must guarantee that real interest rates will move in the same direction as inflation. We beg to differ with this line of reasoning, at least within the context of the NKM.

Our argument is as follows. “Backward-looking” models (such as that by Taylor (1999) himself) indeed require the Taylor Principle to hold for “stability” – meaning here where projected paths eventually approach long-run or steady-state solutions of the endogenous variables. But one should avoid conflating the Taylor Principle with the determinacy condition in the forward-looking NKM. The determinacy condition in NKM, when met, assures us that the model’s solution is unique. The dynamics are “stable” in the above sense by construction (all
shocks are AR(1)). If the determinacy condition is not met, the solution is “immediately explosive” (no finite solution exists) or may result in non-explosive sunspot solutions.¹

To illustrate our point, we begin by showing how the two concepts apply in simple, univariate backward- and forward-looking models. Since many readers will be familiar with the points in the univariate models, we relegate this to Appendix 1.8.1. Next and directly in the text, we turn to two representative bivariate (inflation and output gap) backward- and forward-looking models. The backward-looking model is Taylor’s (1999) model; the forward-looking model is the baseline NKM as exposited by Clarida, Gali and Gertler (1999), Woodford (2001, 2003) and others. As a kind of coup de grace to the proposition that determinacy means Taylor Principle, we produce two examples of solutions where the determinacy condition is met in the baseline NKM with Taylor rule, but in which real interest rates move in the opposite direction of inflation.

### 1.2 The Backward-Looking Taylor (1999)

We start with the bivariate backward-looking model by Taylor (1999):

1. \[ x_t = -\beta (i_t - \pi_t - r) + g_t \]
2. \[ \pi_t = \pi_{t-1} + \alpha x_{t-1} + u_t \]
3. \[ i_t = \varphi_0 + \varphi_{\pi} \pi_t + \varphi_x x_t \]

¹ This paper does not analyze “sunspot” solutions. For sunspot we refer readers to Chiappori and Guesnerie (1991), Lubik and Schorfheide (2002), Evans and McGough (2005) and Chadha and Corrado (2006). These are rational solutions that reflect certain arbitrary and “non-fundamental” disturbances to expectations which produce self-fulfilling and non-explosive solutions when the mathematical determinacy conditions are not met. When the determinacy conditions are met, the effect is to rule out (in some cases “cancel out”) the effects of the sunspots. Our focus will be on whether meeting the determinacy conditions has dynamic implications apart from ruling out sunspots and/or being instantly explosive. We will argue that determinacy conditions do not have such implications for the baseline New Keynesian Model, notwithstanding widespread claims they do.
where \( x \) represent the output gap in logs; \( \pi \) represents the inflation rate; \( i \) represents the short

term nominal interest rate. \( g \) and \( u \) represent (independent and not auto-correlated) shocks with

zero mean. The model parameters \( \alpha \) and \( \beta \) are positive. \( r \) represents the natural rate of interest.

\( \varphi_0, \varphi_\pi \) and \( \varphi_x \) are the policy parameters of the Taylor rule.

The solution path of inflation can be written as:

\[
\pi_{t+i} = \Lambda^{i-1}\pi_t + \alpha \Lambda^{i-1}x_t + \sum_{i=1}^{n} \Lambda^{i-1}g_{t+i} + \frac{\alpha}{1+\beta \varphi_x} \sum_{i=1}^{n} \Lambda^{i-1}u_{t+i}, \quad i = 1, 2, ... n
\]

where the subscript \( t \) denotes time and \( \Lambda = \frac{\alpha \beta (1-\varphi_\pi) + (1+\beta \varphi_x)}{1+\beta \varphi_x} \). The key parameter is \( \Lambda \), which if

inside the unit circle will mean that projections of \( \pi_{t+i} \) will eventually approach the model’s

steady-state value \( \left( \frac{\varphi_0-r}{1-\varphi_\pi} \right) \) – hence, the model is “stable” in the sense Taylor intended. If \( \Lambda \) lies

outside the unit circle the model’s projections will depart continuously from steady-state values;

if \( \Lambda \) lies on the unit circle the model’s projection will approach a value that differs from the

steady-state value. In these cases the model is “unstable” in the Taylor sense, though not

“explosive” in the immediate sense. Each period’s projection is well-defined.
Since \( \Lambda = 1 + \frac{\alpha \beta (1 - \varphi_\pi)}{1 + \beta \varphi_x} \), the Taylor stability condition can be simplified to \( 1 < \varphi_\pi < 1 + \frac{2(1 + \beta \varphi_x)}{\alpha \beta} \). This establishes the Taylor Principle and its role in this model. Taylor (1999) and Cochrane (2011) in his description of the Taylor (1999) model neglect the upper bound required on \( \varphi_\pi \), emphasizing only the requirement that \( \varphi_\pi > 1 \) for Taylor stability. 3

1.3 The Forward-Looking NKM Model: Clarida, Gali and Gertler (1999)

Now while the Taylor Principle governs whether a backward-looking model like Taylor’s above will be “stable” in terms of tending toward steady-state, there is no similar implication for forward-looking models. To illustrate this point, we turn to the bivariate forward-looking model of Clarida, Gali and Gertler (1999, henceforth CGG(1999)). Their baseline NKM consists of the familiar equations for the output gap (“IS”) and inflation (“Phillips curve”):

\[
\begin{align*}
(5) & \quad x_t = E_t[x_{t+1}] - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}]) + g_t \\
(6) & \quad \pi_t = k x_t + \beta E_t[\pi_{t+1}] + u_t
\end{align*}
\]

2 \(-1 < \Lambda < 1\), so\( -1 < 1 + \frac{\alpha \beta (1 - \varphi_\pi)}{1 + \beta \varphi_x} < 1 \)
\(-2 < \frac{\alpha \beta (1 - \varphi_\pi)}{1 + \beta \varphi_x} < 0 \).

If \( 1 + \beta \varphi_x > 1 \):
\(-2(1 + \beta \varphi_x) < \alpha \beta (1 - \varphi_\pi) < 0 \).

If \( \alpha \beta > 0 \):
\(-2(1 + \beta \varphi_x) \alpha \beta < 1 - \varphi_\pi < 0 \)
\(0 < \varphi_\pi - 1 < \frac{2(1 + \beta \varphi_x)}{\alpha \beta} \)
\(1 < \varphi_\pi < 1 + \frac{2(1 + \beta \varphi_x)}{\alpha \beta} \).

Since \( \alpha, \beta \) and \( \varphi_x \) are assumed to be positive from Taylor (1999), we obtain \( 1 < \varphi_\pi < 1 + \frac{2(1 + \beta \varphi_x)}{\alpha \beta} \).

3 This does not seem to be a particularly important omission, since the Fed is presumably free to increase \( \varphi_x \) as needed to maintain \( 1 + \frac{2(1 + \beta \varphi_x)}{\alpha \beta} > \varphi_\pi \) so as to prevent Taylor “instability.”
where \(x\) again represents output gap in logs; \(\pi\) represents inflation (log-deviation from steady-state); and \(i\) represents the nominal interest rate (deviation from steady-state); \(0 < \beta < 1\) is a discount factor; \(k > 0\) is the Phillips curve parameter reflecting the degree of price flexibility (higher means more), and \(\sigma > 0\) is the consumption-elasticity of utility. \(g_t\) and \(u_t\) are shocks of AR(1) form: 
\[
g_t = \lambda g_{t-1} + \varepsilon_t \quad (0 < \lambda < 1) \quad \text{and} \quad u_t = \rho u_{t-1} + \eta_t \quad (0 < \rho < 1).
\]

This model needs to be closed by one of two means. CGG (1999, p.1668) employed, in their wording, a welfare measure in the form of a quadratic loss function

\[
W = -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t E_t [\pi^2_{t+t} + \Gamma x^2_{t+t}]
\]

where \(\Gamma\) is a relative weight on preferences for the output gap \((x)\) relative to inflation \((\pi)\). This provides a first order condition constraint which, together with (5) and (6), permits the derivation of optimal paths for the endogenous variables. The optimal paths themselves do not address the determinacy issue, but it is possible to derive a monetary policy rule (which usually but not necessarily means a Taylor rule) that is consistent with the optimal paths. Once the rule is imposed it is possible to set conditions for determinacy for this rule. More details are provided in Appendix 1.8.1.

More generally, a monetary policy rule (again usually a Taylor rule) can simply be added to (5) and (6) so that the endogenous variables depend on the parameters of the rule. These parameters will indicate whether the solutions are determinate.
1.4 Determinacy with the Taylor Rule

Let the particular Taylor rule be written as

\[
i_t = \varphi_\pi \pi_t + \varphi_x x_t + \varphi_g g_t
\]

which includes no intercept (the model is derived as log-linearized deviations from steady-state) and contains a \( \varphi_g \) which can be used to offset \( g_t \) shocks.\(^4\) Putting (8) into (5) and (6), the model can be written in the matrix form \( M_t = AM_{t+1} + e_t \), where \( M_t' = [x_t \ \pi_t] \), \( M_{t+1}' = [E_t[x_{t+1}] \ E_t[\pi_{t+1}]] \), \( A \) is the two-by-two coefficient matrix and \( e \) is a vector of exogenous shock terms. The solution of the model will be linear in \( u_t \) and \( g_t \). Bullard and Mitra (2002(2000)) showed that determinacy of this model requires that the eigenvalues of \( A \) lie within the unit circle, which in turn requires

\[
k(\varphi_\pi - 1) + (1 - \beta) \varphi_x > 0.
\]

It turns out that in the general solutions, for any set of \( \varphi \)'s the first order conditions’ still make \( x_t \) and \( \pi_t \) proportional; therefore, the optimal ratio of \( x_t \) and \( \pi_t \) can be replaced by any arbitrary values \( R \) depends on which combination of \( \varphi_x \) and \( \varphi_\pi \) we pick.\(^5\)

---

\(^4\) The optimal Taylor rule, as pointed out by Woodford (2001) would have \( \varphi_g \) term (= \( \sigma \)), but some people may object or think that it is impractical or “unrealistic” to expect the Fed bases a rule on a shock term. In chapter 2 and chapter 3, I examine the “simple Taylor Rule” which does not have that term.

\(^5\) In other words, any set of \( \varphi \)'s suggests a particular \( \Gamma \) and other coefficients. The optimal set of \( \varphi \)'s of course will be obtained only if we use the “true” \( \Gamma \). The general solution of course does not in itself reveal the true \( \Gamma \).
1.5 Two Examples Having Unique Solutions But Still Violate the Taylor Principle

The Taylor Principle is not necessary for a forward-looking model to have uniqueness. The following examples show that using Taylor Principle as the economic reasoning for the uniqueness of the forward-looking model is inappropriate.

An Example for u-shock Only

For this case, we either assume $g_t = 0$ for all $t$, or posit that the authorities set the Taylor rule parameter $\varphi_g = \sigma$, which essentially nullifies these demand shocks. Having only one shock means the solution for $x_t$ and $\pi_t$ will be proportional to $u_t$, which means that the expected $t + 1$ values of these variables can be written as $\rho$ times their current values. Then taking the total differentials of (5) with $i_t$ represented by the Taylor rule and (6), one arrives at a constraint between movements in the real interest rate $r_t$ and $E_t[\pi_{t+1}]$ as

\begin{equation}
    r_t = \left\{ \frac{\sigma(1-\rho)}{\sigma(1-\rho)+\varphi_x} (\varphi_\pi - \rho) \right\} E_t[\pi_{t+1}].
\end{equation}

Details can be found in Appendix 1.8.2. The term in the brackets of (10) must be negative if $\varphi_\pi < \rho$ for any positive $\varphi_x$, and $\varphi_x > 0$ is then also required by the determinacy condition
For any policy setting that is not explosive or accommodative of sunspots, the real interest rate must decline in expected inflation when $\varphi_\pi < \rho$, rise in expected inflation when $\varphi_\pi > \rho$ or indeed not change in expected inflation (interest rate change matches inflation) when $\varphi_\pi = \rho$. Figure 1.1 presents an illustrative simulation of the period $t$ effects of $u_t$ on the real interest rate for various settings of $\varphi_\pi$. In addition, Figure 1.2 shows the impulse responses to the unit of $u_t$ in DYNARE/MATLAB with the parameter values of $\rho = 0.5$, $\varphi_\pi = 0.3$, $\varphi_x = 22$, $\varphi_g = \sigma = 1$, $k = 0.3$, $\beta = 0.99$. The effect of $u_t$ is to change $r_t$ in the opposite direction after which it returns gradually to the steady-state.

---

6 From equation (9): 

\[ k(\varphi_\pi - \rho + \rho - 1) + (1 - \beta)\varphi_x > 0 \]
\[ k(\varphi_\pi - \rho) + k(\rho - 1) + (1 - \beta)\varphi_x > 0 \]
\[ (1 - \beta)\varphi_x > k(\rho - \varphi_\pi) + k(1 - \rho). \]

If $(1 - \beta) > 0$, 
\[ \varphi_x > \frac{k(\rho - \varphi_\pi) + k(1 - \rho)}{(1 - \beta)}. \]

If $\varphi_\pi < \rho$, $\varphi_x$ is always positive. Since $0 < \beta < 1$, $k > 0$ and $0 < \rho < 1$, $\varphi_x > 0$. 

Figure 1.1: Simulated Deviations in the Real Interest Rate Created by a Unit Shock in $u_t$

Note: Assumed parameters: $\rho = 0.5$, $\Gamma = 2$, $k = 0.3$, $\sigma = 1$, $\beta = 0.99$. The $\phi_{\pi}$ term is set in each case so that the right side of the determinacy condition (9) is equal to 0.05.
Figure 1.2 Impulse Responses to $u_t$: Real Interest Rates Decline with Inflation

Assumed parameters: $\rho = 0.5$, $\varphi_\pi = 0.3$, $\varphi_x = 22$, $\varphi_g = \sigma = 1$, $k = 0.3$, $\beta = 0.99$

It should be noted that setting $\varphi_\pi < \rho$ (with $\varphi_x$ high enough to assure determinacy) will never provide an optimal outcome in this baseline NKM.$^7$ Figure 1.3 illustrates the effect of

$^7$ As a qualification, we mention here that Thurston (2012) demonstrates that if the welfare function maximized by the authorities contains a term reflecting the variance of the interest rate, and if the weight on this variance approaches infinity, the optimal Taylor rule will result in this limit to a constant. Inflation and projected inflation respond positively to $u$-(and $g$-)shocks—thus, another example of the real interest rate declining in inflation. More to the point, this is a case where the optimal policy requires that the real interest rate decline in inflation.
constraining $\varphi_\pi$ to equal or below $\rho$. Determinate Taylor rule settings are restricted to a range that cannot include the optimal setting.

**Figure 1.3. When $\varphi_\pi$ is constrained to be less or equal to $\rho$**

Note: The solid line denotes the optimal condition $\varphi_\pi = \rho + \frac{k(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \varphi_x$. The dotted line indicates the boundaries of the determinacy condition $k(\varphi_\pi - 1) + \varphi_x(1 - \beta) > 0$. The intercept of solid line at the vertical axis must be greater than $\rho$.

It is worth noting also that the optimal setting of the Taylor rule must involve the Taylor Principle (real interest rates rising in inflation), which conforms to the result of CGG (1999, p. 1672) for this model in which the optimal interest rate is (using our notation)
where the term in brackets must be greater than unity. To say that the Taylor Principle should apply is of course using “should” in the normative sense rather than a condition for determinacy.

An Example for g-shocks Only

What if the origin of the inflation in question comes from the demand side (through \( g \))? Does the determinacy condition (9) mean the real interest rate must move in the same direction as inflation? The answer is no, the real interest rate and inflation rate can move in different, opposite direction, although the conditions are less straightforward than in the previous example. We derive two boundary conditions relating \( \varphi_\pi \) and \( \varphi_x \), in addition to one shown earlier for determinacy. The first additional boundary:

\[
\varphi_\pi = \lambda + \frac{(\beta \lambda - 1) \varphi_x}{k},
\]

which implies that the real interest rate declines in expected inflation given a positive g-shock. The second additional boundary:

\[
\varphi_\pi = \lambda + \frac{(\beta \lambda - 1) \varphi_x}{k} + \frac{\sigma(1 - \lambda)(\beta \lambda - 1)}{k},
\]

which implies that the inflation rate and g-shocks are positively relative. Details also can be found in Appendix 1.8.2.

These boundary conditions are illustrated in Figure 1.4. As before, determinacy requires the combination of \( \varphi_\pi \) and \( \varphi_x \) lie to the Northeast of the dotted boundary. The first new boundary is for the condition \( \varphi_\pi < \lambda + \frac{(\beta \lambda - 1) \varphi_x}{k} \) which assures that real interest rates will
decline as $g$ and inflation rise. (Recall that $\lambda$ is the autoregressive parameter in the $g$ process,

$$g_t = \lambda g_{t-1} + \epsilon_t, \; 0 < \lambda < 1.$$  

The second new boundary is $\varphi \pi > \lambda + \frac{(\beta\lambda - 1)\psi_k}{k} + \frac{\sigma(1-\lambda)(\beta\lambda - 1)}{k}$,  

which is required in order for inflation to rise with the $g$-shocks. The shadowed area to the left in the Figure 1.4 represents combinations that provide determinacy, positive impacts of $g_t$ on inflation, and declines in the real interest rates as $g_t$ and inflation both rise. Figure 1.5 illustrates the impulse responses from DYNARE/MATLAB calculations for a downward movement in real interest rates, in response to a positive shock of $g_t$, followed by a gradual increase in the real interest rate back to equilibrium.
Figure 1.4. Regions Where the Real Interest Rates Decline when Inflation Increases Upon Arrival of a $g$ shock

Note: Shadowed area is where demand shocks increase expected inflation which leads to negatively real interest rate effect. Assumed parameters: $\lambda = 0.5, \varphi_\pi = 1.02, \varphi_x = -0.31, \varphi_g = 1, k = 0.3, \beta = 0.99$ and $\sigma = 1$. 
1.6 Conflating the Taylor Rule with the Taylor Principle

The interpretation of the NKM-Taylor rule determinacy condition by Woodford (2001, p. 233) is probably the most elegantly expressed and most widely cited:

The determinacy condition…has a simple interpretation. A feedback rule satisfies the Taylor Principle if it implies that in the event of a sustained increase in the
inflation rate by k percent, the nominal interest rate will eventually be raised by more than k percent. In the context of the model sketched above, each percentage point of permanent increase in the inflation rate implies an increase in the long-run average output gap of $\frac{1-\beta}{k}$ percent; thus a rule of the form conforms to the Taylor Principle if an only if the coefficients $\phi_\pi$ and $\phi_x$ satisfy $\phi_\pi + \frac{1-\beta}{k} \phi_x > 1$. In particular, the coefficient values necessarily satisfy the criterion, regardless of the size of $\beta$ and $k$. Thus the kind of feedback prescribed in the Taylor rule $[\phi_x = 0.5, \phi_\pi = 1.5]$ suffices to determine an equilibrium price level.

What is our objection to the statement above? First, the paragraph implies that the determinacy condition influences the dynamic paths of $\pi_t$ (and presumably $x_t$), whereas we noted earlier that (9) ensures unique paths for $\pi$ and $x$ which are reached instantly according to NKM. Second, Woodford’s statement suggests that in order for these unique paths to be obtained the Taylor Principle must be applied - i.e., real interest rates must rise with inflation. It has become nearly universal to assert that positive co-variation between the real interest rate and inflation is critical in avoiding explosive solutions and sunspots. Table 1.8.3 in Appendix lists recent articles and is grouped according to the level of explicitness with which they conflate the Taylor Principle (real interest rates rising in inflation) with the standard determinacy condition (9).

The important result we found was that the standard eigenvalue condition (9) for this model is sufficient but not necessary in the case of the standard baseline NKM with Taylor rule.8

8 Rising real interest in inflation may be necessary for optimality, but not for uniqueness and stability.
In particular, when the model has only one shock (9) is less “stringent.” For examples, the conditions for only \(u\) shock case:

\[(1 - \beta)\varphi_x + k(\varphi_\pi - \rho) > 0\]

and for only \(g\) shock case:

\[(1 - \beta \lambda)\varphi_x + k(\varphi_\pi - \lambda) > 0.\]

The latter applies over a certain range that cannot have a “too negative” value of \(\varphi_x\). In Woodford (2001), only the \(g\) shock is considered. In CGG (1999), only the \(u\) shock is part of the optimal paths. Thurston (2010, 2012) further demonstrates another essential point in CGG (1999) which the Taylor rule which gives optimal paths subject to being able to have the interest rate jump to offset \(g\) shocks. The result is that \(g\) shocks are effectively removed from the solution. In this case, the solution is a function of only one shock, and the model can be written as two univariate forward equations each having the same “\(b\)”.\(^9\) The condition \(b\) in the unit circle implies the conditions above, which are “inside” the standard one. These are examples for demonstrating the determinacy condition intuitively.

\(^9\) Thurston (2010, 2012) derived an optimal Taylor rule for the model (5), (6), and (8) as

\[\varphi_g = \sigma,\]

and

\[\varphi_\pi = \frac{k}{\Gamma} \varphi_x + \frac{k \sigma (1 - \rho)}{\Gamma} + \rho,\]

which obviously provides multiple possible values for \(\varphi_x\) and \(\varphi_\pi\). As illustrated in Figure 1.6 in Appendix 1.8.1., the unique, optimal paths

\[x_t = -\frac{k}{k^2 + \Gamma (1 - \beta \rho)} u_t,\]

and

\[\pi_t = \frac{\Gamma}{k^2 + \Gamma (1 - \beta \rho)} u_t\]

will necessarily be reached for all combinations of \(\varphi_x\) and \(\varphi_\pi\) that lie to the Northeast of the borderline for determinacy (9).
1.7 Conclusion

The literature has erroneously conflated the Taylor Principle with the NKM determinacy condition. They are two different kettles of fish.

1.8 Appendixes

1.8.1 The Interpretation of Determinacy Condition in CGG(1999)

To demonstrate the optimized model of Clarida, Gali and Gertler (1999) has a less “stringent” determinacy condition than the standard determinacy one: first, solve the model for $\pi_t$ and $x_t$ as function of their two expected values, $E_t[\pi_{t+1}]$ and $E_t[x_{t+1}]$; second, convert these into a two-equation, univariate model using the first order condition, $x_t = -\frac{k}{\Gamma} \pi_t$ and $E_t[x_{t+1}] = -\frac{k}{\Gamma} E_t[\pi_{t+1}]$; third, constrain the $\phi$s to follow the optimal locus ($\phi_{\pi} = \rho + \frac{k(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \phi_x$).

Taking the univariate equations above for $\pi_t$ and $x_t$ as functions of $E_t[\pi_{t+1}]$ and $E_t[x_{t+1}]$, respectively, find a value for $\phi_x$ that makes the coefficient on the expected future values just equal to unity in each equation (the expression will be the same in both equations). The value of $\phi'_x$ for $\pi_t$ shown in Figure 1.6 is:

\[(12) \quad \phi'_x = \frac{(\rho-1)k\Gamma+(2-\rho)k^2\sigma+(1-\beta)\sigma\Gamma}{-k^2-\Gamma+\beta\Gamma}.\]

This establishes the “borderline” $\phi'_x$ and the associated $\phi'_\pi$ which will meet the optimality condition and above which make the coefficients on the respective expected values less than unity.
Second, find the intersection of the optimality condition \( \varphi_\pi = \rho + \frac{k\sigma(1-\rho)}{\Gamma} + \frac{k}{\Gamma} \varphi_x \) and the lower boundary of the determinacy condition expressed as \( k(\varphi_\pi - 1) + (1 - \beta)\varphi_x = 0 \).

The intersection solution for \( \varphi_x \) is \( \varphi''_x \) is:

\[
(13) \quad \varphi''_x = \frac{(\rho - 1)(k\Gamma - k^2\sigma)}{-k^2\Gamma + \beta\Gamma}.
\]

Values of \( \varphi_\pi \) and \( \varphi_x \) must exceed \( \varphi''_x \) and its associated \( \varphi''_\pi \) in Figure 1.6 in order to meet the standard determinacy condition.

Third, subtract (13) from (12):

\[
(14) \quad \varphi''_x - \varphi'_x = \frac{-k^2\sigma - (1-\beta)\sigma\Gamma}{-k^2 - (1-\beta)\Gamma}.
\]

The value of (14) is positive given the fact that both numerator and denominator are negative.

The borderline combinations for \( \varphi_\pi \) and \( \varphi_x \) in the standard case are larger, respectively, than those for the optimized model case.
Figure 1.6. The Optimality Condition and the Determinacy Condition

Note: \([\varphi'_x, \varphi'_\pi]\) is the intercept of borderline and the optimal locus which meets optimality/time-consistency and make the coefficient on the future value equal to one. 
\([\varphi''_x, \varphi''_\pi]\) is the intercept whose values are optimal and meet the standard determinacy lower-bound, 
\[k(\varphi_\pi - 1) + (1 - \beta) = 0.\]

We can observe that two lines, the borderline and the standard determinacy lower-bound, have the same slope. This can be shown analytically by letting the coefficient on the expected future value less than unity in either one of two univariate equations from CGG(1999) model. Rearranging the inequality, we obtain:

\[k(\varphi_\pi - 1) + (1 - \beta)\varphi_\pi > \frac{\Gamma(\beta - 1) - k^2}{\Gamma} \sigma.\]

Since \(\beta\) has the value between zero and one, the right hand side of the inequality always has
negative value.\textsuperscript{10} The determinacy condition of the optimized CGG(1999) implies that the coefficient on the expected future value is less than one. Thus, the determinacy condition for the optimized model of CGG(1999) is less stringent than the standard determinacy condition.

### 1.8.2 The Comovement of Real Interest Rate and Inflation

To derive the condition for which the real interest rate will move in the opposite direction of inflation while the model still satisfies the NKM determinacy condition, first, consider the case where inflation is driven exclusively by \( u_t \) as in (10). Noting that \( \pi_t = \frac{E_t[\pi_{t+1}]}{\rho} \) write the real interest rate \( r_t = i_t - E_t[\pi_{t+1}] = \varphi_{\pi}\pi_t + \varphi_x x_t - E_t[\pi_{t+1}] = \left(\frac{\varphi_{\pi}}{\rho} - 1\right) E_t[\pi_{t+1}] + \varphi_x x_t. \)

Substitute this last expression into (5) and, noting that \( E_t[x_{t+1}] = \rho x_t \), derive (10). The NKM determinacy condition will be met with \( \varphi_{\pi} < \rho \) provided that \( \varphi_x \) is large enough.

To obtain conditions where inflation is exclusively driven by \( g_t \), note that \( E_t[x_{t+1}] = \lambda x_t \) and \( E_t[\pi_{t+1}] = \lambda \pi_t \) and that the real interest rate is proportional to expected inflation:

\[
(15) \quad r_t = \left\{ \varphi_{\pi} - \left(\frac{\beta \lambda - 1}{k} + \lambda\right) \right\} E_t[\pi_{t+1}].
\]

We obtain (15) by substituting out \( E_t[x_{t+1}] \) and \( E_t[\pi_{t+1}] \) in (5) and (6) by \( E_t[x_{t+1}] = \lambda x_t \) and \( E_t[\pi_{t+1}] = \lambda \pi_t \), and then we substitutes them into Taylor rule \( i_t = \varphi_x x_t + \varphi_{\pi}\pi_t \) (note that we let \( \varphi_g = 0 \), so \( g_t \)-shocks are not nullified by \( \varphi_g \)), the real interest rate is that the nominal

\textsuperscript{10} It is also true when the value of \( \Gamma \) either approaches infinity or equals zero. Using L’Hôpital’s rule, we can find the limit of right hand side of inequality is an negative value: \( \sigma(\beta - 1) \), when \( \Gamma \) approaches infinity or equals zero.
interest rate subtracts the expected inflation. One boundary condition used in Figure 1.4 is that
the real interest rate declines in expected inflation; this requires $\varphi_{\pi} < \left(\frac{(\beta\lambda-1)\varphi_X}{k} + \lambda\right)$. Next, substitute (15) and (5) into (6) to derive an expression for inflation (and $E_t [x_{t+1}]$) that is a
function of $g_t$, $\pi_t = \left\{ \varphi_{\pi} - \left[ \lambda + \frac{\sigma(1-\lambda)(\beta\lambda-1)}{k} \right] + \left(\frac{(\beta\lambda-1)\varphi_X}{k}\right) \right\} g_t$. For the coefficient on $g_t$ to be
positive, this requires the another boundary condition that $\varphi_{\pi} > \left[ \lambda + \frac{\sigma(1-\lambda)(\beta\lambda-1)}{k} \right] + \left(\frac{(\beta\lambda-1)\varphi_X}{k}\right)$
which also is illustrated in Figure 7. Finally, the NKM determinacy condition (9) boundary as
illustrated in Figure 7 can be written $\varphi_{\pi} > 1 - \left(\frac{(1-\beta)}{k}\right) \varphi_X$. 
### Three Levels of Conflation

<table>
<thead>
<tr>
<th>Level</th>
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| 1. | Clear-cut statements are made that “real interest rate rises with inflation” and that condition leads to determinacy in NKM | Walsh (2010, p.342): “a policy that raised the nominal interest rate when inflation rose, and raised \( i_t \) enough to increase the real interest rate so that the output gap fell, would be sufficient to ensure a unique equilibrium.”
| | Woodford (2001, p.233), Walsh (2010, p.342), Gali (2008, pp.78-79), *Cochrane (2011, pp. 572-573, p.583) | Gali (2008, pp. 78-79): “the equilibrium will be unique under interest rate rule whenever \( \phi_x \) and \( \phi_\pi \) are sufficiently large enough to guarantee that the real rate eventually rises in the face of an increase in inflation.” |
| 2. | The NKM’s determinacy condition is simply defined as the “long-run” or “generalized” Taylor Principle. No direct statement is provided that the real interest rate must rise in inflation in order to achieve determinacy, but this is implied. | Misleading, in that an implication is made that in the “long run,” or “eventually,” the real rate must rise in order to imply determinacy |
| 3. | Model is simply constructed so that the model will exhibit the Taylor Principle and determinacy provides for a certain condition to hold (for example, \( \phi_\pi > 1 \)). | Misleading also, in that it hints that somehow the fact that the Taylor Principle holds is responsible for determinacy, when in fact it is by coincidence that both Taylor Principle and determinacy occur |

*Cochrane unlike others in this table, criticizes the NKM on the grounds that its mechanism toward arriving at determinate equilibrium is ill-conceived. However, his characterization of the NKM position on this issue is basically the same as the others in Level 1 of this table.
2  THE SOCIAL WELFARE LOSS AND THE SIMPLE TAYLOR RULE
   IN THE BASELINE NEW KEYNESIAN MODEL
Abstract

Taylor rules, in general, have the symbolic expression $i_t = \phi_x x_t + \phi_\pi \pi_t + \phi_g g_t$. This three-term Taylor rule is the most efficient Taylor rule in terms of the social welfare loss measurement (i.e., the minimized social welfare loss involved with the three-term Taylor rule is the smallest value when we compare it with the minimized social welfare loss involved with a one-term Taylor rule ($i_t = \phi_\pi \pi_t$) or a two-term Taylor rule ($i_t = \phi_x x_t + \phi_\pi \pi_t$).) The three-term Taylor rule is used in this paper as the benchmark for comparing the performance of these kinds of Taylor rules. However, some people may object or think that it is impractical or “unrealistic” to expect the central bank (“the Fed”) bases a rule on a shock term ($\phi_g g_t$), and because of this reason, my focus is only on two-term (“simple”) Taylor rules which do not have $\phi_g g_t$ term—i.e., $i_t = \phi_x x_t + \phi_\pi \pi_t$—, in this chapter and next chapter. From now on and through the rest of the dissertation, the two-term Taylor rules are called the simple Taylor rules for distinguishing them from the three-term Taylor rules.

This chapter is a study of the linear relationship of $\phi_x$ and $\phi_\pi$ when the simple Taylor rules are specified as the optimal interest rate rules, which I call the “optimal simple” Taylor rule, for governing the optimal paths of output gap ($x_t$) and inflation rate ($\pi_t$). The first part of this study
develops an approach to obtain the linear relationship of $\varphi_x$ and $\varphi_\pi$—i.e., $\varphi_\pi = -\frac{k}{r(\beta\rho-1)} \varphi_x + \frac{k\sigma(\rho-1)}{r(\beta\rho-1)} + \rho$, which meets the first order condition for minimum social welfare loss, $L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2]$.  

The second part is the discussion of two properties of the optimal linear relationship of $\varphi_x$ and $\varphi_\pi$ that are observed by comparing with the three-term Taylor rules: (a) the linear relationship is the same for governing the optimal paths of $x_t$ and $\pi_t$ whether $g$-shocks are nullified by containing $\varphi_g = \sigma$ in the baseline new Keynesian model or not; (b) the limit of the social welfare loss containing the simple Taylor rule ($i_t = \varphi_x x_t + \varphi_\pi \pi_t$) is at the minimum when the values of $\varphi_x$ and $\varphi_\pi$ are very big (or approaching infinity), and such minimum is the same as the social welfare loss containing the three-term Taylor rule. This implies the three-term Taylor rule with $\varphi_g (= \sigma)$ suggested by Woodford (2001), whose model has different setup but it works out with the same result, is more efficient than the simple (two-term) Taylor rule.
2.1 Introduction

In Chapter 1 we show that a particular (three-term) Taylor rule,

\[ i_t = \varphi_xx_t + \varphi_{\pi}\pi_t + \varphi_gg_t, \]

is specified as the optimal interest rate rule (or the optimal Taylor rule) for governing the optimal paths of output gap \((x_t)\) and inflation rate \((\pi_t)\) when the coefficient values of \(\varphi_x\) and \(\varphi_{\pi}\) has a linear relationship,

\[ \varphi_{\pi} = \frac{k}{\Gamma} \varphi_x + \frac{k\sigma(1 - \rho)}{\Gamma} + \rho, \]

which is the first order condition for minimum social welfare loss whose specification is similar to what is introduced in section 1.3 equation (7),

\[ L_p = \frac{1}{2} \sum_{i=0}^{\infty} \beta^i E_t [\pi_{t+i}^2 + \Gamma x_{t+i}^2], \]

where the subscript \(p\) denotes the present value of the social welfare loss over time\(^{11}\), and when \(\varphi_g = \sigma\)

for nullifying \(g\)-shocks in the baseline new Keynesian model introduced in section 1.3 equation (5):

\[ x_t = E_t [x_{t+1}] - \frac{1}{\sigma} (i_t - E_t [\pi_{t+1}]) + g_t \]

and equation (6):

\[ \pi_t = kx_t + \beta E_t [\pi_{t+1}] + u_t. \]

\(^{11}\) \(L_p\) is a positive value but the \(W\) function (i.e., equation (7) in section 1.3) is a negative value. \(L_p\) and \(W\) both have multiple terms over time. I will only need the first term of \(L_p\) in chapter 2 and chapter 3 because the Fed’s policy strategy is assumed to be the commitment to the Taylor rules and the solution expectation for all period is the same as the solution for period \(t\). The detail explanation for only using the first term of \(L_p\) is in the section 2.2.
The derivation of the linear relationship of $\varphi_x$ and $\varphi_\pi$ shown above is not a difficult task once we nullify $g$-shocks. See the details of the derivation in Appendix 2.5.1. In addition, the linear relationship of $\varphi_x$ and $\varphi_\pi$ for the particular Taylor rule in the case of a central bank’s commitment to the particular Taylor rule can be derived as

$$
\varphi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k \sigma (\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho.
$$

(1)

Appendix 2.5.2 shows the Maple commands for obtaining (1).

However, the derivation of the linear relationship of $\varphi_x$ and $\varphi_\pi$ in the case of a simple Taylor rule,

$$
i_t = \varphi_x x_t + \varphi_\pi \pi_t,
$$

(2)

is not an easy task when the $g$-shocks are not nullified in the baseline new Keynesian model. The first part of this chapter provides an alternative approach to avoid messy analytical result or no result. The alternative approach involves three components: (a) Thurston’s (2010, 2012) method for obtaining general solutions of $x_t$ and $\pi_t$ in the central-bank loss function which is presented in Appendix 2.5.4; (b) the first and second partial derivative tests; and (c) the mathematical concepts of limit.

The second part of this chapter is the discussion of two properties of the optimal linear relationship of $\varphi_x$ and $\varphi_\pi$ in the simple Taylor rule (2): (a) the linear relationship obtained from the case of commitment to the simple Taylor rule (2) is the same as the relationship (1) from the case of commitment to the three-term Taylor rule; (b) the limit of the social welfare loss

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12 In Appendix 2.5.3, I demonstrate the result of the linear relationship of $\varphi_x$ and $\varphi_\pi$ for (2) is either a mess or void when the same approach from Appendix 2.5.2 is applied.
containing the simple Taylor rule \((i_t = \varphi_x x_t + \varphi_{\pi} \pi_t)\) is at the minimum when the values of \(\varphi_x\) and \(\varphi_{\pi}\) are very big (or approaching infinity), and such minimum is the same as the social welfare loss containing the three-term Taylor rule. This implies the three-term Taylor rule with \(\varphi_g (= \sigma)\) suggested by Woodford (2001), whose model has different setup but it works out with the same result, is more efficient than the simple Taylor rule. The first property implies that the optimal linear relationship between \(\varphi_x\) and \(\varphi_{\pi}\) is the same for governing the optimal paths of output gap \((x_t)\) and inflation rate \((\pi_t)\) whether the Taylor rule nullifies \(g\)-shocks or not. The second property implies that the three-term Taylor rule is more efficient because it allows us to perfectly eliminate the \(g\)-shocks while the simple Taylor rule cannot.

The remainder of the paper is organized as follows. Section 2.2 derives the linear relationship of \(\varphi_x\) and \(\varphi_{\pi}\) in the simple Taylor rule (2). The relationship is the constraint for determining the values of \(\varphi_x\) and \(\varphi_{\pi}\) when the social welfare loss is at minimum. Section 2.3 compares the social welfare loss containing the simple Taylor rule and the social welfare loss containing the three-term Taylor rule. I observe that the minimized social welfare loss containing the three-term Taylor rule is smaller than the minimized social welfare loss containing the simple Taylor rule. Furthermore, when \(\varphi_x\) and \(\varphi_{\pi}\) are approaching infinity, the limiting value of the minimized social welfare loss containing the simple Taylor rule is the same as the minimized social welfare loss containing the three-term Taylor rule. Note that I am not suggesting that the Fed can force \(\varphi_x\) and \(\varphi_{\pi}\) to approach infinity. It is only for the comparison purpose not for the practical reason by letting \(\varphi_x\) and \(\varphi_{\pi}\) approach infinity when we want to know the performance of different Taylor rules. When the performance of a Taylor rule is improved, the minimized social welfare loss containing such Taylor rule will be reduced and
move close to the minimized social welfare loss containing the three-term Taylor rule. I will discuss this application in details in Chapter 3. Section 2.4 contains the conclusion. Section 2.5 contains the appendices.

2.2 An Optimal Monetary Policy is Specified by a Simple Taylor Rule

The analytical derivation remains unsolved for the linear relationship which is the constraint for determining the values of $\varphi_x$ and $\varphi_\pi$ when the social welfare loss containing the simple Taylor rule is at minimum. On the other hand, Woodford (2001, p.235) and Thurston (2010, 2012) both solve the linear relationship of $\varphi_x$ and $\varphi_\pi$ in the case of three-term Taylor rule which $g$-shocks are nullified\(^{13}\) by $\varphi_g = \sigma$, whose results are the same as the equation (1) shown in the Section 2.1. This paper provides the method for determining the analytical values of the optimal\(^{14}\) $\varphi_x$ and $\varphi_\pi$ in the NKM from the viewpoint of the central-bank policy actions following from its commitment to the simple Taylor rule (2). The specification of equation (2) is the first step. The second step, I specify the model economy in the new Keynesian theories which is the baseline NKM well defined in Chapter 1 section 1.3 included the equation of output gap $x_t$ from a forward-looking IS curve shown as

$$x_t = E_t x_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + g_t \quad (3a)$$

and the equation of inflation $\pi_t$ from a New Keynesian Philips curve shown as

\[\]  

\(^{13}\) Walsh (2003, p.549) also had the description for using $\varphi_g$ to nullify $g_t$ (my expression of output gap shocks). Note that Walsh used $u_t$ for the expression of output gap shocks in his book.

\(^{14}\) At this point, these are only optimal conditional on the central bank following the simple Taylor rule which guarantee the extremum of loss function. When the central bank commits to the rule, the fully optimal policy are those been studied in Walsh (2003) section 11.3.
\[ \pi_t = kx_t + \beta E_t \pi_{t+1} + u_t, \quad (3b) \]

where \( g_t \) is the demand shocks and \( u_t \) is the inflation shocks. \( 0 < \beta < 1, k, \sigma > 0, 0 < \rho < 1 \) and \( 0 < \lambda < 1 \). For the ease of exposition, the shocks in this paper follow the AR(1) processes\(^{15} \), so

\[ g_t = \lambda g_{t-1} + \varepsilon_t \]  
\[ u_t = \rho u_{t-1} + \eta_t, \]

where \( \varepsilon_t \) and \( \eta_t \) are i.i.d. normally distributed with variances \( \sigma^2_\varepsilon \) and \( \sigma^2_\eta \). The shocks are interrelated by construction in the baseline NKM. 

Inflation rate is correlated to \( g \)-shocks because of \( kx_t \) term in the (3b). Output gap is correlated to \( u \)-shocks because of \( E_t \pi_{t+1} \) term in the (3a). Macroeconomists usually assumed the innovations \( \varepsilon \) and \( \eta \) of \( g \)-shocks and \( u \)-shocks are i.i.d. for simplicity and based on the specification of \( g \)-shocks and \( u \)-shocks. For \( \varepsilon_t \): \( g_t \equiv E_t \hat{y}^f_{t+1} - y^f_{t+1} \) and \( y^f \) is the flexible-price output. \( y^f \) is derived and determined from Calvo’s model of price stickiness, which output is determined by labor input and aggregate productivity disturbance (See Walsh 2003, p.234 and p.244).\(^1 \) For \( \eta_t \): \( u_t \) is assumed, unless it is permanent, to ultimately affect only the price level, it is called price shock (See Walsh 2003, pp.253-4). It is reasonable to assume that \( \varepsilon \) and \( \eta \) are uncorrelated: \( \sigma_{\varepsilon \eta} = 0 \). The productivity disturbance \( \varepsilon \) and price shock \( \eta \) do not necessarily have correlation because productivity related to innovation and R&D and price shock can be arbitrary and pure nominal phenomenon which has no real impact to productivity.

The third step is obtaining the general solutions of \( x_t \) and \( \pi_t \) in equations (3). It starts with substituting \( i_t \) by (2) in (3a), according to the assumption and the suggestion from Walsh (2003) that the central bank commits to the simple Taylor rule (2), and apply the method of

\(^{15}\) We can have a simplifying assumption for having common degree of serial correlation \( \rho = \lambda \) in order to construct an example. For the details of the discussion of serial correlation, see Woodford (2003, pp.514-7).
undetermined coefficients by setting $x_t$ and $\pi_t$ as functions of $g$-shocks and $u$-shocks with undetermined coefficients:

$$x_t = a_1(\lambda g_{t-1} + \varepsilon_t) + a_2(\rho u_{t-1} + \eta_t)$$  \hspace{1cm} (4a)

and

$$\pi_t = b_1(\lambda g_{t-1} + \varepsilon_t) + b_2(\rho u_{t-1} + \eta_t),$$  \hspace{1cm} (4b)

where $a_1, a_2, b_1$ and $b_2$ are undetermined coefficients. The solutions of these undetermined coefficients have the values which $a_1 = -(\beta \lambda - 1)\sigma/\Psi_1$, $a_2 = \rho - \varphi_\pi/\Psi_2$, $b_1 = k\sigma/\Psi_1$, and $b_2 = -(\rho\sigma - \sigma - \varphi_x)/\Psi_2$, where $\Psi_1 = \beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \lambda \varphi_x - k\lambda + k\varphi_\pi - \lambda \sigma + \sigma + \varphi_x$ and $\Psi_2 = \beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \varphi_x - k\rho + k\varphi_\pi - \rho \sigma + \sigma + \varphi_x$. $\Psi_1 \neq 0$ and $\Psi_2 \neq 0$ are required. I will discuss the properties of $\Psi_1$ and $\Psi_2$ later in this section. Then substituting these analytical values into equations (4), we obtain the general solutions of the baseline new Keynesian model:

$$x_t = -\frac{(\beta \lambda - 1)\sigma}{\Psi_1} g_t + \frac{\rho - \varphi_\pi}{\Psi_2} u_t$$  \hspace{1cm} (5a)

and

$$\pi_t = \frac{k\sigma}{\Psi_1} g_t + \left(-\frac{\rho\sigma - \sigma - \varphi_x}{\Psi_2}\right) u_t.$$  \hspace{1cm} (5b)

Appendix 2.5.4 provides Maple commands for a quick review of the derivations of (4) and (5).

In step 4, I assume the central bank’s objective is to minimize a loss function that balances inflation stability against output gap variability (e.g., Walsh (2003, p.533)). Commitment to the simple Taylor rule implies that the central bank adjusts its interest rate in response to $x_t$ and $\pi_t$ in each period (see equation (2)) in response to realized shocks. Furthermore, since I use only $x_t$ and $\pi_t$ in Taylor rules instead of using $x_t$ and $p_t - p_{t-1}$ (i.e., the difference between price levels
in two periods) in Taylor rules, the Taylor rules do not cause optimal inertia.\(^{16}\) In other words, the Fed has once-and-for-all response to the new development in either \(x_t\) or \(\pi_t\), or both. Such new developments are either from realized output gap shocks or inflation rate shocks or both. The monetary policy objective function is then treated as a periodically quadratic social welfare loss function for two target variables and their coefficients because the solution expectation for all periods is the same as the solution for period \(t\). The symbolic expression of the periodically expected quadratic loss function is:

\[
L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2],
\]

where \(E\) is the expectational operator and \(\Gamma\) is the weights on output gap.\(^{17}\) I treat \(\Gamma\) as a structure parameter\(^{18}\) reflecting the society’s preference. Substituting the right-hand side of equations (5a-b) into (6) for \(x_t\) and \(\pi_t\) and taking expectation, we obtain the social welfare loss expressed as a function of the variances of \(g_t\) and \(u_t\), i.e., \(\sigma_\varepsilon^2\) and \(\sigma_\eta^2\), and \(\sigma_{\varepsilon\eta} = 0\) for simplicity, as

\[
L = \frac{1}{2} \left\{ \left( \frac{k \sigma}{\psi_1} \right)^2 \sigma_\varepsilon^2 + \left( \frac{\rho \sigma - \sigma - \varphi \sigma_x}{\psi_2^2} \right)^2 \sigma_\eta^2 \right\} + \Gamma \left[ \left( \frac{(\beta \lambda - 1) \sigma}{\psi_1} \right)^2 \sigma_\varepsilon^2 + \left( \frac{\rho - \varphi \pi}{\psi_2} \right)^2 \sigma_\eta^2 \right]\]

\(^{16}\) See Marest and Thurston (2017) for Taylor rules include price level, “global” optimum and optimal inertia.

\(^{17}\) The reason I use (6) instead of a social welfare loss function includes some lifecycle of outcome variabilities like the one in Chapter 1 section 1.3 equation (7), which is also shown in the previous section (section 2.1) on page 26, because the results in Chapter 2 are prepared and will be applied to the Chapter 3 in a case when a central bank commit to the simple Taylor rule (2) in the beginning of the period \(t\) but he cannot adjust interest rate for unobservable shocks when he sets up the interest rate in the beginning of the period \(t\). To be consistent with the study in the Chapter 3, I use a period quadratic loss function same as the one I will use in the Chapter 3.

\(^{18}\) See Walsh (2003, p.555): \(\Gamma\) is a ratio of deep parameters, \(\Gamma = \left[ \frac{(1-\omega)(1-\omega\theta)}{\omega} \right] \left( \frac{\sigma + \eta \theta}{1 + \eta \theta} \right)\) in Walsh’s notation, which reflects risk aversion \(\sigma\), demand elasticity \(\eta\), markup \(\theta\) from output, discount rate \(\beta\) and fraction \(1-\omega\) of all firms’ optimally adjusting price each period.
\[
\frac{1}{2} \left\{ \left[ \frac{(k^2 + \Gamma(\beta\lambda - 1)^2)(\sigma)}{\psi_1^2} \right] \sigma_\xi^2 + \left[ \frac{(\rho\sigma - \varphi_x)^2 + \Gamma(\rho - \varphi_\pi)^2}{\psi_2^2} \right] \sigma_\eta^2 \right\}.
\]

(7)

I then determine the “optimal” values of \( \varphi_x \) and \( \varphi_\pi \) for the extremum of \( L \) in (7) by using the first partial derivative test. The first partial derivatives with respect to \( \varphi_x \) and \( \varphi_\pi \) of (7) are

\[
\frac{\partial L}{\partial \varphi_x} = \left\{ \frac{k(\rho\sigma - \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta\rho - 1)(\rho - \varphi_\pi)}{\psi_2^3} \right\} \sigma_\eta^2
\]

\[
\left\{ \left[ \frac{(k\sigma)^2 + \Gamma((\beta\lambda - 1)\sigma)^2}{\psi_1^3} \right] (\beta\lambda - 1) \right\} \sigma_\xi^2
\]

(8a)

and

\[
\frac{\partial L}{\partial \varphi_\pi} = \left\{ - \frac{k(\rho\sigma - \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta\rho - 1)(\rho - \varphi_\pi)}{\psi_2^3} \right\} \sigma_\eta^2
\]

\[
\left\{ - \left[ \frac{(k\sigma)^2 + \Gamma((\beta\lambda - 1)\sigma)^2}{\psi_1^3} \right] k \right\} \sigma_\xi^2.
\]

(8b)

Equations (8a) and (8b) tell us how much changes in social welfare loss from the changes in \( \varphi_x \) and \( \varphi_\pi \).

According to the first partial derivative test, the extremum happens when \( L_{\varphi_x} (\varphi_x, \varphi_\pi) = 0 \) and \( L_{\varphi_\pi} (\varphi_x, \varphi_\pi) = 0 \), and the point \( (\varphi_x^*, \varphi_\pi^*) \) is called a stationary point for being the extremum of \( L \) in (7). Let equations (8a) and (8b) jointly equal to zero, I have the simultaneous equation for deriving the stationary point \( (\varphi_x^*, \varphi_\pi^*) \)

\[
\begin{align*}
\left\{ \frac{(k\sigma)^2 + \Gamma((\beta\lambda - 1)\sigma)^2}{\psi_1^3} \right\} \sigma_\xi^2 &= 0 \quad \text{(9a)} \\
\left\{ \frac{(k\sigma)^2 + \Gamma((\beta\lambda - 1)\sigma)^2}{\psi_2^3} (\beta\lambda - 1) \right\} \sigma_\eta^2 &= 0 \quad \text{(9b)}
\end{align*}
\]
Multiplying \( \frac{k}{(\beta \lambda - 1)} \) on both sides of (9a)

\[
\begin{align*}
\left( \frac{k}{(\beta \lambda - 1)} \right) \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\rho - \varphi_\pi)}}{\psi_2^3} \right) \sigma_\eta^2 & + \left( \frac{(k \sigma + \Gamma((\beta \lambda - 1) \sigma)^2) k}{\psi_1^3} \right) \sigma_\zeta^2 = 0 \quad (9c) \\
\left( \frac{k}{(\beta \lambda - 1)} \right) \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\varphi_x + \sigma - \rho \sigma)}}{\psi_2^3} \right) \sigma_\eta^2 & + \left( \frac{(k \sigma + \Gamma((\beta \lambda - 1) \sigma)^2) k}{\psi_1^3} \right) \sigma_\zeta^2 = 0 \quad (9b)
\end{align*}
\]

and then adding up (9c) and (9b), I obtain the equation

\[
\left( \frac{k}{(\beta \lambda - 1)} \right) \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\rho - \varphi_\pi)}}{\psi_2^3} \right) \sigma_\eta^2 + \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\varphi_x + \sigma - \rho \sigma)}}{\psi_2^3} \right) \sigma_\eta^2 = 0 \quad (10).
\]

Equation (10) is solved for \( \varphi_\pi \) as a function of \( \varphi_x \):

\[
\left( \frac{k}{(\beta \lambda - 1)} \right) \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\rho - \varphi_\pi)}}{\psi_2^3} \right) \sigma_\eta^2 + \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\varphi_x + \sigma - \rho \sigma)}}{\psi_2^3} \right) \sigma_\eta^2 = 0
\]

\[
\left( \frac{k}{(\beta \lambda - 1)} \right) \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\rho - \varphi_\pi)}}{\psi_2^3} \right) \sigma_\eta^2 + \left( \frac{\sqrt[3]{(\rho \sigma - \varphi_x) + \Gamma(\rho - \varphi_\pi)(\beta \rho - 1)(\varphi_x + \sigma - \rho \sigma)}}{\psi_2^3} \right) \sigma_\eta^2 = 0
\]

Therefore, I derive two possible coefficient relationships between \( \varphi_x \) and \( \varphi_\pi \) either from the first square bracket of the left-hand side of equation,

\[
\varphi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k \sigma (\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho \quad (11a)
\]
or from the second square bracket of the left-hand side of equation,

\[(\varphi_\pi - \rho)k + (1 - \beta \lambda)\varphi_x = (1 - \rho)(\beta \lambda - 1)\sigma.\]  \hspace{1cm} (11b)

These two possible relationships each will produce a set of stationary point \((\varphi_x^*, \varphi_\pi^*)\), but only \((11a)\) is valid because \((11b)\) violates the NKM determinacy condition, \((\varphi_\pi - 1)k + (1 - \beta)\varphi_x > 0\), for being an negative value\(^{19}\) in the right-hand side. The valid linear relationship \((11a)\) is the constraint for determining the values of the coefficients in the simple Taylor rule \((2)\).

Next I determine \(\varphi_x^*\) and \(\varphi_\pi^*\) by using \((11a)\) and \((11b)\). Substituting \((11a)\) into \((11b)\) and solving \((11b)\) for \(\varphi_x\), I obtain

\[\varphi_x^* = (\rho - 1)\sigma.\] \hspace{1cm} (12)

Substituting \((12)\) into \((11a)\) and solving \((11a)\) for \(\varphi_\pi\), I obtain

\[\varphi_\pi^* = \rho.\] \hspace{1cm} (13)

Substituting \((12)\) and \((13)\) into \((7)\), and note that

\[\left[ (\rho\sigma - \sigma - \varphi_x)^2 + \Gamma(\rho - \varphi_\pi)^2 \right] \sigma_{\eta}^2\]

from the second square bracket of \((7)\) can be further simplified and factored as:

\[\]

\[^{19}\]Rearrange the NKM determinacy condition:

\[(\varphi_\pi - 1)k + (1 - \beta)\varphi_x > 0\]

\[\Rightarrow (\varphi_\pi - 1 + \rho - \rho)k + (1 - \beta + \beta \lambda - \beta \lambda)\varphi_x > 0\]

\[\Rightarrow (\varphi_\pi - \rho)k + (1 - \beta \lambda)\varphi_x + (\rho - 1)k + (\beta \lambda - \beta)\varphi_x > 0\]

\[\Rightarrow (\varphi_\pi - \rho)k + (1 - \beta \lambda)\varphi_x > (1 - \rho)k + \beta (1 - \lambda)\varphi_x > 0, \hspace{1cm} (\text{if } \varphi_x > 0)\]

but \((\varphi_\pi - \rho)k + (1 - \beta \lambda)\varphi_x = (1 - \rho)(\beta \lambda - 1)\sigma < 0,\]

therefore \((11b)\) is invalid.

\[^{20}\]To verify \((12)\) and \((13)\), I substitute \((12)\) into \((11b)\) and solve \((11b)\) for \(\varphi_\pi\), which I obtain the same solution of \(\varphi_\pi\) as \((13)\).
so it becomes \( \frac{0}{0} \) when \( \varphi_x \) and \( \varphi_\pi \) are substituted out by \( \varphi_x^* = (\rho - 1)\sigma \) and \( \varphi_\pi^* = \rho \) at the same time. I obtain the extremum of the social welfare loss:

\[
0 = \frac{\sigma_\eta^2 + \frac{(\Gamma'(\beta - 1)^2 + k^2)\sigma^2}{2((\beta - 1)\sigma - k)^2(\rho - \lambda)^2}\sigma_\varepsilon^2}{((\beta - 1)\sigma - k)^2 + k(\rho - \varphi_\pi)^2},
\]

(14)

However, the above extreme is not possible to achieve because \( \varphi_x^* \) and \( \varphi_\pi^* \) cannot be applied to the simple Taylor rule (2). Substitute (12) and (13) into the NKM determinacy condition, \((\varphi_\pi - 1)k + (1 - \beta)\varphi_x > 0\), I obtain

\[
(\rho - 1)\sigma(1 - \beta) + k(\rho - 1) > 0,
\]

because

\[
(\rho - 1)\sigma(1 - \beta) < 0
\]

and

\[
k(\rho - 1) < 0,
\]

therefore, \( \varphi_x^* \) and \( \varphi_\pi^* \) cannot both be the coefficient values of the simple Taylor rule (2) at the same time for governing the optimal paths of \( x_t \) and \( \pi_t \). This can be proved by substituting (12) and (13) into (5a) and (5b) for obtaining the optimal paths of \( x_t \) and \( \pi_t \), and I obtain no result for \( \pi_t \) because of division by zero. Since (12) and (13) are derived from (11a) and (11b), this unfortunate result is obvious because (11b) violates the NKM determinacy condition. Thus, (11a) is the only linear relationship of \( \varphi_x \) and \( \varphi_\pi \) which produces a set of stationary point for obtaining the optimal paths of \( x_t \) and \( \pi_t \) when the simple Taylor rule (2) is specified as the optimal Taylor rule. For example, I can obtain the optimal paths of \( x_t \) and \( \pi_t \) by substituting (11a) and (12) into (5a) for optimal \( x_t \):
\[ x_t = -\frac{k}{\Gamma(\beta \rho - 1)^2 + k^2} u_t - \frac{\sigma(\beta \lambda - 1)}{(\rho - \lambda)(k + \sigma - \beta \lambda \sigma)} g_t, \]  

and by substituting (11a) and (12) into (5b) for optimal \( \pi_t \):

\[ \pi_t = \frac{\Gamma(1 - \beta \rho)}{\Gamma(\beta \rho - 1)^2 + k^2} u_t + \frac{k \sigma}{(\rho - \lambda)(k + \sigma - \beta \lambda \sigma)} g_t. \]

(16)

In addition, I can obtain the extremum of social welfare loss by substituting (11a) and (12) into (7):

\[ L = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2 + \frac{(\Gamma(\beta \lambda - 1)^2 + k^2) \sigma^2}{2((\beta \lambda - 1) \sigma - k)^2 (\rho - \lambda)^2} \sigma_\varepsilon^2. \]

(17)

With this, I complete the derivation.

**A Numerical Experiment:**

This experiment shows that the minimum \( L \) occurs when the points on \( L \) is satisfied by (11a). Note the previous analysis is based on equations (5a-b). Since equations (5a-b) have the same expressions as the case involve the particular Taylor rule \( i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t \) with \( \varphi_g = 0 \), for comparison purpose, this experiment is conducted based on the case involve the particular Taylor rule. I will show that (11a) holds for different values of \( \varphi_g \). In Figure 2.1, the values of \( \varphi_x \) and \( \varphi_\pi \) are plotted along the horizontal axis and the values of \( L \) are plotted along the vertical axis. The value of \( \varphi_g \) is presumed to equal \( \sigma \) for nullifying \( g \)-shocks. The parameter values used in Figure 2.1 are \( k = 0.03, \beta = 0.99, \Gamma = 0.05, \sigma = 1, \lambda = 0.9, \) and \( \rho = 0.9 \). The shocks are assumed to be unity so \( \sigma_\varepsilon^2 = 1 \) and \( \sigma_\eta^2 = 1 \). The social welfare loss (which has the \( \varphi_g = \sigma \) built in) is the yellow surface with contours, which is
\[ L_{PTR} = \frac{1}{2} \left\{ \left( \frac{k(\sigma - \varphi_g)}{\psi_1} \right)^2 \sigma_\varepsilon^2 + \left( \frac{\rho \sigma - \sigma - \varphi_x}{\psi_2} \right)^2 \sigma_\eta^2 \right\} \\
+ \Gamma \left\{ \left( \frac{(\beta \lambda - 1)(\sigma - \varphi_g)}{\psi_1} \right)^2 \sigma_\varepsilon^2 + \left( \frac{\rho - \varphi_\pi}{\psi_2} \right)^2 \sigma_\eta^2 \right\} \right\}
\]
\[ = \frac{1}{2} \left\{ \left( \frac{k^2 + \Gamma(\beta \lambda - 1)^2}{\psi_1} \right)(\sigma - \varphi_g)^2 \sigma_\varepsilon^2 + \left[ \frac{(\rho \sigma - \sigma - \varphi_x)^2 + \Gamma(\rho - \varphi_\pi)^2}{\psi_2^2} \right] \sigma_\eta^2 \right\} \]

—i.e.,
\[ L_{PTR} = \frac{0.025(0.9 - \varphi_\pi)^2}{(0.0161 - 0.109\varphi_x - 0.03\varphi_\pi)^2} + \frac{1}{2} \left( \frac{(-0.1 - \varphi_x)^2}{(0.0161 - 0.109\varphi_x - 0.03\varphi_\pi)^2} \right), \]

where the subscript \( PTR \) denotes the particular Taylor rule—i.e., \( i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t \).

Note that the difference between (7) and \( L_{PTR} \) is that the coefficient term on the \( \sigma_\varepsilon^2 \) has \((\sigma)^2\) with the simple Taylor rule and the coefficient term on the \( \sigma_\varepsilon^2 \) has \((\sigma - \varphi_g)^2\) with the particular Taylor rule. In other words, we can treat the case with the simple Taylor rule has the same welfare loss as the case with the particular Taylor rule with the particular Taylor rule and \( \varphi_g = 0 \).

The green, bottom plane of the social welfare loss has the (11a) and \( \varphi_g = \sigma \) built in, which is
\[ L_{PTR, \min} = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2 \]

—i.e.,
\[ L_{PTR, \min} = 16.73304106, \]

where the subscript \( PTR \) denotes particular Taylor rule and the subscript \( min \) denotes minimum value. The grey plane is tracing out (11a)—i.e.,
\[ \varphi_\pi = 1.450458716 + 5.504587156\varphi_x. \]
In other words, (11a) is the line where the yellow, top surface tangent to the green, bottom plane.
So when (11a) is satisfied, $L$ is minimized.

**Figure 2.1. The Optimal Constraint of $\phi_x$ and $\phi_\pi$**

Does different values of $\phi_g$ change the relationship between minimum $L$ and (11a)? Put it differently, does minimum $L$ occurs according to (11a) when $g$-shocks are not nullified? Does the value of $\phi_g$ affect the linear relationship of $\phi_x$ and $\phi_\pi$? No, the value of $\phi_g$ does not affect
the linear relationship of $\varphi_x$ and $\varphi_\pi$. This is because (11a) can be derived directly from solving

$$\frac{\partial f}{\partial \varphi_x} = 0 \text{ and } \frac{\partial f}{\partial \varphi_\pi} = 0$$

together for $\varphi_x$ and $\varphi_\pi$. Recall that Figure 2.1 has $\varphi_g = \sigma$ built in, so I first produce three additional plots with different values of $\varphi_g$ built in for comparing with Figure 2.1. Figure 2.2 shows the results of the comparison for four different values of $\varphi_g$, (a) has $\varphi_g = \sigma = 1$, (b) has $\varphi_g = 0$, (c) has $\varphi_g > \sigma$—i.e., $\varphi_g = 2$, and (d) has $\varphi_g < \sigma$—i.e., $\varphi_g = 0.5$.

In addition, Figure 2.2(a) implies optimal paths of $x_t$ and $\pi_t$. Figure 2.2(b) implies the policy-rate path is based on the simple Taylor rule, $i_t = \varphi_x x_t + \varphi_\pi \pi_t$. All four sub-figures have the same result that the grey plane is tracing out (11a) where the yellow, above surface tangents to the green, below plane which has (11a) built in.

It is clear that (11a) holds whether $g_t$ is nullified by $\varphi_g$ or not. To make it more convincing, Appendix 2.5.5 shows the results of 2-D plots that minimum $L$ occurs where the points satisfied (11a) on $L$ (7) by using different parametrization ($k = 0.03, \beta = 0.99, \Gamma = 0.05, \sigma = 1, \lambda = 0.3, \rho = 0.7, \sigma_\epsilon^2 = 1$ and $\sigma_\eta^2 = 1$) for the case involving the simple Taylor rule.
Figure 2.2. The Optimal Constraint of $\varphi_x$ and $\varphi_\pi$ with Different Values of $\varphi_g$
2.2.1 The Validity of (11a) for the Optimal Simple Taylor Rule

I now consider the validity of (11a) for optimal policy—the optimal equilibrium relationship between \( x_t \) and \( \pi_t \) in equations (3). For (2) to be a policy rule, it must satisfy the NKM determinacy condition. Do (11a) satisfy the NKM determinacy condition? Since (11a) is derived based on the values of \( x_t \) and \( \pi_t \), we should start from equation (5a-b). The denominators \( \Psi_1 \) and \( \Psi_2 \) in equation (5a-b) must be non-zero values, so the values of \( x_t \) and \( \pi_t \) are defined by the right-hand sides of equation (5a-b). Therefore, \( \Psi_1 \) can be rearranged as

\[
\left\{ \left[ \varphi_\pi + \varphi_x \left( \frac{1 - \beta \lambda}{k} \right) \right] + \left[ \frac{\sigma}{k} (\lambda - 1)(\beta \lambda - 1) \right] - \lambda \right\} k. \tag{18}
\]

The terms in the first square bracket is similar to the determinacy condition of the baseline new Keynesian model, \( \varphi_x (1 - \beta) + k (\varphi_\pi - 1) > 0 \). The NKM determinacy condition implies that

\( \varphi_\pi + \varphi_x \left( \frac{1 - \beta}{k} \right) > 1 \), and since \( \left( \frac{1 - \beta \lambda}{k} \right) \geq \left( \frac{1 - \beta}{k} \right) \), we know that the value of the first square bracket is greater than one when the determinacy condition is satisfied.\(^{21}\) The value of the second square bracket is always positive.\(^{22}\) Thus (18) shows that \( \Psi_1 > 0 \) when the determinacy condition is satisfied. The difference between \( \Psi_1 \) and \( \Psi_2 \) is that \( \Psi_1 \) includes the serial

---

\(^{21}\) \( \varphi_x + \varphi_x \left( \frac{1 - \beta}{k} \right) > 1 \) do not imply that \( \varphi_x \) and \( \varphi_\pi \) both have to be positive. First, the derivation of it from the NKM determinacy condition does not require that \( \varphi_x \) and \( \varphi_\pi \) both have to be positive, which is shown below,

\[
\begin{align*}
\varphi_x (1 - \beta) + k (\varphi_\pi - 1) &> 0, \\
\varphi_x (1 - \beta) &> -k (\varphi_\pi - 1), \\
\varphi_x (1 - \beta) &> k (-\varphi_\pi + 1), \\
\varphi_x \left( \frac{1 - \beta}{k} \right) &> -\varphi_\pi + 1, \\
\varphi_\pi + \varphi_x \left( \frac{1 - \beta}{k} \right) &> 1.
\end{align*}
\]

Second, \( \varphi_\pi + \varphi_x \left( \frac{1 - \beta}{k} \right) > 1 \) is satisfied by having either \( \varphi_x < 0 \) or \( \varphi_\pi < 0 \), see Appendix 2.5.7 for a plot illustration.

\(^{22}\) \((\lambda - 1) < 0 \) and \((\beta \lambda - 1) < 0 \), so \((\lambda - 1)(\beta \lambda - 1) > 0 \).
correlation parameter $\lambda$ and $\Psi_2$ includes the serial correlation parameter $\rho$. The same conclusion also applies to $\Psi_2$. Now, substituting (11a) into the NKM determinacy condition and the condition for the optimal simple Taylor rule is then determined by

$$\varphi_x > \frac{(\rho - 1)(-\beta^2\lambda^2\sigma + (k\rho + 2\lambda\sigma)\beta - k - \sigma)}{((-\lambda^2 + \rho)\beta - \rho + 2\lambda - 1)\beta}, \text{if } \beta < 1 + \frac{(\lambda\beta - 1)^2}{\rho\beta - 1}$$

or

$$\varphi_x < \frac{(\rho - 1)(-\beta^2\lambda^2\sigma + (k\rho + 2\lambda\sigma)\beta - k - \sigma)}{((-\lambda^2 + \rho)\beta - \rho + 2\lambda - 1)\beta}, \text{if } \beta > 1 + \frac{(\lambda\beta - 1)^2}{\rho\beta - 1}.$$ 

These two inequalities has an important implication when I determine whether the extremum is a minimum as the result I want to have for the social welfare loss (7) or a maximum which is an unwanted result for the social welfare loss (7). The implication is found when the common degree of serial correlation $\rho = \lambda$ is assumed and these two inequalities is simplified as

$$\varphi_x > \rho\sigma - \frac{k + \sigma}{\beta}, \text{if } \rho > 1$$

or

$$\varphi_x < \rho\sigma - \frac{k + \sigma}{\beta}, \text{if } \rho < 1.$$ 

Since the value of serial correlation $\rho$ is between zero and one, I obtain an interesting condition for the paths of $x_t$ and $\pi_t$ to be unique, existent, and optimal under the commitment to the optimal simple Taylor rule as

$$\varphi_x < \rho\sigma - \frac{k + \sigma}{\beta}. \quad (19)$$
It is easy to see that the value of $\varphi_x$ in (19) is always negative. In other words, when the common degree of serial correlations are the same, i.e., $\rho = \lambda$, the value of $\varphi_x$ must be negative for having the optimal paths of $x_t$ and $\pi_t$ under the commitment to the optimal simple Taylor rule. In the next section, I will apply the implication for determining the limit value of $L$ in (7) is at the minimum.

We can also easily observe from (18) that $\Psi_1$ and $\Psi_2$ are positively affected by higher $\varphi_x$ and $\varphi_\pi$, if only (11a) is satisfied and $\Gamma$ is assumed to be positive. This observation implies that at least there is the limit value of $L$ in (7) as either $\varphi_x$ or $\varphi_\pi$ approaches infinity. If this limit value is the same value when the economy is at its optimality, it suggests that the extremum of the loss function by commitment to the simple Taylor rule may only have a limit and does not have an unconstrained solution when the economy is not at its optimality. Does the value of $L$ at the stationary point $(\varphi_x^*, \varphi_\pi^*)$ in (7) constitute a minimum, or at least, is its limit a minimum? This is the question of determining the extremum a minimum or a maximum using the second partial derivative test. Here we turn to Section 2.3 for determining the extremum of $L$ in (7). In addition, Section 2.3 will discuss the efficiency in terms of the extremum of the loss function under both the commitment to the simple Taylor rule using the general-solution approach and under the three-term Taylor rule using the common approach.

---

23 $\rho \sigma - \frac{k+\sigma}{\beta} = \left(\rho - \frac{1}{\beta}\right) \sigma - \frac{k}{\beta} < 0$ because $0 < \beta < 1$, $0 < \rho < 1$ and $\left(\rho - \frac{1}{\beta}\right) < 0$.

24 This condition is consistent with the Figure 1.4 in Chapter 1.

25 This is no longer the case under the optimal simple Taylor rule (2) and the relation of $\varphi_x$ and $\varphi_\pi$ implied by (19) is therefore not applied. This is the case which demand shocks are not neutralized completely and cause the output gap and inflation to fluctuate inefficiently, see Walsh (2003, p.549) for details.
2.3 The Efficiency of Monetary Policy Under Commitment to the Simple Taylor Rule

The second partial derivative test shows that the second-order sufficient condition of (7) equal to zero, i.e.,

\[ \frac{\partial^2 L}{\partial \varphi_x^2} + \frac{\partial^2 L}{\partial \varphi_\pi^2} - \left( \frac{\partial^2 L}{\partial \varphi_x \partial \varphi_\pi} \right)^2 = 0. \]

The details of the result of the second partial derivative test are in the Appendix 2.5.6. This implies the stationary value of \( L \) can be a relative minimum, a relative maximum or a saddle point. This section I show how to determine the extremum at stationary points. When the three-term Taylor rule (1) is implemented to the baseline new Keynesian model and \( g \)-shocks are nullified by having \( \varphi_g = \sigma \), the minimum social welfare loss function is a function of \( \sigma_\eta^2 \), i.e.,

\[ L_0 = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2, \tag{20} \]

as Thurston (2010, 2012) shows which is an unconstrained solution. We can get the similar result of minimized loss function from (7). Substituting (11a) into the right hand side of (7) for \( \varphi_x \) and \( \varphi_\pi \), I obtain the extremum of the loss function \( L_1 \) under the commitment to the simple Taylor rule,

\[ L_1 = \frac{1}{2} \left( \frac{\Gamma(\beta \lambda - 1)^2 + k^2}{D^2} \right) \sigma_\varepsilon^2 + \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2, \tag{21} \]

where \( D = k \left( -\frac{k}{\Gamma(\beta \rho - 1)} \varphi_x + \frac{k\sigma(\rho-1)}{\Gamma(\beta \rho - 1)} + \rho \right) + (1 - \beta \lambda) \varphi_x + (\lambda - 1)(\beta \lambda - 1)\sigma - k\rho. \)

The difference between (20) and (21) is because the three-term Taylor rule (1) neutralizes the effect of \( g_t \) while the simple Taylor rule (2) does not. On the other hand, for the case of
commitment to the simple Taylor rule, although the simple Taylor rule (2) includes the optimal linear relationship of $\varphi_x$ and $\varphi_\pi$ (11a), it cannot neutralize the effect of $g_t$.

Second, from the previous section we learn $\Psi_1$ and $\Psi_2$ are positively affected by higher $\varphi_x$ and $\varphi_\pi$. Set the limit of $L$ (7), as $\varphi_\pi$ is approaching infinity by letting $\varphi_x$ approach infinity according to (11a), shown by the following equation:

$$
\lim_{\varphi_\pi \to \infty} L = \frac{1}{2} \lim_{\varphi_\pi \to \infty} \left\{ \left( \frac{(k\sigma + \Gamma((\beta\lambda - 1)\sigma)}{\psi_1^2} \right)^2 \sigma_\varepsilon^2 + \left( \frac{(\rho\sigma - \sigma - \varphi_x)^2 + \Gamma(\rho - \varphi_\pi)^2}{\psi_2^2} \right)^2 \sigma_\eta^2 \right\}
$$

$$
= \frac{1}{2} \lim_{\varphi_\pi \to \infty} \left\{ \left( \frac{(\rho\sigma - \sigma - \varphi_x)^2 + \Gamma(\rho - \varphi_\pi)^2}{\psi_2^2} \right)^2 \sigma_\eta^2 \right\}. \quad (22)
$$

Using the L'Hôpital's rule twice, we obtain the limit of (22):

$$
\lim_{\varphi_\pi \to \infty} L = \frac{1}{2} \left( \frac{\Gamma}{(\beta\rho - 1)^2 + k^2} \right) \sigma_\eta^2. \quad (23)
$$

The same result of (23) can be derived by taking the limit of $L_1$ (21) as $\varphi_x$ and $\varphi_\pi$ are approaching infinity.

$$
\lim_{\varphi_\pi \to \infty} L_1 = \frac{1}{2} \lim_{\varphi_\pi \to \infty} \left( \frac{(\Gamma(\beta\lambda - 1)^2 + k^2)\sigma^2}{D^2} \right) \sigma_\varepsilon^2 + \frac{1}{2} \lim_{\varphi_\pi \to \infty} \left( \frac{\Gamma}{(\beta\rho - 1)^2 + k^2} \right) \sigma_\eta^2
$$

$$
= \frac{1}{2} \left( \frac{\Gamma}{(\beta\rho - 1)^2 + k^2} \right) \sigma_\eta^2. \quad (24)
$$

In other words, jointly, when the optimal values of $\varphi_x$ and $\varphi_\pi$ in the simple Taylor rule (i.e., (11a)) is met and these $\varphi_x$ and $\varphi_\pi$ are approaching infinity according to (11a), $L_1$ is approaching
L0. Figure 2.3 illustrates this property\(^{26}\) of optimal values of \(\varphi_x\) and \(\varphi_\pi\) with arbitrary parameter values that \(k = 0.3, \beta = 0.99, \Gamma = 2, \sigma = 1, \lambda = 0.5, \rho = 0.5, \sigma_\varepsilon^2 = 1\) and \(\sigma_\eta^2 = 1\).

To determine (24) is a minimum, I compare (24) and (21) for all \(\varphi_x > -\infty\). The logic\(^{27}\) is as follow: once I determine the function \(L = L(\varphi_x^*, \varphi_\pi^*)\) and \(\varphi_\pi^* = f(\varphi_x^*)\) according to (11a), I can write \(L = L(\varphi_x^*, f(\varphi_x^*)) = \tilde{L}(\varphi_x^*)\). Also let \(\tilde{L}(-\infty)\) is the symbolic expression when the value of equation (21) is at \(\varphi_x\) approaching infinity. If \(\tilde{L}(-\infty) < \tilde{L}(\varphi_x^*)\) for all \(\varphi_x \in (-\infty, \infty)\), \(\tilde{L}(-\infty)\) is the minimum. \(\tilde{L}(-\infty)\) is determined by (24) with the assumption of \(\rho = \lambda\) so (19) is applied for having a negative \(\varphi_x\). Thus,

\[
\tilde{L}(-\infty) = \frac{1}{2} \left( \frac{\Gamma}{(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2.
\]

\(\tilde{L}(\varphi_x^*)\) is identified by (21):

\[
\tilde{L}(\varphi_x^*) = \frac{1}{2} \left( \frac{(\Gamma (\beta \lambda - 1)^2 + k^2) \sigma_\varepsilon^2}{D^2} \right) \sigma_\varepsilon^2 + \frac{1}{2} \left( \frac{\Gamma}{(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2.
\]

\(\varphi_x\) is only in the first bracket of the right-hand side of (21) associated with \(D\) and the value of the first bracket is always positive for all \(\varphi_x > -\infty\) because the terms in the first bracket are squared except \(\Gamma\). Thus, (24) is a minimum because \(\tilde{L}(-\infty) < \tilde{L}(\varphi_x^*)\). Furthermore, since \(\tilde{L}(-\infty) = L0\), we learn that the three-term Taylor rule (1) is more efficient than the simple Taylor rule although

\(^{26}\) In Chapter 3, I will apply this property of \(\varphi_x\) and \(\varphi_\pi\) in the case which the central bank commit to the simple Taylor rule but its target variables (output gap and inflation) cannot be observed at the time when the decision must be made by the central bank.

\(^{27}\) I would like to thank Professor Wim Vijverberg for pointing out this logic to me. All the errors of this approach are mine.
nullifying $g$-shocks is not necessary for the specification of the optimal Taylor rule for governing the optimal paths of $x_t$ and $\pi_t$ with the linear relationship (11a) of $\varphi_x$ and $\varphi_\pi$.

**Figure 2.3. $L1$ Approaches $L0$**

2.4 Conclusion

This paper shows how to derive the linear relationship of the $\varphi_x$ and $\varphi_\pi$ in the simple Taylor rule and studies the properties of the linear relationship of $\varphi_x$ and $\varphi_\pi$ under the commitment to the simple Taylor rule (2). In summary, the steps of the derivations of the optimal values of $\varphi_x$ and $\varphi_\pi$ in the simple Taylor rule are as follows:

1. Specify the simple Taylor rule, which is an instrument rule;
(2) Assume that the central bank commits to the simple Taylor rule, therefore, we substitute the simple Taylor rule into NK-IS for $i_t$;

(3) Derive Thurston’s (2010,2012) general solutions of target variables output gap and inflation rate;

(4) Specify the policy objective function as a loss function described by periodical weighted sum of the variances of the general solutions of output gap and inflation. Periodical because interest rate is adjusted period-by-period according to the values of $x$ and $\pi$ in each period;

(5) Derive the optimal linear relationship of $\varphi_x$ and $\varphi_\pi$ in the simple Taylor rule by using the first partial derivative test;

(6) Determine the limit of optimal $L$ (as expressed by equation (24)) at $\varphi_x$ approaching infinity is a minimum by comparing it with the optimal $L$ (as expressed by equation (21)) for all $\varphi_x \in (-\infty, \infty)$.

I then find the optimal linear relationship between $\varphi_x$ and $\varphi_\pi$ under the commitment to the simple Taylor rule is the same as the one under the three-term Taylor rules.

When $\varphi_x$ and $\varphi_\pi$ approach infinity, the limit value of the central-bank loss function under the commitment to the simple Taylor is the same as the minimized loss function under the three-term Taylor rule. Note that I am not suggesting that the Fed can force $\varphi_x$ and $\varphi_\pi$ to approach infinity. It is only for the comparison purpose not for the practical reason by letting $\varphi_x$ and $\varphi_\pi$ approach infinity when we want to know the performance of different Taylor rules. When the performance of a Taylor rule is improved, the minimized social welfare loss containing such
Taylor rule will be reduced and move close to the minimized social welfare loss containing the three-term Taylor rule. I will discuss this application in details in Chapter 3.

2.5 Appendixes

2.5.1 Deriving the Optimal Loci for $\varphi_x$ and $\varphi_\pi$

Using the optimal paths of output gap,

$$x_t = -\frac{k}{k^2 + \Gamma (1 - \beta \rho)} u_t,$$

and inflation rate,

$$\pi_t = \frac{\Gamma}{k^2 + \Gamma (1 - \beta \rho)} u_t$$

for the discretionary case from Thurston (2010, 2012),\(^{28}\) and let

$$\varphi_g = \sigma$$

for nullifying $g$-shocks, the current output gap and inflation rate is then only affected by $u$-

----------------------------------------

\(^{28}\) These optimal paths are derived by using the first order condition

$$x_t = -\frac{k}{\Gamma} \pi_t,$$

which is the marginal rate of substitution for each period between output gap and inflation in the social welfare loss function,

$$L = \frac{1}{2} \sum_{i=1}^{\infty} \beta^i E_t[\pi_{t+i}^2 + \Gamma x_{t+i}^2],$$

and the forward looking expectation of inflation rate at steady state,

$$E_t[\pi_{t+1}] = \rho \pi_t.$$

Substituting $x_t = -\frac{k}{\Gamma} \pi_t$ and $E_t[\pi_{t+1}] = \rho \pi_t$ into the Phillips equation $\pi_t = k x_t + \beta E_t[\pi_{t+1}] + u_t$, and solving for $\pi_t$, I obtain the optimal path of inflation rate for discretionary case:

$$\pi_t = \frac{k^2 + \Gamma (1 - \beta \rho)}{\Gamma} u_t.$$

I then also obtain the optimal path of output gap for discretionary case:

$$x_t = -\frac{k}{k^2 + \Gamma (1 - \beta \rho)} u_t$$

by multiplying the optimal path of inflation rate by $-\frac{1}{\Gamma}$. 

53
shocks, so I can assume $E_t[x_{t+1}] = \rho x_t$ and $E_t[\pi_{t+1}] = \rho \pi_t$ at steady state. Substituting all of the above equations into the baseline new Keynesian IS equation described in Chapter 1 section 1.3, and applying the particular Taylor rule ($i_t = \phi_x x_t + \phi_\pi \pi_t + \phi_g g_t$) described in Chapter 1 section 1.4,

$$\phi_\pi = \frac{k}{\Gamma} \phi_x + \frac{k\sigma(1-\rho)}{\Gamma} + \rho$$

is obtained by simplifying and rearranging the equation.  

2.5.2 Deriving the (Pre-commitment) Optimal Loci for $\phi_x$ and $\phi_\pi$ in Maple

The Maple commands in italics for obtaining the linear relationship of $\phi_x$ and $\phi_\pi$ in the case of a central bank’s commitment to the particular Taylor rule are as follow:

```
The NK-IS:
BIS := x_t = EX - \frac{1}{\sigma}(i_t - EP) + g_t

Note: EX is $E_t[x_{t+1}]$ and EP is $E_t[\pi_{t+1}]$, EX and EP are forward-looking variables. The demand shock, $g_t$, is an AR(1) process $g_t = \lambda g_{t-1} + \epsilon_t$.

The NKPC:
BLM := \pi_t = k \cdot x_t + \beta \cdot EP + u_t

Note: The cost shock, $u_t$, is an AR(1) process $u_t = \rho u_{t-1} + \eta_t$.

The particular Taylor Rule:
BTR := i_t = \phi_x \cdot x_t + \phi_{\pi} \cdot \pi_t + \phi_g \cdot g_t

Solving Baseline NKM by Undetermined Coefficient Method:
```

\[
x_t = \rho x_t - \frac{1}{\sigma}(\phi_x x_t + \phi_\pi \pi_t + \sigma g_t - \rho \pi_t)
\]

\[
\Rightarrow [(1-\rho)\sigma + \phi_x]x_t = (\rho - \phi_\pi)\pi_t
\]

\[
\Rightarrow [(1-\rho)\sigma + \phi_x] \left( \frac{-k}{k^2 + \Gamma(1-\beta\rho)} \right) u_t = (\rho - \phi_\pi) \left( \frac{\Gamma}{k^2 + \Gamma(1-\beta\rho)} \right) u_t
\]

\[
\Rightarrow [(1-\rho)\sigma + \phi_x] k = (\phi_\pi - \rho) \Gamma
\]

\[
\Rightarrow \phi_\pi = \frac{k}{\Gamma} \phi_x + \frac{k\sigma(1-\rho)}{\Gamma} + \rho.
\]
$$BIS1 \; := \; x_t = a1 \cdot g_t + a2 \cdot u_t$$

$$BLM1 \; := \; \pi_t = b1 \cdot g_t + b2 \cdot u_t$$

$$BIS2 \; := \; x_{t+1} = a1 \cdot g_{t+1} + a2 \cdot u_{t+1}$$

$$BLM2 \; := \; \pi_{t+1} = b1 \cdot g_{t+1} + b2 \cdot u_{t+1}$$

$$BIS3 \; := \; \text{subs}\left( g_{t+1} = \lambda \cdot g_t + \varepsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BIS2 \right)$$

$$BLM3 \; := \; \text{subs}\left( g_{t+1} = \lambda \cdot g_t + \varepsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BLM2 \right)$$

$$EBIS3 \; := \; EX = a1 \cdot \lambda \cdot g_t + a2 \cdot \rho \cdot u_t$$

$$EBLM3 \; := \; EP = b1 \cdot \lambda \cdot g_t + b2 \cdot \rho \cdot u_t$$

$$BTR1 \; := \; \text{subs}(BIS1, BLM1, BTR)$$

$$sol01 \; := \; \text{subs}(BIS1, BTR1, EBIS3, EBLM3, BIS)$$

$$sol02 \; := \; \text{subs}(BIS1, BLM1, EBLM3, BLM)$$

$$con1 \; := \; \text{subs}\left( g_t = 0, u_t = 0, sol01 \right)$$

$$con2 \; := \; \text{subs}\left( g_t = 1, u_t = 0, sol01 \right)$$

$$con3 \; := \; \text{subs}\left( g_t = 0, u_t = 1, sol01 \right)$$

$$con4 \; := \; \text{subs}\left( g_t = 0, u_t = 0, sol02 \right)$$

$$con5 \; := \; \text{subs}\left( g_t = 1, u_t = 0, sol02 \right)$$

$$con6 \; := \; \text{subs}\left( g_t = 0, u_t = 1, sol02 \right)$$

$$OP \; := \; \text{solve}\left( \{ con2, con3, con5, con6 \}, \{ a1, a2, b1, b2 \} \right)$$

$$solx \; := \; \text{subs}(OP, BIS1)$$

$$solp \; := \; \text{subs}(OP, BLM1)$$

$$solx1 \; := \; x_t = -\frac{\left( \beta \lambda - 1 \right) \left( \sigma - \phi_g \right) g_t}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \phi_x - k \lambda + k \phi - \lambda \sigma + \sigma + \phi_x}$$

$$+ \frac{\left( \rho - \phi_{\pi} \right) u_t}{\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi - \rho \sigma + \sigma + \phi_x}$$

$$solx1 \; := \; x_t = -\frac{\left( \beta \lambda - 1 \right) \left( \sigma - \phi_g \right) g_t}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \phi_x - k \lambda + k \phi - \lambda \sigma + \sigma + \phi_x}$$

$$+ \frac{\left( \rho - \phi_{\pi} \right) u_t}{\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi - \rho \sigma + \sigma + \phi_x}$$

$$solx1 \; := \; x_t = -\frac{\left( \beta \lambda - 1 \right) \left( \sigma - \phi_g \right) g_t}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \phi_x - k \lambda + k \phi - \lambda \sigma + \sigma + \phi_x}$$

$$+ \frac{\left( \rho - \phi_{\pi} \right) u_t}{\beta \rho^2 \sigma - \beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi - \rho \sigma + \sigma + \phi_x}$$

Obtaining Optimal Loci of $\phi$ :
2.5.3 The Messy Results of the Optimal Loci for $\phi_x$ and $\phi_\pi$ in Maple

Figure 2.4 below is the screenshot of the messy result of the linear relationship of $\phi_x$ and $\phi_\pi$ in Maple:
Figure 2.4. The Maple Result (1) of the Optimal Loci for $\varphi_x$ and $\varphi_\pi$

![Maple Result Image]

Maple produces the messy result of the linear relationship of $\varphi_x$ and $\varphi_\pi$.

Figure 2.5 below is the screenshot which Maple produces no result for the linear relationship of $\varphi_x$ and $\varphi_\pi$:
2.5.4 Deriving the General Solutions in Maple

This appendix shows the Maple commands for obtaining the general solutions of $x_t$ and $\pi_t$ by using the baseline NKM which has the simple Taylor rule. The Maple commands in italics for the general solutions of $x_t$ and $\pi_t$ are as follow:

In this case, the simple Taylor rule (STR) is $\phi_s x_t + \phi_t \pi_t$, which does not have $\phi[g]$ in it.

Baseline IS curve (BIS) expresses a NK-IS equation and is presented by (1) below:

$$BIS := x_t = EX - \frac{1}{\sigma} (i_t - EP) + g_t$$

Note: $EX$ is $E_x[x_{t+1}]$ and $EP$ is $E_x[\pi_{t+1}]$, $EX$ and $EP$ are forward-looking variables. The demand shock, $g_t$, is an AR(1) process $g_t = \lambda g_{t-1} + \epsilon_t$.

Baseline Philips curve (BPC) expresses a NK-PC equation and is presented by (2) below:

$$BPC := \pi_t = k x_t + \beta \cdot EP + u_t$$

Note: The cost shock, $u_t$, is an AR(1) process $u_t = \rho u_{t-1} + \eta_t$. 
The simple Taylor Rule (STR) expresses a Taylor rule using actual values of $x$ and $\pi$ and is presented by (3) below:

$$STR := i_t = \phi_x \cdot x_t + \phi_\pi \cdot \pi_t$$

The undetermined coefficients equations of $x$ and $\pi$ are represented as (4) and (5) below:

$$BIS1 := x_t = a1 \cdot g_t + a2 \cdot u_t$$

$$BPC1 := \pi_t = b1 \cdot g_t + b2 \cdot u_t$$

Obtaining the general solutions of $x$ and $\pi$ by solving (4) and (5) using the Baseline NKM (1) and (2), and the simple Taylor Rule (3):

Equation (6) below shows $x$ is updated one period ahead:

$$BIS2 := x_{t+1} = a1 \cdot g_{t+1} + a2 \cdot u_{t+1}$$

Equation (7) below shows $\pi$ is updated one period ahead:

$$BPC2 := \pi_{t+1} = b1 \cdot g_{t+1} + b2 \cdot u_{t+1}$$

Equation (8) below shows $g[t+1]$ and $u[t+1]$ in (6) are replaced by $\lambda g[t] + \varepsilon[t+1]$ and $\rho u[t] + \eta[t+1]$:

$$BIS3 := \text{subs}\left(g_{t+1} = \lambda \cdot g_t + \varepsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BIS2\right)$$

Equation (9) below shows $g[t+1]$ and $u[t+1]$ in (7) are replaced by $\lambda g[t] + \varepsilon[t+1]$ and $\rho u[t] + \eta[t+1]$:

$$BPC3 := \text{subs}\left(g_{t+1} = \lambda \cdot g_t + \varepsilon_{t+1}, u_{t+1} = \rho \cdot u_t + \eta_{t+1}, BPC2\right)$$

Equation (10) below shows the undetermined coefficient equation of $EX$ in the Baseline NKM:

$$EBIS3 := EX = a1 \cdot \lambda \cdot g_t + a2 \cdot \rho \cdot u_t$$

Note: The expected values of $\varepsilon_{t+1}$ and $\eta_{t+1}$ are equal to zero.

Equation (11) below shows the undetermined coefficient equation of $EP$ in the Baseline NKM:

$$EBPC3 := EP = b1 \cdot \lambda \cdot g_t + b2 \cdot \rho \cdot u_t$$

Note: The expected values of $\varepsilon_{t+1}$ and $\eta_{t+1}$ are equal to zero.

Equation (12) below shows the $x$ and $\pi$ in STR (3) are replaced by (6) and (7):

$$STR1 := \text{subs}(BIS1, BPC1, STR)$$

Equation (13) below shows $x$, $i$, $EX$ and $EP$ in Baseline IS (1) are replaced by (6), (12), (10) and (11):

$$sol01 := \text{subs}(BIS1, STR1, EBIS3, EBPC3, BIS)$$

Equation (14) below shows $x$, $\pi$ and $EP$ in Baseline Philips curve (2) are replaced by (6), (7) and (11):

$$sol02 := \text{subs}(BIS1, BPC1, EBPC3, BPC)$$

Then I can solve (13) and (14) by the method of undetermined coefficient:

$$con1 := \text{subs}(g_t = 0, u_t = 0, sol01)$$

$$con2 := \text{subs}(g_t = 1, u_t = 0, sol01)$$

$$con3 := \text{subs}(g_t = 0, u_t = 1, sol01)$$

$$con4 := \text{subs}(g_t = 0, u_t = 0, sol02)$$

$$con5 := \text{subs}(g_t = 1, u_t = 0, sol02)$$
Below is the general solution of $x$ when the simple TR does not have $\phi[g]$ in it. 

\[ \text{solx expresses the general solution of } x: \]

\[ x_t = \frac{\beta \lambda - 1}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \lambda \phi_x - k \lambda + k \phi_\pi - \lambda \sigma + \sigma + \phi_x} \quad \frac{\sigma g_t}{\beta \sigma - \phi_x} + \frac{\rho - \phi_\pi}{\beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi_\pi - \rho \sigma + \sigma + \phi_x} \]

Below is the general solution of $\pi$ when the simple TR does not have $\phi[g]$ in it.

\[ \text{solp express the general solution of } \pi: \]

\[ \pi_t = \frac{k \sigma g_t}{\beta \lambda^2 \sigma - \beta \lambda \sigma - \beta \lambda \phi_x - k \lambda + k \phi_\pi - \lambda \sigma + \sigma + \phi_x} \quad \frac{\sigma g_t}{\beta \sigma - \phi_x} + \frac{\rho \sigma - \phi_x}{\beta \rho \sigma - \beta \rho \phi_x - k \rho + k \phi_\pi - \rho \sigma + \sigma + \phi_x} \]

To run the commands and produce equations (4) and equations (5) in Maple, simply copy and paste all the commands into Maple and run the commands.

### 2.5.5 The Relationship between (11a) and minimum $L$: Illustrations from 2-D Plots

I take $\phi_x$ fixed to the yellow surface and then increase $\phi_\pi$ to see where the minimum is, and I reproduce the u-shaped curve with different values of $\phi_x$. I also found the curve repeatedly narrowed down the range of movement of $\phi_\pi$. Note that because $L$ is a fraction, there are rounding problems in the results, the points where minimum $L$ occurs can be a little off from the predictive point as (11a) suggests.
First is an example with $\varphi_x = 2$ and the Maple command:

```
Plot(subs(phi[x]=2, Lsimu1), phi[pi]=5.1..5.3)
```

The minimum is a little short off $\varphi_{\pi} = 5.20$. If I calculate the implied value by (11a), it is a little bit off.

$$
\text{subs}\left(\phi_x = 2, \phi_{\pi} = 1.954397394\phi_x + 1.286319218\right)
$$

$$
\phi_{\pi} = 5.195114006
$$

Third, I try the first derivative with respect to $\varphi_{\pi}$ to see where it crosses the zero line. The Maple command and result are:

```maple
> plot(diff(subs(\phi_x = 2, Lsimu1), \phi_{\pi}), \phi_{\pi} = 5.19 .. 5.2)
```
Then inserted (11a)'s predicted value of $\varphi_\pi$ into the first derivative:

$$-subs(\phi_\pi = 5.195114006, \text{diff}\left(subs(\phi_x = 2, Lsimu1), \phi_\pi\right))$$

\[ -0.0001796875378 \]

This is very close to zero.

Next I check the “inverse ridge” case for showing that (11a) is working in both directions. First I rearrange (11a) to (11a'):

$$\phi_x = \frac{\Gamma \beta \rho^{2} - \Gamma \beta \rho \phi_{\pi} + k \rho \sigma - \Gamma \rho + \Gamma \phi_{\pi} - k \sigma}{k}.$$
Then the parameterization of (11a') is $\phi_x = -0.6581666662 + 0.5116666666 \phi_{\pi}$. I found a u-shaped curve and repeatedly narrowed down the range of movement of $\varphi_x$. To be consistent with the previous example of taking $\varphi_x$ fixed to the yellow (unconstrained) surface, I chose $\varphi_{\pi} = 5.195114006$ as the inverse example and command:

> plot(subs(\(\phi_{\pi} = 5.195114006, Lsimu1\), \(\phi_x = 1.97 .. 2.05\))

![Plot](image)

The minimum is a little short of $\varphi_x = 2.01$. I then calculate the implied value by (11a'), it is a bit off.

> subs(\(\phi_{\pi} = 5.195114006, \phi_x = -0.6581666662 + 0.5116666666 \phi_{\pi}\))

$\phi_x = 1.99999997$
Thirdly, I tried the first derivative with respect to $\varphi_x$ to see where it crosses the 0 line:

$$> \text{plot}\left(\text{diff}\left(\text{subs}\left(\varphi_\pi = 5.195114006, LsimuL\right), \varphi_x\right), \varphi_x = 1.97..2.05\right)$$

It seems like the $\varphi_x$ crosses the 0 line a bit off 2.01. Then insert (11a’)'s predicted value of $\varphi_x$ into the first derivative:

$$> \text{subs}\left(\text{phi} = 1.999999997, \text{diff}\left(\text{subs}\left(\varphi_\pi = 5.195114006, LsimuL\right), \varphi_x\right)\right)$$

$$-0.004210680327$$

Again, this seems pretty close to zero.

Now, I repeat above steps using big value of $\varphi_x$ for the example of taking $\varphi_x$ fixed and big value of $\varphi_\pi$ for the inverse ridge example. First is an example with $\varphi_x = 20$ and command:
The minimum is between $\varphi_\pi = 40.35$ and $\varphi_\pi = 40.4$. If I calculate the implied value by (11a), it is:

\[ \text{subs}(\phi_x = 20, LsimuL, \varphi_\pi = 40..40.75) \]

Then, I try the first derivative with respect to $\varphi_\pi$ to see where it crosses the zero line. The Maple command and result are:

\[ \text{plot}\left(\text{diff}\left(\text{subs}(\phi_x = 20, LsimuL, \varphi_\pi), \varphi_\pi = 40..40.75\right)\right) \]
Then insert (11a)’s predicted value of $\varphi_\pi$ into the first derivative:

$$\text{subs}\left(\varphi_\pi = 40.37426710, \text{diff}\left(\text{subs}\left(\varphi_x = 20, Lsimu1\right), \varphi_\pi\right)\right)$$

$$= -3.929585173 \times 10^{-7}$$

The result $-3.929585173 \times 10^{-7}$ is very close to zero. The predicted value of $\varphi_\pi$ for taking $\varphi_x$ fixed at big value is more accurate than the previous predicted value of $\varphi_\pi$ for taking $\varphi_x$ fixed at small value.

Second is the inverse ridge example with $\varphi_\pi = 40.37426710$ and command:

$$\text{plot}\left(\text{subs}\left(\varphi_\pi = 40.37426710, Lsimu1\right), \varphi_x = 19.5 \ldots 20.5\right)$$
The minimum is a little bit short of $\varphi_x = 20$. I then calculate the implied value by (11a'), it is a little bit off:

\[ \text{subs}\left(\phi_\pi = 40.37426710, \phi_x = -0.6581666662 + 0.5116666666 \phi_\pi\right) \]

\[ \phi_x = 19.99999997 \]

Then I try the first derivative with respect to $\varphi_x$ to see where it crosses the zero line:

\[ \text{plot}\left(\text{diff}\left(\text{subs}\left(\phi_\pi = 40.37426710, Lsimu1\right), \phi_x\right), \phi_x = 19.5 \ldots 20.5\right) \]
It seems like the $\varphi_x$ crosses the zero line at 20. Then insert (11a)'s predicted value of $\varphi_x$ into the first derivative:

$$> \text{subs}(\varphi_x = 19.99999997, \text{diff}(\text{subs}(\varphi_\pi = 40.37426710, L_{simu}), \varphi_x)))$$

-0.00009209196625

Again, this seems pretty close to zero.

Finally, I show a proof that (11a) is the condition for having a minimum loss of equation (7).

First, I pick $\varphi_{x,max} = 2$ and treat it as exogenous. Then I derive the first derivative of equation (7) with respect to $\varphi_\pi$ by substituting $\varphi_{x,max}$ into equation (7), i.e., $\frac{dL}{d\varphi_\pi}$:
Second, I solve \( \frac{dL}{d\varphi_\pi} = 0 \) for minimizing \( \varphi_\pi \) and the Maple command:

\[
> \text{prophi} := \text{simplify} \left( \text{diff} \left( \text{subs} \left( \Phi_x = 2, \text{Lsimu} \right), \Phi_\pi \right), \text{size} \right)
\]

\[
0 = \frac{1}{\left( 0.6851 + 0.03 \Phi_\pi \right)^3 \left( 1.8891 + 0.03 \Phi_\pi \right)^3} \left( -2.47350563100000 + 0.000001906470 \Phi_\pi^4 \\
+ 0.000350205429300000 \Phi_\pi^3 + 0.020804854870000 \Phi_\pi^2 + 0.35814295000000 \Phi_\pi \right)
\]

Third, I calculate predicted value of \( \varphi_\pi \) of (11a) when \( \varphi_x = 2 \):

\[
> \text{subs} \left( \Phi_x = 2, \Phi_\pi = 1.954397394\Phi_x + 1.286319218 \right)
\]

\[
\Phi_\pi = 5.195114006
\]

The predicted value of \( \varphi_\pi \) is a bit off.

Next, I pick \( \varphi_{\pi,max} = 2 \) and treat it as exogenous. Then I derive the first derivative of equation (7) with respect to \( \varphi_x \) by substituting \( \varphi_{\pi,max} \) into equation (7), i.e., \( \frac{dL}{d\varphi_x} \):

\[
> \text{profx} := \text{simplify} \left( \text{diff} \left( \text{subs} \left( \Phi_\pi = 2, \text{Lsimu} \right), \Phi_x \right), \text{size} \right)
\]

\[
0 = \frac{1}{\left( 0.1311 + 0.307 \Phi_x \right)^3 \left( 0.5431 + 0.703 \Phi_x \right)^3} \left( -0.00464386930900000 + 0.0270994563 \\
\Phi_x^4 + 0.0518690504000000 \Phi_x^3 + 0.024251425600000 \Phi_x^2 - 0.005793349600000 \Phi_x \right)
\]

Then I solve \( \frac{dL}{d\varphi_x} = 0 \) for minimizing \( \varphi_x \) and the Maple command:
Finally, I calculate predicted value of $\varphi_x$ of (11a) when $\varphi_\pi = 2$:

> \texttt{solve(profx, }\{\varphi\})
\[
\{\varphi_x = 0.3783418474, \varphi_x = -0.8029593852 + 0.12279485161, \varphi_x = -0.6864485771, \varphi_x = -0.8029593852 - 0.12279485161\}
\]

The predicted value of $\varphi_x$ is a bit off. So I pick $\varphi_\pi, s_{mx} = 40$ and repeat the above steps. The command is:

> \texttt{subs(\varphi = 2, \varphi = -0.6581666662 + 0.511666666\varphi_\pi)}
\[
\varphi_x = 0.3651666658
\]

The predicted value of $\varphi_x$ is still a bit off, but the two values of $\varphi_x$ are more close to each other.

I then try a very big $\varphi_{\pi,max}, \varphi_{\pi,max} = 200000$. The command is:

> \texttt{profx := simplify(diff(subs(\varphi = 40, Lsimu), \varphi_x), size)}
\[
0 = \frac{1}{(1.2711 + 0.307\varphi_x)^3 (1.6831 + 0.703\varphi_x)^3}\left(0.819237410 (\varphi_x + 2.45892303958213) \varphi_x
\right.
\]
\[
- 19.8100960379188 \left(\varphi_x^2 + 4.723904780 \varphi_x + 5.582491174\right)
\]
\[
\texttt{solve(profx, }\{\varphi\})
\[
\{\varphi_x = -2.458923040, \varphi_x = 19.81009604, \varphi_x = -2.361952390 + 0.060597701061, \varphi_x = -2.361952390 - 0.060597701061\}
\]

\texttt{subs(\varphi = 40, \varphi = -0.6581666662 + 0.511666666\varphi_\pi)}
\[
\varphi_x = 19.80849997
\]

The predicted value of $\varphi_x$ is still a bit off, but the two values of $\varphi_x$ are more close to each other.
The predicted value of $\varphi_x$ is almost identical to the solution of $\frac{dL}{d\varphi_x} = 0$ for minimizing $\varphi_x$.

I then check for negative value of $\varphi_{\pi,\text{max}}$. The command is:

```
> profx := simplify(diff(subs($\pi = -2000, Lsimu$, $\varphi_x$), $\varphi_x$), size)
```

\[
0 = \left(41.70606326 (\varphi_x + 1023.99153283064) (\varphi_x - 84.9721664336499) (\varphi_x^2
- 169.0117677\varphi_x + 7141.317747) \right) \left( ( -59.9289 + 0.307 \varphi_x )^3 ( -59.5169 \\
+ 0.703 \varphi_x )^3 \right)
\]

```
solve(profx, [$\varphi_x$])
```

\[
\{\varphi_x = -1023.991533\}, \{\varphi_x = 84.97216643\}, \{\varphi_x = 84.50588385 + 0.27081678371\}, \{\varphi_x
= 84.50588385 - 0.27081678371\}
\]

```
subs($\pi = -2000, \varphi_x = -0.6581666662 + 0.5116666666 \varphi_{\pi}$)
```

$\varphi_x = -1023.991499$

I also I then check for negative value of $\varphi_{x,\text{max}}$. The command is:

```
> profpi := simplify(diff(subs($\pi = -2000, Lsimu$, $\varphi_{\pi}$), $\varphi_{\pi}$), size)
```
Two examples of negative values of $\phi_\pi$, $s_{mx}$ and $\phi_x$, $s_{mx}$ also show that (11a) is the condition for having a minimum loss of equation (7).

2.5.6 The Plot of $\phi_\pi + \phi_x \frac{(1-\beta)}{k} > 1$

The shadowed area is the area of the point $(\phi_\pi, \phi_x)$ which satisfies $\phi_\pi + \phi_x \frac{(1-\beta)}{k} > 1$. 

\[
0 = - \left( 0.0165777513 \left( \phi_\pi + 39086.6615616129 \right) \left( \phi_\pi - 4.68307096686261 \times 10^5 \right) \left( \phi_\pi^2 - 9.376445962 \times 10^5 \phi_\pi + 2.19794349 \times 10^{11} \right) \right) \left( -6139.9289 + 0.03 \phi_\pi \right) \left( -14059.5169 + 0.03 \phi_\pi \right)^3
\]

\[
solve\left( \{ \phi_\pi \} \right)
\[
\left\{ \phi_\pi = -39086.66156 \right\}, \left\{ \phi_\pi = 4.683070967 \times 10^5 \right\}, \left\{ \phi_\pi = 4.688222981 \times 10^5 + 296.1490077 \right\}, \left\{ \phi_\pi = 4.688222981 \times 10^5 - 296.1490077 \right\} \]

\[
\text{subs}\left( \phi_x = -20000, \phi_\pi = 1.95497394 \phi_x + 1.286319218 \right)
\phi_\pi = -39086.66156
\]

Two examples of negative values of $\phi_{\pi,max}$ and $\phi_{x,max}$ also show that (11a) is the condition for having a minimum loss of equation (7).
2.5.7 The Second Partial Derivative Test of the Social Welfare Loss (7)

Substituting (11a) into the second partial derivatives with respect to $\varphi_x$ and $\varphi_\pi$, I obtain

\[
\frac{\partial^2 L}{\partial \varphi_x^2} = - \frac{(\beta \rho - 1)^2 (\beta \lambda - 1)^2}{(-2 + (\lambda + \rho) \beta)^3 (\rho - \lambda)^3 (\rho \sigma - \sigma - \varphi_x)^2 \beta^3 \sigma_\eta^2} 
\]

and

\[
\frac{\partial^2 L}{\partial \varphi_\pi^2} = - \frac{k^2 (\beta \rho - 1)^4}{(\beta \lambda - 1)^2 (-2 + (\lambda + \rho) \beta)^3 (\rho - \lambda)^3 (\rho \sigma - \sigma - \varphi_x)^2 \beta^3 \sigma_\eta^2},
\]

and also substituting (11a) into the cross partial derivative with respect to $\varphi_x$ and $\varphi_\pi$, I obtain

\[
\frac{\partial^2 L}{\partial \varphi_x \partial \varphi_\pi} = \frac{k(\beta \rho - 1)^3}{(-2 + (\lambda + \rho) \beta)^3 (\rho - \lambda)^3 (\rho \sigma - \sigma - \varphi_x)^2 \beta^3 \sigma_\eta^2}. 30
\]

If the stationary value of $L$ is a relative minimum, the value of $\left( \frac{\partial^2 L}{\partial \varphi_x^2} \right) \left( \frac{\partial^2 L}{\partial \varphi_\pi^2} - \left( \frac{\partial^2 L}{\partial \varphi_x \partial \varphi_\pi} \right)^2 \right)$ must be positive. However, the value of $\left( \frac{\partial^2 L}{\partial \varphi_x^2} \right) \left( \frac{\partial^2 L}{\partial \varphi_\pi^2} - \left( \frac{\partial^2 L}{\partial \varphi_x \partial \varphi_\pi} \right)^2 \right)$ from the above derivatives equals zero.

---

30 $g$-shocks are nullified for the simplicity of the expression of equations.
3 USING MONEY SIGNALS TO IMPROVE TAYLOR RULE PERFORMANCE IN THE NEW KEYNESIAN MODEL
Abstract

This chapter is the study of a combination monetary policy rule where the central bank ("the Fed") sets the interest rate before observing current shocks of output gap \( x_t \) and inflation \( \pi_t \). The missing information is \( \varepsilon_t \) in the shock term of \( x_t \) equation—i.e., \( g_t = \lambda g_{t-1} + \varepsilon_t \) where \( \varepsilon_t \sim iid \ N \left( 0, \sigma^2 \right) \), and \( \eta_t \) in the shock term of \( \pi_t \) equation—i.e., \( u_t = \rho u_{t-1} + \eta_t \) where \( \eta_t \sim iid \ N \left( 0, \sigma^2 \right) \). Thus, the Fed cannot adjust their interest rate for those shocks because the Fed cannot observe \( \varepsilon_t \) and \( \eta_t \). On the other hand, the information of money is immediately available to the Fed when I use a model as the abstract representation of the Fed’s observation of money surprise, so the Fed can use signals about money to adjust their interest rate. My model of the Fed’s operation on how they observe money surprise is a simplified model for making a theoretical point, not for the purpose of improving what the Fed is actually doing. The combination policy of a Taylor rule and money signal can improve the social welfare loss when the Fed sets their monetary policy with unobservable shocks. Chapter 3 uses an inverted version of Poole’s (1970) combination policy analysis and shows that the social welfare loss is improved from the money signals.
3.1 Introduction

In the previous two chapters, I discuss the mechanism of the baseline new Keynesian monetary model (“NKM”) for the analysis of monetary policy and potentially the performance of monetary policy. Conventionally speaking, the baseline NKM has a full-information Taylor rule built in and all the shocks are assumed to be known when a central bank (“the Fed”) sets its monetary policy, as well as the model parameters, so it is less complicated for not using money demand and supply analysis. CGG (1999) argues that (a): “Using observable intermediate targets, such as broad money aggregates are a possibility, but experience suggests that these indirect indicators are generally too unstable to be used in practice.” (CGG 1999, p.1686) (b): “Large unobservable shocks to money demand produce high volatility of interest rates when a monetary aggregate is used as the policy instrument. It is largely for this reason that an interest rate instrument may be preferable.” (CGG 1999, p.1687) Their two arguments are based on their summary that the policy results are the same whether the Fed uses a money-based or interest rate based model. There is no necessary choice between money based on interest rate or interest rate based on money. (CGG 1999, p.1667)31

However when the Fed only has limited observation about target variables—i.e., output gaps and inflation rates—, the Fed cannot adjust their interest rate for the missing information in those target variables because they cannot observe them. Basically the missing information is

31 If the shocks are fully observed then it does not matter whether the monetary policy model is money based on interest rate or interest rate based on money. I am not claiming that high volatility of interest rates is a bad thing. In this context the only thing those authors did not prefer about monetary aggregate policy is that it requires more information and is more complicated.
the components of output gap and inflation and those are not observed, so using a Taylor rule the
Fed is not able to respond to those components of output gap and inflation. For example,
observed target variables cannot take into account of unobservable shocks and estimated target
variables are simply ignore the unobservable shocks. Obviously Taylor rule performance will be
worse if the Fed has limited information about \( x_t \) and \( \pi_t \). In this chapter I want to know how the
Fed can improve the performance of limited-information Taylor rule.

William Poole (1970) proposed a combination policy when a certain relationship was
maintained between interest rate and money stock using a fixed-price, IS-LM model:

\[
\begin{align*}
\{ & y = a_0 - a_1 i + \epsilon_y \\
y = b_0 + b_1 M + b_2 i + \epsilon_m, \}
\end{align*}
\]

where \( y \) is output, \( i \) is interest rate, \( M \) is money, \( \epsilon_y \) is output shock and \( \epsilon_m \) is money shock. His
combination policy is depending on the optimal decision, which he proposed (p.204), from “the
policy that minimizes the expected loss from failure of the level of income to equal the desired
level.” He suggests that using a combination policy, so that the Fed respond to the information it
receives from the variation of interest rate in order to adjust money target. Let \( M = B i \), where \( M \)
is money, \( i \) is interest rate, \( B \) is the coefficient and \( B i \) is the response of \( M \) to the new information
about \( i \), so that the Fed can use information from the variation of the interest rate. Find the value
of \( B \) (and call it optimal \( B \)) that minimizes the variance of output \( y \) around the desired level. The
optimal \( B \):

\[
B^* = \frac{a_1 \sigma_m^2 - b_2 \sigma_y^2}{b_1 \sigma_y^2}.
\]

Note that there are two special cases of this optimal \( B \) which I left out in this chapter. When
there is no IS shock \((\sigma_y^2 = 0)\): \(B^* \to \infty\), this is the pure interest rate targeting. When there is no LM shock \((\sigma_m^2 = 0)\): \(B^* = -\frac{b_2}{b_1}\), this is an aggressive money supply targeting.

On the other hand, we can have the “inverted” Poole combination policy that the Fed knows\(^{32}\) the amount of money (e.g., nonborrowed reserves) it is letting in and out in order to reach a particular interest rate objective. That is we let \(i = \Theta M\) where \(\Theta\) is the coefficient and \(\Theta M\) is the response of \(i\) to the new information about \(M\)—so that the Fed adjusts interest rate relative to its planned value when the Fed observes the money movement from its expected value. In other words, the Fed uses its observation of how much it is unexpectedly changing the money supply aggregate in order to reach the target.\(^{33}\) Find the value of \(\Theta\) (and call it optimal \(\Theta\)) that minimizes the variance of output \(y\) around the desired level. The optimal \(\Theta\):

\[
\Theta = \frac{b_1 \sigma_y^2}{a_1 \sigma_m^2 - b_2 \sigma_y^2}.
\]

It is actually the reciprocal of the optimal \(B\). Furthermore, the minimum variance of output \(y\) is exactly the same as the original Poole combination policy. In other words, “Poole’s combination policy can be turned on its head,”—i.e., an inverted version of original Poole’s (1970) combination policy—, and get the same result.

In this chapter, I apply the inverted version of Poole’s (1970) combination policy analysis from traditional IS-LM model to the new Keynesian Model. I formulate a combination Taylor

\(^{32}\) This is an abstract representation of the Fed’s observation of money surprise. The assumption of the Fed’s operation on how they observe money surprise is simplified for making theoretical point, not for the purpose of improving what the Fed is actually doing.

\(^{33}\) The value of money supply is determined by the condition that money demand equals money supply (i.e., an LM curve). When money demand is changed from realized shocks so does the value of the money supply.
rule where $E_{t-1}[i_t]$ is the Taylor rule value of $i_t$ based on $t-1$ projection of inflation and output, i.e., $E_{t-1}[x_t]$ and $E_{t-1}[\pi_t]$. But there is an optimal adjustment based on the $M_t - E_{t-1}[M_t]$ which the Fed finds it has to use in order to keep the interest rate on the desire path. Since the unobserved shocks are i.i.d, this combination Taylor rule component is orthogonal to $t-1$ information, so is just an “add on” to the Taylor rule with this new component.

Note that the model of $M_t$ derived in the section 3.4 is an abstract representation of the Fed’s observation of money surprise, so the Fed can use signals about money to adjust their interest rate. My model of the Fed’s operation on how it observes money surprise is a simplified model for making theoretical point, not for the purpose of improving what the Fed is actually doing.

In short, this paper is the analysis of signal problem when the Fed cannot adjust interest rate according to a Taylor rule and only the signals from money are immediately available for the Fed. In this paper solutions to the questions are determined within the context of the baseline new Keynesian model. The questions arise as a result of the fact that the Fed operates through a Taylor rule but only is able to observe the lag components of output gap and inflation. Since the Fed is not able to respond to the missing components of output gap and inflation because those components are not observed by the Fed, and as the result, the expected social welfare loss is higher than its minimum value.

The analysis produces two major findings. First, it is possible to construct a combination rule in which the constraint on the parameters of the two rule variables—output gap and inflation—maintained in a certain relationship to each other and to show that the optimal
combination rule is as good as or superior to the simple Taylor rule which has the signal problem. Later I will show how much benefit we get from the combination rule. Second, signals coming from unexpected quantity changes in money (i.e., the money signals) help to improve the simple Taylor rule performance. CGG (1999)’s two arguments (a-b) cannot be the reasons for not using the money signals. High variance of money demand and supply shocks will clearly reduce the information value of money surprises, but the Fed can still improve the Taylor rule performance from using money signals with high variance.

In Chapter 2, I have discussed the properties of full-information Taylor rules: a three-term Taylor rule and a simple (“two-term”) Taylor rule. In Section 3.2, I continue the use of the symbolic expression of Taylor rules from Chapter 2 for comparing the performance of three Taylor rules which are two full-information Taylor rules and one limited-information Taylor rule. The performance is measured by the social welfare losses of these three rules. \( L_{\text{standard}} \) is the social welfare loss with the (full-information) three-term Taylor rule built in, which I used in the Chapter 2. \( L_{\text{simple}} \) is the social welfare loss with the (full-information) two-term Taylor rule built in, which is the simple Taylor rule in the Chapter 2. \( L_{\text{unobs}} \) is the social welfare loss with the (limited-information) two-term Taylor rule built in, which is the simple Taylor rule in Chapter 2 with expected values of \( x_t \) and \( \pi_t \) built in. See Table 3.1 for specifications of these labels of social welfare loss. Since the three-term Taylor rule is impractical or “unrealistic” to expect the Fed bases a rule on a shock term. \( L_{\text{standard}} \) is only a benchmark for comparison purpose. The case for (limited-information) three-term Taylor rule is left in the Appendix 3.9.2 for interested readers. The focus in this chapter is the simple Taylor rule. Section 3.3 is the section of the cause of the difference in loss between \( L_{\text{simple}} \) and \( L_{\text{unobs}} \). I then show how to
derive the signals from money for constructing a combination rule in Section 3.4. In Section 3.5, I show the construction of the combination rule and its performance as well as the impact of the interest elasticity to the money signals. In Section 3.6, I will discuss the influence of the variance of money demand to the money signals and to the performance of the combination rule. Section 3.7 presents an example for measuring Taylor rule improvement from the money signals with the parameters used in literature. Section 3.8 is the conclusion. Section 3.9 consists of the appendixes.

### Table 3.1. The Specifications of Three Social Welfare Losses

<table>
<thead>
<tr>
<th>Social Welfare Loss</th>
<th>Period Quadratic Loss Function</th>
<th>Taylor Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{standard}$</td>
<td>$L = \frac{1}{2}(\pi_t^2 + \Gamma x_t^2)$</td>
<td>$i_t = \varphi_x x_t + \varphi_{\pi \pi_t} + \varphi_g g_t$</td>
</tr>
<tr>
<td>$L_{simple}$</td>
<td>$L = \frac{1}{2}(\pi_t^2 + \Gamma x_t^2)$</td>
<td>$i_t = \varphi_x x_t + \varphi_{\pi \pi_t}$</td>
</tr>
<tr>
<td>$L_{unobs}$</td>
<td>$L = \frac{1}{2}(\pi_t^2 + \Gamma x_t^2)$</td>
<td>$i_t = \varphi_x E_{t-1} x_t + \varphi_{\pi \pi_{t-1}} \pi_t$</td>
</tr>
</tbody>
</table>

Note: $x_t = E_t x_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + g_t$ and $\pi_t = k x_t + \beta E_t \pi_{t+1} + u_t$

#### 3.2 The Performance of Three Taylor Rules in the Baseline NKM

I start from comparing the performance of three different Taylor rules in the context of the baseline new Keynesian model for the analysis of monetary policy. This model contains two equations. The first is the equation of output gap (i.e., the difference between current output $y_t$ and the output in full-employment $y_{t}^{f}$) from the new Keynesian IS curve,

$$x_t = E_t x_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + g_t,$$

where $g_t = \lambda g_{t-1} + \varepsilon_t$ is the first order autocorrelation process with the innovation.
\( \varepsilon_t \sim iid \mathcal{N}(0, \sigma^2_\varepsilon) \). The second is the equation of inflation (i.e., the difference between price in current period \( p_t \) and price in previous period \( p_{t-1} \)) from the new Keynesian Phillips Curve,

\[
\pi_t = k x_t + \beta E_t \pi_{t+1} + u_t, \tag{2}
\]

where \( u_t = \rho u_{t-1} + \eta_t \) is the first order autocorrelation process with the innovation \( \eta_t \sim iid \mathcal{N}(0, \sigma^2_\eta) \).

Then we substitute three different Taylor rules into (1) for \( i_t \) based on the assumption of commitment to the simple Taylor rule as we showed in Chapter 2. The first is the three-term Taylor rule (\( TR_{\text{standard}} \)), in which Woodford (2001) argued that \( \varphi_g \) (= \( \sigma \)) term should be added:

\[
i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t, \tag{3}
\]

the second is the simple Taylor rule (\( TR_{\text{simple}} \)) from Chapter 2:

\[
\bar{i}_t = \varphi_x x_t + \varphi_\pi \pi_t. \tag{4}
\]

These two rules are full-information rules for which use actual values of output gap and inflation. The third is an expected simple Taylor rule (\( TR_{\text{unobs}} \)):

\[
\tilde{i}_t = \varphi_x E_{t-1} x_t + \varphi_\pi E_{t-1} \pi_t, \tag{5}
\]

(5) is the limited-information Taylor rule for which uses expected values of output gap and inflation. In appendix 3.9.2, I discuss a case which involves a Taylor rule which is equation (5) plus an additional term \( \varphi_g E_{t-1} g_t \). After separately substituting (3), (4) and (5) into (1) for \( i_t \), we substitute the general solutions\(^{34}\) of \( x_t \) and \( \pi_t \) from solving (1) and (2) into the period quadratic loss function

\(^{34}\) For the discussions of the general solutions, see Thurston (2010, 2012) and Huang (2017).
\[ L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2] \]  

(6)

and we obtain three social welfare functions \( L_{\text{standard}}, L_{\text{simple}} \) and \( L_{\text{unobs}} \) in terms of (3), (4) and (5). We can then obtain an unique constraint on the \( \phi_x \) and \( \phi_\pi \),

\[
\phi_\pi = -\frac{k}{\Gamma(\beta \rho - 1)} \phi_x + \frac{k\sigma(\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho,
\]

(7)

for the minimum social welfare losses of \( L_{\text{standard}}, L_{\text{simple}} \) and \( L_{\text{unobs}} \) by using the first order conditions from differentiating \( L_{\text{standard}}, L_{\text{simple}} \) and \( L_{\text{unobs}} \) with respect to \( \phi_x \) and \( \phi_\pi \).

Next, we substitute (7) into \( L_{\text{standard}}, L_{\text{simple}} \) and \( L_{\text{unobs}} \) and we obtain three minimum social welfare losses as shown

\[
L_{\text{standard},\min} = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2,
\]

\[
L_{\text{simple},\min} = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2 + \frac{1}{2} \left( \frac{(\Gamma(\beta \lambda - 1)^2 + k^2)\sigma_\varepsilon^2}{D^2} \right) \sigma_\varepsilon^2,
\]

(8a)

\[
L_{\text{unobs},\min} = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2 + 1} \right) \sigma_\eta^2 + \frac{1}{2} \left( \frac{(\Gamma(\beta \lambda - 1)^2 + k^2)\sigma_\varepsilon^2}{D^2} + k^2 + \Gamma \right) \sigma_\varepsilon^2,
\]

(8b)

\[
\text{where } D = k \left( -\frac{k}{\Gamma(\beta \rho - 1)} \phi_x + \frac{k\sigma(\rho - 1)}{\Gamma(\beta \rho - 1)} + \rho \right) + (1 - \beta \lambda) \phi_x + (\lambda - 1)(\beta \lambda - 1)\sigma - k\rho.
\]

Figure 3.1 compares the differences of these three minimum social welfare losses as \( \phi_x \) and \( \phi_\pi \) approach infinity. Note that I am not suggesting that the Fed can force \( \phi_x \) and \( \phi_\pi \) to approach infinity. It is for the comparison purpose not for the practical reason by letting \( \phi_x \) and \( \phi_\pi \)
approach infinity. I want to see whether any optimal\textsuperscript{35} combination of $\varphi_x$ and $\varphi_{\pi}$, especially when the values of such $\varphi_x$ and $\varphi_{\pi}$ are very big, the performance of Taylor rules are improved or not. If $\varphi_x$ is approaching infinity then $D^2$ term in the denominator of (8c) is approaching infinity, and the second term in (8c) approaches zero, as the figure 3.1 shows the difference in loss between $L_{\text{standard,min}}$ and $L_{\text{simple,min}}$ can be completely eliminated by having $\varphi_x$ and $\varphi_{\pi}$ approach infinity.\textsuperscript{36} However, the figure 3.1 also shows that we cannot reduce the difference in loss between $L_{\text{simple,min}}$ and $L_{\text{unobs,min}}$ by using the same trick.

\textsuperscript{35} That is the linear relationship between $\varphi_x$ and $\varphi_{\pi}$ which guarantee optimal paths of $x_t$ and $\pi_t$, as well as minimum social welfare loss.

\textsuperscript{36} For the discussion of the relationship between $L_{\text{standard,min}}$ and $L_{\text{simple,min}}$ see Huang (2017).
3.3 What Makes the Difference between $L_{\text{simple, min}}$ and $L_{\text{unobs, min}}$

The comparison of $L_{\text{standard, min}}$ and $L_{\text{simple, min}}$ using two full-information Taylor rules, $TR_{\text{standard}}$ and $TR_{\text{simple}}$, were studied in the previous chapter. Here the focus is on the loss difference between $L_{\text{simple, min}}$ and $L_{\text{unobs, min}}$. Let $\hat{x}_t$ and $\hat{\pi}_t$ are the output gap and inflation when the Fed does not adjust its interest rate to $\epsilon_t$ and $\eta_t$. The following two equations must be true since none of the $t-1$ variables changes, and because $\epsilon$ and $\eta$ are orthogonal to everything, and because the shocks to output gap and inflation will be $\epsilon$ and $k\epsilon + \eta$ when $\epsilon$ and $\eta$ are unobserved:

$$x_t - \hat{x}_t = \epsilon_t,$$

(9a)
and

$$\pi_t - \hat{\pi}_t = k\varepsilon_t + \eta_t, \quad (9b)$$

where $\hat{x}_t$ is the expected value of current period’s output gap and $\hat{\pi}_t$ is the expected value of current period’s inflation. We then name $x_t - \hat{x}_t = DX$ and $\pi_t - \hat{\pi}_t = DP$ and substitute $DX$ and $DP$ into (6) for $x_t$ and $\pi_t$ to obtain the loss difference ($DL$) between $L_{simple}$ and $L_{unobs}$ as:

$$DL = \frac{1}{2}E[(DP)^2 + \Gamma(DX)^2]$$

$$= \frac{1}{2}E[(k\varepsilon_t + \eta_t)^2 + \Gamma(\varepsilon_t)^2]$$

$$= \frac{1}{2}(k^2 + \Gamma)\sigma^2_\varepsilon + \frac{1}{2}\sigma^2_\eta. \quad (10)$$

This loss difference, as we show in Figure 3.1, is the result when the Fed sets the interest rate before observing current variables of output gap and inflation which contains information of shocks $\varepsilon_t$ and $\eta_t$. In other words, the Fed commits to $TR_{unobs}$. In this situation the Fed cannot adjust its interest rate for those shocks according to $TR_{unobs}$. On the other hand, the information of money is immediately available to the Fed, so the Fed can use the signals coming from unexpected quantity changes in money, which we can simply call it the money signal, to adjust their interest rate. In the next section, we will show how to derive the money signal and use it to develop a combination policy rule for reducing $DL$.

### 3.4 Derivation of the Money Signal

As Figure 3.1 shows, the difference in loss between $L_{simple,min}$ and $L_{unobs,min}$ cannot be reduced by using different set of optimal values of $\varphi_x$ and $\varphi_\pi$, or even when these values are forced to approach infinity. Thus using the money signals are the simple and direct way to
reduce the difference in loss between \( L_{simple, min} \) and \( L_{unobs, min} \) when new information about \( \varepsilon_t \) and \( \eta_t \) are unobserved. Here we show that the money signals are actually a linear function of \( \varepsilon_t \) and \( \eta_t \). First we recall the output gap is defined as the difference between current output \( (y_t) \) and the output in full-employment \( (y_t^f) \), i.e., \( x_t = y_t - y_t^f \), and inflation is defined as the difference between price in current period \( (p_t) \) and in previous period \( (p_{t-1}) \), i.e., \( \pi_t = p_t - p_{t-1} \).

Second, we apply the log-linearized Euler condition of money and consumption:

\[
m_t - p_t = \frac{\sigma}{b} y_t - \frac{1}{b} i_t + \Omega_t, \tag{11}
\]

where \( 1/b \) is the semi interest elasticity\(^{37} \) of money demand and \( \Omega \) is the money demand shock which is also the first order autocorrelation process with the innovation \( \omega \sim iid \ N(0, \sigma^2) \).

Substituting \( p_t \) in (11) by \( \pi_t + p_{t-1} \) and \( y_t \) in (11) by \( x_t + y_t^f \) from our definition of output gap and inflation, and then substituting \( x_t \) by \( \hat{x}_t + DX \) and \( \pi_t \) by \( \hat{\pi}_t + DP \) from the discussion in section 3.3, we then obtain the money equation as

\[
m_t = \hat{\pi}_t + DP + p_{t-1} + \frac{\sigma}{b}(\hat{x}_t + DX + y_t^f) - \frac{1}{b} i_t + \omega_t. \tag{12}
\]

Finally, we obtain the money signals by subtracting the expected value of (12) from (12) as

\[
m_t - E_{t-1} m_t = DP + \frac{\sigma}{b}(DX) + \omega_t
\]

\[
= k \varepsilon_t + \eta_t + \frac{\sigma}{b} \varepsilon_t + \omega_t
\]

\[
= \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t, \tag{13}
\]

\(^{37} \text{Empirically, we use semielasticity of money demand with respect to nominal interest rate: } \frac{1}{b} \left( \frac{1}{1+i_t} \right). \text{ Walsh (2003, p.57): “Empirical work often estimates money-demand equations in which the log of real money balances is a function of log income and the level of nominal interest rate. The coefficient on the nominal interest is then equal to the semielasticity of money demand with respect to nominal interest rate.”} \)
where all the shock terms $\varepsilon_t$, $\eta_t$ and $\omega_t$ are independent. (13) is the difference in value between actual money demand and the expectation of money demand, and it will be used to improve $TR_{unobs}$’s performance because the money signals are constituted by the shocks which are not included in $TR_{unobs}$.

We are now ready to turn to the next section for the combination policy rule using the money signals—i.e., I add the money signals and its coefficient $\varphi_m$ in the simple Taylor rule. The discussion in the next section is a way of explaining how the money signals permits the Fed to improve their perception of the actual $\varepsilon_t$ and $\eta_t$. In the next section I am finding the optimal value of $\varphi_m$ at its optimal setting. This is telling us the optimal response to the interest rate from new information that comes from the money surprise. We could less elegantly do it in two steps: find the “best” revision of $\varepsilon_t$ and $\eta_t$, then insert them into the $x_t$ and $\pi_t$ solutions to find the optimal interest rate adjustment. By deriving the optimal $\varphi_m$ I am doing all this in one step.

### 3.5 The Combination Policy Rule and its Performance

Before I get into details, one thing needs to be clarified. The focus is on the simple Taylor rule for this chapter, and the reason is that it is impractical or “unrealistic” to expect the Fed bases a rule on a shock term. It is the same reason for discussing the properties of the simple Taylor rule in Chapter 2. The three-term Taylor rule is only a benchmark for comparison purpose. For interested readers, Appendix 3.9.2 provides the discussion of a combination policy with the three-term Taylor rule built in.
Let $TR_{combine}$ express a combination policy rule from combining (5) and (13) which its equation is shown as

$$i_t = \varphi_x E_{t-1} x_t + \varphi_{\pi} E_{t-1} \pi_t + \varphi_m \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t. \quad (14)$$

When we substitute (14) into (1) for $i_t$, we can observe that the previous (9a) becomes

$$-\frac{1}{\sigma} \varphi_m \left( \left( k + \frac{\sigma}{b} \right) \epsilon_t + \eta_t + \omega_t \right) + \epsilon_t,$$

and the previous (10) becomes

$$\overline{DL} = \frac{1}{2} \left( k^2 + \Gamma \right) \left( \frac{\varphi_m^2}{\sigma^2} \sigma_{\eta}^2 + \frac{(\varphi_m(bk + \sigma) - \sigma b)^2}{\sigma^2 b^2} \sigma_{\epsilon}^2 + \frac{\varphi_m^2}{\sigma^2 \sigma_{\omega}^2} \right) + \frac{1}{2} \sigma_{\omega}^2. \quad (15)$$

The symbol $^\wedge$ on $DL$ is for distinguishing (15) from (10). Since the constraint on $\varphi_x$ and $\varphi_{\pi}$ is unique, we can simply substitute this difference in loss $\overline{DL}$ for $DL$ into $L_{unobs}$, and then we obtain the social welfare loss $L_{combine}$ as

$$L_{combine} = L_{simple,min} + \frac{1}{2} \left( 1 + \frac{(k^2 + \Gamma)}{\sigma^2} \varphi_m^2 \right) \sigma_{\eta}^2 + \frac{1}{2} \left( \frac{(k^2 + \Gamma)(\varphi_m(bk + \sigma) - \sigma b)^2}{\sigma^2 b^2} \right) \sigma_{\epsilon}^2 + \frac{1}{2} \left( \frac{(k^2 + \Gamma)}{\sigma^2} \varphi_m^2 \right) \sigma_{\omega}^2. \quad (16)$$

If the value of $\varphi_m$ is zero in (16), $L_{combine}$ becomes $L_{unobs,min}$ (i.e., equation (8c)). The minimum value of (16), i.e., $L_{combine,min}$ will be obtained when

$$\varphi_m = \frac{\sigma b (bk + \sigma) \sigma_{\epsilon}^2}{(bk + \sigma)^2 \sigma_{\epsilon}^2 + b^2 \sigma_{\eta}^2 + b^2 \sigma_{\omega}^2}$$

$$= \frac{\sigma b}{(bk + \sigma) \left( 1 + \frac{b^2}{(bk + \sigma)^2} \frac{\sigma_{\eta}^2 + \sigma_{\omega}^2}{\sigma_{\epsilon}^2} \right)}, \quad (17)$$
and this analytical value of $\varphi_m$\textsuperscript{38}, as expected from the demonstration of the “inverted” Poole combination policy in section 3.1, has $\varphi_m$ to vary inversely with the variance of $\omega, \eta$ and $\varepsilon$. When the value of $\sigma^2_\omega$ is very large so the value of $\varphi_m$ is very close to zero, in this situation, there are not much of money signals for the improvement of the Fed’s perception of the actual $\varepsilon_t$ and $\eta_t$. The validity of equation (17) can be examined by assuming there is only $g_t$ shock (this implied that $\sigma^2_\eta = 0$ and $\sigma^2_\omega = 0$) and then the loss is eliminated when I substitute the equation (17) in the equation (14) for $\varphi_m$. This implies that the money signals are able to eliminate the difference in the social welfare losses which caused by unobserved $\varepsilon_t$. In other words, when money signals of $\varepsilon_t$ is used with the optimal $\varphi_m$ (17), the value of equation (10) is equal to zero for only having one shock $\varepsilon_t$. A similar result also applies to the case when only unobserved $\eta_t$ happens in the model economy.

Figure 3.2 shows the improvement of social welfare loss from the money signals. $L_{\text{combine, min}}$ is the minimum social welfare loss with the money signals when the unit shock\textsuperscript{39} of money demand exists. $L_{\text{combine, min}}$ is smaller than $L_{\text{unobs, min}}$ which is the social welfare loss with the expected simple Taylor rule described in the section 3.2.

\textsuperscript{38} Solving $\frac{\partial L_{\text{combine}}}{\partial \varphi_m} = 0$ for $\varphi_m$, we obtain (17). Since all variables have optimal settings, (17) is optimal.

\textsuperscript{39} The variance of money demand $\sigma^2_\omega$ equals to one.
Figure 3.2. The Improvement of Social Welfare Loss

![Graph showing the improvement of social welfare loss with different values of $\phi_\pi$.]

Note: $b$ is assumed with value equal to 0.5. All parameter values are the same as used in the section 2.

It is clear from Figure 3.2 that we cannot bring down $L_{unobs, min}$ to $L_{standard, min}$ by using different sets of optimal $\phi$’s or forcing the optimal $\phi$’s higher. The only way to improve the social welfare loss when using the expected simple Taylor rule is by applying the money signals for improving the Fed’s perception of actual $\epsilon_t$ and $\eta_t$. In appendix 3.9.2, I compare and discuss the improvement of social welfare losses for a case in which the Taylor rule contains the $\phi_g g_t$ term.

For simplicity, the special case for the $\phi_m$ can be also shown as

$$\phi_m = \frac{\sigma b (bk + \sigma)}{(bk + \sigma)^2 + b^2}$$  \hspace{1cm} (18)
when we arbitrarily let \( \frac{\sigma_\eta^2 + \sigma_\omega^2}{\sigma_\epsilon^2} = 1 \) in (17).\(^{40}\) This special case can be used for understanding the relationship among minimized social welfare losses, \( \varphi_m \) and \( b \). The special case is only for comparison purpose. In reality the value of \( \varphi_m \) is affected by the variance of shocks. Figure 3.9.1 in the Appendix 3.9 shows that \( \varphi_m \) is always greater than zero and its value has a limit when \( b \) approaches infinity by using equation (18).\(^{41}\) Since \( b \) is the inverse of interest semi-elasticity of money demand, the bigger the value of \( b \) the smaller the interest semi-elasticity will be. The long time debate on the stability of money demand\(^{42}\) through the different estimated values of interest elasticity of money demand should not affect the improvement of the Fed’s perception of actual \( \epsilon_t \) and \( \eta_t \) by using money signals when the coefficient of money

\[
\varphi_m = \frac{ab}{(bk + \sigma) \left( 1 + \frac{b^2}{(bk + \sigma)^2} \right)} = \frac{ab}{(bk + \sigma) + \frac{b^2}{(bk + \sigma)}} = \frac{ab}{(bk + \sigma)^2 + b^2}.
\]

\(^{40}\) The value of \( \varphi_m \) is determined by the inverse of the interest rate semi-elasticity \( b \), which is the factor for the stability of money demand. \( \varphi_m \) will equal to zero only when \( b = 0 \) or \( -\frac{\sigma}{\epsilon} \). Since \( b > 0 \) from our derivation of money demand in section 3.4, \( \varphi_m > 0 \). The unstable money demand is associated with high interest elasticity according to Teles and Zhou (2005, p.52)'s observation of Ball (2001): “Ball (2001) argues that the data after 1987 represent evidence against a stable money demand. He estimates a linear relationship between logarithm of real money, the logarithm of output, and a nominal interest rate for subperiods of 1903-94. For the period 1903-87 the evidence is consistent with a stable relationship with a unitary income elasticity and a relatively high interest elasticity, as shown by Lucas (1988) and Stock and Watson (1993). However, the need to account for the low reaction of M1 to lower interest rates and higher output after 1980 lowers both the estimated interest elasticity and income elasticity. The relatively low income and interest elasticity in the postwar period (1974-94) are significantly different from the unitary income elasticity and relatively high interest elasticity in the prewar period (1903-45), leading Ball to argue against a stable long run money demand.”

\(^{41}\) Walsh (2010, pp. 48-52)
signal \( \varphi_m \) has different values of \( b \) built in. The relatively interest inelastic money demand\(^{43} \) implied by \( b \) approaching infinity does not play an important role for stopping using the money signals.

### 3.6 The Effectiveness of Money Signals

Whether the Fed can respond to unobservable shocks through the money signals is important for the performance of its monetary policy rule. The effectiveness of the money signals is therefore the key to the combination policy performance. At the end of Section 3.5, I discussed the value of \( b \) only have limited impact to the money signals. In this section, I study the impact of \( \sigma_{\omega}^2 \) to the money signals and to the performance of the combination policy rule.

#### 3.6.1 How is \( \varphi_m \) Working in the Combination Policy?

The \( \varphi_m \) is the optimal response coefficient of the interest rate to the money surprise. How does the Fed infer unobserved \( \varepsilon_t \) and \( \eta_t \) from the money signals? The Fed controls the money supply so it is capable of calculating the money signals. Recalled that \( m_t - E_{t-1}[m_t] = \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t \). The information content of the money signals are then used to indicate \( \varepsilon_t \) and \( \eta_t \) as follows. The estimated coefficient of \( \sigma_{\varepsilon}^2 \) conditional on \( \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t \) is

\[
\hat{\alpha}_1 = \frac{\text{Cov} \left[ \varepsilon_t, \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t \right]}{\text{Var} \left[ \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t \right]},
\]

\[
= \left( k + \frac{\sigma}{b} \right) \sigma_{\varepsilon}^2 / \left[ \left( k + \frac{\sigma}{b} \right)^2 \sigma_{\varepsilon}^2 + \sigma_\eta^2 + \sigma_\omega^2 \right],
\]

\(^{43}\) “The key point is that money demand shocks can induce volatile behavior of interest rates. This is particularly true if money demand is relatively interest inelastic in the short run, as is the case for bank reserves….It is for this reason that in practice central banks use interbank lending rates as the policy instrument,…” CGG (1999, p.1686)
and
\[ \hat{\sigma}_\varepsilon^2 = E \left[ \left( 1 - \hat{\alpha}_1 \left( (k + \frac{\sigma}{b}) \varepsilon_t + \eta_t + \omega_t \right) \right)^2 \right], \]
and the estimated coefficient of \( \hat{\sigma}_\eta^2 \) conditional on \((k + \frac{\sigma}{b}) \varepsilon_t + \eta_t + \omega_t\) is that
\[ \hat{\alpha}_2 = \text{Cov} \left[ \eta_t, (k + \frac{\sigma}{b}) \varepsilon_t + \eta_t + \omega_t \right] / \text{Var} \left[ (k + \frac{\sigma}{b}) \varepsilon_t + \eta_t + \omega_t \right], \]
\[ = \sigma_\eta^2 / \left[ \left( k + \frac{\sigma}{b} \right)^2 \sigma_\varepsilon^2 + \sigma_\eta^2 + \sigma_\omega^2 \right], \]
and
\[ \hat{\sigma}_\eta^2 = E \left[ \left( 1 - \hat{\alpha}_2 \left( \left( k + \frac{\sigma}{b} \right)^2 \sigma_\varepsilon^2 + \sigma_\eta^2 + \sigma_\omega^2 \right) \right)^2 \right], \]
where \( \sigma_\varepsilon^2, \sigma_\eta^2 \) and \( \sigma_\omega^2 \) are the variances of output gap, inflation and money demand, which are described in the equation (17). \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) are the parameter values which can be estimated by the central bank. When the money signals are changed by the proportional to \( m_1 \frac{\sigma}{b} \), this indicates that there is changes in \( \varepsilon_t \) through changes in \( \left( k + \frac{\sigma}{b} \right)^2 \sigma_\varepsilon^2 + \sigma_\eta^2 + \sigma_\omega^2 \). Then the Fed can adjust its interest rates in response to \( \varepsilon_t \) by the amount which is determined by \( \varphi_m \) from the section 3.4. The Fed can also adjust its interest rates in response to \( \eta_t \) through \( \hat{\alpha}_2 \) as well. As long as the variance of money demand is small, the money signals are a good tool for the Fed to adjust interest rates in response to the unobserved \( \varepsilon_t \) and \( \eta_t \). The bigger the variance of money demand, the less effective the money signals will be. When the variance of money demand approaches to infinity, the numerical value of the money signals are zero. This implies that the money signals may not be used as signal for unobserved \( \varepsilon_t \) and \( \eta_t \) when the variance of money
demand is big. Figure 3.3 shows this negative relation among $\sigma^2_{\omega}$, $\hat{a}_1$ and $\hat{a}_2$. From the two expressions of information content for $\varepsilon_t$ and $\eta_t$, I can find the values changes in $\hat{a}_1$ and $\hat{a}_2$ when $\sigma^2_{\omega}$ changes by taking the first partial derivatives of $\hat{a}_1$ and $\hat{a}_2$ with respect to $\sigma^2_{\omega}$:

\[
\frac{\partial \hat{a}_1}{\partial \sigma^2_{\omega}} < 0
\]

and

\[
\frac{\partial \hat{a}_2}{\partial \sigma^2_{\omega}} < 0,
\]

so the bigger the $\sigma^2_{\omega}$, the smaller the values of $\hat{a}_1$ and $\hat{a}_2$ will be. Thus, when $\sigma^2_{\omega}$ is high, the improvement of the Fed’s perception of actual $\varepsilon_t$ and $\eta_t$ through money signals is low. Figure 3.3 shows that the values of $\hat{a}_1$ and $\hat{a}_2$ decrease when the value of $\sigma^2_{\omega}$ increases.

**Figure 3.3. The Money Signals and The Noise in Money Demand**
3.6.2 The Social Welfare Loss and the “Noise” in Money Demand

The money demand shocks affect the Fed’s perception of actual $\varepsilon_t$ and $\eta_t$ and this is reflected on the minimized social welfare loss $L_{\text{combine, min}}$ as well. When there are no money demand shocks, the value of $\sigma^2_\omega$ equals zero. Then the Fed’s will have the best perception of actual $\varepsilon_t$ and $\eta_t$ by using the money signals, and $L_{\text{combine, min}}$ will be smaller. Figure 3.4 below shows that when $\sigma^2_\omega = 0$, the minimized social welfare loss is below the minimized social welfare loss when $\sigma^2_\omega \neq 0$.

Figure 3.4. The Impact of $\sigma^2_\omega$ on the Improvement of Social Welfare Losses
3.7 A Numerical Example: Measuring Improvement from the Money Signal

The effectiveness of the money signals is determined by optimized $\varphi_m$, which is determined by parameters and $\sigma_\varepsilon^2$, $\sigma_\eta^2$ and $\sigma_\theta^2$. And the value of $\varphi_m$ is in part determined by the inverse of the interest rate elasticity $b$. I examine the values of $b$ and $\sigma_\theta^2$ to $L$ using Walsh (2010)’s parameters which $k = 0.05, \beta = 0.99, \sigma = 1, \rho = 0.5$ and $\Gamma = 0.25$.\(^{44}\) For the value of $\sigma_\theta^2$, I use Cooley and Leroy (1981)’s standard error of M1 demand $\sigma_\theta = 0.028$ which is the largest one according to Baba, Hendry and Starr (1992)’s survey paper.\(^{45}\) The values of $\sigma_\varepsilon^2$ and $\sigma_\eta^2$ are from Ireland (2011)’s table 1 where he reports $\sigma_\varepsilon = 0.0868$ and $\sigma_\eta = 0.0017$. The value of $b$ is set to be 2 based on Lucas (2000)’s estimation of 0.5 for the interest rate elasticity of money demand (M1).

First, Figure 3.5 shows that $\varphi_m$ is nonzero positive value at $b = 2$. This figure also implies that the interest rate elasticity of money demand (M1) may play less important role in monetary policy then conventional beliefs. Whether the money signals are measured with a stable money demand to interest rates or not, the social welfare loss should be still improved by the money signals.

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\(^{44}\) Marest and Thurston (2017) reported the parameter estimations in the literature of NKM. Their report also includes Walsh (2010)’s parameters. See Table 1 in the Appendix 3.9.3 for details.

\(^{45}\) See Table 2 in the Appendix 3.9.3 for details.
Figure 3.5. The Relationship between $\varphi_m$ and $b$

![Graph showing the relationship between $\varphi_m$ and $b$.]

Note: The values of $\sigma^2_\varphi$ and $\sigma^2_\eta$ are from Peter Ireland (2011)’s table 1, p.42. The value of $\sigma^2_\omega$ is from Baba, Hendry and Starr (1992) table 2, p.44.

Figure 3.6 shows the improvement of minimized social welfare loss from the money signals. All the social welfare losses $L_0$, $L_1$, $L_2$, and $L_3$ are from the definition in previous sections. $L_0$ curve is the optimal social welfare loss defined in section 3.2. $L_1$ curve shows the outcome of social welfare loss when $\varphi_g = 0$ and the existence of g-shocks defined in section 3.2. $L_2$ is the loss when the Fed’s Taylor rule uses only expected values of $x_t$ and $\pi_t$ defined in section 3.2. $L_3$ is the loss when the Fed uses the combination policy rule (with the money signals) defined in section 3.5. $L_3$ is below $L_2$ which indicates the improvement of social welfare loss from the money signals. Table 3 in the appendix 3.9.3 provides the definitions of $L_0$, $L_1$, $L_2$, and $L_3$. The values of $\sigma^2_\varphi$ and $\sigma^2_\eta$ are from Peter Ireland (2011)’s table 1, p.42. The value of $\sigma^2_\omega$ is from Baba, Hendry and Starr (1992) table 2, p.44.
L1, L2 and L3. Table 4 in the appendix 3.9.3 compares the improvement of the minimized social welfare loss to different values of \( b \) and \( \Gamma \) from empirical studies. Note that \( b \) and \( \Gamma \) are parameter values that cannot be used as instrument.

**Figure 3.6. The Improvements of L from the Money Signals**

The money demand shocks affects the Fed’s perception of actual \( \epsilon_t \) and \( \eta_t \) and this is reflected on the minimized social welfare loss \( L_{\text{combine, min}} \) as well. When demand shocks are big, the Fed’s perception of actual \( \epsilon_t \) and \( \eta_t \) is worse. Figure 3.7 shows the perception of actual \( \epsilon_t \) represented by \( \hat{a}_1 \) and the perception of actual \( \eta_t \) represented by \( \hat{a}_2 \) has small values when the value of \( \sigma_{\tau}^2 \) is big. Also Figure 3.7 implies that the Fed can use money signals to improve their perception of actual \( \epsilon_t \) and \( \eta_t \) because the biggest value of the \( \sigma_{\tau} \) is 0.028. Note that the \( \hat{a}_1 \)
curve is lower than $\hat{a}_2$ curve in the Figure 3.7 but the $\hat{a}_1$ curve is higher than $\hat{a}_2$ curve in the Figure 3.3. This is because $b = 2$ in the Figure 3.7 and $b = 0.5$ in the Figure 3.3.

**Figure 3.7. The Social Welfare Loss and the Money Signals**

![Figure 3.7](image.png)

### 3.8 Conclusion

The combination policy can improve the social welfare loss when the Fed sets its monetary policy with unobservable shocks. This paper uses an inverted version of Poole’s (1970) combination policy analysis and shows that the social welfare loss is improved from the money signals. $b$ affects the effectiveness of the money signals, but the money signals always have non-zero values even when $b$ has very big values. $\sigma_m^2$ affects the money signals and the social welfare loss. $\sigma_m^2$ will decrease $\phi_m$ and raise loss to some limit. We should use combination
policy as long as there is any perceptible gain, $\varphi_m > 0$, and the social welfare loss will be reduced by at least a little.

3.9 Appendixes

3.9.1 The Figure for the value of $\varphi_m$

Figure 3.8.1. shows the value of $\varphi_m$ when $b$ is approaching positive (negative) infinity with $k = 0.3, \beta = 0.99$ and $\sigma = 1$.

Figure 3.8.1. The Values of $\varphi_m$
3.9.2 The Example of Using the Money Signals to Improve $L$ when the Quantity of $\varphi_g$ in Taylor Rules are not Zero

In this example I apply the steps from section 3.2 to section 3.5 for Taylor rules whose have the quantity of $\varphi_g$ not equal to zero. These Taylor rules are $TR0$:

$$i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t,$$  \hspace{1cm} (A1)

a limited-information Taylor rule $TR0'$:

$$i'_t = \varphi_x E_{t-1} x_t + \varphi_\pi E_{t-1} \pi_t + \varphi_g E_{t-1} g_t,$$  \hspace{1cm} (A2)

and a combination policy rule using the money signals $TR0''$:

$$i''_t = \varphi_x E_{t-1} x_t + \varphi_\pi E_{t-1} \pi_t + \varphi_g E_{t-1} g_t + \varphi_m \left( \left( k + \frac{\sigma}{b} \right) \varepsilon_t + \eta_t + \omega_t \right).$$  \hspace{1cm} (A3)

Substituting these three rules (A1), (A2) and (A3) separately into (1) for $i_t$ and then solving (1) and (2) for $x_t$ and $\pi_t$ accordingly. Next I substitute each set of general solutions of $x_t$ and $\pi_t$ associated with (A1), (A2) and (A3) into the period quadratic loss function: $L = \frac{1}{2} E[\pi_t^2 + \Gamma x_t^2]$ and we obtain three social welfare loss functions $L0$, $L0'$ and $L0''$ associated with (A1), (A2) and (A3). Separately minimizing these three social welfare loss function and solving the first-order conditions for $\varphi_x$ and $\varphi_\pi$, I obtain the same unique constraint (7) of $\varphi_x$ and $\varphi_\pi$:

$$\varphi_\pi = -\frac{k}{r(\beta \rho - 1)} \varphi_x + \frac{k \sigma (\rho - 1)}{r(\beta \rho - 1)} + \rho,$$ and furthermore solving the first order condition for $\varphi_m$, I obtain the same optimal value (17) of $\varphi_m$:

$$\varphi_m = \frac{\sigma b}{(bk + \sigma) \left( 1 + \frac{b^2}{(bk + \sigma)^2} \frac{\sigma^2 \eta^2 + \sigma^2 \omega}{\sigma^2 \epsilon} \right)}.$$  

Next, substituting (7) into $L0$, $L0'$ and $L0''$ and substituting (17) into $L0''$, I obtain three minimum social welfare losses:
\[ L_0 = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2} \right) \sigma_\eta^2, \quad (A4) \]

\[ L_0' = \frac{1}{2} \left( \frac{\Gamma}{\Gamma(\beta \rho - 1)^2 + k^2 + 1} \right) \sigma_\eta^2 + \frac{1}{2} (k^2 + \Gamma) \sigma_\xi^2, \quad (A5) \]

and

\[ L_0'' = L_0' + \frac{1}{2} \left( 1 + \frac{(k^2 + \Gamma)}{\sigma^2} \varphi_m^2 \right) \sigma_\eta^2 + \frac{1}{2} \left( \frac{(k^2 + \Gamma)(\varphi_m (bk + \sigma) - \sigma b)^2}{\sigma^2 b^2} \right) \sigma_\xi^2 \]

\[ + \frac{1}{2} \left( \frac{(k^2 + \Gamma)}{\sigma^2} \varphi_m^2 \right) \sigma_\omega^2, \quad (A6) \]

where \( \varphi_m = \frac{\sigma b}{(bk + \sigma)(1 + \frac{b^2}{(bk + \sigma)^2} \frac{\sigma_\eta^2 + \sigma_\omega^2}{\sigma_\xi^2})} \).

Table 3.8.2 shows the comparison of \( L_0, L_0' \) and \( L_0'' \) under the special cases. The first column of Table 3.8.2 shows the performance of the combination policy rule is as good as the full-information Taylor rule when there are only \( g \)-shocks, so we should use the combination policy rule when there are only unobservable \( g \)-shocks. The second column of Table 3.8.2 shows that the performance of the combination policy is the same as the limited-information Taylor rule when there are only unobservable \( u \)-shocks. The third column of Table 3.8.2 shows that the performance of combination policy rule are the same as the full-information Taylor rule and the limited-information Taylor rule when there are only \( \omega \)-shocks. The fourth column of Table 3.8.2 shows that the performance of combination policy rule is better than the limited-information Taylor rule, so we should use combination policy rule when current shocks are unobservable.
Table 3.8.2. Comparison of $L_0$, $L_0'$ and $L_0''$

<table>
<thead>
<tr>
<th></th>
<th>$\sigma^2_\varepsilon = 1, \sigma^2_\eta = 0, \sigma^2_\omega = 0$</th>
<th>$\sigma^2_\varepsilon = 0, \sigma^2_\eta = 1, \sigma^2_\omega = 0$</th>
<th>$\sigma^2_\varepsilon = 0, \sigma^2_\eta = 0, \sigma^2_\omega = 1$</th>
<th>$\sigma^2_\varepsilon = 1, \sigma^2_\eta = 1, \sigma^2_\omega = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_m$</td>
<td>0.435</td>
<td>0.000</td>
<td>0.000</td>
<td>0.316</td>
</tr>
<tr>
<td>$L_0$</td>
<td>0</td>
<td>1.000</td>
<td>0.000</td>
<td>1.667</td>
</tr>
<tr>
<td>$L_0'$</td>
<td>1.045</td>
<td>2.000</td>
<td>0.000</td>
<td>3.211</td>
</tr>
<tr>
<td>$L_0''$</td>
<td>0</td>
<td>2.000</td>
<td>0.000</td>
<td>4.563</td>
</tr>
</tbody>
</table>

Note: Assumed parameters are $k = 0.3, \beta = 0.99, \Gamma = 2, \sigma = 1, \lambda = 0.5, \rho = 0.5, \varphi_\pi = 1.5$ and $\varphi_x = -2.18 + 3.367\varphi_\pi$

3.9.3 Tables of Empirical Resources

I reproduce Marest and Thurston (2017)’s table 3 here and I give a comment on the use of $\Gamma$ in the Note.

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Woodford</td>
<td>2003</td>
<td>$0.024, 0.99$</td>
</tr>
<tr>
<td>Billi</td>
<td>2008</td>
<td>$0.024, 0.9926$</td>
</tr>
<tr>
<td>Walsh</td>
<td>2010</td>
<td>$0.05, 0.99$</td>
</tr>
</tbody>
</table>

Note: We should ignore the small values of $\Gamma$ when we compare the improvement of $L$. The smaller the $\Gamma$, the smaller the $L$ will be. Small $\Gamma$ implies that the weight on output gap is small and $L$ is less affected by the changes in output gap from $g_t$. Since the money signals improved $L$ mainly by improving output gap, an almost zero weight on $x$ sends a wrong message that the money signals were ineffective.

I reproduce Baba, Hendry and Starr (1992) table 2 here.

<table>
<thead>
<tr>
<th>Author</th>
<th>Year</th>
<th>Standard Error of M1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldfeld</td>
<td>1973</td>
<td>0.43%</td>
</tr>
<tr>
<td>Garcia and Pak</td>
<td>1979</td>
<td>0.63%</td>
</tr>
<tr>
<td>Rose</td>
<td>1985</td>
<td>0.48%</td>
</tr>
<tr>
<td>Gordon</td>
<td>1984</td>
<td>0.43%</td>
</tr>
<tr>
<td>McAleer, Pagan and Volker</td>
<td>1985</td>
<td>0.31%</td>
</tr>
<tr>
<td>Simpson and Porter</td>
<td>1980</td>
<td>0.52%-0.59%</td>
</tr>
<tr>
<td>Cooley and Leroy</td>
<td>1981</td>
<td>2.80%</td>
</tr>
<tr>
<td>Baba, Hendry and Starr</td>
<td>1992</td>
<td>0.38%</td>
</tr>
</tbody>
</table>
Table 3 shows the definitions of $L_0$, $L_1$, $L_2$ and $L_3$.

<table>
<thead>
<tr>
<th>$L_0$</th>
<th>The social welfare loss with Woodford’s Taylor rule $(i_t = \varphi_x x_t + \varphi_\pi \pi_t + \varphi_g g_t)$ built in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>The social welfare loss with the simple Taylor rule $(i_t = \varphi_x x_t + \varphi_\pi \pi_t)$ built in.</td>
</tr>
<tr>
<td>$L_2$</td>
<td>The social welfare loss with the simple (unobservable shocks) Taylor rule $(i_t = \varphi_x E_{t-1} x_t + \varphi_\pi E_{t-1} \pi_t)$ built in.</td>
</tr>
<tr>
<td>$L_3$</td>
<td>The social welfare loss with the combination policy rule built in.</td>
</tr>
</tbody>
</table>

Here I show the relationship between $b$, $\Gamma$ and The Improvement of $L$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$\Gamma$</th>
<th>The Improvement of $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>big</td>
<td>big</td>
</tr>
<tr>
<td>big</td>
<td>big ($\Gamma &gt; b$)</td>
<td>small</td>
</tr>
<tr>
<td>small</td>
<td>small</td>
<td>almost no improvement</td>
</tr>
</tbody>
</table>
**BIBLIOGRAPHY**


Ireland, Peter N. 2011. “A New Keynesian Perspective on the Great Recession.” *Journal of Money, Credit and Banking* 43: 31-54


