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Frontiers of Conditional Logic

Yale Weiss

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FRONTIERS OF CONDITIONAL LOGIC

by

YALE WEISS

A dissertation submitted to the Graduate Faculty in Philosophy in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2019
This manuscript has been read and accepted by the Graduate Faculty in Philosophy in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

FRONTIERS OF CONDITIONAL LOGIC

by

YALE WEISS

Adviser: Professor Graham Priest

Conditional logics were originally developed for the purpose of modeling intuitively correct modes of reasoning involving conditional—especially counterfactual—expressions in natural language. While the debate over the logic of conditionals is as old as propositional logic, it was the development of worlds semantics for modal logic in the past century that catalyzed the rapid maturation of the field. Moreover, like modal logic, conditional logic has subsequently found a wide array of uses, from the traditional (e.g. counterfactuals) to the exotic (e.g. conditional obligation). Despite the close connections between conditional and modal logic, both the technical development and philosophical exploitation of the latter has outstripped that of the former, with the result that noticeable lacunae exist in the literature on conditional logic. My dissertation addresses a number of these underdeveloped frontiers, producing new technical insights and philosophical applications.

I contribute to the solution of a problem posed by Priest of finding sound and complete labeled tableaux for systems of conditional logic from Lewis’ V-family. To develop these tableaux, I draw on previous work on labeled tableaux for modal and conditional logic; errors and shortcomings in recent work on this problem are identified and corrected. While modal logic has by now been thoroughly studied in non-classical contexts, e.g. intuitionistic and relevant logic, the literature on conditional logic is still overwhelmingly classical. Another contribution of my dissertation is a thorough analysis of intuitionistic conditional logic, in which I utilize both algebraic and worlds semantics, and investigate how several novel
embedding results might shed light on the philosophical interpretation of both intuitionistic logic and conditional logic extensions thereof.

My dissertation examines deontic and connexive conditional logic as well as the underappreciated history of connexive notions in the analysis of conditional obligation. The possibility of interpreting deontic modal logics in such systems (via embedding results) serves as an important theoretical guide. A philosophically motivated proscription on impossible obligations is shown to correspond to, and justify, certain (weak) connexive theses. Finally, I contribute to the intensifying debate over counterpossibles, counterfactuals with impossible antecedents, and take—in contrast to Lewis and Williamson—a non-vacuous line. Thus, in my view, a counterpossible like “If there had been a counterexample to the law of the excluded middle, Brouwer would not have been vindicated” is false, not (vacuously) true, although it has an impossible antecedent. I exploit impossible (non-normal) worlds—originally developed to model non-normal modal logics—to provide non-vacuous semantics for counterpossibles. I buttress the case for non-vacuous semantics by making recourse to both novel technical results and theoretical considerations.
Acknowledgments and Permissions

I would like to express my gratitude, first and foremost, to Professor Graham Priest. I first began thinking seriously about the philosophy and logic of conditionals in his course with Hartry Field on the subject in the fall semester of 2014—my first semester as a graduate student—and have yet to stop. At every stage of my work on the subject, from just learning it to producing original research, Graham Priest made himself available to provide feedback and advice. This dissertation, beginning with the subject to which it is devoted, owes much to his influence.

I owe special thanks, as well, to the other members of my committee. I am grateful to Professor Melvin Fitting for his comments, not only on this dissertation, but on the various papers and presentations over the years that built up to it. I thank Professor Edwin Mares for agreeing to serve on my committee (from far off New Zealand!) and for his valuable remarks on matters technical and philosophical. Finally, I owe thanks to Professor Gary Ostertag, especially for critical comments which have helped hone and clarify the philosophical content of chapter 5.

Thanks are also due to, among others: the anonymous referees and editors of various papers of mine on conditional logic (selections of some of which are included herein), the members of the “Priest Club” and the members of the Logic and Metaphysics Workshop (for listening to me present some of this material and providing helpful feedback), Professor Consuelo Preti (without whose influence I might never have ended up at the Graduate
Center), Professor Branden Fitelson (for introducing me to an automated theorem proving tool used herein), and my parents, Veronica and Wayne Weiss.

This dissertation incorporates material that I have previously published and I am pleased to acknowledge the publishers’ permissions to reuse this material. Some of the material from chapter 4 has been adapted from:


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Chapter 1

Introduction

In what follows, I introduce the subject of conditional logic and explain what this dissertation contributes to it. The history and scope of conditional logic is canvassed in section 1.1. An overview of the results and aims of the main chapters of this dissertation is given in section 1.2.

1.1 The Scope of Conditional Logic

Debate over the logic of conditionals is as old as propositional logic. Sextus famously reports that in third century BCE Alexandria, “even the crows on the roof tops” cawed about the truth conditions for conditionals (Adv. Math. I, 309-310) [77, pp. 42-3]. More recently, during the hegemony of classical logic in the early 20th century, the interpretation of natural language conditionals as material conditionals faced criticism from C. I. Lewis [65] and others. Although defenses of the material conditional have been made (at least as an interpretation of indicative conditionals; more on this anon), most notably by Grice [42] and

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1For an overview of the debate over conditionals in antiquity, see Kneale and Kneale [53, Ch. 3].

2Incidentally, the material conditional and the paradoxes thereof date back to the Megarian Philo at least (Sextus, Hyp. Pyrrh. II, 110 ff.).
Jackson [48], these cannot be considered wholly successful.  

If the classical material conditional is unsatisfactory as an interpretation of the natural language conditional, what conditional is satisfactory? This question, however, presupposes that there is a *unique* natural language conditional. Following Adams’ famous Oswald/Kennedy pair [1], it has been conventional, if not entirely uncontroversial, to hold that there are two distinct types of conditional, distinguished according to the use of the indicative or subjunctive mood. Consider the following two conditionals from [1, p. 90]:

(Sub) If Oswald hadn’t shot Kennedy in Dallas, then no one else would have

(Ind) If Oswald didn’t shot Kennedy in Dallas, then no one else did

Since the antecedent and consequent are the same in each (modulo tense and mood) but (Sub) is true whereas (Ind) is false, they must express different *types* of conditionals, or so it is claimed.

Unfortunately, matters get even more complicated from here. Another distinction, at most approximately coincident with that between indicative and subjunctive, is frequently drawn between counterfactual and non-counterfactual conditionals. Although ‘subjunctive’ and ‘counterfactual’ are both used to label ‘would’-type conditionals, neither, arguably, is entirely appropriate, nor are the two really interchangeable.

For the purposes of this dissertation, I mostly prefer to stay aloof from this debate (for some applications discussed below, I will take an explicit position on which fragment of natural language uses of conditional expressions, if any, the logic is intended to capture). Whether there is a multiplicity of conditionals or not, there is certainly a multiplicity of conditional logics. The earliest of these were either intended as general accounts of conditionality or else were uniquely tailored to counterfactual (if you prefer, subjunctive) conditionals.

---

3 For critical discussion, see Bennett [7, Ch. 2, 3], Priest [103], and Whitaker [127].
4 Adams' paper has generated a considerable amount of literature. For several criticisms, see Lowe [72] and Priest [103, 104].
5 For further discussion, see Lewis [67, pp. 3-4] and Bennett [7, pp. 11-2].
Stalnaker [114] exemplifies the first approach. Although counterfactuals are given much
attention by Stalnaker, his account is explicitly ecumenical in that it is supposed to apply to
conditionals in general, rather than counterfactuals alone [115, p. 274]. Stalnaker [114] and
Stalnaker and Thomason [117] aim to give a non-truth functional account of conditionals
and draw heavily on worlds semantics for modal logic for that purpose.\(^6\) In approximation,
Stalnaker [114, p. 102] holds that a conditional is true at a world if the consequent holds
at the (unique) world which differs minimally from the world of evaluation except that the
antecedent obtains there.

Interest in the logical analysis of counterfactuals specifically can be traced at least as far
back as Chisholm [15] and Goodman [41]. But the most ambitious instance of the second
approach is clearly Lewis [66, 67]. Famously, Lewis holds (loosely) that a counterfactual is
true at a world if the worlds most similar to it which satisfy the antecedent also satisfy the
consequent [67, p. 16]. Lewis proposes to analyze the key notion in this truth condition—
similarity—primarily in terms of an apparatus of nested spheres, but an equivalent semantics
using indexed preorders indicating comparative similarity turns out to be more germane for
many technical results (see chapter 3) and perhaps better suited to philosophical purposes
as well.\(^7\)

The pioneer work of Lewis and Stalnaker established conditional logic as a subject in its
own right, and lent it its name, but did not, ultimately, dictate its scope. As logicians and
philosophers began investigating systems contained in, and containing, the Lewis-Stalnaker
systems, applications quite unrelated to natural language conditionality arose.

Probably the best example of this is the deontic interpretation of conditional logic. Deont-
tic conditional logics (or “dyadic deontic logics” as they are sometimes called) were developed

\(^6\)More than anything else, it was the development of worlds semantics for modal logic, especially in Kripke
[57, 58, 60], that paved the way for penetrating intensional analyses of conditionality.

\(^7\)I agree with Kit Fine, who notes, “The rationale and status of these two formulations [of the semantics]
are not altogether clear. It is surely the formulation in terms of [comparative] similarity that is the more
basic” [25, p. 457].
and investigated using logical resources from, or related to, conditional logic by Hansson [43], van Fraassen [33], Lewis [67, 70], and Chellas [13, 14]. I develop deontic applications of certain conditional logics of my own in chapter 4. In chapter 6, I discuss a possible epistemic interpretation of the conditional connective in intuitionistic conditional logic; such interpretations are, of course, far removed from concerns about the logic of counterfactuals.

Because of the variety of applications of conditional logic, one should not read too much into the name. Even formally specifying what a conditional logic is—beyond stipulating the language—turns out to be a rather fruitless task. Conditional logics without any unique validities naturally arise in the examination of counterpossibles (see chapter 5) and conditional logics which are not subsystems of any Lewis-Stalnaker systems naturally arise in the examination of conditional obligation (see chapter 4). Therefore, let a conditional logic be, in the first instance, a system from one of Stalnaker’s or Lewis’ works on the subject; and in the second instance, a system bearing a family resemblance to one of those systems.

Throughout this dissertation, I use Lewis’ symbol $\Box \rightarrow$ for the novel conditional connective (in some chapters, I also use $\lozenge \rightarrow$) and treat this as a primitive. If a counterfactual interpretation is intended, $\phi \Box \rightarrow \psi$ is to be read: if $\phi$ were the case, then $\psi$ would be the case. If no particular natural language conditional is intended, it may simply be read: if $\phi$, then $\psi$. More exotic interpretations of the connective will be discussed in due course.

1.2 Overview of the Dissertation

This dissertation has, in addition to its introduction and conclusion chapters, five chapters. Of these five chapters, one is primarily a review of standard results, one presents purely technical results, and three aim to make both philosophical and technical contributions to

---

8Some philosophers, notably Åqvist [4], have sought to define $\Box \rightarrow$ (understood to be a subjunctive conditional) in terms of unary connectives, e.g. a modal box and selection operator, and the material conditional. I will not pursue any development along these lines here.
CHAPTER 1. INTRODUCTION

the subject. I review the material to be covered in each of these chapters below.

Chapter 2 is primarily dedicated to a review of the major conditional logics from the
literature. These include Stalnaker’s [114] $C_2 (= \text{VCS})$, Lewis’ [66, 67] $C_1 (= \text{VC})$, and
Chellas’ [13] $\text{CK}$. I examine these systems both syntactically and semantically. On the
syntactic side, I taxonomize and axiomatize various systems of conditional logic and discuss
embeddings of modal logics into these, drawing on previous work by Williamson [128, 129] (cf.
Lewis [67, Ch. 6.3]). Adapting previous work by Chellas [13], Nute [92], and Segerberg [111],
uniform neighborhood-type and algebraic semantics are presented for the major systems
and determination—soundness and completeness—results are proved. I also review more
specialized kinds of semantics, including the sphere and preorder semantics of Lewis [66, 67].
Although most of the material presented in this chapter is not novel, it is essential to include
a review of this sort since all subsequent chapters will draw on and modify this material in
various ways.

The proof theory of certain strong conditional logics—those from the $\text{V}$-family of Lewis
[67]—is investigated in chapter 3. In connection with this subject, it is worth pointing out
that, by far, the most popular way of presenting conditional logics proof theoretically in
the literature is with axiom (Hilbert) systems. While this has many drawbacks, I mention
two specifically. First, axiom systems, unlike tableaux and sequent calculi, cannot be used
to obtain many interesting proof-theoretic results nor are they particularly amenable to
automated proof search techniques. Second, axiom systems are poor pedagogical tools and
make the teaching and learning of derivations in conditional logic needlessly difficult.

While the proof theory of modal logic is by now quite mature, the proof theory of
conditional logic is comparably underdeveloped. In this chapter, I develop sound and


\textsuperscript{9}Following Chellas [14, p. 60], I sometimes say that a system is determined by a semantics (more
particularly, a class of interpretations) if it is sound and complete with respect to it.

\textsuperscript{10}For (labeled) sequent calculi for a range of modal logics, see Negri [85, 86, 87]. For modal tableaux, see
Fitting [30] and Priest [102]. For modal natural deduction systems, see Roy [110].

\textsuperscript{11}After axiom systems, sequent calculi are the most popular proof systems for conditional logic. Sequen-
complete analytic tableaux for the basic Lewis system from the \( \mathbf{V} \)-family, \( \mathbf{C}_0 \) (=\( \mathbf{V} \)), and many of its extensions, including \( \mathbf{C}_1 \). Stalnaker’s \( \mathbf{C}_2 \) is also given sound and complete tableaux, but not all of the rules are analytic. The tableaux developed in this chapter are closely related to a sequent calculus developed for \( \mathbf{C}_1 \) by Negri and Sbardolini [86]. However, while their calculus is seen to be unfaithful to \( \mathbf{C}_1 \), the calculus I develop for \( \mathbf{C}_1 \) does not suffer from this problem.

Chapter 4 is dedicated to the investigation of connexive conditional logic.\(^{12}\) Contemporary connexive logic, which has its roots in work by Angell [2] and McCall [78], is characterized by heterodox principles of conditionality like Aristotle’s and Boethius’ theses.\(^{13}\) These distinctly non-classical principles engender inconsistency in even quite weak conditional logics (see, e.g., Unterhuber [121]). They do, however, have a certain measure of intuitive plausibility for a variety of applications.

As chapter 4 will make clear, (weak versions of) Boethius’ theses, in particular, arise naturally from the deontic interpretation of conditional logic as the logic of conditional obligation (for which, see Lewis [67, Ch. 5.1] and Chellas [14, Ch. 10]). Deontically, \( \phi \rightarrow \psi \) can be read: given \( \phi \), \( \psi \) is obligatory (i.e. it ought to be that \( \psi \) is the case). This chapter investigates systems of conditional obligation, determines which of these is best, and examines the relationship between such systems and systems of unconditional obligation (i.e. deontic modal logics) via embedding results. In addition, several non-deontic connexive and partially connexive conditional logics (including systems and subsystems of Lowe [71]) type calculi for \( \mathbf{C}_1 \) and \( \mathbf{C}_2 \) have been given by de Swart [118] and Gent [38]; more recently, Negri and Sbardolini [89] developed a sequent calculus for \( \mathbf{C}_1 \), but it turns out to be unfaithful. Labeled sequent calculi for conditional logics in the neighborhood of \( \mathbf{CK} \) have been developed by Pozzato [98] and Poggiolesi [97]. Fitch-style natural deduction systems for several conditional logics have been given by Thomason [119] and Roy [110]. Tableaux for some conditional logics not stronger than \( \mathbf{CK}+(\text{ID})+(\text{CMP}) \) were given by Priest [102, Ch. 5]. Rönneal [109] and Zach [134] extended tableaux methods to additional conditional logics, but faced difficulties in obtaining (cut-free) completeness results for the strong systems.

\(^{12}\)The term ‘connexive’ was introduced into contemporary logic by McCall [78, p. 415], drawing inspiration from the third account of conditionals surveyed in a famous passage of Sextus (\textit{Hyp. Pyrrh.} II, 111).

\(^ {13}\)Aristotle’s theses, for example, are \( \neg(\phi \rightarrow \neg\phi) \) and \( \neg(\neg\phi \rightarrow \phi) \). For the names, see McCall [78, 80].
are axiomatized and discussed. Finally, determination and independence results are proved for these systems using neighborhood-type and algebraic semantics. This chapter extends previous work of mine in [126].

As is well known, counterpossibles, counterfactuals with impossible antecedents, are rendered vacuously true in the semantics for most extant conditional logics. Although vacuousness has had its defenders, including, most recently, Williamson [130], it is increasingly a minority position.\textsuperscript{14} Within the last several years, criticisms of vacuousness have been made by, among others, Berto et al. [8], Bjerring [9], Brogaard and Salerno [11], Jago [49, 50], and Krakauer [55, 56]. Despite the emerging philosophical consensus against vacuousness, technical developments have not kept pace. Few critics of vacuousness have proposed detailed replacement semantics, and those that have (e.g. Berto et al. [8] and Mares [74]) have largely shunned proof theory.\textsuperscript{15}

I propose a detailed study of systems of counterpossible logic and their semantics in chapter 5. The most characteristic feature of the non-vacuous approach to counterpossibles is the utilization of impossible, or non-normal, worlds.\textsuperscript{16} Such worlds make available new and interesting model constraints governing the relation between possible and impossible worlds, including the famous ‘strangeness of impossibility’ constraint discussed by Nolan [90]. These non-vacuous semantics are used to prove determination results for various systems of counterpossible logic discussed in the chapter.

Besides the previously mentioned technical developments and results, chapter 5 also engages in the philosophical debate over vacuousness and what, if anything, should replace it. I take a non-vacuous line but also endorse a system for counterpossibles that is considerably weaker than those which many opponents of vacuousness defend. This chapter further

\textsuperscript{14}Which is not to say that it’s false!

\textsuperscript{15}Mares and Fuhrmann [76] is one notable exception to this.

\textsuperscript{16}It should be noted that there is, at least technically speaking, no problem whatsoever about impossible worlds. As Priest [100, p. 291] observes, such worlds were developed and used by Kripke himself in [60] to model weak systems of modal logic including C. I. Lewis’ S2.
develops views of mine previously published in [124].

Conditional logic is studied in a non-classical context, viz. intuitionistic logic, in chapter 6. Given the overwhelmingly classical orientation of the literature on conditional logic, a detailed investigation of the subject from a non-classical perspective should be most welcome. In [13], Chellas developed the basic (normal) classical conditional logic \( \text{CK} \). This chapter, extending previous work of mine in [125], investigates its intuitionistic counterpart, \( \text{ICK} \), as well as various extensions thereof. A number of systems are axiomatized and given both algebraic and worlds semantics. These semantics are shown to be equivalent and determination results are proved.

Chapter 6 not only examines intuitionistic conditional logic, but examines intuitionistic logic via conditional logic, giving a Gödel-McKinsey-Tarski inspired embedding of it into a natural extension of Lewis’ main logic of counterfactuals, viz. \( \text{C1} \). This embedding, and also embeddings of certain intuitionistic epistemic logics into intuitionistic conditional logics, suggest various philosophically interesting (and exotic!) interpretations of \( \square \rightarrow \) which I discuss as appropriate. Moreover, each of these interpretations relate to the BHK (Brouwer-Heyting-Kolmogorov) interpretation of the standard connectives of intuitionistic logic.

---

17 Previous work on paraconsistent or relevant conditional logic has been done by Mares [73, 74, 75], Mares and Fuhrmann [76], and Priest [102, pp. 208-11]. The only previous work on intuitionistic conditional logic that I am aware of is Genovese et al. [36, 37] (which is primarily proof-theoretically oriented) and Weiss [125].

18 It is worth noting that it is exceptionally easy to give algebraic semantics for many intuitionistic conditional logics by exploiting previous work by Nute [91, 92]. Nute’s algebraic semantics, which uses boolean algebras with operators (cf. the work of McKinsey [81], McKinsey and Tarski [82], and Lemmon [63, 64] on classical modal logic), can be easily “Heyting-ized” to give algebraic semantics for intuitionistic conditional logics (cf. the work of Bull [12] and Fischer Servi [28] on intuitionistic modal logic).

19 For the Gödel-McKinsey-Tarski embedding, see Gödel [40] and McKinsey and Tarski [82].

20 For the BHK interpretation of the standard connectives of intuitionistic logic, see (for example) Dummett [23] and section 6.1 below.
Chapter 2

Conditional Logic

Contemporary conditional logic has its origins in the pioneer work of Stalnaker [114, 117] and Lewis [66, 67]. It attained maturity in the work of Chellas [13, 14], Nute [92], and finally Segerberg [111]. The primary purpose of this chapter is to review the main systems of conditional logic both semantically and axiomatically. Reviewing this material is essential since all subsequent chapters will adapt and repurpose it in various ways. Most, though not all, of the results and ideas discussed in this chapter are well-established. Nevertheless, I have found it beneficial for what follows to introduce some generalizations and modifications.

In section 2.1, I discuss systems of conditional logic syntactically and present axiomatizations for the major systems (table 2.3). The close relationship between modal and conditional logic is discussed in subsection 2.1.2, where embeddings of various modal logics into conditional logics are examined.

Uniform semantic treatments of conditional logic are turned to in section 2.2. An apparently novel neighborhood variant of Chellas-Segerberg relational semantics [13, 111, 122] is introduced in subsection 2.2.1. The relation of this semantics to the more standard relational version as well as the algebraic semantics due to Nute [92] is discussed in subsections 2.2.1 and 2.2.2. Finally, determination—soundness and completeness—results for the major systems
of conditional logic with respect to this semantics are sketched in subsection 2.2.3.

Three other kinds of more specialized semantics for conditional logic are discussed in section 2.3. Semantics for systems for which substitution of logical equivalents does not generally hold is discussed in subsection 2.3.1. In subsection 2.3.2, two intuitively rich kinds of semantics developed by Lewis [66, 67, 68] are examined and shown to be equivalent.

2.1 Syntax

Let $L$ be the language of classical propositional logic augmented with the binary conditional connective $\rightarrow$. The formation rules are standard (in particular, arbitrary nesting of $\rightarrow$ is allowed). $\Pi$ denotes the set of all propositional variables ($p, q, ...$) and $\Phi$ the set of all formulae ($\phi, \psi, ...$).

2.1.1 Axiom Systems

This section reviews the terminology and taxonomy of systems of conditional logic. The following definitions are standard (cf. [13, 14], [92]):

**DEFINITION 1.** A set of formulae $L$ (in the language $L$) is a system of conditional logic if it contains all classical propositional tautologies and is closed under modus ponens (for the material conditional):

\[
\frac{\phi, \phi \rightarrow \psi}{\psi} \quad \text{(MP)}
\]

**DEFINITION 2.** $\vdash_L \phi$ ($\phi$ is a theorem of $L$) if and only if $\phi \in L$. $\Gamma \vdash_L \phi$ if and only if there is a set $\{\phi_1, \ldots, \phi_n\} \subseteq \Gamma(n \geq 0)$ such that $\vdash_L (\phi_1 \land \ldots \land \phi_n) \rightarrow \phi$. $L$ is consistent if and only if $\not\vdash_L \bot$. $\Gamma$ is $L$ consistent if and only if $\Gamma \not\vdash_L \bot$ and $\Gamma$ is maximally $L$ consistent if it is $L$ consistent and has no $L$ consistent proper extensions.

\footnote{Take $\rightarrow$ and $\neg$ as primitive; the other connectives, including the constants $\top$ and $\bot$, are introduced using the usual definitions.}
It is obvious from definition 1 that every system of conditional logic contains classical propositional logic (PL). As such, PL can be appealed to in axiomatic proofs wherever classical inferences or tautologies are used. Nontrivial systems of conditional logic are obtained by closing under rules from table 2.1 and adding axioms from table 2.2.

Table 2.1: Rules

\[
\begin{align*}
\phi & \leftrightarrow \psi \\
(\phi \rightarrow \chi) & \leftrightarrow (\psi \rightarrow \chi) \quad \text{(RCEA)} \\
\phi & \leftrightarrow \psi \\
(\chi \rightarrow \phi) & \leftrightarrow (\chi \rightarrow \psi) \quad \text{(RCEC)} \\
(\phi_1 \land \ldots \land \phi_n) & \rightarrow \phi \\
((\psi \rightarrow \phi_1) \land \ldots \land (\psi \rightarrow \phi_n)) & \rightarrow (\psi \rightarrow \phi) \quad \text{(RCK)}
\end{align*}
\]

Each rule from table 2.1 is taken to apply only if the premise is a theorem. The standard convention is adopted that when \( n = 0 \), (RCK) licenses the inference from \( \phi \) to \( \psi \rightarrow \phi \).
CHAPTER 2. CONDITIONAL LOGIC

Table 2.2: Axiom Schemata

\[(\phi \rightarrow (\psi \land \theta)) \rightarrow ((\phi \rightarrow \psi) \land (\phi \rightarrow \theta))\]  \hspace{1cm} (CM)

\[((\phi \rightarrow \psi) \land (\phi \rightarrow \theta)) \rightarrow (\phi \rightarrow (\psi \land \theta))\]  \hspace{1cm} (CC)

\[\phi \rightarrow \top\]  \hspace{1cm} (CN)

\[\phi \rightarrow \phi\]  \hspace{1cm} (ID)

\[(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)\]  \hspace{1cm} (CMP)

\[(\phi \land \psi) \rightarrow (\phi \rightarrow \psi)\]  \hspace{1cm} (CS)

\[\neg\psi \rightarrow \psi \rightarrow (\phi \rightarrow \psi)\]  \hspace{1cm} (MOD)

\[(\phi \rightarrow \psi) \lor (\phi \rightarrow \neg\psi)\]  \hspace{1cm} (CEM)

\[((\phi \rightarrow \psi) \land \neg(\phi \rightarrow \neg\theta)) \rightarrow ((\phi \land \theta) \rightarrow \psi)\]  \hspace{1cm} (CV)

\[((\phi \rightarrow \psi) \land (\psi \rightarrow \phi)) \rightarrow ((\phi \rightarrow \theta) \leftrightarrow (\psi \rightarrow \theta))\]  \hspace{1cm} (CSO)

**DEFINITION 3.** A system of conditional logic \(L\) closed under (RCEC) is *half-classical*. A half-classical system of conditional logic \(L\) closed under (RCEA) is *classical*. Let \(L\) be a classical system: \(L\) is *monotonic* if it contains all instances of (CM), *regular* if it contains all instances of (CM) and (CC), and *normal* if it contains all instances of (CM), (CC), and (CN).

In the literature, most normal systems of conditional logics are axiomatized using closure under (RCK) and (RCEA) rather than the schemes indicated in definition 3. It is not difficult to show that a classical system of conditional logic is normal if and only if it is closed under
(RCK).\textsuperscript{2} Although this result is well-known (see, e.g., Chellas [13, pp. 137-8]), it is worth reviewing here so as to give a feel for proofs in systems of conditional logic.

**PROPOSITION 1.** *Every classical system of conditional logic closed under (RCK) is normal*

Proof is omitted.

**PROPOSITION 2.** *Every monotonic system of conditional logic is closed under the following rule (the rule applies only if the premise is a theorem):*

\[
\frac{\phi \rightarrow \psi}{(\theta \square \phi) \rightarrow (\theta \square \psi)} \tag{RCM}
\]

1. \(\phi \rightarrow \psi\) \hspace{1em} \vdash \text{(Assumption)}
2. \((\phi \land \psi) \leftrightarrow \phi\) \hspace{1em} \text{PL 1}
3. \((\theta \square (\phi \land \psi)) \leftrightarrow (\theta \square \phi)\) \hspace{1em} \text{RCEC 2}
4. \((\theta \square (\phi \land \psi)) \rightarrow ((\theta \square \phi) \land (\theta \square \psi))\) \hspace{1em} \text{CM}
5. \((\theta \square \phi) \rightarrow (\theta \square \psi)\) \hspace{1em} \text{PL 3, 4}

**PROPOSITION 3.** *Every regular system of conditional logic contains all instances of the following scheme:*

\[(\phi \square (\psi \rightarrow \theta)) \rightarrow ((\phi \square \psi) \rightarrow (\phi \square \theta))\] \hspace{1em} \tag{CK}

\textsuperscript{2}More generally, a system of conditional logic is normal if and only if it is closed under (RCK) and (RCEA).
PROPOSITION 4. Every normal system of conditional logic is closed under (RCK)

The proof is by induction on \( n \) in (RCK). When \( n = 0 \), it must be shown that from the theorem \( \phi \) the result \( \psi \rightarrow \phi \) follows. This is immediate from (CN) and (RCEC). (CK) is then used with the induction hypothesis to complete the proof.

THEOREM 1. Let \( L \) be a classical system of conditional logic: \( L \) is normal if and only if \( L \) is closed under (RCK)

Immediate from propositions 1 and 4.

An incomplete list of major systems of conditional logic, in increasing logical strength, is given in table 2.3. Each system is taken to be the smallest set of formulae containing all instances of the listed axiom schemata and closed under the listed rules. The axiomatizations used generally follow Nute [92, pp. 129-30].

\[^{3}\text{I am not aware of any proof, nor have I been able to prove, that Nute's axiomatization of Lewis' C1 is equivalent to Lewis' own axiomatization of it in [67, p. 132]. I have been able to show that Lewis' system contains Nute's, but the converse seems to be more difficult.}\]
Table 2.3: Major Systems of Conditional Logic

<table>
<thead>
<tr>
<th>System</th>
<th>Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>CE</td>
<td>(RCEA), (RCEC)</td>
</tr>
<tr>
<td>CM</td>
<td>CE, (CM)</td>
</tr>
<tr>
<td>CR</td>
<td>CM, (CC)</td>
</tr>
<tr>
<td>CK</td>
<td>CR, (CN)</td>
</tr>
<tr>
<td>C0</td>
<td>CK, (ID), (MOD), (CSO), (CV)</td>
</tr>
<tr>
<td>C1</td>
<td>C0, (CMP), (CS)</td>
</tr>
<tr>
<td>C2</td>
<td>C0, (CMP), (CEM)</td>
</tr>
</tbody>
</table>

For “two-letter” systems, the convention is adopted that $Xy$ is $XY$ where closure under (RCEA) has been dropped (i.e. $Xy$ is the merely half-classical version of $XY$). Chronologically, the first of these is $C2$ (=VCS), introduced by Stalnaker [114]. $C0$ (=V) and $C1$ (=VC) were first introduced by Lewis [66]. Chellas [13] characterized the remaining four (and developed the terminology for describing them used in definition 3). Their little-letter variants (e.g. $Ck$) seem to have first appeared in Chellas [13, n. 14] and Nute [92].

**DEFINITION 4.** Let $\Phi_c$ be the set of all formulae of classical propositional logic. Define a function $\tau : \Phi \rightarrow \Phi_c$ as follows:

1. $\tau(p) = p$

2. $\tau(\neg \phi) = \neg \tau(\phi)$

3. $\tau(\phi \rightarrow \psi) = \tau(\phi \square \rightarrow \psi) = \tau(\phi) \rightarrow \tau(\psi)$

**THEOREM 2** (Consistency). *Each of the systems in table 2.3 is consistent*

The proof is essentially as in Nute [92, pp. 24-5]. It is trivial to verify that every axiom in table 2.2 is mapped to a classical tautology by $\tau$. Moreover, if the premise of any rule in
table 2.1 is mapped by $\tau$ to a tautology, so is the conclusion. Consequently, every theorem of every system, under $\tau$, is a tautology. Since $\tau(\bot)$ is not a tautology, $\bot$ is not a theorem of any of the systems.

\[ \square \]

### 2.1.2 Modal Logic in Conditional Logic

Given the close relationship between modal and conditional logic, some remarks are in order about how to interpret modal logic in conditional logic.\(^4\) I begin by briefly reminding the reader of some of the basic features of modal logic. Let the language of modal logic, $L_{\square}$, be the same as $L$ except that there is a unary connective $\square$ in place of $\Box \rightarrow$. Let $\Phi_{\square}$ be the set of formulae in $L_{\square}$ (the formation rules are standard). Systems of modal logic, theoremhood, etc. are defined essentially as in definitions 1 and 2. Some of the major normal modal logics are then obtained by closing under the rule,\(^5\)

\[ \frac{\phi}{\square \phi} \]  

(NEC)

and adding all instances of (K) and of some selection of the other axiom schemata listed in table 2.4.

---

\(^4\)For comprehensive treatments of modal logic, the reader is referred to Hughes and Cresswell [45, 46], Chellas [14], and Fitting and Mendelsohn [32].

\(^5\)(NEC) only applies if the premise is a theorem.
Normal systems of modal logic will simply be named by listing their axioms in bold, where it is understood that each system is to be closed under (NEC). Thus, $\textbf{K4}$ (for example) is the smallest system of modal logic closed under (NEC) which contains all instances of (K) and (4).

Beginning with Stalnaker [114, p. 105], it has been conventional to introduce $\Box$ into a given conditional logic via the following definition:

$$\Box \phi \equiv (\neg \phi \rightarrow \Box \neg \phi) \quad (\Box \text{Df. 1})$$

It should be clear that (\Box \text{Df. 1}) is especially closely related to one scheme from table 2.2 in particular, viz. (MOD). But how exactly are they related?

Let $\mathbf{L}$ be the smallest normal system of conditional logic containing all instances of (MOD). Following Williamson [129, pp. 88-90] (see also Williamson [128, pp. 298-300]), with some adjustments, consider the following results:

**DEFINITION 5.** Define a function $\sigma : \Phi \rightarrow \Phi$ as follows:

$$\Box \phi \equiv (\neg \phi \rightarrow \Box \neg \phi) \quad (\Box \text{Df. 1})$$

These results—in particular, theorem 3—actually represent a slight refinement of Williamson’s; more on this anon.
1. \( \sigma(p) = p \)

2. \( \sigma(\neg \phi) = \neg \sigma(\phi) \)

3. \( \sigma(\phi \rightarrow \psi) = \sigma(\phi) \rightarrow \sigma(\psi) \)

4. \( \sigma(\Box \phi) = \neg \sigma(\phi) \rightarrow \sigma(\phi) \)

**Lemma 1.** If \( \vdash_K \phi \), then \( \vdash_L \sigma(\phi) \)

By induction on the length of proof. The only somewhat difficult part is showing that \( \vdash_L \sigma(\Box) \). Here is a (condensed) proof:

1. \( \neg(\sigma(\phi) \rightarrow \sigma(\psi)) \rightarrow (\sigma(\phi) \rightarrow \sigma(\psi)) \) Assumption
2. \( \neg \sigma(\phi) \rightarrow \sigma(\phi) \) Assumption
3. \( \neg \sigma(\psi) \rightarrow \sigma(\phi) \) MOD, PL 2
4. \( \neg \sigma(\psi) \rightarrow (\sigma(\phi) \rightarrow \sigma(\psi)) \) MOD, PL 1
5. \( (\neg \sigma(\psi) \rightarrow \sigma(\phi)) \rightarrow (\neg \sigma(\psi) \rightarrow \sigma(\psi)) \) CK, PL 4
6. \( \neg \sigma(\psi) \rightarrow \sigma(\psi) \) PL 3, 5

Thus, by conditional proof and definition 5, \( \vdash_L \sigma(\Box(\phi \rightarrow \psi)) \rightarrow (\sigma(\Box \phi) \rightarrow \sigma(\Box \psi)) = \sigma(\Box) \).

**Definition 6.** Define a function \( \sigma^{-1} : \Phi \rightarrow \Phi_{\Box} \) as follows:

1. \( \sigma^{-1}(p) = p \)

2. \( \sigma^{-1}(\neg \phi) = \neg \sigma^{-1}(\phi) \)

3. \( \sigma^{-1}(\phi \rightarrow \psi) = \sigma^{-1}(\phi) \rightarrow \sigma^{-1}(\psi) \)

4. \( \sigma^{-1}(\phi \rightarrow \psi) = \Box(\sigma^{-1}(\phi) \rightarrow \sigma^{-1}(\psi)) \)

---

8The inverse notation is not intended literally, but it is suggestive; see lemma 3 below.
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LEMMA 2. If $\vdash_L \phi$, then $\vdash_K \sigma^{-1}(\phi)$

Again, the result is by induction on the length of proof.

\[ \square \]

LEMMA 3. $\vdash_K \phi \iff \sigma^{-1}(\sigma(\phi))$

The proof is by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\square \psi$. Then: $\sigma^{-1}(\sigma(\square \psi)) = \sigma^{-1}(\neg \sigma(\psi) \dashv \vdash \sigma(\psi)) = \square(\neg \sigma^{-1}(\sigma(\psi)) \to \sigma^{-1}(\sigma(\psi)))$. In $K$, by the induction hypothesis, this is equivalent to $\square(\neg \psi \to \psi)$, which is clearly equivalent to $\square \psi$.

\[ \square \]

THEOREM 3 (Modal Embedding). $\vdash_K \phi$ if and only if $\vdash_L \sigma(\phi)$

The result follows immediately from lemmata 1, 2, and 3.

\[ \square \]

Results akin to theorem 3 for stronger systems of modal logic can be obtained by considering stronger systems of conditional logic. Note that Williamson [129, p. 85] does not consider the system $L$, but rather $L$ extended by (ID). In that system, $\neg \phi \dashv \vdash \phi$ and $\neg \phi \dashv \vdash \bot$ are equivalent [129, p. 87], and Williamson proves his main results using the latter as a definition of $\square \phi$.

$K$ can be embedded into even weaker systems of conditional logic using definitions besides ($\square$Df. 1). Consider the following interesting definition (cf. Lowe [71, p. 360]):

\[ \square \phi \equiv \top \dashv \vdash \phi \] (\(\square\)Df. 2)

Let $\sigma^*$ be the same as $\sigma$ except that $\sigma^*(\square \phi) = \top \dashv \vdash \sigma^*(\phi)$. Then the following result can

\[ ^9 \text{That these formulae are not equivalent in L can be shown semantically (it is perhaps most convenient to use the algebraic semantics from subsection 2.2.2).} \]
be proved with little modification to the proof of theorem 3 (in particular, the same function $\sigma^{-1}$ can be used):

**THEOREM 4** (Modal Embedding). $\vdash_K \phi$ if and only if $\vdash_{CK} \sigma^*(\phi)$

The result follows from suitably modified versions of lemmata 1, 2, and 3.

\[\square\]

### 2.2 Uniform Semantics

While the most popular semantics for conditional logic is, at base, a worlds semantics, other sorts of semantics have been developed. Nute, building on previous work for modal logic (e.g. Lemmon [63]), provided algebraic semantics for various conditional logics in [91, 92]. More recently, Fine [27] has developed a truthmaker semantics for certain conditional logics, specifically to avoid problems associated with the substitution of logical equivalents (for which, see also Fine [26]). I will have nothing to say about truthmaker semantics for conditional logic here; I will, however, discuss both worlds semantics and algebraic semantics in detail.

Under the umbrella of worlds semantics, there are many variations in the literature on conditional logic (some of these are canvassed by Nute in [92, Ch. 3]). In subsection 2.2.1, I present a neighborhood-type worlds semantics which is intended to be able to characterize any classical system of conditional logic (“classical” in the sense of definition 3). Algebraic semantics suitable for any classical system are presented in subsection 2.2.2. Finally, determination results are proved for each of the systems listed in table 2.3 in subsection 2.2.3.
2.2.1 Proposition Indexed Interpretations

The semantics described in this section is a hybrid of semantics taken from Chellas [13] and Segerberg [111]. It is, in effect, a neighborhood variant of so-called Chellas-Segerberg semantics (for which, see Unterhuber [120] and Unterhuber and Schurz [122]). The reader will note its similarities to the more familiar neighborhood semantics for modal logic.\textsuperscript{10}

**DEFINITION 7.** A proposition indexed interpretation is a structure \( \mathfrak{I} = \langle W, P, \{ f_X : X \in P \}, V \rangle \) such that (cf. Segerberg [111]):

1. \( W \) is a nonempty set of worlds

2. \( P \subseteq \mathcal{P}(W) \) such that:
   
   \begin{enumerate}
   
   \item \( \emptyset \in P \)
   \item If \( S \in P \), then \( -S = W - S \in P \)
   \item If \( S \in P \) and \( T \in P \), then \( S \cap T \in P \)
   \item If \( S \in P \) and \( T \in P \), then \( \{ x \in W : T \in f_S(x) \} \in P \)
   \end{enumerate}

3. \( f_X : W \to \mathcal{P}(P) \)

4. \( V : \Pi \to \mathcal{P}(W) \) such that for all \( p \in \Pi \), \( V(p) \in P \)

Given a proposition indexed interpretation \( \mathfrak{I} = \langle W, P, \{ f_X : X \in P \}, V \rangle \), the truth of a formula at a world of the interpretation (\( \models_w^3 \)) is defined as follows (let \([\phi]\) be \( \{ x \in W : \models_w^3 \phi \} \)):

1. \( \models_w^3 p \) if and only if \( w \in V(p) \)

2. \( \models_w^3 \neg \phi \) if and only if \( \not\models_w^3 \phi \)

3. \( \models_w^3 \phi \to \psi \) if and only if \( \not\models_w^3 \phi \) or \( \models_w^3 \psi \)

\textsuperscript{10}For neighborhood semantics, see Chellas [14, Ch. 7] and Pacuit [94].
4. $\models_w \phi \rightarrow \psi$ if and only if $[\psi] \in f_{[\phi]}(w)$

Logics of any real interest require imposing additional constraints on the function $f_X$. Some of the most important constraints are given below in table 2.5 ($X, S, T \in P$):

Table 2.5: Function Constraints

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(cm)</td>
<td>If $S \cap T \in f_X(w)$, then $S \in f_X(w)$ and $T \in f_X(w)$</td>
</tr>
<tr>
<td>(cc)</td>
<td>If $S \in f_X(w)$ and $T \in f_X(w)$, then $S \cap T \in f_X(w)$</td>
</tr>
<tr>
<td>(cn)</td>
<td>$W \in f_X(w)$</td>
</tr>
<tr>
<td>(id)</td>
<td>$X \in f_X(w)$</td>
</tr>
<tr>
<td>(cmp)</td>
<td>If $S \in f_X(w)$ and $w \in X$, then $w \in S$</td>
</tr>
<tr>
<td>(cs)</td>
<td>If $w \in S \cap T$, then $S \in f_T(w)$</td>
</tr>
<tr>
<td>(mod)</td>
<td>If $X \in f_X(w)$, then $X \in f_S(w)$</td>
</tr>
<tr>
<td>(cem)</td>
<td>Either $S \in f_X(w)$ or $-S \in f_X(w)$</td>
</tr>
<tr>
<td>(cv)</td>
<td>If $S \in f_X(w)$ and $-T \notin f_X(w)$, then $S \in f_{X \cap T}(w)$</td>
</tr>
<tr>
<td>(cso)</td>
<td>If $S \in f_X(w)$ and $X \in f_S(w)$, then $T \in f_X(w)$ iff $T \in f_S(w)$</td>
</tr>
</tbody>
</table>

The semantic constraints listed in table 2.5 correspond to the axiom schemata listed in table 2.2 in the obvious way. Let $\mathcal{C}_L$ designate the class of all proposition indexed interpretations satisfying the constraints corresponding to the system of conditional logic $L$.\footnote{Many the constraints listed in table 2.5 are identical with those given by Arlo-Costa in [5, §3.1.2].} \footnote{Example: $\mathcal{C}^{CK}$ is the class of all proposition indexed interpretations in which $f_X$ satisfies (cm), (cc), and (cn).}
Proofs that these correspondences in fact hold (i.e. determination results) will be given in subsection 2.2.3.

**DEFINITION 8.** Where $\Sigma$ is a set of formulae and $C$ is a class of proposition indexed interpretations, $\Sigma \models_C \phi$ if and only if for all worlds $w$ of all interpretations $I \in C$, if $\models^3_w \psi$ for each $\psi \in \Sigma$, then $\models^3_w \phi$. If $\Sigma \models_C \phi$, the inference is called valid (in $C$). $\phi$ is a valid formula (in $C$) if $\emptyset \models_C \phi$.

**LEMMA 4.** Given a proposition indexed interpretation $I = \langle W, P, \{f_X : X \in P\}, V \rangle$, for all $\phi \in \Phi$, $[\phi] \in P$

The proof is by induction on the complexity of $\phi$. If $\phi$ is a propositional variable $p \in \Pi$, then $[p] = V(p) \in P$ by definition 7. Suppose the result holds for $\psi$ and $\theta$. If $\phi$ is of the form $\neg \psi$, then $[-\psi] = -[\psi] \in P$ since $[\psi] \in P$. If $\phi$ is of the form $\psi \rightarrow \theta$, then $[\psi \rightarrow \theta] = -[\psi] \cup [\theta] = -([\psi] \cap [-\theta]) \in P$. Finally, if $\phi$ is of the form $\psi \mathcal{R} \theta$, then $[\psi \mathcal{R} \theta] = \{x \in W : \models^3_x \psi \mathcal{R} \theta\} = \{x \in W : [\theta] \in f_\phi(x)\} \in P$ since $[\theta] \in P$ and $[\psi] \in P$.

It is worth briefly examining the relation of the semantics offered above to its better known counterpart in the literature.

**DEFINITION 9.** A relational proposition indexed interpretation (cf. [111, p. 160]) is a structure $I_R = \langle W, P, \{R_X : X \in P\}, V \rangle$ where everything is defined as in definition 7 except:

1. In place of $f_X$, there is a relation $R_X \subseteq W \times W$

2. In place of 2(d), if $S \in P$ and $T \in P$, then $\{x \in W : \forall y(xR_S y \Rightarrow y \in T)\} \in P$

Then the truth conditions for complex formulae at a world $w$ are the same as above except:

\[
\models^3_w \phi \mathcal{R} \psi \text{ if and only if } \{y \in W : wR_{[\phi]} y\} \subseteq \psi
\]
A number of constraints can be put on the relation $R_X$ in order to semantically characterize a range of logics; many of these can be found in [111, p. 163]. I will, however, confine my attention to the class of all relational proposition indexed interpretations, which I denote by $C^{\text{CK}}_R$.$^{13}$

A natural question is: what class of proposition indexed interpretations determines the same validities as $C^{\text{CK}}_R$? Using purely semantic methods, $C^{\text{CK}}_R$ can be shown to be equivalent (modulo validity) to a subset of $C^{\text{CK}}$ (the class of all proposition indexed interpretations satisfying the constraints (cm), (cc), and (cn)), viz. the subset of augmented proposition indexed interpretations:

**DEFINITION 10.** Call a proposition indexed interpretation $I = \langle W, P, \{f_X : X \in P\}, V \rangle$ *augmented* (cf. Chellas [14, p. 220]) if it satisfies the condition $(X, Y \in P, w \in W)$:

$$Y \in f_X(w) \text{ if and only if } \bigcap f_X(w) \subseteq Y$$

Let $C_A$ denote the class of all augmented proposition indexed interpretations.

**LEMMA 5.** $C_A \subseteq C^{\text{CK}}$

Consider an arbitrary augmented proposition indexed interpretation $I = \langle W, P, \{f_X : X \in P\}, V \rangle$; it suffices to show that $I$ satisfies (cm), (cc), and (cn). For (cm), suppose $S \cap T \in f_X(w)$; then, since $I$ is augmented, $\bigcap f_X(w) \subseteq S \cap T$. Since $\bigcap f_X(w) \subseteq S$ and $\bigcap f_X(w) \subseteq T$, it follows that $S \in f_X(w)$ and $T \in f_X(w)$, as desired. For (cc), suppose that $S \in f_X(w)$ and $T \in f_X(w)$; from $\bigcap f_X(w) \subseteq S$ and $\bigcap f_X(w) \subseteq T$, it follows that $\bigcap f_X(w) \subseteq S \cap T$. Consequently, $S \cap T \in f_X(w)$. Since $\bigcap f_X(w) \subseteq W$, $W \in f_X(w)$, as required by (cn).

$^{13}$Of course, $\text{CK}$ is sound and complete with respect to this class; for a proof sketch, see Segerberg [111, p. 162].
A straightforward adaptation of proofs relating neighborhood and relational models for modal logic from Chellas [14, pp. 220-22] suffices to show that $\mathcal{C}_R^{\text{CK}}$ and $\mathcal{C}_A$ determine the same validities.

**Lemma 6.** Given an augmented proposition indexed interpretation $\mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle$, there is a relational proposition indexed interpretation $\mathcal{I}_R = \langle W, P_R, \{R_X : X \in P_R\}, V \rangle$ such that $\models^{3R}_w \phi$ if and only if $\models^3_w \phi$

Construct $\mathcal{I}_R = \langle W, P_R, \{R_X : X \in P_R\}, V \rangle$ from $\mathcal{I}$ by taking $W$ and $V$ to be the same, making $P_R$ the closure of $P$ under the required operations, and setting $xR_Xy$ if and only if $y \in \bigcap f_X(x)$ for $X \in P$; else, set $R_X = \emptyset$. It is obvious that this structure meets the conditions of definition 9.

It remains to show, by induction, that $\models^{3R}_w \phi$ if and only if $\models^3_w \phi$. The only case of interest is that in which $\phi$ is of the form $\psi \rightarrow \theta$. By the induction hypothesis and lemma 4, $[\theta]^{3R} = [\theta]^3 \in P$ and $[\psi]^{3R} = [\psi]^3 \in P$. Consequently, $xR_{[\psi]^{3R}}y$ if and only if $y \in \bigcap f_{[\psi]^3}(x)$. Therefore:

$$\models^3_w \psi \rightarrow \theta \text{ if and only if } [\theta]^3 \in f_{[\psi]^3}(w)$$

$$\bigcap f_{[\psi]^3}(w) \subseteq [\theta]^3$$

$$\{y \in W : wR_{[\psi]^{3R}}y\} \subseteq [\theta]^{3R}$$

$$\models^{3R}_w \psi \rightarrow \theta$$

**Lemma 7.** Given a relational proposition indexed interpretation $\mathcal{I}_R = \langle W, P_R, \{R_X : X \in P_R\}, V \rangle$, there is an augmented proposition indexed interpretation $\mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle$ such that $\models^{3R}_w \phi$ if and only if $\models^3_w \phi$

Construct $\mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle$ from $\mathcal{I}_R$ by taking $W$ and $V$ to be the same, making
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\[ P \text{ the closure of } P_R \text{ under the required operations, and setting } f_X(w) = \{ Y \subseteq W : \{ x \in W : wR_X x \} \subseteq Y \} \text{ for } X \in P_R; \text{ else, set } f_X = \{ W \}. \]

If \( X \in P_R \), then it is obvious that \( \bigcap f_X(w) = \{ x \in W : wR_X x \} \). Hence, it follows immediately that \( Y \in f_X(w) \) if and only if \( \bigcap f_X(w) \subseteq Y \). Moreover, since \( Y \in \{ W \} \) if and only if \( \bigcap \{ W \} \subseteq Y \), the result also holds for \( f_X \) when \( X \notin P_R \). Therefore, \( J \) meets the conditions of definition 10. That \( \models_{\text{w}}^2 \phi \) if and only if \( \models_{\text{w}}^2 \phi \) now follows by a routine induction argument, as above.

\[ \square \]

THEOREM 5 (Equivalence). \( \Gamma \models_{\mathcal{A}} \phi \) if and only if \( \Gamma \models_{\mathcal{C}_R} \phi \)

The result follows directly from lemmata 6 and 7.

\[ \square \]

It follows from theorem 5, the completeness of \( \mathcal{C}_K \) with respect to \( \mathcal{C}_R \), and the soundness of \( \mathcal{C}_K \) with respect to \( \mathcal{C}_R \) (see subsection 2.2.3) that if \( \Gamma \models_{\mathcal{C}_A} \phi \), \( \Gamma \models_{\mathcal{C}_K} \phi \). By lemma 5, if \( \Gamma \models_{\mathcal{C}_K} \phi \), then \( \Gamma \models_{\mathcal{C}_A} \phi \). Consequently, \( \mathcal{C}_R \), \( \mathcal{C}_A \), and \( \mathcal{C}_K \) all determine the same validities. It would be nice to have a purely semantic proof of this fact; I have none to offer here.

2.2.2 Algebraic Semantics

In this subsection, I briefly review algebraic semantics for classical systems of conditional logic. The semantics I present here is essentially due to Nute [92, Ch. 7], although I freely modify his notation. The main interest here will be to establish a correspondence between certain classes of algebraic interpretations and proposition indexed interpretations. Such a correspondence is useful because it is often easier to establish certain results (e.g. independence and decidability) using algebraic resources than using worlds semantics.

DEFINITION 11. A conditional algebra is a structure \( \mathcal{A} = \langle B, * \rangle \) in which \( B = \langle B, 1, 0, -, \cup, \cap \rangle \) is a boolean algebra and \( * \) is a binary operation on \( B \).
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Conditional algebras corresponding to logics of any real interest must impose various constraints on \( \ast \). Some of the most important of these are given in table 2.6. They correspond in the obvious way to the constraints listed in table 2.5.\(^{14}\)

Table 2.6: Algebraic Constraints

\[
\begin{align*}
  a \ast (b \cap c) & \leq (a \ast b) \cap (a \ast c) & (cm\ast) \\
  (a \ast b) \cap (a \ast c) & \leq a \ast (b \cap c) & (cc\ast) \\
  a \ast 1 & = 1 & (cn\ast) \\
  a \ast a & = 1 & (id\ast) \\
  a \ast b & \leq -a \cup b & (cmp\ast) \\
  a \cap b & \leq a \ast b & (cs\ast) \\
  -a \ast a & \leq b \ast a & (mod\ast) \\
  -(a \ast b) & \leq a \ast -b & (cem\ast) \\
  (a \ast b) \cap -(a \ast -c) & \leq (a \cap c) \ast b & (cv\ast) \\
  (a \ast b) \cap (b \ast a) \cap (a \ast c) & \leq (b \ast c) & (cso\ast)
\end{align*}
\]

\textbf{DEFINITION 12.} A \textit{conditional algebraic interpretation} is a structure \( \mathcal{J}_* = \langle \mathcal{A}, g \rangle \) in which \( \mathcal{A} \) is a conditional algebra and \( g : \Pi \rightarrow \mathbb{B} \) is extended in such a way that:

1. \( g(\neg \phi) = -g(\phi) \)
2. \( g(\phi \rightarrow \psi) = -g(\phi) \cup g(\psi) \)

\(^{14}\)Most of the constraints listed in table 2.6 can be found in Nute [92, pp. 132-3].
3. \( g(\phi \Box \psi) = g(\phi) \ast g(\psi) \)

**DEFINITION 13.** For \( \mathcal{J}_* = \langle \mathcal{A}, g \rangle \), write \( |\phi| = \mathcal{J}_* \) if and only if \( g(\phi) = 1 \). Where \( \mathcal{C}_* \) is a class of conditional algebraic interpretations, write \( |\phi| = \mathcal{C}_* \) if for all \( \mathcal{J}_* \in \mathcal{C}_* \), in which case \( \phi \) is said to be valid (in \( \mathcal{C}_* \)).

Let \( \mathcal{C}_L^* \) designate the class of all conditional algebraic interpretations satisfying the constraints corresponding to the system of conditional logic \( \mathcal{L} \). For any system \( \mathcal{L} \) from table 2.3, I show that \( \mathcal{C}_L \) (a class of proposition indexed interpretations) and \( \mathcal{C}_L^* \) (a class of conditional algebraic interpretations) characterize the same valid formulae (cf. Nute [92, pp. 138-46]).

A few preliminary lemmata and definitions will expedite the proof (for what follows, if \( \mathcal{I} \) is a proposition indexed interpretation, I write \( |\phi| = \mathcal{I} \phi \) if and only if \( |\phi| = \mathcal{I} \phi \) for all worlds \( w \) of the interpretation).

**LEMMA 8.** In any conditional algebra satisfying \( (cm*) \), \( a \leq b \) implies \( c \ast a \leq c \ast b \)

This short proof is due to Nute [92, p. 133]. Suppose \( a \leq b \); then \( a = a \cap b \). Thus, \( c \ast a = c \ast (a \cap b) \leq (c \ast a) \cap (c \ast b) \Rightarrow c \ast a \leq c \ast b \).

**LEMMA 9.** Given a proposition indexed interpretation \( \mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle \), there is a conditional algebraic interpretation \( \mathcal{J}_* = \langle \mathcal{A}, g \rangle \) such that \( |\phi| = \mathcal{J}_* \phi \) if and only if \( |\phi| = \mathcal{J}_* \phi \)

A conditional algebra must be constructed from \( \mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle \). Set \( \mathbb{B} = P \) (the set of propositions in \( \mathcal{I} \)) and take each of the boolean algebraic operations to be the obvious set-theoretical operations. Finally, set \( a \ast b = \{w \in W : b \in f_a(w)\} \). It is immediate from the conditions imposed on \( P \) in definition 7 that \( \mathcal{A} = \langle \langle \mathbb{B}, 1, 0, -, \cup, \cap, \ast \rangle \rangle \), so defined, is a conditional algebra.

It must be shown that if \( \mathcal{I} \) satisfies a constraint from table 2.5, \( \mathcal{A} \) satisfies the corresponding constraint from table 2.6. For most constraints, this is trivial to establish; I take
(cmp*) as an example. Suppose \( I \) satisfies (cmp) and \( x \in a \ast b = \{ w \in W : b \in f_a(w) \} \); then \( b \in f_a(x) \). Either \( x \in a \) or \( x \in -a = W - a \). In the first case, by (cmp), \( x \in b \); in the second case, \( x \in -a \). So, in any case, \( x \in -a \cup b \). Therefore, \( a \ast b \leq -a \cup b \), as desired.

Having constructed a suitable algebra \( A \), a conditional algebraic interpretation \( I^* = \langle A, g \rangle \) is defined by putting \( g(\phi) = [\phi] \) (recall that \([\phi] \) is the proposition expressed by \( \phi \) in \( I \)). Note that, for all \( \phi, g(\phi) \in B \) by lemma 4. It only remains to be shown that the truth conditions specified in definition 12 are satisfied; the verification of this simple fact is left to the reader.

\( \square \)

I remind the reader of some terminology from lattice theory for what follows.\(^\text{15}\) A filter in a bounded distributive lattice \( \langle D, 1, 0, -, \cup, \cap \rangle \) is a set \( \emptyset \neq \nabla \subseteq D \) such that \( a \cap b \in \nabla \) if and only if \( a, b \in \nabla \) (it is immediate from the definition that every filter contains 1 and is upward closed). A filter \( \nabla \) is proper if \( \nabla \neq D \). A proper filter \( \nabla \) is prime if whenever \( a \cup b \in \nabla \), either \( a \in \nabla \) or \( b \in \nabla \). Note that every prime filter in a boolean algebra is maximal in the sense that, for every \( a \in D \), either \( a \in \nabla \) or \( -a \in \nabla \) (this is immediate from the definition of primeness and the fact that \( 1 = a \cup -a \) for all \( a \) in a boolean algebra).

**Lemma 10.** Given a distributive lattice \( \langle D, 1, 0, \cup, \cap \rangle \) with \( a, b \in D \), if \( b \not\leq a \), then there exists a prime filter \( \nabla \) such that \( a \notin \nabla \) and \( b \in \nabla \)

The proof of this result can be found in, e.g., Rasiowa and Sikorski [106, p. 49].

\( \square \)

**Lemma 11.** Given a conditional algebraic interpretation \( I^* = \langle A, g \rangle \), there is a proposition indexed interpretation \( I = \langle W, P, \{ f_X : X \in P \}, V \rangle \) such that \( \models^I \phi \) if and only if \( \models^{I^*} \phi \)

To construct a proposition indexed interpretation from \( I^* = \langle A, g \rangle \), take \( W \) to be the set of all prime filters in \( A \) and write \( [a] = \{ w \in W : a \in w \} \). Then put \( P = \{ [a] : a \in B \} \),

\(^{15}\)These definitions basically follow Rasiowa and Sikorski [106, pp. 44-50].
$[b] \in f_{[a]}(w)$ if and only if $w \in [a \ast b]$, and $w \models^3 \phi$ if and only if $w \in [g(\phi)]$; since $w \in V(p)$ if and only if $w \models^3 p$ if and only if $w \in [g(p)]$, note that $V(p) = [g(p)] \in P$.

It must be shown that the structure $\mathcal{J} = \langle W, P, \{f_X : X \in P\}, V \rangle$ is a proposition indexed interpretation. That $W \neq \emptyset$ follows from the fact that $1 \not\leq 0$ and lemma 10. Since 0 is not in any prime filter (else it would not be proper), $[0] = \emptyset \in P$, as desired. Showing that the boolean closure conditions on $P$ obtain is straightforward. Suppose $[a]$ and $[b]$ are arbitrary in $P$; then since $a, b \in \mathbb{B}$, $a \ast b \in \mathbb{B}$. Therefore, $\{w \in W : [b] \in f_{[a]}(w)\} = [a \ast b] \in P$, as desired.

It must also be shown that if $\mathcal{A}$ satisfies a constraint from table 2.6, $\mathcal{J}$ satisfies the corresponding constraint from table 2.5. Take $(cm\ast)$ as an example; pick an arbitrary $w \in W$ such that $[b] \cap [c] = [b \cap c] \in f_{[a]}(w)$. Then $a \ast (b \cap c) \in w$, from which it follows by $(cm\ast)$ and the fact that $w$ is a filter that $a \ast b, a \ast c \in w$. Consequently, $[b], [c] \in f_{[a]}(w)$, as desired.

Lastly, it must be shown that the truth conditions are satisfied. The only case I examine is that concerning $\Box \rightarrow$. $w \models^3 \phi \Box \rightarrow \psi$ if and only if $w \in [g(\phi \Box \rightarrow \psi)] = [g(\phi) \ast g(\psi)] = \{x \in W : [g(\psi)] \in f_{[g(\phi)]}(x)\}$, as desired.

Now, if $w \models^3 \phi$, $g(\phi) = 1 \in w$ for all $w \in W$ (since each $w$ is a filter); that is, $w \models^3 \phi$. Conversely, if $w \not\models^3 \phi$, then since $g(\phi) \neq 1$, it is clear that $1 \not\leq g(\phi)$. Hence, by lemma 10, there is a prime filter $w \in W$ such that $g(\phi) \not\in w$; therefore, $w \not\models^3 \phi$, as desired.

\[\Box\]

**THEOREM 6** (Equivalence). $\models_{cL} \phi$ if and only if $\models_{cL} \phi$

The result follows directly from lemmata 9 and 11.

\[\Box\]
2.2.3 Determination Results

Throughout this subsection, let $L$ be any classical system of conditional logic from table 2.3. I prove soundness and completeness results for each such $L$. Note that for the basic system (CE), the relevant class of proposition indexed interpretations ($C^{CE}$) is simply the class of all proposition indexed interpretations.

**THEOREM 7** (Soundness). $\Sigma \vdash_L \phi$ implies $\Sigma \models_{CL} \phi$

It must be shown that all of the axioms of $L$ are valid in $CL$ and that each of the rules preserve this property. I take (RCEA), (CM), and (CN) as representative cases. Note that lemma 4 will frequently be appealed to, but only implicitly.

If $L$ is closed under (RCEA), suppose that $\models_{CL} \phi \leftrightarrow \psi$ but (without loss of generality) $\not\models_{CL} (\phi \rightarrow \chi) \rightarrow (\psi \rightarrow \chi)$. Then there is a world $w$ of an interpretation $I \in CL$ such that $\models_w \phi \rightarrow \chi$ but $\not\models_w \psi \rightarrow \chi$. Therefore, $[\chi] \in f_{[\phi]}(w)$ but $[\chi] \not\in f_{[\psi]}(w)$. Since by the assumption $[\phi] = [\psi]$, it follows that $f_{[\phi]}(w) = f_{[\psi]}(w)$, which is a contradiction. The case of (RCEC) is similar.

If $L$ contains (CM), then for any $I = \langle W, P, \{f_X : X \in P\}, V \rangle \in CL$, $f_X$ satisfies (cm). Take an arbitrary world $w$ of an arbitrary $I \in CL$ such that $\models_w \phi \rightarrow (\psi \land \theta)$. By (cm), $[\psi] \in f_{[\phi]}(w)$ and $[\theta] \in f_{[\phi]}(w)$. Consequently, $\models_w (\phi \rightarrow \psi) \land (\phi \rightarrow \theta)$. The case of (CC), which uses (cc), is essentially the reverse. If $L$ contains (CN), then for any $I = \langle W, P, \{f_X : X \in P\}, V \rangle \in CL$, $f_X$ satisfies (cn). Take an arbitrary world $w$ of an arbitrary $I \in CL$. By (cn), $[\top] = W \in f_{[\phi]}(w)$. Therefore, $\models_w \phi \rightarrow \top$, as desired.

**COROLLARY 1** (Algebraic Soundness). $\vdash_L \phi$ implies $\models_{CL} \phi$

The (weak) soundness of $L$ with respect to $CL$ is an immediate consequence of theorems 6 and 7.
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LEMMA 12 (Lindenbaum’s Lemma). Given a set of formulae \( \Gamma \), if \( \Gamma \not\vdash_L \bot \), then there is a maximally \( L \) consistent set of formulae \( \Delta \) such that \( \Gamma \subseteq \Delta \).

The proof of this lemma is standard and is omitted; see, e.g., [14].

I now prove completeness results for each of the systems. As usual, this is more complicated than proving soundness, though not by much.

DEFINITION 14. \( [\phi]^L = \{ \Gamma \subseteq \Phi : \Gamma \text{ is maximally } L \text{ consistent}, \phi \in \Gamma \} \).

Except where necessary to disambiguate, I write simply \([\phi]\). Using definition 14, canonical models can be defined:

DEFINITION 15. The canonical model for \( L \) is a structure \( J^L = \langle W, P, \{ f_X : X \in P \}, V \rangle \) such that:

1. \( W = \{ \Gamma \subseteq \Phi : \Gamma \text{ is maximally } L \text{ consistent} \} \)
2. \( P = \{ [\phi] : \phi \in \Phi \} \)
3. \( f_{[\phi]}(w) = \{ [\psi] \in P : (\phi \Box \rightarrow \psi) \in w \} \)
4. \( V(p) = [p] \) for all \( p \in \Pi \)

LEMMA 13. Let \( J^L = \langle W, P, \{ f_X : X \in P \}, V \rangle \) be the canonical model for \( L \). Then \( J^L \) is well-defined and \( J^L \in C^L \).

Observe that (RCEA) guarantees the correctness of definition 15. If \([\phi] = [\psi]\), then \( \vdash_L \phi \leftrightarrow \psi \) by lemma 12. By (RCEA), \( \vdash_L (\phi \Box \rightarrow \theta) \leftrightarrow (\psi \Box \rightarrow \theta) \). Therefore, because \((\phi \Box \rightarrow \theta) \in w \) if and only if \((\psi \Box \rightarrow \theta) \in w \), \([\theta] \in f_{[\phi]}(w) \) if and only if \([\theta] \in f_{[\psi]}(w) \), and so \( f_{[\phi]}(w) = f_{[\psi]}(w) \).

Verifying that \( J^L \) meets the general conditions imposed by definition 7 is fairly straightforward. That \( W \) is nonempty is a consequence of theorem 2 and lemma 12. I check one of
the conditions on $P$: for $[\phi], [\psi] \in P$,\footnote{Observe that any $X \in P$ must be of the form $[\phi]$ for some $\phi \in \Phi$.} it must be shown that $\{x \in W : [\psi] \in f_{[\phi]}(x)\} \in P$.

It suffices to show that $\{x \in W : [\psi] \in f_{[\phi]}(x)\} = [\phi \rightarrow \psi]$, but this is obvious given definition 15.

Turning to the special conditions, all of the semantic constraints from table 2.5 associated with $\mathbf{CL}$ must be shown to hold of $\mathcal{IL}$. I take (cm) and (cn) as representative cases. For (cm), suppose $L$ contains (CM) and $[\alpha] \cap [\beta] \in f_{[\gamma]}(w)$ (for $[\alpha], [\beta], [\gamma] \in P$). Since $[\alpha] \cap [\beta] = [\alpha \land \beta] \in f_{[\gamma]}(w)$, it follows that $\gamma \rightarrow (\alpha \land \beta) \in w$. By PL and (CM), $\gamma \rightarrow \alpha \in w$ and $\gamma \rightarrow \beta \in w$. Consequently, $[\alpha] \in f_{[\gamma]}(w)$ and $[\beta] \in f_{[\gamma]}(w)$, which establishes (cm). For (cn), suppose $L$ contains (CN) and take some $[\phi] \in P$. Since $\phi \rightarrow T \in w$, $W = [T] \in f_{[\phi]}(w)$, as required.

\begin{lemma}[Truth Lemma] Let $\mathcal{IL}$ be the canonical model for $L$. Then for all $\phi \in \Phi$ and all $w \in W$: $\models_w^L \phi$ if and only if $\phi \in w$ (i.e. $[\phi] = [\phi]$)
\end{lemma}

The proof is by induction on the complexity of $\phi$. Since it presents no special complications, I omit it.

\begin{theorem}[Completeness] $\Sigma \models_{CL} \phi$ implies $\Sigma \vdash_L \phi$
\end{theorem}

Suppose that $\Sigma \not\vdash_L \phi$. Then $\Sigma \cup \{\neg \phi\}$ is $L$ consistent. By lemma 12, there is a maximally $L$ consistent set $w$ such that $\Sigma, \neg \phi \subseteq w$. Where $\mathcal{IL}$ is the canonical model for $L$, it is clear that $w \in W$. By lemma 14, for all $\psi \in \Sigma, \models_w^L \psi$, but $\not\models_w^L \phi$. By lemma 13, $\mathcal{IL} \in \mathcal{CL}$. Therefore, $\Sigma \not\models_{CL} \phi$.

\begin{corollary}[Algebraic Completeness] $\models_{CL} \phi$ implies $\vdash_L \phi$
\end{corollary}
The (weak) completeness of $\mathbf{L}$ with respect to $C^L_*$ follows directly from theorems 6 and 8.

2.3 Niche Semantics

Later chapters—especially chapters 3 and 5—will make indispensable use of non-uniform worlds semantics. Three kinds of specialized semantics are reviewed below along with basic determination results concerning them.

2.3.1 Half-Classical Semantics

A system of conditional logic $\mathbf{L}$ is non-classical either if it fails to be closed under (RCEA) or (RCEC). It is easy to see that neither the neighborhood-type nor the algebraic semantics of section 2.2.1 can be used to characterize any non-classical system.\(^{17}\) Since non-classical systems will be important in chapter 5, I examine how to semantically characterize some of them here.

In this section, I restrict my attention to half-classical systems, i.e. systems where closure under (RCEA) may fail. More specifically, I restrict my attention to half-classical systems closed under (RCK).\(^{18}\) The semantics of this section is ultimately due to Chellas [13, pp. 149-150, n. 14].

**DEFINITION 16.** A *formula indexed interpretation* is a structure $\mathcal{I}_\Phi = \langle W, \{f_\phi : \phi \in \Phi\}, V \rangle$ such that:

1. $W$ is a nonempty set of worlds
2. $f_\phi : W \to \mathcal{P}(W)$

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\(^{17}\) Although, see Nute [92, pp. 152-56] for an algebraic approach to non-classical systems.

\(^{18}\) Nute [92, p. 53] calls such systems “half-normal.”
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3. $V : \Pi \to \mathcal{P}(W)$

Given a formula indexed interpretation $\mathcal{I}_\Phi = \langle W, \{f_\phi : \phi \in \Phi\} , V \rangle$, the truth conditions for complex formulae at a world $w$ are the same as before except:

$$\models^w_\Phi \phi \square \rightarrow \psi \text{ if and only if } f_\phi(w) \subseteq [\psi]$$

Here, as before, $[\psi]$ is nothing more than the proposition expressed by $\psi$ in $\mathcal{I}_\Phi = \langle W, \{f_\phi : \phi \in \Phi\} , V \rangle$. Intuitively, $f_\phi(w)$ can be understood as the set of worlds ceteris paribus the same as $w$ such that $\phi$ holds. Then to say that $\phi \square \rightarrow \psi$ is true at $w$ is just to say that $\psi$ is true at all of those worlds.

Many constraints might be imposed on formula indexed interpretations, and I will discuss a number of them in chapter 5. In this section, however, I mention just one (for all $w \in W$, all $\phi, \psi \in \Phi$):

$$\text{If } [\phi] = [\psi], \text{ then } f_\phi(w) = f_\psi(w) \text{ (rcea)}$$

The constraint (rcea) allows fully classical conditional logics to be characterized using this semantics. Let $C^\mathcal{Ck}_\Phi$ be the class of all formula indexed interpretations and let $C^{\mathcal{Ck}}_\Phi$ be the class of all formula indexed interpretations satisfying (rcea). Then validity with respect to classes such as these is defined essentially as before (definition 8).

**Theorem 9** (Soundness). $\Sigma \vdash_{\mathcal{Ck}} \phi$ implies $\Sigma \models_{C^\mathcal{Ck}_\Phi} \phi$

Proof is omitted.

For what comes later, it will be useful to review the proof of the completeness of these systems with respect to their “half-classical” semantics. The argument makes use of many

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19For this interpretation, see Priest [102, p. 85] (also [103]).
of the same ideas as in subsection 2.2.3 and I adapt these accordingly. In particular, then, let 
\([\phi]^{Ck} = \{\Gamma \subseteq \Phi : \Gamma \text{ is maximally } Ck \text{ consistent}, \phi \in \Gamma\} \). 

**DEFINITION 17.** The canonical model for Ck is a structure \( \mathcal{I}_{Ck}^{\Phi} = \langle W, \{f_\phi : \phi \in \Phi\}, V \rangle \) such that:

1. \( W = \{\Gamma \subseteq \Phi : \Gamma \text{ is maximally } Ck \text{ consistent}\} \)
2. \( f_\phi(w) = \{x \in W : \{\psi : (\phi \rightarrow \psi) \in w\} \subseteq x\} \)
3. \( V(p) = [p] \)

Since it is obvious that \( \mathcal{I}_{Ck}^{\Phi} \) is a formula indexed interpretation, it only remains to prove the truth lemma.

**LEMMA 15** (Truth Lemma). Let \( \mathcal{I}_{Ck}^{\Phi} \) be the canonical model for Ck. Then for all \( \phi \in \Phi \) and all \( w \in W \): \( \models_w \phi \) if and only if \( \phi \in w \) (i.e. \( [\phi] = [\phi] \)).

The proof is by induction on the complexity of \( \phi \). The result holds for the basis case by definition 17. The only case of interest is that in which \( \phi \) is of the form \( \alpha \rightarrow \beta \). The induction hypothesis is that \( [\alpha] = [\alpha] \) and \( [\beta] = [\beta] \).

Suppose that \( \alpha \rightarrow \beta \in w \) and \( x \in f_\alpha(w) \). Then by the definition of \( f_\alpha(w) \), \( \beta \in x \). By the induction hypothesis, \( x \in [\beta] \). Since \( x \) is arbitrary in \( f_\alpha(w) \), \( f_\alpha(w) \subseteq [\beta] \). Thus, \( \models_w \alpha \rightarrow \beta \), as desired.

Conversely, suppose that \( \alpha \rightarrow \beta \notin w \). Since \( w \) is maximally consistent, \( \neg(\alpha \rightarrow \beta) \in w \). Let \( S = \{\chi : (\alpha \rightarrow \chi) \in w\} \cup \{\neg\beta\} \). I prove that \( S \) is consistent. Suppose otherwise; then
∃χ₀,...,χₙ ∈ S such that χ₀,...,χₙ ⊬_{CK} ⊥. Then:

\[ \chi₀,...,\chiₙ,¬β ⊬_{CK} ⊥ \]
\[ ⊬_{CK} (\chi₀ ∧ ... ∧ \chiₙ ∧ ¬β) → ⊥ \]
\[ ⊬_{CK} (\chi₀ ∧ ... ∧ \chiₙ) → β \]
\[ ⊬_{CK} ((α □→ χ₀) ∧ ... ∧ (α □→ χₙ)) → (α □→ β) \]
\[ w ⊬_{CK} α □→ β \]
\[ w ⊬_{CK} ⊥ \]

But this is impossible because \( w \) is maximally consistent. Therefore, \( S \) is consistent. By lemma 12, extend \( S \) to a maximally \( CK \) consistent set \( x \). Then \( x ∈ W, \{χ : (α □→ χ) ∈ w\} ⊆ x \), and \( β \notin x \). By the induction hypothesis and definition 17, \( x ∈ f_α(w) \) and \( x \notin [β] \). Therefore, \( \not\models_{w}^{CK} α □→ β \), as desired.

\[ \square \]

**THEOREM 10** (Completeness). \( Σ \models_{C_φ} φ \) implies \( Σ \models_{CK} φ \)

The result follows straightforwardly from lemma 15 as in the proof of theorem 8.

\[ \square \]

Essentially the same argument works for \( CK \). The only significant difference is that the canonical model for \( CK, J_φ^{CK} \), must be shown to be in the appropriate class.

**LEMMA 16.** The canonical model \( J_φ^{CK} \in C_φ^{CK} \)

Suppose that \( [φ] = [ψ] \) in \( J_φ^{CK} \). Then, by lemma 15, \( [φ]^{CK} = [ψ]^{CK} \). Since this implies that \( \models_{CK} φ \leftrightarrow ψ \), it follows by (RCEA) that \( \models_{CK} (φ □→ θ) \leftrightarrow (ψ □→ θ) \). Since \( φ □→ θ ∈ w \) if and only if \( ψ □→ θ ∈ w \), by definition 17, it is clear that \( f_φ(w) = f_ψ(w) \).

\[ \square \]
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THEOREM 11 (Determination). \( \Sigma \vdash_{\mathsf{CK}} \phi \) if and only if \( \Sigma \vdash_{\mathsf{C}} \phi \)

Proof follows from previously mentioned results and lemma 16.

2.3.2 Preorder and Sphere Semantics

Lewis [66] defines three types of equivalent interpretations using selection functions, nested spheres, and preorders. The last of these is most germane to the project of chapter 3, and will accordingly receive the lion’s share of my attention. However, since the Lewis’ sphere semantics is more familiar, and will be of some use in chapter 6, I also review it here.

Lewis developed two different versions of his preorder semantics in [66] and later in [67, pp. 48-50]. While I (generally) follow the former version below, my presentation borrows some notation and terminology developed or discussed more fully elsewhere, for example in Lewis [69] and Friedman and Halpern [34].

DEFINITION 18. A preorder interpretation is a structure \( \mathcal{J}_\leq = \langle W, \{ \leq_i : i \in W \}, V \rangle \), where \( W \) is a non-empty set of worlds and \( \leq_i \) is a subset of \( W \times W \) with \( S_i \subseteq W \) as its field.\(^{20}\) The following conditions are imposed:

\[ \forall x \in S_i, x \leq_i x \] (refl)

\[ \forall x, y, z \in S_i, x \leq_i y \text{ and } y \leq_i z \text{ imply } x \leq_i z \] (trans)

\[ \forall x, y \in S_i, x \leq_i y \text{ or } y \leq_i x \] (tot)

Thus, each \( \leq_i \) is a total preorder of \( S_i \). Finally, \( V \) is a function from propositional variables to sets of worlds.

\(^{20}\)In other words, \( S_i = \{ x \in W : \exists y \in W, x \leq_i y \} \cup \{ y \in W : \exists x \in W, x \leq_i y \} \).
Truth conditions for complex formulae at a world \( w \) are the same as before except:

\[
|{3^w}^\phi \Box \psi | \text{ if and only if either } S_w \cap [\phi] = \emptyset, \text{ or } \\
\exists x \in S_w \cap [\phi] \text{ such that } \forall y \in W, \text{ if } y \leq w x, \text{ then } y \in [\phi \rightarrow \psi]
\]

**Observation.** For any preorder interpretation \( \mathfrak{I} \leq = \langle W, \{\leq_i: i \in W\}, V \rangle \) and \( i, j \in W, j \leq_i j \) if and only if \( j \in S_i \).

For the analysis of counterfactuals, a simple intuitive gloss of this semantics is possible. The meaning of \( x \leq_i y \) is that world \( x \) is at least as close (as similar) to world \( i \) as world \( y \) is; \( S_i \) is the set of pertinent, accessible worlds for \( i \). Then the truth condition for \( \phi \Box \psi \) comes to this: \( \phi \Box \psi \) is true at a world \( w \) if either there are no accessible \( \phi \) worlds (the vacuous case) or there is some accessible \( \phi \) world \( x \) such that for every \( \phi \) world \( y \) at least as close to \( w \) as \( x \), \( y \) satisfies \( \psi \).

Besides the constraints on preorder interpretations mentioned above in definition 18, several additional constraints that might be imposed on preorder interpretations are listed in table 2.7 (for all \( i, j \in W, \) all \( \phi \)):\(^{21}\)

\(^{21}\)These more or less correspond to a number of conditions discussed by Lewis in [66, pp. 76-77] and [67, pp. 118-121].
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Table 2.7: Preorder Constraints

\[ S_i \neq \emptyset \quad \text{(norm)} \]
\[ i \in S_i \quad \text{(sa)} \]
\[ i \in S_i \text{ and } \forall x \in S_i, i \leq_i x \quad \text{(wc)} \]
\[ j \leq_i i \text{ only if } j = i \quad \text{(cs)} \]
\[ S_i = W \quad \text{(univ)} \]
\[ \leq_i = \leq_j \quad \text{(abs)} \]

If \( [\phi] \cap S_i \neq \emptyset \), \( \exists x \in [\phi] \cap S_i \) such that \( \forall y \in [\phi], y \leq_i x \text{ only if } x = y \quad \text{(stal)} \)

Validity is defined essentially as before. Let \( C_{\preceq} \) denote the class of all preorder interpretations. Of special importance is the class \( C_{\preceq C1} \), the class of all preorder interpretations satisfying, in addition to the basic constraints, those of weak and strong centering: (wc) and (cs). This class characterizes Lewis’ preferred logic of counterfactuals, \( C_1 \).

THEOREM 12 (Determination). \( \Sigma \models_{C_{\preceq C1}} \phi \text{ if and only if } \Sigma \vdash_{C_1} \phi \)

For the proof of this, see Lewis [66].

I now turn to a presentation of sphere semantics. Here, I focus simply on a basic version of the semantics. For a more detailed presentation, the reader is referred to Lewis [66, 67].

DEFINITION 19. A sphere interpretation is a structure \( \mathcal{I}_S = \langle W, \{S_i : i \in W\}, V \rangle \), where \( W \) is a non-empty set of worlds and \( S_i \subseteq \mathcal{P}(W) \) is such that:

\[ \forall S, T \in S_i, S \subseteq T \text{ or } T \subseteq S \quad \text{(nest)} \]
CHAPTER 2. CONDITIONAL LOGIC

If $S \subseteq S_i$, then $\bigcup S \subseteq S_i$ (uni)

If $\emptyset \neq S \subseteq S_i$, then $\bigcap S \subseteq S_i$ (int)

Finally, $V$ is a function from propositional variables to sets of worlds.

Truth conditions for complex formulae at a world $w$ are the same as before except:

\[ \models_w^3 \phi \iff \psi \text{ if and only if either } \bigcup S_w \cap [\phi] = \emptyset, \text{ or } \exists S \in S_w \text{ such that } S \cap [\phi] \neq \emptyset \text{ and } S \subseteq [\phi \rightarrow \psi] \]

Let $C_3$ be the class of all sphere interpretations and define validity as usual. Then, following Lewis [67, pp. 48-9], with adjustments, it can be shown that $C_3$ and $C_3^*$ determine the same validities.

**Lemma 17.** Given a preorder interpretation $I_3 = \langle W, \{\leq_i : i \in W\}, V \rangle$, there is a sphere interpretation $I_3^* = \langle W, \{S_i : i \in W\}, V \rangle$ such that $\models_w^3 \phi \iff \phi \models_w^3 \phi$

Given a preorder interpretation $I_3 = \langle W, \{\leq_i : i \in W\}, V \rangle$, a sphere interpretation $I_3^* = \langle W, \{S_i : i \in W\}, V \rangle$ is constructed where $W$ and $V$ are the same and, for each $i \in W$, set $S_i = \{ T \subseteq S_i : \forall j, k \in S_i (j \in T \text{ and } k \notin T \Rightarrow j <_i k) \}$.\(^{22}\)

It must be verified that $I_3^*$ satisfies the conditions of definition 19. Suppose that $X, Y \in S_i$ and $X \nsubseteq Y$; I show that $Y \subseteq X$. Since $X \nsubseteq Y$, $\exists x \in X \subseteq S_i$ such that $x \notin Y$. Pick an arbitrary $y \in Y \subseteq S_i$; since $y \in Y$ and $x \notin Y$, $y <_i x$. Now, if $y \notin X$, it would follow by parallel reasoning that $x <_i y$, which is impossible. Hence, $y \in X$, as required by (nest).

For (uni), suppose that $T \subseteq S_i$ and $\bigcup T \not\subseteq S_i$. Then $\exists x, y \in S_i$ such that $x \in \bigcup T$, $y \not\in \bigcup T$, and $x \not<_i y$. Since $x \in \bigcup T$, $\exists T \in T \subseteq S_i$ such that $x \in T$; note that $y \not\in T$ since $y \not\in \bigcup T$. Then it follows from the definition of $S_i$ that $x <_i y$, which is impossible. The case of (int) is similar.

\(^{22}\)I write $j <_i k$ if and only if $j \leq_i k$ and $j \neq k$. 

It remains to show that $\models_{w}^{3_{i}} \phi$ if and only if $\models_{w}^{3_{i}} \phi$. The proof is by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\theta \Box \psi$, which subdivides into two cases depending on whether there are antecedent worlds. Since $\bigcup S_{w} = S_{w}$, it is clear that the vacuous cases are equivalent. Write $R_{w}^{x} = \{y : y \leq_{w} x\}$ and observe that, for all $x \in S_{w}$, $R_{w}^{x} \in S_{w}$.\footnote{Suppose some $R_{w}^{x} \notin S_{w}$; then $\exists m, n \in S_{w}$ such that $m \in R_{w}^{x}$, $n \notin R_{w}^{x}$, and $m \leq_{w} n$, which implies $n \leq_{w} m$ by (tot). Since $m \in R_{w}^{x}$, $m \leq_{w} x$, from which it follows that $n \leq_{w} x$, contradicting the claim that $n \notin R_{w}^{x}$.} Now, suppose $\exists x \in S_{w} \cap [\theta]^{3_{i}}$ such that $\forall y \in W$, if $y \leq_{w} x$, then $y \in [\theta \rightarrow \psi]^{3_{i}}$. Then $R_{w}^{x} \in S_{w}$, $x \in R_{w}^{x} \cap [\theta]^{3_{i}} \neq \emptyset$, and $R_{w}^{x} \subseteq [\theta \rightarrow \psi]^{3_{i}}$, that is, $\models_{w}^{3_{i}} \theta \Box \psi$. Conversely, suppose that $\exists S \in S_{w}$ such that $S \cap [\theta]^{3_{i}} \neq \emptyset$ and $S \subseteq [\theta \rightarrow \psi]^{3_{i}}$; then $\exists y \in S \cap [\theta]^{3_{i}}$. Since $R_{w}^{y} \subseteq S$,\footnote{Observe that every $T \in S_{w}$ is downward closed: if $x \in T$ and $y \leq_{w} x$, then $y \in T$. For suppose that $x \in T$, $y \leq_{w} x$, and $y \notin T$; then it follows from $T \in S_{w}$ that $x <_{w} y$, which is impossible.} it follows easily that $\models_{w}^{3_{i}} \theta \Box \psi$, as desired.

\[\square\]

**Lemma 18.** Given a sphere interpretation $I_{\mathcal{S}} = \langle W, \{S_{i} : i \in W\}, V \rangle$, there is a preorder interpretation $I_{\preceq} = \langle W, \{\preceq_{i} : i \in W\}, V \rangle$ such that $\models_{w}^{3_{i}} \phi$ if and only if $\models_{w}^{3_{i}} \phi$

Construct $I_{\preceq} = \langle W, \{\preceq_{i} : i \in W\}, V \rangle$ from $I_{\mathcal{S}} = \langle W, \{S_{i} : i \in W\}, V \rangle$ by putting $j \preceq_{i} k$ if and only if, for all $T \in S_{i}$, if $k \in T$, then $j \in T$ (where $i \in W$ and $j, k \in \bigcup S_{i}$). Then it is obvious that $S_{i} = \bigcup S_{i}$.

Verifying that $I_{\preceq}$ satisfies the conditions of definition 18 is mostly straightforward; I examine only the case of (tot). Let $j, k \in S_{i}$ and suppose $j \preceq_{i} k$; I show $k \preceq_{i} j$. Take an arbitrary $T \in S_{i}$ such that $j \in T$. Since $j \preceq_{i} k$, $\exists X \in S_{i}$ such that $k \in X$ and $j \notin X$, by the specification of $\preceq_{i}$. It is clear that $T \not\subseteq X$, since $j \in T$; therefore, by (nest), $X \subseteq T$, from which it follows that $k \in T$, which was to be proved. I omit the routine induction proof that $\models_{w}^{3_{i}} \phi$ if and only if $\models_{w}^{3_{i}} \phi$.

\[\square\]

**Theorem 13** (Equivalence). $\Gamma \models_{\preceq} \phi$ if and only if $\Gamma \models_{\mathcal{S}} \phi$
The result follows directly from lemmata 17 and 18.
Chapter 3

Tableaux for Lewis-Stalnaker Logics

In his *Introduction to Non-Classical Logic*, Graham Priest notes that “there are presently no known tableau systems of the kind used in this book for $S$ [basic sphere semantics]” [102, p. 93]. The aim of this chapter is to provide tableaux in the style of Priest’s work for several conditional logics which have traditionally been given a sphere semantics à la Lewis [67]. The systems given tableaux here include Lewis’ preferred logic of counterfactuals ($C_1/VC$) and Stalnaker’s conditional logic ($C_2/VCS$).

Systems in the neighborhood of $CK$ have already been given sound and complete tableaux by Priest [102], Rønnesdal [109], and Zach [134]. Sequent calculi or tableaux for some of the stronger systems have been given by de Swart [118], Gent [38], Negri and Sbardolini [89], and Zach [134]. However, problems have regularly been encountered in proof-theoretic treatments of conditional logic. For example, the tableau system developed for Lewis’ $C_1$ by Zach is non-analytic in that it makes use of rules in which the conclusion-formulae are not generally subformulae of the premise-formulae [134, §3]. In particular, his tableaux require a version of Gentzen’s Cut rule, which is in turn apparently ineliminable.¹

Moreover, Negri and Sbardolini [89], *pace* the authors’ claims therein, do not in fact

¹For Gentzen’s sequent calculus and the Cut rule, see [39, 88]. For a discussion of Cut in a tableaux setting, see Smullyan [113] or Fitting [31].
provide a sequent calculus for Lewis’ $\mathbf{C1}$. For it can be shown that the following version of the (4) axiom,$^2$

$$(\neg\phi \Box \Rightarrow \phi) \Rightarrow (\neg(\neg\phi \Box \Rightarrow \phi) \Box \Rightarrow (\neg\phi \Box \Rightarrow \phi))$$

is derivable in their calculus. However, (4C) is independent of Lewis’ $\mathbf{C1}$. It is, along with a conditional version of the (5) axiom, one of the characteristic axioms of Lewis’ $\mathbf{VCU}$, a lesser known system in Lewis’ family of $\mathbf{V}$-logics (for which, see Lewis [67, Ch. 6]).$^3$

The crux of the problem in the work of Negri and Sbardolini [89] is that they prove determination results for their sequent calculus with respect to a semantics that is implicitly universalized—that implicitly satisfies (univ)—in addition to the constraints required by $\mathbf{C1}$. Since $\mathbf{C1}$ is determined by a class of preorder interpretations that need not satisfy this condition (see subsection 2.3.2 above), their calculus is unfaithful. This problem notwithstanding, I believe that their approach can be modified in such a way as to capture not only Lewis’ $\mathbf{C1}$, but almost every logic in the $\mathbf{V}$-family. The principal asset of their approach lies in its modularity. Briefly, the sequent calculus they present is labeled (or prefixed) and so integrates semantic structure—worlds and ordering—into the proof theoretic apparatus. That labeled sequent calculi have important advantages over their unlabeled counterparts in terms of modularity and semantic insight is clear from the history of proof theory: while older sequent calculi for modal logic are all unlabeled (e.g. Ohnishi and Matsumoto [93]), more recent work has tended to appreciate and utilize labels (e.g. Negri [85]).

Sticking to the problem posed by Priest [102], quoted above, I will not pursue the development of sequent calculi here; instead, I will focus squarely on tableaux. Nevertheless, as is well known, tableaux and sequent calculi are generally isomorphic.$^4$ Note, moreover,

$^2$Why do I call this a version of the (4) axiom? Because in many conditional logics, $\Box \phi$ can be defined as $\neg\phi \Box \Rightarrow \phi$ (see the discussion in section 2.1.2).

$^3$I have communicated this observation to the authors and I am grateful for their acknowledging it.

$^4$The conversion techniques discussed by Smullyan [113] are easily generalized to labeled tableaux and sequent calculi.
that most of the tableaux presented below are analytic (contrast Zach [134]), as are all of the tableaux for conditional logic presented by Priest [102, Ch. 5].

The plan of the chapter is as follows. Tableaux in the style of Priest [102] are presented for various conditional logics from the V-family (not including Stalnaker’s C2 or its extensions) in section 3.1. These tableaux rules effectively mirror the preorder constraints and structure outlined in subsection 2.3.2. Thus, for example, tableaux for C1 are obtained by taking the general rules as well as rules specifically corresponding to weak and strong centering, i.e. (wc) and (cs). Soundness and completeness results for these tableaux are proved in subsection 3.1.2.

Tableaux for Stalnaker’s C2 have proven significantly more difficult to obtain. I discuss some of these difficulties in section 3.2 before proposing a system of tableaux that is sound and complete with respect to the preorder semantics for C2; unfortunately, not all of its rules are analytic. Nevertheless, I conjecture (but have been unable to prove) that (Cut) is eliminable from this system.

## 3.1 Tableaux for Lewis Systems

In subsection 3.1.1, tableaux in the style of Priest [102] are given for conditional logics (not including Stalnaker’s C2) determined by various classes of preorder interpretations (for which, see subsection 2.3.2). Soundness and completeness results for these systems with respect to their semantics are presented in subsection 3.1.2.

### 3.1.1 Tableaux Systems

I begin by briefly reviewing some basic terminology for tableaux systems (for a fuller treatment, see Smullyan [113] or Priest [102]).

**Definition 20.** A tree is a structure such as this:
A, B, etc. are nodes. Nodes can have any of the following eight forms (where \( i, j, \) and \( k \) are non-negative integers and \( \phi \in \Phi \)):

1. \( j \preceq_i k \)
2. \( j = k \)
3. \( \square_i \phi, +j \)
4. \( \square_i \phi, -j \)
5. \( \Box \phi, +j \)
6. \( \Box \phi, -j \)
7. \( \phi, +i \)
8. \( \phi, -i \)

A branch is a maximal path, e.g. \{A, B, D\}. A branch is closed if it contains nodes of the form \( \phi, +i \) and \( \phi, -i \). A tableau is closed if each branch is closed. A tableau is complete if every applicable rule has been applied.

Tableaux rules are presented below. The following notational conventions should be noted: \( \alpha(j) \) is an arbitrary \( j \)-containing node and \( \alpha(i/j) \) is the result of replacing some occurrences of \( j \) by \( i \) in \( \alpha \). \( \alpha_i(j) \) is any relational node in which \( j \) occurs and \( i \) occurs as a subscript (example: \( j \preceq_i k \)).
1. Extensional Rules. Any branch containing the first node can be extended in any of the following ways:

\[ (\phi \rightarrow \psi), +i \]
\[ (\phi \land \psi), +i \]
\[ (\phi \lor \psi), +i \]
\[ \neg \phi, +i \]
\[ \phi, -i \]
\[ \psi, +i \]
\[ \psi, +i \]

2. Order Rules. (REFL) is applied to any branch containing the first node. (TRANS) and (TOT) are applied to any branch containing the first two nodes.

\[ \alpha_i(j) \quad \text{(REFL)} \]
\[ j \preceq_i j \]

\[ h \preceq_i j \quad \text{(TRANS)} \]
\[ j \preceq_i k \]
\[ h \preceq_i k \]
3. Identity Rules. (IDE) is applied to any branch containing $i$. (SUB) is applied to any branch containing the first two nodes.

\[
\begin{align*}
\alpha_i(j) \\
| \\
\alpha_i(k) \\
\end{align*}
\]

\[j \preceq_i k \quad k \preceq_i j\]

4. Subscript Box Rules.\(^5\) (□−) is applied to any branch containing the first node with $k$ new. (□+) is applied to any branch containing the first two nodes.

\[
\begin{align*}
\cdot \\
| \\
i = i \\
\end{align*}
\]

\[i = j\]

\[
\begin{align*}
| \\
\end{align*}
\]

\[\alpha(j)\]

\[
\begin{align*}
| \\
\alpha(i/j) \\
\end{align*}
\]

\(^5\)A similar device to this is employed by Negri and Sbardolini [89, pp. 48-50] in their sequent calculus. For the intuition behind these rules, consider (□+) and imagine a world $j$ and an ordering $\preceq_j$ of some set of worlds by their similarity to $j$. If $\square_j \phi$ is true at some world $i$, then every world at least as similar to $j$ as $i$ must satisfy $\phi$. Informally, this is what (□+) says.
5. Dot Box Rules.\(^6\) \((\Box^-)\) is applied to any branch containing the first node with \(k\) new. \((\Box^+)\) is applied to any branch containing the first two nodes.

\[
\begin{align*}
\Box_j &\phi, -i \\
&\downarrow \\
&k \preceq_j i \\
&\downarrow \\
&\phi, -k
\end{align*}
\]

\[
\begin{align*}
\Box_j &\phi, +i \\
&\downarrow \\
&k \preceq_j i \\
&\downarrow \\
&\phi, +k
\end{align*}
\]

\(^6\)For the intuition behind these rules, consider \((\Box^+)\) and imagine a world \(k\) that is in the “similarity field” of \(i\). Informally, \((\Box^+)\) says that if \(\Box \phi\) is true at \(i\), \(\phi\) is true at \(k\). That is, if \(\Box \phi\) is true at a world, \(\phi\) is true at any world similar to it (i.e. all worlds in the given world’s similarity ordering).
6. Fundamental Preorder Rules. To obtain tableaux faithful to basic preorder semantics, add to the rules of sections 1, 2, 4, and 5 the following rules. \((\Box \rightarrow _1 -)\) is applied to any branch containing the first node. \((\Box \rightarrow _2 -)\) is applied to any branch containing the first two nodes. \((\Box \rightarrow ^+ -)\) is applied to any branch containing the first node for \(k\) new.

\[
\begin{align*}
\Box \phi, +i & \quad (\Box^+) \\
\mid & \\
\alpha_i(k) & \\
\mid & \\
\phi, +k
\end{align*}
\]

\[
\begin{align*}
(\phi \Box \rightarrow \psi), -i & \quad (\Box \rightarrow _1 -) \\
\mid & \\
\Box \neg \phi, -i
\end{align*}
\]

\[
\begin{align*}
\alpha_i(j) & \quad (\Box \rightarrow _2 -) \\
\mid & \\
(\phi \Box \rightarrow \psi), -i \\
\mid & \\
\phi, -j \quad \Box_i(\phi \rightarrow \psi), -j
\end{align*}
\]

\[
\begin{align*}
(\phi \Box \rightarrow \psi), +i & \quad (\Box \rightarrow ^+ -) \\
\mid & \\
\Box \neg \phi, +i \quad \phi, +k \\
\mid & \\
k \preceq_i k \\
\mid & \\
\Box_i(\phi \rightarrow \psi), +k
\end{align*}
\]
7. Special Rules. The rules of this section correspond to special semantic constraints considered above in table 2.7. Note that some of these semantic constraints require multiple rules.

(a) Normality. (NORM) is applied to any branch containing $i$ with $j$ new.

```
\begin{center}
\begin{array}{c}
\alpha_i(j) \\
\mid \\
\alpha_i j \\
\end{array}
\end{center}
```

\begin{center}(NORM)\end{center}


(b) Self-Accessibility. (SA) is applied to any branch containing $i$.

```
\begin{center}
\begin{array}{c}
\alpha_i j \\
\mid \\
\alpha_i i \\
\end{array}
\end{center}
```

\begin{center}(SA)\end{center}


(c) Weak-Centering. The Weak-Centering rules are (SA) and (WC), which is applied to any branch containing the first node.

```
\begin{center}
\begin{array}{c}
\alpha_i(j) \\
\mid \\
\alpha_i j \\
\end{array}
\end{center}
```

\begin{center}(WC)\end{center}

(d) Strong-Centering. The Strong-Centering rules are the identity rules of section 3 and (CS), which is applied to any branch containing the first node.

```
\begin{center}
\begin{array}{c}
\alpha_i j \\
\mid \\
\alpha i j \\
\end{array}
\end{center}
```

\begin{center}(CS)\end{center}
(e) Universality. (UNIV) is applied to any branch containing \( i \) and \( j \).

\[
\begin{array}{c}
. \\
| \\
j \preceq_i j \\
\end{array}
\]

(UNIV)

(f) Absoluteness. (ABS) is applied to any branch containing the first node for any \( i \) on the branch.

\[
\begin{array}{c}
j \preceq_k l \\
| \\
j \preceq_i l \\
\end{array}
\]

(ABS)

**DEFINITION 21.** A *system of preorder tableaux* \( L^T \) is any collection of tableau rules which extends the set of tableau rules from sections 1, 2, 4, 5, and 6, and which respects dependencies (for example, it doesn’t contain (CS) unless it also contains the identity rules from section 3).

**Example.** \( C1^T \) is the set of tableau rules from sections 1–6 and 7b–d.

**DEFINITION 22.** Where \( \Sigma \) is a (finite) set of formulae and \( L^T \) is a system of preorder tableaux, \( \Sigma \vdash_{L^T} \phi \) if and only if there is a closed tableau (constructed using the appropriate rules) whose initial list consists of nodes of the form \( \psi, +0 \) for all \( \psi \in \Sigma \) and \( \phi, -0 \).

**Example.** \( \phi, \psi \vdash_{C1^T} \phi \square \rightarrow \psi \)
Explanation: the ‘initial list’ consists of the first three nodes. The next node is introduced by (SA). The tree branches by an application of \((\phi \rightarrow \psi)\) where \(i = j = 0\); the left branch is closed. On the right branch, the next two nodes follow by \((\Box \neg)\). The node \(0 = 1\) is introduced by (CS). Consequently, the use of (SUB) and the extensional rules closes the branch.
3.1.2 Determination Results

If $L^T$ is a system of preorder tableaux, $C^L_\leq$ is the corresponding class of preorder interpretations (the reader should carefully review definition 18 for what follows). Thus, if (for example) $L^T$ contains (NORM), $C^L_\leq$ is taken to be a class of preorder interpretations satisfying (norm). In this section, I prove soundness and completeness for systems of preorder tableaux with respect to various classes of preorder interpretations. To avoid unnecessary clutter, preorder subscripts are suppressed and I write (for example) $I$ for $I^L_\leq$ and $C^L$ for $C^L_\leq$. Given a preorder interpretation $I = \langle W, \{\leq_w: w \in W\}, V \rangle$, recall that for any $w \in W$, $S_w$ is defined to be the field of $\leq_w$.

**DEFINITION 23.** Let $\mathfrak{I} = \langle W, \{\leq_w: w \in W\}, V \rangle$ be a preorder interpretation ($\mathfrak{I} \in C^L_\leq$) and $b$ be a branch of an $L^T$ tableau. $\mathfrak{I}$ is faithful to $b$ if and only if there is a function $g : Z \rightarrow W$ such that:

1. If $\phi, +i$ is on $b$, $g(i) \in [\phi]$ in $\mathfrak{I}$
2. If $\phi, -i$ is on $b$, $g(i) \notin [\phi]$ in $\mathfrak{I}$
3. If $i \preceq_j k$ is on $b$, $g(i) \leq g(j) g(k)$ in $\mathfrak{I}$
4. If $i = j$ is on $b$, $g(i) = g(j)$ in $\mathfrak{I}$
5. If $\Box_j \phi, +i$ is on $b$, $\forall x \in W$, if $x \leq g(j) g(i)$, then $x \in [\phi]$ in $\mathfrak{I}$
6. If $\Box_j \phi, -i$ is on $b$, $\exists x \in W$ such that $x \leq g(j) g(i)$ and $x \notin [\phi]$ in $\mathfrak{I}$
7. If $\Box \phi, +i$ is on $b$, $\forall x \in S_{g(i)}$, $x \in [\phi]$ in $\mathfrak{I}$
8. If $\Box \phi, -i$ is on $b$, $\exists x \in S_{g(i)}$ such that $x \notin [\phi]$ in $\mathfrak{I}$

**LEMMA 19.** If $\mathfrak{I}$ is faithful to $b$ and an $L^T$ tableau rule is applied to $b$, then $\mathfrak{I}$ is faithful to at least one of the resulting branches.
The proof is by cases of the rule applied. Since the extensional cases are routine, I only examine the rules which pertain to intensional connectives.

Suppose that $\mathcal{I}$ is faithful to $b$ and $\Box_j \phi, -i$ occurs on $b$. If $(\Box^-)$ is applied, then $b$ is extended to a branch $b^*$ with nodes $k \leq_j i$ and $\phi, -k$, for $k$ new. By the faithfulness of $\mathcal{I}$ to $b$, there is an $x \in W$ such that $x \leq_{g(j)} g(i)$ and $x \notin [\phi]$. If $f$ is the same as $g$ except that $f(k) = x$, $f$ shows that $\mathcal{I}$ is faithful to $b^*$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $\Box \phi, +i$ and $k \preceq_j i$ occur on $b$. By the faithfulness of $\mathcal{I}$, $g(k) \leq_{g(i)} g(i)$ and, consequently, $g(k) \in [\phi]$. Then $g$ shows $\mathcal{I}$ is faithful to the branch $b^*$ which results from applying $(\Box^+)$ and appending the node $\phi, +k$ to $b$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $\Box \phi, -i$ occurs on $b$. If $(\Box^-)$ is applied, then $b$ is extended to a branch $b^*$ with nodes $k \leq_i k$ and $\phi, -k$, for $k$ new. By the faithfulness of $\mathcal{I}$ to $b$, there is an $x \in S_{g(i)}$ such that $x \notin [\phi]$. If $f$ is the same as $g$ except that $f(k) = x$, $f$ shows that $\mathcal{I}$ is faithful to $b^*$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $\Box \phi, +i$ and $\alpha_i(k)$ occur on $b$. By the faithfulness of $\mathcal{I}$, $g(k) \in S_{g(i)}$ and $g(k) \in [\phi]$. Consequently, $g$ shows $\mathcal{I}$ is faithful to the branch $b^*$ which results from applying $(\Box^+)$ and appending the node $\phi, +k$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $(\phi \Box \psi), -i$ occurs on $b$. By the faithfulness of $\mathcal{I}$, there is a $z \in S_{g(i)} \cap [\phi]$, i.e. $S_{g(i)} \not\subseteq [\neg \phi]$. If $(\Box \to^-)$ is applied, $b$ is extended to a branch $b^*$ with node $\Box \neg \phi, -i$. Then $g$ shows that $\mathcal{I}$ is faithful to $b^*$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $(\phi \Box \psi), -i$ and $\alpha_i(j)$ occur on $b$. By the faithfulness of $\mathcal{I}$, $\forall x \in S_{g(i)}$, either $x \notin [\phi]$ or $\exists y \in W$ such that $y \leq_{g(i)} x$ and $y \notin [\phi \to \psi]$. If $(\Box \to^-)$ is applied, $b$ is extended to a branch $b^*$ with node $\phi, -j$ and a branch $b^{**}$ with node $\Box_i (\phi \to \psi), -j$. Since $g(j) \in S_{g(i)}$, if $g(j) \not\in [\phi]$, $g$ shows that $\mathcal{I}$ is faithful to $b^*$. Alternatively, $g$ shows that $\mathcal{I}$ is faithful to $b^{**}$ because $\exists y \in W$ such that $y \leq_{g(i)} g(j)$ and $y \notin [\phi \to \psi]$.

Suppose that $\mathcal{I}$ is faithful to $b$ and $(\phi \Box \psi), +i$ occurs on $b$. $(\Box \to^+)$ is applied and $b$
is extended to a branch $b^*$ with node $\Box \neg \phi, +i$ and a branch $b^{**}$ with nodes $\phi, +k, k \leq_i k$, and $\Box_i (\phi \rightarrow \psi), +k$, where $k$ is new. By the faithfulness of $J$, either $S_{g(i)} \cap [\phi] = \emptyset$ or there is an $x \in S_{g(i)} \cap [\phi]$ such that $\forall y \in W$, if $y \leq g(i) x$, then $y \in [\phi \rightarrow \psi]$. In the first case, $g$ shows $J$ is faithful to $b^*$. In the second case, if $f$ is the same as $g$ except that $f(k) = x$, $f$ shows that $J$ is faithful to $b^{**}$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (NORM). Suppose that $J$ is faithful to $b$ and $i$ occurs on $b$. Since $CL$ is subject to (norm), $\exists x \in S_{g(i)} \neq \emptyset$. If (NORM) is applied, $b$ is extended to a branch $b^*$ with node $j \leq_i j$, for $j$ new. Then if $f$ is the same as $g$ except that $f(j) = x$, $f$ shows that $J$ is faithful to $b^*$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (SA). Suppose that $J$ is faithful to $b$ and $i$ occurs on $b$. Since $CL$ is subject to (sa), $g(i) \in S_{g(i)}$. If (SA) is applied, $b$ is extended to a branch $b^*$ with node $i \leq_i i$. Then $g$ shows that $J$ is faithful to $b^*$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (WC). Suppose that $J$ is faithful to $b$ and $\alpha_i(j)$ occurs on $b$. Since $CL$ is subject to (wc) and, by the faithfulness of $J$ to $b$, $g(j) \in S_{g(i)}$, it follows that $g(i) \leq g(i) g(j)$. If (WC) is applied, $b$ is extended to a branch $b^*$ with node $i \leq_i j$. Then $g$ shows that $J$ is faithful to $b^*$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (CS). Suppose that $J$ is faithful to $b$ and $j \leq_i i$ occurs on $b$. Since $CL$ is subject to (cs) and, by the faithfulness of $J$ to $b$, $g(j) \leq g(i) g(i)$, it follows that $g(j) = g(i)$. If (CS) is applied, $b$ is extended to a branch $b^*$ with node $i = j$. Then $g$ shows that $J$ is faithful to $b^*$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (UNIV). Suppose that $J$ is faithful to $b$ and $i$ and $j$ occur on $b$. Since $CL$ is subject to (univ), $S_{g(i)} = S_{g(j)} = W$. Then $g(j) \in S_{g(i)}$. If (UNIV) is applied, $b$ is extended to a branch $b^*$ with node $j \leq_i j$. Then $g$ shows that $J$ is faithful to $b^*$.

Let $b$ be a branch of an $LT$ tableau where $LT$ includes (ABS). Suppose that $J$ is faithful to $b$ and $j \leq_k l$ and $i$ occur on $b$. By faithfulness, $g(j) \leq g(k) g(l)$ in $J$. Since $CL$ is subject
to (abs), \( \leq_{g(k)} \leq_{g(i)} \) in \( \mathcal{J} \), whence \( g(j) \leq_{g(i)} g(l) \). If (ABS) is applied, \( b \) is extended to a branch \( b^* \) with node \( j \preceq_i l \). Then \( g \) shows that \( \mathcal{J} \) is faithful to \( b^* \).

\[ \square \]

**THEOREM 14** (Soundness). Where \( L^T \) is a system of preorder tableaux, if \( \Sigma \vdash_{L^T} \phi \), then \( \Sigma \models_{CL} \phi \)

Suppose that \( \Sigma \vdash_{L^T} \phi \) but \( \Sigma \not\models_{CL} \phi \). Then there is an interpretation \( \mathcal{J} \in CL \) such that \( \exists w \in W \) for which \( \forall \chi \in \Sigma, \models_{w}^{3} \chi \) and \( \not\models_{w}^{3} \phi \). As \( \Sigma \vdash_{L^T} \phi \), there is a closed tableau \( T \) and \( \mathcal{J} \) is faithful to the initial segment of \( T \) (i.e. the initial list, the nodes antecedent to rule applications) because of the function \( g(0) = w \). By repeated application of lemma 19, there is a branch \( b \) of \( T \) such that \( \mathcal{J} \) is faithful to each segment of \( b \). If \( T \) is closed, however, \( b \) must be as well; then \( b \) has nodes of the form \( \psi, +k \) and \( \psi, -k \). Then by definition 23, \( \models_{w}^{3} \psi \) and \( \not\models_{w}^{3} \psi \), which is impossible.

\[ \square \]

**DEFINITION 24.** Let \( L^T \) be a system of preorder tableaux excluding the identity rules. Then a preorder interpretation \( \mathcal{I} = \langle W, \{ \leq_i : i \in W \}, V \rangle \) induced by an open branch \( b \) of an \( L^T \) tableau is such that:

1. \( W = \{ w_i : i \text{ on } b \} \)

2. \( w_k \leq_{w_i} w_j \) if and only if \( k \geq_i j \) is on \( b \)

3. For propositional variables \( p, w_i \in V(p) \) if \( p, +i \) is on \( b \)

4. For propositional variables \( p, w_i \not\in V(p) \) if \( p, -i \) is on \( b \)

Note that for any propositional variables which do not occur on the branch, \( V \) can be arbitrary.
**Lemma 20 (Truth Lemma).** Let $L^T$ be a system of preorder tableaux excluding the identity rules. If $b$ is an open, complete branch of an $L^T$ tableau and $I$ is a preorder interpretation induced by $b$, then:

1. If $\phi, +i$ occurs on $b$, $w_i \in [\phi]$ in $I$

2. If $\phi, -i$ occurs on $b$, $w_i \not\in [\phi]$ in $I$

The proof is by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\psi \Box \rightarrow \theta$.

Suppose $(\psi \Box \rightarrow \theta), +i$ is on $b$; since $b$ is complete and $(\Box \rightarrow ^+)$ has been applied, either $\Box \neg \psi, +i$ is on $b$ or (for some $k$) $\psi, +k, k \preceq i$, and $\Box_i (\psi \rightarrow \theta), +k$ are on $b$. In the first case, by the completeness of $b$, $(\Box^+)$ has been applied for every $j$ such that $\alpha_i (j)$ is on $b$. Hence, if $w_j \in S_{w_i}, \psi, -j$ appears on $b$, from which it follows (by the induction hypothesis) that $w_j \not\in [\psi]$. Thus, since $S_{w_i} \cap [\psi] = \emptyset$, $w_i \in [\psi \Box \rightarrow \theta]$. In the second case, for every non-negative integer $l$ such that $l \leq i$, $k$ is on $b$, $(\psi \rightarrow \theta), +l$ is on $b$ by the completeness of $b$ and rule $(\Box^+)$. Therefore, by the induction hypothesis and definition 24, $w_k \in [\psi] \cap S_{w_i}$ and $\{w_l \in W : w_l \leq w_i, w_k \} \subseteq [\psi \rightarrow \theta]$, from which it follows that $w_i \in [\psi \Box \rightarrow \theta]$.

Suppose $(\psi \Box \rightarrow \theta), -i$ is on $b$; since $b$ is complete and $(\Box \rightarrow _-)$ has been applied, $\Box \neg \psi, -i$ is on $b$. Consequently, (for some $k$) $k \preceq_i k$ and $\psi, +k$ are on $b$, again by the completeness of $b$. By the induction hypothesis, $w_k \in S_{w_i} \cap [\psi] \neq \emptyset$. By the completeness of $b$, $(\Box \rightarrow _-)$ has been applied for every non-negative integer $l$ such that $\alpha_i (l)$ appears on $b$; hence, for every such $l$, either $\psi, -l$ or $\Box_i (\psi \rightarrow \theta), -l$ is on $b$. In the latter case, by the completeness of $b$ and $(\Box^-)$, for each such $l$ there is some $h$ such that $h \preceq_i l$ and $\psi \rightarrow \theta, -h$ are on $b$. Now consider an arbitrary $w_l \in S_{w_i} \cap [\psi]$; it is clear from the construction and induction hypothesis that $\psi, -l$ does not occur on $b$, so it follows that $\Box_i (\psi \rightarrow \theta), -l$ is on $b$. Consequently, by definition 24 and the induction hypothesis, there is some $w_h$ such that $w_h \leq w_i, w_l$ and $w_h \not\in [\psi \rightarrow \theta]$. This suffices to show that $w_i \not\in [\psi \Box \rightarrow \theta]$. 
LEMMA 21. Let $L^T$ be a system of preorder tableaux excluding the identity rules. If $b$ is an open, complete branch of an $L^T$ tableau and $\mathcal{I}$ is a preorder interpretation induced by $b$, then $\mathcal{I} \in C^L$.

It must be verified that $\mathcal{I}$ satisfies the basic constraints on preorder interpretations (see definition 18) in addition to any constraints particular to $L$ (see table 2.7).

For (refl), consider an arbitrary $w_j \in S_{w_i}$. It is clear that some $\alpha_i(j)$ must occur on $b$, from which it follows that $j \preceq_i j$ occurs on $b$ by (REFL). Thus, by definition 24, $w_j \leq_w w_j$, as required.

For (trans), suppose $w_j \leq_w w_k$ and $w_k \leq_w w_l$; then $j \preceq_i k$ and $k \preceq_i l$ occur on $b$, from which it follows by (TRANS) that $j \preceq_i l$ occurs on $b$, and consequently that $w_j \leq_w w_l$ in $\mathcal{I}$, as desired.

For (tot), suppose that $w_j, w_k \in S_{w_i}$ and $w_j \nleq_w w_k$. By (TOT), either $j \preceq_i k$ or $k \preceq_i j$ are on $b$. By definition 24, it cannot be the first; therefore, $k \preceq_i j$ is on $b$ and $w_k \leq_w w_j$ in $\mathcal{I}$, as desired.

Suppose that $L^T$ includes (NORM); it must be shown that $\mathcal{I}$ satisfies (norm). Take an arbitrary $w_i \in W$. Since $i$ occurs on $b$, by (NORM), $j \preceq_i j$ occurs on $b$ for some $j$. Thus, $w_j \in S_{w_i} \neq \emptyset$.

Suppose that $L^T$ includes (SA); it must be shown that $\mathcal{I}$ satisfies (sa). Take an arbitrary $w_i \in W$. Since $i$ occurs on $b$, by (SA), $i \preceq_i i$ occurs on $b$. Thus, $w_i \in S_{w_i}$.

Suppose that $L^T$ includes (WC); it must be shown that $\mathcal{I}$ satisfies (wc). Insofar as the proof requires showing that (sa) holds, the proof proceeds as before with the observation that (SA) is a dependency of (WC). For the remainder, consider an arbitrary $w_j \in S_{w_i}$; then it is clear that some $\alpha_i(j)$ occurs on $b$. By (WC), $i \preceq_i j$ occurs on $b$, from which it follows that $w_i \leq_w w_j$, as desired.
Suppose that $L^T$ includes (UNIV); it must be shown that $I$ satisfies (univ). For arbitrary $w_i$, it is obvious that $S_{w_i} \subseteq W$, hence it remains to show the converse. Pick an arbitrary $w_j \in W$; then $j$ occurs on $b$, as does $i$. By (UNIV), $j \preceq_i j$ occurs on $b$, from which it follows that $w_j \in S_{w_i}$.

Suppose that $L^T$ includes (ABS); it must be shown that $I$ satisfies (abs). Suppose that $w_k \leq_{w_i} w_j$; for $w_l \in W$, I show that $w_i \leq_{w_i} w_j$. Since $k \preceq_i j$ occurs on $b$ and $l$ occurs on $b$, by (ABS), $k \preceq_i j$ occurs on $b$, from which the desired result follows. The converse direction (required to show that $\leq_{w_i} \preceq_{w_i}$) is the same.

\[\square\]

**Theorem 15** (Completeness). Let $L^T$ be a system of preorder tableaux excluding the identity rules. If $\Sigma \models_{CL} \phi$, then $\Sigma \vdash_{L^T} \phi$.

Suppose that $\Sigma \not\models_{L^T} \phi$. Then the attempted proof results in a completed open tableau with at least one complete open branch $b$. Let $I$ be a preorder interpretation induced by $b$ (it is obvious that such interpretations exist). Then in $I$, $\forall \psi \in \Sigma$, $\models_\Sigma^\psi \psi$ but $\not\models_\Sigma^\psi \phi$ by lemma 20. Moreover, by lemma 21, $I \in C^L$. Therefore, $\Sigma \not\models_{CL} \phi$.

\[\square\]

To prove the completeness of systems of preorder tableaux including the identity rules, definition 24 must be modified. Let $b$ be a complete branch of an $L^T$ tableau where $L^T$ includes the identity rules. If $i$ and $j$ are non-negative integers on $b$, say $i \sim j$ if and only if $i = j$ occurs on $b$.

**Lemma 22.** If $b$ is a complete branch of an $L^T$ tableau where $L^T$ includes the identity rules, then $\sim$ is an equivalence relation on \(\{i \in \mathbb{Z} : i \text{ occurs on } b\}\)

By (IDE), if $i$ occurs on $b$, then $i = i$ occurs on $b$. Hence, for any such $i$, $i \sim i$. Suppose that $i \sim j$ and $j \sim k$; then $i = j$ and $j = k$ occur on $b$. By (SUB), $i = k$ occurs on $b$, from
which it follows that $i \sim k$. Finally, suppose that $i \sim j$; since $i = j$ occurs on $b$, it follows by (SUB) and (IDE) that $j = i$ occurs on $b$. Thus, $j \sim i$, as desired.

Fixing a complete branch $b$, lemma 22 shows that $\{i \in \mathbb{Z} : i \text{ occurs on } b\}$ can be partitioned by $\sim$; write $\|x\| = \{y : x \sim y\}$.

**DEFINITION 25.** Let $L^T$ be a system of preorder tableaux including the identity rules. Then a preorder interpretation $I = \langle W, \{\leq_i : i \in W\}, V \rangle$ induced by an open, complete branch $b$ of an $L^T$ tableau is such that:

1. $W = \{w_{\|i\|} : i \text{ on } b\}$

2. $w_{\|k\|} \leq w_{\|i\|} w_{\|j\|}$ if and only if $k \leq_i j$ is on $b$

3. For propositional variables $p$, $w_{\|i\|} \in V(p)$ if $p, +i$ is on $b$

4. For propositional variables $p$, $w_{\|i\|} \notin V(p)$ if $p, -i$ is on $b$

Note that such induced interpretations are well-defined. For example, suppose that $w_{\|k\|} \leq w_{\|i\|} w_{\|k\|} = w_{\|k'\|}$, $w_{\|i\|} = w_{\|i'\|}$, and $w_{\|j\|} = w_{\|j'\|}$. Then $k \leq_i j$, $k = k'$, $i = i'$, and $j = j'$ are on $b$. Consequently, by the completeness of $b$ and the identity rules, $k' \leq_{i'} j'$ is on $b$. Thus, $w_{\|k\|} \leq w_{\|i'\|} w_{\|j'\|}$, which was to be proved.

Using definition 25, proofs of versions of the truth lemma (lemma 20) and constraints lemma (lemma 21) go through essentially as before. I omit the truth lemma and consider only one case of the constraints lemma, viz. that pertaining to (CS).

**LEMMA 23.** Let $L^T$ be a system of preorder tableaux containing the identity rules. If $b$ is an open, complete branch of an $L^T$ tableau and $I$ is a preorder interpretation induced by $b$, then $I \in C^L$. 

Suppose that $L^T$ includes (CS); it must be shown that $I$ satisfies (cs). Suppose that $w_{\|j\|} \leq w_{\|i\|}$; then $j \preceq_i i$ occurs on $b$. By (CS), $j = i$ occurs on $b$. Since $j \sim i$, it follows that $w_{\|j\|} = w_{\|i\|}$, as desired.

THEOREM 16 (Completeness). Let $L^T$ be a system of preorder tableaux including the identity rules. If $\Sigma \models_{cl} \phi$, then $\Sigma \vdash_{L^T} \phi$

The proof is just like that of theorem 15, using lemma 23.

3.2 Tableaux for Stalnaker’s System

In the previous section, I gave sound and complete analytic tableaux for a number of systems of conditional logic, including Lewis’ preferred logic of counterfactuals, $C1$. However, the approach of the previous section cannot, it seems, be made to work for Stalnaker’s $C2$. It is worth briefly considering why. Recall that Stalnaker’s condition, (stal), is the following monstrosity:\footnote{A somewhat less ugly condition would be to require that $\leq_i$ well-order $S_i$, i.e. to impose the stated condition on all nonempty subsets of $S_i$ rather than just formula-indexed ones (cf. Lewis [67, p. 79]).}

If $[\phi] \cap S_i \neq \emptyset$, $\exists x \in [\phi] \cap S_i$ such that $\forall y \in [\phi]$, $y \leq_i x$ only if $x = y$ (stal)

Tableaux rules which are sound with respect to this condition can be concocted without too much difficulty. Nodes of the form $\oplus_i \phi, +j$, where $i$ and $j$ are non-negative integers and $\phi$ is a formula, are now allowed. Then the rules corresponding to (stal) are:
8. Subscript At Rule. \((@^+)\) is applied to any branch containing the first two nodes.

\[
\begin{array}{c}
\neg_i \phi, +j \\
\downarrow \\
\neg_i j \\
\downarrow \\
\neg_i -\phi, +k
\end{array}
\]

\( (@^+) \)

9. Antecedent Well-Ordering. \((\Box \rightarrow ^\pm)\) is applied to any branch containing the first node for \(k\) new.

\[
\begin{array}{c}
\phi \Box \rightarrow \psi, \pm i \\
\downarrow \\
\neg_i \phi, +i \\
\downarrow \\
\neg_i -\phi, +k
\end{array}
\]

\( (\Box \rightarrow ^\pm) \)

The proof that \((@^+)\) and \((\Box \rightarrow ^\pm)\) are sound with respect to classes of preorder interpretations satisfying (stal) proceeds just as in subsection 3.1.2. For what follows, let \(L^T\) be any system of preorder tableaux including the Stalnaker rules.

**DEFINITION 26.** Let \(J = (W, \{\leq_i: i \in W\}, V)\) be a preorder interpretation \((J \in C^L)\) and \(b\) be a branch of an \(L^T\) tableau. \(J\) is *faithful* to \(b\) if and only if there is a function \(g : \mathbb{Z} \rightarrow W\) such that \(g\) is just as in definition 23 but, in addition:

- If \(@_i \phi, +j\) is on \(b\), \(\forall x \in W\), if \(x \leq g(i) g(j)\), then either \(x = g(j)\) or \(x \in [\phi]\)

**LEMMA 24.** If \(J\) is faithful to \(b\) and an \(L^T\) tableau rule is applied to \(b\), then \(J\) is faithful to at least one of the resulting branches.
The proof is by cases of the rule applied. I only examine the two cases which differ from those covered in the proof of lemma 19.

Suppose that \( J \) is faithful to \( b \) and \( @, \psi, +j \) and \( k \preceq_i j \) occur on \( b \). \((@^+)\) is applied and \( b \) is extended to a branch \( b^* \) with node \( j = k \) and a branch \( b^{**} \) with node \( \phi, +k \). By the faithfulness of \( J \) to \( b \), \( g(k) \leq g(j) \) and \( \forall x \in W \), if \( x \leq g(i) \) \( g(j) \), then either \( x = g(j) \) or \( x \in [\phi] \). Thus, either \( g(k) = g(j) \) (in which case \( g \) shows that \( J \) is faithful to \( b^* \)) or \( g(k) \in [\phi] \) (in which case \( g \) shows that \( J \) is faithful to \( b^{**} \)).

Suppose that \( J \) is faithful to \( b \) and \( \phi \rightarrow \psi, \pm i \) occurs on \( b \). If \((\rightarrow^{\pm})\) is applied, \( b \) is extended to a branch \( b^* \) with node \( \square \neg \phi, +i \) and a branch \( b^{**} \) with nodes \( \phi, +k, k \preceq_i k, \) and \( @i \neg \phi, +k \). Since \( C^L \) is (by assumption) subject to \((\text{stal})\), either \( [\phi] \cap S_{g(i)} = \emptyset \) or \( \exists x \in [\phi] \cap S_{g(i)} \) such that \( \forall y \in [\phi], y \leq g(i), x \) only if \( x = y \). In the first case, \( g \) shows that \( J \) is faithful to \( b^* \). In the second, if \( h \) is the same as \( g \) except that \( h(k) = x \), \( h \) shows that \( J \) is faithful to \( b^{**} \).

\[ \square \]

**THEOREM 17** (Soundness). Where \( L^T \) is a system of preorder tableaux including the Stalnaker rules, if \( \Sigma \vdash_{L^T} \phi \), then \( \Sigma \models_{C^L} \phi \)

From lemma 24, as in the proof of theorem 14.

\[ \square \]

**DEFINITION 27.** Let \( C^{2T} \) be \( C^{1T} \) (the rules of sections 1–6 and 7b–d) extended by the rules of sections 8 and 9, i.e. \((@^+)\) and \((\rightarrow^{\pm})\).

It is immediate from theorem 17 that \( C^{2T} \) is sound with respect to the class of preorder interpretations which determines Stalnaker’s \( C^2 \). But is it complete? One reason to suspect that it is is that (CEM), the characteristic scheme of Stalnaker’s \( C^2 \), is a theorem of \( C^{2T} \):

**PROPOSITION 5.** \( \vdash_{C^{2T}} (\phi \rightarrow \psi) \lor (\phi \rightarrow \neg \psi) \)
\[
(\phi \Box \rightarrow \psi) \lor (\phi \Box \rightarrow \neg \psi), -0
\]

\[
\vdash \\
\phi \Box \rightarrow \psi, -0
\]

\[
\vdash \\
\phi \Box \rightarrow \neg \psi, -0
\]

\[
\vdash \\
\neg \phi, -0
\]

\[
\vdash \\
1 \preceq 1
\]

\[
\vdash \\
\neg \phi, -1
\]

\[
\vdash \\
\phi, +1
\]

\[
\vdash \\
\Box \neg \phi, +0 \quad \phi, +2
\]

\[
\vdash \\
\neg \phi, +1 \quad 2 \preceq 2
\]

\[
\vdash \\
\phi, -1 \quad \Box_0 \neg \phi, +2
\]

\[
\vdash \\
\phi, -2 \quad \Box_0 (\phi \rightarrow \psi), -2
\]

\[
\vdash \\
3 \preceq 2
\]

\[
\vdash \\
\phi \rightarrow \psi, -3
\]

\[
\vdash \\
1
\]

\[
\vdash \\
\phi, +3
\]

\[
\vdash \\
\psi, -3
\]

\[
\vdash \\
\Box \neg \phi, +2
\]

\[
\vdash \\
\neg \phi, +3
\]

\[
\vdash \\
\phi, +3
\]

\[
\vdash \\
\psi, -3
\]

\[
\vdash \\
2 = 3
\]

\[
2 = 3
\]

\[
\neg \phi, +3
\]

\[
\vdash \\
\phi, -3
\]

\[
\Box_0 (\phi \rightarrow \psi), -3 \quad \phi, -3
\]

\[
\vdash \\
4 \preceq 3
\]

\[
\vdash \\
\phi \rightarrow \neg \psi, -4
\]

\[
\vdash \\
\phi, +4
\]

\[
\vdash \\
\neg \psi, -4
\]

\[
\vdash \\
\psi, +4
\]

\[
\vdash \\
4 \preceq 2
\]

\[
4 = 2 \quad \neg \phi, +4
\]

\[
\vdash \\
4 = 3 \quad \phi, -4
\]

\[
\vdash \\
\psi, +3
\]
In virtue of proposition 5, it is prima facie plausible to expect that $C_2^T$ is complete with respect to the validities of $C^{C2}$. However, there is a serious mismatch between what the characteristic rules of $C_2^T$ establish and what must hold in any interpretation induced by an open branch, i.e. interpretations used to show completeness. For the rules given in sections (8) and (9) above only establish that Stalnaker’s condition obtains for formulae which occur as antecedents of $\Box \rightarrow$ conditionals on the branch. But Stalnaker’s condition must be shown to obtain for all formulae, including those which do not occur anywhere on the branch. It may be possible to define induced interpretations in such a way that they automatically satisfy (stal) for $\phi$ which do not occur on the branch, but given the complexity of the condition, it is not at all clear how one might do this.

Now, completeness can be proved for an extension of $C_2^T$ with a special sort of non-analytic tableau rule. (Cut) is a rule which can be applied to any branch, for any formula $\phi$ (including $\phi$ not previously on the branch), and for any non-negative integer on the branch $i$: \footnote{This is simply a “modalized” analogue of Fitting [31, p. 227] (cf. Zach [134, p. 612]).}

\[
\phi, +i \quad \phi, -i
\]

(Cut)

The significance of (Cut) lies in the following observation: if a system of preorder tableaux $L^T$ includes (Cut), then the set of formulae provable in $L^T$ (i.e. its theorems), or the set of formulae derivable in $L^T$ from some (finite) set of formulae $\Sigma$, is closed under (MP). For suppose that (Cut) is included in $L^T$, $\Sigma \vdash_{L^T} \phi$, and $\Sigma \vdash_{L^T} \phi \rightarrow \psi$. Then there are closed trees, $T_0$ and $T_1$, corresponding (respectively) to each of those inferences. It follows that $\Sigma \vdash_{L^T} \psi$ (let $\Sigma = \{\sigma_0, \ldots, \sigma_n\}$):
CHAPTER 3. TABLEAUX

\[ \sigma_0, +0 \]
\[ \vdots \]
\[ \sigma_n, +0 \]
\[ \psi, -0 \]
\[ \phi \rightarrow \psi, +0 \quad \phi \rightarrow \psi, -0 \]
\[ \phi, -0 \quad \psi, +0 \quad \mathcal{T}_i \]
\[ \mathcal{T}_0 \]

Using this observation, it is easily shown that the system of preorder tableaux \( \mathbf{C}^2 \mathcal{T} + \text{(Cut)} \) is sound and complete with respect to \( \mathbf{C}^2 \): 

THEOREM 18 (Soundness). If \( \Sigma \vdash_{\mathbf{C}^2 \mathcal{T} + \text{(Cut)}} \phi \), then \( \Sigma \models_{\mathbf{C}^2} \phi \)

It is obvious that (Cut) is sound with respect to \( \mathbf{C}^2 \); that is, if an interpretation \( \mathcal{I} \in \mathbf{C}^2 \) is faithful to a tableau branch \( b \) and (Cut) is applied, then \( \mathcal{I} \) must be faithful to one of the resulting branches. Thus, the proof of soundness proceeds as before.

\[ \square \]

THEOREM 19 (Completeness). If \( \Sigma \models_{\mathbf{C}^2} \phi \), then \( \Sigma \vdash_{\mathbf{C}^2 \mathcal{T} + \text{(Cut)}} \phi \)

Suppose that \( \Sigma \models_{\mathbf{C}^2} \phi \). By the completeness of the axiom system \( \mathbf{C}^2 \) with respect to \( \mathbf{C}^2 \), it follows that \( \Sigma \vdash_{\mathbf{C}^2} \phi \). Now, every axiom of \( \mathbf{C}^2 \) is provable in \( \mathbf{C}^2 \mathcal{T} \): this follows from proposition 5 and the fact that \( \mathbf{C}^2 \mathcal{T} \) includes all the rules of a system of preorder tableaux that is complete with respect to \( \mathbf{C}^1 \). Moreover, the set of formulae provable in \( \mathbf{C}^2 \mathcal{T} \) is closed.
under (RCEA) and (RCEC). Finally, closure under (MP) is a consequence of (Cut). Thus, if $\Sigma \vdash_{C2} \phi$, it follows that $\Sigma \vdash_{C2}^{\tau+\text{(Cut)}} \phi$.

While I have given a system of tableaux for $C2$ that is sound and complete, this system is quite unlike the other tableaux systems given in this chapter. $C2^T$ contains a non-analytic rule, (Cut), which I appealed to in proving completeness. It would be very nice to show that the use of (Cut) is not, in fact, essential.

**DEFINITION 28.** If (R) is a tableau rule and $L^T$ is a system of preorder tableaux including (R), then (R) is *eliminable* from $L^T$ if whenever $\Sigma \vdash_{L^T} \phi$, $\Sigma \vdash_{L^T}^{\tau-(R)} \phi$. In other words, a tableau rule is eliminable in a given system if everything provable using it can be proved without it.

It is easily shown that the addition of (Cut) to any of the sound and complete systems discussed in section 3.1 has no effect on what’s provable. In other words, (Cut) is eliminable from all such extensions:

**COROLLARY 3 (Hauptsatz).** Where $L^T$ is a system of preorder tableaux for which theorems 14 and 15 (16) apply, if $\Sigma \vdash_{L^T}^{\tau+(\text{Cut})} \phi$, then $\Sigma \vdash_{L^T} \phi$

Suppose $\Sigma \vdash_{L^T}^{\tau+(\text{Cut})} \phi$. By soundness, and the fact that (Cut) is obviously sound with respect to any class of interpretations, $\Sigma \models_{\text{cL}} \phi$. Then, by completeness, $\Sigma \vdash_{L^T} \phi$.

The methods used in the proof of corollary 3 are non-constructive: no method is provided for turning a proof using (Cut) into one which does not use it. In fact, constructive proofs of (Cut) elimination are forthcoming for at least some of the systems surveyed in section 3.1.

---

9It is a good exercise to show that if there is a closed tree for $\phi \leftrightarrow \psi$, there is also one for $(\theta \supset \phi) \leftrightarrow (\theta \supset \psi)$, as required by (RCEC). An analogous result holds for (RCEA).

10If you prefer, (Cut) is *admissible* for all the systems discussed there.

11Basically, the approach of Negri and Sbardolini [89, §3] works, with slight modifications, and everything must be flipped upside down.
Unfortunately, I have been unable to discover a proof of this kind for the tableaux I have devised for $\textbf{C}_2$. Nevertheless, I conjecture that (Cut) is in fact eliminable:

**CONJECTURE 1** (Hauptsatz). If $\Sigma \vdash _{\textbf{C}_2^{\tau+(\text{Cut})}} \phi$, then $\Sigma \vdash _{\textbf{C}_2^{\tau}} \phi$

I leave for future work either a proof, or a refutation, of this conjecture concerning $\textbf{C}_2^{T}$. 
Chapter 4

Connexive Conditional Logic

What is most characteristic of connexive logic are the following distinctively non-classical conditional theses, typically referred to as Aristotle’s and Boethius’ theses:

\[
\neg(\phi \implies \neg\phi) \quad \text{(AT1)}
\]

\[
\neg(\neg\phi \implies \phi) \quad \text{(AT2)}
\]

\[
(\phi \implies \psi) \implies \neg(\phi \implies \neg\psi) \quad \text{(BT1)}
\]

\[
(\phi \implies \neg\psi) \implies \neg(\phi \implies \psi) \quad \text{(BT2)}
\]

Putting aside the ancient history of connexive logic, its modern history, interestingly, is often traced to a paper on subjunctive conditionals by Angell [2]. Angell proposes a system \textbf{PA1} which contains what he calls the \textit{principle of subjunctive contrariety}, that is:

The principle that ‘If \( p \) were true then \( q \) would be true’ and ‘If \( p \) were true then \( q \) would be false’ are incompatible [2, p. 327]

In Angell’s axiomatization of \textbf{PA1}, this principle corresponds to (BT1) above [2, p. 328].

\footnote{For the names, and their historical credentials, see McCall [78, pp. 415-6].}
However, the statement of the principle is ambiguous. To say that $\phi \rightarrow \psi$ and $\phi \rightarrow \neg \psi$ are incompatible could just be to say that $\neg ((\phi \rightarrow \psi) \land (\phi \rightarrow \neg \psi))$ is true. That is, the principle could reasonably be read as making the following claims:\footnote{Incidentally, both (WBT1) and (WBT2) are theorems of \textbf{PA1} [2, p. 336].}

\begin{align*}
(\phi \rightarrow \psi) & \rightarrow \neg (\phi \rightarrow \neg \psi) \\
(\phi \rightarrow \neg \psi) & \rightarrow \neg (\phi \rightarrow \psi)
\end{align*}

(WBT1)  
(WBT2)

I follow Pizzi and Williamson \[96, \text{pp. 569-70}\] in referring to (WBT1) and (WBT2) as \textit{weak} Boethius’ theses. All of what I have to say in the sequel will concern (WBT1) and (WBT2) rather than (BT1) and (BT2).

While connexive logic was originally developed alongside conditional logic, the two have largely grown apart. To speculate, this seems to mainly have been due to the difficulties faced in semantically modeling connexive principles and, relatedly, in identifying intuitively plausible interpretations of logics containing them. Thus, Routley and Montgomery write disparagingly,

The effect of \textit{Boethius} [(BT1)] or even of \textit{Aristotle} [(AT2)] alone, in quite weak sentential logics, is sufficient to cast serious doubt both on the merit of proceeding in the directions Angell suggests and on the value of connexive logics [84, p. 82]

While connexive logic languished in the heyday of conditional logic (at least in relative terms), there has since been a resurgence of interest in it. Connexive theses have by now been studied in a variety of settings: consistent and inconsistent, classical and nonclassical, etc.

This chapter belongs to a tradition which examines connexivism in a consistent classical context. More particularly, I intend to examine connexive principles situated in (weak)
systems of classical conditional logic, as developed in chapter 2.

The plan of the chapter is as follows. I motivate interest in connexive conditional logic by proposing a deontic interpretation of $\Box \rightarrow$ in section 4.1. There is a long history of deontic applications of conditional logic, an unappreciated aspect of which includes the examination and use of connexive theses. Where appropriate, I review this history. Under a deontic interpretation of $\Box \rightarrow$, I argue both that connexive principles like (WBT1) and (WBT2) are quite natural and that principles which must be abandoned to maintain consistency in the presence of such connexive theses are quite unnatural.

Systems of connexive conditional logic are presented in section 4.2. There, among other things, I show how to develop deontic modal logic within deontic conditional logic. Semantics along the lines of those from section 2.2 are presented in section 4.3. Determination and independence results are also proved there.

4.1 Obligation: Absolute and Conditional

Deontic logic is concerned with reasoning about obligations, prohibitions, and permissions. Note, however, that each of these come in two subspecies. Some obligations (for example) are unconditional: it is (unconditionally) obligatory not to steal. Other obligations, however, are conditional: it is obligatory to return the merchandise given that you stole it.\footnote{One might wonder whether there are any unconditional obligations. To adapt an example of Plato’s (Rep. I, 331c), it might seem to be obligatory to return what one has borrowed, but should one return borrowed weapons to a friend if they have become deranged? I take the question of the existence of unconditional obligations to be a substantive philosophical issue about which logic should remain neutral (see the brief remarks on necessitation in subsection 4.1.1).} This section discusses logical approaches to modeling both types. The reader will find that connexive ideas arise quite naturally in developing formal systems of conditional obligation.
4.1.1 Deontic Modal Logic

Contemporary deontic logic has its roots in von Wright [131], in which the deontic modalities obligatory and permitted were treated as applying to symbols denoting act-types. Subsequent work largely shifted the emphasis from what agents ought to do to what ought to be the case. As a result, the deontic modalities (like the alethic modalities) are now predominantly viewed as applying to arbitrary formulae. I follow this approach below.4

For the purposes of this chapter, let □φ be read: it is obligatory that (or: it ought to be the case that) φ. Consequently, via the definition ♦φ ≡ ¬□¬φ, it is natural to read ♦φ as: it is permissible that φ (i.e. it is not obligatory that not φ). Given these interpretations, the logic of obligation and permissibility can be explored within the syntactic and semantic framework of modal logic.

An important early work in this vein is Lemmon’s “New Foundations” [62]. What is most characteristic of Lemmon’s five deontic systems—the D-systems—is the axiom:

\[ \neg □ \bot \]  

(D)

(D) encapsulates the claim that what is impossible cannot be obligatory, i.e. that whatever ought to be can be. Since plausible deontic logics tend to be quite weak, it is a matter of contention what other axioms should be adopted. Lemmon (I think rightly) holds that (NEC) has little plausibility in deontic logic. After all, why should logic say that there are any obligations [62, p. 185]? Consequently, Lemmon’s deontic systems are not generally normal (i.e. do not generally contain the system K).

One of Lemmon’s subnormal systems deserves special mention. Following Lemmon [62] (also [63, p. 47]), the system D2 can be axiomatized by adding to classical propositional

4For a more detailed account of the relevant history (and also for a different approach), see Horty [44, Ch. 1].
5Or, in Lemmon’s notation, CLpNLp [62, p. 184].
logic (K) and (D) and closing under the rule,\(^6\)

\[
\frac{\phi \rightarrow \psi}{\square \phi \rightarrow \square \psi} \tag{RM}
\]

To borrow the terminological conventions of Chellas [14], **D2** is the smallest regular system of modal logic containing all instances of (D), i.e. **EMCD**.

By weakening or strengthening **D2**, other important deontic systems can be obtained. Dropping (K) yields Chellas’ ‘minimal deontic logic’ [14, p. 202]. The system known widely as ‘standard deontic logic’ is just **KD**, that is, **D2** closed under (NEC); or, if you prefer, **K** extended by (D).\(^7\)

Concerns about (NEC) notwithstanding, **KD** fares reasonably well under the deontic interpretation of the modal connectives. However, even if **KD** is adequate for analyzing unconditional obligation, it is is infamously inadequate for analyzing conditional obligation. It will help motivate what follows to examine why in some detail.

Under its deontic interpretation, \(\phi \square \rightarrow \psi\) is read: given \(\phi\), \(\psi\) is obligatory. It may be speculated that \(\square \rightarrow\) is definable using the resources of **KD** or some other deontic modal logic. However, a paradox due to Chisholm [16] suggests that this is not the case (my presentation of this paradox generally follows McNamara [83, §4.5]). Observe that the following four statements seem to be jointly consistent:

1. It is obligatory that (it be the case that) John go to the assistance of his neighbors

2. Given that John is going to the assistance of his neighbors, it is obligatory that (it be the case that) he tell them he is coming

3. If it is not the case that John is going to the assistance of his neighbors, it is obligatory that (it be the case that) he not tell them he is coming

\(^6\) (RM) is applicable only if the premise is a theorem.

\(^7\) For standard deontic logic, see Chellas [14] and McNamara [83].
4. John is not going to the assistance of his neighbors

The most natural ways of symbolizing (1), (3), and (4) are as □φ, ¬φ → □¬ψ, and ¬φ respectively. The issue is how to interpret (2). If φ □→ ψ ≡ □(φ → ψ), then (2) is interpreted, à la Chisholm [16, p. 35], as □(φ → ψ). By (K) and (1), this implies □ψ. (3) and (4) imply □¬ψ. □ψ and □¬ψ jointly implies □⊥, which contradicts (D).

Other symbolizations of the four claims are possible, but as McNamara [83] (cf. Chellas [14, p. 201]) points out, each of them engender significant problems. For example, if (2) and (3) are translated analogously, i.e. if the definition φ □→ ψ ≡ φ → □ψ is endorsed, then (2) simply follows from (4). But this is not plausible: it is false that there is rampant starvation in Ireland right now, but it does not follow from that that given rampant starvation in Ireland, it is obligatory that they increase food exports.

The upshot of this paradox is that no plausible definition of □→ can be given within deontic modal logic: all definitions result in inconsistency or other unreasonable consequences. Accordingly, the connective □→ must be taken as primitive. Therefore, a logic must be described for this connective, only after which can the details of its connections to the deontic box be examined. Before turning to this project in subsection 4.1.2, one desideratum on the connection between deontic modal and conditional logic should be addressed.

Von Wright [132], in discussing a logic for “relative” (i.e. conditional) permissibility and obligation, notes that the system, in a sense, includes his old system of unconditional deontic logic. More specifically, the proposed system of conditional obligation contains,

The old system of “absolute” permission, prohibition, and obligation by virtue of the fact that the laws, which hold in the old system, appear in the new system in the form of laws for permission, prohibition, and obligation under tautologous conditions (emphasis his) [132, p. 509]

In effect, von Wright’s (unary modal) logic of unconditional obligation is recoverable in his
(dyadic) conditional logic of obligation under definition (□Df. 2): □φ ≡ ⊤ □→ φ.

This definition shows up occasionally in the general conditional logic literature (see, e.g., Lowe [71, p. 360]), but frequently in the more specialized deontic conditional logic literature (see, e.g., Chellas [14, p. 275]), and with good reason. ⊤ □→ φ states that, given ⊤, φ is obligatory. But since ⊤ always obtains, φ is always obligatory. Therefore, ⊤ □→ φ states that φ is simply (i.e. unconditionally) obligatory. I take it to be a desideratum that for any conditional logic to qualify as a proper logic of conditional obligation, it must determine a plausible deontic modal logic under (□Df. 2).

## 4.1.2 Deontic Conditional Logic

I have argued that □→ is primitive; now, its logic must be investigated. I proceed by freely exploiting the definitions, rules, and axioms discussed in section 2.1.

Observe that the logic of conditional obligation must be classical in the sense of definition 3. Intuitively, this is because what determines an obligation (or what an obligation is determined by) is something deeper than a formula (cf. [14, p. 273]). Take (RCEA) as an example; if the state of affairs described by φ gives rise to the obligation described by χ, it should do so under any equivalent description (say, ψ). Therefore, the logic of conditional obligation must be closed under (RCEA) and (RCEC). Moreover, the logic of conditional obligation should be monotonic, i.e. contain all instances of (CM). If ψ ∧ θ is obligatory given φ, then each of ψ and θ are independently obligatory given φ. This much, at least, should be uncontroversial.

The status of (CC) is somewhat more controversial. Chellas [14, p. 274] does not include it among the axioms of his system of minimal conditional deontic logic. This, however, seems to be mainly because he wants to construct a close analogue of his minimal (unconditional)

---

8Note that in classical systems of conditional logic, (AT1) and (AT2) are equivalent, as are (WBT1) and (WBT2). I prove this in section 4.2.

9Equivalently, be closed under (RCM).
deontic logic, which rejects a comparable principle of regularity for □ [14, pp. 201-2].

If one were to generalize the reasons given there to the conditional case, the concern would seem to be that regular systems of conditional logic cannot distinguish between, on the one hand, controversial theses like (WBT1) and (WBT2), and on the other hand, the purportedly more innocuous:

\[
\neg(\phi \rightarrow \bot)
\]  

That these are indistinguishable in regular systems of conditional logic is indeed the case, as I verify in proposition 6 (cf. Weiss [126]):

**PROPOSITION 6.** Given a regular system of conditional logic \( L \), \( L \) contains all instances of (CD) if and only if \( L \) contains all instances of (WBT1)

If \( L \) contains (CD):

1. \( \phi \rightarrow \psi \)  
2. \( \neg(\phi \rightarrow \bot) \)  
3. \( \bot \leftrightarrow (\psi \land \neg \psi) \)  
4. \( (\phi \rightarrow \bot) \leftrightarrow (\phi \rightarrow (\psi \land \neg \psi)) \)  
5. \( \neg(\phi \rightarrow (\psi \land \neg \psi)) \)  
6. \( ((\phi \rightarrow \psi) \land (\phi \rightarrow \neg \psi)) \rightarrow (\phi \rightarrow (\psi \land \neg \psi)) \)  
7. \( \neg((\phi \rightarrow \psi) \land (\phi \rightarrow \neg \psi)) \)  
8. \( \neg(\phi \rightarrow \neg \psi) \)  
9. \( (\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg \psi) \)

If \( L \) contains (WBT1):

\footnote{In particular, it lacks the scheme (C): \( \square \phi \land \square \psi \rightarrow \square (\phi \land \psi) \).}
CHAPTER 4. CONNEXIVE CONDITIONAL LOGIC

1. $\phi \Box \rightarrow \bot$ Assumption
2. $\bot \leftrightarrow (\psi \land \neg \psi)$ ⊢
3. $(\phi \Box \rightarrow \bot) \leftrightarrow (\phi \Box \rightarrow (\psi \land \neg \psi))$ RCEC 2
4. $(\phi \Box \rightarrow (\psi \land \neg \psi)) \rightarrow ((\phi \Box \rightarrow \psi) \land (\phi \Box \rightarrow \neg \psi))$ CM
5. $(\phi \Box \rightarrow \psi) \land (\phi \Box \rightarrow \neg \psi)$ PL 1, 3, 4
6. $(\phi \Box \rightarrow \psi) \rightarrow \neg(\phi \Box \rightarrow \neg \psi)$ WBT1
7. $\bot$ PL 5, 6
8. $\neg(\phi \Box \rightarrow \bot)$ PL 1-7

Since Chellas does not consider (WBT1) or (WBT2) and explicitly disavows (CD) (which he calls (COD+)) at [14, pp. 273-4], this is clearly not his reason for rejecting (CC). In any case, the possibility of distinguishing these should be weighed as a factor of importance only if at least one of them is to be endorsed as a principle of conditional obligation. Since my own position is that (WBT1), (WBT2), and (CD) should all be endorsed as principles of conditional obligation, again, the inability to distinguish between these is unimportant. Therefore, I endorse (CC) and move on to considering defenses of weak Boethius’ theses and (CD).

Before I do so, however, some remarks are in order about conditional permissibility. Consider a binary connective $\Diamond \rightarrow$ such that $\phi \Diamond \rightarrow \psi$ is read: given $\phi$, $\psi$ is permissible. I take only $\Box \rightarrow$ as primitive and adopt the definition,\(^{11}\)

$$\phi \Diamond \rightarrow \psi \equiv \neg(\phi \Box \rightarrow \neg \psi) \quad (\Diamond \rightarrow \text{Df.})$$

It is clear that ($\Diamond \rightarrow \text{Df.}$) directly parallels the definition of (unconditional) permissibility in

\(^{11}\)Under the counterfactual interpretation of these connectives, this equivalence is of course famously endorsed by Lewis [67, p. 2]. It shows up under their deontic interpretation (in reverse) at least as far back as von Wright [132, p. 509].
deontic modal logics like $\textbf{KD}$. Now observe that, under $(\circarrow\text{Df.})$, (WBT1) is equivalent to the scheme: $(\phi \circarrow \psi) \rightarrow (\phi \diamondarrow \psi)$. That is, (WBT1) encapsulates the claim that what is obligatory given some condition is also permissible given that condition.

Before saying a few words in defense of (WBT1) as a principle of conditional obligation, the prevalence of it (and axioms equivalent to it) in the literature on deontic conditional logic deserves comment. Von Wright, who states it in terms of conditional permissibility, claims that its self-evidence “can hardly be disputed” [132, p. 509]. (WBT1) is also endorsed by van Fraassen [33] and features in all of the systems surveyed by Lewis [70]. Despite this, there is (to my knowledge) not a single comment in any major work of deontic logic on connexive logic, nor in connexive logic on deontic logic.

In any case, (WBT1) should be endorsed as a principle of conditional obligation. Since it seems highly plausible to regard the set of obligatory states as a subset of the set of permissible states, and more generally, the set of obligatory states conditional on $\phi$ as a subset of the set of permissible states conditional on $\phi$, (WBT1) (and for the same reasons (WBT2)) is unobjectionable. To deny it is to hold that there is some $\phi$ which occasions obligations that are impermissible on the same basis; I hold this to be incoherent.

Turning now to (CD), observe that it is the direct conditional analogue of (D). Whereas (D) represents the thesis that there are no impossible obligations (full stop), (CD) represents the claim that no circumstance gives rise to impossible obligations. So why does Chellas, who endorses (D), take issue with (CD)? His main concern seems to be that, given an impossible situation, there might be impossible obligations. For this reason, he holds that $\neg(\bot \circarrow \bot)$, an instance of (CD), should not be a theorem [14, p. 274].

Suppose, for the sake of argument, that $\bot \circarrow \bot$ were true. Then it easily follows that $\neg(\bot \diamondarrow \bot)$:
As I argued above, it is incoherent for there to be obligations which are impermissible. If \( \bot \) were obligatory given \( \bot \), then it would also be impermissible given \( \bot \) by the foregoing argument. Hence, \( \bot \rightarrow \bot \) cannot possibly be true. Consequently, (CD) should be endorsed without qualification.

One consequence of (CD) is that \( \bot \rightarrow \top \) is a theorem: given \( \bot \), \( \top \) is permissible. More generally, it might be held that, given \( \bot \), everything is permissible. This seems at least prima facie plausible; to appeal to authority, Lewis [70, pp. 9-10] takes this position, albeit somewhat arbitrarily. But if everything is permissible given \( \bot \), nothing is obligatory given \( \bot \). This last statement corresponds to the scheme:

\[
\neg(\bot \rightarrow \phi)
\]

(CAD)

Observe that (CD) and (CAD) yield, respectively, \( \neg(\neg \bot \rightarrow \bot) \) and \( \neg(\bot \rightarrow \neg \bot) \), special instances of (AT2) and (AT1). I endorse (CAD) as a principle of conditional obligation.

The principles of conditional obligation so far endorsed determine a weak regular system of conditional logic, the exact properties of which will be more precisely investigated in section 4.2. While this system is deontically plausible, most extensions of it by other axioms from table 2.2 are not.

Note, for example, that the logic of conditional obligation should not be normal (definition 3), i.e. not contain (CN). (CN) is deontically implausible because it mandates the
existence of (trivial) obligations in arbitrary circumstances. Surely, however, given that Amsterdam is in the Netherlands, it is not obligatory that I either read the Times or do not read the Times. (ID) is even more absurd: it is not the case that given that there is starvation, this ought to be so!

While some connexive principles are, as I have argued, eminently plausible on the deontic interpretation of $\Box \rightarrow$, others are not. Aristotle’s theses, for instance, are not (in general) deontically credible, for (AT1) would have it that:

Given that there is murder, it is obligatory that there not be murder

is false. Despite the deontic implausibility of Aristotle’s theses, these too can in fact be found in the literature on conditional obligation, though of course not named as such (see, e.g., T3 in van Fraassen [33, p. 422]). For my part, I hold that all proper logics of conditional obligation are only partially connexive.

### 4.2 Axiom Systems

In this section, I axiomatize several connexive (and half-connexive) systems of conditional logic. I divide these systems into two categories: deontic and non-deontic. I prove various results about systems of both types and show how to recapture deontic modal logics in systems of deontic conditional logic.

Before examining specific systems and results concerning them, it will be useful to formally define two types of connexive system and prove some general results for these.

**DEFINITION 29.** A system of conditional logic $\mathbf{L}$ is *half-connexive* if it contains all instances of (WBT1) and (WBT2). $\mathbf{L}$ is *connexive* if it contains all instances of (WBT1), (WBT2), (AT1), and (AT2).
The following results will help axiomatically characterize half-connexive and connexive systems of conditional logic. They should be compared with results of Unterhuber [121] and Weiss [126].

**PROPOSITION 7.** Every classical system of conditional logic containing all instances of (AT1) also contains all instances of (AT2)

\[
\begin{align*}
1 & \phi \leftrightarrow \neg \neg \phi \quad \vdash \\
2 & (\neg \phi \quad \rightarrow \quad \phi) \leftrightarrow (\neg \phi \quad \rightarrow \quad \neg \neg \phi) \quad \text{RCEC 1} \\
3 & \neg (\neg \phi \quad \rightarrow \quad \neg \neg \phi) \quad \text{AT1} \\
4 & \neg (\neg \phi \quad \rightarrow \quad \phi) \quad \text{PL 2, 3}
\end{align*}
\]

Note that the converse of proposition 7 also holds by an analogous proof.

**PROPOSITION 8.** Every classical system of conditional logic containing all instances of (WBT1) also contains all instances of (WBT2)

\[
\begin{align*}
1 & \phi \quad \rightarrow \quad \neg \psi \quad \text{Assumption} \\
2 & (\phi \quad \rightarrow \quad \neg \psi) \rightarrow \neg (\phi \quad \rightarrow \quad \neg \neg \psi) \quad \text{WBT1} \\
3 & \neg (\phi \quad \rightarrow \quad \neg \neg \psi) \quad \text{PL 1, 2} \\
4 & \psi \leftrightarrow \neg \neg \psi \quad \vdash \\
5 & (\phi \quad \rightarrow \quad \psi) \leftrightarrow (\phi \quad \rightarrow \quad \neg \neg \psi) \quad \text{RCEC 4} \\
6 & \neg (\phi \quad \rightarrow \quad \psi) \quad \text{PL 3, 5} \\
7 & (\phi \quad \rightarrow \quad \neg \psi) \rightarrow \neg (\phi \quad \rightarrow \quad \psi) \quad \text{PL 1-6}
\end{align*}
\]

Again, note that the converse of proposition 8 holds by an analogous proof.

**PROPOSITION 9.** Every regular system of conditional logic containing all instances of (AT1) also contains all instances of (WBT1)
CHAPTER 4. CONNEXIVE CONDITIONAL LOGIC

1  $\bot \leftrightarrow (\phi \land \neg \phi)$ ⊤

2  $(\phi \square \rightarrow (\phi \land \neg \phi)) \rightarrow ((\phi \square \rightarrow \phi) \land (\phi \square \rightarrow \neg \phi))$ CM

3  $(\phi \square \rightarrow \bot) \leftrightarrow (\phi \square \rightarrow (\phi \land \neg \phi))$ RCEC 1

4  $(\phi \square \rightarrow \bot) \rightarrow ((\phi \square \rightarrow \phi) \land (\phi \square \rightarrow \neg \phi))$ PL 2, 3

5  $(\phi \square \rightarrow \bot) \rightarrow (\phi \square \rightarrow \neg \phi)$ PL 4

6  $\bot \leftrightarrow (\psi \land \neg \psi)$ ⊤

7  $(\phi \square \rightarrow \bot) \leftrightarrow (\phi \square \rightarrow (\psi \land \neg \psi))$ RCEC 6

8  $\neg (\phi \square \rightarrow \neg \phi)$ AT1

9  $\neg (\phi \square \rightarrow \bot)$ PL 5, 8

10  $\neg (\phi \square \rightarrow (\psi \land \neg \psi))$ PL 7, 9

11  $((\phi \square \rightarrow \psi) \land (\phi \square \rightarrow \neg \psi)) \rightarrow (\phi \square \rightarrow (\psi \land \neg \psi))$ CC

12  $\neg ((\phi \square \rightarrow \psi) \land (\phi \square \rightarrow \neg \psi))$ PL 10, 11

13  $(\phi \square \rightarrow \psi) \rightarrow \neg (\phi \square \rightarrow \neg \psi)$ PL 12

It is important to note that the converse of proposition 9 does not hold. This is a corollary of some results obtained using semantic methods in section 4.3.

In the previous section, I noted that (CN) and (ID) are deontically implausible. In fact, however, these schemes cannot generally be consistently combined with connexive principles. More specifically, note the following limitations:

PROPOSITION 10. Every normal connexive system of conditional logic is inconsistent

1  $\neg \bot$ ⊤

2  $\bot \square \rightarrow \neg \bot$ RCK 1

3  $\neg (\bot \square \rightarrow \neg \bot)$ AT1

4  $\bot$ PL 2, 3
**PROPOSITION 11.** Every monotonic half-connexive system of conditional logic containing (ID) is inconsistent

Every monotonic half-connexive system of conditional logic contains all instances of (CD) as theorems (see the second half of the proof of proposition 6). Therefore, if \( L \) is a monotonic half-connexive system of conditional logic, since by (CD) \( \vdash_L \neg(\bot \rightarrow \bot) \) and by (ID) \( \vdash_L \bot \rightarrow \bot \), \( L \) is inconsistent.

\[ \square \]

### 4.2.1 Deontic Connexive Systems

I argued in subsection 4.1.2 that (WBT1) and (CD) are principles of conditional obligation, but that (AT1) is not. Consequently, deontic systems of conditional logic are half-connexive, but not (fully) connexive. In this subsection, I examine two half-connexive systems with plausible deontic interpretations.

**DEFINITION 30.** A system of conditional logic \( L \) is **deontic** if it is regular and half-connexive, but not (fully) connexive. Moreover, \( L \) must determine a deontic modal logic under (□Df. 2).

It is immediate from definition 30 that all deontic systems of conditional logic extend the basic regular system of conditional logic \( CR \) (recall table 2.3). While some philosophers may advocate for weaker systems (see the discussion in subsection 4.1.2), I have found no good reason to reject (CC).

To axiomatize the systems I will be interested in, one more axiom, not yet encountered, is needed:

\[ \top \rightarrow \top \]  \hspace{1cm} (CN*)

(CN*) should be carefully distinguished from (CN). (CN*) is equivalent in any classical system of conditional logic to rule (RC3) from van Fraassen [33, p. 421] and rule (RN) from
Lowe [71, p. 360]. It corresponds to the claim that what is necessary is obligatory given what is necessary.

Table 4.1: Deontic Systems of Conditional Logic

<table>
<thead>
<tr>
<th>System</th>
<th>Condition</th>
<th>Deontic</th>
<th>Logics</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRD</td>
<td>CR, (CD), (CAD)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CND</td>
<td>CRD, (CN*)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Observation. By proposition 6, CRD and CND contain all instances of (WBT1). By proposition 8, CRD and CND are half-connexive.

CRD is what I take to be the correct logic of conditional obligation: it is the system corresponding to all of the principles of conditional obligation endorsed in subsection 4.1.2. CND is a slight strengthening of it, important mainly due to its connection to the deontic modal logic KD (more on this anon).

DEFINITION 31. Let $\tau' : \Phi \rightarrow \Phi_c$ be the same function as $\tau$ from definition 4 except:

3. $\tau'(\phi \square \rightarrow \psi) = \tau'(\phi) \land \tau'(\psi)$

THEOREM 20 (Consistency). Each of the systems in table 4.1 is consistent

The proof is essentially as in that of theorem 2, but using the function $\tau'$ from definition 31 (cf. Lowe [71, p. 360]).

The proof that the systems in table 4.1 are deontic, in the sense of definition 30, requires showing that (AT1) is independent of them. I show this using semantic methods in section 4.3. Here, I verify that each of these systems in fact determines a deontic modal logic. I show that CRD determines Lemmon’s D2, and then indicate how the result can be extended to show that CND determines KD.
**DEFINITION 32.** Let $\sigma' : \Phi_\Box \to \Phi$ be the same function as $\sigma$ from definition 5 except:

4. $\sigma'(\Box \phi) = \top \rightarrow \sigma'(\phi)$

**LEMMA 25.** If $\vdash_{D2} \phi$, then $\vdash_{\text{CRD}} \sigma'(\phi)$

The result follows by induction on the length of proof. For (D), note that $\sigma'(-\Box \bot) = \neg(\top \rightarrow \sigma'(\bot))$, which can be obtained from (CD). For (RM), suppose that $\vdash_{\text{CRD}} \sigma'(\phi \to \psi) = \sigma'(\phi) \to \sigma'(\psi)$. Then it must be shown that $\sigma'(\Box \phi \to \Box \psi) = (\top \rightarrow \sigma'(\phi)) \rightarrow (\top \rightarrow \sigma'(\psi))$ is provable in CRD; this follows immediately from the assumption and (RCM).

**DEFINITION 33.** Let $\sigma'^{-1} : \Phi \to \Phi_\Box$ be the same function as $\sigma^{-1}$ from definition 6 except:

4. $\sigma'^{-1}(\phi \rightarrow \psi) = \Box(\sigma'^{-1}(\phi) \land \sigma'^{-1}(\psi))$

**LEMMA 26.** If $\vdash_{\text{CRD}} \phi$, then $\vdash_{D2} \sigma'^{-1}(\phi)$

Again, the result is obtained by induction on the length of proof. As a representative case, consider (CD): $\sigma'^{-1}(\neg(\phi \rightarrow \bot)) = \neg(\Box(\sigma'^{-1}(\phi) \land \sigma'^{-1}(\bot)))$, which is equivalent to $\neg(\Box(\sigma'^{-1}(\phi) \land \bot))$. The proof of this in D2 is easy:

1. $\neg \Box \bot$ D
2. $(\sigma'^{-1}(\phi) \land \bot) \rightarrow \bot$ ⊢
3. $\Box(\sigma'^{-1}(\phi) \land \bot) \rightarrow \Box \bot$ RM 2
4. $\neg \Box(\sigma'^{-1}(\phi) \land \bot)$ PL 1, 3

**LEMMA 27.** $\vdash_{D2} \phi \leftrightarrow \sigma'^{-1}(\sigma'(\phi))$

The proof is by induction on the complexity of $\phi$. Consider the case where $\phi$ is of the form $\Box \psi$; the induction hypothesis is that $\vdash_{D2} \psi \leftrightarrow \sigma'^{-1}(\sigma'(\psi))$. Then: $\sigma'^{-1}(\sigma'(\Box \psi)) = \sigma'^{-1}(\top \rightarrow \neg(\Box \bot))$
\[ \sigma'(\psi) = \Box(\sigma'^{-1}(\top) \land \sigma'^{-1}(\sigma'(\psi))) \], which (by the induction hypothesis) is equivalent to 
\[ \Box(\top \land \psi) \]. It must be shown that \( \vdash_{D_2} \Box \psi \leftrightarrow \Box(\top \land \psi) \). But this is easily proved:

1. \( \psi \rightarrow (\top \land \psi) \) \( \vdash \)
2. \( \Box \psi \rightarrow \Box(\top \land \psi) \) \( \text{RM 2} \)
3. \( (\top \land \psi) \rightarrow \psi \) \( \vdash \)
4. \( \Box(\top \land \psi) \rightarrow \Box \psi \) \( \text{RM 3} \)
5. \( \Box \psi \leftrightarrow \Box(\top \land \psi) \) \( \text{PL 2, 4} \)

**THEOREM 21** (Modal Embedding). \( \vdash_{D_2} \phi \) if and only if \( \vdash_{CRD} \sigma'(\phi) \)

The result follows immediately from lemmata 25, 26, and 27.

To show that \( \vdash_{KD} \phi \) if and only if \( \vdash_{CND} \sigma'(\phi) \), only one more case must be examined in lemmata 25 and 26. For the first, since \( \vdash_{CND} \sigma'(\Box \top) = \top \rightarrow \top \), it is clear that (NEC) will present no problems. For the second, since \( \vdash_{KD} \Box(\top \land \top) \), it is clear that (CN*) is provable under \( \sigma'^{-1} \).

### 4.2.2 Non-Deontic Connexive Systems

In this subsection, I discuss connexive and half-connexive systems without plausible deontic interpretations. Of these, special interest attaches to two systems of Lowe [71], which are intended to formalize nonmaterial natural language conditionals.\(^{12}\)

In addition to the schemes that I discussed above, two more must be introduced:

\[
((\phi \rightarrow \psi) \land (\psi \rightarrow \theta)) \rightarrow (\phi \rightarrow \theta)
\] (CT)

\(^{12}\)Lowe [71, p. 357] does not (logically) distinguish between indicative/non-counterfactual and subjunctive/counterfactual conditionals. Though it will not be a concern of this chapter, Lowe's ultimate project is to reduce conditional logic to modal logic. Interestingly, Angell’s **PA1** admits of such a reduction [3].
CHAPTER 4. CONNEXIVE CONDITIONAL LOGIC

\[
((\phi \rightarrow \psi) \lor (\top \rightarrow \neg \phi)) \leftrightarrow (\top \rightarrow (\phi \rightarrow \psi)) \quad (\text{CL})
\]

Each of these correspond to axioms from Lowe [71] (D1.7 and D2.9, respectively).

The systems that I will be concerned with for the remainder of this section are listed in table 4.2.

Table 4.2: Non-Deontic Systems of Connexive Conditional Logic

<table>
<thead>
<tr>
<th>CX</th>
<th>CR, (AT1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LD1</td>
<td>CR, (WBT1), (CMP), (CT)</td>
</tr>
<tr>
<td>LD2</td>
<td>LD1, (CN*), (CL)</td>
</tr>
</tbody>
</table>

Observation. By propositions 7, 8, and 9, CX is (fully) connexive. By proposition 8, LD1 and LD2 are half-connexive.

CX is the smallest (fully) connexive regular system of conditional logic. LD1 and LD2 are the systems Lowe [71] calls D1 and D2 (to avoid confusion with the Lemmon systems, I have changed their names). By proposition 6, LD1 (LD2) could be equivalently axiomatized using (CD).

**THEOREM 22** (Consistency). *Each of the systems in table 4.2 is consistent*

The proof is exactly as in the proof of theorem 20. The result for LD1 and LD2 is due to Lowe [71, p. 360-1].

\[\square\]

It is clear that none of the systems from table 4.2 are deontically plausible. CX is not a plausible deontic logic due to its inclusion of (AT1). LD1 and LD2 are deontically implausible due to their inclusion of (CMP).\(^{13}\) None of these systems really seem satisfactory

\(^{13}\)Here's a deontic counterexample to (CMP): from “given that there is global warming, it is obligatory that we reduce carbon emissions” and “there is global warming,” “we are reducing carbon emissions” does not follow.
as an account of any sort of natural language conditional either (pace Lowe). Observe that all of these systems contain (CD) as a theorem; but surely, it is natural to hold that if \( \bot \) were the case, then \( \bot \) would be the case. Therefore, it remains opaque what philosophical significance these systems have, if any.

It is worth emphasizing that \( \text{LD2} \) is a particularly unusual system of conditional logic, in that the conditional \( \Box \rightarrow \) is supposed to be \textit{stronger} than the strict conditional, which it is when \( \Box \) is defined in accordance with (\( \Box \text{Df. 2} \)).\(^{14}\) Indeed, under (\( \Box \text{Df. 2} \)), it follows straightforwardly from (CL) that \( \vdash_{\text{LD2}} (\phi \rightarrow \psi) \rightarrow \Box(\phi \rightarrow \psi) \). Again, it is hard to see what philosophical sense this makes.

### 4.3 Semantics

In this section, I semantically investigate the systems from tables 4.1 and 4.2. Determination results for each of these systems with respect to their semantics are proved. Finally, I prove some independence results which establish that the systems from table 4.1 satisfy a constraint from definition 30.

#### 4.3.1 Proposition Indexed Interpretations

Since all of the systems listed in tables 4.1 and 4.2 are classical, the semantic framework from subsection 2.2.1 can be utilized. The reader should recall the definition of a proposition indexed interpretation (definition 7) and the constraints listed in table 2.5. Table 4.3 lists additional constraints that may be imposed on an interpretation \( \mathfrak{J} = \langle W, P, \{ f_{X} : X \in P \}, V \rangle \).

\(^{14}\)Lowe [71, p. 363] both acknowledges and endorses this feature.
### Table 4.3: Deontic and Connexive Function Constraints

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>∅ ∉ ( f_X(w) )</td>
<td>(cd)</td>
</tr>
<tr>
<td>( f_{\emptyset}(w) = \emptyset )</td>
<td>(cad)</td>
</tr>
<tr>
<td>If ( Y \in f_X(w) ), then ( -Y \not\in f_X(w) )</td>
<td>(wbt1)</td>
</tr>
<tr>
<td>( -X \not\in f_X(w) )</td>
<td>(at1)</td>
</tr>
<tr>
<td>If ( S \in f_X(w) ) and ( T \in f_S(w) ), then ( T \in f_X(w) )</td>
<td>(ct)</td>
</tr>
<tr>
<td>( W \in f_W(w) )</td>
<td>(cn(^*))</td>
</tr>
<tr>
<td>((S \in f_X(w) \text{ or } -X \in f_W(w)) \iff (-X \cup S) \in f_W(w))</td>
<td>(cl)</td>
</tr>
</tbody>
</table>

The semantic constraints listed in table 4.3 correspond to the axioms discussed in this chapter in the obvious way. I adopt the same conventions here as in chapter 2 concerning the naming of classes of proposition indexed interpretations. Thus, for example, \( C^{CRD} \) is the class of proposition indexed interpretations in which \( f_X \) satisfies (cm), (cc), (cd), and (cad).

**PROPOSITION 12.** Given a proposition indexed interpretation \( \mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle \) which satisfies (cm) and (cc), \( \mathcal{I} \) satisfies (cd) if and only if it satisfies (wbt1).

It is obvious that this is the semantic counterpart of proposition 6. Consider an interpretation \( \mathcal{I} = \langle W, P, \{f_X : X \in P\}, V \rangle \) satisfying the hypothesis of the proposition. Suppose \( \mathcal{I} \) satisfies (cd) and \( Y \in f_X(w) \); if \( -Y \in f_X(w) \), then by (cc), \( Y \cap -Y = \emptyset \in f_X(w) \), which is impossible. Thus, \( -Y \not\in f_X(w) \), and \( \mathcal{I} \) satisfies (wbt1). Conversely, suppose that \( \mathcal{I} \) satisfies (wbt1); if \( \emptyset \in f_X(w) \), then \( Y \cap -Y \in f_X(w) \), so by (cm), \( Y \in f_X(w) \) and \( -Y \in f_X(w) \), which is impossible. Therefore, \( \mathcal{I} \) satisfies (cd).
Soundness and completeness proofs for the systems from tables 4.1 and 4.2 with respect to the semantics sketched above are just extensions of the proofs given in subsection 2.2.3. For what follows, let \( L \) be any system from table 4.1 or 4.2.

**THEOREM 23** (Soundness). \( \Sigma \vdash_L \phi \) implies \( \Sigma \models_L \phi \)

I take (CD), (AT1), and (CT) as representative cases. If \( L \) contains (CD), then for any \( \mathcal{I} = \langle W, P, \{ f_X : X \in P \}, V \rangle \in \mathcal{C}_L \), \( f_X \) satisfies (cd). Take an arbitrary world \( w \) of an arbitrary \( \mathcal{I} \in \mathcal{C}_L \). Then by (cd), \( [\bot] = \emptyset \not\in f_{[\phi]}(w) \). Therefore, since \( \not\models^3_w \phi \rightarrow \bot \), it follows that \( \models^3_w \neg (\phi \rightarrow \bot) \), as desired. If \( L \) contains (AT1), then for any \( \mathcal{I} \in \mathcal{C}_L \), \( f_X \) satisfies (at1). Consequently, given an arbitrary world \( w \) of an arbitrary interpretation \( \mathcal{I} \in \mathcal{C}_L \), by (at1), \( -[\phi] = [\neg \phi] \not\in f_{[\phi]}(w) \). Therefore, \( \models^3_w \neg (\phi \rightarrow \neg \phi) \), as desired. Finally, if \( L \) contains (CT), take an arbitrary world \( w \) of an arbitrary interpretation \( \mathcal{I} \in \mathcal{C}_L \) such that \( \models^3_w (\phi \rightarrow \psi) \land (\psi \rightarrow \theta) \). Since \( [\psi] \in f_{[\phi]}(w) \) and \( [\theta] \in f_{[\psi]}(w) \), by (ct), \( [\theta] \in f_{[\phi]}(w) \), whence \( \models^3_w (\phi \rightarrow \theta) \), as desired.

\[ \Box \]

Canonical models for these systems are defined as in definition 15 and the truth lemma (lemma 14) goes through as before. For completeness, it simply remains to verify that the canonical model for each system is in the appropriate class of interpretations. Recall that \( [\phi]^L = \{ \Gamma \subseteq \Phi : \Gamma \text{ is maximally } L \text{ consistent}, \phi \in \Gamma \} \) (again, superscripts are suppressed except if needed to disambiguate).

**LEMMA 28.** Let \( \mathcal{J}^L \) be the canonical model for \( L \). Then \( \mathcal{J}^L \in \mathcal{C}_L \)

I take (cad) and (cl) as representative cases. For (cad), suppose that \( L \) contains (CAD) and take some arbitrary \( w \in W \). By (CAD) and the maximal consistency of \( w \), \( f_\emptyset(w) = f_{[\bot]}(w) = \{ [\psi] \in P : (\bot \rightarrow \psi) \in w \} = \emptyset \), as desired. For (cl), suppose that \( L \) contains (CL) and take some arbitrary \( w \in W \) and \( [\phi], [\psi] \in P \). \( -[\phi] \cup [\psi] = [\phi \rightarrow \psi] \in f_W(w) = f_{[\top]}(w) \) if and only if \( \top \rightarrow (\phi \rightarrow \psi) \in w \) if and only if (by \( \text{PL} \) and (CL)) \( (\phi \rightarrow \psi) \lor (\top \rightarrow \psi) \rightarrow \psi \in w \).


\[ \neg \phi \] \in w \text{ if and only if either } \lceil \psi \rceil \in f_{\lceil \phi \rceil}(w) \text{ or } \neg \lceil \phi \rceil = \lceil \neg \phi \rceil \in f_{\lceil \top \rceil}(w) = f_{W}(w), \text{ which was to be proved.} \]

\[ \square \]

**THEOREM 24** (Completeness). \( \Sigma \models_{cl} \phi \) implies \( \Sigma \vdash_{L} \phi \)

The result follows from lemma 28 as in the proof of theorem 8.

\[ \square \]

### 4.3.2 Conditional Algebraic Interpretations

Algebraic semantics can easily be given for the systems from tables 4.1 and 4.2 (the reader should recall the discussion of such semantics from subsection 2.2.2).\(^{15}\) For the sake of establishing independence results, the algebraic semantics is to be preferred. Table 4.4 lists a number of constraints (in addition to those listed in table 2.6) that might be imposed on a conditional algebraic interpretation \( I_{*} = \langle A, g \rangle \).\(^{16}\)

---

\(^{15}\)I note that Pizzi [95] used algebraic semantics to characterize a connexive conditional logic, albeit of a kind rather different than those examined above. See also Weiss [126].

\(^{16}\)These correspond in the obvious way to the constraints listed in table 4.3.
Table 4.4: Deontic and Connexive Algebraic Constraints

\[
\begin{align*}
  a \ast 0 &= 0 & (cd\ast) \\
  0 \ast a &= 0 & (cad\ast) \\
  a \ast b &\leq -(a \ast -b) & (wbt1\ast) \\
  a \ast -a &= 0 & (at1\ast) \\
  (a \ast b) \cap (b \ast c) &\leq (a \ast c) & (ct\ast) \\
  1 \ast 1 &= 1 & (cn^\ast \ast) \\
  (a \ast b) \cup (1 \ast -a) &= 1 \ast (-a \cup b) & (cl\ast)
\end{align*}
\]

I follow the established convention of denoting classes of conditional algebraic interpretations using the \( \ast \) subscript. Observe that the equivalence theorem (theorem 6) can be extended and (weak) soundness and completeness results can be obtained for the systems from tables 4.1 and 4.2 with respect to classes of conditional algebraic interpretations using the results of subsection 4.3.1 (cf. corollaries 1 and 2). However, since the main concern of this section is with independence results, I will simply sketch soundness proofs and display pertinent countermodels.

**THEOREM 25** (Soundness). For \( \mathbf{L} \) from tables 4.1 and 4.2, \( \vdash \mathbf{L} \phi \) implies \( \models_{C^L} \phi \)

I take (CAD) and (AT1) as representative cases. If \( \mathbf{L} \) contains (CAD), then take an arbitrary \( \mathfrak{I}_* \in \mathcal{C}^\mathbf{L}_{\ast} \). Since \( 0 \ast g(\phi) = 0 \) by (cad\ast), it follows that \( g(\bot \square \phi) = 0 \). Therefore, \( \models^{3\ast} \neg(\bot \square \phi) \), as desired. For (AT1), let \( \mathfrak{I}_* \in \mathcal{C}^\mathbf{L}_{\ast} \) be arbitrary and note that \( g(\neg(\phi \square \neg\phi)) = \neg(g(\phi) \ast -g(\phi)) = -0 = 1 \) by (at1\ast). Therefore, \( \models^{3\ast} \neg(\phi \square \neg\phi) \), as desired.

\[ \square \]
I now complete the demonstration that \textit{CRD} and \textit{CND} are deontic in the sense of definition 30. I do this by producing algebraic countermodels to (AT1) and (AT2). To obtain these countermodels, I used the automated theorem proving program Vampire [123].

\textbf{PROPOSITION 13. (AT1) and (AT2) are independent of CRD}

Since (AT1) and (AT2) are equivalent in \textit{CRD} (see proposition 7), it suffices to show that (AT1) is independent. Consider the four element boolean algebra \(\mathcal{B}\) based on the set \(\mathbb{B} = \{0, a, b, 1\}\) with Hasse diagram \((-a = b, \text{ etc.})\):

Now consider the conditional algebra \(\mathcal{A}\) based on \(\mathcal{B}\) with \(*\) given by the following table:

\[
\begin{array}{c|cccc}
  & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & a & a \\
b & 0 & 0 & 0 & b \\
1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

It is tedious, though not hard, to verify that \(*\) satisfies all of the conditions imposed on \(\mathcal{C}^\text{CRD}_*\). Now consider a conditional algebraic interpretation \(\mathcal{I}_* = \langle \mathcal{A}, g \rangle\) in which \(g(p) = a\). Then \(g(\neg(p \supset \neg p)) = -(g(p) * g(p)) = -(a * -a) = -a = b\). Therefore, by theorem 25, \(\not\vdash_{\text{CRD}} \neg(\phi \supset \neg \phi)\).

\(\square\)

\textbf{PROPOSITION 14. (AT1) and (AT2) are independent of CND}

The proof proceeds just as before, but the operation \(*\) is now defined by the table:
Consider a conditional algebraic interpretation $I = (\mathcal{A}, g)$ in which $g(p) = a$. Then, once more, $g(\neg(\square p \rightarrow \neg p)) = b$. Consequently, $\not\vdash_{\text{CND}} \neg(\phi \rightarrow \neg \phi)$.
Chapter 5

Counterpossible Logic

This chapter deals with *counterpossibles*—counterfactuals with impossible antecedents. Such conditionals arise frequently in ordinary discourse, as well as mathematical and philosophical discourse. I take all of the following to be examples of counterpossibles:

1. If Tully weren’t Cicero, Cicero wouldn’t have been Tully
2. If Hippasus had discovered that the square-root of two is rational, he would not have drowned at sea
3. Were there a counterexample to the law of the excluded middle, Brouwer would have been wrong to mistrust that law

As the range of these examples indicates, finer distinctions could be drawn within the genus of counterpossibles. The first could be classified as a countermetaphysical, the middle as a countermathematical, and the last as a counterlogical. Whatever the benefits of such a taxonomy, it won’t concern me here.

---

1. In recent work on counteridenticals, Kocurek [54] has defended contingent identity and, consequently, would deny that the Cicero conditional is really a counterpossible. I have implicitly assumed, with Kripke [61], that identities are (metaphysically) necessary if true at all.
2. The class of counterlogicals is an exception to this rule, for reasons that will become clear in section 5.1.
CHAPTER 5. COUNTERPOSSIBLE LOGIC

What will concern me is classifying counterpossibles according to their truth or falsity. I claim that there are true counterpossibles (for example the first) and that there are false counterpossibles (for example the last). A venerable tradition, including Lewis [67], Stalnaker [116], and Williamson [128, 129, 130], holds that all counterpossibles are *vacuously* true. I argue that this tradition is mistaken and subsequently develop counterpossible tolerant conditional logics, or simply, counterpossible logics.

Prominent arguments for and against “vacuousness” are discussed in section 5.1. I claim that the arguments against vacuous truth for counterpossibles are decisive. In subsection 5.1.2, I explore a semantic framework for handling counterpossibles nontrivially. Systems of counterpossible logic are examined in section 5.2. Finally, determination results are given in section 5.3.

5.1 Vacuousness and Counterpossibles

This section discusses and weighs in on the debate between proponents and opponents of vacuousness for counterpossibles. Before diving into the specifics, however, I should clarify exactly how vacuousness arises in standard treatments of counterfactuals.

First, it should be observed that the problem of vacuousness (to the extent that it is a problem) is really fundamentally a semantic one. Vacuousness appears in proof theory only if certain counterlogicals are being considered:

**PROPOSITION 15.** If $L$ is a monotonic system of conditional logic containing all instances of (ID), then $\vdash_L \bot \rightarrow \phi$

1. $\bot \rightarrow \phi$  $\vdash$
2. $(\bot \rightarrow \bot) \rightarrow (\bot \rightarrow \phi)$  RCM 1
3. $\bot \rightarrow \bot$  ID
4. $\bot \rightarrow \phi$  PL 2, 3
While proposition 15 may not seem terribly illuminating, it does gesture at how vacuousness arises in standard semantic treatments of counterfactuals: it has something to do with the conditions required for (CM) and (ID).\footnote{But note that the condition required for (CM) is obscured in most semantic treatments and the condition required for (ID) is obscured in many.}

Now consider the uniform semantics from subsection 2.2.1 and take some arbitrary world $w$ of some arbitrary interpretation $\mathcal{J} = \langle W, P, \{ f_X : X \in P \}, V \rangle$ for which $f_X$ satisfies (cm) and (id). Let $\phi$ be impossible (in whatever sense you like); then $[\phi]$, the set of worlds where $\phi$ is true, is empty. By (id), $[\phi] = \emptyset \in f_\phi(w) = f_{[\phi]}(w)$. Take some arbitrary $\psi$; then $[\psi] \in P$ and, since $\emptyset = \emptyset \cap [\psi] \in f_{[\phi]}(w)$, $[\psi] \in f_{[\phi]}(w)$ by (cm). Therefore, $\models^w \phi \rightarrow \psi$.

While the uniform neighborhood-type semantics shows how the problem arises for any counterpossible given minimal background assumptions, it doesn’t really provide a clear sense of the issue. A more intuitive picture can be obtained by considering formula indexed interpretations (see subsection 2.3.1). Recall that the intuitive idea behind this semantics is that counterfactuals are evaluated by looking at worlds that are ceteris paribus the same as a given world except that the antecedent holds. This guiding idea essentially requires the addition of a semantic constraint corresponding to (ID):

$$f_\phi(w) \subseteq [\phi]$$

(id)

Take some arbitrary world $w$ of an arbitrary formula indexed interpretation $\mathcal{I}_\Phi = \langle W, \{ f_\phi : \phi \in \Phi \}, V \rangle$ such that $f_\phi$ satisfies (id). If $\phi$ is impossible, then by (id), $f_\phi(w) \subseteq [\phi] = \emptyset$. Since $\emptyset \subseteq [\psi]$ for arbitrary $\psi$, $\models^w_\Phi \phi \rightarrow \psi$.

Finally, the problem can be considered, as it most often is, using either variant of Lewis semantics. Since there is not, however, any technical mystery about where the vacuousness comes from here—it’s built into the truth conditions (see subsection 2.3.2)—I will not belabor
So much for where vacuousness comes from. Why does it matter? It matters because, arguably, it’s both the case that counterpossibles satisfy properties like (ID) and that not all counterpossibles are true. If this is so, then the standard semantic treatments of counterfactuals must be reexamined so as to accommodate counterpossibles. The now well-established way to go is to augment the worlds machinery with impossible worlds; then counterpossibles get evaluated at these worlds, worlds at which their antecedents can obtain.4 But before doing all that, I first must show that not all counterpossibles are true. I turn to this issue now.

5.1.1 The Case against Vacuousness

At the beginning of this chapter, I gave an example of a counterpossible that seems to me to be intuitively false: “Were there a counterexample to the law of the excluded middle, Brouwer would have been wrong to mistrust that law.” Why do I think this conditional is false? In thinking about it, I imagine scenarios in which there is a counterexample to the law of the excluded middle—say, neither the continuum hypothesis nor its negation are true—but everything else is basically as is. In particular, Brouwer still holds his skeptical views about it and is, I take it, not wrong to mistrust it. Thus, I think the conditional is false. The fundamental argument against vacuousness is simply this: many counterpossibles are intuitively false.

The advocate of vacuousness has two basic options for responding to this claim. The first is to deny that there are such intuitions. The second is to concede that such intuitions exist, but reject them as unsound.5 Lewis [67] is an instance of the first approach while

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4For some recent work in line with this approach, see Berto et al. [8] and Weiss [124].

5What does one imagine when one envisions a world satisfying an impossible antecedent—say, that Hesperus is not Phosphorus? One might envision a situation in which there are two distinct celestial objects satisfying many of the observational properties which Venus in fact satisfies (perhaps they are even called by the names ‘Hesperus’ and ‘Phosphorus’). Nevertheless, according to Kripke [61, p. 102], this is not a situation
CHAPTER 5. COUNTERPOSSIBLE LOGIC

Williamson [130] is an instance of the second.

Lewis [67] notes that at least some counterpossibles must be true since they are used and asserted in proofs by contradiction.\(^6\) Why, then, should they all be vacuously true? Apparently because there is no inclination to ever assert the negation of a counterpossible [67, p. 25]. This is an empirical claim: people do not want to assert that counterpossibles are false. However, experimental data collected by Ripley [108] suggests that this is not so. Ripley found that (to take one example) the conditional “If Stephen Curry had been both exactly five feet tall and exactly six feet tall, then ants would have had ten legs” was, on average, rated between probably false and definitely false. Therefore, it should be conceded that non-vacuous intuitions exist.\(^7\)

One possible deflationary explanation of these intuitions goes as follows. Consider the following variation on an intuitively false counterpossible from Nolan [90, p. 544]:

If Hobbes had squared the circle, then sick children in the Andes would have been impressed

in which Hesperus is not Phosphorus, and to describe it as such is to misdescribe it. More generally, then, the intuitions against vacuousness might be unsound because, in envisioning a counterpossible situation where the consequent does not intuitively hold, one might be envisioning the “wrong scenario” and, consequently, draw the wrong conclusions. I think this sort of skepticism is a general worry for counterfactuals (when I imagine what is consequent upon my throwing the rock at the glass, am I imagining the right scenarios?), so I will only offer two remarks in response here. First, the concern is epistemic in nature: if one regularly draws conclusions about the truth or falsity of counterpossibles based on faulty imaginings, this may undermine one’s justification for saying whether any counterpossible is false, but it would not show that none are false. However, other reasons (e.g. theoretical reasons) might still militate in favor of some counterpossibles being false. Second, I think this sort of skepticism is unwarranted against counterpossibles primarily concerned with logical facts, e.g. “If intuitionistic logic were correct, then the law of the excluded middle would be valid.” Surely we are in a position to say that conditionals like this are false (you might mis-imagine an intuitionistic world in all sorts of ways but you don’t even need to imagine such worlds to see that this conditional is false). Of course, one might take this to really be expressing a fact about a logical consequence relation, but the burden is on the skeptic to say why that fact can’t be expressed using a counterpossible.\(^8\)

Or rather, even if counterpossibles are not used in such mathematical reasoning per se, they are consequent upon it (cf. Cohen [17, p. 92]). Thus, were the counterpossible summarizing such a reductio false, the corresponding implication would intuitively be false as well.

It may be objected that there is an important difference between asserting that something is false and rating something as false (as part of a questionnaire, say). Consequently, it may be held that this data says little about how people actually use language, which is what really matters. Later, in considering a case from Jenny [52, 51], I will argue that there are examples of counterpossibles which are rejected in more natural contexts.
I think the intuition that this is false is rather strong. However, it may be claimed that in judging this conditional to be false, speakers are not really reacting to it, but rather to the more general conditional:

If Hobbes had proved X, then sick children in the Andes would have been impressed

This latter conditional can be evaluated using possible worlds (even the actual world, assuming Hobbes succeeded in proving something) and is, of course, false. The objection implicit in this explanation is clear: in evaluating counterpossibles, speakers sometimes react to a distinct but related conditional which is not a counterpossible, and therefore is possibly false. Thus, speaker’s intuitions can go awry.

An explanation like this has some prima facie appeal for the counterpossible concerning Hobbes, but is inadequate as a general explanation of non-vacuous intuitions. It is hard to discern what alternative conditional speakers might be reacting to in judging the counterpossible about Brouwer false, as I suspect most would. Moreover, just about any candidate for a related conditional would still be an intuitively false counterpossible (“If there were a counterexample to logical law X, then…”). Therefore, I do not think that a response such as this has much force against non-vacuous intuitions generally.\(^8\)

However, it is also not clear that speakers react to the counterpossible about Hobbes in the way this explanation suggests. The base hypothesis must be that speakers, when assessing a given conditional, assess that conditional, as opposed to some other one. It may be that the base hypothesis should be rejected, but this should not be done without good reason. The fact that the standard semantics for counterfactuals—which has much to recommend it—endorses vacuousness is not a good reason to reject the base hypothesis: it seems to be little more than a historical accident (one which is rapidly being corrected to

\(^8\)Its bears emphasis that the same point can be made using counterpossibles other than counterlogicals, e.g. “If Hobbes had disproved Euclid’s theorem, Wallis wouldn’t have checked for mistakes.”
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judge by recent publications on the subject) that vacuousness is part of the standard theory.\footnote{It definitely seems to be a historical accident that vacuous truth for counterpossibles is part of the standard view. Lewis is extremely lukewarm in his endorsement of vacuous truth over vacuous falsity, writing, “I am fairly content to let counterfactuals with impossible antecedents be vacuously true. But my reasons are less than decisive” [67, p. 25].}

Williamson [130] offers a different explanation of non-vacuous intuitions. He suggests that a heuristic is at play when one (uncritically) assesses counterfactuals. When $\beta$ and $\gamma$ are inconsistent, we incline to treat $\alpha \square \rightarrow \beta$ and $\alpha \square \rightarrow \gamma$ as inconsistent; indeed, in the simplest case, Williamson states the heuristic as, “If you accept one of $\alpha \square \rightarrow \beta$ and $\alpha \square \rightarrow \neg \beta$, reject the other” [130, p. 364].\footnote{The reader will note that this is a heuristic analogue of weak Boethius’ theses; see chapter 4.}

As Berto et al. [8, §4.2] point out, the existence of such a heuristic seems rather implausible. For example, there is an immediate inclination to accept both “If it were round and not round, it would be round” and “If it were round and not round, it would be not round.” Williamson need not hold that the heuristic applies in all cases (indeed, for him, one must be able to critically override it), but its failing to apply in even such a simple case does not speak in its favor.

Williamson seeks to buttress his heuristic hypothesis by discussing vacuous quantification. There is a disposition to treat “Every $\sigma \phi$’s” and “Every $\sigma \neg \phi$’s” as inconsistent, but this goes wrong when there are no $\sigma$’s [130, p. 366]. Hence, both “Every round square is round” and “Every round square is not round” are true. Just as this quantificational heuristic goes wrong in the limit case, the suggestion goes, the heuristic concerned with counterfactuals can too in its own limit case. Therefore, the inclination to rule some counterpossibles false should be resisted.\footnote{It bears remarking that the connection between these conditional and quantificational heuristics is mirrored in connexive logics by the connection between weak Boethius’ theses and existential import (see especially McCall [79]).}

But again, Berto et al. [8, §4.3] give reasons to be skeptical about the existence of such a quantificational heuristic. Even if there were such a fallible heuristic, though, it is not clear
how this would help Williamson’s argument. Why shouldn’t the defender of non-vacuous semantics simply deny the analogy? It is presumably consistent to hold that all vacuous universal claims are true but not all counterpossibles are true.

In addition to intuitions, there are compelling methodological and metaphilosophical reasons which militate against vacuousness. Jenny [51, 52] has built a case against vacuousness from relative computability theory. Relative computability theory studies the reducibility of various decision problems to others. For example, the following conditional, expressing such a reduction, is taken to be true:¹²

\[ \text{If the validity problem were algorithmically decidable, then the halting problem would also be algorithmically decidable [52, p. 533]} \]

Note that under standard assumptions,¹³ this conditional is a counterpossible: if there is no algorithm for deciding validity, it is (at least) metaphysically necessary that this is so. Consequently, this is also a counterpossible:

\[ \text{If the validity problem were algorithmically decidable, then arithmetical truth would also be algorithmically decidable [52, p. 533]} \]

This conditional, however, is generally thought to be false by practitioners in the field. Consequently, to endorse vacuousness is to challenge such judgments and disregard philosophical humility [52, p. 536]. Jenny argues convincingly that such a transgression cannot be justified.

Therefore, both intuitive and theoretical reasons motivate a non-vacuous semantics for counterpossibles. Nevertheless, it may be objected that actually implementing such a semantics results in technical or philosophical incoherence. I now consider several objections in this vein.

¹²Loosely, the question of the validity problem is whether the set of validities of predicate logic is algorithmically decidable; it is, famously, not [52, p. 533]. For the other problems, see Jenny [52, 51] or any standard mathematical logic text, e.g. Hunter [47].

¹³For a discussion, see Jenny [52, pp. 534-5].
Lewis, with an eye towards counterlogicals, notes that “a counterfactual in which the antecedent logically implies the consequent ought always to be true” [67, p. 24]. Such a claim is highly plausible on its face. However, since a contradiction logically implies anything, this has the effect of rendering a large class of counterlogicals vacuously true. Moreover, the argument goes, if so many counterpossibles are trivially true, why hold that any counterpossibles fail to be trivially true? What is the significant difference between those counterpossibles and others?

The main problem with this line of argument is that the pertinent notion of logical implication is ambiguous. Consider, for example, the following conditional:

If it were the case that the liar is true, its negation is true, and Priest’s LP is the one true logic, then it would be the case that cats can fly.

It is obvious that the antecedent—something of the form $\lambda \land \neg \lambda \land \phi$—classically implies anything. But the antecedent also explicitly invites consideration of scenarios which are not classically closed, that is, scenarios in which the paraconsistent logic of Priest [99] holds. Therefore, this argument doesn’t seem very persuasive, even restricted to the class of counterfactuals with contradictory antecedents.\(^{14}\)

Williamson, like Lewis, has appealed to proofs by contradiction to defend the orthodoxy. In [130, §3] (cf. Lewis [67, p. 25]), Williamson rehearses a proof that there is no largest prime number. It goes something like this:

1. If there were a largest prime $p$, $p! + 1$ would be prime

2. If there were a largest prime $p$, $p! + 1$ would be composite

3. If there were a largest prime $p$, $p! + 1$ would be prime and composite

4. There is no largest prime

\(^{14}\)For another criticism of this point, see Brogaard and Salerno [11, pp. 648-9].
(1) is supported by general features of factorials and divisibility. (2) is immediate from the antecedent. (3) follows from (1) and (2) by (CC). (4) follows from (3) by implicit appeal to (CMP). It seems like all the conditionals used in this argument are true, a feature which the orthodoxy captures. Thus, the argument is sound (given a reasonably strong logic of counterfactuals).

Williamson [130, p. 363] suggests, however, that this is not so unless vacuousness is endorsed. The basic point is that, were there a largest prime, the whole theory of natural numbers would be very different, so it’s hard to discern if any of (1)-(3) are true.

Following Berto et al. [8, §3.3], context can be brought in to answer this objection. The basic point is that context plays a role in determining what the relevant worlds are for evaluating a counterfactual: to make the point using formula indexed interpretations, \( f_\phi(w) \) is sensitive not only to the antecedent (\( \phi \)) and base world (\( w \)), but to the situation in which a counterfactual with \( \phi \) as antecedent is uttered. Thus, to use a cliché example from Lewis [67, pp. 66-7], both of the following conditionals can be read as true:

If Caesar had been in command [in Korea] he would have used the atom bomb

If Caesar had been in command he would have used catapults

In a context where the focus is on the (merciless) character of Caesar and the sort of weapons available for use during the actual Korean war, the worlds selected are such that the first conditional is true. In a context where the focus is on the historical Caesar’s knowledge of war, worlds are selected such that the second conditional is true.\(^{15}\)

Therefore, it can be responded that, in the context of a mathematical proof, (many) features of arithmetic are fixed so that (1)-(3) continue to come out true [8, p. 704]. Williamson complains that a contextualist response like this requires more specific motivation, without

\(^{15}\)For additional discussion of issues of context and counterfactuals, see Priest [104, §2.3] and Berto et al. [8, §3.3].
which “the appeal to context-dependence is just an all-purpose objection to any valid argument” [130, p. 364]. Williamson seems to doubt whether there are (reasonable?) contexts in which, for example, (3) can be heard as false (or untrue), a datum against the contextualist thesis.

But (3) can surely be heard as untrue in any situation in which either (1) or (2) are heard as untrue, and it seems that there are plausible situations that can be imagined for this. For example (cf. [8, p. 704]), suppose the context is a discussion of ultrafinitistic mathematics (there are only finitely many integers) with a “gappy” background logic; it’s plausible that any statement about \( p! + 1 \) (where \( p \) is the largest prime) would be without a truth value in such a situation, so none of (1)-(3) would be true.

Another objection to consider concerns the technical resources used for developing a non-vacuous semantics for counterpossibles. Such semantics tend to make liberal use of non-normal, or impossible, worlds. The idea is that counterpossibles are evaluated just like counterfactuals are, except that the pertinent antecedent worlds are impossible, and so not subject to at least some significant logical constraints. But it may be objected that impossible worlds are incoherent or philosophically illegitimate.

A comprehensive discussion and defense of impossible worlds is beyond the scope of this chapter. Let it at least be noted that impossible worlds present no special technical difficulties. From a technical standpoint, an impossible or non-normal world is just one where the truth conditions for at least some formulae are deviant. As Priest [100, p. 291] observes, such worlds were originally devised by Kripke [60] for modeling non-normal modal logics. Thus, they are technically legitimate and have been around since practically the beginning of worlds semantics.

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16See, for example, Mares [74], Nolan [90], Brogaard and Salerno [11], Berto et al. [8], and Weiss [124].

17For thorough treatments of impossible worlds, the reader is referred to Yagisawa [133], Krakauer [56], and Jago [49, 50].

18Note that only the truth conditions associated with the special modal connectives are treated as deviant at non-normal worlds in [60, p. 211].
Of course, there are grades of deviance. At one extreme, all formulae can be assigned truth values randomly at impossible worlds. In this case, impossible worlds will fail to be closed under any significant consequence relation. On the other hand, impossible worlds may be taken to be closed under some weak paraconsistent consequence relation; this will render them inconsistency tolerant without thereby being trivial. The first approach is sometimes characterized as ‘North American,’ while the latter is sometimes characterized as ‘Australasian’ [101, p. 484].

While some may hold that closure under a significant consequence relation is essential to being a world of any type (see, e.g., Stalnaker [116, p. 59]), and therefore find the North American approach objectionable, it is the approach I take below. Since any Australasian world can be modeled by a North American world, but not conversely, the latter approach is to be favored for its generality.\footnote{Note that this technically motivated preference need not imply any position on the philosophical issue raised above. Whether or not closure under a substantial consequence relation is philosophically desirable for all worlds, impossible or not, the North American approach can be used (just restrict your attention to the subclass of North American models that are Australasian models of the desired kind as philosophically required).}

Finally, it may be objected that accommodating counterpossibles leads to a logic that is too weak to have any interest. Such an objection is suggested by Williamson [129, p. 95] (see also the discussion in Berto et al. [8, §3.1]). I will bracket this concern for now, and revisit it once a number of systems of counterpossible logic have been developed. At that point, it will be more clear whether and to what extent this objection is fair.

In sum, I motivated non-vacuous semantics for counterpossibles by appeal both to intuitions and theoretical principles. I considered and rebutted most of the substantial objections against non-vacuous semantics for counterpossibles. Since vacuousness has now been dealt with, its replacement must be found. I turn to this now.
CHAPTER 5. COUNTERPOSSIBLE LOGIC

5.1.2 Non-vacuous Semantics

In what follows, I develop non-vacuous semantics for counterpossibles. The basic semantic framework I use modifies that of formula indexed interpretations (recall definition 16). I elaborate on the motivation for using this sort of semantics following its formal presentation.

The logics to be examined for the remainder of this chapter are to be constructed in a language $L_{\perp}$ which extends $L$ with the unary connective $\Box$. Let $\Phi_{\perp}$ be the set of all formulae in $L_{\perp}$ (the formation rules are standard).

**DEFINITION 34.** A counterpossible interpretation is a structure $\mathcal{I}_{\Phi_{\perp}} = \langle W, N, \{ f_\phi : \phi \in \Phi_{\perp} \}, R, V \rangle$ such that:

1. $W$ is a nonempty set of worlds
2. $N \subseteq W$ is a nonempty set of normal worlds
3. $f_\phi : N \rightarrow \mathcal{P}(W)$
4. $R \subseteq N \times N$ such that $R$ is an equivalence relation
5. $V : \Phi_{\perp} \rightarrow \mathcal{P}(W)$

Given a counterpossible interpretation $\mathcal{I}_{\Phi_{\perp}} = \langle W, N, \{ f_\phi : \phi \in \Phi_{\perp} \}, R, V \rangle$, truth conditions for complex formulae at normal worlds are given just as before, with one new addition. If $w \in N$, then for $p \in \Pi = \Pi_{\perp}$:

1. $\models_{w_{\perp}}^{3} p$ if and only if $w \in V(p)$
2. $\models_{w_{\perp}}^{3} \neg \phi$ if and only if $\not\models_{w_{\perp}}^{3} \phi$
3. $\models_{w_{\perp}}^{3} \phi \rightarrow \psi$ if and only if $\not\models_{w_{\perp}}^{3} \phi$ or $\models_{w_{\perp}}^{3} \psi$

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20Once again, $[\phi] = \{ x \in W : \models_{x_{\perp}}^{3} \phi \}$. 
4. $\models_{w}^{3} \square \phi$ if and only if $\{x : wR x\} \subseteq [\phi]$

5. $\models_{w}^{3} \phi \square \rightarrow \psi$ if and only if $f_{\phi}(w) \subseteq [\psi]$

On the other hand, if $w \not\in N$ (that is, if $w \in W - N$), then the value of every formula is assigned by $V$ (for $\phi \in \Phi_{\perp}$):

6. $\models_{w}^{3} \phi$ if and only if $w \in V(\phi)$

**DEFINITION 35.** Where $\Sigma$ is a set of formulae and $C_{\Phi_{\perp}}$ is a class of counterpossible interpretations, $\Sigma \models_{C_{\Phi_{\perp}}} \phi$ if and only if for all worlds $w \in N$ of all interpretations $I_{\Phi_{\perp}} \in C_{\Phi_{\perp}}$, if $\models_{w}^{3} \psi$ for each $\psi \in \Sigma$, then $\models_{w}^{3} \phi$. If $\Sigma \models_{C_{\Phi_{\perp}}} \phi$, the inference is called valid (in $C_{\Phi_{\perp}}$).

$\phi$ is a valid formula (in $C_{\Phi_{\perp}}$) if $\emptyset \models_{C_{\Phi_{\perp}}} \phi$.

For the remainder of this chapter, I suppress subscripts (except if necessary to disambiguate) to avoid unnecessary clutter. For example, a class of counterpossible interpretations will be written as $C$, rather than $C_{\Phi_{\perp}}$.

Counterpossible interpretations contain a separate accessibility relation corresponding to the modal connective $\square$. $R$ is taken to be an equivalence relation (reflexive, transitive, and symmetric) on $N$. Therefore, the modal logic determined by $R$ is $S5$ (=$KT45$) (see, e.g., Chellas [14, Ch. 5]). The inclusion of an accessibility relation for $\square$ is desirable because reasonable definitions of $\square$ in terms of $\rightarrow$ will not, in general, be possible in counterpossible tolerant conditional logics. I have made $R$ an equivalence relation, partly for technical convenience, but also because $S5$ is the standardly assumed modal logic in most of the literature on counterpossibles (see, e.g., Berto et al. [8] and Weiss [124]).

Truth-functionality features only at normal worlds. While $V$ is taken to be a function defined on all formulae, for normal worlds, it only enters the truth conditions for propositional variables. On the other hand, every formula is evaluated using only $V$ at non-normal worlds. This is how ‘North American’ impossible worlds are realized in the semantics.
A number of constraints might be imposed on counterpossible interpretations. I list those constraints which have been most prominent in the recent literature on counterpossibles in table 5.1.

### Table 5.1: Counterpossible Semantic Constraints

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_\phi(w) \subseteq [\phi]$</td>
<td>(id) If $w \in [\phi]$, then $w \in f_\phi(w)$</td>
</tr>
<tr>
<td>If ${x : w R x} \cap [\phi] \neq \emptyset$, then $f_\phi(w) \subseteq N$</td>
<td>(sic)</td>
</tr>
</tbody>
</table>

For the most part, it should be fairly obvious which axiom schemata from table 2.2 correspond to the semantic constraints listed in table 5.1. Before commenting on the non-obvious case—(sic)—let me preempt a concern the reader may have about the austere nature of table 5.1. While many constraints in addition to those listed here might be considered, few of them have much plausibility once counterpossibles are taken seriously (that being said, I will justify some of the omissions in section 5.2). Moreover, it would be needlessly tedious to review all variations of conditional logic in this setting.

Moving on, then, (sic) corresponds, approximately, to the condition given the same name by Nolan [90, p. 566]. The idea behind (sic) is that, if there are any accessible, normal, antecedent-satisfying worlds, then all the worlds which are pertinent for evaluating a counterfactual with that antecedent should be normal as well. Adopting the definition $\diamond \phi \equiv \neg \Box \neg \phi$, (sic) corresponds to the following (RCK)-like rule of inference (which applies—semantically speaking—only if the premise is valid):\(^{22}\)

\(^{21}\)Nolan [90] works with a Lewis-style sphere semantics in which there is no separate accessibility relation. The correspondence, to the extent there is one, consists in the characteristic validities associated with this condition.

\(^{22}\)The convention is adopted that when $n = 0$, (SIC) licenses the inference from $\phi$ to $\diamond \psi \rightarrow (\psi \Box \rightarrow \phi)$. 
(φ₁ ∧ ... ∧ φₙ) → φ

◊ψ → (((ψ □→ φ₁) ∧ ... ∧ (ψ □→ φₙ)) → (ψ □→ φ))

(SIC)

PROPOSITION 16. (SIC) is validity preserving in any class of counterpossible interpretations satisfying (sic)

The proof is by induction on n. For the basis case, suppose that |=C φ, and let w ∈ N be an arbitrary normal world of an arbitrary J ∈ C such that |=wᵦψ. Since {x : wRx} ∩ [ψ] ≠ ∅, it follows by (sic) that fψ(w) ⊆ N. Since N ⊆ [φ] (by assumption), fψ(w) ⊆ [φ]. Therefore, |=wᵦψ □→ φ.

The induction hypothesis is that if |=C (φ₁ ∧ ... ∧ φₖ) → φ, then |=C ◊ψ → (((ψ □→ φ₁) ∧ ... ∧ (ψ □→ φₖ)) → (ψ □→ φ)). Suppose that |=C (φ₁ ∧ ... ∧ φₖ ∧ φₖ₊₁) → φ; then since |=C (φ₁ ∧ ... ∧ φₖ) → (φₖ₊₁ → φ), by the induction hypothesis it follows that |=C ◊ψ → (((ψ □→ φ₁) ∧ ... ∧ (ψ □→ φₖ)) → (ψ □→ (φₖ₊₁ → φ))). By (sic), it follows that |=C ◊ψ → (((ψ □→ φ₁) ∧ ... ∧ (ψ □→ φₖ)) → ((ψ □→ φₖ₊₁) → (ψ □→ φ))). The desired result then follows by importation.

5.2 Axiom Systems

This section has two purposes. The first is to discuss principles of counterpossible logic from an axiomatic standpoint, and evaluate how plausible these are. Once this has been done, several systems of counterpossible logic will be axiomatized and evaluated.

5.2.1 Principles of Counterpossible Logic

Many of the standard axioms of conditional logic have little plausibility once counterpossibles are taken seriously. To evaluate different counterpossible logics, and ultimately characterize
what it is for a system of conditional logic to be a system of counterpossible logic, I discuss a number of axioms and rules below.

To begin with, systems of counterpossible logic should be neither half-classical nor classical (recall definition 3). That is, they should neither be closed under (RCEA) nor (RCEC). The reason for this is simple: closing under either trivializes a large class of counterlogicals. For consider the following two conditionals:

If the Liar were true, then the Liar would be true and not true

If the Liar were true, then cats would be mammals and would not be mammals

The first conditional is a plausibly true, non-vacuous counterpossible. Its consequent, a contradiction, is logically equivalent to the consequent of the second conditional. Consequently, by (RCEC), both conditionals should be equivalent. Similar considerations suffice to show that (RCEA) is inappropriate.

A rule for antecedent exchange can be restored by adopting (CSO) but, as I argued in [124], (CSO) is itself objectionable. It is worth briefly reviewing the argument against it. Consider the following scenario:

Fred is teaching George arithmetic. Fred asks George what $5 + 7$ is, and George mistakenly responds $13$. Fred snidely remarks, “if $5 + 7$ were $13$, you would have answered correctly.” This is true. What else might be the case if $5 + 7 = 13$? Plausibly, $5 + 6 = 12$. Conversely, if $5 + 6 = 12$, it would seem reasonable to expect that $5 + 7 = 13$ [124, p. 390]

In the context of this scenario, it is claimed that each of the following three conditionals is true:

$$5 + 7 = 13 \rightarrow 5 + 6 = 12$$
(CSO) then licenses the inference from these three conditionals to:

\[ 5 + 6 = 12 \implies 5 + 7 = 13 \]

However, the last conditional does not appear to be true; after all, George isn’t answering a question about that sum. Since I advise against adopting (CSO),\(^{23}\) it is my position that none of the Lewis systems—\(C0\) or its extensions—have plausible counterpossible companions, at least insofar as that entails endorsing (CSO).

Perhaps the most objectionable thesis of standard conditional logic, from the perspective of counterpossibles, is (CN): \(\phi \implies \top\). If (CN) were adopted, then every instance of the following scheme would be a theorem (for any formula \(\phi\)):

\[ \text{If the law of the excluded middle failed, then } \phi \lor \neg \phi \]

Since this is obviously unacceptable, no proper system of counterpossible logic can contain (CN). For theorems to hold given arbitrary antecedents is characteristic of non-counterpossible conditional logics.

(MOD) also must be rejected by any reasonable analysis of counterpossibles. For suppose you are considering what would be the case were some instance of the law of non-contradiction false. If \(\neg \neg (\phi \land \neg \phi)\) were the case, then it should also be the case that \(\neg (\phi \land \neg \phi)\) (for if there were true contradictions, then said contradictions would also, presumably, be false). By (MOD), it follows that, for any \(\psi\), \(\psi \implies \neg (\phi \land \neg \phi)\). But for many instances of \(\psi\), this is implausible (for example, consider the case where \(\psi\) states that \(\phi\) is without a truth value).

Other theses of conditional logic are perhaps not as objectionable, but still questionable. (SIC) is a case in point. On the one hand, as Nolan [90, p. 566] notes, such a condition

\(^{23}\)For an objection to (CSO) that doesn’t make use of counterpossibles, see Gabbay [35, p. 101].
is desirable for maintaining a level of formal familiarity for conditionals with satisfiable
antecedents. However, (SIC) also seems to license some intuitively invalid inferences. Let
φ be the claim that “there are or there are not counterexamples to classic tautologies.” By
(SIC), this is a theorem:

\[ \Diamond \phi \rightarrow (\phi \Box \rightarrow \top) \]

Since φ is itself a classical tautology, by elementary modal logic, φ \( \Box \rightarrow \top \) is a theorem. But it seems bizarre to say that any arbitrary classical tautology would be the case were
there counterexamples to classical tautologies or not; surely, in evaluating this conditional,
consideration should be given to worlds where either disjunct holds, that is, both normal
and non-normal worlds (cf. [124, p. 388]).

There is little to say about (ID) or (CMP). Both are, arguably, essential to any theory
of counterfactual conditionality. Both may be subject to contrived counterexamples, but
neither seem to be subject to any objections as weighty as those considered above. I am
happy to endorse them both as principles of counterpossible logic.

### 5.2.2 Systems of Counterpossible Logic

In subsection 5.2.1, I examined a number of theses of conditional logic and assessed these
for plausibility. Below, I define what it is for a system of conditional logic to be a system
of counterpossible logic and axiomatize several systems. These are readily shown to be
consistent.

**DEFINITION 36.** A system of conditional logic L is *counterpossible* if it contains KT45
and \( \not \vdash_L \bot \rightarrow \phi \).\(^{24}\)

\(^{24}\)It would also be reasonable to require that \( \not \vdash_L \neg(\bot \rightarrow \phi) \), i.e. that counterlogicals with contradictory
antecedents not be treated as vacuously false (to put it semantically). While this condition will be satisfied
by all the systems I examine, I know of no philosopher who endorses vacuous falsity for counterpossibles
and, consequently, do not require its avoidance as part of definition 36.
A couple remarks are in order about definition 36. First, the definition centers around what I take to be the absolutely minimal condition of being a counterpossible system of conditional logic, viz. not proving all counterfactuals with contradictory antecedents. While a number of systems which handle counterpossibles poorly—for example, CE augmented by KT45—meet the conditions of the definition, there is still some justice in labeling them as counterpossible systems of conditional logic insofar as they avoid the problem of vacuousness (as it is traditionally stated and understood). Of course, I will mainly be interested in systems which not only meet the conditions of this definition, but also avoid the principles found to be objectionable in subsection 5.2.1.

Second, it is pretty much immediate from definition 36 that the smallest counterpossible system of conditional logic is simply KT45 (recall that the language includes both □ and □→). While this may seem like nothing more than a technical artifact, as will become apparent in section 5.3, this is not the case.

While KT45 may be the smallest system of counterpossible logic, systems of any real interest will extend it with various axioms discussed in subsection 5.2.1. The main systems I will be interested in for the remainder of this chapter are listed in table 5.2.

Table 5.2: Counterpossible Systems of Conditional Logic

<table>
<thead>
<tr>
<th>System</th>
<th>Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>KT45</td>
<td>K, (T), (4), (5)</td>
</tr>
<tr>
<td>CP1</td>
<td>KT45, (ID), (CMP)</td>
</tr>
<tr>
<td>CP2</td>
<td>CP1, (SIC)</td>
</tr>
</tbody>
</table>

KT45, CP1, and CP2 axiomatize several logics familiar from the literature on counterpossibles. KT45 is characterized by the basic “no-constraints” semantics of Berto et al. [8] and axiomatizes the logic I call C# in [124]. CP1 is what I take to be the correct logic of counterpossibles: it embraces all and only the principles endorsed in subsection 5.2.1. CP1 axiomatizes C1# from [124] and another of the logics discussed by Berto et al. [8]. Finally,
CHAPTER 5. COUNTERPOSSIBLE LOGIC

CP2 corresponds to the final $C^\#$ logic discussed in [124], viz. $C^\#_3$. It also corresponds to the strongest logic discussed by Berto et al. [8].

THEOREM 26 (Consistency). Each of the systems in table 5.2 is consistent

The proof is trivial. Let $L$ be the system which results from combining KT45 and C1 in $L_{\perp}$. It is clear that $L$ is consistent: all of its theorems can be mapped to classical tautologies using a function $\tau$ which erases boxes (cf. Chellas [14, pp. 22-3]) and turns $\square\rightarrow$ conditionals into material conditionals. Now observe that all the systems listed in table 5.2 are subsystems of $L$; in particular, note that any system closed under (RCK) is also closed under (SIC). Therefore, all of these systems are consistent.

\[\square \]

5.3 Determination Results

In this section, I establish soundness and completeness results for the systems from table 5.2 with respect to appropriate classes of counterpossible interpretations (subsection 5.1.2). I again follow the convention of denoting by $C^L$ the class of all interpretations satisfying the constraints associated with the characteristic axiom schemata of $L$. Note that $C^{KT45}$ is just the class of all counterpossible interpretations. For the remainder of this chapter, let $L$ be any system from table 5.2.

THEOREM 27 (Soundness). $\Sigma \vdash_L \phi$ implies $\Sigma \models_{C^L} \phi$

The proof is trivial. The only difficult case involves (SIC), for which see proposition 16 above.

\[\text{Some of these axiomatic equivalences might be considered surprising since the semantics developed for the } C^\# \text{ family in my [124] is superficially weaker than the corresponding semantics in Berto et al. [8]. In particular, I used relational rather than functional interpretations in [124], à la FDE (cf. Dunn [24]). However, since I defined validity over the normal worlds, made the relational assignment functional at normal worlds, and only let it be arbitrary for all formulae at non-normal worlds, the same effect can be achieved with a function that is arbitrary at only non-normal worlds.}\]
As an application of theorem 27, it can be shown that each system from table 5.2 meets the negative condition of definition 36, viz. $\not\models_L \perp \supset \phi$.

**Proposition 17.** $\not\models_L \perp \supset \phi$

For concreteness, take $\perp \equiv p \land \neg p$ (the result will go through however $\perp$ is defined). Consider the following interpretation $\mathcal{I} = \langle W, N, \{f_\phi : \phi \in \Phi\}, \mathcal{R}, V \rangle$: $W = \{w_0, w_1\}$, $N = \{w_0\}$, $f_\phi(p \land \neg p)(w_0) = \{w_1\}$, $\mathcal{R} = \{(w_0, w_0)\}$, $V(p) = V(q) = \emptyset$, $V(p \land \neg p) = \{w_1\}$, and for all remaining formulae $\phi$ and $w \in W$, let $V$ be arbitrary and put $w \in f_\phi(w_0)$ if and only if $w \in N \cap [\phi]$. The reader can verify that $\mathcal{I} \in CL$ for any system $L$. Moreover, it is clear that $f_\phi(p \land \neg p)(w_0) \not\subseteq [q]$, hence $\not\models^{3}_{w_0} p \land \neg p \supset q$. The desired result follows from theorem 27.

Proving completeness requires some straightforward modifications to the argument used for formula indexed interpretations (subsection 2.3.1). One complication is that different canonical models must be used depending on whether $L$ is closed under (SIC). Let $[\phi] = \{\Gamma \subseteq \Phi : \phi \in \Gamma\}$ (i.e. $[\phi]$ is the set of all subsets of formulae containing $\phi$).

**Definition 37.** The canonical model for a system $L$ not closed under (SIC) is a structure $\mathcal{J}^L = \langle W, N, \{f_\phi : \phi \in \Phi\}, \mathcal{R}, V \rangle$ such that:

1. $W = \mathcal{P}(\Phi)$
2. $N = \{\Gamma \subseteq \Phi : \Gamma$ is maximally $L$ consistent $\}$
3. For $w \in N$, $f_\phi(w) = \{x \in W : \{\psi : (\phi \supset \psi) \in w\} \subseteq x\}$
4. For $x, y \in N$, $x \mathcal{R} y$ if and only if $\{\phi : \Box \phi \in x\} \subseteq y$
5. $V(\phi) = [\phi]$
Lemma 29 (Truth Lemma). Let $J^L$ be the canonical model for a system $L$ not closed under (SIC). Then for all $\phi \in \Phi$ and all $w \in W$: $\models_{w}^{J^L} \phi$ if and only if $\phi \in w$ (i.e. $[\phi] = [\phi]$)

The proof divides into two cases depending on whether $w \in N$ or $w \in W - N$. If $w \in W - N$, then $\models_{w}^{J^L} \phi$ if and only if $w \in V(\phi)$ if and only if $\phi \in w$. If $w \in N$, the proof proceeds by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\alpha \rightarrow \beta$.

Suppose that $\alpha \rightarrow \beta \in w \in N$ and $x \in f_\alpha(w)$. Then by definition 37, $\beta \in x$. By the induction hypothesis, it follows (as in the proof of lemma 15) that $\models_{w}^{J^L} \alpha \rightarrow \beta$. Conversely, suppose that $\alpha \rightarrow \beta \not\in w$ and consider the set $x = \{ \chi : (\alpha \rightarrow \chi) \in w \}$. It is clear that $x \in W = P(\Phi)$, $x \in f_\alpha(w)$, and $\beta \not\in x$. By the induction hypothesis, $x \not\in [\beta]$. Therefore, $f_\alpha(w) \not\subseteq [\beta]$, that is, $\not\models_{w}^{J^L} \alpha \rightarrow \beta$.

\[\square\]

Lemma 30. Let $J^L$ be the canonical model for a system $L$ not closed under (SIC). Then $J^L \in C^L$

I examine both of (id) and (cmp). For (id), suppose that $L$ contains (ID) and suppose that $w \in N$ and $x \in f_\phi(w)$. Since $w$ is maximal, $\phi \rightarrow \phi \in w$, from which it follows that $\phi \in x$. Then in $J^L$, by lemma 29, $x \in [\phi]$. By the arbitrariness of $x$, $f_\phi(w) \subseteq [\phi]$. For (cmp), suppose $w \in N$, $w \in [\phi] = [\phi]$, and $\phi \rightarrow \psi \in w$. By the closure of $w$ under $L$, which contains (CMP), $\psi \in w$. Since $\psi$ is arbitrary such that $\phi \rightarrow \psi \in w$, $w \in f_\phi(w)$.

\[\square\]

Theorem 28 (Completeness). Let $L$ not be closed under (SIC). $\Sigma \models_{cL} \phi$ implies $\Sigma \vdash_{L} \phi$

The result follows directly from lemmata 29 and 30.
Observation. Theorems 27 and 28 jointly imply that \( \text{KT45} \) is determined by the class of all counterpossible interpretations. This is a sort of triviality result: the only valid formulae and inferences involving \( \Box \rightarrow \) in the basic counterpossible semantics are those which are instances of patterns already valid in (the usual semantics for) \( \text{KT45} \).

DEFINITION 38. The canonical model for a system \( L \) closed under (SIC) is a structure \( \mathcal{J}^L = \langle W, N, \{f_\phi : \phi \in \Phi \}, R, V \rangle \) defined exactly as in definition 37 except:

For \( w \in N, f_\phi(w) = \begin{cases} \{ x \in N : \{ \psi : (\phi \Box \rightarrow \psi) \in w \} \subseteq x \} & \text{if } \Box \phi \in w \\ \{ x \in W : \{ \psi : (\phi \Box \rightarrow \psi) \in w \} \subseteq x \} & \text{otherwise} \end{cases} \)

LEMMA 31 (Truth Lemma). Let \( \mathcal{J}^L \) be the canonical model for a system \( L \) closed under (SIC). Then for all \( \phi \in \Phi \) and all \( w \in W : \models_{\mathcal{J}^L_w} \phi \) if and only if \( \phi \in w \) (i.e. \([\phi] = [\phi]^L\))

The proof is the same as the proof of lemma 29 except for the converse direction of the case concerning normal worlds and \( \Box \rightarrow \) conditionals. Suppose that \( w \in N \) and \( (\alpha \Box \rightarrow \beta) \not\in w \). Either \( \Box \alpha \not\in w \) or \( \Box \alpha \in w \). If \( \Box \alpha \not\in w \), the proof proceeds as in lemma 29. If \( \Box \alpha \in w \), the proof proceeds by modifying the argument used in the proof of lemma 15. Let \( S = \{ \chi : (\alpha \Box \rightarrow \chi) \in w \} \cup \{ \neg \beta \} \). I prove that \( S \) is consistent. If it were not, then \( \exists \chi_0, \ldots, \chi_n \in S \) such that \( \chi_0, \ldots, \chi_n \vdash_L \bot \). Then:

\[
\chi_0, \ldots, \chi_n, \neg \beta \vdash_L \bot
\]

\[
\vdash_L (\chi_0 \land \ldots \land \chi_n) \rightarrow \beta
\]

\[
\vdash_L \Box \alpha \rightarrow (((\alpha \Box \rightarrow \chi_0) \land \ldots \land (\alpha \Box \rightarrow \chi_n)) \rightarrow (\alpha \Box \rightarrow \beta))
\]

\[
w \vdash_L \alpha \Box \rightarrow \beta
\]

Since this is impossible, \( S \) is consistent. By lemma 12, \( S \) can be extended to a maximally \( L \) consistent set (i.e. a normal world). Then it clearly follows that \( \not\models_{\mathcal{J}^L_{\Box \mathcal{J}^L}} \alpha \Box \rightarrow \beta \), as desired.
The proof that the canonical model for $L$ is in the appropriate class ($\mathcal{M}^L \in C^L$) proceeds just as before.

**THEOREM 29 (Completeness).** Let $L$ be closed under (SIC). $\Sigma \models_{cL} \phi$ implies $\Sigma \vdash L \phi$

The result follows immediately from lemma 31.

At this point, it is worth revisiting Williamson’s complaint that counterpossible logics are too weak to be interesting (see subsection 5.1.1 above). The point seems to be fair, insofar as the weakest “natural” logic of counterpossibles is being considered. For, as I observed above, the weakest natural semantics for counterpossibles—a semantics found, in variant forms, in both Berto et al. [8] and Weiss [124]—determines a logic no stronger than its background modal logic.

Nevertheless, the objection fails to be decisive because almost all advocates of non-vacuous semantics actually support stronger semantic constraints which, in turn, determine more robust systems. That being said, I support a fairly weak system, namely $CP1$. In response to Williamson’s objection, my inclination is to concede that very little is valid on my preferred semantics, but that this is how it ought to be. Weakness is not, in itself, problematic; given the aberrance of impossible situations, one should expect few hard and fast rules. Many formulae Williamson would call valid, I am happy to call merely true (or, true in a context). For example, consider the following conditional, spoken in the context of a discussion of intuitionistic logic:

If it were the case that there is a counterexample to the law of the excluded middle and two is prime, then it would be the case that two is prime.

---

26Hence, I am largely in agreement with Nolan, who writes, “Since most principles concerning the conditional have counterexamples, when sufficiently strange antecedents are employed, I think that there will be almost no distinctive theorems which hold of conditionals regardless of what propositions make up their antecedents and the consequents” [90, p. 554].
I claim that this conditional is true because conjunction simplification holds at all the relevant impossible worlds (i.e. intuitionistic worlds satisfying the antecedent), though not all impossible worlds. But if the reader prefers a stronger logic and does not find my concerns over (SIC) convincing, **CP2** is an option as well.

---

27 Once again, I find myself in agreement with Nolan, who writes, “There are few exceptionless principles, but restrictions of context mean that many inference moves which are not formally valid will be acceptable in a wide range of circumstances” [90, p. 555].
Chapter 6

Intuitionistic Conditional Logic

Despite the variety of conditional logics and uses thereof, almost all extant conditional logics extend classical logic.\(^1\) The project of this chapter is to examine both intuitionistic logic via conditional logic and conditional logic extensions of intuitionistic logic.

First, I review intuitionistic logic both axiomatically and semantically in section 6.1. In addition, I touch on the BHK interpretation of the connectives. The principal result of this section is that intuitionistic logic can be embedded into a natural conditional logic using a Gödel-McKinsey-Tarski inspired translation. I offer some brief remarks on what the philosophical significance of this result might be in light of the BHK interpretation.

In section 6.2, I axiomatize a number of intuitionistic systems of conditional logic. I discuss how to embed various intuitionistic modal logics into these and propose an epistemic reading of \( \Box \to \) in two systems based on such embeddings. In subsection 6.2.2, both worlds semantics and algebraic semantics are given for a number of systems of intuitionistic conditional logic. The principal result of this subsection is theorem 34, which establishes equivalences between various classes of algebraic and worlds interpretations. In section 6.3,

\(^1\)A few notable exceptions are Mares [74], Mares and Fuhrmann [76], Genovese et al. [36, 37], and Weiss [125]. Of these, the work of Genovese et al. is the only work on intuitionistic conditional logic that I am aware of besides my own. Note, however, that there is little fundamentally in common between their work and mine. For a fuller discussion of the relationship of their work to mine, see my [125].
soundness and completeness results are proved.

6.1 Intuitionistic Logic

Let $\mathcal{L}_i$ be the language of propositional intuitionistic logic, which is given as follows: there is a denumerable set of propositional variables $\Pi$, a unary connective $\neg$, a set of binary connectives $\{\to, \lor, \land\}$, and the parentheses. The formation rules are standard; let $\Phi_i$ denote the set of intuitionistic formulae. The connectives $\leftrightarrow, \bot,$ and $\top$ are defined in the usual way.

Informally, just as the significance of a statement of classical logic is explicated by its truth conditions, the significance of a statement of intuitionistic logic can be explicated by its proof conditions. The provability interpretation of intuitionistic formulae is typically referred to as the Brouwer-Heyting-Kolmogorov (BHK) interpretation. There are several subtly different versions of this; I present the version from Priest [102, p. 104] (the notation has been modified as needed):

1. A proof of $\phi \land \psi$ is a pair comprising a proof of $\phi$ and a proof of $\psi$
2. A proof of $\phi \lor \psi$ is a proof of $\phi$ or a proof of $\psi$
3. A proof of $\neg \phi$ is a proof that there is no proof of $\phi$
4. A proof of $\phi \to \psi$ is a construction that, given any proof of $\phi$, can be applied to give a proof of $\psi$

It must be stressed that the BHK conditions are informal as stated. Ambiguity attaches to several points, including whether ‘is a proof of’ is decidable (in which case the condition for $\to$ ought to include a verification clause) and what the pertinent notion of ‘construction’ is.\(^2\)

The inexactness of the BHK conditions notwithstanding, they do adequately motivate a number of non-classical conclusions. There is no proof of the Riemann hypothesis, nor

\(^2\)For a discussion of these and related points, see van Dalen [18, pp. 231-2] and Dummett [23, pp. 12-3].
is there any proof that there is no proof of the Riemann hypothesis. Consequently, where
\( \phi \) states the Riemann hypothesis, there is no proof of \( \phi \lor \neg \phi \), and the law of the excluded
middle fails.\(^3\)

6.1.1 Axiomatics and Semantics

In this subsection, I review an axiom system for intuitionistic propositional logic (IPL) and
its Kripke semantics. For concreteness, I use an axiomatization of IPL from Božić and
Došen [10, p. 219]. IPL is characterized by the axioms (axiom schemata) and rules listed
in table 6.1.

\(^3\)Clearly, this will work for any undecided conjecture (cf. Priest [102, p. 104]).
CHAPTER 6. INTUITIONISTIC CONDITIONAL LOGIC

Table 6.1: Intuitionistic Axiom Schemata and Rules

\[
\begin{align*}
\phi & \rightarrow (\psi \rightarrow \phi) \quad \text{(H1)} \\
(\phi \rightarrow (\psi \rightarrow \theta)) & \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta)) \quad \text{(H2)} \\
(\phi \rightarrow \psi) & \rightarrow ((\phi \rightarrow \theta) \rightarrow (\phi \rightarrow (\psi \land \theta))) \quad \text{(H3)} \\
(\phi \land \psi) & \rightarrow \phi \quad \text{(H4)} \\
(\phi \land \psi) & \rightarrow \psi \quad \text{(H5)} \\
\phi & \rightarrow (\phi \lor \psi) \quad \text{(H6)} \\
\psi & \rightarrow (\phi \lor \psi) \quad \text{(H7)} \\
(\phi \rightarrow \psi) & \rightarrow ((\theta \rightarrow \psi) \rightarrow ((\phi \lor \theta) \rightarrow \psi)) \quad \text{(H8)} \\
(\phi \rightarrow \neg \psi) & \rightarrow (\psi \rightarrow \neg \phi) \quad \text{(H9)} \\
\neg \phi & \rightarrow (\phi \rightarrow \psi) \quad \text{(H10)} \\
\phi, \phi \rightarrow \psi & \frac{\psi}{\psi} \quad \text{(MP)}
\end{align*}
\]

DEFINITION 39. IPL is the smallest system (set of formulae) closed under (MP) which contains every instance of (H1)-(H10).

DEFINITION 40. $\vdash_{\text{IPL}} \phi$ (\(\phi\) is a theorem of IPL) if and only if $\phi \in \text{IPL}$. $\Gamma \vdash_{\text{IPL}} \phi$ if and only if there is a sequence $\phi_1, \ldots, \phi_n$ such that each formula in $\phi_1, \ldots, \phi_n, \phi$ is a theorem of IPL, an element of $\Gamma$, or is obtained by (MP) from preceding formulae.

It is transparently obvious that IPL is consistent, and a subsystem of classical propositional logic (PL). There are many different semantic treatments of intuitionistic logic. I will
primarily be concerned with its Kripke semantics, originally developed in [59].

**DEFINITION 41.** An IPL interpretation is a structure $\mathcal{J} = \langle W, \mathcal{R}, V \rangle$ such that:

1. $W$ is a nonempty set of worlds
2. $\mathcal{R} \subseteq W \times W$ such that $\mathcal{R}$ is reflexive and transitive
3. $V : \Pi \rightarrow \mathcal{P}(W)$ such that if $w \in V(p)$ and $w \mathcal{R} x$, then $x \in V(p)$

Given an IPL interpretation $\mathcal{J} = \langle W, \mathcal{R}, V \rangle$, truth conditions for complex formulae are given as follows ($w \in W$):

1. $\models^3_w p$ if and only if $w \in V(p)$
2. $\models^3_w \phi \lor \psi$ if and only if $\models^3_w \phi$ or $\models^3_w \psi$
3. $\models^3_w \phi \land \psi$ if and only if $\models^3_w \phi$ and $\models^3_w \psi$
4. $\models^3_w \neg \phi$ if and only if $\forall x$ such that $w \mathcal{R} x$, $\neg \models^3_x \phi$
5. $\models^3_w \phi \rightarrow \psi$ if and only if $\forall x$ such that $w \mathcal{R} x$, either $\neg \models^3_x \phi$ or $\models^3_x \psi$

Let $C_{\text{IPL}}$ be the class of all IPL interpretations. Then validity ($\models_{C_{\text{IPL}}}$) is defined as usual.

**THEOREM 30** (Determination). $\Gamma \models_{C_{\text{IPL}}} \phi$ if and only if $\Gamma \vdash_{\text{IPL}} \phi$

The proof of this result, which is standard and well-known, is omitted.

\(^4\)Once more, observe that $[\phi] = \{ x \in W : \models^3_x \phi \}$. 


CHAPTER 6. INTUITIONISTIC CONDITIONAL LOGIC

6.1.2 A Conditional Embedding of Intuitionistic Logic

In this subsection, I show that intuitionistic propositional logic can be embedded into a natural extension of Lewis’ counterfactual logic $C_1$. The result is inspired by the famous Gödel-McKinsey-Tarski embedding [40, 82] of intuitionistic logic into $S_4$ (=KT4), in conjunction with the interpretability of various modal logics in conditional logic (for a discussion, see subsection 2.1.2).

For the purposes of proving this embedding result, it is most convenient to utilize Lewis’ sphere semantics (recall definition 19 and the associated truth conditions for $\Box \rightarrow$). Where $C_s$ is the class of all sphere interpretations, I wish to consider a narrower class of sphere interpretations $I = \langle W, \{S_i : i \in W\}, V \rangle$ subject to the following constraints (for all $i, j \in W$):

\[
\{i\} \in S_i \quad \text{(cent)}
\]

\[
\text{If } j \in \bigcup S_i, \text{ then } \bigcup S_j \subseteq \bigcup S_i \quad \text{(4s)}
\]

Let the class of all sphere interpretations satisfying (cent) and (4s) be denoted by $C_{sC4}$, and call such interpretations $C_4$ sphere interpretations. Accordingly, I call the system characterized by $C_{sC4} C_4$. It is not hard to see that $C_{sC4}$ is equivalent, in the sense of theorem 13, to the class of all preorder interpretations satisfying (we), (cs), and this condition:

\[
\text{If } j \in S_i, \text{ then } S_j \subseteq S_i \quad \text{(4≤)}
\]

As is well known, (cent) corresponds, axiomatically, to (CMP) and (CS). Thus, Lewis’ $C_1$ is a subsystem of $C_4$. (4s), on the other hand, corresponds to (4C), which the reader will recall from the beginning of chapter 3. I am not particularly interested in treating $C_4$ proof theoretically, however, and will only consider it semantically hereafter.

Define a mapping from the set of formulae of intuitionistic propositional logic into the
set of formulae of conditional logic as follows:

**DEFINITION 42.** The function $\tau : \Phi_i \rightarrow \Phi$ is given by:

1. $\tau(p) = \neg p \rightarrow \bot$
2. $\tau(\neg \phi) = \tau(\phi) \rightarrow \bot$
3. $\tau(\phi \lor \psi) = \tau(\phi) \lor \tau(\psi)$
4. $\tau(\phi \land \psi) = \tau(\phi) \land \tau(\psi)$
5. $\tau(\phi \rightarrow \psi) = (\tau(\phi) \land \neg \tau(\psi)) \rightarrow \bot$

I prove that $\tau$ embeds intuitionistic propositional logic into $\mathbf{C4}$. The result is proved semantically and makes use of two lemmata.

**LEMMA 32.** If $\not \vDash_{\mathbf{C4}}^w \tau(\phi)$, then $\not \vDash_{\mathbf{CPL}} \phi$

Suppose that $\not \vDash_{\mathbf{C4}}^w \tau(\phi)$; then there exists a world $w$ of a $\mathbf{C4}$ sphere interpretation $\mathcal{I}_s = \langle W, \{s_i : i \in W\}, V \rangle$ such that $\not \vDash_{\mathbf{CHPL}}^w \tau(\phi)$. From $\mathcal{I}_s$, an intuitionistic interpretation $\mathcal{J} = \langle W, \mathcal{R}, V_3 \rangle$ is constructed with the same set of worlds $W$ but such that $i \mathcal{R} j$ if and only if $j \in \bigcup s_i$ and $V_3(p) = [\tau(p)]^{3s}$.

It must be verified that $\mathcal{J} = \langle W, \mathcal{R}, V_3 \rangle$, so defined, is an intuitionistic interpretation. By (cent), $i \in \bigcup s_i$, hence $\mathcal{R}$ is reflexive. Suppose that $i \mathcal{R} j$ and $j \mathcal{R} k$; then $j \in \bigcup s_i$ and $k \in \bigcup s_j$. By (4s), $k \in \bigcup s_j \subseteq \bigcup s_i$, hence $i \mathcal{R} k$; therefore, $\mathcal{R}$ is transitive. For intuitionistic heredity, suppose that $i \mathcal{R} j$ and $i \in V_3(p) = [\tau(p)]^{3s} = [\neg p \rightarrow \bot]^{3s}$. Since $j \in \bigcup s_i$, by (4s), $\bigcup s_j \subseteq \bigcup s_i$. Since $i \in [\neg p \rightarrow \bot]^{3s}$, there are two cases to consider. If $\bigcup s_i \cap [\neg p]^{3s} = \emptyset$, then $\bigcup s_j \cap [\neg p]^{3s} = \emptyset$, and so $j \in [\neg p \rightarrow \bot]^{3s} = V_3(p)$. Otherwise, $\exists S \in s_i$ such that $\exists k \in S \cap [\neg p]^{3s}$ and $S \subseteq [\neg p \rightarrow \bot]^{3s}$; but then $k \in [\bot]^{3s}$, which is impossible.

It remains to show that, for all $w \in W$, $\vDash_{w}^{3s} \tau(\phi)$ if and only if $\vDash_{w}^{3s} \phi$. The proof of this is by induction on the complexity of $\phi$. I consider only the case where $\phi$ is of the form
proof is by induction on $\phi$ $R$ transitivity of $\mathsf{IPL}$ suffices for (int). Finally, suppose $j$ such that $jR$ it is not the case that $jR$ by contraposition; suppose that $\bigcup h$ (since $S,T \in \subset$ $\mathcal{I}$ such that $S \cap [\tau(\theta)]^3 = \emptyset$ and $S \subseteq [\tau(\theta) \rightarrow \bot]^3$. Since the latter case is impossible, $\bigcup S \cap [\tau(\theta)]^3 = \emptyset$, whence by the induction hypothesis, $\bigcup S \cap [\theta]^3 = \emptyset$. Pick an arbitrary $j$ such that $wRj$; since $j \in \bigcup S$, $j \notin [\theta]^3$. Therefore, $\models^3_w \neg \theta$, as desired.

\[ \square \]

**LEMMA 33.** If $\not\models_{\mathsf{IPL}} \phi$, then $\not\models_{\mathsf{C4}} \tau(\phi)$

Suppose that $\not\models_{\mathsf{IPL}} \phi$; then there exists a world $w$ of an $\mathsf{IPL}$ interpretation $\mathcal{I} = \langle W, \mathcal{R}, V \rangle$ such that $\not\models^3_w \phi$. From $\mathcal{I}$, a $\mathsf{C4}$ sphere interpretation $\mathcal{I}_S = \langle W, \{ S_i : i \in W \}, V \rangle$ is constructed as follows. $W$ is the same and $V(p) = V_3(p)$. For $w \in W$, let $R_w$ be any well-order of the set $\mathcal{Y}_w = \{ x : wR x \}$ such that $w$ is the least element in $R_w$.\(^5\) Then, following Lewis [67, p. 139], define $S_w = \{ X \subseteq \mathcal{Y}_w : \text{for } j,k \in \mathcal{Y}_w, \text{if } j \in X \text{ and } k \notin X, \text{then } jR_k \}.$

It must be verified that $\mathcal{I}_S = \langle W, \{ S_i : i \in W \}, V \rangle$ is a $\mathsf{C4}$ sphere interpretation. Since $i$ is the least element in $R_i$, it is clear that $\{ i \} \in S_i$, as required by (cent). Suppose that, for $S,T \in S_i$, $S \not\subseteq T$; then $\exists k \in S$ such that $k \notin T$ and, consequently, for any $j \in T$, $jR_k$. Then $T \subseteq S$; for if not, then $\exists h \in T$ such that $h \notin S$ and both $kR_h$ and $hR_k$, which is impossible (since $h \neq k$). This suffices to show that (nest) is satisfied. For (uni), the argument proceeds by contraposition; suppose that $\bigcup S \not\subseteq S_i$. Then $\exists j,k \in \mathcal{Y}_i$ such that $j \in \bigcup S$, $k \notin \bigcup S$, and it is not the case that $jR_k$. Since $j \in \bigcup S$, $\exists S \in S \subseteq S_i$ such that $j \in S$, $k \notin S$, and it is not the case that $jR_k$, which is impossible. Thus, $S \not\subseteq S_i$, as desired. A similar argument suffices for (int). Finally, suppose $j \in \bigcup S_i$ and $k \in \bigcup S_j$. Then since $iR_j$ and $jR_k$, by the transitivity of $\mathcal{R}$, $iR_k$, from which it follows that $k \in \bigcup S_i$, as required by (4s).\(^6\)

Finally, it must be shown that, for all $w \in W$, $\models^3_w \tau(\phi)$ if and only if $\models^3_w \phi$. Again, the proof is by induction on $\phi$. I consider the cases where $\phi$ is either of the form $p \in \Pi$ or $\psi \square \rightarrow \theta$.

\[ ^5 \text{Since } \mathcal{R} \text{ is reflexive, it is clear that } w \in \mathcal{Y}_w. \]

\[ ^6 \text{In general, observe that for } i,j \in W, iR_j \text{ if and only if } j \in \bigcup S_i. \text{ If } iR_j, \text{ then consider the set } \mathcal{Y}_i \text{ itself. It is clear that } j \in \mathcal{Y}_i \in S_i, \text{ thus } j \in \bigcup S_i. \text{ Conversely, if } j \in \bigcup S_i, \text{ then } \exists S \in S_i \text{ such that } j \in S \subseteq \mathcal{Y}_i, \text{ from which it follows that } iR_j. \]
For the first, $\models^\mathbb{S}_w^3 \tau(p)$ if and only if $\models^\mathbb{S}_w^3 \neg p \square \bot$ if and only if $\bigcup S_w \cap \neg p^\mathbb{S}_w^3 = \emptyset$ (the other case is impossible) if and only if $\forall x \in W$ such that $wR x$, $x \in V(p) = V_\mathbb{S}_w(\tau(p))$ if and only if $w \in V_\mathbb{S}_w(\tau(p))$ (by reflexivity and heredity) if and only if $\models^\mathbb{S}_w^3 p$. For the second, $\models^\mathbb{S}_w^3 \tau(\psi \rightarrow \theta)$ if and only if $\models^\mathbb{S}_w^3 (\tau(\psi) \land \neg \tau(\theta)) \square \bot$ if and only if $\bigcup S_w \cap [\tau(\psi) \land \neg \tau(\theta)]^\mathbb{S}_w^3 = \emptyset$ (the other case is impossible) if and only if $\forall x \in W$ such that $wR x$, $x \notin [\tau(\psi)]^\mathbb{S}_w^3 \cup [\theta]^\mathbb{S}_w^3$ (by the induction hypothesis) if and only if $\models^\mathbb{S}_w^3 \neg \psi \rightarrow \theta$.

\[\square\]

**THEOREM 31.** $\models_{\text{IPL}} \phi$ if and only if $\models_{\text{C}_4} \tau(\phi)$

Directly from lemmata 32 and 33.

\[\square\]

The Gödel embedding of intuitionistic logic into $\textbf{S}_4$ realizes the BHK interpretation in the sense that $\textbf{S}_4$ can be interpreted as a provability logic. Where $\square \phi$ is intuitively understood as expressing that $\phi$ is provable (in some mathematical theory, say), the characteristic theses of $\textbf{S}_4$ seem fairly plausible. With this interpretation of $\square$ in mind, it is natural to read $\phi \square \rightarrow \psi$ as expressing that $\psi$ is provable from $\phi$. Under such an interpretation, $\square \phi$ can reasonably be defined as $\neg \phi \square \rightarrow \bot$: for to show that $\phi$ is provable, it suffices—in classical logic—to show that its negation proves a contradiction.

It is implausible to hold that $\textbf{C}_4$ exhausts the logic of $\square \rightarrow$, understood as a binary provability connective. For instance, transitivity would seem to hold of such a relation. Nevertheless, it appears that (almost) all of the theses of $\textbf{C}_4$ respect such an interpretation.\(^7\)

Thus, theorem 31 can be viewed as a natural generalization of the Gödel embedding result.

\(^7\)Admittedly, (CS) does not appear to respect such an interpretation. I suspect $\text{IPL}$ can be embedded into $\textbf{C}_4$ sans (CS), since this system still suffices to capture $\textbf{S}_4$, but I have not actually verified this.
6.2 Intuitionistic Conditional Logic

In this section, I turn to conditional logic extensions of intuitionistic logic. Let the language \( \mathcal{L}_{\rightarrow} \) be an extension of \( \mathcal{L}_i \) by \( \rightarrow \). The set of formulae (the formation rules are standard) in \( \mathcal{L}_{\rightarrow} \) is denoted by \( \Phi_{\rightarrow} \). I begin by examining intuitionistic conditional logics axiomatically in subsection 6.2.1. In subsection 6.2.2, I continue this investigation semantically. Results relating the two approaches are presented later on in section 6.3.

6.2.1 Axiom Systems

In this subsection, I examine intuitionistic conditional logic from a syntactic perspective. Where \( L \) is a classical system of conditional logic,\(^8\) I denote its intuitionistic companion by \( IL \) (for a formal characterization of companionship, see definition 43 below). Thus, for example, \( ICK \) is \( IPL \) extended by all instances of (CM), (CC), (CN), and closed under (RCEA) and (RCEC), where the pertinent notion of theoremhood suitably modifies definition 40.

DEFINITION 43. A set of formulae is an *intuitionistic system of conditional logic* if it contains \( IPL \). An intuitionistic system of conditional logic is *proper* if it does not contain \( PL \) (classical propositional logic). An intuitionistic system of conditional logic is *nontrivial* if it contains axioms which are not already contained in \( IPL \). The *intuitionistic companion* of a classical system of conditional logic \( L \) is a system \( IL \) extending \( IPL \) with all instances of the same conditional schemata and closing under the same conditional rules (modulo the pertinent notion of theoremhood); classical companions are defined dually.

In this chapter, I will have occasion to discuss one schema that I haven’t yet explicitly mentioned, although it corresponds to a previously discussed semantic constraint, viz.

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\(^8\)Throughout this chapter, by ‘classical system of conditional logic’ I mean a system of conditional logic that extends classical propositional logic.
(norm). It is:

\[ \neg (\top \rightarrow \bot) \]  

(NORM)

Now, the main intuitionistic systems of conditional logic that I will be concerned with for the remainder of this chapter are listed in table 6.2.

Table 6.2: Intuitionistic Systems of Conditional Logic

<table>
<thead>
<tr>
<th>System</th>
<th>Extensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICK</td>
<td>IPL, (RCEA), (RCEC), (CM), (CC), (CN)</td>
</tr>
<tr>
<td>ICK+</td>
<td>ICK, (ID), (CMP)</td>
</tr>
<tr>
<td>ICE−</td>
<td>ICK, (CS)</td>
</tr>
<tr>
<td>ICE</td>
<td>ICE−, (NORM)</td>
</tr>
</tbody>
</table>

Table 6.2 requires a couple comments. First, I will not be interested in any subnormal intuitionistic systems of conditional logic in this chapter. That is, all systems of intuitionistic conditional logic I examine will contain ICK. I record the following important fact about extensions of ICK.

**PROPOSITION 18.** Any extension of ICK is closed under (RCK)

Since the argument given for proposition 4 in chapter 2 is intuitionistically valid, it also suffices to establish this result; see also Weiss [125, §3.1].

Second, I will, for the most part, not be interested in the intuitionistic companions of stronger conditional logics like Lewis’ C1. The main reason for this is that a certain ambiguity attaches to IC0 and its extensions. Recall the schema (CV), which I reproduce below for the reader’s convenience:

\[ (((\phi \Box \rightarrow \psi) \land \neg (\phi \Box \rightarrow \neg \theta)) \rightarrow ((\phi \land \theta) \Box \rightarrow \psi) \]  

(CV)
Under definition (◊→Df.), where φ ◊→ ψ is defined as ¬(φ □→ ¬ψ) (cf. Lewis [67, p. 2]), (CV) is equivalent to:

\[(\phi \circrightarrow \psi) \land (\phi \circleftarrow \theta) \rightarrow ((\phi \land \theta) \circrightarrow \psi)\]  

(CV*)

Now, □→ and ◊→ will not generally be interdefinable in systems of intuitionistic conditional logic (compare with ∀ and ∃ in intuitionistic predicate logic), at least not if they are given their expected truth conditions. Consequently, these two versions of (CV) will come apart. The original version, which is fit for a language including only □→, seems to correspond to no particularly enlightening semantic condition; the latter version may, but at the cost of augmenting the language. While a detailed examination of intuitionistic systems of conditional logic containing both □→ and ◊→ is desirable, due to the complexities such an undertaking presents in its details, it is beyond the scope of this chapter. Consequently, I mostly bracket consideration of strong systems of intuitionistic conditional logic for future work. Nevertheless, one comment must be made about IC2.

It is a fact deserving of note that not all classical systems of conditional logic have proper intuitionistic companions. That is, for some classical systems of conditional logic L, L and IL coincide. The paradigmatic example of this is Stalnaker’s C2:

**PROPOSITION 19.** Any intuitionistic system of conditional logic containing all instances of both (CMP) and (CEM) also contains all instances of:

\[\phi \lor \neg\phi\]  

(LEM)
1 \((\top \rightarrow \phi) \lor (\top \rightarrow \neg \phi)\) CEM
2 \((\top \rightarrow \phi) \rightarrow (\top \rightarrow \phi)\) CMP
3 \((\top \rightarrow \neg \phi) \rightarrow (\top \rightarrow \neg \phi)\) CMP
4 \((\top \rightarrow \phi) \lor (\top \rightarrow \neg \phi)\) IPL 1-3
5 \(\top\) ⊢
6 \(\phi \lor \neg \phi\) IPL 4, 5

COROLLARY 4. IC2 is not proper: IC2 and C2 coincide

In the axiom schemata and rules from tables 2.1 and 2.2, if \(\rightarrow\) is replaced by the intuitionistic conditional \(\rightarrow\), not all of the resulting formulae are theorems of IPL. For example, (MOD), under such a replacement, is not a theorem. Nevertheless, since IPL is a subsystem of PL, consistency can be proved simply using translations into classical logic (I mention only those schemata below that will be of concern hereafter):

THEOREM 32 (Consistency). Let \(L\) be any extension of ICK by (ID), (CMP), (CS), (MOD), or (NORM). Then \(L\) is consistent

Let \(\Phi_c\) be the set of all formulae of classical propositional logic. I define a function \(\mu : \Phi_{\rightarrow} \to \Phi_c\) which maps each standard intuitionistic connective to its classical counterpart (e.g. \(\mu(\neg \phi) = \neg \mu(\phi)\)) and \(\mu(\phi \rightarrow \psi) = \mu(\phi) \rightarrow \mu(\psi)\). Under \(\mu\), every axiom of \(L\) is mapped to a classical tautology, and the rules preserve this property. Then \(\bot\) is not a theorem of \(L\), since \(\mu(\bot)\) is not a tautology.

I turn now to the development of intuitionistic modal logic within intuitionistic conditional logic. The language of intuitionistic modal logic is \(\mathcal{L}_i\) extended by \(\Box\), \(\mathcal{L}_i\Box\), and the set of its formulae is \(\Phi_{i\Box}\). I adopt the same conventions for referring to these modal systems as in subsection 2.1.2, but prefixing an ‘I’ as above. Thus, IKT (for example) is the smallest
system closed under (NEC) and (MP) which contains IPL and all instances of (K) and (T). With minimal modification to the proofs given in subsection 2.1.2, it can be shown that, under (□Df. 2), ICK and ICK⁺ exactly realize the intuitionistic companions of K and KT, viz. IK and IKT [125, §4.2].

With a view to potential applications of some of the intuitionistic systems of conditional logic listed in table 6.2, two heterodox systems of intuitionistic modal logic recently discussed by Artemov and Protopopescu [6, 105] deserve comment. Both systems include the so-called “co-reflection” scheme:

\[ \phi \rightarrow \Box \phi \]  

(CO-R)

The system IEL⁻ is IK extended by (CO-R); IEL is obtained by extending IEL⁻ with (D) [105, pp. 9-10].

IEL⁻ and IEL are named as such because they are supposed to be intuitionistic epistemic logics, that is, logics of knowledge or verification. Intuitively, \( \Box \phi \) is to be read as expressing that \( \phi \) is verified, where verification is a more liberal notion than proof. Since any proof yields a verification, (CO-R) is intuitively valid given the BHK semantics for intuitionistic logic: if \( \phi \) is true, i.e. proved, then it is verified [105, p. 9]. Conversely, precisely because not every verification is a proof, (T) fails. (D), then, is also intuitively satisfactory since \( \bot \) cannot be verified.

As the reader may have guessed, IEL⁻ and IEL can be embedded into ICE⁻ and ICE respectively under (□Df. 2). This, in turn, suggests an epistemic reading of \( \Box \rightarrow \) in these systems. Before examining that possibility, I sketch the embedding result; I prove this for IEL and ICE, thereby also establishing the result for the weaker systems.

**Definition 44.** Let \( \lambda : \Phi_\Box \rightarrow \Phi_\rightarrow \) be the function:

1. \( \lambda(p) = p \)

---

9Božić and Došen [10, 22] refer to these systems as HK\( \Box \) and HT\( \Box \) respectively.
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2. \( \lambda(\neg \phi) = \neg \lambda(\phi) \)

3. \( \lambda(\phi \lor \psi) = \lambda(\phi) \lor \lambda(\psi) \)

4. \( \lambda(\phi \land \psi) = \lambda(\phi) \land \lambda(\psi) \)

5. \( \lambda(\phi \rightarrow \psi) = \lambda(\phi) \rightarrow \lambda(\psi) \)

6. \( \lambda(\Box \phi) = \top \rightarrow \lambda(\phi) \)

**LEMMA 34.** If \( \vdash_{\text{IEL}} \phi \), then \( \vdash_{\text{ICE}} \lambda(\phi) \)

By induction on the length of proof. I show the only interesting case, viz. \( \vdash_{\text{ICE}} \lambda(\phi \rightarrow \Box \phi) \).

Note that \( \lambda(\phi \rightarrow \Box \phi) = \lambda(\phi) \rightarrow (\top \rightarrow \lambda(\phi)) \).

<table>
<thead>
<tr>
<th>Step</th>
<th>Line</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \lambda(\phi) )</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>( \top \land \lambda(\phi) )</td>
<td><strong>IPL</strong> 1</td>
</tr>
<tr>
<td>3</td>
<td>( \top \land \lambda(\phi) )</td>
<td><strong>CS</strong> 2</td>
</tr>
<tr>
<td>4</td>
<td>( \lambda(\phi) \rightarrow (\top \land \lambda(\phi)) )</td>
<td><strong>IPL</strong> 1-3</td>
</tr>
</tbody>
</table>

**DEFINITION 45.** Let \( \lambda^{-1} : \Phi_{\rightarrow} \rightarrow \Phi_{\Box} \) be the same as \( \lambda \) except:

6. \( \lambda^{-1}(\phi \rightarrow \psi) = \Box(\lambda^{-1}(\phi) \rightarrow \lambda^{-1}(\psi)) \)

**LEMMA 35.** If \( \vdash_{\text{ICE}} \phi \), then \( \vdash_{\text{IEL}} \lambda^{-1}(\phi) \)

By induction on the length of proof.

**LEMMA 36.** \( \vdash_{\text{IEL}} \phi \leftrightarrow \lambda^{-1}(\lambda(\phi)) \)

Exactly as in [125, §4.2].
THEOREM 33 (Modal Embedding). $\vdash_{\text{IEL}} \phi$ if and only if $\vdash_{\text{ICE}} \lambda(\phi)$

The result follows immediately from lemmata 34, 35, and 36.

In light of theorem 33, epistemic readings of $\square \rightarrow$ in the systems ICE$^-$ and ICE are worth considering. For suppose that $\phi \rightarrow \psi$ is understood to express that, given $\phi$ is true (or proved), $\psi$ is verified. Then, since $\top$ is clearly proved, $\top \rightarrow \phi$ intuitively expresses that $\phi$ is verified, i.e. $\square \phi$. The axioms of ICE (ICE$^-$) generally make good sense on this interpretation, as do at least some of the omissions. (CS) can be justified in by noting that if $\phi \land \psi$ is proved, then both of the conjuncts are verified, and therefore verified given either conjunct. But (CMP) is not justified, since the fact that $\psi$ is verified given a proof that $\phi$ does not entail that $\psi$ has a proof given a proof that $\phi$, because verification is weaker than proof.

6.2.2 Semantics

In this section, I examine intuitionistic conditional logic from a semantic perspective. I develop worlds semantics for such logics using the framework of relational proposition indexed interpretations, discussed in subsection 2.2.1. The motivation for using this sort of semantics, as opposed to the neighborhood variant, is that the constraint required for intuitionistic heredity can be more perspicuously stated in relational terms. In addition, I develop algebraic semantics for these logics and present equivalence results.\(^{10}\)

DEFINITION 46. An intuitionistic proposition indexed interpretation is a structure $\mathcal{J}_R = \langle W, \mathcal{R}, P, \{R_X : X \in P\}, V \rangle$ where $\langle W, \mathcal{R}, V \rangle$ is an IPL interpretation (definition 41) and in addition:\(^{11}\)

\(^{10}\)The results of this section generalize and expand on those I presented in [125].

\(^{11}\)For $R, S \subseteq W$, define $R \circ S = \{(x, y) : \exists z(x R z S y)\}$. I typically just write $RS$ for $R \circ S$. I follow Božić and Došen [10] in employing this device in the relational semantics. Incidentally, many tools from intuitionistic
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4. \( R_X \subseteq W \times W \) such that \( R \circ R_X \subseteq R_X \circ R \)

5. \( P \subseteq \mathcal{P}(W) \) such that:
   
   (a) \( \emptyset \in P \)
   
   (b) If \( S \in P \) and \( T \in P \), then \( S \cup T \in P \) and \( S \cap T \in P \)
   
   (c) If \( S \in P \) and \( T \in P \), then \( \{ x \in W : \forall y(x \mathcal{R} y \text{ and } y \in S \Rightarrow y \in T) \} \in P \)
   
   (d) If \( S \in P \) and \( T \in P \), then \( \{ x \in W : \forall y(x \mathcal{R}_S y \Rightarrow y \in T) \} \in P \)

6. For all \( p \in \Pi \), \( V(p) \in P \)

Then the truth conditions for complex formulae at a world \( w \) are the same as above except:

\[ \models_{w}^{J_R} \phi \square \psi \text{ if and only if } \{ y \in W : wR[\phi]y \} \subseteq [\psi] \]

**LEMMA 37.** Given an intuitionistic proposition indexed interpretation \( J_R = \langle W, \mathcal{R}, P, \{ R_X : X \in P \}, V \rangle \), for all \( \phi \in \Phi_{\Pi^+} \), \( [\phi] \in P \)

The proof is by induction on the complexity of \( \phi \). The result holds when \( \phi \) is a propositional variable by definition 46. If \( S, T \in P \), write \( S \rightarrow T = \{ x \in W : \forall y(x \mathcal{R} y \text{ and } y \in S \Rightarrow y \in T) \} \). Then if \( \phi \) is of the form \( \psi \rightarrow \theta \), it is clear that \( [\psi \rightarrow \theta] = [\psi] \rightarrow [\theta] \in P \) by the induction hypothesis and definition 46. Where \( \phi \) is of the form \( \neg \psi \), to show that \( [\neg \psi] \in P \) it suffices to show that \( [\neg \psi] = [\psi] \rightarrow \emptyset \), since the latter is in \( P \) by the induction hypothesis and definition 46. Suppose \( w \in [\neg \psi] \) and \( w\mathcal{R}x \); then \( x \not\in [\psi] \). Therefore, it is vacuously the case that for all \( x \) such that \( w\mathcal{R}x \) and \( x \in [\psi], x \notin \emptyset \). That is, \( w \in [\psi] \rightarrow \emptyset \). Conversely, suppose that \( w \in [\psi] \rightarrow \emptyset \) and \( w\mathcal{R}x \); if \( x \in [\psi], \) it would follow that \( x \in \emptyset \), which is impossible. Thus, since it must be the case that \( x \not\in [\psi], w \in [\neg \psi] \). The other cases are all straightforward.

---

modal logic can be adapted to intuitionistic conditional logic with ease. For an overview of intuitionistic modal logic, see Simpson [112].
LEMMA 38 (Intuitionistic Heredity). Given an intuitionistic proposition indexed interpretation $\mathfrak{I}_R = (W,\mathcal{R},P,\{R_X : X \in P\},V)$, for all $\phi \in \Phi_{\rightarrow}$, $[\phi]$ is closed under $\mathcal{R}$ (i.e. if $w \in [\phi]$ and $wR x$, then $x \in [\phi]$)

Again, the proof is by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\psi \rightarrow \theta$. Suppose $w \in [\psi \rightarrow \theta]$, $wR x$, and $xR_{\psi}y$. Since $wR_{\psi}zR y$, there is a $z$ such that $wR_{\psi}zR y$. Then $z \in [\theta]$, from which it follows by the induction hypothesis that $y \in [\theta]$, which was to be proved.

\[\square\]

Stronger systems of intuitionistic conditional logic can be modeled by imposing various constraints on intuitionistic proposition indexed interpretations. If $S$ is a proposition in $P$, write $\rightarrow S$ for $S \mapsto \emptyset = \{x \in W : \forall y (xR y \Rightarrow y \notin S)\}$. Some constraints of significance are listed in table 6.3 below.

Table 6.3: Intuitionistic Relational Constraints

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${y : xR_{\psi}y} \subseteq S$</td>
<td>(id)</td>
</tr>
<tr>
<td>If $x \in S$, then $x \in {y : xR_{\psi}y}$</td>
<td>(cmp)</td>
</tr>
<tr>
<td>If $x \in S$, then ${y : xR_{\psi}y} \subseteq {y : xR y}$</td>
<td>(sc')</td>
</tr>
<tr>
<td>If ${y : xR_{\rightarrow\psi}y} \subseteq S$, then ${y : xR_{\rightarrow}y} \subseteq S$</td>
<td>(mod')</td>
</tr>
<tr>
<td>${y : xR_{\rightarrow\psi}y} \neq \emptyset$</td>
<td>(norm)</td>
</tr>
</tbody>
</table>

Many of the constraints listed in table 6.3 can be found (in somewhat different notation) in Segerberg [111, p. 163] and Unterhuber and Schurz [122, p. 905]. The conditions marked with an apostrophe (’) are those which depart from the classical conditions in some nontrivial way.
Let \( L \) be any extension of \( \text{ICK} \) by (ID), (CMP), (CS), (MOD), or (NORM). I use \( C^L_H \) to designate the class of all intuitionistic proposition indexed interpretations satisfying the constraints (from table 6.3) associated with \( L \). Validity is defined, in the usual way, with respect to each such class.

**DEFINITION 47.** A Heyting conditional algebra is a structure \( H = \langle A, * \rangle \) in which \( A = \langle A, 1, 0, \cup, \cap, \rightarrow \rangle \) is a Heyting algebra and * is a binary operation on \( A \) subject to the following conditions:

1. \( a * (b \cap c) = (a * b) \cap (a * c) \)
2. \( a * 1 = 1 \)

Recall that a Heyting algebra is a bounded distributive lattice such that \( \forall a, b, \exists (a \rightarrow b) \) such that \( \forall c, a \cap c \leq b \) if and only if \( c \leq a \rightarrow b \). Define the pseudocomplement of \( a, \div a \), as \( a \rightarrow 0 \).

**DEFINITION 48.** A Heyting conditional interpretation is a structure \( J_H = \langle H, f \rangle \) in which \( H \) is a Heyting conditional algebra and \( f : \Pi \rightarrow A \) is extended in such a way that:

1. \( f(\phi \lor \psi) = f(\phi) \cup f(\psi) \)
2. \( f(\phi \land \psi) = f(\phi) \cap f(\psi) \)
3. \( f(\phi \rightarrow \psi) = f(\phi) \rightarrow f(\psi) \)
4. \( f(\neg \phi) = \div f(\phi) \)
5. \( f(\phi \rightarrow \psi) = f(\phi) * f(\psi) \)
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Table 6.4: Heyting Algebraic Constraints

\[
a \ast a = 1 \quad (\text{id*})
\]

\[
a \ast b \leq a \hookrightarrow b \quad (\text{cmp*})
\]

\[
a \cap b \leq a \ast b \quad (\text{cs*})
\]

\[
\div a \ast a \leq b \ast a \quad (\text{mod*})
\]

\[
1 \ast 0 = 0 \quad (\text{norm*})
\]

**DEFINITION 49.** For \( \mathfrak{J}_H = \langle H, f \rangle \), write \( \models_{\mathfrak{J}_H} \phi \) if and only if \( f(\phi) = 1 \). Let \( L \) be any extension of \( \text{ICK} \) by (ID), (CMP), (CS), (MOD), or (NORM); write \( C^L_H \) for the class of all Heyting conditional interpretations satisfying the constraints (from table 6.4) associated with \( L \). Then \( \models_{C^L_H} \phi \) if \( \models_{\mathfrak{J}_H} \phi \) for all \( \mathfrak{J}_H \in C^L_H \), in which case \( \phi \) is said to be valid (in \( C^L_H \)).

Where \( L \) is any extension of \( \text{ICK} \) by (ID), (CMP), (CS), (MOD), or (NORM), I wish to show that \( C^L_H \) and \( C^L_R \) characterize the same validities. The proof of this fact modifies that of theorem 6. It should be compared with the proof of theorem 1 from my [125, \S 2.3] and related results in Nute [92, Ch. 7].

**LEMMA 39.** Given an intuitionistic proposition indexed interpretation \( \mathfrak{J}_R = \langle W, R, P, \{R_X : X \in P\}, V \rangle \in C^L_R \), there is a Heyting conditional interpretation \( \mathfrak{J}_H = \langle H, f \rangle \in C^L_H \) such that \( \models_{\mathfrak{J}_H} \phi \) if and only if \( \models_{\mathfrak{J}_R} \phi \).

A Heyting conditional interpretation \( \mathfrak{J}_H = \langle \mathcal{A}, 1, 0, \cup, \cap, \hookrightarrow, \ast \rangle, f \rangle \) is constructed from the given intuitionistic proposition indexed interpretation \( \mathfrak{J}_R = \langle W, R, P, \{R_X : X \in P\}, V \rangle \). Set \( \mathcal{A} = \{Q \in P : Q \text{ is } R\text{-closed}\} \) and take \( \cup \) and \( \cap \) to be ordinary set union and intersection respectively. \( \hookrightarrow \) is \( \hookrightarrow \), \( \leq \) is \( \subseteq \), 0 is \( \emptyset \), and 1 is \( W \). It is not difficult to verify that \( \langle \mathcal{A}, 1, 0, \cup, \cap, \hookrightarrow \rangle \), so defined, is a Heyting algebra. In particular, for \( S, T \in P \), note that
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S \mapsto T is \mathcal{R}\text{-closed whenever its arguments are; hence, if } a, b \in \mathbb{A}, a \leftrightarrow b \in \mathbb{A}. To show that 
\mapsto satisfies the required properties, observe that if } S, T, U \in P \text{ are } \mathcal{R}\text{-closed, if } S \cap U \subseteq T, 
\text{then } S \subseteq U \mapsto T; \text{ for if } w \in S, wR x, \text{ and } x \in U, \text{ then } x \in S, \text{ from which it follows by the } 
\text{assumption that } x \in T, \text{ which was to be proved. Conversely, if } w \in S \cap U \text{ and } S \subseteq U \mapsto T, 
\text{then since } wRw, w \in T, \text{ as desired.}

To obtain a Heyting conditional algebra, set } a \ast b = \{ w \in W : \forall x(wR_a x \Rightarrow x \in b) \}. \text{ It must be verified that } a \ast b \in P \text{ is } \mathcal{R}\text{-closed and that } \ast \text{ satisfies the conditions specified in } 
definition 47. \text{ Suppose } w \in a \ast b, wR x, \text{ and } xR_a y. \text{ Since } \mathcal{R} \circ R_a \subseteq R_a \circ \mathcal{R}, \exists z \text{ such that } 
wR_a zR_y. \text{ Then, since } z \in b, \text{ it follows from the fact that } b \text{ is } \mathcal{R}\text{-closed that } y \in b, \text{ which was to be proved. That } \ast \text{ satisfies the basic conditions is straightforward to show and is omitted; I turn now to the specialized conditions.}

If \mathfrak{J}_R \text{ satisfies (id), it must be shown that } \mathfrak{J}_H \text{ satisfies (id\ast). Since by (id), } \{ x : wR_a x \} \subseteq a 
\text{ for all } w, \text{ it is clear that } a \ast a = \{ w \in W : \{ x : wR_a x \} \subseteq a \} = W = 1, \text{ as required.}

If \mathfrak{J}_R \text{ satisfies (cmp), it must be shown that } \mathfrak{J}_H \text{ satisfies (cmp\ast). Suppose } w \in a \ast b = 
\{ x \in W : \forall y(xR_a y \Rightarrow y \in b) \}, wR z, \text{ and } z \in a. \text{ Since } a \ast b \text{ is } \mathcal{R}\text{-closed, } z \in a \ast b. \text{ By (cmp), } 
z \in \{ x : zR_a x \}. \text{ Thus, } z \in b, \text{ which suffices to show that } w \in a \leftrightarrow b.

If \mathfrak{J}_R \text{ satisfies (sc'), it must be shown that } \mathfrak{J}_H \text{ satisfies (cs\ast). Suppose } w \in a \cap b \text{ and } 
wR_a x; \text{ by (sc'), since } w \in a \text{ and } wR_a x, wR x. \text{ Thus, since } w \in b \text{ and } b \text{ is } \mathcal{R}\text{-closed, } x \in b, \text{ which was to be proved.}

If \mathfrak{J}_R \text{ satisfies (mod'), it must be shown that } \mathfrak{J}_H \text{ satisfies (mod\ast). Suppose that } w \in 
\mathbin{\downarrow} a \ast a = \{ x \in W : \{ y : xR_{\downarrow a} y \} \subseteq a \}; \text{ then since } \{ y : wR_{\downarrow a} y \} \subseteq a, \text{ by } 
\text{(mod'), } \{ y : wR_b y \} \subseteq a. \text{ That is, } w \in b \ast a, \text{ which was to be proved.}

If \mathfrak{J}_R \text{ satisfies (norm), it must be shown that } \mathfrak{J}_H \text{ satisfies (norm\ast). Suppose, for contradiction, that there were a } w \in 1 \ast 0 = \{ w \in W : \{ x : wR_1 x \} \subseteq 0 \}. \text{ By (norm), } \exists y \text{ such that } 
wR_1 y. \text{ Hence, } y \in 0 = \emptyset, \text{ which is impossible.}

To obtain the Heyting conditional interpretation } \mathfrak{J}_H = \langle \mathcal{H}, f \rangle, \text{ set } f(\phi) = [\phi] \text{ for all } \phi.
It is clear from lemma 38 that each \( \phi \in A \). It only remains to verify that the conditions of definition 48 are met; but this is trivial from the definitions of the operations. Finally, since \( f(\phi) = 1 \) if and only if \( \phi = W \), it is clear that the result obtains.

Recall the definitions of filters, proper filters, and prime filters from subsection 2.2.2. Given a lattice \( \langle D, 1, 0, \cup, \cap \rangle \), the filter generated by a nonempty set \( S \subseteq D \) is the set \( \{ x \in D : \exists y_0, \ldots, y_n \in S ( x \geq y_0 \cap \ldots \cap y_n ) \} \).

**LEMMA 40.** Given a filter \( \nabla \) such that \( a \ast c \notin \nabla \), the filter generated by \( \{ b : a \ast b \in \nabla \} \) does not include \( c \)

The proof is the same as in my [125, §2.3], but I repeat it here. Suppose the filter generated by \( \{ b : a \ast b \in \nabla \} \) does contain \( c \). Then \( \exists d_0, \ldots, d_n \in \{ b : a \ast b \in \nabla \} \) such that \( d_0 \cap \ldots \cap d_n \leq c \). Since lemma 8 also holds for any Heyting conditional algebra, it follows that \( (a \ast d_0) \cap \ldots \cap (a \ast d_n) = a \ast (d_0 \cap \ldots \cap d_n) \leq a \ast c \). Since \( \nabla \) is a filter and each \( (a \ast d_i) \in \nabla \), \( a \ast (d_0 \cap \ldots \cap d_n) \in \nabla \). But then, since \( \nabla \) is upward closed, \( a \ast c \in \nabla \).

**LEMMA 41.** Given a filter \( \nabla \) such that \( a \notin \nabla \), \( \nabla \) can be extended to a prime filter \( \nabla^* \) such that \( a \notin \nabla^* \)

Proof is omitted; see [29], [106].

**LEMMA 42.** Given a Heyting conditional interpretation \( \mathfrak{J}_H = \langle \mathcal{H}, f \rangle \in \mathcal{C}_H \), there is an intuitionistic proposition indexed interpretation \( \mathfrak{J}_R = \langle W, \mathcal{R}, P, \{ R_X : X \in P \}, V \rangle \in \mathcal{C}_R \) such that \( \models^{\mathfrak{J}_H} \phi \) if and only if \( \models^{\mathfrak{J}_R} \phi \)

An intuitionistic proposition indexed interpretation \( \mathfrak{J}_R = \langle W, \mathcal{R}, P, \{ R_X : X \in P \}, V \rangle \) must be constructed from \( \mathfrak{J}_H \). Let \( W \) be the set of all prime filters in \( \mathcal{H} \), let \( \mathcal{R} \) be \( \subseteq \), and
write $[a] = \{ w \in W : a \in w \}$. Then put $P = \{ [a] : a \in A \}$, $xR_{[a]}y$ if and only if $\{ b : a * b \in x \} \subseteq y$, and $[\phi] = [f(\phi)]$ for all $\phi$. This completes the construction; it remains to verify that it satisfies all the requisite constraints.

It is obvious that $R$ is reflexive and transitive since $\subseteq$ is. If $w \in V(p) = [p] = [f(p)]$ and $w \subseteq x$, then clearly $f(p) \in x$, as required for heredity. Suppose that $wR_0yR_{[a]}x$; since $w \subseteq w$ and $\{ b : a * b \in w \} \subseteq \{ b : a * b \in y \} \subseteq x$, $wR_0wR_{[a]}x$, as required.

It is somewhat more complicated to verify that $P$ satisfies its required conditions; I show a few cases here. Note that $\emptyset \in P$, because $\emptyset = [0] \in P$. If $[a], [b] \in P$, then $[a] \cup [b] \in P$ because $[a] \cup [b] = [a \cup b]$. For if $x \in [a \cup b]$, since $x$ is prime, $x$ must be in either $[a]$ or $[b]$; and the converse follows from the fact that $x$ is upward closed. To show that $[a] \mapsto [b] \in P$, it suffices to show that $[a] \mapsto [b] = [a \mapsto b]$. If $a \mapsto b \in w$, it is easily shown that $w \in [a] \mapsto [b]$ using the properties of filters and Heyting algebras. Conversely, suppose $a \mapsto b \not\in w$; then lemma 41 can be used to show that $w \not\in [a] \mapsto [b]$. I turn now to the special conditions.

If $J_H$ satisfies $(id^*)$, it must be shown that $J_R$ satisfies $(id)$. Suppose that $wR_{[a]}x$, i.e. $\{ b : a * b \in w \} \subseteq x$; by $(id^*)$, $a * a = 1 \in w$, hence $a \in x$, i.e. $x \in [a]$.

If $J_H$ satisfies $(cmp^*)$, it must be shown that $J_R$ satisfies $(cmp)$. Suppose $w \in [a]$ and $a * b \in w$; to show $wR_{[a]}w$, it suffices to show that $b \in w$. Since $a * b \in w$, by $(cmp^*)$, $a \mapsto b \in w$; since $a \in w$, it follows that $b \in w$, which was to be proved.

If $J_H$ satisfies $(cs^*)$, it must be shown that $J_R$ satisfies $(sc')$. Suppose $w \in [a]$ and $wR_{[a]}x$; then $a \cap a = a \in w$, so by $(cs^*)$, $a * a \in w$. Accordingly, $a \in x$ since $wR_{[a]}x$. Since $a \in w$ is arbitrary, $w \subseteq x$, which was to be proved.

If $J_H$ satisfies $(mod^*)$, it must be shown that $J_R$ satisfies $(mod')$. Suppose $\{ x : wR_{[a]}x \} \subseteq [a]$ and $wR_{[b]}y$. It is clear, using lemmata 40 and 41, that $\vdash a * a \in w$. Hence, by $(mod^*)$, $b * a \in w$, from which it follows that $a \in y$, as desired.

If $J_H$ satisfies $(norm^*)$, it must be shown that $J_R$ satisfies $(norm)$. Then $1 * 0 = 0 \not\in w$. 

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Using lemmata 40 and 41, there is a prime filter \( y \) such that \( \{ b : 1 \ast b \in w \} \subseteq y \), i.e. \( wR_{[1]}y \).

The truth conditions must be verified; I examine only the case of \( \phi \rightarrow \psi \) (for the other cases, see Fitting [29]). Suppose \( w \in [\phi \rightarrow \psi] = [f(\phi \rightarrow \psi)] = [f(\phi) \ast f(\psi)] \) and \( wR_{f(\phi)}x \). Then since \( \{ b : f(\phi) \ast b \in w \} \subseteq x \), \( f(\psi) \in x \). Thus, \( x \in [f(\psi)] = [\psi] \), as desired. Conversely, suppose that \( w \not\in [\phi \rightarrow \psi] \); then \( f(\phi) \ast f(\psi) \not\in w \). Let \( \nabla \) be the filter generated by \( \{ b : f(\phi) \ast b \in w \} \); by lemma 40, \( f(\psi) \not\in \nabla \). Then, by lemma 41, there is a prime filter \( x \) such that \( \{ b : f(\phi) \ast b \in w \} \subseteq \nabla \subseteq x \) and \( f(\psi) \not\in x \). It is clear that \( wR_{[\phi]}x \) and \( x \not\in [\psi] \), which was to be proved.

Finally, if \( f(\phi) = 1 \), \( [\phi] = [f(\phi)] = W \), since every world is a filter. Conversely, if \( f(\phi) \neq 1 \), since \( \{1\} \) is a filter and \( f(\phi) \not\in \{1\} \), by lemma 41 there is a world \( w \) such that \( w \not\in [\phi] \), which was to be proved.

\[ \square \]

**THEOREM 34** (Equivalence). \( \models_{c_R} \phi \) if and only if \( \models_{c_R} \phi \)

The result follows directly from lemmata 39 and 42.

\[ \square \]

Theorem 34 has useful applications for obtaining decidability and independence results. I will not address these topics here, but the interested reader can find a discussion of both topics in my [125].

### 6.3 Determination Results

Throughout this section, let \( L \) be any extension of \( \textbf{ICK} \) by \( \text{(ID)} \), \( \text{(CMP)} \), \( \text{(CS)} \), \( \text{(MOD)} \), or \( \text{(NORM)} \). I prove soundness and completeness for all such \( L \) (including the systems described in table 6.2) with respect to \( C_R^L \). (Weak) soundness and completeness results with respect to classes of Heyting conditional interpretations follow as corollaries.
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THEOREM 35 (Soundness). \( \Gamma \vdash \phi \) implies \( \Gamma \models_{\mathcal{L}_R} \phi \)

All the axioms of \( \mathcal{L} \) must be shown to be valid in \( \mathcal{C}_R^\mathcal{L} \) and all the rules of \( \mathcal{L} \) must be shown to be validity preserving in \( \mathcal{C}_R^\mathcal{L} \). I take (RCEC), (CMP), and (MOD) as representative cases.

For (RCEC), suppose that \( \models_{\mathcal{C}_R^\mathcal{L}} \phi \leftrightarrow \psi \), but \( \not\models_{\mathcal{C}_R^\mathcal{L}} (\theta \Box \phi) \leftrightarrow (\theta \Box \psi) \). Without loss of generality, there exists some world \( w \) of an interpretation \( \mathfrak{I}_R \in \mathcal{C}_R^\mathcal{L} \) such that \( wRx, \models_{\mathfrak{I}_R} \theta \Box \phi \), and \( \not\models_{\mathfrak{I}_R} \theta \Box \psi \). Then there is a \( y \) such that \( xR[y]y, y \not\in [\psi] \), and \( y \in [\phi] \). But since, by the assumption, \([\phi] = [\psi]\), this is impossible.

For (CMP), suppose that \( \not\models_{\mathcal{C}_R^\mathcal{L}} (\phi \Box \rightarrow \psi) \rightarrow (\phi \rightarrow \psi) \). Then there is some \( w \) of some \( \mathfrak{I}_R \in \mathcal{C}_R^\mathcal{L} \) such that \( wRx, \models_{\mathfrak{I}_R} \psi \), and \( \not\models_{\mathfrak{I}_R} \phi \rightarrow \psi \). Hence, \( \exists y \) such that \( xR[y], \models_{\mathfrak{I}_R} \phi \), and \( \not\models_{\mathfrak{I}_R} \psi \). By (cmp), \( yR[\phi]y \). By lemma 38, \( \models_{\mathfrak{I}_R} \phi \Box \rightarrow \psi \). Thus, \( y \in [\psi] \), which is impossible.

For (MOD), suppose that \( \not\models_{\mathcal{C}_R^\mathcal{L}} (\neg \phi \Box \rightarrow \phi) \rightarrow (\psi \Box \rightarrow \phi) \). Then there is some \( w \) of some \( \mathfrak{I}_R \in \mathcal{C}_R^\mathcal{L} \) such that \( wRx, \models_{\mathfrak{I}_R} \neg \phi \Box \rightarrow \phi \), and \( \not\models_{\mathfrak{I}_R} \psi \Box \rightarrow \phi \). Hence, \( \exists y \) such that \( xR[y]y \) and \( y \not\in [\phi] \), and also \( \{ z : xR_{[\phi]} z \} = \{ z : xR_{[\neg \phi]} z \} \subseteq [\phi] \). Then by (mod'), \( y \in [\phi] \), which is impossible.

COROLLARY 5 (Algebraic Soundness). \( \vdash \phi \) implies \( \models_{\mathcal{C}_R^\mathcal{L}} \phi \)

The (weak) soundness of \( \mathcal{L} \) with respect to \( \mathcal{C}_R^\mathcal{L} \) is a straightforward consequence of theorems 34 and 35.

DEFINITION 50. A set of formulae \( \Gamma \) is nice if and only if:

1. \( \Gamma \) is consistent

2. \( \Gamma \) is closed under \( \vdash_\mathcal{L} \) (i.e. if \( \Gamma \vdash_\mathcal{L} \phi \), then \( \phi \in \Gamma \))

3. \( \Gamma \) is prime (i.e. if \( \phi \lor \psi \in \Gamma \), then either \( \phi \in \Gamma \) or \( \psi \in \Gamma \))
LEMMA 43. Given a set of formulae $\Gamma$ and a formula $\phi$, if $\Gamma \not\vdash_L \phi$, then there exists a nice set $\Delta$ such that $\Gamma \subseteq \Delta$ and $\phi \notin \Delta$.

The proof of this lemma is standard and is omitted; see, e.g., [10, p. 225].

□

DEFINITION 51. $[\phi]^L = \{ \Gamma \subseteq \Phi : \Gamma \text{ is nice, } \phi \in \Gamma \}$.\(^{12}\)

DEFINITION 52. The canonical model for $L$ is a structure $J^L_R = \langle W, R, P, \{ R_X : X \in P \}, V \rangle$ defined as follows:

1. $W = \{ \Gamma \subseteq \Phi : \Gamma \text{ is nice} \}$
2. $P = \{ [\phi] : \phi \in \Phi \}$
3. For $x, y \in W$, $xRy$ if and only if $x \subseteq y$
4. For $x, y \in W$, $xR[\phi]y$ if and only if $\{ \psi : (\phi \rightarrow \psi) \in x \} \subseteq y$
5. $V(p) = [p]$ for all $p \in \Pi$

LEMMA 44. Let $J^L_R = \langle W, R, P, \{ R_X : X \in P \}, V \rangle$ be the canonical model for $L$. Then $J^L_R$ is well-defined and $J^L_R \in C^L_R$

To show that $R_{[\phi]}$ is well-defined in $J^L_R$, suppose that $[\phi] = [\psi]$. If $\forall_L \phi \leftrightarrow \psi$, then by lemma 43, $[\phi] \neq [\psi]$. Hence, $\vdash_L \phi \leftrightarrow \psi$, from which it follows by (RCEA) that $\vdash_L (\phi \rightarrow \theta) \leftrightarrow (\psi \rightarrow \theta)$. Since every $w \in W$ is closed under $\vdash_L$, $\phi \rightarrow \theta \in w$ if and only if $\psi \rightarrow \theta \in w$. It follows immediately that $R_{[\phi]} = R_{[\psi]}$.

It must be verified that $J^L_R$ meets the conditions imposed by definition 46. Since $L$ is consistent, $\not\vdash_L \bot$; hence, by lemma 43, $W \neq \emptyset$. $R$ is reflexive and transitive because $\subseteq$ is. To see that $P$ satisfies the required conditions, note that $\emptyset = [\bot] \in P$. Otherwise,

\[^{12}\text{Unless it is needed to disambiguate, I omit the superscript and write } \Phi \text{ for } \Phi^{\phi\varphi}.\]
suppose $S, T \in P$; then for some $\phi$ and $\psi$, $S = [\phi]$ and $T = [\psi]$. Then it is obvious that $S \cap T = [\phi \land \psi] \in P$ and $S \cup T = [\phi \lor \psi] \in P$ (for the latter, note that any $w \in [\phi \lor \psi]$ is prime). The proofs of the remaining cases parallel cases that must be proved in lemma 45, so I only survey one of these. To show that $[\phi] \mapsto [\psi] \in P$, it suffices to show that $[\phi] \mapsto [\psi] = [\phi \rightarrow \psi]$. Suppose $w \in [\phi \rightarrow \psi]$, $w R y$ (i.e. $w \subseteq y$), and $y \in [\phi]$; then, since $\phi, \phi \rightarrow \psi \in y$ and $y$ is closed under $\vdash_L$, $y \in [\psi]$, which establishes that $w \in [\phi] \mapsto [\psi]$.

Conversely, suppose that $\phi \rightarrow \psi \not\in w$; then it is clear that $w, \phi \not\vdash_L \psi$. By lemma 43, there is a nice set $y$ such that $w, \phi \subseteq y$ and $\psi \not\in y$. That is, since $w R y$, $y \in [\phi]$, and $y \not\in [\psi]$, $w \not\in [\phi] \mapsto [\psi]$, which was to be proved.

To show that $R \circ R \subseteq R \circ R$, suppose that $x R [\phi] y \ (X \text{ must be some such } [\phi])$; then there is some $z$ such that $x \subseteq z R [\phi] y$, so clearly $x R [\phi] y$, by definition 52, from which it follows that $x R [\phi] y R y$, since $y R y$. Finally, if $w \in V(p) = [p]$ and $w \subseteq x$, it is clear that $x \in V(p)$ as well, so all of the conditions have been met. I turn now to the special conditions.

Suppose $L$ contains (ID); it must be shown that $J^L_R$ satisfies (id). Suppose $w R [\phi] x$; it must be shown that $x \in [\phi]$. Since $w$ is closed under $\vdash_L$ and $L$ contains (ID), $\phi \Box \rightarrow \phi \in w$. Therefore, by definition 52, $\phi \in x$, which was to be proved.

Suppose $L$ contains (CMP); it must be shown that $J^L_R$ satisfies (cmp). Suppose $w \in [\phi]$ and $w R [\phi] \psi w$. By the closure of $w$ under $\vdash_L$, $\psi \in w$. Since $\phi \Box \rightarrow \psi$ is arbitrary in $w$, $w R [\phi] w$, which was to be proved.

Suppose $L$ contains (CS); it must be shown that $J^L_R$ satisfies (sc'). Suppose $w \in [\phi]$ and $w R [\phi] x$; it suffices to show that $w R x$. Since $\phi \in w$ and $w$ is closed under $\vdash_L$, $\phi \land \phi \in w$, and consequently $\phi \Box \rightarrow \phi \in w$ by (CS). Thus, $\phi \in x$ since $w R [\phi] x$. Since this will hold for any $\phi \in w$, it is clear that $w \subseteq x$, i.e. $w R x$.

Suppose $L$ contains (MOD); it must be shown that $J^L_R$ satisfies (mod'). Suppose that $\{x : w R_{[\phi]} x\} \subseteq [\phi]$ and $w R [\psi] y$; it suffices to show that $y \in [\phi]$. Observe that $\neg \phi \Box \rightarrow
$\phi \in w$; then by (MOD) and the closure of $w$ under $\vdash_L$, $\psi \Box \phi \in w$. Then, since $wR[\psi]\gamma$, $\phi \in y$, which was to be proved.

Suppose $L$ contains (NORM); it must be shown that $3^L_R$ satisfies (norm). Since $w$ is $L$ consistent and $L$ contains (NORM), $\top \Box \perp \not\in w$, and consequently, $\perp \not\in \{\psi : (\top \Box \psi) \in w\}$. It follows (by an argument rehearsed in the proof of lemma 45) that $\{\psi : (\top \Box \psi) \in w\} \not\vdash_L \perp$. Consequently, by lemma 43, there is a world $x$ such that $\{\psi : (\top \Box \psi) \in w\} \subseteq x$ and $\perp \not\in x$. Therefore, by definition 52, $wR[\top]x$.

LEMA 45 (Truth Lemma). Let $3^L_R = \langle W, R, P, \{R_X : X \in P\}, V \rangle$ be the canonical model for $L$. Then for all $\phi \in \Phi$ and all $w \in W$: $\models^L_w \phi$ if and only if $\phi \in w$ (i.e. $[\phi] = [\phi]$)

The proof is by induction on the complexity of $\phi$. The only case of interest is that in which $\phi$ is of the form $\psi \Box \theta$.

Suppose that $\psi \Box \theta \in w$ and $wR[\psi]x$ (for $x$ arbitrary). By the induction hypothesis and definition 52, $wR[\psi]x$, that is, $\{\chi : (\psi \Box \chi) \in w\} \subseteq x$. Then it is obvious that $\theta \in x$, from which it follows (by the induction hypothesis) that $x \in [\theta]$, which was to be proved.

Conversely, suppose that $\psi \Box \theta \not\in w$; I show that $\{\chi : (\psi \Box \chi) \in w\} \not\vdash_L \theta$. Suppose otherwise; then for some set $S = \{\chi_0, \ldots, \chi_n\} \subseteq \{\chi : (\psi \Box \chi) \in w\}, S \vdash_L \theta$. That is, using the deduction theorem and rule (RCK):

\[
S \vdash_L \theta \\
\vdash_L (\chi_0 \land \ldots \land \chi_n) \rightarrow \theta \\
\vdash_L ((\psi \Box \chi_0) \land \ldots \land (\psi \Box \chi_n)) \rightarrow (\psi \Box \theta) \\
w \vdash_L \psi \Box \theta
\]

But this is impossible since $w$ is nice and $\psi \Box \theta \not\in w$. Since $\{\chi : (\psi \Box \chi) \in w\} \not\vdash_L \theta$, it

The essentials of the argument why are discussed in detail in the proof of lemma 45.
follows by lemma 43 that there is a nice set \( x \) such that \( \{ \chi : (\psi \supset \chi) \in w \} \subseteq x \) and \( \theta \not\in x \).

By the induction hypothesis, it follows that \( w \not\in [\psi \supset \theta] \).

\[\square\]

**THEOREM 36 (Completeness).** \( \Gamma \models_{cR} \phi \) implies \( \Gamma \vdash_{L} \phi \)

Suppose that \( \Gamma \not\models_{L} \phi \). By lemma 43, there is a nice set \( w \) such that \( \Gamma \subseteq w \) and \( \phi \not\in w \). Where \( \mathcal{J}_{R}^{L} = \langle W, \mathcal{R}, P, \{ R_{X} : X \in P \}, V \rangle \) is the canonical model for \( L \), \( w \in W \) by definition 52 and \( \mathcal{J}_{R}^{L} \in \mathcal{C}_{R}^{L} \) by lemma 44. By lemma 45, for all \( \psi \in \Gamma \), \( w \in [\psi] \), but \( w \not\in [\phi] \). Therefore, \( \Gamma \not\models_{cR} \phi \), which was to be proved.

\[\square\]

**COROLLARY 6 (Algebraic Completeness).** \( \models_{cL} \phi \) implies \( \vdash_{L} \phi \)

The (weak) completeness of \( L \) with respect to \( c_{R}^{L} \) is an immediate consequence of theorems 34 and 36.

\[\square\]
Chapter 7

Concluding Remarks

In this dissertation, I have aimed to develop several “frontiers” of conditional logic. In chapter 3, I gave sound and complete tableaux for a number of conditional logics in Lewis’ V-family, rounding out a notable gap in the literature. Connexive conditional logics and applications of them to deontic reasoning were considered in chapter 4. I examined several systems of counterpossible logic in chapter 5 and discussed the relationship of this work to recently proposed non-vacuous semantics and debates over the legitimacy of a logic of counterpossibles. Finally, in chapter 6, I pursued and motivated the development of conditional logic in a non-classical context, viz. intuitionistic logic. Nevertheless, many frontiers of conditional logic remain underdeveloped. In these concluding remarks, I would like to mention some of these, and discuss how the issues, results, and ideas explored in this dissertation intersect with them.

Throughout this dissertation, I have focused entirely on propositional conditional logic. This is not a unique feature of this dissertation; there are exceptionally few treatments of quantificational conditional logic in the literature.1 While this may be because philosophers and logicians have generally thought quantifiers present no special problems for conditional

1A couple notable examples are Stalnaker and Thomason [117], Delgrande [19], and Priest [102, Ch. 19].
logic that do not already occur in modal logic, this is not entirely obvious. For one, as noted in Berto et al. [8, §2.3] (but see also Kocurek [54, §5]), issues specific to identity can be used as another motivation for a non-vacuous semantics for counterpossibles. For if there were only possible worlds and names are rigid designators (à la Kripke [61]), then the following inference (substituting ‘Cicero’ for ‘Tully’) would be valid, though it is not:

If Cicero had not been Tully, British classicists would have called him Marky.

Therefore, if Cicero had not been Cicero, British classicists would have called him Marky.

Quantifiers also raise interesting issues in connection with connexive conditional logic, given the close link between connexive theses on the one hand and existential import for universal quantification on the other. It is beyond the scope of this conclusion section to say anything detailed on the subject, but I note that Boethius’ theses allow the derivation of ‘some’ from ‘all’ with minimal additional assumptions about the conditional and quantifiers [79, p. 352]. Thus, a detailed examination of quantified conditional logic would be most welcome and connect in interesting ways to some of the topics discussed above.

Another topic which deserves a close examination is the ‘might’ conditional/counterfactual, typically symbolized using Lewis’ ◊→. I indicated, in chapter 6, that this connective must generally be treated as independent of □→ in intuitionistic conditional logics. Accordingly, an examination of the logic of ◊→ in intuitionistic logic is arguably required for an adequate treatment of strong intuitionistic conditional logics, given the (covert) occurrence of ◊→ in the scheme (CV). Lewis’ view that □→ and ◊→ are interdefinable also comes under pressure from non-vacuous treatments of counterpossibles; for a discussion, see my [124, p. 389 n. 9]. Thus, a clarification of the connection between □→ and ◊→ is needed for many contexts, including counterfactuals.

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2For other criticisms Lewis’ so-called duality thesis, see DeRose [20, 21].
A third topic deserving more attention is non-classical conditional logic. Chapter 6 has only scratched the surface of this field, but the techniques and tools developed in it are actually widely applicable. To take one example, just as the algebraic semantics of Nute [92] was “Heyting-ized” to characterize intuitionistic conditional logics, it can be adapted to characterize many other non-classical conditional logics as well. Future work might try to develop relevant conditional logics using a version of this algebraic semantics adapted to (say) De Morgan monoids and link this to previous work on the subject by Mares and Fuhrmann [76] and Priest [102, Ch. 10.7] via equivalence results.\footnote{A large number of interesting algebraic structures for non-classical logics are canvassed by Restall [107], and it could be an interesting project to explore the whole range.}

In summary, many interesting avenues for the future development of conditional logic remain. These projects, I have suggested, intersect with many of the developments of this dissertation, and can both motivate and be motivated by them. Therefore, it is hoped that this dissertation’s contributions both inspire future work along new lines and that that work also inspires deeper work along the lines pursued here.
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