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One-Dimensional Excited Random Walk with Unboundedly Many Excitations Per Site

Omar Chakhtoun
The Graduate Center, City University of New York

ONE-DIMENSIONAL EXCITED RANDOM WALK WITH UNBOUNDEDLY MANY EXCITATIONS PER SITE

by

Omar Chakhtoun

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Professor Elena Kosygina

Date

Chair of Examining Committee

Professor Ara Basmajian

Date

Executive Officer

Professor Louis-Pierre Arguin

Professor Olympia Hadjiliadis

Professor Jay Rosen

Supervisory Committee

The City University of New York

Abstract

We consider a discrete time nearest neighbor random walk on the one-dimensional integer lattice $\mathbb{Z}$ for which the jump probabilities from the current site depend on the number of prior visits to this site. More precisely, we assume that the probability to jump from z to $z+1$ upon the i-th visit to z is given by $\omega(z,i) \in (0,1)$, where $\omega(z,i) = 1/2$ for all sufficiently large i, and the sequences of jump probabilities $(\omega(z,\cdot))_{z\in\mathbb{Z}}$ are i.i.d.. The number of jump probabilities from z to $z+1$ that are different from $1/2$ is referred to as the number of excitations at z giving rise to the name "excited random walk" (ERW). This thesis studies the long term behavior of ERWs with finite but possibly random and unbounded numbers of excitations per site.

The above model belongs to a large class of so-called self-interacting random walks, which are non-Markovian by nature. This renders many standard methods inapplicable. Our approach relies on an analog of a well-known pathwise mapping, at certain stopping times, between random walks and Galton-Watson type trees and on generalized Ray-Knight type theorems. This approach was previously successfully applied by H. Kesten, M. V. Kozlov, and F. Spitzer to the study of limit laws of random walks in random environments and later by B. Tóth for self-interacting random walks of other type than ERWs. The connection between ERWs and branching processes was first observed by A.-L. Basdevant and A. Singh and then developed in many directions by other authors.

This work extends many of the existing methods and results for ERW with a bounded number of excitations to a much wider class of ERW models by requiring only appropriate moment bounds on the (random) number of excitations per site. Main results include criteria for recurrence versus transience, ballisticity versus zero linear speed, a complete classification of limit laws in the transient regime, and functional limit theorems in the recurrent regime.

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Contents

Chapter 1

Introduction

1.1 Excited random walks (ERWs) and common assumptions

The excited random walk (ERW), a.k.a. the "cookie random walk", on the integer lattice $\mathbb{Z}$ is a self-interacting discrete time process such that the jump probabilities at each site and each time step depend on the number of prior visits to the current site. An informal description can be given as follows. Place an infinite stack of cookies at each site of the lattice $\mathbb{Z}$; each cookie within the stack has a "strength". The random walker eats a cookie from the bottom of the stack at his current location and moves one step to the right or to the left with probabilities prescribed by the "strength" of that cookie. If a site happens to have no cookies left then the walker chooses one of the two neighboring sites at random and moves there on the next step.

The following standard notation will be used throughout the thesis: the collection of all infinite cookies stacks on $\mathbb{Z}$ is denoted by Ω_∞ and a generic element ω of Ω_∞ is called a cookie environment. The elements of Ω_∞ are written as $\omega = \Big(\omega(z,i)\Big)_{z\in\mathbb{Z},i\in\mathbb{N}}$ where $\omega(z,i)$ denotes the probability to jump from z to $z+1$ upon the i-th visit to z. That is, for a fixed

$\omega \in \Omega_\infty$ and $x \in \mathbb{Z}$, the ERW starting at x in the environment ω is a process $X := (X_n)_{n\geq 0}$ defined on a suitable probability space $(\Sigma, \mathcal{G}, P_{x,\omega})$ such that

$$P_{x,\omega}(X_0 = x) = 1,$$
$$P_{x,\omega}\Big(X_{n+1} = X_n + 1 \mid X_0, ..., X_n\Big) = \omega\Big(X_n, \sum_{j=0}^{n} \mathbb{1}_{\{X_j = X_n\}}\Big),$$
$$P_{x,\omega}\Big(X_{n+1} = X_n - 1 \mid X_0, ..., X_n\Big) = 1 - \omega\Big(X_n, \sum_{j=0}^{n} \mathbb{1}_{\{X_j = X_n\}}\Big).$$

The cookie environment ω can also be chosen at random according to some probability measure $\mathbb{P}_\infty$ on the space $\Big(\Omega_\infty, \mathcal{F}\Big)$, where $\Omega_\infty = [0,1]^{\mathbb{N}\times\mathbb{Z}}$ and $\mathcal{F}$ is the canonical product Borel σ-field generated by the cylinder sets.

We refer to $P_{x,\omega}$ defined on the space $(\Sigma, \mathcal{G})$ as the quenched probability measure of the ERW, while the annealed, or averaged, probability measure P_x of the ERW is a probability measure on $\Big(\Omega_\infty \times \Sigma, \mathcal{F} \bigotimes \mathcal{G}\Big)$ defined by averaging over all cookie environments

$$P_x(A) := \int_{\Omega_\infty} P_{x,\omega}\Big(\Big\{s \in \Sigma : (s,\omega) \in A\Big\}\Big)\mathbb{P}_\infty(d\omega) \quad \forall A \in \mathcal{F} \bigotimes \mathcal{G}.$$

The expectation operators corresponding to $P_{x,\omega}, \mathbb{P}_\infty$, and P_x are denoted by $E_{x,\omega}, \mathbb{E}_\infty$, and E_x, respectively. The most common assumptions considered in the literature about the measure $\mathbb{P}_\infty$ are:

(IID) the cookie stacks $(\omega(z,\cdot))_{z\in\mathbb{Z}}$ are i.i.d under $\mathbb{P}_\infty$; or a weaker one:

(SE) the cookie stacks $(\omega(z,\cdot))_{z\in\mathbb{Z}}$ are under $\mathbb{P}_\infty$ stationary and ergodic with respect to the shift operators on $\mathbb{Z}$;

(BD) there is a constant $M \in \mathbb{N}$ such that $\mathbb{P}_\infty$-a.s.: $\omega(z) = \frac{1}{2}$ $\forall i > M$ and $\forall z \in \mathbb{Z}$;

(POS) $\forall z \in \mathbb{Z}, \forall i \in \mathbb{N}, \omega(z,i) \geq \frac{1}{2}$ $\mathbb{P}_\infty$-a.s. That is every cookie induces a nonnegative drift of size $2\omega(z,i)-1\geq 0$; such cookies are called non-negative or, with a slight abuse of notation, positive.

To avoid some degenerate situations, one can assume one of the following ellipticity conditions, called weak ellipticity, ellipticity, and uniform ellipticity respectively.

(WEL) $\forall z\in\mathbb{Z}:\mathbb{P}_\infty\Big[\forall i\in\mathbb{N}:\omega(z,i)>0\Big]>0.$

(ELL) $\forall z\in\mathbb{Z}$ and $\forall i\in\mathbb{N}:\mathbb{P}_\infty-$ a.s. $\omega(z,i)>0.$

(UEL) There is $\kappa>0$ such that $\forall z\in\mathbb{Z}$ and $\forall i\in\mathbb{N}:\kappa\leq\omega(z,i)\leq 1-\kappa\quad\mathbb{P}_\infty$-a.s.

Note that (**UEL**) $\Longrightarrow$ (**ELL**) $\Longrightarrow$ (**WEL**).

If the measure $\mathbb{P}_\infty$ satisfies (**SE**) and either (**POS**) or (**BD**) then the total expected drift per site $z\in\mathbb{Z}$

$$\delta=\delta(z)=\mathbb{E}_\infty\left[\sum_{i\geq 1}\Big(2\omega(z,i)-1\Big)\right]. \tag{1.1}$$

is well-defined and does not depend on z. The parameter δ plays an important role in the classification of the asymptotic behavior of ERWs.

In this work we shall consider ERWs in environments, which satisfy (**IID**) and (**UEL**) under $\mathbb{P}_\infty$ and for which the number of excitations per site is a random variable satisfying a tail decay condition. After a brief excursion to the history of various ERW models and some of the known results we shall give a precise description of the model we study.

1.2 Brief history of excited random walks

ERWs on $\mathbb{Z}^d$ were introduced by I. Benjamini and D. Wilson in [11]. Denote by e_j the j-th unit coordinate vector. For a fixed $p\in[1/2,1]$ the probability $\omega(z,\pm e_j,i)$ to jump from

z to $z \pm e_j$ upon the i-th visit to z is given by

$$\begin{aligned}
\omega(z, e_1, 1) &= \frac{p}{d}; \\
\omega(z, -e_1, 1) &= \frac{1-p}{d}; \\
\omega(z, \pm e_j, i) &= \frac{1}{2d}, \quad \text{if } i \geq 2 \text{ or if } i = 1 \text{ and } e_j \neq \pm 1.
\end{aligned}$$

The main results of the paper are that the walks are recurrent for $d = 1$, and transient for $d \geq 2$. Moreover, the walk is ballistic in the first coordinate direction for $d \geq 4$, that is $(X_n \cdot e_1)/n \to v > 0$ P_x-a.s. as $n \to \infty$. Ballisticity was extended to all $d \geq 2$ by J. Bérard and A. Ramírez ([8]). They also proved the strong law of large numbers (**LLN**) and the Central Limit Theorem (**CLT**) for this model in all dimensions $d \geq 2$. A substantial generalization of the original once-excited random walk under a version of the condition (**POS**) was studied later in [50].

Multi-excited random walks were introduced by M. Zerner in [73]. The model allowed more than one excitation per site (including infinite number) but required that all drifts were non-negative. Under conditions (**SE**) and (**POS**) M. Zerner gave a criterion for recurrence and transience and proved the strong law of large numbers for his model on $\mathbb{Z}$. In this paper he proposed the "cookie" description of the model, which later became very popular. He also proved that in environments with at most 2 cookies per site the ERW always has zero speed and posed an open question about the minimum number of cookies that will generate a positive speed. This question was partially answered by T. Mountford, L. Pimentel, and G. Valle in [49]. Moreover, they have constructed environments with arbitrarily large δ satisfying (**SE**) and (**POS**) in which the ERW had zero linear speed.

In [74] M. Zerner extended the model to several dimensions and gave criteria for recurrence and transience of ERWs on $\mathbb{Z}^d$ and strips under (**IID**), (**UEL**), and a multi-dimensional version of (**POS**).

ERW among bounded cookies stacks on $\mathbb{Z}$. The open question posed by M. Zerner about the conditions under which the speed of ERW on $\mathbb{Z}$ is positive was answered by A. Basdevant and A. Singh in [9] for a class of ERWs among deterministic identical finite stacks of positive cookies on $\mathbb{Z}$. They gave a criterion for the positivity of the speed of ERW. Moreover, they restated the problem in terms of branching processes. This connection was previously used in [37] for the study of limit laws of random walks in random environments (see also [66]) and proved to be an indispensable tool for ERWs as well. In a subsequent paper [10] they obtained results on the rate of growth of the ERW for their model. Unfortunately the connection with branching processes is limited to $d = 1$.

This model was significantly generalized by E. Kosygina and M. Zerner in [44]. They allowed random cookie environments which satisfy (**IID**), (**WEL**), (**BD**) and removed the positivity assumption (**POS**) altogether. They gave criteria for recurrence and transience, proved the strong **LLN**, gave criteria for ballisticity, and established a **CLT** for the case $|\delta| > 4$. Currently this model is the most studied and well understood model of ERWs on $\mathbb{Z}$. Further results about this model were obtained in [18–20, 39, 45, 46, 53]. They cover limit laws and functional limit theorems in both transient and recurrent regimes, large deviations as well as present a study of excursions and occupation times. For a comprehensive survey of ERW models on $\mathbb{Z}^d$ and results up to 2012 we refer the reader to [45].

ERW with identical periodic cookie stacks on $\mathbb{Z}$. This model features identical infinite cookie stacks at every site of $\mathbb{Z}$ but requires that the cookie strengths within one stack be periodic and be strictly between 0 and 1. This model was introduced and studied by G. Kozma, T. Orenshtein, and I. Shinkar in [41]. The authors used Lamperti's approach to give criteria for recurrence and transience. As an application, they provided a different argument for recurrence and transience conditions for some models of ERWs with boundedly many cookies per site. Questions about ballisticity and limit laws remained open until this model was extended to Markovian cookie stacks (see below). The results can be found in [42] and [43].

ERW with Markovian cookie stacks model was introduced by E. Kosygina and J. Peterson in [42] as a natural generalization of the previous one. The measure $\mathbb{P}_\infty$ is assumed to satisfy (**IID**), (**UEL**), and the jump probabilities within the cookie stack $(\omega(z,i))_{i\geq 1}$ at site z follow a Markov chain on a finite state space. It is assumed that this Markov chain has a unique irreducible closed set. The model exhibits two regimes, namely critical and non-critical, depending on whether the total drift under the invariant measure of the Markov chain is equal to 0 or differs from 0. In the non-critical regime the ERW is always transient, ballistic with non-zero linear speed, and satisfies the classical central limit theorem, whereas the critical regime gives rise to many different types of limit laws (just as for ERWs with bounded cookie stacks). The authors obtained criteria for recurrence versus transience and ballisticity of the walk as well as a characterization of the limiting behavior in the transient case and a functional limit theorem in the borderline of recurrent regime.

ERW in a Markovian environment with bounded cookie stacks was proposed and studied by Nicholas Travers in [69]. The main novelty of this work is in the weakening of the (**IID**) condition to a one that is between (**IID**) and (**SE**). Namely, from site z to site $z+1$ the cookie environment follows a uniformly ergodic Markov chain. The author was able to show that many the results proved for ERWs among bounded (**IID**) cookie stacks hold for this much more general model.

The present work generalizes ERWs among i.i.d. bounded cookie stacks by replacing the boundedness assumption, (**BD**), with the tail decay condition on the number of cookies in a stack. No structural conditions (such as periodicity or Markov property) are imposed on the cookies within a single stack. This thesis extends many of the results known for ERWs among i.i.d. bounded cookie stacks to this much wider class of models. Our model also includes some of the ERWs with Markovian cookie stacks, namely when $\frac{1}{2}$ is an absorbing state of the Markov chain.

Chapter 2

Model and main results

2.1 Our model

Let $\kappa \in (0, 1/2)$ and

$$\Omega := \left\{ \Big(\omega(z,i)\Big)_{i\in\mathbb{N}, z\in\mathbb{Z}} \in [\kappa, 1-\kappa]^{\mathbb{Z}\times\mathbb{N}} \,\Big|\, \forall z \in \mathbb{Z},\ \sup\left\{ i \in \mathbb{N} : \omega(z,i) \neq \frac{1}{2} \right\} < \infty \right\}.$$

If the set $\mathcal{C}(z,\omega) := \left\{ i \in \mathbb{N} : \omega(z,i) \neq \frac{1}{2} \right\}$ is empty then we put $\sup \mathcal{C}(z,\omega) = \max \mathcal{C}(z,\omega) = 0$. For each $z \in \mathbb{Z}$ and $\omega \in \Omega$ we define the number of cookies at each site $z \in \mathbb{Z}$ by

$$M(z,\omega) := \max \mathcal{C}(z,\omega) = \max\left\{ i \in \mathbb{N} : \omega(z,i) \neq \frac{1}{2} \right\}. \tag{2.1}$$

We will assume throughout that $\mathbb{P}(\Omega) := \mathbb{P}_\infty(\Omega) = 1$. This automatically implies that $\mathbb{P}$ satisfies (**UEL**) and $M(z,\omega) < \infty$ for all $z \in \mathbb{Z}$ and $\omega \in \Omega$. We shall also suppose that (**IID**) holds and the cookie stack heights $\Big\{M(z,\omega)\Big\}_{z\in\mathbb{Z}}$, defined by (2.1), have sufficiently fast decaying tails so that δ (see (1.1)) is well-defined and, moreover,

$$(\textbf{TDE}) \qquad H(n) := \mathbb{P}\Big(M(z,\omega) > n\Big) \leq \frac{C}{n^\alpha} \quad \text{for some } C > 0 \text{ and } \alpha > |\delta| \vee 4. \tag{2.2}$$

Note that the family of random variables $(M(z,\omega))_{z\in\mathbb{Z}}$ is i.i.d. under $\mathbb{P}$. We shall often drop ω from the notation and write simply $M(z)$.

We remark that without loss of generality we can assume that $\delta \geq 0$ due to the following symmetry. If the environment $(\omega(z,i))_{z\in\mathbb{Z}}$ is replaced by $(\omega'(z,i))_{z\in\mathbb{Z}}$ where $\omega'(z,i) = 1 - \omega(z,i)$ for all $i \in \mathbb{N}, z \in \mathbb{Z}$, then $X' := \{X'_n\}_{n\geq 0}$, the ERW corresponding to this new environment, satisfies :

$$X' \stackrel{D}{=} -X \tag{2.3}$$

where $\stackrel{D}{=}$ is the equality in distribution.

2.2 Examples

Example 2.1. Let $\mathbb{P}_\infty$ be a probability measure on $\left([\kappa, 1-\kappa]^{\mathbb{N}} \times \mathbb{N}_0\right)^{\mathbb{Z}}$ such that the quantities $(((\bar{\omega}(z,i))_{i\in\mathbb{N}}, M(z)))_{z\in\mathbb{Z}}$ are i.i.d. under $\mathbb{P}_\infty$. Assume also that $M(z)$ is independent from the cookie stack $(\bar{\omega}(z,i))_{i\in\mathbb{N}}$ and satisfies (**TDE**). To obtain a cookie environment from Ω_∞ we cut the cookie stack at z at the level $M(z)$ and set $\omega(z,i) = 1/2$ for all $i > M(z)$, $z \in \mathbb{Z}$. Then we consider the measure $\mathbb{P}$ on the "trimmed" cookie stacks. By construction $\mathbb{P}$ satisfies all the required conditions. The total expected drift δ at each site z can be easily computed.

$$\begin{aligned}
\delta &= \mathbb{E}\left[\sum_{i\geq 1}(2\omega(z,j)-1)\right] \\
&= \sum_{N=0}^{\infty}\mathbb{E}\left[\sum_{j=1}^{N}(2\omega(z,j)-1)\Big| M(z)=N\right]\mathbb{P}\Big(M(z)=N\Big) \\
&= \sum_{N=0}^{\infty}\mathbb{E}\left[\sum_{j=1}^{N}(2\omega(z,j)-1)\right]\mathbb{P}\Big(M(z)=N\Big) \\
&= \sum_{N=0}^{\infty}\delta_N\mathbb{P}\Big(M(z)=N\Big) = \mathbb{E}\left[\delta_{M(z)}\right]
\end{aligned}$$

where $\delta_N := \mathbb{E}\left[\sum_{j=1}^{N}(2\omega(z,j)-1)\right]$. Let us mention the following special cases.

Suppose that the environment is deterministic and the cookies have the same strength for some $p \in (0,1)$. That is

$$\omega(z,j) = \begin{cases} p & j = 1,2,...,M(z),\ \forall z \in \mathbb{Z}, \\ \frac{1}{2} & j > M(z). \end{cases}$$

Then the total expected drift is

$$\delta = (2p-1)\mathbb{E}(M(z)).$$

Note that in other cases our results are new even for this simple example.

If we also suppose that $M(z)$ has a geometric distribution with parameter $q \in (0,1)$ then we recover the example of ERW with Markovain cookie stacks with two states given in [42, Example 1.2]. The transition matrix

$$\mathbf{K} = \begin{pmatrix} 1-q & q \\ 0 & 1 \end{pmatrix}$$

and the initial distribution $\eta = (\eta(1), \eta(2)) = (1,0)$ such that $p(1) = 1$ and $p(2) = \frac{1}{2}$. The total expected drift per site is

$$\delta \ = \ \frac{2p-1}{q}.$$

In the above example, the number of cookies $M(z)$ at each site z and the cookie strengths $\Big(\omega(z,j)\Big)_{j\geq 1}$ were assumed to be independent. The next two classes of examples allow dependence between $\Big(M(z)\Big)_{z\in\mathbb{Z}}$ and $\Big(\omega(z,j)\Big)_{j\geq 1}$ and are built from two different perspectives.

Example 2.2. Construct an environment $\omega \in \Omega$ at each site $z \in \mathbb{Z}$ by first sampling the number of cookies per site, $M(z)$, according to some distribution F on $\mathbb{N}_0$ and then, given $M(z) = m \neq 0$, sample the cookie strengths $\Big(\omega(z,i)\Big)_{i\geq 1}$ according to a distribution π_m on $[\kappa, 1-\kappa]^m$ from some family of distributions π_m, $m \in \mathbb{N}$. If $m = 0$ then we set $\omega(z,i) = \frac{1}{2}$ $\forall i \in \mathbb{N}$. For a specific example we can sample $M(z)$ according to a geometric distribution with parameter p, $p \in [\kappa, 1-\kappa]$, and let

$$\pi_m = \frac{1}{2}\delta_{\bar{\omega}_a} + \frac{1}{2}\delta_{\bar{\omega}_b},\ a \in [0, 1-2\kappa],\ b \in [\kappa, 1-\kappa], \text{ where}$$

$$\omega_a(z,j) = \kappa + \frac{a}{j} \text{ if } j \in \{1,2,...,m\} \text{ and } \omega_a(z,j) = \tfrac{1}{2} \text{ if } j > m;$$

$$\omega_b(z,j) = b \text{ if } i \in \{1,2,...,m\} \text{ and } \omega_b(z,j) = \tfrac{1}{2} \text{ if } j > m.$$

Then

$$\begin{aligned}
\delta &= \mathbb{E}\left[\sum_{i=1}^{M(z)} (2\omega(z,j) - 1)\right] \\
&= \sum_{m=0}^{\infty} \mathbb{E}\left[\sum_{j=1}^{M(z)} (2\omega(z,j)-1)\Big| M(z) = m\right] \mathbb{P}\Big(M(z) = m\Big) \\
&= \sum_{m=0}^{\infty} p(1-p)^m \Big[\frac{1}{2}\, m(2b-1) + \frac{1}{2}\sum_{j=1}^{m}\Big(2\Big(\kappa + \frac{a}{j}\Big) - 1\Big)\Big] \\
&= \sum_{m=0}^{\infty} p(1-p)^m \Big[\frac{m(2b-1+2\kappa-1)}{2} + \sum_{j=1}^{m}\frac{a}{j}\Big] \\
&= \frac{[(b+\kappa)-1](1-p)}{p} + ap\sum_{j=1}^{\infty}\sum_{m=j}^{\infty}(1-p)^m\frac{1}{j} \\
&= \frac{[(b+\kappa)-1](1-p)}{p} + ap\sum_{j=1}^{\infty}\frac{1}{j}\sum_{m=j}^{\infty}(1-p)^m \\
&= \frac{[(b+\kappa)-1](1-p)}{p} + ap\sum_{j=1}^{\infty}\frac{1}{j}\frac{(1-p)^j}{p} \\
&= \frac{[(b+\kappa)-1](1-p)}{p} + a\ln\frac{1}{p} = -\frac{[1-(b+\kappa)](1-p)}{p} + a\ln\frac{1}{p}
\end{aligned}$$

- The **RHS** is maximized when $b = 1-\kappa,\ a = 1-2\kappa,\ p = \kappa$ and then

$$\lim_{\kappa\to 0^+} -(1-2k)\ln\frac{1}{\kappa} = +\infty$$

- The **RHS** is minimized when $b = \kappa,\ a = 0$, and then we have to minimize

$$-\frac{(1-2\kappa)(1-p)}{p}$$

over $p \in [\kappa, 1-\kappa]$. The minimum is attained at $p = \kappa$. Thus,

$$-\lim_{\kappa\to 0^+}\frac{(1-2\kappa)(1-\kappa)}{\kappa} = -\infty.$$

This shows that we can construct a model of this type with any $\delta \in \mathbb{R}$.

Example 2.3. In this class of examples we first sample an infinite cookie stack from some distribution on $[\kappa, 1-\kappa]^{\mathbb{N}}$ and then, given a realization of the cookie stack $\Big(\omega(z,j)\Big)_{j\geq 1}$, choose the distribution of $M(z)$ as a function of $\Big(\omega(z,j)\Big)_{j\geq 1}$ and slash the infinite cookie stack at each $z \in \mathbb{Z}$ to the level $M(z)$. The simplest example of this type can be constructed as follows. Let f be a probability density function supported on $[\kappa, 1-\kappa]$, $F_s,\ s \in S \subset \mathbb{R}$, be a one parameter set of distributions on $\mathbb{N}_0$ with finite expectations $m_s, s \in S$, and $g : [\kappa, 1-\kappa] \longrightarrow S$ be an arbitrary function. The cookie environment at site z is constructed in two steps.

- Step 1. Sample $\omega(z,1)$ according to the density f and compute $s = g(\omega(z,1))$.

- Step 2. Sample $M(z)$ from the distribution $F_s,\ s = g(\omega(z,1))$. Set

$$\begin{aligned}\omega(z,j) &= \omega(z,1) \quad \forall j = 1,2,...,M(z) \quad \text{and,}\\ \omega(z,j) &= \frac{1}{2} \quad \forall j > M(z).\end{aligned}$$

Then

$$\begin{aligned}\delta &= \mathbb{E}\left[\sum_{i=1}^{M(z)}(2\omega(z,i)-1)\right] = \mathbb{E}\left[(2\omega(z,1)-1)M(z)\right] \\ &= \mathbb{E}\left[(2\omega(z,1)-1)\mathbb{E}\left[M(z)|\omega(z,1)\right]\right] = \mathbb{E}\left[(2\omega(z,1)-1)m_{g(\omega(z,1))}\right] \\ &= \int_{\kappa}^{1-\kappa}(2p-1)m_{g(p)}f(p)dp\end{aligned}$$

A specific example can be obtained by taking f to be a uniform on $[\kappa, 1-\kappa]$, F_s be the family of geometric distributions with parameter s, and $g(p) = p$. Then

$$\delta = \frac{1}{1-2\kappa}\int_{\kappa}^{1-\kappa}(2p-1)\frac{1-p}{p}dp = 2 - \frac{\ln(\frac{1-\kappa}{\kappa})}{1-2\kappa}.$$

$\lim_{\kappa\searrow 0}\delta = -\infty$ and $\lim_{\kappa\nearrow\frac{1}{2}}\delta = 0$. With $g(p) = 1-p$, we get

$$\delta = \frac{1}{1-2\kappa}\int_{\kappa}^{1-\kappa}(2p-1)\frac{p}{1-p}dp = \frac{\ln(\frac{1-\kappa}{\kappa})}{1-2\kappa} - 2,$$

so that $\lim_{\kappa\searrow 0}\delta = +\infty$ and $\lim_{\kappa\nearrow\frac{1}{2}}\delta = 0$. This allows us to get examples with any given $\delta \neq 0$.

Another possibility is to keep f as before and let F_s be the family of Zipf's distributions with parameter s: the probability mass function is $\frac{k^{-s}}{\zeta(s)}$, where $\zeta(s)$ is Riemann's zeta function, $k \in \mathbb{N},\ s > 2$. Then $m_s = \dfrac{\zeta(s-1)}{\zeta(s)}$ and for an arbitrary continuous function $g : [\kappa, 1-\kappa] \to (2, \infty)$ we get

$$\delta = \frac{1}{1-2\kappa}\int_{\kappa}^{1-\kappa}(2p-1)\frac{\zeta(g(p)-1)}{\zeta(g(p))}dp.$$

The above examples illustrate that our model includes many different types of examples which are both new and interesting. We were able to extend many of the existing results (see [9, 10, 18, 39, 44] or [45] obtained under the (**BD**) condition to our much more general setting.

2.3 Main results

All results of this section are extensions of the corresponding results obtained for the case of bounded cookie stacks. We note that even though we employ overall the same methods, most of the technical work have been redone from scratch and many steps required a lot of additional work. The technical results are the content of the last two chapters.

2.3.1 Recurrence versus transience

Theorem 2.1. ***(Recurrence vs transience)***

(i) *If $\delta \in [-1,1]$, then the walk is recurrent, i.e., for $\mathbb{P}$-a.a. environments ω it returns $P_{0,\omega}$-a.s. infinitely often to its starting point.*

(ii) *If $\delta > 1$, then the walk is transient to the right, i.e., for $\mathbb{P}$-a.a. ω, $X_n \to \infty$ as $n \to \infty$ $P_{0,\omega}$-a.s.*

(iii) *If $\delta < -1$, then the walk is transient to the left, i.e., for $\mathbb{P}$-a.a. ω, $X_n \to -\infty$ as $n \to \infty$ $P_{0,\omega}$-a.s.*

2.3.2 Strong law of large numbers and ballisticity

Theorem 2.2. *There is a deterministic $v \in [-1,1]$ such that the ERW satisfies for $\mathbb{P}$-a.a. ω the strong law of large numbers,*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \qquad P_{0,\omega}\text{-a.s.}. \tag{2.4}$$

Moreover, $v < 0$ for $\delta < -2$, $v = 0$ for $\delta \in [-2,2]$ and $v > 0$ for $\delta > 2$.

2.3.3 Limit Theorems for transient ERW.

To describe limit laws of the ERW we need the following notation. For $\theta \in (0,2]$ and $b > 0$, denote by $Z_{\theta,b}$ a random variable on some probability space whose characteristic function is given by:

$$\log \mathbb{E}e^{iuZ_{\theta,b}} = \begin{cases} -b|u|^{\theta}\Big(1 - i\frac{u}{|u|}\tan(\frac{\pi\theta}{2})\Big) & \text{if } \theta \neq 1; \\ -b|u|\Big(1 + \frac{2i}{\pi}\frac{u}{|u|}\log|u|\tan(\frac{\pi\theta}{2})\Big) & \text{if } \theta = 1. \end{cases} \tag{2.5}$$

Note that $Z_{2,b}$ is just a centered normal random variable with variance $2b$. Recall that it is enough to consider only $\delta > 1$, since $\delta < -1$ can be treated by symmetry discussed in Section 2.1.

Theorem 2.3. *Let $T_n = \inf\{j \geq 0 \mid X_j = n\}$ and v be the speed of the ERW. The following statements hold under the averaged measure P_0.*

(i) *If $\delta \in (1,2)$ then there is a constant $b > 0$ such that as $n \longrightarrow \infty$*

(a) $\dfrac{T_n}{n^{\frac{2}{\delta}}} \Rightarrow Z_{\frac{\delta}{2},b}$ *and* *(b)* $\dfrac{X_n}{n^{\frac{\delta}{2}}} \Rightarrow \Big(Z_{\frac{\delta}{2},b}\Big)^{-\frac{\delta}{2}}$.

(ii) *If $\delta = 2$ then there is a constant $c_1 > 0$ such that as $n \longrightarrow \infty$*

(a) $\dfrac{T_n}{n\log n} \longrightarrow \dfrac{1}{c_1}$ *and* *(b)* $\dfrac{X_n}{\frac{n}{\log n}} \longrightarrow c_1$.

Note that the convergence is in probability. Moreover, there are a positive constant b and the functions $D(n) \sim \log n$ and $\Gamma(n) \sim \frac{1}{\log n}$ such that as $n \longrightarrow \infty$

(a) $\dfrac{T_n - c_1^{-1}nD(n)}{n} \Rightarrow Z_{1,b}$ *and* *(b)* $\dfrac{X_n - c_1 n\Gamma(n)}{c_1^2 n \log^{-2} n} \Rightarrow Z_{1,b}$.

(iii) *If $\delta \in (2,4)$ then there is a constant $b > 0$ such that as $n \longrightarrow \infty$,*

$$(a)\ \frac{T_n - v^{-1}n}{n^{\frac{2}{\delta}}} \Rightarrow Z_{\frac{\delta}{2},b} \qquad \text{and} \qquad (b)\ \frac{X_n - vn}{n^{\frac{2}{\delta}}} \Rightarrow -v^{1+\frac{\delta}{2}} Z_{\frac{\delta}{2},b}.$$

(iv) *If $\delta = 4$ then there is a constant $b > 0$ such that as $n \longrightarrow \infty$*

$$(a)\ \frac{T_n - v^{-1}n}{\sqrt{n \log n}} \Rightarrow Z_{2,b} \qquad \text{and} \qquad (b)\ \frac{X_n - vn}{\sqrt{n \log n}} \Rightarrow -v^{\frac{3}{2}} Z_{2,b}.$$

(iv) *If $\delta > 4$ then there is a constant $b > 0$ such that as $n \longrightarrow \infty$*

$$(a)\ \frac{T_n - v^{-1}n}{\sqrt{n}} \Rightarrow Z_{2,b} \qquad \text{and} \qquad (b)\ \frac{X_n - vn}{\sqrt{n}} \Rightarrow -v^{\frac{3}{2}} Z_{2,b}.$$

Moreover, everywhere above X_n can be replaced by $\sup_{i \le n} X_i$ or $\inf_{i \ge n} X_i$.

Note that Functional Limit Theorems (**FLT**) in the transient regime under the (**BD**) assumption were obtained in [44, 45] and can be obtained as well for our model along the same lines but we do not pursue this here.

2.3.4 Functional Limit Theorems in the recurrent regime

Let $D([0,\infty))$ be the space of càdlàg function on $[0,\infty)$ and denote by $\xrightarrow{J_1}$ the weak convergence in the standard (J_1) Skorokhod topology on $D([0,\infty))$. Let $B = (B(t))$, $t \ge 0$ standard Brownian motion started at the origin at time 0 and $W_{\theta,\tilde{\theta}} = (W_{\theta,\tilde{\theta}}(t)), t \ge 0$, be an $(\theta, \tilde{\theta})$ perturbed Brownian motion, i.e. the solution of the equation

$$W_{\theta,\tilde{\theta}}(t) = B(t) + \theta \sup_{s \le t} W_{\theta,\tilde{\theta}}(s) + \tilde{\theta} \inf_{s \le t} W_{\theta,\tilde{\theta}}(s) \tag{2.6}$$

We state results from the literature [13, Theorem 1 and 2] about the solutions of equation (2.6)

(i) The equation (2.6) has no solution if either $\theta \ge 1$, or $\tilde{\theta} \ge 1$,

(ii) For $\theta < 1$, and $\tilde{\theta} < 1$ the equation (2.6) has a unique pathwise solution, and this solution is adapted to the filtration generated by the driving Brownian motion B.

(iii) For continuity of paths of $W_{\theta,\tilde{\theta}}$ and the strong Markov property of the process

$$\Big(B(t),\ \theta \sup_{s\le t} W_{\theta,\tilde{\theta}}(s),\ \tilde{\theta} \inf_{s\le t} W_{\theta,\tilde{\theta}}(s)\Big)$$

we refer the reader to [57].

Without loss of generality we assume that $\delta \ge 0$.

Theorem 2.4 (**Non-boundary case**). *If* $\delta \in [0,1)$ *then under* P_0

$$\frac{X_{[n\cdot]}}{\sqrt{n}} \xrightarrow{J_1} W_{\delta,-\delta}(\cdot) \quad as \quad n \longrightarrow \infty. \tag{2.7}$$

Theorem 2.5 (**Boundary case**). *Let* $\delta = 1$ *and* $B^*(t) = \max_{s\le t} B(s)$. *Then there exists a constant* $b \in (0,\infty)$ *such that under* P_0

$$\frac{X_{[n\cdot]}}{b\sqrt{n}\log n} \xrightarrow{J_1} B^*(\cdot) \quad as \quad n \longrightarrow \infty. \tag{2.8}$$

Chapter 3

Branching processes associated to ERWs

In this chapter we shall explain a known relation between the ERW on $\mathbb{Z}$ and two classes of modified branching processes. The latter are Markovian processes. This relation is a powerful device in the analysis of ERWs. Recurrence, transience and limits laws of ERWs can be obtained by studying the mentioned above Markovian processes. This approach was previously successfully applied by H. Kesten, M. V. Kozlov, and F. Spitzer ([37]) in the study of limit laws for random walks in random environments and later by B. Tóth ([66, 68]) for several types of self-interacting random walks with a different mechanism of self-inteaction. The connection between ERWs and branching processes was first observed by A.-L. Basdevant and A. Singh ([9, 10]) and then developed in many directions by other authors. The interested reader is referred to ([3, 18, 19, 32, 33, 39, 42–46, 53–55]).

3.1 Coin toss construction

For a fixed cookie environment $\omega \in \Omega$ let $(B_k(i))_{k\in\mathbb{Z},i\in\mathbb{N}}$ be a family of independent Bernoulli random variables such that

$$P_\omega(B_k(i) = 1) = 1 - P_\omega(B_k(i) = 0) = \omega(k, i).$$

Events $\{B_k(i) = 1\}$ will be referred to as "successes" and events $\{B_k(i) = 0\}$ will be referred as "failures". For a subset A of environments and realizations of Bernoulli random variables we let

$$P(A) = \mathbb{E}(P_\omega(A)). \tag{3.1}$$

Bernoulli random variables $\Big(B_k(i)\Big)_{i\geq 1}$ will be used to construct both the paths of the excited random walk and the associated branching processes. This will provide a natural coupling which we will use throughout the thesis. A path of ERW in a given environment ω can be constructed recursively as follows: if $X_n = k$ and $\sum_{r=0}^{n} \mathbb{1}_{\{X_r=k\}} = j$, then $X_{n+1} := X_n + 2B_k(j) - 1$, for $n \geq 0$. This construction defines simultaneously all paths of ERW starting from every $k \in \mathbb{Z}$: the walk starts at k and follows the path constructed from the given ω and a realization of Bernoulli random variables $(B_k(j))_{k\in\mathbb{Z},j\in\mathbb{N}}$. From now on we shall use this construction of ERW and keep denoting the resulting averaged probability measure of ERW with a starting point at z by P_z.

3.2 Forward branching process

The forward branching process is constructed by looking at the right excursions of the ERW from the origin. Left excursions from the origin can be considered by the symmetry (2.3).

Consider an ERW path $(X_n)_{n\geq 0}$, which starts at the origin and define hitting times

$$T_k \;:=\; \inf\Big\{n>0\Big|X_n=k\Big\}\in\mathbb{N}\cup\Big\{\infty\Big\}\quad k\in\mathbb{Z}.$$

Note that T_0 is the time of the first return of the ERW to the origin. On the event $\{X_1=1\}$ consider the right excursion, i.e. $\Big(X_n\Big)_{0\leq n<T_0}$. For every $k\geq 0$, let $U_k^{(0)}$ be the number of up-crossings from k to to $k+1$ by the ERW prior to time T_0. That is,

$$U_0^{(0)}:=1\quad\text{and}\quad U_k^{(0)}:=\sum_{j=0}^{T_0-1}\mathbb{1}_{\{X_j=k,X_{j+1}=k+1\}}.$$

Note that if $T_0<\infty$ and $U_{k-1}=m$, then the walk makes m down-crossings from k before time T_0. Hence, $U_k^{(0)}$ is the number of up-crossings that the ERW makes from k before the m-th down-crossing from k. Moreover, the number of up-crossings $U_k^{(0)}$ can be computed from the Bernoulli random variables $\Big(B_k(j)\Big)_{j\geq 1}$. If we refer to the coin toss construction of the ERW path, then $U_k^{(0)}$ is the number of successes in the Bernoulli sequence $\Big(B_k(j)\Big)_{j\geq 1}$ before the m-th failure. Denoting by $S_m^{(k)}$ the number of successes before the m-th failure in the sequence $\Big(B_k(j)\Big)_{j\geq 1}$ and setting

$$S_0^{(k)}\;:=\;0\quad\text{and}\quad S_m^{(k)}:=\inf\left\{r\geq 0:\sum_{j=1}^{r+m}(1-B_k(j))=m\right\}.$$

It easy to see that on the event $\{T_0<\infty\}\cap\{U_{k-1}^{(0)}=m\}$ we have $U_k^{(0)}=S_m^{(k)}$.

Let $\zeta_m^{(k)}=S_m^{(k)}-S_{m-1}^{(k)}$ be the number of successes between the $(m-1)$-th failure and the m-th failure in the Bernoulli sequence $\Big(B_k(j)\Big)_{j\geq 1}$. We define recursively the forward branching process $(Z_k)_{k\geq 0}$ started at $z\geq 1$ by

$$Z_0:=\;z,\quad\text{and}\quad Z_{k+1}:=S_{Z_k}^{(k+1)}=\sum_{m=1}^{Z_k}\zeta_m^{(k+1)}\quad\text{for } k\geq 1. \tag{3.2}$$

Lemma 3.1. *The following statements hold under P (see (3.1)):*

(i) $(Z_k)_{k\geq 0}$ *is a time homogeneous Markov chain;*

(ii) $U_k^{(0)} = Z_k \quad \forall k \geq 0$ *on the event* $\{T_0 < \infty\}$*;*

(iii) $U_k^{(0)} \leq Z_k \quad \forall k \geq 0$ *on the event* $\{T_0 = \infty\}$*.*

Proof. Statement (i) follows from (3.2), the (**IID**) assumption on the environment, and the conditional independence of Bernoulli random variables $\Big(B_k(j)\Big)_{j\geq 1}$ given a realization ω of the environment. The proofs of (ii) and (iii) are the same as in [44, p. 1961] and are, therefore, omitted. □

We can rewrite (3.2) as

$$Z_{k+1} \;=\; Z_k + \sum_{m=1}^{M(k)} (\zeta_m^{(k+1)} - 1) + \sum_{m=M(k)+1}^{Z_k} (\zeta_m^{(k+1)} - 1) \tag{3.3}$$

with understanding that if the upper limit of the summation is less than the lower one then the sum is set to be 0. Here $\zeta_m^{(k+1)}$ can be interpreted as the number of children of the m-th individual in the k-th generation. Note that the following holds under P:

(3.2.a) the random quantities $\left(\zeta_1^{(k)}, ..., \zeta_{M(k)}^{(k)}\right)$, $\left(\zeta_{M(k)+j}^{(k)}\right)_{j\in\mathbb{N}}$, $k \in \mathbb{N}$, are independent;

(3.2.b) the random vectors of random dimension $\left(\zeta_1^{(k)}, ..., \zeta_{M(k)}^{(k)}\right)$, $k \in \mathbb{N}$, which take values in $\cup_{m=1}^{\infty} \mathbb{N}_0^m \cup \Delta$, where Δ represents a "vector" of dimension 0, are i.i.d.;

(3.2.c) the random variables $\left(\zeta_{M(k)+j}^{(k)}\right)_{j,k\in\mathbb{N}}$ are i.i.d. geometric with parameter $\frac{1}{2}$.

Lemma 3.2.

$$E_x\left[\sum_{m=1}^{M(k)} \left(\zeta_m^{(k+1)} - 1\right)\right] = \delta. \tag{3.4}$$

The proof is the same as the one in [44, Lemma 14] and is omitted.

The following two observations will allow us to study right excursions of ERW using forward branching processes: (**1**) the maximum of the right excursion of ERW is equal to the extinction time of the associated forward branching process; (**2**) the duration of the right excursion from the origin is twice the size of the total progeny of the associated forward branching process. Define the extinction time and the total progeny of the forward branching process Z respectively by

$$\sigma_0^Z = \inf\{j > 0 \mid Z_j = 0\}, \quad S^Z = \sum_{j=0}^{\sigma_0^Z - 1} Z_j.$$

Theorem 3.1. *Let $z \in \mathbb{N}$ and $Z = (Z_n)_{n\geq 0}$ be defined by (3.2). Denote by P_z^Z the probability measure associated to this branching process.*

(i) If $\delta > 1$, then

$$P_z^Z(\sigma_0^Z = \infty) > 0. \tag{3.5}$$

(ii) If $\delta < 1$, then there are $C_1, C_2 \in (0,\infty)$ such that

$$\lim_{n\to\infty} n^{1-\delta} P_z^Z(\sigma_0^Z > n) = C_1(z); \qquad \lim_{n\to\infty} n^{\frac{1-\delta}{2}} P_z^Z(S_Z > n) = C_2(z).$$

(iii) If $\delta = 1$, then there are $C_3, C_4 \in (0,\infty)$ such that

$$\lim_{n\to\infty} (\log n) P_z^Z(\sigma_0^Z > n) = C_3(z); \qquad \lim_{n\to\infty} (\log n) P_z^Z(S_Z > n) = C_4(z).$$

The proof of this theorem depends on a number of technical results and is deferred to Chapter 8.

3.3 Backward branching process

The backward branching process is related to down-crossings of the ERW when the random walk reaches a fixed level $n \in \mathbb{N}$, to the right of the origin. To be precise, let

$$D_k^{(n)} = \sum_{j=0}^{T_n-1} \mathbb{1}_{\{X_j=k, X_{j+1}=k-1\}} \quad \text{for} \quad n \geq 1,\ k \leq n,$$

be the number of down-crossing of the walk from k to $k-1$ before time T_n.

Note that the number of down-crossings from $k+1$ to k prior to T_n is 1 less than the number of up-crossings from k to $k+1$ prior to T_n and also that the ERW can make a few down-crossings from k to $k-1$ before making an up-crossing from k to $k+1$. In fact, we can consider down-crossings from k to $k-1$ before the first up-crossing from k to $k+1$ as "children" of the first up-crossing, down-crossings from k to $k-1$ before the second up-crossing from k to $k+1$ as "children" of the second up-crossing, etc. Thus, given the number of down-crossings from $k+1$ to k, we add 1 "immigrant" to get the correct number of up-crossings from k to $k+1$ up to time T_n. That is, for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$

$$U_k^{(n)} \;=\; D_{k+1}^{(n)} + \mathbb{1}_{\{0 \leq k < n\}} \quad \text{where} \quad U_k^{(n)} = \sum_{j=0}^{T_n-1} \mathbb{1}_{\{X_j=k,\ X_{j+1}=k+1\}}.$$

Therefore

$$T_n \;=\; \sum_{k\in\mathbb{Z}} U_k^{(n)} + D_k^{(n)} = n + 2\sum_{k\leq n} D_k^{(n)} = n + 2\sum_{k=0}^{n} D_k^{(n)} + 2\sum_{k<0} D_k^{(n)}$$

From this information we can compute the distribution of the number of down-crossings from k to $k-1$, add an "immigrant" to get the number of up-crossings from $k-1$ to k, and so on. Thus, looking at the numbers of down-crossings from n to $n-1$ to $n-2$ etc., we obtain a branching process, which we shall now describe more formally.

The number of down-crossings $D_k^{(n)}$ can be computed from the Bernoulli random variables $\Big(B_k(j)\Big)_{j\geq 1}$. If we denote by

$$F_0^{(k)} := 0 \quad \text{and} \quad F_m^{(k)} := \min\left\{ r \geq 0 : \sum_{j=1}^{r+m} B_k(j) = m \right\}$$

the number of failures in the sequence $\Big(B_k(j)\Big)_{j\geq 1}$ before the m-th success, then the following holds.

Claim 3.3.1. *On the event $\{T_n < \infty\}$ we have $D_n^{(n)} = 0$ and for $0 \leq k < n$*

$$\text{if } D_{k+1}^{(n)} = m, \text{ then } D_k^{(n)} = F_{m+1}^{(k)}.$$

For a proof of this claim we refer the reader to [42, Section 2] and reference therein.

Let $\xi_m^{(k)} = F_m^{(k)} - F_{m-1}^{(k)}$ be the number of failures between the $m-1$-th and the m-th successes in the Bernoulli sequence $\Big(B_k(j)\Big)_{j\geq 1}$. We define the backward branching process $(V_k)_{k\geq 0}$ started at $x \geq 0$ by

$$V_0 := 0, \quad \text{and} \quad V_{k+1} := F_{V_k+1}^{(k)} = \sum_{m=1}^{V_k+1} \xi_m^{(k)} \quad \text{for } k \in \mathbb{N}_0. \tag{3.6}$$

Lemma 3.3. *Let $n \geq 1$. On the event $\{T_n < \infty\}$ we have*

$$(D_n^{(n)}, D_{n-1}^{(n)}, ..., D_0^{(n)}) \stackrel{D}{=} (V_0, V_1, ..., V_{n-1}, V_n).$$

The proof is similar to the one in [37]. For full details we refer the reader to [9, Proposition 2.2]. Rewriting equation (3.6) we get

$$V_{k+1} \;=\; V_k + 1 + \sum_{m=1}^{M(k)} (\xi_m^{(k)} - 1) + \sum_{m=M(k)+1}^{V_k+1} (\xi_m^{(k)} - 1) \tag{3.7}$$

Note that the following holds under P:

(3.3.a) the random quantities $(\xi_1^{(k)}, ..., \xi_{M(k)}^{(k)})$, $\left(\xi_{M(k)+j}^{(k)}\right)_{j\in\mathbb{N}}$, $k \in \mathbb{N}$, are independent;

(3.3.b) the random vectors of random length $(\xi_1^{(k)}, ..., \xi_{M(k)}^{(k)})$ $(k \geq 1)$ which take values in $\cup_{m=1}^{\infty}\mathbb{N}_0^m \cup \Delta$, where Δ represents a "vector" of dimension 0 are i.i.d.;

(3.3.c) the random variables $\left(\xi_{M(k)+j}^{(k)}\right)_{j\in\mathbb{N}}$ are i.i.d. geometric with parameter $\frac{1}{2}$.

Lemma 3.4.

$$E\left[\sum_{m=1}^{M(k)} \xi_m^{(k)} - M(k) + 1\right] = 1 - \delta. \tag{3.8}$$

For the proof we shall need another lemma.

Lemma 3.5. *For a fixed $\omega \in \Omega$*

$$E_\omega\left[F_{M(k)}^{(k)}\right] = M(k) - \sum_{j=1}^{M(k)} (2\omega(k,j) - 1). \tag{3.9}$$

Proof. Let S denote the number of successes in the first $M(k)$ trials.

$$E_\omega\left[F_{M(k)}^{(k)}\right] = E_\omega\left[E_\omega\left[F_{M(k)}^{(k)}\middle|S\right]\right] \quad \text{and} \quad F_{M(k)}^{(k)}|S \overset{D}{=} M(k) - S + \widetilde{F_{M(k)-S}^{(k)}},$$

where $M(k)-S$ is the number of failures in the first $M(k)$ trials and $\widetilde{F_{M(k)-S}^{(k)}}$ is the number of failures before the $M(k)-S$ successes in the i.i.d Bernoulli sequence with success probability $\frac{1}{2}$ i.e it is just negative binomial distribution with parameters $(M(k) - S, \frac{1}{2})$.

$$\begin{aligned} E_\omega\left[F_{M(k)}^{(k)}\right] &= M(k) - E_\omega[S] + E_{x,\omega}\left[E_\omega\left[F_{M(k)}^{(k)}\middle|S\right]\right] \\ &= M(k) - E_\omega[S] + E_\omega[M(k) - S] \\ &= 2E_{x,\omega}[M(k) - S] \quad \text{and,} \end{aligned}$$

$$E_\omega[S] = E_\omega\left[\sum_{j=1}^{M(k)} 1_{\{B_k(j)=1\}}\right] = \sum_{j=1}^{M(i)} P_\omega(B_k(j)=1) = \sum_{j=1}^{M(k)} \omega(k,j).$$

Hence,

$$E_\omega\left[F^{(k)}_{M(k)}\right] = 2\Big(M(k) - \sum_{j=1}^{M(k)} \omega(k,j)\Big) = M(k) - \sum_{j=1}^{M(k)} (2\omega(k,j)-1). \quad \square$$

Proof of Lemma 3.4. Taking expectation of both sides in (3.9) of Lemma 3.5

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^{M(k)} \xi^{(k)}_m - M(k) + 1\right] &= \mathbb{E}\left[E_\omega\left[\sum_{m=1}^{M(k)} \zeta^{(k)}_m - M(k) + 1\right]\right] \\ &= \mathbb{E}\left[E_\omega\left[F^{(k)}_{M(k)} - M(k) + 1\right]\right] \\ &= 1-\delta. \quad \square\end{aligned}$$

We shall use the information about life times and total progeny of the backward branching process to construct and analyze a regeneration structure for transient ERWs. This analysis is the key to the proofs of ballisticity and limit laws for transient ERWs.

Theorem 3.2. *Let $x \in \mathbb{N}_0$ and $V = (V_n)_{n\ge 0}$ be the backward branching process defined in (3.6) and P^V_x be the probability measure associated with this branching process.*

Define the life time and the total progeny of the backward branching process V respectively by

$$\sigma^V_0 = \inf\{j > 0 \mid V_j = 0\}, \quad S^V = \sum_{j=0}^{\sigma^V_0 - 1} V_j.$$

(i) If $\delta < 0$ then

$$P^V_x(\sigma^V_0 = \infty) > 0. \tag{3.10}$$

(ii) If $\delta > 0$ then there are $C_5, C_6 \in (0,\infty)$ such that

$$\lim_{n\to\infty} n^{\delta} P_x^V(\sigma_0^V > n) = C_5(x); \qquad \lim_{n\to\infty} n^{\frac{\delta}{2}} P_x^V(S^V > n) = C_6(x).$$

(iii) If $\delta = 0$*, then there are* $C_7, C_8 \in (0, \infty)$ *such that*

$$\lim_{n\to\infty} (\log n) P_x^V(\sigma_0^V > n) = C_7(x) \qquad \lim_{n\to\infty} (\log n) P_x^V(S_V > n) = C_8(x).$$

The proof of this theorem depends on a number of technical results and is deferred to Chapter 8.

Chapter 4

Proof of Theorems 2.1 and 2.2

4.1 Preliminaries

We shall start by stating a definition and some lemmas that are needed for the proofs of recurrence versus transience , law of larges numbers and ballisticity.

Definition 4.1. *Let $x \in \mathbb{Z}$ and $\omega \in \Omega$ we write $x \xrightarrow{\omega} x+1$ if and only if $\sum_{i=i}^{\infty} \omega(x,i) = \infty$ and $x \xrightarrow{\omega} x-1$ if and only if $\sum_{i=i}^{\infty}(1-\omega(x,i)) = \infty$.*
Define

$$b_1 \;:=\; \mathbb{P}\Big(\sum_{i=i}^{\infty} \omega(0,i) < \infty\Big) \quad \text{and} \quad b_{-1} := \mathbb{P}\Big(\sum_{i=i}^{\infty}(1-\omega(0,i)) < \infty\Big).$$

The meaning of the above relation $x \xrightarrow{\omega}$ is illustrated in the content of the following lemma, which follows by a straightforward application of the second Borel-Cantelli Lemma.

Lemma 4.1. *Let $\omega \in \Omega$ and $x, y \in \mathbb{Z}$ with $x \xrightarrow{\omega} y$. Then on the event that ERW visits x infinitely often, y is P_0-a.s. visited infinitely often as well.*

Proof. Let $I_x = \Big\{ \sum \mathbb{1}_{\{X_n = x\}} = \infty \Big\}$ and we shall show that, $\forall x \neq y, \quad P_0\Big(I_x \Delta I_y\Big) = 0.$

W.l.o.g we may assume that $x > y$. It is enough to show that $P_0\Big(I_x \Delta I_y\Big) = 0$. Since $P_0\Big(I_x \Delta I_y\Big) \leq \sum_{k=y-1}^{x} P_0\Big(I_k \Delta I_{k-1}\Big)$, it suffices to prove that

$$I_x \overset{P_{0,\omega}}{=} I_{x-1} \quad \text{or} \quad P_0\Big(I_x \Delta I_{x-1}\Big) = 0.$$

Fix an environment $\omega \in \Omega$ and let $\mathcal{F}_n$ be the sigma algebra generated by $X_0, X_1, ..., X_n$. Consider the event $A_n = \{X_{n-1} = x, X_n = x-1\} \in \mathcal{F}_n$ and denote by $L_x(n) = \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = x\}}$ the number of visits to x prior to time n. Then,

$$\begin{aligned} P_{0,\omega}\Big(A_n \Big| \mathcal{F}_{n-1}\Big) &= P_{0,\omega}\Big(X_n = x-1, X_{n-1} = x \Big| \mathcal{F}_{n-1}\Big) \\ &= \mathbb{1}_{\{X_{n-1}=x\}} \omega(X_{n-1}, L_{X_{n-1}}(n-1)). \end{aligned}$$

Let $T_x^{(k)}$ the time of the k-th return to x. Then on the event I_x, $T_x^{(k)} < \infty$. Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} P_{0,\omega}\Big(A_n \Big| \mathcal{F}_{n-1}\Big) &\geq \sum_{n > T_x^{(M(x)+1)}}^{\infty} P_{0,\omega}\Big(A_n \Big| \mathcal{F}_{n-1}\Big) \\ &= \frac{1}{2} \sum_{n > T_x^{(M(x)+1)}}^{\infty} \mathbb{1}_{\{X_{n-1}=x\}} = \infty. \end{aligned}$$

Therefore on the event $I_x = \Big\{ \sum_{n=1}^{\infty} \mathbb{1}_{\{X_{n-1}=x\}} = \infty \Big\}$ we have

$$\sum_{n=1}^{\infty} P_{0,\omega}\Big(A_n \Big| \mathcal{F}_{n-1}\Big) = \infty.$$

By the second Borel-Cantelli Lemma [21, Lemma 5.3.2] we have

$$\Big\{A_n \quad i.o.\Big\} = \left\{ \sum_{n=1}^{\infty} \mathbb{1}_{A_n} = \infty \right\} \overset{P_{0,\omega}}{=} \left\{ \sum_{n=1}^{\infty} P_{0,\omega}\Big(A_n \Big| \mathcal{F}_{n-1}\Big) = \infty \right\},$$

where $A \stackrel{P_0}{=} B$ means that $P_0(A\Delta B) = 0$. Note we have the following inclusions:

$$I_x = \left\{ \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = x\}} = \infty \right\} \subseteq \left\{ A_n \quad i.o. \right\} \subseteq I_{x-1} = \left\{ \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = x-1\}} = \infty \right\} \quad \forall \omega \in \Omega.$$

We conclude that

$$I_x \stackrel{P_{0,\omega}}{\subseteq} I_{x-1}, \quad \forall \omega \in \Omega.$$

Similarly, $I_{x-1} \stackrel{P_{x,\omega}}{\subseteq} I_x, \quad \forall \omega \in \Omega$. Finally $P_{0,\omega}(I_x \Delta I_y) = 0$ forall $\omega \in \Omega$, hence

$$P_0(I_x \Delta I_y) = \int_\Omega P_{0,\omega}(I_x \Delta I_y)\mathbb{P}(d\omega) = 0.$$

Thus,

$$\left\{ \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = x\}} = \infty \right\} \stackrel{P_0}{=} \left\{ \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = y\}} = \infty \right\}.$$

□

Corollary 4.1. *Let $\omega \in \Omega$. Then $P_{0,\omega}$-a.s.*

$$\liminf X_n \in \{-\infty, +\infty\} \quad \textit{and} \quad \limsup X_n \in \{-\infty, +\infty\}.$$

Proof. Suppose that $\limsup_{n\to\infty} X_n \notin \{-\infty, +\infty\}$ then there exists $R = R(\omega, s)$ where $(\omega, s) \in \Omega \times \Sigma$ such that

$$\limsup_{n\to\infty} X_n \quad := \quad R < \infty.$$

On the event $\{R = r\}$ the ERW visits the site r infinitely often and by lemma 4.1, it will visit $r+1$ infinitely often as well and this contradicts the fact that r is a $\limsup$. Thus $\limsup_{n\to\infty} X_n = \infty$. Similar argument for the $\liminf X_n$. $\square$

Theorem 4.1. *[45, Theorem 2.3] Assume* (**SE**) *and* (**ELL**).

(a) *If $b_1 > 0$ and $b_{-1} > 0$ then the range is P_0-a.s. finite.*

(b) *If $b_1 = 0$ and $b_{-1} > 0$ then $P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = 1$.*

(c) *If $b_1 > 0$ and $b_{-1} = 0$ then $P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 1$.*

(d) *If $b_1 = 0$ and $b_{-1} = 0$ then the range is P_0-a.s. infinite.*

Under our assumptions $x \xrightarrow{\omega} x \pm 1$ for all x and all ω both b_1 and b_{-1} are 0 and we have the following

Lemma 4.2. *Under* (**SE**) *and* (**ELL**), *the following statements are equivalent.*

(i) $P_0\Big(\lim_{n\to\infty} X_n = \infty\Big)$, $P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) \in \Big\{0, 1\Big\}$.

(ii) $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big)$, $P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) \in \Big\{0, 1\Big\}$.

Proof. We shall show that $(i) \Longrightarrow (ii)$. There are only 3 cases to consider

- If $P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = 1$, then

$$P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = 1 \quad \text{and} \quad P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 0.$$

- If $P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 1$, then

$$P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = 0 \quad \text{and} \quad P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 1.$$

- If $P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 0$, then only cases (a) or (d) of Theorem 4.1 are possible.

In case (a) the ERW has finite range P_0-a.s. and thus

$$P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 0.$$

In case (d) we argue as follows. Note that all inclusions below are up to sets of , P_0, measure zero. Since we assumed that $P_0\Big(X_n \not\longrightarrow -\infty\Big) = 1$, we have

$$\begin{aligned}&\Big\{\sup_{n\geq 0} X_n < \infty\Big\} \subseteq \Big\{\sup_{n\geq 0} X_n < \infty\Big\} \bigcap \Big\{X_n \not\longrightarrow -\infty\Big\} \subseteq \bigcup_{z\in\mathbb{Z}} \Big\{X_n = z \text{ i.o.}\Big\}\\ \subseteq\ &\Big\{\sup_{n\geq 0} X_n = \infty\Big\},\end{aligned}$$

where the last inclusion is due to (d) and Lemma 4.1. Hence $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = 1$. Similarly,

$$\begin{aligned}&\Big\{\inf_{n\geq 0} X_n > -\infty\Big\} \subseteq \Big\{\inf_{n\geq 0} X_n > -\infty\Big\} \bigcap \Big\{X_n \not\longrightarrow \infty\Big\} \subseteq \bigcup_{z\in\mathbb{Z}} \Big\{X_n = z \text{ i.o.}\Big\}\\ \subseteq\ &\Big\{\inf_{n\geq 0} X_n = -\infty\Big\},\end{aligned}$$

and we have $P_0\Big(\inf_{n\geq 0} X_n = \infty\Big) = 1$. □

Next we show the converse: $(ii) \Longrightarrow (i)$. We shall consider 4 cases.

- If $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 0$, then the ERW has finite range and thus $P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 0$.
- If $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = 1$ and $P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 0$, then only cases (b) or (d) of theorem 4.1 are possible. In case (b) we have that

$$P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = 1 \quad \text{and} \quad P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 0.$$

In case (d); since we assumed that $P_0\Big(\inf_{n\geq 0} X_n > -\infty\Big) = 1$, we have

$$\Big\{\inf_{n\geq 0} X_n > -\infty\Big\} \bigcap \Big\{X_n \not\longrightarrow \infty\Big\} \subseteq \bigcup_{z\in\mathbb{Z}} \Big\{X_n = z \text{ i.o.}\Big\} \subseteq \Big\{\inf_{n\geq 0} X_n = -\infty\Big\},$$

where the last inclusion is due to lemma 4.1. Thus, $P_0\Big(X_n \not\longrightarrow \infty\Big) = 0$.

- If $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = 0$ and $P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 1$, then only cases (c) and (d) of Theorem 4.1 are possible. If case (c) we have

$$P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = 0 \quad \text{and} \quad P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 1.$$

In case (d),

$$\Big\{\sup_{n\geq 0} X_n < \infty\Big\} \bigcap \Big\{X_n \not\longrightarrow -\infty\Big\} \subseteq \bigcup_{z\in\mathbb{Z}} \Big\{X_n = z \text{ i.o.}\Big\} \subseteq \Big\{\sup_{n\geq 0} X_n = \infty\Big\}.$$

where the last inclusion is due to Lemma 4.1 and thus $P_0\Big(X_n \not\longrightarrow -\infty\Big) = 0$.

- If $P_0\Big(\sup_{n\geq 0} X_n = \infty\Big) = P_0\Big(\inf_{n\geq 0} X_n = -\infty\Big) = 1$, then, obviously,

$$P_0\Big(\lim_{n\to\infty} X_n = \infty\Big) = P_0\Big(\lim_{n\to\infty} X_n = -\infty\Big) = 0.$$

$\square$

4.2 Proof of recurrence

We shall modify the environment in order to allow the ERW to jump to the right after every visit to the origin in the following manner

$$\omega^{+}(z,j) = \begin{cases} \omega(z,j) & \text{if} \quad z \neq 0 \quad \textit{and} \quad j \geq 1 \\ 1 & \text{if} \quad z = 0 \quad \textit{and} \quad \forall j \geq 1. \end{cases}$$

Let $T_0^{(0)} = 0$ and for $n \geq 1$ and $z \geq 0$, let

$$T_0^{(n)} = \inf\Big\{k > T_0^{(n-1)} : X_{k-1} > 0 \quad \text{and} \quad X_k = 0\Big\}.$$

be the time of the $n-$th return from the right by the ERW to the origin. It follows from Lemma 3.1 that

$$\begin{aligned} & P_{0,\omega^+}\Big(T_0^{(n)} < \infty\Big) \\ = \quad & P_{0,\omega^+}\Big(U_z^{(n)} = 0 \quad \text{for all z sufficienty large}\Big) \leq P_{n,\omega}\Big(\sigma_0^Z < \infty\Big) \end{aligned} \tag{4.1}$$

where $U_z^{(n)}$ is the number of up-crossings of the ERW from z to $z+1$ prior to time $T_0^{(n)}$ and Z the forward branching process associated to the ERW. Note that in the last probability on the **RHS** of (4.1) we can change the cookie environment from ω^+ to ω since our forward branching process Z is generated by using the same Bernoulli random variables $(B_z(j))_{j\in\mathbb{N}}$ with $z \geq 1$.

- If $T_0^{(n)} = \infty$, then since the $\omega(z,j)$ are uniformly bounded away from 0 and 1 the walk cannot stay bounded for the first n excursions to the right of 0. Therefore,

$$P_{0,\omega^+}\big(T_0^{(0)} = \infty\Big) = P_{0,\omega^+}\Big(U_z^{(n)} \geq 1 \quad \forall \quad z \geq 1\Big) \leq P_{n,\omega}^{Z}\Big(\sigma_0^Z = \infty\Big) \tag{4.2}$$

where the last inequality follows from Lemma 3.1. By combining (4.1) and (4.2) we conclude that

$$P_{0,\omega^+}\Big(T_0^{(n)} < \infty\Big) \;=\; P^Z_{n,\omega}\Big(\sigma^Z_0 < \infty\Big), \quad \forall n \geq 1. \tag{4.3}$$

Suppose that $\delta \leq 1$. Then it follows from (4.3) and Theorem 3.1 that

$$\begin{aligned} P_0\Big(T_0^{(n)} < \infty\Big) &= \mathbb{E}\Big[P_{0,\omega^+}\Big(T_0^{(n)} < \infty\Big)\Big] = \mathbb{E}\Big[P^Z_{n,\omega}\Big(\sigma^Z_0 < \infty\Big)\Big] \\ &= P^Z_n\Big(\sigma^Z_0 < \infty\Big) = 1 \qquad \forall n \geq 1. \end{aligned}$$

That is , with probability one every excursion of the ERW to the right of 0 will eventually return to 0. By symmetry of our model if $\delta \geq -1$ then all excursions to the left of 0 eventually return to 0. Thus, if $\delta \in [-1, 1]$ then all excursions from 0 are almost surely finite and so the ERW will returns to 0 infinitely often and by Lemma 4.1 the walk will visits every site infinitely often.

4.3 Proof of transience

Claim 4.3.1. *If $\delta > 1$, then $P_1(T_0 = \infty) > 0$.*

If $\delta > 1$ then the ERW is recurrent from the left. Consider the first excursion from the right. If $P_1(T_0 < \infty) = 1$, then $P_1(\sigma^Z_0 < \infty) = 1$ which contradicts that $P_1(\sigma^Z_0 = \infty) > 0$ by theorem 3.1. Thus $P_1(T_0 = \infty) > 0$. □

Without loss of generality, we may consider the first right excursion from 0. Let

$$p \;:=\; P_1(T_0 = \infty),$$

and let $T_0^{(i)}$ be the time of the $i-$th return to 0 by the ERW. Consider consecutive right excursions of 0 and let $K_i = \max\{X_n, 0 \le n \le T_0^{(i)}\}$. Since we are assuming that $\delta > 1$, the walk is recurrent from the left and all left excursions of 0 are P_0 a.s finite and return infinitely often to zero. That is $P_1(\limsup_{n\to\infty} X_n \ge 0) = 1$.

Claim 4.3.2.

$$P_0(\limsup_{n\to\infty} X_n \ge 0) = 1 \implies P_0(\limsup_{n\to\infty} X_n = \infty) = 1.$$

Suppose that $\limsup_{n\to\infty} X_n =: R < \infty$. On the event $\{R = r\}$ the ERW visits the site r infinitely often and it will visit $r+1$ infinitely often as well and this contradicts the fact r is a $\limsup$. Thus $\limsup_{n\to\infty} X_n = \infty$ with probability one. □

Claim 4.3.3. *There exists an infinite excursion to the right of* 0.

Since $P_0(\limsup_{n\to\infty} X_n = \infty) = 1$, the ERW will always hit $K_i + 1$ at some time after $T_0^{(i)}$ with probability one. At that hitting time, the law of its excursion to the right from that level is the same as for the ERW starting from 1 (by stationarity of the environment starting from K_i). In addition, this right excursions starting from $K_i + 1$ is independent of all previous right excursions and has probability $P_1(T_0 = \infty)$ to be infinite. We conclude that with probability one there exists an i such that the right excursion starting from K_i+1 is infinite.Thus, there exists an infinite excursion to the right from 0. □

Claim 4.3.4. *On the event that there is an infinite right excursion we have*

$$P_0\Big(X_n \to \infty\Big) = 1.$$

Indeed, we have

$$\Big\{X_n \not\to \infty\Big\} \overset{P_{0,\omega}}{=} \Big\{X_n \not\to \infty, \text{there is an infinite right excursion}\Big\}.$$

On the other hand, on the event $\Big\{ \limsup_{n\to\infty} X_n \geq 0 \Big\}$

$$\Big\{ X_n \not\to \infty \Big\} \subset \bigcup_{x\in\mathbb{Z}} \Big\{ \sum_{n\geq 1} \mathbb{1}_{\{X_n = x\}} = \infty \Big\} \subset \Big\{ \sum_{n\geq 0} \mathbb{1}_{\{X_n = 0\}} = \infty \Big\}.$$

the last inclusion is due to lemma 4.1. But, almost surely, we have the following inclusion

$$\begin{aligned} &\Big\{ X_n \not\to \infty \Big\} \bigcap \Big\{ \exists \text{ an infinite right excursion} \Big\} \\ \subset \; &\Big\{ \sum_{n\geq 0} \mathbb{1}_{\{X_n = 0\}} = \infty \Big\} \bigcap \Big\{ \exists \text{ an infinite right excursion} \Big\}. \end{aligned}$$

and the intersection in the **LHS** is empty, this implies that $P_0(X_n \not\to \infty) = 0$. Thus, $P_0(X_n \to \infty) = 1$. The ERW is transient to the right when $\delta > 1$. By symmetry, the ERW is transient to the left when $\delta < -1$. □

4.4 Proof of law of large numbers and ballisticity

- **Proof of law of large numbers :** Since our environment satisfies (**IID**) and (**UEL**), it also satisfies (**SE**) and (**ELL**). It follows from [3, Theorem 1.2] that the $0-1$ law (i) of Lemma 4.2 holds. Since both $0-1$ laws of Lemma 4.2 are equivalent , the ERW satisfies the law of large numbers with deterministic speed $v \in [-1.1]$ by [44, Proposition 13]. The idea of the proof is exactly the same as the one given in [73, Theorem 13] by showing that $\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v}$ and the statement of Theorem 2.1 follows by using a standard argument from [72, Lemma 2.1.17]. To this end, Let k_n be the smallest positive integer such that, $T_{k_n} \leq n < T_{k_n+1}$. Then $\lim_{n\to\infty} \frac{n}{k_n} = \frac{1}{v}$. Note that by definition of T_n we have $k_n - (n - T_{k_n}) \leq X_n \leq k_n$. Thus,

$$\frac{k_n}{n} - \Big(1 - \frac{T_{k_n}}{n}\Big) \leq \frac{X_n}{n} \leq \frac{k_n}{n}$$

But, $\lim_{n\to\infty} \frac{k_n}{n} = \lim_{n\to\infty} \frac{n}{T_n} = v$. Therefore,

$$v \geq \limsup_{n\to\infty} \frac{X_n}{n} \geq \liminf_{n\to\infty} \frac{X_n}{n} \geq v$$

□

We shall introduce some notations and facts that are needed for the proof of ballisticity and throughout the sequel as well. Let $S^V = \sum_{j=0}^{\sigma_0^V - 1} V_j$ and define the consecutive times when $V_0 = 0$ by

$$\sigma_{0,0} = 0, \qquad \sigma_{0,i} = \inf\{j > \sigma_{0,i-1} | V_j = 0\}, \qquad i \in \mathbb{N}. \tag{4.4}$$

Let $S_i := \sum_{j=\sigma_{0,i-1}}^{\sigma_{0,i}-1} V_j, i \in \mathbb{N}$ the total progeny of V over each lifetime, and by

$$N_n = \max\{i \geq 0 | \sigma_{0,i} \leq n\} \tag{4.5}$$

the number of renewals up to time n. The sequence $(\sigma_{0,i} - \sigma_{0,i-1}, S_i)_{i\geq 1}$ is i.i.d under P_0^V. Moreover

$$\sigma_{0,i} - \sigma_{0,i-1} \overset{D}{=} \sigma_0^V \quad \text{and} \quad S_i \overset{D}{=} S^V, \ i \in \mathbb{N}. \tag{4.6}$$

By The Strong Law of large numbers (**SLLN**) for Counting Process [28, Theorem 5.1] that

$$\lim_{n\to\infty} \frac{N_n}{n} = \lambda := \frac{1}{E_0^V(\sigma_0^V)} > 0 \quad \text{a.s.} \tag{4.7}$$

and if the first and the second moments are finite, it follows from The Central Limit

Theorem for Counting Process [28, Theorem 5.2]that

$$\frac{N_n - \lambda n}{\sqrt{\frac{E_0^V(\sigma_0^V)^2 n}{\lambda^3}}} \quad \Rightarrow \quad \mathcal{N}(0,1) \quad \text{as} \quad n \longrightarrow \infty. \tag{4.8}$$

- **Proof of ballisticity :** Without loss of generality we may assume that $\delta > 1$ due to the symmetry of our model. Recall that for every $n \in \mathbb{N}$:

$$T_n = n + 2\sum_{k=0}^{n} D_k^{(n)} + 2\sum_{k<0} D_k^{(n)}.$$

Since the ERW is transient to the right, it will spend only a finite time on the negative integers. Then,

$$\lim_{n\to\infty} \frac{\sum_{k<0} D_k^{(n)}}{n} \quad = \quad 0 \quad a.s.$$

and therefore

$$\frac{1}{v} = \lim_{n\to\infty} \frac{T_n}{n} = 1 + \lim_{n\to\infty} \frac{2\sum_{k=0}^{n} D_k^{(n)}}{n}.$$

Invoking Lemma 3.3 and what have been said above

$$\frac{N_n}{n} \cdot \frac{2\sum_{i=1}^{N_n} S_i}{N_n} \leq \frac{2\sum_{j=1}^{n} V_j}{n} = \frac{2\sum_{i=1}^{N_n} S_i + 2\sum_{\sigma_{0,N_n}}^{n} V_i}{n} \leq \frac{2\sum_{i=1}^{N_n+1} S_i}{N_n+1} \cdot \frac{N_n+1}{n} \tag{4.9}$$

Since $N_n \to \infty$ with probability one as $n \to \infty$ and using (4.8)

$$\lim_{n\to\infty} \frac{2\sum_{j=0}^{n} V_j}{n} = 2\lambda E S_i = 2\lambda E S^V \quad \text{and} \quad v = \left(1 + \frac{2E_0^V(S^V)}{E_0^V(\sigma_0^V)}\right)^{-1}.$$

Now we can conclude that

$$v > 0 \iff E_0^V(S^V) < \infty \iff \delta > 2 \quad \text{(by Theorem 3.2)}$$

and by symmetry $v < 0 \Leftrightarrow \delta < -2$.

Note that

$$E_0^V(S^V) = \infty \iff \delta \leq 2 \quad \text{(by Theorem 3.2)},$$

it follows from [21, Theorem 2.5.9] that

$$\limsup_{n\to\infty} \frac{S_n}{n} = \infty,$$

Therefore one can deduce from (4.9) that $v = 0$ if and only if $\delta \leq 2$ and by symmetry $v = 0 \iff \delta \geq -2$.

□

Chapter 5

Proof of Theorem 2.3

The life time and the total progeny of the backward branching defined over one single lifetime cycle, namely the two quantities

$$\sigma_0^V := \{j \geq 1 | V_j = 0\} \quad \text{and} \quad S^V := \sum_{j=0}^{\sigma_0^V - 1} V_j,$$

partially characterize the regenerative structure of the ERW in the transient regime and the idea of the proof of Theorem 2.3 relies on this regenerative structure. It was already used in [37, 38] to derive the limit laws of T_n and it was for the first time adapted to multi-dimensional RWRE in [60] in order to give a proof for a law of large numbers. The i.i.d structure of the regenerative times $(\sigma_{0,i} - \sigma_{0,i-1})_{i \geq 1}$ (see 4.6) and the life time of the total progeny S_i of the branching process prior to each regeneration combined with theorem 3.1 will enable us to give limit laws of the T_n when the ERW is transient.

Recall that

$$T_n = n + 2\sum_{k=0}^{n} D_k^{(n)} + 2\sum_{k<0} D_k^{(n)}. \tag{5.1}$$

- **Proof of** :

$$\frac{T_n}{n^{\frac{2}{\delta}}} \Rightarrow Z_{\frac{\delta}{2},b} \quad \text{when} \quad \delta \in (1,2).$$

Since $\delta > 1$, the random walk is transient to the right and it will spend only a finite time on the negative integers. With this in mind and Lemma 3.3 one can see that the fluctuations of T_n are determined by those of $\sum_{k=0}^{n} D_k^{(n)}$. Thus, it suffices to show that

$$\frac{T_n}{n^{\frac{2}{\delta}}} - \frac{2\sum_{j=0}^{n} V_j}{n^{\frac{2}{\delta}}} \Rightarrow 0 \quad \text{as} \quad n \to \infty.$$

We have by Theorem 3.2 that for $\delta > 0$

$$\lim_{n\to\infty} n^{\frac{\delta}{2}} P_0^V \Big(S^V > n \Big) = C_6(0) \in (0,\infty).$$

It follows from [21, Theorem 3.7.2.] that the distribution of S^V is in the domain of attraction of a stable law with index $\frac{\delta}{2} \in (\frac{1}{2},1)$. As $n \longrightarrow \infty$, $\frac{S^V - b_n}{a_n} \Rightarrow Y_1$ where Y_1 has a non degenerate distribution. The scaling constant a_n and the centering constant b_n can be chosen as follow

$$a_n := \inf \Big\{ y : P_0^V \Big(S_V > y \Big) \leq \frac{1}{n} \Big\} \quad \text{and} \quad b_n :\sim n E_0^V \Big(S^V \mathbb{1}_{\{S_V \leq a_n\}} \Big).$$

Thus we can take $a_n = n^{\frac{2}{\delta}}$ and since the index $\frac{\delta}{2} < 1$, the centering constant b_n can be chosen equal to zero for all $n > 0$. While the sequence $(\sigma_{0,k} - \sigma_{0,k-1})_{i\geq 1}$ is in the domain of attraction of a stable law of index $\delta \in (1,2)$. The scaling is of the order $n^{\frac{1}{\delta}}$ and the centering is of size $n\bar{\sigma}$ where $\bar{\sigma} = E_0^V(\sigma_{0,1}^V)$. Therefore

$$\frac{\sum_{i=0}^{n} S_i}{n^{\frac{2}{\delta}}} \quad \Rightarrow \quad Y_1 = Z_{\frac{\delta}{2},b} \quad \text{as n} \to \infty. \tag{5.2}$$

$$\frac{\sum_{k=1}^{n}(\sigma_{0,k}-\sigma_{0,k-1})-n\bar{\sigma}}{n^{\frac{1}{\delta}}} \Rightarrow Y_2 \quad \text{as n} \to \infty. \tag{5.3}$$

(5.2), (5.3) and the Portmanteau Theorem [7, Theorem 2.1. pp16] imply respectively that for all $\nu > 0$ there exists a constant $c_2(\nu) > 0$ and an integer N_ν^1 such that

$$\forall n \geq N_\nu^1 : \qquad P\Big(\Big|\sum_{k=1}^{n}(\sigma_{0,k}-\sigma_{0,k-1})-n\bar{\sigma}\Big| > c_2 n^{\frac{1}{\delta}}\Big) < \frac{\nu}{2} \quad \text{and,} \tag{5.4}$$

for all $\nu > 0$ there exists a constant $c_3(\nu) > 0$ and an integer N_ν^2 such that

$$\forall n \geq N_\nu^2 : \qquad P\Big(\Big|\sum_{i=1}^{n} S_i\Big| > c_3 n^{\frac{2}{\delta}}\Big) < \frac{\nu}{2}. \tag{5.5}$$

One can rewrite

$$\frac{\sum_{k=1}^{n} V_k}{n^{\frac{2}{\delta}}} = \frac{\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n^{\frac{2}{\delta}}} + \frac{\sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n^{\frac{2}{\delta}}}.$$

In order to get the claim, it's enough to show that the term :

$$\frac{\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n^{\frac{2}{\delta}}} \xrightarrow{P} 0 \quad \text{as n} \to \infty. \tag{5.6}$$

For $\epsilon > 0$ and $\epsilon' > 0$ fixed and very small

$$\begin{aligned} P_0^V\Bigg(\Bigg|\sum_{k=1}^{n} V_k - \sum_{j=1}^{\lceil\frac{n}{\bar{\sigma}}\rceil} S_j\Bigg| > \epsilon n^{\frac{2}{\delta}}\Bigg) &\leq P_0^V\Bigg(\Bigg|\sum_{k=1}^{\lceil\frac{n}{\bar{\sigma}}\rceil}(\sigma_{0,k}-\sigma_{0,k-1}) - \frac{n}{\bar{\sigma}}\bar{\sigma}\Bigg| > n^{\frac{1}{\delta}+\epsilon'}\Bigg) \\ + \qquad P_0^V\Bigg(\Bigg|\sum_{k=1}^{\lceil\frac{n}{\bar{\sigma}}\rceil}(\sigma_{0,k}-\sigma_{0,k-1}) - \frac{n}{\bar{\sigma}}\bar{\sigma}\Bigg| &\leq n^{\frac{1}{\delta}+\epsilon'}, \Bigg|\sum_{k=1}^{n} V_k - \sum_{j=1}^{\lceil\frac{n}{\bar{\sigma}}\rceil} S_j\Bigg| > \epsilon n^{\frac{2}{\delta}}\Bigg) \\ \leq \qquad P_0^V\Bigg(\Bigg|\sum_{k=1}^{\lceil\frac{n}{\bar{\sigma}}\rceil}(\sigma_{0,k}-\sigma_{0,k-1}) - \frac{n}{\bar{\sigma}}\bar{\sigma}\Bigg| &> n^{\frac{1}{\delta}+\epsilon'}\Bigg) + P_0^V\Bigg(\sum_{|j-\lceil\frac{n}{\bar{\sigma}}\rceil| \leq n^{\frac{1}{\delta}+\epsilon'}} S_j > \epsilon n^{\frac{2}{\delta}}\Bigg) \end{aligned}$$

$$\leq \quad P_0^V\left(\left|\sum_{k=1}^{\lceil \frac{n}{\bar\sigma}\rceil}(\sigma_{0,k}-\sigma_{0,k-1})-\frac{n}{\bar\sigma}\bar\sigma\right|>n^{\frac{1}{\delta}+\epsilon'}\right)+P_0^V\left(\sum_{j=1}^{\lceil 2n^{\frac{1}{\delta}+\epsilon'}+2\rceil}S_j>\epsilon n^{\frac{2}{\delta}}\right)$$

$$\overset{(5.4)\text{ and }(5.5)}{\leq} \quad \nu.$$

where we replaced n by $\lceil \frac{n}{\bar\sigma}\rceil$ in (5.4) and by $\lceil 2n^{\frac{1}{\delta}+\epsilon'}+2\rceil$ in (5.5) with $\frac{2}{\delta}-\frac{2}{\delta^2}-\frac{2\epsilon'}{\delta}>0$. □

- **Proof of:**

$$\frac{X_n}{n^{\frac{2}{\delta}}}\Rightarrow\left(Z_{\frac{\delta}{2},b}\right)^{-\frac{\delta}{2}}\quad\text{when}\quad\delta\in(1,2).$$

We shall show that $\dfrac{\sup_{i\leq n}X_i}{n^{\frac{2}{\delta}}}$ and $\dfrac{\inf_{i\geq n}X_i}{n^{\frac{2}{\delta}}}$ have the same limiting distribution and then deduce the limiting distribution of $\dfrac{X_n}{n^{\frac{2}{\delta}}}$. To this end, note that

$$\left\{\sup_{i\leq n}X_i<m\right\} = \left\{T_m>n\right\}\quad\text{and,}\tag{5.7}$$

$$\left\{\sup_{i\leq n}X_i<m\right\} \subset \left\{\inf_{i\geq n}X_i<m\right\}\subset\left\{\sup_{i\leq n}X_i<m+r\right\}\cup\left\{\inf_{i\geq T_{m+r}}X_i<m\right\}\tag{5.8}$$

Due to [39, Lemma 9.1] the last event on (**RHS**) of (5.6) has zero probability i.e.

$$\lim_{r\to\infty}\sup_{n\geq1}P_0\Big(\inf_{i\geq T_{m+r}}X_i<m\Big)=0.\tag{5.9}$$

By taking $m=\lceil xn^{\frac{\delta}{2}}\rceil$ and $r=\lceil\log n\rceil$ together with (5.7) and (5.8)

$$\lim_{n\to\infty}P_0\left\{\frac{\sup_{i\leq n}X_i}{n^{\frac{2}{\delta}}}<\frac{\lceil xn^{\frac{\delta}{2}}\rceil}{n^{\frac{2}{\delta}}}\right\}=\lim_{n\to\infty}P_0\left\{\sup_{i\leq n}X_i<\lceil xn^{\frac{\delta}{2}}\rceil\right\}$$

$$\overset{(5.7)}{=}\quad\lim_{n\to\infty}P_0\left\{\frac{T_{\lceil xn^{\frac{\delta}{2}}\rceil}}{\lceil xn^{\frac{\delta}{2}}\rceil)^{\frac{2}{\delta}}}>\frac{n}{(\lceil xn^{\frac{\delta}{2}}\rceil)^{\frac{2}{\delta}}}\right\}=P_0\left\{2\left[E_x^V(\sigma_0^V)\right]^{-\frac{2}{\delta}}Z_{\frac{\delta}{2},b}>x^{-\frac{2}{\delta}}\right\}$$

$$\overset{(5.8)}{\leq}\quad\lim_{n\to\infty}P_0\left\{\frac{\inf_{i\geq n}X_i}{n^{\frac{2}{\delta}}}<\frac{\lceil xn^{\frac{\delta}{2}}\rceil}{n^{\frac{2}{\delta}}}\right\}$$

$$\stackrel{(5.8)\text{and}(5.9)}{\leq} \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i}{n^{\frac{2}{\delta}}} < \frac{\lceil xn^{\frac{2}{\delta}}\rceil + \log n}{n^{\frac{2}{\delta}}}\right\}$$

$$= P_0\left\{2^{-\frac{\delta}{2}} E_x^V(\sigma_0^V)\left(Z_{\frac{\delta}{2},b}\right)^{-\frac{\delta}{2}} < x\right\}.$$

and we conclude that

$$\lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i}{n^{\frac{\delta}{2}}} < x\right\} = \lim_{n\to\infty} P_0\left\{\frac{\inf_{i\geq n} X_i}{n^{\frac{2}{\delta}}} < x\right\}$$

$$= P_0\left\{2^{-\frac{\delta}{2}} E_x^V(\sigma_0^V)\left(Z_{\frac{\delta}{2},b}\right)^{-\frac{\delta}{2}} < x\right\}.$$

Since $\inf_{i\geq n} X_i \leq X_i \leq \sup_{i\leq n} X_i$, it follows that

$$\lim_{n\to\infty} P_0\left\{\frac{X_n}{n^{\frac{\delta}{2}}} < x\right\} = P_0\left\{2^{-\frac{\delta}{2}} E_0^V(\sigma_0^V)\left(Z_{\frac{\delta}{2},b}\right)^{-\frac{\delta}{2}} < x\right\}.$$

□

- **Proof of :**

$$\frac{T_n}{n\log n} \xrightarrow{P} \frac{1}{c_1} \quad \text{as} \quad n \longrightarrow \infty \quad \text{when} \quad \delta = 2.$$

By Lemma 3.3 and (5.1)

$$\frac{T_n}{n\log n} - \frac{2\sum_{j=0}^{n} V_j}{n\log n} \Rightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{5.10}$$

It suffices to show that the second term in the **LHS** of (5.10) converges in distribution to a constant.

Since $\delta = 2$, it follows from Theorem 3.2 that the the sequence $(\sigma_{0,k} - \sigma_{0,k-1})_{i\geq 1}$ is in the domain of attraction of the normal distribution [29, Theorem 3.3]. The normalizing constant a_n is of the order $\sqrt{C_5 n \log n}$ by [29, Remark 3.5. pp. 338] and the centering $b_n = nE_0^V(\sigma_{0,1}^V)$. Then,

$$\frac{\sum_{k=1}^{n}(\sigma_{0,k} - \sigma_{0,k-1}) - nE_0^V(\sigma_{0,1}^V)}{\sqrt{C_5 n \log n}} \Rightarrow Z_{2,b} \quad \text{as n} \to \infty \quad \text{and,} \tag{5.11}$$

for the i.i.d sequence $(S_i)_{i\geq 1}$ the scaling $a_n \sim n$ and the centering $b_n \sim C_5 n \log n$. Then

$$\frac{\sum_{i=1}^{n} S_i - b_n}{n} \Rightarrow Y_3 \quad \text{as n} \to \infty. \tag{5.12}$$

where Y_3 has non degenerate distribution. We have the following

$$\frac{2\sum_{k=1}^{n} V_k}{n \log n} = \frac{2\Big(\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j\Big)}{n \log n} + \frac{2\sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n \log n} \quad \text{and,} \tag{5.13}$$

$$\frac{2\sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n \log n} = \frac{2\Big(\sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j - b_{\frac{n}{\bar{\sigma}}}\Big)}{n \log n} + \frac{2b_{\frac{n}{\bar{\sigma}}}}{n \log n} \xrightarrow{P} 2C_5\bar{\sigma}^{-1}. \tag{5.14}$$

Since the first term in the **RHS** of (5.14) goes to zero in probability, we only need to show that the term:

$$\frac{\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j}{n \log n} \xrightarrow{P} 0 \quad \text{as n} \to \infty. \tag{5.15}$$

For $\epsilon > 0$ and for n sufficiently large

$$P_0^V\left(\left|\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j\right| > \epsilon n \log n\right) \leq P_x^V\left(\left|\sum_{k=1}^{[\frac{n}{\bar{\sigma}}]}(\sigma_{0,k} - \sigma_{0,k-1}) - \frac{n}{\bar{\sigma}}\bar{\sigma}\right| > n^{\frac{3}{4}}\right)$$
$$+P_0^V\left(\left|\sum_{k=1}^{[\frac{n}{\bar{\sigma}}]}(\sigma_{0,k} - \sigma_{0,k-1}) - \frac{n}{\bar{\sigma}}\bar{\sigma}\right| \leq n^{\frac{3}{4}}, \left|\sum_{k=1}^{n} V_k - \sum_{j=1}^{[\frac{n}{\bar{\sigma}}]} S_j\right| > \epsilon n \log n\right)$$

$$\le P_x^V\left(\left|\sum_{k=1}^{\lceil \frac{n}{\bar\sigma}\rceil}(\sigma_{0,k}-\sigma_{0,k-1})-\frac{n}{\bar\sigma}\bar\sigma\right|>n^{\frac{3}{4}}\right)+P_0^V\left(\sum_{|j-\lceil\frac{n}{\bar\sigma}\rceil|\le\epsilon_2 n^{\frac{3}{4}}} S_j>\epsilon n\log n\right)$$

$$\le P_x^V\left(\left|\sum_{k=1}^{\lceil \frac{n}{\bar\sigma}\rceil}(\sigma_{0,k}-\sigma_{0,k-1})-\frac{n}{\bar\sigma}\bar\sigma\right|>n^{\frac{3}{4}}\right)+P_0^V\left(\sum_{j=1}^{\lceil 2\epsilon_2 n^{\frac{3}{4}}+2\rceil} S_j>\epsilon n\log n\right)$$

$$\overset{(5.11)\text{ and }(5.12)}{\le} 2\nu.$$

□

- **Proof of :**

$$\frac{X_n}{\frac{n}{\log n}}\overset{P}{\longrightarrow} c_1 \quad \text{as} \quad n\longrightarrow\infty, \quad \text{when} \quad \delta=2.$$

We shall show that $\frac{\sup_{i\le n}X_i}{\frac{n}{\log n}}$ and $\frac{\inf_{i\ge n}X_i}{\frac{n}{\log n}}$ have the same limiting distribution and then derive the limit law of $\frac{X_n}{\frac{n}{\log n}}$. It follows from [39, Lemma 9.1] by choosing $m=\lceil\frac{xn}{\log n}\rceil$ and $r=\lceil\sqrt{n}\rceil$ and using the inclusions (5.7) and (5.8)

$$\lim_{n\to\infty} P_0\left\{\frac{\sup_{i\le n}X_i}{\frac{n}{\log n}}<\frac{\lceil\frac{xn}{\log n}\rceil}{\frac{n}{\log n}}\right\}$$

$$= \lim_{n\to\infty} P_0\left\{\sup_{i\le n}X_i<\lceil\frac{xn}{\log n}\rceil\right\}$$

$$\overset{(5.7)}{=} \lim_{n\to\infty} P_0\left\{\frac{T_{\lceil\frac{xn}{\log n}\rceil}}{\lceil\frac{xn}{\log n}\rceil\log(\lceil\frac{xn}{\log n}\rceil)}>\frac{n}{\lceil\frac{xn}{\log n}\rceil\log(\lceil\frac{xn}{\log n}\rceil)}\right\}$$

$$\overset{(5.8)}{\le} \lim_{n\to\infty} P_0\left\{\frac{\inf_{i\ge n}X_i}{\frac{n}{\log n}}<\frac{\lceil\frac{xn}{\log n}\rceil}{\frac{n}{\log n}}\right\}$$

$$\overset{(5.8)\text{and}(5.9)}{\le} \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\le n}X_i}{\frac{n}{\log n}}<\frac{\lceil\frac{xn}{\log n}\rceil+\lceil\sqrt{n}\rceil}{\frac{n}{\log n}}\right\}.$$

Since,

$$\lim_{n\to\infty} \frac{n}{\lceil \frac{xn}{\log n}\rceil \log(\lceil \frac{xn}{\log n}\rceil)} = \lim_{n\to\infty} \frac{n}{\frac{xn}{\log n}\log(\frac{xn}{\log n})} = \frac{1}{x}$$

we conclude that

$$\frac{\sup_{i\le n} X_n}{\frac{n}{\log n}} \xrightarrow{P} c_1 = \frac{E_0^V(\sigma_0^V)}{2C_5} \quad \text{as} \quad n \longrightarrow \infty.$$

Since $\inf_{i\ge n} X_i \le X_i \le \sup_{i\le n} X_i$, it follows that $\frac{X_n}{\frac{n}{\log n}} \xrightarrow{P} c_1$. □

- **Proof of :**

$$\frac{T_n - c^{-1}nD(n)}{n} \Rightarrow Z_{1,b}, \quad \text{and} \quad \frac{X_n - cn\Gamma(n)}{c^2 n \log^{-2} n} \Rightarrow Z_{1,b} \quad \text{when} \quad \delta = 2.$$

For $\delta = 2$ the centering and the normalizing constants for the limiting distribution are not quiet obvious, some work must be done. The idea of the proof is similar to the one given in [37, Page 166-168] for one dimensional random walk in random environment and recently it was proven and fully written with all the details [43] and the proof is omitted. □

- **Proof of :**

$$\frac{T_n - v^{-1}n}{n^{\frac{2}{\delta}}} \Rightarrow Z_{\frac{\delta}{2},b} \quad \text{when} \quad 2 < \delta < 4.$$

The proof is exactly the same as the one given in [39]. □

- **Proof of:**

$$\frac{X_n - nv}{n^{\frac{2}{\delta}}} \Rightarrow -v^{1+\frac{2}{\delta}} Z_{\frac{\delta}{2},b} \quad \text{when} \quad 2 < \delta < 4.$$

We shall show that $\left\{\frac{\sup_{i\leq n} X_n - nv}{n^{\frac{2}{\delta}}}\right\}$ and $\left\{\frac{\inf_{i\geq n} X_n - nv}{n^{\frac{2}{\delta}}}\right\}$ have the same limiting distribution, and then we derive the limit law of $\frac{X_n}{n^{\frac{2}{\delta}}}$. We apply [39, Lemma 9.1] by choosing $m = \lceil xn^{\frac{2}{\delta}} + nv\rceil$ and $r = \lceil\sqrt{n}\rceil$ in (5.7) and (5.8)

$$\begin{aligned}
& \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{n^{\frac{2}{\delta}}} < \frac{\lceil xn^{\frac{2}{\delta}} + nv\rceil - nv}{n^{\frac{2}{\delta}}}\right\} \\
= \quad & \lim_{n\to\infty} P_0\left\{\sup_{i\leq n} X_i < \lceil xn^{\frac{2}{\delta}} + nv\rceil\right\} \\
\overset{(5.7)}{=} \quad & \lim_{n\to\infty} P_0\left\{\frac{T_{\lceil xn^{\frac{2}{\delta}}+nv\rceil} - \lceil(xn^{\frac{2}{\delta}} + nv)\rceil v^{-1}}{(\lceil xn^{\frac{2}{\delta}} + nv\rceil)^{\frac{2}{\delta}}} > \frac{n - \lceil(xn^{\frac{2}{\delta}} + nv)\rceil v^{-1}}{(\lceil xn^{\frac{2}{\delta}} + nv\rceil)^{\frac{2}{\delta}}}\right\} \\
= \quad & P_0\left\{Z_{\frac{\delta}{2},b} > -xv^{-1-\frac{2}{\delta}}\right\} \\
\overset{(5.8)}{\leq} \quad & \lim_{n\to\infty} P_0\left\{\frac{\inf_{i\geq n} X_i - nv}{n^{\frac{2}{\delta}}} < \frac{\lceil xn^{\frac{2}{\delta}} + nv\rceil - nv}{n^{\frac{2}{\delta}}}\right\} \\
\overset{(5.8)\text{and}(5.9)}{\leq} \quad & \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{n^{\frac{2}{\delta}}} < \frac{\lceil xn^{\frac{2}{\delta}} + nv\rceil + \lceil\sqrt{n}\rceil - nv}{n^{\frac{2}{\delta}}}\right\} \\
= \quad & P_0\left\{-v^{1+\frac{2}{\delta}} Z_{\frac{\delta}{2},b} < x\right\}.
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} P_0\left\{\frac{X_n - vn}{n^{\frac{2}{\delta}}} < x\right\} = P_0\left\{-v^{1+\frac{2}{\delta}} Z_{\frac{\delta}{2},b} < x\right\}.$$

□

- **Proof of :**

$$\frac{T_n - v^{-1}n}{\sqrt{n\log n}} \Rightarrow Z_{2,b} \quad \text{when} \quad \delta = 4.$$

By Theorem 3.2 the distribution of S^V in the domain of attraction of the normal distribution. Since the variance of S^v is infinite in this case, the right normalizing constant is of the order $a_n \sim \sqrt{C_6 n \log n}$ see ([29, Remark 3.5 page 438]). By using Lemma 3.3 and (5.1), it's enough to show that as $n \longrightarrow \infty$

$$\frac{T_n - v^{-1}n}{\sqrt{n\log n}} = \frac{2\Big(\sum_{k=0}^{n} D_k^{(n)} - (v^{-1}-1)\frac{n}{2}\Big)}{\sqrt{n\log n}} \overset{D}{=} \frac{2\Big(\sum_{j=0}^{n} V_j - (v^{-1}-1)\frac{n}{2}\Big)}{\sqrt{n\log n}} \tag{5.16}$$

The last term in (5.16) is equal to

$$= \frac{\sum_{i=1}^{N_n}\Big(S_i - E_x^V(S_i)\Big)}{\sqrt{n\log n}} + E_x^V(S^V)\frac{N_n - \lambda n}{\sqrt{n\log n}} + \frac{\sum_{j=\sigma_{0,N_n}}^{n} V_j}{\sqrt{n\log n}} \tag{5.17}$$

The first term in (5.17) converges in distribution to a stable law $Z_{2,b}$. It follows from (4.5) and the Portmanteau Theorem [7, Theorem 2.1. pp. 16] that for all $\epsilon > 0$ there exists $c_4 > 0$ such that

$$P_0^V\Big(|N_n - \lambda n| > c_4\sqrt{n\log n}\Big) \;\le\; P_0^V\Big(|N_n - \lambda n| > c_4\sqrt{n}\Big) < \epsilon.$$

It remains to show that the last term in (5.17) goes to zero in probability, which is bounded by

$$\frac{\sum_{j=\sigma_{0,N_n}}^{n} V_j}{\sqrt{n\log n}} \le \frac{S_{N_n+1}}{\sqrt{n\log n}}.$$

Then for every $\nu > 0$ and n sufficiently large

$$P_0^V\left(S_{N_n+1} > \nu\sqrt{n\log n}\right) \le P_0^V\left(S_{N_n+1} > \nu n^{\frac{3}{4}}\right)$$
$$\le \; P_0^V\left(\max_{m\in[\lambda n - c_5\sqrt{n},\lambda n + c_5\sqrt{n}]} S_{m+1} > \nu n^{\frac{3}{4}}, \Big|N_n - \lambda n\Big| < c_5\sqrt{n}\right)$$

$$
\begin{aligned}
& +P_0^V\left(\left|N_n-\lambda n\right|>c_5\sqrt{n}\right)\\
\le\ & P_0^V\left(\max_{m\in[\lambda n-c_5\sqrt{n},\lambda n+c_5\sqrt{n}]} S_{m+1}>\nu n^{\frac{3}{4}}\right)+P_0^V\left(\left|N_n-\lambda n\right|>c_5\sqrt{n}\right)\\
\le\ & 1-\left[1-P_0^V\left(S_1>\nu n^{\frac{3}{4}}\right)\right]^{2c_5\sqrt{n}+2}+\epsilon\\
\le\ & 1-\left(1-\frac{2C_6}{\nu^2 n^{\frac{3}{2}}}\right)^{2c_5\sqrt{n}+2}+\epsilon\\
<\ & 2\epsilon.
\end{aligned}
$$

□

- **Proof of :**

$$\frac{X_n-vn}{\sqrt{n\log n}}\Rightarrow -v^{\frac{3}{2}}Z_{2,b}\quad\text{when}\quad\delta=4.$$

We will show that $\frac{\sup_{i\le n}X_n-vn}{\sqrt{n\log n}}$ and $\frac{\inf_{i\ge n}X_n-vn}{\sqrt{n\log n}}$ have the same limiting distribution by using [39, Lemma 9.1] with $m=\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil$ and $r=\lceil\sqrt{n}\rceil$ that

$$
\begin{aligned}
& \lim_{n\to\infty}P_0\left\{\frac{\sup_{i\le n}X_i-nv}{v^{\frac{3}{2}}\sqrt{n\log n}}<\frac{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil-nv}{v^{\frac{3}{2}}\sqrt{n\log n}}\right\}\\
=\ & \lim_{n\to\infty}P_0\left\{\sup_{i\le n}X_i<\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil\right\}\\
\overset{(5.7)}{=}\ & \lim_{n\to\infty}P_0\left\{\frac{T_{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil}-\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil v^{-1}}{\sqrt{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil\log\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil}}>\right.\\
& \left.\frac{n-\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil v^{-1}}{\sqrt{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil\log\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil}}\right\}=P_0\left\{-Z_{2,b}<x\right\}.
\end{aligned}
$$

For all sufficiently large n, the term below is of the order

$$\frac{-xv^{\frac{1}{2}}\sqrt{n\log n}}{\sqrt{(xv^{\frac{3}{2}}\sqrt{n\log n}+nv)\log{(xv^{\frac{3}{2}}\sqrt{n\log n}+nv)}}} \sim \frac{-xv^{\frac{1}{2}}\sqrt{n\log n}}{\sqrt{nv\log n}} \longrightarrow_{n\to\infty} -x.$$

By applying (5.8) and (5.9) we have

$$\begin{aligned}
& \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil - nv}{v^{\frac{3}{2}}\sqrt{n\log n}}\right\} \\
\stackrel{(5.8)}{\leq}\quad & \lim_{n\to\infty} P_0\left\{\frac{\inf_{\geq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil - nv}{v^{\frac{3}{2}}\sqrt{n\log n}}\right\} \\
\stackrel{(5.8)\text{and}(5.9)}{\leq}\quad & \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n\log n}+nv\rceil - nv + \lceil\sqrt{n}\rceil}{v^{\frac{3}{2}}\sqrt{n\log n}}\right\} \\
=\quad & P_0\left\{-Z_{2,b} < x\right\}.
\end{aligned}$$

Since $\inf\limits_{i\geq n} X_i \leq X_i \leq \sup\limits_{i\leq n} X_i$, it follows that

$$\begin{aligned}
& \lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < x\right\} = \lim_{n\to\infty} P_0\left\{\frac{\inf_{i\geq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < x\right\} \\
= \ & \lim_{n\to\infty} P_0\left\{\frac{X_i - nv}{v^{\frac{3}{2}}\sqrt{n\log n}} < x\right) = P_0\left\{-Z_{2,b} < x\right\}.
\end{aligned}$$

□

- **Proof of :**

$$\frac{T_n - v^{-1}n}{\sqrt{n}} \Rightarrow Z_{2,b} \quad \text{when} \quad \delta > 4.$$

Since both the first and second moments of S^V are finite by Theorem 3.2, it follows that the distribution of S^V is in the domain of attraction of a normal distribution.

Choose

$$a_n = \sqrt{var(S^V)n} \quad \text{and} \quad b_n = nE(S^V)$$

By using Lemma 3.3 and (5.1), it's enough to show that as $n \longrightarrow \infty$

$$\frac{T_n - v^{-1}n}{\sqrt{n}} = \frac{2\Big(\sum_{k=0}^{n} D_k^{(n)} - (v^{-1}-1)\frac{n}{2}\Big)}{\sqrt{n}} \overset{D}{=} \frac{2\Big(\sum_{j=0}^{n} V_j - (v^{-1}-1)\frac{n}{2}\Big)}{\sqrt{n}} \tag{5.18}$$

The last term on the **RHS** of (5.18) can be rewritten

$$\frac{\sum_{i=1}^{N_n}\Big(S_i - E_x^V(S_i)\Big)}{\sqrt{n}} + E_x^V(S^V)\frac{N_n - \lambda n}{\sqrt{n}} + \frac{\sum_{j=\sigma_{0,N_n}}^{n} V_j}{\sqrt{n}}. \tag{5.19}$$

The first term converges in distribution to a table law $Z_{2,b}$ and the last term is bounded by $\frac{S_{N_n+1}}{\sqrt{n}}$ and goes to zero in probability. Indeed for every $\nu > 0$ and n sufficiently large

$$\begin{aligned}
& P_0^V\Bigg(S_{N_n+1} > \nu\sqrt{n}\Bigg) \\
\leq\ & P_0^V\Bigg(\max_{m\in[\lambda n - c_5\sqrt{n}, \lambda n + c_5\sqrt{n}]} S_{m+1} > \nu\sqrt{n}, \Big|N_n - \lambda n\Big| < c_5\sqrt{n}\Bigg) \\
& + P_0^V\Bigg(\Big|N_n - \lambda n\Big| > c_5\sqrt{n}\Bigg) \\
\leq\ & P_0^V\Bigg(\max_{m\in[\lambda n - c_5\sqrt{n}, \lambda n + c_5\sqrt{n}]} S_{m+1} > \nu\sqrt{n}\Bigg) + P_0^V\Bigg(\Big|N_n - \lambda n\Big| > c_5\sqrt{n}\Bigg) \\
\leq\ & 1 - \Bigg[1 - P_0^V\Big(S_1 > \nu\sqrt{n}\Big)\Bigg]^{2c_5\sqrt{n}+2} + \epsilon \\
\leq\ & 1 - \Bigg(1 - \frac{2C_6}{\nu^{\frac{\delta}{2}} n^{\frac{\delta}{4}}}\Bigg)^{2c_4\sqrt{n}+2} + \epsilon \\
<\ & 2\epsilon.
\end{aligned}$$

□

- **Proof of :**

$$\frac{X_n - vn}{\sqrt{n}} \Rightarrow -v^{\frac{3}{2}} Z_{2,b} \quad \text{when} \quad \delta > 4.$$

We will show that $\frac{\sup_{i\leq n} X_n - vn}{\sqrt{n}}$ and $\frac{\inf_{i\geq n} X_n - vn}{\sqrt{n}}$ have the same limiting distribution, and then we deduce the limit law of $\frac{X_n - nv}{\sqrt{n}}$. By applying [39, Lemma 9.1] with $m = \lceil xv^{\frac{3}{2}}\sqrt{n} + nv \rceil$ and $r = \lceil \log n \rceil$

$$\begin{aligned}
& \lim_{n\to\infty} P_0\left\{ \frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil - nv}{v^{\frac{3}{2}}\sqrt{n}} \right\} \\
= \quad & \lim_{n\to\infty} P_0\left\{ \sup_{i\leq n} X_i < \lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil \right\} \\
\overset{(5.7)}{=} \quad & \lim_{n\to\infty} P_0\left\{ \frac{T_{\lceil xv^{\frac{3}{2}}\sqrt{n}+nv\rceil} - \lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil v^{-1}}{\sqrt{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil}} > \frac{n - \lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil v^{-1}}{\sqrt{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil}} \right\} \\
= \quad & P_0\left\{ -\left(\frac{1}{E_x^V \sigma_0^V}\right)^{\frac{1}{2}} Z_{2,b} < x \right\} \\
\overset{(5.8)}{\leq} \quad & \lim_{n\to\infty} P_0\left\{ \frac{\inf_{i\geq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil}{v^{\frac{3}{2}}\sqrt{n}} \right\} \\
\overset{(5.8)\text{and}(5.9)}{\leq} \quad & \lim_{n\to\infty} P_0\left\{ \sup_{i\leq n} X_i < \lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil + \lceil \log n\rceil \right\} \\
= \quad & \lim_{n\to\infty} P_0\left\{ \frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n}} < \frac{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil - nv + \lceil \log n\rceil}{v^{\frac{3}{2}}\sqrt{n}} \right\}.
\end{aligned}$$

When n is sufficient large, the term below is of the order

$$\frac{\lceil xv^{\frac{3}{2}}\sqrt{n} + nv\rceil - nv + \lceil \log n\rceil}{v^{\frac{3}{2}}\sqrt{n}} \longrightarrow_{n\to\infty} -x,$$

we conclude that

$$\begin{aligned}\lim_{n\to\infty} P_0\left\{\frac{\sup_{i\leq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n}} < x\right\} &= \lim_{n\to\infty} P_0\left\{\frac{\inf_{i\geq n} X_i - nv}{v^{\frac{3}{2}}\sqrt{n}} < x\right\} \\ &= P_0\left\{-\left(\frac{1}{E_x^V \sigma_0^V}\right)^{\frac{1}{2}} Z_{2,b} < x\right\}\end{aligned}$$

□

Chapter 6

Proof of Theorems 2.4 and 2.5

The scaling limits of ERW on $\mathbb{Z}$ in i.i.d. cookie environment with bounded number of excitations per site in the recurrent regime was proved in [18]. It was shown, when the total drift $|\delta| < 1$, that the right scaling is of the order $\sqrt{n}$ and the functional limiting distribution of the scaled ERW is a $(\delta, -\delta)$- perturbed Brownian motion. In the boundary case $|\delta| = 1$, the scaling is of the order $\sqrt{n} \log n$ and the functional limiting distribution of the scaled ERW is a multiplicative constant of the running maximum of standard Brownian motion. In contrast to our model where the number of excitations per site is random almost surely finite and satisfying a tail decay estimate (**TDE**) , we obtain the same results but the novelty of our work lies in how to handle the randomness of these excitations in the technical part of the proofs specially in the non-boundary case, whereas in the boundary case the proofs go word to word as in [18].

Without loss of generality, we shall assume that the total drift $\delta \geq 0$ and the ERW starts at the origin $X_0 = 0$ and let $T_x = \inf\{j \geq 0 : X_j = x\}$ be the first time the walk hits the level $x \in \mathbb{Z}$. Set

$$S_n = \max_{k \leq n} X_k, \quad I_n = \min_{k \leq n} X_k, \quad R_n = S_n - I_n + 1, \quad n \geq 0.$$

Define the local time of the ERW by $L_m(n) := \sum_{j=0}^{n} \mathbb{1}_{\{X_j=m\}}$.

6.1 Non-boundary case: Two main lemmas.

Let $\delta \in [0,1)$. The goal is to show that by time n the ERW consumes all the drift between I_n and S_n. More precisely the claim states the existence of a small probability of the order a power decay that the weighted average number of points visited by the ERW that are less than $M(z)_{z\in\mathbb{Z}}$ will be greater than some power of n. That is

Lemma 6.1. *Assume that $\delta \in [0,1)$. Given $\gamma_1 > \delta$ there exist a positive constants K_1 and a β satisfying $\frac{1}{\alpha} < \beta < \frac{\gamma_1-\delta}{2}$ such that for all $1 \le \ell \le n$*

$$P_0\left(\sum_{m=n-\ell}^{n-1} M(m)\mathbb{1}_{\{L_m(T_n)<M(m)\}} > \ell^{\gamma_1}\right) \le \frac{K_1}{\ell^{\alpha\beta-1}} \quad \textit{and,} \tag{6.1}$$

$$P_0\left(\sum_{m=-(n-1)}^{-(n-\ell)} M(m)\mathbb{1}_{\{L_m(T_{-n})<M(m)\}} > \ell^{\gamma_1}\right) \le \frac{K_1}{\ell^{\alpha\beta-1}} \tag{6.2}$$

Proof. This is an extended version of a statement in [18, Lemma 4] for ERW with boundedly many cookies per site. The idea of the proof is exactly the same but a control of the randomness created by the number of cookies $(M(z))_{z\in\mathbb{Z}}$ satisfying **TDE** assumtion is needed.

We shall show (6.1) by using the backward branching process connection to the ERW. Since the events that we are trying to estimate their probabilities depends only on the environment and the behavior of the walk on $\{n-\ell, n-\ell+1, ...\}$, we may assume without loss of generality that the process starts at $n-\ell$ and, therefore, by translation invariance we shall consider only the case when $\ell = n$. Let $L_k^V(n) := \sum_{j=0}^{n} \mathbb{1}_{\{V_j=k\}}$ we have

$$\begin{aligned}
& P_0^V\left(\sum_{m=0}^{n-1} M(m)\mathbb{1}_{\{L_m(T_n)<M(m)\}} > n^{\gamma_1}\right) \\
= \ & P_0\left(\sum_{m=0}^{n-1} M(m)\mathbb{1}_{\{L_m(T_n)<M(m)\}} > n^{\gamma_1}, \max_{m\le n} M(m) \le N(n)\right) \\
+ \ & P_0\left(\sum_{m=0}^{n-1} M(m)\mathbb{1}_{\{L_m(T_n)<M(m)\}} > n^{\gamma_1}, \max_{m\le n} M(m) > N(n)\right) \\
\le \ & P_0\left(\sum_{m=0}^{n} M(m)\mathbb{1}_{\{D_m^{(n)}<N(n)\}} > n^{\gamma_1} \,\middle|\, \max_{m\le n} M(m) \le N(n)\right) P_0\left(\max_{m\le n} M(m) \le N(n)\right) \\
+ \ & P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & P_0\left(\sum_{m=0}^{n} \mathbb{1}_{\{V_m<N(n)\}} > n^{\gamma_1}\right) P_0\left(\max_{m\le n} M(n) \le N(n)\right) + P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
= \ & P_0\left(\sum_{m=0}^{n} \sum_{k=0}^{N(n)-1} \mathbb{1}_{\{V_m=k\}} > \frac{n^{\gamma_1}}{N(n)}\right) P_0\left(\max_{m\le n} M(n) \le N(n)\right) + P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & P_0\left(\sum_{k=0}^{N(n)-1} \max_{0\le k<N(n)} \left(\sum_{m=0}^{n} \mathbb{1}_{\{V_m=k\}}\right) > \frac{n^{\gamma_1}}{N(n)}\right) P_0\left(\max_{m\le n} M(n) \le N(n)\right) \\
+ \ & P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & P_0\left(N(n) \max_{0\le k<N(n)} \left(\sum_{m=0}^{n} \mathbb{1}_{\{V_m=k\}}\right) > \frac{n^{\gamma_1}}{N(n)}\right) P_0\left(\max_{m\le n} M(n) \le N(n)\right) \\
+ \ & P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & \sum_{k=0}^{N(n)-1} P_0\left(\sum_{m=0}^{n} \mathbb{1}_{\{V_m=k\}} > \frac{n^{\gamma_1}}{N(n)^2}\right) P_0\left(\max_{m\le n} M(n) \le N(n)\right) + P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & N(n) \max_{0\le k<N(n)} P_0^V\left(\sum_{m=0}^{n} \mathbb{1}_{\{V_m=k\}} > \frac{n^{\gamma_1}}{N(n)^2}\right) \Big(1 - H(N(n))\Big)^n + P_0\left(\max_{m\le n} M(m) > N(n)\right) \\
\le \ & N(n) \max_{0\le k<N(n)} P_0^V\left(L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2}\right) \Big(1 - H(N(n))\Big)^n + 1 - \Big(1 - H(N(n))\Big)^n
\end{aligned}$$

• **Case 1** If $\delta \in (0,1)$. Let $k = 0$ and by Theorem 3.2 we get

$$P_0^V\left(L_0^V(n) > \frac{n^{\gamma_1}}{N(n)^2}\right) \leq \prod_{j=1}^{\left\lfloor \frac{n^{\gamma_1}}{N(n)^2} \right\rfloor} P_0^V\left(\sigma_j^V - \sigma_{j-1}^V \leq n\right) \leq \left(1 - \frac{C_5}{2n^\delta}\right)^{\left\lfloor \frac{n^{\gamma_1}}{N(n)2} \right\rfloor}.$$

By putting terms together we get the estimate

$$P_0^V\left(\sum_{m=0}^{n-1} M(m)\mathbb{1}_{\{L_m(T_n)<M(m)\}} > n^{\gamma_1}\right) \leq N(n)\left(1 - \frac{C_5}{2n^\delta}\right)^{\left\lfloor \frac{n^{\gamma_1}}{N(n)2} \right\rfloor} + 1 - \Big(1 - H(N(n))\Big)^n.$$

Note that cutting off $N(n)$ at the level n^β for some $\beta > 0$

$$\begin{aligned} 1 - \Big(1 - H(N(n)))\Big)^n \leq nH(N(n)) \quad &\leq \quad \frac{C}{n^{\alpha\beta-1}} \quad \text{and,} \\ \lim_{n\to\infty} n^{\alpha\beta-1}n^\beta\left(1 - \frac{C_5}{2n^\delta}\right)^{\left\lfloor \frac{n^{\gamma_1}}{n^{2\beta}} \right\rfloor} \quad &= \quad 0 \quad \text{for} \quad \beta < \frac{\gamma_1 - \delta}{2}. \end{aligned}$$

Therefore $\forall \nu > 0 \quad \exists n_0 \quad$ such that $\quad \forall n \geq n_0 : n^\beta\left(1 - \frac{C_5}{2n^\delta}\right)^{\left\lfloor \frac{n^{\gamma_1}}{N(n)^2} \right\rfloor} < \frac{\nu}{n^{\alpha\beta-1}}$, and the claim of the lemma is proved for the case k=0.

• Suppose now that $k \in \{1, 2, ..., \lfloor N(n) \rfloor\}$. Then for any $\epsilon > 0$

$$\begin{aligned} & P_0^V\left(L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2}\right) \\ = \quad & P_0^V\left(L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2}, L_0^V(n) > \frac{\epsilon n^{\gamma_1}}{2N(n)^2}\right) + P_0^V\left(L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2}, L_0^V(n) \leq \frac{\epsilon n^{\gamma_1}}{2N(n)^2}\right) \\ \leq \quad & P_0^V\left(L_0^V(n) > \frac{\epsilon n^{\gamma_1}}{2N(n)^2}\right) + P_0^V\left(L_0^V(n) \leq \frac{\epsilon n^{\gamma_1}}{2N(n)^2}\middle| L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2}\right) \end{aligned}$$

We only need to estimate the second term on **RHS** in the above inequality. By the uniform ellipticity (**UFL**) there exists $\kappa > 0$ such that:

$$P_0^V(V_{j+1} = 0 | V_j = k) \geq \kappa^{k+1} := \epsilon(\kappa) > 0 \quad \text{for all} \quad k \in \Big\{1, 2, ..., \lfloor N(n) \rfloor \Big\}.$$

Choose now $\epsilon = \epsilon(\kappa)$ and on the event $\left\{ L_0^V(n) \leq \frac{\epsilon(\kappa) n^{\gamma_1}}{2N(n)^2} \middle| L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2} \right\}$ there are at least $\lfloor \frac{n^{\gamma_1}}{N(n)^2} \rfloor$ independent Bernoulli trials with a probability of success in each trial of at least $\epsilon(\kappa)$ and there are at most $\left\lfloor \dfrac{\epsilon(\kappa) n^{\gamma_1}}{2N(n)^2} \right\rfloor$ such successes. Thus,

$$\begin{aligned} & P_0^V\left(L_0^V(n) \leq \frac{\epsilon(\kappa) n^{\gamma_1}}{2N(n)^2} \middle| L_k^V(n) > \frac{n^{\gamma_1}}{N(n)^2} \right) \leq \Big(\epsilon(\kappa)\Big)^{\left\lfloor \frac{\epsilon(\kappa) n^{\gamma_1}}{2N(n)} \right\rfloor} \\ \leq \quad & e^{\frac{n^{\gamma_1}}{N(n)^2} \epsilon(\kappa) \ln(\epsilon(\kappa))}. \\ \leq \quad & e^{n^{\gamma_1 - 2\beta} \epsilon(\kappa) \ln(\epsilon(\kappa))} \quad \text{where} \quad \epsilon(\kappa) \in (0,1). \end{aligned}$$

This complete the proof for the case $\delta > 0$.

• **Case 2** If $\delta = 0$. We shall modify the environment by increasing the drift to the right in the first $(M(z))_{z \in \mathbb{Z}}$ excitations at each site in the following manner. Suppose that $\omega = \Big(\omega(z,j)\Big)_{z \in \mathbb{Z}, j \in \mathbb{N}}$ is the deterministic cookie environment with the cookie stacks $\omega_z = \Big(\omega(z,1), \omega(z,2), ..., \omega(z,M), \frac{1}{2}, \frac{1}{2}, ...\Big)$ and pick U uniformly at random in $[0,1]$ by setting

$$B_z(i) = \mathbb{1}_{\{U \leq \omega(z,i)\}} \quad \text{and} \quad B_z^\epsilon(z,i) = \mathbb{1}_{\{U \leq \omega(z,i) + \epsilon\}}, i \in \mathbb{N}, z \in \mathbb{Z}, \ \epsilon > 0.$$

and the two coupled branching processes V_i and V_i^ϵ satisfy that $V_i^\epsilon \leq V_i$ for all $i \in \{0, 1, ..., n\}$. Accordingly to this coupling we have

$$\sum_{j=0}^{n} M(j) \mathbb{1}_{\{V_j < M(j)\}} \leq \sum_{j=0}^{n} M(j) \mathbb{1}_{\{V_j^\epsilon < M(j)\}}$$

and the total drift in this case $\delta^\epsilon = \delta + 2\epsilon EM(0) > 0$ and we repeat the argument as in the case of $\delta > 0$. Next we use the symmetry of the ERW by replacing X by $-X$ to get (6.2) by reducing it to prove (6.1) when $\delta \leq 0$ and $\gamma_1 > 0$. Thus, the result for $\delta \leq 0$ can be deduced from the result for $\delta \in (0, \gamma_1)$ by using the coupling described above. $\square$

The next lemma emphasizes that the right scaling in Theorem 2.4 is of the order $\sqrt{n}$.

Lemma 6.2. *[43, Lemma 3.1] If $\delta \in [0, 1)$, then there exist positive constants c_5, c_6 such that*

$$P_0\left(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\right) \leq c_6 e^{-c_5\sqrt{L}} \quad \textit{and} \quad P_0\left(T_{-\ell-n} - T_{-\ell} \leq \frac{n^2}{L}\right) \leq c_6 e^{-c_5\sqrt{L}},$$

for for all integers $\ell \geq 0, n \geq 1$, and $L \in (0, \infty)$.

Proof. We shall prove the first inequality for $\delta \in (0, 1)$. The case $\delta = 0$ and the second inequality are handled in exactly the same way as in the proof of Lemma 6.1.

Since $T_{\ell+n} - T_\ell \geq \sum_{k=\ell}^{n+\ell} D_k^{(n+\ell)} \overset{D}{=} \sum_{j=0}^{n} V_j$ we have

$$p_{\ell,n,L} \quad := \quad P_0\left(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\right) \leq P_0^V\left(\sum_{j=0}^{n} V_j \leq \frac{n^2}{2L}\right).$$

Using the same argument as in [18, Lemma 3.2] , we can show that there is $c_5 > 0$, $L_0 < \infty$ and $n_0 \in \mathbb{N}$ such that

$$q_{n,L} \quad := \quad P_0^V\left(\sum_{j=0}^{n} V_j \leq \frac{n^2}{2L}\right) \leq e^{-c_5\sqrt{L}} \quad \forall L \geq L_0 \quad \text{and} \quad \forall n \geq \sqrt{L} n_0. \tag{6.3}$$

Indeed, when $n < L$ the lemma holds trivially since $P_0\Big(T_{\ell+n} - T_\ell \geq n\Big) = 1$ and we have

$$0 \quad = \quad P_0\Big(T_{\ell+n} - T_\ell \leq n\Big) \leq P_0\Big(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\Big) = 0 \leq e^{-c_5\sqrt{L}}.$$

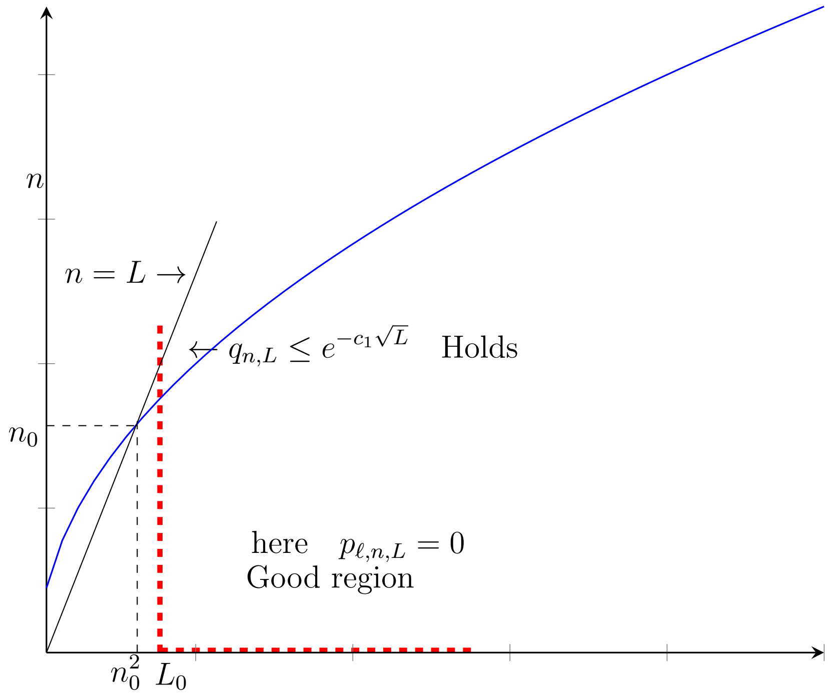

Figure 6.1: The region $L \ge L_0$

Without loss of generality, by increasing L_0 if necessary, we can assume that $L_0 \ge n_0^2$, so

$$P_0^V\left(\sum_{j=0}^{n} V_j \le \frac{n^2}{2L}\right) \;\; \le \;\; e^{-c_5\sqrt{L}} \quad \forall L \ge L_0 \ge n_0^2 \qquad \forall n \ge \sqrt{L}n_0$$

Thus, in the region $L \ge L_0$ and $n \in \mathbb{N}$

$$P_0\Big(T_{\ell+n} - T_\ell \le \frac{n^2}{L}\Big) \;\; \le \;\; e^{-c_5\sqrt{L}} \quad \forall \ell \in \mathbb{N}_0.$$

Now we consider the leftover region: $n \in \mathbb{N}$ and $0 < L < L_0$. The estimate is trivial because $\forall \ell \in \mathbb{N}_0, n \in \mathbb{N}, L \in (0, L_0)$ we have

$$P_0\Big(T_{\ell+n} - T_\ell \le \frac{n^2}{L}\Big) \le 1 = e^{c_5\sqrt{L_0}}e^{-c_5\sqrt{L_0}} \le e^{-c_5\sqrt{L_0}}e^{-c_5\sqrt{L}} = c_6 e^{-c_5\sqrt{L}}$$

where $c_6 := e^{c_5\sqrt{L_0}} > 1$. So for all $n \in \mathbb{N}, \ell \in \mathbb{N}_0, L \geq L_0$ we have

$$\begin{aligned} P_0\Big(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\Big) &\leq e^{-c_5\sqrt{L}} \quad \forall n \in \mathbb{N}_0, \forall \ell \in \mathbb{N}_0, \forall L \geq L_0 \quad \text{and,} \\ P_0\Big(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\Big) &\leq c_6 e^{-c_5\sqrt{L}} \quad \forall n \in \mathbb{N}_0, \forall \ell \in \mathbb{N}_0, \forall L \in (0, L_0). \end{aligned}$$

By combining both estimates, we get that

$$P_0\Big(T_{\ell+n} - T_\ell \leq \frac{n^2}{L}\Big) \leq c_6 e^{-c_5\sqrt{L}} \quad \forall n \in \mathbb{N}_0, \forall \ell \in \mathbb{N}_0, \forall L \in (0, \infty).$$

- If $\delta \in (0,1)$, then by the Markov property and the monotonicity of the process V

$$\begin{aligned} P_0^V\left(\sum_{j=0}^{m+k} V_j \leq n\right) &\leq P_0^V\left(\sum_{j=m+1}^{m+k} V_j \leq n \,\middle|\, \sum_{j=0}^{m} V_j \leq n\right) P_0^V\left(\sum_{j=0}^{m} V_j \leq n\right) \\ &\leq P_0^V\left(\sum_{j=0}^{k} V_j \leq n\right) P_0^V\left(\sum_{j=0}^{m} V_j \leq n\right). \end{aligned}$$

If the following claim

Claim 6.1.1. *There exists $K, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$P_0^V\left(\sum_{j=0}^{Kn} V_j \leq n^2\right) \leq \frac{1}{2}, \tag{6.4}$$

holds, then for all $L > 4K^2$ and $n \geq \sqrt{L} n_0$

$$\begin{aligned} P_0^V\left(\sum_{j=0}^{n} V_j \leq \frac{n^2}{L}\right) &= P_0^V\left(\sum_{j=0}^{\frac{\sqrt{L}}{2K}\cdot\frac{2Kn}{\sqrt{L}}} V_j \leq \frac{n^2}{L}\right) \leq P_0^V\left(\sum_{j=0}^{\left\lfloor\frac{\sqrt{L}}{2K}\right\rfloor\cdot\left\lfloor\frac{2Kn}{\sqrt{L}}\right\rfloor} V_j \leq \frac{n^2}{L}\right) \\ &\leq \left[P_0^V\left(\sum_{j=0}^{\left\lfloor\frac{2Kn}{\sqrt{L}}\right\rfloor} V_j \leq \frac{n^2}{L}\right)\right]^{\left\lfloor\frac{\sqrt{L}}{2K}\right\rfloor} \leq \left[P_0^V\left(\sum_{j=0}^{2K\left\lfloor\frac{n}{\sqrt{L}}\right\rfloor} V_j \leq 4\Big(\frac{n}{\sqrt{L}}\Big)^2\right)\right]^{\left\lfloor\frac{\sqrt{L}}{2K}\right\rfloor} \stackrel{(6.4)}{\leq} \left(\Big(\frac{1}{2}\Big)^{\frac{1}{4K}}\right)^{\sqrt{L}}. \end{aligned}$$

□

To prove the claim in (6.4), note that due to Theorem 3.2 the sequence $\frac{\sigma_m^V}{m^{\frac{1}{\delta}}}, m \in \mathbb{N}$ converges in distribution to a non degenerate distribution [21, lemma 3.7.2]. Thus we can choose K so large such that

$$P_0^V\left(\sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor} > Kn\right) \leq \frac{1}{4} \quad \text{for all large enough n.}$$

It follows that $\exists\, n_0, \forall n \geq n_0$

$$\begin{aligned}
& P_0^V\left(\sum_{j=0}^{Kn} V_j \leq n^2\right) \\
= \;& P_0^V\left(\sum_{j=0}^{Kn} V_j \leq n^2, \sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor} > Kn\right) + P_0^V\left(\sum_{j=0}^{Kn} V_j \leq n^2, \sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor} \leq Kn\right) \\
\leq \;& P_0^V\left(\sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor} > Kn\right) + P_0^V\left(\sum_{j=0}^{\sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor}} V_j \leq n^2, \sigma_{\lfloor(\sqrt{K}n)^\delta\rfloor} \leq Kn\right) \\
\leq \;& \frac{1}{4} + P_0^V\left(\sum_{j=0}^{\lfloor(\sqrt{K}n)^\delta\rfloor} S_j^V \leq n^2\right) \\
\leq \;& \frac{1}{4} + \prod_{j=1}^{\lfloor(\sqrt{K}n)^\delta\rfloor} P_0^V\left(S_j^V \leq n^2\right) \\
\overset{(3.2)}{\leq} \;& \frac{1}{4} + \left(1 - \frac{C_6}{2n^\delta}\right)^{\lfloor(\sqrt{K}n)^\delta\rfloor} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

□

Corollary 6.1. *[43, Corollary 3.2] If $\delta \in [0,1)$, then*

$$P_0\Big(\sup_{k\leq n} |X_k| > K\sqrt{n}\Big) \;\leq\; 2c_6 e^{-c_5 K} > 0, \quad \forall n \geq 1, K > 0.$$

where c_5 and c_6 are the same constants in Lemma 6.2

Proof.

$$P_0\Big(\sup_{k\le n}|X_k|>K\sqrt{n}\Big) \;\le\; P_0\Big(T_{\lfloor K\sqrt{n}\rfloor}\le\frac{\lfloor K\sqrt{n})\rfloor^2}{K^2}\Big)+P_0\Big(T_{-\lfloor K\sqrt{n}\rfloor}\le\frac{\lfloor K\sqrt{n})\rfloor^2}{K^2}\Big)$$
$$\stackrel{(6.2)}{\le}\; 2c_6e^{-c_5K}.$$

Corollary 6.2. *[43, Corollary 3.3] Let $S_n=\sup_{k\le n}X_k$ and $I_n=\inf_{k\le n}X_k$ be the running maximum and minimum, respectively, of the excited random walk. If $\delta\in[0,1)$, then for any $\epsilon>0$ and $t<\infty$,*

$$\lim_{\nu\to 0}\limsup_{n\to\infty}P\Bigg(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|S_k-S_\ell\Big|\ge 2\epsilon\sqrt{n}\Bigg) = 0 \quad \textit{and,}$$
$$\lim_{\nu\to 0}\limsup_{n\to\infty}P\Bigg(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|I_k-I_\ell\Big|\ge 2\epsilon\sqrt{n}\Bigg) = 0.$$

Proof. it suffices to show the limit for the running maximum because the proof is the same for the running minimum. If the running maximum increases by at least $2\epsilon\sqrt{n}$ over some time interval of length less than $n\nu$, then it follows that some time interval of the form $\Big[(m-1)\lfloor\epsilon\sqrt{n}\rfloor,m\lfloor\epsilon\sqrt{n}\rfloor\Big]$ is crossed in less that $n\nu$ steps. Therefore,

$$P\Bigg(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|S_k-S_\ell\Big|\ge\epsilon\sqrt{n}\Bigg)$$
$$\le\; P\Big(S_n\ge\frac{\epsilon\sqrt{n}}{\nu}\Big)+\sum_{m=1}^{\lfloor\frac{1}{\nu}\rfloor}P\Big(T_{m\lfloor\epsilon\sqrt{n}\rfloor}-T_{(m-1)\lfloor\epsilon\sqrt{n}\rfloor}\le n\nu\Big)$$
$$\le\; 2c_6e^{-c_5\frac{\epsilon}{\nu}}+\sum_{m=1}^{\lfloor\frac{1}{\nu}\rfloor}c_6\mathbf{e}^{\frac{-c_5m\epsilon}{\sqrt{\nu}}}.$$

By taking limits

$$\lim_{\nu\to 0}\limsup_{n\to\infty} P\left(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|S_k - S_\ell\Big| \ge 2\epsilon\sqrt{n}\right) \le \lim_{\nu\to 0}\Big(2c_6 e^{-c_5\frac{\epsilon}{\nu}} + \sum_{m=1}^{\lfloor\frac{1}{\nu}\rfloor} c_6 \mathbf{e}^{\frac{-c_5 m\epsilon}{\sqrt{\nu}}}\Big) = 0.$$

□

6.1.1 Convergence of the martingale part and tightness

Let $\Delta_n = X_{n+1} - X_n$ and set

$$B_n = \sum_{k=0}^{n-1}\Big(\Delta_k - E_{0,\omega}(\Delta_k|\mathcal{F}_k)\Big) \quad \text{and} \quad C_n = \sum_{k=0}^{n-1} E_{0,\omega}(\Delta_k|\mathcal{F}_k) \tag{6.5}$$

Then $X_n = B_n + C_n$. Define

$$X^{(n)}(t) := \frac{X_{\lfloor nt\rfloor}}{\sqrt{n}}, \quad B^{(n)}(t) := \frac{B_{\lfloor nt\rfloor}}{\sqrt{n}}, \quad C^{(n)}(t) := \frac{C_{\lfloor nt\rfloor}}{\sqrt{n}} \quad t \ge 0, n \in \mathbb{N}.$$

Note that B_n is a martingale with respect to the filtration $\mathcal{F}_k$ with bounded increment. Indeed,

$$\begin{aligned} & E_{0,\omega}\Big(B_{n+1}\Big|\mathcal{F}_n\Big) = E_{0,\omega}\Big(X_{n+1} - C_{n+1}\Big|\mathcal{F}_n\Big) \\ = \;& E_{0,\omega}\Big(X_{n+1} - C_n - E_{0,\omega}\Big(X_{n+1} - X_n\Big|\mathcal{F}_n\Big)\Big|\mathcal{F}_n\Big) \\ = \;& E_{0,\omega}\Big(X_n - C_n + X_{n+1} - E_{0,\omega}\Big(X_{n+1}\Big|\mathcal{F}_n\Big)\Big|\mathcal{F}_n\Big) \\ = \;& X_n - C_n + E_{0,\omega}\Big(X_{n+1}\Big|\mathcal{F}_n\Big) - E_{0,\omega}\Big(X_{n+1}\Big|\mathcal{F}_n\Big) = B_n. \end{aligned}$$

Bounded increment follows easily by using Jensen"s inequality

$$\Big|B_{n+1} - B_n\Big| = \Big|X_{n+1} - X_n - E_{0,\omega}\Big(X_{n+1} - X_n\Big|\mathcal{F}_n\Big)\Big|$$

$$\leq \left|X_{n+1} - X_n\right| + \left|E_{0,\omega}\Big(X_{n+1} - X_n\Big|\mathcal{F}_n\Big)\right| \leq 1 + E_{0,\omega}\Big(|X_{n+1} - X_n|\Big|\mathcal{F}_n\Big) \leq 2. \quad \square$$

Let

$$\xi_{nk} = \frac{1}{\sqrt{n}}\Big(\Delta_{k-1} - E_{0,\omega}(\Delta_{k-1}|\mathcal{F}_k)\Big), k, n \in \mathbb{N}.$$

We shall state a Theorem from the literature that is needed to prove the next lemma below.

Theorem 6.1. *[7, Theorem 18.2] Assume that*

$$\sum_{k\leq nt} E_{0,\omega}\Big(\xi_{nk}^2\Big|\mathcal{F}_{k-1}\Big) \Rightarrow t \tag{6.6}$$

for every t and that

$$\sum_{k\leq nt} E_{0,\omega}\Big(\xi_{nk}^2 \mathbb{1}_{\{|\xi_{nk}|>\epsilon\}}\Big) \longrightarrow 0. \tag{6.7}$$

for every t and $\epsilon > 0$. If $B^n(t) = \sum_{k\leq nt} \xi_{nk}$, then $B^n(t) \Rightarrow B$ in the sense of $D[0,\infty)$ where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion.

Lemma 6.3. *Let $B = (B_t)_{t\geq 0}$ be a standard brownian motion. Then under the quenched measure $P_{0,\omega}$ we have*

$$B^{(n)} \overset{J_1}{\Rightarrow} B \quad as \quad n \to \infty.$$

Proof. Note that this is a functional limit theorem (**FLT**) for martingale differences and the proof is a straight forward application of the stated above Theorem by checking the conditions (6.6) and (6.7).

- Condition (6.6)

$$\begin{aligned}
&\sum_{k\le nt} E_{0,\omega}\left[\xi_{nk}^2 \mathbb{1}_{\{|\xi_{nk}|>\epsilon\}}\right] \le \frac{2}{n} E_{0,\omega}\left[\left(\Delta_{k-1}^2 + \left(E_{0,\omega}(\Delta_{k-1}|\mathcal{F}_k)\right)^2\right)\mathbb{1}_{\{|\xi_{nk}|>\epsilon\}}\right] \\
\le\ & \sum_{k\le nt} \frac{2}{n} E_{0,\omega}\left[\left(\Delta_{k-1}^2 + (E_{0,\omega}(\Delta_{k-1}^2|\mathcal{F}_k)\right)\mathbb{1}_{\{|\xi_{nk}|>\epsilon\}}\right] \\
\le\ & \sum_{k\le nt} \frac{4}{n} P_{0,\omega}\left(\left|\Delta_{k-1} - E_{0,\omega}(\Delta_{k-1}|\mathcal{F}_k)\right| \ge \epsilon\sqrt{n}\right) \\
\le\ & \sum_{k\le nt} \frac{4E_{0,\omega}\left(\left|\Delta_{k-1} - E_{0,\omega}(\Delta_{k-1}|\mathcal{F}_k)\right|\right)}{n^2\epsilon^2} \\
\le\ & \frac{8\lfloor nt\rfloor}{n^2\epsilon^2} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

□

- Condition (6.7). This is just the convergence of the quadratic variation of our martingale B_n. For $\mathbb{P}-$ a .e. environment ω and for each $t>0$ we have

$$\begin{aligned}
\sum_{k\le nt} E_{0,\omega}\left[\xi_{nk}^2\middle|\mathcal{F}_{k-1}\right] &= \frac{\lfloor nt\rfloor}{n} - \frac{1}{n}\sum_{k\le nt}\left[E_{0,\omega}\left(\Delta_{k-1}|\mathcal{F}_k\right)\right]^2 \\
\le\ \frac{\lfloor nt\rfloor}{n} - \frac{1}{n}\sum_{k=\min_{k\le\lfloor nt\rfloor} X_k}^{\max_{k\le\lfloor nt\rfloor} X_k} M(k).
\end{aligned}$$

It only remains to show that $\forall\epsilon_1>0$:

$$\lim_{n\to\infty} P_{0,\omega}\left(\frac{1}{n}\sum_{k=\min_{k\le\lfloor nt\rfloor} X_k}^{\max_{k\le\lfloor nt\rfloor} X_k} M(k) > \nu\right) \;=\; 0.$$

Indeed,

$$\begin{aligned}
& P_{0,\omega}\left(\frac{1}{n}\sum_{k=\min_{k\le\lfloor nt\rfloor}X_k}^{\max_{k\le\lfloor nt\rfloor}X_k} M(k) > \epsilon_1\right) = P_{0,\omega}\left(\frac{1}{n}\sum_{k=\min_{k\le\lfloor nt\rfloor}X_k}^{\max_{k\le\lfloor nt\rfloor}X_k} M(k) > \epsilon_1, R_{\lfloor nt\rfloor} \le n\nu\right) \\
+ \quad & P_{0,\omega}\left(\frac{1}{n}\sum_{k=\min_{k\le\lfloor nt\rfloor}X_k}^{\max_{k\le\lfloor nt\rfloor}X_k} M(k) > \epsilon_1, R_{\lfloor nt\rfloor} > n\nu\right) \\
\le \quad & P_{0,\omega}\left(\frac{1}{n}\sum_{k=-n\nu}^{n\nu} M(k) > \epsilon_1\right) + P_{0,\omega}\left(R_{\lfloor nt\rfloor} > n\nu\right) \\
\le \quad & P_{0,\omega}\left(\frac{1}{n}\sum_{k=0}^{2n\nu+1} M(k) > \epsilon_1\right) + P_{0,\omega}\left(R_{\lfloor nt\rfloor} > n\nu\right)
\end{aligned}$$

By the Strong Law of Large Numbers (**SLLN**)

$$\frac{1}{2n\nu+1}\sum_{k=0}^{2n\nu+1} M(k) \xrightarrow[n\to\infty]{a.s} E(M(0))$$

Therefore, for almost everywhere an environment ω $\forall\eta > 0, \exists N(\eta,\omega)$ such that:

$$\forall n \ge N(\eta,\omega) : P_{0,\omega}\left(\left|\frac{1}{2n\nu+1}\sum_{k=0}^{2n\nu+1} M(k) - EM(0)\right| > \epsilon_1\right) < \frac{\eta}{2}.$$

Thus

$$\begin{aligned}
& P_{0,\omega}\left(\frac{1}{n}\sum_{k=0}^{2n\nu+1} M(k) > \epsilon_1\right) \\
\le \quad & P_{0,\omega}\left(\left|\frac{1}{n}\sum_{k=0}^{2n\nu+1} M(k) - EM(0)\right| + \left|EM(0)\right| > \epsilon_1 n\right) \\
\le \quad & P_{0,\omega}\left(\left|\frac{1}{2n\nu+1}\sum_{k=0}^{2n\nu+1} M(k) - EM(0)\right| > \frac{\epsilon_1 n}{2(2n\nu+1)}\right) + P_{0,\omega}\left(\left|EM(0)\right| > \frac{\epsilon_1 n}{2}\right) \\
< \quad & \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}$$

It only remains to show that $P_{0,\omega}\Big(R_{\lfloor nt\rfloor} > n\nu\Big) \xrightarrow{a.s.} 0$ for each $\epsilon > 0$. Indeed we have,

$$P_{0,\omega}\Big(R_{\lfloor nt\rfloor} > n\nu\Big) \le P_{0,\omega}\Big(T_{\lfloor n\nu\rfloor} \le nt\Big) + P_{0,\omega}\Big(T_{-\lfloor n\nu\rfloor} \le nt\Big) \quad =: \quad g_{n,\epsilon}(\omega,t).$$

By Fubini's Theorem and Lemma 6.2.

$$\begin{aligned}\mathbb{E}\Bigg(\sum_{n=1}^{\infty} g_{n,\epsilon}(\omega,t)\Bigg) &= \sum_{n=1}^{\infty}\mathbb{E}\Big[g_{n,\epsilon}(\omega,t)\Big] \\ &= \sum_{n=1}^{\infty}\Bigg[P_{0,\omega}\Big(T_{\lfloor\frac{n\epsilon}{6}\rfloor} \le nt\Big) + P_0\Big(T_{-\lfloor\frac{n\epsilon}{6}\rfloor} \le nt\Big)\Bigg] < \infty.\end{aligned}$$

It follows that $g_{n,\epsilon}(\omega,t) \xrightarrow[n\to\infty]{a.s} 0$. □

Lemma 6.4. *For each $t \ge 1$ and $\epsilon > 0$*

$$P_0\Bigg(\sup_{k\le nt}\frac{\Big|C_k - \delta R_k\Big|}{\sqrt{n}} > \epsilon\Bigg) \to 0.$$

Proof. Let $d(m) = \sum_{i=1}^{M(m)}\Big(2\omega_m(i) - 1\Big)$ denotes the total drift stored at site m, $m \in \mathbb{Z}$.

$$\begin{aligned}&C_k - \delta R_k \\ &= \sum_{j=0}^{k-1}\Bigg(2\omega_{X_j}\Big(L_{X_j}(j+1)\Big) - 1\Bigg) - \delta R_k = \sum_{m=I_k}^{S_k}\sum_{j=1}^{L_m(k)}\Big(2\omega_m(j) - 1\Big) - \sum_{m=I_k}^{S_k}\delta \\ &= \sum_{m=I_k}^{S_k}\Bigg(d(m) - \sum_{j=L_m(k)+1}^{M(m)}\Big(2\omega_m(j) - 1\Big)\Bigg)\mathbb{1}_{\{L_m(k)<M(m)\}} - \sum_{m=I_k}^{S_k}\delta \\ &= \sum_{m=I_k}^{S_k}\Big(d(m) - \delta\Big) - \sum_{m=I_k}^{S_k}\mathbb{1}_{\{L_m(k)<M(m)\}}\sum_{j=L_m(k)+1}^{M(m)}\Big(2\omega_m(j) - 1\Big).\end{aligned}$$

By Lemma 6.2 , given $\nu > 0$, choose K sufficiently large so that $P_0\Big(R_{\lfloor nt\rfloor} > K\sqrt{n}\Big) < \dfrac{\nu}{2}$ for all $n \in \mathbb{N}$. we have

$$
\begin{aligned}
& P_0\left(\sup_{k\le nt}\frac{\left|C_k-\delta R_k\right|}{\sqrt{n}}>\epsilon\right)=P_0\left(\sup_{k\le nt}\frac{\left|C_k-\delta R_k\right|}{\sqrt{n}}>\epsilon, R_{\lfloor nt\rfloor}\le K\sqrt{n}\right)\\
+ \quad & P_0\left(\sup_{k\le nt}\frac{\left|C_k-\delta R_k\right|}{\sqrt{n}}>\epsilon, R_{\lfloor nt\rfloor}> K\sqrt{n}\right)\\
\le \quad & P_0\left(\max_{k\le nt}\frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{\sqrt{n}}>\frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\right)\\
+ \quad & P_0\left(\max_{k\le nt}\frac{\left|\sum_{m=I_k}^{S_k}\mathbb{1}_{\{L_m(k)<M(m)\}}\sum_{j=L_m(k)+1}^{M(m)}\Big(2\omega_m(j)-1\Big)\right|}{\sqrt{n}}>\frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\right)\\
+ \quad & P_0\left(R_{\lfloor nt\rfloor}> K\sqrt{n}\right)\\
\le \quad & P_0\left(\max_{k\le nt}\frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\frac{R_k}{\sqrt{n}}>\frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\right)\\
+ \quad & P_0\left(\frac{\sum_{m=I_{\lfloor nt\rfloor}}^{S_{\lfloor nt\rfloor}}M(m)\mathbb{1}_{\{L_m(\lfloor nt\rfloor)<M(m)\}}}{\sqrt{n}}>\frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\right)+P_0\left(R_{\lfloor nt\rfloor}> K\sqrt{n}\right)
\end{aligned}
$$

Since the environment is i.i.d., by **SLLN** we have:

$$
\begin{aligned}
& \mathbb{P}\left(\lim_{R_k\to\infty}\frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}=0\right)=1\\
\Leftrightarrow \quad & \mathbb{P}\left(\forall\epsilon>0\ \exists r(\omega)\ \forall R_k\ge r(\omega):\frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\le\frac{\epsilon}{2K}\right)=1.
\end{aligned}
$$

In other words $\mathbb{P}$ almost surely an environment ω there exists an $r(\omega)$ such that

$$
\forall R_k\ge r(\omega):\frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\le\frac{\epsilon}{2K} \tag{6.8}
$$

Denote by

$$
\begin{aligned}
I_1 &= \max_{k\le\lfloor nt\rfloor, R_k\le r(\omega)} \frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\frac{R_k}{\sqrt{n}} \quad \textit{and}\\
I_2 &= \max_{k\le\lfloor nt\rfloor, R_k\ge r(\omega)} \frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\frac{R_k}{\sqrt{n}}.
\end{aligned}
$$

Then we have the estimate

$$
\begin{aligned}
& P_0\Bigg(\max_{k\le\lfloor nt\rfloor} \frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\frac{R_k}{\sqrt{n}} > \frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\Bigg)\\
=\ & P_0\Bigg(I_1\vee I_2 > \frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\Bigg)\\
=\ & P_0\Bigg(\max_{k\le\lfloor nt\rfloor, R_k\le r(\omega)} \frac{\left|\sum_{m=I_k}^{S_k}\Big(d(m)-\delta\Big)\right|}{R_k}\frac{R_k}{\sqrt{n}}\vee\frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\Bigg)\\
=\ & P_0\Bigg(\frac{\max_{k\le nt, R_k\le r(\omega)}(M(m)+1)(r(\omega)+1)}{\sqrt{n}} > \frac{\epsilon}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\Bigg)\\
\le\ & P_0\Bigg(\max_{-r(\omega)\le m\le r(\omega)}(M(m)+1)(r(\omega)+1) > \frac{\epsilon\sqrt{n}}{2}, R_{\lfloor nt\rfloor}\le K\sqrt{n}\Bigg)\\
=\ & \mathbb{E}\Bigg[P_{0,\omega}\Bigg(\max_{-r(\omega)\le m\le r(\omega)} M(m,\omega)(r(\omega)+1) > \frac{\epsilon\sqrt{n}}{2}\Bigg)\Bigg]
\end{aligned}
$$

Since

$$
\begin{aligned}
& P_{0,\omega}\Bigg(\max_{-r(\omega)\le m\le r(\omega)} M(m,\omega)(r(\omega)+1) > \frac{\epsilon\sqrt{n}}{2}\Bigg)\le 1 \quad \text{and}\\
& P_{0,\omega}\Bigg(\max_{-r(\omega)\le m\le r(\omega)} M(m,\omega)(r(\omega)+1) > \frac{\epsilon\sqrt{n}}{2}\Bigg) \xrightarrow[n\to\infty]{\mathbb{P}} 0.
\end{aligned}
$$

it follows from the Bounded Convergence Theorem (**BCF**) that

$$\mathbb{E}\left[P_{0,\omega}\left(\max_{-r(\omega)\leq m\leq r(\omega)} M(m,\omega)(r(\omega)+1) > \frac{\epsilon\sqrt{n}}{2}\right)\right] \xrightarrow[n\to\infty]{a.s} 0.$$

For the second term we proceed as follows : Divide the interval $\left[I_{\lfloor nt\rfloor}, S_{\lfloor nt\rfloor}\right]$ into subinterval of equal length $n^{\frac{1}{4}}$. So there are at most $Kn^{\frac{1}{4}}$ such subintervals:

$$N = \frac{S_{\lfloor nt\rfloor} - I_{\lfloor nt\rfloor}}{n^{\frac{1}{4}}} \leq \frac{K\sqrt{n}}{n^{\frac{1}{4}}} = Kn^{\frac{1}{4}}$$

and the endpoints of each subinterval satisfy:

$$x_j = I_{\lfloor nt\rfloor} + jn^{\frac{1}{4}} \qquad \forall j = 0,1,2,..., \frac{S_{\lfloor nt\rfloor} - I_{\lfloor nt\rfloor}}{n^{\frac{1}{4}}} \quad \text{and} \quad x_j - x_{j-1} = n^{\frac{1}{4}}.$$

Let $\gamma_1 \in (0,1)$ such that $n^{\frac{\gamma_1+1}{4}} < n^{\frac{1}{2}}$ and apply Lemma 6.1 to each subinterval except for the extreme ones because we do not know if $n^{\frac{1}{4}}$ is either greater than or less than $I_{\lfloor nt\rfloor}$ and $S_{\lfloor nt\rfloor}$ respectively. It follows from Lemma 6.1 that there is a probability of at most $\frac{K_1}{\ell^{\alpha\beta-1}}$ that the weighted average number of points that are visited is less than $M(0)$ will be greater than $\lfloor n^{\frac{\gamma_1}{4}}\rfloor$. That is,

$$\begin{aligned}
& P_0\left(\sum_{m=I_{\lfloor nt\rfloor}}^{S_{\lfloor nt\rfloor}} M(m)\mathbb{1}_{\{L_m(\lfloor nt\rfloor)<M(m)\}} > \frac{\epsilon\sqrt{n}}{2}, R_{\lfloor nt\rfloor} \leq K\sqrt{n}\right) \\
\leq\ & P_0\left(\sum_{m=I_{\lfloor nt\rfloor}}^{S_{\lfloor nt\rfloor}} M(m)\mathbb{1}_{\{L_m(\lfloor nt\rfloor)<M(m)\}} > n^{\frac{\gamma_1+1}{4}} + 2n^{\frac{1}{4}}, R_{\lfloor nt\rfloor} \leq K\sqrt{n}\right) \\
\leq\ & P_0\left(2n^{\frac{1}{4}} + \sum_{m=x_{j-1}, j=2,\dots,Kn^{\frac{1}{4}}-1}^{x_j} M(m)\mathbb{1}_{\{L_m(\lfloor nt\rfloor)<M(m)\}} > n^{\frac{\gamma_1+1}{4}} + 2n^{\frac{1}{4}}\right) \\
=\ & P_0\left(\sum_{m=x_{j-1}, j=2,\dots,Kn^{\frac{1}{4}}-1}^{x_j} M(m)\mathbb{1}_{\{L_m(\lfloor nt\rfloor)<M(m)\}} > n^{\frac{\gamma_1+1}{4}}\right)
\end{aligned}$$

$$\leq \sum_{j=0}^{Kn^{\frac{1}{4}}} P_0\Bigg(\sum_{m=0}^{n^{\frac{1}{4}}} M(m)\mathbb{1}_{\{L_m(\lfloor n^{\frac{1}{4}}t\rfloor)<M(m)\}} > \frac{n^{\frac{\gamma_1+1}{4}}}{Kn^{\frac{1}{4}}}\Bigg)$$
$$= \sum_{j=0}^{Kn^{\frac{1}{4}}} P_0\Bigg(\sum_{m=0}^{n^{\frac{1}{4}}} M(m)\mathbb{1}_{\{L_m(\lfloor n^{\frac{1}{4}}t\rfloor)<M(m)\}} > \frac{n^{\frac{\gamma_1}{4}}}{K}\Bigg)$$
$$\leq \frac{K_2}{n^{\frac{\alpha\beta-2}{4}}} \xrightarrow[n\to\infty]{a.s} 0 \quad \text{if and only if} \quad \alpha\beta > 2.$$

Lemma 6.5. *The sequence* $\left\{\frac{X_{\lfloor n\cdot\rfloor}}{\sqrt{n}}\right\}_{n\geq 1}$ *is tight in the space* $D([0,\infty))$ *of cádlág paths equipped with the* $\mathbf{J_1}$ *Skorokhod topology. Moreover if* X *is a limit point of this sequence and* P *is the corresponding measure on* $D([0,\infty))$ *then* $P\Big(X \in C([0,\infty))\Big) = 1$.

Proof. The claim in the above lemma 6.5 will follow as soon as we show that

$$\lim_{\nu\to 0}\limsup_{n\to\infty} P\Bigg(\sup_{k;\ell\leq nt,\ |k-\ell|\leq n\nu}\Big|X_k - X_\ell\Big| \geq \epsilon\sqrt{n}\Bigg) = 0, \quad \forall\epsilon > 0, \quad t < \infty \tag{6.9}$$

Indeed,

$$P\Bigg(\sup_{k;\ell\leq nt,\ |k-\ell|\leq n\nu}\Big|X_k - X_\ell\Big| \geq \epsilon\sqrt{n}\Bigg)$$
$$\leq P\Bigg(\sup_{k;\ell\leq nt,\ |k-\ell|\leq n\nu}\Big|B_k - B_\ell\Big| \geq \frac{\epsilon\sqrt{n}}{2}\Bigg) + P\Bigg(\sup_{k;\ell\leq nt,\ |k-\ell|\leq n\nu}\Big|C_k - C_\ell\Big| \geq \frac{\epsilon\sqrt{n}}{2}\Bigg).$$

By Lemma 6.4

$$\lim_{\nu\to 0}\limsup_{n\to\infty} P\Bigg(\sup_{k;\ell\leq nt,\ |k-\ell|\leq n\nu}\Big|B_k - B_\ell\Big| \geq \frac{\epsilon\sqrt{n}}{2}\Bigg) = 0, \quad \forall\epsilon > 0, \quad t < \infty.$$

For the second term on the **RHS** can be rewritten

$$\Big|C_k - C_\ell\Big| \leq \Big|C_k - \delta R_k\Big| + \delta\Big|S_k - S_\ell\Big| + |1-\delta|\Big|I_k - I_\ell\Big| + \Big|C_\ell - \delta R_\ell\Big|$$

Hence,

$$\begin{aligned}
&\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|C_k - C_\ell\Big| \\
\le\ & 2\sup_{k;\ell\le nt}\Big|C_k-\delta R_k\Big| + \sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\delta\Big|S_k-S_\ell\Big| + \sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}|1-\delta|\Big|I_k-I_\ell\Big|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\left(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|C_k-C_\ell\Big| \ge \frac{\epsilon\sqrt{n}}{2}\right) \\
\le\ & P\left(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|C_k-\delta R_k\Big| \ge \frac{\epsilon\sqrt{n}}{12}\right) + P\left(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|S_k-S_\ell\Big| \ge \frac{\epsilon\sqrt{n}}{6}\right) \\
+\ & P\left(\sup_{k;\ell\le nt,\ |k-\ell|\le n\nu}\Big|I_k-I_\ell\Big| \ge \frac{\epsilon\sqrt{n}}{6}\right) \\
\le\ & \frac{\epsilon_1}{3}+\frac{\epsilon_1}{3}+\frac{\epsilon_1}{3} = \epsilon_1 \quad \Big(\text{by Lemma 6.4 and Corollary 6.2}\Big).
\end{aligned}$$

It follows from [7, Theorem 16.8] that the sequence $\left\{\frac{X_{\lfloor n\cdot\rfloor}}{\sqrt{n}}\right\}_{n\ge1}$ is tight. Since the size of jumps of the rescaled excited random walk is $\pm\frac{1}{\sqrt{n}}$ converges in distribution to zero, it follows from [7, Theorem 13.3] that any sub-sequential limit of $\frac{X_{\lfloor n\cdot\rfloor}}{\sqrt{n}}$ in the space $D([0,\infty))$ has continuous path.

6.1.2 Proof of Theorem 2.4.

We are now able to show that the rescaled path of the excited random walk converges in distribution to a $(-\delta,\delta)$- perturbed Brownian motion by using all the results from the first three steps. To this end, we begin by defining rescaled versions of the processes X_n, B_n and C_n

$$X_n(t) = \frac{X_{\lfloor nt\rfloor}}{\sqrt{n}} \quad B_n(t) = \frac{B_{\lfloor nt\rfloor}}{\sqrt{n}} \quad \text{and} \quad C_n(t) = \frac{C_{\lfloor nt\rfloor}}{\sqrt{n}}$$

such that we have $X_n = B_n + C_n$. Note that Lemmas 6.3 and 6.5 imply that the joint sequence $\Big(X_n, B_n, C_n\Big)$ is tight in the space $\Big(D([0,\infty))\Big)^3$ such that any subsequence that converges in distribution has continuous paths in $\Big(C([0,\infty))\Big)^3$. Note also that in Lemma 6.5 it was implicitly shown that the sequence of paths is tight with a sub-sequential limit that has continuous paths in $C([0,\infty))$.

Now, $\Psi : D([0,\infty)) \longrightarrow D([0,\infty))$ be the mapping defined by

$$\Psi(x)(t) = \delta \sup_{s\leq t} x(s) - \delta \inf_{s\leq t} x(s)$$

Then Lemma 6.4 is equivalent to the statement that

$$\lim_{n\to\infty} P\left(\sup_{s\leq t}\Big|C_n - \Psi(X_n)\Big| \geq \epsilon\right) = 0, \quad \forall\epsilon > 0 \quad t < \infty. \tag{6.10}$$

Since the map Ψ is continuous on a subset of $C([0,\infty))$ of continuous functions, it follows from (6.10), the Continuous Mapping Theorem, and lemmas 6.3 and 6.5 that if n_k is a subsequence on which the joint sequence $\Big(X_n, B_n, C_n\Big)$ converges in distribution it must converges to a joint process of the form $\Big(X, B, \Psi(X)\Big)$, where X is a continuous process and B is a standard Brownian motion. Since $X_n = B_n + C_n$ the limiting process must satisfies $X = B + \Psi(X)$, that is

$$X(t) = B(t) + \delta \sup_{s\leq t} X(s) - \delta \inf_{s\leq t} X(s) \quad \forall t \geq 0,$$

and so X must be a $(\delta, -\delta)$-perturbed Brownian motion and since X_n is tight it follows that X_n converges in distribution to a $(\delta, -\delta)$-perturbed Brownian motion.

6.2 Boundary case: Proof of Theorem 2.5

The proof of this case goes word to word like in [18], which we will present again in this section. Let $\delta = 1$. For $t \geq 0$ and $n \geq 2$ set

$$T^{(n)}(x) = \frac{T_{\lfloor nx \rfloor}}{\frac{n^2}{\log n}}, \quad X^{(n)}(t) = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}\log n}, \quad S^{(n)}(t) = \frac{M_{\lfloor nt \rfloor}}{\sqrt{n}\log n}$$

Recall the notations introduced in Chapter 4, Section 4.4 $\sigma_0^V \in [1.\infty]$ and S^V respectively the life time and the total progeny over the life time of V i.e

$$\sigma_0^V = \inf\{k > 0 : V_k = 0\} \quad \text{and} \quad S^V = \sum_{j=0}^{\sigma_0^V - 1} V_j$$

and we shall consider the process V over many life times by defining $\sigma_{0,0} = 0, S_i = 0$

$$\sigma_{0,i} = \inf\{j > \sigma_{0,i-1} | V_j = 0\}, \quad S_i = \sum_{j=\sigma_{0,i-1}}^{\sigma_{0,i}-1} V_j \quad i \in \mathbb{N}. \tag{6.11}$$

Then $\Big(\sigma_{0,i} - \sigma_{0,i-1}, S_i\Big)_{i\in\mathbb{N}}$ are i.i.d. under P_0^V and $\Big(\sigma_{0,i} - \sigma_{0,i-1}, S_i\Big) \overset{D}{=} \Big(\sigma_0^V, S^V\Big), i \in \mathbb{N}$. It follows from [70, Chapter 4, Section 5, Theorem 4.5.3] that

$$\frac{\sum_{j=0}^{\lfloor n\cdot \rfloor} S_j}{n^2} \overset{J_1}{\Rightarrow} aH(\cdot) \tag{6.12}$$

where $a > 0$ and $H := (H(x)), x \geq 0$ is a stable subordinator with index $\frac{1}{2}$ i.e.

$$H(x) = \inf\Big\{t > 0 : B(t) = x\Big\}.$$

where $B = (B(t))_{t\geq 0}$ is a standard Brownian motion. The proof of Theorem 2.5 is an easy consequence of the following two lemmas.

Lemma 6.6. *The finite dimensional distribution of $T^{(n)}$ converges to those of $c_7 H$, where $c_7 > 0$ is a constant and H is given by (6.12).*

Proof. Let $k \in \mathbb{N}$ and $0 = x_0 < x_1 < ... < x_k$. We need to show that for any $0 = t_0 < t_1 < ... < t_k$ that

$$\begin{aligned} & P_0\bigg(T^{(n)}(x_k) - T^{(n)}(x_i) \leq t_{k-1}, \forall i = 0, 1, 2, ..., k-1\bigg) \\ \underset{n\to\infty}{\longrightarrow} \quad & P_0\bigg(T(x_k) - T(x_i) \leq t_{k-1}, \forall i = 0, 1, 2, ..., k-1\bigg) \end{aligned}$$

At time $T_{\lfloor nx_k \rfloor}$ we consider the corresponding backward branching process by looking back from $\lfloor nx_k \rfloor$. Notice that $D^{(j)}_{\lfloor nx_i \rfloor} \leq D^{(j)}_{\lfloor nx_k \rfloor}$ for $i \leq k$ and $\forall j$. This remark will enables us to get bounds on $T_{\lfloor nx_i \rfloor}$, $i = 1, 2, ..., k-1$, in terms of down crossings. Let $N^{(0)} = 0$,

$$N^{(k-i)} = \inf\Big\{m \in \mathbb{N} : \sigma^V_m \geq \lfloor nx_k \rfloor - \lfloor nx_i \rfloor\Big\}, \quad i = 0, 1, 2, ..., k-1$$

Since

$$2 \sum_{j=1}^{N^{(k-i)}-1} S^V_j \leq T_{\lfloor nx_k \rfloor} - T_{\lfloor nx_i \rfloor} \leq nx_k - nx_i + 2 \sum_{j=1}^{N^{(k-i)}-1} S^V_j$$

It follows that

$$\begin{aligned} & P_0\bigg(T^{(n)}(x_k) - T^{(n)}(x_i) \leq t_{k-1}, \quad \forall i = 0, 1, 2, ..., k-1\bigg) \\ & \leq P\bigg(\lfloor nx_k \rfloor - \lfloor nx_i \rfloor + 2 \sum_{j=1}^{N^{(k-i)}-1} S^V_j \leq \frac{n^2 t_{k-1}}{\log^2 n}, \quad \forall i = 0, 1, 2, ..., k-1\bigg) \end{aligned} \tag{6.13}$$

and

$$
\begin{aligned}
&P_0\left(T^{(n)}(x_k) - T^{(n)}(x_i) \le t_{k-1}, \quad \forall i = 0,1,2,...,k-1\right) \\
&\ge P\left(2\sum_{j=1}^{N^{(k-i)}-1} S_j^V \le \frac{n^2 t_{k-1}}{\log^2 n}, \quad \forall i = 0,1,2,...,k-1\right)
\end{aligned}
\tag{6.14}
$$

Next we need to have some control on the $N^{(k-i)}, i = 0,1,2,...,k-1$, and on the maximal lifetime over $\lfloor nx_k \rfloor$ generations. It follows from Theorem 3.2 and [21, Theorem 3.7.2] that $\frac{\sigma_n}{n \log n} \Rightarrow \frac{1}{b}$ for some positive constant b. It can be easily deduced form this that

$$
\frac{\min\left\{m \in \mathbb{N} : \sigma_m > n\right\}}{\frac{nb}{\log n}} \Rightarrow 1 \quad \text{as} \quad n \to \infty, \tag{6.15}
$$

which implies convergence in probability. Using the definition of $N^{(k-i)}$ we get that for every $\epsilon, \nu > 0$ there exists an n_0 such that $\forall n \ge n_0$

$$
P\left(1 - \nu \le \frac{N^{(k-1)}}{\bar{N}^{(k-i)}} \le 1 + \nu, \quad i = 0,1,2,...,k-1\right) > 1 - \epsilon,
$$

where $\bar{N}^{(k-i)} = \frac{b(x_k - x_i)n}{\log n}$. By choosing $C_9 = (1+\nu)bx_k$ we get

$$
P\left(N^{(k)} \le \frac{C_9 n}{\log n}\right) > 1 - \epsilon.
$$

define $\lambda_n = \frac{1}{\sqrt{\log n}}$ or any sequence $\lambda_n, n \in \mathbb{N}$ such that $\lambda_n \to 0$ and $\lambda_n \log n \to \infty$ will work. So by Theorem 3.2 there exists n_1 such that $\forall n \ge n_1$

$$
P\left(\max_{1 \le i \le \frac{C_9 n}{\log n}} \left(\sigma_i - \sigma_{i-1}\right) \le n\lambda_n\right) \ge \left(1 - \frac{2C_5}{n\lambda_n}\right)^{\frac{C_9 n}{\log n}} > 1 - \epsilon.
$$

Therefore, on a set Ω_ϵ of measure at least $2\epsilon - 1$ for all $n \ge n_0 \vee n_1$, the number of lifetimes of the backward branching process V covering $\Big(\lfloor nx_k\rfloor - \lfloor nx_i\rfloor\Big)$ generations, i=0,1,2,..., k-1, is well controlled and the maximal lifetime over $\lfloor nx_k\rfloor$ does not exceed $n\lambda_n$. In particular, on the event Ω_ϵ, the number of lifetimes in any interval $\Big(\lfloor nx_i\rfloor, \lfloor nx_{i+1}\rfloor\Big)$, i=0,1,2,..., k-1, goes to infinity as $n \to \infty$. Hence, on the event Ω_ϵ and using (6.12) and (6.13) we get the following lower bound

$$\begin{aligned}
& P_0\Bigg(T^{(n)}(x_k) - T^{(n)}(x_i) \le t_{k-1}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
\le \quad & P\Bigg(2\sum_{j=1}^{(1-\nu)\bar{N}^{(k-i)}-1} S_j^V \le \frac{n^2 t_{k-1}}{\log^2 n}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
= \quad & P\Bigg(\frac{2\sum_{j=1}^{(1-\nu)\bar{N}^{(k-i)}-1} S_j^V}{\left(\frac{(1-\nu)n}{\log n}\right)^2} \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
\xrightarrow{n\to\infty} \quad & P\Bigg(aH(b(x_k - x_i)) \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
= \quad & P\Bigg(2ab^2 H((x_k - x_i)) \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg)
\end{aligned}$$

Similarly we get the upper bound

$$\begin{aligned}
& P_0\Bigg(T^{(n)}(x_k) - T^{(n)}(x_i) \le t_{k-1}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
\ge \quad & P\Bigg(\lfloor nx_k\rfloor - \lfloor nx_i\rfloor + 2\sum_{j=1}^{(1+\nu)\bar{N}^{(k-i)}-1} S_j^V \le \frac{n^2 t_{k-1}}{\log^2 n}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
= \quad & P\Bigg(\frac{\lfloor nx_k\rfloor - \lfloor nx_i\rfloor}{\left(\frac{(1+\nu)n}{\log n}\right)^2} + \frac{2\sum_{j=1}^{(1+\nu)\bar{N}^{(k-i)}-1} S_j^V}{\left(\frac{(1+\nu)n}{\log n}\right)^2} \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
\xrightarrow{n\to\infty} \quad & P\Bigg(aH(b(x_k - x_i)) \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg) \\
= \quad & P\Bigg(2ab^2 H((x_k - x_i)) \le \frac{t_{k-1}}{2(1-\nu)^2}, \quad \forall i = 0,1,2,...,k-1\Bigg)
\end{aligned}$$

Letting $\nu \to 0$ and then $\epsilon \to 0$ we obtain the claim in Lemma 6.6

$$T(\cdot) = 2ab^2 H(\cdot) := cH(\cdot).$$

Lemma 6.7. *For every $\epsilon > 0$ and $T > 0$*

$$\lim_{n\to\infty} P_0\left(\sup_{t\in[0,T]}\left(S^{(n)}(t) - X^{(n)}(t)\right) > \epsilon\right) = 0.$$

Without loss of generality we may assume that $t \in [0,1]$. Fix some $\nu > 0$. we have

$$\begin{aligned} &P_0\left(\sup_{t\in[0,1]}\left(S^{(n)}(t) - X^{(n)}(t)\right) > \epsilon\right) \\ &\le P_0\left(S_n \ge K_4\sqrt{n}\log n\right) \\ &+ P_0\left(\max_{0\le m\le n}\left(S_m - X_m\right) > \epsilon\sqrt{n}\log n, S_n < K_5\sqrt{n}\log n\right) \end{aligned} \tag{6.16}$$

Lemma 6.6 ensures the existence of a such $K_3 > 0$ and for all large n the first term on **RHS** of (6.16) is bounded above by

$$P_0\left(S_n \ge K_5\sqrt{n}\log n\right) \le P_0\left(T_{\lfloor K_4\sqrt{n}\log n\rfloor} \le n\right) < \nu.$$

To estimate the last term on the **RHS** of (6.16) we will use the properties of the backward branching process V. Define

$$N(n) = \min\left\{m \in \mathbb{N} : \sigma_m > K_4\sqrt{n}\log n\right\}$$

Then the last term in (6.16) is bounded by

$$\begin{aligned}
& P_0\left(\max_{0\le m\le n}\left(S_m - X_m\right) > \epsilon\sqrt{n}\log n, S_n < K_4\sqrt{n}\log n\right) \\
\le\quad & P_0^V\left(\max_{i\le N(n)}\left(\sigma_i - \sigma_{i-1}\right) > \epsilon\sqrt{n}\log n\right) \\
\le\quad & P_0^V\left(N > C_9\sqrt{n}\right) + P_0^V\left(\max_{i\le C_9\sqrt{n}}\left(\sigma_i - \sigma_{i-1}\right) > \epsilon\sqrt{n}\log n, N \le C_9\sqrt{n}\right) \\
\overset{(6.14),(3.2)}{\le}\quad & \nu + 1 - \left(1 - \frac{2C_5}{\epsilon\sqrt{n}\log n}\right)^{\lfloor \frac{C_9}{\log n}\rfloor} \\
<\quad & 2\nu \quad \text{for all large n.}
\end{aligned}$$

• **Proof of Theorem 2.5.** It follows from Lemma 6.6 that the finite dimensional distribution of the process $S^{(n)}$ converges to those of $D\max_{s\le t} B(s)$ where $D>0$ is a constant and $B(t)$ is standard Brownian motion. Indeed,

$$\begin{aligned}
& P_0\left(S^{(n)}(t_i) < x_i,, \quad \forall i = 0,1,2,...,k\right) \\
=\quad & P_0\left(S_{\lfloor nt_i\rfloor} < x_i\sqrt{n}\log n,, \quad \forall i = 0,1,2,...,k\right) \\
=\quad & P_0\left(T_{x_i\sqrt{n}\log n} > nt_i,, \quad \forall i = 0,1,2,...,k\right) \\
=\quad & P_0\left(\frac{T_{x_i\sqrt{n}\log n}}{\frac{(\sqrt{n}\log n)^2}{(\log\sqrt{n}\log n)^2}} > \frac{nt_i}{\frac{(\sqrt{n}\log n)^2}{(\log\sqrt{n}\log n)^2}},, \quad \forall i = 0,1,2,...,k\right) \\
=\quad & P_0\left(\frac{T_{x_i\sqrt{n}\log n}}{\frac{(\sqrt{n}\log n)^2}{(\log\sqrt{n}\log n)^2}} > \frac{t_i}{4} + \frac{\log(\log n)}{\log n} + \left(\frac{\log(\log n)}{\log n}\right)^2 \quad \forall i = 0,1,2,...,k\right) \\
\xrightarrow{n\to\infty}\quad & P_0\left(aH(x_i) > \frac{t_i}{4}, \quad \forall i = 0,1,2,...,k-1\right) \\
=\quad & P_0\left(H(x_i) > \frac{t_i}{4a}, \quad \forall i = 0,1,2,...,k\right)
\end{aligned}$$

$$= \quad P_0\left(\max_{s\leq\frac{t_i}{4a}} B(s) > x_i, \quad \forall i = 0,1,2,...,k\right)$$
$$= \quad P_0\left(\frac{1}{2\sqrt{a}}\max_{s\leq t_i} B(s) > x_i, \quad \forall i = 0,1,2,...,k\right) \quad \Big(\text{Scaling property}\Big).$$

Since $S^{(n)}$ has continuous paths and are monotone and the limiting process $\max_{s\leq t} B(s)$ is continuous, it follows from [4, Corollary 1.3 and Remark (c) on p.588] that $S^{(n)}$ converges weakly and locally to $D\max_{s\leq t} B(s)$ in the uniform topology. Since we have by Lemma 6.7 that for each $T > 0$

$$\left(\sup_{t\in[0,T]}\Big(S^{(n)}(t) - X^{(n)}(t)\Big) > \epsilon\right) \xrightarrow[n\to\infty]{P_0} 0 \quad \text{and} \quad S^{(n)} \Rightarrow D\max_{s\leq t} B(s)$$

It follows from [7, Theorem 3.1] that $X^{(n)} \Rightarrow D\max_{s\leq t} B(s)$ locally in the uniform topology, and, thus, in J_1.

Chapter 7

Diffusion approximation

The evolution of the branching processes (**BPs**) associated to the ERW killed upon reaching the origin can be described by a simple diffusion process. The diffusion limit of the backward branching process V stopped at $\sigma_{\epsilon n}^V = \inf\{j > 0 | V_j \leq \epsilon n\}$ for some $\epsilon > 0$ was studied in [39] and then it was extended to the forward branching process stopped all the way up to $\sigma_0^V = \inf\{j > 0 | V_j = 0\}$. The diffusion limits of these processes will enable us to obtain informations about tails distribution of the life time and of the total progeny of the branching processes , which are the contents of both Theorems 3.1 and 3.2 and reference therein [18, 39, 46]. For our purpose the evolution of our branching processes upon entering the interval $[0, \infty)$ can be approximated as a limiting processes of a diffusion process defined in terms of solutions of the stochastic differential equation (**SDE**).

$$dY(t) \quad = \quad \nu dt + \sqrt{2Y(t)} dB(t), \quad Y = 0, \quad t \in [0, \tau_0^Y] \tag{7.1}$$

where we set for $x \geq 0$

$$\tau_x^Y = \inf\{t \geq 0 | Y(t) = x\} \quad \text{and}$$

$B(t)$ is a standard Brownian motion. Note that

(a) $\nu = \delta$ corresponds to the diffusion of the forward branching process.

(b) $\nu = 1 - \delta$ corresponds to the diffusion of the backward branching process.

(c) $\nu = 1$ corresponds to the boundary cases δ is 1 or 0 of the forward or backward processes respectively.

7.1 The diffusion approximation of the backward branching process

Throughout this chapter we shall only consider the diffusion of the backward branching process and study its properties, whereas the properties of the diffusion of the forward branching process can be similarly obtained by just changing ν by δ in (7.1). Before starting this diffusion approximation we will state a theorem from the literature, which is needed to prove that the limiting process converges in distribution to the solution of the martingale problem of the **SDE** (7.1). That is,

Theorem 7.1. *[23, Theorem. 4.1 p. 354] Let $a = \Big(a_{ij}\Big)$ be a continuous, symmetric, non-negative definite, $d \times d$ matrix-valued function on $\mathbb{R}^d$ and let $b : \mathbb{R}^d \to \mathbb{R}^d$ be continuous. Let*

$$A = \left\{ \left(f, Gf = \frac{1}{2} \sum a_{ij} \partial_i \partial_j f + \sum b_i \partial_i f \right) : f \in \ C_c^{\infty} \mathbb{R}^d \right\}$$

and suppose that the $C_{\mathbb{R}^d}[0, \infty)$ martingale problem is well-posed.

For $n = 1, 2, ...,$ let X_n and B_n be processes with sample paths in $D_{\mathbb{R}^d}[0, \infty)$, and let $A_n = (A_n^{ij})$ be a symmetric $d \times d$ matrix-valued process such that A_n^{ij} has sample paths in $D_{\mathbb{R}}[0, \infty)$

and $A_n(t) - A_n(s)$ is nonnegative definite for $t > s \geq 0$. Set $\mathcal{F}_t^n = \sigma(X_n(s), B_n(s), A_n(s) : s \leq t)$. Let

$$\tau_{n,r} = \inf\Big\{t : |X_n(t)| \vee |X_n(t-)| \geq r\Big\}, \tag{7.2}$$

and suppose that

$$M_n = X_n - B_n \quad \text{and} \tag{7.3}$$

$$M_n^i M_n^j - A_n^{ij}, i, j = 1, 2, ..., d, \tag{7.4}$$

are $\mathcal{F}_t^n-$ local martingales, and that for each $r > 0, T > 0$, and $i, j = 1, 2, ..., d$,

$$\lim_{n\to\infty} E\bigg[\sup_{t\leq T\wedge\tau_{n,r}} \Big|\widetilde{X}_n(t) - \widetilde{X}_n(t-)\Big|^2\bigg] = 0 \tag{7.5}$$

$$\lim_{n\to\infty} E\bigg[\sup_{t\leq T\wedge\tau_{n,r}} \Big|B_n(t) - B_n(t-)\Big|^2\bigg] = 0 \tag{7.6}$$

$$\lim_{n\to\infty} E\bigg[\sup_{t\leq T\wedge\tau_{n,r}} |A_n(t) - A_n(t-)|\bigg] = 0 \tag{7.7}$$

$$\sup_{t\leq T\wedge\tau_{n,r}} \left|B_n^i(t) - \int_0^t b_i(X(s))ds\right| \xrightarrow{P} 0 \quad \text{and} \tag{7.8}$$

$$\sup_{t\leq T\wedge\tau_{n,r}} \left|A_n(t) - \int_0^t a_{ij}(X_n^(s))ds\right| \xrightarrow{P} 0 \tag{7.9}$$

Suppose that $P\Big(X_n^{-1}(0)\Big) \Rightarrow \mu \in \mathcal{P}(\mathbb{R}^d)$. Then X_n converges in distribution to the solution of the martingale problem for (A, μ).

Lemma 7.1. *(Diffusion approximation, [39, Lemma 3.1], [19, Lemma 3.4]) Fix an arbitrary $\epsilon > 0, y > \epsilon$, and a sequence $y_n \to y$ as $n \to \infty$. Define $\widetilde{Y}_n(t) = \frac{V_{\lfloor nt\rfloor\wedge\tau_{\epsilon n}}}{n}, t \geq 0$. Then, under $P_{ny_n}^V$ the process $\widetilde{Y}_n$ converges in the Skorokhod (J_1) topology to $Y(\cdot \wedge \tau_\epsilon^Y)$ where Y is the* *solution of*

$$dY(t) = (1-\delta)dt + \sqrt{2Y^+(t)}dB(t), \quad Y(0) = y \quad \textit{and} \tag{7.10}$$

$\tau_\epsilon^Y = \inf\Big\{t \geq 0 | Y(t) \geq \epsilon\Big\}$.

It follows from [23, Th. 3.10] that the **SDE** (7.10) has a weak solution $Y = (Y(t))_{t\geq 0}$ for any initial distribution μ on $\mathbb{R}$ and any $\delta \in \mathbb{R}$. Due to Yamada-Watanabe uniqueness theorem [71, Th. 1] (see also [59, Th.40.1]) pathwise uniqueness holds for (7.10). By [71, Prop. 1] (see also [23, Th. 3.6]) uniqueness in distribution holds as well. Therefore by [23, Prop 3.1] the martingale problem (A, μ) corresponding to the full generator

$$A = \Big\{\Big(f, Gf = x^+\frac{\partial^2}{\partial^2 x} + (1-\delta)\frac{\partial}{\partial x}\Big) : f \in C_c^\infty(\mathbb{R})\Big\}$$

is well -posed where μ is any initial probability distribution on $\mathbb{R}$.
Note that for $\delta \leq 1$, $2Y$ is a squared Bessel process of dimension $2(1-\delta)$ and reference therein [58, Ch. XI, §1]. For $\delta > 1$, $2Y$ coincides with squared Bessel process with negative dimension. See for example [30]. In order to obtain the diffusion approximation we will modify our original backward branching V to $\widetilde{V}$ upon the the first entry of the intervals $(-\infty, \epsilon]$ and $[\epsilon, \infty)$ for a fixed $\epsilon > 0$, which has some nice martingales, and then state a functional limit theorem corresponding to $\widetilde{V}$. That is the modified process $\widetilde{V}$ is the content of the following Lemma. Recall the construction of our backward branching process in chapter 3 defined by (3.6) can be rewritten as

$$V_{k+1} = V_k + 1 + \sum_{m=1}^{V_k+1}(\xi_m^{(k)} - 1)$$

The modified recursion is defined below in (7.11).

Lemma 7.2. *let* $y \in \mathbb{Z}, \widetilde{V}_0 := y$ *and let* ξ *satisfy* (**3.3.a**) *through* (**3.3.c**) *and* (3.7)) *under some probability measure. Set*

$$v := Var\Big[1 + \sum_{m=1}^{M(k)} (\xi_m^{(k)} - 1)\Big]$$

For $k \in \mathbb{N}_0$ *define*

$$\widetilde{V}_{k+1} := \widetilde{V}_k + 1 + \sum_{m=1}^{(\widetilde{V}_k+1)\vee M(k)} (\xi_m^{(k)} - 1) \tag{7.11}$$

$$\mathcal{M}_k := \widetilde{V}_k - (1-\delta)k, \quad and \tag{7.12}$$

$$\mathcal{A}_k := vk + 2\sum_{m=0}^{k-1} E\Bigg[\bigg((\widetilde{V}_k + 1) - M(m)\bigg)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\bigg|\widetilde{V}_m\Bigg]. \tag{7.13}$$

Then $(\mathcal{M}_k)_{k\geq 0}$ *and* $(\mathcal{M}_k^2 - \mathcal{A}_k)_{k\geq 0}$ *are martingales with respect to the filtration* $(\mathcal{F}_k)_{k\geq 0}$ *where* $\mathcal{F}_k$ *is generated by* $\xi_m^{(i)}, m \geq 1, 1 \leq i \leq k$.

Proof.

$$
\begin{aligned}
E\Big[\mathcal{M}_{k+1}\Big|\mathcal{F}_k\Big] &= E\Big[\widetilde{V}_k - (k+1)(1-\delta)\Big|\mathcal{F}_k\Big] \\
&= E_{\widetilde{V}_k}\Big[\widetilde{V}_k - (k+1)(1-\delta) + 1 + \sum_{m=1}^{(\widetilde{V}_k+1)\vee M(k)} (\xi_m^{(k)} - 1)\Big] \\
&= \mathcal{M}_k - (1-\delta) + E\Bigg[\bigg(1 + \sum_{m=1}^{(\widetilde{V}_k+1)\vee M(k)} (\xi_m^{(k)} - 1)\bigg)\Bigg] \\
&= \mathcal{M}_k - (1-\delta) + E_{\widetilde{V}_k}\Bigg[\bigg(1 + \sum_{m=1}^{\widetilde{V}_k+1} \Big(\xi_m^{(k)} - 1\Big)\bigg)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\Bigg] \\
&+ E_{\widetilde{V}_k}\Bigg[\bigg(1 + \sum_{m=1}^{M(k)} \Big(\xi_m^{(k)} - 1\Big)\bigg)\mathbb{1}_{\{M(k)\geq\widetilde{V}_k+1\}}\Bigg] \\
&= \mathcal{M}_k - (1-\delta)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad E_{\widetilde{V}_k}\left[\left(1+\sum_{m=1}^{M(k)}\left(\xi_m^{(k)}-1\right)\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}} + \sum_{m=M(k)+1}^{\widetilde{V}_k+1}\left(\xi_m^{(k)}-1\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k\}}\right] \\
&+ \quad E_{\widetilde{V}_k}\left[\left(1+\sum_{m=1}^{M(k)}\left(\xi_m^{(k)}-1\right)\right)\mathbb{1}_{\{M(k)\geq\widetilde{V}_k+1\}}\right] \\
&= \quad \mathcal{M}_k-(1-\delta)+E_{\widetilde{V}_k}\left[\left(1+\sum_{m=1}^{M(k)}\left(\xi_m^{(k)}-1\right)\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\right] \\
&+ \quad E_{\widetilde{V}_k}\left[\left(1+\sum_{m=1}^{M(k)}\left(\xi_m^{(k)}-1\right)\right)\mathbb{1}_{\{M(k)\geq\widetilde{V}_k+1\}}\right] \\
&+ \quad E_{\widetilde{V}_k}\left[\sum_{m=M(k)+1}^{\widetilde{V}_k+1}\left(\xi_m^{(k)}-1\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\right] \\
&= \quad \mathcal{M}_k-(1-\delta)+E_{\widetilde{V}_k}\left[\left(1+\sum_{m=1}^{M(k)}\left(\xi_m^{(k)}-1\right)\right)\right] \\
&+ \quad \sum_{p=1}^{\widetilde{V}_k}E_{\widetilde{V}_k}\left[\sum_{m=p+1}^{\widetilde{V}_k+1}\left(\xi_m^{(k)}-1\right)\mathbb{1}_{\{M(k)\leq\widetilde{V}_k+1\}}\middle|M(k)=p\right]P_{V_k^\epsilon}\Big(M(k)=p\Big) \\
&= \quad \mathcal{M}_k-(1-\delta)+(1-\delta)+\sum_{p=1}^{\widetilde{V}_k}E_{\widetilde{V}_k}\left[\sum_{m=p+1}^{\widetilde{V}_k+1}\left(\xi_m^{(k)}-1\right)\right]P_{\widetilde{V}_k}\Big(M(k)=p\Big) \\
&= \quad \mathcal{M}_k-(1-\delta)+(1-\delta)+0=\mathcal{M}_k.
\end{aligned}
$$

So $(\mathcal{M}_k)_{k\geq 0}$ is a martingale. For the second part is just a Doob's decomposition

$$
\begin{aligned}
& E\left[\mathcal{M}_{k+1}^2-\mathcal{M}_k^2\middle|\mathcal{F}_k\right] \\
= \quad & E\left[\Big(\mathcal{M}_{k+1}-\mathcal{M}_k+\mathcal{M}_k\Big)^2-\mathcal{M}_k^2\middle|\widetilde{V}_k\right] \\
= \quad & E_{\widetilde{V}_k}\left[\Big(\mathcal{M}_{k+1}-\mathcal{M}_k\Big)^2\right]+2\mathcal{M}_kE_{\widetilde{V}_k}\Big[\mathcal{M}_{k+1}-\mathcal{M}_k\Big] \\
= \quad & E_{\widetilde{V}_k}\left[\left(\left(\sum_{m=1}^{(\widetilde{V}_k+1)\vee M(k)}(\xi_m^{(k)}-1)\right)-(1-\delta)\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= E_{\widetilde{V}_k}\left[\left(\left(1+\sum_{m=1}^{M(k)}(\xi_m^{(k)}-1)\right)-(1-\delta)\right)^2 \mathbb{1}_{\{M(k)\geq \widetilde{V}_k+1\}}\right] \\
&+ E_{\widetilde{V}_k}\left[\left(\left(1+\sum_{m=1}^{\widetilde{V}_k+1}(\xi_m^{(k)}-1)\right)-(1-\delta)\right)^2 \mathbb{1}_{\{\widetilde{V}_k+1> M(k)\}}\right] \\
&= E_{\widetilde{V}_k}\left[\left(\left(1+\sum_{m=1}^{M(k)}(\xi_m^{(k)}-1)\right)-(1-\delta)\right)^2 \mathbb{1}_{\{M(k)\geq \widetilde{V}_k+1\}}\right] \\
&+ E_{\widetilde{V}_k}\left[\left(\left(1+\sum_{m=1}^{M(k)}(\xi_m^{(k)}-1)\right)-(1-\delta)+\left(\sum_{m=M(k)+1}^{\widetilde{V}_k+1}(\xi_m^{(k)}-1)\right)\right)^2 \mathbb{1}_{\{M(k)\}<\widetilde{V}_k+1}\right] \\
&= v+E_{\widetilde{V}_k}\left[\left(\sum_{m=M(k)+1}^{\widetilde{V}_k+1}(\xi_m^{(k)}-1)\right)^2 \mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\right] \\
&= v+\sum_{p=1}^{\widetilde{V}_k} E_{V_k^\epsilon}\left[\left(\sum_{m=M(k)+1}^{\widetilde{V}_k+1}(\xi_m^{(k)}-1)\right)^2 \middle| M(k)=p\right] P_{\widetilde{V}_k}\Big(M(k)=p\Big) \\
&= v+\sum_{p=1}^{\widetilde{V}_k} E_{\widetilde{V}_k}\left[\left(\sum_{m=p+1}^{\widetilde{V}_k+1}(\xi_m^{(k)}-1)\right)^2\right] P_{\widetilde{V}_k}\Big(M(k)=p\Big) \\
&= v+2\sum_{p=1}^{\widetilde{V}_k}(\widetilde{V}_k+1-p)P_{V_k^\epsilon}\Big(M(k)=p\Big) \\
&= v+2(\widetilde{V}_k+1)P_{\widetilde{V}_k}\Big(M(k)<\widetilde{V}_k+1\Big)-2E_{\widetilde{V}_k}\Big(M(k)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\Big) \\
&= v+2E\left[\left(\widetilde{V}_k+1-M(k)\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\middle|\widetilde{V}_k\right].
\end{aligned}
$$

Now it follows from above that

$$
\begin{aligned}
& E\left[\mathcal{M}_{k+1}^2-\mathcal{A}_{k+1}\middle|\mathcal{F}_k\right] \\
&= E\left[\Big(\mathcal{M}_{k+1}^2-\mathcal{M}_k^2+\mathcal{M}_k^2-\mathcal{A}_k+\mathcal{A}_k-\mathcal{A}_{k+1}\Big)\middle|\mathcal{F}_k\right] \\
&= v+2E\left[\left(\widetilde{V}_k+1)-M(k)\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\middle|\widetilde{V}_k\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad M_k^2 - A_k - \upsilon - 2E\left[\left(\widetilde{V}_k + 1 - M(k)\right)\mathbb{1}_{\{M(k)<\widetilde{V}_k+1\}}\middle|\widetilde{V}_k\right] \\
&= \quad \mathcal{M}_k^2 - \mathcal{A}_k.
\end{aligned}
$$

Hence $\mathcal{M}_k^2 - \mathcal{A}_k$ is a martingale with respect to the filtration $\mathcal{F}_k$. □

Rescaling our modified process we have the following proposition

Proposition 7.1. *Let $(y_n)_{n\geq 1}$ be a sequence of positive numbers which converges to $y > 0$, $\delta \in \mathbb{R}$ and ξ, satisfy **(3.3.a)** through **(3.3.c)** and (3.7). For each $n \in \mathbb{N}$ define $\widetilde{V}_n = (\widetilde{V}_{n,k})_{k\geq 0}$ and $\widetilde{Y}_n = ((Y_n(t))_{t\geq 0}$ by setting $\widetilde{V}_{n,0}; = \lfloor ny_n \rfloor$ and*

$$
\begin{aligned}
\widetilde{V}_{n,k+1} &:= \widetilde{V}_{n,k} + 1 + \sum_{m=1}^{(\widetilde{V}_{n,k}+1)\vee M(k)} (\xi_m^{(k)} - 1) \quad \textit{for} \quad k \geq 0 \quad \textit{and} \\
\widetilde{Y}_n(t) &:= \frac{\widetilde{V}_{n,\lfloor nt \rfloor}}{n}, \quad \textit{for } t \in [0,\infty)
\end{aligned}
$$

Let $Y = (Y(t))_{t\geq 0}$ solve the **SDE** *(7.10) with $Y(0) = y$. Then $\widetilde{Y}_n \overset{J_1}{\to} Y$ as $n \to \infty$.*

Proof. We shall apply Theorem 7.1 by checking that all the conditions (7.3) through (7.9) are satisfied. Define for each $n \in \mathbb{N}$,$(\mathcal{M}_{n,k})_{k\geq 0}$ and $(\mathcal{A}_{n,k})_{k\geq 0}$ in terms of $\widetilde{V}_{n,k}$ as in Lemma 7.2, respectively. For $t \in [0,\infty)$ set

$$
\mathcal{M}_n(t) := \frac{\mathcal{M}_{n,\lfloor tn \rfloor}}{n}, \qquad \mathcal{A}_n(t) := \frac{\mathcal{A}_{n,\lfloor tn \rfloor}}{n^2}, \qquad \mathcal{B}_n(t) := \frac{(1-\delta)\lfloor nt \rfloor}{n}.
$$

- Both conditions (7.3) and (7.4) are satisfied because both $\mathcal{M}_n$ and $\mathcal{M}_n^2 - A_n$ are martingales due to Lemma 7.2. To check the remaining conditions (7.5) through (7.9) we fix r,T$\in (0,\infty)$ and consider the stopping time $\tau_{n,r}$ defined by (7.2) .

- Condition (7.5) .

$$
\begin{aligned}
& E\left[\sup_{t\le T\wedge\tau_{n,r}}\left|\widetilde{Y}_n(t)-\widetilde{Y}_n(t-)\right|^2\right] \\
= \; & \frac{1}{n^2}E\left[\max_{1\le k\le(nT)\wedge\tau^{\widetilde{V}_n}_{\lfloor tn\rfloor}}\left|1+\sum_{m=1}^{(\widetilde{V}_{n,k-1}+1)\vee M(k-1)}(\xi_m^{(k)}-1)\right|^2\right] \\
= \; & \frac{1}{n^2}E\left[\max_{1\le k\le(nT)\wedge\tau^{\widetilde{V}_n}_{\lfloor tn\rfloor}}\left|1+\sum_{m=1}^{M(k-1)}(\xi_m^{(k)}-1)\right|^2 \mathbb{1}_{\{M(k-1)\ge\widetilde{V}_{n,k-1}+1\}}\right] \\
+ \; & \frac{1}{n^2}E\left[\max_{1\le k\le(nT)\wedge\tau^{\widetilde{V}_n}_{\lfloor tn\rfloor}}\left|1+\sum_{m=1}^{\widetilde{V}_{n,k-1}+1}(\xi_m^{(k)}-1)\right|^2 \mathbb{1}_{\{\widetilde{V}_{n,k-1}+1>M(k-1)\}}\right] \\
\le \; & \frac{1}{n^2}E\left[\max_{1\le k\le(nT)\wedge\tau^{\widetilde{V}_n}_{\lfloor tn\rfloor}}\left|1+\sum_{m=1}^{M(k-1)}(\xi_m^{(k)}-1)\right|^2 \mathbb{1}_{\{M(k-1)\ge\widetilde{V}_{n,k-1}+1\}}\right] \\
+ \; & \frac{2}{n^2}E\left[\max_{1\le k\le(nT)}\left|1+\sum_{m=1}^{M(k-1)}(\xi_m^{(k)}-1)\right|^2 \mathbb{1}_{\{\widetilde{V}_{n,k-1}+1>M(k-1)\}}\right] \\
+ \; & \frac{2}{n^2}E\left[\max_{1\le k\le(nT)\wedge\tau^{\widetilde{V}_n}_{\lfloor tn\rfloor}}\left|\sum_{m=M(k-1)+1}^{\widetilde{V}_{n,k-1}+1}(\xi_m^{(k)}-1)\right|^2 \mathbb{1}_{\{\widetilde{V}_{n,k-1}+1>M(k-1)\}}\right] \\
\le \; & \frac{2T}{n}\left[E\left(\left|1+\sum_{m=1}^{M(0)}(\xi_m^{(0)}-1)\right|^2\right)\right] \\
+ \; & \frac{2}{n^2}\sum_{p=1}^{j}E\left[\max_{1\le k\le nT}\max_{p+1\le j\le nr}\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2 \middle| M(k-1)=p\right]P(M(k-1)=p) \\
\le \; & \frac{2Tv}{n}+\frac{2}{n^2}\sum_{p=1}^{j}E\left[\max_{1\le k\le nT}\max_{p+1\le j\le nr}\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2\right]P\Big(M(k-1)=p\Big).
\end{aligned}
$$

It follows from lemma [46, Lemma 7.4], which we shall state in chapter 8 as lemma 8.1, that we have the estimate of the second term on the right in the above inequality

$$
\begin{aligned}
&\frac{2}{n^2}\sum_{y=0}^{\infty} P\left[\max_{1\le k\le (nT)}\max_{p+1\le j\le nr}\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2 > y\right]\\
=\ & \frac{2}{n^2}\sum_{y=0}^{\lfloor (nr)^{\frac{3}{2}}\rfloor -1} P\left[\max_{1\le k\le (nT)}\max_{p+1\le j\le nr}\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2 > y\right]\\
+\ & \frac{2}{n^2}\sum_{y=\lfloor (nr))^{\frac{3}{2}}\rfloor}^{\infty} P\left[\max_{1\le k\le (nT)}\max_{p+1\le j\le nr}\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2 > y\right]\\
\le\ & \frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{2}{n^2}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor}^{\infty}\sum_{k=1}^{nr}\sum_{m=p+1}^{j+1} P\left[\left|\sum_{m=p+1}^{j+1}(\xi_m^{(k)}-1)\right|^2 > y\right]\\
\le\ & \frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{2}{n^2}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor}^{\infty} nT(nr+1-p)P\left[\left|\sum_{m=p+1}^{j+1}(\xi_m^{(0)}-1)\right|^2 > y\right]\\
\le\ & \frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{4T}{n}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor}^{\infty}(nr+1-p)e^{-\frac{y}{6[((j+1)\vee\sqrt{y}]}} \qquad \Big(\text{[46], lemma 11}\Big)\\
\le\ & \frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{4T}{n}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor}^{\infty}(nr+1-p)e^{-\frac{y}{6[((nr+1)\vee\sqrt{y}]}}\\
\le\ & \frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{4T}{n}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor+1}^{\infty}(nr+1-p)e^{-\frac{1}{6}y^{\frac{1}{3}}}
\end{aligned}
$$

Now putting everything together we get

$$
\begin{aligned}
&I \le \frac{2Tv}{n}\\
+\ & \sum_{p=1}^{j}\left(\frac{2r^{\frac{3}{2}}}{\sqrt{n}}+\frac{4T}{n}\sum_{y=\lfloor (nr)^{\frac{3}{2}}\rfloor+1}^{\infty}(nr+1-p)e^{-\frac{1}{6}y^{\frac{1}{3}}}\right)P\Big(M(k-1)=p\Big)\\
\le\ & \frac{2Tv}{n}+\frac{2r^{\frac{3}{2}}P(M(0)<nr+1)}{\sqrt{n}}
\end{aligned}
$$

$$+ \quad \frac{4T(nr+1)P(M(0)<nr+1)}{n} \times \sum_{y=\lfloor (nr)^{\frac{3}{2}} \rfloor+1}^{\infty} e^{-\frac{1}{6}y^{\frac{1}{3}}}$$
$$- \quad E\Bigg(M(0)\mathbb{1}_{\{M(0)<nr+1\}} \Bigg) \times \sum_{y=\lfloor (nr)^{\frac{3}{2}} \rfloor+1}^{\infty} e^{-\frac{1}{6}y^{\frac{1}{3}}}.$$

and all terms go to zero as $n \to \infty$.

- Condition (7.6).

$$\begin{aligned}
&\lim_{n\to\infty} E\Bigg[\sup_{t\leq T\wedge \tau_{n,r}} \Big| \mathcal{B}_n(t) - \mathcal{B}_n(t-) \Big|^2 \Bigg] \\
&= \lim_{n\to\infty} \frac{1}{n^2} E\Bigg[\max_{1\leq k\leq (nT)\wedge \tau^{\widetilde{V}_n}_{\lfloor tn \rfloor}} \Big| k(1-\delta) - (k-1)(1-\delta) \Big|^2 \Bigg] \\
&= \lim_{n\to\infty} \frac{(1-\delta)^2}{n^2} = 0
\end{aligned}$$

- Condition (7.7) .

$$\begin{aligned}
&\sup_{t\leq T\wedge \tau_{n,r}} \Big| \mathcal{A}_n(t) - \mathcal{A}_n(t-) \Big| \\
&= \frac{1}{n^2} \max_{1\leq k\leq (nT)\wedge \tau^{\widetilde{V}_n}_{\lfloor tn \rfloor}} \Bigg| v + 2E\Bigg[(\widetilde{V}_{n,k-1} + 1 - M(k-1))\mathbb{1}_{\{\widetilde{V}_{n,k-1}+1 > M(k-1)\}} \Bigg] \Bigg| \\
&\leq \frac{1}{n^2} \Bigg[v + 2E\Big[\Big(nr + 1 + M(0) \Big) \mathbb{1}_{\{M(0)<nr+1} \Big] \Bigg] \\
&\leq \frac{1}{n^2} \Bigg[v + 2\Big(nr + 1 + E(M(0)) \Big) \Bigg] \to 0 \quad \text{as} \quad \to \infty.
\end{aligned}$$

- Condition (7.8) . Let $\nu > 0$

$$\sup_{t\leq T\wedge \tau_{n,r}} \Big| \mathcal{B}_n(t) - \int_0^t (1-\delta)ds \Big| \quad = \quad \frac{1-\delta}{n} \sup_{t\leq T\wedge \tau^{\widetilde{V}_n}_{n,r}} \Big| \lfloor nt \rfloor - nt \Big| < \frac{|1-\delta|}{n} < \nu.$$

Hence

$$\lim_{n\to 0} P\left(\sup_{t\leq T\wedge \tau_{n,r}^{\widetilde{V}_n}} \left| \mathcal{B}_n(t) - \int_0^t (1-\delta)ds \right| > \nu \right) = 0.$$

- Condition (7.9).

$$\begin{aligned}
&\left| \mathcal{A}_n(t) - 2\int_0^t Y_n^+(s)ds \right| \\
=& \left| \frac{v\lfloor tn \rfloor}{n^2} + \frac{2}{n^2} \sum_{m=0}^{\lfloor tn \rfloor -1} E\left[(\widetilde{V}_{n,m}^+ + 1 - M(m)) \mathbb{1}_{\{\widetilde{V}_{n,m}^+ + 1 > M(m)\}} \middle| \widetilde{V}_{n,m}^+ \right] \right. \\
& \left. -\frac{2}{n} \int_0^{\frac{\lfloor tn \rfloor}{n}} \widetilde{V}_{n,\lfloor sn \rfloor}^+ ds - 2\int_{\frac{\lfloor tn \rfloor}{n}}^t Y_n^+(s)ds \right| \\
\leq& \frac{vt}{n} + \frac{2}{n^2} \sum_{m=0}^{\lfloor tn \rfloor -1} E\left| \left[(\widetilde{V}_{n,m} + 1 - M(m)) \mathbb{1}_{\{\widetilde{V}_{n,m} + 1 > M(m)\}} - \widetilde{V}_{n,m} \middle| \widetilde{V}_{n,m}^+ \right] \right| \\
+& \frac{2}{n} \sup_{s<t} Y_n^+(s) \\
\leq& \frac{vt}{n} + \frac{2}{n^2} \sum_{m=0}^{\lfloor tn \rfloor -1} E\left| \left[(V_{n,m}^+ + 1 - M(m)) - V_{n,m}^+ \middle| V_{n,m}^+ \right] \right| \\
+& \frac{2}{n} \sup_{s<t} Y_n^+(s) \\
\leq& \frac{(v+2+2EM(0))T + 2r}{n} \quad \text{which converges to 0 as} \quad n\to\infty
\end{aligned}$$

□

In order to get the claim of Lemma 7.1, we need to apply the Continuous Mapping Theorem to proposition 7.1. To this end we invoke the following general statement from the literature. Define for every $h \in D[0,\infty)$ and $y \in \mathbb{R}$ by $\varphi_y(h) := h(\cdot \wedge \tau_y^h)$. The function is stopped after entering the interval $[y,\infty)$ where

$$\sigma_y^h; = \inf\{t \in I : h(t) \leq y\}, I \subseteq [0,\infty), h : I \to \mathbb{R}.$$

Lemma 7.3. *[46, Lemma 3.3] Let $\delta \in \mathbb{R}, 0 < \epsilon < y < \infty$ and let ψ be any of the following* *mappings defined on* $D[0,\infty)$:

$$h \mapsto \sigma_\epsilon^h, \quad h \mapsto \varphi_\epsilon(h) \quad h \mapsto \int_0^{\sigma_\epsilon^h} h^+(s)ds \in [0,\infty].$$

Denote by $Cont(\psi) := \{h \in D[0,\infty) : \psi \text{ is continuous at } h\}$ *the set of continuity of* ψ*. Then the solution* Y *of* (7.10) *satisfies* $P_y\Big(Y \in Con(\psi)\Big) = 1$.

Proof. for $0 < \epsilon < y < \infty$ let

$$H = \Big\{h \in C[0,\infty) \Big| h(0) = y, \quad \sigma_\epsilon^h < \infty \Longrightarrow \text{h has no local maximun at } \sigma_\epsilon^h \Big\}$$

First we will show that on the set $\sigma_y^h < \infty$: $Y \in H$ with probability one. the claim follows by using the strong Markov property and the fact [59, Lemma 46 (i)]

$$P_\epsilon\Big(\exists \quad \nu > 0 \quad \text{such that} \quad Y(t) \le \epsilon \quad \forall \quad t \in [0,\nu]\Big) = 0.$$

Indeed,

$$\begin{aligned}
P(h \in H) &= P_y\Bigg(\forall \quad \nu > 0 \quad \exists \quad t \in [0,\nu] : Y(t + \sigma_\epsilon^Y) < \epsilon\Bigg) \\
&= E\Bigg[E_y\Bigg(\mathbb{1}_{\{\forall \quad \nu>0 \quad \exists \quad t\in[0,\nu]:Y(t)<\epsilon\}} \circ \mathbb{1}_{\{\theta_{\sigma_\epsilon^Y}\}}\Bigg|\mathcal{F}_{\sigma_\epsilon^Y}\Bigg)\Bigg] \\
&= P_{Y(\sigma_\epsilon^Y)}\Bigg(\forall \quad \nu > 0 \quad \exists \quad t \in [0,\nu] : Y(t) < \epsilon\Bigg) \\
&= 1 - P_{Y(\sigma_\epsilon^Y)}\Bigg(\exists \quad \nu > 0 \quad \forall \quad t \in [0,\nu] : Y(t) \ge \epsilon\Bigg) \\
&\ge 1 - P_\epsilon\Bigg(\exists \quad \nu > 0 \quad \forall \quad t \in [0,\nu] : Y(t) \ge \epsilon\Bigg) = 1.
\end{aligned}$$

and thus $P_y(h \in H) = 1$.

Consequently, it remains to show that $H \subset Cont(\psi)$. For $\psi = \sigma_\epsilon$ and $\psi = \phi_\epsilon$ this follows from [36, Ch. VI, Prop. 2.11] and [36, Ch. VI, Prop. 2.12], respectively. Note that in the notation of [36, Ch. VI, 2.9] , $\sigma_\epsilon = S_\alpha(\alpha)$ with $\alpha := e^{-h}$ and $a := e^{-\epsilon}$ given that $h \mapsto e^{-h}$ is continuous with respect to the J_1-topology.

For the continuity of of the functional $\psi(h) = \int_0^{\sigma_\epsilon^h} h^+(s)ds$, choose a sequence $h_n \in D[0,\infty)$ such that $h_n \xrightarrow{J_1} h \in H$. We need to show that $\psi(h_n) \longrightarrow \psi(h)$.

- If $\sigma_\epsilon^h < \infty$, there $\exists \quad n_0 \in \mathbb{N}$ such that $\forall \quad n \geq n_0 : \sigma_\epsilon^{h_n} < \sigma_\epsilon^h + 1 := T$. and therefore

$$\Big|\psi(h_n) - \psi(h)\Big| \leq T \sup_{[0,T]} \Big|h_n(t) - h(t)\Big|$$

  which converges locally uniformly to zero as $n \longrightarrow \infty$ by [[36], Ch. VI, Prop. 1.17b].

- If $\sigma_\epsilon^h = \infty$, then for any $T < \infty, \sigma_\epsilon^{h_n} \geq T$ for all $n \geq n_0$ and hence

$$\psi(h_n) \geq \int_0^T h_n^+(s)ds \underset{n\to\infty}{\longrightarrow} \int_0^T h^+(s)ds \underset{T\to\infty}{\longrightarrow} \int_0^\infty h^+(s)ds = \infty.$$

  since $h(s) > \epsilon \quad \forall \quad s \geq 0$.

7.2 Properties the limiting diffusion process

The rest of this section is devoted to state several facts about the process Y. We denote by Y^y to specify that the process starts at y at time zero. Suppose that Y_t is a solution of the martingale problem $MP(1-\delta, \sqrt{y^+})$. Then the natural scaling function is (see for example [59, 48 pp 284]), given by

$$\varphi(x) = \begin{cases} x^\delta & \text{if} \quad \delta \neq 0 \\ \log x & \text{if} \quad \delta = 0. \end{cases}$$

Define

$$\tau_x = \inf\{t \geq 0 | Y(t) = x\} \quad \text{for} \quad x \geq 0 \quad \text{and let} \quad T = \tau_a \wedge \tau_b. \tag{7.14}$$

Lemma 7.4. *If $a < y < b$ then $P_y(T < \infty) = 1$.*

Proof. $X_t = \varphi(Y_t)$ is local martingale with infinite quadratic variation, which is a time change of Brownian motion. More precisely by Itô formula

$$dX_t = g(X_t)dB_t \quad \text{and} \quad \langle X\rangle_t = \int_0^t (\varphi'(X_s)^2(2X_s^+)ds.$$

where $g = (\varphi'\sigma) \circ \varphi^{-1}$ and $\sigma(x) = \sqrt{2x^+}$. If we let $\gamma = \inf\{t : \langle X\rangle_t > u\}$, then $B_u = X_{\gamma(u)}$ is a Brownian motion and $X_t = B_{\langle X\rangle_t}$. Since Brownian exists the interval $(\varphi(a), \varphi(b))$ with probability one, the process X_t will exit the interval $(\varphi(a), \varphi(b))$ with probability one, and Y_t will exit the interval (a, b) with probability one.

Lemma 7.5. *[39, Lemma 3.2] Fix $y > 0$*

- **(i)** ***(Scaling property)*** *Let $\tilde{Y} = \{\tilde{Y}_t\}_{t\geq 0}$, where $\tilde{Y}_t = \frac{Y^y_{ty}}{y}$. Then $\tilde{Y} \overset{D}{=} Y^1$*

- **(ii)** ***(Hitting probabilities)*** *Let $0 \leq a < y < b$. Then*

$$P_y^Y(\tau_{a<}\tau_b) = \begin{cases} \frac{b^\delta - y^\delta}{b^\delta - a^\delta} & \textit{if} \quad \delta \neq 0 \\ \frac{\log b - \log y}{\log b - \log a} & \textit{if} \quad \delta = 0. \end{cases}$$

Proof. For (i) we use equation (7.10)

$$\begin{aligned} Y_{ty} &= (1-\delta)d(ty) + \sqrt{(2Y_{ty} \vee 0)}dB_{ty} \\ &= y(1-\delta)dt + \sqrt{(2Y_{ty} \vee 0)}dB_{ty}, \end{aligned}$$

and by the scaling property for Brownian motion $B_t \overset{D}{=} \frac{1}{\sqrt{y}} B_{ty}$ we get,

$$dY_{ty} = (1-\delta)d(ty) + \sqrt{\left(2Y_{ty} \vee 0\right)} dB_{ty} = y(1-\delta)d(ty) + \sqrt{y}\sqrt{\left(2Y_{ty} \vee 0\right)} dB_t$$

divide both side by y

$$d\left(\frac{Y_{ty}}{y}\right) = (1-\delta)dt + \sqrt{2\left(\frac{Y_{ty}}{y}\right)} dB_t = (1-\delta)dt + \sqrt{\left(2\tilde{Y}_t \vee 0\right)} dB_t = d\tilde{Y}_t$$

Hence $\tilde{Y}_t$ is also a solution of the **SDE** (7.10) and by the uniqueness in distribution of the solution we have $\tilde{Y} \overset{D}{=} Y^1$. For the second assertion $\psi(Y_T)$ is a uniformly bounded martingale, the optimal stopping theorem implies

$$\varphi(y) = E_y^Y\left[\varphi\left(Y_{\tau_a \wedge \tau_b}\right)\right] = \varphi(a) P_y^Y(\tau_a < \tau_b) + \varphi(b)\left(1 - P_y^Y(\tau_a < \tau_b)\right)$$

and solving we get (ii) of the lemma. □

Lemma 7.6. *[39, Lemma 3.3] Let Y be the diffusion process defined by (7.10).*

$$\lim_{x\to\infty} x^\delta P_1^Y(\tau_0 > x) = c_8 \in (0,\infty).$$

The proof relies heavily on the following lemma, which is stated and not proved in [39].

Lemma 7.7 ([39], Lemma 3.4)**.** *Let Y be the diffusion process defined by (7.10). Then*

$$\lim_{x\to\infty} x^\delta P_1^Y(\tau_0 > x) < \infty.$$

Proof. On the event $A = \{\tau_0 > x, , \tau_{\epsilon x} > \tau_0\}$ means that the diffusion process Y starting at 1 spends more than x units of time in the interval $(0, \epsilon x]$. So in order to estimate the probability of the event A , we rewrite $(0, \epsilon x]$ as disjoint union of intervals by using a dyadic expansion:

$$(0, \epsilon x] \quad =: \quad \bigcup_{i=1}^{\infty} \left(\epsilon x 2^{-i}, \epsilon x 2^{-i+1}\right] =: \bigcup_{i=1}^{\infty} I_i$$

Let A_i be the event that the time spent in the interval I_i is at least $\frac{x}{i(i+1)}$:

$$A_i \quad = \quad \left\{ \int_0^{\tau_0} 1_{\{Y_t \in I_i\}} dt \geq \frac{x}{i(i+1)} \right\}$$

Now we need to show that the set $A \subset \bigcup_{i=1}^{\infty} A_i$. If $(\bigcup_{i=1}^{\infty} A_i)^c \subset A^c$, then for every generic element $\omega \in \Omega$ and for all i the time spent in the set A_i is less than $\frac{x}{i(i+1)}$. Therefore the total time spent in the interval $(x, \epsilon x]$ is less than $\sum_{i=1}^{\infty} \frac{x}{i(i+1)} = x$, which implies that $\omega \in A^c$. Thus $A \subset \bigcup_{i=1}^{\infty} A_i$. Now we shall estimate the probability of the set A_i for a fixed index i. To this end set $u_i := \epsilon x 2^{-1}$ and use u_i as a time unit at scale i. Let

$$r \quad =: \quad \left[\frac{x}{i(i+1)\epsilon x 2^{-1}}\right] =: \left[\frac{2^i}{\epsilon i(i+1)}\right]$$

be the number of full units of time $\frac{x}{i(i+1)}$. Choosing $\epsilon < \frac{1}{3}$ just to make sure that $r \geq 2$ for all $i \geq 1$. Denote by ρ_0 the first entrance time to the interval $\overline{I}_i := (u_i, 2u_i]$ and set:

$$\rho_k = \inf\{t \geq \rho_{k-1} + u_i : Y_t \in \overline{I}_i\}, k = 1, 2, ..., r.$$

Notice that for each $k = 1, 2, ..., r$ during the time interval $[\rho_{k-1}, \rho_k]$ the process Y_t spends no more than 1 unit of time in $\overline{I}_i$. For the event A_i to happen the process Y_t have to spend at least r units of time in $\overline{I}_i$ before hitting level zero at time τ_0, which implies that $\rho_r < \rho_0$. Therefore

$$P_1^Y(A_i) \leq P_1^Y(\rho_r < \tau_0, \tau_0 < \tau_{\epsilon x}) < P_1^Y(\rho_r < \tau_0 \big| \rho_{r-1} < \tau_0) P_1^Y(\rho_{r-1} < \tau_0).$$

If $\rho_{r-1} < \tau_0$, then the process Y_t at time ρ_{r-1} is in the interval $\overline{I}_i$ and in order to get the time $\rho_r < \tau_0$ it should survive for at least 1 additional unit of time. Thus,

$$\begin{aligned} & P_1^Y(\rho_r < \tau_0 \big| \rho_{r-1} < \tau_0) \leq \max_{y \in \overline{I}_i} P_y^Y(\tau_0 > u) \\ = \quad & P^Y_{\epsilon x 2^{1-i}}(\tau_0 > \epsilon x 2^{-i}) = P_1^Y(\tau_0 > \frac{1}{2}) := c_8 < 1. \end{aligned}$$

The last line is due to the monotonicity and the scaling property of the process. By induction we get:

$$P_1^Y(\rho_r < \tau_0 \big| \rho_{r-1} < \tau_0) \quad < \quad c_2^r P_1^Y(\rho_0 < \tau_0).$$

Remark 7.1. *ρ_k and r depend on i and let's indicate this dependence by a superscript. By Lemma 7.5 we have:*

$$P_1^Y(\rho_0^{(i)} < \tau_0) < \begin{cases} (\epsilon x 2^{-i})^{-\delta}, & \text{if} \quad \epsilon x 2^{-i} > 1 \\ \\ 1, & \text{if} \quad \epsilon x 2^{-i} < 1 \end{cases}$$

Multiplying by $c_2^{r^{(i)}}$ and summing over i gives:

$$\begin{aligned} x^\delta P_1^Y(A) &\leq \sum_{i < \log_2(\epsilon x)} (\epsilon x 2^{-i})^{-\delta} c_2^{c_2^{(i)}} + \sum_{i > \log_2(\epsilon x)} x^\delta c_2^{r^{(i)}} \\ &\leq \frac{1}{\epsilon^\delta} \sum_{i < \log_2(\epsilon x)} 2^{i\delta} c_2^{r^{(i)}} + \frac{1}{\epsilon^\delta} \sum_{i > \log_2(\epsilon x)} 2^{i\delta} c_2^{r^{(i)}} \\ &\leq \frac{1}{\epsilon^\delta} \sum_{i=1}^{\infty} 2^{i\delta} c_2^{r \left[\frac{2^i}{\epsilon i(i+1)}\right]} < \infty, \qquad \text{since } c_2 < 1. \end{aligned}$$

□

Lemma 7.8. *[39, Lemma 3.5] Let Y be the diffusion process defined by* (7.10)*. Then*

$$\lim_{y\to\infty} y^{\delta} P_1^Y\left(\int_0^{\tau_0} Y_t dt > y^2\right) = c_9$$

The proof is exactly the same as in [39] and is omitted. The next lemma is stated and not proved in [19].

Lemma 7.9. *[19, Lemma A.8] If $\delta = 0$ then for every $h > 0$ and $y > 0$*

$$\lim_{\epsilon\to 0} P_y^Y\Big(\tau_\epsilon > h\Big) \;=\; 1 \quad \textit{and} \quad \lim_{\epsilon\to 0} P_y^Y\left(\int_0^{\tau_\epsilon} Y(t)dt > h\right) = 1.$$

Proof. By the scaling property (see Lemma 7.5(i)) it is sufficient to show the result for $y = 1$ and an arbitrary $h > 0$. For all $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} & P_1^Y\Big(\tau_\epsilon > h\Big) \geq P_1^Y\Big(\tau_\epsilon > h, \tau_\epsilon > \tau_n\Big) = P_1^Y\Big(\tau_\epsilon > h\Big|\tau_\epsilon > \tau_n\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big) \\ = \;& P_n^Y\Big(\tau_\epsilon > h\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big) \geq P_n^Y\Big(\tau_1 > h\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} & P_1^Y\Big(\int_0^{\tau_\epsilon} Y(t)dt > h\Big) \geq P_1^Y\Big(\int_0^{\tau_\epsilon} Y(t)dt > h\Big|\tau_\epsilon > \tau_n\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big) \\ = \;& P_n^Y\Big(\int_0^{\tau_\epsilon} Y(t)dt > h\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big) \geq P_n^Y\Big(\tau_1 > h\Big)P_1^Y\Big(\tau_\epsilon > \tau_n\Big). \end{aligned}$$

Since Y is $\frac{1}{2}$ of a squared Bessel process of dimension 2, it has the same distribution as the process $\dfrac{1}{2}\left(B_1^2 + B_2^2\right)$ where B_1 and B_2 are independent Brownian motions. The starting point $Y(0) = n$ can be translated into the starting point $B_1(0) = \sqrt{2n}$ and $B_2(0) = 0$ for the Brownian motion. Thus,

$$
\begin{aligned}
& P_n^Y\Big(\tau_1 > h\Big) = P_n^Y\Big(\min_{0\le t\le h} Y(t) > 1\Big) \ge P_{\sqrt{2n}}^{B_1}\Big(\min_{0\le t\le h} B_t(t) > \sqrt{2}\Big) \\
= \;& P_0\Big(\max_{0\le t\le h} B(t) < \sqrt{2n} - \sqrt{2}\Big) = 1 - 2P\Big(Z \ge \frac{\sqrt{2n}-\sqrt{2}}{\sqrt{h}}\Big) \to 1 \quad \text{as} \quad n\to\infty,
\end{aligned}
$$

where B is a standard one dimensional Brownian motion and Z is a standard normal random variable. To finish the proof, fix an $\nu > 0$, choose and fix n large enough so that

$$
P_n^Y\Big(\tau_1 > h\Big) \;\ge\; 1 - \frac{\nu}{2}
$$

and choose an ϵ_0 small enough so that

$$
P_1^Y\Big(\tau_{\epsilon_0} > \tau_n\Big) = \frac{\log\left(\frac{1}{\epsilon_0}\right)}{\log n + \log\left(\frac{1}{\epsilon_0}\right)} > 1 - \frac{\nu}{2} \quad \Big(\text{By Lemma 7.5(ii)}\Big).
$$

Therefore, $\forall \epsilon \in (0, \epsilon_0)$

$$
P_y^Y\Big(\tau_\epsilon > h\Big) \wedge P_y^Y\Big(\int_0^{\tau_\epsilon} Y(t)dt > h\Big) \ge \Big(1 - \frac{\nu}{2}\Big)^2 > 1 - \nu.
$$

□

Chapter 8

Main technical lemmas

This is the core chapter of this thesis since most of all our results rely on technical statements namely the overshoot lemma, which enables us to get the tail asymptotics of the branching processes associated to the ERW, and the martingale approximation lemma that gives estimates for the exit probabilities in the Main Lemma, a discrete version of lemma 7.4. We shall begin by stating two lemmas that are needed in the sequel.

Lemma 8.1. *[46, Lemma 28] Let $(\xi_i)_{i\in\mathbb{N}}$ be independent random variables , which are geometrically distributed with parameter $\frac{1}{2}$ and $E\xi_i = 1$. Then for all $x, y \in \mathbb{N}$,*

$$P\left(\sum_{i=1}^{x}(\xi_1 - 1) \geq y\right) \leq \exp\left(-\frac{y^2}{6\left(x \vee y\right)}\right) \leq e^{-\frac{y^2}{6x}} + e^{-\frac{y}{6}}.$$

Similarly, for $x \geq y$,

$$P\left(\sum_{i=1}^{x}(\xi_i - 1) \leq -y\right) \leq \exp\left(-\frac{y^2}{6\left(x \vee y\right)}\right) \leq e^{-\frac{y}{6x}}.$$

Note that if $x < y$, then

$$P\left(\sum_{i=1}^{x}(\xi_i - 1) \leq -y\right) = 0.$$

Proof. We shall use a result from [6, Theorem . A.1.1] which states if $(S_n)_{n\geq 0}$ is a simple symmetric random walk on the integers and $S_0 = 0$ then for any $a, n \geq 0$ we have the bound:

$$P\Big(S_n \geq a\Big) \leq e^{-\frac{a^2}{2n}}.$$

Let $(Y_i)_{i\geq 1}$ be a sequence of independent Bernoulli random variables with parameter $\frac{1}{2}$. By interpreting $\xi_i + 1$ as the time of the first appearance of heads in a sequence of independent fair coin tosses we get

$$\begin{aligned} P\left(\sum_{i=1}^{x}(\xi_i - 1) \geq y\right) &= P\left(\sum_{i=1}^{x}(\xi_i + 1) \geq 2x + y\right) = P\left(\sum_{i=1}^{2x+y-1} Y_i < x\right) \\ = \quad P\left(\sum_{i=1}^{2x+y-1}(2Y_i - 1) < -(y-1)\right) &= P\Big(S_{2x+y-1} \geq y\Big) \leq e^{-\frac{y^2}{2(2x+y-1)}} \leq e^{-\frac{y^2}{6(x\vee y)}}. \end{aligned}$$

The second equality is due to the fact that the sum $\sum_{i=1}^{x}(\xi_i + 1)$ is the number of trials needed for the x-th success to occur. So the event in the left hand side is "it takes at least $2x + y$ trials to get the x-th success" the same as the event in the right hand side is " fewer than x successes to occur". Therefore the two probabilities are the same. For the second inequality given x and y such that $x \geq y$ we obtain in similar fashion that

$$\begin{aligned} P\left(\sum_{i=1}^{x}(\xi_i - 1) \leq -y\right) &= P\left(\sum_{i=1}^{x}(\xi_i + 1) \leq 2x - y\right) = P\left(\sum_{i=1}^{2x-y} Y_i \geq x\right) \\ = \quad P\left(\sum_{i=1}^{2x-y}(2Y_i - 1) \geq y\right) &= P\Big(S_{2x-y} \geq y\Big) \leq e^{-\frac{y^2}{2(2x-y)}} \leq e^{-\frac{y^2}{6(x\vee y)}}. \end{aligned}$$

□

Lemma 8.2. *Let $(M(z))_{z\in\mathbb{Z}}$ be non negative valued random variable satisfying* (**TDE**)*, and $(X_i)_{i\geq 1}$ be an i.i.d sequence of geometric random variables of parameter $\frac{1}{2}$, and $(\widetilde{X}_i)_{i\geq 1}$ be an i.i.d sequence of geometric random variables of parameter p. Assume $(M(z)_{z\in\mathbb{Z}}), (\widetilde{X}_i)_{i\geq 1}, (X_i)_{i\geq 1}$ are all independent. Then,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)+\frac{1}{\sqrt{n}}\sum_{i=M(0)+1}^{n}(X_i-1)\Rightarrow\sqrt{2}Z \quad as \quad n\to\infty.$$

where Z is a standard normal random variable.

Proof. We can rewrite

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)+\frac{1}{\sqrt{n}}\sum_{i=M(0)+1}^{n}(X_i-1)\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)-\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(X_i-1)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-1) \qquad (8.1)
\end{aligned}$$

It follows from the Central Limit Theorem (**CLT**) that the third term in (8.1) converges in distribution to $\sqrt{2}Z$. It suffices to show now that both the first and the second terms in (8.1) converges to 0 in probability. To this end let $\nu>0$

$$\begin{aligned}
&P\left(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)\Big|>\nu\right)\\
=\;&P\left(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)\Big|>\nu, M(0)\leq n^{\frac{1}{4}}\right)+P\left(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i-1)\Big|>\nu, M(0)>n^{\frac{1}{4}}\right)\\
\leq\;&P\left(\Big|\sum_{i=1}^{\lfloor n^{\frac{1}{4}}\rfloor}(\widetilde{X}_i-1)\Big|>\sqrt{n}\nu\right)+P\left(M(0)>n^{\frac{1}{4}}\right)\\
\leq\;&P\left(\sum_{i=1}^{\lfloor n^{\frac{1}{4}}\rfloor}\widetilde{X}_i>\sqrt{n}\nu-\lfloor n^{\frac{1}{4}}\rfloor\right)+H\left(\lfloor n^{\frac{1}{4}}\rfloor\right)
\end{aligned}$$

$$\leq \frac{\lfloor n^{\frac{1}{4}}\rfloor E\widetilde{X}_1}{\sqrt{n}\nu - \lfloor n^{\frac{1}{4}}\rfloor} + H\Big(\lfloor n^{\frac{1}{4}}\rfloor\Big) \overset{P}{\longrightarrow} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Similarly one can show that $\lim_{n\to\infty} P\left(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(X_i - 1)\Big| > \nu\right) = 0$. Note that

$$\begin{aligned} & P\Big(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i - 1) - \frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(X_i - 1)\Big| > \nu\Big) \\ \leq\ & P\Big(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(\widetilde{X}_i - 1)\Big| > \frac{\nu}{2}\Big) + P\Big(\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{M(0)\wedge n}(X_i - 1)\Big| > \frac{\nu}{2}\Big) \longrightarrow 0. \end{aligned}$$

and by converging together lemma [21] the claim is proved. □

8.1 Overshoot Lemma

Lemma 8.3. *There are constants $c_{10}, c_{11} > 0$ such that for every $x \geq 1$ and $y \geq 1$,*

$$\sup_{0\leq z<x} P_z^V\Big(V_{\tau_x} > x + y\Big|\tau_x < \sigma_0\Big) \ \leq\ c_{10}\left[e^{-c_{11}\frac{y^2}{x}} + e^{-c_{11}y} + H\Big(\lfloor\frac{y}{4}\rfloor\Big)\right] \tag{8.2}$$

$$\max_{x<z<4x} P_z\Big(V_{\sigma_x\wedge\tau_{4x}} < x - y\Big) \ \leq\ c_{10}\left(e^{-c_{11}\frac{y^2}{x}} + H\Big(\lfloor\frac{y}{4}\rfloor\Big)\right) \tag{8.3}$$

Proof. We shall show (8.2) .

$$\begin{aligned} & P_z(V_{\tau_x} > x + y, \tau_x < \sigma_0) = \sum_{n=1}^{\infty} P_z(V_n > x + y, \tau_x = n, \tau_x < \sigma_0) \\ =\ & \sum_{n=1}^{\infty} P_z(V_n > x + y, V_n \geq x, \tau_x = n, \tau_x < \sigma_0) \\ =\ & \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z(V_n > x + y, V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, ..., n-2\}) \\ =\ & \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z(V_n > x + y)|V_n \geq x, V_{n-1} = r)P_z\left(V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, ..., n-2\}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} \frac{P_z(V_n > x+y|V_{n-1}=r)}{P_z(V_n \geq x|V_{n-1}=r)} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1}=r) \\
&= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1}=r) \\
&\leq \max_{0\leq r<x} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1}=r) \\
&\leq \max_{0\leq r<x} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} P_z(\tau_x < \sigma_0).
\end{aligned}
$$

Thus,

$$
\max_{0\leq z<x} P_z\left(V_{\tau_x} > x+y \,\middle|\, \tau_x < \sigma_0\right) \leq \max_{0\leq z<x} \frac{P_z(V_1 > x+y)}{P_z(V_1 \geq x)}
$$

We need to estimate the term on the **RHS** of the above inequality. Recall that V_1 is the sum of the number of offspring produced by each z particle and the immigrant particle. The offspring distribution can be affected by at most $M(\cdot) \stackrel{D}{=} M(0)$ cookies.

$$
V_1 = z+1+\sum_{m=1}^{z+1}(\xi_m^{(0)}-1).
$$

Let $N_1(r)$ be the stopping time defined by $N_1(r) = \inf\left\{n \geq 1 : \sum_{m=1}^{n}(\xi_m^{(0)}-1) \geq r\right\}$. Then,

$$
\begin{aligned}
\max_{0\leq z<x} \frac{P_z\left(V_1 > x+y\right)}{P_z\left(V_1 \geq x\right)} &= \max_{0\leq z<x} \frac{P_z\left(z+1+\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) > x+y\right)}{P_z\left(z+1+\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) \geq x\right)} \\
= \max_{0\leq z<x} \frac{P_z\left(\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) > y+x-z-1\right)}{P_z\left(\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) \geq x-z-1\right)} &= \max_{0\leq r<x} \frac{P\left(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1) > y+r\right)}{P\left(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1) \geq r\right)}.
\end{aligned}
$$

- Estimate of the denominator

$$
\begin{aligned}
& P\Bigg(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1)\geq r\Bigg) = \sum_{n=1}^{x-r} P\Bigg(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1)\geq r, N_1(r)=n\Bigg) \\
= & \sum_{n=1}^{x-r} P\Bigg(\sum_{m=1}^{n}(\xi_m^{(0)}-1) + \sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1)\geq r, N_1(r)=n\Bigg) \\
\geq & \sum_{n=1}^{x-r} P\Bigg(\sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1)\geq 0, N_1(r)=n\Bigg) \\
= & \sum_{n=1}^{x-r} P\Bigg(\sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1)\geq 0 \Bigg| N_1(r)=n\Bigg) P\Bigg(N_1(r)=n\Bigg) \\
\geq & \min_{n,k\geq 1} P\Bigg(\sum_{m=n+1}^{n+k}(\xi_m^{(0)}-1)\geq 0 \Bigg| N_1(r)=n\Bigg) P\Bigg(N_1(r)\leq x-r\Bigg).
\end{aligned}
$$

This would imply directly the estimate of the denominator if we could show that

Lemma 8.4.

$$
\min_{n,k\geq 1} P\Bigg(\sum_{m=n+1}^{n+k}(\xi_m^{(0)}-1)\geq 0 \Bigg| N_1(r)=n\Bigg) \geq c_{12} > 0.
$$

Proof. By the uniform ellipticity of the cookie environment (**UEL**), there exist $\kappa > 0$ such that $\forall z \in \mathbb{Z}, \forall i \in \mathbb{N} : \kappa := p_0 < \omega(z,i) < p_1 := 1-\kappa \quad \mathbb{P}-a.s.$ Note that on the event $\Big\{N_1(r)=n\Big\}$ there are at most $M(0)$ cookies left of strength no more than p_1 and no less than p_0. Therefore $\forall n,k$ we have the bounds

$$
\begin{aligned}
& P\Bigg(\sum_{m=n+1}^{(n+k)\wedge M(0)}(\zeta_m^{(0)}-1) + \sum_{m=(M(0)\vee n)+1}^{n+k}(\xi_m^{(0)}-1)\geq 0\Bigg) \\
\leq & \qquad P\Bigg(\sum_{m=n+1}^{n+k}(\xi_m^{(0)}-1)\geq 0 \Bigg| N_1(r)=n\Bigg) \qquad (8.4) \\
\leq & \; P\Bigg(\sum_{m=n+1}^{(n+k)\wedge M(0)}(\zeta_m^{(1)}-1) + \sum_{m=(M(0)\vee n)+1}^{n+k}(\xi_m^{(0)}-1)\geq 0\Bigg).
\end{aligned}
$$

where $\left(\xi_m^{(0)}\right)_{m>M(0)}$ are i.i.d geometric random variables of parameter $\frac{1}{2}$, $\zeta_m^{(0)}$ are i.i.d geometric random variables of parameter p_0, $\zeta_m^{(1)}$ are i.i.d geometric random variables of parameter p_1, all independent and independent of $M(0)$. It follows from lemma 8.2 that both the first and the last probabilities in (8.4) converges to $\frac{1}{2}$ as $k \to \infty$ for every $n \in \mathbb{N}$. Therefore for every $n \in \mathbb{N}$

$$\min_{k\geq 1} P\left(\sum_{m=n+1}^{(n+k)\wedge M(0)} (\zeta_m^{(0)} - 1) + \sum_{m=(M(0)\vee n)+1}^{n+k} (\xi_m^{(0)} - 1) \geq 0\right) := c_n > 0 \tag{8.5}$$

and the minimum is attained at some k_n. Since $M(0)$ is finite with probability 1 we have,

$$\begin{aligned} &\liminf_{n\to\infty} P\left(\sum_{m=n+1}^{(n+k_n)\wedge M(0)} (\zeta_m^{(0)} - 1) + \sum_{m=(M(0)\vee n)+1}^{n+k_n} (\xi_m^{(0)} - 1) \geq 0\right) \\ \geq\ &\liminf_{n\to\infty} P\left(\sum_{m=n+1}^{(n+k_n)\wedge M(0)} (\zeta_m^{(0)} - 1) + \sum_{m=(M(0)\vee n)+1}^{n+k_n} (\xi_m^{(0)} - 1) \geq 0, M(0) \leq n\right) \\ \geq\ &\min_{k\geq 1} P\left(\sum_{m=n+1}^{k+n} (\bar{\xi}_m - 1) \geq 0\right) \end{aligned}$$

where $\bar{\xi}_m$ are geometric with parameter $\frac{1}{2}$. By the **CLT** and the same reasoning as in (8.5), the last minimum right above is strictly positive i.e. equals to $\bar{c} > 0$. Since $c_n > 0$ for all $n \in \mathbb{N}$ and $\lim\inf_{n\to\infty} c_n = \bar{c} > 0$, we conclude that $\min_{n\geq 1} c_n := c_{12}$.

□

And thus

$$P\left(\sum_{m=1}^{x-r}(\xi_m^{(0)} - 1) \geq r\right) \geq c_{12} P\left(N_1(r) \leq x - r\right).$$

Estimate of the numerator. Note that we have the following event inclusions

$$\begin{aligned} & \left\{\sum_{m=1}^{x-r}(\xi_m^{(0)}-1) > y+r\right\} \\ \subset \ & \left\{\sum_{m=1}^{N_1(r)}(\xi_m^{(0)}-1) > \frac{y}{2}+r\right\} \bigcup \left\{\sum_{m=N_1(r)+1}^{x-r}(\xi_m^{(0)}-1) > \frac{y}{2}\right\}. \end{aligned} \tag{8.6}$$

The probability estimate of the second set on the **RHS** of (8.6) by using the upper bound in (8.4) and Lemma 8.1.

$$\begin{aligned} & P\left(\sum_{m=N_1(r)+1}^{x-r}(\xi_m^{(0)}-1) > \frac{y}{2}\right) = \sum_{n=1}^{x-r} P\left(\sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1) > \frac{y}{2}, N_1(r)=n\right) \\ = \ & \sum_{n=1}^{x-r} P\left(\sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1) > \frac{y}{2} \middle| N(r)=n\right) P\left(N_1(r)=n\right) \\ \leq \ & \sum_{n=1}^{x-r}\left[P\left(\sum_{m=n+1}^{(x-r)\wedge M(0)}(\zeta_m^{(0,1)}-1) + \sum_{m=(M(0)\vee n)+1}^{x-r}(\xi_m^{(0)}-1) \geq \frac{y}{2}\right) P\left(N(r)=n\right)\right. \\ \leq \ & \sum_{n=1}^{x-r}\left[P\left(\sum_{m=n+1}^{x-r}(\zeta_m^{(1)}-1) > \frac{y}{2}\right) + P\left(M(0) + \sum_{m=M(0)+1}^{x-r}(\xi_m^{(0)}-1) \geq \frac{y}{2}\right)\right] \\ \times \ & P\left(N_1(r)=n\right) \\ \leq \ & \sum_{n=1}^{x-r}\left[p_1^{\lfloor \frac{y}{2}\rfloor} + P\left(\sum_{m=M(0)+1}^{x-r}(\xi_m^{(0)}-1) \geq \frac{y}{4}\right) + P\left(M(0) \geq \frac{y}{4}\right)\right] P\left(N_1(r)=n\right) \\ \leq \ & \sum_{n=1}^{x-r}\left[p_1^{\lfloor \frac{y}{2}\rfloor} + \max_{1\leq k\leq x} P\left(\sum_{m=M(0)+1}^{M(0)+k}(\xi_m^{(0)}-1) \geq \frac{y}{4}\right) + H\left(\lfloor \frac{y}{4}\rfloor\right)\right] P_z\left(N_1(r)=n\right) \\ \leq \ & \sum_{n=1}^{x-r}\left[p_1^{\lfloor \frac{y}{2}\rfloor} + \max_{1\leq k\leq x} P\left(\sum_{m=1}^{k}(\xi_m^{(0)}-1) \geq \frac{y}{4}\right) + H\left(\lfloor \frac{y}{4}\rfloor\right)\right)\right] P\left(N_1(r)=n\right) \\ \leq \ & \sum_{n=1}^{x-r}\left[p_1^{\lfloor \frac{y}{2}\rfloor} + e^{-\frac{y^2}{96x}} + e^{-\frac{y}{24}} + H\left(\lfloor \frac{y}{4}\rfloor\right)\right] P\left(N_1(r)=n\right) \\ \leq \ & \left[p_1^{\lfloor \frac{y}{2}\rfloor} + e^{-\frac{y^2}{96x}} + e^{-\frac{y}{24}} + H\left(\lfloor \frac{y}{4}\rfloor\right)\right] P\left(N_1(r)\leq x-r\right). \end{aligned}$$

The probability estimate of the first set on the **RHS** of (8.6).

$$
\begin{aligned}
& P\left(\sum_{m=1}^{N_1(r)}(\xi_m^{(0)}-1) \geq \frac{y}{2}+r\right) \\
= & \sum_{n=1}^{x-r} P\left(\sum_{m=1}^{n}(\xi_m^{(0)}-1) \geq \frac{y}{2}+r, N_1(r)=n\right) \\
= & \sum_{n=1}^{x-r}\sum_{\ell=1-n}^{r-1} P_z\left(\sum_{m=1}^{n}(\xi_m^{(0)}-1) \geq \frac{y}{2}, N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right) \\
= & \sum_{n=1}^{x-r}\sum_{l=1-n}^{r-1} P_z\left(\xi_n^{(0)}-1 \geq \frac{y}{2}+r-\ell \middle| N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right) \\
\times & P\left(N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{0}-1)=\ell\right) \\
= & \sum_{n=1}^{x-r}\sum_{\ell=1-n}^{r-1} P\left(\xi_n^{(0)} \geq \frac{y}{2}+r-\ell+1 \middle| \zeta_n^{(0)} \geq r-\ell+1, N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right) \\
\times & P\left(N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right) \\
\leq & P\left(\xi_k^{(0)}+M(0) \geq \frac{y}{2}\right)\sum_{n=1}^{z+1}\sum_{\ell=1-n}^{x-z-2} P\left(N_1(r)=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right), \forall k \geq M(0)+1 \\
\leq & P\left(\xi_k^{(0)}+M(0)\right) \geq \frac{y}{2}\right) P\left(N_x \leq x-r\right), \forall k \geq M(0)+1 \\
\leq & \left[P\left(\xi_k^{(0)} \geq \frac{y}{4}\right)+P\left(M(0) \geq \frac{y}{4}\right)\right] P\left(N_1(r) \leq x-r\right), \forall k \geq M(0)+1 \\
\leq & \left(2^{-\lfloor \frac{y}{4}\rfloor}+H\left(\lfloor\frac{y}{4}\rfloor\right)\right) P\left(N_1(r) \leq x-r\right).
\end{aligned}
$$

We conclude that,

$$
\begin{aligned}
& P\left(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1) > y+r\right) \\
\leq & \left[p_1^{\lfloor \frac{y}{2}\rfloor}+e^{-\frac{y^2}{96x}}+e^{-\frac{y}{24}}+2^{-\lfloor\frac{y}{4}\rfloor}+2H\left(\lfloor\frac{y}{4}\rfloor\right)\right] P\left(N_1(r) \leq x-r\right).
\end{aligned}
$$

□

- Proof of (8.3)

$$
\begin{aligned}
& P_z^V\Big(V_{\sigma_x\wedge\tau_{4x}} < x-y\Big) = P_z^V\Big(V_{\sigma_x} < x-y, \sigma_x < \tau_{4x}\Big) \\
= \quad & \sum_{n=1}^{\infty} P_z\Big(V_{\sigma_x} < x-y, \sigma_x = n, \sigma_x < \tau_{4x}\Big) \\
= \quad & \sum_{n=1}^{\infty} P_z\Big(V_n < x-y, V_n \le x, \sigma_x = n, \sigma_x < \tau_{4x}\Big) \\
= \quad & \sum_{n=1}^{\infty}\sum_{r=x+1}^{\infty} P_z\Big(V_n < x-y, V_{n-1} = r, V_j > x, j \in \{1,...,n-2\}\Big) \\
= \quad & \sum_{n=1}^{\infty}\sum_{r=x+1}^{\infty} \frac{P_z\Big(V_n < x-y\Big|V_{n-1}=r\Big)}{P_z\Big(V_n \le x\Big|V_{n-1}=r\Big)} P_z\Big(\sigma_x < \tau_{4x}, \sigma_x = n, V_{n-1} = r\Big) \\
= \quad & \sum_{n=1}^{\infty}\sum_{r=x+1}^{\infty} \frac{P_r\Big(V_1 < x-y\Big)}{P_r\Big(V_1 \le x\Big)} P_z\Big(\sigma_x < \tau_{4x}, \sigma_x = n, V_{n-1} = r\Big) \\
\le \quad & \max_{x<r<4x} \frac{P_r\Big(V_1 < x-y\Big)}{P_r\Big(V_1 \le x\Big)} \sum_{n=1}^{\infty}\sum_{r=x+1}^{\infty} P_z\Big(\sigma_x < \tau_{4x},, \sigma_x = n, V_{n-1} = r\Big) \\
\le \quad & \max_{x<r<4x} \frac{P_r\Big(V_1 < x-y\Big)}{P_r\Big(V_1 \le x\Big)} P_z\Big(\sigma_x < \tau_{4x}\Big).
\end{aligned}
$$

Let $N_2(r)$ be the stopping time defined by $N_2(r) = \inf\left\{n \ge 1 : \sum_{m=1}^{n}(\xi_m^{(0)} - 1) \le -r\right\}$. Then,

$$
\max_{x<z<4x} P_z\Big(V_{\sigma_x\wedge\tau_{4x}} < x-y\Big) \le \max_{x<z<4x} \frac{P_z\Big(V_1 < x-y\Big)}{P_z\Big(V_1 \le x\Big)}
$$

$$= \max_{x<z<4x} \frac{P_z\Big(z+1+\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) < x-y\Big)}{P_z\Big(z+1+\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) \le x\Big)}$$

$$= \max_{x<z<4x} \frac{P_z\Big(\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) < x-z-1-y\Big)}{P_z\Big(\sum_{m=1}^{z+1}(\xi_m^{(0)}-1) \le x-z-1\Big)} = \max_{0<r\le 3x} \frac{P\Big(\sum_{m=1}^{x-r}(\xi_m^{(0)}-1) < -r-y\Big)}{P\Big(\sum_{m=1}^{x+r}(\xi_m^{(0)}-1) \le -r\Big)}.$$

- Estimate of the denominator by using Lemma 8.4

$$\begin{aligned}
&P\left(\sum_{m=1}^{x+r}(\xi_m^{(0)}-1) \le -r\right) = \sum_{n=1}^{x+r} P\left(\sum_{m=1}^{x+r}(\xi_m^{(0)}-1) \le -r, N_2(r)=n\right)\\
=\ &\sum_{n=1}^{x+r} P\left(\sum_{m=1}^{n}(\xi_m^{(0)}-1) + \sum_{m=n+1}^{x+r}(\xi_m^{(0)}-1) \le -r, N_2(r)=n\right)\\
\ge\ &\sum_{n=1}^{x+r} P\left(\sum_{m=n+1}^{x+r}(\xi_m^{(0)}-1) \le 0, N_2(r)=n\right)\\
=\ &\sum_{n=1}^{x+r} P\left(\sum_{m=n+1}^{x-r}(\xi_m^{(0)}-1) \le 0 \middle| N_2(r)=n\right) P\Big(N_2(r)=n\Big)\\
\ge\ &\min_{n,k\ge 1} P\left(\sum_{m=n+1}^{n+k}(\xi_m^{(0)}-1) \le 0 \middle| N_2(r)=n\right) P\Big(N_2(r)\le x+r\Big)\\
&\ge c_{12} P\Big(N_2(r)\le x+r\Big).
\end{aligned}$$

- Estimate of the numerator. We proceed as before by considering the following event inclusions:

$$\begin{aligned}
&\left\{\sum_{m=1}^{x+r}(\xi_m^{(0)}-1) < -r-y\right\}\\
\subset\ &\left\{\sum_{m=1}^{N_2(r)}(\xi_m^{(0)}-1) \le -\frac{y}{2}-r\right\} \bigcup \left\{\sum_{m=N_2(r)+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}\right\}
\end{aligned} \tag{8.7}$$

- The probability estimate of the second set on the **RHS** of (8.7) by using the lower bound in (8.4) and Lemma (8.1)

$$
\begin{aligned}
& P\left(\sum_{m=N_2(r)+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}\right) = \sum_{n=1}^{x+r} P\left(\sum_{m=n+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}, N_2(r)=n\right) \\
= & \sum_{n=1}^{x-r} P\left(\sum_{m=n+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}\Big| N_2(r)=n\right) P_z\left(N_2(r)=n\right) \\
\le & \sum_{n=1}^{x+r} P\left(\sum_{m=n+1}^{(x+r)\wedge M(0)}(\xi_m^{(0)}-1) + \sum_{m=(M(0)\vee n)+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}\right)\Bigg] P\left(N_2(r)=n\right) \\
\le & \sum_{n=1}^{x+r}\left[P\left(\sum_{m=n+1}^{x+r}(\zeta_m^{(0)}-1) \le -\frac{y}{2}\right) + P\left(-M(0) + \sum_{m=M(0)+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{2}\right)\right] \\
\times & \ P\left(N_2(r)=n\right) \\
\le & \sum_{n=1}^{x+r}\left[P\left(\sum_{m=M(0)+1}^{x+r}(\xi_m^{(0)}-1) \le -\frac{y}{4}\right) + P\left(-M(0) \le -\frac{y}{4}\right)\right] P\left(N_2(r)=n\right) \\
\le & \sum_{n=1}^{x+r}\left[\min_{1\le k\le x} P\left(\sum_{m=M(0)+1}^{M(0)+k}(\xi_m^{(0)}-1) \le -\frac{y}{4}\right) + P\left(M(0) \ge \frac{y}{4}\right)\right] P\left(N_2(r)=n\right) \\
\le & \left[e^{-c_{11}\frac{y^2}{x}} + H\left(\lfloor\frac{y}{4}\rfloor\right)\right] P\left(N_2(r) \le x+r\right).
\end{aligned}
$$

- The probability estimate for the first set on the **RHS** of (8.7).

$$
\begin{aligned}
& P\left(\sum_{m=1}^{N_2}(\xi_m^{(0)}-1) \le -r-\frac{y}{2}\right) \\
= & \sum_{n=1}^{x+r} P\left(\sum_{m=1}^{N_2}(\xi_m^{(0)}-1) \le -r-\frac{y}{2}, N_2(r)=n\right) \\
= & \sum_{n=1}^{x+r}\sum_{\ell=x-r-1}^{1-n} P\left(\sum_{m=1}^{n}(\xi_n^{(0)}-1) \le -r-\frac{y}{2}, N_2=n, \sum_{m=1}^{n-1}(\zeta_m^{(0)}-1)=\ell\right) \\
= & \sum_{n=1}^{x+r}\sum_{\ell=1-r}^{1-n} P\left((\xi_n^{(0)}-1) \le -r-\ell-\frac{y}{2}\Big| N_2=n, \sum_{m=1}^{n-1}(\xi_m^{(0)}-1)=\ell\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad P\Bigg(N_2 = n, \sum_{m=1}^{n-1}(\xi_m^{(0)} - 1) = \ell\Bigg) \\
&\leq \quad \sum_{n=1}^{x+r}\sum_{\ell=1-r}^{1-n} P\Bigg(\zeta_n^{(0)} \leq -r - \ell + 1 - \frac{y}{2}\Bigg|\xi_n^{(0)} \leq -r - \ell + 1, N_2 = n, \sum_{m=1}^{n-1}(\xi_m^{(0)} - 1) = \ell\Bigg) \\
&\times \quad P\Bigg(N_2 = n, \sum_{m=1}^{n-1}(\xi_m^{(0)} - 1) = \ell\Bigg) \\
&\leq \quad P\Bigg(- M(0) + \xi_k^{(0)} \leq -\frac{y}{2}\Bigg)\sum_{n=1}^{x-r}\sum_{\ell=r-1}^{1-n} P\Bigg(N_2 = n, \sum_{m=1}^{n-1}(\xi_m^{(0)} - 1) = \ell\Bigg), \forall k \geq M(0) + 1 \\
&\leq \quad \Bigg[P\Bigg(\xi_k^{(0)} \leq -\frac{y}{4}\Bigg) + P_z\Bigg(M(0) \geq \frac{y}{4}\Bigg)\Bigg]P\Bigg(N_2 \leq x + r\Bigg) \\
&\leq \quad H\Big(\lfloor\frac{y}{4}\rfloor\Big)P\Bigg(N_2 \leq x + r\Bigg).
\end{aligned}
$$

The last term on the right of the inequality is due to the fact that $P\Bigg(\xi_k^{(0)} \leq -\frac{y}{4}\Bigg) = 0.$ Therefore by putting terms together we have

$$
\max_{x<z<4x} P_z\Bigg(V_{\sigma_x \wedge \tau_{4x}} < x - y\Bigg) \quad \leq \quad c_{10}\Bigg(e^{-c_{11}\frac{y^2}{x}} + H\Big(\lfloor\frac{y}{4}\rfloor\Big)\Bigg).
$$

□

8.2 Martingale Approximation Lemma

The main idea of the section is a martingale approximation to estimate the exit probability of the backward branching process form the interval $[a^{n-1}, a^n]$ for some $a \in [1, 2)$. At first we shall state some auxiliary lemmas needed for this section.

Lemma 8.5. *For all $p \geq 1$ we have*

$$E\left[\max_{1\leq N\leq M(k)}\left|\sum_{m=1}^{N}(\xi_m^{(k)}-1)\right|^p\right] \;\leq\; C_p E\Big(M(k)^p\Big).$$

First recall the construction of our backward branching process in chapter 3 section 3.3. Let $\xi_m^{(k)} = F_m^{(k)} - F_{m-1}^{(k)}$ to be the number of failures between the $(m-1)$-st success and the m-th success in the Bernoulli sequence $B_k = \Big(B_k(j)\Big)_{j\geq 1}$. Let $\mathbf{U}$ be unifom supported on the interval $[0,1]$ and define the coupling

$$B_k(j) = \begin{cases} \mathbb{1}_{\{\mathbf{U}=1\}}, & j = 1, ..., M(k), k \geq 0 \\ \mathbb{1}_{\{\mathbf{U}=\omega(k,j)\}}, & j = M(k)+1, ..., M(k)+1+N, k \geq 0 \end{cases}$$

Proof. The coupling above gives that

$$-N \leq \sum_{m=1}^{N}(\xi_m^{(k)}-1) \leq M(k) + \sum_{m=1}^{N}(\xi_m^{(k)}-1)$$

and thus we have the bound

$$\begin{aligned}\left|\sum_{m=1}^{N}(\xi_m^{(k)}-1)\right|^p \;&\leq\; \max\left(N^p, \left|M(k)+\sum_{m=1}^{N}(\xi_m^{(k)}-1\right|^p\right) \\ &\leq\; 2^{p-1}\left(M(k)^p + \left|\sum_{m=1}^{N}(\xi_m^{(k)}-1\right|^p\right).\end{aligned}$$

By taking the maximum and then expectation on both sides we get the estimate

$$E\left[\max_{1\leq N\leq M(k)}\left|\sum_{m=1}^{N}(\xi_m^{(k)}-1)\right|^p\right] \leq 2^{p-1}E\Big(M(k)^p\Big) + 2^{p-1}E\left[\max_{1\leq N\leq M(k)}\left|\sum_{m=1}^{N}(\xi_m^{(k)}-1)\right|^p\right]$$

But the second term on the on the **RHS** in the above inequality can be estimated as follows

$$
\begin{aligned}
& E\bigg(\max_{1\le N\le M(k)}\bigg|\sum_{m=1}^{N}(\xi_m^{(k)}-1)\bigg|^p\bigg) \le E\bigg(\Big(\max_{i\le N\le M(k)}\sum_{m=1}^{N}\Big|\xi_m^{(k)}-1\Big|\Big)^p\bigg)\\
= \;& E\bigg(\Big(\sum_{m=1}^{M(k)}\Big|\xi_m^{(k)}-1\Big|\Big)^p\bigg)\\
\le \;& E\bigg(\Big(M(k)\Big)^{p-1}\sum_{m=1}^{M(k)}\Big|\xi_m^{(k)}-1\Big|^p\bigg) \quad (\text{ Holder inequality})\\
= \;& E\bigg[\Big(M(k)\Big)^{p-1}E\bigg(\sum_{m=1}^{M(k)}\Big|\xi_m^{(k)}-1\Big|^p\bigg|M(k)\bigg)\bigg]\\
= \;& E\bigg[\Big(M(k)\Big)^{p-1}M(k)E\bigg(\Big|\xi_1^{(0)}-1\Big|^p\bigg)\bigg] \quad (\text{ i.i.d. of the } \xi_m^{(k)}, m > M(k))\\
= \;& C_pE\Big(M(k)\Big)^p. \quad \square
\end{aligned}
$$

Remark 8.1. *If $X_1, X_2, ..., X_n$ are i.i.d. centered random variables then*

$$E\bigg[\bigg(\sum_{j=1}^{n}X_j\bigg)^4\bigg] = nEX_1^4 + 3n(n-1)\Big(EX_1^2\Big)^2 \le c_{13}n^2$$

Lemma 8.6. *Fix $a \in (1, 2]$, let $\frac{1}{2} < \epsilon < 1$. Consider the process V with $|V_0 - a^n| \le a^{\epsilon n}$ and define the stopping time*

$$\gamma = \inf\{k \ge 0 | V_k \notin (a^{n-1}, a^{n+1})\}.$$

If $\delta > 0$, then for all sufficiently large n

$$P^V\bigg(dist\Big(V_\gamma, (a^{n-1}, a^{n+1})\Big) \ge a^{\epsilon(n-1)}\bigg) \le c_{14}\bigg(3e^{-c_{15}a^{(2\epsilon-1)n}} + 2H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\bigg) \tag{8.8}$$

$$\left|P^V(V_\gamma \le a^{n-1}) - \frac{a^\delta}{a^\delta+1}\right| \le K_{14}\left[\frac{1}{a^{(1-\epsilon)n}} + H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\right]. \tag{8.9}$$

If $\delta = 0$, then set $a = 2$ and for all sufficiently large n

$$P^V\left(dist\Big(V_\gamma,(2^{n-1},2^{n+1})\Big) \ge 2^{\epsilon(n-1)}\right) \le c_{14}\left[3e^{-c_{15}2^{(2\epsilon-1)n}} + 2H\Big(\lfloor\frac{2^{\epsilon(n-1)}}{4}\rfloor\Big)\right] \tag{8.10}$$

$$\left|P^V(V_\gamma \le 2^{n-1}) - \frac{1}{2}\right| \le K_{14}\left[\frac{1}{2^{(1-\epsilon)n}} + H\Big(\lfloor\frac{4}{2^{\epsilon(n-1)}}\rfloor\Big)\right] \tag{8.11}$$

Proof. We shall give the proof of the lemma only in the case when $\delta > 0$ and for $\delta = 0$ the proof is exactly the same by taking $a = 2$ and the scaling function for the martingale argument will be $s(x) = \log x$ defined on the interval $(\frac{2}{3}, \frac{3}{2})$. It follows from Lemma 8.3 that

$$\begin{aligned} & P^V\left(dist(V_\gamma,(a^{n-1},a^{n+1})) \ge a^{\epsilon(n-1)}\right) \\ \le\ & P^V\left(V_\gamma \le a^{n-1} - a^{\epsilon(n-1)}\right) + P^V\left(V_\gamma \ge a^{n+1} + a^{\epsilon(n-1)}\right) \\ \le\ & c_{14}\left(2e^{-c_{15}a^{(2\epsilon-1)n-2\epsilon-1}} + e^{-c_{11}a^{(2\epsilon-1)(n-1)}} + 2H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\right) \\ \le\ & c_{14}\left(3e^{-c_{15}a^{(2\epsilon-1)n}} + 2H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\right) \end{aligned}$$

and the inequality (8.8) is proved. To show (8.9) we proceed as follows: Let $s \in C_0^\infty([0,\infty))$ be a non-negative function such that $s(x) = x^\delta$ on $(\frac{2}{3a}, \frac{3a}{2})$. Fix an n and define the process $U^n := (U_k^n)_{k\ge 0)}$ by

$$U_k^n = s\left(\frac{V_{k\wedge\gamma}}{a^n}\right).$$

We would like to show for large n that U_k^n is close to be a martingale with respect to its natural filtration $(\mathcal{F}_k)_{k\geq 0}$. On the event $\{\gamma > k\}$ and Taylor expansion of the second order

$$E\Bigg(U^n_{k+1}|\mathcal{F}_k\Bigg) = E\Bigg[s\Bigg(\frac{V_k}{a^n} + \frac{V_{k+1}-V_k}{a^n}\Bigg)\Bigg|V_k\Bigg] = E_{V_k}\Bigg[s\Bigg(\frac{V_k}{a^n} + \frac{V_{k+1}-V_k}{a^n}\Bigg)\Bigg]$$

$$= s\Big(\frac{V_k}{a^n}\Big) + E_{V_k}\Bigg[s'\Bigg(\frac{V_k}{a^n}\Bigg)\Bigg(\frac{V_{k+1}-V_k}{a^n}\Bigg)\Bigg] + \frac{1}{2}E_{V_k}\Bigg[s''\Bigg(\frac{V_k}{a^n}\Bigg)\Bigg(\frac{(V_{k+1}-V_k)^2}{a^{2n}}\Bigg)\Bigg] + r^n_k$$

$$= U^n_k - \frac{\delta-1}{a^n}s'\Bigg(\frac{V_k}{a^n}\Bigg) + \frac{1}{2a^{2n}}s''\Bigg(\frac{V_k}{a^n}\Bigg)E_{V_k}\Bigg[\Bigg(V_{k+1}-V_k\Bigg)^2\Bigg] + r^n_k$$

where,

$$r^n_k = \frac{1}{6}s^{(3)}\Bigg(\frac{V_k}{a^n}\Bigg)E_{V_k}\Bigg[\Bigg(\frac{V_{k+1}-V_k}{a^n}\Bigg)^3\Bigg]$$

It follows from Lemma (8.5) that

$$v = E_{V_k}\Bigg[\Bigg(1 + \sum_{m=1}^{(V_k+1)\wedge M(k)}(\zeta^{(k)}_m - 1)\Bigg)^2\Bigg] \leq 2 + 2E_{V_k}\Bigg[\Bigg(\sum_{m=1}^{(V_k+1)\wedge M(k)}(\zeta^{(k)}_m - 1)\Bigg)^2\Bigg]$$

$$\leq 2 + 2E\Bigg[\max_{1\leq N\leq M(k)}\Bigg(\sum_{m=1}^{N}(\zeta^{(k)}_m - 1)\Bigg)^2\Bigg].$$

Claim 8.2.1.

$$E_{V_k}\Bigg[\Bigg(V_{k+1}-V_k\Bigg)^2\Bigg] = v + 2\Big(V_k+1\Big) - 2E_{V_k}\Bigg((V_k+1)\wedge M(k)\Bigg).$$

Proof.

$$E_{V_k}\Bigg[\Bigg(V_{k+1}-V_k\Bigg)^2\Bigg]$$

$$= E_{V_k}\Bigg[\Bigg(V_{k+1}-V_k\Bigg)^2\mathbb{1}_{\{V_k+1\leq M(k)\}}\Bigg] + E_{V_k}\Bigg[\Bigg(V_{k+1}-V_k\Bigg)^2\mathbb{1}_{\{V_k+1> M(k)\}}\Bigg]$$

$$
\begin{aligned}
&= E_{V_k}\left[\left(1+\sum_{m=1}^{V_k+1}(\zeta_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1\le M(k)\}}\right] + E_{V_k}\left[\left(1+\sum_{m=1}^{V_k+1}(\zeta_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&= E_{V_k}\left[\left(1+\sum_{m=1}^{V_k+1}(\zeta_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1\le M(k)\}}\right]\\
&+ E_{V_k}\left[\left(1+\sum_{m=1}^{M(k)}(\zeta_m^{(k)}-1)+\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{k}-1)\right)^2 \mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&= E_{V_k}\left[\left(1+\sum_{m=1}^{V_k+1}(\zeta_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1\le M(k)\}}\right] + E_{V_k}\left[\left(1+\sum_{m=1}^{M(k)}(\zeta_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&+ 2E_{V_k}\left[\left(1+\sum_{m=1}^{M(k)}(\zeta_m^{(k)}-1)\right)\left(\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{k}-1)\right)\mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&+ E_{V_k}\left[\left(\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&= E_{V_k}\left[\left(1+\sum_{m=1}^{(V_k+1)\wedge M(k)}(\zeta_m^{(k)}-1)\right)^2\right] + E_{V_k}\left[\left(\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{(k)}-1)\right)^2 \mathbb{1}_{\{V_k+1> M(k)\}}\right]\\
&= v+0+\sum_{p=0}^{V_k}E_{V_k}\left[\left(\sum_{p=M(k)+1}^{V_k+1}(\xi_m^{(k)}-1)\right)^2\middle| M(k)=p\right]P_{V_k}\Big(M(k)=p\Big)\\
&= v+2\sum_{p=0}^{V_k}\Big(V_k+1-p\Big)P_{V_k}\Big(M(k)=p\Big)\\
&= v+2\Big(V_k+1\Big)P_{V_k}\Big(M(k)<V_k+1\Big)-2\sum_{p=0}^{V_k}pP_{V_k}\Big(M(k)=p\Big)\\
&= v+2\Big(V_k+1\Big)-2\Big(V_k+1\Big)P_{V_k}\Big(M(k)\ge V_k+1\Big)-2E_{V_k}(M(k))\\
&- 2E_{V_k}\Big(M(k)\mathbb{1}_{\{M(k)\ge V_k+1\}}\Big)\\
&= v+2\Big(V_k+1\Big)-2E_{V_k}\Big((V_k+1)\mathbb{1}_{\{M(k)\ge V_k+1\}}\Big)-2E_{V_k}\Big[\Big(M(k)\mathbb{1}_{\{M(k)< V_k+1\}}\Big)\Big]\\
&= v+2\Big(V_k+1\Big)-2E_{V_k}\Big((V_k+1)\wedge M(k)\Big).
\end{aligned}
$$

□

Note that the function $s(x)$ satisfies the differential equation $xs''(x)-(1-\delta)s'(x)=0$, with this in mind we get

$$
\begin{aligned}
& E\Big(U_{k+1}^n|\mathcal{F}_k\Big) - U_k^n \\
= \quad & -\frac{\delta-1}{a^n}s'\Big(\frac{V_k}{a^n}\Big) + \frac{1}{2a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\Big(v + 2\Big(V_k+1\Big) - 2E_{V_k}\Big((V_k+1)\wedge M(k)\Big) + r_k^n \\
= \quad & -\frac{\delta-1}{a^n}s'\Big(\frac{V_k}{a^n}\Big) + \frac{1}{a^n}\big(\frac{V_k}{a^n}\big)s''\Big(\frac{V_k}{a^n}\Big) \\
+ \quad & \frac{1}{a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\Big[\frac{v}{2} + 1 - E_{V_k}\Big((V_k+1)\wedge M(k)\Big)\Big] + r_k^n \\
= \quad & \frac{1}{a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\Big(\frac{v}{2} + 1 - E_{V_k}\Big((V_k+1)\wedge M(k)\Big) + r_k^n.
\end{aligned}
$$

It's easy to see that the term $\left|E_{V_k}\Big((V_k+1)\wedge M(k)\Big)\right| \leq EM(k) < \infty$. By putting all terms together we get

$$
\left|\frac{1}{a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\Big(v + 2\Big(V_k+1\Big) - 2E_{V_k}\Big((V_k+1)\wedge M(k)\Big)\Big)\right| \quad \leq \quad \frac{K_5}{a^{2n}} \tag{8.12}
$$

Now we need to estimate the Taylor's remainder r_k^n

$$
|r_k^n| \leq \frac{1}{6}||s''||_\infty E\Big[\big(\frac{|V_{k+1}-V_k|^3}{a^n}\big)\Big|\mathcal{F}_k\Big] \leq \frac{1}{6}||s''||_\infty\Big(E\Big[\big(\frac{V_{k+1}-V_k}{a^n}\big)^4\Big|\mathcal{F}_k\Big]\Big)^{\frac{3}{4}}
$$

Similar computations like the ones for the second moment and remark 8.2 we get the estimate

$$
\begin{aligned}
& E\Big[\Big(V_{k+1}-V_k\Big)^4\Big|\mathcal{F}_k\Big] \\
\leq \quad & 8E_{V_k}\Big[\Big(1 + \sum_{m=1}^{M(k)\wedge(V_k+1)}(\zeta_m^{(k)}-1)\Big)^4\Big] + 8E_{V_k}\Big[\Big(\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{(k)}-1)\Big)^4 \mathbb{1}_{\{V_k+1>M(k)\}}\Big] \\
\leq \quad & 8E_{V_k}\Big[\Big(1 + \sum_{m=1}^{M(k)\wedge(V_k+1)}(\zeta_m^{(k)}-1)\Big)^4\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 8\sum_{p=1}^{V_k} E_{V_k}\left[\left(\sum_{m=M(k)+1}^{V_k+1}(\xi_m^{(k)}-1)\right)^4 \middle| M(k)=p\right] P_{V_k}\Big(M(k)=p\Big) \\
&\leq \quad 64 + c_{16}E\Big(M(k)^4\Big) + c_{17}\sum_{p=1}^{V_k}\Big(V_k-p+1\Big)^2 P_{V_k}\Big(M(k)=p\Big) \\
&\leq \quad 64 + c_{16}E\Big(M(k)^4\Big) + c_{17}E\Big(V_k-M(k)+1\Big)^2 \leq c_{18}+c_{19}(V_k+1)^2 \leq c_{18}+c_{19}a^{2(n+1)}.
\end{aligned}
$$

By putting all terms together and using Claim 8.2.1 we get the bounds

$$
E\left[\Big(\frac{V_{k+1}-V_k}{a^n}\Big)^4 \middle| \mathcal{F}_k\right] \leq \frac{K_6}{a^{2n}} \quad \text{and} \quad \big|r_n^k\big| \leq \frac{K_7}{a^{\frac{3n}{2}}}. \tag{8.13}
$$

Let $R_0^n = 0$ and for $k \geq 1$ set

$$
\begin{aligned}
R_k^n \quad = \quad & \sum_{j=0}^{k-1}\left[\frac{1}{a^{2n}}s''\Big(\frac{V_j}{a^n}\Big)\Big(\frac{1}{2}E_{V_j}\left[\Big(1+\sum_{m=1}^{(V_j+1)\wedge M(j)}(\zeta_m^{(k)}-1)\Big)^2\right]\right. \\
+ \quad & \left. 1 - E_{V_j}\left[\Big((V_j+1)\wedge M(j)\Big)\right] + r_j^n\right].
\end{aligned}
$$

We shall show that on the event $\{\gamma > k\}$ the process $U_k^n - R_{k-1}^n$ is a martingale with initial value U_0^n. Indeed,

$$
\begin{aligned}
& E\Big(U_{k+1}^n - R_{k+1}^n|\mathcal{F}_k\Big) = U_n^k + \frac{1}{a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\left(\frac{v_j}{2} + -E_{V_j}\left[\Big((V_k+1)\wedge M(k)\Big)\right]\right) \\
+ \quad & r_k^n - \frac{1}{a^{2n}}s''\Big(\frac{V_k}{a^n}\Big)\left(\frac{v_j}{2}+1-E_{V_j}\left[\Big((V_k+1)\wedge M(k)\Big)\right]\right) \\
- \quad & r_k^n - \sum_{j=1}^{k}\left[\frac{1}{a^{2n}}s''\Big(\frac{V_j}{a^n}\Big)\left(\frac{v_j}{2}+1-E_{V_j}\left[\Big((V_j+1)\wedge M(j)\Big)\right]\right)+r_j^n\right] \\
= \quad & U_k^n - R_k^n
\end{aligned}
$$

where $v_j = E_{V_j}\left[\Big(1+\sum_{m=1}^{(V_j+1)\wedge M(j)}(\zeta_m^{(j)}-1)\Big)^2\right]$. In order to use the optimal stopping theorem we need to check that $EU_\gamma^n < \infty$ and $ER_\gamma^n < \infty$. Since by [39, Proposition A.2] $E\gamma < c_{20}a^n$ and using the estimates (8.12) and (8.13) we get

$$\Big|ER_\gamma^n\Big| \le E\Big|R_\gamma^n\Big| \le \Big(\frac{K_7}{a^{2n}} + \frac{K_6}{a^{\frac{3n}{2}}}\Big)E\gamma \le \frac{K_8}{a^{\frac{n}{2}}}.$$

This allows us to pass to the limit as $k \to \infty$ and conclude that $U_0^n = EU_\gamma^n - ER_\gamma^n$. Thus,

$$\begin{aligned} & U_0^n - \frac{K_9}{a^{\frac{n}{2}}} \le EU_\gamma^n \\ \le \;& P\Big(V_\gamma \in [a^{n+1}, a^{n+1} + a^{\epsilon(n-1)})\Big) s(a + a^{(\epsilon-1)n-\epsilon}) \\ + \;& P\Big(V_\gamma \in (a^{n-1} - a^{\epsilon(n-1)}, a^{n-1}]\Big) s(a^{-1}) + E\Big(U_\gamma 1_{\{d(V_\gamma,(a^{n-1},a^{n+1}) \ge a^{\epsilon(n-1)}\}}\Big). \end{aligned}$$

It follows from Lemma 8.5 that

$$\begin{aligned} E\Big(U_\gamma^n 1_{\{d(V_\gamma,(a^{n-1},a^{n+1}) \ge a^{\epsilon(n-1)}\}}\Big) \;&\le\; ||s||_\infty \Bigg(P^V\Bigg(dist(V_\gamma, (a^{n-1}, a^{n+1}) \ge a^{\epsilon(n-1)}\Bigg) \\ &\le\; K_9 \Bigg(e^{-c_{11}a^{(2\epsilon-1)n}} + H\Big(\lfloor a^{\epsilon(n-1)} \rfloor\Big)\Bigg). \end{aligned}$$

Since s is C^∞ Taylor expansion to the first order gives:

$$s(a + a^{(\epsilon-1)n-\epsilon}) = s(a) + a^{(\epsilon-1)n-\epsilon}s'(\theta) = a^\delta + \frac{K_{10}}{a^{(1-\epsilon)n}}.$$

Therefore,

$$U_0^n - K_{11}\left[\frac{1}{a^{(1-\epsilon)n}} + H\Big(\lfloor a^{\epsilon(n-1)} \rfloor\Big)\right] \le P\Big(V_\gamma \ge a^{n+1}\Big)a^\delta + P\Big(V_\gamma \le a^{n-1}\Big)a^{-\delta}.$$

Similarly we get:

$$P\Big(V_\gamma \geq a^{n+1}\Big)a^{\delta} + P\Big(V_\gamma \leq a^{n-1}\Big)a^{-\delta} \quad \leq \quad U_0^n + K_{12}\left[\frac{1}{a^{(1-\epsilon)n}} + H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\right].$$

and this ensures and complete the proof of

$$\left|P\Big(V_\gamma \leq a^{n-1}\Big) - \frac{a^{\delta}}{1+a^{\delta}}\right| \quad \leq \quad K_{13}\left[\frac{1}{a^{(1-\epsilon)n}} + H\Big(\lfloor a^{\epsilon(n-1)}\rfloor\Big)\right].$$

8.3 Main technical Lemma

Lemma 8.7. *([39, Lemma 5.3] and [19, Lemma A.1]] For each $a \in (1,2]$ there is an $l \in \mathbb{N}$ such that if $\ell, m, u, x \in \mathbb{N}$ satisfy $\ell_0 \leq \ell < m < u$ and $|x - a^m| \leq a^{\epsilon n}$ for all $\frac{1}{2} < \epsilon < 1$ then*

$$\frac{h_a^-(m) - h_a^-(\ell)}{h_a^-(u) - h_a^-(\ell)} \leq P_x^V\Big(\sigma_{a^\ell} > \tau_{a^u}\Big) \leq \frac{h_a^+(m) - h_a^+(\ell)}{h_a^+(u) - h_a^+(\ell)},$$

where for $j \geq 1$

$$h_a^{\pm}(i) = \begin{cases} \displaystyle\prod_{r=\ell+1}^{i} \Big(a^{\delta} \mp a^{-\lambda r}\Big), & \text{if } \delta > 0; \\ \displaystyle\prod_{r=\ell+1}^{i} \Big(a^{-\delta} \mp a^{-\lambda r}\Big)^{-1}, & \text{if } \delta < 0; \\ i \pm \dfrac{1}{i}, & \text{if } \delta = 0. \end{cases}$$

and λ is some positive number not depending on ℓ.

Remark 8.2. *When $\delta \neq 0$ and for a fixed ℓ there are $K_1(\ell)$ and $K_2(\ell)$ such that*

$$K_1(\ell) \leq \frac{h_a^{\pm}(i)}{a^{(i-\ell)\delta}} \leq K_2(\ell) \qquad \text{for all } i > \ell \quad \text{and} \quad K_j(\ell) \longrightarrow 1 \quad \text{as} \quad \ell \longrightarrow \infty, j = 1,2.$$

Proof. The idea of the proof is exactly the same as the one given in [39, Lemma 5.3], which is consisting of two steps. For the clarity and the completeness of the proof we shall give more details about these steps. To begin with, the first step is the same as in [39, Lemma 5.3] however, the second step is quite different in the estimate part that gives rise to the constraint of the (**TDE**) condition. In order to get the upper bound of this lemma we shall construct another process $\widetilde{V}$, which dominates our backward branching process V and whose exit time probabilities from the interval (a^{n-1}, a^{n+1}) behaves like an exit problem for a Birth-and-Death process. To this end, for $i \in \mathbb{N}$ set $x_i = [a^i + a^{\epsilon i}]$ for some $\epsilon \in (\frac{1}{2}, 1)$ and by the monotonicity property, it is enough to prove the upper bound when the starting point x is equal to x_m. Thus, we set $V_0 = x_m$. The first step is the construction of the process $\widetilde{V}$ that dominates V and the second step is a supermartingale argument to get the upper bound and a submartingale argument to get the lower bound.

- **Step 1**. We begin by constructing a sequence of stopping times $\gamma_i, i \geq 0$, and a dominating process $\widetilde{V} = (\widetilde{V}_k)_{k\geq 0}$ with an absorbing state ℓ at the point x_ℓ such that $\widetilde{V}_k \geq V_k$ for all k before the absorption. Define the successive exit times from the interval (a^{m-1}, a^{m+1}) γ_i by letting $\gamma_0 = 0, \gamma_i = \inf\left\{k > 0 : V_k \notin (a^{m-1}, a^{m+1})\right\}$ and $\widetilde{V}_k = V_k$ for $k = 0, 1, ..., \gamma_i - 1$. At time γ_1 add to V_{γ_1} the necessary number of particles in order to get

$$\widetilde{V}_{\gamma_i} = \begin{cases} x_{m-1}, & \text{if} \quad V_{\gamma_i} \leq a^{m-1} \\ x_{m+j}, & \text{if} \quad x_{m+j-1} < V_{\gamma_1} \leq x_{m+j}, j \in \mathbb{N}. \end{cases}$$

Clearly, it follows from this construction that,

$$\widetilde{V}_{\gamma_1} \geq V_{\gamma_1}, \quad V_{\gamma_i} = x_n \quad \text{for some} \quad n \geq m-1, n \neq m.$$

The process will be stopped once it reaches the absorbing state i.e $\widetilde{V}_{\gamma_i} = x_\ell$. Assume by induction hypothesis that we already have defined the stopping times $\gamma_r, r = 0, 1, ..., i$, and the process $\widetilde{V}_k$ for all $k \leq \gamma_i$ such that $\widetilde{V}_{\gamma_i} = x_n$ for some $n > \ell$. Then, We define $\widetilde{V}_k$ for $k > \gamma_i$ the same way we constructed our backward branching process V, namely

$$\widetilde{V}_{k+1} = \widetilde{V}_k + 1 + \sum_{m=1}^{\widetilde{V}_k+1} (\zeta_m^{(k)} - 1) \quad for \quad k \geq \gamma_i.$$

Denote by γ_{i+1} the first time after the stopping time γ_i when $\widetilde{V}$ exits the interval (a^{n-1}, a^{n+1}). At time γ_{i+1}, if the process exited through the lower end of the interval then we set $\widetilde{V}_{\gamma_{i+1}} = x_{n-1}$, if the process exited through the upper end we add to $\widetilde{V}$ the minimal number of particles needed to get $\widetilde{V}_{\gamma_{i+1}} = x_s$ for some $s > n$. If $\widetilde{V}_{\gamma_{i+1}} = x_\ell$, then we stop the process. Thus, we obtain a sequence of stopping times $\gamma_i, i \geq 0$, and the desired dominating process $\widetilde{V}$ absorbed at x_ℓ such that $\widetilde{V}_{\gamma_i} \in \{x_\ell, x_{\ell+1}, ...\}, i \geq 0$.

- **Step 2**. Define a Markov chain $R = (R_j)_{j\geq 0}$ on the set $\{\ell, \ell+1, ...\}$ by setting

$$\begin{aligned} R_j &= n \quad if \quad \widetilde{V}_{\gamma_j} = x_n, \quad j \geq 0. \\ \sigma_\ell^R &= \inf\left\{ j \geq 0 \middle| R_j = \ell \right\} \quad \text{and,} \\ \tau_u^R &= \inf\left\{ j \geq 0 \middle| R_j \geq u \right\} \end{aligned}$$

By construction we have

$$P_{x_m}^V\left(\sigma_{a^l}^V > \tau_{a^u}^V\right) \leq P_{x_m}^{\widetilde{V}}\left(\sigma_{x_\ell}^{\widetilde{V}} > \tau_{x_u}^{\widetilde{V}}\right) = P_m^R\left(\sigma_\ell^R > \tau_u^R\right).$$

- **Case 1** Suppose that $\delta > 0$. We shall show that $\left(h_a^+(R)\right)_{j\geq 0}$ is a supermartingale with respect to its natural filtration. We set $h_a^+(l) = 1$, the optimal stopping theorem and monotonicity of the function h_a^+ will immediately imply the upper bound in the statement of the lemma. For $i > \ell$ we have

$$E_i^R\Big(h_a^+(R_1)\Big) = h_a^+(i-1)P_i^R(R_1 = i-1)$$
$$+ \quad h_a^+(i+1)P_i^R(R_1 = i+1) + \sum_{n=i+2}^{\infty} h_a^+(n)P_i^R(R_1 = n)$$

By using the definition of the function h_a^+ this is less or equal to

$$h_a^+(i)\Bigg[\Big(a^\delta - a^{-\lambda i}\Big)^{-1} P_i^R(R_1 = i-1) + \Big(a^\delta - a^{-\lambda(i+1)}\Big)P_i^R\Big(R_1 = i+1\Big)$$
$$+ \quad \sum_{n=i+2}^{\infty} a^{\delta(n-i)}P_i^R\Big(R_1 = n\Big)\Bigg]. \tag{8.14}$$

It follows from lemma 8.6 and **TDE** that for all $i > \ell$ and ℓ is sufficiently large,

$$P_i^R(R_1 = i-1) = \frac{a^\delta}{a^\delta + 1} + O\Big(\frac{1}{a^{(1-\epsilon)i}}\Big)$$
$$P_i^R(R_1 = i+1) = \frac{1}{a^\delta + 1} + O\Big(\frac{1}{a^{(1-\epsilon)i}}\Big) \tag{8.15}$$
$$P_i^R(R_1 = n) \leq \quad P_i^R(R_1 \leq n) = O\Big(\frac{1}{a^{n\alpha}}\Big) \quad \text{for all} \quad n \geq i+1.$$

The last inequality is due to (8.2) of Lemma 8.3. Indeed for all $n \geq i+1$

$$P_i^R\Big(R_1 = n\Big) \leq P_i^R\Big(R_1 \geq n\Big) = P_i^R\Big(R_1 \geq n, \tau_n^R < \sigma_\ell^R\Big) \leq P_i^R\Big(R_{\tau_n^R} \geq n\Big|\tau_n^R < \sigma_\ell^R\Big)$$
$$= \quad P_{x_i}^R\Big(\widetilde{V}_{\tau_n^{\tilde{V}}} \geq x_n\Big|\tau_{x_n}^R < \sigma_{x_\ell}^R\Big) = P_{x_i}^R\Big(\widetilde{V}_{\tau_n^{\tilde{V}}} \geq x_{i+1} + x_n - x_{i+1}\Big|\tau_{x_n}^R < \sigma_{x_\ell}^R\Big)$$
$$\leq \quad c_8\Bigg[e^{-c_9\frac{(x_n - x_{i+1})^2}{x_i}} + e^{-c_9(x_n - x_{i+1})} + H\Big(\lfloor\frac{x_n - x_{i+1}}{4}\rfloor\Big)\Bigg]$$
$$\leq \quad c_8\Bigg[\Big(\mathbf{e}^{-c_9\frac{(a^n - a^{i+1})^2}{x_i}} + e^{-c_9(a^n - a^{i+1})} + H\Big(\lfloor\frac{a^n - a^{i+1}}{4}\rfloor\Big)\Bigg]$$
$$\leq \quad \frac{K_{14}}{(a^n - a^{i+1})^\alpha}.$$

It follows that

$$\begin{aligned}\sum_{n=i+2}^{\infty} a^{\delta(n-i)} P_i^R\Big(R_1 = n\Big) &\leq \sum_{n=i+2}^{\infty} a^{\delta(n-i)} \frac{K_{15}}{a^{(i+1)\alpha}(a^{n-i-1}-1)^{\alpha}} \\ = \quad \frac{K_{15}}{a^{(\delta+\alpha)i+\alpha}} \sum_{n=1}^{\infty} \frac{a^{\delta n}}{(a^n-1)^{\alpha}} &< \infty \quad \text{if and only if} \quad \alpha > \delta.\end{aligned}$$

Using (8.14) by substituting in (8.13) and some elementary algebra we obtain

$$E_i^R\Big(h_a^+(R_1)\Big) \leq h_a^+(i)\left[1 - \frac{a^{-\lambda i}}{a^{\delta}+1}\Big(a^{-\lambda} - a^{-\delta}\Big) + O(a^{-2\lambda i})\right] \leq h_a^+(i). \quad (8.16)$$

provided that $\lambda \leq \inf\left\{\frac{1-\epsilon}{2}, \delta\right\}$ and ℓ is sufficiently large. Hence $h_a^+(R_j)$ is a super-martingale and by the optimal stopping time we have

$$\begin{aligned}E_m^R\Big(R_{\sigma_\ell^R \wedge \tau_u^R}\Big) &= h_a^+(u) P_m^R(\sigma_l^R > \tau_u^R) + h_a^+(\ell) P_m^R(\sigma_l^R < \tau_u^R) \\ &= (h_a^+(u) - 1) P_m^R(\sigma_l^R > \tau_u^R) \leq (h_a^+(m) - 1).\end{aligned}$$

For the lower bound of this lemma we argue in similar fashion as in step 1 by constructing a process $\widetilde{V}$ that will be dominated by our backward branching process V. Indeed, for $i \in \mathbb{N}$ set $x_i = [a^i - a^{\epsilon i}] + 1$ for some $\epsilon \in (\frac{1}{2}, 1)$ and using the monotonicity property, it is enough to prove the lower bound when the starting point x is equal to x_m. That is, we set $V_0 = x_m = [a^m - a^{\epsilon m}] + 1$. We begin by constructing a sequence of stopping times $\gamma_i, i \geq 0$, and a dominated process $\widetilde{V} = (\widetilde{V}_k)_{k\geq 0}$ with an absorbing state ℓ at x_ℓ such that $\widetilde{V}_k \leq V_k$ for all k before the absorption. Define the successive exit times from the interval (a^{m-1}, a^{m+1}) γ_i by setting $\gamma_0 = 0$

$$\gamma_i = \inf\left\{k > 0 \Big| V_k \notin (a^{m-1}, a^{m+1})\right\}, \qquad \widetilde{V}_k = V_k \text{ for } k = 0, 1, ..., \gamma_i - 1,$$

and at time γ_1 remove from V_{γ_1} the necessary number of particles in order to get

$$\widetilde{V}_{\gamma_i} = \begin{cases} x_{m+1}, & \text{if} \quad V_{\gamma_i} \geq a^{m+1} \\ x_{m+j}, & \text{if} \quad x_{m+j} \leq V_{\gamma_1} < x_{m+j+1}, j \in \mathbb{N}. \end{cases}$$

Clearly, it follows from this construction that,

$$\widetilde{V}_{\gamma_1} \leq V_{\gamma_1}, \quad V_{\gamma_i} = x_n \quad \text{for some} \quad n \geq m+1, n \neq m$$

The process will be stopped once it reaches the absorbing state i.e. $\widetilde{V}_{\gamma_1} = x_\ell$. Assume by induction hypothesis that we already have defined the stopping times $\gamma_r, r = 0, 1, ..., i$, and the process $\widetilde{V}_k$ for all $k \leq \gamma_i$ such that $\widetilde{V}_{\gamma_i} = x_n$ for some $n > \ell$. Then, We define $\widetilde{V}_k$ for $k > \gamma_i$ the same way as we constructed our backward branching process V, namely

$$\widetilde{V}_{k+1} = \widetilde{V}_k + 1 + \sum_{m=1}^{\widetilde{V}_k+1} (\zeta_m^{(k)} - 1) \quad for \quad k \geq \gamma_i.$$

Denote by γ_{i+1} the first time after the stopping time γ_i when $\widetilde{V}$ exits the interval (a^{n-1}, a^{n+1}). At time γ_{i+1}, if the process exited through the upper end of the interval then we set $\widetilde{V}_{\gamma_{i+1}} = x_{n+1}$, if the process exited through the lower end we reduce the number of particles by removing the minimal number of particles to ensure that $\widetilde{V}_{\gamma_{i+1}} = x_s$ for some $s < n$. If $\widetilde{V}_{\gamma_{i+1}} \leq x_\ell$, then we stop the process by redefining $\widetilde{V}_{\gamma_{i+1}} = x_\ell$. Thus, we obtain a sequence of stopping times $\gamma_i, i \geq 0$, and the desired dominated process $\widetilde{V}$ absorbed at x_ℓ such that $\widetilde{V}_{\gamma_i} \in \{x_\ell, x_{\ell+1}, ...\}, i \geq 0$.

Define the Markov chain $R = (R_j)_{j \geq 0}$ on the set $\{\ell, \ell+1, ...\}$ by setting $R_j = n$ if $\widetilde{V}_{\gamma_j} = x_n, \quad j \geq 0$. Let

$$\sigma_\ell^R = \inf\Big\{j \geq 0 \Big| R_j = \ell\Big\} \quad \text{and} \quad \tau_u^R = \inf\Big\{j \geq 0 \Big| R_j \geq u\Big\}.$$

By construction we have,

$$P^V_{x_m}\left(\sigma^V_{a^\ell} > \tau^V_{a^u}\right) \geq P^{\widetilde{V}}_{x_m}\left(\sigma^{\widetilde{V}}_{x_l} > \tau^{\widetilde{V}}_{x_u}\right) = P^R_m\left(\sigma^R_\ell > \tau^R_u\right).$$

We will show that $\left(h_a^-(R)\right)_{j\geq 0}$ is a submartingale with respect to its natural filtration. We set $h_a^-(\ell) = 1$, the optimal stopping theorem and monotonicity of the function h_a^- will immediately imply the lower bound in the statement of the lemma . That is, for $i > \ell$ we have

$$\begin{aligned} E^R_i\left(h_a^-(R_1)\right) &= h_a^-(i-1)P^R_i(R_1 = i-1) + h_a^-(i+1)P^R_i(R_1 = i+1) \\ &+ \sum_{n=i+2}^{\infty} h_a^-(n)P^R_i(R_1 = n). \end{aligned} \tag{8.17}$$

By using the definition of the function h_a^- this is greater or equal to

$$\begin{aligned} & h_a^-(i)\Bigg[\left(a^\delta + a^{-\lambda i}\right)^{-1}P^R_i(R_1 = i-1) + \left(a^\delta + a^{-\lambda(i+1)}\right)P^R_i(R_1 = i+1) \\ + & \sum_{n=i+2}^{\infty} a^{\delta(n-i)}P^R_i(R_1 = n)\Bigg]. \end{aligned} \tag{8.18}$$

Using (8.15) and substituting into (8.18) we obtain

$$\mathbb{E}^R_i\left(h_a^-(R_1)\right) \geq h_a^-(i)\left[1 + \frac{a^{-\lambda i}}{a^\delta + 1}\left(a^{-\lambda} - a^{-\delta}\right) + O(a^{-2\lambda i})\right] \geq h_a^-(i), \tag{8.19}$$

provided that $\lambda \leq \inf\left\{\frac{(1-\epsilon)}{2}, \delta\right\}$, $\alpha > \delta$ and ℓ is sufficiently large. Hence $h_a^-(R_j)$ is a submartingale and by the optimal stopping time

$$E^R_m\left(R_{\sigma^R_\ell \wedge \tau^R_u}\right) \;\; \geq \;\; h_a^-(m).$$

and similar computations as before we get the lower bound.

- **Case2** Suppose that $\delta < 0$. The proof is exactly the same by using the function

$$h_a^{\pm}(i) \;=\; \prod_{r=\ell+1}^{i} \Big(a^{-\delta} \mp a^{-\lambda r}\Big)^{-1}$$

The same construction as in **step 1** in the case of $\delta > 0$ and for $i > \ell$ we have

$$\begin{aligned} E_i^R\Big(h_a^+(R_1)\Big) &= h_a^+(i-1)P_i^R(R_1 = i-1) + h_a^+(i+1)P_i^R(R_1 = i+1) \\ &+ \sum_{n=i+2}^{\infty} h_a^+(n)P_i^R(R_1 = n) \end{aligned} \tag{8.20}$$

By definition of the function h_a^+ this is less or equal to

$$\begin{aligned} & h_a^+(i)\Bigg[\Big(a^{-\delta} - a^{-\lambda i}\Big)P_i^R(R_1 = i-1) + \Big(a^{-\delta} - a^{-\lambda(i+1)}\Big)^{-1} P_i^R\Big(R_1 = i+1\Big) \\ + & \sum_{n=i+2}^{\infty} K_2(i)a^{\delta(n-i)}P_i^R\Big(R_1 = n\Big)\Bigg]. \end{aligned} \tag{8.21}$$

Using (8.15) by substituting in (8.21) and some elementary computations we obtain

$$E_i^R\Big(h_a^+(R_1)\Big) \le h_a^+(i)\left[1 - \frac{a^{-\lambda i+\delta}}{a^{\delta}+1}\Big(1 - a^{-\lambda}\Big) + O(a^{-2\lambda i})\right] \le h_a^+(i).$$

provided that $\lambda \le \frac{1-\epsilon}{2}$ and ℓ is sufficiently large. Thus $h_a^+(R_j)$ is a supermartingale and by the optimal stopping time theorem we get the upper bound.

For the lower bound we proceed similarly as in **step2** by showing that $\Big(h_a^-(R)\Big)_{j\ge 0}$ is a submartingale with respect to its natural filtration.(We set $h_a^-(\ell) = 1$). The optimal stopping theorem and monotonicity of the function h_a^- will immediately imply the lower bound in the statement of the lemma. That is, for $i > \ell$ we have

$$E_i^R\Big(h_a^-(R_1)\Big) \;=\; h_a^-(i-1)P_i^R(R_1 = i-1) + h_a^-(i+1)P_i^R(R_1 = i+1) + \sum_{n=i+2}^{\infty} h_a^-(n)P_i^R(R_1 = n).$$

By using the definition of the function h_a^- this is greater or equal to

$$h_a^-(i)\Bigg[\Big(a^{-\delta}+a^{-\lambda i}\Big)P_i^R(R_1 = i-1) + \Big(a^{-\delta}+a^{-\lambda(i+1)}\Big)^{-1}P_i^R(R_1 = i+1) + \sum_{n=i+2}^{\infty} K_1(i)a^{\delta(n-i)}P_i^R(R_1 = n)\Bigg]. \tag{8.22}$$

Using (8.15) by substituting in (8.22) we obtain

$$\mathbb{E}_i^R\Big(h_a^-(R_1)\Big) \geq h_a^-(i)\left[1+\frac{a^{-\lambda i+\delta}}{a^{\delta}+1}\Big(1-a^{-\lambda}\Big)+O(a^{-2\lambda i})\right] \geq h_a^-(i), \tag{8.23}$$

provided that $\lambda \leq \frac{(1-\epsilon)}{2}$ and ℓ is sufficiently large. Hence $h_a^+(R_j)$ is a supermartingale and by the optimal stopping time theorem

$$E_m^R\Big(R_{\sigma_\ell^R\wedge\tau_u^R}\Big) \;\geq\; h_a^-(m).$$

and similar computations as before we get the lower bound.

- **Case 2 when** $\delta = 0$. This proof of this case was was done in [19] by considering the forward branching process associated to our excited random walk in the recurrent regime when $\delta = 1$ and reference therein.

8.4 Proof of Theorems 3.1 and 3.2

The main tools for the proof either of theorem 3.1 or theorem 3.2 is the main lemma and the properties of our diffusion approximation. By symmetry of our model we will sketch the proof only of theorem 3.2.

- **Case** $\delta < 0$. The proof is exactly the same as in [[42], section 6, 6.4].
- **Case** $\delta > 0$ It suffices to substitute the lemmas of the above sections and results from the diffusion approximation in [39,46] and repeat the proof given in the paper for $\delta > 0$.
- **Case** $\delta = 0$. The proof is identical to the one given in [[19] , Theorem 2.1].

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