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A DIFFERENTIAL ALGEBRA APPROACH TO COMMUTING POLYNOMIAL VECTOR FIELDS AND TO PARAMETER IDENTIFIABILITY IN ODE MODELS

by

PETER A. THOMPSON

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2019

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This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

A DIFFERENTIAL ALGEBRA APPROACH TO COMMUTING POLYNOMIAL VECTOR FIELDS AND TO PARAMETER IDENTIFIABILITY IN ODE MODELS

by

PETER A. THOMPSON

Adviser: Professor Alexey Ovchinnikov

In the first part, we study the problem of characterizing polynomial vector fields that commute with a given polynomial vector field. One motivating factor is that we can write down solution formulas for an ODE that corresponds to a planar vector field that possesses a linearly independent commuting vector field. This problem is also central to the question of linearizability of vector fields. We first show that a linear vector field admits a full complement of commuting vector fields. Then we study a type of planar vector field for which there exists an upper bound on the degree of a commuting polynomial vector field. Finally, we turn our attention to conservative Newton systems, which form a special class of Hamiltonian systems, and show the following result. Let $f \in K[x]$, where K is a field of characteristic zero, and d the derivation that corresponds to the differential equation $\ddot{x} = f(x)$ in a standard way. We show that if $\deg f \geqslant 2$, then any K-derivation commuting with d is equal to d multiplied by a conserved quantity. For example, the classical elliptic equation $\ddot{x} = 6x^2 + a$, where $a \in \mathbb{C}$, falls into this category.

In the second part, we study structural identifiability of parameterized ordinary differential equation models of physical systems, for example, systems arising in biology and medicine. A parameter is said to be structurally identifiable if its numerical value can be determined from perfect observation of the observable variables in the model. Structural identifiability is necessary for practical identifiability. We study structural identifiability via differential algebra. In particular, we use characteristic sets. A system of ODEs can be viewed as a set of differential polynomials in a

differential ring, and the consequences of this system form a differential ideal. This differential ideal can be described by a finite set of differential equations called a characteristic set. The technique of studying identifiability via a set of special equations, sometimes called “input-output” equations, has been in use for the past thirty years. However it is still a challenge to provide rigorous justification for some conclusions that have been drawn in published studies. Our main result is on linear systems, which are a topic of current interest. We show that for a linear system of ODEs with one output, the coefficients of a monic characteristic set are identifiable. This result is then generalized, with additional hypotheses, to nonlinear systems.

Acknowledgments

The results of chapter Chapter 1 are joint work with Alexey Ovchinnikov and Joel Nagloo. The results of Chapter 2 are joint work with Alexey Ovchinnikov and Gleb Pogudin.

This work was partially supported by the NSF grants CCF-1563942, CCF-0952591, DMS-1700336, DMS-1606334, and DMS-1760448, by the NSA grant #H98230-15-1-0245, by CUNY CIRG #2248, and by PSC-CUNY grants #69827-00 47, #60456-00 48, and #60098-00 48. I am grateful to the CCiS at CUNY Queens College for the computational resources.

Contents

Chapter 1

Commuting polynomial vector fields

1.1 Introduction

We study the problem of characterizing polynomial vector fields that commute with a given polynomial vector field. One motivating factor is that we can write down solution formulas for an ODE that corresponds to a planar vector field that possesses a linearly independent (transversal) commuting vector field (see Theorem 1.2.1). This problem is also central to the question of linearizability of vectors fields (cf. Giné and Grau (2006) and Sabatini (1997)). In what follows, we will use the standard correspondence between (polynomial) vector fields and derivations on (polynomial) rings.

In Section 1.3, we show that a K-derivation on $K[x_1,\ldots,x_n]$ defined by linear polynomials admits a full complement of commuting K-linearly independent K-derivations. In Section 1.4, we prove a degree bound on the degree of any derivation commuting with a K-derivation on $K[x,y]$ of the form

$$d = f_1 \cdot \frac{\partial}{\partial x} + f_2 \cdot \frac{\partial}{\partial y}$$

satisfying $f_1 f_2 \neq 0$, $\deg_y \dfrac{\partial f_2}{\partial x} < \deg_y f_2$, $\deg_y(y f_1) < \deg_y f_2$, $\deg_x \dfrac{\partial f_1}{\partial y} < \deg_x f_1$, and $\deg_x(x f_2) <$

$\deg_x f_1$. In Section 1.5, we show that a nonlinear planar polynomial derivation corresponding to a conservative Newton system does not admit a linearly independent commuting derivation. Let

$$d = y\frac{\partial}{\partial x} + f(x)\frac{\partial}{\partial y} \tag{1.1}$$

be a K-derivation, where f is a polynomial with coefficients in a field K of zero characteristic. This derivation corresponds to a conservative Newton system, and so to the differential equation $\ddot{x} = f(x)$. Observe that d is a special type of Hamiltonian derivation. That is, $d(x) = \frac{\partial H}{\partial y}$ and $d(y) = -\frac{\partial H}{\partial x}$, where $H = \frac{1}{2}y^2 - \int f(x)dx$. It is shown in (Nowicki, 1994, Corollary 7.1.5) that the set of all polynomial derivations that commute with d forms a $K[H]$-module. In this paper, we show that, for every such d, the module M_d is of rank 1 if and only if $\deg f \geqslant 2$. For example, the classical elliptic equation $\ddot{x} = 6x^2 + a$, where $a \in \mathbb{C}$, falls into this category.

A characterization of commuting planar derivations in terms of a common Darboux polynomial is given in (Petravchuk (2010)). This was generalized to higher dimensions in (Li and Du (2012)). In (Choudhury and Guha (2013)), Darboux polynomials are used to find linearly independent commuting vector fields and to construct linearizations of the vector fields. In the case in which K is the real numbers, our result generalizes a result on conservative Newton systems with a center to the case in which a center may or may not be present. A vector field has a center at point P if there is a punctured neighborhood of P in which every solution curve is a closed loop. A center is called isochronous if every such loop has the same period. It was proven in (Villarini, 1992, Theorem 4.5) that, if D_1 and D_2 are commuting vector fields orthogonal at noncritical points, then any center of D_1 is isochronous. The hypothesis of this result can be relaxed to the case in which D_2 is transversal to D_1 at noncritical points (cf. (Sabatini, 1997, Theorem, p. 92)). In light of this result, one approach to showing the nonexistence of a vector field commuting with D is to show that D has a non-isochronous center. In fact, Amel'kin (Amel'kin, 1977, Theorem 11) has shown that if the system of ordinary differential equations (ODEs) corresponding to derivation (1.1) is not linear and has a center at the origin, then there is no transversal vector field that commutes with d.

As far as we are aware, there has not been a standard method to show the nonexistence of a transversal polynomial vector field in the absence of a nonisochronous center. We develop our own method to do this, which includes building a triangular system of differential equations. One technique we use in approaching this system involves constructing a family of pairs of commuting derivations on rings of the form $K[x^{1/t}, x^{-1/t}, y]$ (see Lemma 1.5.7) and using recurrence relations.

It is impossible to remove the condition $\deg f \geqslant 2$ from the statement of our main result, as every non-zero derivation of degree less than 2 commutes with another transversal derivation (see Proposition 1.2.1). The form of d in our main result implies that d is divergence free (which is the same as Hamiltonian in the planar case). It is not possible to strengthen our result to the case in which d is merely assumed to be divergence free of degree at least 2, as shown in Example 1.2.1 and Proposition 1.2.2.

1.2 Basic terminology and related results

We direct the reader to Kaplansky (1957) and Kolchin (1973) for the basics of a ring with a derivation.

Definition 1.2.1. An S-derivation on a commutative ring R with subring S is a map $d\colon R \to R$ such that $d(S) = 0$ and for all a, $b \in R$,

$$d(a+b) = d(a) + d(b) \quad \text{and} \quad d(ab) = d(a) \cdot b + a \cdot d(b).$$

Definition 1.2.2. Let K be a field. A non-zero K-derivation d on $K[x_1, \dots, x_n]$ is called *integrable* if there exist commuting K-derivations $\delta_1, \dots, \delta_{n-1}$ on $K[x_1, \dots, x_n]$ that are linearly independent from

d over $K(x_1,\ldots,x_n)$, and commute with d, that is, for all $a \in K[x_1,\ldots,x_n]$ and i,j, $1 \leqslant i,j \leqslant n-1$,

$$d(\delta_i(a)) = \delta_i(d(a)) \quad \text{and} \quad \delta_i(\delta_j(a)) = \delta_j(\delta_i(a)).$$

The following is a result that follows easily from classical theory, although to the best of our knowledge it is not explicitly stated in this form.

Theorem 1.2.1. *Let d and δ be $\mathbb{R}$-derivations on $\mathbb{R}(x,y)$ defined by*

$$d(x) = f_1(x,y),\ d(y) = f_2(x,y),\ \delta(x) = g_1(x,y),\ \delta(y) = g_2(x,y).$$

Let $(x_0,y_0) \in \mathbb{R}^2$. Suppose that d and δ commute and there is no $(\lambda_1,\lambda_2) \in \mathbb{R}^2 \backslash \{(0,0)\}$ such that

$$\lambda_1 \begin{pmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{pmatrix} = \lambda_2 \begin{pmatrix} g_1(x_0,y_0) \\ g_2(x_0,y_0) \end{pmatrix}.$$

Then the initial value problem

$$\dot{x} = f_1(x,y),\ \dot{y} = f_2(x,y),\ x(0) = x_0,\ y(0) = y_0$$

has a solution given by

$$(x(t),y(t)) = F^{-1}(t,0),$$

where

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{x_0}^{x} \frac{g_2(r,y)}{\Delta(r,y)}dr + \int_{y_0}^{y} \frac{-g_1(x_0,s)}{\Delta(x_0,s)}ds \\ \displaystyle\int_{x_0}^{x} \frac{-f_2(r,y)}{\Delta(r,y)}dr + \int_{y_0}^{y} \frac{f_1(x_0,s)}{\Delta(x_0,s)}ds \end{pmatrix},$$

and $\Delta(x,y) = f_1(x,y)g_2(x,y) - f_2(x,y)g_1(x,y)$.

Proof. Suppose $(x(t),y(t))$ is a solution to the initial value problem. A straightforward calculation shows that $F(x(t),y(t)) = (t,0)$. Observing that the Jacobian determinant of F does not vanish at (x_0,y_0), we see that F is a diffeomorphism in a neighborhood of (x_0,y_0). We conclude that $(x(t),y(t)) = F^{-1}(t,0)$. □

Example 1.2.1. Consider the initial value problem

$$\dot{x} = 1+x^2, \quad \dot{y} = -2xy, \quad x(0) = x_0, \quad y(0) = y_0,$$

where x_0 and y_0 are real numbers and $y_0 \neq 0$. The corresponding derivation is

$$d(x) = 1+x^2, \quad d(y) = -2xy,$$

and we observe that the derivation

$$\delta(x) = 0, \quad \delta(y) = y$$

commutes with d, and that d and δ are independent at (x_0,y_0). Using the above formula, we obtain the solution

$$x(t) = \tan(t+\tan^{-1}x_0), \quad y(t) = y_0(1+x_0^2)\cos^2(t+\tan^{-1}x_0).$$

We make some observations, in the form of the following propositions:

Proposition 1.2.1. *Let K be a field. Every non-zero K-derivation of degree less than or equal to* 1 *on $K[x,y]$ is integrable.*

A proof for n variables is given in 1.3.1. We give a more explicit proof for the case of 2 variables here.

Proof. We will consider the following cases. The symbols a, b, c, e, f, and g are taken to be elements of K.

Case 0 : $d(x) = c, \quad d(y) = g$. Observe that d commutes with any constant derivation.

Case 1 : $d(x) = ax, \quad d(y) = ay, \quad a \neq 0$. Observe that d commutes with δ, where $\delta(x) = y$, $\delta(y) = x$.

Case 2 : $d(x) = ax + by, \quad d(y) = ex + fy,$ different from Case 1. Observe that d commutes with δ, where $\delta(x) = x, \quad \delta(y) = y$.

Case 3 : $d(x) = ax + by + c, \quad d(y) = ex + fy + g, \quad af - be \neq 0$. In this case, d is equivalent to a derivation from Case 1 or Case 2 via a linear change of coordinates. Let (x_0, y_0) be the solution to the system $ax + by + c = ex + fy + g = 0$. Now let $u = x - x_0$ and $v = y - y_0$, so that $d(u) = au + bv$ and $d(v) = eu + fv$.

Case 4 : $d(x) = ax + by + c, \quad d(y) = ex + fy + g, \quad af - be = 0$

(a) $a = b = 0$, different from Case 0. If $e \neq 0$, then d commutes with and is transversal to δ given by $\delta(x) = -\frac{g}{e}$, $\delta(y) = 0$. If $f \neq 0$, then d commutes with and is transversal to δ given by $\delta(x) = 0$, $\delta(y) = -\frac{g}{f}$.

(b) at least one of a and b is not 0. First assume $a \neq 0$. If $f = e = 0$, then this is equivalent to Case 4a by swapping the roles of x and y. Assume at least one of f and e is not 0. By the condition $af - be = 0$, it must be that $e \neq 0$. Using the coordinate $z = ex - ay$ instead of x puts this into the form of Case 4a. Next, assume $b \neq 0$. If $f = e = 0$, then this is equivalent to Case 4a. Assume at least one of f and e is not 0. By the condition $af - be = 0$, it must be that $f \neq 0$. Using the coordinate $z = fx - by$ instead of x puts this into the form of Case 4a. □

Definition 1.2.3. Let K be a field and let d be a K-derivation on $K[x_1, \dots, x_n]$. We say d is *divergence-free* if

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} d(x_i) = 0.$$

Proposition 1.2.2. *Let K be a field of characteristic 0. There exist integrable divergence-free K-derivations on $K[x,y]$ that are not coordinate-change equivalent to a derivation of degree less than or equal to 1.*

Proof. The K-derivation defined by the same equations as d from Example 1.2.1 is divergence-free and integrable. Note that the vector field corresponding to d vanishes only at the points $(\sqrt{-1},0)$ and $(-\sqrt{-1},0)$ in $\overline{K}^2$. Since $\mathrm{char} K = 0$, these points are distinct. After a coordinate change, the number of points in $\overline{K}^2$ at which a vector field vanishes does not change. The vector field of any derivation of degree less than or equal to 1 vanishes at zero, one, or infinitely many points. We conclude that d is not coordinate-change equivalent to a derivation of degree no greater than 1. □

1.3 The linear case

We show in Proposition 1.3.1 that every nonzero K-derivation defined by polynomials of degree no greater than 1 on $K[x_1,...,x_n]$ is integrable (see Definition 1.2.2). We will make use of the following lemma.

Lemma 1.3.1. *Let K be a field. Let ∂ be a non-zero K-derivation on the polynomial ring $K[x_1,\dots,x_n]$ such that*

$$\partial(x) = Cx + a,$$

where $x = (x_1,\dots,x_n)^T$, C is the companion matrix of a polynomial over K of degree n, and a is an $n\times 1$ matrix with entries in K. Then there exist K-derivations $\delta_2,\dots,\delta_n$ such that

1. *$\forall i,j$ $\delta_i(x_j)$ has degree at most 1,*
2. *$\forall i$ $\delta_i \circ \partial = \partial \circ \delta_i$,*
3. *$\forall i,j$ $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, and*
4. *$\{\partial,\delta_2,\dots,\delta_n\}$ is K-linearly independent.*

Proof. Write

$$C = \begin{pmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & \ddots & \ddots & \vdots \\ & & 1 & c_{n-1} \end{pmatrix}, \qquad a = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

Case 1: $a_0 = 0$ or $c_0 \neq 0$

If $c_0 \neq 0$, let $v = C^{-1}a$. If $c_0 = 0$ let $v = (a_1, a_2, \ldots, a_{n-1}, 0)^T$. Observe that in either case, $Cv = a$. Now for $i = 0, \ldots, n-1$ define δ_i to be the K-derivation given by

$$\delta_i(x) = C^i x + C^i v$$

and note that $\partial = \delta_1$.

We first show that for all i and j $\delta_i \circ \delta_j = \delta_j \circ \delta_i$. We have $\delta_i(\delta_j(x)) = \delta_i(C^j x + C^j v) = C^j(C^i x + C^i v) = C^{i+j}x + C^{i+j}v$. We also have $\delta_j(\delta_i(x)) = \delta_j(C^i x + C^i v) = C^i(C^j x + C^j v) = C^{i+j}x + C^{i+j}v$.

We now show that $\{\delta_0, \ldots, \delta_{n-1}\}$ is K-linearly independent. Suppose $C^0x, Cx, \ldots, C^{n-1}x$ are not K-linearly independent. Then there exist $b_0, \ldots, b_{n-1} \in K$ not all 0 such that $b_0C^0x + \ldots b_{n-1}C^{n-1}x = (b_0C^0 + \ldots + b_{n-1}C^{n-1})x = (0, \ldots, 0)^T$. Since $x_1, \ldots, x_n$ are algebraically independent over K, the only way this could happen is if $b_0C^0 + \ldots + b_{n-1}C^{n-1}$ is the zero matrix. Since C is a companion matrix of a degree n polynomial, the minimal polynomial of C has degree n (cf. (Hoffman and Kunze, 1971, Corollary, p. 230)). Therefore $b_0 = \ldots = b_{n-1} = 0$. We conclude that $\{C^0x, \ldots, C^{n-1}x\}$ is K-linearly independent. It follows that $\{C^0x + C^0v, \ldots, C^{n-1}x + C^{n-1}v\}$ is K-linearly independent.

Define δ_n to be δ_0. Now we have shown that $\{\delta_2, \ldots, \delta_n\}$ satisfy the properties in the statement of the lemma.

Case 2: $a_0 \neq 0$ and $c_0 = 0$

For $i = 1,\ldots,n$ let δ_i be the K-derivation defined by

$$\delta_i(x) = C^i x + C^{i-1} a$$

and note that $\delta_1 = \partial$.

We show that for all i and j $\delta_i \circ \delta_j = \delta_j \circ \delta_i$. We have $\delta_i(\delta_j(x)) = \delta_i(C^j x + C^{j-1}a) = C^j(C^i x + C^{i-1}a) = C^{i+j}x + C^{i+j-1}a$. We also have $\delta_j(\delta_i(x)) = \delta_j(C^i x + C^{i-1}a) = C^i(C^j x + C^{j-1}a) = C^{i+j}x + C^{i+j-1}a$.

Next we show that the set $\{\delta_1,\ldots,\delta_n\}$ is K-linearly independent. Suppose $(b_1,\ldots,b_n) \in K^n \backslash \{(0,\ldots,0)\}$ is such that

$$b_1(Cx+a) + b_2(C^2x+Ca) + \ldots + b_n(C^n x + C^{n-1}a) = 0^{n\times 1}. \tag{1.2}$$

It follows that

$$b_1 Cx + b_2 C^2 x + \ldots + b_n C^n x = 0^{n\times 1}.$$

Since $x_1,\ldots,x_n$ are K-algebraically independent, and hence K-linearly independent, it follows that $b_1 C + \ldots + b_n C^n = 0^{n\times n}$. Since C is a companion matrix and $c_0 = 0$, the minimal polynomial of C is $p(X) = X^n - c_{n-1}X^{n-1} - \ldots - c_1 X$. Hence there exists $r \in K\backslash\{0\}$ such that $b_n = r$ and for $i = 1,\ldots,n-1$ $b_i = -rc_i$. It follows from this and (1.2) that

$$(-c_1 I - c_2 C - \ldots + C^{n-1})a = 0^{n\times 1},$$

where I is the $n\times n$ identity matrix. Let $D = -c_1 I - c_2 C - \ldots + C^{n-1}$. Since CD is the 0 matrix, we see that the image of D, as a K-linear map from $K^{n\times 1}$ to $K^{n\times 1}$, lies in the kernel of C. Observe that since $c_0 = 0$, the kernel of C has dimension 1. Because D is a K-linear combination of $C^0,\ldots,C^{n-1}$, D is not the zero matrix. Hence, the image of D has positive dimension and thus the image of D

has dimension 1. Therefore the kernel of D has dimension $n-1$. Let $e_1,\ldots,e_n$ be the basis for $K^{n\times 1}$ where e_i has 1 in the i-th entry and 0 elsewhere. Observe that since the first column of C^i has a 1 in the $(i+1)$-th entry and 0 in all other entries, $De_1 = (-c_1,-c_2,\ldots,-c_{n-1},1)^T \neq 0^{n\times 1}$. We now argue that for $i=2,\ldots,n$ $De_i = 0^{n\times 1}$. To do this, we work over the field $L := K(\tilde{c}_1,\ldots,\tilde{c}_{n-1})$, where $\tilde{c}_1,\ldots,\tilde{c}_{n-1}$ are K-algebraically independent, and consider the matrices $\tilde{C}$ defined as the companion matrix of $X^n - \tilde{c}_{n-1}X^{n-1} - \ldots - \tilde{c}_1 X$, and $\tilde{D} := -\tilde{c}_1 I - \tilde{c}_2\tilde{C} - \ldots + \tilde{C}^{n-1}$. Viewing $\tilde{C}$ and $\tilde{D}$ as L-linear maps on L^n, we have that $\ker\tilde{C}$ is the L-span of $(-\tilde{c}_1,-\tilde{c}_2,\ldots,-\tilde{c}_{n-1},1)^T$ and that $\operatorname{im}\tilde{D} = \ker\tilde{C}$. Thus, each column of $\tilde{D}$ is of the form $(-\tilde{c}_1 r,-\tilde{c}_2 r,\ldots,-\tilde{c}_{n-1}r,r)^T$, where $r\in L$. Since for $i\geq 1$ each element of the top row of $\tilde{C}^i$ is 0, we see that the top row of $\tilde{D}$ is $(-\tilde{c}_1,0,\ldots,0)$. Thus, we have

$$\tilde{D} = \begin{pmatrix} -\tilde{c}_1 & 0 & \cdots & 0 \\ -\tilde{c}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Observing that D is the specialization of $\tilde{D}$ at $\tilde{c}_1 = c_1,\ldots,\tilde{c}_{n-1} = c_{n-1}$ gives us

$$D = \begin{pmatrix} -c_1 & 0 & \cdots & 0 \\ -c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

and therefore $De_i = 0^{n\times 1}$ for $i>1$. Writing $a = a_0e_1 + a_1e_2 + \ldots + a_{n-1}e_n$ and recalling that $a_0\neq 0$, we see that $Da \neq 0^{n\times 1}$. This contradicts that (1.2) holds. Therefore $\{\delta_1,\ldots,\delta_n\}$ is K-linearly independent.

□

Proposition 1.3.1. *Let K be a field. Let ∂ be a non-zero K-derivation on the polynomial ring*

$R = K[x_1,...,x_n]$ such that each $\partial(x_i)$ has degree at most 1. Then there exist K-derivations $\delta_2,...,\delta_n$ on R such that

1. *$\forall i, j$ $\delta_i(x_j)$ has degree at most 1,*
2. *$\forall i$ $\delta_i \circ \partial = \partial \circ \delta_i$,*
3. *$\forall i, j$ $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, and*
4. *$\{\partial, \delta_2, \ldots, \delta_n\}$ is K-linearly independent.*

Proof. Write

$$\partial x = Ax + a,$$

where $A \in K^{n\times n}$ and $a \in K^{n\times 1}$. First, we show that without loss of generality we can assume A is in rational canonical form. By (Hungerford, 1974, Theorem 4.6(ii), p. 360), there exists $P \in GL_n(K)$ such that $\hat{A} = PAP^{-1}$ is in rational canonical form. Letting $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T = Px$, we have $K[x_1, \ldots, x_n] = K[\hat{x}_1, \ldots, \hat{x}_n]$ and $\partial(\hat{x}) = \hat{A}\hat{x} + Pa$.

Henceforth, we assume that A is in rational canonical form. Write

$$A = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_k \end{pmatrix},$$

where for all i C_i is the companion matrix of a polynomial of degree d_i. For $i = 1, \ldots, k$ define l_i as follows. Let $l_1 = 0$ and for $i > 1$ let $l_i = l_{i-1} + d_{i-1}$. For $i = 1, \ldots, k$ and for $j = 1, \ldots, d_i$ we define the K-derivation $\delta_{i,j}$ as follows. Lemma 1.3.1 for the ring $K[x_{l_i+1}, \ldots, x_{l_i+d_i}]$ and the K-derivation

$$\partial_i(x_{l_i+1}, \ldots, x_{l_i+d_i})^T = C_i(x_{l_i+1}, \ldots, x_{l_i+d_i})^T + (a_{l_i+1}, \ldots, a_{l_i+d_i})^T$$

guarantees the existence of K-derivations $\delta_2, \ldots, \delta_{d_i}$ on $K[x_{l_i+1}, \ldots, x_{l_i+d_i}]$ such that the set

$\{\partial_i, \delta_2, \ldots, \delta_{d_i}\}$ is commutative and K-linearly independent. Let $\delta_{i,1}$ be the extension of ∂_i to $K[x]$ by

$$\delta_{i,1}(x_r) = \begin{cases} \partial_i(x_r) & \text{if } l_i < r \leq l_i + d_i \\ 0 & \text{otherwise} \end{cases}$$

and for $j = 2, \ldots, d_i$ let $\delta_{i,j}$ be the extension of δ_j to $K[x]$ by

$$\delta_{i,j}(x_r) = \begin{cases} \delta_j(x_r) & \text{if } l_i < r \leq l_i + d_i \\ 0 & \text{otherwise} \end{cases}.$$

Observe that $\partial = \delta_{1,1} + \ldots + \delta_{k,1}$. If $k = 1$, then the theorem is proven by Lemma 1.3.1. Assume $k > 1$. Now consider the set

$$\begin{aligned} S &:= \{\partial, \delta_{1,1}, \ldots, \delta_{1,d_1}, \delta_{2,1}, \ldots, \delta_{2,d_2}, \ldots, \delta_{k-1,1}, \ldots, \delta_{k-1,d_{k-1}}, \delta_{k,2}, \ldots, \delta_{k,d_k}\} \\ &= \{\partial\} \cup \{\delta_{i,j} \mid i = 1, \ldots, k;\ j = 1, \ldots, d_i\} \backslash \{\delta_{k,1}\}. \end{aligned}$$

Observe that S contains n elements. We now show that S is commutative. Fix i, j, p, q, r such that $1 \leq i \leq k$, $1 \leq j \leq d_j$, $1 \leq p \leq k$, $1 \leq q \leq d_p$, and $1 \leq r \leq n$. If $i = p$, then $\delta_{p,q} \circ \delta_{i,j} = \delta_{i,j} \circ \delta_{p,q}$. Suppose $i \neq p$. Since $\delta_{i,j}(x_r) \in K[x_{l_i}, \ldots, x_{l_i+d_i}]$ we have $\delta_{p,q}(\delta_{i,j}(x_r)) = 0$. Similarly, $\delta_{p,q}(x_r) \in K[x_{l_p}, \ldots, x_{l_p+d_p}]$ and hence $\delta_{i,j}(\delta_{p,q}(x_r)) = 0$. We conclude that $\delta_{i,j}$ commutes with $\delta_{p,q}$. Since $\partial = \delta_{1,1} + \ldots + \delta_{k,1}$, we see that ∂ commutes with $\delta_{i,j}$.

Now we show that S is K-linearly independent. Suppose $b, b_{1,1}, \ldots, b_{1,d_1}, b_{2,1}, \ldots, b_{k,d_k} \in K$ are such that

$$b\partial + b_{1,1}\delta_{1,1} + \ldots + b_{k,d_k}\delta_{k,d_k} = 0. \quad \text{(Note that } \delta_{k,1} \text{ is not included.)}$$

Since $\partial = \delta_{1,1} + \ldots + \delta_{k,1}$, this implies

$$(b_{1,1}+b)\delta_{1,1}+\ldots+(b_{k-1,d_{k-1}}+b)\delta_{k-1,d_{k-1}}+b\delta_{k,1}+(b_{k,2}+b)\delta_{k,2}+\ldots+(b_{k,d_k}+b)\delta_{k,d_k}=0. \tag{1.3}$$

Equation (1.3) implies that for all $i = 1,\ldots,k-1$ and for all r such that $l_i < r \leq l_i + d_i$

$$(b_{i,1}+b)\delta_{i,1}(x_r)+\ldots+(b_{i,d_1}+b)\delta_{i,d_1}(x_r)=0.$$

It follows that

$$\forall i=1,\ldots,k-1 \quad (b_{i,1}+b)\delta_{i,1}+\ldots+(b_{i,d_i}+b)\delta_{i,d_i}=0. \tag{1.4}$$

Equation (1.3) also implies that for all r such that $l_k < r \leq l_k + d_k$

$$b\delta_{k,1}(x_r)+(b_{k,2}+b)\delta_{k,2}(x_r)+\ldots+(b_{k,d_k}+b)\delta_{k,d_k}(x_r)=0.$$

It follows that

$$b\delta_{k,1}+(b_{k,2}+b)\delta_{k,2}+\ldots+(b_{k,d_k}+b)\delta_{k,d_k}=0. \tag{1.5}$$

Since for all i $\delta_{i,1},\ldots,\delta_{i,d_i}$ are K-linearly independent, (1.4) implies that $b_{i,j} = -b$ for $i = 1,\ldots,k-1$ and $j = 1,\ldots,d_i$ and (1.5) implies that $b = 0$ and $b_{k,2} = \ldots = b_{k,d_k} = -b$. We conclude that $b = 0$ and $b_{i,j} = 0$ for all i and j. Therefore S is K-linearly independent.

□

1.4 A class of derivations admitting upper bounds on the degree of a commuting derivation

1.4.1 The utility of upper bounds

Let $d(x,y) = (f_1, f_2)$ be a K-derivation on $K[x,y]$. Suppose $b \in \mathbb{N}$ is such that the following statement is true: "If $\delta(x,y) = (g_1, g_2)$ is a K-derivation on $K[x,y]$ that commutes with and is transversal to d, then the degrees of g_1 and g_2 are no greater than b." Such a b is sometimes called an *upper bound*. We can use this information to determine whether d is integrable. Write $g_i = \sum_{j,k;j+k \leqslant b} a_{i,j,k} x^j y^k$. Now the equations $d(\delta(x)) = \delta(d(x))$ and $d(\delta(y)) = \delta(d(y))$ form a system of two equations of polynomials, and thus a finite system of equations on elements of K obtained by equating like coefficients. These equations are linear in the variables $a_{i,j,k}$. Hence the problem of determining whether d is integrable has been reduced to studying a finite system of linear equations over K.

1.4.2 Main result

We present a class of derivations and give an upper bound for each element of this class.

Notation.

- Define $\deg_y(0) := -\infty$, so that for all $n \in \mathbb{Z}$ $\deg_y(0) < n$.
- Let P and Q be elements of $K[x,y]$. Define $\deg_y(P/Q) = \deg_y(P/\gcd(P,Q)) - \deg_y(Q/\gcd(P,Q))$.
- Let U be a matrix with entries in $K(x,y)$. Define $deg_y(U) := \max\{\deg_y(u) \mid u \text{ is an entry of } U\}$.

Proposition 1.4.1. *Let K be a field of characteristic 0. Let d be a K-derivation on $K[x,y]$ given by*

$$d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

satisfying the conditions

- $f_2 \neq 0$,
- $\deg_y \dfrac{\partial f_2}{\partial x} < \deg_y f_2$, *and*
- $\deg_y(yf_1) < \deg_y f_2$.

If δ *is a* K*-derivation on* $K[x,y]$ *defined by*

$$\delta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and δ *commutes with* d*, then* $\max\{\deg_y g_1, \deg_y g_2\} \leqslant \deg_y f_2$.

Proof. The equations

$$d(\delta(x)) = \delta(d(x)) \quad \text{and} \quad d(\delta(y)) = \delta(d(y))$$

yield

$$f_1\frac{\partial g_1}{\partial x} + f_2\frac{\partial g_1}{\partial y} = g_1\frac{\partial f_1}{\partial x} + g_2\frac{\partial f_1}{\partial y} \quad \text{and} \quad f_1\frac{\partial g_2}{\partial x} + f_2\frac{\partial g_2}{\partial y} = g_1\frac{\partial f_2}{\partial x} + g_2\frac{\partial f_2}{\partial y}, \tag{1.6}$$

which we rearrange as

$$\begin{pmatrix} -\frac{yf_1}{f_2}\frac{\partial g_1}{\partial x} \\ -\frac{yf_1}{f_2}\frac{\partial g_2}{\partial x} \end{pmatrix} - y\frac{\partial}{\partial y}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} \frac{y}{f_2}\frac{\partial f_1}{\partial x} & \frac{y}{f_2}\frac{\partial f_1}{\partial y} \\ \frac{y}{f_2}\frac{\partial f_2}{\partial x} & \frac{y}{f_2}\frac{\partial f_2}{\partial y} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For conciseness of notation, we define the matrices

- $g := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$,

- $N := \begin{pmatrix} -\frac{yf_1}{f_2}\frac{\partial g_1}{\partial x} \\ -\frac{yf_1}{f_2}\frac{\partial g_2}{\partial x} \end{pmatrix}$, and
- $M := \begin{pmatrix} \frac{y}{f_2}\frac{\partial f_1}{\partial x} & \frac{y}{f_2}\frac{\partial f_1}{\partial y} \\ \frac{y}{f_2}\frac{\partial f_2}{\partial x} & \frac{y}{f_2}\frac{\partial f_2}{\partial y} \end{pmatrix}$.

so that this equation is written

$$N - y \cdot \frac{\partial}{\partial y} g + M \cdot g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let M^i denote the i-th row of M, and let

$$\alpha_i = \max\{\deg_y(M^i), 0\}.$$

Let

$$D = \operatorname{diag}(y^{-\alpha_1}, y^{-\alpha_2}), \quad A = D \cdot M, \quad \text{and} \quad B = D \cdot N.$$

Now we have

$$B - D \cdot y \cdot \frac{\partial}{\partial y} g + A \cdot g = 0. \tag{1.7}$$

Note that by the construction of D, $\deg_y(A) \leqslant 0$, so D and A are both elements of $K(x)[\![\frac{1}{y}]\!]$. Hence we can write

$$D = D_0 + \frac{D_1}{y} + \dots, \quad A = A_0 + \frac{A_1}{y} + \dots,$$

where each D_i is in $M^{2\times 2}(K)$, each $A_{y,i}$ is in $M^{2\times 2}(K(x))$, and the series for A is possibly infinite. Let $\mu = \deg_y(g)$ and $\nu = \deg_y(B)$. Recall that since the entries of g are polynomials, $\mu \geqslant 0$, whereas

ν may be negative. Thus, we can write

$$g = \begin{pmatrix} c_\mu \\ d_\mu \end{pmatrix} y^\mu + \text{lower degree terms},$$

where $\begin{pmatrix} c_\mu \\ d_\mu \end{pmatrix} \in M^{2\times 1}(K[x])$ and at least one of c_μ and d_μ is non-zero. Now equation (1.7) becomes

$$\operatorname{lc}(B)\cdot y^\nu - (\mu \cdot D_0 - A_0)\cdot \begin{pmatrix} c_\mu \\ d_\mu \end{pmatrix} \cdot y^\mu + \text{terms of degree lower than } \max\{\nu,\mu\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\gamma = \deg_y \dfrac{yf_1}{f_2} = \deg_y(yf_1) - \deg_y(f_2)$. We see from the definition of B that $\delta_y \leqslant \gamma + \mu$. Since we have assumed $\gamma < 0$, we have that $\nu < \mu$. It follows that $(c_\mu, d_\mu)^T$ is a non-zero element of the null space of $\mu D_0 - A_0$, so $\det(\mu D_0 - A_0) = 0$. Therefore μ belongs to the set

$$R = \{n \in \mathbb{N} : \det(n \cdot D_0 - A_0) = 0\}.$$

Observe that if

$$\det(\lambda D_{y,0} - A_{y,0}) \neq 0,$$

then R is finite and $\deg_y g \in R$.

We first examine the first row of M. It follows from the hypotheses that

$$\deg_y \frac{y}{f_2}\cdot\frac{\partial f_1}{\partial x} < 0 \quad \text{and} \quad \deg_y \frac{y}{f_2}\cdot\frac{\partial f_1}{\partial y} < 0.$$

Hence, $\alpha_1 = 0$.

Now we consider the second row. Observe that $\gamma < 0$ implies $\deg_y f_2 \geqslant 2$, so

$$\deg_y \frac{y}{f_2}\frac{\partial f_2}{\partial y} = 0.$$

Since $\deg_y \frac{\partial f_2}{\partial x} < \deg_y f_2$, it follows that

$$\deg_y \frac{y}{f_2}\frac{\partial f_2}{\partial x} \leqslant 0.$$

Thus $\alpha_2 = 0$ and it follows that $D = \mathrm{diag}(1,1)$ and $A = M$.

Write $f_2 = ay^b + \text{terms of lower degree in } y$, where $b \in \mathbb{N}$ and $a \in K$. We see that

$$A_0 = \begin{pmatrix} 0 & 0 \\ * & b \end{pmatrix}.$$

Now

$$\lambda D_0 - A_0 = \begin{pmatrix} \lambda & 0 \\ * & \lambda - b \end{pmatrix}.$$

Now $R = \{0, b\}$, so $\deg_y g = b$ or 0. □

Corollary 1.4.1. *Let K be a field of characteristic 0. Let d be a K-derivation on $K[x,y]$ given by*

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

satisfying the conditions

- $f_1 \neq 0$,
- $\deg_x \dfrac{\partial f_1}{\partial y} < \deg_x f_1$, *and*
- $\deg_x(xf_2) < \deg_x f_1$.

If δ is a K-derivation on $K[x,y]$ defined by

$$\delta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

and δ commutes with d, then $\max\{\deg_x g_1, \deg_x g_2\} \leqslant \deg_x f_1$.

Proof. This is identical to Proposition 1.4.1 but with the roles of x and y switched. $\square$

Corollary 1.4.2. *Let K be a field of characteristic 0. Let $d = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y}$ be a K-derivation on $K[x,y]$ satisfying the conditions*

- $f_1 f_2 \neq 0$,
- $\deg_y \dfrac{\partial f_2}{\partial x} < \deg_y f_2$,
- $\deg_y (y f_1) < \deg_y f_2$,
- $\deg_x \dfrac{\partial f_1}{\partial y} < \deg_x f_1$, *and*
- $\deg_x (x f_2) < \deg_x f_1$.

Then there is no $r \in K[x,y] \setminus K$ such that $d(r) = 0$, that is, the system of ODEs

$$\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

does not have a polynomial first integral.

Proof. Suppose there is an $r \in K[x,y] \setminus K$ such that $d(r) = 0$. Suppose without loss of generality that $\deg_y r = a > 0$. Now the derivation $r \cdot d$ is a derivation that commutes with d and has degree in y greater than $\deg_y d$, contradicting Theorem 1.4.1. Hence, no such r exists. $\square$

Example 1.4.1. As an example, consider the K-derivation

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^3+y \\ x+y^3 \end{pmatrix}$$

on the ring $K[x,y]$. We verify that we have satisfied the hypotheses above. First, $f_1, f_2 \neq 0$. Now

$$\gamma_x = \deg_x \frac{x(x+y^3)}{x^3+y} = -1 < 0, \quad \gamma_y = \deg_y \frac{y(x^3+y)}{x+y^3} = -1 < 0.$$

Next, we check that

$$\deg_x \frac{\partial f_1}{\partial y} = 0 < 3 = \deg_x f_1, \quad \deg_y \frac{\partial f_2}{\partial y} = 0 < 3 = \deg_y f_2.$$

We conclude that any K-derivation on $K[x,y]$ that commutes with d is defined by polynomials of degree no greater than 3.

1.5 Conservative Newton Systems

Fix a field K of characteristic 0. Suppose δ_f represents a second-order differential equation of the form

$$\ddot{x} = f,$$

where $f \in K[x] \setminus K$, which corresponds to a conservative Newton system. That is,

$$\delta_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f \end{pmatrix} \tag{1.8}$$

If $\deg f = 1$, then δ_f is integrable by Proposition 1.2.1. The following theorem, which is our main result, addresses the case of $\deg f \geqslant 2$.

Theorem 1.5.1. *For every*

- $f \in K[x]$ *such that* $\deg f \geqslant 2$ *and*
- K*-derivation* γ *on* $K[x,y]$ *that commutes with* δ_f*, where* δ_f *is the* K*-derivation defined by* (1.8)*,*

there exists $q \in K[H]$ *such that*

$$\gamma = q \cdot \delta_f,$$

where $H = y^2 - 2\int f dx$ *and* $\int f dx$ *has* 0 *as the constant term.*

As a corollary, we recover the following result on conservative Newton systems with a center at the origin. This result was first proven in (Amel'kin, 1977, Theorem 11) and was given new proofs in (Chicone and Jacobs, 1989, Theorem 4.1) and (Cima et al., 1999, Corollary 2.6) (see also (Volokitin and Ivanov, 1999, p. 30)).

Corollary 1.5.1. *The real system*

$$\dot{x} = -y$$
$$\dot{y} = f(x),$$

with $f(0) = 0$*,* $f'(0) = 1$*, has a transversal commuting polynomial derivation if and only if* $f(x) = x$*.*

Proof of Theorem 1.5.1. Fix $f \in K[x]$ such that $\deg f \geqslant 2$. Fix a K-derivation δ so that $\delta(x) = y$ and $\delta(y) = f$. Fix a K-derivation γ such that $[\delta, \gamma] = 0$. First consider the case in which $\deg_y \gamma \leqslant 1$.

Lemma 1.5.1. *If*

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 y + c_0 \\ d_1 y + d_0 \end{pmatrix},$$

where $c_1, c_0, d_1, d_0 \in K[x]$*, and* $[\delta, \gamma] = 0$*, then*

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \delta.$$

Proof. The equations $\delta(\gamma(x)) = \gamma(\delta(x))$ and $\delta(\gamma(y)) = \gamma(\delta(y))$ yield

$$\begin{cases} c_1' y^2 + c_0' y + f c_1 = d_1 y + d_0 \\ d_1' y^2 + d_0' y + f d_1 = f' c_1 y + f' c_0. \end{cases}$$

Equating coefficients of like powers of y, we obtain the two independent systems

$$c_1' = 0, \quad d_0' = c_1 f', \quad f c_1 = d_0 \tag{1.9}$$

and

$$d_1' = 0, \quad c_0' = d_1, \quad f d_1 = c_0 f'. \tag{1.10}$$

The solution set of (1.9) is $c_1 = \text{constant}, d_0 = c_1 f$. System (1.10) has no non-zero solution, which we deduce as follows. We have

$$\left(\frac{c_0}{f}\right)' = \frac{c_0' f - f' c_0}{f^2} = 0,$$

so $c_0 = (\text{const}) f$. Therefore, $d_1 = (\text{const}) f'$, which implies $d_1' = (\text{const}) f'' = 0$. Since we assume $\deg f \geqslant 2$, the constant must be 0. Therefore,

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} y \\ f \end{pmatrix}. \qquad \square$$

Now assume $\deg_y \gamma = M \geqslant 2$. Write

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_M y^M + \ldots + c_0 \\ d_M y^M + \ldots + d_0 \end{pmatrix}, \tag{1.11}$$

where for all i, $c_i, d_i \in K[x]$. Since $M = \deg_y \gamma$, at least one of c_M and d_M is non-zero. Now the system

$$\begin{pmatrix} \delta(\gamma(x)) \\ \delta(\gamma(y)) \end{pmatrix} = \begin{pmatrix} \gamma(\delta(x)) \\ \gamma(\delta(y)) \end{pmatrix}$$

becomes

$$\begin{pmatrix} c_M' y^{M+1} + c_{M-1}' y^M + \ldots + c_0' y \\ d_M' y^{M+1} + d_{M-1}' y^M + \ldots + d_0' y \end{pmatrix} + \begin{pmatrix} M f c_M y^{M-1} + \ldots + f c_1 \\ M f d_M y^{M-1} + \ldots + f d_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ f' & 0 \end{pmatrix} \begin{pmatrix} c_M y^M + \ldots + c_0 \\ d_M y^M + \ldots + d_0 \end{pmatrix}. \tag{1.12}$$

Viewing these matrix entries as polynomials in y and equating coefficients yields the following system of first-order ODEs

$$\begin{array}{cc}
c_M' = 0 & d_M' = 0 \\
c_{M-1}' = d_M & d_{M-1}' = f' c_M \\
c_{M-2}' + M f c_M = d_{M-1} & d_{M-2}' + M f d_M = f' c_{M-1} \\
c_{M-3}' + (M-1) f c_{M-1} = d_{M-2} & d_{M-3}' + (M-1) f d_{M-1} = f' c_{M-2} \\
c_{M-4}' + (M-2) f c_{M-2} = d_{M-3} & d_{M-4}' + (M-2) f d_{M-2} = f' c_{M-3} \\
c_{M-5}' + (M-3) f c_{M-3} = d_{M-4} & d_{M-5}' + (M-3) f d_{M-3} = f' c_{M-4} \\
\vdots & \vdots \\
c_0' + 2 f c_2 = d_1 & d_0' + 2 f d_2 = f' c_1 \\
f c_1 = d_0 & f d_1 = f' c_0
\end{array}$$

as well as the condition

$$c_M \neq 0 \text{ or } d_M \neq 0.$$

In each equation, it is the case that if c_i and d_j both appear, then i and j have opposite parities. Thus, we see that this system consists of two independent systems. If M is odd, these systems are:

$$\begin{array}{cc}
(Io)_M & (IIo)_M \\
c'_M = 0 & d'_M = 0 \\
d'_{M-1} = f'c_M & c'_{M-1} = d_M \\
c'_{M-2} + Mfc_M = d_{M-1} & d'_{M-2} + Mfd_M = f'c_{M-1} \\
d'_{M-3} + (M-1)fd_{M-1} = f'c_{M-2} & c'_{M-3} + (M-1)fc_{M-1} = d_{M-2} \\
c'_{M-4} + (M-2)fc_{M-2} = d_{M-3} & d'_{M-4} + (M-2)fd_{M-2} = f'c_{M-3} \\
d'_{M-5} + (M-3)fd_{M-3} = f'c_{M-4} & c'_{M-5} + (M-3)fc_{M-3} = d_{M-4} \\
\vdots & \vdots \\
c'_1 + 3fc_3 = d_2 & d'_1 + 3fd_3 = f'c_2 \\
d'_0 + 2fd_2 = f'c_1 & c'_0 + 2fc_2 = d_1 \\
fc_1 = d_0 & fd_1 = f'c_0
\end{array}$$

If M is even, the systems are:

$$
\begin{array}{cc}
(IIe)_M & (Ie)_M \\
c'_M = 0 & d'_M = 0 \\
d'_{M-1} = f'c_M & c'_{M-1} = d_M \\
c'_{M-2} + Mfc_M = d_{M-1} & d'_{M-2} + Mfd_M = f'c_{M-1} \\
d'_{M-3} + (M-1)fd_{M-1} = f'c_{M-2} & c'_{M-3} + (M-1)fc_{M-1} = d_{M-2} \\
c'_{M-4} + (M-2)fc_{M-2} = d_{M-3} & d'_{M-4} + (M-2)fd_{M-2} = f'c_{M-3} \\
d'_{M-5} + (M-3)fd_{M-3} = f'c_{M-4} & c'_{M-5} + (M-3)fc_{M-3} = d_{M-4} \\
\vdots & \vdots \\
c'_0 + 2fc_2 = d_1 & d'_0 + 2fd_2 = f'c_1 \\
fd_1 = f'c_0 & fc_1 = d_0
\end{array}
$$

In light of these observations, let

$$n = \max\{i \mid i \text{ odd and } c_i \neq 0 \text{ or } i \text{ even and } d_i \neq 0\},$$

$$p = \max\{i \mid i \text{ even and } c_i \neq 0 \text{ or } i \text{ odd and } d_i \neq 0\}.$$

Note that n or p may be undefined. Now write $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1(x)$ contains the terms of $\gamma(x)$ of odd degree in y, $\gamma_1(y)$ contains the terms of $\gamma(y)$ of even degree in y, $\gamma_2(x)$ contains the terms of

$\gamma(x)$ of even degree in y, and $\gamma_2(y)$ contains the terms of $\gamma(y)$ of odd degree in y. Explicitly,

$$\gamma_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} c_n y^n + c_{n-2} y^{n-2} + \ldots + c_1 y \\ d_{n-1} y^{n-1} + d_{n-3} y^{n-3} + \ldots + d_0 \end{pmatrix} & \text{if } n \text{ is odd,} \\ \begin{pmatrix} c_{n-1} y^{n-1} + c_{n-3} y^{n-3} + \ldots + c_1 y \\ d_n y^n + d_{n-2} y^{n-2} + \ldots + d_0 \end{pmatrix} & \text{if } n \text{ is even,} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n \text{ is undefined,} \end{cases}$$

and

$$\gamma_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} c_{p-1} y^{p-1} + c_{p-3} y^{p-3} + \ldots + c_0 \\ d_p y^p + d_{p-2} y^{p-2} + \ldots + d_1 y \end{pmatrix} & \text{if } p \text{ is odd,} \\ \begin{pmatrix} c_p y^p + c_{p-2} y^{p-2} + \ldots + c_0 \\ d_{p-1} y^{p-1} + d_{p-3} y^{p-3} + \ldots + d_1 y \end{pmatrix} & \text{if } p \text{ is even,} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } p \text{ is undefined.} \end{cases}$$

As we have seen, the criterion $[\delta, \gamma] = 0$ is equivalent to the conjunction of two systems of equations in which one system only involves the terms of γ_1 and the other only involves the terms of γ_2. Hence, $[\delta, \gamma_1] = [\delta, \gamma_2] = 0$.

Let us examine the possible values of n. If n is undefined, then $\gamma_1(x,y) = (0,0)$. If $n = 0$, then γ_1 is the same as the γ of Lemma 1.5.1 with $c_1 = c_0 = d_1 = 0$. Thus, by Lemma 1.5.1, $\gamma_1 = 0$, which contradicts that $n = 0$. If $n = 1$, then γ_1 is the same as the γ of Lemma 1.5.1 with $c_0 = d_1 = 0$. Thus by Lemma 1.5.1, $\gamma_1 = c_1\delta$, and, in the proof of Lemma 1.5.1, it is shown that $c_1 \in K$. If $n \geqslant 2$ is even, the coefficients of γ_1 must satisfy $(Ie)_n$ and $d_n \neq 0$. We will show in Lemma 1.5.4 and Corollary 1.5.2 that this is impossible. If n is odd, the coefficients of γ_1 must satisfy $(Io)_n$ and $c_n \neq 0$. We will show in Lemma 1.5.2 and Lemma 1.5.3 that this implies $\gamma_1 = q\delta$ for some $q \in K[H]$. In summary,

- If n is undefined, then $\gamma_1 = 0 \cdot \delta$.
- It is impossible that $n = 0$.
- If $n = 1$, then $\gamma_1 = c_1 \cdot \delta$ and $c_1 \in K$.
- It is impossible that $n \geqslant 2$ is even. (Lemma 1.5.4, Corollary 1.5.2)
- If $n \geqslant 3$ is odd, then $\gamma_1 = q \cdot \delta$ for some $q \in K[H]$. (Lemmas 1.5.2, 1.5.3)

Let us examine the possible values of p. If p is undefined, then $\gamma_2(x,y) = (0,0)$. If $p = 0$, then γ_2 is the same as the γ from Lemma 1.5.1 with $c_1 = d_1 = d_0 = 0$. Thus, by Lemma 1.5.1, $\gamma_2 = 0$, which contradicts that $p = 0$. If $p = 1$, then γ_2 is the same as the γ of Lemma 1.5.1 with $c_1 = d_0 = 0$. Thus, by Lemma 1.5.1, $\gamma_2 = 0$, which contradicts that $p = 1$. If $p \geqslant 2$ is even, the coefficients of γ_2 must satisfy $(IIe)_p$ and $c_p \neq 0$. We will show in Lemma 1.5.5 and Corollary 1.5.3 that this is impossible. If $p \geqslant 3$ is odd, the coefficients of γ_2 must satisfy $(IIo)_p$ and $d_p \neq 0$. We will show in Lemma 1.5.6, Lemma 1.5.7, Lemma 1.5.9, and Corollary 1.5.4 that this is impossible. We summarize these results as follows:

- If p is undefined, then $\gamma_2 = 0 \cdot \delta$.
- It is impossible that $p = 0$.

- It is impossible that $p = 1$.
- It is impossible that $p \geqslant 2$ is even. (Lemma 1.5.5, Corollary 1.5.3)
- It is impossible that $p \geqslant 3$ is odd. (Lemmas 1.5.6, 1.5.7, 1.5.9, Corollary 1.5.4)

From the bulleted statements, it follows that $\gamma_1 = q\delta$ for some $q \in K[H]$ and $\gamma_2 = 0$. These lemmas and their corollaries constitute the rest of the proof of Theorem 1.5.1.

Definition 1.5.1. Let $a \in K[x,y]$. We define $\int a dx$ to be the element of $K[x,y]$ whose partial derivative with respect to x is a and whose constant term is 0.

Lemma 1.5.2. *For every odd integer $m \geqslant 3$, the solution set of $(Io)_m$, with $c_0, \dots, d_m$ treated as variables, is an $\frac{m+1}{2}$-dimensional K-vector space.*

Proof. Fix $m \geqslant 3$. Label the equations of $(Io)_m$ as follows:

$$\begin{array}{cc}
e_{m+1} & c'_m = 0 \\
e_m & d'_{m-1} = f'c_m \\
e_{m-1} & c'_{m-2} + mfc_m = d_{m-1} \\
\vdots & \vdots \\
e_1 & d'_0 + 2fd_2 = f'c_1 \\
e_0 & fc_1 = d_0
\end{array}$$

We show the following by induction on k, $0 \leqslant k \leqslant \frac{m-3}{2}$:

$$\text{The solution set of } \{e_{m+1}, e_m, \dots, e_{m-2k-2}, d_{m-2k-3} = fc_{m-2k-2}\} \text{ is a } K\text{-vector space of dimension } k+2. \tag{1.13}$$

<u>Base Case: $k = 0$</u>

The system

$$\{e_{m+1}, e_m, e_{m-1}, e_{m-2}, d_{m-3} = fc_{m-2}\} \tag{1.14}$$

is

$$\begin{aligned}
e_{m+1}: &\qquad c_m' = 0\\
e_m: &\qquad d_{m-1}' = f'c_m\\
e_{m-1}: &\qquad c_{m-2}' = -mfc_m + d_{m-1}\\
e_{m-2}: &\qquad d_{m-3}' = -(m-1)fd_{m-1} + f'c_{m-2}\\
&\qquad d_{m-3} = fc_{m-2}
\end{aligned}$$

Let $\left(\tilde{d}_{m-3}, \tilde{c}_{m-2}, \tilde{d}_{m-1}, \tilde{c}_m\right)$ be a solution of (1.14). By e_{m+1}, $\tilde{c}_m = a_1$ for some $a_1 \in K$. It follows that

$$f'\tilde{c}_{m-2} + f\tilde{c}_{m-2}' = -(m-1)f\tilde{d}_{m-1} + f'\tilde{c}_{m-2},$$

and hence

$$\tilde{c}_{m-2}' = -(m-1)\tilde{d}_{m-1},$$

and so

$$\tilde{d}_{m-1} = mf\tilde{c}_m + \tilde{c}_{m-2}' = mf\tilde{c}_m - (m-1)\tilde{d}_{m-1}.$$

Thus

$$\tilde{d}_{m-1} = f\tilde{c}_m = a_1 f.$$

It follows from this and e_{m-1} that

$$\tilde{c}_{m-2}' = -(m-1)a_1 f,$$

and hence

$$\tilde{c}_{m-2} = -\int (m-1)a_1 f dx + a_2$$

for some $a_2 \in K$. From this and the condition $\tilde{d}_{m-3} = f\tilde{c}_{m-2}$ it follows that

$$\tilde{d}_{m-3} = f\left(-\int (m-1)a_1 f\,dx + a_2\right).$$

One can verify that

$$\Big(f\big(-\int (m-1)a_1 f\,dx + a_2\big), \quad -\int (m-1)a_1 f\,dx + a_2, \quad a_1 f, \quad a_1\Big) \tag{1.15}$$

is indeed a solution of (1.14). We have just shown that the solution set of (1.14) is exactly the elements of $K[x]^4$ of the form (1.15) with $a_1, a_2 \in K$. This set is the K-span of the tuples

$$\Big(f\big(-\int (m-1)f\,dx\big),\ -\int (m-1)f\,dx,\ f,\ 1\Big) \quad \text{and} \quad (f,\ 1,\ 0,\ 0).$$

Hence, the solution space is a two-dimensional K-vector space.

Inductive Step: Fix k, $0 \leqslant k < \frac{m-3}{2}$. Consider

$$\{e_{m+1}, e_m, \ldots, e_{m-2k-2}, d_{m-2k-3} = fc_{m-2k-2}\} \tag{1.16}$$

$$\{e_{m+1}, e_m, \ldots, e_{m-2k-4}, d_{m-2k-5} = fc_{m-2k-4}\} \tag{1.17}$$

Assume

The solution set of (1.16) is a K-vector space of dimension $k+2$. (1.18)

We will show

The solution set of (1.17) is a K-vector space of dimension $k+3$. (1.19)

We first show that

The solution set of (1.17) is the solution set of (1.20)

$$\{e_{m+1},\ldots,e_{m-2k-2},e_{m-2k-3},d_{m-2k-3}=fc_{m-2k-2},d_{m-2k-5}=fc_{m-2k-4}\}. \tag{1.21}$$

For ease of reference, we write the equations e_{m-2k-3} and e_{m-2k-4}:

$$\begin{aligned} e_{m-2k-3}: &\quad c'_{m-2k-4}=-(m-2k-2)fc_{m-2k-2}+d_{m-2k-3} \\ e_{m-2k-4}: &\quad d'_{m-2k-5}=-(m-2k-3)fd_{m-2k-3}+f'c_{m-2k-4} \end{aligned}$$

Suppose $(\tilde{d}_{m-2k-5},\ldots,\tilde{c}_m)$ is a solution of

$$\{e_{m+1},\ldots,e_{m-2k-4},d_{m-2k-5}=fc_{m-2k-4}\}.$$

Then $(\tilde{d}_{m-2k-3},\ldots,\tilde{c}_m)$ is a solution of $\{e_{m+1},\ldots,e_{m-2k-2}\}$. We now show that

$$\tilde{d}_{m-2k-3}=f\tilde{c}_{m-2k-2}. \tag{1.22}$$

Since $(\tilde{d}_{m-2k-5},\ldots,\tilde{c}_m)$ satisfies e_{m-2k-4}, we have

$$\tilde{d}'_{m-2k-5}=-(m-2k-3)f\tilde{d}_{m-2k-3}+f'\tilde{c}_{m-2k-4}. \tag{1.23}$$

Since $\tilde{d}_{m-2k-5}=f\tilde{c}_{m-2k-4}$, it follows that

$$\tilde{d}'_{m-2k-5}=f'\tilde{c}_{m-2k-4}+f\tilde{c}'_{m-2k-4}.$$

Combining this with (1.23), we get

$$f\tilde{c}'_{m-2k-4}=-(m-2k-3)f\tilde{d}_{m-2k-3},$$

and hence

$$\tilde{c}'_{m-2k-4} = -(m-2k-3)\tilde{d}_{m-2k-3}. \tag{1.24}$$

Since $(\tilde{d}_{m-2k-5}, \ldots, \tilde{c}_m)$ satisfies e_{m-2k-3}, we have

$$\tilde{c}'_{m-2k-4} + (m-2k-2)f\tilde{c}_{m-2k-2} = \tilde{d}_{m-2k-3},$$

and combining this with (1.24) gives us (1.22).

We now show the opposite inclusion. Suppose $(\tilde{d}_{m-2k-5}, \ldots, \tilde{c}_m)$ satisfies (1.21). Since the tuple satisfies $d_{m-2k-5} = fc_{m-2k-4}$, e_{m-2k-3}, and $d_{m-2k-3} = fc_{m-2k-2}$, we have

$$\begin{aligned}
\tilde{d}'_{m-2k-5} &= f'\tilde{c}_{m-2k-4} + f\tilde{c}'_{m-2k-4} \\
&= f'\tilde{c}_{m-2k-4} + f(-(m-2k-2)f\tilde{c}_{m-2k-2} + \tilde{d}_{m-2k-3}) \\
&= f'\tilde{c}_{m-2k-4} + f(-(m-2k-2)\tilde{d}_{m-2k-3} + \tilde{d}_{m-2k-3}) \\
&= f'\tilde{c}_{m-2k-4} - (m-2k-3)f\tilde{d}_{m-2k-3}.
\end{aligned}$$

Thus the tuple also satisfies e_{m-2k-4}. This completes the proof of (1.20).

Now we show (1.19). Since (1.17) is a system consisting of homogeneous linear differential equations and a homogeneous linear equation in $2k+6$ variables, the solution set is a K-vector subspace of $K[x]^{2k+6}$. Let W denote this vector space, let $\pi_i \colon K[x]^{2k+6} \to K[x]$ be projection onto the i-th coordinate, and let $\pi \colon K[x]^{2k+6} \to K[x]^{2k+4}$ be the projection onto the last $2k+4$ coordinates. Similarly, the solution set of (1.16) is a K-vector subspace of $K[x]^{2k+4}$. Call this space V. By (1.18), $\dim V = k+2$. Let $p_i \colon K[x]^{2k+4} \to K[x]$ be the projection onto the i-th coordinate.

Let $a_1, \ldots, a_{k+2} \in K[x]^{2k+4}$ be a basis for V. For each $i = 1, \ldots, k+2$, we define $b_i \in K[x]^{2k+6}$ as follows. Let

$$\pi(b_i) = a_i, \quad \pi_2(b_i) = \int(-(m-2k-2)fp_2(a_i) + p_1(a_i))dx, \quad \pi_1(b_i) = f\pi_2(b_i).$$

By (1.20), each b_i is a solution of (1.17). Since d_{m-2k-5} and c_{m-2k-4} only appear in the equations

$$c'_{m-2k-4} + (m-2k-2)fc_{m-2k-2} = d_{m-2k-3},$$
$$d'_{m-2k-5} + (m-2k-3)fd_{m-2k-3} = f'c_{m-2k-4},$$
$$d_{m-2k-5} = fc_{m-2k-4}$$

of (1.17), we observe that

$$b_{k+3} := (f, 1, 0, \ldots, 0) \in W.$$

We show that

$$\operatorname{span}_K\{b_1, \ldots, b_{k+3}\} = W.$$

Suppose $w \in W$. By (1.20), $\pi(w) \in V$, so there exist $\alpha_i \in K$, $1 \leqslant i \leqslant k+2$, such that

$$\pi(w) = \sum_{i=1}^{k+2} \alpha_i \pi(b_i).$$

Also by (1.20), there is a $\beta \in K$ such that

$$\begin{aligned}
\pi_2(w) &= \int \Big(-(m-2k-2)f\pi_4(w) + \pi_3(w) \Big) dx + \beta \\
&= \int \Big(-(m-2k-2)f \sum_{i=1}^{k+2} \alpha_i \pi_4(b_i) + \sum_{i=1}^{k+2} \alpha_i \pi_3(b_i) \Big) dx + \beta \\
&= \sum_{i=1}^{k+2} \alpha_i \int \Big(-(m-2k-2)f\pi_4(b_i) + \pi_3(b_i) \Big) dx + \beta = \sum_{i=1}^{k+2} \alpha_i \pi_2(b_i) + \beta.
\end{aligned}$$

By (1.20), we have $\pi_1(w) = f\pi_2(w)$. Using the fact that $\pi_1(b_i) = f\pi_2(b_i)$, we get

$$\pi_1(w) = \sum_{i=1}^{k+2} \alpha_i \pi_1(b_i) + f\beta.$$

Thus,

$$w = \sum_{i=1}^{k+2} \alpha_i b_i + \beta b_{k+3}.$$

We conclude that $\operatorname{span}_K\{b_1, \ldots, b_{k+3}\} = W$.

Since $\{\pi(b_1), \ldots, \pi(b_{k+2})\}$ is K-linearly independent, $\{b_1, \ldots, b_{k+2}\}$ is K-linearly independent. Since the constant term of $\pi_2(b_i)$ is 0 for $i = 1, \ldots, k+2$, it is clear that

$$b_{k+3} \notin \operatorname{span}_K\{b_1, \ldots, b_{k+2}\}.$$

We conclude that $\dim_K W = k+3$. This completes the inductive step.

Setting $k = \frac{m-3}{2}$ in (1.13) proves the lemma. □

Lemma 1.5.3. *If $n \geqslant 3$ is odd, then $\gamma_1 = q\delta$ for some $q \in K[H]$.*

Proof. Recall that, if $n \geqslant 3$ is odd, the coefficients of γ_1 must satisfy $(Io)_n$. Observe that $\delta(H) = 0$. Hence, any K-derivation D of the form

$$D\begin{pmatrix} x \\ y \end{pmatrix} = \left(a_{\frac{n-1}{2}} H^{\frac{n-1}{2}} + a_{\frac{n-1}{2}-1} H^{\frac{n-1}{2}-1} + \ldots + a_0\right) \cdot \begin{pmatrix} y \\ f \end{pmatrix}, \quad a_i \in K,$$

commutes with δ. Writing D in the form of (1.11), we see that $c_i = 0$ for even i and $d_i = 0$ for odd i, so a choice of

$$a_0, \ldots, a_{\frac{n-1}{2}}$$

provides a solution to $(Io)_n$. Moreover, two distinct choices of $a_0, \ldots, a_{\frac{n-1}{2}}$ provide two distinct solutions of $(Io)_n$. Thus, the set of solutions of $(Io)_n$ that correspond to derivations of the form $q\delta$, where $q \in K[H]$, is a K-vector space of dimension $\frac{n+1}{2}$. Since this vector space is contained in the vector space of solutions to $(Io)_n$, which by Lemma 1.5.2 has dimension $\frac{n+1}{2}$, the spaces must be equal. □

Lemma 1.5.4. *For all even $m \geqslant 2$, the system $(Ie)_m$ implies $d_m = 0$.*

Proof. Fix even $m \geqslant 2$. Label the equations in $(Ie)_m$ as follows:

$$\begin{aligned}
e_{m+1}: &\quad d'_m = 0 \\
e_m: &\quad c'_{m-1} = d_m \\
e_{m-1}: &\quad d'_{m-2} + mfd_m = f'c_{m-1} \\
e_{m-2}: &\quad c'_{m-3} + (m-1)fc_{m-1} = d_{m-2} \\
&\quad \vdots \\
e_1: &\quad d'_0 + 2fd_2 = f'c_1 \\
e_0: &\quad fc_1 = d_0
\end{aligned}$$

We show by induction on k, $0 \leqslant k \leqslant \frac{m-2}{2}$, that

$$\{e_0, e_1, \ldots, e_{2k+1}\} \quad \text{implies} \quad c'_{2k+1} = -(2k+2)d_{2k+2}. \tag{1.25}$$

The case $k = 0$ is straightforward. For the inductive hypothesis, fix k, $0 \leqslant k < \frac{m-2}{2}$, and assume (1.25). Now assume $\{e_0, e_1, \ldots, e_{2k+3}\}$. Equations e_{2k+2} and e_{2k+3} are

$$c'_{2k+1} = -(2k+3)fc_{2k+3} + d_{2k+2} \quad \text{and} \quad d'_{2k+2} = -(2k+4)fd_{2k+4} + f'c_{2k+3},$$

and the inductive hypothesis gives us

$$c'_{2k+1} = -(2k+2)d_{k+2}.$$

Equating the two expressions for c'_{2k+1}, we obtain $d_{2k+2} = fc_{2k+3}$. Differentiating this and equating the two expressions for d'_{2k+2} gives us

$$f'c_{2k+3} + fc'_{2k+3} = -(2k+4)fd_{2k+4} + f'c_{2k+3},$$

which implies

$$c'_{2k+3} = -(2k+4)d_{2k+4}.$$

This completes the inductive step. This shows that a consequence of $(Ie)_m$ is

$$c'_{m-1} = -md_m.$$

Since m was assumed to be even, we have $m \neq -1$. In order that e_m and $c'_{m-1} = -md_m$ both be satisfied, it is necessary that $d_m = 0$. □

Corollary 1.5.2. *It is impossible that n is an even integer greater than or equal to 2.*

Proof. Suppose $n \geqslant 2$ and n is even. Then the coefficients of γ_1 must satisfy $(Ie)_n$, and also $d_n \neq 0$. But by Lemma 1.5.4, $d_n = 0$ is a consequence of $(Ie)_n$. □

Lemma 1.5.5. *For all even $m \geqslant 2$, the system $(IIe)_m$ implies $c_m = 0$.*

Proof. Fix even $m \geqslant 2$. Label the equations of $(IIe)_m$ as follows:

$$\begin{aligned}
e_{m+1}: &\quad c'_m = 0 \\
e_m: &\quad d'_{m-1} = f'c_m \\
e_{m-1}: &\quad c'_{m-2} + mfc_m = d_{m-1} \\
e_{m-2}: &\quad d'_{m-3} + (m-1)fd_{m-1} = f'c_{m-2} \\
&\quad \vdots \\
e_1: &\quad c'_0 + 2fc_2 = d_1 \\
e_0: &\quad fd_1 = f'c_0
\end{aligned}$$

We first show the following by induction on k, $0 \leqslant k \leqslant \frac{m-2}{2}$:

$$\begin{aligned}
&\text{If } (\tilde{d}_{m-2k-1}, \ldots, \tilde{c}_m) \text{ is a solution of } \{e_{m+1}, \ldots, e_{m-2k}\} \text{ with } \tilde{c}_m \neq 0, \\
&\text{then } \tilde{d}_{m-2k-1} \neq 0,\ \deg(\tilde{d}_{m-2k-1}) = \deg(f \cdot \tilde{c}_{m-2k}), \text{ and } \mathrm{lc}(\tilde{d}_{m-2k-1}) = \mathrm{lc}(f \cdot \tilde{c}_{m-2k}).
\end{aligned} \tag{1.26}$$

Base Case, $k = 0$:

Suppose $(\tilde{d}_{m-1}, \tilde{c}_m)$ is a solution of $\{c'_m = 0, d'_{m-1} = f' c_m\}$ and $c_m \neq 0$. Since $\deg f \geqslant 2$ and $\tilde{c}_m$ is a non-zero constant,

$$\tilde{d}_{m-1} \neq 0 \quad \text{and} \quad \deg \tilde{d}_{m-1} = \deg(f\tilde{c}_m) = \deg f.$$

We have $\operatorname{lc}(\tilde{d}'_{m-1}) = \deg f \cdot \operatorname{lc} f \cdot \tilde{c}_m$. Since $\tilde{c}_m$ is a constant and $\deg \tilde{d}_{m-1} = \deg f$, we have

$$\operatorname{lc}(\tilde{d}_{m-1}) = \operatorname{lc}(f\tilde{c}_m).$$

Inductive Step:

Fix k, $0 \leqslant k < \frac{m-2}{2}$. Assume (1.26) for this k. Suppose $(\tilde{d}_{m-2k-3}, \ldots, \tilde{c}_m)$ is a solution of $\{e_{m+1}, \ldots, e_{m-2k-2}\}$ such that $\tilde{c}_m \neq 0$. For ease of reference, we write:

$$\begin{aligned} e_{m-2k-1}: &\quad c'_{m-2k-2} + (m-2k) \cdot f \cdot c_{m-2k} = d_{m-2k-1} \\ e_{m-2k-2}: &\quad d'_{m-2k-3} + (m-2k-1) \cdot f \cdot d_{m-2k-1} = f' \cdot c_{m-2k-2} \end{aligned}$$

Then

$$\tilde{c}'_{m-2k-2} = \tilde{d}_{m-2k-1} - (m-2k) f \cdot \tilde{c}_{m-2k}.$$

Since m is even, $m - 2k - 1 \neq 0$. Therefore, by the inductive hypothesis,

$$\deg(\tilde{c}'_{m-2k-2}) = \deg(\tilde{d}_{m-2k-1}) \geqslant 0 \tag{1.27}$$

and we have

$$\operatorname{lc}(\tilde{c}'_{m-2k-2}) = -(m-2k-1) \cdot \operatorname{lc}(\tilde{d}_{m-2k-1}),$$

and hence

$$\deg \tilde{c}_{m-2k-2} \cdot \operatorname{lc}(\tilde{c}_{m-2k-2}) = -(m-2k-1) \cdot \operatorname{lc}(\tilde{d}_{m-2k-1}). \tag{1.28}$$

By equation e_{m-2k-2}, we have

$$\tilde{d}'_{m-2k-3} = f' \cdot \tilde{c}_{m-2k-2} - (m-2k-1) \cdot f \cdot \tilde{d}_{m-2k-1}. \tag{1.29}$$

We will show that the degrees of the two terms on the right-hand side of (1.29) are equal and that their leading coefficients do not cancel. From (1.27), it follows that

$$\deg \tilde{c}_{m-2k-2} = \deg \tilde{d}_{m-2k-1} + 1,$$

so that

$$\deg(f' \cdot \tilde{c}_{m-2k-2}) = \deg(f \cdot \tilde{d}_{m-2k-1}). \tag{1.30}$$

Observe that

$$\mathrm{lc}(f' \cdot \tilde{c}_{m-2k-2}) = \deg f \cdot \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_{m-2k-2})$$

and, using (1.28),

$$\mathrm{lc}(f \cdot \tilde{d}_{m-2k-1}) = \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{d}_{m-2k-1}) = \mathrm{lc}\, f \cdot \tfrac{-1}{m-2k-1} \cdot \mathrm{lc}(\tilde{c}_{m-2k-2}) \cdot \deg \tilde{c}_{m-2k-2}.$$

It follows that

$$\mathrm{lc}(f' \cdot \tilde{c}_{m-2k-2}) \neq (m-2k-1) \cdot \mathrm{lc}(f \cdot \tilde{d}_{m-2k-1}), \tag{1.31}$$

and, together with (1.29) and (1.30), this gives us

$$\mathrm{lc}(\tilde{d}'_{m-2k-3}) = \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_{m-2k-2}) \cdot (\deg f + \deg \tilde{c}_{m-2k-2}). \tag{1.32}$$

By (1.29), (1.30), and (1.31), we have

$$\deg(\tilde{d}_{m-2k-3}) = \deg f + \deg \tilde{c}_{m-2k-2}. \tag{1.33}$$

Combining (1.32) and (1.33) gives us

$$\mathrm{lc}(\tilde{d}_{m-2k-3}) = \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_{m-2k-2}).$$

This completes the inductive step.

We proceed with the proof of the lemma. Let $(\tilde{c}_0, \dots, \tilde{c}_m)$ be a solution of $(IIe)_m$ with $\tilde{c}_m \neq 0$. We will derive a contradiction. It follows immediately that $(\tilde{d}_1, \dots, \tilde{c}_m)$ is a solution of $\{e_{m+1}, \dots, e_1\}$. Setting $k = \frac{m-2}{2}$ in (1.26), we have that $\deg(\tilde{d}_1) = \deg(f \cdot \tilde{c}_2) \geqslant 0$ and

$$\mathrm{lc}(\tilde{d}_1) = \mathrm{lc}(f) \cdot \mathrm{lc}(\tilde{c}_2). \tag{1.34}$$

From e_0, we see that

$$\deg(\tilde{d}_1) = \deg(\tilde{c}_0) - 1 = \deg(\tilde{c}_0').$$

By equation e_1, we have

$$\mathrm{lc}(\tilde{d}_1) = 2 \cdot \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_2) + \deg \tilde{c}_0 \cdot \mathrm{lc}(\tilde{c}_0).$$

Therefore, by (1.34), we have

$$\mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_2) = 2 \cdot \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_2) + \deg \tilde{c}_0 \cdot \mathrm{lc}(\tilde{c}_0)$$

and hence

$$\mathrm{lc}(\tilde{c}_0) = \frac{-\mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_2)}{\deg \tilde{c}_0}.$$

By equation e_0, we have

$$\mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{d}_1) = \deg f \cdot \mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_0) = \deg f \cdot \mathrm{lc}\, f \cdot \left(\frac{-\mathrm{lc}\, f \cdot \mathrm{lc}(\tilde{c}_2)}{\deg \tilde{c}_0} \right).$$

By (1.34),

$$\operatorname{lc} f \cdot \operatorname{lc} f \cdot \operatorname{lc}(\tilde{c}_2) = \deg f \cdot \operatorname{lc} f \cdot \left(\frac{-\operatorname{lc} f \cdot \operatorname{lc}(\tilde{c}_2)}{\deg \tilde{c}_0} \right).$$

It follows that

$$\deg \tilde{c}_0 = -\deg f,$$

which is a contradiction, since $\deg f > 0$. $\square$

Corollary 1.5.3. *It is impossible that p is an even integer greater than or equal to 2.*

Proof. Suppose $p \geqslant 2$ and p is even. Then the coefficients of γ_2 must satisfy $(IIe)_p$, together with $c_p \neq 0$. But by Lemma 1.5.5, $(IIe)_p$ implies $c_p = 0$. $\square$

In the lemmas that follow, we refer to K-derivations on the ring $K[x^{1/t}, x^{-1/t}, y]$, where t is a positive integer. We view this ring as isomorphic to

$$K[x,y,z,w]/(z^t - x, zw - 1).$$

By (Kolchin, 1973, Lemma II.2.1), since $\operatorname{char} K = 0$, any K-derivation on $K[x,y]$ extends uniquely to a K-derivation on $K[x^{1/t}, x^{-1/t}, y]$. One consequence of this is that a K-derivation on $K[x^{1/t}, x^{-1/t}, y]$ can be defined by stating its action on x and y.

Lemma 1.5.6. *For every odd integer m greater than or equal to 3, there exists $P_m(X) \in \mathbb{Z}[X] \setminus \{0\}$ such that:*

- $\deg P_m \leqslant \frac{m+1}{2}$
- *for every*
  - *positive integer t*
  - $h \in K[x^{1/t}, x^{-1/t}] \setminus \{0\}$,

if the K-derivation

$$\beta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_{m-1}y^{m-1} + c_{m-3}y^{m-3} + \ldots + c_0 \\ d_m y^m + d_{m-2}y^{m-2} + \ldots + d_1 y \end{pmatrix}$$

on $K[x^{1/t}, x^{-1/t}, y]$ commutes with the K-derivation

$$\alpha\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ h \end{pmatrix}$$

on $K[x^{1/t}, x^{-1/t}, y]$, then

$$P_m(N) = 0 \quad or \quad N \in \{-1\} \cup \left\{ -\tfrac{k}{k-1} \,\middle|\, 2 \leqslant k \leqslant \tfrac{m+1}{2} \right\},$$

where $N = \deg h$, each $c_i, d_i \in K[x^{1/t}, x^{-1/t}]$ and $d_m \neq 0$.

Proof. Fix $m \geqslant 3$. For $i = 0, \ldots, m$, we define $T_i(X) \in \mathbb{Z}[X]$ as follows. Let

$$T_m(X) = T_{m-1}(X) = 1.$$

For $1 \leqslant k \leqslant \frac{m-1}{2}$, let

$$T_{m-2k}(X) = X \cdot T_{m-(2k-1)}(X) - (m-(2k-2)) \cdot ((k-1)\cdot(X+1)+1) \cdot T_{m-(2k-2)}(X) \tag{1.35}$$

and let

$$T_{m-(2k+1)}(X) = T_{m-2k}(X) - (m-(2k-1)) \cdot k \cdot (X+1) \cdot T_{m-(2k-1)}(X). \tag{1.36}$$

Let

$$P_m(X) = \left(\tfrac{m-1}{2} \cdot (X+1) + 1\right) \cdot T_1(X) - X \cdot T_0(X). \tag{1.37}$$

We first prove that

$$\deg P_m(X) \leqslant \tfrac{m+1}{2}. \tag{1.38}$$

We show by induction on k, $0 \leqslant k \leqslant \frac{m-1}{2}$, that

$$\deg T_{m-2k}(X) \leqslant k \ \text{ and } \ \deg T_{m-(2k+1)}(X) \leqslant k. \tag{1.39}$$

For the base case, $k = 0$, we have

$$\deg T_m(X) = \deg T_{m-1}(X) = 0.$$

For the inductive step, fix k, $0 \leqslant k < \frac{m-1}{2}$, and assume (1.39). It follows from (1.35) and the inductive hypothesis that

$$\deg T_{m-(2k+2)}(X) \leqslant k+1,$$

and it follows from (1.36) and the inductive hypothesis that

$$\deg T_{m-(2k+3)}(X) \leqslant k+1.$$

This completes the proof by induction. As a consequence, we have

$$\deg T_1(X) \leqslant \tfrac{m-1}{2} \quad \text{and} \quad \deg T_0(X) \leqslant \tfrac{m-1}{2}.$$

Therefore, (1.38) holds. Next, we show that $P_m(X)$ is not the zero polynomial. To this end, we first prove by induction on k, $0 \leqslant k \leqslant \frac{m-1}{2}$, that

$$T_{m-2k}(-1) \neq 0 \quad \text{and} \quad T_{m-(2k+1)}(-1) \neq 0. \tag{1.40}$$

The base case, $k = 0$, is trivial, since $T_m(X) = T_{m-1}(X) = 1$. For the inductive hypothesis, fix k,

$0 \leqslant k < \frac{m-1}{2}$, and assume

$$T_{m-2k}(-1) \cdot T_{m-(2k+1)}(-1) \neq 0.$$

Equation (1.36) shows that

$$T_{m-(2k+1)}(-1) = T_{m-2k}(-1).$$

Replacing k with $k+1$ in (1.35) gives us

$$T_{m-(2k+2)}(-1) = -1 \cdot T_{m-(2k+1)}(-1) - (m-2k) \cdot T_{m-2k}(-1) = -(m-2k+1) \cdot T_{m-2k}(-1).$$

Since $k < \frac{m-1}{2}$, it must be that $m-2k+1 \neq 0$. Now by the inductive hypothesis,

$$T_{m-(2k+2)}(-1) \neq 0.$$

Replacing k with $k+1$ in (1.36) yields

$$T_{m-(2k+3)}(-1) = T_{m-(2k+2)}(-1) \neq 0.$$

This completes the proof of (1.40). By (1.37), we have

$$P_m(-1) = T_1(-1) + T_0(-1).$$

Replacing k with $\frac{m-1}{2}$ in (1.36) gives

$$T_0(-1) = T_1(-1),$$

and hence

$$P_m(-1) = 2 \cdot T_1(-1) \neq 0.$$

This completes the proof that $P_m(X)$ is not the zero polynomial.

We proceed to show that $P_m(X)$ satisfies the remaining property stated in the lemma. Fix $t \in \mathbb{Z}^{\geqslant 1}$, fix $h \in K[x^{1/t}, x^{-1/t}] \setminus \{0\}$, and define α as in the statement of the lemma. Fix β as in the statement of the lemma. Note that c_i and d_i must satisfy the equations of system $(IIo)_m$, with f replaced by h. Label these equations as follows:

$$\begin{array}{ll}
e_{m+1}: & d'_m = 0 \\
e_m: & c'_{m-1} = d_m \\
e_{m-1}: & d'_{m-2} + mhd_m = h'c_{m-1} \\
e_{m-2}: & c'_{m-3} + (m-1)hc_{m-1} = d_{m-2} \\
\vdots & \vdots \\
e_{m-(2k-1)}: & d'_{m-2k} + (m-(2k-2))hd_{m-(2k-2)} = h'c_{m-(2k-1)} \\
e_{m-2k}: & c'_{m-(2k+1)} + (m-(2k-1))hc_{m-(2k-1)} = d_{m-2k} \\
e_{m-(2k+1)}: & d'_{m-(2k+2)} + (m-2k)hd_{m-2k} = h'c_{m-(2k+1)} \\
e_{m-(2k+2)}: & c'_{m-(2k+3)} + (m-(2k+1))hc_{m-(2k+1)} = d_{m-(2k+2)} \\
\vdots & \vdots \\
e_0: & hd_1 = h'c_0
\end{array}$$

Let $N = \deg h$ and let $L = \mathrm{lc}(h)$. Assume that

$$N \notin \{-1\} \cup \left\{ -\tfrac{k}{k-1} \,\middle|\, 2 \leqslant k \leqslant \tfrac{m+1}{2} \right\}.$$

We first show by induction that for all k, $0 \leqslant k \leqslant \frac{m-1}{2}$,

$$\deg d_{m-2k} \leqslant k(N+1) \quad \text{and} \quad \deg c_{m-(2k+1)} \leqslant k(N+1)+1. \tag{1.41}$$

We first treat the base case, $k = 0$. By equations e_{m+1} and e_m, $\deg d_m \leqslant 0$ and $\deg c_{m-1} \leqslant 1$.

For the inductive hypothesis, fix k, $0 \leqslant k < \frac{m-1}{2}$ and assume (1.41). Consider $e_{m-(2k+1)}$. By the inductive hypothesis, we have

$$\deg(hd_{m-2k}) \leqslant k(N+1)+N \quad \text{and} \quad \deg(h'c_{m-(2k+1)}) \leqslant k(N+1)+N.$$

It follows that

$$\deg d_{m-(2k+2)} \leqslant (k+1)(N+1). \tag{1.42}$$

Now consider $e_{m-(2k+2)}$. By the inductive hypothesis,

$$\deg(hc_{m-(2k+1)}) \leqslant (k+1)(N+1).$$

It follows from this and (1.42) that

$$\deg c_{m-(2k+3)} \leqslant (k+1)(N+1)+1.$$

This concludes the proof of (1.41) for all k, $0 \leqslant k \leqslant \frac{m-1}{2}$.

Define $a_m, a_{m-1}, \ldots, a_0$ as follows. Let

$$a_{m-2k} = \text{ the coefficient of } x^{k(N+1)} \text{ in } d_{m-2k},$$
$$a_{m-(2k+1)} = \text{ the coefficient of } x^{k(N+1)+1} \text{ in } c_{m-(2k+1)}.$$

Equations e_{m+1} and e_m and the requirement that $d_m \neq 0$ imply that $a_{m-1} = a_m$. Now we prove that, for all k, $1 \leqslant k \leqslant \frac{m-1}{2}$,

$$a_{m-(2k+1)} = \big(a_{m-2k} - (m-(2k-1)) \cdot L \cdot a_{m-(2k-1)}\big) \cdot \tfrac{1}{k(N+1)+1} \tag{1.43}$$

and

$$a_{m-2k} = (L \cdot N \cdot a_{m-(2k-1)} - (m-(2k-2)) \cdot L \cdot a_{m-(2k-2)}) \cdot \tfrac{1}{k(N+1)}. \tag{1.44}$$

Fix k, $1 \leqslant k \leqslant \frac{m-1}{2}$. By equation $e_{m-(2k-1)}$, we have

$$d'_{m-2k} = h'c_{m-(2k-1)} - (m-(2k-2)) \cdot h \cdot d_{m-(2k-2)}. \tag{1.45}$$

Let us write an equation equating the coefficients of $x^{k(N+1)-1}$ on both sides of (1.45). First, observe that the coefficient of $x^{k(N+1)-1}$ in d'_{m-2k} is $k(N+1) \cdot a_{m-2k}$. Next consider $h'c_{m-(2k-1)}$. First consider the case $N \neq 0$. It follows that $\deg h' = N-1$. By (1.41), we have

$$\deg c_{m-(2k-1)} \leqslant k(N+1) - N. \tag{1.46}$$

Thus, the coefficient of $x^{k(N+1)-1}$ in $h'c_{m-(2k-1)}$ is $N \cdot L \cdot a_{m-(2k-1)}$. Now consider the case $N = 0$. Either $h' = 0$, or $h' \neq 0$ and $\deg h' < N-1$. If $h' = 0$, then $h'c_{m-(2k-1)} = 0$ and the coefficient of $x^{k(N+1)-1}$ in $h'c_{m-(2k-1)}$ is 0, which is equal to $L \cdot N \cdot a_{m-(2k-1)}$. If $N = 0$ and $h' \neq 0$, then, since $\deg h' < N-1$ and by (1.46), the coefficient of $x^{k(N+1)-1}$ in $h'c_{m-(2k-1)}$ is 0, which is equal to $L \cdot N \cdot a_{m-(2k-1)}$. Finally, consider $h \cdot d_{m-(2k-2)}$. Since $\deg h = N$ and, by (1.41),

$$\deg d_{m-(2k-2)} \leqslant k(N+1) - N - 1,$$

we see that the coefficient of $x^{k(N+1)-1}$ in $h \cdot d_{m-(2k-2)}$ is $L \cdot a_{m-(2k-2)}$. Since $N \neq -1$, we have $k(N+1) \neq 0$. Thus, equating the coefficients of $x^{k(N+1)-1}$ in (1.44) yields

$$a_{m-2k} = \left(L \cdot N \cdot a_{m-(2k-1)} - (m-(2k-2)) \cdot L \cdot a_{m-(2k-2)}\right) \cdot \tfrac{1}{k(N+1)}.$$

By equation e_{m-2k}, we have

$$c'_{m-(2k+1)} = d_{m-2k} - (m-(2k-1)) \cdot h \cdot c_{m-(2k-1)}. \tag{1.47}$$

Let us write an equation equating the coefficients of $x^{k(N+1)}$ on either side of (1.47). The coefficient of $x^{k(N+1)}$ in $c'_{m-(2k+1)}$ is $(k(N+1)+1)\cdot a_{m-(2k+1)}$. The coefficient of $x^{k(N+1)}$ in d_{m-2k} is a_{m-2k}. By (1.41), we have

$$\deg c_{m-(2k-1)} \leqslant k(N+1)-N,$$

and, since $\deg h = N$, the coefficient of $x^{k(N+1)}$ in $hc_{m-(2k-1)}$ is $L\cdot a_{m-(2k-1)}$. Since $N \neq -\frac{k+1}{k}$, we have $k(N+1)+1 \neq 0$. Thus, equating the coefficients of $x^{k(N+1)}$ on either side of (1.47) yields

$$a_{m-(2k+1)} = \big(a_{m-2k} - (m-(2k-1))\cdot L\cdot a_{m-(2k-1)}\big)\cdot \tfrac{1}{k(N+1)+1}.$$

This concludes the proof of (1.43) and (1.44).

For $i = 0,\ldots,m$, define $S_i \in \mathbb{Z}$ as follows. Let

$$S_m = S_{m-1} = 1.$$

For every k, $1 \leqslant k \leqslant \frac{m-1}{2}$, let

$$S_{m-2k} = k(N+1)\cdot S_{m-(2k-1)} \quad \text{and} \quad S_{m-(2k+1)} = (k(N+1)+1)\cdot S_{m-2k}.$$

Next, we prove by induction that for all k, $0 \leqslant k \leqslant \frac{m-1}{2}$, we have

$$T_{m-2k}(N) = S_{m-2k}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-2k} \ \text{ and } \ T_{m-(2k+1)}(N) = S_{m-(2k+1)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+1)}. \tag{1.48}$$

Recall that by our assumption on the form of β, we have $a_m \neq 0$.

The base case, $k = 0$, is proved immediately by noting that $a_m = a_{m-1}$ follows from e_{m+1} and e_m.

For the inductive hypothesis, fix k, $0 \leqslant k < \frac{m-1}{2}$ and assume (1.48) holds. We have from (1.35), (1.48), and the definition of S_i that

$$\begin{aligned}
T_{m-(2k+2)}(N) &= N\cdot T_{m-(2k+1)}(N) - (m-2k)(k(N+1)+1)\cdot T_{m-2k}(N)\\
&= N\cdot S_{m-(2k+1)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+1)} - (m-2k)(k(N+1)+1)\cdot S_{m-2k}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-2k}\\
&= N\cdot \tfrac{S_{m-(2k+2)}}{(k+1)(N+1)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+1)} - (m-2k)\cdot \tfrac{S_{m-(2k+2)}}{(k+1)(N+1)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-2k}\\
&= S_{m-(2k+2)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot \left(Na_{m-(2k+1)} - (m-2k)a_{m-2k}\right)\cdot \tfrac{1}{(k+1)(N+1)}.
\end{aligned}$$

From (1.44) with k replaced by $k+1$, we see that

$$T_{m-(2k+2)}(N) = S_{m-(2k+2)}\cdot \tfrac{1}{L^{k+1}}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+2)}. \tag{1.49}$$

We have from (1.36), (1.49), (1.48), and the definition of S_i that

$$\begin{aligned}
&T_{m-(2k+3)}(N) = T_{m-(2k+2)}(N) - (m-(2k+1))\cdot(k+1)(N+1)\cdot T_{m-(2k+1)}(N)\\
&= S_{m-(2k+2)}\cdot \tfrac{1}{L^{k+1}}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+2)} - (m-(2k+1))\cdot(k+1)(N+1)\cdot S_{m-(2k+1)}\cdot \tfrac{1}{L^k}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+1)}\\
&= S_{m-(2k+3)}\cdot \tfrac{1}{L^{k+1}}\cdot \tfrac{1}{a_m}\cdot \left(a_{m-(2k+2)} - L(m-(2k+1))a_{m-(2k+1)}\right)\cdot \tfrac{1}{(k+1)(N+1)+1}.
\end{aligned}$$

From (1.43) with k replaced by $k+1$, we see that

$$T_{m-(2k+3)}(N) = S_{m-(2k+3)}\cdot \tfrac{1}{L^{k+1}}\cdot \tfrac{1}{a_m}\cdot a_{m-(2k+3)}.$$

This completes the proof of (1.48).

Now we show that $P_m(N)=0$. Using $k=\frac{m-1}{2}$ in (1.48) and $S_0 = (\frac{m-1}{2}(N+1)+1)S_1$, we have

$$\begin{aligned}
P_m(N) &= \left(\tfrac{m-1}{2}(N+1)+1\right)\cdot T_1(N) - N\cdot T_0(N)\\
&= \left(\tfrac{m-1}{2}(N+1)+1\right)S_1\cdot \tfrac{1}{L^{(m-1)/2}}\cdot \tfrac{1}{a_m}\cdot a_1 - N\cdot S_0\cdot \tfrac{1}{L^{(m-1)/2}}\cdot \tfrac{1}{a_m}\cdot a_0 = \tfrac{S_0}{L^{(m-1)/2}}\cdot \tfrac{1}{a_m}\cdot (a_1 - Na_0).
\end{aligned}$$

Consider equation e_0:

$$hd_1 = h'c_0.$$

Equating the coefficients of $x^{(N+1)((m-1)/2)+N}$ in e_0, recalling (1.41), yields

$$a_1 = Na_0.$$

We conclude that $P_m(N) = 0$. $\square$

Lemma 1.5.7. *For every positive integer k, the K-derivation*

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x^{-\frac{2k+1}{2k-1}} \end{pmatrix}$$

of the ring $K[x^{-\frac{1}{2k-1}}, x^{\frac{1}{2k-1}}, y]$ *commutes with the K-derivation*

$$\beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum_{l=0}^{k} a_{2(k-l)} x^{1+(1-\frac{2k+1}{2k-1})l} y^{2(k-l)} \\ \sum_{l=0}^{k} a_{2(k-l)+1} x^{(1-\frac{2k+1}{2k-1})l} y^{2(k-l)+1} \end{pmatrix},$$

where the $a_i \in K$ *are defined recursively as follows:* $a_{2k+1} \in K \setminus \{0\}$ *is arbitrary,* $a_{2k} = a_{2k+1}$*, and for* $0 < l \leqslant k$*,*

$$a_{2(k-l)+1} = \left(-\tfrac{2k+1}{2k-1} a_{2(k-l)+2} - (2(k-l)+3) a_{2(k-l)+3}\right) \left((1 - \tfrac{2k+1}{2k-1}) l\right)^{-1}$$
$$a_{2(k-l)} = \left(a_{2(k-l)+1} - (2(k-l)+2) a_{2(k-l)+2}\right) \left((1 - \tfrac{2k+1}{2k-1}) l + 1\right)^{-1}.$$

Proof. We first show that

$$\beta(\alpha(x)) = \alpha(\beta(x)).$$

We have $\beta(\alpha(x)) = \beta(y)$. Note that, in $\beta(y)$, only odd powers of y with exponents less than or equal to $2k+1$ appear, and for all l, $0 \leqslant l \leqslant k$, the coefficient of $y^{2(k-l)+1}$ is

$$a_{2(k-l)+1}x^{(1-\frac{2k+1}{2k-1})l}. \tag{1.50}$$

In $\alpha(\beta(x))$, only odd powers of y with exponents less than or equal to $2k+1$ appear. The coefficient of y^{2k+1} is a_{2k}, which equals a_{2k+1}, which is the coefficient of y^{2k+1} in $\beta(\alpha(x))$. For all l, $1 \leqslant l \leqslant k$, the coefficient of $y^{2(k-l)+1}$ in $\alpha(\beta(x))$ is

$$a_{2(k-l)}x^{\left(1-\frac{2k+1}{2k-1}\right)l}\left(1+(1-\tfrac{2k+1}{2k-1})l\right)+a_{2(k-l)+2}x^{\left(1-\frac{2k+1}{2k-1}\right)l}(2(k-l)+2).$$

By the definition of $a_{2(k-l)}$, this equals (1.50). Now we show that

$$\beta(\alpha(y)) = \alpha(\beta(y)).$$

We have

$$\beta(\alpha(y)) = \beta\left(x^{-\frac{2k+1}{2k-1}}\right) = -\tfrac{2k+1}{2k-1}x^{-\frac{2k+1}{2k-1}-1}\beta(x).$$

This expression contains only even powers of y from y^0 to y^{2k}. For all l, $0 \leqslant l \leqslant k$, the coefficient of $y^{2(k-l)}$ in $\beta(\alpha(y))$ is

$$-\tfrac{2k+1}{2k-1}a_{2(k-l)}x^{\left(1-\frac{2k+1}{2k-1}\right)l-\frac{2k+1}{2k-1}}. \tag{1.51}$$

We see that $\alpha(\beta(y))$ contains only even powers of y from y^0 to y^{2k}. For $l<k$, the coefficient of $y^{2(k-l)}$ in $\alpha(\beta(y))$ is

$$a_{2(k-l)+1}x^{\left(1-\frac{2k+1}{2k-1}\right)l-\frac{2k+1}{2k-1}}(2(k-l)+1)+a_{2(k-l)-1}x^{\left(1-\frac{2k+1}{2k-1}\right)l-\frac{2k+1}{2k-1}}\left(1-\tfrac{2k+1}{2k-1}\right)(l+1).$$

By definition,

$$a_{2(k-l)-1} = \left(-\tfrac{2k+1}{2k-1}a_{2(k-l)} - (2(k-l)+1)a_{2(k-l)+1}\right)\left(\left(1-\tfrac{2k+1}{2k-1}\right)(l+1)\right)^{-1}.$$

Hence, the coefficient of $y^{2(k-l)}$ in $\alpha(\beta(y))$ is (1.51). The coefficient of y^0 in $\alpha(\beta(y))$ is

$$a_1 x^{\left(1-\frac{2k+1}{2k-1}\right)k-\frac{2k+1}{2k-1}}.$$

It remains to show that

$$a_1 = -\tfrac{2k+1}{2k-1}a_0. \tag{1.52}$$

This is an immediate consequence of the following lemma.

Lemma 1.5.8. *In the notation of Lemma 1.5.7, for all l, $0 \leqslant l \leqslant k$,*

$$\tfrac{2k+1}{2k-1}a_{2(k-l)} = \tfrac{2(k-l)+1}{2(k-l)-1}a_{2(k-l)+1}.$$

Proof. We proceed by induction on l. The base case $l = 0$ is immediate, since by definition $a_{2k} = a_{2k+1}$. For the inductive hypothesis, fix $l < k$ and assume

$$\tfrac{2k+1}{2k-1}a_{2(k-l)} = \tfrac{2(k-l)+1}{2(k-l)-1}a_{2(k-l)+1}.$$

We want to show that

$$\tfrac{2k+1}{2k-1}a_{2(k-l)-2} = \tfrac{2(k-l)-1}{2(k-l)-3}a_{2(k-l)-1}. \tag{1.53}$$

The left-hand side of (1.53) is, by the definition of $a_{2(k-l)-2}$,

$$\tfrac{2k+1}{2k-1}\cdot\frac{\left(a_{2(k-l)-1} - 2(k-l)a_{2(k-l)}\right)}{\left(1-\tfrac{2k+1}{2k-1}\right)(l+1)+1}.$$

By the definition of $a_{2(k-l)-1}$, this equals

$$\frac{2k+1}{2k-1}\cdot\frac{\left(\frac{-\frac{2k+1}{2k-1}a_{2(k-l)}-(2(k-l)+1)a_{2(k-l)+1}}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)}-2(k-l)a_{2(k-l)}\right)}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)+1}.$$

By the inductive hypothesis, this is equal to

$$\frac{\left(\frac{-\frac{2k+1}{2k-1}\cdot\frac{2(k-l)+1}{2(k-l)-1}-\frac{2k+1}{2k-1}(2(k-l)+1)}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)}-2(k-l)\frac{2(k-l)+1}{2(k-l)-1}\right)}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)+1}\cdot a_{2(k-l)+1}. \tag{1.54}$$

The right-hand side of (1.53) is, using the definition of $a_{2(k-l)-1}$,

$$\frac{2(k-l)-1}{2(k-l)-3}\cdot\frac{-\frac{2k+1}{2k-1}a_{2(k-l)}-(2(k-l)+1)a_{2(k-l)+1}}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)}.$$

By the inductive hypothesis, this equals

$$\frac{2(k-l)-1}{2(k-l)-3}\frac{-\frac{2(k-l)+1}{2(k-l)-1}-(2(k-l)+1)}{\left(1-\frac{2k+1}{2k-1}\right)(l+1)}\cdot a_{2(k-l)+1},$$

which is equal to (1.54), as a computation shows. $\square$

By letting $l=k$ in Lemma 1.5.8, we see that (1.52) holds. $\square$

Lemma 1.5.9. *For every*

- *positive integer t,*
- $h\in K[x^{1/t},x^{-1/t}]\setminus\{0\}$,

if there exists a K-derivation β on $K[x^{1/t},x^{-1/t},y]$ such that

- β *commutes with the K-derivation*

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ h \end{pmatrix}$$

on $K[x^{-1/t}, x^{1/t}, y]$ and

- β *is of the form*

$$\beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_{m-1}y^{m-1} + c_{m-3}y^{m-3} + \ldots + c_0 \\ d_m y^m + d_{m-2}y^{m-2} + \ldots + d_1 y \end{pmatrix},$$

where $m \geqslant 3$ is odd, c_i, $d_i \in K[x^{-1/t}, x^{1/t}]$, and $d_m \neq 0$,

then

$$N := \deg h \in S \cup T,$$

where

$$S = \{1\} \cup \left\{ -\tfrac{2k+1}{2k-1} \,\middle|\, k \in \mathbb{Z}, 1 \leqslant k \leqslant \tfrac{m-1}{2} \right\},$$
$$T = \{-1\} \cup \left\{ -\tfrac{k}{k-1} \,\middle|\, k \in \mathbb{Z}, k \geqslant 2 \right\}.$$

Proof. Fix $t \in \mathbb{Z}^{\geqslant 1}$. Fix $h \in K[x^{-1/t}, x^{1/t}] \backslash \{0\}$ and hence α of the form stated in the lemma. Let $N = \deg h$ and assume $N \notin T$. Suppose a K-derivation β satisfying the properties stated in the lemma exists and let m be the least odd integer greater than or equal to 3 such that there exists such a β. By Lemma 1.5.6, $P_m(N) = 0$, and P_m has at most $\frac{m+1}{2}$ zeros. We show that these zeros are exactly the elements of S.

We show that $P_m(1) = 0$. The K-derivations

$$\partial_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{and} \quad \partial_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

on $K[x,x^{-1},y]$ commute and ∂_1 has the form of α in the statement of Lemma 1.5.6. The polynomial $r := y^2 - x^2$ is a first integral of ∂_1, and so $r^{(m-1)/2}\partial_2$ is a K-derivation commuting with ∂_1 of the form of β in the statement of Lemma 1.5.6. Therefore, by Lemma 1.5.6, $P_m(1) = 0$.

We show that, for all k, $1 \leqslant k \leqslant \frac{m-1}{2}$,

$$P_m\left(-\tfrac{2k+1}{2k-1}\right) = 0. \tag{1.55}$$

Fix k. Let K-derivations ∂_1 and ∂_2 on $K[x^{\frac{1}{2k-1}}, x^{-\frac{1}{2k-1}}, y]$ be defined as α and β are in Lemma 1.5.7. Now

$$r = y^2 + 2\left(\tfrac{2k-1}{2}\right)x^{-2/(2k-1)}$$

is a first integral of ∂_1. Note that $\deg_y \partial_2(y) = 2k+1$. Now $r^{(m-(2k+1))/2}\partial_2$ is a derivation commuting with ∂_1 of the form of β of Lemma 1.5.6. Hence, we have (1.55).

The set S consists of $\frac{m+1}{2}$ elements, and we have shown that each is a zero of P_m, which is nonzero of degree at most $\frac{m+1}{2}$. It follows that S is exactly the zero set of P_m. □

Corollary 1.5.4. *It is impossible that p is an odd integer greater than or equal to* 3.

Proof. Suppose $p \geqslant 3$ and p is odd. Let $N = \deg f$. Recall that $p = \deg_y \gamma_2$. Consider Lemma 1.5.9. Since the extensions of δ and γ_2 to K-derivations on $K[x,x^{-1},y]$ are of the forms of α and β, it follows that $N \in S \cup T$. Since N is assumed to be an integer greater than or equal to 2, this is a contradiction. □

This finishes the proof of Theorem 1.5.1. □

Chapter 2

Identifiability for polynomial ODE models

2.1 Introduction

The question of parameter identifiability is of great importance in modeling, e.g. in biological systems (Muñoz-Tamayo et al. (2018)). Recent work studies identifiability in oncology (Brouwer et al. (2016)), phylogeny (Durden and Sullivant (2019), Long and Sullivant (2015)), Gaussian graphical models (Leung et al. (2016)), and cardiovascular models (Mahdi et al. (2014)). Various techniques have been used to study identifiability, and the use of differential algebra in particular extends back 30 years (see, e.g., Walter and Pronzato (1996)).

We illustrate the problem of parameter identifiability with the following toy example.

Example 2.1.1. Consider the ordinary differential equation

$$x' = \mu_1 + \mu_2. \tag{2.1}$$

The symbols have the following interpretations:

symbol	meaning
x	observable variable
μ_1, μ_2	unknown constant parameter

Question: Assume we can perfectly measure x in some time interval. Can we identify the numerical value of μ_1?

Answer: No.

Explanation: The solution to the system is $x(t) = (\mu_1 + \mu_2)t + x_0$. Replacing μ_1 with $\mu_1 + c$ and μ_2 with $\mu_2 - c$ for some number c will yield the same input-output data (t- and x-values), but the numerical value of μ_1 is different. In other words, for a given set of output data, there are multiple values of μ_1 that could have resulted in this data.

We say that the variable μ_1 is not *structurally identifiable*. No matter how accurately we can measure data, it is impossible for us to determine the value of μ_1.

Systems of ODEs with unknown parameters are often used to model real-world systems with the goal of using input-output data to determine the numerical values of some of the parameters. Sometimes this is impossible because of imperfect data, that is, the parameter in question is not *practically identifiable*. In other cases this is impossible because of the form of the equations, as in Example 2.1.1, that is, the parameter in question is not *structurally identifiable*. Note that structural identifiability is necessary for practical identifiability. Examples of such models are the Lotka-Volterra system, which models predator-prey populations, and the two-stage clonal expansion model shown below, which models cancer cell progression.

$$\begin{aligned} x' &= \mu_1 x - \mu_2 xy \\ y' &= -\mu_3 y + \mu_4 xy \end{aligned}$$

Lotka-Volterra system (predator-prey)

$$\begin{aligned} x' &= -\mu_0 x(1 - x_1) \\ x_1' &= -(\alpha_1 + \beta_1 + \mu_1)x_1 + \beta_1 + \alpha_1 x_1^2 \end{aligned}$$

Two-stage clonal expansion (Brouwer et al. (2016))
(cancer cell progression)

It was shown in (Hong et al., 2018, Example 5, p. 7) that if x is observable and y is not observable, then μ_1, μ_3, and μ_4 are identifiable but μ_2 is not. It was shown in (Brouwer et al. (2016)) that $\mu_0\mu_1$, $\alpha_1\mu_1$, and $\alpha_1 - \beta_1 - \mu_1$ are identifiable functions of parameters.

If it is known that a particular parameter in a given model is not structurally identifiable, then resources spent trying to identify that parameter would be wasted. Several techniques have been used to modify the course of action when it is known that a parameter of interest is not structurally identifiable. We list some of these here to illustrate the role that structural identifiability plays in modeling.

1. Use an accepted numerical value for some parameter. For example, substituting a numerical value for μ_2 in Example 2.1.1 results in μ_1 being identifiable.

2. Observe a variable that was not observable in the original model. In the Lotka-Volterra system above, it is clear from the result in (Hong et al., 2018, Example 5, p. 7) and the symmetry of the system that observing y as well as x will make μ_2 identifiable.

3. Modify the modeling process so that we are only interested in identifiable combinations of parameters. For example, in Example 2.1.1 we may find a way to obtain useful results from the fact that $\mu_1 + \mu_2$ and x_0 are identifiable. Variations on this idea are sometimes referred to as *reparametrization*. Although there is no standard definition of this term, it is prevalent in the literature (see Meshkat and Sullivant (2014), Dasgupta et al. (2007), Little et al. (2009)).

4. Introduce an input to some part of the system to obtain a modified system of equations in which the parameter of interest is identifiable (see Example 2.1.3).

There are several subtleties involved in the definition of identifiability that we will present. To justify these, we first look at some examples that illustrate these subtleties. Then we will give a rigorous definition and revisit these examples.

Example 2.1.2. Consider the system

$$\begin{cases} x' = \mu x \\ x(0) = x^*, \end{cases} \tag{2.2}$$

We ask the question: "If x can be measured perfectly at all times, can we determine the numerical values of μ and x^*?" Here, the solution to the system is

$$X(t) = x^* e^{\mu t}.$$

Certainly, x^* can be identified from perfect observation of x, since x^* equals $X(0)$, so x^* is identifiable. If x^* is not equal to 0, then we can identify μ by observing $X'(0)/X(0)$. However, if $x^* = 0$, then the value of μ is not unique. Thus, we would like to say that μ is generically identifiable. That is, it is identifiable as long as $x^* \neq 0$.

From this example, we want that our definition of identifiability

- should account for the notion of "generic" identifiability, that is, a parameter's being identifiable unless some equation among the parameters is satisfied.

Example 2.1.3. Consider a spring fixed at one end and attached to an object of mass m at the other end. For simplicity, assume the mass moves with no friction or air resistance. A typical system of equations modeling the displacement x from equilibrium is

$$\begin{cases} mx'' - kx = 0 \\ x(0) = x^* \\ x'(0) = v^*, \end{cases} \tag{2.3}$$

where k is the constant of the spring force. If $m \neq 0$, which is a natural assumption for a model of this type, and if the parameters satisfy the inequality

$$x^* \neq 0, \tag{2.4}$$

then perfect knowledge of x is sufficient to determine numerical value of k/m. To see this, observe that the solution to (2.3) satisfies the equation

$$\frac{k}{m} = \frac{x''}{x}$$

at all times at which x does not vanish. The condition (2.4) is sufficient to avoid the denominator vanishing at $t = 0$. Therefore if we have perfect knowledge of x, we can determine the numerical value of k/m.

However, perfect observation of x satisfying (2.3) is not enough to determine the numerical value of m. If we want to know m, one technique we can use is the introduction of an applied force on the spring. Thus, we obtain the following modified system:

$$\begin{cases} mx'' - kx - u = 0 \\ x(0) = x^* \\ x'(0) = v^*, \end{cases} \tag{2.5}$$

where u is the time-dependent applied force. One can show that a consequence of system (2.5) is

$$m = \frac{xu' - x'u}{xx''' - x'x''}.$$

Thus, at any time when the denominator is non-zero, we can use perfect knowledge of x and u to find the numerical value of m. A sufficient condition for guaranteeing the identifiability of m is thus $(xx''' - x'x'')|_{t=0} \neq 0$. Using the relation $mx'' = kx + u$ shows us that this is equivalent to the condition $x^*u'(0) - v^*u(0) \neq 0$.

From this example, we want that our definition of identifiability

- should allow for functions of parameters, not only individual parameters, to be identifiable
- should allow for an "input" variable
- should allow the initial values of the input variable to be used in the inequality determining genericity.

2.2 Notation and definitions

Based on the preceding examples, we would like to define the notion of identifiability for systems of differential equations that include state, input, and output variables, and allow constant parameters both in the coefficients of the differential equations and as initial conditions. We want to say that a function of parameters is identifiable if it can be determined uniquely from perfect input and output data as long as the parameters and the initial values of the input and its derivatives lie outside of some small set. We follow the notation of (Hong et al. (2018)), as follows.

Let

- $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\lambda)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$, $\boldsymbol{x^*} = (x_1^*, \ldots, x_n^*)$,
- $\boldsymbol{\theta} = \boldsymbol{x^*} \cup \boldsymbol{\mu}$, $s = n + \lambda$, and
- $R = \mathbb{C}[\boldsymbol{\mu}]\{\boldsymbol{x}, \boldsymbol{y}, u\}$.

We call $\boldsymbol{x}$ the state variables, $\boldsymbol{y}$ the output variables, and $\boldsymbol{\theta}$ the parameters. We will use u to represent an input variable. Let $\mathbb{C}^\infty(0)$ denote the set of functions from $\mathbb{C}$ to $\mathbb{C}$ that are complex analytic in some neighborhood of 0.

Let Σ be the system of differential equations with initial conditions:

$$\Sigma = \begin{cases} x_i' = \dfrac{F_i}{Q} \\ y_j = \dfrac{G_j}{Q} \\ x_i(0) = x_i^*, \end{cases} \tag{2.6}$$

where $F_1, \ldots, F_n, G_1, \ldots, G_m, Q \in \mathbb{C}[\boldsymbol{\mu}, \boldsymbol{x}, u]$. Let

$$\Omega = \{(\hat{\boldsymbol{\theta}}, \hat{u}) \in \mathbb{C}^s \times \mathbb{C}^\infty(0) \mid Q(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{x}}^*, \hat{u})|_{t=0} \neq 0\}. \tag{2.7}$$

Let $(\hat{\boldsymbol{\theta}}, \hat{u}) \in \Omega$. The system of ODEs obtained from (2.6) by inserting the components of $\hat{\boldsymbol{\theta}}$ into $\boldsymbol{\mu}$ and $\boldsymbol{x}$ and $\hat{u}$ into u in S has a unique solution. We denote this solution $X(\hat{\boldsymbol{\theta}}, \hat{u}), Y(\hat{\boldsymbol{\theta}}, \hat{u})$.

To accommodate the notion of "generic" identifiability, we will use the following definitions:

Definition 2.2.1. (see (Hong et al., 2018, Notation 1 (e), p. 4))

- Let $U \subset \mathbb{C}^s$. We say that U is Zariski open if there exists some $P \in \mathbb{C}[X_1, \ldots, X_s] \backslash 0$ such that

$$U = \{(a_1, \ldots, a_s) \in \mathbb{C}^s \mid P(a_1, \ldots, a_s) \neq 0\}.$$

- Let $U \subset \mathbb{C}^\infty(0)$. We say that U is Zariski open if there exists $h \in \mathbb{Z}_{\geqslant 0}$ and $P \in \mathbb{C}[X_0, \ldots, X_h] \backslash 0$ such that

$$U = \{\hat{u} \in \mathbb{C}^\infty(0) \mid P(\hat{u}, \hat{u}', \ldots, \hat{u}^{(h)})|_{t=0} \neq 0\}.$$

- Let $\tau(\mathbb{C}^s)$ denote the set of non-empty Zariski open subsets of $\mathbb{C}^s$.

- Let $\tau(\mathbb{C}^\infty(0))$ be the set of non-empty Zariski open subset of $\mathbb{C}^\infty(0)$.

Definition 2.2.2. Let Σ and Ω be as in (2.6) and (2.7). Let $f \in \mathbb{C}(\boldsymbol{\theta})$ and let $\mathrm{dom} f$ be the domain of f. We say that f is Σ-*identifiable* if

$$\exists \Theta \in \tau(\mathbb{C}^s)\ \exists U \in \tau(\mathbb{C}^\infty(0))\ \forall (\hat{\boldsymbol{\theta}}, \hat{u}) \in (\Omega \cap \Theta \cap \mathrm{dom} f) \times U\ |S_f(\hat{\boldsymbol{\theta}}, \hat{u})| = 1,$$

where $S_f(\hat{\boldsymbol{\theta}}, \hat{u}) = \{f(\tilde{\boldsymbol{\theta}}) \mid (\tilde{\boldsymbol{\theta}}, \hat{u}) \in \Omega \cap \mathrm{dom} f \text{ and } Y(\tilde{\boldsymbol{\theta}}, \hat{u}) = Y(\hat{\boldsymbol{\theta}}, \hat{u})\}$. We say f is *identifiable* if Σ is clear from context.

Note. Our definition of identifiability is sometimes referred to as *global identifiability* in the literature. This is in contrast to *local identifiability*, which differs from Definition 2.2.2 in that "$|S_f(\hat{\boldsymbol{\theta}}, \hat{u})| = 1$" is replaced by "$|S_f(\hat{\boldsymbol{\theta}}, \hat{u})|$ is finite."

To illustrate the definition, we return to our earlier examples.

Example 2.2.1. We revisit Example 2.1.1. To make this conform to our convention on labeling state and output variables, we write this system as

$$\Sigma = \begin{cases} x' = \mu_1 + \mu_2 \\ y = x \\ x(0) = x^*. \end{cases} \tag{2.8}$$

We have that $s = 3$ and $\Omega = \mathbb{C}^3$. To see that x^* is Σ-identifiable, take $\Theta = \mathbb{C}^3$ and $U = \mathbb{C}^\infty(0)$. Note that although u does not appear in Σ, we discuss the role of U to illustrate the definition of identifiability. For $\hat{\boldsymbol{\theta}} = (\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*)$ and $\tilde{\boldsymbol{\theta}} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{x}^*) \in \mathbb{C}^3$ and $\hat{u} \in \mathbb{C}^\infty(0)$, we have $Y(\hat{\boldsymbol{\theta}}, \hat{u})(t) = (\hat{\mu}_1 + \hat{\mu}_2)t + \hat{x}^*$ and $Y(\tilde{\boldsymbol{\theta}}, \hat{u})(t) = (\tilde{\mu}_1 + \tilde{\mu}_2)t + \tilde{x}^*$. If $Y(\hat{\boldsymbol{\theta}}, \hat{u})(t) = Y(\tilde{\boldsymbol{\theta}}, \hat{u})(t)$ for all values of t, it must be that $\hat{x}^* = \tilde{x}^*$. So $|S_{x^*}(\hat{\boldsymbol{\theta}}, \hat{u})| = 1$.

To see that μ_1 is not Σ-identifiable, choose any Θ and U and choose any $\hat{\boldsymbol{\theta}} = (\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*) \in \Theta$ and $\hat{u} \in U$. Since Θ is Zariski open and non-empty, the set $L := \{(\hat{\mu}_1 + c, \hat{\mu}_2 - c, \hat{x}^*) \mid c \in \mathbb{C}\}$ is such that either $L \cap \Theta = \emptyset$ or $L \cap \Theta$ is infinite. Since $L \cap \Theta$ contains $\hat{\boldsymbol{\theta}}$, it must be that $L \cap \Theta$ is infinite.

Hence there is some $c \neq 0$ such that $(\hat{\mu}_1 + c, \hat{\mu}_2 - c, \hat{x}^*) \in \Theta$. We have $Y((\hat{\mu}_1 + c, \hat{\mu}_2 - c, \hat{x}^*), \hat{u}) = Y((\hat{\mu}_1, \hat{\mu}_2, \hat{x}^*), \hat{u})$. Hence $|S_{\mu_1}(\hat{\boldsymbol{\theta}}, \hat{u})| > 1$ and μ_1 is not Σ-identifiable.

Example 2.2.2. We revisit Example 2.1.2. We have

$$\Sigma = \begin{cases} x' = \mu x \\ y = x \\ x(0) = x^*. \end{cases} \tag{2.9}$$

We have $s = 2$ and $\Omega = \mathbb{C}$. To see that μ is identifiable, take $\Theta = \{(\hat{\mu}, \hat{x}^*) \in \mathbb{C}^2 \mid \hat{x}^* \neq 0\}$ and $U = \mathbb{C}^\infty(0)$. Now suppose $\hat{\boldsymbol{\theta}} \in \Theta$, $\tilde{\boldsymbol{\theta}} \in \Omega$, and $\hat{u}$ is in U. If $Y(\hat{\boldsymbol{\theta}}, \hat{u})(t) = Y(\tilde{\boldsymbol{\theta}}, \hat{u})(t)$, then we have $\hat{x}^* e^{\hat{\mu}t} = \tilde{x}^* e^{\tilde{\mu}t}$ and, since $\hat{x}^* \neq 0$, $\hat{x}^* \tilde{x}^* \neq 0$, which implies $\hat{\mu} = \tilde{\mu}$. Therefore $|S_\mu(\hat{\boldsymbol{\theta}}, \hat{u})| = 1$.

Example 2.2.3. We revisit the model (2.5) of Example 2.1.3. We have

$$\Sigma = \begin{cases} x_1' = x_2 \\ x_2' = \dfrac{\mu_1 x_1 + u}{\mu_2} \\ y = x_1 \\ x_1(0) = x_1^* \\ x_2(0) = x_2^*. \end{cases} \tag{2.10}$$

Note that we have used μ_1 and μ_2 instead of k and m. We have $s = 4$ and $\Omega = \{(\hat{\mu}_1, \hat{\mu}_2, \hat{x}_1^*, \hat{x}_2^*) \in \mathbb{C}^4 \mid \hat{\mu}_2 \neq 0\}$.

To see that μ_2 is identifiable, take $\Theta = \{(\hat{\mu}_1, \hat{\mu}_2, \hat{x}_1^*, \hat{x}_2^*) \in \mathbb{C}^4 \mid \hat{x}_1^* \neq 0\}$ and $U = \{\hat{u} \in \mathbb{C}^\infty(0) \mid \hat{u}(0)\hat{u}'''(0) - \hat{u}'(0)\hat{u}''(0) \neq 0\}$. Let $(\hat{\boldsymbol{\theta}}, \hat{u}) \in (\Omega \cap \Theta) \times U$ and suppose $(\tilde{\boldsymbol{\theta}}, \hat{u}) \in \Omega \times U$ is such that $Y(\hat{\boldsymbol{\theta}},\hat{u}) = Y(\tilde{\boldsymbol{\theta}},\hat{u})$. As described above, we have

$$\hat{\mu}_2 = \frac{Y(\hat{\boldsymbol{\theta}},\hat{u})(t)\cdot\hat{u}'(t) - Y(\hat{\boldsymbol{\theta}},\hat{u})'(t)\cdot\hat{u}(t)}{Y(\hat{\boldsymbol{\theta}},\hat{u})(t)\cdot Y(\hat{\boldsymbol{\theta}},\hat{u})'''(t) - Y(\hat{\boldsymbol{\theta}},\hat{u})'(t)\cdot Y(\hat{\boldsymbol{\theta}},\hat{u})''(t)} \tag{2.11}$$

and

$$\tilde{\mu}_2 = \frac{Y(\tilde{\boldsymbol{\theta}},\hat{u})(t)\cdot\hat{u}'(t) - Y(\tilde{\boldsymbol{\theta}},\hat{u})'(t)\cdot\hat{u}(t)}{Y(\tilde{\boldsymbol{\theta}},\hat{u})(t)\cdot Y(\tilde{\boldsymbol{\theta}},\hat{u})'''(t) - Y(\tilde{\boldsymbol{\theta}},\hat{u})'(t)\cdot Y(\tilde{\boldsymbol{\theta}},\hat{u})''(t)}$$

$$= \frac{Y(\hat{\boldsymbol{\theta}},\hat{u})(t)\cdot\hat{u}'(t) - Y(\hat{\boldsymbol{\theta}},\hat{u})'(t)\cdot\hat{u}(t)}{Y(\hat{\boldsymbol{\theta}},\hat{u})(t)\cdot Y(\hat{\boldsymbol{\theta}},\hat{u})'''(t) - Y(\hat{\boldsymbol{\theta}},\hat{u})'(t)\cdot Y(\hat{\boldsymbol{\theta}},\hat{u})''(t)} \tag{2.12}$$

for all times t that do not make the denominator of (2.11) and (2.12) vanish.

We show that there is at least one value of t at which the denominator does not vanish. For conciseness, we use Y, Y', Y'', and Y''' for $Y(\hat{\boldsymbol{\mu}},\hat{u})(t)$ and its first, second, and third derivatives, respectively. Suppose $YY''' - Y'Y'' = 0$. Since the left-hand side is the Wronskian of Y and Y'', this implies that there is some $c_1 \in \mathbb{C}$ such that $Y'' = c_1 Y$. Using the relation $Y'' = \frac{\mu_1}{\mu_2}Y + \frac{1}{\mu_2}\hat{u}$, it follows that $(\mu_2 c_1 - \mu_1)Y = \hat{u}$. It follows that $\hat{u}'' = (\mu_2 c_1 - \mu_1)Y'' = (\mu_2 c_1 - \mu_1)c_1 Y = c_1\hat{u}$. Thus $\hat{u}''' = c_1\hat{u}'$, and we have that $\hat{u}\hat{u}''' - \hat{u}'\hat{u}'' = 0$. However, since $\hat{u} \in U$, this is impossible.

The techniques used to prove identifiability in Example 2.1.3 illustrate that if a function of parameters can be expressed as a function of input and output variables alone, then that function of parameters is identifiable. Moreover, the converse is true. The statement for individual parameters was proven in (Hong et al., 2018, Proposition 1, p. 13). We generalize this to rational functions of parameters. To facilitate the proof, we introduce some new notation:

- Let S be the set $\{Qx_i - F_i, Qy_j - G_j\}_{i=1,\ldots,n}^{j=1,\ldots,m}$.

- Let J be the ideal $\mathbb{C}[\boldsymbol{\mu}]\{\boldsymbol{x},\boldsymbol{y},u\}([S]:Q^\infty)$. It is known J is a prime ideal (Hong et al., 2018, eqn. 8, Lemma 1, and Lemma 2).

- Let $\mathcal{F} = \mathrm{Frac}(R/J)$.

- Let $\mathcal{E}$ be the image of $\mathrm{Frac}(\mathbb{C}\{\mathbf{y}, u\})$ in $\mathrm{Frac}(R/J)$.

Proposition 2.2.1. *Let Σ be as in* (2.6). *Let $f \in \mathbb{C}(\boldsymbol{\theta})$. Then f is Σ-identifiable if and only if $f \in \mathcal{E}$.*

Proof. Write $f = f_1/f_2$, where $f_1, f_2 \in \mathbb{C}[\boldsymbol{\theta}]$. Let Σ_1 be the system of equations obtained by adding

$$x'_{n+1} = 0,$$
$$y_{m+1} = x_{n+1} - f,$$
$$x_{n+1}(0) = x^*_{n+1}$$

to Σ, where x_{n+1} and y_{m+1} are new indeterminates and x^*_{n+1} is a new parameter. Accordingly, let Ω_1 be the complement of the vanishing of Qf_2 in $\mathbb{C}^{s+1} \times \mathbb{C}^\infty(0)$. Let

$$J_1 := [S \cup \{x'_{n+1}, f_2 y_{m+1} - f_2 x_{n+1} + f_1\}] : Qf_2^\infty \subset R\{x_{n+1}, y_{m+1}\}.$$

We will talk about Σ-identifiability of an element of $\mathbb{C}(\boldsymbol{\theta})$ and Σ_1-identifiability of an element of $\mathbb{C}(\boldsymbol{\theta}, x^*_{n+1})$. We present the proof in four parts.

1. *f is not Σ-identifiable $\implies$ f is not Σ_1-identifiable.* Suppose f is not Σ-identifiable. Let $\Theta_1 \in \tau(\mathbb{C}^{s+1})$ and let $U \in \tau(\mathbb{C}^\infty(0))$. Let $\mathrm{proj}_{\mathbb{C}^s} \colon \mathbb{C}^{s+1} \to \mathbb{C}^s$ send $(\hat{\boldsymbol{\theta}}, \hat{x}^*_{n+1})$ to $\hat{\boldsymbol{\theta}}$ and let $\mathrm{proj}_{\mathbb{C}^s \times \mathbb{C}^\infty(0)} \colon \mathbb{C}^{s+1} \times \mathbb{C}^\infty(0) \to \mathbb{C}^s \times \mathbb{C}^\infty(0)$ send $(\hat{\boldsymbol{\theta}}, \hat{x}^*_{n+1}, \hat{u})$ to $(\hat{\boldsymbol{\theta}}, \hat{u})$. Let $\Theta = \mathrm{proj}_{\mathbb{C}^s}(\Theta_1)$. Because f is not Σ-identifiable, there exist $(\hat{\boldsymbol{\theta}}, \hat{u}) \in \mathrm{proj}_{\mathbb{C}^s \times \mathbb{C}^\infty(0)}(\Omega_1) \cap \Theta \times U$ and $\tilde{\boldsymbol{\theta}} \in \mathbb{C}^s$ such that $f(\hat{\boldsymbol{\theta}}) \neq f(\tilde{\boldsymbol{\theta}})$ and $Y_\Sigma(\hat{\boldsymbol{\theta}}, \hat{u}) = Y_\Sigma(\tilde{\boldsymbol{\theta}}, \hat{u})$. Let $\hat{x}^*_{n+1}$ be such that $(\hat{\boldsymbol{\theta}}, \hat{x}^*_{n+1}) \in \Theta_1$. Let $\hat{\boldsymbol{\theta}}_\mathbf{1} = (\hat{\boldsymbol{\theta}}, \hat{x}^*_{n+1})$ and let $\tilde{\boldsymbol{\theta}}_\mathbf{1} = (\tilde{\boldsymbol{\theta}}, \hat{x}^*_{n+1} - f(\hat{\boldsymbol{\theta}}) + f(\tilde{\boldsymbol{\theta}}))$. Since $\tilde{\boldsymbol{\theta}}$ is such that $Q(\tilde{\boldsymbol{\theta}}, \hat{u})$ and $f(\tilde{\boldsymbol{\theta}})$ are well-defined and x^*_{n+1} does not appear in f it follows that $(\tilde{\boldsymbol{\theta}}_\mathbf{1}, \hat{u}) \in \Omega_1$. Hence we have $f(\hat{\boldsymbol{\theta}}_\mathbf{1}) = f(\hat{\boldsymbol{\theta}}) \neq f(\tilde{\boldsymbol{\theta}}) = f(\tilde{\boldsymbol{\theta}}_\mathbf{1})$. Note that the first m components of $Y_{\Sigma_1}(\hat{\boldsymbol{\theta}}_\mathbf{1}, \hat{u})$ are exactly $Y_\Sigma(\hat{\boldsymbol{\theta}}, \hat{u})$ and the last component is $y(t) = \hat{x}^*_{n+1} - f(\hat{\boldsymbol{\theta}})$. Similarly, the first m components

of $Y_{\Sigma_1}(\tilde{\boldsymbol{\theta}}_{\mathbf{1}},\hat{u})$ are exactly $Y_\Sigma(\tilde{\boldsymbol{\theta}},\hat{u})$ and the last component is $y(t)=\hat{x}^*_{n+1}-f(\hat{\boldsymbol{\theta}})$. Therefore $Y_{\Sigma_1}(\hat{\boldsymbol{\theta}}_{\mathbf{1}})=Y_{\Sigma_1}(\tilde{\boldsymbol{\theta}}_{\mathbf{1}})$. We conclude that f is not Σ_1-identifiable.

2. *f is Σ-identifiable $\implies$ f is Σ_1-identifiable.* Suppose f is Σ-identifiable. Let $\Theta\in\tau(\mathbb{C}^s)$ and $U\in\tau(\mathbb{C}^\infty(0))$ be such that $\forall(\hat{\boldsymbol{\theta}},\hat{u})\in\Theta\times U\ |S_f(\hat{\boldsymbol{\theta}},\hat{u})|=1$. Let $\Theta_1=\Theta\times\mathbb{C}$. Let $(\hat{\boldsymbol{\theta}}_{\mathbf{1}},\hat{u})\in\Omega_1\cap\Theta_1\times U$ and let $\tilde{\boldsymbol{\theta}}_{\mathbf{1}}\in\Omega_1$. Let $\hat{\boldsymbol{\theta}}=\operatorname{proj}_{\mathbb{C}^s}(\hat{\boldsymbol{\theta}}_{\mathbf{1}})$ and let $\tilde{\boldsymbol{\theta}}=\operatorname{proj}_{\mathbb{C}^s}(\tilde{\boldsymbol{\theta}}_{\mathbf{1}})$. Suppose $Y_{\Sigma_1}(\hat{\boldsymbol{\theta}}_{\mathbf{1}},\hat{u})=Y_{\Sigma_1}(\tilde{\boldsymbol{\theta}}_{\mathbf{1}},\hat{u})$. Since the first m components of Y_{Σ_1} do not depend on x^*_{n+1}, it follows that $Y_\Sigma(\hat{\boldsymbol{\theta}},\hat{u})=Y_\Sigma(\tilde{\boldsymbol{\theta}},\hat{u})$. Because of the choice of Θ and the fact that f does not depend on x^*_{n+1}, we have $f(\hat{\boldsymbol{\theta}}_{\mathbf{1}})=f(\hat{\boldsymbol{\theta}})=f(\tilde{\boldsymbol{\theta}})=f(\tilde{\boldsymbol{\theta}}_{\mathbf{1}})$. We conclude that f is Σ_1-identifiable.

3. *f is Σ_1-identifiable $\iff$ x^*_{n+1} is Σ_1-identifiable.* Suppose x^*_{n+1} is Σ_1-identifiable as witnessed by Θ and U. Let $(\hat{\boldsymbol{\theta}},\hat{u})\in(\Theta\cap\Omega_1)\times U$, and let $\tilde{\boldsymbol{\theta}}\in\Omega_1$ be such that $Y_{\Sigma_1}(\hat{\boldsymbol{\theta}},\hat{u})=Y_{\Sigma_1}(\tilde{\boldsymbol{\theta}},\hat{u})$. Since x^*_{n+1} is Σ_1-identifiable and by our choice of Θ and U, it follows that $\tilde{x}^*_{n+1}=\hat{x}^*_{n+1}$. We have that $f(\tilde{\boldsymbol{\theta}})=-Y_{\Sigma_1}(\tilde{\boldsymbol{\theta}},\hat{u})_{m+1}+\tilde{x}^*_{n+1}=-Y_{\Sigma_1}(\hat{\boldsymbol{\theta}},\hat{u})_{m+1}+\hat{x}^*_{n+1}=f(\hat{\boldsymbol{\theta}})$. Thus $|S_f(\hat{\boldsymbol{\theta}},\hat{u})|=1$ and we conclude that f is Σ_1-identifiable. A similar argument proves the other direction.

4. *x^*_{n+1} is Σ_1-identifiable $\iff$ $f\in\mathcal{E}$.*

   Let $\mathcal{F}_1=\operatorname{Frac}(R\{x_{n+1},y_{m+1}\}/J_1)$. We have an injection $\iota\colon\mathcal{F}\hookrightarrow\mathcal{F}_1$. Let $\mathcal{E}_1$ equal $\iota(\mathcal{E})$. By (Hong et al., 2018, Proposition 1 (a) $\Leftrightarrow$ (c), p. 13), x^*_{n+1} is Σ_1-identifiable if and only if $x_{n+1}\in\mathcal{E}_1(y_{m+1})$. We also have $x_{n+1}\in\mathcal{E}_1(y_{m+1})$ if and only if $f=x_{n+1}-y_{m+1}\in\mathcal{E}_1(y_{m+1})$. We now show that $f\in\mathcal{E}_1(y_{m+1})$ if and only if $f\in\mathcal{E}_1$. One direction is trivial. We address the other direction.

   Consider the differential ring homomorphism $\rho\colon R\{x_{n+1},y_{m+1}\}\to R\{x_{n+1},y_{m+1}\}$ defined by $\rho|_R=\mathrm{id}|_R$, $\rho(y_{m+1})=y_{m+1}+1$, and $\rho(x_{n+1})=x_{n+1}+1$. We show that $a\in J_1$ if and only if $\rho(a)\in J_1$. Since $S\subset R$, ρ fixes S. Now $\rho(x_{n+1})'=(\rho(x_{n+1}))'=(x_{n+1}+1)'=x'_{n+1}$, and similarly $\rho(f_2y_{m+1}-f_2x_{n+1}+f_1)=f_2y_{m+1}-f_2x_{n+1}+f_1$. Suppose $a\in J_1$. Then there exists ν and an element $g\in[S\cup\{x'_{n+1},f_2y_{m+1}-f_2x_{n+1}+f_1\}]$ such that $(Qf_2)^\nu a=g$. Since

ρ fixes Q, f_2, and g, it follows that $(Qf_2)^{\nu}\rho(a) = g$, and therefore $\rho(a) \in J_1$. Since ρ^{-1} fixes $S \cup \{x'_{n+1}, f_2 y_{m+1} - f_2 x_{n+1} + f_1\}$, a similar argument shows that if $\rho(a) \in J_1$ then $a \in J_1$.

Thus, ρ induces an automorphism σ on $\mathcal{F}_1$ such that σ fixes $\iota(\mathcal{F})$, $\sigma(x_{n+1}) = x_{n+1} + 1$, and $\sigma(y_{m+1}) = y_{m+1} + 1$. The fixed field of $\sigma|_{\mathcal{E}_1(y_{m+1})}$ is $\mathcal{E}_1$. Note that $\sigma(f) = \sigma(x_{n+1} - y_{m+1}) = f$. Therefore if $f \in \mathcal{E}_1(y_{m+1})$, then $f \in \mathcal{E}_1$. Since the map $\iota|_{\mathcal{E}} \colon \mathcal{E} \hookrightarrow \mathcal{E}_1$ is a bijection, we conclude that $f \in \mathcal{E}$. $\square$

2.3 Identifiability of coefficients of a characteristic set

In this section, we discuss the use of characteristic sets in determining identifiability.

2.3.1 Definitions and basic properties

Definition 2.3.1. Let R be a ring. A function $\delta \colon R \to R$ is called a *derivation* if it satisfies

- $\forall a, b \in R \quad \delta(a+b) = \delta(a) + \delta(b)$ and
- $\forall a, b \in R \quad \delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b)$.

Definition 2.3.2. A ring R equipped with one or more derivations is called a *differential ring*. Let (R, δ) be a differential ring, and let I be a non-empty subset of R. We say I is a *differential ideal* if I is an ideal and $\forall a \in I \ \delta(a) \in I$. If S is a subset of R, we denote the smallest differential ideal containing S by $[S]$.

Every element of $[S]$ can be written in the form $\sum_{i=1}^{N} \sum_{j=0}^{M_N} r_{i,j} s_i^{(j)}$, where N and $M_1, \ldots, M_N$ are natural numbers, each $r_{i,j}$ is in R, and each s_i is in S.

We present the definition and basic properties of characteristic sets for a differential polynomial ring over a field of characteristic 0 and one derivation. For the case of several derivations, see (Hubert (2000)).

Let K be a field of characteristic 0.

Definition 2.3.3. The differential polynomial ring $\mathcal{R} := K\{X_1, \ldots, X_n\}$ with derivation δ is isomorphic to the polynomial ring $K[X_1, \ldots, X_n, \delta X_1, \ldots, \delta X_n, \delta^2 X_1, \ldots, \delta^2 X_n, \ldots]$ with derivation δ defined by $\delta(K) = 0$ and $\delta(\delta^i X_j) = \delta^{i+1} X_j$.

Let $\{\delta^i X_j\}$ stand for $\{\delta^i X_j \mid i \geqslant 0, 1 \leqslant j \leqslant n\}$.

Definition 2.3.4. A ranking on $\mathcal{R}$ is a total order $>$ on $\{\delta^i X_j\}$ that satisfies the following properties:

- $\forall a \in \{\delta^i X_j\}\ \delta a > a$
- $\forall a, b \in \{\delta^i X_j\}\ a > b \to \delta a > \delta b$.

Definition 2.3.5. Let $>$ be a ranking on $\mathcal{R}$. Let $p \in \mathcal{R} \backslash K$.

- The *leader* of p, denoted $\mathrm{ld}(p)$, is the element of $\{\delta^i X_j\}$ of highest rank appearing in p.
- The *initial* of p, denoted $\mathrm{in}(p)$, is the leading coefficient of p when p is viewed as a polynomial in $\mathrm{ld}(p)$ with coefficients in $K[\delta^i X_j < \mathrm{ld}(p)]$.
- The *separant* of p, denoted $\mathrm{sep}\, p$, is the partial derivative of p with respect to $\mathrm{ld}(p)$.

Definition 2.3.6. Let $>$ be a ranking on $\mathcal{R}$. Let $p, q \in \mathcal{R} \backslash K$ and let A be a subset of $\mathcal{R} \backslash K$.

- The *rank* of p, denoted $\mathrm{rank}\, p$, is $\mathrm{ld}(p)^{\deg_{\mathrm{ld}(p)} p}$.
- We say $p > q$ if either
  - $\mathrm{ld}(p) > \mathrm{ld}(q)$ or
  - $\mathrm{ld}(p) = \mathrm{ld}(q)$ and $\deg_{\mathrm{ld}(p)} p > \deg_{\mathrm{ld}(q)} q$.

  Note that $>$ does not induce a total order on $\mathcal{R}$.
- We say *q is reduced with respect to p* if
  - no proper derivative of $\mathrm{ld}(p)$ appears in q and

– $\deg_{\mathrm{ld}(p)} q < \deg_{\mathrm{ld}(p)} p$.

- We say q *is reduced with respect to* A if q is reduced with respect to every element of A.
- We say A is *autoreduced* if every element of A is reduced with respect to every other element of A.

Definition 2.3.7. Let $A = a_1 < \ldots < a_r$ and $B = b_1 < \ldots < b_s$ be autoreduced subsets of $\mathcal{R} \backslash K$. Note that such a strict ordering is possible by the definition of an autoreduced set and every autoreduced set is finite by (Kolchin, 1973, p. 77). We say $A < B$ if either

- $\exists k,\ 1 \leqslant k \leqslant \min\{r,s\}$, such that $\forall i < k\ \mathrm{rank}(a_i) = \mathrm{rank}(b_i)$ and $\mathrm{rank}(a_k) < \mathrm{rank}(b_k)$ or
- $\forall i \leqslant s\ \mathrm{rank}(a_i) = \mathrm{rank}(b_i)$ and $r > s$.

Definition 2.3.8. If A is autoreduced and I is a differential ideal of $\mathcal{R}$, we say A is a *characteristic set* of I if no autoreduced subset B of I is such that $B < A$ (cf. Definition 2.3.7).

We will need the following lemmas for our results:

Lemma 2.3.1. *Let $1 < r \leqslant n$ and let $>$ be a differential ranking on $\mathcal{R}$ such that $k \geqslant r > l$ implies $\forall i, j\ \delta^i X_k > \delta^j X_l$. Let I be a differential ideal of $\mathcal{R}$. If A is a characteristic set of I, then $B := A \cap K\{X_1, \ldots, X_{r-1}\}$ is a characteristic set of $J := I \cap K\{X_1, \ldots, X_{r-1}\}$.*

Proof. Write $B = B_1 < \ldots < B_b$ and

$$A = B_1 < \ldots < B_b < A_1 < \ldots < A_a.$$

Suppose $C = C_1 < \ldots < C_c$ is an autoreduced subset of J. We show that it is impossible that $C < B$.

Suppose there exists $k,\ 1 \le k \le \min\{b,c\}$, such that $(\forall i < k\ \mathrm{rank}(C_i) = \mathrm{rank}(B_i))$ and $\mathrm{rank}(C_k) < \mathrm{rank}(B_k)$. Then $C_1, \ldots, C_k$ is an autoreduced subset of I that is less than A, which contradicts that A is a characteristic set of I.

Suppose that $(\forall i\ \mathrm{rank}(C_i) = \mathrm{rank}(B_i))$ and $c > b$. Note that $C_1, \ldots, C_{b+1}$ is an autoreduced subset of I. Furthermore, since $C_{b+1} \in K\{X_1, \ldots, X_{r-1}\}$ and $A_1 \notin K\{X_1, \ldots, X_{r-1}\}$, it follows that $\mathrm{rank}(C_{b+1}) < \mathrm{rank}(A_1)$. Therefore $C_1 \ldots, C_{b+1}$ is less than A, which contradicts that A is a characteristic set of I. □

Lemma 2.3.2. *(Kolchin, 1973, Proposition 3, p. 81; Lemma 2, p. 167) Let $P \subset \mathcal{R}$ be a prime differential ideal and let $>$ be a ranking on $\mathcal{R}$. Then there exists a subset $A \subset P$ that is a characteristic set of P. Furthermore, $P = [A] : H^\infty$, where H is the product of the initials and separants of the elements of A. Recall that $[A]$ is the smallest differential ideal containing A. Explicitly,*

$$P = \{r \in \mathcal{R} \mid \exists \nu \in \mathbb{N}\ H^\nu \cdot r \in [A]\}.$$

Lemma 2.3.3. *Let $>$ be a ranking on $\mathcal{R}$ and let A be an autoreduced set. Then $|A| \leqslant n$.*

Proof. Let $\delta^i X_j$ and $\delta^k X_l$ be leaders of elements of A. By the definition of reduced, we have that $j \neq l$. Thus the leaders of the elements of A are derivatives of distinct X_i. □

2.3.2 Results on identifiability

We establish the following notation.

Notation. Let

- $R = \mathbb{C}[\boldsymbol{\mu}]\{\boldsymbol{x}, \boldsymbol{y}, u\}$,
- $\mathcal{R} = \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{x}, \boldsymbol{y}, u\}$,
- Σ be as in (2.6) with $Q \in \mathbb{C}[\boldsymbol{\mu}]\backslash 0$,
- S be the set $\{Qx_i - F_i, Qy_j - G_j\}_{i=1,\ldots,n}^{j=1,\ldots,m} \subset R$,
- J be the ideal $[S] : Q^\infty$ in R,

- $\mathcal{J}$ be the ideal $\mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{x},\boldsymbol{y},u\}([S]:Q^\infty) = \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{x},\boldsymbol{y},u\}[S]$ in $\mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{x},\boldsymbol{y},u\}$.

We will be using characteristic sets of a differential polynomial ring over a field, so we will work over $\mathcal{R}$ instead of R. We have thus far only defined identifiability in terms of R. The following lemma will allow us to work with $\mathcal{R}$.

Lemma 2.3.4. *If $f \in \mathbb{C}(\boldsymbol{\mu})$ belongs to the image of* $\operatorname{Frac}(\mathbb{C}\{\boldsymbol{y},u\})$ *in* $\operatorname{Frac}(\mathcal{R}/\mathcal{J})$*, then f is Σ-identifiable.*

Proof. Write $f = f_1/f_2$, where $f_1, f_2 \in \mathbb{C}\{\boldsymbol{y},u\}$ satisfy $\gcd(f_1, f_2) = 1$. Now $f_2 f - f_1 \in \mathcal{J}$. Now there exists $g \in \mathbb{C}[\boldsymbol{\mu}]$ such that $g \cdot (f_2 f - f_1) \in J$. Since J is prime and $J \cap \mathbb{C}[\boldsymbol{\mu}] = \emptyset$, we have $f_2 f - f_1 \in J$. Therefore $f \in \operatorname{Frac}(R/J)$. Now f is Σ-identifiable by Proposition 2.2.1. □

We now present a condition under which the coefficients of a monic element of $\mathcal{J}$ are identifiable.

Proposition 2.3.1. *Let $g \in \mathcal{J}$. Write $g = \sum_{j=1}^{N} a_j z_j$, where $a_j \in \mathbb{C}(\boldsymbol{\mu})$, $a_1 = 1$, and $z_1, \ldots, z_N$ are distinct monomials in $\mathbb{C}\{\boldsymbol{y},u\}$.*

If for all non-empty $Z \subsetneq \{z_1, \ldots, z_N\}$ it holds that $Wr(Z) \notin \mathcal{J}$,

then a_j is identifiable for all $j = 1, \ldots, N$.

Proof. Modulo $\mathcal{J}$, we have

$$\sum_{j=2}^{N} a_j z_j = -z_1$$

Since $\mathcal{J}$ is a differential ideal, the derivatives of the above equation are also true. Hence, we obtain the system

$$\begin{aligned} \sum a_j z_j &= -z_1 \\ \sum a_j z_j' &= -z_1' \\ &\vdots \\ \sum a_j z_j^{(N-1)} &= -z_1^{(N-1)}. \end{aligned} \tag{2.13}$$

Let M be the $N \times N$ matrix whose (k,ℓ) entry is $z_{\ell+1}^{(k-1)}$. Now (2.13) can be written

$$M \begin{pmatrix} a_2 \\ \vdots \\ a_N \end{pmatrix} = - \begin{pmatrix} z_1 \\ \vdots \\ z_1^{(N-1)} \end{pmatrix}.$$

We can use Cramer's rule to solve for a_j. For each j, let M_j be M but with the j-th column replaced by $-(z_1,\ldots,z_1^{(N-1)})^T$. It follows that

$$\det(M)a_j = \det(M_j).$$

Because M is the Wronskian of $z_2,\ldots,z_N$ and we have assumed such an element does not belong to $\mathcal{I}$, it follows that $\det(M) \neq 0$. Since $\det(M)$ and $\det(M_j)$ are in $\mathbb{C}\{\boldsymbol{y},u\}$, the element a_j is identifiable by Lemma 2.3.4. $\square$

Proposition 2.3.1 implies that it is sufficient to check global identifiability of coefficients by checking the Wronskian of all subsets of monomials of size $N-1$. It turns out that it is sufficient to check just a single subset of size $N-1$. This will be shown in Proposition 2.3.2, but first we will need the following lemma:

Lemma 2.3.5. *Let K be a field and let V be a K-vector space. Let $a_1,\ldots,a_N \in K\backslash\{0\}$ and $v_1,\ldots,v_N \in V$ be such that $\sum_{\ell=1}^{N} a_\ell v_\ell = 0$.*

If there is a subset of $\{v_1,\ldots,v_N\}$ of size $N-1$ that is K-linearly dependent, then all subsets of $\{v_1,\ldots,v_N\}$ of size $N-1$ are K-linearly dependent.

Proof. Without loss of generality, we show that if $\{v_1,\ldots,v_{N-1}\}$ is K-linearly dependent then all subsets of $\{v_1,\ldots,v_N\}$ of size $N-1$ are K-linearly dependent. Suppose there exist $b_1,\ldots,b_{N-1} \in K$ such that $b_1v_1+\ldots+b_{N-1}v_{N-1} = 0$.

Case 1: For all i,j it holds that $\frac{b_i}{a_i} = \frac{b_j}{a_j}$. It follows that $a_Nv_N = 0$ and since $a_N \neq 0$ it must be that $v_N = 0$. Now for all k such that $1 \leqslant k \leqslant N-1$, the set $\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_N\}$ is K-linearly dependent.

Case 2: There exist i, j such that $\frac{b_i}{a_i} \neq \frac{b_j}{a_j}$. Fix these i and j. Let k be such that $1 \leqslant k \leqslant N-1$. Then

$$\frac{b_k}{a_k} \cdot \sum_{\ell=1}^{N} a_\ell v_\ell - \sum_{\ell=1}^{N-1} b_\ell v_\ell = 0. \tag{2.14}$$

On the left-hand side of (2.14), the coefficient of v_k is 0. Also, the coefficients of v_i and v_j are $\dfrac{b_k a_i}{a_k} - b_i$ and $\dfrac{b_k a_j}{a_k} - b_j$, respectively. If both of these are 0, then $\dfrac{b_i}{a_i} = \dfrac{b_j}{a_j}$, which is not true. Hence at least one coefficient in (2.14) is non-zero and this gives a K-linear dependence of $\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_N\}$. $\square$

Proposition 2.3.2. *Let $p = \sum_{i=1}^{N} a_i z_i$, where $a_i \in \mathbb{C}(\boldsymbol{\mu})\backslash 0$ and $z_1, \ldots, z_N$ are distinct monomials in $\mathbb{C}\{\mathbf{y}, u\}$. Let j and k be in $\{1, \ldots, N\}$.*

If $\mathrm{Wr}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_N) \notin \mathcal{I}$, then $\mathrm{Wr}(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_N) \notin \mathcal{I}$.

Proof. Suppose $\mathrm{Wr}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_N) \notin \mathcal{I}$ and $\mathrm{Wr}(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_N) \in \mathcal{I}$. Let K be the field of constants of $\mathrm{Frac}(\mathcal{R}/\mathcal{I})$. For an element $a \in R$, denote by $\bar{a}$ the image of a in $\mathrm{Frac}(\mathcal{R}/\mathcal{I})$. By Lemma 2.4.2, $\{\bar{z}_1, \ldots, \bar{z}_{k-1}, \bar{z}_{k+1}, \ldots, \bar{z}_N\}$ is K-linearly dependent. By Lemma 2.3.5, $\{\bar{z}_1, \ldots, \bar{z}_{j-1}, \bar{z}_{j+1}, \ldots, \bar{z}_N\}$ is K-linearly dependent. It follows that $\mathrm{Wr}(\bar{z}_1, \ldots, \bar{z}_{j-1}, \bar{z}_{j+1}, \ldots, \bar{z}_N) = 0$, and hence $\mathrm{Wr}(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_N\} \in \mathcal{I}$. But this contradicts our assumption. $\square$

Combining Proposition 2.3.1 and Proposition 2.3.2 gives the following corollary:

Corollary 2.3.1. *Let $g \in \mathcal{I}$. Write $g = \sum_{j=1}^{N} a_j z_j$, where $a_j \in \mathbb{C}(\boldsymbol{\mu})$, $a_1 = 1$, and $z_1, \ldots, z_N$ are distinct monomials in $\mathbb{C}\{\mathbf{y}, u\}$.*

If for some non-empty $Z \subsetneq \{z_1, \ldots, z_N\}$ of size $N-1$ it holds that $\mathrm{Wr}(Z) \notin \mathcal{I}$, then a_j is identifiable for all $j = 1, \ldots, N$.

We have shown that if a monic element of the differential ideal generated by a system of ODEs satisfies a certain condition, then its coefficients are identifiable. The following proposition gives a sort of converse to this.

Proposition 2.3.3. *Let $>$ be a differential ranking on $\mathcal{R}$ such that $x_i > y_j^{(k)}$ for all i,k; $x_i > u^{(j)}$ for all i,j; and $x_i^{(k)} > x_j$ for all i,j and all $k \geqslant 1$.*

Let A be a characteristic set of $\mathcal{J}$ under $>$ with monic elements. Let $\{A_1,\ldots,A_\rho\} = A \cap \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}$. For each i write $A_i = \sum_{j=1}^{n_i} a_{i,j} z_{i,j}$, where $a_{i,j} \in \mathbb{C}(\boldsymbol{\mu})$, $a_{i,1} = 1$, and the $z_{i,j}$ are distinct monomials in $\mathbb{C}\{\boldsymbol{y},u\}$. Let $\Xi = \{a_{i,j}\}_{i=1,\ldots,\rho}^{j=1,\ldots,n_i}$.

If $f \in \mathbb{C}(\boldsymbol{\mu})$ is identifiable, then $f \in \mathbb{C}(\Xi)$.

Proof. Suppose $f \in \mathbb{C}(\boldsymbol{\mu})$ is identifiable. By Lemma 2.3.4, $f \in \mathcal{E}$. Hence, there exist $P_1, P_2 \in \mathbb{C}\{\boldsymbol{y},u\}$ where $P_1 \notin \mathcal{J}$ such that

$$P_1 \cdot f - P_2 \in \mathcal{J}.$$

We also have

$$P_1 \cdot f - P_2 \in \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}.$$

The ranking $<$ is assumed to be such that $x_j^{(k)}$ ranks higher than any differential variable in $\mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}$ for all j and all $k \geqslant 0$. By Lemma 2.3.1, $A_1,\ldots,A_\rho$ forms a characteristic set for $\mathcal{J} \cap \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}$. Let H be the product of the initials and separants of $A_1,\ldots,A_\rho$. By Lemma 2.3.2, we can write

$$H^\nu (P_1 \cdot f - P_2) = \sum_{i,j} B_{i,j} A_i^{(j)}, \tag{2.15}$$

where $\nu \in \mathbb{N}$ and $B_{i,j} \in \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}$.

Suppose that $f \notin \mathbb{C}(\Xi)$. By (Milne, 2018, Theorem 9.29, p.117) there exists an automorphism α_0 of $\overline{\mathbb{C}(\boldsymbol{\mu})}$ such that α_0 acts on $\mathbb{C}(\Xi)$ as the identity and $\alpha_0(f) \neq f$. We can extend α_0 to a function α from $\overline{\mathbb{C}(\boldsymbol{\mu})}\{\boldsymbol{x},\boldsymbol{y},u\}$ to $\overline{\mathbb{C}(\boldsymbol{\mu})}\{\boldsymbol{x},\boldsymbol{y},u\}$ that fixes $\boldsymbol{x}$, $\boldsymbol{y}$, and u. We apply α to both sides of (2.15) and obtain

$$H^\nu (P_1 \alpha(f) - P_2) = \sum_{i,j} \alpha(B_{i,j}) A_i^{(j)}. \tag{2.16}$$

Subtracting (2.16) from (2.15), we obtain

$$H^{\nu}P_1(f-\alpha(f)) = \sum_{i,j}(B_{i,j}-\alpha(B_{i,j}))A_i^{(j)}. \tag{2.17}$$

We show that $P_1 \in \mathcal{J}$. Dividing both sides of (2.17) by $f-\alpha(f)$, we obtain

$$H^{\nu}\cdot P_1 = \sum \frac{B_{i,j}-\alpha(B_{i,j})}{f-\alpha(f)}\cdot A_i^{(j)}. \tag{2.18}$$

It follows that $H^{\nu}\cdot P_1 \in (\overline{\mathbb{C}(\boldsymbol{\mu})}\{\boldsymbol{y},u\})[A_1,\ldots,A_\rho]$, the differential ideal generated by $A_1,\ldots,A_\rho$ in $\overline{\mathbb{C}(\boldsymbol{\mu})}\{\boldsymbol{y},u\}$. But we also have $H^{\nu}\cdot P_1 \in \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\}$. By Lemma 2.3.6 (found below), we have that $H^{\nu}\cdot P_1 \in (\mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{y},u\})\,[A_1,\ldots,A_\rho]$. Since $\mathcal{J}$ is prime and $H \notin \mathcal{J}$, it follows that $P_1 \in \mathcal{J}$. Thus we have a contradiction, and we conclude that our assumption that $f \notin \mathbb{C}(\Xi)$ is false. $\square$

For completeness, we provide a proof of the following well-known result.

Lemma 2.3.6. *Let K/k be a field extension. Let I be an ideal of $k[Z]$, a polynomial ring over k, where Z is a possibly infinite set of indeterminates. If $f \in K[Z]I \cap k[Z]$, then $f \in I$.*

Proof. There exists a basis $\{b_i\}_i$ for K as a k-vector space with $b_0 = 1$. Let $\{M_i\}_i$ be the set of monomials in $k[Z]$. Now $\{b_iM_j\}_{i,j}$ forms a basis for $K[Z]$ as a k-vector space. For every $g \in K[Z]$, there exist unique $g_{i,j} \in k$ such that $g = \sum_{i,j} g_{i,j}b_iM_j = \sum_i b_i\left(\sum_j g_{i,j}M_j\right)$. Hence there exist unique $f_i \in k[Z]$ such that $f = \sum_i f_ib_i$. Since f can be written as fb_0, it must be that $f \in I$. $\square$

Corollary 2.3.2. *Let Ξ be as defined in Proposition 2.3.3. If for all $i = 1,\ldots,\rho$ we have that some non-empty $Z \subsetneq \{z_{i,1},\ldots,z_{i,n_i}\}$ of size $n_i - 1$ satisfies $Wr(Z) \notin \mathcal{J}$, then*

$$\{f \in \mathbb{C}(\boldsymbol{\mu}) \mid f \textit{ is identifiable }\} = \mathbb{C}(\Xi).$$

Proof. The $\subseteq$ direction is proven by Proposition 2.3.3. The $\supseteq$ direction is proven by Corollary 2.3.1.

$\square$

2.4 Identifiability for input-output equations in systems with one output

In this section, we discuss the case where only one output is present. In this case, any characteristic set contains at most one element in the ring $\mathbb{C}(\boldsymbol{\mu})\{y,u\}$ (see the first paragraph of the proof of Theorem 2.4.1).

Definition 2.4.1. If a monic characteristic set of $[S]$ contains an element of $\mathbb{C}(\boldsymbol{\mu})\{y,u\}$, we call this element the *input-output equation* of Σ.

Theorem 2.4.1 says that if the input-output equation for a monic characteristic set is the sum of a $\mathbb{C}(\boldsymbol{\mu})$-linear combination of the derivatives of y and an element of $\mathbb{C}(\boldsymbol{\mu})\{u\}$, then all of its coefficients are identifiable. Theorem 2.4.2 shows that the input-output equation of a linear system has a factor of this form of rank equal to the rank of the input-output equation, and thus its coefficients are identifiable. The identifiability of linear models continues to be a topic of interest (see Meshkat and Sullivant (2014), Meshkat et al. (2014), Gross et al. (2017), Gross et al. (2018), Yates et al. (2009), Baaijens and Draisma (2016)).

We use the following setup for this section. Since there is one output ($m = 1$), we use the variable y instead of y_1. We work in the ring $\mathcal{R} = \mathbb{C}(\boldsymbol{\mu})\{\boldsymbol{x},y,u\}$. Let $<$ be a differential ranking such that $x_i > y^{(j)} > u^{(k)}$ for all i,j,k. Let C be a characteristic set of $\mathcal{I}$ for $<$.

Theorem 2.4.1. *If the input-output equation of Σ is of the form $p = \sum_{i=0}^{N} a_i y^{(i)} + h$, where $a_i \in \mathbb{C}(\boldsymbol{\mu})$, $a_N = 1$, and $h \in \mathbb{C}(\boldsymbol{\mu})\{u\}$, then the coefficients of the input-output equation are identifiable.*

Proof. By Lemma 2.3.1, $C \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$ is a characteristic set of $[S] \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$. Now by Lemma 2.3.3, $C \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$ can have at most two elements. If it has two elements, then one of them is contained in $\mathbb{C}(\boldsymbol{\mu})\{u\}$, but by (Hong et al., 2018, Lemma 1, p. 12) this is impossible. Therefore $C \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\} = \{p\}$. Since the initial and separant of p are equal to 1, we have that $\mathcal{I} \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\} = [p] : H_p^\infty \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\} = [p] \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$.

Let $\{z_1,\ldots,z_\rho\}$ be the monomials of p. We will assume that some subset of this has its Wronskian in $\mathcal{J}$ and then show that this leads to a contradiction. Assume $\nu < \rho$ is such that $\mathrm{Wr}(z_1,\ldots,z_\nu) \in \mathcal{J}$ and either $\nu = 1$ or $\mathrm{Wr}(z_1,\ldots,z_{\nu-1}) \notin \mathcal{J}$. Note that $\mathrm{Wr}(z_1,\ldots,z_\nu)$ and $\mathrm{Wr}(z_1,\ldots,z_{\nu-1})$ belong to $\mathbb{C}(\boldsymbol{\mu})\{y,u\}$ and $\mathcal{J}\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\} = [p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\}$. Hence we have that $\mathrm{Wr}(z_1,\ldots,z_\nu) \in [p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\}$ and either $\nu = 1$ or $\mathrm{Wr}(z_1,\ldots,z_{\nu-1}) \notin [p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\}$. By Lemma 2.4.2 with $A = \mathbb{C}(\boldsymbol{\mu})\{y,u\}/([p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\})$, there exist $c_1,\ldots,c_{\nu-1}$ in the field of constants of $\mathrm{Frac}(\mathbb{C}(\boldsymbol{\mu})\{y,u\}/([p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\}))$ such that $c_1z_1+\ldots+c_{\nu-1}z_{\nu-1}+z_\nu = 0$. By Lemma 2.4.3 with $K = \overline{\mathbb{C}(\boldsymbol{\mu})}$ there is an element c in the constants of $\mathrm{Frac}(\overline{\mathbb{C}(\boldsymbol{\mu})}\{y,u\}/\overline{\mathbb{C}(\boldsymbol{\mu})}([p]\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\}))$ such that for $i = 1,\ldots,\rho$ we have that c times the coefficient of z_i in p equals the coefficient of z_i in $c_1z_1+\ldots+c_{\nu-1}z_{\nu-1}+z_\nu$. Since $\nu < \rho$, it must be that $c = 0$. But since the coefficient of z_ν in $c_1z_1+\ldots+c_{\nu-1}z_{\nu-1}+z_\nu$ is 1, it must be that $c \neq 0$. This is a contradiction. Hence our assumption that there exists a proper subset of $\{z_1,\ldots,z_\rho\}$ whose Wronskian lies in $\mathcal{J}$ is false. By Proposition 2.3.1, the coefficients of p are identifiable. □

Theorem 2.4.2. *Suppose F_i and G are such that the equations of Σ can be written*

$$x' = Mx+f, \quad y = bx+g,$$

where $x = (x_1,\ldots,x_n)^T$, $M \in \mathbb{C}(\boldsymbol{\mu})^{n\times n}$, $b \in \mathbb{C}(\boldsymbol{\mu})^{1\times n}$, $f \in \mathbb{C}(\boldsymbol{\mu})[u]^{n\times 1}$, $g \in \mathbb{C}(\boldsymbol{\mu})[u]$.

If A is an input-output equation, then the coefficients of $A/\mathrm{in}(A)$ are identifiable.

Proof. By an argument similar to that in the first paragraph of the proof of Theorem 2.4.1, we have $C\cap\mathbb{C}(\boldsymbol{\mu})\{y,u\} = \{A\}$.

We show that $A = \mathrm{in}(A)\cdot p$, where $p = \sum_{i=0}^{N} a_i y^{(i)} + h$, where $N \leq n$, $a_i \in \mathbb{C}(\boldsymbol{\mu})$, $a_N = 1$, and $h \in \mathbb{C}(\boldsymbol{\mu})\{u\}$. Observe that for each i there exists an $h_i \in \mathbb{C}(\boldsymbol{\mu})\{u\}$ such that $y^{(i)} = bM^ix+h_i$ modulo $\mathcal{J}$. Let N be minimal such that $bx, bMx,\ldots,bM^Nx$ are linearly dependent over $\mathbb{C}(\boldsymbol{\mu})$. Since M is an $n\times n$ matrix with entries in $\mathbb{C}(\boldsymbol{\mu})$, by the Cayley-Hamilton Theorem $M^0,\ldots,M^n$ are $\mathbb{C}(\boldsymbol{\mu})$-linearly dependent. Therefore $bM^0x,\ldots,bM^nx$ are $\mathbb{C}(\boldsymbol{\mu})$-linearly dependent. It follows that $N \leq n$. Then there exist $a_i \in \mathbb{C}(\boldsymbol{\mu})$, $a_N = 1$, and $h \in \mathbb{C}(\boldsymbol{\mu})\{u\}$ such that $p := \sum_{i=0}^{N} a_i y^{(i)} + h = 0 \mod \mathcal{J}$. We know by (Hong et al., 2018, Lemma 1, p. 12) that $x_1, \ldots, x_n$ are algebraically independent over $K := \operatorname{Frac}(\mathbb{C}(\boldsymbol{\mu})\{u\}) \mod \mathcal{J}$. Now we show that $bx, \ldots, bM^{N-1}x$ are linearly independent over K mod $\mathcal{J}$. Suppose there exist $a_0, \ldots, a_{N-1} \in K$ such that

$$a_0 bx + \ldots + a_{N-1} bM^{N-1}x = 0 \mod \mathcal{J}.$$

Multiplying this by the least common multiple of the demonimators of $a_0, \ldots, a_{N-1}$, it follows that

$$\sum_v f_{0,v} v bx + \ldots + \sum_v f_{N-1,v} v bM^{N-1}x = 0 \mod \mathcal{J},$$

where in each sum, v ranges over the monomials of $\mathbb{C}\{u\}$, each $f_{i,v} \in \mathbb{C}(\boldsymbol{\mu})$, and only finitely many $f_{i,v}$ are non-zero. Therefore

$$\sum_v (f_{0,v} bx + \ldots + f_{N-1,v} bM^{N-1}x) v = 0 \mod \mathcal{J}.$$

By (Hong et al., 2018, Lemma 1, p. 12), we have that

$$\sum_v (f_{0,v} bx + \ldots + f_{N-1,v} bM^{N-1}x) v = 0. \quad (\text{in } \mathcal{R}).$$

Therefore,

$$\forall v \; f_{0,v} bx + \ldots + f_{N-1,v} bM^{N-1}x = 0.$$

Since $bM^0x, \ldots, bM^{N-1}x$ are linearly independent over $\mathbb{C}(\boldsymbol{\mu})$, it follows that $f_{0,v} = \ldots = f_{N-1,v} = 0$, and hence that $a_0 = \ldots = a_{N-1} = 0$. We conclude that $bx, \ldots, bM^{N-1}x$ are linearly independent over K mod $\mathcal{J}$. It follows from Lemma 2.4.1 (found below) that $bx, \ldots, bM^{N-1}x$ are algebraically independent over K mod $\mathcal{J}$. It follows that $bx + h_0, \ldots, bM^{N-1}x + h_{N-1}$ are algebraically independent over K mod $\mathcal{J}$ and hence $y, \ldots, y^{(N-1)}$ are algebraically independent over K mod $\mathcal{J}$.

Now $\mathrm{ord}_y A \geqslant N$, where $\mathrm{ord}_y A = \max(\{0\} \cup \{i \mid y^{(i)} \text{ appears in } A\})$. Since $p \in \mathcal{J} \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\} = [A] : (\mathrm{in}(A) \cdot \mathrm{sep}(A))^\infty$, it must be that $\mathrm{ord}_y A = N$ and the degree of A in $y^{(N)}$ is 1. It follows that $A - (\mathrm{in}(A))p$ either is 0 or has lower rank than A. By Lemma 2.3.1, A is a characteristic set of $\mathcal{J} \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$. If $A - (\mathrm{in}(A))p \neq 0$, then $A - (\mathrm{in}(A))p$ is an autoreduced subset of $\mathcal{J} \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$ that is less than A, which contradicts that A is a characteristic set of $\mathcal{J} \cap \mathbb{C}(\boldsymbol{\mu})\{y,u\}$. Therefore it must be that $A = (\mathrm{in}(A))p$.

Let C_1 be the set $(C \setminus \{A\}) \cup \{p\}$. Now C_1 is a characteristic set of $\mathcal{J}$ under $<$. By Theorem 2.4.1, the coefficients of p are identifiable. □

Lemma 2.4.1. *Let K be a field and consider the polynomial ring $K[z_1, \ldots, z_r]$. Let $t_1, \ldots, t_s$ be K-linear combinations of $z_1, \ldots, z_r$. If $t_1, \ldots, t_s$ are K-linearly independent, then for any l, the set of products of l elements of $\{t_1, \ldots, t_s\}$ is K-linearly independent.*

Proof. It must be that $s \leqslant r$. Let σ be an automorphism of $K[z_1, \ldots, z_r]$ fixing K and sending t_i to z_i. Such an automorphism can be obtained as follows. The set of homogeneous linear polynomials of $K[z_1, \ldots, z_r]$ is an r-dimensional K-vector space V. Since $t_1, \ldots, t_s$ is K-linearly independent, there exist $t_{s+1}, \ldots, t_r$ such that $t_1, \ldots, t_r$ is a basis for V. Now let τ be the K-linear automorphism on V such that $\forall i\ \sigma(t_i) = z_i$. Now τ extends uniquely to a K-automorphism σ on $K[z_1, \ldots, z_r]$. We introduce the notation $t^\alpha := t_1^{\alpha(1)} \cdot \ldots \cdot t_s^{\alpha(s)}$, where $\alpha \in (\mathbb{Z}_{\geqslant 0})^s$. Consider a collection of elements $t^{\alpha_1}, \ldots, t^{\alpha_k}$, where $\alpha_1, \ldots, \alpha_k \in (\mathbb{Z}_{\geqslant 0})^s$ are distinct and for each i, $\alpha_i(1) + \ldots + \alpha_i(s) = l$. Now the elements $\sigma(t^{\alpha_1}), \ldots, \sigma(t^{\alpha_k})$ are distinct products of l elements of $\{z_1, \ldots, z_s\}$, which are clearly K-linearly independent. Since σ is a K-automorphism, it follows that $t^{\alpha_1}, \ldots, t^{\alpha_k}$ are K-linearly independent. □

Lemma 2.4.2. *Let A be a differential ring that is also an integral domain. Let $a_1, \ldots, a_N \in A$. If $\mathrm{Wr}(a_1, \ldots, a_N) = 0$ and either $N = 1$ or $\mathrm{Wr}(a_1, \ldots, a_{N-1}) \neq 0$, then there exist $c_1, \ldots, c_{N-1}$ in the field of constants of Frac(A) such that $c_1 a_1 + \ldots + c_{N-1} a_{N-1} + a_N = 0$.*

Proof. If $N = 1$, then $\mathrm{Wr}(a_1) = a_1$. Hence $a_1 = 0$.

Assume $N > 1$. By (Kaplansky, 1957, Theorem 3.7, p. 21), there exist $b_1, \dots, b_N$ in the field of constants of $\mathrm{Frac}(A)$, not all 0, such that $b_1 a_1 + \ldots + b_{N-1} a_{N-1} + b_N a_N = 0$. Since $\mathrm{Wr}(a_1, \dots, a_{N-1}) \neq 0$, by (Kaplansky, 1957, Theorem 3.7, p. 21) $b_N \neq 0$. Hence, $\frac{b_1}{b_N} a_1 + \ldots + \frac{b_{N-1}}{b_N} a_{N-1} + a_N = 0$. □

Lemma 2.4.3. *Let K be an algebraically closed field. Consider the differential polynomial ring $K\{y,u\}$ with derivation ∂ satisfying $\partial(K) = 0$. Consider $P \in K\{y,u\}$ of the form*

$$P = D_P(y) + U_P(u),$$

where $D_P \in K[\partial]$ is a monic linear differential operator over K and $U_P \in K\{u\}$. Let $L := \mathrm{Frac}(K\{y,u\}/[P])$ and by $C(L)$ we denote the field of constants of L. Then every linear dependence of the images of the monomials of P in L over $C(L)$ is proportional to the one given by the coefficients of P.

Proof. We describe the notation used in this proof. An element W of $K\{y,u\}$ is considered to be a differential polynomial in y and u with coefficients in K. We consider $K\{y,u\}$ to be a subset of $C(L)\{y,u\}$, the ring of differential polynomials in y and u with coefficients in $C(L)$. There is a $C(L)$-algebra homomorphism from $C(L)\{y,u\}$ to L that sends y and u to their respective images in L. If W equals u, we denote the image of W under this map by $\overline{u}$. Otherwise, we denote the image of W by $W(\overline{y},\overline{u})$ (or, e.g., $W(\overline{u})$ if y does not appear in W). An element of D of $C(L)[\partial]$ is considered to be a map from $C(L)\{y,u\}$ to $C(L)\{y,u\}$ or from L to L. We consider $K[\partial]$ to be a subset of $C(L)[\partial]$.

Note that P is a characteristic set of $[P]$ with respect to any elimination ranking with $y > u$. Hence, $\overline{u}$ is differentially independent over K. Since $C(L)$ is a differential algebraic extension of K, it follows that $\overline{u}$ is differentially independent over $C(L)$.

Assume that the statement of the lemma is not true. Then there exists a nonzero polynomial $Q \in C(L)\{y,u\}$ such that

- $Q(\overline{y}, \overline{u}) = 0$,
- every monomial in Q appears in P, and
- P and Q are not proportional.

Without loss of generality, we may assume that

$$Q = D_Q(y) + U_Q,$$

where $D_Q \in C(L)[\partial]$ and $U_Q \in C(L)\{u\}$. Let D_0 be the monic gcd of D_P and D_Q. If $D_P = D_Q$ then $U_P(\overline{u}) - U_Q(\overline{u})$ gives a differential relation of $\overline{u}$ over $C(L)$, which is impossible. Thus, $D_P \neq D_Q$, so $\operatorname{ord} D_0 < \operatorname{ord} D_P$.

If F is an algebraically closed field and p is an element of the univariate polynomial ring $F[X]$ and p factors as $p = qr$, then q and r belong to $F[X]$. Therefore, since D_0 divides D_P and K is algebraically closed, $D_0 \in K[\partial]$ and there exists monic $D_1 \in K[\partial]$ such that $D_P = D_1 D_0$. There also exist $A, B \in C(L)[\partial]$ such that $D_0 = AD_P + BD_Q$. Consider

$$R := A(P) + B(Q) = D_0(y) + U_R,$$

where $U_R = A(U_P) + B(U_Q)$. Then $R(\overline{y}, \overline{u}) = 0$. Since $P - D_1(R) \in C(L)\{u\}$ vanishes on $\overline{u}$ and $\overline{u}$ is differentially independent over $C(L)$, it follows that $P = D_1(R)$.

Considering a basis of $C(L)$ over K, we can write

$$U_R = U_0 + e_1 U_1 + \ldots + e_N U_N,$$

where $U_0, \ldots, U_N \in K\{u\}$ and $1, e_1, e_2, \ldots, e_N \in C(L)$ are linearly independent over K. Since $D_1(U_R) = U_P$ and $D_1 \in K[\partial]$, $U_1, \ldots, U_N \in \ker D_1$, where we consider D_1 as a function from $C(L)\{y,u\}$ to $C(L)\{y,u\}$. There are two cases:

- D_1 is not divisible by ∂. Then $\ker D_1 = \{0\}$. Hence $U_1 = \ldots = U_N = 0$.

- D_1 is divisible by ∂. Then $\ker D_1 = C(L)$. Thus, $U_1, \ldots, U_N$ belong to K. However, since $U_P = D_1(U_R)$, U_P does not contain a term in K. Hence U_Q does not contain a term in $C(L)$ and, consequently, U_R does not contain a term in $C(L)$. Thus, $U_1 = \ldots = U_N = 0$.

In both cases, we have shown that $U_R \in K\{u\}$. Thus, $R \in K\{y,u\}$ and $R \in [P]$. But this is impossible because P is a characteristic set with respect to any elimination ranking with $y > u$ and $\operatorname{ord} D_0 < \operatorname{ord} D_P$. $\square$

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