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Hermitian Maass lift for General Level

by

An Hoa Vu

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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Abstract

Hermitian Maass lift for General Level

by

An Hoa Vu

Advisor: Krzysztof Klosin

For an imaginary quadratic field K of discriminant $-D$, let $\chi = \chi_K$ be the associated quadratic character. We will show that the space of special hermitian Jacobi forms of level N is isomorphic to the space of plus forms of level DN and nebentypus χ (the hermitian analogue of Kohnen's plus space) for any integer N prime to D . This generalizes the results of Krieg from $N = 1$ to arbitrary level. Combining this isomorphism with the recent work of Berger and Klosin and a modification of Ikeda's construction we prove the existence of a lift from the space of elliptic modular forms to the space of hermitian modular forms of level N which can be viewed as a generalization of the classical hermitian Maaß lift to arbitrary level.

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Chapter 1

Introduction

The Langlands program is a vast array of conjectures in number theory and representation theory, many of which remain open, that have very far-reaching consequences in modern number theory. For instance, it gives a partial generalization of class field theory to (certain) non-abelian extensions of local and global fields. In essence, it predicts the existence of various correspondences between some arithmetic data and analytic objects; more precisely, a *functorial* correspondence¹ between the automorphic/admissible representations

$$\mathcal{A}(G) := \{\text{automorphic/admissible representations of } G\}$$

¹This is not really a bijection in general; but a surjection $\mathcal{A}(G) \rightarrow \Phi(G)$ that partitions $\mathcal{A}(G)$ into L -packets of representations whose L -functions are indistinguishable. For G with multiplicity-one like GL_n , each L -packet is a singleton and we do expect a bijection.

and certain admissible group homomorphisms

$$\Phi(G) := \{\text{admissible } L\text{-homomorphism } L_k \rightarrow {}^L G\}$$

for any reductive group G defined over a (global or local) field k . Here, L_k is the Langlands group² associated with the field k and ${}^L G$ is the Langlands dual group³. Langlands' principle of functoriality⁴ asserts that if G and H are reductive groups (e.g. $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_m$) over a number field F and

$${}^L G \rightarrow {}^L H$$

is an L -group homomorphism between the corresponding L -groups then an automorphic representation π of G gives rise (i.e. lifts) to an automorphic representation Π of H in which the local data correspond in the appropriate way.

One of the well-known instances of functoriality holds for $\mathrm{GL}_1 \times \mathrm{GL}_1$ and GL_2 : Classical Eisenstein series can be viewed as lifts of a pair of Dirichlet characters, i.e., automorphic representations of GL_1 .

The special case when $G = 1, H = \mathrm{GL}_n$ is of particular interest to algebraic number theorists: When $n = 1$, it recovers the results of class field

²This group is known to be the Weil or the Weil-Deligne group when k is a local field while its existence is only hypothetical when k is global field.

³Despite the notation, this group depends on both G and k .

⁴Functoriality is predicted from the above Langlands correspondence: An L -group homomorphism $\varphi : {}^L G \rightarrow {}^L H$ gives rise to $\Phi(G) \rightarrow \Phi(H)$ by composition with φ ; hence inducing a correspondence $\mathcal{A}(G) \rightarrow \mathcal{A}(H)$.

theory; one of the most important achievements of the 20th century mathematics. For $n \geq 2$, we get a generalization of class field theory to some non-abelian extensions which we alluded to earlier.

1.1 Saito-Kurokawa lift

An instance of functoriality in the case $G = \mathrm{PGL}_2 \times \mathrm{PGL}_2$ and $H = \mathrm{PGSp}_4$ and the natural embedding of the corresponding L -groups $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{Sp}_4(\mathbb{C})$ was discovered experimentally and conjectured independently by Saito and Kurokawa in late 1970s. In the classical language of modular forms, it can be stated the following:

Theorem 1.1. *Let k be even. Then for each normalized Hecke eigenform $f \in \mathfrak{S}_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$, there is a Siegel eigenform $F \in \mathfrak{S}_k^{\mathrm{Siegel}}(\mathrm{Sp}_4(\mathbb{Z}))$ (uniquely determined up to a non-zero scalar) such that*

$$L_{\mathrm{spin}}(F, s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s)$$

where $L(f, s)$ is the Hecke L -function of f and $L_{\mathrm{spin}}(F, s)$ is the spin L -function of F .

A form F in the theorem is called the *Saito-Kurokawa lift* of f . It was first constructed in the work of H. Maass, A. Andrianov, and Eichler-Zagier

as a composition of three maps

$$\mathfrak{M}_{2k-2}(1) \longrightarrow \mathfrak{M}_{k-1/2}^+(4) \longrightarrow J_{k,1}^{Siegel} \longrightarrow \mathfrak{M}_k^{Siegel}(\mathrm{Sp}4(\mathbb{Z}))$$

where the first map is the Shintani's map, the second one lifts a half-integer weight modular form to a certain Jacobi form, and the last map is usually referred to as the (classical) Maass lift (obtained by certain index shifting operator). The space of all such lifts is known as the *Maass Spezialschar*. This result was reinterpreted in the language of automorphic representations by Piatetski-Shapiro [25] and Ralf Schmidt [28], [29].

The Saito-Kurokawa lift was of special interest in its own right as well as for its various consequences in number theory, coming from congruences between a lift and a non-lift. Congruences between modular forms, in particular between an Eisenstein series and a cusp form, were exploited by Ribet [26] to prove the converse of Herbrand's theorem. His influential method was adapted by Mazur and Wiles [22] to establish the Main Conjecture of Iwasawa theory for $\mathbb{Q}$ and subsequently by Wiles [33] for all totally real fields. More recently, it was used again by Skinner-Urban [32], by J. Brown [7], [1] and by K. Klosin [16], [17] to give evidence⁵ for the Bloch-Kato conjecture for

⁵Skinner-Urban's work [32] concerns the rank part whereas Brown and Klosin concerns the interpretation of the 'constant term' part of the Bloch-Kato conjecture. Under suitable hypotheses, the former showed that if the order of vanishing of $L(f, s)$ is odd then the Selmer group is infinite and the latter showed one divisibility relation between Tamagawa number, the order of a Selmer group and L -values.

certain motives attached to modular forms. Skinner-Urban's method for instance used deformation theory of representations attached to modular forms, a theory originally developed by H. Hida [12], [11] based on p -adic modular forms. Their result gives new evidence for the Birch and Swinnerton-Dyer conjecture; namely, that if E is an elliptic curve over $\mathbb{Q}$ with ordinary reduction at an odd prime p and order of vanishing $\text{ord}_{s=1} L(E, s)$ is odd, then $\text{Sel}_{\mathbb{Q}}(E)_p$ is infinite.

1.2 Hermitian Maaß lift

The Hermitian analogue of Saito-Kurokawa lift is a lifting of elliptic modular forms to hermitian modular forms of degree 2, typically referred to as the Maaß lift in the literature. It has been studied since 1980 with the paper of Kojima [20]. Such a lift is of arithmetic interest, for example, in providing evidence for the Bloch-Kato conjecture by Klosin [16], [17] and for the p -adic theory. For these arithmetic applications, there is a need for the generalization of the Hermitian Maaß lift.

As with the classical Saito-Kurokawa lift, the first construction of the Hermitian Maaß lift by Kojima was a composition of the three lifts: Let $K = \mathbb{Q}(i)$ and assume that k is divisible by 4. Then we have

$$\mathfrak{S}_{k-1}(4, \chi) \longrightarrow \mathfrak{S}_{k-1}^+(4, \chi) \longrightarrow \mathfrak{J}_{k,1}^*(1) \longrightarrow \mathfrak{M}_{k,2}(1) \quad (1.2.1)$$

where

(i) the lift from $\mathfrak{S}_{k-1}(4, \chi)$ to $\mathfrak{S}_{k-1}^+(4, \chi)$ is an analogue of Shimura–Shitani isomorphism (but it is not an isomorphism here though and unlike Shitani’s map, this map is not obtained by certain cycle integrals) and $\mathfrak{S}_{k-1}^+(4, \chi)$ is an analogue of Kohnen’s plus space [19];

(ii) the lift from $\mathfrak{S}_{k-1}^+(4, \chi)$ to the space of special Hermitian Jacobi forms $\mathfrak{J}_{k,1}^*(1)$; and finally,

(iii) the Hermitian analogue of the original Maaß lift from $\mathfrak{J}_{k,1}^*(1)$ to $\mathfrak{M}_{k,2}(1)$.

See chapter 3 for our definition and notation of the spaces and groups mentioned.

In [21], Krieg generalized Kojima’s result to general imaginary quadratic fields K . He established the maps

$$\mathfrak{M}_{k-1}^+(D, \chi) \longrightarrow \mathfrak{J}_{k,1}^*(1) \longrightarrow \mathfrak{M}_{k,2}(1) \quad (1.2.2)$$

Moreover, Krieg defined the Hermitian Maaß space $\mathfrak{M}_{k,2}^*(1) \subset \mathfrak{M}_{k,2}(1)$ and he showed that the image of the second map is $\mathfrak{M}_{k,2}^*(1)$. Also, he proved that $\mathfrak{J}_{k,1}^*(1) \rightarrow \mathfrak{M}_{k-1}^+(D, \chi)$ is an isomorphism. As for the map $\mathfrak{S}_{k-1}(D, \chi)$ to $\mathfrak{S}_{k-1}^+(D, \chi)$, Krieg remarked that if the discriminant D is prime, the space of cusp forms $\mathfrak{S}_{k-1}(D, \chi)$ has a basis consisting of primitive form

$f_1, \dots, f_a, f_{a+1}, \dots, f_{a+b}$ with $f_i^\rho = f_i$ for all $i = a+1, \dots, a+b$ and the forms $f_i - f_i^\rho$ is a basis for cusp forms in $\mathfrak{M}_{k-1}^+(D, \chi)$. Here, $f^\rho(\tau) = \overline{f(-\bar{\tau})}$ is the form obtained by complex conjugating all Fourier coefficients of f . Ikeda [14] gives a generalization of this construction to arbitrary discriminant D , which we modify to accommodate forms of higher level (c.f. section 4.4). We remark that for level $N = 1$, there are alternative approaches to the Hermitian Maaß lift via the theory of compatible Eisenstein series by Ikeda [14], via Imai's Converse Theorem [15] proved recently by Matthes [27] (but only for $K = \mathbb{Q}(i)$), and theta lifting by Atobe [4].

This thesis concerns a generalization of this result to higher level: Let K be an imaginary quadratic field of discriminant $-D$, we describe the (first two) maps

$$\mathfrak{M}_{k-1}(DN, \chi) \longrightarrow \mathfrak{M}_{k-1}^+(DN, \chi) \longrightarrow \mathfrak{J}_{k,1}^*(N) \longrightarrow \mathfrak{M}_{k,2}(N). \quad (1.2.3)$$

For general level N such that $(D, N) = 1$, Berger and Klosin, in their recent work [5], generalized the definition of Krieg's Maaß space $\mathfrak{M}_{k,2}^*(1)$ to that of higher level $\mathfrak{M}_{k,2}^*(N)$. Using a modification of Kojima and Krieg's result and Ibukiyama's construction of the classical Maaß lift for general level, they then constructed the Hermitian analogue of the Maaß lift $\mathfrak{J}_{k,1}^*(N) \rightarrow \mathfrak{M}_{k,2}(N)$ i.e. the last map in the above diagram (1.2.3) and showed that its image is the

Maaß space $\mathfrak{M}_{k,2}^*(N)$. When D is prime, they defined $\mathfrak{M}_{k-1}^+(DN, \chi)$ and constructed a map $\mathfrak{J}_{k,1}^*(N) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi)$ which they showed is injective and that it is surjective onto the space of p -old cusps forms when $N = p$ is a prime split in K .

In this thesis, we generalize Krieg's method to higher level to provide the surjectivity for the map in the second main result of Berger and Klosin [5] i.e. to show that in fact, $\mathfrak{J}_{k,1}^*(N) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi)$ is always an isomorphism, under appropriate modification of the space $\mathfrak{M}_{k-1}^+(DN, \chi)$ defined in [5].

The main technical challenge to overcome is the lack of a simple set of generators for the congruence subgroup $\Gamma_0(N)$ (even though there are known generators given in [8]), typically used to reduce the verification of the functional equation satisfied by a Jacobi form to a finite set of functional equations that can be verified by analytical mean. A minor challenge is to avoid the computation of the inverse of a certain matrix $M_{u,v}(\sigma)$ which Krieg [21] and Berger-Klosin [5] used. (In Krieg's case, the matrix involved $M_{u,v}(\sigma)$ is not too complicated since he only needs the one for $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the reflection matrix. In Berger-Klosin's case, they need to exploit a relationship between the matrix $M_{\pi u, \pi v}(\sigma)$ and $M_{u,v}(\sigma)$; c.f. Section 3.2 of [5] for details.)

1.3 Outline

In chapter 2, we briefly recount the theory of classical (elliptic) modular forms. In chapter 3, we introduce Hermitian modular forms, the Hermitian Maaß space and Hermitian Jacobi forms.

We prove our main result (Theorem 4.12) in chapter 4 after we generalize Krieg's work, namely giving a characterization of the space of plus forms in section 4.1 and obtain an arithmetic criterion for the lift we define to be a Hermitian Jacobi form in section 4.2. We also explain our modification of Ikeda's construction to get a surjective map

$$\mathfrak{M}_{k-1}(DN, \chi) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi),$$

thus completing the Hermitian Maaß lift for general level depicted in (1.2.3).

As a bonus, we also show that the Maaß space is stable under Hecke operator T_p for inert prime p in K .

Chapter 2

Modular forms

In this chapter, we review the theory of modular forms, typically referred to as *classical* or *elliptic* modular forms to disambiguate with other types of modular forms such as Siegel modular forms or Hermitian modular forms. The theory of modular forms is well-known and there are many textbooks available on the subject such as [24], [18], [30], [9], etc. so we shall only briefly review it.

2.1 Definition and example

Let

$$\mathfrak{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

denote the complex upper half plane, which we shall view as a natural Riemann surface to talk about holomorphic functions on it. The subgroup of

orientation preserving linear transformations

$$\mathrm{GL}_2^+(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) > 0\}$$

of $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ transitively by the formula

$$\gamma \cdot \tau := \frac{a\tau + b}{c\tau + d} \quad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as well as on complex valued functions on $\mathfrak{H}$ by the slash- k action

$$(f|_k \gamma)(\tau) := (c\tau + d)^{-k} f(\gamma \cdot \tau)$$

for each integer k .

Let $\Gamma \subset \mathrm{GL}_2^+(\mathbb{R})$ be a discrete subgroup such as the standard Hecke congruence subgroups

$$\begin{aligned} \Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \\ \Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid d \equiv 1, c \equiv 0 \pmod{N} \right\} \\ \Gamma(N) &:= \ker(\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N)). \end{aligned}$$

(More generally, a discrete subgroup is called a *congruence subgroup* if it contains $\Gamma(N)$ for some N .)

Definition 2.1. A holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ is called a modular form of weight k and level Γ if

(i) f is invariant under $|_k\gamma$ action for all $\gamma \in \Gamma$ i.e.

$$f|_k\gamma = f \quad \text{for all } \gamma \in \Gamma$$

and that

(ii) f is holomorphic at the cusps.

We denote by $\mathfrak{M}_k(\Gamma)$ the space of such forms.

See [24], page 40 for the meaning of the second condition in general; for congruence subgroups, it is equivalent to

$$\lim_{\tau \rightarrow i\infty} f|_k\alpha \quad \text{exists for all } \alpha \in \mathrm{SL}_2(\mathbb{Z}).$$

A modular form of level $\Gamma_0(N)$ or $\Gamma_1(N)$ is necessary periodic of period 1 i.e. $f(\tau + 1) = f(\tau)$ from the functional equation for the translation matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It thus admits *Fourier expansion* at the cusp ∞ of the form

$$f(\tau) = \sum_{n=m}^{\infty} a_n(f)q^n$$

where $q := e^{2\pi i\tau}$ and $m \geq 0$.

As a typical practice, we shall also refer to modular forms of level $\Gamma_0(N)$ as “modular forms of level N ”. Thus, we also denote $\mathfrak{M}_k(N) := \mathfrak{M}_k(\Gamma_0(N))$.

Definition 2.2. A modular form f is called a *cusp form* if $a_0(f|_k\alpha) = 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

The first examples of modular forms are *Eisenstein series*

$$G_k(\tau) := \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} \frac{1}{(c\tau + d)^k}$$

for $k \geq 4$ even. It is easy to check that G_k is a modular form of weight k and level 1 i.e. $\mathrm{SL}_2(\mathbb{Z})$ and its Fourier expansion is given by

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \\ &= -\frac{B_k(2\pi i)^k}{k!} + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned}$$

where $\zeta(z)$ is Riemann's zeta function, B_k is the k -th Bernoulli number i.e. the coefficients in the Taylor series expansion of the function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k t^k$$

and

$$\sigma_k(n) := \sum_{0 < d|n} d^k$$

is the sum of power k divisors function.

The first important result in the theory of modular form is the fact that $\mathfrak{M}_k(\Gamma)$ (hence, $\mathfrak{M}_k(\Gamma_1(N))$ and later $\mathfrak{M}_k(N, \chi)$) is finite dimensional $\mathbb{C}$ -vector space for any Fuchsian group of the first kind Γ and the dimension can be explicitly computed (except when $k = 1$) using Riemann-Roch theorem. (Roughly, it is linear in k .) See [24], Theorem 2.5.2 and 2.5.3.

Moreover, there is an inner product on $\mathfrak{M}_k(\Gamma, \chi)$ called the *Petterson's inner product*: If $f, g \in \mathfrak{M}_k(\Gamma, \chi)$ such that one of them is a cusp form then we define the inner product

$$(f, g) := \frac{1}{\text{vol}(\Gamma \backslash \mathfrak{H})} \int_{\Gamma \backslash \mathfrak{H}} f(z) \overline{g(z)} \text{Im}(z)^k dv(z)$$

where $dv(z) = \frac{dx dy}{y^2}$ is hyperbolic volume form on $\mathfrak{H}$ if we write $z = x + iy$ with $x, y \in \mathbb{R}$.

2.2 Modular forms with Nebentypus

Next, we are going to introduce modular forms with nebentypus. Observe that we have a surjective group homomorphism

$$\Gamma_0(N) \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$$

whose kernel is precisely $\Gamma_1(N)$. In other words, $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and the quotient is isomorphic to the abelian group $(\mathbb{Z}/N\mathbb{Z})^\times$. There is a natural representation of the quotient $G_N := \Gamma_0(N)/\Gamma_1(N)$ on the $\mathbb{C}$ -vector space $\mathfrak{M}_k(\Gamma_1(N))$, namely by $f \mapsto f|_k \gamma$. (It is easy to check that this is a well-defined action.) Now basic representation theory shows that this representation splits as a direct sum of irreducible representations of $G_N \cong (\mathbb{Z}/N\mathbb{Z})^\times$ which are Dirichlet characters, in other words, we have

direct sum decomposition of the vector space

$$\mathfrak{M}_k(\Gamma_1(N)) = \bigoplus_{\chi: (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}} \mathfrak{M}_k(\Gamma_1(N))^\chi$$

where $\mathfrak{M}_k(\Gamma_1(N))^\chi$ is the χ -eigensubspace, explicitly

$$\mathfrak{M}_k(\Gamma_1(N))^\chi := \{f \in \mathfrak{M}_k(\Gamma_1(N)) \text{ s.t. } \forall \gamma \in \Gamma_0(N) : f|_k \gamma = \chi(d_\gamma) f\}.$$

Here, d_γ denotes the lower right entry of the matrix γ .

Definition 2.3. An element of $\mathfrak{M}_k(\Gamma_1(N))^\chi$ is called a modular form of weight k , level N and nebentypus χ .

We denote $\mathfrak{M}_k(N, \chi)$ the space of such modular forms.

We remark that a modular form of level N is exactly a modular form with trivial nebentypus.

2.3 Hecke operators

Let G be a group. Two subgroups Γ, Γ' are called *commensurable*, denoted by $\Gamma \approx \Gamma'$, if their intersection $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ' . We define the *commensurator* of Γ in G to be

$$\tilde{\Gamma}_G := \{g \in G | g\Gamma g^{-1} \approx \Gamma\}.$$

From now on, G is fixed and so implicitly understood; hence, we shall drop the subscript from the notation. It is easy to check that commensurability

is an equivalence relation.

For any pair $\Gamma \approx \Gamma'$, any semi-subgroup $\Gamma \subset \Delta \subset \tilde{\Gamma}$, and any commutative ring K , we define

$$R_K(\Gamma, \Gamma'; \Delta) := \left\{ \sum a_\alpha \Gamma \alpha \Gamma' \mid \alpha \in \Delta, a_\alpha \in K \right\}$$

the free module generated by the double cosets $\Gamma \alpha \Gamma'$ for $\alpha \in \Delta$. We shall write $T(\alpha)$ for the double coset $\Gamma \alpha \Gamma'$ when Γ, Γ' are implicitly understood.

There is a natural product

$$R_K(\Gamma, \Gamma'; \Delta) \times R_K(\Gamma', \Gamma''; \Delta) \rightarrow R_K(\Gamma, \Gamma''; \Delta)$$

which turns $R_K(\Gamma, \Gamma; \Delta)$ into a K -algebra, called the *Hecke algebra* associated to the pair (Γ, Δ) . We shall see that elements of this algebra act naturally on the space of modular forms: they transform a form of level Γ to a form of level Γ' ; and so they are called *Hecke operators*.

From now, let $G = \mathrm{GL}_2^+(\mathbb{R})$ and fix a commutative ring K such that $\mathbb{Z} \subset K \subset \mathbb{C}$. Suppose χ is a character of Γ of finite order that could be extended to a character of Δ and such that if $\alpha \gamma \alpha^{-1} \in \Gamma$ then $\chi(\alpha \gamma \alpha^{-1}) = \chi(\gamma)$. If Γ_1, Γ_2 are subgroups of Γ of finite index then for any $f \in \mathfrak{M}_k(\Gamma_1, \chi)$, we define

$$f|_{\Gamma_1 \alpha \Gamma_2} := \det(\alpha)^{k/2-1} \sum_{m=1}^d \overline{\chi(a_m)} (f|_k \alpha_m)$$

$$= \det(\alpha)^{k-1} \sum_{m=1}^d \overline{\chi(a_m)} j(\alpha_m, z)^{-k} f(\alpha_m z)$$

where $\{\alpha_m \mid 1 \leq m \leq d\}$ is a complete set of representatives of $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$ i.e.

such that $\Gamma_1 \alpha \Gamma_2 = \sqcup_{m=1}^d \Gamma_1 \alpha_m$. Then $f|_{\Gamma_1 \alpha \Gamma_2} \in \mathfrak{M}_k(\Gamma_2, \chi)$.

In the case of $\Gamma = \Gamma_0(N)$, one has $\tilde{\Gamma} = \mathbf{GL}_2(\mathbb{Q})$ by Lemma 4.5.1 of [24]

and we take for Δ the following semigroup

$$\begin{aligned} \Delta_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap \mathbf{GL}_2(\mathbb{Q}) \mid c \equiv 0 \pmod{N}, (a, N) = 1 \right\} \\ \Delta_0^*(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \cap \mathbf{GL}_2(\mathbb{Q}) \mid c \equiv 0 \pmod{N}, (d, N) = 1 \right\} \end{aligned}$$

Let $R(N) := R_{\mathbb{Z}}(\Gamma_0(N), \Delta_0(N))$. Then $R(N)$ is polynomial ring over $\mathbb{Z}$

generated by $T_p, T(pI)$ for all primes $p \nmid N$ and T_q for primes $q|N$ where

$$T_n := \sum_{\det \alpha = n} T(\alpha)$$

where the sum is over double-coset representatives of $\Gamma_0(N) \backslash \Delta_0(N) / \Gamma_0(N)$

of determinant n . Note that $T_p = T \begin{pmatrix} p & \\ & 1 \end{pmatrix}$ if p is a prime. In other words,

any matrix in $\Delta_0(N)$ whose determinant is p must be of the form $\alpha \begin{pmatrix} p & \\ & 1 \end{pmatrix} \beta$

for some $\alpha, \beta \in \Gamma_0(N)$.

Similar results hold for $R^*(N) := R_{\mathbb{Z}}(\Gamma_0(N), \Delta_0^*(N))$.

2.4 Atkin-Lehner's theory of new forms

Observe that if $f \in \mathfrak{M}_k(N, \chi)$ ($f \in \mathfrak{S}_k(N, \chi)$) then we have $g(z) := f(\ell z) \in \mathfrak{M}_k(\ell N, \chi)$ ($g \in \mathfrak{S}_k(\ell N, \chi)$, respectively). This phenomenon leads to the notion of *old forms* in $\mathfrak{M}_k(N, \chi)$: those that are obtained as $f(\ell z)$ for some form of smaller level $\frac{N}{\ell}$. In other words, they are not genuine forms of level N . We define

$$\mathfrak{S}_k^{old}(N, \chi) := \bigcup_{\mathfrak{f}_\chi | M | N} \bigcup_{\ell | \frac{N}{M}} \{f(\ell z) \mid f \in \mathfrak{S}_k(M, \chi)\}$$

where $\mathfrak{f}_\chi$ denotes the conductor of χ and let $\mathfrak{S}_k^{new}(N, \chi)$ to be the orthogonal complement of $\mathfrak{S}_k^{old}(N, \chi)$ with respect to the Pettersson's inner product. A form $f \in \mathfrak{S}_k^{new}(N, \chi)$ is called a *primitive form* if f is eigen-function for all Hecke operators T_n with $(n, N) = 1$ and that its first Fourier coefficient $a_1(f) = 1$.

A primitive form is automatically eigenfunction for all Hecke operators in $R(N) \cup R^*(N)$ and it is well-known that $\mathfrak{S}_k^{new}(N, \chi)$ has a basis consisting of primitive forms. As a consequence, any form in $\mathfrak{S}_k^{new}(N, \chi)$ can be written as

$$\sum_M \sum_\ell a_{M,\ell} f_{M,\ell}(\ell z)$$

where the sum extends over M, ℓ such that $M\ell | N$ and $f_{M,\ell} \in \mathfrak{S}_k(M, \chi)$ are primitive forms.

Chapter 3

Hermitian modular forms and Jacobi forms

For the rest of this thesis, we fix an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D = -D_K$. We fix an embedding $K \rightarrow \mathbb{C}$ to treat K as a subfield of $\mathbb{C}$ and identify the non-trivial automorphism of $\text{Gal}(K/\mathbb{Q})$ as the restriction of complex conjugation of $\mathbb{C}$ to K . Thus, if $x \in K$ then its norm $|x|^2 = x\bar{x} \in \mathbb{Q}$ makes sense. We denote by $\mathfrak{o}_K$ the ring of integers of K and by $\mathfrak{d}_K := \frac{\mathfrak{o}_K}{\sqrt{-D}}$ the inverse different of K .

All rings are commutative with unities unless otherwise specified.

3.1 Unitary group

Let R be a ring and S an R -algebra equipped with an involution $\iota : S \rightarrow S$. If A is any R -algebra then $S \otimes_R A$ has an obvious induced involution $\iota \otimes \text{Id}$. For each matrix $g \in M_n(S \otimes_R A)$, we denote g^ι the matrix $(\iota \otimes \text{Id})(g_{ij})$ obtained

by applying the induced involution to every entry of g . We also denote g^t the transpose of g ; and when ι could be inferred without ambiguity, g^* is used to denote $(g^t)^\iota$.

If the triple $[R, S, \iota]$ is reasonable (namely, so that the subsequently mentioned Weil restriction of scalars $\text{Res}_{S/R} \mathbf{GL}_{2n/S}$ exists, for instance, S is a finitely generated projective R -module as in [23], Proposition 2.35), we have the unitary similitude group schemes defined over R and its subgroups:

$$\mathbf{GU}(n, n)_{[R, S, \iota]} := \{g \in \text{Res}_{S/R} \mathbf{GL}_{2n/S} \mid g^* J_{2n} g = \mu(g) J_{2n}, \mu(g) \in \mathbf{G}_m/R\}$$

$$\mathbf{U}(n, n)_{[R, S, \iota]} := \{g \in \mathbf{GU}(n, n) \mid \mu(g) = 1\}$$

$$\mathbf{SU}(n, n)_{[R, S, \iota]} := \{g \in \mathbf{U}(n, n) \mid \det g = 1\}$$

where $J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the $n \times n$ identity matrix and $\mathbf{GL}_{2n/S}$, $\mathbf{G}_m/R$ are the general linear group (over S) and multiplicative group (over R) respectively. Here, $\text{Res}_{S/R}$ denotes Weil restriction of scalars.

The case we are interested in is $R = \mathbb{Z}$, $A = \mathfrak{o}_K$ and ι being complex conjugation i.e. the restriction of the non-trivial element of $\text{Gal}(K/\mathbb{Q})$ to $\mathfrak{o}_K$. Thus, we omit the subscript $[R, S, \iota]$ if $R = \mathbb{Z}$, $S = \mathfrak{o}_K$ and ι is complex conjugation. Also, due to frequent usage, we also denote $G_n = \mathbf{GU}(n, n)$, $U_n = \mathbf{U}(n, n)$.

Just like $\mathbf{GL}_2^+(\mathbb{R})$ and $\mathbf{SL}_2(\mathbb{R})$ acting on the upper half plane, the group

$G_n^+(\mathbb{R})$ and its subgroup $U_n(\mathbb{R})$ act transitively on the *Hermitian half space*

$$\mathbb{H}_n := \left\{ Z \in M_n(\mathbb{C}) \mid \frac{1}{2i}(Z - Z^*) > 0 \right\}$$

by a similar formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

(note that the $AZ + B$ and $CZ + D$ are matrices so the order of multiplication matters) as well as on functions $F : \mathbb{H}_n \rightarrow \mathbb{C}$ by

$$(F|_{k,\nu} g)(Z) := \det(g)^{-\nu} j(g, Z)^{-k} F(g \cdot Z)$$

for any $k, \nu \in \mathbb{Z}$. Here, the factor of automorphy $j(g, Z) := \det(CZ + D)$.

Here is a proof for the above claims about the action being well-defined and transitive. Let us make a convention that whenever a complex number λ is used and an $n \times n$ matrix is expected, the implicit matrix is the diagonal matrix λI_n . For instance, i should mean $iI_n \in \mathbb{H}_n$ in $g \cdot i$ where $g \in G_n^+(\mathbb{R})$. First, we note the following fact about positive semi-definite matrices:

Lemma 3.1 (Cholesky decomposition c.f. [6], [13] Corollary 7.2.9). *A complex matrix A is positive semi-definite if and only if $A = T^*T$ for some upper triangular matrix T with nonnegative diagonal entries.*

If A is positive definite, then T is unique, has positive diagonal entries and the factorization $A = T^*T$ is called the Cholesky decomposition of A .

Observe that

- any matrix of the form T^*T (or $TT^* = (T^*)^*T^*$) is always positive semi-definite since for any column vector v , one has $\langle v, T^*Tv \rangle = v^*(T^*T)v = (Tv)^*(Tv) = |Tv|^2 \geq 0$. (Note also that $\langle v, T^*Tv \rangle = 0$ if and only if $Tv = 0$.)

More generally, if A is positive semi-definite then T^*AT is also positive semi-definite.

- Lemma 3.1 implies a positive semi-definite matrix A is necessarily Hermitian i.e. $A^* = A$ and that A is positive definite if and only if A is positive semi-definite and is invertible. In fact, A is positive semi-definite (definite) if and only if A is Hermitian and has non-negative (positive, resp.) eigenvalues.

Lemma 3.2. *For any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$, the matrix $Ci + D$ is invertible.*

Proof. Explicitly, one has

$$G_n(\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_{2n}(\mathbb{C}) \left| \begin{array}{l} AB^* = BA^*, \\ CD^* = DC^*, \\ AD^* - BC^* = \mu I_n \text{ for } \mu \in \mathbb{R}^\times \end{array} \right. \right\}.$$

To show that $Ci + D$ is invertible, it suffices to show that the positive semi-definite matrix

$$(Ci + D)(Ci + D)^* = (Ci + D)(-iC^* + D^*)$$

$$\begin{aligned}
&= CC^* + \underbrace{(CD^* - DC^*)i}_0 \text{ by assumption } g \in G_n(\mathbb{R}) + DD^* \\
&= CC^* + DD^*
\end{aligned}$$

is in fact positive definite.

Now, suppose on the contrary that we have some vector $v \neq 0$ such that $\langle v, (Ci + D)(Ci + D)^*v \rangle = 0$. That means $\langle v, CC^*v \rangle + \langle v, DD^*v \rangle = |C^*v|^2 + |D^*v|^2 = 0$ and so we must have $C^*v = D^*v = 0$. But then from $g \in G_n(\mathbb{R})$, we have $AD^* - BC^* = \mu I_n$ for some $\mu \in \mathbb{R}^\times$ and so multiplying both sides by v , we deduce

$$A \underbrace{D^*v}_0 - B \underbrace{C^*v}_0 = \mu v \neq 0$$

which is a contradiction. So $Ci + D$ is invertible. $\square$

Lemma 3.3. *For any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$ with similitude μ , let*

$$Z := (Ai + B)(Ci + D)^{-1}$$

$$E := (C^*i + D^*)^{-1}.$$

Then

$$\frac{1}{2i}(Z - Z^*) = \mu EE^* = \mu(DD^* + CC^*)^{-1}$$

is positive definite if and only if $\mu > 0$. In other words, if $\mu > 0$ then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot i \in \mathbb{H}_n.$$

Proof. Since $Ci + D$ is invertible, one has

$$\begin{aligned} (Ai + B)(Ci + D)^{-1} = Z &\iff Ai + B = Z(Ci + D) \\ &\iff B = -Ai + ZCi + ZD \end{aligned}$$

The assumption about $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n(\mathbb{R})$ with similitude μ is equivalent to

$$AD^* - BC^* = \mu I_n \tag{3.1.1}$$

$$AB^* = BA^* \tag{3.1.2}$$

$$CD^* = DC^* \tag{3.1.3}$$

Substituting the expression for B into (3.1.1), we find that

$$\begin{aligned} \mu I_n &= AD^* - (-Ai + ZCi + ZD)C^* \\ &= AD^* + AC^*i - ZCC^*i - ZDC^* \\ &= AD^* + AC^*i - ZCC^*i - ZCD^* && \text{by (3.1.3)} \\ &= (A - ZC)(D^* + C^*i) \end{aligned}$$

So both

$$A = \mu(D^* + C^*i)^{-1} + ZC$$

$$= ZC + \mu E$$

and

$$B = (ZC - A)i + ZD$$

$$= ZD - \mu Ei$$

are completely determined from C, D and Z .

Now (3.1.2) becomes

$$\begin{aligned} & (ZC + \mu E)(D^*Z^* + \mu E^*i) = (ZD - \mu Ei)(C^*Z^* + \mu E^*) \\ \iff & ZCD^*Z^* + \mu ZCE^*i + \mu ED^*Z^* + \mu^2 EE^*i \\ & = ZDC^*Z^* + \mu ZDE^* - \mu EC^*Z^*i - \mu^2 EE^*i \\ \iff & ZCE^*i - ZDE^* + EC^*Z^*i + ED^*Z^* + 2\mu EE^*i = 0 \\ \iff & Z \underbrace{(Ci - D)E^*}_{-I_n} + \underbrace{E(C^*i + D^*)}_{I_n} Z^* + 2\mu EE^*i = 0 \\ \iff & \mu EE^* = \frac{1}{2i}(Z - Z^*). \end{aligned}$$

It is clear that

$$\begin{aligned} EE^* &= (D^* + C^*i)^{-1}(D - Ci)^{-1} \\ &= [(D - Ci)(D^* + C^*i)]^{-1} \\ &= (DD^* + CC^*)^{-1}. \end{aligned}$$

□

Lemma 3.4. *For any $Z \in \mathbb{H}_n$, there exists an element $g \in U_n(\mathbb{R})$ such that $g \cdot i = Z$.*

Proof. The proof of previous lemma identifies

$$\{g \in U_n(\mathbb{R}) \mid g \cdot i = Z\}$$

with the set of pairs of matrices

$$\left\{ (C, D) \mid C, D \in M_n(\mathbb{C}), CD^* = DC^*, (DD^* + CC^*)^{-1} = \frac{1}{2i}(Z - Z^*) \right\}.$$

If $Z \in \mathbb{H}_n$ then $\frac{1}{2i}(Z - Z^*) > 0$ is positive definite and the second set is non-empty: We can take Cholesky decomposition $T^*T = \frac{1}{2i}(Z - Z^*)$ and then take

$$C = 0, D = T^{-1}.$$

Then

$$A = E = (D^*)^{-1} = T^*$$

$$B = ZD - Ei = ZT^{-1} - T^*i$$

and it is easy to check directly that the matrix

$$\begin{pmatrix} T^* & ZT^{-1} - T^*i \\ 0 & T^{-1} \end{pmatrix} \in U_n(\mathbb{R})$$

sends i to Z . □

Proposition 3.5. *The formula*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}$$

gives a well-defined action of $G_n^+(\mathbb{R})$ on $\mathbb{H}_n$.

Proof. Lemma 3.2 and Lemma 3.3 establishes the result for $Z = iI_n$, namely that $Ci + D$ is invertible and $(Ai + B)(Ci + D)^{-1} \in \mathbb{H}_n$. We now extend it to all $Z \in \mathbb{H}_n$ using transitivity attained in Lemma 3.4.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n^+(\mathbb{R})$ and $Z \in \mathbb{H}_n$ be arbitrary. Let $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in U_n(\mathbb{R})$ such that

$$(A'i + B')(C'i + D')^{-1} = Z$$

as in Lemma 3.4. Then we have

$$CZ + D = C(A'i + B')(C'i + D')^{-1} + D$$

is invertible if and only if

$$(CZ + D)(C'i + D') = C(A'i + B') + D(C'i + D')$$

is invertible (multiply both sides by the invertible matrix $C'i + D'$). But $(CA' + DC')i + (CB' + DD')$ is of the form $C''i + D''$ where C'', D'' are the two lower blocks of the product matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} * & * \\ C'' & D'' \end{pmatrix} \in G_n^+(\mathbb{R})$$

and hence, is invertible by Lemma 3.2.

Similarly, upon realizing that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} i$$

and using Lemma 3.3 for the product matrix, one easily deduces that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z \in \mathbb{H}_n.$$

□

Proposition 3.6. *The Lie group $U_n(\mathbb{R})$ acts transitively on $\mathbb{H}_n$ and the isotropy (i.e. stabilizer) subgroup of iI_n is precisely*

$$\mathcal{K}_{n,\infty} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in M_n(\mathbb{C}), AB^* = BA^*, AA^* + BB^* = I_n \right\}$$

which is the maximal compact subgroup of $U_n(\mathbb{R})$. Thus, we have isomorphism

$$U_n(\mathbb{R})/\mathcal{K}_\infty \cong \mathbb{H}_n$$

of smooth manifolds.

Proof. By the argument of Lemma 3.4 in the particular case where $Z = i$, for any matrix such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} i = i$, one has $CD^* = DC^*$ and $DD^* + CC^* = I_n$ and the A, B block are uniquely determined from C, D and Z . Evidently,

$$\begin{pmatrix} D & -C \\ C & D \end{pmatrix} i = i$$

so we must have $A = D, B = -C$. □

3.2 Hermitian modular forms

For $n > 1$, let $\mathbf{i}_n := iI_n$ and define the hermitian upper half-plane of degree n as the complex manifold

$$\mathbb{H}_n := \{Z \in M_n(\mathbb{C}) \mid -\mathbf{i}_n(Z - \overline{Z}^t) > 0\}$$

where for a matrix $A \in M_n(\mathbb{C})$, the notation $A > 0$ means that $A \in M_n(\mathbb{R})$ and A is positive-definite. The subgroup

$$G_n^+(\mathbb{R}) := \{g \in G_n(\mathbb{R}) \mid \mu(g) > 0\}$$

acts transitively on $\mathbb{H}_n$ via the familiar formula

$$g \cdot Z := (AZ + B)(CZ + D)^{-1}$$

if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_n(\mathbb{C})$.

A subgroup $\Gamma \subset G_n^+(\mathbb{R})$ is called a *congruence subgroup* if

- (i) Γ is *commensurable* with $U_n(\mathbb{Z})$, and
- (ii) $\Gamma \supseteq \Gamma_n(N) := \{g \in U(n, n)(\mathbb{Z}) \mid g \equiv I_{2n} \pmod{N}\}$ for some positive $N \in \mathbb{Z}$.

For $g \in G_n^+(\mathbb{R})$ and $Z \in \mathbb{H}_n$, we define the factor of automorphy

$$j(g; Z) := \det(CZ + D)$$

and for integer $k \geq 0$ and function $F : \mathbb{H}_n \rightarrow \mathbb{C}$, we define the action $|_k g$ by

$$F|_k g(Z) := j(g; Z)^{-k} F(gZ).$$

Definition 3.7. A function $F : \mathbb{H}_n \rightarrow \mathbb{C}$ is called a *Hermitian semi-modular form* of weight k and level Γ if

$$F|_k g = F \quad \text{for all } g \in \Gamma.$$

If F is further holomorphic then it is called a *Hermitian modular form*. We denote $\mathfrak{M}'_k(\Gamma)$ for the space of all such Hermitian semi-modular forms and by $\mathfrak{M}_k(\Gamma) \subset \mathfrak{M}'_k(\Gamma)$ for the subspace of Hermitian modular forms.

Subsequently, we are only interested in the case where congruence subgroup $\Gamma = \Gamma_{0,n}(N)$ with

$$\Gamma_{0,n}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n, n)(\mathbb{Z}) \mid C \equiv 0 \pmod{N\mathfrak{o}_K} \right\}$$

so we denote

$$\mathfrak{M}_{k,n}(N) := \mathfrak{M}_k(\Gamma_{0,n}(N))$$

Any Hermitian modular form of level N possesses Fourier expansion

$$F(Z) = \sum_{T \in S_n(\mathbb{Q})} c_F(T) \mathbf{e}[\mathrm{Tr}(TZ)]$$

where $c_F(T) \in \mathbb{C}$,

$$S_n = L_n^\vee := \{x \in \mathbb{S}_n(\mathbb{Q}) \mid \mathrm{Tr}(xL) \subset \mathbb{Z}\}$$

is the dual lattice of the lattice $L_n = \mathbb{S}_n(\mathbb{Z}) \cap M_n(\mathfrak{o}_K)$ of integral Hermitian matrices (with respect to the trace pairing Tr) and

$$\mathbb{S}_n := \{h \in \mathrm{Res}_{\mathfrak{o}_K/\mathbb{Z}} M_{n/\mathfrak{o}_K} \mid h^* = h\}$$

is the $\mathbb{Z}$ -group scheme of Hermitian matrices. Also, we adopt the notation $\mathbf{e}[x] := e^{2\pi i x}$. Explicitly, one has, for example

$$S_2 = \left\{ \begin{pmatrix} \ell & t \\ \bar{t} & m \end{pmatrix} \in M_2(K) \mid \ell, m \in \mathbb{Z}; t \in \mathfrak{d}_K \right\}.$$

Definition 3.8. The *Maaß space* $\mathfrak{M}_k^*(N) \subseteq \mathfrak{M}_{k,2}(N)$ consists of forms $F \in \mathfrak{M}_{k,2}(N)$ such that there exists a function $\alpha_F : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ so that the Fourier coefficients of F can be described using α_F in the following way

$$c_F(T) = \sum_{\substack{d \in \mathbb{Z}_+, d \mid \epsilon(T) \\ (d, N) = 1}} d^{k-1} \alpha_F \left(\frac{D_K \det T}{d^2} \right)$$

for all $T \in S_2(\mathbb{Z}), T \geq 0, T \neq 0$ where $c_F(T)$ is the T -th Fourier coefficient of F and

$$\epsilon(T) := \max \left\{ q \in \mathbb{Z}_+ \mid \frac{1}{q} T \in S_2(\mathbb{Z}) \right\}.$$

Writing a matrix $Z \in \mathbb{H}_2$ as $Z = \begin{pmatrix} \tau & z \\ w & \tau^* \end{pmatrix}$, we can think of a Hermitian modular form as a function of four complex variables (τ, z, w, τ^*) where we

can collect its Fourier expansion to express

$$F(Z) = \sum_{m=0}^{\infty} \varphi_m^F(\tau, z, w) \mathbf{e}[m\tau^*].$$

It is well-known (c.f. [21] Section 3 for level $N = 1$ and [10] Satz 7.1 for general level) that the functions φ_m^F 's are Jacobi forms of index m , normally referred to as Fourier–Jacobi coefficients of F . We define Jacobi forms in the next section.

3.3 Hermitian Jacobi forms

We recall the notion of Jacobi forms from [5]. First, any $M \in \mathbf{U}(1, 1)(\mathbb{Z})$ can be written as ϵA for some $A \in \mathbf{SL}_2(\mathbb{Z})$ and $\epsilon \in \mathfrak{o}_K^\times$. Let $\mathfrak{H}$ denote the complex upper half plane. We define the action of the Jacobi group $\mathbf{U}(1, 1)(\mathbb{Z}) \ltimes \mathfrak{o}_K^2$ on the functions $\varphi : \mathfrak{H} \times \mathbb{C}^2 \rightarrow \mathbb{C}$ by

$$\begin{aligned} \varphi|_{k,m}[\epsilon A] &:= (c\tau + d)^{-k} \mathbf{e}\left[-m \frac{czw}{c\tau + d}\right] \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\epsilon z}{c\tau + d}, \frac{\bar{\epsilon} w}{c\tau + d}\right) \\ \varphi|_{k,m}[\lambda, \mu] &:= \mathbf{e}\left[m(|\lambda|^2\tau + \bar{\lambda}z + \lambda w)\right] \varphi(\tau, z + \lambda\tau + \mu, w + \bar{\lambda}\tau + \bar{\mu}) \end{aligned}$$

As noted in [5], the ϵ in the expression $M = \epsilon A$ is not unique but the action described here is well-defined as long as k is divisible by $|\mathfrak{o}_K^\times|$.

Definition 3.9. A *Hermitian Jacobi form of weight k , index m and level*

N is a holomorphic function $\varphi : \mathfrak{H} \times \mathbb{C}^2 \rightarrow \mathbb{C}$ such that

- (i) $\varphi|_{k,m}[\epsilon A] = \varphi$ for all $A \in \Gamma_0(N)$;
- (ii) $\varphi|_{k,m}[\lambda, \mu] = \varphi$ for all $\lambda, \mu \in \mathfrak{o}_K$; and
- (iii) for each $M \in \mathrm{SL}_2(\mathbb{Z})$, the function $\varphi|_{k,m}[M]$ has a Fourier expansion of the form

$$\varphi|_{k,m}[M](\tau, z, w) = \sum_{\substack{\ell \in \mathbb{Z}_{\geq 0}, t \in \mathfrak{d}_K^{-1} \\ \nu|t|^2 \leq \ell m}} c_\varphi^M(\ell, t) \mathbf{e} \left[\frac{\ell}{\nu} \tau + \bar{t}z + tw \right]$$

where $\nu = \nu(M) \in \mathbb{Z}_+$ depends on M and $\nu(I_2) = 1$. The $c_\varphi^M(\ell, t) \in \mathbb{C}$ are the Fourier coefficients. We drop the superscript for $M = I_2$ i.e. $c_\varphi(\ell, t) = c_\varphi^{I_2}(\ell, t)$.

We denote by $\mathfrak{J}_{k,m}(N)$ for the space of Hermitian Jacobi forms of weight k , index m and level N .

Proposition 2.4 of [5] says that just like the Siegel case, a form F in the Hermitian Maaß space $\mathfrak{M}_k^*(N)$ is completely determined by its first Jacobi form φ_1^F and one of the major results of the same paper (Theorem 2.8) shows that in fact $\mathfrak{M}_k^*(N) \cong \mathfrak{J}_{k,1}^*(N)$ where $\mathfrak{J}_{k,1}^*(N) \subset \mathfrak{J}_{k,1}(N)$ is the subspace of special Jacobi forms.

Definition 3.10. A form $\varphi \in \mathfrak{J}_{k,1}(N)$ is called a *special Jacobi form* if its Fourier coefficients $c_\varphi(\ell, t)$ only depend on $\ell - |t|^2$.

Thus, we have the analogue of the classical Maaß lift for Hermitian modular forms.

3.4 Theta decomposition

Recall that $\mathfrak{d}_K = \frac{i}{\sqrt{D}}\mathfrak{o}_K$ is the different ideal of $\mathfrak{o}_K$. Let $[\mathfrak{d}_K] := \frac{i}{\sqrt{D}}\mathfrak{o}_K/\mathfrak{o}_K$.

For each $u \in [\mathfrak{d}_K]$, we define the theta function

$$\vartheta_u(\tau, z, w) := \sum_{a \in u + \mathfrak{o}_K} \mathbf{e} [|a|^2 \tau + \bar{a}z + aw].$$

For fixed τ , the theta functions are linearly independent (c.f. Haverkamp [10] Proposition 5.1) and they transformed in the following way:

$$\vartheta_u|_{1,1} \sigma = \sum_{v \in [\mathfrak{d}_K]} M_{u,v}(\sigma) \vartheta_v$$

for each $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ where

$$M_{u,v} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} \frac{-i}{c\sqrt{D}} \sum_{\gamma \in u + \mathfrak{o}_K/c\mathfrak{o}_K} \mathbf{e} \left[\frac{a|\gamma|^2 - \gamma\bar{v} - \bar{\gamma}v + d|v|^2}{c} \right] & \text{if } c \neq 0; \\ \mathrm{sign}[a] \delta_{u,av} \mathbf{e}[ab|u|^2] & \text{otherwise} \end{cases} \quad (3.4.1)$$

according to [5] Lemma 3.1 or the Lemma in Section 4 of [21]. Since the transformation $|_{k,m}\sigma$ is associative, one could think of M as a group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{GL}_d(\mathbb{C})$ where d is the number of classes in $[\mathfrak{d}_K]$ sending each σ to the matrix $(M_{u,v}(\sigma))_{u,v \in [\mathfrak{d}_K]}$ whose rows and columns are indexed by $[\mathfrak{d}_K]$ should we fix an ordering of $[\mathfrak{d}_K]$.

By section 3 of [5], any special Jacobi form $\varphi \in \mathfrak{J}_k^*(N)$ can be expressed uniquely as a combination

$$\varphi = \sum_{u \in [\mathfrak{d}_K]} f_u \vartheta_u \quad \text{where} \quad f_u(\tau) = \sum_{\substack{\ell \geq 0 \\ \ell \equiv -D|u|^2 \pmod{D}}} \alpha_\varphi^*(\ell) \mathbf{e} \left[\frac{\ell \tau}{D} \right]. \quad (3.4.2)$$

Here, $\alpha_\varphi^*(D(m - |u|^2)) = c_\varphi^{I_2}(m, u)$ are the Fourier coefficients of φ as in definition 3.9. We call (3.4.2) the *theta decomposition* of φ .

3.5 The injection $\mathfrak{J}_k^*(N) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi)$

Following Krieg, we decompose the character $\chi_K = \prod_{p \mid D} \chi_p$. More concretely, $\chi_p(n) = (n \mid p)$ is just the Legendre symbol mod p for odd prime $p \mid D$ whereas

$$\chi_2(n) = \begin{cases} (-4 \mid n) = (-1)^{(n-1)/2} & \text{if } 4 \mid D \text{ and } 8 \nmid D, \\ (-8 \mid n) = (-1)^{(n^2-4n+3)/8} & \text{if } 8 \mid D \text{ and } \frac{D}{8} \equiv 1 \pmod{4}, \\ (8 \mid n) = (-1)^{(n^2-1)/8} & \text{if } 8 \mid D \text{ and } \frac{D}{8} \equiv 3 \pmod{4}. \end{cases}$$

Then we define

$$a_D(\ell) := |\{u \in [\mathfrak{d}_K] \mid D|u|^2 \equiv -\ell \pmod{D}\}|$$

which could be computed as

$$a_D(\ell) = \prod_{p \mid D} (1 + \chi_p(-\ell))$$

according to Krieg [21], Section 4, equation (5). Now, we define the space

$$\mathfrak{M}_{k-1}^+(DN, \chi) := \{g \in \mathfrak{M}_{k-1}(DN, \chi) \mid \alpha_\ell(g) = 0 \text{ if } a_D(-\ell) = 0\}$$

similar to that of [21] but for arbitrary N as opposed to $N = 1$ in [21]. When D is prime, this is the same as the space defined in [5]. We state the following generalization of proposition 3.5 of [5]:

Proposition 3.11. *Suppose that $(D, N) = 1$. Then the map*

$$\mathfrak{J}_k^*(N) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi) \quad \varphi \mapsto f := f_0|_{k-1} W_D$$

is an injection where

$$W_D = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} D & y \\ 0 & 1 \end{pmatrix}$$

and y is any integer such that $y \equiv N^{-1} \pmod{D}$. The Fourier coefficients of f satisfy

$$a_\ell(f) = i \frac{a_D(\ell)}{\sqrt{D}} \chi(N) \alpha_\varphi^*(\ell).$$

Note that we are basically dropping the assumptions D prime and N odd in [5]. The proof is the same as the proof given in [5]. One only needs to check that the simple lemma used in the proof also works for non-prime D and N being even number:

Lemma 3.12 (Lemma 3.3 in [5]). *Suppose $(D, N) = 1$ and $t \in \mathbb{Z}$ such that $(t, N) = 1$. Then*

$$\sum_{\gamma \in \mathfrak{o}_K / N \mathfrak{o}_K} \mathbf{e} \left[\frac{t|\gamma|^2}{N} \right] = \chi(N)N.$$

Proof. By Chinese Remainder Theorem, the statement can be reduced to the case where N is a prime power. When N is an odd prime power, the left hand side is a product of two Gauss sum as in Haverkamp [10], Lemma 0.4. When $N = 2^r$ is a power of 2, one can use induction on r . $\square$

Notice that Proposition 3.11 already tells us what the lifting should be, if it exists. Suppose we have $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$ and $\varphi_g \in \mathfrak{J}_k^*(N)$ is its pre-image. Write $\varphi_g = \sum f_{u,g} \vartheta_u$ as in (3.4.2). Then by Proposition 3.11, $g = f_{0,g}|_{k-1} W_D$ and the Fourier coefficients of g satisfies $a_\ell(g) = i \frac{a_D(\ell)}{\sqrt{D}} \chi(N) \alpha_{\varphi_g}^*(\ell)$ which leads to

$$\alpha_{\varphi_g}^*(\ell) = \frac{-i\sqrt{D}}{a_D(\ell)\chi(N)} a_\ell(g)$$

and the decomposition (3.4.2) tells us that

$$\begin{aligned} f_{u,g}(\tau) &= \sum_{\substack{\ell \geq 0 \\ \ell \equiv -D|u|^2 \pmod{D}}} \alpha_{\varphi_g}^*(\ell) e\left[\frac{\ell\tau}{D}\right] \\ &= \sum_{\substack{\ell \geq 0 \\ \ell \equiv -D|u|^2 \pmod{D}}} \frac{-i\sqrt{D}}{a_D(\ell)\chi(N)} a_\ell(g) e\left[\frac{\ell\tau}{D}\right] \\ &= \chi(N) \frac{-i\sqrt{D}}{a_D(-D|u|^2)} \sum_{\substack{\ell=0 \\ \ell \equiv -D|u|^2 \pmod{D}}}^{\infty} a_\ell(g) e\left[\frac{\ell\tau}{D}\right]. \end{aligned}$$

Note that this is the lift given by Krieg in Section 6 of [21] for level $N = 1$, twisted by the factor of $\chi(N)$ to account for the level.

In section 4.3, we will show that for any $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$, the function $\varphi_g = \sum f_{u,g} \vartheta_u$ with the $f_{u,g}$ as derived above is indeed a Hermitian Jacobi form; and so the map in Proposition 3.11 is an isomorphism. To do that, we generalize Krieg's characterization of plus forms and obtain the arithmetical condition for φ_g to be a Hermitian Jacobi form.

Chapter 4

The Maaß lift

4.1 Characterization of plus forms

For each $m \mid D$, set $\psi_m := \prod_{p \mid m} \chi_p$ as in section 5 of [21]. For a character ψ of modulus M , the (twisted) Gauss sum associated to ψ is defined as

$$\mathbf{G}(\psi; b) := \sum_{a \in \mathbb{Z}/M} \psi(a) \mathbf{e} \left[b \frac{a}{M} \right].$$

and the standard Gauss sum $\mathbf{G}(\psi) := \mathbf{G}(\psi; 1)$. We remark that χ_K is of conductor D and that if $(m, \frac{D}{m}) = 1$ then ψ_m is of conductor m . It is well-known from one of Gauss' proofs of quadratic reciprocity (c.f. [24], Lemma 4.8.1) that

$$\mathbf{G}(\psi_m) := \varepsilon(\psi_m) \sqrt{m}$$

where

$$\varepsilon(\psi_m) := \begin{cases} 1 & \text{if } \psi_m(-1) = 1, \\ i & \text{if } \psi_m(-1) = -1. \end{cases}$$

A fancier proof is available in Shimura's book [31], Appendix A, Corollary A6.4.

Fix natural numbers m, n such that $D = mn$ and $(m, n) = 1$. We consider the Hecke operator (without normalization factor)

$$g|_{k-1}U_m := \sum_{j=0}^{m-1} g|_{k-1} \begin{pmatrix} 1 & j \\ 0 & m \end{pmatrix}$$

and choose a matrix $P_m \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$P_m \equiv \begin{cases} J \bmod m^2 \\ I \bmod (nN)^2 \end{cases}$$

where $J := J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are the usual generators for $\mathrm{SL}_2(\mathbb{Z})$. Set

$$Q_m := P_m \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$$

and define the operator V_m by

$$g|_{k-1}V_m := g|_{k-1}U_m|_{k-1}Q_m.$$

Similar to [21] Section 5, one has

Lemma 4.1. $Q_m \Gamma_0(ND) Q_m^{-1} = \Gamma_0(ND)$ and so $g \mapsto g|_{k-1}Q_m$ is an involution of $\mathfrak{M}_{k-1}(DN, \chi)$.

Proof. Suppose that $x, y, z, t \in \mathbb{Z}$ such that

$$\gamma = \begin{pmatrix} x & y \\ NDz & t \end{pmatrix} \in \Gamma_0(ND)$$

and we need to check that $Q_m \gamma Q_m^{-1} \in \Gamma_0(ND)$. To do that, it suffices to check that

$$Q_m \gamma Q_m^{-1} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{M}$$

for $M = m$ and $M = nN$ by Chinese Remainder Theorem. One has

$$\begin{aligned} Q_m \gamma Q_m^{-1} &= P_m \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ NDz & t \end{pmatrix} \begin{pmatrix} 1/m & 0 \\ 0 & 1 \end{pmatrix} P_m^{-1} \\ &= P_m \begin{pmatrix} x & my \\ Nnz & t \end{pmatrix} P_m^{-1} \\ &\equiv \begin{cases} J \begin{pmatrix} x & my \\ Nnz & t \end{pmatrix} J^{-1} \pmod{m} \\ I \begin{pmatrix} x & my \\ Nnz & t \end{pmatrix} I^{-1} \pmod{nN} \end{cases} \quad \text{by definition of } P_m \\ &\equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & my \\ Nnz & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pmod{m} \\ \begin{pmatrix} x & my \\ 0 & t \end{pmatrix} \pmod{nN} \end{cases} \\ &\equiv \begin{cases} \begin{pmatrix} t & -Nnz \\ -my & x \end{pmatrix} \pmod{m} \\ \begin{pmatrix} x & my \\ 0 & t \end{pmatrix} \pmod{nN} \end{cases} \\ &\equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{m} \text{ and } \pmod{nN} \end{aligned}$$

To check the part about involution: for any $\gamma \in \Gamma_0(ND)$, one has $Q_m \gamma Q_m^{-1} =$

γ_1 (equivalently $Q_m\gamma = \gamma_1 Q_m$) for some $\gamma_1 \in \Gamma_0(ND)$ and so

$$(g|_{k-1}Q_m)|_{k-1}\gamma = g|_{k-1}Q_m\gamma = g|_{k-1}\gamma_1 Q_m = \chi(\gamma_1)g|_{k-1}Q_m.$$

It remains to check that $\chi(\gamma_1) = \chi(\gamma)$. And this follows from the computation above. Note that it is not true for general character χ since the lower right entry of γ_1 is not congruent to $t \pmod{D}$: It is congruent to $t \pmod{Nn}$ and to $x \pmod{m}$. However, since our character χ is real i.e. take values ± 1 , $\chi(x) = \frac{1}{\chi(t)} = \chi(t)$ as $\frac{1}{\pm 1} = \pm 1$. Decompose $\chi = \psi_m\psi_n$ and we find that $\chi(\gamma_1) = \psi_m(x)\psi_n(t) = \psi_m(t)\psi_n(t) = \chi(t) = \chi(\gamma)$. $\square$

Proposition 4.2. *If k is even and $g \in \mathfrak{M}_{k-1}(DN, \chi)$ then we have*

$$g|_{k-1}V_m = \mathbf{G}(\psi_m)m^{1-k}g$$

if and only if

$$\alpha_\ell(g) = 0 \text{ for all } \ell \text{ such that } \psi_m(-\ell) = -1.$$

Proof. The proof is verbatim as in [21] with appropriate modification to take care of the level. $\square$

Corollary 4.3. *Let k be even and $D = 2^e d$ with $2 \nmid d$. Then*

$$\mathfrak{M}_{k-1}^+(DN, \chi) = \left\{ g \in \mathfrak{M}_{k-1}(DN, \chi) \left| \begin{array}{l} g|_{k-1} V_p = \mathbf{G}(\psi_p) p^{1-k} g \\ \text{for } p = 2^e \text{ and every prime } p|d \end{array} \right. \right\}.$$

Furthhermore, if $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$ and $(m, \frac{D}{m}) = 1$ then

$$g|_{k-1} V_m = \mathbf{G}(\psi_m) m^{1-k} g.$$

Note that Proposition 4.2 and Corollary 4.3 are generalizations of Proposition and Corollary A (resp.) in Section 5 of [21].

For any $j \in \mathbb{Z}/D\mathbb{Z}$, set $\sigma_j := \begin{pmatrix} 1 & j \\ 0 & D \end{pmatrix}$. Our goal is to compute $g|_{k-1} \sigma_j \sigma$ for $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$ and any $\sigma \in \Gamma_0(N)$. Fix such a $\sigma = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. For any $j \in \mathbb{Z}$, let $\mu = \mu_\sigma(j) := (x + zj, D)$ and $m = m_\sigma(j) := D_\mu$ be the μ -component of D i.e.

$$m = \prod_{p|\mu \text{ prime}} p^{\text{val}_p(D)}$$

and let $n := \frac{D}{m}$. Then we have

Lemma 4.4. (i) $(x + zj, n) = 1$;

(ii) $(r(x + zj) - nz, m) = 1$ for any $r \in \mathbb{Z}$; in particular, $(m, z) = 1$;

(iii) $(x + zj, y + tj) = 1$;

(iv) $(y + tj, m) = 1$.

Proof. It is obvious from the definition that $(m, n) = 1$ and that if m and μ has the same set of prime divisors.

(i) We have $(x + zj, n) \mid (x + zj, D) = \mu \mid m$ while on the other hand, we obviously have $(x + zj, n) \mid n$ and so $(x + zj, n) \mid (m, n) = 1$.

(ii) If we have a prime $p \mid (r(x + zj) - nz, m)$ then $p \mid m$ so $p \mid \mu$ by construction so $p \mid x + zj$. We also have $p \mid r(x + zj) - nz$ so that $p \mid nz$ which implies $p \mid z$ since $(m, n) = 1$. But then we get from $p \mid x + zj$ that $p \mid x$. So p divides both x, z ; contradicting the fact that $(x, z) = 1$ from the assumption $\sigma \in \Gamma_0(N)$.

(iii) This is obvious since $x + zj$ and $y + jt$ are the first row of the matrix $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ in $\mathbf{SL}_2(\mathbb{Z})$.

(iv) Suppose that we have a prime divisor $p \mid (y + jt, m)$. Then as in (ii), $p \mid m$ so $p \mid \mu$; hence $p \mid x + zj$. But then $p \mid (x + zj, y + jt) = 1$ and we get a contradiction with (iii).

□

Proposition 4.5. *For any $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$, we have*

$$g|_{k-1}\sigma_j\sigma = \sum_{s=0}^{m-1} \frac{\mathbf{G}(\psi_m; (x + zj)s - \lambda)}{m \psi_n(x + zj)} g|_{k-1} \begin{pmatrix} 1 & ns + \kappa \\ 0 & D \end{pmatrix}$$

where $\kappa = \kappa_\sigma(j) \in \mathbb{Z}$ is any integer such that

$$\kappa \equiv \frac{y + tj}{x + zj} \pmod{n}$$

and

$$\lambda := \frac{y + jt - \kappa(x + zj)}{n} \in \mathbb{Z}.$$

In case D is square-free, the formula can be simplified to

$$g|_{k-1}\sigma_j\sigma = \frac{\mathbf{G}(\psi_m; nz)}{m \psi_n(x + zj)} \sum_{s=0}^{m-1} g|_{k-1} \begin{pmatrix} 1 & ns + \kappa \\ 0 & D \end{pmatrix}.$$

Proof. Let $P_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\begin{aligned} g|_{k-1}\sigma_j\sigma &= \frac{m^{k-1}}{\mathbf{G}(\psi_m)} g|_{k-1} V_m |_{k-1} \begin{pmatrix} x + zj & y + jt \\ Dz & Dt \end{pmatrix} \text{ by Corollary 4.3} \\ &= \frac{m^{k-1}}{\mathbf{G}(\psi_m)} \sum_{r=0}^{m-1} g|_{k-1} \begin{pmatrix} a + rc & b + rd \\ mc & md \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x + zj & y + jt \\ Dz & Dt \end{pmatrix} \\ &= \frac{1}{\mathbf{G}(\psi_m)} \sum_{r=0}^{m-1} g|_{k-1} \begin{pmatrix} (a + rc)(x + zj) + (b + rd)nz & (a + rc)(y + jt) + (b + rd)nt \\ mc(x + zj) + dDz & mc(y + jt) + (b + rd)mdnt \end{pmatrix} \end{aligned}$$

Let us temporarily denote the above matrix by $\gamma = \gamma_r$ and denote by A, B

the entries of the first column of γ i.e.

$$A = (a + rc)(x + zj) + (b + rd)nz$$

$$B = mc(x + zj) + dDz$$

Observe that $\det \gamma = D$ and that the lower row of γ is divisible by D . Thus, the matrix obtained by dividing both entries on the lower row of γ by D , namely the matrix

$$\begin{pmatrix} A & (a+rc)(y+jt) + (b+rd)nt \\ \frac{B}{D} & \frac{mc(y+jt) + (b+rd)mdnt}{D} \end{pmatrix}$$

is in $\mathrm{SL}_2(\mathbb{Z})$ whence $(A, \frac{B}{D}) = 1$ and so $(A, B) = (A, mn\frac{B}{D}) = (A, m)(A, n) = 1$ by Lemma 4.4 for $A \equiv \begin{cases} r(x+zj) - nz & \text{mod } m \\ x + zj & \text{mod } n \end{cases}$. Since $(A, B) = 1$ and $ND \mid B$, there exists a matrix $\begin{pmatrix} A & * \\ B & * \end{pmatrix} \in \Gamma_0(ND)$ and so we have

$$g|_{k-1}\gamma = \frac{\psi_n(x+zj)}{\psi_m(r(x+zj) - nz)} g|_{k-1} \begin{pmatrix} 1 & \frac{(a+rc)(y+jt) + (b+rd)nt}{(a+rc)(x+zj) + (b+rd)nz} \text{ mod } D \\ 0 & D \end{pmatrix}$$

Let us denote the upper right entry

$$F_r = \frac{(a+rc)(y+jt) + (b+rd)nt}{(a+rc)(x+zj) + (b+rd)nz} \text{ mod } D \equiv \begin{cases} \frac{r(y+jt) - nt}{r(x+zj) - nz} & \text{mod } m \\ \frac{y+jt}{x+zj} & \text{mod } n \end{cases}$$

so that we get

$$g|_{k-1}\sigma_j\sigma = \sum_{r=0}^{m-1} \frac{\psi_n(x+zj)}{\mathbf{G}(\psi_m; r(x+zj) - nz)} g|_{k-1} \begin{pmatrix} 1 & F_r \\ 0 & D \end{pmatrix}$$

As r runs through the residues mod m , the F_r also goes through the subset of residues

$$\{ns + \kappa \mid 0 \leq s \leq m-1\} \subseteq \mathbb{Z}/D\mathbb{Z}.$$

Using a simple change-of-variable from r to s and an observation that $\frac{1}{\mathbf{G}(\psi_m; b)} = \frac{\mathbf{G}(\psi_m; -b)}{m}$ to move the Gauss sum to the numerator, we derive the formula claimed in the proposition.

With the assumption that D is square-free, we always have $\mu = m$ so $m \mid x + zj$ and the factor $\mathbf{G}(\psi_m; r(x + zj) - nz) = \mathbf{G}(\psi_m; -nz)$ is independent of s . $\square$

4.2 Arithmetic Criterion

To simplify a lot of subsequent formulas, we shall also denote $a_u := a_D(-D|u|^2)$ for $u \in [\mathfrak{d}_K]$. Given a plus form

$$g = \sum_{\ell=0}^{\infty} a_{\ell}(g) \mathbf{e}[\ell\tau] \in \mathfrak{M}_{k-1}^{+}(DN, \chi),$$

we define

$$g_u := \frac{-i\sqrt{D}}{a_u} \sum_{\substack{\ell=0 \\ \ell \equiv -D|u|^2 \pmod{D}}}^{\infty} a_{\ell}(g) \mathbf{e}\left[\frac{\ell\tau}{D}\right] \quad \text{for all } u \in [\mathfrak{d}_K]$$

and

$$\varphi_g := \sum g_u \vartheta_u.$$

If φ_g is a Jacobi form of level N then it will be the lifting of $\chi(N)g = \pm g$, by the argument at the end of section 3.

In this section, we shall derive transformation formula for g_u using result

of the previous section and then deduce an *arithmetic condition* for φ_g to be a Hermitian Jacobi form (Proposition 4.8).

Proposition 4.6. *Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. For each $j \in \mathbb{Z}$, we choose $\kappa = \kappa_\sigma(j) \in \mathbb{Z}$ such that*

$$\kappa \equiv \frac{b + dj}{a + cj} \pmod{n}$$

as before; in addition that if $m \neq \mu$ (i.e. $m = 2\mu$ or $m = 4\mu$), then

$$\kappa \equiv \frac{(b + dj + c)/2^f}{(a + cj)/2^f} \pmod{\frac{m}{\mu}}$$

where $f = \text{val}_2(a + cj) = \text{val}_2(\mu)$ is the maximum power of two dividing μ .

Then

$$\begin{aligned} & g_u|_{k-1\sigma} \\ &= \frac{1}{Da_u} \sum_{v \in [\mathfrak{d}_K]} \sum_{\substack{j=0 \\ (D|v|^2, m)=\mu}}^{D-1} R_\sigma(v, j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) \mathbf{e} [|u|^2 j - |v|^2 \kappa] g_v \end{aligned}$$

where

$$R_\sigma(v, j) = \begin{cases} \frac{1}{2} \left(1 + \mathbf{e} \left[-\frac{(a+cj)D|v|^2}{2m} \right] \chi_2(5 - 2nc) \right) & \text{if } m = 4\mu, \\ 1 & \text{if } m \neq 4\mu. \end{cases}$$

It is not difficult to see that if $m \neq \mu$ then $2^f | b + dj + c$ from $ad - bc = 1$ and $2^f | a + cj$ and so $\frac{(b+dj+c)/2^f}{(a+cj)/2^f} \pmod{\frac{m}{\mu}}$ is well-defined. Also, note that such

κ always exists for $(n, \frac{m}{\mu}) \mid (n, m) = 1$ so we can solve for κ by Chinese

Remainder Theorem.

Proof. For $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$, observe the following

$$\begin{aligned} \text{(i)} \quad g_u &= \frac{-iD^{k-3/2}}{a_u} \sum_{j=0}^{D-1} \mathbf{e}[|u|^2 j] g|_{k-1} \sigma_j \\ \text{(ii)} \quad g &= \frac{i}{\sqrt{D}} \sum_{v \in [\mathfrak{d}_K]} g_v|_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\ \text{(iii)} \quad g_v|_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \sigma_j &= D^{1-k} \mathbf{e}[-|v|^2 j] g_v \end{aligned}$$

and then use Proposition 4.5, we arrive at

$$\begin{aligned} g_u|_{k-1} \sigma &= \frac{-iD^{k-3/2}}{a_u} \sum_{j=0}^{D-1} \sum_{s=0}^{m-1} \frac{\mathbf{e}[|u|^2 j] \mathbf{G}(\psi_m; s(a+cj) - \lambda)}{m \psi_n(a+cj)} g|_{k-1} \sigma_{ns+\kappa} \text{ by (i) and Proposition 4.5} \\ &= \frac{-iD^{k-3/2}}{a_u} \sum_{j,s} \frac{\mathbf{e}[|u|^2 j] \mathbf{G}(\psi_m; s(a+cj) - \lambda)}{m \psi_n(a+cj)} \frac{i}{\sqrt{D}} \sum_{v \in [\mathfrak{d}_K]} g_v|_{k-1} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} |_{k-1} \sigma_{ns+\kappa} \text{ by (ii)} \\ &= \frac{-iD^{k-3/2}}{a_u} \sum_{j,s} \frac{\mathbf{e}[|u|^2 j] \mathbf{G}(\psi_m; s(a+cj) - \lambda)}{m \psi_n(a+cj)} \frac{i}{\sqrt{D}} \sum_{v \in [\mathfrak{d}_K]} D^{1-k} \mathbf{e}[-|v|^2 (ns+\kappa)] g_v \text{ by (iii)} \\ &= \frac{1}{Da_u} \sum_{v \in [\mathfrak{d}_K]} \sum_{j,s} \frac{\mathbf{G}(\psi_m; s(a+cj) - \lambda) \mathbf{e}[|u|^2 j - |v|^2 (ns+\kappa)]}{m \psi_n(a+cj)} g_v \\ &= \frac{1}{Da_u} \sum_v \sum_{j=1}^{D-1} \frac{\mathbf{G}(\psi_m) \mathbf{e}[|u|^2 j - |v|^2 \kappa]}{m \psi_n(a+cj)} \left(\sum_{s=0}^{m-1} \psi_m(s(a+cj) - \lambda) \mathbf{e}\left[-D|v|^2 \frac{s}{m}\right] \right) g_v \end{aligned}$$

With the appropriate requirement for κ as in the statement of the proposition,

a careful case-by-case analysis yields the inner most sum

$$\sum_{s=0}^{m-1} \psi_m(s(a+cj) - \lambda) \mathbf{e}\left[-D|v|^2 \frac{s}{m}\right]$$

$$= \begin{cases} \frac{m}{2} \psi_m(nc) \left(1 + \mathbf{e} \left[-\frac{(a+cj)D|v|^2}{2m} \right] \chi_2(5-2nc) \right) \delta_{\mu, (D|v|^2, m)} & \text{if } m = 4\mu, \\ m \psi_m(nc) \delta_{\mu, (D|v|^2, m)} & \text{if } m \neq 4\mu. \end{cases}$$

where

$$\delta_{X,Y} := \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{otherwise} \end{cases}$$

is the usual Kronecker delta.

□

Remark 4.7. We remark that in case $m = 4\mu$ (which can only happen if $8 \mid D$) and $(D|v|^2, m) = \mu$, we have $\mathbf{e} \left[-\frac{(a+cj)D|v|^2}{2m} \right] = \mathbf{e} \left[-\frac{\frac{a+cj}{2} \frac{D|v|^2}{\mu}}{4} \right] \in \{\pm i\}$ only depends on $j \bmod 8$. Also, since $\chi_2(5) = -1$ for both possibilities of $\chi_2 = (\pm 8 \mid *)$, we have

$$\chi_2(5-2nc) = (-1)^{\frac{nc+1}{4}} \chi_2(-1) = \begin{cases} -\chi_2(-1) & \text{if } nc \equiv 1 \bmod 4, \\ \chi_2(-1) & \text{if } nc \equiv -1 \bmod 4. \end{cases}$$

Proposition 4.8. *Let $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$. Then $\varphi_g = \sum g_u \vartheta_u$ is a Hermitian*

Jacobi form of level N if and only if the equation

$$\sum_u \frac{M_{u,v}(\sigma) a_w}{D a_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a+cj) \mathbf{e} [|u|^2 j - |w|^2 \kappa] = \delta_{D|w|^2, D|v|^2}^{\bmod D} \quad (4.2.1)$$

holds for any $\sigma \in \Gamma_0(N)$ and any $v, w \in [\mathfrak{d}_K]$ such that $g_w \neq 0$.

Here, for any integers X, Y, M ,

$$\delta_{X,Y}^{\bmod M} := \begin{cases} 1 & \text{if } X \equiv Y \bmod M, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From transformation formula for ϑ_u , we find that φ_g is Jacobi form of level N if and only if

$$\sum_{u \in [\mathfrak{d}_K]} M_{u,v}(\sigma) g_u|_{k-1} \sigma = g_v \quad \text{for all } v \in [\mathfrak{d}_K] \text{ and } \sigma \in \Gamma_0(N)$$

By Proposition 4.6, the above equation is the same as

$$\sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{D a_u} \sum_{w \in [\mathfrak{d}_K]} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a+cj) \mathbf{e} [|u|^2 j - |w|^2 \kappa] g_w = g_v.$$

Now observe that $g_w = g_{w'}$ if $D|w|^2 \equiv D|w'|^2 \pmod{D}$ and conversely, if $D|w|^2 \not\equiv D|w'|^2 \pmod{D}$ (or equivalently, $|w|^2 \not\equiv |w'|^2 \pmod{D}$ in $\mathbb{Q}/\mathbb{Z}$) then g_w and $g_{w'}$ have disjoint Fourier expansion so if they are non-zero, they must be linearly independent. So grouping the summation over w by $D|w|^2 \pmod{D}$, we find that the above equation holds if and only if

$$\sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{D a_u} \sum_{\substack{w' \in [\mathfrak{d}_K] \\ D|w'|^2 \equiv D|w|^2 \pmod{D} \\ (D|w'|^2, m)=\mu}} \sum_{j=0}^{D-1} R_\sigma(w', j) \mathbf{G}(\psi_m; nc) \psi_n(a+cj) \mathbf{e} [|u|^2 j - |w'|^2 \kappa] = \delta_{D|w|^2, D|v|^2}^{\pmod{D}} \quad (4.2.2)$$

for all $w \in [\mathfrak{d}_K]$ such that $g_w \neq 0$. Now if $D|w'|^2 \equiv D|w|^2 \pmod{D}$ then

- $\mathbf{e} [-|w'|^2 \kappa] = \mathbf{e} [-|w|^2 \kappa]$ as $\kappa \in \mathbb{Z}$;
- $(D|w'|^2, m) = (D|w|^2, m)$ so the sum over j such that $(D|w'|^2, m) = \mu$ goes through the same set of j 's independent of $w' \in [\mathfrak{d}_K]$; and also

- $R_\sigma(w', j) = R_\sigma(w, j)$: obviously if $m \neq 4\mu$ and in case $m = 4\mu$, the only term that involve the w in the definition of R_σ is

$$e\left[-\frac{(a+cj)D|w'|^2}{2m}\right] \quad \text{which equals} \quad e\left[-\frac{(a+cj)D|w|^2}{2m}\right]$$

as long as $D|w'|^2 \equiv D|w|^2 \pmod{D}$.

Thus, we can replace all occurrences of w' in the summand of (4.2.2) with w and the (4.2.2) reduces to the equivalent

$$\sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{Da_u} \sum_{\substack{w' \in [\mathfrak{d}_K] \\ D|w'|^2 \equiv D|w|^2 \pmod{D}}} \underbrace{\sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a+cj) e\left[|u|^2 j - |w|^2 \kappa\right]}_S = \delta_{D|w|^2, D|v|^2}^{\text{mod } D}$$

One can now see that the summand S is free of the summation variable w' and so the middle sum over w' is just $a_w S$ where a_w is the number of $w' \in [\mathfrak{d}_K]$ such that $D|w'|^2 \equiv D|w|^2 \pmod{D}$. Hence, we have (4.2.1). $\square$

Remark 4.9. Observe that the equation (4.2.1) does not have modular forms in it: It purely depends on the arithmetic of the field K . Thus, if it is true, one could expect that it is true in general for all $\sigma \in \mathbf{SL}_2(\mathbb{Z})$ and $v, w \in [\mathfrak{d}_K]$. (Note that even if the equation holds for all $\mathbf{SL}_2(\mathbb{Z})$, this does not imply that φ_g is Jacobi form of level 1; since the computation of $g|_{k-1}\sigma_j\sigma$ we used along the way is only valid for $\sigma \in \Gamma_0(N)$.)

Remark 4.10. We remark here that one technique to show (4.2.1) is true for higher level is to use existence of lifting for level 1 given by Krieg. In section 5 of [21], Krieg defined the Eisenstein series

$$E_{k-1}^*(\tau) = \sum_{\substack{D=mn \\ (m,n)=1}} \psi_m(-1) f_{k-1}(\tau; \psi_m, \psi_n)$$

and he showed that

$$\mathfrak{M}_{k-1}^+(D, \chi) = \mathbb{C}E_{k-1}^* \oplus \mathfrak{S}_{k-1}^+(D, \chi)$$

for all even $k > 2$. The Fourier coefficient of E_{k-1}^* is given by

$$\alpha_\ell(E_{k-1}^*) = a_D(\ell) \sum_{d \mid \ell} \chi(d) d^{k-2}$$

for $(\ell, D) = 1$. Thus, $(E_{k-1}^*)_w \neq 0$ for all $w \in [\mathfrak{d}_K]$ such that $(D|w|^2, D) = 1$. Thus, by Krieg's isomorphism for the case $N = 1$, we know that (4.2.1) is true for all $\sigma \in \Gamma$ and all w, v such that $(D|w|^2, D) = 1$. When D is prime, the remaining class $w \in [\mathfrak{d}_K]$ such that $(D|w|^2, D) \neq 1$ is $w = 0$ and one easily works out the equation to show surjectivity in the case of interest in [5]. We will not pursue this direction.

Our next result reduces the verification of the equation in Proposition 4.8 to that of representatives of each equivalence class of $\Gamma_0(N)$ under the equivalence relation induced by $\Gamma_0(D)$, namely $\alpha \sim \beta$ if $\alpha = \beta\gamma$ for some $\gamma \in \Gamma_0(D)$.

Proposition 4.11. *If the equation (4.2.1) in Proposition 4.8 holds for $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ and every $w, v \in [\mathfrak{d}_K]$ then it also holds for $\sigma\gamma$ and every $w, v \in [\mathfrak{d}_K]$ for any $\gamma \in \Gamma_0(D)$. In particular, (4.2.1) holds for all $\sigma \in \Gamma_0(D)$.*

Proof. Fix a matrix $\gamma = \begin{pmatrix} x & y \\ Dz & t \end{pmatrix} \in \Gamma_0(D)$. From

$$\sigma\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ Dz & t \end{pmatrix} = \begin{pmatrix} ax + bDz & ay + bt \\ cx + dDz & cy + dt \end{pmatrix}$$

we find that $\mu_{\sigma\gamma}(j) = \mu_\sigma(j)$ which implies $m_{\sigma\gamma}(j) = m_\sigma(j)$ and $n_{\sigma\gamma}(j) = n_\sigma(j)$. For any $w \in [\mathfrak{d}_K]$ such that $(D|w|^2, m) = \mu$, it is easy to check that

$$R_{\sigma\gamma}(w, j) = R_\sigma(w, j)$$

and finally, we can take

$$\kappa_{\sigma\gamma}(j) = ty + t^2\kappa_\sigma(j).$$

First, assume $z > 0$. In Lemma 3.1 of [5], it was mentioned that $M_{u,v}(\gamma) = \delta_{u,tv} \mathbf{e}[xy|u|^2] \chi(t)$ and so from the fact that M is a group homomorphism $M(\sigma\gamma) = M(\sigma)M(\gamma)$, one has $M_{u,v}(\sigma\gamma) = M_{u,tv}(\sigma) \mathbf{e}[xy|tv|^2] \chi(t)$. So the left hand side of (4.2.1) for $\sigma\gamma$ is

$$\begin{aligned} \sum_u \frac{M_{u,v}(\sigma\gamma)a_w}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_{\sigma\gamma}(w, j) \mathbf{G}(\psi_m; n(cx + dDz)) \\ \times \psi_n((ax + bDz) + (cx + dDz)j) \mathbf{e}[|u|^2j - |w|^2\kappa_{\sigma\gamma}] \end{aligned}$$

$$\begin{aligned}
&= \sum_u \frac{M_{u,tv}(\sigma) \mathbf{e}[xy|tv|^2] \chi(t)a_w}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; ncx) \psi_n(ax + cxj) \\
&\quad \times \mathbf{e}[|u|^2 j - |w|^2 (ty + t^2 \kappa_\sigma)] \\
&= \sum_u \frac{M_{u,tv}(\sigma) \mathbf{e}[xyt^2|v|^2 - ty|w|^2] \chi(tx)a_w}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} \mathbf{G}(\psi_m; nc) \psi_n(a + cj) \mathbf{e}[|u|^2 j - |tw|^2 \kappa_\sigma] \\
&= \mathbf{e}[ty(|v|^2 - |w|^2)] \underbrace{\sum_u \frac{M_{u,tv}(\sigma)a_{tw}}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} \mathbf{G}(\psi_m; nc) \psi_n(a + cj) \mathbf{e}[|u|^2 j - |tw|^2 \kappa_\sigma]}_{\text{this is LHS of (4.2.1) with } tv, tw \text{ replacing } v, w \text{ respectively}}
\end{aligned}$$

by observing that $xt \equiv 1 \pmod D$ whence $\mathbf{e}[xyt^2|v|^2 - ty|w|^2] = \mathbf{e}[ty(|v|^2 - |w|^2)]$ ■

as $(xt - 1)|v|^2 \in \mathbb{Z}$. Since $(t, D) = 1$, multiplication by t induces automorphism of $[\mathfrak{d}_K]$ so that we have $a_{tw} = a_w$, $(D|tw|^2, m) = (D|w|^2, m)$.

As indicated, the last sum is the left hand side of (4.2.1) for σ and tv, tw in place of v, w . So we get $\delta_{D|tw|^2, D|tv|^2}^{\text{mod } D}$ for the answer by the hypothesis that the equation holds for σ . Note that the same as $\delta_{D|tw|^2, D|tv|^2}^{\text{mod } D} = \delta_{D|w|^2, D|v|^2}^{\text{mod } D}$ again by $(t, D) = 1$. Evidently, if $D|w|^2 \equiv D|v|^2 \pmod D$ then $\mathbf{e}[ty(|v|^2 - |w|^2)] = 1$. So (4.2.1) holds for $\sigma\gamma$ and v, w .

The case $z = 0$ can be handled similarly. Note that this case includes the matrix $-I_2$ which accounts for the case where $z < 0$. Recall that $M_{w',v}(\gamma) = \text{sign}[x] \delta_{w',xv} \mathbf{e}[xy|w'|^2]$ from the definition so that

$$M_{u,v}(\sigma\gamma) = \sum_{w'} M_{u,w'}(\sigma) M_{w',v}(\gamma)$$

$$\begin{aligned}
&= \sum_{w'} M_{u,w'}(\sigma) \text{sign}[x] \delta_{w',xv} \mathbf{e}[xy|w'|^2] \\
&= \text{sign}[x] M_{u,xv}(\sigma) \mathbf{e}[xy|xv|^2]
\end{aligned}$$

and we get

$$\begin{aligned}
&\sum_u \frac{\text{sign}[x] M_{u,xv}(\sigma) \mathbf{e}[xy|xv|^2] a_w}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; ncx) \psi_n(ax + cxj) \times \\
&\quad \times \mathbf{e}[|u|^2 j - |w|^2 (ty + t^2 \kappa_\sigma)] \\
&= \text{sign}[x] \chi(x) \mathbf{e}[xy|xv|^2 - |w|^2 ty] \sum_u \frac{M_{u,xv}(\sigma) a_w}{Da_u} \sum_{\substack{j=0 \\ (D|w|^2, m)=\mu}}^{D-1} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) \times \\
&\quad \times \mathbf{e}[|u|^2 j - |tw|^2 \kappa_\sigma] \quad \blacksquare
\end{aligned}$$

The inner sum is again the left hand side of (4.2.1) with xv and tw in place of v, w respectively so we get $\delta_{D|xv|^2, D|tw|^2}^{\text{mod } D}$ as a result by assumption. Note that $z = 0$ implies $x = t = \pm 1$ so $|xv|^2 = |x|^2$ and $|tw|^2 = |w|^2$. Thus, if $D|xv|^2 \equiv D|tw|^2 \pmod{D}$ then if $x = t = 1$ then $\text{sign}[x] \chi(x) \mathbf{e}[xy|xv|^2 - |w|^2 ty] = \text{sign}[1] \chi(1) \mathbf{e}[y|v|^2 - |w|^2 y] = 1$ obviously. Otherwise, if $x = t = -1$ then $\text{sign}[x] \chi(x) \mathbf{e}[xy|xv|^2 - |w|^2 ty] = \text{sign}[-1] \chi(-1) \mathbf{e}[-y|v|^2 + |w|^2 y] = 1$.

For the remaining statement, one easily check that the equation holds for $\sigma = I_2$ and all $v, w \in [\mathfrak{d}_K]$. $\square$

4.3 Hermitian Maaß lift

Now we can prove our main theorem

Theorem 4.12. *Suppose that $(D, N) = 1$. Let $g \in \mathfrak{M}_{k-1}^+(DN, \chi)$. For any $u \in [\mathfrak{d}_K]$, define*

$$g_u(\tau) := \chi(N) \frac{-i\sqrt{D}}{a_D(-D|u|^2)} \sum_{\substack{\ell=0 \\ \ell \equiv -D|u|^2 \pmod{D}}}^{\infty} a_\ell(g) \mathbf{e} \left[\frac{\ell\tau}{D} \right]$$

and

$$\varphi_g := \sum_{u \in [\mathfrak{d}_K]} g_u \vartheta_u \quad \text{where} \quad \vartheta_u(\tau, z, w) := \sum_{a \in u + \mathfrak{o}_K} \mathbf{e} \left[|a|^2 \tau + \bar{a}z + aw \right].$$

Then $\varphi_g \in \mathfrak{J}_{k,1}^*(N)$ and $\varphi_g \mapsto g$ under the injective map $\mathfrak{J}_k^*(N) \rightarrow \mathfrak{M}_{k-1}^+(DN, \chi)$ ■

in Proposition 3.11.

In particular, we have an isomorphism $\mathfrak{J}_k^*(N) \cong \mathfrak{M}_{k-1}^+(DN, \chi)$.

By the results of previous section, we only need to show the equation (4.2.1) holds for all representatives $\sigma \in \Gamma_0(1) = \mathbf{SL}_2(\mathbb{Z})$ of $\Gamma_0(1)/\Gamma_0(D)$ and all $v, w \in [\mathfrak{d}_K]$. (This is stronger than the criterion in Proposition 4.8.) From the well-known representatives for $\Gamma_0(1)/\Gamma_0(D)$ and the fact that the class of the identity matrix is already settled in Proposition 4.11, we reduce the verification to that of $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \mid D$ and $c > 0$. Fix such an arbitrary σ and two arbitrary classes $v, w \in [\mathfrak{d}_K]$. Let

$$A_u := \sum_{\substack{j \pmod{D} \\ (D|w|^2, m) = \mu}} \frac{a_w}{a_u} R_\sigma(w, j) \mathbf{G}(\psi_m; nc) \psi_n(a + cj) \mathbf{e} \left[|u|^2 j - |w|^2 \kappa \right]$$

and

$$A := \sum_{u \in [\mathfrak{d}_K]} \frac{M_{u,v}(\sigma)}{D} A_u.$$

be a factor of the inner summand and the left hand side (respectively) of (4.2.1). Under our assumption on the lower left entry c of σ , we shall evaluate A_u and then A explicitly to establish (4.2.1), namely to show that $A = \delta_{D|w|^2, D|v|^2}^{\text{mod } D}$.

For clarity and due to the fact that the complete set of representatives of $[\mathfrak{d}_K]$ are of different forms, we consider the two cases of odd and even discriminant D separately. The proof strategy is pretty much the same; except for the fact that the even discriminant case is technically much more complicated.

In this section, to simplify our formula, we recycle the previous notation $\mathbf{e}[\bullet]$ as follow: For any integer $M \neq 0$ and any rational number $r \in \mathbb{Z}_{(M)}$ where

$$\mathbb{Z}_{(M)} := \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, (a, b) = 1 \text{ and } (b, M) = 1 \right\},$$

the notation $\mathbf{e} \left[\frac{r}{M} \right]$ will always mean the root of unity $\mathbf{e} \left[\frac{s}{M} \right] = e^{2\pi i s/M}$ where s is any integer such that $r \equiv s \pmod{M\mathbb{Z}_{(M)}}$. For example, if r is odd, then $\mathbf{e} \left[\frac{\frac{1}{r}}{2^e} \right] = \mathbf{e} \left[\frac{s}{2^e} \right]$ where s is a multiplicative inverse of $r \pmod{2^e}$ i.e. s is any integer such that $sr \equiv 1 \pmod{2^e}$. This way, we do not have to write the

longer notation $e\left[\frac{r^{-1} \bmod 2^e}{2^e}\right]$ or resort to defining a lot of new symbol for multiplicative inverse.

4.3.1 The odd discriminant case

Let us consider the case where the discriminant $-D$ is odd. In this case, $D \equiv 3 \bmod 4$ and we also know that D is square-free. The representatives for $[\mathfrak{d}_K]$ can be taken from

$$\left\{ \frac{ix}{\sqrt{D}} \mid x \in \mathbb{Z}/D\mathbb{Z} \right\}.$$

Here, $x \in \mathbb{Z}/D\mathbb{Z}$ refers to the choice of $x \in \mathbb{Z}$ from any complete sets of representatives of $\mathbb{Z}/D\mathbb{Z}$. Note that $D|u|^2 \bmod D$ is independent of the representative for u .

We record the following lemma which is a slight generalization of Lemma A in [21] Section 6.

Lemma 4.13. *Given odd prime p and $x, y, z \in \mathbb{Z}$, $p \nmid z$, let $\psi = (*|p)$ be the Legendre symbol for the prime p . Then one has*

$$\sum_{j=1}^{p-1} \psi(j) e\left[z \frac{jx^2 + j^{-1}y^2}{p}\right] = \mathbf{G}(\psi; z) \frac{\psi(x^2) + \psi(y^2)}{1 + \psi(y^2)} \sum_{\substack{\gamma \bmod p \\ \gamma^2 \equiv y^2 \bmod p}} e\left[\frac{2xz\gamma}{p}\right]$$

Proof. If $p \mid x, y$ then the left hand side is just $\sum_{j=1}^{p-1} (j|p) = 0$ and $\psi(x^2) + \psi(y^2) = 0$ on the right hand side.

When $p \mid x$ but $p \nmid y$, we can remove jx^2 and by a change of variable $t = j^{-1}y^2z$, we get Gauss sum:

$$\begin{aligned} \sum_{j=1}^{p-1} \psi(j) \mathbf{e} \left[z \frac{j^{-1}y^2}{p} \right] &= \sum_{t=1}^{p-1} \psi(ty^{-2}z^{-1}) \mathbf{e} \left[\frac{t}{p} \right] \\ &= \psi(z) \sum_{t=1}^{p-1} \psi(t) \mathbf{e} \left[\frac{t}{p} \right] \\ &= \mathbf{G}(\psi; z) \end{aligned}$$

One can check that the remaining factor on the RHS is 1. Likewise for the case $p \mid y$ but $p \nmid x$.

For the remaining case $p \nmid xy$, we get a Salie sum $S(zx^2; zy^2)$ where

$$\begin{aligned} S(a; b) &= \sum_{j=1}^{p-1} (j|p) \mathbf{e} \left[\frac{aj + bj^{-1}}{p} \right] \\ &= \begin{cases} 0 & \text{if } (ab|p) = -1 \\ (ac|p)S(c; c) & \text{if } (ab|p) = 1 \text{ and } ab = c^2 \pmod{p} \end{cases} \\ S(c; c) &= (c|p) i^{((p-1)/2)^2} p^{1/2} (\mathbf{e}[2c] + \mathbf{e}[-2c]) \\ &= \mathbf{G}((*)|p; c) \left(\mathbf{e} \left[\frac{2c}{p} \right] + \mathbf{e} \left[\frac{-2c}{p} \right] \right) \end{aligned}$$

explicitly evaluated according to [34]. Thus, we get

$$\begin{aligned} S(zx^2; zy^2) &= (zx^2xyz|p) \mathbf{G}((*)|p; xyz) (\mathbf{e}[2xyz/p] + \mathbf{e}[-2xyz/p]) \\ &= \mathbf{G}(\psi; z) \left(\mathbf{e} \left[\frac{2xyz}{p} \right] + \mathbf{e} \left[\frac{-2xyz}{p} \right] \right). \end{aligned}$$

The remaining factor matches the right hand side. □

Set $D^* = \frac{D}{c}$. We proceed to evaluate the left hand side of (4.2.1). Under our assumption that $c|D$ and $c > 0$, it is easy to evaluate the matrix explicitly

$$M_{u,v}(\sigma) = \frac{-i}{\sqrt{D}} e^{\left[\frac{ax^2 - 2xy + dy^2}{Dc} \right]} \mathbf{G}(\psi_c; a) \delta_{x,dy}^{\text{mod } c} \quad (4.3.1)$$

if $u, v \in [\mathfrak{d}_K]$ are represented by $u = [\frac{ix}{\sqrt{D}}], v = [\frac{iy}{\sqrt{D}}]$ for some $x, y \in \mathbb{Z}$.

Proof. Note that u, v are purely imaginary so that $\bar{u} = -u, \bar{v} = -v$. Also, $D|u|^2 = x^2$ and $D|v|^2 = y^2$. We first simplify the summand in the definition of $M_{u,v}(\sigma)$ as

$$\begin{aligned} & e^{\left[\frac{1}{c} (a|\gamma + u|^2 - (\gamma + u)\bar{v} - \overline{\gamma + uv} + d|v|^2) \right]} \\ &= e^{\left[\frac{1}{c} (a|\gamma|^2 + a\bar{\gamma}u + a\gamma\bar{u} + a|u|^2 - \gamma\bar{v} - u\bar{v} - \bar{\gamma}v - \bar{u}v + d|v|^2) \right]} \\ &= e^{\left[\frac{1}{c} (a|\gamma|^2 + a\bar{\gamma}u - a\gamma u + a|u|^2 + \gamma v + uv - \bar{\gamma}v + uv + d|v|^2) \right]} \\ &= e^{\left[\frac{1}{c} (a|\gamma|^2 - (au - v)(\gamma - \bar{\gamma}) + a|u|^2 + 2uv + d|v|^2) \right]} \\ &= e^{\left[\frac{1}{c} (a|u|^2 + 2uv + d|v|^2) \right]} e^{\left[\frac{a|\gamma|^2 - (au - v)(\gamma - \bar{\gamma})}{c} \right]} \end{aligned}$$

Now let $\omega = \frac{1+i\sqrt{D}}{2}$ and identify $\mathfrak{o}_K = \mathbb{Z}[\omega]$, we get

$$\begin{aligned} & \sum_{\gamma \in \mathfrak{o}_K / c\mathfrak{o}_K} e^{\left[\frac{a|\gamma|^2 - (au - v)(\gamma - \bar{\gamma})}{c} \right]} \\ &= \sum_{\substack{\alpha, \beta \in \mathbb{Z}/c \\ \gamma = \alpha + \beta\omega}} e^{\left[\frac{a(\alpha^2 + \alpha\beta + \frac{D+1}{4}\beta^2) - (a\frac{ix}{\sqrt{D}} - \frac{iy}{\sqrt{D}})(i\beta\sqrt{D})}{c} \right]} \\ &= \sum_{\alpha, \beta \in \mathbb{Z}/c} e^{\left[\frac{a(\alpha^2 + \alpha\beta + \frac{D+1}{4}\beta^2) + (ax - y)\beta}{c} \right]} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha, \beta \in \mathbb{Z}/c} e \left[\frac{a(\alpha^2 + 2\alpha\beta + (D+1)\beta^2) + 2(ax-y)\beta}{c} \right] \text{ since } 2 \nmid c, \\
&\quad \text{we can make change of variable } \beta \text{ to } 2\beta \\
&= \sum_{\beta \in \mathbb{Z}/c} \sum_{\alpha \in \mathbb{Z}/c} e \left[\frac{a(\alpha + \beta)^2 + 2(ax-y)\beta}{c} \right] \text{ for } c \mid D \\
&= \sum_{\alpha, \beta \in \mathbb{Z}/c} e \left[\frac{a\alpha^2 + 2(ax-y)\beta}{c} \right] \text{ change of variable } \alpha \text{ for } \alpha + \beta \\
&= \left(\sum_{\alpha \in \mathbb{Z}/c} e \left[\frac{a\alpha^2}{c} \right] \right) \underbrace{\left(\sum_{\beta \in \mathbb{Z}/c} e \left[\frac{2(ax-y)\beta}{c} \right] \right)}_{c \delta_{ax,y}^{\text{mod } c}}
\end{aligned}$$

By Chinese Remainder Theorem,

$$\begin{aligned}
\sum_{\alpha \in \mathbb{Z}/c} e \left[\frac{a\alpha^2}{c} \right] &= \prod_{p \mid c} \left(\sum_{\alpha \in \mathbb{Z}/p} e \left[\frac{a((\frac{c}{p})^{-1} \text{ mod } p)\alpha^2}{p} \right] \right) \\
&= \prod_{p \mid c} \mathbf{G} \left(\psi_p; a((\frac{c}{p})^{-1} \text{ mod } p) \right) \\
&= \mathbf{G}(\psi_c; a)
\end{aligned}$$

and thus

$$\begin{aligned}
M_{u,v}(\sigma) &= \frac{-i}{c\sqrt{D}} e \left[\frac{1}{c}(a|u|^2 + 2uv + d|v|^2) \right] \sum(\dots) \\
&= \frac{-i}{c\sqrt{D}} e \left[\frac{aD|u|^2 + 2Duv + dD|v|^2}{Dc} \right] \mathbf{G}(\psi_c; a) c \delta_{ax,y}^{\text{mod } c} \\
&= \frac{-i}{\sqrt{D}} e \left[\frac{ax^2 - 2xy + dy^2}{Dc} \right] \mathbf{G}(\psi_c; a) \delta_{x,dy}^{\text{mod } c}
\end{aligned}$$

as claimed. □

Choose any $h \in \mathbb{Z}$ such that $h \equiv \begin{cases} b & \text{mod } c \\ c^{-1} & \text{mod } D^* \end{cases}$ and for any $u \in [\mathfrak{d}_K]$,

we set

$$F_u := \sum_{\substack{\gamma \bmod D^* \\ \gamma^2 \equiv D|w|^2 \bmod D^*}} \mathbf{e} \left[\frac{2xh^2\gamma}{D^*} \right] \quad \text{if } u = \left[\frac{ix}{\sqrt{D}} \right].$$

(The right hand side is the same for any $x \in \mathbb{Z}$ as long as $x^2 \equiv D|u|^2 \bmod D$.)

Also, recall that we fix $w \in [\mathfrak{d}_K]$.

Then using Lemma 4.13, we find that for any fixed $u \in [\mathfrak{d}_K]$, the inner sum over j on the left hand side of (4.2.1) equals

$$A_u = c \delta_{D|u|^2, D|dw|^2}^{\text{mod } c} F_{u,w} \mathbf{G}(\psi_{D^*}) \psi_c(a) \mathbf{e} \left[-|w|^2 dh - \frac{D|u|^2 ah^2}{D^*} \right]. \quad (4.3.2)$$

Take representative $v = [\frac{iy}{\sqrt{D}}] \in [\mathfrak{o}_K]$. By mean of (4.3.1) and (4.3.2) and change of variable from $u = [\frac{ix}{\sqrt{D}}]$ to $x \bmod D$, one has

$$\begin{aligned} A &= \sum_{\substack{x \bmod D \\ x^2 \equiv D|dw|^2 \bmod c}} \frac{\frac{-i}{\sqrt{D}} \mathbf{e} \left[\frac{ax^2 - 2xy + dy^2}{Dc} \right] \mathbf{G}(\psi_c; a) \delta_{x, dy}^{\text{mod } c}}{D} c F_{[\frac{ix}{\sqrt{D}}]} \mathbf{G}(\psi_{D^*}) \times \\ &\quad \times \psi_c(a) \mathbf{e} \left[-|w|^2 dh - \frac{x^2 ah^2}{D^*} \right] \\ &= \sum_{\substack{x \bmod D \\ x \equiv dy \bmod c \\ x^2 \equiv D|dw|^2 \bmod c}} \frac{1}{D^*} \mathbf{e} \left[\frac{ax^2 - 2xy + dy^2}{Dc} - |w|^2 dh - \frac{x^2 ah^2}{D^*} \right] F_{[\frac{ix}{\sqrt{D}}]} \end{aligned}$$

Here, note that $\mathbf{G}(\psi_c) \mathbf{G}(\psi_{D^*}) = i\sqrt{D}$ since $\psi_c(-1) \psi_{D^*}(-1) = \chi(-1) = -1$.

Evidently, if $D|v|^2 \equiv d^2 y^2 \not\equiv D|dw|^2 \bmod c$ then the last sum is an empty sum (there is no x satisfying the two conditions) and thus is zero which is the same as $\delta_{D|v|^2, D|dw|^2}^{\text{mod } D}$.

Let us consider the case $D|v|^2 \equiv D|dw|^2 \pmod{c}$ and the sum reduces to just sum over $x \pmod{D}$ such that $x \equiv dy \pmod{c}$. Making change of summation variable to $x \pmod{D^*}$ by identifying x with $cx + dy$ and we continue with

$$\begin{aligned}
A &= \sum_{x \pmod{D^*}} \frac{1}{D^*} e \left[\frac{a(cx + dy)^2 - 2(cx + dy)y + dy^2}{Dc} - |w|^2 dh \right] \times \\
&\quad \times e \left[-\frac{(cx + dy)^2 ah^2}{D^*} \right] F_{\left[\frac{i(cx + dy)}{\sqrt{D}}\right]} \\
&= \sum_{x \pmod{D^*}} \frac{1}{D^*} e \left[\frac{a(cx + dy)^2 - 2(cx + dy)y + dy^2}{Dc} - |w|^2 dh \right] \times \\
&\quad \times e \left[-\frac{(cx + dy)^2 ah^2}{D^*} \right] \left(\sum_{\substack{\gamma \pmod{D^*} \\ \gamma^2 \equiv D|w|^2 \pmod{D^*}}} e \left[\frac{2(cx + dy)h^2 \gamma}{D^*} \right] \right) \\
&= \sum_{\substack{\gamma \pmod{D^*} \\ \gamma^2 \equiv D|w|^2 \pmod{D^*}}} e \left[-\frac{D|w|^2 dh}{D} + \frac{bdy^2}{D} + \frac{2dyh^2 \gamma - ad^2 y^2 h^2}{D^*} \right] \times \\
&\quad \times \frac{1}{D^*} \sum_{x \pmod{D^*}} e \left[(2by + 2h\gamma - 2dyah) \frac{x}{D^*} \right]
\end{aligned}$$

The inner sum over $x \pmod{D^*}$ is either D^* or 0 depending on whether $D^* \mid 2by + 2h\gamma - 2dyah$ or not. Note that $D^* \mid 2by + 2h\gamma - 2dyah$ is equivalent to $\gamma \equiv y \pmod{D^*}$. And so the outer sum over γ is empty i.e. $A = 0$ if $y^2 = D|v|^2 \not\equiv D|w|^2 \pmod{D^*}$. We have thus shown that $A = 0$ if $D|v|^2 \not\equiv D|w|^2 \pmod{D}$ i.e. if either $D|v|^2 \not\equiv D|w|^2 \pmod{D^*}$ or $D|v|^2 \not\equiv D|w|^2 \pmod{c}$; which matches the right hand side $\delta_{D|w|^2, D|v|^2}^{\pmod{D}}$ of (4.2.1) by Chinese Remainder Theorem.

In case $D|v|^2 \equiv D|w|^2 \pmod{D}$, the sum over γ is just the summand at $\gamma = y$ i.e.

$$\begin{aligned} A &= e \left[-\frac{D|w|^2 dh}{D} + \frac{bdy^2}{D} + \frac{2dyh^2y - ad^2y^2h^2}{D^*} \right] \\ &= e \left[\frac{-hd + bd + 2cdh^2 - acd^2h^2}{D} y^2 \right] \\ &= 1 \end{aligned}$$

because $-hd + bd + 2cdh^2 - acd^2h^2 \equiv 0 \pmod{D}$ by definition of h . It is obvious $\pmod{c}$ since $h \equiv b \pmod{c}$. We have $c(-hd + bd + 2cdh^2 - acd^2h^2) \equiv -d + cbd + 2d - ad^2 \equiv d - (ad - bc)d \equiv 0 \pmod{D^*}$ and so $-hd + bd + 2cdh^2 - acd^2h^2 \equiv 0 \pmod{D^*}$ as well.

This concludes our verification of (4.2.1); thus establishing the lifting when D is odd.

4.3.2 The even discriminant case

Now we move on to the even discriminant case. In this case the representatives for $[\mathfrak{d}_K]$ can be taken from

$$\left\{ \frac{x_1}{2} + \frac{ix_2}{\sqrt{D}} \mid x_1 \in \mathbb{Z}/2\mathbb{Z} \text{ and } x_2 \in \mathbb{Z}/\left(\frac{D}{2}\right)\mathbb{Z} \right\}$$

Decompose $D = 2^e D'$, $c = 2^f c'$ where $e = \text{val}_2(D)$, $f = \text{val}_2(c)$ are 2-adic valuation of D and c respectively; in other words, $2 \nmid c'$ and $2 \nmid D'$. Let c^* be the c component of D i.e. $c^* = 2^e c'$ if $f \geq 1$ and $c^* = c = c'$ if $f = 0$ i.e.

$2 \nmid c$. (Note that D' is square-free so c' is also square-free.) Let $D^* := \frac{D}{c^*}$,

$$f_1 := \text{val}_2((D^*, D|w|^2)),$$

$$f_2 := e - \log_2 \left(\frac{c^*}{c'} \right) = \begin{cases} 0 & \text{if } 2 \mid c, \\ e & \text{if } 2 \nmid c. \end{cases}$$

Note that f_2 is the power of 2 dividing D^* for

$$D^* = \frac{D}{c^*} = \frac{2^e D'}{2^{e-f_2} c'} = 2^{f_2} \frac{D'}{c'}$$

so

$$f_1 = \min(f_2, \text{val}_2(D|w|^2)).$$

For any $u \in [\mathfrak{d}_K]$, set

$$E_u := \begin{cases} \mathbf{G}(\chi_2; D'c) \chi_2(D|u|^2 + D|w|^2) & \text{if } 2 \nmid c \\ \delta_{D|u|^2 - D|w|^2, c}^{2^{e-1}} 2^{e-1} \chi_2(a) \mathbf{e} \left[-\frac{D' D |w|^2 ab}{2^e} \right] \\ \quad \times \left(1 + \mathbf{e} \left[\frac{D'(D|u|^2 - D|w|^2(2bc+1) - D|w|^2 cd)}{2^e} \right] \chi_2(1+ac) \right) & \text{if } 2 \mid c \end{cases}$$

$$F_u := \sum_{\substack{\gamma \bmod \frac{D'}{c'} \\ \gamma^2 \equiv D|u|^2 \bmod \frac{D'}{c'}}} \mathbf{e} \left[\frac{2x \frac{1}{cc^*} 2^{-f_2} \gamma}{\frac{D'}{c'}} \right] \quad \text{for any } x \text{ such that } x^2 \equiv D|u|^2 \bmod \frac{D'}{c'}$$

$$E'_u := \begin{cases} 1 & \text{if } e - f_1 = 0, \\ (-1)^{D|u|^2} & \text{if } e - f_1 = 1, \\ (-1)^{D|u|^2/2} & \text{if } e - f_1 = 2 \text{ and } D|u|^2 \equiv 0 \bmod 2, \\ (-1)^{\frac{D|u|^2 + D|w|^2}{2}} \chi_2(-1) & \text{if } e - f_1 = 2 \text{ and } D|u|^2 \equiv 1 \bmod 2 \end{cases}$$

$$K_u := \mathbf{e} \left[-\frac{D|u|^2 \frac{ab}{D/c'}}{c'} - \frac{D|u|^2 \frac{a}{cc^*}}{D^*} - \frac{D|w|^2 \frac{d}{cc^*}}{D^*} \right]$$

$$= \mathbf{e} \left[-\frac{D|u|^2 \frac{ab}{D/c'}}{c'} - \frac{\frac{1}{2^{f_2}} D|u|^2 \frac{a}{cc^*}}{D'/c'} - \frac{\frac{1}{2^{f_2}} D|w|^2 \frac{d}{cc^*}}{D'/c'} - \frac{\frac{1}{D'/c'} D|u|^2 \frac{a}{cc^*}}{2^{f_2}} - \frac{\frac{1}{D'/c'} D|w|^2 \frac{d}{cc^*}}{2^{f_2}} \right]$$

Then with Lemma 4.13 and a lot of computations (to be performed in chapter ??), we find that the inner sum over j on the left hand side of (4.2.1) is likewise given by

$$A_u = B_u + (1 - \delta_{f_1,0})C_u$$

where

$$B_u = c' \delta_{D|u|^2, D|dw|^2}^{\text{mod } c'} F_u K_u E_u \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \mathbf{G}(\psi_{\frac{D^*}{2^{f_2}}}; 2^{f_2} c^* c) \psi_{c'}(a)$$

$$C_u = c \delta_{D|u|^2, D|dw|^2}^{\text{mod } c} F_u K_u E'_u \frac{1}{1 + \chi_2(D|u|^2)} \mathbf{G}(\psi_{D^*}) \psi_c(a)$$

In the formula above, the B_u and $(1 - \delta_{f_1,0})C_u$ are the partition of the sum over j in the definition of A_u into those such that $2 \nmid (D|w|^2, m)$ and $2 \mid (D|w|^2, m)$ respectively. Certainly, there exists j such that $2 \mid (D|w|^2, m)$ only when $2 \mid (D|w|^2, D^*)$ i.e. $f_1 \neq 0$; the fact we indicated by the factor $(1 - \delta_{f_1,0})$ in the formula.

Note that $1 - \delta_{0,f_1} \neq 0$ only when c is odd and $2 \mid D|w|^2$; in which case, $f_2 = e$, $\mathbf{G}(\psi_{\frac{D^*}{2^{f_2}}}; 2^{f_2} c^* c) = \mathbf{G}(\psi_{\frac{D^*}{2^e}}; 2^e)$, $\mathbf{G}(\psi_{D^*}) = \mathbf{G}(\psi_{\frac{D^*}{2^e}}; 2^e) \mathbf{G}(\chi_2; \frac{D^*}{2^e})$ and notice that $\frac{D^*}{2^e} = \frac{D}{c^* 2^e} = \frac{D'}{c'}$ so if $1 - \delta_{0,f_1} \neq 0$, we have a simpler formula

$$A_u = c' \delta_{D|u|^2, D|dw|^2}^{\text{mod } c'} F_u K_u \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \mathbf{G}(\psi_{\frac{D^*}{2^{f_2}}}; 2^{f_2} c^* c) \psi_{c'}(a) [E_u + (1 - \delta_{f_1,0}) E'_u \mathbf{G}(\chi_2; D'/c')].$$

As for the matrix, let $u = [\frac{x_1}{2} + \frac{ix_2}{\sqrt{D}}]$ and $v = [\frac{y_1}{2} + \frac{iy_2}{\sqrt{D}}]$ be representatives then we have

$$M_{u,v}(\sigma) = e^{\left[\frac{\frac{1}{D/c'} b d y_2^2}{c'} \right]} \frac{\delta_{x_2, d y_2}^{\text{mod } c'} \mathbf{G}(\psi_{c'}; a 2^f)}{i \sqrt{D}} e^{\left[\frac{\frac{1}{2^{e+f} c'^2} (a x_2^2 - 2 x_2 y_2 + d y_2^2)}{D'/c'} \right]} M_{u,v}^*(\sigma) \blacksquare$$

where the factor

$$M_{u,v}^*(\sigma) := 2^{f-2} \delta_{x_2, d y_2}^{\text{mod } 2^{f-1}} e^{\left[\frac{\frac{1}{c'} (a x_1^2 - 2 x_1 y_1 + d y_1^2)}{2^{f+2}} + \frac{\frac{1}{D' c'} (a x_2^2 - 2 x_2 y_2 + d y_2^2)}{2^{e+f}} \right]} \blacksquare$$

Here, we take a convention that $\delta_{X,Y}^{\text{mod } 1/2} = 1$ for all $X, Y \in \mathbb{Z}$.

Since $D = 2^e c' (\frac{D'}{c'})$ and the three factors are relatively prime,

$$\delta_{D|v|^2, D|w|^2}^{\text{mod } D} = \delta_{D|v|^2, D|w|^2}^{\text{mod } 2^e} \times \delta_{D|v|^2, D|w|^2}^{\text{mod } c'} \times \delta_{D|v|^2, D|w|^2}^{\text{mod } D'/c'}.$$

Similar to the odd discriminant case, we show that $A = \delta_{D|v|^2, D|w|^2}^{\text{mod } D}$ by showing that $A = 1$ only when $D|v|^2 \equiv D|w|^2 \pmod{c'}$, $D|v|^2 \equiv D|w|^2 \pmod{\frac{D'}{c'}}$ and $D|v|^2 \equiv D|w|^2 \pmod{2^e}$; and $A = 0$ otherwise.

In all cases, we see that in order that $\frac{M_{u,v}(\sigma)}{D} A_u \neq 0$, u must satisfies the two congruences $D|u|^2 \equiv D|dw|^2 \pmod{c'}$ and $x_2 \equiv d y_2 \pmod{c'}$. The second condition implies $D|u|^2 \equiv D|dv|^2 \pmod{c'}$. Thus, if $D|v|^2 \not\equiv D|w|^2 \pmod{c'}$, $A = 0$. So let us now assume that $D|v|^2 \equiv D|w|^2 \pmod{c'}$ and the sum reduces to over u such that $x_2 \equiv d y_2 \pmod{c'}$ (for then $D|u|^2 \equiv D|dw|^2 \pmod{c'}$ holds automatically). Identifying u with a pair $(x_1, x_2) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(\frac{D}{2})\mathbb{Z}$,

we can rewrite the sum

$$A = \frac{\mathbf{G}(\psi_{c'}; 2^f) \mathbf{G}(\psi_{\frac{D'}{c'}}; 2^{e+f})}{i2^e \sqrt{D}} A^+ A^-$$

where

$$\begin{aligned} A^+ &= \frac{1}{D'/c'} \sum_{\substack{x_2 \bmod D' \\ x_2 \equiv dy_2 \bmod c'}} F_u e \left[\frac{\frac{1}{2^{e+f} c'^2} (ax_2^2 - 2x_2 y_2 + dy_2^2)}{D'/c'} - \frac{\frac{1}{2^{f_2}} D |u|^2 \frac{a}{cc^*}}{D'/c'} - \frac{\frac{1}{2^{f_2}} D |w|^2 \frac{d}{cc^*}}{D'/c'} \right] \\ A^- &= \sum_{\substack{x_1 \bmod 2 \\ x_2 \bmod 2^{e-1}}} e \left[-\frac{\frac{1}{D'/c'} D |u|^2 \frac{a}{cc^*}}{2^{f_2}} - \frac{\frac{1}{D'/c'} D |w|^2 \frac{d}{cc^*}}{2^{f_2}} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \times \\ &\quad \times [E_u + (1 - \delta_{f_1,0}) E'_u \mathbf{G}(\chi_2; D'/c')] M_{u,v}^*(\sigma) \end{aligned}$$

We shall show that

$$A^+ = \delta_{D|v|^2, D|w|^2}^{\bmod D'/c'} \quad (4.3.3)$$

and

$$\frac{\mathbf{G}(\psi_{c'}; 2^f) \mathbf{G}(\psi_{\frac{D'}{c'}}; 2^{e+f})}{i2^e \sqrt{D}} A^- = \delta_{D|v|^2, D|w|^2}^{\bmod 2^e} \quad (4.3.4)$$

which completes the proof that $A = \delta_{D|v|^2, D|w|^2}^D$. But first, using property of

Gauss sum and quadratic reciprocity, one can simplify (4.3.4) to

$$A^- = \delta_{D|v|^2, D|w|^2}^{\bmod 2^e} (-1)^{\frac{(c'-1)(D'/c'-1)}{4}} \psi_{D'}(2)^f \psi_{c'}(2^e) 2^e \mathbf{G}(\chi_2; D') \quad (4.3.5)$$

We compute A^+ to establish (4.3.3):

$$A^+ = \frac{1}{D'/c'} \sum_{\substack{x_2 \bmod D' \\ x_2 \equiv dy_2 \bmod c'}} \sum_{\substack{\gamma \bmod \frac{D'}{c'} \\ \gamma^2 \equiv D|w|^2 \bmod \frac{D'}{c'}}} e \left[\frac{2x_2 \frac{1}{cc^*} 2^{-f_2} \gamma}{D'/c'} \right] \times$$

$$\begin{aligned}
& \times e \left[\frac{\frac{1}{2^{e+f}c'^2}(ax_2^2 - 2x_2y_2 + dy_2^2)}{D'/c'} - \frac{\frac{1}{2^{f_2}}x_2^2 \cancel{\frac{d}{cc^*}}}{\cancel{D'/c'}} - \frac{\frac{1}{2^{f_2}}D|w|^2 \frac{d}{cc^*}}{D'/c'} \right] \\
& \quad \text{since } 2^{f_2}cc^* = 2^{f_2}2^f c' 2^{e-f_2} c' = 2^e c'^2 \\
& = \frac{1}{D'/c'} e \left[\frac{\frac{1}{2^{e+f}c'^2}dy_2^2}{D'/c'} - \frac{\frac{1}{2^{e+f}c'^2}dD|w|^2}{D'/c'} \right] \sum_{\substack{\gamma \bmod \frac{D'}{c'} \\ \gamma^2 \equiv D|w|^2 \bmod \frac{D'}{c'}}} \sum_{\substack{x_2 \bmod D' \\ x_2 \equiv dy_2 \bmod c'}} e \left[\frac{2\frac{1}{2^{e+f}c'^2}(\gamma - y_2)x_2}{D'/c'} \right] \\
& = \frac{1}{D'/c'} e \left[\frac{\frac{1}{2^{e+f}c'^2}dy_2^2}{D'/c'} - \frac{\frac{1}{2^{e+f}c'^2}dD|w|^2}{D'/c'} \right] \sum_{\substack{\gamma \bmod \frac{D'}{c'} \\ \gamma^2 \equiv D|w|^2 \bmod \frac{D'}{c'}}} \frac{D'}{c'} \delta_{\gamma, y_2}^{\bmod D'/c'} \\
& = \frac{1}{D'/c'} \frac{D'}{c'} \delta_{y_2^2, D|w|^2}^{\bmod D'/c'} e \left[\frac{\frac{1}{2^{e+f}c'^2}dy_2^2}{D'/c'} - \frac{\frac{1}{2^{e+f}c'^2}dD|w|^2}{D'/c'} \right] \\
& = \delta_{D|v|^2, D|w|^2}^{\bmod D'/c'}
\end{aligned}$$

Thus, we showed that $A = 0$ if $D|v|^2 \not\equiv D|w|^2 \bmod D'/c'$.

To prove (4.3.5), observe that it is in principle a finite check since we have finitely many (but a lot of) possibilities for e, f and $a, b, c, d, \frac{D'}{c'}, D|w|^2, D|v|^2 \bmod 2^e$. Note that $D|u|^2 \equiv 2^{e-2}D'x_1^2 + x_2^2 \bmod 2^e$ only depend on $x_1 \bmod 2$ and $x_2 \bmod 2^{e-1}$. Also, we must have $\chi_2(D|u|^2) \neq -1$ so $\chi_2(D|u|^2) = 0$ if $2 \mid D|u|^2$ and is 1 otherwise. In other words, $\chi_2(D|u|^2) = \delta_{D|u|^2, 1}^{\bmod 2}$. In particular, if $e = 2$ then $D|u|^2 \equiv 1 \bmod 4$ if $2 \nmid D|u|^2$. For illustration, we establish (4.3.5) in the case $e = 2$. The case $e = 3$ is similarly solved by case-by-case analysis.

From now on, we assume $e = 2$ and “left hand side” (LHS) and “right hand side” (RHS) refers to that of equation (4.3.5). As the matrix σ is fixed,

we drop it in the notation $M_{u,v}^*(\sigma)$.

Note that when $e = 2$, we must have $D' \equiv 1 \pmod{4}$ so that $D = 2^e D' = 4D'$ is a fundamental discriminant.

The case $f = 2$ and $e = 2$

Note that in this case, c is even so a, d are both odd. Then

$$A^- = \sum_{\substack{x_1 \pmod{2} \\ x_2 \pmod{2^{e-1}}}} \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} E_u M_{u,v}^*(\sigma). \quad (4.3.6)$$

since $f_2 = 0$ and $1 - \delta_{f_1,0} = 0$. (Note that the above holds as long as $2 \mid c$ i.e. $f \geq 1$.)

By definition, $M_{u,v}^*(\sigma) \neq 0$ only when $x_1 \equiv dy_1 \pmod{2}$ and $x_2 \equiv dy_2 \pmod{2}$ in which case $\chi_2(D|u|^2) = \chi_2(d^2 D|v|^2) = \chi_2(D|v|^2)$ and (4.3.6) reduces to

$$\begin{aligned} A^- &= \sum_{\substack{x_2 \pmod{2^{e-1}} \\ x_1 = dy_1, x_2 \equiv dy_2 \pmod{2}}} \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|v|^2)} E_u M_{u,v}^*(\sigma) \\ &= \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|v|^2)} \sum_{\substack{x_2 \pmod{2^{e-1}} \\ x_1 = dy_1, x_2 \equiv dy_2 \pmod{2}}} E_u M_{u,v}^*(\sigma) \end{aligned} \quad (4.3.7)$$

Likewise, observe that $E_u \neq 0$ only when $D|u|^2 - D|w|^2 \equiv c \pmod{2^{e-1}}$ so we must have $D|v|^2 \equiv D|w|^2 \pmod{2^{e-1}}$ in order that $A^- \neq 0$. It also follows from the definition of

$$E_u = \delta_{D|u|^2 - D|w|^2, c} 2^{e-1} \chi_2(a) \mathbf{e} \left[-\frac{D' D|w|^2 ab}{2^e} \right] \times$$

$$\times \left(1 + \mathbf{e} \left[\frac{D'(D|u|^2 - D|w|^2(2bc + 1) - D|w|^2cd)}{2^e} \right] \chi_2(1 + ac) \right) \blacksquare$$

that we further need $D|v|^2 \equiv D|w|^2 \pmod{2^e}$ in order that the last factor

$1 + \mathbf{e}[\dots] \chi_2(1 + ac) \neq 0$. Hence, $A^- = 0$ unless $D|v|^2 \equiv D|w|^2 \pmod{2^e}$.

Now let us assume $D|v|^2 \equiv D|w|^2 \pmod{2^e}$. We find that

$$E_u = 2^e \chi_2(a) \mathbf{e} \left[-\frac{D'D|w|^2 ab}{2^e} \right]$$

is independent of u (if its parameter satisfies the summation condition $x_1 \equiv$

$dy_1 \pmod{2}$ and $x_2 \equiv dy_2 \pmod{2}$) and so in case $e = f = 2$, one has

$$\begin{aligned} A^- &= 2^e \chi_2(a) \mathbf{e} \left[-\frac{D'D|w|^2 ab}{2^e} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|v|^2)} \sum_{\substack{x_2 \pmod{2^{e-1}} \\ x_1 = dy_1, x_2 \equiv dy_2 \pmod{2}}} M_{u,v}^*(\sigma) \\ &= 2^e \chi_2(a) \mathbf{e} \left[-\frac{D'D|v|^2 ab}{2^e} \right] \mathbf{e} \left[\frac{\frac{1}{c'}(a(dy_1)^2 - 2(dy_1)y_1 + dy_1^2)}{16} + \frac{\frac{1}{D'c'}(a(dy_2)^2 - 2(dy_2)y_2 + dy_2^2)}{16} \right] \times \\ &\quad \times \left(1 + \mathbf{e} \left[\frac{ac'D'}{4} \right] \right) \left(1 + \mathbf{e} \left[\frac{ac'}{4} \right] \right) \\ &= 2^e \chi_2(a) \mathbf{e} \left[-\frac{D'D|v|^2 ab}{2^e} \right] \mathbf{e} \left[\frac{bdy_1^2}{4} + \frac{\frac{1}{D'}bdy_2^2}{4} \right] \left(1 + \mathbf{e} \left[\frac{ac'}{4} \right] \right)^2 \\ &= 2^e \chi_2(a) \left(1 + \mathbf{e} \left[\frac{ac'}{4} \right] \right)^2 \\ &= 2^e \chi_2(a) \left(1 + 2\mathbf{e} \left[\frac{ac'}{4} \right] + \underbrace{\mathbf{e} \left[\frac{ac'}{2} \right]}_{=(-1)^{ac'} = -1} \right) \\ &= 2^e 2 \chi_2(a) \mathbf{e} \left[\frac{ac'}{4} \right] = 2i \chi_2(a) \chi_2(ac') \\ &= 2^{e+1} i \chi_2(c'). \end{aligned}$$

In this case, the right hand side of (4.3.5) is $2^e(-1)^{(c'-1)/2}2i = 2^e2i\chi_2(c')$.

That is because $\mathbf{G}(\psi_2, D') = 2i$ and since $D' \equiv 1 \pmod{4}$, we find that $\frac{D'}{c'} \equiv c' \pmod{4}$ and so

$$(-1)^{(c'-1)(D'/c'-1)/4} = (-1)^{(c'-1)^2/4} = (-1)^{(c'-1)/2}.$$

So we are done.

The case $f = 1$ and $e = 2$

For the case $f = 1$, we have

$$M_{u,v}^* = 2\delta_{2^{e-2}D', x_2-dy_2}^{\text{mod } 2} \delta_{1, x_1-dy_1}^{\text{mod } 2} \mathbf{e} \left[\frac{\frac{1}{c'}(ax_1^2 - 2x_1y_1 + dy_1^2)}{8} + \frac{\frac{1}{D'c'}(ax_2^2 - 2x_2y_2 + dy_2^2)}{2^{e+1}} \right] \blacksquare$$

So $M_{u,v}^* \neq 0$ only for $x_2 = dy_2 + D' \equiv dy_2 + 1 \pmod{2}$ and $x_1 \equiv dy_1 + 1 \pmod{2}$

(then $D|u|^2 \equiv (dy_1 + 1)^2 + (dy_2 + 1)^2 \equiv D|v|^2 + 2d(y_1 + y_2) + 2 \pmod{4}$ so

$D|u|^2 \equiv D|v|^2 \pmod{2}$ and we still have $\chi_2(D|u|^2) = \chi_2(D|v|^2)$. We likewise

find that with such conditions on x_1, x_2 , the factor $E_u \neq 0$ only when $D|v|^2 \equiv$

$D|w|^2 \pmod{4}$: first, we need $D|u|^2 \equiv D|w|^2 \pmod{2}$ so $D|v|^2 \equiv D|w|^2 \pmod{2}$

and given that, the factor

$$\begin{aligned} & 1 + \mathbf{e} \left[\frac{D'(D|u|^2 - D|w|^2(2bc + 1) - D|w|^2cd)}{2^e} \right] \chi_2(1 + ac) \\ &= 1 + \mathbf{e} \left[\frac{D|v|^2 + 2d(y_1 + y_2) + 2 - D|w|^2 - 2D|w|^2c'd}{4} \right] (-1)^{ac'} \text{ since } D' \equiv 1 \pmod{4} \text{ and } c = 2c' \\ &= 1 - \mathbf{e} \left[\frac{D|v|^2 - D|w|^2}{4} + \frac{d(y_1 + y_2) + 1 - D|w|^2c'd}{2} \right] \end{aligned}$$

$$= 1 + \mathbf{e} \left[\frac{D|v|^2 - D|w|^2}{4} \right]$$

in the definition of E_u is zero unless $D|v|^2 \equiv D|w|^2 \pmod{4}$. When $D|v|^2 \equiv D|w|^2 \pmod{4}$, the value of the above factor is 2. So if $D|v|^2 \equiv D|w|^2 \pmod{4}$ then the summation in (4.3.7) is just a singleton where $x_1 = dy_1 + 1$ and $x_2 = dy_2 + 1$:

$$\begin{aligned} A^- &= 2^e \chi_2(a) \mathbf{e} \left[-\frac{D'D|w|^2 ab}{2^e} \right] 2\mathbf{e} \left[\frac{\frac{1}{c'}(a(dy_1 + 1)^2 - 2(dy_1 + 1)y_1 + dy_1^2)}{8} \right] \times \\ &\quad \times \mathbf{e} \left[\frac{\frac{1}{D'c'}(a(dy_2 + 1)^2 - 2(dy_2 + 1)y_2 + dy_2^2)}{2^{e+1}} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[-\frac{D|v|^2 ab}{4} + \frac{\frac{1}{c'}(2(ad - 1)y_1 + (ad - 1)dy_1^2 + a)}{8} \right] \times \\ &\quad \times \mathbf{e} \left[\frac{\frac{D'}{c'}(2(ad - 1)y_2 + (ad - 1)dy_2^2 + a)}{8} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[-\frac{D|v|^2 ab}{4} + \frac{\frac{1}{c'}(2 \cdot 2bc' \cdot y_1 + 2bc'dy_1^2 + a) + \frac{D'}{c'}(2 \cdot 2bc' \cdot y_2 + 2bc'dy_2^2 + a)}{8} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[-\frac{D|v|^2 ab}{4} + \frac{4by_1 + 2bdy_1^2 + D'(4by_2 + 2bdy_2^2) + ac' + a\frac{D'}{c'}}{8} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[-\frac{D|v|^2 ab}{4} + \frac{by_1 + D'by_2}{2} + \frac{bdy_1^2 + D'bdy_2^2}{4} + \frac{ac'(D' + 1)}{8} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[\frac{b((d - a)D|v|^2 + 2(y_1 + y_2))}{4} + \frac{ac'(D' + 1)}{8} \right] \\ &= 2^{e+1} \chi_2(a) \mathbf{e} \left[\frac{c'a(D' + 1)}{8} \right] \end{aligned}$$

Note here to derive the last equation above, we reason that $a \equiv d \equiv 1 \pmod{2}$ so we either have $a - d \equiv 2 \pmod{4}$ whence $\frac{d-a}{2}D|v|^2 + y_1 + y_2 \equiv D|v|^2 + y_1 + y_2 \equiv y_1^2 + y_2^2 + y_1 + y_2 \equiv 0 \pmod{2}$ and so $b((d - a)D|v|^2 + 2(y_1 + y_2)) \equiv 0 \pmod{4}$;

or $a - d \equiv 0 \pmod{4}$ whence $ad \equiv 1 \pmod{4}$ and so we must have $2 \mid b$ which also leads to $b((d - a)D|v|^2 + 2(y_1 + y_2)) \equiv 0 \pmod{4}$ since the other factor is clearly divisible by 2 and so

$$\frac{b((d - a)D|v|^2 + 2(y_1 + y_2))}{4} \in \mathbb{Z}$$

always.

Once again, we match the RHS perfectly as in the previous case for $e = f = 2$. The difference is the involvement of the $\psi_{D'}(2)$ since the exponent f is odd:

$$(-1)^{\frac{(c'-1)(D'/c'-1)}{4}} \psi_{D'}(2)^f \psi_{c'}(2^e) 2^e \mathbf{G}(\chi_2; D') = (-1)^{\frac{(c'-1)(D'/c'-1)}{4}} \psi_{D'}(2) 2^{e+1} i$$

and we observe that

$$\begin{aligned} \mathbf{e}\left[\frac{c'a(D' + 1)}{8}\right] &= \mathbf{e}\left[\frac{c'a\frac{D'+1}{2}}{4}\right] \\ &= \begin{cases} \mathbf{e}\left[\frac{c'a}{4}\right] & \text{if } \frac{D'+1}{2} \equiv 1 \pmod{4} \iff D' \equiv 1 \pmod{8} \\ \mathbf{e}\left[-\frac{c'a}{4}\right] & \text{if } \frac{D'+1}{2} \equiv -1 \pmod{4} \iff D' \equiv 5 \pmod{8} \end{cases} \\ &= \psi_{D'}(2) \mathbf{e}\left[\frac{c'a}{4}\right] \text{ since } \mathbf{e}\left[\frac{c'a}{4}\right] \in \{\pm i\} \end{aligned}$$

and so (4.3.5) reduces to

$$\chi_2(a) \mathbf{e}\left[\frac{c'a}{4}\right] = (-1)^{\frac{(c'-1)(D'/c'-1)}{4}} i = \chi_2(c') i$$

which was established in the previous case.

The case $f = 0$ and $e = 2$

If $f = 0$ then the equation (4.3.5) reads

$$\begin{aligned} & \sum_{\substack{x_1 \bmod 2 \\ x_2 \bmod 2^{e-1}}} \mathbf{e} \left[-\frac{\frac{1}{D'/c'} D|u|^2 \frac{a}{cc^*}}{2^e} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \times \\ & \quad \times [\mathbf{G}(\chi_2; D'c) \chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f_1,0}) E'_u \mathbf{G}(\chi_2; D'/c')] M_{u,v}^* \\ & = \delta_{D|v|^2, D|w|^2}^{\bmod 2^e} (-1)^{\frac{(c'-1)(D'/c'-1)}{4}} \psi_{c'}(2^e) 2^e \mathbf{G}(\chi_2; D') \mathbf{e} \left[\frac{\frac{1}{D'/c'} D|w|^2 \frac{d}{cc^*}}{2^e} \right] \end{aligned}$$

We can cancel the Gauss sum on both sides to get an equivalent equation

$$\begin{aligned} & \sum_{\substack{x_1 \bmod 2 \\ x_2 \bmod 2^{e-1}}} \mathbf{e} \left[-\frac{\frac{1}{D'/c'} (2^{e-2} \cancel{D'x_1^2} + \cancel{x_2^2}) a}{2^e} - \frac{\frac{1}{D'/c'} D|w|^2 d}{2^e} \right] \frac{1 + \chi_2(D|w|^2)}{1 + \chi_2(D|u|^2)} \times \\ & \quad \times [\chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f_1,0}) E'_u] \times \\ & \quad \times \mathbf{e} \left[\frac{\frac{1}{c'} (q \cancel{x_1^2} - 2x_1 y_1 + dy_1^2)}{4} + \frac{\frac{1}{D'c'} (q \cancel{x_2^2} - 2x_2 y_2 + dy_2^2)}{2^e} \right] \\ & = \delta_{D|v|^2, D|w|^2}^{\bmod 2^e} (-1)^{\frac{(c'-1)(D'/c'-1)}{4}} \psi_{c'}(2^e) 2^e \chi_2(c) \end{aligned} \quad (4.3.8)$$

whose left hand side can be further simplified to

$$\begin{aligned} & (1 + \chi_2(D|w|^2)) \mathbf{e} \left[-\frac{\frac{d}{D'/c'} (D|v|^2 - D|w|^2)}{2^e} \right] \times \\ & \quad \times \sum_{\substack{x_1 \bmod 2 \\ x_2 \bmod 2^{e-1}}} \frac{\chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f_1,0}) E'_u}{1 + \chi_2(D|u|^2)} \mathbf{e} \left[-\frac{\frac{1}{c'} x_1 y_1}{2} - \frac{\frac{1}{D'c'} x_2 y_2}{2^{e-1}} \right] \end{aligned}$$

since

$$\mathbf{e} \left[\frac{\frac{1}{c'} (dy_1^2)}{4} + \frac{\frac{1}{D'c'} (dy_2^2)}{2^e} \right] = \mathbf{e} \left[\frac{\frac{1}{c'} (dy_1^2) 2^{e-2} + \frac{1}{D'c'} (dy_2^2)}{2^e} \right] = \mathbf{e} \left[\frac{\frac{d}{D'/c'} [2^{e-2} D' y_1^2 + y_2^2]}{2^e} \right] = \mathbf{e} \left[\frac{\frac{d}{D'/c'} D|v|^2}{2^e} \right]$$

If $2 \nmid D|w|^2$ then $\chi_2(D|u|^2 + D|w|^2) + (1 - \delta_{f_1,0})E'_u = \chi_2(D|u|^2 + D|w|^2)$ is non-zero only when $D|u|^2 \equiv 0 \pmod{2}$; in other words, we need $x_1 \equiv x_2 \pmod{2}$. In this case, we always have $\chi_2(D|w|^2) = 1$ and $\chi_2(D|u|^2) = 0$. So the left hand side of (4.3.8) is just

$$2e \left[\frac{\frac{1}{D'/c'}(D|v|^2 - D|w|^2)d}{2^e} \right] \sum_{\substack{x_1 \pmod{2} \\ x_2 \pmod{2^{e-1}} \\ 2 \mid D|u|^2}} \chi_2(D|u|^2 + D|w|^2) e \left[\frac{\frac{1}{c'}(-2x_1y_1)}{4} + \frac{\frac{1}{D'c'}(-2x_2y_2)}{2^e} \right]$$

If $e = 2$ then note that $D' \equiv 1 \pmod{4}$ and $D|w|^2 \equiv 1 \pmod{4}$ and so

$$\begin{aligned} & \sum_{\substack{x_1 \pmod{2} \\ x_2 \pmod{2^{e-1}} \\ 2 \mid D|u|^2}} \chi_2(D|u|^2 + D|w|^2) e \left[\frac{\frac{1}{c'}(-2x_1y_1)}{4} + \frac{\frac{1}{D'c'}(-2x_2y_2)}{2^e} \right] \\ &= \sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} \chi_2(2x_1^2 + D|w|^2) e \left[\frac{\frac{1}{c'}(-2x_1y_1)}{4} + \frac{\frac{1}{c'}(-2x_2y_2)}{4} \right] \\ &= \sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} (-1)^{\frac{2x_1^2 + D|w|^2 - 1}{2}} e \left[\frac{\frac{-1}{c'}(y_1 + y_2)x_1}{2} \right] \\ &= \sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} (-1)^{\frac{2x_1^2}{2}} (-1)^{(y_1 + y_2)x_1} \\ &= 1 - (-1)^{y_1 + y_2} \end{aligned}$$

so $A^- = 0$ unless $y_1 + y_2 \equiv 1 \pmod{2}$, in which case $D|v|^2 \equiv 1 \pmod{2}$ and so

$D|v|^2 \equiv D|w|^2 \equiv 1 \pmod{4}$. When that happen the left hand side is just 4.

And the remaining factor on the right hand side $(-1)^{(c'-1)(D'/c'-1)/4} \psi_{c'}(2^e) \chi_2(c) =$

1. That is because we must have $c' \equiv D'/c' \pmod{4}$ here and so $(-1)^{(c'-1)(D'/c'-1)/4} =$

$(-1)^{(c'-1)/2} = \chi_2(c)$.

Now let us consider $2 \mid D|w|^2$. We either have $e - f_1 = 0$ i.e. $D|w|^2 \equiv 0 \pmod{4}$ whence $E'_u = 1$ or $e - f_1 = 1$ i.e. $D|w|^2 \equiv 2 \pmod{4}$ whence $E'_u = (-1)^{D|u|^2}$. And so

$$\begin{aligned}
\text{LHS}_{(4.3.8)} &= \mathbf{e} \left[-\frac{\frac{d}{D'/c'}(D|v|^2 - D|w|^2)}{2^e} \right] \sum_{x_1, x_2 \pmod{2}} \frac{\chi_2(D|u|^2 + D|w|^2) + E'_u}{1 + \chi_2(D|u|^2)} \mathbf{e} \left[-\frac{\frac{1}{c'}x_1y_1}{2} - \frac{\frac{1}{D'c'}x_2y_2}{2} \right] \\
&= \mathbf{e} [\dots] \sum_{x_1, x_2 \pmod{2}} \frac{\chi_2(D|u|^2 + D|w|^2) + E'_u}{1 + \chi_2(D|u|^2)} (-1)^{x_1y_1 + x_2y_2} \text{ since } D' \equiv 1 \pmod{4} \\
&= \mathbf{e} [\dots] \left\{ \sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} \frac{\chi_2(D|u|^2 + D|w|^2) + E'_u}{1 + \chi_2(D|u|^2)} (-1)^{x_1y_1 + x_2y_2} + \right. \\
&\quad \left. + \sum_{\substack{x_1 \pmod{2} \\ x_2 = 1 - x_1}} \frac{\chi_2(D|u|^2 + D|w|^2) + E'_u}{1 + \chi_2(D|u|^2)} (-1)^{x_1y_1 + x_2y_2} \right\} \\
&= \mathbf{e} [\dots] \left(\sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} E'_u (-1)^{x_1(y_1 + y_2)} + \sum_{\substack{x_1 \pmod{2} \\ x_2 = 1 - x_1}} \frac{(-1)^{(D|u|^2 + D|w|^2 - 1)/2} + E'_u}{2} (-1)^{x_1(y_1 - y_2) + y_2} \right) \\
&= \mathbf{e} [\dots] \left(\sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} E'_u (-1)^{x_1(y_1 + y_2)} + \frac{(-1)^{y_2}}{2} \sum_{\substack{x_1 \pmod{2} \\ x_2 = 1 - x_1}} [(-1)^{D|w|^2/2} + E'_u] (-1)^{x_1(y_1 + y_2)} \right) \\
&= \mathbf{e} [\dots] \begin{cases} \left(\sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} (-1)^{x_1(y_1 + y_2)} + (-1)^{y_2} \sum_{\substack{x_1 \pmod{2} \\ x_2 = 1 - x_1}} (-1)^{x_1(y_1 + y_2)} \right) & \text{if } D|w|^2 \equiv 0 \pmod{4}, \\ \left(\sum_{\substack{x_1 \pmod{2} \\ x_2 = x_1}} (-1)^{x_1(y_1 + y_2)} + \frac{(-1)^{y_2}}{2} \sum_{\substack{x_1 \pmod{2} \\ x_2 = 1 - x_1}} (-1 - 1) (-1)^{x_1(y_1 + y_2)} \right) & \text{if } D|w|^2 \equiv 2 \pmod{4} \end{cases} \\
&= \mathbf{e} [\dots] \begin{cases} (1 + (-1)^{y_2}) \left(\sum_{x_1 \pmod{2}} (-1)^{x_1(y_1 + y_2)} \right) & \text{if } D|w|^2 \equiv 0 \pmod{4}, \\ (1 - (-1)^{y_2}) \left(\sum_{x_1 \pmod{2}} (-1)^{x_1(y_1 + y_2)} \right) & \text{if } D|w|^2 \equiv 2 \pmod{4} \end{cases} \\
&= \mathbf{e} [\dots] (1 + (-1)^{D|w|^2/2} (-1)^{y_2}) (1 + (-1)^{y_1 + y_2})
\end{aligned}$$

is non-zero only when $y_1 + y_2 \equiv 0 \pmod{2}$ and $y_2 + \frac{D|w|^2}{2} \equiv 0 \pmod{2}$. The first implies $y_1 \equiv y_2 \pmod{2}$; together with the second condition, we get $D|v|^2 \equiv D|w|^2 \pmod{4}$. So we find that LHS of (4.3.8) is 0 if $D|v|^2 \not\equiv D|w|^2 \pmod{4}$ and is 4 if $D|v|^2 \equiv D|w|^2 \pmod{4}$ in case $2 \mid D|w|^2$ as well.

4.4 The map from elliptic forms to plus forms

With Berger-Klosin's result, we have completed most of the Saito-Kurokawa lift. All that remains is to construct a surjective linear map

$$\mathfrak{S}_{k-1}(DN, \chi) \rightarrow \mathfrak{S}_{k-1}^+(DN, \chi).$$

For this linear map, we modify Ikeda's construction in [14]. We recall the notation of [14]. For any positive integer M and prime q , we denote

$$M_q = q^{\text{val}_q(M)}$$

where val_q is the q -adic valuation and for a Dirichlet character ψ modulo M , we denote ψ_q to be the character mod M_q defined by $\psi_q(n) = \psi(n')$ for $(n, q) = 1$ where n' is any integer such that

$$n' \equiv \begin{cases} n & \pmod{M_q} \\ 1 & \pmod{M/M_q} \end{cases}$$

and $\psi_q(n) = 0$ in case $(n, q) > 1$. Let ψ'_q be the character ψ/ψ_q . (When $\psi = \chi_K$, this notation agrees with what we used previously in section 3.)

We remark that our weight $k - 1$ corresponds to the weight $2k + 1$ in Ikeda's paper.

As in [14], we observe that for a primitive form $f \in \mathfrak{S}_{k-1}(Dm, \chi)$ and every subset $Q \subset Q_D := \{\text{prime divisors of } D\}$, there exists a primitive form f_Q whose Fourier coefficients at prime p is given by

$$a_{f_Q}(p) = \begin{cases} \chi_Q(p) a_f(p) & \text{if } p \notin Q, \\ \chi'_Q(p) \overline{a_f(p)} & \text{otherwise.} \end{cases}$$

where

$$\chi_Q := \prod_{q \in Q} \chi_q \quad \text{and} \quad \chi'_Q := \prod_{q \in Q_D \setminus Q} \chi_q = \frac{\chi}{\chi_Q}.$$

For any primitive form $f \in \mathfrak{S}_{k-1}(Dm, \chi)$ where $m \mid N$ and for any ℓ such that $\ell m \mid N$, we define

$$f^*[\ell] := \sum_{Q \subset Q_D} \chi_Q(-\ell) f_Q$$

Let us recycle the notation $\sigma_\ell := \begin{pmatrix} \ell & \\ & 1 \end{pmatrix}$. Then any general form $f \in \mathfrak{S}_{k-1}(DN, \chi)$ can be expressed uniquely as

$$f = \sum_{i \in I} \alpha_i (f_i|_{k-1} \sigma_{\ell_i})$$

where I is some finite indexing set, $\alpha_i \in \mathbb{C}^\times$, $f_i \in \mathfrak{S}_{k-1}(Dm_i, \chi)$ are primitive forms (under appropriate independence assumption) and m_i, ℓ_i are natural numbers such that $m_i \ell_i \mid N$ and we set

$$f^* := \sum \alpha_i (f_i^*[\ell_i] | \sigma_{\ell_i}).$$

It is easy to verify that $f^* \in \mathfrak{S}_{k-1}^+(DN, \chi)$. This is because Lemma 15.4 in [14] (which we restate below) holds verbatim and Corollary 15.5 follows with a minor twist:

Lemma 4.14 (Lemma 15.4 for higher level). *Let $f \in \mathfrak{S}_{k-1}(Dm, \chi)$ be a primitive form with $(m, D) = 1$. For any $M \in \mathbb{Z}$, we split the prime factorization of M into primes not dividing D , primes dividing D that are not in Q , and primes dividing D in Q :*

$$M' = \prod_{p \nmid D} p^{\text{val}_p(M)} \quad M'_Q = \prod_{p \mid D, p \notin Q} p^{\text{val}_p(M)} \quad M_Q = \prod_{p \mid D, p \in Q} p^{\text{val}_p(M)}$$

as in the notation of Ikeda. Then the M -th Fourier coefficient of f_Q are related to that of f by

$$a_{f_Q}(M) = a_f(M' M'_Q) \overline{a_f(M_Q)} \prod_{q \in Q} \underline{\chi}_q(M)$$

where $\underline{\chi} = \bigotimes' \underline{\chi}_q$ is the idele character corresponding to the quadratic character χ .

Corollary 4.15. *Let f be as in previous lemma. Then*

$$\begin{aligned} a_{f^*[\ell]}(M) &= a_f(M') \prod_{q \mid D} (a_f(M_q) + \chi_q(\ell) \underline{\chi}_q(-M) \overline{a_f(M_q)}) \\ &= a_f(M') a_D(\ell M) \prod_{q \mid (D, M)} (a_f(M_q) + \chi_q(-1) \chi_q(\ell) \overline{a_f(M_q)} \underline{\chi}_q(M)) \end{aligned}$$

With this computation, it is evident that the form $f^*[\ell] \sigma_\ell \in \mathfrak{S}_{k-1}^+(Dm\ell, \chi)$ ■

as $a_D(\ell^2 M) = a_D(M)$ as long as $(D, \ell) = 1$.

Proposition 4.16 (Ikeda [14] Proposition 15.17 for higher level). *The map*

$\mathfrak{S}_{k-1}(DN, \chi) \rightarrow \mathfrak{S}_{k-1}^+(DN, \chi)$ *where* $f \mapsto f^*$ *is surjective.*

Proof. Suppose that $g \in \mathfrak{S}_{k-1}^+(DN, \chi)$ which we can express as

$$g = \sum_{i \in I} \alpha_i g_i |_{k-1} \sigma_{\ell_i}$$

with similar meaning for I, α_i ; for example, g_i 's are primitive forms in $\mathfrak{S}_{k-1}(Dm_i, \chi)$ ■

and $m_i \ell_i \mid N$. Following Ikeda, for each subset $Q \subset Q_D$, we consider the form

$$g' = \sum \alpha_i \chi_Q(-\ell_i)(g_i)_Q | \sigma_{\ell_i}$$

and observe that $g - g'$ has its n -th Fourier coefficient vanishing for all $(n, D_Q) = 1$ (i.e. $q \nmid n$ for all $q \in Q$; equivalently, $n_Q = 1$). To see that, one has ¹

$$\begin{aligned} a_{g-g'}(n) &= \sum \alpha_i a_{g_i}(n/\ell_i) - \sum \alpha_i \chi_Q(-\ell_i) a_{(g_i)_Q}(n/\ell_i) \\ &= \sum_{\substack{i \\ \ell_i \mid n}} \alpha_i [a_{g_i}(n/\ell_i) - \chi_Q(-\ell_i) a_{g_i}((n/\ell_i)'(n/\ell_i)'_Q) \overline{a_{g_i}((n/\ell_i)_Q)} \prod_{q \in Q} \chi_q(n/\ell_i)] \\ &\hspace{15em} \text{by Lemma 4.14} \\ &= \sum_{\substack{i \\ \ell_i \mid n}} \alpha_i [a_{g_i}(n/\ell_i) - \chi_Q(-\ell_i) a_{g_i}(\underbrace{(n/\ell_i)'(n/\ell_i)'_Q}_{n/\ell_i}) \overline{a_{g_i}(\underbrace{(n/\ell_i)_Q}_1)} \prod_{q \in Q} \chi_q(n/\ell_i)] \\ &\hspace{15em} \text{by Lemma 15.1 of [14]} \\ &= \sum_{\substack{i \\ \ell_i \mid n}} \alpha_i [a_{g_i}(n/\ell_i) - \chi_Q(-n) a_{g_i}(n/\ell_i)] \end{aligned}$$

¹Here, we take a convention that $a_g(r) = 0$ if $r \notin \mathbb{Z}$.

$$\begin{aligned}
&= \sum_{\substack{i \\ \ell_i | n}} \alpha_i [1 - \chi_Q(-n)] a_{g_i}(n/\ell_i) \\
&= [1 - \chi_Q(-n)] \left(\sum \alpha_i a_{g_i}(n/\ell_i) \right) \\
&= [1 - \chi_Q(-n)] a_g(n)
\end{aligned}$$

Note the fact that $(n/\ell_i)_Q = 1$ due to $(n, Q) = n_Q = 1$ whence $(n/\ell_i)'(n/\ell_i)'_Q = n/\ell_i$. Now by assumption $a_g(n) = 0$ whenever $a_D(n) = 0$, we find that $a_{g-g'}(n) = 0$ if $a_D(n) = 0$. If $(n, Q) = 1$ and $a_D(n) \neq 0$ then we must have $\chi_q(-n) \neq -1$ for all $q|D$; in other words, either $\chi_q(-n) = 0$ or $\chi_q(-n) = 1$ for all $q|D$. We can't conclude that $\chi_q(-n) = 1$ for all $q|D$ but this should be the case for all $q \in Q$ by assumption $(n, Q) = 1$. And so the factor $1 - \chi_Q(-n) = 0$ and we still have $a_{g-g'}(n) = 0$.

We have shown that $g - g' \in \mathfrak{S}_{k-1}(DN, \chi)$ has vanishing n -th Fourier coefficient for all $(n, D_Q) = 1$. By Miyake [24] Theorem 4.6.8, we find that $g - g' = 0$. Thus, we proved that

$$g = \sum \alpha_i \chi_Q(-\ell_i) (g_i)_Q | \sigma_{\ell_i}$$

for every $Q \subset Q_D$. Summing both sides over all Q , we get

$$\begin{aligned}
2^{|Q_D|} g &= \sum_Q \sum \alpha_i \chi_Q(-\ell_i) (g_i)_Q | \sigma_{\ell_i} \\
&= \sum \alpha_i g_i^*[\ell_i] | \sigma_{\ell_i}
\end{aligned}$$

so g is in the image of the map. □

Remark 4.17. We remark that when D is prime and $N = 1$, we get Krieg's remark mentioned in the introduction. For then, $Q = \{D\}$, $\mathfrak{S}_{k-1}(DN, \chi)$ has a basis of primitive form and we have $f_\emptyset = f$ and $f_Q = f^\rho$ for any primitive form f . And we find that $f^* = f_\emptyset - f_Q = f - f^\rho$.

Chapter 5

Hecke equivariant

Now we establish Hecke equivariance of our Maass lift for certain Hecke operators.

First, we characterize the Maaß space, similar to Andrianov's characterization of Maaß space in [2]. The following lemma is generalization of the Lemma in [21], Section 7 for higher level Maaß space. Intuitively, it expresses the fact that the Fourier coefficients of the hermitian modular forms in the Maaß space only depends on $\epsilon(T)$ and $\det(T)$, which is evident in Definition 3.8.

Lemma 5.1. *Suppose that $F \in \mathfrak{M}_{k,2}(N)$ with Fourier expansion*

$$F(Z) = \sum_{T \in S_2(\mathbb{Q})} c_F(T) \mathbf{e}[\mathrm{Tr}(TZ)].$$

Then $F \in \mathfrak{M}_k^(N)$ if and only if there exists a function $\beta : \mathbb{Z}^+ \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ such that*

(i) for all $T \in S_2(\mathbb{Q}), T \geq 0, T \neq 0$:

$$c_F(T) = \beta \left(\epsilon(T), D \frac{\det(T)}{\epsilon(T)^2} \right)$$

(ii) for all $d \in \mathbb{Z}_{\geq 0}, q \in \mathbb{Z}^+$ and all primes $p \nmid Nq$:

$$(1 - p^{k-1}W) \sum_{v=0}^{\infty} \beta(p^v q, d) W^v = \sum_{v=0}^{\infty} \beta(q, p^{2v} d) W^v$$

holds as a formal power series in W ; in other words, $\beta(p^v q, d) - p^{k-1} \beta(p^{v-1} q, d) = \beta(q, dp^{2v})$ for all $v \geq 0$.

(iii) $\beta(u, v) = \beta(1, vu^2)$ for all $u \mid N^\infty$ i.e. if every prime divisor of u divides N .

The main utility of the lemma is the fact that the function β relates directly to the Fourier coefficients of the form.

Now we recall the Hecke operator for Hermitian modular forms. The argument in Lemma 3.1 and Lemma 3.2 of [3] can be used to show that the commesurator of $\Gamma_{0,n}(N)$ in $\mathbf{GU}(n, n)(\mathbb{Q})^+$ is the whole group $\mathbf{GU}(n, n)(\mathbb{Q})^+$. (This was also claimed in [21], Section 1.) Modifying Andrianov's definition for Siegel modular forms, we define the following sub-semigroup of $\mathbf{GU}(n, n)(\mathbb{Q})^+$

$$\Delta_{0,n}(N) := \Gamma_{0,n}(N) \left\{ g \in \mathbf{GU}(n, n)(\mathbb{Q})^+ \cap \mathbf{GL}_{2n}(K_{(N)}) \left| g \equiv \begin{pmatrix} I_n & 0 \\ 0 & \mu(g)I_n \end{pmatrix} \pmod{N} \right. \right\} \Gamma_{0,n}(N) \blacksquare$$

where $K_{(N)} = \bigcap_{\mathfrak{p} \mid N} (\mathfrak{o}_K)_{\mathfrak{p}} \subset K$ denotes the subring of our imaginary quadratic field K consisting of N -integral elements.

Then $(\Delta_{0,n}(N), \Gamma_{0,n}(N))$ is a Shimura pair and we have the associated $\mathbb{Z}$ -ring of Hecke operators as in [24], Chapter 2 or [30], Chapter 3. For a rational prime p , let us consider the Hecke operator

$$T_p = T_p(N) := \Gamma_{0,2}(N) \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \Gamma_{0,2}(N).$$

In the proof of Theorem in Section 7 of [21], Krieg showed that when $N = 1$ and p is inert in K , one has the decomposition

$$\Gamma_{0,2}(1) \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \Gamma_{0,2}(1) = \bigsqcup_{(D,B) \in R} \Gamma_{0,2}(1) \begin{pmatrix} p\widehat{D} & B \\ 0 & D \end{pmatrix} \quad (5.0.1)$$

where $\widehat{D} := (\overline{D^t})^{-1} = (D^*)^{-1}$ and R is the finite set of following pair of matrices (D, B) where

- $D = I_2, B = 0$;
- $D = pI_2, B = \begin{pmatrix} \gamma & b \\ \bar{b} & \delta \end{pmatrix}$ where $b \in \mathfrak{o}_K/p\mathfrak{o}_K$ and $\gamma, \delta = 1, 2, \dots, p$;
- $D = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$ where $\gamma = 1, 2, \dots, p$; and
- $D = \begin{pmatrix} 1 & d \\ 0 & p \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$ where $d \in \mathfrak{o}_K/p\mathfrak{o}_K$ and $\gamma = 1, 2, \dots, p$.

We extend his result to higher level.

Lemma 5.2. *Suppose that $p \nmid N$ is inert in K and let $\xi, \lambda \in \mathbb{Z}$ be such that $p\xi - N\lambda = 1$. Then the set*

$$R_N := \left\{ \begin{array}{l} \begin{pmatrix} \xi p I_2 & \lambda I_2 \\ N I_2 & I_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \gamma & b \\ 0 & 1 & \bar{b} & \delta \\ N & 0 & 1 + N\gamma & Nb \\ 0 & N & N\bar{b} & 1 + N\delta \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & \gamma & 0 \\ 0 & \xi p & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & N & 0 & 1 \end{pmatrix}, \begin{pmatrix} \xi p & 0 & \lambda & \lambda d \\ -\bar{d} & 1 & 0 & \gamma \\ N & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \text{where } \gamma, \delta = 1, 2, \dots, p \text{ and } b, d \in \mathfrak{o}_K/p\mathfrak{o}_K \end{array} \right\}$$

is a complete set of representatives for

$$\underbrace{\Gamma_{0,2}(N) \cap \alpha^{-1}\Gamma_{0,2}(N)\alpha \setminus \Gamma_{0,2}(N)}_{\alpha^{-1}\Gamma_{0,2}(Np)\alpha} \quad \text{where} \quad \alpha = \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix}.$$

Proof. It follows from (5.0.1) and Shimura [30], Proposition 3.1 that R_N is a complete set of representatives for $\alpha^{-1}\Gamma_{0,2}(p)\alpha \setminus \Gamma_{0,2}(1)$.

These representatives for $\alpha^{-1}\Gamma_{0,2}(p)\alpha \setminus \Gamma_{0,2}(1)$ must be thus distinct classes in $\alpha^{-1}\Gamma_{0,2}(Np)\alpha \setminus \Gamma_{0,2}(N)$ as well since the condition for two matrices to be in the same class is independent of the level.

It remains to see that there are no more classes. In other words, any matrix $\begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in \Gamma_{0,2}(N)$ lies in one of the right $\alpha^{-1}\Gamma_{0,2}(Np)\alpha$ -coset of some matrix in R_N : Let $\rho \in \Gamma_{0,2}(N)$ be arbitrary and let $\rho' \in R_N$ be the element

such that ρ lies in the right coset $(\alpha^{-1}\Gamma_{0,2}(p)\alpha) \rho'$. (Such an element exists because R_N is a complete set of representatives for $(\alpha^{-1}\Gamma_{0,2}(p)\alpha)\backslash\Gamma_{0,2}(1)$ that we extracted from Krieg.) Then there exists a matrix $\rho'' \in \Gamma_{0,2}(p)$ such that $\rho = \alpha^{-1}\rho''\alpha\rho'$. We then have $\alpha^{-1}\rho''\alpha = \rho(\rho')^{-1} \in \Gamma_{0,2}(N)$ so that $\rho'' \in \alpha\Gamma_{0,2}(N)\alpha^{-1}$. From a simple observation that

$$\alpha \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \alpha^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & \frac{1}{p}I_n \end{pmatrix} = \begin{pmatrix} A & \frac{1}{p}B \\ pNC & D \end{pmatrix}$$

is in $\Gamma_{0,2}(p)$ if and only if $B \equiv 0 \pmod{p}$ and we find $\alpha\Gamma_{0,2}(N)\alpha^{-1} \cap \Gamma_{0,2}(p) = \Gamma_{0,2}(Np)$ under the assumption that $p \nmid N$. This proves $\rho'' \in \Gamma_{0,2}(Np)$; in other words, ρ is in the right $\alpha^{-1}\Gamma_{0,2}(Np)\alpha$ -coset of ρ' .

Thus, our set R_N is the complete set of representatives for $\alpha^{-1}\Gamma_{0,2}(Np)\alpha\backslash\Gamma_{0,2}(N)$. ■

□

Corollary 5.3. *Suppose that $p \nmid N$ is inert in K . We have the following decomposition*

$$\Gamma_{0,2}(N) \alpha \Gamma_{0,2}(N) = \bigsqcup_{(D,B) \in R} \Gamma_{0,2}(N) \begin{pmatrix} p\widehat{D} & B \\ 0 & D \end{pmatrix}$$

Theorem 5.4. *Suppose that p is an inert prime in K and $p \nmid N$. Then the Maaß space $\mathfrak{M}_k^*(N)$ is stable under T_p . In other words, if $F \in \mathfrak{M}_k^*(N)$ then $G = F|_k T_p$ is also in the Maaß space.*

Proof. Let β_F be the function associated to F by Lemma 5.1. Following [21], Section 7, we define¹

$$\begin{aligned} \beta_G(p^v q, r) &:= \beta_F(p^{v-1} q, r) + p^{4-2k} \beta_F(p^{v+1} q, r) \\ &+ \begin{cases} p^{1-k} \beta_F(p^{v+1} q, p^{-2} r) + p^{3-k} \beta_F(p^{v-1} q, p^2 r) & \text{if } p^2 \mid r \\ p^{1-k} \beta_F(p^v q, r) + p^{3-k} \beta_F(p^{v-1} q, p^2 r) & \text{if } p \mid r, p^2 \nmid r \\ p^{1-k}(p+1) \beta_F(p^v q, r) + p^{1-k}(p^2 - p) \beta_F(p^{v-1} q, p^2 r) & \text{if } p \nmid r \end{cases} \quad \blacksquare \end{aligned}$$

where $p \nmid q$. Then Krieg [21] already showed that β_G satisfies (i) and (ii) in Lemma 5.1 for G . The last property is evident from the definition of β_G and the corresponding property (ii) and (iii) of β_F .

□

Remark 5.5. The Maaß space is not stable under all Hecke operators but its *adelic* analogue should be stable under all Hecke operators; as illustrated by Klosin [17], Section 5.

¹There appears to be a mistake in Krieg's paper [21], it should be p^{4-2k} , not p^{6-2k} .

Bibliography

- [1] AGARWAL, M., AND BROWN, J. On the Bloch-Kato conjecture for elliptic modular forms of square-free level. *Math. Z.* 276, 3-4 (2014), 889–924.
- [2] ANDRIANOV, A. N. Modular descent and the Saito-Kurokawa conjecture. *Invent. Math.* 53, 3 (1979), 267–280.
- [3] ANDRIANOV, A. N., AND ZHURAVLĚV, V. G. *Modular forms and Hecke operators*, vol. 145 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1995. Translated from the 1990 Russian original by Neal Koblitz.
- [4] ATOBE, H. Pullbacks of Hermitian Maass lifts. *J. Number Theory* 153 (2015), 158–229.
- [5] BERGER, T., AND KLOSIN, K. A p -adic hermitian Maass lift (to appear). *Glasg. Math. J.* (2018). arXiv:1602.07987.
- [6] BHATIA, R. *Positive definite matrices*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007. [2015] paperback edition of the 2007 original [MR2284176].
- [7] BROWN, J. Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture. *Compos. Math.* 143, 2 (2007), 290–322.
- [8] CHUMAN, Y. Generators and relations of $\Gamma_0(N)$. *J. Math. Kyoto Univ.* 13 (1973), 381–390.
- [9] DIAMOND, F., AND SHURMAN, J. *A first course in modular forms*, vol. 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

- [10] HAVERKAMP, K. Hermitesche Jacobiformen. *Schriftenreihe des Mathematischen Instituts der Universität Münster. 3. Serie 15* (1995), 105.
- [11] HIDA, H. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.* 85, 3 (1986), 545–613.
- [12] HIDA, H. Iwasawa modules attached to congruences of cusp forms. *Ann. Sci. École Norm. Sup. (4)* 19, 2 (1986), 231–273.
- [13] HORN, R. A., AND JOHNSON, C. R. *Matrix analysis*, second ed. Cambridge University Press, Cambridge, 2013.
- [14] IKEDA, T. On the lifting of Hermitian modular forms. *Compos. Math.* 144, 5 (2008), 1107–1154.
- [15] IMAI, K. Generalization of Hecke’s correspondence to Siegel modular forms. *Amer. J. Math.* 102, 5 (1980), 903–936.
- [16] KLOSIN, K. Congruences among modular forms on $U(2, 2)$ and the Bloch-Kato conjecture. *Ann. Inst. Fourier (Grenoble)* 59, 1 (2009), 81–166.
- [17] KLOSIN, K. The Maass space for $U(2, 2)$ and the Bloch-Kato conjecture for the symmetric square motive of a modular form. *J. Math. Soc. Japan* 67, 2 (2015), 797–860.
- [18] KOBLITZ, N. *Introduction to elliptic curves and modular forms*, second ed., vol. 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.
- [19] KOHNEN, W. Modular forms of half-integral weight on $\Gamma_0(4)$. *Math. Ann.* 248, 3 (1980), 249–266.
- [20] KOJIMA, H. An arithmetic of Hermitian modular forms of degree two. *Invent. Math.* 69, 2 (1982), 217–227.
- [21] KRIEG, A. The Maaß spaces on the Hermitian half-space of degree 2. *Math. Ann.* 289, 4 (1991), 663–681.
- [22] MAZUR, B., AND WILES, A. Class fields of abelian extensions of $\mathbf{Q}$. *Invent. Math.* 76, 2 (1984), 179–330.

- [23] MILNE, J. S. *Algebraic groups*, vol. 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.
- [24] MIYAKE, T. *Modular forms*, English ed. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.
- [25] PIATETSKI-SHAPIRO, I. I. On the Saito-Kurokawa lifting. *Invent. Math.* 71, 2 (1983), 309–338.
- [26] RIBET, K. A. A modular construction of unramified p -extension of $\mathbb{Q}(\mu_p)$. *Inventiones mathematicae* 34 (1976), 151–162.
- [27] ROLAND, M. A note on the Saito-Kurokawa lift for Hermitian forms (preprint).
- [28] SCHMIDT, R. The Saito-Kurokawa lifting and functoriality. *Amer. J. Math.* 127, 1 (2005), 209–240.
- [29] SCHMIDT, R. On classical Saito-Kurokawa liftings. *J. Reine Angew. Math.* 604 (2007), 211–236.
- [30] SHIMURA, G. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Kanô Memorial Lectures, No. 1.
- [31] SHIMURA, G. *Euler Products and Eisenstein Series*. No. no. 93 in CBMS Regional Conference Series. American Mathematical Society, 1997.
- [32] SKINNER, C., AND URBAN, E. Sur les déformations p -adiques de certaines représentations automorphes. *J. Inst. Math. Jussieu* 5, 4 (2006), 629–698.
- [33] WILES, A. The Iwasawa conjecture for totally real fields. *Ann. of Math.* (2) 131, 3 (1990), 493–540.
- [34] WILLIAMS, K. S. Note on Salié’s sum. *Proc. Amer. Math. Soc.* 30 (1971), 393–394.