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Modest Automorphisms of Presburger Arithmetic

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by Simon Heller

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ABSTRACT

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It is interesting to consider whether a structure can be expanded by an automorphism so that one obtains a nice description of the expanded structure's first-order properties. In this dissertation, we study some such expansions of models of Presburger arithmetic. Building on some of the work of Harnik [5] and Llewellyn-Jones [9], in Chapter 2 we use a back-and-forth construction to obtain two automorphisms of sufficiently saturated models of Presburger arithmetic. These constructions are done first in the quotient of the Presburger structure by $\mathbb{Z}$ (which is a divisible ordered abelian group with some added structure), and then lifted to the full Presburger structure.

The first automorphism we construct (which we call σ) has special tightly controlled properties that enable us in Chapters 4 and 5 to prove quantifier elimination, decidability, and axiomatizability for both the quotient and the Presburger structure expanded by this automorphism, with explicit axiomatizations given in Chapter 3. The second automorphism is maximal in the sense that its fixed-point set consists only of $\mathbb{Z}$, and has certain properties in common with those of σ, but we have not attempted to prove quantifier elimination for structures expanded by this automorphism.

In Chapters 6 and 7, we use the quantifier elimination results to describe the definable sets and algebraic closure of the quotient structure and Presburger structure expanded by σ. This allows us in Chapter 8 to show that the DP-rank in both cases is 2. Finally, in the concluding chapter, we describe some areas of possible future research.

Contents

Chapter 1

Introduction

The study of automorphism groups of first-order structures lies at the intersection of model theory and permutation group theory [6]. With respect to totally-ordered sets without additional structure, [4] gives a systematic treatment of automorphisms (that is, order-preserving permutations). One highlight of this work is the result that the theory of such a group of permutations is undecidable [4, Theorem 2.2.8], proved by interpreting the integers with addition and multiplication in the automorphism group of a totally-ordered structure.

The structure of the automorphism group of a totally-ordered set is relevant to the structure of automorphism groups of structures containing additional relations or functions. For example, one connection between such groups and the automorphism groups of totally-ordered structures in expanded languages is provided by two propositions, one stating that for recursively saturated models $\mathcal{M}$ of Peano arithmetic, the automorphism group $\mathrm{Aut}(\mathbb{Q}, <)$ embeds into $\mathrm{Aut}(\mathcal{M})$, and the other stating that $\mathrm{Aut}(\mathcal{M})$ embeds onto a dense subgroup of $\mathrm{Aut}(\mathbb{Q}, <)$ [7, Propositions 9.5.1, 9.5.2].

Another way in which automorphism groups have been studied is by examining how theories change if they are expanded by an automorphism. For example, Laskowski and

Pal [8] showed the existence of a model companion to the theory of divisible ordered abelian groups together with an automorphism satisfying extra properties, while in general there is no model companion to the theory of divisible ordered abelian groups with an automorphism.

This dissertation examines the theory of a nonstandard model of Presburger arithmetic expanded by a specific automorphism with nice properties, in the sense that in the expanded theory we still have quantifier elimination, axiomatizability, and a number of other model-theoretic properties that are similar to the properties of Presburger arithmetic itself. The automorphism by which we expand Presburger arithmetic – which we term *modest* – is increasing on all positive elements, and moves elements that are not fixed to "nearby" elements, in a sense to be made precise below. Open questions for further research include which expansions of the theory of Presburger arithmetic by other automorphisms also yield similar nice properties, and whether expansions by other automorphisms give, for example, theories with the independence property, or theories with arbitrarily large finite DP-rank.

After introducing basic definitions useful in working with models of Presburger arithmetic (standard parts and magnitude classes) in this chapter, in chapter 2 we construct a specific automorphism σ of a countable model $\mathcal{M}$ of Presburger arithmetic; this is the automorphism by which we will expand the theory of such a model. The method of construction also yields a related automorphism, which we also denote by σ, of the divisible ordered abelian group obtained by forming the quotient of $\mathcal{M}$ by $\mathbb{Z}$. We also construct a maximal automorphism τ of both $\mathcal{M}$ and its quotient which may be useful in further research providing a contrast to the "nice" properties of the expansion by σ.

In chapter 3, we record collections of axioms T and T^* that are satisfied by the Presburger structure and its quotient, both expanded by σ, respectively. In chapters 4 and 5, we prove quantifier elimination for both expanded theories. The steps involved are similar, though more involved for the expanded Presburger theory because of additional predicates we define in the language. Then, in chapters 6 and 7, we describe the definable sets in both expanded theories, which, in chapter 8, allows us to find the DP-rank of both. Finally, in chapter 9, we conclude by discussing a couple of areas of related future research.

1.1 The theory of Presburger arithmetic

Mojzesz Presburger proved quantifier elimination for the theory of Presburger arithmetic in 1929 by adding divisibility predicates $P_n(n = 2, 3, \ldots)$ to the language $(<, +, 0, 1)$, and his proof was published in 1930 [12]. Skolem gave a different proof in 1930 by expanding the language $(<, +, 0, 1)$ with rational multipliers q $(q \in \mathbb{Q})$ instead of by the divisibility predicates P_n; his proof was published in 1931 [14]. Because both the construction of a modest increasing automorphism and the proof of quantifier elimination will be implemented in the language with divisibility predicates, we start with the axioms of Presburger arithmetic in the language $\mathcal{L}_{Pr} := (+, <, 0, 1, P_n(n = 2, 3, \ldots))$.
Following Marker [11, p. 82], the axioms of the theory of Presburger arithmetic, denoted Pr, which is the theory of $\mathbb{Z}$ with the usual ordering, addition, constants 0 and 1, and predicates for divisibility by positive integers greater than 1, are as follows. (We abbreviate $\underbrace{1 + \cdots + 1}_{k \text{ times}}$ by k, and we abbreviate $\underbrace{y + \cdots + y}_{k \text{ times}}$ by ky.)

1. the axioms for a discrete ordered abelian group, with identity 0
2. $0 < 1$
3. (scheme, for $n = 2, 3, \ldots$) $\forall x (P_n(x) \leftrightarrow \exists y (x = ny))$
4. (scheme, for $n = 2, 3, \ldots$) $\forall x \bigvee_{i=0}^{n-1} \left(P_n(x + i) \wedge \bigwedge_{j=0, j \neq i}^{n-1} \neg P_n(x + j) \right)$

By virtue of axiom (4), for each element of a model $\mathcal{M} \models \Pr$ and each predicate P_n, there is a unique $i \in \{0, \ldots, n-1\}$ such that $\mathcal{M} \models P_n(x+i)$; this i is the **residue of x mod n**, and is a useful starting point for reviewing the structure of such models and the structure of the quotient of such models by $\mathbb{Z}$.

1.2 Basic definitions

In this section, we give basic definitions used to describe Presburger arithmetic and its automorphisms; we will use these in constructing a modest increasing automorphism. Throughout this subsection, $\mathcal{M}$ is a countable model of Presburger arithmetic. All of the definitions and theorems are contained in Harnik [5] and Llewellyn-Jones[9].

To any element $a \in M$, we can associate the sequence of residues modulo n:

Definition 1. Let $a \in M$. The *divisibility type* $\rho(a)$ of a is the sequence $(r_2, r_3, \ldots, r_n, \ldots)$ where r_n is the residue of a mod n.

We use the following repeatedly below:

Definition 2. For any $a \in M$, the *$\mathbb{Z}$-chain containing* a is the set $\{a+k | k \in \mathbb{Z}\}$. Thus two elements $a, b \in M$ are said to be in the same $\mathbb{Z}$-chain if $a - b \in \mathbb{Z}$.

Divisibility type for $\mathbb{Z}$-chains. Now, given $a \in M$, the divisibility type of any $b \in M$ that is in the same $\mathbb{Z}$-chain as a is determined by the divisibility type of a. That is, if

$\rho(a) = (r_2, r_2, \ldots)$ and $b - a = m (m \in \mathbb{Z})$, then

$\rho(b) = (r_2 + m \pmod 2, r_3 + m \pmod 3, \ldots)$. So the set of divisibility types of a single $\mathbb{Z}$-chain is determined by the divisibility type of any element in that $\mathbb{Z}$-chain. Following Harnik, we can define the *color* of a $\mathbb{Z}$-chain A, denoted $c[A]$, as the set of divisibility types of elements of A. We can enumerate the colors of $\mathcal{M}$ as $c_0, c_1, \ldots$ (fixing c_0 as the color of the standard part $\mathbb{Z}$).

The quotient structure $\mathcal{M}/\mathbb{Z}$. The set of all $\mathbb{Z}$-chains in $\mathcal{M}$ forms a divisible ordered abelian group that is useful in constructing the modest increasing automorphism, and is interesting in its own right. Indeed, because the standard integers $\mathbb{Z}$ form a subgroup of $\mathcal{M}$ (which is itself an abelian group), it is clear $\mathcal{M}/\mathbb{Z}$ is an abelian group, whose elements are the collapsed $\mathbb{Z}$-chains of $\mathcal{M}$. This quotient group inherits the ordering and addition from $\mathcal{M}$ in the natural way. It is a divisible group as well: given any $\mathbb{Z}+a$ in $\mathcal{M}/\mathbb{Z}$ and any $n \in \{2, 3, \ldots\}$, axiom (3) for Presburger arithmetic says that there is a $\mathbb{Z}+b$ and an $i \in \{0, \ldots, n-1\}$ such that $\mathbb{Z}+a = n(\mathbb{Z}+b)+i = n(\mathbb{Z}+b)$. For elements a^* in the quotient, we can define the *unary color predicates* $C_0, C_1, \ldots$ by $C_i(a^*) \Leftrightarrow c[A] = c_i$, where $c[A]$ is the color of the $\mathbb{Z}$-chain whose image in the quotient is a^*. In the construction of the modest automorphism, we will use only the single color predicate C_0.

The next two definitions are essential for constructing the modest automorphism. The first definition allows us to describe elements whose absolute values are within rational multiples of each other:

Definition 3. Suppose $a, b \in M/\mathbb{Z}$. Then a and b are in the same *magnitude class* iff there are positive rational numbers q, r such that $q|a| < |b| < r|a|$. The property of being in the same magnitude class is clearly an equivalence relation; and we denote the equivalence class of an element a by $v(a)$, the *value* (or magnitude class) of a. Thus, we also say that a and b have the same value, and write $v(a) = v(b)$. For $a, b \in M$, a and b are in the same magnitude class and have the same value iff their images under the quotient homomorphism $\alpha : M \to M/\mathbb{Z}$ are in the same magnitude class in $M/\mathbb{Z}$.

The automorphism we will construct below moves no element outside its magnitude class, and is termed a *value-preserving* automorphism. The next definition provides a way to measure how far apart elements are within a magnitude class.

The *standard part* of two elements is a measure of how far apart the elements are, and is also defined for both $\mathcal{M}$ and $\mathcal{M}/\mathbb{Z}$:

Definition 4. Let $\mathcal{M} \models \mathrm{Pr}$.

(1) For $0 < a, b \in M$ both nonstandard and in the same magnitude class, the *standard part of a over b* is

$$\mathrm{st}\left(\frac{a}{b}\right) = \sup\{q : qb < a, 0 < q \in \mathbb{Q}\}.$$

(2) For a, b in the same magnitude class and $a < 0 < b$,

$$\mathrm{st}\left(\frac{a}{b}\right) = -\mathrm{st}\left(\frac{|a|}{b}\right).$$

(3) For $0 < a < b$ in distinct magnitude classes,

$$\mathrm{st}\left(\frac{a}{b}\right) = 0 \text{ and } \mathrm{st}\left(\frac{b}{a}\right) = \infty.$$

This definition extends naturally to a definition of standard part for pairs of elements of the quotient $M/\mathbb{Z}$:

Definition 5. (1) For $0 < a, b \in M$

$$\mathrm{st}\left(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\right) = \mathrm{st}\left(\frac{a}{b}\right)$$

(2) For $a, b \in M$ and $b \neq 0$,

$$\mathrm{st}\left(\frac{\mathbb{Z}}{\mathbb{Z}+b}\right) = 0$$

and

$$\mathrm{st}\Big(\frac{\mathbb{Z}+a}{\mathbb{Z}}\Big) = \begin{cases} \infty, & \text{if } a > \mathbb{Z} \\ 0, & \text{if } a \in \mathbb{Z} \\ -\infty & \text{if } a < \mathbb{Z} \end{cases}$$

and

$$\mathrm{st}\Big(\frac{\mathbb{Z}-a}{\mathbb{Z}+b}\Big) = -\mathrm{st}\Big(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\Big) = \mathrm{st}\Big(\frac{\mathbb{Z}+a}{\mathbb{Z}-b}\Big)$$

and

$$\mathrm{st}\Big(\frac{\mathbb{Z}-a}{\mathbb{Z}-b}\Big) = \mathrm{st}\Big(\frac{\mathbb{Z}+a}{\mathbb{Z}+b}\Big).$$

The following proposition allows us to compute standard parts just as the notation suggests:

Proposition 1. *[5, Lemma 11] [9, Lemma 8.1.5] For $a, b, c \in M/\mathbb{Z}$, $q \in \mathbb{Q}$, the following hold, so long as multiplication is not of the form $0 \cdot \pm\infty$ and addition is not of the form $\infty + (-\infty)$:*

1. $\mathrm{st}\big(\frac{a}{b}\big) \cdot \mathrm{st}\big(\frac{b}{c}\big) = \mathrm{st}\big(\frac{a}{c}\big)$

2. $\mathrm{st}\big(\frac{qa}{b}\big) = q \cdot \mathrm{st}\big(\frac{a}{b}\big)$

3. $\mathrm{st}\big(\frac{a+b}{c}\big) = \mathrm{st}\big(\frac{a}{c}\big) + \mathrm{st}\big(\frac{b}{c}\big)$

4. if $c > 0$ and $0 \leq a \leq b$, then $\mathrm{st}\big(\frac{a}{c}\big) \leq \mathrm{st}\big(\frac{b}{c}\big)$

5. if $\mathrm{st}\big(\frac{a}{b}\big) \notin \{0, \pm\infty\}$, then $\mathrm{st}\big(\frac{a}{b}\big) = \mathrm{st}\big(\frac{b}{a}\big)^{-1}$.

For purposes of constructing a modest automorphism, the following definition, applicable to both $\mathcal{M}$ and $\mathcal{M}/\mathbb{Z}$, based on the definition of standard part is useful:

Definition 6. We say that a is *near* b if

$$\mathrm{st}\left(\frac{a}{b}\right) = 1.$$

Equivalently, a is near b if they are in the same magnitude class and $|a - b|$ is in a lower magnitude class.

We can now define a modest automorphism:

Definition 7. An automorphism g of either $\mathcal{M}$ or $\mathcal{M}/\mathbb{Z}$ is *modest* if a is near $g(a)$ for all $a \in M$ (or $a \in M/\mathbb{Z}$).

1.3 Main results

Theorem 2. Let $\mathcal{M} \models Pr$ be a countable, pseudo-recursively saturated model. Then there is an automorphism σ of $\mathcal{M}$ satisfying the following:

(1) the fixed point set F of σ is a convex, dense set of magnitude classes containing the standard integers;

(2) σ is modest and strictly increasing on the positive part of $M \setminus F$; and

(3) the $\mathbb{Z}$-chains containing elements of the set of differences $D = \{x|\exists w(\sigma(w) - w = x)\}$ are dense and co-dense in the $\mathbb{Z}$-chains in F.

Theorem 3. Let $\mathcal{M} \models Pr$ be a countable, pseudo-recursively saturated model. Then there is an automorphism τ of $\mathcal{M}$ satisfying the following:

(1) the fixed point set F of τ is $\mathbb{Z}$;

(2) τ is modest and strictly increasing on the positive part of M above $\mathbb{Z}$; and

(3) the set of $\mathbb{Z}$-chains containing an element of the set of differences $D = \{x|\exists w(\tau(w) - w = x\}$ is dense in the $\mathbb{Z}$-chains in M.

Theorem 4. Let $\mathcal{M}^* \models T^*$, where T^* is the set of sentences specified in Section 3.1, and let $\phi(x, \bar{y})$ be a quantifier-free formula that is a conjunction of literals in the language

$\mathcal{L}_q = (+, -, \sigma, \sigma^{-1}, <, D, 0, q \text{ (for } q \in \mathbb{Q}))$. Then there is a quantifier-free formula θ such that $\mathcal{M}^* \models \exists x \phi(x, \bar{y}) \leftrightarrow \theta(\bar{y})$.

Theorem 5. Let $\mathcal{M} \models T$, where T is the set of sentences specified in section 3.2, and let $\phi(x, \bar{y})$ be a quantifier-free formula that is a conjunction of literals in the language $\mathcal{L} = (+, -, \sigma, \sigma^{-1}, <, P_n \ (n = 2, 3, \ldots), D, D^+, D^-, Z, 0, 1)$. Then there is a quantifier-free formula θ such that $\mathcal{M} \models \exists x \phi(x, \bar{y}) \leftrightarrow \theta(\bar{y})$.

Corollaries 2-3. The theories T^* and T are complete and decidable.

Theorems 11-12. The theories of the quotient model $\mathcal{M}/\mathbb{Z}$ and $\mathcal{M}$ have DP-rank 2.

Chapter 2

Construction of a modest increasing automorphism

The properties of the specific modest automorphism σ of a countable, sufficiently saturated model $\mathcal{M}$ of Presburger arithmetic we will construct are:

1. σ is *modest*, that is, for all $a \in M$, $\mathrm{st}\frac{\sigma(a)}{a} = 1$;

2. σ fixes a convex set F properly containing the set of standard integers and closed under addition and additive inverses;

3. σ is increasing for all positive a, and strictly increasing for all positive a above F: $0 \leq a \rightarrow a \leq \sigma(a)$ and $F < a \rightarrow a < \sigma(a)$; and

4. the set $D = \{w : \exists x(\sigma(x) - x = w)\}$ of *differences* of σ is a subset of F, and the $\mathbb{Z}$-chains meeting D are dense and codense in the $\mathbb{Z}$-chains of F.

2.1 Significant results used in the construction

The specific definitions and results used in the construction are from [5] and [9]. We construct σ by the back-and-forth method, first on the quotient group $\mathcal{M}/\mathbb{Z}$, and then lifting that automorphism to an automorphism of $\mathcal{M}$ itself, via the following proposition:

Proposition 2. *[9, Proposition 3.2.3] Let $\alpha : \mathcal{M}/\mathbb{Z} \to \mathcal{M}/\mathbb{Z}$ be an automorphism that preserves color predicates. Then α lifts to an automorphism of $\mathcal{M}$.*

Remark. Because we are constructing a modest automorphism, given any element $a \in M/\mathbb{Z}$ such that $C_i(a)$, its image $\sigma(a)$ must also have color C_i. So the difference $d = \sigma(a) - a$ must have color C_0 (the color of the standard part). In addition, for $\mathrm{st}\left(\frac{\sigma(a)}{a}\right) = 1$, d must be in a magnitude class lower than the magnitude class of a in the ordering of magnitude classes. The model for which we construct σ is *pseudo-recursively saturated*, to be defined below. Pseudo-recursive saturation is weaker than and implied by recursive saturation, and extracts specific features of a recursively saturated model of Presburger arithmetic that facilitate the construction of automorphisms.

Definition 8. [9, Definition 9.3.5] A countable model $\mathcal{M}$ of Presburger arithmetic is *pseudo-recursively saturated* if $\mathcal{M} \neq \mathbb{Z}$ and

1. (PRS1) the colors are dense in $\mathcal{M}/\mathbb{Z}$;
2. (PRS2) for $x, y, z \in \mathcal{M}/\mathbb{Z}$ with $z \neq 0$, there is some $w \neq 0$ for which $\mathrm{st}(w/z) = \mathrm{st}(x/y)$; and
3. (PRS3) the set of magnitude classes in $\mathcal{M}/\mathbb{Z}$ is a dense linear order with respect to the ordering $<$ with least element 0 and no greatest element.

As is shown in [5], recursive saturation implies each of the three conditions PRS1-PRS3. On the other hand, a pseudo-recursively saturated model is not necessarily recursively saturated: a recursively saturated model realizes every recursive divisibility type, and the standard parts occurring in a recursively saturated model include every recursive element of $\mathbb{R}$ [9, Lemma 9.2.3], but a pseudo-recursively saturated model need not.

In a countable, pseudo-recursively saturated model of Presburger arithmetic, we define a stronger form of linear independence of elements that is essential to constructing automorphisms.

Definition 9. [9, Definition 8.2.1] A subset B of $M/\mathbb{Z}$ is *strongly independent* if $0 \notin B$ and

$$\left\{\mathrm{st}\left(\frac{b}{a}\right) : b \in B\right\} \setminus \{0, \pm\infty\}$$

is linearly independent over $\mathbb{Q}$ for all $a \in M/\mathbb{Z}$. A subset $B \subset M$ is strongly independent if it contains at most one representative of each coset of $\mathbb{Z}$, and these cosets satisfy the definition of strongly independent for $M/\mathbb{Z}$.

Two propositions about strong independence are particularly useful in establishing that sets are strongly independent.

Proposition 3. *[9, Lemma 8.2.5] $B \subseteq M/\mathbb{Z}$ is strongly independent if and only if $0 \notin B$ and every nontrivial $\mathbb{Q}$-linear combination*

$$a = q_1 b_1 + \cdots + q_n b_n$$

has value

$$v(a) = max\{v(b_j) : 1 \leq j \leq n, q_j \neq 0\}$$

where $\{q_1, \ldots, q_n\} \subset \mathbb{Q}$ and $\{b_1, \ldots, b_n\} \subseteq B$.

Proposition 4. *[9, Proposition 8.2.7] For $b_1, b_2 \in M/\mathbb{Z}$ the set $\{b_1, b_2\}$ is strongly independent if and only if $st(\frac{b_1}{b_2}) \notin \mathbb{Q}$.*

The theorem used to construct σ is the following; the proof is in [9].

Theorem 1. *[9, Theorem 10.1.1] Suppose $\mathcal{M}$ is a countable pseudo-recursively saturated model of Presburger arithmetic, and that we have strongly independent subsets of $\mathcal{M}$*

$$A = \{a_1, a_2, \dots, a_n\}, \quad B = \{b_1, b_2, \dots, b_n\}$$

which satisfy the following:

1. *$\rho(a_i) = \rho(b_i)$ for $1 \leq i \leq n$, where $\rho(x)$ is the divisibility type of x; and*
2. *$st\left(\frac{a_i}{a_j}\right) = st\left(\frac{b_i}{b_j}\right)$*

Then there exists an automorphism $\theta : \mathcal{M} \to \mathcal{M}$ which maps a_i to b_i for $1 \leq i \leq n$.

The requirements of this theorem will be satisfied by the lift of a back-and-forth construction on the quotient structure.

2.2 The back-and-forth construction of σ

We begin with a pseudo-recursively saturated model $\mathcal{M} \models \mathrm{Pr}$, and choose an initial positive nonstandard segment I of its quotient $\mathcal{M}/\mathbb{Z}$ closed under addition. I will be the nonnegative part of the fixed point set F of the automorphism σ. Furthermore, we choose I so that it contains no last magnitude class, and so that $M/\mathbb{Z} \setminus I$ has no first magnitude class; that is, we consider the positive nonstandard segment of $\mathcal{M}/\mathbb{Z}$ as a dense linear order of magnitude classes isomorphic to the order of $\mathbb{Q}$, and choose I so that it corresponds to an irrational cut in $\mathbb{Q}$. More formally, we choose an isomorphism β from the dense linear order of positive rational numbers $\mathbb{Q}^+$ to the dense linear order of positive magnitude classes of $\mathcal{M}/\mathbb{Z}$, and choose any irrational cut r in $\mathbb{Q}^+$. Then $I = \{m|\beta(m) < r\}$ for all magnitude classes m.

We then have two dense linear orders of magnitude classes without endpoints: I and $\mathcal{M}/\mathbb{Z} \setminus I$. We fix an isomorphism $\alpha : \mathcal{M}/\mathbb{Z} \setminus I \to I$ of these two dense linear orders of magnitude classes. We will use α in the construction of the automorphism σ to ensure that the set of differences $D = \{x|\exists w(\sigma(w) - w = x)\}$ is dense in F.

We next fix an enumeration $a'_1, a'_2, \ldots$, of all the positive elements of $\mathcal{M}/\mathbb{Z}$, and construct the automorphism using this enumeration.

At stage m of the back-and-forth construction (when we have considered a'_m) we obtain sets $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $D = \{d_1, \ldots, d_n\}$, of positive elements which satisfy the following conditions:

1. the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are strongly independent;

2. for each $i, j \in \{1, \ldots, n\}$, $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$;

3. for each $i \in \{1, \ldots, n\}$, the color of a_i is the same as the color of b_i (this ensures that the automorphism is a residue automorphism);

4. for each $i \in \{1, \ldots, n\}$, if $a_i \notin I$ (that is, a_i is not in the fixed-point set), then $a_i < b_i$; in particular, we require that $b_i - a_i = d_i \in I$ (this ensures that the automorphism is increasing for positive elements above the fixed point set F, and that differences are contained in F); for each $a_i \in I$, we require that $a_i = b_i$;

5. for each $i \in \{1, \ldots, n\}$, $\mathrm{st}\left(\frac{b_i}{a_i}\right) = 1$;

6. for any pair $\{a_{j_0}, a_{j_1}\} \subseteq \{a_1, \ldots, a_n\}$,
   if $v(a_{j_0}) < v(a_{j_1})$ and $a_{j_0}, a_{j_1} \notin I$, then $v(d_{j_0}) < v(d_{j_1})$; recall that $v(a)$ is the magnitude class of a;

7. for any pair $\{a_{j_0}, a_{j_1}\} \subseteq \{a_1, \ldots, a_n\}$,
   if $v(a_{j_0}) = v(a_{j_1})$, then $v(d_{j_0}) = v(d_{j_1})$ and

$$\operatorname{st}\left(\frac{a_{j_t}}{a_{j_u}}\right) = \operatorname{st}\left(\frac{d_{j_t}}{d_{j_u}}\right) \in \mathbb{R} \setminus \mathbb{Q}$$

   for all $j_t, j_u \in \{j_0, \ldots, j_s\}$;

8. for any $a_i \in A \setminus I, b_i \in B$ and difference $d_i = a_i - b_i$, $v(d_i) = \alpha(v(a_i))$, where α is the isomorphism between magnitude classes of $\mathcal{M}/\mathbb{Z} \setminus I$ and magnitude classes of I; and

9. the elements $a'_1, a'_2, \ldots, a'_m$ are in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n\}$.

Conditions (6) and (7) will ensure that the automorphism we construct will be increasing not only on the elements of the strongly independent sets, but also on linear combinations of these elements. Condition (8) will ensure that $D = \{d_1, d_2, \ldots\}$ will be dense in F. Finally, condition (9) ensures that the set A will span the entire structure once we have completed the construction.

2.2.1 Forth

We begin by setting $a_1 = a'_1$, the first element in the enumeration. If $a_1 \in I$, we put $b_1 = a_1$ and $d_1 = 0$. If not, by PRS3, we can find $c_1 \in I$ such that $v(0) \neq v(c_1)$ and $v(c_1) = \alpha(v(a_1))$. Clearly $a_1 + c_1 \neq a_1$, so by PRS1 we can find $0 < d_1 < c_1$ such that the color of $(a_1 + d_1)$ is the same as the color of a_1, and such that $v(d_1) = v(c_1)$. Note that $\mathcal{M}/\mathbb{Z} \models C_0(d_1)$. Set $b_1 = a_1 + d_1$. To see that a_1 and b_1 satisfy the conditions above, observe that:

1. The singletons a_1 and b_1 are trivially strongly independent.

2. Clearly $\operatorname{st}\left(\frac{a_1}{a_1}\right) = 1 = \operatorname{st}\left(\frac{b_1}{b_1}\right)$

3. The colors of a_1 and b_1 are equal by construction.

4. If $a_1 \notin I$, then $a_1 < a_1 + d_1 = b_1$ because $0 < d_1$.

5. $\mathrm{st}\left(\frac{b_1}{a_1}\right) = \mathrm{st}\left(\frac{a_1+d_1}{a_1}\right) = \mathrm{st}\left(\frac{a_1}{a_1}\right) + \mathrm{st}\left(\frac{d_1}{a_1}\right) = 1 + 0 = 1$, since $v(d_1) < v(a_1)$.

Conditions (6)-(9) are obviously satisfied.

Next, suppose that at stage $(m-1)'$ and for some $n < m$ we have sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ meeting the conditions. Let a'_m be the next element in the enumeration. If a'_m is in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n\}$, we leave the sets as they are.

If not, then we use the following lemma (the Exchange Lemma) to find a new strongly independent element a_{n+1} such that a'_m is in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n, a_{n+1}\}$:

Lemma 1. *[9, Lemma 8.2.10] If $a_1, \ldots, a_n$ are strongly independent in $\mathcal{M}/\mathbb{Z}$ and $a \in \mathcal{M}/\mathbb{Z}$, then either $a \in < a_1, \ldots, a_n >$ or there is a_{n+1} such that $a_1, \ldots, a_n, a_{n+1}$ are strongly independent and $a_{n+1} \in < a_1, \ldots, a_n, a >$ and $a \in < a_1, \ldots, a_n, a_{n+1} >$.*

If $a_{n+1} \in I$, we put $b_{n+1} = a_{n+1}$. If $a_{n+1} \notin I$, there are two cases, depending on whether the magnitude class of a_{n+1} is already represented among $a_1, \ldots, a_n$, or not.

Case 1: $v = v(a_{n+1}) = v(a_i)$ for some $i \in \{1, \ldots, n\}$. In this case, let $a_{j_1}, \ldots, a_{j_r} \in \{a_1, \ldots, a_n\}$ be those elements of magnitude v.

Now let

$$\mathrm{st}\left(\frac{a_{n+1}}{a_{j_s}}\right) = r_s \text{ for } j_s \in \{j_1, \ldots, j_r\}.$$

These standard parts order the elements of A of magnitude v within their magnitude class.

Now either a_{n+1} is between two elements with magnitude v, or not.

Suppose first that a_{n+1} is less than any of the elements of A of magnitude v, and let a_j be the next largest element, with $\mathrm{st}\left(\frac{a_{n+1}}{a_j}\right) = r < 1$. By PRS2, there is c_{n+1} such that

$\mathrm{st}\left(\frac{c_{n+1}}{d_j}\right) = r$, and this implies also that $v(c_{n+1}) = v(d_j)$. Then by PRS1, there is d_{n+1} near c_{n+1} such that the color of $(a_{n+1} + d_{n+1})$ is the same as the color of (a_{n+1}). Because d_{n+1} is near c_{n+1}, $\mathrm{st}\left(\frac{d_{n+1}}{d_j}\right) = r$ also. Put $b_{n+1} = a_{n+1} + d_{n+1}$. The case where a_{n+1} is greater than any of the elements in A of magnitude v is the same, except that $\mathrm{st}\left(\frac{a_{n+1}}{a_j}\right) > 1$. The case where a_{n+1} is between two elements is almost the same; we simply find d_{n+1} using either of the elements in A of the same magnitude class that is adjacent to a_{n+1} in the ordering established by the standard parts.

Case 2: $v(a_{n+1}) \neq v(a_i)$ for any $i \in \{1, \ldots, n\}$. There are three subcases.

(1) If $v(a_{n+1}) < v(a_i)$ for all $i \in \{1, \ldots, n\}$, by PRS 3 we can find $c_{n+1} \in I$ such that $v(c_{n+1}) = \alpha(v(a_{n+1}))$. Because α is an order-preserving isomorphism on magnitude classes, $v(c_{n+1}) < v(d_i)$ for all $d_i \in D$. By PRS1, we can find d_{n+1} such that $v(d_{n+1}) = v(c_{n+1})$ and such that the color of $(a_{n+1} + d_{n+1})$ is the same as the color of (a_{n+1}). Put $b_{n+1} = a_{n+1} + d_{n+1}$.

(2) Similarly, if $v(a_{n+1}) > v(a_i)$ for all $i \in \{1, \ldots, n\}$, by PRS 3 we can find $c_{n+1} \in I$ such that $v(c_{n+1}) = \alpha(v(a_{n+1}))$, and then $v(c_{n+1}) > v(d_i)$ for all $d_i \in D$. By PRS1, we can find d_{n+1} such that $v(d_{n+1}) = v(c_{n+1})$ and such that the color of $(a_{n+1} + d_{n+1})$ is the same as the color of (a_{n+1}). Put $b_{n+1} = a_{n+1} + d_{n+1}$.

(3) If $v(a_j) < v(a_{n+1}) < v(a_{j'})$ for $j, j' \in \{1, \ldots, n\}$ such that $v(a_j)$ and $v(a_{j'})$ are consecutive in the ordering of the magnitude classes of A, by PRS 3 we can find $c_{n+1} \in I$ such that $v(c_{n+1}) = \alpha(v(a_{n+1}))$, and then $v(d_j) < v(c_{n+1}) < v(d_{j'})$. By PRS1, we can find d_{n+1} such that $v(d_{n+1}) = v(c_{n+1})$ and such that the color of $(a_{n+1} + d_{n+1})$ is the same as the color of $= (a_{n+1})$. Put $b_{n+1} = a_{n+1} + d_{n+1}$.

In each case, the element d_{n+1} we add to a_{n+1} has zero residue, that is, $\mathcal{M}/\mathbb{Z} \models C_0(d_{n+1})$. We now check the conditions on the sets A and B with the new elements a_{n+1} and b_{n+1} included, respectively:

1. Because we used the Exchange Lemma, the set $\{a_1, \dots, a_{n+1}\}$ is strongly independent. To see that the set $\{b_1, \dots, b_{n+1}\}$ is also strongly independent, suppose not. Then there is some $\gamma \in \mathcal{M}/\mathbb{Z}$ such that the set of standard parts

$$\left\{\mathrm{st}\left(\frac{b_1}{\gamma}\right), \mathrm{st}\left(\frac{b_2}{\gamma}\right), \dots, \mathrm{st}\left(\frac{b_{n+1}}{\gamma}\right)\right\} \setminus \{0, \infty\} = \left\{\mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right), \dots, \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right)\right\}$$

is not linearly independent over $\mathbb{Q}$. So there are $q_1, \dots, q_m \in \mathbb{Q}$ not all equal to 0 such that

$$q_1 \mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right) + \cdots + q_m \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right) = 0$$

But then

$$q_1 \mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right)\mathrm{st}\left(\frac{a_{i_1}}{b_{i_1}}\right) + \cdots + q_m \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right)\mathrm{st}\left(\frac{a_{i_m}}{b_{i_m}}\right) = 0$$

because $\mathrm{st}\left(\frac{a_i}{b_i}\right) = 1$ for $i \in \{1, \dots, n+1\}$. Using property 1 of Proposition 1, we have:

$$q_1 \mathrm{st}\left(\frac{a_{i_1}}{\gamma}\right) + \cdots + q_m \mathrm{st}\left(\frac{a_{i_m}}{\gamma}\right) = 0$$

This contradicts the strong independence of $\{a_1, \dots, a_{n+1}\}$.

2. $\mathrm{st}\left(\frac{b_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{a_{n+1}+d_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_{n+1}}\right) + \mathrm{st}\left(\frac{d_{n+1}}{a_{n+1}}\right) = 1 + 0 = 1$, since $v(d_{n+1}) < v(a_{n+1})$.

3. The colors of a_{n+1} and b_{n+1} are equal by construction.

4. If $a_{n+1} \notin I$, then $a_{n+1} < a_{n+1} + d_{n+1} = b_{n+1}$ because $0 < d_{n+1}$.

5. For $i \in \{1, \dots, n\}$,

$$\mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) \cdot \mathrm{st}\left(\frac{a_{n+1}}{b_{n+1}}\right) \cdot \mathrm{st}\left(\frac{b_i}{a_i}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_i}\right)$$

because $\mathrm{st}\left(\frac{a_{n+1}}{b_{n+1}}\right) = \mathrm{st}\left(\frac{b_i}{a_i}\right) = 1$.

Conditions (6) and (7) are clearly satisfied by the construction. Condition (8) is satisfied by the construction as well because we used α to locate differences for elements of $A \setminus I$. Condition (9) is satisfied because, by use of the Exchange Lemma, a'_m is in the $\mathbb{Q}$-span of the $\{a_1, \ldots, a_n, a_{n+1}\}$ (or of $\{a_1, \ldots, a_n\}$ in the case where a'_m is already in the $\mathbb{Q}$-span we started with at stage m.)

The back step is essentially the same, except that, given a new strongly independent element b_{m+1}, we subtract the element d_{m+1} using the same criteria above.

Next, we need to verify that the back-and-forth enumerations produce an automorphism σ of $\mathcal{M}/\mathbb{Z}$ with the required properties. So let σ be the map defined by the back-and-forth construction, so $\sigma : a_i \mapsto b_i$ for all $i \in \omega$.

First, σ is well-defined. Let $a \in \mathcal{M}/\mathbb{Z}$. Since the a'_i's enumerate all the elements of $\mathcal{M}/\mathbb{Z}$, $a = a'_m$ for some $m \in \omega$. By the construction of the back-and-forth sets, we know that a'_m is in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_r\}$ for some $r \leq m$, so

$$a'_m = q_1 a_1 + q_2 a_2 + \cdots + q_r a_r$$

for some $q_1, \cdots, q_r$. So the mapping of a'_m under σ is determined by the mapping of the a_i's by σ, that is,

$$\sigma : a'_m \mapsto q_1 b_1 + \cdots + q_r b_r.$$

Second, by Lemma 8.2.11 in [9], because the colors of (a_i) and (b_i) are the same and $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$ for all $i, j \in \{1, \ldots, r\}$, and because the sets $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ are strongly independent, $\mathrm{tp}(a_1, \ldots, a_r) = \mathrm{tp}(b_1, \ldots, b_r)$; hence $\mathrm{tp}(a'_m) = \mathrm{tp}(\sigma(a'_m))$. So σ is an automorphism so long as it is bijective.

Surjectivity: for any $\gamma \in M/\mathbb{Z}$, by the back step, there are $b_1, \ldots, b_t$ in the enumeration such that for some $q_1, \ldots, q_t \in \mathbb{Q}$, $\gamma = q_1 b_1 + \cdots + q_t b_t$. Thus γ is the image under σ of $q_1 a_1 + \cdots + q_t a_t$.

Injectivity: Suppose $\gamma_r \neq \gamma_s$. In our enumeration of the elements of $\mathcal{M}/\mathbb{Z}$, assume γ_s comes later, say as γ_m. Then for some $q_{r_1}, \ldots, q_{r_m}, q_{s_1}, \ldots, q_{s_m}$, with $q_{r_j} \neq q_{s_j}$ for some j, we have
$\gamma_r = q_{r_1} a_1 + \cdots + q_{r_m} a_m$ and
$\gamma_s = q_{s_1} a_1 + \cdots + q_{s_m} a_m$. Hence

$$\sigma(\gamma_r) = q_{r_1} b_1 + \cdots + q_{r_m} b_m \neq q_{s_1} b_1 + \cdots + q_{s_m} b_m = \sigma(\gamma_s)$$

because the b_i's are linearly independent.

Next, we need to show that σ satisfies the conditions that elements outside the fixed-point set I are near their automorphic images, and that elements in the fixed-point set are indeed fixed.

First, suppose $0 < a = q_1 a_1 + \cdots + q_n a_n$, with $a_1, \ldots, a_n$ in the strongly independent set A determined by the construction, $a \in I$, and all coefficients nonzero. Then by Proposition 3, $v(a) = \max\{v(a_j) : 1 \leq j \leq n\}$. Thus, each $a_1, \ldots, a_n \in I$, since a lies below the magnitude classes not fixed by σ. But then, by the construction of σ, $a_1, \ldots, a_n$ are all fixed by σ. Hence a is also fixed by σ.

Second, let $0 < a = q_1 a_1 + \cdots + q_n a_n$ for some $q_1, \ldots, q_n \in \mathbb{Q}$, and $a_1, \ldots, a_n \in A$ with $a \in M/\mathbb{Z} \setminus I$. Since at least one of the a_i's is moved by σ, we can group the terms in this sum as follows

$$a = q_{f_1} a_{f_1} + \cdots + q_{f_j} a_{f_j} + q_{g_1} a_{g_1} + \cdots + q_{g_k} a_{g_k}$$

where $a_{f_1}, \ldots, a_{f_j}$ are fixed by σ, $a_{g_1}, \ldots, a_{g_k}$ are moved by σ, $j + k = n$, $k \geq 1$ and j is possibly 0.

Then

$$\sigma(a) = \sigma(q_1 a_1 + \cdots + q_n a_n)$$
$$= q_{f_1} a_{f_1} + \cdots + q_{f_j} a_{f_j} + q_{g_1}(a_{g_1} + d_{g_1}) + \cdots + q_{g_k}(a_{g_k} + d_{g_k})$$
$$= a + q_{g_1} d_{g_1} + \cdots + q_{g_k} d_{g_k}.$$

By the construction and Proposition 3, $v(a) > v(d_i)$ for $d_1, \ldots d_n$; hence

$$\operatorname{st}\Big(\frac{\sigma(a)}{a}\Big) = \operatorname{st}\Big(\frac{a}{a}\Big) + \operatorname{st}\Big(\frac{q_{g_1} d_{g_1}}{\gamma}\Big) + \cdots + \operatorname{st}\Big(\frac{q_{g_k} d_{g_k}}{\gamma}\Big) = 1 + 0 = 1.$$

We need to also show that σ is increasing on the entire positive part of $\mathcal{M}/\mathbb{Z} \setminus I$, that is, that $\sigma(a) - a = q_{g_1} d_{g_1} + \cdots + q_{g_k} d_{g_k} > 0$:

Lemma 2. *The automorphism σ constructed above is strictly increasing for all positive elements of $M/\mathbb{Z} \setminus I$, and trivial for all elements of I.*

Proof. 1. If $a \in I$, then we can represent a as a $\mathbb{Q}$-linear combination of strongly independent elements that occur in the back-and-forth construction, so $a = q_1 a_1 + q_2 a_2 + \cdots + q_n a_n$. By Proposition 3, $v(a) = \max\{v(a_1), \ldots, v(a_n)\}$. Thus, the magnitude classes of $a_1, \ldots, a_n$ are all in I. Hence $\sigma(a_i) = a_i$ for $i \in \{1, \ldots, n\}$, and the linear combination is fixed by σ as well.

2. If a is above I, we can again represent a as a $\mathbb{Q}$-linear combination of positive strongly independent elements from the back-and-forth construction:

$$0 < a = q_1 a_1 + q_2 a_2 + \cdots + q_n a_n.$$

By the construction,

$$\sigma(a) - a = q_1 d_1 + q_2 d_2 + \cdots + q_n d_n$$

where the $\{d_1, \ldots, d_n\}$ are the nonnegative differences from the construction. By strong independence of $\{a_1, \ldots, a_n\}$ and Proposition 3, the magnitude class of the linear combination is equal to the largest magnitude class of any a_i occurring in the sum. Similarly, the sign of the difference $\sigma(a) - a$ is determined by the sum taken over the differences of maximal magnitude, which by the construction correspond to the elements of maximal magnitude in $\{a_1, \ldots, a_n\}$. Thus, without loss of generality we may assume that $v(a) = v(a_1) = v(a_2) = \cdots = v(a_n)$, and that $v(d_1) = v(d_2) = \cdots = v(d_n)$.

By strong independence and Proposition 4, there are $r_2, r_3, \ldots, r_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$\mathrm{st}\left(\frac{a_1}{a_j}\right) = r_j = \mathrm{st}\left(\frac{d_1}{d_j}\right) \quad \text{for } j \in \{2, \ldots, n\}.$$

Note that because these standard parts are all irrational, $v(a_1 - a_j) = v(a)$ for $j \in \{2, \ldots, n\}$. The strategy for the proof will be to approximate the r_j's by rational numbers, so that we can express a as near a rational multiple qa_1 of a_1, and $\sigma(a) - a$ as near a rational multiple qd_1 of d_1.

Fact: For irrational standard parts, it follows from the definition of standard parts and the density of $\mathbb{Q}$ in $\mathbb{R}$ that for all $0 < \epsilon$, we may approximate the irrational standard part r of two elements in the same magnitude class by a rational number u such that either (1) $0 < u - r < \epsilon$, or so that (2) $0 < r - u < \epsilon$.

Now, in the sum

$$0 < a = q_1 a_1 + q_2 a_2 + \cdots + q_n a_n$$

for some summand $q_j a_j$, the coefficient q_j is positive (some q_j must be positive, else a is negative). We will now approximate a by replacing $a_2, a_3, \ldots, a_n$ by rational multiples of a_1 in such a way that the linear combination remains positive, but less than a. To do so, we will choose a rational multiple of a_1 that is slightly greater than a_j for those summands

$q_j a_j$ whose coefficients q_j are negative (thus decreasing the linear combination), and choose a rational multiple of a_1 that is slightly less than a_j for those summands whose coefficients are positive. Then a will be greater than this linear combination of rational multiples of a_1, and we can transfer this approximating process to the linear combination of the differences d_j to show that indeed $\sigma(a) - a$ is positive.

Specifically, for each $j \in \{2, \ldots, n\}$ such that q_j is positive, we can find a rational $p_j < r_j$ so that

$$0 < r_j - p_j < \frac{a}{2nq_j} \quad \text{where as above } r_j = \mathrm{st}\left(\frac{a_1}{a_j}\right).$$

Then

$$q_j a_j > q_j(p_j a_1)$$

and if we substitute $q_j p_j a_1$ in the sum for $q_j a_j$, we have reduced the sum by less than

$$q_j \cdot \frac{a}{2nq_j} = \frac{a}{2n}.$$

Similarly, if the coefficient q_j is negative, we can find a rational number p_j so that $0 < p_j - r_j < \frac{a}{2n(-q_j)}$, and again if we substitute $q_j p_j a_1$ for $q_j a_j$, we have reduced the sum by less than

$$q_j \cdot \frac{a}{2nq_j} = \frac{a}{2n}.$$

Thus, the sum $(q_1 a_1 + q_2 p_2 a_1 + \cdots + q_n p_n a_1) > a - n \cdot \frac{a}{2n} > a/2 > 0$. Hence, since $(q_1 a_1 + q_2 p_2 a_1 + \cdots + q_n p_n a_1) = a_1(q_1 + q_2 p_2 + \cdots + q_n p_n)$, and $a_1 > 0$, the sum $(q_1 + q_2 p_2 + \cdots + q_n p_n)$ is also positive.

Then, since d_1 is positive, we have

$$d_1(q_1 + q_2p_2 + \cdots + q_np_n) > 0$$

and when we replace $q_jp_jd_1$ by q_jd_j, we *increase* the sum. Hence the difference $q_1d_1 + \cdots + q_nd_n$ is positive, as required. □

Finally, we want to show that all possible differences, that is, all elements of the fixed point set F with color C_0, actually occur as differences. This will give immediately that the set D of differences is dense in F because elements with color C_0 are dense in F.

Lemma 3. *For σ as constructed above, every zero-residue element d in the fixed-point set F is a difference. That is,*

$$c \in (F \cap C_0) \leftrightarrow \exists x(\sigma(x) - x = c).$$

Proof. In constructing σ, we obtained a set D of differences corresponding to the strongly independent set A. It suffices to show that the $\mathbb{Q}$-span of D contains $F \cap C_0$. First, the elements of D are strongly independent. This follows from condition (7) in the construction of σ, and Proposition 4: for any pair of strongly independent elements $a_1, a_2 \in A$, we required that the corresponding differences $d_1 = \sigma(a_1) - a_1, d_2 = \sigma(a_2) - a_2$ satisfy:

$$\mathrm{st}\left(\frac{a_1}{a_2}\right) = \mathrm{st}\left(\frac{d_1}{d_2}\right) \in \mathbb{R} \setminus \mathbb{Q}.$$

Now suppose towards a contradiction that $c \in F \cap C_0$, but $c \notin \langle D \rangle$, the $\mathbb{Q}$-span of D. Then c is linearly independent of every finite subset of D, and is either itself strongly independent of D or, if not, by the Exchange Lemma (Lemma 1) we can find an element c' that is strongly independent of D such that $c \in \langle D \cup c' \rangle$. By strong independence and Proposition 4, the standard part of c' with respect to any $d \in D$ such that $v(d) = v(c')$ is irrational, say

$$\mathrm{st}\left(\frac{c'}{d}\right) = r \in \mathbb{R} \setminus \mathbb{Q}.$$

In particular, there is no $d' \in D$ such that

$$\mathrm{st}\left(\frac{d'}{d}\right) = r \in \mathbb{R} \setminus \mathbb{Q},$$

for then

$$\mathrm{st}\left(\frac{c'}{d}\right) \cdot \mathrm{st}\left(\frac{d}{d'}\right) = r \cdot \frac{1}{r} = 1,$$

in which case c' is not strongly independent of D. Because the (irrational) standard parts occurring among elements of $D \cup c'$ of magnitude class equal to $v(c')$ are the same as the standard parts occurring among elements of A of magnitude class v' where $\alpha(v') = v$, there must be some new strongly independent element not already in A of magnitude v'. This contradicts the construction of A, which exhausts all strongly independent elements. $\square$

So we now also have the following, which will be significant for quantifier elimination of the expanded model:

Corollary 1. *The set of differences $D = \{x | \exists w(\sigma(w) - w = x)\}$ is dense in the fixed point set F.*

To obtain the automorphism of $\mathcal{M} \models \mathrm{Pr}$ itself, we now lift σ to $\mathcal{M}$ using Proposition 2 and denote it by σ as well.

The following theorem, stated before in section 1.3, summarizes the properties of σ.

Theorem 2. *Let $\mathcal{M} \models Pr$ be a countable, pseudo-recursively saturated model. Then there is an automorphism σ of $\mathcal{M}$ satisfying the following:*

(1) the fixed point set F of σ is a convex, dense set of magnitude classes containing the standard integers;

(2) σ is modest and strictly increasing on the positive part of $M \setminus F$; and

(3) the $\mathbb{Z}$-chains containing elements of the set of differences $D = \{x | \exists w(\sigma(w) - w = x)\}$ are dense and co-dense in the $\mathbb{Z}$-chains in F.

In the next section, we show that there is a maximal automorphism τ of $\mathcal{M}$, that is, an automorphism with fixed-point set equal to $\mathbb{Z}$. (Of course, $\mathbb{Z}$ must be fixed by any automorphism.)

2.3 A maximal modest increasing automorphism

In this section, we adjust the construction of σ to obtain a new increasing modest automorphism τ of a countable, pseudo-recursively model of Presburger arithmetic that is maximal, that is, every positive nonstandard element a is increased by τ, and $\mathrm{st}(\frac{\tau(a)}{a}) = 1$.

We construct τ first in the quotient $\mathcal{M}/\mathbb{Z}$ by modifying the construction of σ as follows. Because the set of positive magnitude classes is a countable dense linearly ordered set without endpoints, we can choose a strictly increasing automorphism α from this set to itself. For a magnitude class v_i, let $\alpha(w_i) = v_i$, so that $w_i < v_i$. We will use α to match each magnitude class in $\mathcal{M}/\mathbb{Z}$ with a lower magnitude class in which the differences will be located.

Forth: We enumerate all the nonstandard positive elements of $\mathcal{M}/\mathbb{Z}$, as $a'_1, a'_2, \ldots$.
Just as in the construction of σ, at stage k of the back-and-forth construction when we have considerered a'_k in the enumeration, we will obtain sets $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $D = \{d_1, \ldots, d_n\}$, of positive elements which satisfy the following conditions:

1. the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are strongly independent.

2. for each $i, j \in \{1, \ldots, n\}$, $\mathrm{st}\left(\frac{a_i}{a_j}\right) = \mathrm{st}\left(\frac{b_i}{b_j}\right)$

3. for each $i \in \{1, \ldots, n\}$, the color of a_i is the same as the color of b_i

4. for each $i \in \{1, \ldots, n\}$, $a_i < b_i$; in particular, we require that $b_i - a_i = d_i$, where $\alpha(v(d_i)) = v(a_i) = v(b_i)$, that is, the magnitude class of the difference d_i is mapped by α to the magnitude class of a_i and b_i

5. for each $i \in \{1, \ldots, n\}$, $\mathrm{st}\left(\frac{b_i}{a_i}\right) = 1$

6. for any subset $\{a_{j_0}, a_{j_1}, \ldots, a_{j_r}\} \subseteq \{a_1, \ldots, a_n\}$,
   if $v(a_{j_0}) < v(a_{j_1}) < \cdots < v(a_{j_r})$, then $v(d_{j_0}) < v(d_{j_1}) < \cdots < v(d_{j_r})$; this will be an immediate consequence of using α to locate differences

7. for any subset $\{a_{j_0}, a_{j_1}, \ldots, a_{j_s}\} \subseteq \{a_1, \ldots, a_n\}$,
   if $v(a_{j_0}) = v(a_{j_1}) = \cdots = v(a_{j_s})$, then $v(d_{j_0}) = v(d_{j_1}) = \cdots = v(d_{j_s})$ and

   $$\mathrm{st}\left(\frac{a_{j_t}}{a_{j_u}}\right) = \mathrm{st}\left(\frac{d_{j_t}}{d_{j_u}}\right)$$

   for all $j_t, j_u \in \{j_0, \ldots, j_s\}$

8. the elements $a'_1, a'_2, \ldots, a'_k$ are in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n\}$.

We first put $a'_1 = a_1$. We choose difference d_1 according to the DLO automorphism α, that is, so that $\alpha(v(d_1)) = v(a_1)$, and so that the color of $a_1 + d_1$ is the same as the color of a_1. Because $v(d_1) < v(a_1)$,

$$\mathrm{st}\left(\frac{a_1 + d_1}{a_1}\right) = 1$$

and because d_1 is positive, $a_1 + d_1 > a_1$. We put $b_1 = a_1 + d_1$.
Verification that the conditions are satisfied for a_1, b_1, d_1 is immediate:

1. the singletons a_1 and b_1 are trivially strongly independent.

2. $\mathrm{st}\left(\frac{a_1}{a_1}\right) = 1 = \mathrm{st}\left(\frac{b_1}{b_1}\right)$

3. the color of a_1 is the same as the color of b_1 (that is, $C_0(d_1)$).

4. $a_1 < a_1 + d_1 = b_1$ by choice of positive d_1, and $\alpha(v(d_1)) = v(a_1)$.

5. $\mathrm{st}\left(\frac{a_1+d_1}{a_1}\right) = \mathrm{st}\left(\frac{b_1}{a_1}\right) = 1$ because d_1 is in a magnitude class lower than the magnitude class of a_1.

Conditions (6)-(8) are satisfied trivially.

Next, suppose for some $n \leq m$ we have sets $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ and $\{d_1, \ldots, d_n\}$ meeting the conditions. Let a'_m be the next element in the enumeration. If a'_m is in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n\}$, we leave the sets as they are.

If not, then by the Exchange Lemma for strongly independent sets, we can find a_{n+1} that is strongly independent of $a_1, \ldots, a_n$, and such that a'_m is in the $\mathbb{Q}$-span of $\{a_1, \ldots, a_n, a_{n+1}\}$. As in the contraction of σ, there are now two cases: either $v(a_{n+1}) = v(a_i)$ for some $i \in \{1, \ldots, n\}$, or not.

Case1: $v(a_{n+1}) \neq v(a_i)$ for any $i \in \{1, \ldots, n\}$. In this case, we choose d_{n+1} so that $\alpha(v(d_{n+1})) = v(a_{n+1})$, and so that the color of $a_1 + d_1$ is the same as the color of a_1 (that is, so that $C_0(d_1)$).

Case 2: $v(a_{n+1}) = v(a_i)$ for some $i \in \{1, \ldots, n\}$. In this case, let $a_{j_1}, \ldots, a_{j_r} \in \{a_1, \ldots, a_n\}$ be those elements that have the same magnitude v as a_{n+1}.

Now let

$$\mathrm{st}\left(\frac{a_{n+1}}{a_{j_s}}\right) = r_s \in \mathbb{R} \setminus \mathbb{Q}.$$

These standard parts order the elements of A of magnitude v within their magnitude class. Now either a_{n+1} is between two elements with magnitude v, or not.

Suppose first that a_{n+1} is less than any of the elements of A of magnitude v, and let a_j be the next largest element, with $\mathrm{st}\left(\frac{a_{n+1}}{a_j}\right) = r < 1$. By PRS2, there is c_{n+1} such that

$\mathrm{st}\left(\frac{c_{n+1}}{d_j}\right) = r$, and this implies also that $v(c_{n+1}) = v(d_j)$. Then by PRS1, there is d_{n+1} near c_{n+1} such that the color of $a_{n+1} + d_{n+1}$ is the same as the color of a_{n+1}; that is, so that $C_0(d_{n+1})$. Because d_{n+1} is near c_{n+1}, $\mathrm{st}\left(\frac{d_{n+1}}{d_j}\right) = r$ also. Put $b_{n+1} = a_{n+1} + d_{n+1}$. The case where a_{n+1} is greater than any of the elements in A of magnitude v is the same, except that $\mathrm{st}\left(\frac{a_{n+1}}{a_j}\right) > 1$. The case where a_{n+1} is between two elements is virtually the same; we simply find d_{n+1} using either of the elements in A of the same magnitude class that is adjacent to a_{n+1} in the ordering established by the standard parts.

We now check the conditions.

1. Because we used the Exchange Lemma, the set $\{a_1, \ldots, a_{n+1}\}$ is strongly independent. To see that the set $\{b_1, \ldots, b_{n+1}\}$ is also strongly independent, suppose not. We obtain the same contradiction as we did in the construction of σ: there is some $\gamma \in M/\mathbb{Z}$ such that the set of standard parts

$$\left\{\mathrm{st}\left(\frac{b_1}{\gamma}\right), \mathrm{st}\left(\frac{b_2}{\gamma}\right), \ldots, \mathrm{st}\left(\frac{b_{n+1}}{\gamma}\right)\right\} \setminus \{0, \infty\} = \left\{\mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right), \ldots, \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right)\right\}$$

is not linearly independent over $\mathbb{Q}$. So there are $q_1, \ldots, q_m \in \mathbb{Q}$ not all equal to 0 such that

$$q_1 \mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right) + \cdots + q_m \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right) = 0$$

But then

$$q_1 \mathrm{st}\left(\frac{b_{i_1}}{\gamma}\right) \mathrm{st}\left(\frac{a_{i_1}}{b_{i_1}}\right) + \cdots + q_m \mathrm{st}\left(\frac{b_{i_m}}{\gamma}\right) \mathrm{st}\left(\frac{a_{i_m}}{b_{i_m}}\right) = 0$$

because by (5) below, $\mathrm{st}\left(\frac{a_i}{b_i}\right) = 1$ for $i \in \{1, \ldots, n+1\}$. This contradicts the strong independence of $\{a_1, \ldots, a_{n+1}\}$.

2. For $i \in \{1, \ldots, n\}$,

$$\mathrm{st}\left(\frac{b_{n+1}}{b_i}\right) \cdot \mathrm{st}\left(\frac{a_{n+1}}{b_{n+1}}\right) \cdot \mathrm{st}\left(\frac{b_i}{a_i}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_i}\right)$$

   because $\mathrm{st}\left(\frac{a_{n+1}}{b_{n+1}}\right) = \mathrm{st}\left(\frac{b_i}{a_i}\right) = 1$.

3. The colors of a_{n+1} and b_{n+1} are equal by construction, and $\alpha(v_{n+1}) = v(a_{n+1})$ as well by construction.

4. $a_{n+1} < a_{n+1} + d_{n+1} = b_{n+1}$ because $0 < d_{n+1}$.

5. $\mathrm{st}\left(\frac{b_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{a_{n+1}+d_{n+1}}{a_{n+1}}\right) = \mathrm{st}\left(\frac{a_{n+1}}{a_{n+1}}\right) + \mathrm{st}\left(\frac{d_{n+1}}{a_{n+1}}\right) = 1 + 0 = 1$, since $v(d_{n+1}) < v(a_{n+1})$.

   Conditions (6) and (7) are satisfied by the construction, and condition (8) is satisfied by the Exchange Lemma.

Back: As in the construction of σ, at each back step of the construction, given a new strongly independent element b_{m+1}, we choose d_{m+1} (depending on whether the magnitude class of b_{m+1} is represented in $\{b_1, \ldots, b_m\}$ or not) according to Case 2 or Case 1, respectively, and subtract d_{m+1} from b_{m+1}.

The verification that the back-and-forth construction produces an automorphism τ of $\mathcal{M}/\mathbb{Z}$ with the required properties is exactly as in the case above of a modest increasing automorphism σ with a fixed point set that strictly contains the standard integers. Again, we lift τ to an automorphism, also denoted by τ, of $\mathcal{M}$. The same proof as in the case of the automorphism σ show that τ is increasing for all positive elements, and will also show that the set of differences is dense in $\mathcal{M}/\mathbb{Z}$, and that the $\mathbb{Z}$-chains containing a difference are dense in the $\mathbb{Z}$-chains in the fixed-point set of $\mathcal{M}$. We therefore have proved the following:

Theorem 3. *Let $\mathcal{M} \models Pr$ be a countable, pseudo-recursively saturated model. Then there is an automorphism τ of $\mathcal{M}$ satisfying the following:*

(1) the fixed point set F of τ is $\mathbb{Z}$;

(2) τ is modest and strictly increasing on the positive part of M; and

(3) the set of $\mathbb{Z}$-chains containing an element of the set of differences

$D = \{x | \exists w(\tau(w) - w = x\}$ *is dense in the $\mathbb{Z}$-chains in M.*

Chapter 3

Axioms for the expanded structures

Here we exhibit axiomatizations for both the quotient structure and the full Presburger structure, each expanded by the modest automorphism σ constructed above. In chapters 4 and 5, we will prove quantifier elimination and completeness for these sets of axioms. The chief distinctions between the two sets of axioms are that (1) the quotient structure is a dense order, but the Presburger structure is a discrete order; (2) the quotient structure is a divisible group, but elements of the Presburger structure are generally not divisible. The set of differences in the Presburger structure is the intersection of the fixed-point set of σ and zero-residue elements, and the set of $\mathbb{Z}$-chains containing such zero-residue elements is dense in the $\mathbb{Z}$-chains of the fixed-point set. In the quotient structure, the differences correspond to those $\mathbb{Z}$-chains collapsed to single elements, and are thus dense and codense in the fixed-point set.

3.1 Axioms for $(\mathcal{M}/\mathbb{Z}, \sigma)$

The quotient structure expanded by σ satisfies the following axioms T^* in the language $\mathcal{L}_q = (+, -, \sigma, \sigma^{-1}, <, D, 0, q \text{ (for } q \in \mathbb{Q}))$, where the predicate D for the set of differences is defined by

$$D(x) \leftrightarrow \exists w(\sigma(w) - w = x).$$

3.1.1 Axioms for a divisible ordered abelian group

1. $\forall x \neg(x < x)$
2. $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$
3. $\forall x \forall y(x < y \vee x = y \vee y < x)$
4. $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y)$
5. $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
6. $\forall x \forall y(x + y = y + x)$
7. $\forall x \exists y(x + y = 0)$
8. $\forall x(x + 0 = x)$
9. $\forall x \forall y \forall z(x < y \rightarrow x + z < y + z)$
10. (scheme, for n=2,3, ...) $\forall x \exists y(x = \underbrace{y + y + \cdots + y}_{n \text{ times}})$

3.1.2 Axioms for a modest increasing automorphism with convex fixed point set (for $0 < x$)

1. $\forall x \exists y(\sigma(y) = x)$
2. $\forall x \forall y(\sigma(x) = \sigma(y) \rightarrow x = y)$
3. $\forall x \forall y(\sigma(x + y) = \sigma(x) + \sigma(y))$

4. $\forall x \forall y(\sigma(x-y) = \sigma(x) - \sigma(y))$

5. $\forall x \forall y(x < y \rightarrow \sigma(x) < \sigma(y))$

6. $\exists x(\sigma(x) \neq x)$

7. $\forall x \sigma(x) \neq x \rightarrow x < \sigma(x)$

8. $\forall x \forall y(x < y \wedge \sigma(x) \neq x) \rightarrow (\sigma(y) \neq y)$

9. $\sigma(0) = 0$

10. $\exists x(0 < x \wedge \sigma(x) = x)$

11. (scheme for $q \in \mathbb{Q}$) $\forall x(\sigma(qx) = q\sigma(x))$

12. (scheme, for $n = 2, 3, \ldots$) $\forall x(nx \leq n\sigma(x) < (n+1)(x))$

13. $\forall x(\sigma^{-1}(\sigma(x)) = \sigma(\sigma^{-1}(x)) = x)$

3.1.3 Axioms for differences and density/codensity of differences in fixed point set:

1. $D(x) \leftrightarrow \exists w(\sigma(w) - w = x)$

2. $D(0)$

3. $\forall x \forall y(\sigma(x) = x \wedge x < y) \rightarrow \exists z(x < z < y \wedge D(z))$

4. $\forall x \forall y(\sigma(y) = y \wedge x < y) \rightarrow \exists z(x < z < y \wedge D(z))$

5. $\forall x \forall y(\sigma(x) \neq x \wedge \sigma(y) \neq y \wedge x < 0 \wedge y > 0) \rightarrow \exists z(x < z < y \wedge D(z))$

6. $\forall x \forall y(D(x) \wedge D(y) \wedge x < y) \rightarrow \exists z(x < z < y \wedge \neg D(z))$

7. $\forall x(D(x) \rightarrow \sigma(x) = x)$

3.2 Axioms for $(\mathcal{M}, \sigma)$

The Presburger structure expanded by σ satisfies the following set of axioms T in the language $\mathcal{L}_T = (+, -, \sigma, \sigma^{-1}, <, P_n(n = 2, 3, \ldots), D, Z, 0, 1)$ where the P_n are the Presburger divisibility predicates, D is again a predicate for the differences defined by

$$D(x) \leftrightarrow \exists w(\sigma(w) - w = x).$$

and Z is a predicate for standard integers, defined by:

$$Z(x) \leftrightarrow (x < 0 \wedge \forall y(x < y < 0 \rightarrow \neg D(y)) \vee (0 < x \wedge \forall y(0 < y < x \rightarrow \neg D(y)) \vee (x = 0),$$

that is, x is in the standard part (the same $\mathbb{Z}$-chain as 0)).

3.2.1 Axioms for a discrete ordered abelian group:

1. $\forall x \neg(x < x)$
2. $\forall x \forall y \forall z((x < y \wedge y < z) \rightarrow x < z)$
3. $\forall x \forall y(x < y \vee x = y \vee y < x)$
4. $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$
5. $\forall x \forall y(x + y = y + x)$
6. $\forall x \exists y(x + y = 0)$
7. $\forall x(x + 0 = x)$
8. $\forall x \forall y \forall z(x < y \rightarrow x + z < y + z)$
9. $\forall x \exists y(x < y \wedge \forall z(x < z \rightarrow (z = y \vee y < z))$

3.2.2 Presburger axioms

1. $0 < 1$

2. $\forall x(x \leq 0 \vee x \geq 1)$

3. $\forall x \bigvee_{i=0}^{n-1} \Big(P_n(x + \underbrace{1 + 1 + \cdots + 1}_{i \text{ times}}) \wedge \bigwedge_{j=0, j\neq i}^{n-1} \neg P_n(x + \underbrace{1 + 1 + \cdots + 1}_{j \text{ times}}) \Big)$
(scheme, for $n = 2, 3, \ldots$)

4. $\forall x \forall y(P_n(x) \wedge P_n(y) \to P_n(x + y) \wedge P_n(x - y))$

   (scheme, for $n = 2, 3, \ldots$)

5. $\forall x \forall y(\underbrace{y + y + \cdots + y}_{n \text{ times}} = x) \to P_n(x)$
(scheme, for $n = 2, 3, \ldots$)

6. $\forall x(P_n(x) \to P_m(x))$ (for $m|n$)

   (scheme, for $n, m = 2, 3, \ldots$)

7. $\forall x(P_{kn}(\underbrace{x + x + \cdots + x}_{k \text{ times}}) \to P_n(x))$
(scheme, for $n, k = 2, 3, \ldots$)

3.2.3 Axioms for a modest increasing automorphism with convex fixed point set (for $0 < x$)

1. $\forall x \exists y(\sigma(y) = x)$

2. $\forall x \forall y(\sigma(x) = \sigma(y) \to x = y)$

3. $\forall x \forall y(\sigma(x + y) = \sigma(x) + \sigma(y))$

4. $\forall x \forall y(\sigma(x - y) = \sigma(x) - \sigma(y))$

5. $\forall x \forall y(x < y \to \sigma(x) < \sigma(y))$

6. $\exists x(\sigma(x) \neq x)$

7. $\sigma(x) \neq x \to x < \sigma(x)$

8. $\forall x \forall y(x < y \wedge \sigma(x) \neq x) \to (\sigma(y) \neq y)$

9. $\forall x \forall y(x < y \wedge \sigma(y) = y) \to (\sigma(x) = x)$

10. $\sigma(0) = 0 \wedge \sigma(1) = 1$

11. $\exists x(0 < x \wedge \sigma(x) = x)$

12. (scheme for $n \in \mathbb{Z}$) $\forall x(\sigma(nx) = n\sigma(x))$

13. (scheme, for $n = 2, 3, \ldots$) $\forall x(nx \leq n\sigma(x) < (n+1)(x))$

14. (scheme, for $n = 2, 3, \ldots$) $\forall x(P_n(x) \leftrightarrow P_n(\sigma(x)))$

15. $\forall x(\sigma^{-1}(\sigma(x)) = \sigma(\sigma^{-1}(x)) = x)$

3.2.4 Axioms for differences, density of $\mathbb{Z}$-chains containing differences among $\mathbb{Z}$-chains in fixed point set:

1. $D(x) \leftrightarrow \exists w(\sigma(w) - w = x)$

2. $Z(x) \leftrightarrow (x < 0 \wedge \forall y(x < y < 0 \to \neg D(y)) \vee (0 < x \wedge \forall y(0 < y < x \to \neg D(y)) \vee (x = 0)$

3. $D(0)$

4. $\forall x \forall y(\sigma(x) = x \wedge x < y \wedge \neg Z(y - z)) \to \exists z(x < z < y \wedge D(z))$

5. $\forall x \forall y(\sigma(y) = y \wedge x < y \wedge \neg Z(y - z)) \to \exists z(x < z < y \wedge D(z))$

6. $\forall x \forall y(\sigma(x) \neq x \wedge \sigma(y) \neq y \wedge x < 0 \wedge y > 0) \to \exists z(x < z < y \wedge D(z))$

7. $\forall x \forall y(D(x) \wedge D(y) \wedge x < y) \rightarrow \exists z(x < z < y \wedge \neg D(z))$

8. $\forall x \forall y(D(x) \wedge D(y) \wedge x < y) \rightarrow \exists z(x < z < y \wedge D(z))$

9. $\forall x(D(x) \rightarrow \exists z(x < z \wedge D(z))$

10. $\forall x(D(x) \rightarrow \exists z(z < x \wedge D(z))$

11. $\forall x(D(x) \rightarrow \sigma(x) = x)$

Because the standard part $\mathbb{Z}$ is not definable in Presburger arithmetic itself, we note the following fact in a proposition:

Proposition 5. *$\mathbb{Z}$ is definable in $(\mathcal{M}, \sigma)$.*

Proof. In the structure $(\mathcal{M}, \sigma)$, $\mathbb{Z}$-chains containing differences are dense among the $\mathbb{Z}$-chains in the fixed-point set of σ. Thus, if $x < 0$ and if x is in a different $\mathbb{Z}$-chain from the $\mathbb{Z}$-chain containing 0, then there is a difference between x and 0. (Similarly in the case $0 < x$.) The axiom defining the predicate $Z(x)$ states that there is no difference between x and 0 (or $x = 0$). Hence x is in the same $\mathbb{Z}$-chain as 0, i.e., $x \in \mathbb{Z}$. Conversely if $x \in \mathbb{Z}$, then either $x = 0$ or there is no difference between x and 0. Thus, $\mathbb{Z}$ is defined by the formula $\varphi(x) := Z(x)$. $\square$

Chapter 4

Quantifier elimination for $(\mathcal{M}/\mathbb{Z}, \sigma)$

In this chapter and the next, we prove that for countable, pseudo-recursively saturated $\mathcal{M} \models \mathrm{Pr}$, the theories of the expanded structures $(\mathcal{M}/\mathbb{Z}, \sigma)$ and $(\mathcal{M}, \sigma)$ both have quantifier elimination, where σ is the increasing, modest automorphism constructed in Chapter 2. Axioms for these two theories are given in chapter 3, above. Before beginning the quantifier elimination process, recall from, *e.g.* [11, Lemma 3.1.5], that it suffices to show that the quantifier can be eliminated from $\exists x \phi(x; \bar{y})$, where $\phi(x; \bar{y})$ is a conjunction of atomic and negated atomic formulas in a single variable x and parameters $\bar{y}$.

We begin with quantifier elimination for $(\mathcal{M}/\mathbb{Z}, \sigma)$.

To prove quantifier elimination , we use the language

$\mathcal{L}_q = (+, -, \sigma, \sigma^{-1}, <, D, 0, q$ (for $q \in \mathbb{Q}$)), where the predicate D for the set of differences is defined by

$$D(x) \leftrightarrow \exists w(\sigma(w) - w = x)$$

We use the abbreviation $F(x)$ to mean $\sigma(x) = x$, but do not add it to the language. We begin by considering terms in this language that can occur in literals.

Terms. All terms t in $\mathcal{L}$ (except for 0) can be rewritten in the following form:

$$t(\bar{v}) = p_1(v_1) + p_2(v_2) + \cdots + p_n(v_n)$$

where each summand $p_i(v_i)$ is a sum of rational multiples of integer powers of $\sigma(v_i)$, which we term a σ-polynomial.

Definition 10. A *σ-polynomial* in a single variable v_i is a term of the form

$$p_i(v_i) = a_k\sigma^k(v_i) + \cdots + a_1\sigma(v_i) + a_0 v_i + a_{-1}\sigma^{-1}(v_i) + \cdots + a_{-l}\sigma^{-l}(v_i)$$

with coefficients $a_k, \ldots, a_1, a_0, a_{-1}, \ldots, a_{-l} \in \mathbb{Q}$.

Remark. In any formula ϕ in which a σ-polynomial occurs, we may replace that formula with an equivalent formula in which only positive powers of σ occur by applying σ to the formula l times, where $-l$ is the least negative power of σ occurring in ϕ. Thus, in the remainder of this chapter, we assume without loss of generality that all σ-polynomials contain only positive powers of σ.

Literals. Literals are of one of the following forms for terms s and t:

1. $s = t$, $\neg(s = t)$
2. $s < t$, $\neg(s < t)$
3. $D(s)$, $\neg D(s)$.

Remark. Given a σ-polynomial $p(x, \bar{y})$ in variables x and $\bar{y}$, we can always separate $p(x)$ into two σ-polynomials $q(x), r(\bar{y})$ such that $p(x, \bar{y}) = q(x) + r(\bar{y})$. This follows directly from the fact that for any integer n,
$\sigma^n(ax + b_1y_1 + \ldots + b_ny_n) = \sigma^n(ax) + \sigma^n(b_1y_1) + \ldots + \sigma^n(b_ny_n)$. Thus, in equations and

inequalities, we may assume without loss of generality that terms involving x occur only on one side, and terms involving $\bar{y}$ occur only on the other side.

We now consider a formula $\exists x \phi(x, \bar{y})$, where ϕ is a conjunction of literals as above. We need to show that there is a quantifier-free formula ψ such that $(\mathcal{M}/\mathbb{Z}, \sigma) \models \exists x \phi \leftrightarrow \psi$. A few lemmas will be useful.

Notation. In the subsequent chapters, for a σ-polynomial
$p(x) = a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x$, we put $A = \Sigma_{i=0}^n a_i$,
$B = na_n + (n-1)a_{n-1} + \ldots + a_1$, and $B' = (n+1)a_n + na_{n-1} + \ldots + 2a_1 + a_0 = A + B$.

Lemma 4. *(applying positive coefficient σ-polynomials to inequalities) Let*
$p(x) = a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x$*, and such that* $0 < A$*. If*
$0 < a < b \in M^*$*, then* $p(a) < p(b)$*.*

Proof. If both a and b are fixed points, then $p(a) = Aa$, and $p(b) = Ab$, and the result is immediate.

If a is a fixed point and b is not, then $p(a) = Aa$ and $p(b) = Ab + Bd$ where $d = \sigma(b) - b$.
Regardless of B, since b is in a higher magnitude class than a, $p(a) < p(b)$.
If $\sigma(a) \neq a$ and $\sigma(b) \neq b$, put $\sigma(a) = a + c, \sigma(b) = b + d$.
Then

$$p(a) = Aa + (na_n + (n-1)a_{n-1} + \cdots + 2a_2 + a_1)c \tag{4.1}$$

$$p(b) = Ab + (na_n + (n-1)a_{n-1} + \cdots + 2a_2 + a_1)d \tag{4.2}$$

Therefore $p(b) - p(a) = A(b-a) + (na_n + (n-1)a_{n-1} + \cdots a_1)(d-c)$. Note that each of c, d and $d - c$ is in the fixed point set F of σ.
(1) If the magnitude class $v(b)$ of b is greater than the magnitude class of a, then
$p(b) - p(a)$ is obviously positive.

(2) If $v(a) = v(b) = v(b - a)$, then again this difference is positive, because $v(A(b - a)) > v(d - c)$.

(3) If $v(b - a) < v(a) = v(b)$ and $b - a = k \notin F$, then $b = a + k$, and

$$p(b) - p(a) = Ak + (na_n + (n - 1)a_{n-1} + \cdots a_1)(d - c)$$

Again, because $v(k) > v(d)$ and $v(k) > v(c)$, $p(b) - p(a)$ is positive.

(4) If $v(b - a) < v(a) = v(b)$ and $b - a = m \in F$, then $b = a + m$, and $\sigma(b) - b = (\sigma(a + m)) - (a + m) = (\sigma(a) + m) - (a + m) = \sigma(a) - a$, and therefore $c = d$. Hence

$$p(b) - p(a) = Am$$

which is once again positive. □

As an immediate consequence of Lemma 4, if $0 < q(\bar{y}) < x < r(\bar{y})$, and if $p(x)$ is as in the Lemma, then $p(q(\bar{y})) < p(x) < p(r(\bar{y}))$.

Note that if A is negative, then we similarly obtain that $p(b) < p(a)$. Also note that Lemma 4 fails if $A = 0$ and $b - a \in F$; in this case $p(b) = p(a) = 0$.

Lemma 5. *(differences increasing) (1) If $a < b$ then $\sigma(a) - a \leq \sigma(b) - b$, with equality if and only if $b - a \in F$, and (2) if $\sigma(a) - a < \sigma(b) - b$, then $a < b$.*

Proof. (1) Since $a < b$, $0 < b - a$ and so because σ is increasing, $b - a \leq \sigma(b - a)$. Equality holds if and only if $b - a \in F$. Thus, since σ is an automorphism, $b - a \leq \sigma(b) - \sigma(a)$. Upon adding $\sigma(a) - b$ to both sides, we get $\sigma(a) - a \leq \sigma(b) - b$, with equality if and only if $b - a \in F$. (2) Add $b - \sigma(a)$ to both sides of $\sigma(a) - a < \sigma(b) - b$ to get $b - a < \sigma(b) - \sigma(a) = \sigma(b - a)$. So σ increases $b - a$, hence $b - a > 0$ and $a < b$. □

Lemma 6. *(basic fixed-point set facts) If $a, b \in F, c \notin F, q \in \mathbb{Q}$, then the following hold:*

1. $a + b \in F, a - b \in F$
2. $qa \in F$
3. $c + a \notin F$
4. $qc \notin F$.

Proof. Apply σ to $a \pm b$, qa, $c + a$ and qc. □

Lemma 7. *(basic facts about the set of differences and its complement) Suppose* $0 \neq q \in \mathbb{Q}$.

1. $D(a) \leftrightarrow D(qa)$
2. $\neg D(a) \leftrightarrow \neg D(qa)$
3. $D(a) \wedge D(b) \rightarrow D(a + b)$
4. $D(a) \wedge \neg D(b) \rightarrow \neg D(a + b)$

Proof. (1) If $D(a)$, then $a = \sigma(w) - w$ for some w. So $\sigma(qw) - qw = q(\sigma(w) - w) = qa$, and so $D(qa)$. Conversely, if $D(qa)$, then $qa = \sigma(z) - z$ for some z. Since z is divisible, $z = qw$ for some w, and so $qa = \sigma(qw) - qw = q(\sigma(w) - w)$, and hence $D(a)$. (2) Directly from (1). (3) Since $a = \sigma(w) - w$ and $b = \sigma(z) - z$,
$a + b = \sigma(w) + \sigma(z) - (w + z) = \sigma(w + z) - (w + z)$. (4) Suppose not, so $D(a + b)$. Then for some w, $a + b = \sigma(w) - w$. Since $D(a)$, there is z such that $a = \sigma(z) - z$. But then $b = (\sigma(w) - w) - (\sigma(z) - z) = \sigma(w - z) - (w - z)$, and so $D(b)$. □

Remark. Parts (3) and (4) of Lemma 7 hold in the Presburger structure as well; the proofs are identical.

Lemma 8. *(solving equations and inequalities) Suppose that* $p(x) = q(\bar{y})$*, where* p *and* q *are* σ*-polynomials. By clearing denominators of the coefficients, assume without loss of generality that* $a_0, \ldots, a_k \in \mathbb{Z}$*. If* $A > 0$*, we have the following:*

(1) If $B, B' \neq 0$, then $p(x) = q(\bar{y})$ is equivalent to

$$x = \frac{B'(q(\bar{y})) - B(\sigma(q(\bar{y}))}{A^2}$$

(2) If $B = 0$, then $p(x) = q(\bar{y})$ is equivalent to

$$x = \frac{q(\bar{y})}{A}$$

(3) If $B' = 0$, then $p(x) = q(\bar{y})$ is equivalent to

$$x = \frac{\sigma(q(\bar{y}))}{A}.$$

If $p(x) < q(\bar{y})$ and $A > 0$ in $p(x)$, then we have the following:

(1) If $B, B' \neq 0$, then $p(x) < q(\bar{y})$ is equivalent to

$$x < \frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}$$

(2) If $B = 0$, then $p(x) < q(\bar{y})$ is equivalent to

$$x < \frac{q(\bar{y})}{A}$$

(3) If $B' = 0$, then $p(x) < q(\bar{y})$ is equivalent to

$$x < \frac{\sigma(q(\bar{y}))}{A}.$$

Proof. We first treat equations, and then inequalities.

Equations: Because σ is a modest increasing automorphism, the left-hand side of

$$a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x = q(\bar{y}) \quad (*)$$

is equal to:

$$(a_n + a_{n-1} + \cdots + a_1 + a_0)x + (na_n + (n-1)a_{n-1} + \cdots + a_1)d$$

where $\sigma(x) - x = d \in F$.

First case: Assume $B \neq 0$ and $B' \neq 0$. Thus (*) is

$$Ax + Bd = q(\bar{y}).$$

Now apply σ to both sides of (*) to get:

$$\sigma(Ax + Bd) = a_n\sigma^{n+1}(x) + a_{n-1}\sigma^n(x) + \cdots + a_1\sigma^2(x) + a_0\sigma(x) = \sigma(q(\bar{y}))$$

and in $\sigma(Ax + Bd)$ the coefficient of x is still A, but the new coefficient of d is

$$B' = (n+1)a_n + na_{n-1} + \cdots + 2a_1 + a_0,$$

and so $\sigma(Ax + Bd) = Ax + B'd = \sigma(q(\bar{y}))$. Also note that $B' - B = A$.

Since both B and B' are nonzero,

$$B'(Ax + Bd) = B'Ax + B'Bd = B'(q(\bar{y})), \quad B(Ax + B'd) = BAx + BB'd = B(\sigma(q(\bar{y}))$$

Hence

$$(B'Ax + B'Bd) - (BAx + BB'd) = (B' - B)Ax = A^2x = B'(q(\bar{y})) - B(\sigma(q(\bar{y}))$$

so that

$$x = \frac{B'(q(\bar{y})) - B(\sigma(q(\bar{y}))}{A^2}.$$

Second case: Either B or B' is equal to 0. In this case, we have either

$$Ax = q(\bar{y})$$

or

$$Ax = \sigma(q(\bar{y}))$$

hence

$$x = \frac{q(\bar{y})}{A}$$

or

$$x = \frac{\sigma(q(\bar{y}))}{A}.$$

Inequalities: The cases are similar to the cases with equations.

As above, the left hand-side of

$$a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x < q(\bar{y})$$

is equivalent to:

$$Ax + Bd < q(\bar{y})$$

where $0 \neq \sigma(x) - x = d \in F$.

Then there is some positive M such that

$$Ax + Bd + M = q(\bar{y}).$$

and

$$\sigma(Ax + Bd + M) = Ax + B'd + \sigma(M) = \sigma(q(\bar{y})).$$

First case: $B \neq 0$ and $B' \neq 0$. We multiply the two equations above by B' and B respectively to obtain:

$$B'Ax + B'Bd + B'M = B'q(\bar{y}).$$

and

$$BAx + BB'd + B\sigma(M) = B\sigma(q(\bar{y}))$$

Subtracting, we get:

$$(B'Ax - BAx) + (B'M - B\sigma(M)) = B'q(\bar{y}) - B\sigma(q(\bar{y})).$$

If M is a fixed point of σ, we have:

$$(B' - B)Ax + (B' - B)M = B'q(\bar{y}) - B\sigma(q(\bar{y}))$$

and since $B' - B = A > 0$ (but M may be arbitrarily close to 0),

$$(B' - B)Ax < B'q(\bar{y}) - B\sigma(q(\bar{y}))$$

which, upon division by $(B' - B)A = A^2$ gives

$$x < \frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}.$$

If M is not a fixed point of σ, then, since $M > 0$, $\sigma(M) > M$. Let us write $\sigma(M) = M + d_M$. Note that d_M is a difference, hence a fixed point, and hence less than M. So we have:

$$(B'Ax - BAx) + (B'M - B\sigma(M)) = (B'A - BA)x + (B'M - (BM + Bd_M)) = A^2x + (B'M - (BM + Bd_M))$$

Thus,

$$A^2x + (B'M - (BM + Bd_M)) = B'q(\bar{y}) - B\sigma(q(\bar{y}))$$

Now consider the term $(B'M - (BM + Bd_M)) = AM - Bd_M$. AM is positive and at least equal to M in the case when $A = 1$. So $AM - Bd_M \geq M - Bd_m$, and $M - Bd_m > 0$, because Bd_M is a fixed point, hence less than M.
So:

$$A^2x < B'q(\bar{y}) - B\sigma(q(\bar{y})),$$

and again

$$x < \frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}.$$

For the converse, we use Lemma 4 for the cases where x and $q(\bar{y})$ are both positive.

$$x < \frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}.$$

Suppose $q(\bar{y})$ is not fixed by σ. Apply the σ-polynomial to both sides, as in Lemma 4, to get:

$$p(x) < p\Big(\frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}\Big),$$

and, putting $d_q = \sigma(q(\bar{y})) - q(\bar{y}) = \sigma^2(q(\bar{y})) - \sigma(q(\bar{y}))$, we have

$$p(x) < A\Big(\frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}\Big) + B\Big(\frac{B'd_q - Bd_q}{A^2}\Big)$$

and the right-hand side is the same as

$$\begin{gathered}
A\Big(\frac{B'q(\bar{y}) - B(q(\bar{y}) + d_q)}{A^2}\Big) + B\Big(\frac{B'd_q - Bd_q}{A^2}\Big) = \\
A\Big(\frac{(B' - B)q(\bar{y}) + d_q)}{A^2}\Big) + B\Big(\frac{B'd_q - Bd_q}{A^2}\Big) = \\
A\Big(\frac{(Aq(\bar{y}) - d_q)}{A^2}\Big) + B\Big(\frac{B'd_q - Bd_q}{A^2}\Big) = \\
q(\bar{y}) - \frac{Bd_q}{A} + \frac{B(B' - B)d_q}{A^2} = \\
q(\bar{y}) - \frac{Bd_q}{A} + \frac{BAd_q}{A^2} = q(\bar{y}) - \frac{Bd_q}{A} + \frac{Bd_q}{A} = q(\bar{y})
\end{gathered}$$

so

$$p(x) < q(\bar{y}).$$

If $q(\bar{y})$ is fixed by σ, then

$$p(x) < p\Big(\frac{B'q(\bar{y}) - B\sigma(q(\bar{y}))}{A^2}\Big) = A\Big(\frac{B'q(\bar{y}) - B(q(\bar{y}))}{A^2}\Big) = \frac{A(B'-B)q(\bar{y})}{A^2} = q(\bar{y}).$$

Second case: $B = 0$. In this case, our original inequality is just

$$Ax < q(\bar{y})$$

and since $A > 0$,

$$x < \frac{q(\bar{y})}{A}.$$

Third case: $B' = 0$. In this case,

$$Ax < \sigma(q(\bar{y}))$$

and so

$$x < \frac{\sigma(q(\bar{y}))}{A}.$$

The converses in these cases are immediate upon multiplication by A. □

Example of Lemma 8. Suppose θ is $\exists x(7\sigma^2(x) + 3\sigma(x) - 2x = q(\bar{y}))$. Then, following the lemma, $\sigma(q(\bar{y})) = 7\sigma^3(x) + 3\sigma^2(x) - 2\sigma(x)$. In this case, $A = 8, B = 17, B' = 25$, and hence

$$x = \frac{25q(\bar{y}) - 17\sigma(q(\bar{y}))}{64}.$$

Thus,

$$(\mathcal{M}/\mathbb{Z}, \sigma) \models \exists x(7\sigma^2(x) + 3\sigma(x) - 2x = q(\bar{y})) \leftrightarrow x = \frac{25q(\bar{y}) - 17\sigma(q(\bar{y}))}{64}$$

and we have found a quantifier-free formula equivalent to θ.

With these technical lemmas in hand, we can prove that the theory of the expanded structure $\mathcal{M}/\mathbb{Z}$ has quantifier elimination.

Theorem 4. *Let $\mathcal{M}^* \models T^*$, where T^* is the set of sentences specified in Section 3.1 above, and let $\phi(x, \bar{y})$ be a quantifier-free formula that is a conjunction of literals in the language $\mathcal{L}_q$. Then there is a quantifier-free formula θ such that $\mathcal{M}^* \models \exists x \phi(x, \bar{y}) \leftrightarrow \theta(\bar{y})$.*

Proof. The proof proceeds in several steps. We consider $\exists x \phi$, where ϕ is arranged as follows:

$$\bigwedge (p_i(x) = q_i(\bar{y})) \wedge \tag{4.3}$$

$$\bigwedge \neg(p_l(x) < q_l(\bar{y})) \wedge \bigwedge \neg(p_u(x) > q_u(\bar{y})) \wedge \tag{4.4}$$

$$\bigwedge (p_k(x) < q_k(\bar{y})) \wedge \bigwedge (p_w(x) > q_w(\bar{y})) \wedge \tag{4.5}$$

$$\bigwedge (p_j(x) \neq q_j(\bar{y})) \wedge \tag{4.6}$$

$$\bigwedge D(p_m(x) + q_m(\bar{y})) \wedge \bigwedge \neg D(p_n(x) + q_n(\bar{y})). \tag{4.7}$$

Note that in the conjunction above, the term in $\bar{y}$ may be 0.

The goal is to find an equivalent formula not containing x.

First, because each negated strict inequality in (4.4) above is equivalent to a disjunction of an equation and the opposite strict inequality, and because each inequation in (4.6) above is equivalent to a disjunction of two strict inequalities, we may without loss of generality assume that ϕ is of the following form:

$$\bigwedge (p_i(x) = q_i(\bar{y})) \wedge \tag{4.8}$$

$$\bigwedge (p_k(x) < q_k(\bar{y})) \wedge \bigwedge (p_w(x) > q_w(\bar{y})) \wedge \tag{4.9}$$

$$\bigwedge D(p_m(x) + q_m(\bar{y})) \wedge \bigwedge \neg D(p_n(x) + q_n(\bar{y})). \tag{4.10}$$

Outline of the remainder of the proof: (1) If possible, we use a conjunct in (4.8) if there is one to solve for x in terms of $\bar{y}$ using Lemma 8. But even if there are conjuncts in (4.8), such a conjunct yields a solution for x only if $A \neq 0$ for some $p_i(x)$. If $A = 0$, then instead we obtain a solution for $\sigma(x) - x$ in terms of $\bar{y}$, and a solution for $\sigma(x) - x$ imposes constraints on x.

(2) Using Lemma 8, we next solve inequalities in (4.9). Again, if $A \neq 0$ for some $p_k(x)$ or $p_w(x)$, we obtain a solved inequality of the form $x < q_k^*(\bar{y})$, where $q_k^*(\bar{y})$ is a new σ-polynomial in $\bar{y}$ as given in the Lemma. Such inequalities will then be reconciled at the end of the proof. For inequalities in which $A = 0$ for $p_k(x)$ or $p_w(x)$, we obtain an inequality for $\sigma(x) - x$ in terms of $\bar{y}$, and such inequalities again impose constraints on x.

(3) We then find formulas equivalent to difference predicates and negated difference predicates by specifying conditions on bounds on x from inequalities that guarantee that the predicates will be satisfied.

(4) Finally, we combine inequalities to eliminate x in a formula equivalent to the original existential formula.

Step 1, equations: We examine all formulas of the form $p_i(x) = q_i(\bar{y})$ in which $A \neq 0$, if there are any such. We then apply Lemma 8 to solve for x in terms of $q_i(\bar{y})$ in one such equation, obtaining an equivalent formula

$$x = \frac{B_i'(q(\bar{y})) - B_i(\sigma(q(\bar{y}))}{A_i^2} = q_i^*(\bar{y})$$

or

$$x = \frac{q_i(\bar{y})}{A} = q_i^*(\bar{y}).$$

We now replace x in all other formulas by the term $q_i^*(\bar{y})$, and have thus eliminated x from ϕ and have a quantifier-free equivalent formula. (Note that in the special case in which $A \neq 0$ for $p(x)$ and the equation is $p(x) = 0$, we get just $x = 0$ and replace x with 0 throughout.)

If there are no equations in which $A \neq 0$, we examine all equations in which $A = 0$, if there are any such. If $B_i \neq 0$, such a formula can be re-written as

$$B_i d = (na_n + (n-1)a_{n-1} + \cdots + a_1)d = q_i(\bar{y})$$

where $d = \sigma(x) - x$. We thus obtain the solution $\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}$ from such a formula. (If also $B_i = 0$, then the equation is just $0 = q(\bar{y})$, which does not contain x and so we remove it from the scope of the quantifier.) We replace the original equation with this new equation. Thus, at the end of Step 1, we have either eliminated x completely from ϕ and can eliminate the existential quantifier, or, if there were no equations in which $A \neq 0$, we have obtained the following formula ϕ_1 such that $\exists x \phi$ is equivalent to $\exists x \phi_1$:

$$\phi_1 := \bigwedge_i \Big(\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}\Big) \wedge \tag{4.11}$$

$$\bigwedge_k (p_k(x) < q_k(\bar{y})) \wedge \bigwedge_w (p_w(x) > q_w(\bar{y})) \wedge \tag{4.12}$$

$$\bigwedge_m D(p_m(x) + q_m(\bar{y})) \wedge \bigwedge_n \neg D(p_n(x) + q_n(\bar{y})). \tag{4.13}$$

Proof of equivalence: To see that $\exists x \phi \Leftrightarrow \exists x \phi_1$, observe first that if there is an x witnessing ϕ, then such an x obviously satisfies the conjuncts in (4.12) and (4.13); it also satisfies the conjuncts in (4.11) because we have solved the equations in (4.8) (with $A = 0$) to obtain the conjuncts in (4.11). Conversely, if there is an x witnessing ϕ_1, then such an x obviously satisfies the conjuncts in (4.9) and (4.10), and it also satisfies the conjuncts in (4.8) because the steps in obtaining the equations in (4.11) are reversible.

Step 2, inequalities, part 1: Using Lemma 8, we solve each inequality in (4.12) above, obtaining two types of new inequalities.

First, where $A \neq 0$, we get equivalent inequalities $\bigwedge(x < q_k^*(\bar{y})) \wedge \bigwedge(x > q_w^*(\bar{y}))$, where $q_k^*(\bar{y})$ and $q_w^*(\bar{y})$ are the σ-polynomials in the solution provided in the lemma.

Second, where $A = 0$, we get new inequalities $\bigwedge(\sigma(x) - x < q_k^*(\bar{y})) \wedge \bigwedge(\sigma(x) - x > q_w^*(\bar{y}))$.

After re-indexing and renaming q^* as q, we now have the following formula ϕ_2, such that $\exists x\phi_1 \Leftrightarrow \exists x\phi_2$:

$$\phi_2 := \bigwedge_i \Big(\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}\Big) \wedge \tag{4.14}$$

$$\bigwedge_k (x < q_k(\bar{y})) \wedge \bigwedge_w (x > q_w(\bar{y})) \wedge \tag{4.15}$$

$$\bigwedge_p (\sigma(x) - x < q_p(\bar{y})) \wedge \bigwedge_r (\sigma(x) - x > q_r(\bar{y})) \wedge \tag{4.16}$$

$$\bigwedge D(p_m(x) + q_m(\bar{y})) \wedge \bigwedge \neg D(p_n(x) + q_n(\bar{y})). \tag{4.17}$$

Proof of equivalence: First, $\exists x\phi_1 \Rightarrow \exists x\phi_2$, because if x witnesses ϕ_1, then x obviously satisfies the conjuncts in (4.14) and (4.17); it also satisfies the inequalities in (4.15) and (4.16) because we solved the inequalities in (4.12) to obtain them. Conversely, if there is an x witnessing ϕ_2, then such an x obviously satisfies the conjuncts in (4.11) and (4.13). It also witnesses the inequalities in (4.12) by Lemma 8.

Remark: Note that if there is an element satisfying the formulas in (4.14)-(4.16), then there is some convex set of element that satisfies those formulas (as described in Step 4 below). This is because each equation in (4.14) is satisfied by a convex set (a translation of the fixed-point set of σ), and each formula in (4.15) and (4.16) is also satisfied by an (open) interval. Therefore, if there is an element satisfying the formulas in (4.14)-(4.16), there is a dense set of elements satisfying those formulas.

Step 3, difference predicates and negated difference predicates:

Case in which only negated difference predicates occur. If there are no difference predicates but only negated difference predicates in ϕ_2, then we want to show that there is a formula not containing x equivalent to the conjunction of those negated difference predicates and all inequalities. So let $\neg D(p_1(x) + q_1(\bar{y})), \ldots, \neg D(p_n(x) + q_n(\bar{y}))$ be those difference predicates. First, observe that by Lemma 7, if $A \neq 0$ then

$$\neg D(p(x) + q(\bar{y})) \leftrightarrow \neg D(Ax + Bd + q(\bar{y})) \leftrightarrow \neg D(Ax + q(\bar{y})) \leftrightarrow \neg D(x + \frac{q(\bar{y})}{A}),$$

and if $A = 0$, then

$$\neg D(p(x) + q(\bar{y})) \leftrightarrow \neg D(Bd + q(\bar{y})) \leftrightarrow \neg D(q(\bar{y})).$$

In the latter case where $A = 0$, we replace the negated difference predicate with its equivalent not containing x, and remove it from the scope of the quantifier.

In the case where $A \neq 0$, we consider the convex set of elements satisfying the formulas in (4.14)-(4.16) as described in the Remark above. Let (R, S) be an interval contained in that convex set. By density of non-differences in the entire structure, there is an element $R + y \neq 0$ in (R, S) such that y is not a difference and $R + 2y$ is also in (R, S). Then by Lemma 7, none of the $n + 1$ elements

$$y, y + y/2, y + y/2 + y/4, \ldots, y + y/2 + \cdots y/2^n$$

is a difference because each is a rational multiple of y. Also, all the $n+1$ elements

$$R+y, R+y+y/2, R+y+y/2+y/4, \ldots, R+y+y/2+\cdots y/2^n$$

are in the interval because all are between R and $R+2y$, and $R+2y$ is in (R,S). Now observe that for each of the terms $q_i(\bar{y})$ occurring in the negated difference predciates, at most one of the terms

$$q_i(\bar{y})+R+y, \ldots, q_i(\bar{y})+R+y+y/2+\cdots y/2^n$$

is a difference; if any two are a difference, then by subtraction so is some rational multiple of y, which is not possible. By the pigeonhole principle, there is at least one element x among

$$R+y, R+y+y/2, R+y+y/2+y/4, \ldots, R+y+y/2+\cdots y/2^n$$

such that the n sums $x+q_1(\bar{y}), \ldots, x+q_n(\bar{y})$ are all non-differences, and so all the negated difference predicates are satisfied by the same element x contained in the convex set determined by the formulas in (4.14)-(4.16). Thus, if only negated difference predicates are present, they are always satisfied so long as the formulas (4.14)-(4.16) are satisfied.

Case in which difference predicates occur. We now consider the difference predicates $D(p_m(x)+q_m(\bar{y}))$ which occur in ϕ_2. For all predicates in which $A=0$ in $p_m(x)$, by Lemma 7, $D(Ax+Bd+q_m(\bar{y}))$ is equivalent to $D(q_m(\bar{y}))$; we remove these from the scope of the quantifier. If $A \neq 0$, then also by Lemma 7,

$$D(Ax+Bd+q_m(\bar{y})) \leftrightarrow D(Ax+q_m(\bar{y})) \leftrightarrow D\Big(x+\frac{q_m(\bar{y})}{A}\Big).$$

Now by density of the differences in the fixed point set, the existence of an x making the rightmost formula above true is equivalent to $x + \frac{q_m(\bar{y})}{A}$ being either (1) between a fixed point and another point or (2) between two points neither of which is fixed but between which lies the entire fixed point set. We now examine the inequalities in (4.15) and (4.16) and the equations in (4.14) and find formulas not containing x that ensure that this condition is met. For formulas $\bigwedge(x < q_k(\bar{y})) \wedge \bigwedge(x > q_w(\bar{y}))$ that occur in (4.15), the following formula captures the condition:

$$\psi_m(\bar{y}) := \Big(\bigwedge_{w,k} q_w(\bar{y}) + \frac{q_m(\bar{y})}{A} < q_k(\bar{y}) + \frac{q_m(\bar{y})}{A}\Big) \wedge$$
$$\Big(\bigwedge_{w,k} \big(F(q_w(\bar{y}) + \frac{q_m(\bar{y})}{A})\big) \vee \big(F(q_k(\bar{y}) + \frac{q_m(\bar{y})}{A})\big) \vee \big(q_w(\bar{y}) + \frac{q_m(\bar{y})}{A} < 0 < q_k(\bar{y}) + \frac{q_m(\bar{y})}{A})\big)\Big).$$

Thus, if ψ_m holds, then there exists an x that satisfies both the difference predicate $D(p_m(x) + q_m(\bar{y}))$ and each of the inequalities in (4.15).

Next, we consider the formulas in (4.14), if any such occur. If formulas occur in (4.14), we must ensure that the x satisfying the difference predicate $D(x + \frac{q_m(\bar{y})}{A})$ and the inequalities in (4.15) also satisfies $\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}$. Since $x + \frac{q_m(\bar{y})}{A}$ must be a difference, we have

$$\sigma(x + \frac{q_m(\bar{y})}{A}) - (x + \frac{q_m(\bar{y})}{A}) = (\sigma(x) - x) + \Big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A}\Big) = 0$$

because differences are fixed points. Hence

$$(\sigma(x) - x) = \frac{q_i(\bar{y})}{B_i} = -\Big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A}\Big)$$

so we add the conjunction of formulas

$$\bigwedge_{m,i} \frac{q_i(\bar{y})}{B_i} = -\Big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A}\Big)$$

to ψ_m as well.

If there are also inequalities in (4.16), we must also ensure that they allow for the existence of an element x such that $D(x + \frac{q_m(\bar{y})}{A})$ is a difference. As above, there is such an x if and only if the following holds:

$$(\sigma(x) - x) + \Big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A}\Big) = 0$$

and thus, for all bounds in (4.16), if and only if the following holds:

$$\bigwedge_{p,r} \Big(q_r(\bar{y}) + \big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A}\big) < 0 < q_p(\bar{y}) + \big(\sigma(\frac{q_m(\bar{y})}{A}) - \frac{q_m(\bar{y})}{A})\Big).$$

So if there are inequalities in (4.16), we add this conjunction to ψ_m as well.

Finally, we must add formulas ensuring that the same x witnesses each additional difference predicate and each negated difference predicate. So for each additional difference predicate $D(x + \frac{q_{m'}(\bar{y})}{A_{m'}})$, we add a conjunct $\nu_{m'}$:

$$D\Big(\frac{q_{m'}(\bar{y})}{A_{m'}} - \frac{q_m(\bar{y})}{A}\Big).$$

We also add a corresponding conjunct $\neg\nu_n$ specifying that this x does not witness any of the negated difference predicates $\neg D_n(x + \frac{q_n(\bar{y})}{A_n})$:

$$\neg D\Big(\frac{q_n(\bar{y})}{A_n} - \frac{q_m(\bar{y})}{A}\Big).$$

After re-indexing, we have the following formula ϕ_3 such that $\exists x\phi_2 \Leftrightarrow \exists x\phi_3$:

$$\phi_3 := \bigwedge_i \Big(\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}\Big)\wedge \tag{4.18}$$

$$\bigwedge_k (x < q_k(\bar{y})) \wedge \bigwedge_w (x > q_w(\bar{y}))\wedge \tag{4.19}$$

$$\bigwedge_p (\sigma(x) - x < q_p(\bar{y})) \wedge \bigwedge_r (\sigma(x) - x > q_r(\bar{y}))\wedge \tag{4.20}$$

$$\psi_m(\bar{y}) \wedge \bigwedge_{m'} \nu_{m'} \wedge \bigwedge_n \neg\nu_n. \tag{4.21}$$

where m' ranges over those those difference predicates in which $A \neq 0$, and n ranges over those negated difference predicates in which $A \neq 0$.

Proof of equivalence: First, $\exists x\phi_2 \Rightarrow \exists x\phi_3$ because if x witnesses ϕ_2, then x obviously witnesses the conjuncts in (4.18), (4.19) and (4.20). If x witnesses the difference predicate $D(x + \frac{q_m(\bar{y})}{A})$ (resp. negated difference predicate $\neg D(p_n(x) + q_n(\bar{y}))$), with $A \neq 0$, it also witnesses the formulas in (4.21) by the discussion above.

Conversely, $\exists x\phi_3 \Rightarrow \exists x\phi_2$ for the following reasons. If x witnesses ϕ_3, then it obviously witnesses the conjuncts in (4.14), (4.15), and (4.16). Together with ψ_m, the formulas $\nu_{m'}$ ensure that such an x also satisfies all other difference predicates, because by Lemma 7,

$$\Big(D(x + \frac{q_m(\bar{y})}{A}) \wedge D\Big(\frac{q_{m'}(\bar{y})}{A_{m'}} - \frac{q_m(\bar{y})}{A}\Big)\Big) \to D(x + \frac{q_{m'}(\bar{y})}{A_{m'}})$$

and the formulas ν_n ensure that such an x also satisfies all the negated difference predicates because, again by Lemma 7,

$$\Big(D(x + \frac{q_m(\bar{y})}{A}) \wedge \neg D\Big(\frac{q_n(\bar{y})}{A_n} - \frac{q_m(\bar{y})}{A}\Big)\Big) \to \neg D(x + \frac{q_n(\bar{y})}{A_n}).$$

Step 4, inequalities part 2. We first work with the conjunction of inequalities in (4.20):

$\bigwedge(\sigma(x) - x < q_p(\bar{y})) \wedge \bigwedge(\sigma(x) - x > q_r(\bar{y}))$, and the equations in (4.18), if any occur. (As to the equations in (4.18) each of the form $\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}$, in the conjunction θ below, the conjuncts η_i state that all the terms $\frac{q_i(\bar{y})}{B_i}$ are equal and are differences). If no inequalities involving x occur, then the existence of an x witnessing these inequalities is equivalent to the conjunction

$$\theta_{p,r,i}(\bar{y}) := \bigwedge \left(q_r(\bar{y}) < \frac{q_i(\bar{y})}{B_i} < q_p(\bar{y})\right) \wedge \eta_i(\bar{y})$$

because the only constraint on x is that $\sigma(x) - x = \frac{q_i(\bar{y})}{B_i}$.

Remark. In case there are no equations in (4.18) and no inequalities in (4.19), we instead obtain the conjunction

$$\theta_{p,r}(\bar{y}) := \bigwedge(q_r(\bar{y}) < q_p(\bar{y})) \wedge \Big(\bigwedge(F(q_r(\bar{y}) \vee F(q_p(\bar{y}))\vee$$
$$\bigwedge \left((q_r(\bar{y}) < 0 < (q_p(\bar{y}))\right)\Big).$$

A witness x to this conjunction exists precisely if a witness exists to the conjunction in (4.20) alone.

If inequalities involving x also occur, we must also ensure that $\sigma(x) - x$ corresponds to the bounds on x. By Lemma 6, corresponding to the inequality $x < q_k(\bar{y})$, we have $\sigma(x) - x \leq \sigma(q_k(\bar{y})) - q_k(\bar{y})$, and corresponding to the inequality $x > q_w(\bar{y})$, we have $\sigma(x) - x \geq \sigma(q_w(\bar{y})) - q_w(\bar{y})$. (Equality is possible if x differs from $q_k(\bar{y})$ (or $q_w(\bar{y})$) by a fixed point.) Hence, we also add inequalities

$$\bigwedge_{k,r} q_r(\bar{y}) < \sigma(q_k(\bar{y})) - q_k(\bar{y}) \wedge \bigwedge_{w,p} \sigma(q_w(\bar{y})) - q_w(\bar{y}) < q_p(\bar{y})$$

Finally, we eliminate x from the inequalities involving x alone, and replace them by the conjunction:

$$\bigwedge_{k,w} q_w(\bar{y}) < q_k(\bar{y}).$$

In our new formula ϕ_4, we eliminated x; all subformulas contain only the parameters $\bar{y}$:

$$\phi_4 := \psi_m(\bar{y}) \wedge \bigwedge_{m'} \nu_{m'} \wedge \bigwedge_{n} \neg\nu_n \tag{4.22}$$

$$\bigwedge \theta_{p,r,i}(\bar{y}) \wedge \tag{4.23}$$

$$\bigwedge_{k,r} q_r(\bar{y}) < \sigma(q_k(\bar{y})) - q_k(\bar{y}) \wedge \bigwedge_{w,p} \sigma(q_w(\bar{y})) - q_w(\bar{y}) < q_p(\bar{y}) \tag{4.24}$$

$$\bigwedge_{k,w} q_w(\bar{y}) < q_k(\bar{y}). \tag{4.25}$$

Proof of equivalence: $\exists x\phi_3 \Rightarrow \exists x\phi_4$: The formulas in (4.22) are exactly the formulas in (4.21), and do not contain x. If there is an x witnessing the conjuncts in (4.18), (4.19) and (4.20), then the conjuncts in (4.23)-(4.25) must hold, as set forth above. $\exists x\phi_4 \Rightarrow \exists x\phi_3$: If there is an x satisfying the formulas in (4.22), then there is an x satisfying the identical formulas in (4.21). If such an x also satisfies the formulas in (4.23)-(4.25), then such an x satisfies the formulas in (4.18)-(4.20), also by the derivation above.
Since x no longer occurs in the formula $\exists x\phi_4$, we can remove the quantifier and have completed the quantifier elimination for $\exists\phi(x, \bar{y})$. □

As a corollary, we have:

Corollary 2. *The theory T^* is complete and decidable.*

Proof. Any quantifier-free sentences is a Boolean combination of sentences of the following forms, because the only constant in $\mathcal{L}_q$ is 0, and because $\sigma^{\pm 1}(q0) = \sigma^{\pm 1}(0) = 0$ $(q \in \mathbb{Q})$). These atomic sentences are decidable, as follows:

1. $D(0) \leftrightarrow \top, \neg D(0) \leftrightarrow \bot$

2. $0 = 0 \leftrightarrow \top, 0 \neq 0 \leftrightarrow \bot$

3. $0 < 0 \leftrightarrow \bot$

Thus, T^* is complete and decidable. $\square$

Chapter 5

Quantifier elimination for $(\mathcal{M}, \sigma)$

The main additional difficulty in proving quantifier elimination for $(\mathcal{M}, \sigma)$ arises from the fact that in the lift of $(\mathcal{M}/\mathbb{Z}, \sigma)$, the differences are no longer dense in the fixed-point set of M, but instead those $\mathbb{Z}$-chains containing a difference are dense in the set of all $\mathbb{Z}$-chains in the fixed-point set. Apart from this, the method we use to prove quantifier elimination involves lemmas similar to those in Chapter 4. To make the quantifier elimination possible, we add to the language two new definable unary predicates. We start with the language in which the set of axioms for $(\mathcal{M}, \sigma)$ is written in section 3.2 above. These axioms are in the language $\mathcal{L}_T = (+, -, \sigma, \sigma^{-1}, P_n(n = 2, 3, \ldots), D, Z, 0, 1)$, where D is a unary predicate defined by $D(x) \leftrightarrow \exists w(\sigma(w) - w = x)$. We now add to this language the following predicates, and recall the predicate Z:

1. $D^+(x) \leftrightarrow (\exists z(x < z) \wedge (\exists! w(x < w < z \wedge D(w))))$ (x is in the same $\mathbb{Z}$-chain as a difference, and that difference is greater than x)

2. $D^-(x) \leftrightarrow (\exists z(z < x) \wedge (\exists! w(z < w < x \wedge D(w)))))$ (x is in the same $\mathbb{Z}$-chain as a difference, and that difference is less than x)

3. $(Z(x) \leftrightarrow (x < 0 \wedge \forall y(x < y < 0 \rightarrow \neg D(y)) \vee (0 < x \wedge \forall y(0 < y < x \rightarrow \neg D(y)))$ (x is

in the standard part (the same $\mathbb{Z}$-chain as 0))

In addition to the new unary predicates, we use the following abbreviation for convenience: $F(a) \Leftrightarrow \sigma(a) = a$.

5.1 Preliminary lemmas

To begin, we prove some ancillary lemmas analogous to those in Chapter 4.

Lemma 9. *(Closure properties of the set D).*
Let $k \in \mathbb{Z}$. Then:

1. $D(a) \wedge D(b) \rightarrow (D(a+b) \wedge D(a-b))$;

2. $D(a) \leftrightarrow D(ka)$ *if* $k \neq 0$;

3. $D(a) \leftrightarrow (D(\sigma^m(a))(m \in \mathbb{Z})$;

4. $D(a) \wedge \neg D(c) \rightarrow \neg D(a+c) \wedge \neg D(a-c)$; *and*

5. $(\neg D(c) \wedge k \neq 0) \leftrightarrow \neg D(kc)$.

Proof. (1) For some x, y, $a = \sigma(x) - x, b = \sigma(y) - y$. Then
$\sigma(x+y) - (x+y) = \sigma(x) + \sigma(y) - x - y = a + b$, and
$\sigma(x-y) - (x-y) = \sigma(x) - \sigma(y) - x + y = a - b$. (2) ($\rightarrow$) For some x, $\sigma(x) - x = a$. Then $\sigma(kx) - kx = k\sigma(x) - kx = ka$. ($\leftarrow$) For some x, $\sigma(x) - x = ka$. Now one of the elements of $\{x, x+1, \ldots, x+k-1\}$ is divisible by k. Let $y = x + i$ be divisible by k, so $y = kz$. Then:

$$\sigma(y) - y = \sigma(x+i) - (x+i) = \sigma(x) + \sigma(i) - x - i = \sigma(x) - x = ka$$

so $\sigma(kz) - kz = ka$, hence $k(\sigma(z) - z) = ka$, and $\sigma(z) - z = a$, and $D(a)$.

(3) All differences are fixed by σ. (4) If $D(a)$ and $D(a+c)$, then $D(c)$ by (1) and (2). If $D(a)$ and $D(a-c)$, then $D(c)$ by (1) and (2). (5) is the negation of (2). □

Lemma 10. *(Applying positive coefficient polynomials to inequalities).*
Let $p(x) = a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x$, *with* σ *modest and increasing and such that* $0 < A$. *If* $0 < a < b \in M$, *then* $p(a) < p(b)$.

Proof. If a and b are both fixed points, then $p(a) = Aa$, $p(b) = Ab$, and the result is immediate. If a is a fixed point and b is not, then $p(a) = Aa$ and $p(b) = Ab + B(\sigma(b) - b)$, and $p(a) < p(b)$ because, regardless of the coefficient B, $\sigma(b) - b$ is in a magnitude class lower than b, and so $Ab + B(\sigma(b) - b)$ is in a magnitude class above a.
If both a and b are moved by σ, put $\sigma(a) = a + c, \sigma(b) = b + d$.
Then

$$p(a) = Aa + (na_n + (n-1)a_{n-1} + \cdots + 2a_2 + a_1)c \tag{5.1}$$

$$p(b) = Ab + (na_n + (n-1)a_{n-1} + \cdots + 2a_2 + a_1)d \tag{5.2}$$

Therefore $p(b) - p(a) = A(b-a) + (na_n + (n-1)a_{n-1} + \cdots a_1)(d-c)$. Note that each of c, d and $d - c$ is in the fixed point set F of σ.
If the magnitude class $v(b)$ of b is greater than the magnitude class of a, then $p(b) - p(a)$ is obviously positive.
If $v(a) = v(b) = v(b-a)$, then again this difference is positive, because $v(A(b-a)) > v(d-c)$.
If $v(b-a) < v(a) = v(b)$ and $b - a = k \notin F$, then $b = a + k$, and

$$p(b) - p(a) = Ak + (na_n + (n-1)a_{n-1} + \cdots a_1)(d-c)$$

Again, because $v(k) > v(d)$ and $v(k) > v(c)$, this difference is positive.

If $v(b-a) < v(a) = v(b)$ and $b - a = m \in F$, then $b = a + m$, and

$\sigma(b) - b = (\sigma(a+m)) - (a+m) = (\sigma(a) + m) - (a+m) = \sigma(a) - a$, and therefore $c = d$.

Therefore

$$p(b) - p(a) = Am$$

which is once again positive. □

The following lemma is almost identical to the corresponding lemma in Chapter 4, except that we need to impose a divisibility condition for the solution of an equation to exist, arising from the fact that elements are in general not divisible.

Lemma 11. *(Solving equations).*

Suppose that $a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x = q(\bar{y})$, where q is a σ-polynomial in parameters $\bar{y}$. If $A \neq 0$, then this equation has a solution if and only if

$$\Big(A^2x = B'(q(\bar{y})) - B(\sigma(q(\bar{y}))\Big) \wedge P_{A^2}(B'((q(\bar{y})) - B(\sigma(q(\bar{y}))).$$

Proof. Because σ is a modest increasing automorphism, the left-hand side of

$$a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x = q(\bar{y}) \quad (^*)$$

is equal to:

$$(a_n + a_{n-1} + \cdots + a_1 + a_0)x + (na_n + (n-1)a_{n-1} + \cdots + a_1)d$$

where $\sigma(x) - x = d$.

Thus (*) is

$$Ax + Bd = q(\bar{y})$$

Now apply σ to both sides of (*) to get:

$$\sigma(Ax + Bd) = a_n\sigma^{n+1}(x) + a_{n-1}\sigma^n(x) + \cdots + a_1\sigma^2(x) + a_0\sigma(x) = \sigma(q(\bar{y}))$$

and in $\sigma(Ax + Bd)$ the coefficient of x is still A, but the new coefficient of d is

$$B' = (n+1)a_n + na_{n-1} + \cdots + 2a_1 + a_0$$

and so $\sigma(Ax + Bd) = Ax + B'd = \sigma(q(\bar{y}))$.

Now observe that $B' - B = a_n + a_{n_1} + \cdots + a_1 + a_0 = A$, so $B' - B = 0 \implies A = 0$. Since $B' - B \neq 0$,

$$B'(Ax + Bd) = B'Ax + B'Bd = B'(q(\bar{y})), \quad B(Ax + B'd) = BAx + BB'd = B(\sigma(q(\bar{y}))$$

Hence

$$(B'Ax + B'Bd) - (BAx + BB'd) = (B' - B)Ax = A^2x = B'(q(\bar{y})) - B(\sigma(q(\bar{y})).$$

This equation has a solution if and only if the right side is divisible by A^2, so $P_{A^2}(B'(q(\bar{y})) - B(\sigma(q(\bar{y}))))$, and $A^2x = B'(q(\bar{y})) - B(\sigma(q(\bar{y}))$. $\square$

Remark. If $B = 0$, we get the simpler equivalent formula

$$Ax = q(\bar{y}) \wedge P_A(q(\bar{y})),$$

as we see by replacing B by 0, and if $B' = 0$ (hence $A = -B$), we get the simpler equivalent formula

$$Ax = \sigma(q(\bar{y})) \wedge P_A(\sigma(q(\bar{y}))).$$

In the following lemma, we see again a change from the corresponding lemma in Chapter 4, made necessary by the fact the the elements of $\mathcal{M}$ are not in general divisible.

Lemma 12. *(Solving inequalities). Suppose that* $a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x < q(\bar{y})$*, where* q *is a* σ*-polynomial in parameters* $\bar{y}$*. If* $A > 0$*, then the inequality is equivalent to:*

$$A^2x < B'(q(\bar{y})) - B(\sigma(q(\bar{y}))$$

where B, B' *are as above. Similarly, if* $r(\bar{y}) < a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_1\sigma(x) + a_0x$*, then the inequality is equivalent to:*

$$B'(r(\bar{y})) - B(\sigma(r(\bar{y})) < A^2x.$$

If $B = 0$*, then the inequalities above are equivalent to:*

$$Ax < q(\bar{y}), r(\bar{y}) < Ax.$$

If $B' = 0$*, then the inequalities above are equivalent to*

$$Ax < \sigma(q(\bar{y})), \sigma(r(\bar{y})) < Ax.$$

(If $A < 0$*, take the negative of the inequalities above to obtain an equivalent inequality with* $A > 0$*.)*

Proof. For the case $<$: The left-hand-side of the inequality is

$$Ax + Bd < q(\bar{y}).$$

Case 1: $B = 0$. So $Ax < q(\bar{y})$.

Case 2: $B' = 0$. In this case, $\sigma(Ax + Bd) = Ax + B'd = Ax < \sigma(q(\bar{y}))$.

Case 3: $B \neq 0$ and $B' \neq 0$. Since $Ax + Bd < q(\bar{y})$, there is some $M > 0$ such that $Ax + Bd + M = q(\bar{y})$. We multiply both sides of $Ax + Bd + M = q(\bar{y})$ by B' and both sides of $Ax + B'd + \sigma(M) = \sigma(q(\bar{y}))$ by B and subtract to get

$$(B'Ax + B'Bd + B'M) - (BAx + BB'd + B\sigma(M) = B'q(\bar{y}) - B(\sigma(q(\bar{y})))$$

so

$$(B'A - BA)x + (B'M - B\sigma(M)) = A^2x + (B'M - B\sigma(M)) = B'q(\bar{y}) - B(\sigma(q(\bar{y})))$$

Now if $\sigma(M) = M$, then this equation becomes

$$A^2x + AM = B'q(\bar{y}) - B(\sigma(q(\bar{y})))$$

Since $A, M > 0$, also $AM > 0$ so

$$A^2x < B'q(\bar{y}) - B(\sigma(q(\bar{y}))).$$

If $\sigma(M) \neq M$, then since $M > 0$, $\sigma(M) > M$. Put $\sigma(M) = M + d'$. Then the equation becomes:

$$A^2x + (B'M - B(M + d')) = B'q(\bar{y}) - B(\sigma(q(\bar{y})))$$

But $B'M - B(M + d') = B'M - BM - Bd' = (B' - B)M - Bd' = AM - Bd' > 0$, because $A > 0$ and d' is in a lower magnitude class than the magnitude class of M.

So again, we get:

$$A^2 x < B' q(\bar{y}) - B(\sigma(q(\bar{y}))).$$

The proof of the opposite inequality is similar. For the converse, we apply Lemma 10 just as we did in proving the corresponding Lemma on solving inequalities in Chapter 4. □

Lemma 13. *(Differences increasing). If $0 < a < b$ then $\sigma(a) - a \leq \sigma(b) - b$.*

Proof. Since $a < b$, $0 < b - a$ and so because σ is increasing, $b - a \leq \sigma(b - a)$. Equality holds if and only if $b - a$ is fixed by σ. Thus, since σ is an automorphism, $b - a \leq \sigma(b) - \sigma(a)$. Upon adding $\sigma(a) - b$ to both sides, we get $\sigma(a) - a \leq \sigma(b) - b$, with equality if and only if $b - a$ is fixed by σ. □

Finally, some rather trivial properties of the predicate for the standard integers.

Lemma 14. *(Properties of the predicate Z).*

Suppose $Z(a), Z(b)$, and let $p(x)$ be a σ-polynomial with $A \neq 0$. Then:

1. $Z(a + b), Z(a - b)$
2. $Z(ka), k \in \mathbb{Z}$
3. $Z(a + c) \leftrightarrow Z(c)$
4. $Z(\sigma^k(a)), k \in \mathbb{Z}$
5. $Z(a) \leftrightarrow Z(p(a))$.

Proof. (1) If a and b are both standard integers, so is their sum and difference. (2) If a is a standard integer, so is any standard integer multiple of a. (3) If a is a standard integer and $a + c$ is a standard integer, then their difference c must be a standard integer. (4) σ fixes standard integers. (5) If $Z(a)$, then a is fixed by σ, so $p(a) = Aa$, which is a standard integer. For the converse, suppose that $\neg Z(a)$. If a is fixed by σ, then again $p(a) = Aa$, so also $\neg Z(p(a))$. If a is not fixed by σ, then $p(a) = Aa + B(\sigma(a) - a)$, and because $\sigma(a) - a$ is in a magnitude class lower than the magnitude class of a, $p(a)$ is in a magnitude class different from 0 as well. $\square$

Equipped with these lemmas, we prove:

Theorem 5. *Let $\mathcal{M} \models T$, where T is the set of sentences specified in section 3.2 above, and let $\phi(x, \bar{y})$ be a quantifier-free formula that is a conjunction of literals in the language $\mathcal{L} = (+, -, \sigma, \sigma^{-1}, <, P_n\ (n = 2, 3, \ldots), D, D^+, D^-, Z, 0, 1)$. Then there is a quantifier-free formula θ such that $\mathcal{M} \models \exists \phi(x, \bar{y}) \leftrightarrow \theta(\bar{y})$.*

Proof. Outline of the proof. (1) We first solve equations in which $A \neq 0$, if any occur, to immediately achieve quantifier elimination. (2) We next solve inequalities. (3) We next simplify all predicates and implement a change of variable to make the coefficient of x uniform across all subformulas. (4) With the simplified formulas, we impose conditions on the inequalities involving x and $\sigma(x) - x$ that guarantee a witness to all the predicates, and thereby reduce predicates to formulas not containing x. (5) Finally, we eliminate x and $\sigma(x) - x$ from all inequalities, producing a formula equivalent to ϕ in the parameters $\bar{y}$ only, and can then eliminate the existential quantifier.

Remark. In the case in which ϕ contains no inequalities involving x, we may assume without loss of generality that it contains the formula $x < 0$ (or the formula $(x > 0)$). In addition, because a negated divisibility predicate is equivalent to a disjunction of divisibility predicates, we may assume that ϕ contains no negated divisibility predicates. Also, as in Chapter 4, we may assume that ϕ contains no inequations or negated inequalities.

To begin, we arrange ϕ as:

$$\bigwedge (p_i(x) = q_i(\bar{y})) \wedge \tag{5.3}$$

$$\bigwedge (p_l(x) < q_l(\bar{y})) \wedge \bigwedge (p_m(x) > q_m(\bar{y})) \wedge \tag{5.4}$$

$$\bigwedge D(p_t(x) + q_t(\bar{y})) \wedge \bigwedge \neg D(p_u(x) + q_u(\bar{y})) \wedge \tag{5.5}$$

$$\bigwedge D^+(p_\alpha(x) + q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(p_\beta(x) + q_\beta(\bar{y})) \wedge \tag{5.6}$$

$$\bigwedge D^-(p_\delta(x) + q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(p_\zeta(x) + q_\zeta(\bar{y})) \wedge \tag{5.7}$$

$$\bigwedge Z(p_v(x) + q_v(\bar{y})) \wedge \bigwedge \neg Z(p_w(x) + q_w(\bar{y})) \wedge \tag{5.8}$$

$$\bigwedge P_{a_i}(p_j(x) + q_j(\bar{y})) \tag{5.9}$$

where the p's and q's are σ-polynomials with coefficients in $\mathbb{Z}$, with possible constant terms in $\mathbb{Z}$. Without loss of generality, we clear negative powers of σ by acting on terms of the form $p(x) + q(\bar{y})$ that occur in ϕ by the appropriate positive power of σ to obtain equivalent formulas. Thus, we assume no negative powers of σ occur in ϕ. This is the most general form of ϕ; fewer kinds of predicates may be present.

Step 1: equations with $A \neq 0$. If ϕ contains an equation $p(x) = q(\bar{y})$ with $A \neq 0$, this equation is of the form

$$Ax + B(\sigma(x) - x) = q(\bar{y}).$$

Then by Lemma 11, this equation is equivalent to

$$A^2 x = B'(q(\bar{y})) - B(\sigma(q(\bar{y})) \wedge P_{A^2}(B'((q(\bar{y})) - B(\sigma(q(\bar{y}))).$$

We now multiply every other equation, inequality and argument predicate through by A^2.

That is, we consider the following formula:

$$\phi^* := \bigwedge (A^2 p_i(x) = A^2 q_i(\bar{y})) \wedge$$
$$\bigwedge (A^2 p_l(x) < A^2 q_l(\bar{y})) \wedge \bigwedge (A^2 p_m(x) > A^2 q_m(\bar{y})) \wedge$$
$$\bigwedge D(A^2 p_t(x) + A^2 q_t(\bar{y})) \wedge \bigwedge \neg D(A^2 p_u(x) + A^2 q_u(\bar{y})) \wedge$$
$$\bigwedge D^+(A^2 p_\alpha(x) + A^2 q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(A^2 p_\beta(x) + A^2 q_\beta(\bar{y})) \wedge$$
$$\bigwedge D^-(A^2 p_\delta(x) + A^2 q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(A^2 p_\zeta(x) + A^2 q_\zeta(\bar{y})) \wedge$$
$$\bigwedge Z(A^2 p_v(x) + A^2 q_v(\bar{y})) \wedge \bigwedge \neg Z(A^2 p_w(x) + A^2 q_w(\bar{y})) \wedge$$
$$\bigwedge P_{A^2 a_i}(A^2 p_j(x) + A^2 q_j(\bar{y})).$$

In ϕ^*, each subformula is equivalent to the corresponding subformula in ϕ. For equations and inequalities, this is clear because $A^2 > 0$. For difference and standard integer predicates, Lemmas 9 and 14 establish equivalence. For directed difference predicates, because $A^2 > 0$, if $D^+(t)$ for a term t, then $D(t + k)$ for some positive integer k, hence, by Lemma 9, $D(A^2 t + A^2 k)$, so $D^+(A^2 t)$ (similarly for D^- predicates and negated directed difference predicates), and conversely. Finally, for divisibility predicates, if a term $A^2 t$ is divisible by $A^2 a_i$, then the term t is divisible by a_i; and, conversely, if t is divisible by a_i, then $A^2 t$ is divisible by $A^2 a_i$. Each of the terms $A^2 p(x)$ in ϕ^* is in turn equal to $p(A^2 x)$. We now replace $A^2 x$ throughout with $B'(q(\bar{y})) - B(\sigma(q(\bar{y}))$ to obtain a new formula which, together with the predicate $P_{A^2}(B'((q(\bar{y})) - B(\sigma(q(\bar{y})))$, is equivalent to ϕ, and have eliminated x in this equivalent formula. So we may assume that ϕ contains no such conjuncts in (5.3), and proceed to Step 2.

Step 2: inequalities, part 1. Applying Lemma 12, we replace each inequality with $A \neq 0$ in the conjunction $\bigwedge p_l(x) < q_l(\bar{y})$, and each inequality with $A \neq 0$ in the conjunction $\bigwedge p_m(x) > q_m(\bar{y})$, with the equivalent inequalities

$$A^2x < B'(q_l(\bar{y})) - B(\sigma(q_l(\bar{y}))) \text{ or } A^2x < Aq_l(\bar{y}),$$

and

$$B'(q_m(\bar{y})) - B(\sigma(q_m(\bar{y}))) < A^2x \text{ or } Aq_m(\bar{y}) < A^2x,$$

respectively. We re-label these inequalities as $A^2x < q_l(\bar{y})$ and $q_m(\bar{y}) < A^2x$.

If $A = 0$, then the inequality $p_l(x) < q_l(\bar{y})$ is of the form $B_l(\sigma(x) - x) < q_l(\bar{y})$ (in the case $B_l > 0$, else the opposite inequality) by:

$$B_l(\sigma(x) - x) < q_l(\bar{y})$$

Similarly, in the case where $A = 0$, we replace the inequality $p_m(x) > q_m(\bar{y})$ by:

$$q_m(\bar{y}) < B_m(\sigma(x) - x).$$

We re-label these formulas as $B_l(\sigma(x) - x) < q_p(\bar{y})$ and $q_r(\bar{y}) < B_m(\sigma(x) - x)$, respectively. After these replacements and relabeling, we have a new formula ϕ_1

$$\phi_1 := \bigwedge (p_i(x) = q_i(\bar{y})) \wedge \tag{5.10}$$

$$\bigwedge (A_l^2 x < q_l(\bar{y})) \wedge \bigwedge (A_m^2 x > q_m(\bar{y})) \wedge \tag{5.11}$$

$$\bigwedge (B_l(\sigma(x) - x) < q_p(\bar{y})) \wedge \bigwedge B_m((\sigma(x) - x) > q_r(\bar{y})) \wedge \tag{5.12}$$

$$\bigwedge D(p_t(x) + q_t(\bar{y})) \wedge \bigwedge \neg D(p_u(x) + q_u(\bar{y})) \wedge \tag{5.13}$$

$$\bigwedge D^+(p_\alpha(x) + q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(p_\beta(x) + q_\beta(\bar{y})) \wedge \tag{5.14}$$

$$\bigwedge D^-(p_\delta(x) + q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(p_\zeta(x) + q_\zeta(\bar{y})) \wedge \tag{5.15}$$

$$\bigwedge Z(p_v(x) + q_v(\bar{y})) \wedge \bigwedge \neg Z(p_w(x) + q_w(\bar{y})) \wedge \tag{5.16}$$

$$\bigwedge P_{a_i}(p_j(x) + q_j(\bar{y})). \tag{5.17}$$

Note that in ϕ_1, all the equations in (5.10) have $A = 0$.

Proof of equivalence. $\exists x \phi \Rightarrow \exists x \phi_1$: If there is an x witnessing the formulas in (5.3)-(5.9), it clearly witnesses the formulas in (5.10) and (5.13)-(5.17), which occur identically in both ϕ and ϕ_1. Any such x also satisfies the formulas in (5.11) and (5.12) because (5.11) and (5.12) were derived from the formulas in (5.4). $\exists x \phi_1 \Rightarrow \exists x \phi$: Again, if there is an x witnessing the formulas in (5.10) and (5.13)-(5.17), it also witnesses the identical formulas in (5.3) and (5.5)-(5.9). If such an x also witnesses the formulas in (5.11) and (5.12), then we can reverse the steps used in deriving those formulas above to obtain the formulas in (5.4), so such an x witnesses the formulas in (5.4) too. Similarly, we can obtain the original inequalities in (5.4) because the converse holds in Lemma 12.

Step 3: equations with $A = 0$. If ϕ_1 contains at least one equation with $A = 0$, each such equation has the form:

$$B_i(\sigma(x) - x) = q_i(\bar{y}).$$

If $B_i = 0$, then such an equation is $0 = q_i(\bar{y})$ and we remove the equation from the scope of the quantifier. We replace each equation in (5.10) by one of the form above to obtain ϕ_2:

$$\phi_2 := \bigwedge B_i(\sigma(x) - x) = q_i(\bar{y}) \wedge \tag{5.18}$$

$$\bigwedge (A_l^2 x < q_l(\bar{y})) \wedge \bigwedge (A_m^2 x > q_m(\bar{y})) \wedge \tag{5.19}$$

$$\bigwedge (B_l(\sigma(x) - x) < q_p(\bar{y})) \wedge \bigwedge (B_m(\sigma(x) - x) > q_r(\bar{y})) \wedge \tag{5.20}$$

$$\bigwedge D(p_t(x) + q_t(\bar{y})) \wedge \bigwedge \neg D(p_u(x) + q_u(\bar{y})) \wedge \tag{5.21}$$

$$\bigwedge D^+(p_\alpha(x) + q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(p_\beta(x) + q_\beta(\bar{y})) \wedge \tag{5.22}$$

$$\bigwedge D^-(p_\delta(x) + q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(p_\zeta(x) + q_\zeta(\bar{y})) \wedge \tag{5.23}$$

$$\bigwedge Z(p_v(x) + q_v(\bar{y})) \wedge \bigwedge \neg Z(p_w(x) + q_w(\bar{y})) \wedge \tag{5.24}$$

$$\bigwedge P_{a_i}(p_j(x) + q_j(\bar{y})). \tag{5.25}$$

Proof of equivalence. It is clear that $\exists x \phi_1 \Leftrightarrow \exists x \phi_2$. Any x satisfying (5.19)-(5.25) satisfies the identical formulas in (5.11)-(5.17), and conversely. And any such x which also satisfies (5.18) satisfies (5.10).

Step 4: Difference predicates, directed difference predicates, divisibility predicates, and negations of such predicates in which $A = 0$. We now consider all predicates in (5.21), (5.22), (5.23), and (5.25) in which $A = 0$. Because $A = 0$, each term $p_k(x)$ in each such predicate is of the form $B(\sigma(x) - x)$.

Such conjuncts in (5.21), (5.22), (5.23), and (5.25) can be written in the form:

$$\bigwedge D(B_t(\sigma(x)-x)+q_t(\bar{y})) \wedge \bigwedge \neg D(B_u(\sigma(x)-x)+q_u(\bar{y}))\wedge$$
$$\bigwedge D^+(B_\alpha(\sigma(x)-x)+q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(B_\beta(\sigma(x)-x)+q_\beta(\bar{y}))\wedge$$
$$\bigwedge D^-(B_\delta(\sigma(x)-x)+q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(B_\zeta(\sigma(x)-x)+q_\zeta(\bar{y}))\wedge$$
$$\bigwedge P_{a_i}(B_j(\sigma(x)-x)+q_j(\bar{y})).$$

Next, by Lemma 9 and because the differences $B_{a_i}(\sigma(x)-x)$ in the divisibility predicate have zero residue for every $n \in \{2,3,\ldots\}$, this conjunction is equivalent to the following conjunction α not containing x:

$$\alpha(\bar{y}) := \bigwedge D(q_t(\bar{y})) \wedge \bigwedge \neg D(q_u(\bar{y}))\wedge$$
$$\bigwedge D^+(q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(q_\beta(\bar{y}))\wedge$$
$$\bigwedge D^-(q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(q_\zeta(\bar{y}))\wedge$$
$$\bigwedge P_{a_i}(q_j(\bar{y})).$$

Since these conjuncts do not contain x, we remove them from the scope of the quantifier to obtain a new formula ϕ_3 such that $\exists x\phi_3 \Leftrightarrow \exists x\phi_2$:

$$\phi_3 := \bigwedge B_i(\sigma(x) - x) = q_i(\bar{y}) \wedge \tag{5.26}$$

$$\bigwedge (A_l^2 x < q_l(\bar{y})) \wedge \bigwedge (A_m^2 x > q_m(\bar{y})) \wedge \tag{5.27}$$

$$\bigwedge (B_l(\sigma(x) - x) < q_p(\bar{y})) \wedge \bigwedge (B_m(\sigma(x) - x) > q_r(\bar{y})) \wedge \tag{5.28}$$

$$\bigwedge D(p_t(x) + q_t(\bar{y})) \wedge \bigwedge \neg D(p_u(x) + q_u(\bar{y})) \wedge \tag{5.29}$$

$$\bigwedge D^+(p_\alpha(x) + q_\alpha(\bar{y})) \wedge \bigwedge \neg D^+(p_\beta(x) + q_\beta(\bar{y})) \wedge \tag{5.30}$$

$$\bigwedge D^-(p_\delta(x) + q_\delta(\bar{y})) \wedge \bigwedge \neg D^-(p_\zeta(x) + q_\zeta(\bar{y})) \wedge \tag{5.31}$$

$$\bigwedge Z(p_v(x) + q_v(\bar{y})) \wedge \bigwedge \neg Z(p_w(x) + q_w(\bar{y})) \wedge \tag{5.32}$$

$$\bigwedge P_{a_i}(p_j(x) + q_j(\bar{y})) \tag{5.33}$$

where $A \neq 0$ in each of the conjuncts in (5.29), (5.30), (5.31) and (5.33).

Proof of equivalence. We have simply removed formulas from ϕ_2 with equivalents not containing x.

Step 5: Simplifying remaining predicates. We consider next all the predicates in (5.29)-(5.33), with the aim of replacing each with an equivalent, simpler predicate.

1. *Difference predicates and negated difference predicates*:

For each difference predicate in (5.29), we apply Lemma 9 to obtain the following equivalence:

$$D(p_t(x) + q_t(\bar{y})) \leftrightarrow D(Ax + Bd + q_t(\bar{y})) \leftrightarrow D(A_t x + q_t(\bar{y}))$$

where $d = \sigma(x) - x$.

Similarly, by Lemma 9 we obtain the following equivalence for negated difference predicates:

$$\neg D(p_u(x) + q_u(\bar{y})) \leftrightarrow \neg D(A_u x + q_u(\bar{y}))$$

So we replace every difference predicate in ϕ_3 and every negated difference with these simpler equivalents.

2. *Directed difference predicates and negated difference predicates*:

For some positive integer M, the directed difference predicate $D^+(p_\alpha(x) + q_\alpha(\bar{y}))$ is equivalent to the difference predicate $D(p_\alpha(x) + q_\alpha(\bar{y}) + M)$. We now, have, again by Lemma 9, the following equivalence:

$$D(p_\alpha(x) + q_\alpha(\bar{y}) + M) \leftrightarrow D(Ax + Bd + q_\alpha(\bar{y}) + M) \leftrightarrow D(Ax + q_\alpha(\bar{y}) + M)$$

where $d = \sigma(x) - x$. The rightmost predicate is equivalent to $D^+(Ax + q_\alpha(\bar{y}))$. We replace $D^+(p_\alpha(x) + q_\alpha(\bar{y}))$ with this equivalent. It follows as well that $\neg D^+(p_\beta(x) + q_\beta(\bar{y}))$ is equivalent to $\neg D^+(Ax + q_\beta(\bar{y}))$.

In exactly the same way (but with M now some negative integer), we obtain that $D^-(p_\delta(x) + q_\delta(\bar{y}))$ is equivalent to $D^-(Ax + q_\delta(\bar{y})$, and $\neg D^-(p_\zeta(x) + q_\zeta(\bar{y}))$ is equivalent to $\neg D^-(Ax + q_\zeta(\bar{y})$.

So in ϕ_3 we replace each directed difference predicate and each negated directed difference predicate with its equivalent.

3. *Standard integer predicates and negated standard integer predicates*:

We consider the standard integer predicate $Z(p_v(x) + q_v(\bar{y}))$. We have a few cases:

(1) Suppose first that $A = 0$. Then

$$Z(p_v(x) + q_v(\bar{y})) \leftrightarrow Z(B_v(\sigma(x) - x) + q_v(\bar{y})).$$

So we replace the predicate $Z(p_v(x) + q_v(\bar{y}))$ with $Z(B_v(\sigma(x) - x) + q_v(\bar{y}))$. Similarly, in this case we replace $\neg Z(p_w(x) + q_w(\bar{y}))$ with $\neg Z(B_w(\sigma(x) - x) + q_w(\bar{y}))$.

(2) If $A \neq 0$ and $B = 0$, then $Z(p_v(x) + q_v(\bar{y}))$ is again equivalent to $Z(Ax + q_v(\bar{y})$; and $\neg Z(p_w(x) + q_w(\bar{y}))$ is equivalent to $\neg Z(Ax + q_w(\bar{y})$.

(3) If $A \neq 0$ and $B' = 0$, note that since $\sigma(p_v(x) + q_v(\bar{y})) = p_v(x) + q_v(\bar{y}) = Ax + \sigma(q_v(\bar{y}))$, we similarly obtain the equivalent predicate $Z(Ax + \sigma(q_v(\bar{y}))$, and in the case of a negated predicate, we get $\neg Z(Ax + \sigma(q_w(\bar{y}))$.

(4) If $A \neq 0$ and $B, B' \neq 0$, then because $Ax + Bd + q_v(\bar{y})$ is an integer, it is in particular fixed by σ. Hence, for some integer M

$$Ax + B'd + \sigma(q_v(\bar{y})) = M = Ax + Bd + q_v(\bar{y}).$$

Thus, $BAx + BB'd + B\sigma(q_v(\bar{y})) = BM$ and $B'Ax + BB'd + B'q_v(\bar{y}) = B'M$.

Subtracting the former from the latter, we obtain:

$$(B'A - BA)x + (B'q_v(\bar{y}) - B\sigma(q_v(\bar{y}))) = (B' - B)M$$

and since $A = B' - B$, we get

$$A^2x + (B'q_v(\bar{y}) - B\sigma(q_v(\bar{y}))) = AM \quad (*).$$

Thus,

$$A^2x + (B'q_v(\bar{y}) - B\sigma(q_v(\bar{y})))$$

is a standard integer.

So in this case the predicate is equivalent to $Z(A^2x + (B'q_v(\bar{y}) - B\sigma(q_v(\bar{y}))))$, and the negated predicate is equivalent to $\neg Z(A^2x + (B'q_w(\bar{y}) - B\sigma(q_w(\bar{y})))$.

Remark. For the converse, apply the σ-polynomial p_v to the left-hand side of (*) as follows, with $d_Y = \sigma(q_v(\bar{y})) - q_v(\bar{y})$:

$$p_v(A^2x + (B'q_v(\bar{y}) - B\sigma(q_v(\bar{y})))) =$$

$$A^3x + A^2Bd + A((B'q_v(\bar{y}) - B\sigma(q_v(\bar{y}))) + (BB'd_Y - B^2d_Y) =$$

$$A^3x + A^2Bd + AB'q_v(\bar{y}) - AB(q_v(\bar{y}) + d_Y) + (B' - B)Bd_Y =$$

$$A^3x + A^2Bd + A(B' - B)q_v(\bar{y}) - ABd_y + ABd_y =$$

$$A^3x + A^2Bd + A^2q_v(\bar{y})$$

so by Lemma 14, $Z(A^3x + A^2Bd + A^2q_v(\bar{y}))$, hence also $Z(Ax + Bd + q_v(\bar{y}))$, as required.

4. *Divisibility predicates*: Because σ preserves divisibility types, each divisibility predicate $P_{a_i}(p_j(x) + q_j(\bar{y}))$ in (5.33) is of the form

$$P_{a_i}(A_jx + q_j(\bar{y})).$$

We rewrite this predicate as

$$A_jx \equiv -q_j(\bar{y}) \pmod{a_i}.$$

Next, by dividing each congruence through by the greatest common divisor (A_j, a_i) and then multiplying through by the inverse of

$$\frac{A_j}{(A_j, a_i)} \pmod{m_i}$$

where $m_i = \frac{a_i}{(A_j, a_i)}$, we obtain equivalent congruences

$$x \equiv -q_j^*(\bar{y}) \pmod{m_i}$$

where $-q_j^*(\bar{y})$ is the result of the division and multiplication above.

After re-indexing, we obtain the equivalent divisibility predicate

$$P_{m_i}(x + q_j(\bar{y})),$$

and replace the original divisibility predicate with this one.

$$\phi_4 := \bigwedge B_i(\sigma(x) - x) = q_i(\bar{y}) \wedge \tag{5.34}$$

$$\bigwedge (A_l^2 x < q_l(\bar{y})) \wedge \bigwedge (A_m^2 x > q_m(\bar{y})) \wedge \tag{5.35}$$

$$\bigwedge (B_l(\sigma(x) - x) < q_p(\bar{y})) \wedge \bigwedge (B_m(\sigma(x) - x) > q_r(\bar{y})) \wedge \tag{5.36}$$

$$\bigwedge D(A_t x + q_t(\bar{y})) \wedge \bigwedge \neg D(A_u x + q_u(\bar{y})) \wedge \tag{5.37}$$

$$\bigwedge D^+(A_\alpha x + q_\alpha(\bar{y}) \wedge \bigwedge \neg D^+(A_\beta x + q_\beta(\bar{y}) \wedge \tag{5.38}$$

$$\bigwedge D^-(A_\delta x + q_\delta(\bar{y}) \wedge \bigwedge \neg D^-(A_\zeta x + q_\zeta(\bar{y})) \wedge \tag{5.39}$$

$$\bigwedge Z(B_v(\sigma(x) - x)) + q_v(\bar{y})) \wedge \bigwedge \neg Z((B_w(\sigma(x) - x)) + q_w(\bar{y}) \wedge \tag{5.40}$$

$$\bigwedge Z(A^2 x + (B' q_v(\bar{y}) - B\sigma(q_v(\bar{y})))) \wedge \tag{5.41}$$

$$\bigwedge \neg Z(A^2 x + (B' q_w(\bar{y}) - B\sigma(q_w(\bar{y}))) \wedge \tag{5.42}$$

$$\bigwedge P_{m_i}(x + q_j(\bar{y})). \tag{5.43}$$

Proof of equivalence. $\exists x \phi_3 \Rightarrow \exists x \phi_4$: If there is an x witnessing ϕ_3, then x also witnesses all the formulas in ϕ_4, either because they are identical to the formulas in ϕ_3, or derived above from formulas in ϕ_3.

$\exists x \phi_4 \Rightarrow \exists x \phi_3$: If there is an x witnessing all the formulas in ϕ_4, then such an x witnesses the formulas in (5.34), (5.35), (5.36), hence it witnesses the identical formulas in (5.26), (5.27) and (5.28). Because such an x witnesses the formulas in (5.37)-(5.39), it also witnesses the formulas in (5.29)-(5.31), because we simply add differences to the arguments of each predicate to obtain those formulas. Finally, because such an x also witnesses all the formulas in (5.40)-(5.43), it also witnesses the formulas in (5.32) and (5.33), because the steps we used in deriving (5.40)-(5.43) are reversible.

Step 6: Change of variable. We make all the coefficients of x occurring in terms in ϕ_4 uniform as follows. We replace each inequality and equation in (5.34)-(5.36) by a new equivalent inequality or equation so that the new coefficient of x is equal to the (positive) least common multiple L of all coefficients of x (or $\sigma(x) - x$)) occurring in the formula. For example, we replace each equation in (5.34) by

$$L(\sigma(x) - x) = \sigma(Lx) - Lx = \frac{L}{B_i} q_i(\bar{y})$$

and we replace each inequality $A_l^2 x < q_l(\bar{y})$ in (5.35) by

$$Lx < \frac{L}{A_l^2} q_l(\bar{y})$$

if $\frac{L}{A_l^2} > 0$, and the opposite inequality otherwise. Note here that $\frac{L}{B_i}$ (and $\frac{L}{A_l^2}$) are integers because L is divisible by both B_i and A_i^2. These new equations and inequalities are obviously equivalent to the original ones.

We replace each predicate by an equivalent predicate with new coefficient of x also equal to L in a similar way. For example, by Lemma 9, the difference predicate $D(A_t x + q_t(\bar{y}))$ is equivalent to $D(Lx + \frac{L}{A_t} q_t(\bar{y})$; and the directed difference predicate $D^+(A_\alpha x + q_\alpha(\bar{y}))$ is equivalent to $D^+(Lx + \frac{L}{A_\alpha} q_\alpha(\bar{y}))$ if $\frac{L}{A_\alpha} > 0$ and equivalent to $D^-(Lx + \frac{L}{A_\alpha} q_\alpha(\bar{y}))$ if $\frac{L}{A_\alpha} < 0$. Again, the coefficients of the terms in $\bar{y}$ are integers. We then rename Lx as x, rename the terms in $\bar{y}$ by their previous names, and add the predicate $P_L(x)$ to our formula to obtain ϕ_5. Thus, we get:

$$\phi_5 := \bigwedge(\sigma(x) - x) = q_i(\bar{y}) \wedge \tag{5.44}$$

$$\bigwedge(x < q_l(\bar{y})) \wedge \bigwedge(x > q_m(\bar{y})) \wedge \tag{5.45}$$

$$\bigwedge(\sigma(x) - x < q_p(\bar{y})) \wedge \bigwedge(\sigma(x) - x > q_r(\bar{y})) \wedge \tag{5.46}$$

$$\bigwedge D(x + q_t(\bar{y})) \wedge \bigwedge \neg D(x + q_u(\bar{y})) \wedge \tag{5.47}$$

$$\bigwedge D^+(x + q_\alpha(\bar{y}) \wedge \bigwedge \neg D^+(x + q_\beta(\bar{y}) \wedge \tag{5.48}$$

$$\bigwedge D^-(x + q_\delta(\bar{y}) \wedge \bigwedge \neg D^-(x + q_\varsigma(\bar{y})) \wedge \tag{5.49}$$

$$\bigwedge Z((\sigma(x) - x) + q_v(\bar{y})) \wedge \bigwedge \neg Z((\sigma(x) - x) + q_w(\bar{y}) \wedge \tag{5.50}$$

$$\bigwedge Z(x + (B' q_v(\bar{y}) - B\sigma(q_v(\bar{y})))) \wedge \tag{5.51}$$

$$\bigwedge \neg Z(x + (B' q_w(\bar{y}) - B\sigma(q_w(\bar{y}))) \wedge \tag{5.52}$$

$$\bigwedge P_{m_i}(x + q_j(\bar{y})) \wedge P_L(x). \tag{5.53}$$

Proof of equivalence. $\exists x \phi_4 \Leftrightarrow \exists x \phi_5$ is a direct consequence of properties of equations and inequalities, and of Lemmas 9 and 14.

Step 7: Reconciling remaining predicates containing x with inequalities, and with each other. In this step, we find formulas equivalent to all predicates and inequalities in ϕ_5 such that the new predicate formulas no longer contain x. Because there are several possible combinations of predicates and inequalities that may occur in a formula, we proceed by finding "linking" formulas that ensure that a witness for one kind of predicate is also a witness for each other predicate. In particular, as we proceed through the possible predicates we find such linking formulas that connect witnesses for the predicate currently under consideration to those previously considered.

Remark 1. In the inequalities in (5.45), there exists some pair of bounds R, S on x with $R < x < S$ and $S - R$ least. Similarly, there exists some pair of bounds T, U on $\sigma(x) - x$ in

(5.46) with $T < \sigma(x) - x < U$ and $U - T$ least. We will construct our next formula, equivalent to ϕ_5 above, by ensuring that an x exists between each pair of bounds on x and on $\sigma(x) - x$ that witnesses each predicate in (5.47)-(5.53), keeping in mind the equalities in (5.44). In doing so, we will typically need to treat two cases separately, one in which the minimal bounds on x from inequalities contain densely many $\mathbb{Z}$-chains, and one in which the minimal bounds on x are within a single $\mathbb{Z}$-chain. Where bounds on x determine a finite interval and we identify witnesses, we must also ensure that these witness are divisible by L.

Remark 2: If an equation of the form $q(\bar{y}) = \sigma(x) - x$ occurs in ϕ_5 and if a witnesses this equation, so does $a + b$ where b is any fixed point, because
$\sigma(a + b) - (a + b) = (\sigma(a) - a) + (\sigma(b) - b) = \sigma(a) - a$. Thus, a convex set of elements about a satisfies this equation. So no formula in (5.44) or (5.46) forces x to lie within a single $\mathbb{Z}$-chain, but formulas in (5.45) may do so.

Difference predicates and negated difference predicates: We begin by considering the difference predicates $D(x + q_t(\bar{y}))$, if any, which occur in (5.47). Suppose first only one such occurs.

The existence of an x making this formula true is equivalent to $x + q_t(\bar{y})$ being either (1) between a fixed point and a point outside the fixed point set or (2) between two points neither of which is fixed but which contain the entire fixed point set or (3) between two points both of which are fixed and either lie in different $\mathbb{Z}$-chains, or in the same $\mathbb{Z}$-chain with a difference between them.

We now examine the inequalities in (5.45) to capture this in a formula that ensures that the bounds on x respect these conditions. For each pair of lower and upper bounds R_m, S_l, we put $R_t = R_m + q_t(\bar{y}) < x + q_t(\bar{y}) < S_l + q_t(\bar{y}) = S_t$. These bounds respect the conditions if an only if the following formula ψ_t holds:

$$\psi_t := \Big(F(R_t) \wedge \neg F(S_t)\Big) \vee \Big(\neg F(R_t) \wedge F(S_t)\Big) \vee$$
$$\Big(\neg F(R_t) \wedge \neg S(R_t) \wedge R_t < 0 < S_t\Big) \vee \Big(F(R_t) \wedge F(S_t) \wedge \neg Z(S_t - R_t)\Big) \vee$$
$$\Big(F(R_t) \wedge F(S_t) \wedge Z(S_t - R_t) \wedge D^+(R_t) \wedge D^-(S_t)\Big).$$

Thus, if ψ_t holds for every pair of bounds R_m, S_l, then there exists an x that satisfies the difference predicate and each of the inequalities in (5.45). For equations in (5.44) and inequalities in (5.46), if such occur, we also need to specify conditions that guarantee that the difference predicate is satisfied. For equations, there will be an element satisfying the equation and the difference predicate if and only if

$$\sigma(x + q_t(\bar{y})) = \sigma(x) - x + \sigma(q_t(\bar{y})) - q_t(\bar{y}) = q_i(\bar{y}) + \sigma(q_t(\bar{y})) - q_t(\bar{y}) = 0$$

so we add the equation

$$q_i(\bar{y}) + \sigma(q_t(\bar{y})) - q_t(\bar{y}) = 0$$

to ψ_t as well. For each pair of bounds in (5.46), there will be an element satisfying the inequality and the difference predicate if and only if

$$q_r(\bar{y}) + \sigma(q_t(\bar{y})) - q_t(\bar{y}) < 0 < q_p(\bar{y}) + \sigma(q_t(\bar{y})) - q_t(\bar{y})$$

and so we add such an inequality to ψ_t for each inequality in (5.46).

We replace the difference predicate by a conjunction of formulas ψ_t for each pair of bounds on x.

In addition, to specify that the same x witnesses each difference predicate, for each additional difference predicate $D(x + q_{t'}(\bar{y}))$, we add a conjunct $\nu_{t,t'}$:

$$D\Big(q_t(\bar{y}) - q_{t'}(\bar{y})\Big).$$

Together with the formula ψ_t, this is equivalent to the predicate $D(x + q_{t'}(\bar{y}))$, because by Lemma 9, if $D(q_t(\bar{y}) - q_{t'}(\bar{y})$ and $D(x + q_t(\bar{y}))$, then also $D(x + q_{t'}(\bar{y}))$. We also add a corresponding conjunct $\neg\nu_u$ specifying that this x witnesses the negated difference predicates $\neg D(x + q_u(\bar{y}))$:

$$\neg D\Big(q_u(\bar{y}) - q_t(\bar{y})\Big).$$

By Lemma 9, this, together with ψ_t, is equivalent to $\neg D(x + q_u(\bar{y}))$.

If ϕ_5 contains no difference predicates but does contain negated difference predicates, let the negated difference predicates be

$$\neg D(x + q_1(\bar{y})), \ldots, \neg D(x + q_n(\bar{y})).$$

We have two cases.

Case 1: There are bounds on x that are finite, so there is a formula $R < x < S$ in (5.45) such that $Z(S - R)$. In this case, we consider only those elements in the interval divisible by L. Thus, if $R + 1$ is divisible by L and $D(R + 1 + q_1(\bar{y}))$, then $\neg D(R + 1 + kL + q_1(\bar{y}))$ for all integers $k \neq 1$, and each of these is also divisible by L. Hence if the following formula holds:

$$(R + 1 < S) \wedge (R + 1 + L < S) \wedge \cdots \wedge (R + 1 + (n)L < S)$$

then by the pigeonhole principle, one of the elements $R + 1, \ldots, R + 1 + (n)L$ will witness the n negated difference predicates (and all will be divisible by L). But fewer may suffice. So the following formula is equivalent to the existence of an element between these bounds that witnesses all the negated difference predicates:

$$\varphi := \Big(P_L(R+1) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+1+iL+q_j(\bar{y}))) \wedge (R+1+iL < S))\Big) \vee$$
$$\Big(P_L(R+2) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+2+iL+q_j(\bar{y}))) \wedge (R+2+iL < S))\Big) \vee$$
$$\ldots \vee$$
$$\Big(P_L(R+L) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+L+iL+q_j(\bar{y}))) \wedge (R+L+iL < S))\Big).$$

Note here that the indices n and $n+1$ are determined by the number of negated difference predicates, and do not depend on $\bar{y}$, and that L is fixed from our change of variable step. This formula identifies the unique element among the first L elements in the interval that is divisible by L and then, for that one, checks that among the $n+1$ elements that follow it at intervals of length L (so which are also divisible by L), there is a witness to all the negated difference predicates.

Case 2: There are no finite bounds as in Case 1. Then there are densely many $\mathbb{Z}$-chains between the bounds, and we can choose any such. By the same reasoning as in case 1, $n+1$ elements all divisible by L suffice to find a witness to all the negated difference predicates. But every infinite set contains $n+1$ such elements, so in this case there is always a witness to the negated difference predicates.

Hence, if there are no difference predicates in ϕ_5 but there are negated difference predicates, those predicates are equivalent to the formula:

$$Z(S-R) \wedge \varphi$$

for the each pair of bounds R, S that occurs in (5.45). Again, equations in (5.44) and inequalities in (5.46) always determine densely many $\mathbb{Z}$-chains, so we need not consider them.

Directed difference predicates: We next work with the directed difference and negated difference predicates D^+ and $\neg D^+$ in (5.48); the case for the predicates D^- and $\neg D^-$ in (5.49) is similar. Assume first that this is only one such predicate $D^+(x + q(\bar{y}))$. This formula is equivalent to $D(x + q(\bar{y}) + M)$ for some $0 < M \in \mathbb{N}$ because a difference is above and within standard integer distance of $x + q(\bar{y})$. Put $w = q(\bar{y})$.

We now reconcile the predicate with the inequalities in (5.45). Each inequality $R < x < S$ in (5.45) is equivalent to

$$R + w < x + w < S + w$$

We need to find restrictions that force the existence of an x that makes $x + w + M$ a difference.

Subcase 1: No pair of bounds R and S is in the same $\mathbb{Z}$-chain. Let R and S be the pair of bounds on x such that $S - R$ is minimal. Then differences of the form $x + w + M$ occur between $R + w$ and $S + w$ if either (1) one bound is a fixed point; or (2) neither bound is a fixed point and one is negative and the other positive. Choose any difference d between the bounds and not contained in the same $\mathbb{Z}$-chain as either bound, and put $x = d - (w + M)$. Then it is clear that this x makes $D(x + w + M)$ hold, hence $D^+(x + w)$ as well. Because we used the shortest interval bounding x, such an x agrees with all the inequalities in (5.45). So the first part of a formula equivalent to the directed difference predicate, covering this subcase, is the following formula:

$$\Big(\neg Z(S - R) \wedge \big(F(R + w) \vee F(S + w)\big)\Big) \vee \tag{5.54}$$

$$\Big(\neg F(R + w) \wedge \neg F(S + w) \wedge \big(R + w < 0 < S + w\big)\Big). \tag{5.55}$$

Subcase 2: There are bounds on x that are in the same $\mathbb{Z}$-chain. Again, there is some pair R, S of such bounds such that $S - R$ is minimal. In this case, the difference must be between them, and so $(D^+(R + w) \wedge D^-(S + w))$.

Conversely, if there is a difference between $R + w$ and $S + w$, then there is an x satisfying $R < x < S$ such that $x + w + M$ is a difference. Again, because we chose the shortest interval bounding x, x witnesses all the inequalities in (5.45).

So in this case, where the bounds are in the same $\mathbb{Z}$-chain, we get the equivalent formula:

$$\Big(Z(S - R) \wedge (D^+(R + w) \wedge D^-(S + w))\Big). \tag{5.56}$$

Thus, for each inequality $R < x < S$ in (5.45), the predicate $D^+(x + w)$ is equivalent to the disjunction of the formulas in (5.54), (5.55), and (5.56), and we label this disjunction μ_α. (Similarly, a negated directed difference predicate is equivalent to the negation of such a disjunction, $\neg\mu_\beta$.) For directed difference predicates of the form D^-, we include corresponding disjunctions labeled η_δ and $\neg\eta_\zeta$.

We replace each directed difference predicate and negated difference predicate with such disjunctions.

Finally, we need to guarantee that the same x witnesses all the directed difference predicates (if there are more than one), as well as all the difference predicates that were reconciled previously with inequalities. We do so as follows. With respect to the difference predicates D, there is a shortest interval bounding x (with bounds $R < x < S$), and some x in this interval must witness all the difference predicates; in particular it witnesses a difference predicate $D(x + q_t(\bar{y}))$. So $D(R + q(\bar{y}) + y)$, for some $y < S - R$. Now consider, without loss of generality, a difference predicate $D(x + w + M)$, equivalent to $D^+(x + w)$ as above. Rewrite this as $D(R + y + w + M)$. Then by Lemma 9 and subtracting the arguments, we also have $D(w + M - q_t(\bar{y}))$. This is equivalent to $D^+(w - q(\bar{y}))$. So using any of the difference predicates we reconciled, we add as well to our revised formula, for any directed difference predicate D^+_α, a conjunct

$$D^+(q_\alpha(\bar{y}) - q_t(\bar{y})).$$

We add the negation of such a predicate for each negated predicate $\neg D^+_\beta$, and, similarly, we add a conjunct

$$D^-(q_\delta(\bar{y}) - Aq(\bar{y}))$$

for each directed difference predicate D^-_δ (and the negation of such for each negated directed difference predicate D^-_ζ). We label all these conjunctions together as β and add them to our formula.

No difference predicates: If there are no difference predicates but there are negated difference predicates in ϕ_5, we have two cases.

Case 1: the shortest interval bounding x is not finite. Then there is a witness to the directed difference predicates so long as (5.54) and (5.55) hold. Also, if x is a witness to the directed difference predicates, then so is $x - k$ for k a positive standard integer. So if there are n negated difference predicates, we can guarantee that there will be $n + 1$ elements (all divisible by L) that witness the directed difference predicates by choosing $k > L(n + 1)$. This suffices to also witness the negated difference predicates, as set forth above in the discussion of cases where only negated difference predicates occur.

Case 2: the shortest interval bounding x is finite. In this case, the following formula, modified from the discussion of cases where only negated difference predicates occur, is equivalent to the existence of a witness to the negated difference predicates and the directed difference predicates:

$$\Big(P_L(R+1) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+1+iL+q_j(\bar{y}))) \wedge (R+1+iL<S) \wedge D^+(R+1+iL))\Big) \vee$$
$$\Big(P_L(R+2) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+2+iL+q_j(\bar{y}))) \wedge (R+2+iL<S) \wedge D^+(R+2+iL))\Big) \vee$$
$$\ldots \vee$$
$$\Big(P_L(R+L) \wedge \bigvee_{i=0}^{n} (\bigwedge_{j=1}^{n} \neg D(R+L+iL+q_j(\bar{y}))) \wedge (R+L+iL<S) D^+(R+L+iL))\Big).$$

This formula states that there is a witness (divisible by L) to all n negated difference predicates between the bounds on x, and that that same witness also satisfies the directed difference predicate. Note that the indices n and $n+1$ are determined by the number of negated difference predicates, and do not depend on $\bar{y}$, nor does L depend on $\bar{y}$.

No difference predicates or negated difference predicates. Finally, if no difference predicates or negated difference predicates occur in ϕ_5, we need to exhibit a formula that implies that all the directed difference predicates have a common witness. There are again two cases:

Case 1: The minimal bounds R, S on x are such that $Z(S-R)$. In this case we must check that the witness to the predicate is divisible by L. Since no $\mathbb{Z}$-chain contains more than a single difference, the directed difference predicates $D^+(x+w_j)$ are equivalent to

$$\bigvee_{i=1}^{L} (\bigwedge_{j} D^+(R+i+w_j))$$

and the directed difference predicates $D^-(x+w_k)$ are equivalent to

$$\bigvee_{i=1}^{L} (\bigwedge_{k} D^-(S-i+w_k)).$$

So in this case we let β be the conjunction of the two formulas above together with the formulas $D^-(w_k - w_j)$, which guarantee that the same element witnesses all the predicates. and replace the directed difference predicates with it. Negated directed difference predicates $\neg D^+(x + w_j)$ and $\neg D^-(x + w_j)$ (here divisibility by L is unimportant) are similarly equivalent in this case to

$$\bigwedge \neg D^+(R + w_j)) \wedge \bigwedge \neg D^-(S + w_j)),$$

which we also add to β.

Case 2: The minimal bounds on x contain a dense set of $\mathbb{Z}$-chains, and thus include $\mathbb{Z}$-chains containing differences. Let the directed difference predicates in ϕ_5 be:

$$D^+(x + w_1), \dots, D^+(x + w_j), D^-(x + w_{j+1}), \dots, D^-(x + w_n) \quad (*)$$

and choose x such that $D^+(x + w_1)$ holds. (There is certainly such an x, for example, $x = w_1 - K$ for any positive integer K.)

These are equivalent to the following:

$$D(x + w_1 + M_1), \dots, D(x + w_j + M_j), D(x + w_{j+1} - M_{j+1}), \dots, D(x + w_n - M_n)$$

where $M_1, \dots, M_n$ are positive integers.

Observe that by shifting x down in its $\mathbb{Z}$-chain by a sufficiently large positive integer K, we can also witness the directed difference predicates

$$D^+(x + w_1), D^+(x + w_2), \dots, D^+(x + w_j), D^+(x + w_{j+1}), \dots, D^+(x + w_n)$$

and similarly by moving x up in its $\mathbb{Z}$-chain by a sufficiently large positive integer K, we can also witness the directed difference predicates

$$D^-(x+w_1), D^-(x+w_2), \ldots, D^-(x+w_j), D^-(x+w_{j+1}), \ldots, D^-(x+w_n).$$

Finally, observe that if, for a particular pair of predicates $D(x+w_1+N_1), D(x+w_2+N_2)$, with N an integer (positive or negative), we have $D((w_2-w_1)+(N_2-N_1))$. Assume without loss of generality that N_2-N_1 is positive, then $D^+(w_2-w_1)$ holds. In this case, if we shift x down in its $\mathbb{Z}$-chain to ensure that $D^+(x+w_1)$ holds, then certainly also $D^+(x+w_2)$ will hold. And if we move x up in its $\mathbb{Z}$-chain to ensure that $D^-(x+w_1)$ holds, then we can achieve either $D^+(x+w_2)$ or $D^-(x+w_2)$. Thus, we can find an x witnessing (*) above if and only if the following formula holds for each pair of predicates of $D^+(x+w_i), D^-(x+w_j)$:

$$D^-(w_i-w_j).$$

So in this case, we replace the directed difference predicates by the conjunction β of all such formulas and (5.54) and (5.55). Finally, if only predicates $D^+(x+w_i)$ occur, these are witnessed so long as there is a difference anywhere within the $\mathbb{Z}$-chains contained within the bounds on x. So this is equivalent to the conjunction

$$\bigwedge_{i\neq k}\big(D(w_i-w_k)\vee D^+(w_i-w_k)\vee D^-(w_i-w_k)\big).$$

We get the same equivalent formula if only predicates $D^-(x+w_j)$ occur. Depending on which case we are in ((1) if there are difference predicates, (2) if there are only negated difference predicates or (3) if there are neither difference predicates nor negated difference predicates), we replace the directed difference predicates with the appropriate new formula, labelled β, above. We also include in β formulas $\neg D^+(w_i-w_l)$ for each negated directed difference predicate $\neg D^+(x + w_l)$, and similar formulas for negated directed difference predicates $\neg D^-(x + w_m)$.

Only negated directed difference predicates. If only negated directed difference predicates occur, and the shortest interval bounding x is finite with $R < x < S$, then the negated difference predicate $\neg D^+(x + w)$ is equivalent to $\neg D^+(R + w)$, and the negated difference predicate $\neg D^-(x + w)$ is equivalent to $\neg D^-(S + w)$, so we have:

$$Z(S - R) \wedge \bigwedge \neg D^+(x + w_i) \wedge \bigwedge \neg D^-(S + w_j).$$

If only negated directed difference predicates occur and the shortest interval bounding x is not finite, then any one such predicate is satisfied by some x, say $\neg D^+(x + w_l)$ because there is some x in the interval making this predicate hold; and then we can increase x so that the remaining negated directed difference predicates $\neg D^+(x + w_m)$ are also satisfied by this same x.

Standard integer predicates: Part 1: We first consider predicates and negated predicates in (5.50).

If there are equations in (5.44) that determine $\sigma(x) - x$, we replace each standard integer predicate with the following equivalent predicate:

$$Z(q_i(\bar{y}) + q_v(\bar{y}))$$

and replace each negated predicate in (5.50) with the negation

$$\neg Z(q_i(\bar{y}) + q_w(\bar{y})).$$

These are equivalent to the original predicates and do not contain x, so we remove them from the scope of the quantifier.

If there are no equations in (5.44), we have to ensure that the inequalities in ϕ_5 are such that $\sigma(x) - x$ is in the same $\mathbb{Z}$-chain as $w = -q_v(\bar{y})$. First, since w must be in the same $\mathbb{Z}$-chain as a difference, we have

$$(D^-(w) \vee D^+(w) \vee D(w)).$$

Next, we consider inequalities in (5.45) and (5.46). Let $R < x < S$ be a pair of bounds on x and $T < \sigma(x) - x < U$ a pair of bounds on $\sigma(x) - x$. Since w must be in the same $\mathbb{Z}$-chain as $\sigma(x) - x$, we have

$$\sigma(R) - R \leq w + M \leq \sigma(S) - S$$

and

$$T < w + M < U$$

for some $M \in \mathbb{Z}$, where $w + M = \sigma(x) - x$. Assume that all bounds are of the form $\sigma(R) - R \leq w + M \leq \sigma(S) - S$.

We have two subcases.

Subcase 1. $\sigma(R) - R$ and $\sigma(S) - S$ are in different $\mathbb{Z}$-chains, that is, $\neg Z(\sigma(S - R) - (S - R))$.

If also neither bound on $w + M$ is in the same $\mathbb{Z}$-chain as $w + M$, then the inequality is strict:

$$\sigma(R) - R < w + M < \sigma(S) - S$$

and so the predicate is equivalent to:

$$\neg Z(\sigma(R) - R + w) \wedge \neg Z(\sigma(S) - S + w) \wedge (\sigma(R) - R < w < \sigma(S) - S).$$

If, say, the lower bound is in the same $\mathbb{Z}$-chain as $w + M$, then in fact $w + M = \sigma(R) - R$ because no $\mathbb{Z}$-chain contains more than one difference. In this case, there is still an $x > R$ with $\sigma(x) - x$ in the same $\mathbb{Z}$-chain as w: choose $x = R + k$ for any $0 < k \in \mathbb{N}$. Then $\sigma(x) - x = \sigma(R + k) - (R + k) = \sigma(R) - R$. So in this case the predicate is equivalent to:

$$Z(\sigma(R) - R + w).$$

Subcase 2. The bounds are in the same $\mathbb{Z}$-chain, so $Z(\sigma(S - R) - (S - R))$, i.e., $\sigma(S - R) - (S - R) = 0$. In this case, because no $\mathbb{Z}$-chain contains more than one difference, and both must be equal to $\sigma(x) - x = w + M$. Then the predicate is equivalent to $Z(\sigma(R) - R + w)$, because for any x between R and S, the difference $\sigma(x) - x$ will be in the same $\mathbb{Z}$-chain as w.

Taken together we get the following formula $\theta_{v,R,S}$ equivalent to the original predicate:

$$\begin{gathered}
\theta_{v,R,S} := \\
(D^-(w) \vee D^+(w) \vee D(w)) \wedge \\
\Big(\big((\neg Z(\sigma(R) - R + w) \wedge \neg Z(\sigma(S) - S + w) \wedge (\sigma(R) - R < w < \sigma(S) - S)) \big) \vee \\
\big(Z(\sigma(R) - R + w) \vee Z(\sigma(S) - S + w) \big) \Big)
\end{gathered}$$

where v is the original index of the predicate in (5.50), and R and S range over all bounds in (5.45). We get a similar formula for each pair of bounds T, U in (5.46):

$$\theta^*_{v,T,U} :=$$
$$(D^-(w) \vee D^+(w) \vee D(w)) \wedge$$
$$\Big(\big((\neg Z(T+w) \wedge \neg Z(U+w) \wedge (T < w < U)\big) \vee$$
$$\big(Z(T+w) \vee Z(U+w)\big)\Big).$$

The conjunction of all such formulas θ and $\theta*$ replaces the predicate, and we label this conjunction θ_v. Because these formulas are equivalent to the original predicate, the negated predicates in (5.50) are equivalent to $\neg\theta_w$, where w is the original index of the negated predicate. In addition, for each pair of positive predicates $Z(x + w_i)$, $Z(x + w_j)$ in (5.50) we add to θ_v the formulas

$$Z(w_i - w_j)$$

and for each pair of mixed predicates $Z(x + w_i)$, $\neg Z(x + w_k)$ in (5.50), we add the formulas

$$\neg Z(w_i - w_k).$$

These formulas ensure that the same element witnesses all the predicates.

Part 2: We next consider formulas in (5.51)-(5.52). For notational convenience, we write each predicate as $Z(x + w)$ and each negated predicate as $\neg Z(x + w)$, where w is $(B' q_v(\bar{y}) - B\sigma(q_v(\bar{y}))))$.

We will replace this predicate with inequalities ensuring that an x satisfying the inequalities in (5.45) and (5.46) satisfies the predicate, that is, we must add appropriate new inequalities involving x and its finite distance from $-w$. Put $W = -w$. The required inequalities are as follows.

As above, for convenience, let R and S be the lower and upper bounds, respectively, in all inequalities on x in (5.45), and let T and U be the lower and upper bounds on $\sigma(x) - x$ in (5.46). If the bounds R, S on x are themselves in different $\mathbb{Z}$-chains, then there will be such an element between them witnessing the predicate if and only if:

$$(R < W < S) \vee Z(W - R) \vee Z(S - W)$$

for the following reason. If W is not in the same $\mathbb{Z}$-chain as either R or S, then its entire $\mathbb{Z}$-chain lies between them, and there is certainly an element in that $\mathbb{Z}$-chain satisfying the predicate. Otherwise, if W is in the same $\mathbb{Z}$-chain as R, W may be less than R, but still in the same chain as x; similarly, if W is in the same $\mathbb{Z}$-chain as S, it may be greater than S but still in the same chain as x.

So, together, we get one condition in the case that the bounds are in different $\mathbb{Z}$-chains:

$$\neg Z(R - S) \wedge \Big((R < W < S) \vee Z(W - R) \vee Z(S - W)\Big). \tag{5.57}$$

If the bounds are in the same $\mathbb{Z}$-chain, then the formula

$$Z(S - R) \wedge Z(R - W) \tag{5.58}$$

guarantees the existence of an element in the same chain as W, as required.

We label the disjunction of (5.57) and (5.58), for each pair for bounds R, S as $\iota_{v',R,S}$. Each negated standard integer predicate is thus equivalent to the negation of the formula ι. We also need to ensure that an x witnessing one such predicate also witnesses all other such predicates, as well as all the difference and directed difference predicates. First, suppose

$Z(x+w_1)$ is another such predicate in addition to $Z(x+w)$. Then if also $Z(x+w')$, it is immediate that $Z(w-w')$, by Lemma 14. And, conversely, if $Z(x+w)$ and $Z(w-w')$, then $Z(x+w')$. So we add the conjunction

$$\bigwedge Z(w-w') \wedge \bigwedge \neg Z(w-w'')$$

where w' ranges over all other terms occurring in predicates of the form $Z(x+w')$, and w'' ranges over all other terms occurring in predicates of the form $\neg Z(x+w'')$. By adding these predicates we make the witness to a single standard integer predicate also a witness to every other standard integer predicate and to every negated standard integer predicate. If there are also predicates of the form $Z(\sigma(x)-x+w)$ in (5.50), we link these to predicates of the form $Z(x+W)$ by the formula

$$Z(\sigma(W)-W+w),$$

and negated predicates $\neg Z(\sigma(x)-x+w)$ are linked by

$$\neg Z(\sigma(W)-W+w).$$

Next, to ensure that the x witnessing all these standard integer predicates also witnesses the difference predicates and directed difference predicates, we consider the difference predicate of the form $D(x+q_t(\bar{y}))$ which we reconciled above with inequalities and all other difference and directed difference predicates. (Note that if there are no such difference or directed difference predicates, we simply skip this step, because we have a formula guaranteeing a witness to all standard integer predicates and inequalities.)
So we will make sure that the disjunction we found above that guarantees the existence of an element satisfying the standard integer predicate $Z(x+w)$ also guarantees that the

same element also satisfies the difference predicate $D(x + q_t(\bar{y}))$. We restate that formula:

$$\neg Z(S-R) \wedge \Big((R < W < S) \vee Z(W-R) \vee Z(S-W)\Big) \vee$$
$$Z(S-R) \wedge Z(R-W)$$

Because this disjunction specifies bounds on W ensuring the existence of an element in the same $\mathbb{Z}$-chain as W, we modify the condition to the following in the case where the bounds R, S are in different $\mathbb{Z}$-chains to the following:

$$\Big(\big(D^+(W + q_t(\bar{y})) \vee D^-(W + q_t(\bar{y})) \vee D(W + q_t(\bar{y}))\Big) \wedge \tag{5.59}$$
$$\Big(\big(\neg Z(S-R) \wedge \big((R < W < S) \vee Z(W-R) \vee Z(S-W)\big)\Big) \tag{5.60}$$

and the condition where the bounds are in the same $\mathbb{Z}$-chain is modified to:

$$\Big(Z(S-R) \wedge Z(R-W) \wedge \tag{5.61}$$
$$\wedge D^+(R) \wedge D^-(S) \wedge \big(D^+(W + q_t(\bar{y})) \vee D^-(W + q_t(\bar{y})) \vee D(W + q_t(\bar{y}))\big). \tag{5.62}$$

The new conditions in (5.59) ensure that there will be a difference in the same $\mathbb{Z}$-chain as W of the form $x + q_t(\bar{y})$, as required to witness the difference predicate; and the new conditions in (5.60) ensure that there will be a difference of that form between the bounds R and S which lie in the same $\mathbb{Z}$-chain. Together, the disjunction of the formulas in (5.59)-(5.60) and (5.61)-(5.62) guarantees that the witness to the standard integer predicate also witnesses all the difference predicates and directed difference predicates. So if there are both standard integer predicates and difference predicates, we label this disjunction for each pair of bounds R, S as $\iota_{v',R,S}$.

If there are no difference predicates but there are n negated difference predicates in (5.47), in the case where the bounds on x are in different $\mathbb{Z}$-chains, there is always a witness to both the negated difference predicates and all the standard integer predicates because any consecutive set of $n+1$ elements in a $\mathbb{Z}$-chain witnesses the negated difference predicates, and there is certainly such a consecutive set in some $\mathbb{Z}$-chain between R and S. If the bounds on x define a finite interval, then again the formula equivalent to the existence of a witness to those predicates in that interval exhibits a witness that will also witness all the standard integer predicates.

If there are neither difference predicates nor negated difference predicates in (5.47), but there are directed difference predicates in (5.48), we link such a directed difference predicate to any one of the standard integer predicates by the following formula:

$$Z(q_\alpha(\bar{y}) + W).$$

This formula ensures that the witness to the directed difference predicate is also a witness to the standard integer predicate. If there are only negated directed difference predicates in (5.48), we similarly link those to the standard integer predicates by the following formula:

$$Z(q_\beta + W).$$

(If there are no predicates in (5.48), but there are predicates in (5.49), we use exactly the same linking formulas except with indices δ or ζ for predicates D^- and $\neg D^-$, respectively.)

Divisibility predicates: We consider divisibility predicates in (5.53):

$$\bigwedge P_{m_i}(x + q_j(\bar{y})) \wedge P_L(x).$$

The following generalization of the Chinese remainder theorem allows us to determine if this conjunction has a witness:

Theorem 6. *[10] Given integers $n_1, \ldots n_k$ and moduli $a_1, \ldots, a_k$, the system of linear congruences*

$$x \equiv n_1 \pmod{a_1}$$
$$x \equiv n_2 \pmod{a_2}$$
$$* * *$$
$$x \equiv n_k \pmod{a_k}$$

has a solution if and only if for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$,
$n_i \equiv n_j \pmod{(a_i, a_j)}$, where (a_i, a_j) is the greatest common divisor of a_i and a_j. If there is a solution, it is unique $\pmod{[a_1, a_2, \ldots, a_k]}$, where $[a_1, a_2, \ldots, a_k]$ is the least common multiple of $a_1, \ldots, a_k$.

Thus, we replace the conjunction of divisibility predicates in (5.53) by the formula π:

$$\bigwedge_{j \neq k} P_{(m_j, m_k)}(q_j(\bar{y}) - q_k(\bar{y})) \tag{5.63}$$

where (m_j, m_k) is the greatest common divisor of the moduli in $P_{m_j}(x + q_j(\bar{y}))$ and $P_{m_k}(x + q_k(\bar{y}))$, respectively, and where we include the predicate $P_L(x)$ necessitated by the change of variables step among these pairs.

Put $M = [m_1, \ldots, m_k, L]$ for all distinct moduli occurring in the conjunction. Then by the last clause of the theorem, the existence of a witness to this conjunction is unique modulo

M, that is a solution exists in every interval of length M. We now find formulas ensuring that a witness exists to the divisibility predicates and all other possible predicates in ϕ_5.

When we consider formulas linking the conjunction (5.63) to other predicates and inequalities in our next equivalent formulas ϕ_6, we need only consider those formulas whose witnesses must lie in an interval shorter than M, or which impose requirements on the divisibility of witnesses. For example, each inequality in (5.46) and each equation in (5.44) has witnesses throughout an infinite convex set. By contrast, difference predicates in (5.47) determine the divisibility type of any element satisfying them.

Case 1: Inequalities $R < x < S$ occur in (5.45) that restrict x to a finite interval. Then we add the following formulas to our new formula:

$$\bigvee_{y=1}^{M} \bigwedge_{k} P_{m_k}(R + y + q_k(\bar{y})) \wedge (R + y < S).$$

This formula checks that the divisibility predicates are satisfied within the first interval of length M determined by the inequality in (5.35). Note here that M depends only on the divisibility predicates occurring in the formula, not on $\bar{y}$. We add this formula as a conjunct to π.

Case 2: Difference predicates occur in ϕ_4. We choose one such predicate $D(x + q_t(\bar{y}))$ that we used to link to all the other predicates. Observe that for any divisibility predicate P_n,

$$P_n(x) \leftrightarrow P_n(-q_t(\bar{y})).$$

Thus, any x witnessing the difference predicate will also witness all the divisibility predicates as well if and only if the following formula holds:

$$\bigwedge_k P_{m_k}\big(q_k(\bar{y}) - q_t(\bar{y})\big) \tag{5.64}$$

for all divisibility predicates (indexed by k). Because difference predicates were already linked to all other predicates above, we add this formula as a conjunct to π, and have a formula guaranteeing a witness to the divisibility predicates and all other predicates and inequalities.

Case 3: No difference predicates occur, but n negated difference predicates occur. In this case, recall that there is a witness to the negated difference predicates that is also divisible by L in any interval of length $L(n+1)$. But M is itself divisible by L because we included L in calculating M. We thus add the following formula as a conjunct to π:

$$\bigvee_{y=1}^{M(n+1)} \Big(\bigwedge_k P_{m_k}(R + y + q_k(\bar{y})) \wedge \neg D(R + y + q_u(\bar{y})) \wedge (R + y < S)\Big). \tag{5.65}$$

In this formula, the interval we consider has length at most $M(n+1)$, because if there is a witness to all divisibility predicates and negated difference predicates, it must occur in an interval of this length; each subinterval of length M contains a witness to the divisibility predicates, and there are $n+1$ such witnesses, one of which must witness all n negated difference predicates. Not here, again that M is determined only by the divisibility predicates that occur, and $n+1$ is determined by the number of negated difference predicates; neither depends on $\bar{y}$. Because we have linked negated difference predicates to all other predicates occurring in ϕ_4, π guarantees a witness to the divisibility predicates and all other predicates and inequalities.

Case 4: No difference predicates or negated difference predicates occur in ϕ_4, but there are directed difference predicates and/or negated directed difference predicates. First, note that if there are only directed difference predicates D^+ or only directed difference predicates D^-, such predicates do not restrict x to a finite interval, and so we add nothing to π. If both occur and x is bounded by a finite interval $R < x < S$ in (5.45), then we add the following formula to π:

$$\bigvee_{y=1}^{M} \Big(\bigwedge_k P_{m_k}(R + y + q_k(\bar{y})) \wedge \tag{5.66}$$

$$D^+(R + y + q_\alpha(\bar{y})) \wedge D^-(R + y + q_\delta(\bar{y})) \wedge (R + y < S)\Big). \tag{5.67}$$

This formula ensures that all divisibility predicates are satisfied within the interval as well as all directed difference predicates. If both occur and x is bounded by an interval containing a dense set of $\mathbb{Z}$-chains, then we chose any directed difference predicate $D^+(x + w)$ and add the following formula to π:

$$\bigvee_{y=1-M}^{M-1} \Big(\bigwedge D^-(y + q_\delta(\bar{y}) - q_\alpha(\bar{y})) \wedge \bigwedge_k P_{m_k}(y + q_k(\bar{y}))\Big). \tag{5.68}$$

This formula states that there is an interval of length M by which we can shift the witness to the directed difference predicates up or down and still have them hold and witness the divisibility predicates as well.

Case 5: No positive or negated difference predicates, or positive or negated directed difference predicates occur, but standard integer predicates occur. Notice that no standard integer predicate or negated standard integer predicate restricts x to a finite interval. That is, we can find a witness to the divisibility predicates in any $\mathbb{Z}$-chain. So in this case we add no linking formula to π.

At this point we have replaced all predicates containing x with new formulas not containing x and after reindexing obtain a formula ϕ_6 such that $\exists x \phi_6 \Leftrightarrow \exists x \phi_5$:

$$\phi_6 := \bigwedge \sigma(x) - x = q_i(\bar{y}) \wedge \tag{5.69}$$

$$\bigwedge (x < q_l(\bar{y})) \wedge \bigwedge (x > q_m(\bar{y})) \wedge \tag{5.70}$$

$$\bigwedge (\sigma(x) - x < q_p(\bar{y})) \wedge \bigwedge (\sigma(x) - x > q_r(\bar{y})) \wedge \tag{5.71}$$

$$\bigwedge_{R,S} \psi_t \wedge \bigwedge \nu_{t,t'} \wedge \bigwedge \neg \nu_u \wedge \tag{5.72}$$

$$\bigwedge \mu_\alpha \wedge \bigwedge \neg \mu_\beta \wedge \bigwedge \nu_\delta \wedge \bigwedge \neg \nu_\zeta \wedge \bigwedge \beta \wedge \tag{5.73}$$

$$\bigwedge \theta_v \wedge \bigwedge \neg \theta_w \wedge \tag{5.74}$$

$$\bigwedge \iota_{v',R,S} \wedge \tag{5.75}$$

$$\pi. \tag{5.76}$$

Proof of equivalence. $\exists x \phi_5 \Rightarrow \exists x \phi_6$: This direction is clear, because if there is an x witnessing all the conjuncts in (5.44)-(5.53), then such an x also witnesses the formulas in (5.69)-(5.75), which are either the identical formulas, or formulas derived from formulas in (5.44)-(5.53). $\exists x \phi_6 \Rightarrow \exists x \phi_5$: If there is an x satisfying the formulas in ϕ_6, then such an x satisfies the identical formulas in (5.44), (5.45) and (5.46). In addition, such an x will satisfy as well all the formulas in (5.47)-(5.53), because we built the new conjuncts in (5.72) through (5.76) using the inequalities in (5.45) and (5.46) so that some x satisfying all those inequalities will, given the constraints placed on x by the parameters $\bar{y}$ by (5.72) through (5.76), also satisfy the remaining predicates in ϕ_5. In other words, we have linked all the predicates using formulas ensuring that an x witnessing one predicate witnesses all the remaining predicates. For example, if $\mathcal{M} \models \theta_v$, then there is a difference in the same $\mathbb{Z}$-chain as $q_v(\bar{y})$, and that difference satisfies all the inequalities in (5.45) and (5.46).

Hence, there is some x satisfying the standard integer predicate $Z(\sigma(x) - x) + q_v(\bar{y}))$ as well as all the inequalities.

Step 8: inequalities, part 2. We first work with the remaining equations and inequalities involving x in (5.69), (5.70), and (5.71). First, we must ensure that $\sigma(x) - x$ corresponds to the bounds on x. By Lemma 13, corresponding to the inequality $x < q_l(\bar{y})$, we have $\sigma(x) - x \leq \sigma(q_l(\bar{y})) - q_l(\bar{y})$, and corresponding to the inequality $x > q_m(\bar{y})$, we have $\sigma(x) - x \geq \sigma(q_m(\bar{y})) - q_m(\bar{y})$. (Equality is possible if x differs from $q_l(\bar{y})$ (or $q_l(\bar{y})$) by a fixed point.) Hence, we add inequalities

$$\bigwedge_{r,l} q_r(\bar{y}) < \sigma(q_l(\bar{y})) - q_l(\bar{y}) \wedge \bigwedge_{p,m} \sigma(q_m(\bar{y})) - q_m(\bar{y}) < q_p(\bar{y}).$$

We also replace the inequalities in (5.71) and the equations in (5.69), if any, by the following:

$$\bigwedge_{r,i,p} \Big(\big(q_r(\bar{y}) < \frac{q_i(\bar{y})}{B_i} < q_p(\bar{y}) \big) \wedge E_i \Big)$$

where E_i states that all terms $q_i(\bar{y})$ in (5.60), if any, are pairwise equal, and that each is a difference. Finally, we replace the inequalities in (5.61) with

$$\bigwedge_{m,l} q_m(\bar{y}) < q_l(\bar{y}).$$

With these replacements, we now have a formula ϕ_7 such that $\exists x \phi_7 \Leftrightarrow \exists x \phi_6$:

$$\phi_6 := \bigwedge_{r,i,p} \Big(\big(q_r(\bar{y}) < \frac{q_i(\bar{y})}{B_i} < q_p(\bar{y}) \big) \wedge E_i \Big) \wedge \tag{5.77}$$

$$\bigwedge_{r,l} q_r(\bar{y}) < \sigma(q_l(\bar{y})) - q_l(\bar{y}) \wedge \bigwedge_{p,m} \sigma(q_m(\bar{y})) - q_m(\bar{y}) < q_p(\bar{y}) \wedge \tag{5.78}$$

$$\bigwedge_{m,l} q_m(\bar{y}) < q_l(\bar{y}) \wedge \tag{5.79}$$

$$\bigwedge_{R,S} \psi_t \wedge \bigwedge \nu_{t,t'} \wedge \bigwedge \neg \nu_u \wedge \tag{5.80}$$

$$\bigwedge \mu_\alpha \wedge \bigwedge \neg \mu_\beta \wedge \bigwedge \nu_\delta \wedge \bigwedge \neg \nu_\zeta \wedge \bigwedge \beta \wedge \tag{5.81}$$

$$\bigwedge \theta_v \wedge \bigwedge \neg \theta_w \wedge \tag{5.82}$$

$$\bigwedge \iota_{v',R,S} \wedge \tag{5.83}$$

$$\pi. \tag{5.84}$$

Proof of equivalence. $\exists x \phi_6 \Rightarrow \exists x \phi_7$: This direction is clear, because all formulas in ϕ_7 either occur identically in ϕ_6, or are consequences of formulas in (5.69)-(5.71).
$\exists x \phi_7 \Rightarrow \exists x \phi_7$: If $\mathcal{M} \models \phi_7$, then $\mathcal{M}$ satisfies all the formulas in ϕ_6 not containing x, which occur identically in both. And there is an x satisfying all the remaining formulas in ϕ_6 because the inequalities in (5.77)-(5.79) imply that there is an element satisfying the inequalities in (5.70) and (5.71) which also respects the equations, if any, in (5.69). Note in particular that because the witness to ϕ_7 is divisible by L, there is also a witness to our original formula, before we made the coefficients of x uniform in Step 6.
We have now found a formula ϕ_7 such that $\exists x \phi_7 \Leftrightarrow \exists x \phi$ and such that ϕ_7 does not contain x, so we can remove the quantifier. □

As in the case of the theory of the quotient structure, we also have completeness of T:

Corollary 3. *The theory T is complete and decidable.*

Proof. Any quantifier-free sentence is a Boolean combination of sentences each of which can be put into one of the following forms, since $\sigma^{\pm 1}(n) = n$ for all $n \in \mathbb{Z}$:

1. $P_a(n), \neg P_a(n)$

2. $D(n), \neg D(n)$

3. $D^+(n), \neg D^+(n)$

4. $D^-(n), \neg D^-(n)$

5. $Z(n), \neg Z(n)$

6. $m = n, m < n, m \neq n$

for $m, n \in \mathbb{Z}$.

These are all clearly decidable. For example, $Z(n) \leftrightarrow \top$; $D^+(n) \leftrightarrow n < 0$; and $D(n) \leftrightarrow n = 0$. □

Chapter 6

Definable sets in models of T^*

In this chapter and the next, we classify the sets definable by formulas in a single variable with parameters in models of T^* and T, the theories of $(\mathcal{M}/\mathbb{Z}, \sigma)$ and in $(\mathcal{M}, \sigma)$, respectively. We can then use the description of the definable sets to determine some basic model-theoretic properties of both the quotient structure and the Presburger structure, and then determine the DP-rank of both.

6.1 Definable sets in the quotient structure

We first consider the definable sets in T^*. This analysis is somewhat simpler than the case of the corresponding expanded Presburger structure, primarily because we do not need to consider divisibility predicates or the additional predicates (D^+, D^-, Z) defined above that were needed to prove quantifier elimination for the expanded Presburger structure.

Because we have quantifier elimination for T^*, we need only analyze sets defined by quantifier-free formulas. Let $\mathcal{M}^*$ be any model of T^*.

Thus, let ϕ be a quantifier-free formula in the language

$\mathcal{L} = (0, q\ (q \in \mathbb{Q}), +, -, \sigma, \sigma^{-1}, <, D)$. As above, D is a unary predicate for elements of

$D = \{d | \exists w(\sigma(w) - w = d)\}$, that is the differences between elements and their automorphic images. Without loss of generality, we may assume that ϕ is in disjunctive normal form. Thus, each disjunct is a conjunction of literals.

6.2 Sets defined by literals

Because a disjunction of formulas defines the union of the sets defined by each disjunct, we first find the following:

Lemma 15. *An atomic formula in one variable in $\mathcal{M}^*$ defines one of the following categories of sets:*

1. *single elements*
2. *infinite convex sets (including intervals)*
3. *infinite sets dense/codense in convex sets.*

Proof. We fix parameters $\bar{y} = (y_1, \ldots, y_m)$. We want to characterize sets

$$\{x : \mathcal{M}^* \models \phi(x, \bar{y})\}$$

where ϕ is an atomic formula.

Atomic formulas are one of the following for terms s and t:

1. $s = t$
2. $s < t$
3. $D(s)$

Terms s and t are σ-polynomials, that is, sums of expressions of the form:

$$q(y_i) = a_k\sigma^k(y_i) + \cdots + a_1\sigma(y_i) + a_0 y_i + a_{-1}\sigma^{-1}(y_i) + \cdots + a_{-l}\sigma^{-l}(y_i)$$

or of the form

$$p(x) = a_j\sigma^j(x) + \cdots + a_1\sigma(x) + a_0 x + a_{-1}\sigma^{-1}(x) + \cdots + a_{-t}\sigma^{-t}(x)$$

with coefficients $a_i \in \mathbb{Q}$. In each atomic formula of type (1) or (2), we can assume without loss of generality that the left-hand side contains all terms in the variable x, and the right-hand side contains all terms in the parameters $\bar{y}$.

We consider each of the three types of atomic formulas in turn.

Equations: An equation defines either a single element or a convex infinite set. In this case, $s = t$ is of the form $p(x) = q(\bar{y})$. We can clear negative powers of σ by applying σ to each side of the equation m times, where m is the absolute value of the least negative exponent of σ occurring in the equation. By clearing denominators, we may also assume coefficients in the equation are integers. Thus, we may assume that the equation $p(x) = q(\bar{y})$ has only nonnegative exponents and integer coefficients.

Case 1: $A \neq 0$. In this case, we apply Lemma 8 to conclude that the equation defines a single element

$$x = \frac{B'(q(\bar{y})) - B(\sigma(q(\bar{y}))}{A^2} \text{ or } x = \frac{q(\bar{y})}{A} \text{ or } x = \frac{\sigma(q(\bar{y}))}{A}.$$

Case 2: $A = 0$. In this case, the equation determines a specific difference $d = \frac{q(\bar{y})}{B}$. Thus, given any x such that $\sigma(x) - x = d$, the set of elements defined by this equation is all elements whose distance from x is in the fixed point set of σ, and this is a convex infinite set.

Inequalities: An inequality $p(x) < q(\bar{y})$ defines an infinite interval (ray) (of the form $x < a$ or $a < x$), an infinite convex set, or the empty set. If $A \neq 0$, then by Lemma 8 the inequality defines an infinite interval. If $A = 0$, then the inequality is of the form $B(\sigma(x) - x) < q(\bar{y})$. This defines the empty set if $q(\bar{y})$ is below the fixed point set, and otherwise determines a set of differences in increasing order; by Case 2 above for equations, each of these differences defines a convex infinite set, and so the set of elements x defined by the inequality is a convex infinite set.

Difference predicates: A difference predicate defines an infinite set of elements dense and codense in a convex set, or else all elements. Suppose first that $D(p(x))$ is the predicate and that $A \neq 0$. Then $D(p(x)) = D(Ax + Bd)$. $Ax + Bd = d'$ is a difference, and Bd is a difference, so because the differences form a subgroup, Ax is also a difference. By Lemma 8, the equation $Ax + Bd = d'$ has solution

$$x = \frac{B'd' - Bd'}{A^2} = \frac{(B' - B)d'}{A^2}.$$

For each possible difference d', we obtain a distinct solution for x. Each such solution is itself a difference because the set of differences is closed under addition and multiplication. Indeed, every difference can be written in this form, that is, if d^* is a difference, so is

$$\frac{A^2 d^*}{B' - B}.$$

Thus, $D(p(x))$ defines the set of all differences, which is an infinite set dense and codense in the convex fixed point set of σ. Thus, $D(p(x) + q(\bar{y}))$ is a translation of this set, and is also an infinite set dense and codense in an infinite convex set (a translation of the fixed point set).

If $A = 0$, then for each x, $p(x)$ has the form Bd for some integer B and $d = \sigma(x) - x$. Of course $\mathcal{M}^* \models D(Bd)$ because d is a difference. So in this case, $\mathcal{M}^* \models D(p(x))$ for all x, so the formula defines the entire universe. Similarly, in this case $D(p(x) + q(\bar{y}))$ defines the entire universe if $D(q(\bar{y}))$ and the empty set otherwise. $\square$

Having found the sets defined by atomic formulas, the sets defined by negated atomic formulas can be determined immediately:

Lemma 16. *A negated atomic formula in $\mathcal{M}^*$ defines one of the following categories of sets:*

1. *the entire structure minus a single element*
2. *the entire structure minus an infinite convex set or interval*
3. *the entire structure minus an infinite set dense and codense in a convex set.*

6.3 Arbitrary definable sets

From the lemmas above, it is straightforward to determine the sets defined by a conjunction of literals:

Lemma 17. *Sets defined by a conjunction of literals are one of the following:*

1. *the empty set or a single element*
2. *a convex set or finite union of convex sets (including intervals as convex sets)*
3. *a dense/codense infinite set in a convex set or in a finite union of convex sets.*

Proof. Note first that if the conjunction contains an equation with $A \neq 0$, then the conjunction defines either a single element, or the empty set if some other literal in the conjunction defines a set that does not include the single element defined by the equation.

If the conjunction contains one or more negated equations with $A \neq 0$, each such negated equation defines the entire structure with a single element removed, so k such negated equations remove at most k elements from the universe; thus, a conjunction containing k such negated equations will be intervals or convex sets with finitely many elements removed, which are themselves convex sets.

If the conjunction contains literals defining convex sets or pairs of convex sets, then it defines the intersection of those convex sets (with, possibly, finitely many elements removed due to negated equations). Such an intersection is itself a finite union of convex sets.

If the conjunction contains literals defining infinite sets dense in convex sets, then it defines sets dense in a finite union of convex sets or the empty set, for the following reasons. If the two sets are both defined by a difference predicate, then each defines a translation of the set of differences, so their intersection is a subset of such a translation, and is itself a dense/codense set. Similarly, if the two sets are both defined by negated difference predicates, then each is a translation of the set of non-differences, and their intersection is a subset of such a translation, and hence also dense/codense. If one set is defined by a difference predicate and the other by a negated difference predicate, then their intersection is empty. □

It follows that:

Theorem 7. *Any formula ϕ in $\mathcal{M}^*$ defines a union of a finite set, convex sets (including intervals), or infinite sets dense in convex sets.*

6.4 Uniform finiteness

An easy consequence of the classification of definable sets in Section 6.3 above is that the quotient structure $\mathcal{M}/\mathbb{Z}$ satisfies uniform finiteness:

Definition 11. A structure $\mathcal{M}$ satisfies *uniform finiteness* if for any formula $\phi(x, \bar{y})$, there is $N_\phi \in \mathbb{N}$ such that for any $\bar{a}$, if $\phi(M, \bar{a})$ is finite, then $|\phi(M, \bar{a})| \leq N_\phi$.

Corollary 4. *$\mathcal{M}^*$ satsifies uniform finiteness.*

Proof. By Theorem 7, any formula ϕ defines a union of a finite, infinite convex sets, and infinite dense sets. If ϕ defines a finite set, it must be a disjunction of formulas each of which defines either a single element or the empty set. Thus, we can choose N_ϕ equal to the number of disjuncts in the disjunctive normal form of ϕ. □

6.5 Algebraic closure

Definition 12. For S a finite subset of M, we say that a is in the *algebraic closure* of S if there is a formula $\phi(x, \bar{y})$ such that $\mathcal{M} \models \phi(a, \bar{y})$, where $\bar{y}$ is a tuple of elements of S, and where the set $\phi(M, \bar{y})$ is finite.

Again from Theorem 7, we can readily find the algebraic closure of any finite set in M^*.

Corollary 5. *Let $A = \{a_1, \ldots, a_n\} \subset M^*$. Then the algebraic closure of A is precisely the set of $\mathbb{Q}$-linear combinations of the elements of A and their associated differences $d_1, \ldots, d_n$ (where $d_i = \sigma(a_i) - a_i$).*

Proof. From the analysis of definable sets above, we need only consider elements y defined by equations over A; all other sets defined by formulas over A are infinite.
Such an equation is of the form

$$y = p_1(a_1) + p_2(a_2) + \cdots + p_n(a_n)$$

where

$$p_i(a_i) = c_j\sigma^j(a_i) + \cdots + c_1\sigma(a_i) + c_0 a_i + c_{-1}\sigma^{-1}(a_i) + \cdots + c_{-t}\sigma^{-t}(a_i)$$

for $i \in \{1, \ldots, n\}$, $\{c_j, \ldots, c_{-t}\} \subset \mathbb{Q}$, and integer powers of σ decreasing from left to right.

Now we can rewrite $p_i(a_i)$ as follows:

$$p_i(a_i) = c_j(a_i + jd_i) + c_{j-1}(a_i + (j-1)d_i) + \cdots + c_0(a_i) + c_{-1}(a_i - d_i) + \cdots + c_{-t}(a_i - td_i)$$

where $d_i = \sigma(a_i) - a_i$.

Collecting terms in a_i and d_i, we get

$$p_i(a_i) = C_i a_i + B_i d_i$$

where C_i is the sum of the coefficients c and

$$B_i = jc_j + (j-1)c_{j-1} + \cdots + c_1 - c_{-1} - 2c_{-2} - \cdots - tc_{-t}.$$

(Note that either C_i or B_i may equal 0.)

It follows that y is of the form

$$y = C_1 a_1 + B_1 d_1 + C_2 a_2 + B_2 d_2 + \cdots + C_n a_n + B_n d_n$$

Of course, it may happen that some $a_s \in A$ is a rational multiple of a different $a_r \in A$, or that some of the differences associated with elements of A are equal (or additive inverses of one another); in this case, the sum may collapse accordingly.

Thus, y is a $\mathbb{Q}$-linear combination of the elements of A and their associated differences $d_1, \ldots, d_n$.

Conversely, suppose $z_1 = qa_1 + rd_1$ for $q, r \in \mathbb{Q}$. Then z_1 is in the algebraic closure of a_1, because $z_1 = p_1(a_1)$ where

$$p_1(a_1) = r(\sigma(a_1)) + (q - r)a_1 = r(a_1 + d_1) + qa_1 - ra_1 = qa_1 + ra_1$$

Hence any linear combination of elements of A is in the algebraic closure of A. $\square$

Chapter 7

Definable sets in models of T

We now consider sets definable in models of T. Let $\mathcal{M}$ be any model of T.

Before analyzing the definable sets, a useful point of comparison is the following version (see [15]) of a classic theorem of Ginsburg and Spanier [3] on the definable sets in $\mathbb{N}$ in the language of Presburger arithmetic. The theorem uses the following definitions:

Definition 13. Let $X \subseteq \omega$. X is *ultimately periodic* if there exist a positive integer p (a *period* of X) and a natural number x_0 such that, for $x \geq x_0$, $x \in X \leftrightarrow x + p \in X$. X is *semi-linear* if it is a union of the ranges of finitely many arithmetic progressions.

The theorem, which we state without proof, characterizes the sets definable in the additive semigroup $\mathbb{N}$.

Theorem 8. *Let $X \subseteq \mathbb{N}$. Then the following are equivalent:*

1. *X is definable in the language of Presburger arithmetic;*

2. *X is ultimately periodic;*

3. *X is semi-linear.*

When we expand by the modest automorphism σ, we obtain other kinds of definable sets, arising from the fact that the subgroup of differences D and its cosets are all definable, and these cosets are dense and codense in $\mathbb{Z}$-chains, and also not periodic. But we can use the following modified definition:

Definition 14. Let $\mathcal{M} \models T$. Then $X \subseteq M$ is periodic if there exists a positive standard integer p such that $x \in X \leftrightarrow x + p \in X$. We also use *periodic interval* to denote the intersection of an interval and a periodic set.

To account for cosets of the set of differences, we use the following definition:

Definition 15. An infinite $X \subseteq M$ is *$\mathbb{Z}$-sporadic* if no $\mathbb{Z}$-chain contains more than one element of X and if the set of $\mathbb{Z}$-chains containing an element of X is dense and co-dense in a convex subset of M.

We want to characterize sets

$$\{x : \mathcal{M} \models \phi(x, \bar{y})\}$$

where $\mathcal{M} \models T$ with T as in section 3.2. So let ϕ be a quantifier-free formula in the language $\mathcal{L} = (+, -, \sigma, \sigma^{-1}, <, D, D^+, D^-, Z, P_n(n = 2, 3, \ldots), 0, 1)$. Recall that:

1. σ, σ^{-1} is a modest automorphism increasing on positive elements, and its inverse, respectively
2. D is a unary predicate for elements of $D = \{d : \exists w(\sigma(w) - w = d)\}$ (differences)
3. D^+ is a unary predicate meaning that there is a difference greater than and in the same $\mathbb{Z}$-chain
4. D^- is a unary predicate meaning that there is a difference less than and in the same $\mathbb{Z}$-chain

5. Z is a unary predicate for elements in the standard part

6. $P_n (n = 2, 3, \ldots)$ are predicates for divisibility by elements of $\mathbb{N} \setminus \{0, 1\}$.

As in Chapter 6, we may assume that ϕ is in disjunctive normal form, and each disjunct is a literal. We fix parameters $\bar{y} = (y_1, \ldots, y_m)$.

7.1 Sets defined by literals

First, we classify the sets defined by atomic formulas.

Lemma 18. *An atomic formula in $\mathcal{M}$ defines one of the following sets:*

1. *a single element*

2. *an infinite convex set (including intervals and single $\mathbb{Z}$-chains)*

3. *the set of all elements divisible by a single $n \in \{2, 3, \ldots\}$*

4. *a subset of an infinite convex set (such as rays within $\mathbb{Z}$-chains) divisible by a single $n \in \{2, 3, \ldots\}$, that is, a periodic set or periodic interval*

5. *a coset of an infinite set divisible by all $n \in \mathbb{N} \setminus \{0\}$ (that is, a coset of the set D of differences), that is, a $\mathbb{Z}$-*sporadic *set.*

Proof. Fix parameters $\bar{y} = (y_1, \ldots, y_m)$. We want to characterize sets

$$\{x : \mathcal{M} \models \phi(x, \bar{y})\}$$

where ϕ is an atomic formula.

Atomic formulas are one of the following for terms s and t:

1. $s = t$

2. $s < t$
3. $D(s)$
4. $D^+(s)$
5. $D^-(s)$
6. $Z(s)$
7. $P_n(s)$ ($n = 2, 3, \ldots$)

Terms s and t are σ-polynomials, that is, sums of expressions of the form:

$$q(y_i) = a_k\sigma^k(y_i) + \cdots + a_1\sigma(y_i) + a_0y_i + c + a_{-1}\sigma^{-1}(y_i) + \cdots + a_{-l}\sigma^{-l}(y_i)$$

or of the form

$$p(x) = a_j\sigma^j(x) + \cdots + a_1\sigma(x) + a_0x + d + a_{-1}\sigma^{-1}(x) + \cdots + a_{-t}\sigma^{-t}(x)$$

with coefficients $c, d, a_i \in \mathbb{Z}$. (In each atomic formula of type (1) or (2), we may assume without loss of generality that the left-hand side contains all terms in the variable x, and the right-hand side contains all terms in the parameters $\bar{y}$).

Remark: In $\mathcal{M}$, $Z(x)$ may define a set not isomorphic to $\mathbb{Z}$. In this chapter, we use the term $\mathbb{Z}$-chain to refer to a subset $S \subset M$ such that for $a, b \in S$, $\mathcal{M} \models Z(b - a)$. Such a $\mathbb{Z}$-chain is always an infinite convex set, because $\mathcal{M} \models Z(\pm n)$ where n abbreviates $\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}$, and so $a + k$ is in the same $\mathbb{Z}$-chain as a for all $k \in \mathbb{Z}$.

We consider each category of atomic formula in turn.

Equations: An equation defines either a single element, the empty set, or a convex infinite set. Now, $s = t$ is of the form $p(x) = q(\bar{y})$, and we can clear negative powers of σ on both sides of the equation by applying σ to each side of the equation m times, where m is the absolute value of the least negative exponent of σ occurring in the equation. Thus, we may assume that the equation $p(x) = q(\bar{y})$ has only nonnegative exponents.

Case 1: $A \neq 0$. In this case, we apply Lemma 11 to conclude that the equation defines a single element

$$x = \frac{B'(q(\bar{y})) - B(\sigma(q(\bar{y}))}{A^2}$$

if $B'(q(\bar{y})) - B(\sigma(q(\bar{y}))$ is divisible by A^2; otherwise, the equation has no solution and defines the empty set.

Case 2: $A = 0$. In this case, the equation determines a specific difference $d = \frac{q(\bar{y})}{B}$. Thus, given any x such that $\sigma(x) - x = d$, the set of elements defined by this equation is all elements whose distance from x is in the fixed point set of σ, and this is a convex infinite set.

Inequalities: An inequality defines either (1) an infinite interval (ray) (of the form $x < a$ or $a < x$, where a is the solution of the inequality from Lemma 12 or (2) an infinite convex set. If $A \neq 0$, then by Lemma 12, the inequality defines an infinite ray. If $A = 0$, then the inequality is of the form $B(\sigma(x) - x) < q(\bar{y})$. This defines the empty set if $q(\bar{y})$ is below the fixed point set, and otherwise determines a set of differences in increasing order; by Case 2 above for equations, each of these differences defines a convex infinite set, and so the set of elements x defined by the inequality is a convex infinite set.

Difference predicates: A difference predicate defines an infinite set of elements in a convex set, or the entire universe, or the empty set. First, suppose $D(p(x))$ is the predicate, and that $A \neq 0$. So $p(x) = d$ for some difference d. This equation has a single solution

$$x = \frac{B'd - Bd}{A^2} = \frac{(B' - B)d}{A^2},$$

Thus, for each difference d, we obtain a distinct solution for x. Each such solution is itself a difference because the set of differences is closed under addition, multiplication and nonzero division. Indeed, every difference can be written in this form, that is, if d' is a difference, so is

$$\frac{A^2 d'}{B' - B}$$

and thus $D(p(x))$ defines the set of all differences, which is a $\mathbb{Z}$-sporadic set. Thus, $D(p(x) + q(\bar{y}))$ is a translation of this set, and is also a $\mathbb{Z}$-sporadic set.

If $A = 0$, then for each x, $p(x)$ has the form Bd for some integer B and $d = \sigma(x) - x$. This is always true, because Bd is a difference. So the formula defines the entire universe. For $D(Bd + q(\bar{y}))$, note that this is equivalent to $D(q(\bar{y}))$, so the formula defines defines either the entire universe if $D(q(\bar{y}))$, or the empty set if $\neg D(q(\bar{y}))$.

Directed difference predicates: Suppose first that $A \neq 0$. Note first that the difference predicate $D^+(x)$ defines the set of left-hand rays in all $\mathbb{Z}$-chains containing differences. So this is an infinite subset of the fixed point set of σ. (Similarly, $D^-(x)$ defines the set of right-hand rays in all $\mathbb{Z}$-chains containing differences). Thus, directed difference predicates $D^+(x + q(\bar{y})), D^-(x + q(\bar{y}))$ are translations of such collections of left- and right-hand rays in the fixed point set, respectively.

Next, consider $D^+(p(x)) = D^+(Ax + Bd)$, where $A \neq 0$. This formula defines a translation of the set defined by $D^+(Ax)$. Observe that if c is in a left-hand ray in a $\mathbb{Z}$-chain containing a difference d, then Ac is in the left-hand ray $\mathbb{Z}$-chain containing the difference Ad if $A > 0$ (or in the right-hand ray if $A < 0$). Conversely, if Ac is in a left-hand ray $\mathbb{Z}$-chain containing a difference d', then c is in the left-hand ray $\mathbb{Z}$-chain containing the difference d'/A if $A > 0$ (or in the right-hand ray if $A < 0$). Thus $D^+(Ax)$ is the set of elements divisible by A in left-hand rays in $\mathbb{Z}$-chain containing differences. So

$D^+(p(x)+q(\bar{y}))$ and $D^-(p(x)+q(\bar{y}))$ are translations of such collections of left- and right-hand rays in the fixed point set, thinned out according to A.

If $A=0$, consider first $D^+(p(x))=D^+(Bd)$. This formula defines the empty set, because there is no difference to the left of and in the same $\mathbb{Z}$-chain as another difference. Next consider $D^+(p(x)+q(\bar{y}))=D^+(Bd+q(\bar{y}))$. Since Bd is a difference, this predicate defines either the entire universe if $q(\bar{y})$ is contained in a $\mathbb{Z}$-chain that contains a difference and is to the left of that difference; and it defines the empty set otherwise. Similarly, $D^-(p(x)+q(\bar{y}))$ defines either the entire universe if $q(\bar{y})$ is contained in a $\mathbb{Z}$-chain that contains a difference and is to the right of that difference; and the empty set otherwise.

Standard part predicates: Such a predicate defines an infinite subset of a single $\mathbb{Z}$-chain, or a translation of the fixed point set of σ. Consider first $Z(p(x))$, with $A \neq 0$ in $p(x)$. Since $Z(p(x))$ implies that $p(x)$ is fixed by σ, we may suppress σ in $p(x)$, and so $Z(p(x))$ is just $Z(Ax)$. This defines all elements in the $\mathbb{Z}$-chain that are multiples of A. Therefore, $Z(p(x)+q(\bar{y}))$ defines a translation of a $\mathbb{Z}$-chain with the added divisibility condition determined by A.

If the $A=0$ in $p(x)$, we get $Z(Bd)$, whence $d=0$. So any x that is fixed by σ satisfies this predicate, and so the predicate defines the fixed point set. Thus, in the case $A=0$, the predicate $Z(p(x)+q(\bar{y}))$ defines a translation of the fixed point set.

Divisibility predicates: If $A \neq 0$ in $p(x)$, then for fixed n, $P_n(p(x))=P_n(Ax+Bd)$ defines all elements divisible by $\frac{n}{\gcd(n,A)}$. Bd is always divisible by n, so Ax must also be; and Ax is divisible by n precisely when x is divisible by $\frac{n}{\gcd(n,A)}$. (Note that if $\frac{n}{\gcd(n,A)}=1$, then the predicate defines the entire universe.) Hence $P_n(p(x)+q(\bar{y}))$ defines a translation of such a set of elements divisible by $\frac{n}{\gcd(n,A)}$. Note here that there are only finitely many such translations (specifically, there are $\frac{n}{\gcd(n,A)}$ distinct translations of the set defined by $P_n(p(x))$). If $A=0$ in $p(x)$, then $P_n(p(x))=P_n(Bd)$. But Bd is a difference, hence witnesses every divisibility predicate, and again we have a formula that defines the entire universe. And so in this case $P_n(p(x)+q(\bar{y}))$ defines the entire universe as well if $P_n(q(\bar{y}))$, and the empty set otherwise. $\square$

With this categorization of sets defined by atomic formulas in hand, we can readily categorize the set defined by negated atomic formulas. Before doing so, the following observation simplifies the classification of these sets. A negated divisibility predicate $\neg P_n$ defines the complement of the set of all elements divisible by n, or the translation of such a complement (as above, there are only finitely many such translations.) But $\neg P_n(p(x)+q(\bar{y}))$ is equivalent to a disjunction of divisibility predicates; for example,

$$\neg P_3(p(x)+q(\bar{y})) \leftrightarrow \Big(P_3(p(x)+q(\bar{y})+1) \vee P_3(p(x)+q(\bar{y})+2)\Big).$$

Thus, we may replace a negated divisibility predicate by a disjunction of atomic divisibility predicates, and then replace ϕ with an equivalent formula in disjunctive normal form that contains no negated divisibility predicates. For the same reason, the negated atomic formulas that define subsets of infinite convex sets not divisible by a single element $n \in \{2,3,\ldots\}$ (in Lemma 18 part (4)) may be replaced by a disjunction of finitely many atomic formulas that define subsets of infinite convex sets divisible by a single element. We therefore have:

Lemma 19. *A negated atomic formula in $\mathcal{M}$ defines one of the following categories of sets:*

1. *the universe minus a single element (so the union of two intervals)*

2. *the universe minus a convex set (so the union of two convex sets)*

3. *subsets of infinite convex sets not divisible by all $n \in \mathbb{N}$*

4. *the universe minus a $\mathbb{Z}$-sporadic set.*

7.2 Sets defined by a conjunction of literals

Using Lemmas 18 and 19, we can categorize the sets definable by a conjunction of literals.

Lemma 20. *Sets defined by a conjunction of literals in $\mathcal{M}$ are one of the following:*

1. *a single element (or the empty set)*
2. *a finite set contained in a single $\mathbb{Z}$-chain*
3. *an infinite convex set, or finite unions of infinite convex sets*
4. *a periodic set*
5. *a* $\mathbb{Z}$-sporadic *set*
6. *an infinite convex set, or finite unions of infinite convex sets, with a periodic set or a finite number of* $\mathbb{Z}$-sporadic *sets removed.*

Proof. First, if the conjunction contains an equation $p(x) = q(\bar{y})$ with $A \neq 0$, then the conjunction defines either a single element or the empty set. (This is case (1) in the Lemma.)

Second, if the conjunction contains *only* negated inequalities, we have:

$$\bigwedge_{i=0}^{j} x < a_i \wedge \bigwedge_{k=0}^{l} x > a_k.$$

This conjunction of inequalities defines either a finite set or an infinite set, depending on the interval between the least a_i and the greatest a_k. If it is finite, it is contained within a single $\mathbb{Z}$-chain. (These fall within cases (2) and (3) of the Lemma.)

Third, If the conjunction contains *only* one or more formulas defining infinite convex sets (equations with $A = 0$, or predicates D^+, D^- or Z), then their conjunction define the intersection of several such infinite convex sets, which is either empty or itself an infinite convex set. This falls in case (3) of the Lemma.

Fourth, if the conjunction contains *only* one or more formulas defining complements of infinite convex sets (so negated equations with $A = 0$, or $\neg D^+$, $\neg D^-$, or $\neg Z$), then each such negated formula defines a union of two disjoint convex infinite sets. Several such negated equations define the universe with finitely many infinite convex sets removed, yielding finitely many disjoint infinite convex sets. These fall within case (3) of the Lemma.

Fifth, if the conjunction contains a negated equation with $A \neq 0$, then that equation defines either the entire universe or the entire universe minus a single element. So each such conjunct removes at most a single element from the conjunction. The effect of removing a single element is (possibly) to subdivide a convex set into two convex sets, or reduce the size of a finite set. The result of removing finitely many single elements falls within cases (2) and (3) of the Lemma.

Sixth, if the conjunction contains one or more conjuncts that are divisibility predicates; those conjuncts are equivalent either to a single divisibility predicate or to the empty set. This follows from the generalization of the Chinese remainder theorem given above as Theorem 6.

Combining these six possibilities, we have sets listed in parts (1), (2), and (3) of the Lemma: single elements, finite sets, and finite unions of convex sets. If the conjunction in addition contains divisibility predicates, negated divisibility predicates, difference predicates, or negated difference predicates, these conjuncts each remove elements from the sets defined by remaining conjuncts: they "thin out" such sets. These fall within part (4) of the Lemma. □

7.3 Sets defined by arbitrary formulas

From Lemma 20, it follows immediately that any formula ϕ defines a union of sets of the types listed in that Lemma. By including single elements as convex sets, we obtain the following:

Theorem 9. *Every formula in $\mathcal{M}$ defines a finite union of sets of the following kinds:*

1. *convex sets*
2. *periodic sets and periodic intervals*
3. $\mathbb{Z}$-sporadic *sets*
4. *convex sets minus periodic sets and* $\mathbb{Z}$-sporadic *sets.*

Since intervals and convex sets are themselves periodic intervals and periodic sets (with period $p = 1$), we can make the theorem slightly more concise, and closer in spirit to the Ginsburg-Spanier Theorem:

Theorem 10. *Every formula in $\mathcal{M}$ defines a finite union of sets of the following kinds:*

1. *periodic sets and periodic intervals*
2. $\mathbb{Z}$-sporadic *sets*
3. *periodic sets and periodic intervals with* $\mathbb{Z}$-sporadic *sets removed.*

7.4 Uniform finiteness

From the classification of definable sets, it is immediate that the full Presburger model does not have uniform finiteness. For example, for each parameter y the set of those x satisfying $Z(y - x) \wedge (0 < x < y)$ is always finite, but there is no $N \in \mathbb{N}$ such that for all parameters

y the set of witnesses x is always of size less than N. However, the full Presburger structure does satisfy the following definition of uniform finiteness on $\mathbb{Z}$-chains and $\mathbb{Z}$-rays:

Definition 16. A structure $\mathcal{M}$ that has order type $A\mathbb{Z}$, where A is a dense linear order without endpoints, satisfies $\mathbb{Z}$-*uniform finiteness* if for any formula $\phi(x, \bar{y})$, there is $N_\phi \in \mathbb{N}$ such that for any $\bar{a}$, if $\phi(M, \bar{a})$ defines a finite union Φ of disjoint sets each contained in a single $\mathbb{Z}$-chain, then the number of $\mathbb{Z}$-chains intersecting Φ is less than or equal to N_ϕ.

Corollary 6. *$\mathcal{M}$ satisfies $\mathbb{Z}$-uniform finiteness.*

Proof. Let ϕ be any formula in the language of the full Presburger structure, and assume without loss of generality that ϕ is in disjunctive normal form. If ϕ defines only finitely many $\mathbb{Z}$-chains or $\mathbb{Z}$-rays (or such sets intersected with divisibility predicates or with finitely many elements removed), then ϕ contains no disjuncts defining intervals containing more than a single $\mathbb{Z}$-chain, nor does it contain any disjoints defining convex sets (such as the fixed point set of σ) that contain densely many $\mathbb{Z}$-chains. So all the disjuncts constituting ϕ define either single $\mathbb{Z}$-chains, single $\mathbb{Z}$-rays, or single such sets intersected with divisibility predicates or with finitely may elements removed, or finite sets of elements. So we can take N_ϕ to be the number of disjuncts in ϕ. □

7.5 Algebraic closure

We can now also find the algebraic closure of a finite subset of M.

Corollary 7. *Let $A = \{a_1, \ldots, a_n\} \subset M$. Then the algebraic closure of A is the set of $\mathbb{Z}$-linear combinations of the elements of A and their associated differences $d_1, \ldots, d_n$, where $d_i = \sigma(a_i) - a_i$, and quotients of such linear combinations by standard integers k if the linear combination is divisible by k.*

Proof. From the classification of definable sets in Theorem 9 above, we need only consider elements y defined by (1) equations or (2) finite intervals over A; all other sets defined by formulas over A are infinite or empty.

Equations: An equation is of the form

$$ky = p_1(a_1) + p_2(a_2) + \cdots + p_n(a_n)$$

where

$$p_i(a_i) = c_j\sigma^j(a_i) + \cdots + c_1\sigma(a_i) + c_0 a_i + c + c_{-1}\sigma^{-1}(a_i) + \cdots + c_{-t}\sigma^{-t}(a_i)$$

for $i \in \{1, \ldots, n\}$, $\{c_j, \ldots, c_{-t}\} \subset \mathbb{Z}$, and integer powers of σ decreasing from left to right, and for $k \in \{1, 2, 3, \ldots\}$, and $k \in \mathbb{N}$.

Next, observe that,

$$p_i(a_i) = c_j(a_i + jd_i) + c_{j-1}(a_i + (j-1)d_i) + \cdots + c_0(a_i) + c + c_{-1}(a_i - d_i) + \cdots + c_{-t}(a_i - td_i)$$

where $d_i = \sigma(a_i) - a_i$.

Collecting terms in a_i and d_i, we get

$$p_i(a_i) = C_i a_i + B_i d_i + c$$

where C_i is the sum of the coefficients c_k, $c \in \mathbb{Z}$, and

$$B_i = jc_j + (j-1)c_{j-1} + \cdots + c_1 - c_{-1} - 2c_{-2} - \cdots - tc_{-t}.$$

(Note that either C_i or B_i may equal 0.)

It follows now that ky is of the form

$$ky = C_1a_1 + B_1d_1 + C_2a_2 + B_2d_2 + \cdots + C_na_n + B_nd_n + C$$

where all coefficients are integers, as is C. Of course, it may happen that some $a_s \in A$ is an integer multiple of a different $a_r \in A$, or that some of the differences associated with elements of A may be equal (where elements of A differ by a fixed point) or integer multiples of each other, and then the sum may collapse accordingly.

So ky is a $\mathbb{Z}$-linear combination of the elements of A and their associated differences $d_1, \ldots, d_n$. In addition, y itself is also in the algebraic closure of A, because the linear combination is divisible by k.

Finite intervals: In addition to elements y that are the $\mathbb{Z}$-linear combinations of the elements of A and their associated differences $d_1, \ldots, d_n$ and their quotients, the algebraic closure of A also contains finite intervals bounded by such linear combinations, for example, the finite interval

$$p_1(a_1) - m < y < p_1(a_1) + n \quad (m, n \in \mathbb{Z})$$

or the finite interval

$$p_1(a_1) < y < p_2(a_2) \quad (\text{if } a_1, a_2 \text{ are in the same } \mathbb{Z}\text{-chain}).$$

But each element in such an interval is itself also a $\mathbb{Z}$-linear combinations of elements of A.

Divisibility predicates: The divisibility predicate $P_n(y)$ can further restrict the finite set defined by equations (or intervals), but this same restriction can be achieved by simply eliminating certain equations or altering inequalities defining finite intervals from the disjunction determining the finite subset.

Thus, the algebraic closure of A is contained in the set of all $\mathbb{Z}$-linear combinations of elements of A and their associated differences, and the quotients of such linear combinations by appropriate standard integers k.

Conversely, suppose $kz = pa_1 + qd_1 + m$ for $p, q, m \in \mathbb{Z}, k \in \{2, 3, \ldots\}$. Then z is in the algebraic closure of a_1, because $kz = p_1(a_1)$ where

$$p_1(a_1) = q(\sigma(a_1)) + (p - q)a_1 + m = q(a_1 + d_1) + pa_1 - qa_1 + m = pa_1 + qd_1 + m.$$

Hence $z = \frac{p_1(a_1)}{k}$. Thus, the algebraic closure of A is precisely equal to the set of $\mathbb{Z}$-linear combinations of the elements of A and their associated differences $d_1, \ldots, d_n$, and quotients of such linear combinations by appropriate standard integers k. $\square$

Chapter 8

DP rank

In this chapter, we use the classification of definable sets from Chapters 6 and 7 to determine that the DP-rank of the theories of $\mathcal{M}/\mathbb{Z}$ and $\mathcal{M}$ is 2. (To do so, we work in a sufficiently saturated model of each theory.) A consequence of DP-rank 2 is that the theories of both structures have the *non-independence property*, or NIP [1, p. 270], and are indeed far removed from having the independence property. By contrast, the theory of Peano arithmetic has the independence property [13, p. 8]. To define DP-rank, it is necessary first to define an ICT-pattern:

Definition 17. For any cardinal κ, an *ict-pattern of depth* κ is a sequence of formulas $\langle \phi_\alpha(x;\bar{y}) : \alpha < \kappa \rangle$, and an array of tuples $\langle \bar{b}_i^\alpha : \alpha < \kappa, i < \omega \rangle$, such that for every function $\eta : \kappa \to \omega$, the following conjunction of formulas is consistent:

$$\bigwedge_{\alpha<\kappa} \phi_\alpha(x;\bar{b}^\alpha_{\eta(\alpha)}) \wedge \bigwedge_{\alpha<\kappa, i\neq\eta(\alpha)} \neg\phi_\alpha(x;\bar{b}^\alpha_i).$$

Thus, an ict-pattern is an array of formulas, uniform in each row, such that for every vertical path through the array there is an element that satisfies the formulas along that path, and no other paths.

Note that a formula that defines a finite set cannot occur in an ICT-pattern of depth greater than 1, because it provides for only finitely many possible distinct paths through sets defined in another row.

Definition 18. The *dp-rank* of a theory T is the minimal cardinal κ (if it exists) such that there is no ict-pattern of depth κ^+.

In the next two subsections, it is established that the DP-rank of the theory T^* of the quotient structure and of the theory T of the Presburger structure is 2, that is, an ICT-pattern exists for each of depth 2, but not of depth 3. In each case, the proof exhibits an array with two rows satisfying the definition of an ICT-pattern. Further, the proofs show that adding an additional row no longer satisfies the pattern because some pair of rows that occur in arrays of three or more rows will fail to contain the vertical paths required, that is, there will exist a pair of sets in 2 rows such that any witness to a path through those two sets is also a witness to a path through a different pair of sets in those 2 rows.
Remark. By [2, Fact 2.12], we need not consider any row of consisting of a disjunction $\phi(x, \bar{y}) := \phi_1(x, \bar{y}) \vee \cdots \vee \phi_n(x, \bar{y})$, because if such a disjunction witnesses that two rows lack the requisite vertical paths, then one of its disjuncts already witnesses the failure of requisite vertical paths. Thus, we need only consider formulas in ICT-patterns that are conjunctions of literals.

8.1 Quotient model

By the Remark above, the sets definable in T^* that are finite unions of convex sets (or of finite unions of convex sets with dense sets or complements of dense sets removed) need not be considered, because a formula defining such a finite union can be written as a finite disjunction of formulas. Thus, in proving the theorem below, we need only consider those formulas that define: (1) a convex set; (2) a convex set with finitely many cosets of the set of differences removed; and (3) cosets of the set of differences.

Theorem 11. *The theory T^* has DP-rank 2.*

Proof. 1. *DP-rank is at least 2.* Consider a row of formulas defining disjoint intervals in the fixed-point set of σ, and a row of formulas defining cosets of the subgroup of differences. So row 1 consists of formulas $\phi_i(x; b_{1_i}, b_{2_i}) := b_{1_i} < x < b_{2_i}$ with distinct parameters in each entry in the row, and row 2 consists of formulas $\psi_j(x; b_j) := D(x + b_j)$ with distinct parameters in each entry each defining a different coset. Because each interval contains an element of every coset, and the cosets are disjoint, a witness to a path through an interval in row 1 and a coset in row 2 is not a witness to a path through any other pair of intervals in the first and second rows. Hence these two rows exhibit an ICT-pattern of depth 2.

2. *DP-rank is at most 2.* Fact 1: No ict-pattern can contain more than a single row of convex sets. We may assume that the convex sets in row 1 are all disjoint, and that the convex sets in row 2 are all disjoint. If such a pattern exists, each convex set in row 1 must intersect each convex set in row 2 so that there will be a witness to a path through them. But this is not possible: A set T in row 2 cannot be contained in a set S in row 1, because then a path through any other set S' in row 1 and T will also be a path between S and T. (By symmetry, S cannot be contained in T.) But every every set T in row 2 must meet every set in row 1, but if it meets three disjoint sets S_1, S_2, S_3 in row 1, then one these three must be contained in T, which is not possible.

Fact 2: No ict-pattern can contain more than one row of cosets of the subgroup of differences. Two distinct cosets have empty intersection, so there cannot be a witness to every path through two such rows.

Fact 3: No ict-pattern can contain a row of convex sets and a row of convex sets minus finite unions of cosets of the differences.

We may again assume that the convex sets in row 1 are disjoint. Let S_1 be a convex set in row 1, and let $T_1 = J_1 \setminus K$ be a set in row 2, where K is the union of m many cosets ($m \in \mathbb{N}$), and J_1 is a convex set. Then T_1 cannot be contained in S_1, because then a witness to a path between T_1 and S_1 will also be a witness to a path between T_1 and any other set in row 1.

So, without loss of generality, assume S_1 meets J_1 on the left. Suppose a second set S_2 in row 1 also meets J_1 on the left, and assume without loss of generality that the intersection of S_1 with J_1 is contained in the intersection of S_2 with J_1. Then a witness to a path through T_1 and S_1 is also a witness to a path through T_1 and S_2. So at most one set in row 1 meets T_1 on the left; and, similarly, at most one such meets T_1 on the right. But there are infinitely many disjoint sets in row 1, and each must meet T_1.

Fact 4: No ict-pattern can contain two rows of convex sets minus finite unions of cosets of the subgroup of differences.

Proof. Let $S_1 = I_1 \setminus K$ where I_1 is a convex set and K is the union of m many cosets be in row 1, and let $T_1 = J_1 \setminus L$, where J_1 is a convex set and L is the union of n cosets be in row 2. Note that the number of cosets removed from each convex set in row 1 is m, and the number of cosets removed from each convex set in row 2 is n.

Containment not possible. Suppose first that S_1 is contained in J_1. Then a witness w_2 to a path through S_1 and any other set T_2 in row 2 will also be a witness to a path through S_1 and T_1 unless w_2 is contained in one of the n cosets removed from T_1. Thus, if we take $n+1$ sets $T_2, \ldots, T_{n+2}$ in row 2, then two of the witnesses $w_2, \ldots, w_{n+2}$ to paths through these sets and S_1 must lie in the same coset, and hence be witnesses to two paths. Thus, S_1 cannot be contained in J_1. Similarly, T_1 cannot be contained in I_1.

Second, let S_2 be another set in row 1, and suppose that S_2 is contained in I_1. Then a path through a set T in the second row and S_2 will be a path through S_1 as well unless the witness to the path is in one of the m cosets removed from S_1. Thus, at least one witness to a two paths through $m+1$ distinct sets $T_1, \dots, T_{m+1}$ in the second row and S_2 must fall in the same coset removed from S_1, and hence witnesses paths through two sets in row 2. Similarly no set T in row 2 is contained in a convex set J_i where $T_i = J_i \setminus L_i$ is another set in row 2.

Nested intervals not possible. Thus, S_1 must meet every set T_i in row 2; cannot be contained in any of them; and cannot contain any of them. Therefore, without loss of generality, infinitely many sets in row 2 must intersect S_1 on the right, giving the following picture:

$$\Big[_{S_1} \Big(_{T_1} \Big(_{T_2} \cdots \Big(_{T_q} \cdots \Big]_{S_1} \Big)_{T_1} \Big)_{T_2} \cdots \Big)_{T_q} \cdots$$

Now each of $T_1, T_2, \cdots T_q$ is a convex set with n cosets removed, and assume, say $q > 2n$. So we have

$$T_1 = J_1 \setminus \{a_{1_1}, \dots, a_{1_n}\}, T_2 = J_2 \setminus \{a_{2_1}, \dots, a_{2_n}\}, \dots, T_q = J_q \setminus \{a_{q_1}, \dots, a_{q_n}\}$$

where $a_{1_1}, \dots, a_{q_n}$ are cosets. Consider a witness to a path through T_q and S_1. To avoid also being a path through S_1 and any one of $T_1, \dots, T_{q-1}$, such a witness must be in a coset that has been removed from *all* of the $q-1$ convex sets $J_1, \dots, J_{q-1}$. Thus, without loss of generality, we must have $a_{1_1} = a_{2_1} = \cdots = a_{q_1} = A_1$. So

$$T_1 = J_1 \setminus \{A_1, \dots, a_{1_n}\}, T_2 = J_2 \setminus \{A_1, \dots, a_{2_n}\}, \dots, T_{q-1} = J_{q-1} \setminus \{A_1, \dots, a_{q-1_n}\}.$$

Now consider a witness to a path through T_{q-1} and S_1. To avoid also being a path through and S_1 and any one of $T_1, \dots, T_{q-2}$, such a witness must be in a coset that has been removed from *all* of the $q-2$ convex sets $J_1, \dots, J_{q-2}$. Thus, without loss of generality, we must have $a_{1_2} = a_{2_2} = \cdots = a_{q_2} = A_2$. And thus

$$T_1 = J_1 \setminus \{A_1, A_2, \ldots, a_{1_n}\}, T_2 = J_2 \setminus \{A_1, A_2, \ldots, a_{2_n}\}, \ldots, T_{q-2} = J_{q-2} \setminus \{A_1, A_2 \ldots, a_{q-2_n}\}.$$

It is now clear that as we seek witnesses satisfying an ICT pattern through the sets $T_{q-2}, T_{q-3}, \ldots$, after n stages the remaining $q - n$ sets $J_1, J_2, \ldots J_{q-n}$ convex sets will all have the same n cosets removed; and then at the $n+1$ stage, it is no longer possible to find a path through T_{q-n} and S_1 that is in a coset removed from all the lower index convex sets, because all those cosets are removed from J_{q-n} as well.

Because any ICT-pattern with 3 or more rows must contain one of the pairs of rows we have just ruled out, T^* has DP-rank at most 2. $\square$

8.2 Presburger model

Setting aside divisibility predicates, by Theorem 9 infinite definable sets in $(\mathcal{M}, \sigma) \models T$, where T is as in section 3.2, are finite unions of (1) infinite convex sets; (2) infinite convex sets minus finitely many cosets of the subgroup of differences; (3) infinite convex sets intersected with a single divisibility predicate; (4) infinite convex sets intersected with a single divisibility predicate minus finitely many cosets of the subgroup of differences; and (5) infinite cosets of the subgroup of differences. This classification, together with the methods of proof from Subsection 8.1 above, enable us to prove:

Theorem 12. *The theory T of the Presburger model $\mathcal{M}$ has DP-rank 2.*

Proof. The proof here is somewhat tedious because of the number of cases involved, and essentially involves combining the proofs in the case of the quotient model with a basic property of divisibility predicates: a single predicate with varying parameters defines at most finitely many distinct congruence classes. *First*, the DP-rank of T is at least 2. We can take a row of disjoint intervals contained in the fixed point set each of which contains densely many $\mathbb{Z}$-chains, and a row of disjoint cosets of the subgroup of differences. (For example, as the second row we can take the formulas $D(x+n)$ for $n \in \mathbb{N}$.) Each such interval contains a $\mathbb{Z}$-chain containing a difference, as well as all cosets defined by $D(x+n)$, and thus each path through the two rows has a witness that does not also witness any other path, because the intervals are disjoint.

Second, we need to show that the DP-rank of T is at most 2. As in the proof that the quotient model has DP-rank 2, we can simplify the analysis by observing that finite unions of convex sets defined by a single conjunction may be defined by a disjunction of literals, and so again by [2, Fact 2.12], it suffices to consider rows of formulas that define (1) convex sets, or (2) convex sets intersected with a single divisibility predicate, or (3) convex sets minus finitely many cosets of the subgroup of differences, or (4) convex sets intersected with a single divisibility predicate minus finitely many cosets of the subgroup of differences, or (5) cosets of the subgroup of differences.

Case 1: No ict-pattern contains two rows of convex sets. The proof is exactly the same as in the case of the quotient model.

Case 2: No ict-pattern contains a first row of convex sets and a second row of convex sets intersected with a divisibility predicate. Let P_n be the divisibility predicate occurring in the formula defining the second row. Observe that only n distinct divisibility predicates occur in row 2.

We may assume that the convex sets in the first row are disjoint. Let S_1 be a convex set in row 1 and let $T_1 = I_1 \cap P_n$ be a convex set I_1 intersected with a divisibility predicate P_n in row 2 such that there is at least one other set T_2 whose defining formula includes P_n. T_1 cannot be contained in S_1, because if it is then any witness to a path through T_1 and another set S_2 in row 1 will also be a witness to a path through T_1 and S_1. Conversely, if S_1 is contained in I_1, then any path through S_1 and T_2 will also be a path through S_1 and

T_1. And if S_1, S_2, and S_3 are three sets in row 1, T_1 must meet each of them, but then one of them is contained in T_1, which is not possible.

Case 3: No ict-pattern contains a first row of convex sets and a second row of convex sets with finitely many cosets of the subgroup of differences removed. The proof is exactly the same as in the case of the quotient model.

Case 4: No ict-pattern contains a first row of convex sets and a second row of convex sets intersected with a single divisibility predicate minus finitely many cosets of the subgroup of differences. We can modify the proof for case 3 as follows: We may assume that the convex sets in row 1 are disjoint. Let S_1 be a convex set in row 1, and let $T_1 = J_1 \setminus K$ be a set in row 2 intersected with a divisibility predicate in row 2 where K is the union of m many cosets ($m \in \mathbb{N}$), and J_1 is a convex set, and such that there are infinitely many other sets $T_2, T_3, \ldots$ in row 2 whose defining formula includes that same divisibility predicate. Then T_1 cannot be contained in S_1, because then a witness to a path between T_1 and S_1 will also be a witness to a path between T_1 and every other set in row 1.

Conversely, suppose S_1 is contained in J_1. Then a witness w_2 to a path through S_1 and T_2 in row 2 will also be a witness to a path through S_1 and T_1 unless w_2 is contained in one of the n cosets removed from T_1. Thus, if we consider $n+1$ sets $T_2, \ldots, T_{n+2}$, then two of the witnesses $w_2, \ldots, w_{n+2}$ to paths through these sets and S_1 must lie in the same coset, and hence violate ICT condition. So S_1 cannot be contained in J_1. Finally, if S_1, S_2, and S_3 are three sets in row 1, T_1 must meet each of them, but then one of them is contained in J_1, which is not possible.

Case 5: No ict-pattern contains two rows of convex sets intersected with a divisibility predicate. First, we must assume that the divisibility predicates in rows 1 and 2 are compatible; for example, we cannot have a predicate $P_4(x)$ in the formula for a convex set in row 1, and $P_4(x+1)$ in the formula for a convex set in row 2, because there will be no path between these sets. Next, observe that in each row, the divisibility predicate

P_k ($k \in \mathbb{N} \setminus \{0,1\}$) defines at most k distinct congruence classes. Choose one (P) that occurs infinitely often in row 1, and one (Q) that occurs infinitely often in row 2, and let $S = I \cap P$ and $T = J \cap Q$ be sets in rows 1 and 2 respectively satisfying these predicates (where I and J are convex sets). It is not possible for S to be contained in J because then every witness to a path through S and any other set T' in row 2 with predicate Q will also be a witness to a path through S and T. By symmetry, T cannot be contained in I. Note that we may assume that convex sets with the same divisibility predicates are disjoint. Now T must meet the three disjoint sets S_1, S_2, S_3 all satisfying divisibility predicate P. But then one of these must be contained in T, which is not possible.

Case 6: No ict-pattern contains a row of convex sets intersected with a divisibility predicate and a row of convex sets minus finitely many cosets of the subgroup of differences. We choose a predicate P that occurs infinitely often in row 1, and let $S_1 = I_1 \cap P, S_2 = I_2 \cap P, \ldots$ be sets satisfying predicate P, with $I_1, I_2, \ldots$ convex sets. We may assume that these sets are disjoint. Let $T_1 = J_1 \setminus K$ be a set in row 2, where K is the union of m many cosets ($m \in \mathbb{N}$), and J_1 is a convex set. Then T_1 cannot be contained in I_1, because then a witness to a path between T_1 and S_2 will also be a witness to a path between T_1 and S_1.

Conversely, suppose S_1 is contained in J_1. Then a witness w_2 to a path through S_1 and T_2 in row 2 will also be a witness to a path through S_1 and T_1 unless w_2 is contained in one of the n cosets removed from T_1. Thus, if we consider $n+1$ sets $T_2, \ldots, T_{n+2}$, then two of the witnesses $w_2, \ldots, w_{n+2}$ to paths through these sets and S_1 must lie in the same coset, and hence violate the ict-pattern condition. So S_1 cannot be contained in J_1.

Finally, if S_1, S_2, and S_3 are three sets in row 1, T_1 must meet each of them, but then one of them is contained in J_1, which is not possible.

Case 7: No ict-pattern contains a row of convex sets intersected with a divisibility predicate and a row of convex sets intersected with a divisibility predicate minus finitely many cosets of the subgroup of differences. This is a variation on the proof of the prior case. We choose a divisibility predicate P that occurs infinitely often in row 1, and a divisibility predicate Q that occurs infinitely often in row 2. Let $S_1 = I_1 \cap P, S_2 = I_2 \cap P, \ldots$ be sets satisfying predicate P, with $I_1, I_2, \ldots$ convex sets. We may assume these sets are disjoint. Let $T_1 = J_1 \setminus K_1 \cap Q, T_2 = J_2 \setminus K_2 \cap Q, \ldots$ be sets satisfying Q in row 2, with $|K_1| = |K_2| = \cdots = m \in \mathbb{N}$. Then no set among $T_1, T_2, \ldots$ can be contained in any of the sets $I_1, I_2, \ldots$. For suppose $T_1 \subset I_1$. Then every witness to a path through S_2 and T_1 will also be a witness to a path through S_1 and T_1. Conversely, suppose $S_1 \subset J_1$. If we consider $n+1$ sets $T_2, \ldots, T_{n+2}$, then two of the witnesses $w_2, \ldots, w_{n+2}$ to paths through these sets and S_1 must lie in the same coset, and hence violate the ict-pattern condition. So S_1 cannot be contained in J_1. Finally, if S_1, S_2, and S_3 are three sets in row 1, T_1 must meet each of them, but then one of them is contained in J_1, which is not possible.

Case 8: No ict-pattern contains two rows of convex sets minus finitely many cosets of the subgroup of differences. The proof is the same as in the case of the quotient group.

Case 9: No ict-pattern contains a row of convex sets minus finitely many cosets of the subgroup of differences and a row of convex sets intersected with a divisibility predicate minus finitely many cosets of the subgroup of differences. This is a variation of the proof in the prior case, except that when we demonstrate that nested intervals are not possible, we choose nested intervals $T_1, T_2, \ldots, T_q$, with $q > 2n$ (where n is the number of cosets removed from each convex set) that all satisfy the same divisibility predicate Q.

Case 10: No ict-pattern contains two rows of convex sets intersected with a divisibility predicate minus finitely many cosets of the subgroup of differences. This is a further variation of the proof of the prior case, where we now consider sets in row 1 that all satisfy the same divisibility predicate P, and sets in row 2 that all satisfy the same divisibility predicate Q.

Case 11: No ict-pattern contains two rows of cosets of the subgroup of differences. Any two such cosets either coincide or are disjoint.

These cases exhaust the possibilities that could result in 3 rows forming an ict-pattern. So the DP-rank of T is at most 2.

$\square$

Chapter 9

Conclusion: Further Questions

(1) An immediate general question that is raised by the analysis of the specific modest automorphism σ is the extent to which the results above on quantifier elimination, completeness, and DP-rank can be generalized to expansions by some other automorphism. For example, can these results be generalized to any automorphism that, like σ, is increasing on positive elements? Unlike a modest increasing automorphism such as σ, for a more general increasing automorphism g, we lose the nice feature that all differences $g(x) - x$ can be chosen to lie in the fixed point set of g. Thus, for example, if $g(a)$ is such that

$$\mathrm{st}\left(\frac{g(a)}{a}\right) = 2$$

then $g(a) - a$ is in the same magnitude class as a; thus, the differences of g occur throughout the part of M that is moved by g. On the other hand, we can construct such a g such that $g(a) - 2a$ is in the fixed-point set of g. Further investigation of such automorphisms would be a closely-related next step.

(2) Having shown that the theories of T and T^* have DP-rank 2, another natural question to ask is whether expanding either $\mathcal{M}$ or $\mathcal{M}/\mathbb{Z}$ by some other automorphism gives a theory

with greater DP-rank, or perhaps even a theory with the independence property. Of course, quantifier elimination for $(\mathcal{M}, \sigma)$ and $(\mathcal{M}/\mathbb{Z}, \sigma)$ simplifies the analysis of DP-rank. So one possibility is to look for an automorphism for which quantifier elimination seems unlikely. For example, we could consider the maximal modest increasing automorphism τ constructed above in section 2.3. Eliminating the quantifier from the formula

$$\exists x p(x) = q(\bar{y})$$

where $p(x)$ and $q(\bar{y})$ are τ-polynomials, poses an obstacle. We would like to repeat the method of solving equations given for σ, but for τ-polynomials $p(x)$ with $A \neq 0$, we cannot duplicate that method, in which we eliminated the difference d, because if

$$p(x) = Ax + Bd$$

where $d = \tau(x) - x$, then

$$\tau(p(x)) = Ax + Bd + Bd'$$

where $d' = \tau(d) - d \neq 0$. So it may be that examining such an automorphism will be fruitful in this regard.

(3) The example provided by τ suggests a way in which one could construct an automorphism with greater DP-rank. For example, we may be able to construct a modest automorphism μ whose differences $\mu(x) - x$ are not fixed (and that is not increasing). Then, for a nonstandard, we have

$$\mu(a) = a + d_1,\ \mu^2(a) = a + 2d_1 + d_2,\ \mu^3(a) = a + 3d_1 + 3d_2 + d_3, \ldots, \tag{9.1}$$

$$\mu^n(a) = a + nd_1 + \binom{n}{2} d_2 + \binom{n}{3} d_3 \ldots + nd_{n-1} + d_n \tag{9.2}$$

where $d_1, d_2, \ldots$ are successive differences of smaller magnitude. These equations allow us to solve for the successive differences $d_1, d_2, \ldots$ in terms of just μ and a. For example, $d_2 = \mu^2(a) - 2(\mu(a)) + a$. It may be possible to increase the DP-rank of a Presburger structure expanded by μ by starting with disjoint intervals, each of which contains densely many magnitude classes, and assigning ω many distinct differences d_1 within some set of lower magnitude classes to subintervals of each such interval, and ω many distinct second-order differences d_2 within sub-subintervals, etc., at each stage ensuring that there are densely many magnitude classes within all intervals, and that the differences also decrease in magnitude as their indices increase. For example, it may be possible in an expanded Presburger model $(\mathcal{M}, \mu)$ to have an ICT pattern with three rows as follows:
Row 1: disjoint intervals each containing a dense set of magnitude classes, defined by the formula:

$$\phi_1(x; \bar{b}^1) := b^1_{i,1} < x < b^1_{i,2}.$$

Row 2: disjoint intervals each containing a dense set of magnitude classes, all lying below the magnitude classes occurring in row 1, each of which contains a difference corresponding to an element in each of the intervals in row 1, defined by the formula:

$$\phi_2(x; \bar{b}^2) := b^2_{i,1} < \mu^2(a) - 2(\mu(a)) + a < b^2_{i,2}.$$

Row 3: disjoint intervals each containing a dense set of magnitude classes, all lying below the magnitude classes occurring in row 2, each of which contains a second-order difference corresponding to an element in each of the intervals in row 2, defined by the formula:

$$\phi_3(x;\bar{b}^3) := b^3_{i,1} < \mu^3(a) - 3(\mu^2(a)) + 3(\mu(a)) - a < b^3_{i,2}.$$

If constructing μ is possible, then the disjointness of the intervals in each row will guarantee that we have an ICT-pattern of depth 3. By repeatedly subdividing the initial disjoint intervals in row 1 above, it seems reasonable that we could obtain an ICT-pattern of arbitrary finite rank.

(4) Another direction for additional research would be to examine expansions of Presburger by multiple value-preserving automorphisms, or by automorphisms that do not preserve values, and investigate whether such expansions yield more complex structures.

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