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Zeta Functions of Classical Groups and Class Two Nilpotent Groups

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by

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Abstract

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Advisor: Professor Gautam Chinta

This thesis is concerned with zeta functions and generating series associated with two families of groups that are intimately connected with each other: classical groups and class two nilpotent groups. Indeed, the zeta functions of classical groups count some special subgroups in class two nilpotent groups.

In the first chapter, we provide new expressions for the zeta functions of symplectic groups and even orthogonal groups in terms of the cotype zeta function of the integer lattice. In his paper on universal p-adic zeta functions, J. Igusa computed explicit formulae for the zeta functions of classical algebraic groups. These zeta functions were later extended to more general algebraic groups and their representations by M. du Sautoy and A. Lubotzky. We show computations of local Hecke series by A. Andrianov and later by T. Hina and T. Sugano lead to new expressions for the zeta functions of classical groups.

In the second chapter, we attempt to generalize the notion of a cotype zeta function to subgroup growth zeta functions of class two nilpotent groups. G. Chinta, N. Kaplan and S. Koplewitz recently used the cotype zeta function of the integer lattice to prove interesting

facts about the proportion of sublattices of given corank and to match the distribution of sublattices with a particular Cohen-Lenstra distribution. Motivated by the method of subgroup counting in nilpotent groups in the seminal paper of F. J. Grunewald, D. Segal and G. C. Smith, we give a general definition of the cotype of a subgroup of finite index and use it to compute new multivariable zeta functions. Such zeta functions help in determining the distribution of subgroups of finite index.

In the last chapter, we make use of a generating series in several variables to count finite class two nilpotent groups. In 2009, C. Voll computed the number $g(n, 2, 2)$ of nilpotent groups of order n, of class at most 2 generated by at most 2 generators, by giving an explicit formula for the Dirichlet generating function of $g(n, 2, 2)$. Later in 2012, A. Ahmad, A. Magidin and R. F. Morse gave a direct enumeration of such groups building on the works of M. Bacon, L. Kappe, et al. We use their enumeration to provide a natural multivariable extension of the generating function counting such groups and as a result rederive Voll's explicit formula. Similar formulas or enumerations for finite groups of nilpotency class 2 on more than 2 generators or of at least class 3 on 2 or more generators is currently unknown.

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Contents

Introduction

One of the fundamental quests in mathematics is the classification of objects that one is studying up to some kind of equivalence. For instance, the primary problem in group theory is the classification of groups up to isomorphism. The enumeration of such classes of objects may involve number theory. If you take nilpotent groups for example, which are one of the simplest typical non-abelian groups, they seem to be a very wild category of groups. Even if these groups are "almost abelian," we don't really know very much about these groups. For instance, the quasi-isometry classification of finitely generated nilpotent groups remains one of the major open problems in geometric group theory [FM99]. Even in the finite case, there is yet no complete classification of finite nilpotent groups or equivalently of finite p-groups up to group isomorphism [LM02]. The last chapter of this thesis deals with this interesting class of finite groups.

We can look for some kind of invariants and try to do some kind of classification maybe less delicate than isomorphism. One of the interesting invariants to come up is a tool from number theory which is to realize there is a kind of a natural zeta function associated to something like a nilpotent group. Here, by a zeta function, we mean a Dirichlet series in one complex variable having an Euler product decomposition that is convergent on some right half-plane. The question of whether these zeta functions also satisfy the other usual properties of classical zeta functions such as analytic continuation and functional equations is addressed in [dSW08].

A canonical class of examples of the sorts of groups we will be looking at is the discrete Heisenberg groups, denoted $H_{2d+1}(\mathbb{Z})$, of upper triangular $(2d+1)$ by $(2d+1)$ matrices over the integers with $1'$s on the diagonal, 0's everywhere else except the first row and last column, d a positive integer and matrix multiplication as the group operation. They are finitely generated nonabelian infinite nilpotent groups.

The zeta function we are interested in is constructed in the similar way as the Dedekind zeta function of number fields. For number fields, we sum up reciprocals of norms of ideals raised to a complex power to form the zeta function. The norm of an ideal is its index in the ring of integers of the number field. So, one may ask why not do the same over something which is not commutative. For instance, considering the subgroup structure inside the nilpotent group $H_{2d+1}(\mathbb{Z})$, we take the index of a subgroup of finite index as the norm of the subgroup and raise it to a power of -s where s is a complex variable and sum these expressions over subgroups of finite index in $H_{2d+1}(\mathbb{Z})$. This is a non-commutative version of a zeta function. The fact that it is a zeta function is justified as it has an Euler product decomposition.

In the case we are considering, there is a finite number of subgroups, say $a_n(H_{2d+1})$, of each given index n, bounded by a polynomial in n. So, we can express the zeta function as a Dirichlet series as follows:

$$\zeta_{H_{2d+1}}(s) = \sum_{n\geq 1} \frac{a_n(H_{2d+1})}{n^s}. \tag{0.0.1}$$

This Dirichlet series is called the *subgroup growth zeta function* of H_{2d+1}. It converges for $\text{Re}\{s\} > d-1$. A more precise analogue of the zeta function of number fields is obtained by counting normal subgroups of finite index. We will denote the number of normal subgroups of index n by $a_n^{\triangleleft}(H_{2d+1})$ and the corresponding zeta function by

$$\zeta^{\triangleleft}_{H_{2d+1}}(s) = \sum_{n \geq 1} \frac{a_n^{\triangleleft}(H_{2d+1})}{n^s}, \tag{0.0.2}$$

also known as the *normal subgroup growth zeta function* of $H_n(\mathbb{Z})$.

A finite nilpotent group is a direct product of its Sylow p-subgroups. This implies that both Dirichlet series we defined above have a natural Euler product decomposition [GSS88]. That is, the series pulls apart into local factors. So, for a prime p, we define the local factor (p-part or Euler factor) $\zeta_{H_{2d+1},p}(s)$ (or $\zeta^{\triangleleft}_{H_{2d+1},p}(s)$) to be the sum of p^{-ns} over all subgroups (respectively, normal subgroups) of p-power index p^n inside the group $H_{2d+1}(\mathbb{Z})$:

$$\zeta_{H_{2d+1},p}(s) = \sum_{n \geq 1} \frac{a_{p^n}(H_{2d+1})}{p^{ns}} \tag{0.0.3}$$

$$\zeta^{\triangleleft}_{H_{2d+1},p}(s) = \sum_{n \geq 1} \frac{a^{\triangleleft}_{p^n}(H_{2d+1})}{p^{ns}} \tag{0.0.4}$$

From now on, we will be looking closely at the normal subgroup growth zeta functions of groups and show the tools that have been developed to understand the normal subgroups.

The first major theorem [GSS88] about these functions, proved using the model theory of p-adic integers by Denef [D84], is that the local factors for a finitely generated torsion-free nilpotent group are rational functions in p^{-s}. This implies that the lattice of subgroups is well ordered, meaning there is a recurrence relation on the lattice of subgroups and we also get interesting invariants such as the poles of these functions which tell us about global properties of the product of the local factors [dSG00].

The basic idea used to prove this theorem is the idea of encoding the arithmetic function (local factors) in terms of a p-adic integral. The p-adic integrals are the main tools used here to capture the arithmetic data. They were first introduced to understand arithmetic data coming from counting solutions of polynomial equations modulo p^m. This arithmetic data was defined inside a zeta function by Z. I. Borevich and I. R. Shafarevich [D84]. So, we can define for each polynomial, a sort of local zeta function and study its properties after realizing it as a p-adic integral [I00]. Analogously, we can represent the local factors of a normal subgroup growth zeta function as a p-adic integral and apply this tool to capture the subgroup structure of nilpotent groups. The p-adic integrals are also the motivation for the definition of zeta functions of algebraic groups discussed in the first chapter of this thesis.

For torsion-free infinite abelian groups such as integer lattices of any dimension, these ideas have been generalized to a multivariable setting and further studied by Igusa [I00], V. M. Petrogradsky [P07] and G. Chinta, N. Kaplan and S. Koplewitz [CKK]. Such abelian groups are a special case of $\mathscr{T}$-groups. $\mathscr{T}$-groups are torsion-free finitely generated nilpotent groups and the above abelian groups are $\mathscr{T}$-groups of nilpotency class one. The next level of generality is class two.

One of the goals of this thesis is to bring the one variable subgroup growth zeta function ideas into a multi-variable setting for arithmetically intersting non-abelian infinite groups. This provides more refined invariants than the single-variable zeta functions as discussed in chapter two here. For instance, in the case of discrete Heisenberg groups of all ranks, we use the newly defined cotype zeta functions to determine the distribution of subgroups of finite index such as the density of subgroups with cyclic quotient or other certain cotypes. We also present an example of a multiple integral representation of the local part of the cotype zeta function of $\mathscr{T}$-groups of nilpotency class two. This will help us see whether there is a phenomenon analogous to the Cohen-Lenstra distribution studied in [CKK].

Chapter 1

Zeta functions of classical groups

1.1 The cotype zeta function of $\mathbb{Z}^d$

Let $n \geq 1$ be a positive integer. Denote the number of subgroups of index n in a finitely generated group G by $a_n(G)$. Assume G has a polynomial subgroup growth. That is, there exists some positive integer c for all n such that

$$a_n(G) \leq n^c. \tag{1.1.1}$$

The *subgroup growth* zeta function of G is defined as the Dirichlet series

$$\zeta_G(s) := \sum_{n \geq 1} \frac{a_n(G)}{n^s}. \tag{1.1.2}$$

In this note, we will be concerned with the group $\mathbb{Z}^d$, the integer lattice of dimension d. The subgroup growth zeta function of $\mathbb{Z}^d$ is a product of shifted Riemann zeta functions:

$$\zeta_{\mathbb{Z}^d}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-(d-1)). \tag{1.1.3}$$

There are more than 5 proofs of this fact [LS03; P07], [Sh71, p. 64, eqn 3.2.3] and [H15, Prop. 4.2.1]. One proof proceeds by replacing the count of sublattices (i.e subgroups) by a count of canonical coset representatives.

More precisely, given a sublattice Λ of index n in $\mathbb{Z}^d$, we can choose a basis of vectors and arrange them into the columns of a $d \times d$ matrix g. Then $\Lambda = \mathbb{Z}^d \cdot g$ with $n = |\det(g)|$. Such a matrix is chosen up to a right action of $\mathrm{GL}_d(\mathbb{Z})$, the group of automorphisms of $\mathbb{Z}^d$. Therefore, sublattices of $\mathbb{Z}^d$ are in bijection with $\mathrm{GL}_d(\mathbb{Z})$-orbits of $d \times d$ integral matrices [C13]. In particular, the sublattices of finite index are in bijection with right coset representatives in $\mathrm{GL}_d(\mathbb{Z})\backslash\mathrm{GL}_d^+(\mathbb{Q})$ where $\mathrm{GL}_d^+(\mathbb{Q}) = \mathrm{M}_d(\mathbb{Z}) \cap \mathrm{GL}_d(\mathbb{Q})$. For each right coset, a canonical right coset representative is given by a Hermite normal form.

The sublattice counting problem now translates into a problem of counting Hermite normal forms of given determinant which easily leads to the formula for the zeta function of $\mathbb{Z}^d$ [LS03, p. 311]. This counting argument in a different guise is attributed to Eisenstein by Siegel and later generalized to number rings by Hurwitz [GG06]. These ideas were also taken up further by L. Solomon, C. Bushnell and I. Reiner in their study of zeta functions of orders in semisimple algebras over $\mathbb{Q}$ or $\mathbb{Q}_p$ [BR80].

A more refined count of the sublattices can be obtained via the elementary divisors of the quotient of $\mathbb{Z}^d$ by the sublattices. The theme of enumerating sublattices by their elementary divisor type is a central one in [V10] where a new method was introduced to compute explicit formulae for various zeta functions associated to groups and rings. Let $\Lambda \subset \mathbb{Z}^d$ be a sublattice of finite index. The *cotype* of Λ is defined in [CKK] as the unique d-tuple of integers, i.e., elementary divisors, $(\alpha_1, \ldots, \alpha_d) = (\alpha_1(\Lambda), \ldots, \alpha_d(\Lambda))$ such that the finite abelian group $\mathbb{Z}^d/\Lambda$ is isomorphic to the direct sum of cyclic groups

$$(\mathbb{Z}/\alpha_1\mathbb{Z}) \oplus (\mathbb{Z}/\alpha_2\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/\alpha_d\mathbb{Z}) \tag{1.1.4}$$

where $\alpha_{i+1} \mid \alpha_i$ for $1 \leq i \leq d-1$. Petrogradsky defines in [P07], a multiple zeta function $\zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d)$, called the *cotype zeta function* of $\mathbb{Z}^d$ [CKK].

Definition 1. *Let $a_\alpha(\mathbb{Z}^d)$ denote the number of subgroups $\Lambda \subset \mathbb{Z}^d$ of cotype α. The cotype* *zeta function of $\mathbb{Z}^d$ is the the multivariable Dirichlet series*

$$\zeta_{\mathbb{Z}^d}(s_1,\ldots,s_d) = \sum_{|\mathbb{Z}^d:\Lambda|<\infty} \alpha_1(\Lambda)^{-s_1}\ldots\alpha_d(\Lambda)^{-s_d} = \sum_{\alpha=(\alpha_1,\ldots,\alpha_d)} a_\alpha(\mathbb{Z}^d)\cdot\alpha_1^{-s_1}\ldots\alpha_d^{-s_d}.$$

where the last sum is over all possible cotypes, i.e. $a_i \geq 1$ for all $1 \leq i \leq d$ and $\alpha_1|\alpha_2|\ldots|\alpha_d$.

Setting $s_1 = \cdots = s_d = s$, we get $\zeta_{\mathbb{Z}^d}(s,\ldots,s) = \zeta_{\mathbb{Z}^d}(s)$ which is the subgroup growth zeta function of $\mathbb{Z}^d$.

Furthermore, Petrogradsky showed that the cotype zeta function of $\mathbb{Z}^d$ has an Euler product decomposition

$$\zeta_{\mathbb{Z}^d}(s_1,\ldots,s_d) = \prod_p \zeta_{\mathbb{Z}^d,p}(s_1,\ldots,s_d) \tag{1.1.5}$$

where the Euler p-part is defined as

$$\zeta_{\mathbb{Z}^d,p}(s_1,\ldots,s_d) = \sum_{k_1\geq\ldots\geq k_d\geq 0} a_{(p^{k_1},\ldots,p^{k_d})}(\mathbb{Z}^d)p^{-s_1k_1-\cdots-s_dk_d} \tag{1.1.6}$$

and is given by

$$\zeta_{\mathbb{Z}^d,p}(s_1,\ldots,s_d) = \frac{\sum\limits_{\lambda\subseteq\mathbb{N}_{d-1}} w_\lambda(p^{-1})\prod\limits_{j\in\lambda} t_j}{(1-t_1)(1-t_2)\cdots(1-t_d)}, \tag{1.1.7}$$

where $t_i = p^{-z_i}$, $z_j = s_1+\cdots+s_j - j(d-j)$ and $w_\lambda(p^{-1}) = \sum\limits_{\mu\subseteq\lambda}(-1)^{|\lambda|-|\mu|}\begin{bmatrix} d \\ \mu \end{bmatrix}_{1/p}$. We refer to [P07] for more on the notations and proof of the formula above.

The first few examples are

$$\zeta_{\mathbb{Z},p}(s) = \frac{1}{1-p^{-s}} \tag{1.1.8}$$

$$\zeta_{\mathbb{Z}^2,p}(s_1,s_2) = \frac{1-p^{-2s_1}}{(1-p^{-s_1})(1-p^{1-s_1})(1-p^{-s_1-s_2})} \tag{1.1.9}$$

$$\zeta_{\mathbb{Z}^3,p}(s_1,s_2,s_3) = \frac{1+(p+1)p^{-s_1}+(p+1)p^{-s_1-s_2}+p^{1-2s_1-s_2}}{(1-p^{2-s_1})(1-p^{2-s_1-s_2})(1-p^{-s_1-s_2-s_3})}. \tag{1.1.10}$$

The goal of this section is to show that the zeta functions of classical algebraic groups can be expressed in terms of the cotype zeta function of $\mathbb{Z}^d$ in the sense of Igusa's universal

p-adic zeta functions [I89] and Voll's blueprint result on certain p-adic integrals [V10]. For this we reconsider the cotype zeta function of $\mathbb{Z}^d$ from a different viewpoint. Below we will show how to derive the cotype zeta function of $\mathbb{Z}^d$ from zeta functions associated to local Hecke series of Hecke algebras. To start with, a background on the relevant notions is given below using the setup in [A70], [V08] and [FC90].

The **abstract Hecke algebra** $L = H(\Gamma, S)$ (also known as the Hecke ring or the double coset algebra) of the pair $(\Gamma, S) = (\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n^+(\mathbb{Q}))$ where $\mathrm{GL}_n^+(\mathbb{Q}) = \mathrm{M}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ is the algebra whose elements are the finite formal $\mathbb{Q}$-linear combinations of double cosets $\Gamma s \Gamma$ with $s \in S$. In general, when each double coset $\Gamma s \Gamma$ can be written as a finite disjoint union of right cosets Γs_i, we call the pair (Γ, S) a Hecke pair. The product of elements in the algebra determined by the pair is defined as follows:

$$(\Gamma s \Gamma) \cdot (\Gamma r \Gamma) = \sum_{i,j} \Gamma s_i r_j. \tag{1.1.11}$$

On the other hand, the (spherical) **Hecke algebra** $\mathcal{L} = \mathcal{L}_{\mathbb{Z}}(\Gamma, S)$ of the pair (Γ, S) over $\mathbb{Z}$ is defined as the $\mathbb{Z}$-linear vector space of all continuous (i.e. locally constant) $\mathbb{Z}$-valued functions f on S with compact supports and satisfying $f(ugv) = f(g)$ for any $u, v \in \Gamma$ and $g \in S$. The algebra structure is given by the convolution product

$$(f * \phi)(g) = \int_S f(gh^{-1})\phi(h)dh \qquad (g \in S; f, \phi \in \mathcal{L}) \tag{1.1.12}$$

where dh is the Γ-bi-invariant Haar measure on S normalized by $\int_\Gamma dh = 1$. The spherical Hecke algebra is the image of the abstract Hecke algebra in the endomorphism algebra of a space of automorphic forms [H12], [A87, Prop. 3.1.10].

We will now use both the abstract Hecke algebra L and the spherical Hecke algebra $\mathcal{L}$ to show two methods of obtaining the rational expression for the p-part of the cotype series of $\mathbb{Z}^n$.

1.1.1 Via a local multiple Hecke series

This method works only in the few cases a formal identity can be proved using the multiplication laws of the abstract Hecke algebra elements. Consider the integral local subalgebra L_p generated by integral cosets of the local abstract Hecke algebra $L(\frac{1}{p})$ over $\mathbb{Z}$ associated to the Hecke pair $(\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n^+(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}[1/p]))$. It is generated by the n double coset classes [Sa63; V08]:

$$\mathbf{T}_i(p) = t(\underbrace{1,\ldots,1}_{n-i},\underbrace{p,\ldots,p}_{i}) := \mathrm{diag}(\underbrace{1,\ldots,1}_{n-i},\underbrace{p,\ldots,p}_{i}). \tag{1.1.13}$$

Therefore, we have $L_p = \mathbb{Z}[\mathbf{T}_1(p),\ldots,\mathbf{T}_n(p)]$. A canonical representative of a general double coset in this algebra has the form $\mathrm{diag}(p^{k_1},\ldots,p^{k_n})$ with $0 \le k_1 \le k_2 \le \ldots \le k_n$. Analogous to the Hecke series studied by Andrianov[1] [A87, see p122], we form the formal power series $Z_p(v_1,\ldots,v_n)$ over L_p as follows:

Definition 2. *The local multiple Hecke series of the abstract Hecke algebra L_p is a formal power series given by*

$$Z_p(v_1,\ldots,v_n) = \sum_{0\le k_1\le\ldots\le k_n} t(p^{k_1},\ldots,p^{k_n}) v_1^{k_1}\cdots v_n^{k_n}$$

where $t(p^{k_1},\ldots,p^{k_n}) := diag(p^{k_1},\ldots,p^{k_n})$.

Remark 3. *In contrast to the decreasing order of numbers in a cotype, the tuple representing a double coset has increasing numbers. So, when we compare cotypes and double coset representatives, we will have to reverse the order of the tuple elements.*

Each double coset representative above corresponds to finitely many right cosets which is in agreement with the fact that each cotype corresponds to finitely many sublattices. Hence, we get a counting function giving the number of right cosets with the same double coset

[1] The Hecke pair Andrianov considers is $(\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Q}))$ instead of the one we study here.

representative. In particular, $\mathrm{ind}(t(p^{k_1},\ldots,p^{k_n}))$ equals $a_\alpha(\mathbb{Z}^n)$, the number of sublattices of $\mathbb{Z}^n$ of cotype $\alpha=(p^{k_1},\ldots,p^{k_n})$. This counting function on double coset representatives, called the *index function* and denoted *ind*, is a classical ring homomorphism from L_p to $\mathbb{Z}$ [K90, see Chapter I, Corollary 4.5], [BG].

Since L_p is an integral domain, we can extend *ind* to a homomorphism from the rational function field $Frac(L_p)(v_1,\ldots,v_n)$ to $\mathbb{Q}(v_1,\ldots,v_n)$ by declaring it to be the identity on $v_1,\ldots,v_n$. Thus, the index function maps the multiple Hecke series $Z_p(v_1,\ldots,v_n)$ into the p-part of the cotype series of $\mathbb{Z}^n$ as follows:

$$\begin{aligned}\mathrm{ind}(Z_p(v_1,\ldots,v_n)) &= \sum_{0\le k_1\le\ldots\le k_n}\mathrm{ind}(t(p^{k_1},\ldots,p^{k_n}))v_1^{k_1}\cdots v_n^{k_n}\\ &= \sum_{\alpha=(p^{k_1},\ldots,p^{k_n})} a_\alpha(\mathbb{Z}^n)p^{-k_1s_1}\cdots p^{-k_ns_n}\\ &= \zeta_{\mathbb{Z}^n,p}(s_1,\ldots,s_n), \qquad (1.1.14)\end{aligned}$$

where we make the change of variables $v_i=p^{-s_i}$ for $1\le i\le n$.

An example: $n=2$

The formal power series $Z_p(v_1,v_2)$ over L_p^2 is:

$$Z_p(v_1,v_2)=\sum_{0\le k_1\le k_2} t(p^{k_1},p^{k_2})v_1^{k_1}v_2^{k_2}. \qquad (1.1.15)$$

We now prove that it is a rational function in v_1,v_2 and find the explicit form below. It may be possible to show rationality of the local multiple Hecke series via the method below but finding the explicit rational function does not seem possible when $n\ge 3$ as discussed below.

Proposition 4.

$$Z_p(v_1,v_2)=\frac{1-t(p,p)v_2^2}{(1-t(p,p)v_1v_2)(1-t(1,p)v_2+pt(p,p)v_2^2)}$$

Proof. We sum the formal series using relations satisfied by the coefficients as follows.

$$\begin{aligned}
Z_p(v_1, v_2) &= \sum_{0\le k_1\le k_2} t(p^{k_1}, p^{k_2}) v_1^{k_1} v_2^{k_2} \\
&= \sum_{0\le k\le k+\delta} t(p^k, p^{k+\delta}) v_1^k v_2^{k+\delta} && \text{let } k_1 = k, k_2 = k+\delta, \delta \ge 0 \\
&= \sum_{k,\delta\ge 0} t(p^k, p^k) t(1, p^\delta) v_1^k v_2^{k+\delta}, && \text{by Lemma 3.2.4 in [A87]} \\
&= \sum_{k\ge 0} t(p^k, p^k)(v_1 v_2)^k \sum_{\delta\ge 0} t(1, p^\delta) v_2^\delta \\
&= \sum_{k\ge 0} (t(p,p) v_1 v_2)^k \sum_{b\ge 0} t(1, p^\delta) v_2^\delta \\
&= \frac{1}{1 - t(p,p) v_1 v_2} \sum_{\delta\ge 0} t(1, p^\delta) v_2^\delta
\end{aligned}$$

For the remaining sum above, consider (these are the steps that won't work for $n \ge 3$)

$$\begin{aligned}
\frac{1}{1 - t(p,p)v_2^2} \sum_{\delta\ge 0} t(1, p^\delta) v_2^\delta &= \sum_{a\ge 0} t(p^a, p^a) v_2^{2a} \sum_{\delta\ge 0} t(1, p^\delta) v_2^\delta \\
&= \sum_{a,\delta\ge 0} t(p^a, p^a) t(1, p^\delta) v_2^{2a+\delta} \\
&= \sum_{a,\delta\ge 0} t(p^a, p^{a+\delta}) v_2^{2a+\delta} \\
&= \sum_{\delta\ge 0} \Big(\sum_{0\le a\le \frac{\delta}{2}} t(p^a, p^{\delta-a}) \Big) v_2^\delta \\
&= \sum_{\delta\ge 0} t(p^\delta) v_2^\delta, && \text{by eqn 3.2.10 in [A87]} \\
&= \frac{1}{1 - t(1,p) v_2 + p t(p,p) v_2^2}.
\end{aligned}$$

So, we get

$$\sum_{b\ge 0} t(1, p^b) v_2^b = \frac{1 - t(p,p)v_2^2}{1 - t(1,p)v_2 + p t(p,p) v_2^2}.$$

Therefore, the formal power series has the form

$$Z_p(v_1, v_2) = \frac{1 - t(p,p)v_2^2}{(1 - t(p,p)v_1 v_2)(1 - t(1,p)v_2 + p t(p,p) v_2^2)}$$

□

By Chapter V, Proposition 7.2 in [K90], the indices of the generators are $\mathrm{ind}(t(1,p)) = p+1$ and $\mathrm{ind}(t(p,p)) = 1$. As the discussion around equation 1.1.14, the p-part of the cotype zeta function of $\mathbb{Z}^2$ is obtained as follows.

$$\begin{aligned}\mathrm{ind}(Z_p(v_1,v_2)) &= \sum_{0\le k_1\le k_2} \mathrm{ind}(t(p^{k_1},p^{k_2}))v_1^{k_1}v_2^{k_2}\\ &= \frac{1-\mathrm{ind}(t(p,p))v_2^2}{(1-\mathrm{ind}(t(p,p))v_1v_2)(1-\mathrm{ind}(t(1,p))v_2+p\mathrm{ind}(t(p,p))v_2^2)}\\ &= \frac{1-v_2^2}{(1-v_1v_2)(1-(p+1)v_2+pv_2^2)}\\ &= \frac{1-v_2^2}{(1-v_2)(1-pv_2)(1-v_1v_2)}\end{aligned}$$

where $v_1 = p^{-s_1}, v_2 = p^{-s_2}$. So,

$$\zeta_{\mathbb{Z}^2,p}(s_2,s_1) = \frac{1-p^{-s_2}}{(1-p^{-s_2})(1-p^{1-s_2})(1-p^{-s_1-s_2})},$$

as expected. The ordering of the variables is reversed as explained in Remark 3.

For $n \ge 3$ there is no simple multiplication law for the elements of the form $t(p^{k_1},\dots,p^{k_n})$, cf. [A70, Lemma 9], [A87, pages 132,151]. So, there does not seem to be a way to generalize the above computation to obtain the explicit rational expression when $n \ge 3$. However, we can simplify the computations in the abstract Hecke algebra by using a realization of its image (the Hecke algebra) given by Satake's spherical mapping. This is the second method discussed below. We note that expanding out the rational expression above in powers of v_1, v_2 and matching coefficients with the power series provides the multiplication law of the algebra. So, the rational expression is a finite expression that encodes the product structure of the algebra and the cotype zeta function being derived from it can be seen as an intrinsic invariant of the algebra up to isomorphism and deserves further investigation in the more general setting of commutative algebras of Hecke pairs.

1.1.2 Via Andrianov's summation of Hecke series

Andrianov studied the multiple Hecke series of a Hecke pair such as $(\mathrm{GL}_n(\mathbb{Z}_p), \mathrm{GL}_n(\mathbb{Q}_p))$ in his study of zeta functions associated to GL_n over local fields [A87], [A70]. For the above Hecke pair, he proved rationality of the corresponding Hecke series by using the Satake transform, also known as the Satake spherical mapping. In [HS83], Hina and Sugano adapted Andrianov's method of summation to Hecke series associated with other classical algebraic groups and found explicit formulae. In particular, with the interpretation of the count $a_\alpha(\mathbb{Z}^n)$ of sublattices of cotype $\alpha = (p^{k_1}, \ldots, p^{k_n})$ as the index $\mathrm{ind}(t(p^{k_1}, \ldots, p^{k_n}))$ of a canonical double coset representative in $(\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z}[1/p])))$, we see that the cotype zeta function of $\mathbb{Z}^n$ computed by Petrogradsky [P07] is precisely the multiple index function series computed by Hina and Sugano in [HS83].

We now state Hina and Sugano's result and discuss the consequences for the zeta functions of classical groups. Even if we are not computing any new zeta functions or Hecke series, this viewpoint provides an alternative derivation of the zeta functions of the classical groups which were first computed uniformly by Igusa [I89] and later generalized extensively by du Sautoy and Lubotzky [dSL96]. This also confirms that the technical computations done independently in different contexts by the aforementioned mathematicians are in complete agreement with one another.

Let $\mathbb{N}_d = \{1, \ldots, d\}$ and $I = \{i_1, \ldots, i_r\} \subset \mathbb{N}_{d-1}$ such that $1 \leq i_1 < \cdots < i_r \leq d-1$. Also let $i_0 = 0$ and $i_{r+1} = d$. Then

$$\psi_{(I)}(v) = \frac{\psi_d(v)}{\psi_{i_1 - i_0}(v) \cdots \psi_{i_{r+1} - i_r}(v)},$$

is a polynomial in v where $\psi_i(v) = \prod_{j=1}^{i} (v^j - 1)$ for $i \geq 1$. When $I = \phi$, set $\psi_\phi(v) = 1$. Based on the discussion above, we can deduce the following formula for $\zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d)$ from

Proposition 2 in Hina and Sugano [HS83].

Proposition 5. *Let* $z_r = p^{r(d-r)}v_{d-r}$ *for* $0 \le r \le d-1$. *Put* $v_r = \prod_{i=1}^{r} p^{-s_{r+1-i}}$ *for* $1 \le r \le d$. *Then*

$$\zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d) = \frac{\sum_{I \subset \mathbb{N}_{d-1}} \psi_{(I)}(p^{-1}) \prod_{i \in I} z_i \prod_{j \in \mathbb{N}_{d-1}-I} (1 - z_j)}{\prod_{i=0}^{d-1} (1 - z_i)}$$

where $\mathbb{N}_{d-1} - I$ *denotes the complement of* I *in* $\mathbb{N}_{d-1}$.

This is a another expression for the p-part of the cotype zeta function of $\mathbb{Z}^n$ besides the ones obtained by Petrogradsky in [P07, eqn. (23)] and by Igusa in [I89]. Moreover, it leads to new expressions for zeta functions of classical algebraic groups discussed below.

1.2 Zeta functions of symplectic and even orthogonal groups

Let $\mathcal{G}$ be a linear algebraic group and $\mathcal{G}^+(\mathbb{Q}_p) = M_d(\mathbb{Z}_p) \cap \mathcal{G}(\mathbb{Q}_p)$. Let ν be a (multiplicative) Haar measure on $\mathcal{G}(\mathbb{Q}_p)$ normalized so that $\nu(\mathcal{G}(\mathbb{Z}_p)) = 1$. The local zeta function $\zeta_{\mathcal{G},p}(s)$ associated with the linear algebraic group $\mathcal{G}$ is defined by the following equivalent expressions [LS03, eqn 15.11], [dSW08, Definition 6.1].

Definition 6. *The local zeta function of* $\mathcal{G}$ *is*

$$\zeta_{\mathcal{G},p}(s) = \sum_{g \in \mathcal{G}(\mathbb{Z}_p) \backslash \mathcal{G}^+(\mathbb{Q}_p)} |det(g)|_p^s = \int_{\mathcal{G}^+(\mathbb{Q}_p)} \nu(\mathcal{G}(\mathbb{Z}_p)g)^{-1} |det(g)|_p^s d\nu$$

The determinant formulation shows that the local zeta function is a Dirichlet series counting the distinct right coset representatives with same double coset representative of given determinant for the Hecke pair $(\mathcal{G}(\mathbb{Z}_p), \mathcal{G}^+(\mathbb{Q}_p))$. For instance, when $\mathcal{G}$ is the classical group of root system type A_{d+1}; that is, GL_d, we are counting Hermite normal forms with same Smith normal form for the Hecke pair $(\mathrm{GL}_d(\mathbb{Z}_p), \mathrm{GL}_d^+(\mathbb{Q}_p))$. This count is by definition given by the index function *ind*. The invertible Hermite normal forms correspond to sublattices of finite index in $\mathbb{Z}_p^d$ and the corresponding Smith normal forms are the cotypes of these sublattices. Thus the local zeta function of GL_d is the subgroup growth zeta function of $\mathbb{Z}^d$ and the cotype zeta function of $\mathbb{Z}^d$ is the multivariable generalization of this fact.

In general, when an algebraic group arises as the 'algebraic automorphism group' of a finitely generated nilpotent group, say Γ, the p-adic integral above coincides with the pro-isomorphic zeta function of Γ [GSS88, Proposition 4.2]. The pro-isomorphic zeta function counts the subgroups whose profinite completion is isomorphic to the profinite completion of the original group. In a future work, the multivariable extension, analogous to the GL_d case, will also be considered for the zeta functions of the other classical groups.

One consequence of the computation of local Hecke series for classical groups in [HS83] is that we can easily deduce that the local zeta functions of classical groups of type C_d and D_d, that is GSp_{2d} and GO_{2d}^+ respectively, in terms of the local part of the cotype zeta function of $\mathbb{Z}^d$.

Theorem 7.

$$\zeta_{GSp_{2d},p}(s) = \frac{\zeta_{\mathbb{Z}^d,p}(1-ds,2,3,\ldots,d)}{1-p^{-ds}}$$

and

$$\zeta_{GO_{2d}^+,p}(s) = \frac{\zeta_{\mathbb{Z}^d,p}(-ds,1,2,\ldots,d-1)}{1-p^{-ds}}.$$

Proof. This follows from [HS83, Remark 4] and the formulas on page 147 onward in [HS83].

□

1.2.1 Examples

1. Type C_3. By 1.1.10, we have

$$\frac{\zeta_{\mathbb{Z}^3,p}(1-3s,2,3)}{1-p^{-3s}} = \frac{1+p^{1-3s}+p^{-3s}+p^{3-3s}+p^{4-3s}+p^{5-6s}}{(1-p^{-3s})(1-p^{3-3s})(1-p^{5-3s})(1-p^{6-3s})}$$
$$= \zeta_{\mathrm{GSp}_6,p}(s),$$

which agrees with example (iii) in [dSL96].

2. Type D_2. Similarly, we have

$$\frac{\zeta_{\mathbb{Z}^2,p}(-2s,1)}{1-p^{-2s}} = \frac{1-p^{-4s}}{(1-p^{-2s})(1-p^{-2s})(1-p^{1-2s})(1-p^{1-2s})}$$
$$= \frac{1+p^{-2s}}{(1-p^{-2s})(1-p^{1-2s})^2}$$
$$= \zeta_{\mathrm{GO}_4^+,p}(s),$$

as expected [dSW08, pages 156-157].

1.2.2 Future directions

1. The tuples $(1,2,\ldots,d)$ and $(0,1,\ldots,d-1)$ that appear in the cotype zeta functions in Theorem 7 are the Satake parameters of the index function which is also the trivial spherical character on the corresponding Hecke algebras (cf. p168 of [A87] with slightly different notation). When the index function in our discussion of zeta functions so far is replaced by an arbitrary spherical character, we get a zeta function with a spherical character associated with a Hecke pair (cf. eqn. 3.2 in [A70]). It will be interesting to determine whether the natural boundaries [dSW08, Chapter 6] that arise for the zeta functions of classical groups also persist when considering zeta functions of classical groups with nontrivial spherical characters.

2. Determine the zeta function of odd orthogonal groups GO_{2d+1} in terms of the cotype zeta function of $\mathbb{Z}^d$. For the rest of the linear algebraic groups, determine the Satake parameters and the constants necessary to translate all the local Hecke series in [HS83] into the corresponding zeta functions.

3. Reprove the results on local functional equations (formally inverting p) and natural boundaries for the zeta functions of classical groups using the new expressions in Theorem 7.

Chapter 2

Zeta functions of class two nilpotent groups

2.1 The notion of cotype in nilpotent groups

Let $\Lambda \subset \mathbb{Z}^d$ be a sublattice (subgroup) of finite index. We recall that the cotype of Λ is the unique d-tuple of integers $(\alpha_1, \ldots, \alpha_d) = (\alpha_1(\Lambda), \ldots, \alpha_d(\Lambda))$ such that the finite abelian group $\mathbb{Z}^d/\Lambda$ is isomorphic to the sum of cyclic groups

$$(\mathbb{Z}/\alpha_1\mathbb{Z}) \oplus (\mathbb{Z}/\alpha_2\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/\alpha_d\mathbb{Z})$$

where $\alpha_{i+1} \mid \alpha_i$ for $1 \leq i \leq d-1$ [CKK]. With this, the cotype zeta function of $\mathbb{Z}^d$ was defined as

$$\zeta_{\mathbb{Z}^d}(s_1, \ldots, s_d) = \sum_{|\mathbb{Z}^d : H| < \infty} \alpha_1(H)^{-s_1} \ldots \alpha_d(H)^{-s_d} = \sum_{\alpha = (\alpha_1, \ldots, \alpha_d)} a_\alpha(\mathbb{Z}^d) \cdot \alpha_1^{-s_1} \ldots \alpha_d^{-s_d}.$$

where $a_\alpha(\mathbb{Z}^d)$ is the number of subgroups $\Lambda \subset \mathbb{Z}^d$ of cotype α. Furthermore, setting all variables equal to one another, i.e., $s_1 = \cdots = s_d = s$, we get $\zeta_{\mathbb{Z}^d}(s, \ldots, s) = \zeta_{\mathbb{Z}^d}(s)$ which is the subgroup growth zeta function of $\mathbb{Z}^d$.

In general, if $a_n(G)$ denotes the number of subgroups H of index n in a group G, then the subgroup growth zeta function of G

$$\zeta_G(s) = \sum_{|G:H| < \infty} \frac{1}{[G:H]^s} = \sum_{n \geq 1} a_n(G) n^{-s} \tag{2.1.1}$$

will be well-defined whenever the coefficients $a_n(G)$ grow at most polynomially. Moreover, the work of A. Lubotzky, A. Mann and D. Segal [LMS93] shows that a finitely generated residually finite group has a polynomial subgroup growth if and only if it is virtually solvable of finite rank. Thus, the above definition of the cotype zeta function can be generalized to finitely generated residually finite groups whose quotients by normal subgroups of finite index is abelian. However, this attempt at generalization breaks down when the quotients can also be nonabelian.

In the study of subgroup growth, Heisenberg groups and their various generalizations serve as the first nonabelian test cases for a general theory [SV14]. Therefore, we start investigating the notion of cotype with the first Heisenberg group. For instance, we show below that for the 3-dimensional discrete Heisenberg group $H_3(\mathbb{Z})$ there is a normal subgroup of index 8 whose quotient group is D_4, the nonabelian dihedral group of 8 elements. This won't give us any interesting tuple of numbers except its order or the order sequence of its elements in contrast to the case of finite abelian quotients. So, $H_3(\mathbb{Z})$ (and in general, every Heisenberg group in higher dimensions $H_{2d+1}(\mathbb{Z})$) does not have the cotype property. That is, its quotient by a normal subgroup of finite index is not necessarily abelian.

Remark 8. *The ith layer type $\Omega_i(G)/\Omega_{i-1}(G)$ of an abelian p-group G is an elementary abelian p-group of order, say p^{λ_i}, where $\Omega_i(G)$ is the subgroup of elements of order dividing p^i [GLP04]. The sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ is called the* ***layer type*** *of G. Besides the type of G, the layer type is also a natural tuple of numbers associated with finite abelian p-groups and it will be interesting to investigate finite quotients of lattices via their layer types.*

2.1.1 Example: nonabelian quotient

Consider one of the simplest non-abelian examples of torsion free f.g. nilpotent groups: the integral Heisenberg group. The integral (or discrete) Heisenberg group H_3 is the group of upper triangular 3×3-matrices with integer entries and 1's on the main diagonal. It is alternately denoted in the literature as the unitriangular matrix group $UT(3,\mathbb{Z})$ or $U(3,\mathbb{Z})$ or $H_3(\mathbb{Z})$ or $F_{(2,2)}$ or $N(2,2)$ to mean 2-step nilpotent on two generators. The group H_3 is a torsion-free, nilpotent group of class 2 generated by two elements as follows:

$$H_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a,b,c \in \mathbb{Z} \right\} =< x,y,z | z = [x,y], [z,x] = [z,y] = 1 >$$

where $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proposition 9. *The discrete Heisenberg group H_3 does not have the cotype property in that its quotient by a normal subgroup of finite index is not necessarily abelian.*

Proof. Consider the subgroup

$$N = \begin{pmatrix} 1 & 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 1 & 2\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a,b,c \in 2\mathbb{Z} \right\}.$$

N is a normal subgroup of index 8 in H_3. It is easy to prove that in general

$$\begin{pmatrix} 1 & n_1\mathbb{Z} & n_3\mathbb{Z} \\ 0 & 1 & n_2\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

is a normal subgroup of H_3 as long as $n_3 | n_1, n_3 | n_2$ for positive integers n_1, n_2, n_3. Let (n_1, n_2, n_3) denote the matrix

$$\begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the 8 cosets of N are

$$H_3/N = \{(0,0,0)N, (0,0,1)N, (0,1,0)N, (0,1,1)N, (1,0,0)N, (1,0,1)N, (1,1,0)N, (1,1,1)N\}$$

where $(1,1,0)N$ has order 4 and the rest have order 2 except the identity. Letting $r = (1,1,0)N$ and $s = (0,0,1)N$, we have

$$H_3/N =< r, s | r^4 = s^2 = (sr)^2 = \mathrm{id} >\cong D_4, \tag{2.1.2}$$

the nonabelian dihedral group with 8 elements. □

2.2 Cotype zeta functions of class 2 nilpotent groups

Torsion-free finitely generated nilpotent groups are called $\mathscr{T}$-groups. Such groups that are also abelian, that is, the integer lattices $\mathbb{Z}^n$ of any dimension, are $\mathscr{T}$-groups of nilpotency class one. Their cotype zeta functions are studied in [P07] and [CKK]. The next level of generality is class two. An example of a class 2 $\mathscr{T}$-group is the integral (or discrete) Heisenberg group H_3.

As shown in Subsection 2.1.1 , we don't have a cotype for some of the subgroups of H_3. However, the enumeration of subgroups for free nilpotent groups F of class 2 on d generators ($d \geq 2$) in [GSS88, cf. pp. 208–209] suggests that we can give a new definition of cotype and get a multiple Dirichlet series in $d+1$ complex variables for these groups similar to the cotype zeta function.

2.2.1 Euler product decomposition

For a prime p and a $\mathscr{T}$-group G, let a_n and $a_n^{\lhd}$ denote the number of subgroups and normal subgroups, respectively, of index n in G. Write

$$\zeta_{G,p}(s) = \sum_{n\geq 1} \frac{a_{p^n}}{p^{ns}}$$

$$\zeta^{\lhd}_{G,p}(s) = \sum_{n\geq 1} \frac{a^{\lhd}_{p^n}}{p^{ns}}.$$

The series $\zeta_{G,p}(s)$ and $\zeta^{\triangleleft}_{G,p}(s)$ count subgroups and normal subgroups, respectively, of index a power of p in G. The following Euler product decompositions of the subgroup growth zeta function and normal subgroup growth zeta function of G are well-known.

Proposition 10. *[GSS88, Proposition 1.2]*

$$\zeta_{G,p}(s) = \prod_p \zeta_{G,p}(s)$$
$$\zeta^{\triangleleft}_{G,p}(s) = \prod_p \zeta^{\triangleleft}_{G,p}(s).$$

The factors $\zeta_{G,p}(s)$ and $\zeta^{\triangleleft}_{G,p}(s)$ are called the local or Euler factors or p-parts of the corresponding subgroup growth zeta functions. As a result, it is sufficient to work with the Euler factors and then extend results to the product. From now on we will work with the Euler factors and define a local cotype zeta function for them . We then take the product of the local cotype zeta functions to form the cotype zeta function of the group G.

First let's fix notations. For partitions λ, μ, ν and G an abelian p-group of type λ, G has $g^{\lambda}_{\mu\nu}(p)$ subgroups K such that K has type μ and G/K has type ν where $g^{\lambda}_{\mu\nu}(X) \in \mathbb{Z}[X]$ is a Hall polynomial in the variable X. Let

$$h^{\lambda}_{b} = \sum_{|\nu|=b} \sum_{\mu} g^{\lambda}_{\mu\nu} \tag{2.2.1}$$
$$k^{d}_{\nu} = \sum_{\mu} g^{((\nu_1)^d)}_{\mu\nu} \tag{2.2.2}$$

where $\nu = (\nu_1, \ldots, \nu_l)$ with $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l$ and $|\nu| = \nu_1 + \cdots + \nu_l$.

Then $h^{\lambda}_{b}(p)$ is the number of subgroups of index p^b in an abelian p-group of type λ, and $k^{d}_{\nu}(p)$ is the number of subgroups K in $\mathbb{Z}^d$ such that $\mathbb{Z}^d/K$ is a p-group of type ν. Then an argument in [GSS88] shows that the number of subgroups of index p^n in $F_{2,d}$ is given by

$$a_{p^n} = \sum p^{bd} k^{d}_{\mathbf{a}}(p) h^{\mathbf{a}^{\star}}_{b}(p) \tag{2.2.3}$$

where the summation is over all $(d+1)$-tuples $(a_1, \ldots, a_d, b)$ of nonnegative integers such that $a_1 \geq a_2 \geq \ldots \geq a_d$, $a_1 + \cdots + a_d + b = n$, $\mathbf{a} = (a_1, \ldots, a_d)$, $\mathbf{a}^* = (a_1 + a_2, a_1 + a_3, \ldots, a_1 + a_d, a_2 + a_3, \ldots, a_{d-1} + a_d) = (a_i + a_j)_{1 \leq i < j \leq d}$.

A similar argument in [GSS88] shows that the number of normal subgroups of index p^n in $F_{2,d}$ is given by

$$a_{p^n}^{\triangleleft} = \sum p^{bd} k_{\mathbf{a}}^{d}(p) h_b^{\mathbf{a}^\dagger}(p) \tag{2.2.4}$$

where the range of summation is as before and $\mathbf{a}^\dagger = (a_2, a_3, a_3, a_4, a_4, a_4, \ldots, a_d, \ldots, a_d) =: (a_2, (a_3)^2, \ldots, (a_d)^{d-1})$.

When the quotients by finite index normal subgroups are nonabelian, we don't get a tuple of numbers to form a cotype in the previous sense. We can resolve this issue by considering the product $A = ZH$, where $Z = [G, G]$ is the commutator subgroup and H is a subgroup of finite index in a group G. Now, since A contains Z, it is normal in G and the quotient G/A is finite abelian. This leads to a new general definition of cotype. We define the **cotype** of any subgroup H of finite index in G as the cotype (in the old sense) of $[G, G]H$ in G with one more additional number.

Definition 11. *Let H be a subgroup of finite index n in a group G and let $Z = [G, G]$ be the commutator subgroup. $Z/[ZH, G]$ is finite and say has order b. Moreover, $G/(ZH)$ is a finite abelian group. So, let's denote its type by $(a_1, \ldots, a_d)$. The **cotype** of H in G is defined as the tuple of numbers $(a_1, \ldots, a_d; b)$.*

Note: when the index $|G/H|$ is a prime power, we write the cotype $(p^{e_1}, \ldots, p^{e_d}; p^k)$ simply as $(e_1, \ldots, e_d; k)$.

Now, let's consider free nilpotent groups F of class two on d generators $(d \geq 2)$. Based on the above discussion, we define the subgroup growth cotype zeta function or simply the

cotype zeta function and the normal subgroup growth cotype zeta function or simply the **normal cotype zeta function** of a free nilpotent group $F = F_{2,d}$ of class 2 on d generators via their p-parts as follows.

Definition 12. *The cotype zeta function of $F = F_{2,d}$ is*

$$\zeta_F(s_1, \dots, s_d, w) = \prod_p \zeta_{F,p}(s_1, \dots, s_d, w) \tag{2.2.5}$$

where the p-parts are defined as

$$\zeta_{F,p}(s_1, \dots, s_d, w) = \sum_{\substack{H \subset F \\ (F:H)=p^n, n\geq 0}} a_1(H)^{-s_1} \cdots a_d(H)^{-s_d} b(H)^{-w} \tag{2.2.6}$$

Similarly, the normal cotype zeta function of F is

$$\zeta_F^{\triangleleft}(s_1, \dots, s_d, w) = \prod_p \zeta_{F,p}^{\triangleleft}(s_1, \dots, s_d, w) \tag{2.2.7}$$

where the p-parts are defined as

$$\zeta_{F,p}^{\triangleleft}(s_1, \dots, s_d, w) = \sum_{\substack{H \triangleleft F \\ (F:H)=p^n, n\geq 0}} a_1(H)^{-s_1} \cdots a_d(H)^{-s_d} b(H)^{-w} \tag{2.2.8}$$

By equations 2.2.3 and 2.2.4 for a_{p^n} and $a_{p^n}^{\triangleleft}$, respectively, we have the following expressions for the local cotype zeta function and local normal cotype zeta functions of F.

Lemma 13.

$$\zeta_{F,p}(s_1, \dots, s_d, w) = \sum_{n=0}^{\infty} \sum_{\substack{a_1 \geq a_2 \geq \dots \geq a_d \geq 0 \\ a_1 + \dots + a_d + b = n}} p^{bd} k_{\boldsymbol{a}}^d(p) h_b^{\boldsymbol{a}^\star}(p) p^{-a_1 s_1 - a_2 s_2 - \dots - a_d s_d - bw} \tag{2.2.9}$$

$$\zeta_{F,p}^{\triangleleft}(s_1, \dots, s_d, w) = \sum_{n=0}^{\infty} \sum_{\substack{a_1 \geq a_2 \geq \dots \geq a_d \geq 0 \\ a_1 + \dots + a_d + b = n}} p^{bd} k_{\boldsymbol{a}}^d(p) h_b^{\boldsymbol{a}^\dagger}(p) p^{-a_1 s_1 - a_2 s_2 - \dots - a_d s_d - bw} \tag{2.2.10}$$

These coincide with the subgroup (respectively, normal) subgroup growth zeta functions of $F_{2,d}$ when $s_1 = \dots = s_d = w = s$. When $d = 2$, the group $F_{2,2}$ is isomorphic to the 3-dimensional Heisenberg group $H_3(\mathbb{Z})$ whose cotype zeta function is explicitly given below.

Moreover, for any $d \geq 2$, using a series of contour integrals, the residue theorem and convolution of generating series, there is a method that will produce the rational function expression for the local parts of the cotype zeta function of $F_{2,d}$. However, the p-parts grow quickly for $d \geq 4$. For instance, the p-part of the normal subgroup growth zeta function of $F_{2,4}$ is rational in p^{-s} and the numerator of the rational function has 490 terms [P05]. To illustrate the method, a multiple integral representation of the p-part of the cotype zeta function of $F_{2,3}$ is presented in Section 2.4 with the aid of computations in [SAGE].

2.3 Cotype zeta function of Heisenberg groups

We now turn our attention to the integral Heisenberg groups. They are the first family of nonabelian infinite groups for which we compute the normal subgroup growth cotype zeta functions in a closed form. The free k-step (or class k) finitely generated nilpotent groups are the close nonabelian analogues in structure to finitely generated free abelian groups. The free abelian group of rank m is the 1-step m-generated free nilpotent group. Only $H_3(\mathbb{Z})$ is free 2-step nilpotent. The rest of the integral Heisenberg groups H_{2n+1} for $n \geq 2$ are not free 2-step nilpotent. But they are quotients of free $2n$-generated 2-step nilpotent groups. It turns out the same method used for counting subgroups and defining a cotype for free 2-step nilpotent groups will work here for Heisenberg groups. The only difference is we will have to determine $\mathbf{a}^{\dagger}$, the type of $Z/[ZH, H_{2n+1}]$, that corresponds to the case of Heisenberg groups, which turns out to be easily described.

We show below that the normal subgroup growth cotype zeta function of H_{2d+1} has the following closed form:

$$\zeta^{\triangleleft}_{H_{2d+1}}(s_1, \ldots, s_{2d}, w) = \zeta_{\mathbb{Z}^{2d}}(s_1, \ldots, s_{2d})\zeta(s_1 + \cdots + s_{2d} + w - 2d) \qquad (2.3.1)$$

When we set $s_1 = \cdots = s_d = w = s$, this indeed agrees with the local factor of the normal subgroup growth zeta function of $H_n(\mathbb{Z})$ as computed first by G.C Smith [S83] for $n = 3$, and later by F.J. Grunewald, D. Segal and G.C. Smith [GSS88] for all n.

Examples: Using the first line in the table on page 218 of [GSS88], we can deduce the p-parts of the normal subgroup growth zeta functions of the higher rank Heisenberg groups H_{2m+1}. See [GSS88, page 219] for notations used in the examples below and the groups $G(m, r)$ which include Heisenberg groups. Setting $r = 0$ in $G(m, r)$, we get the Heisenberg groups $H_{2m+1} = G(m, 0)$.

<u>$m = 1$</u>:

$$\begin{aligned}\zeta^{\triangleleft}_{H_3,p}(s) &= Z_{2(1)+0}P^{2(1)+1}_{2(1)+0} = Z_2P^3_2 = P^1_0P^1_1(1 - X^3_2)^{-1}\\ &= \frac{1}{(1 - p^{-s})(1 - p^{1-s})(1 - p^{2-3s})}\\ &= \zeta_{\mathbb{Z}^2}(s)\zeta(3s - 2)\end{aligned}$$

<u>$m = 2$</u>:

$$\begin{aligned}\zeta^{\triangleleft}_{H_5,p}(s) &= Z_{2(2)+0}P^{2(2)+1}_{2(2)+0} = Z_4P^5_4 = P^1_0P^1_1P^1_2P^1_3(1 - X^5_4)^{-1}\\ &= \frac{1}{(1 - p^{-s})(1 - p^{1-s})(1 - p^{2-s})(1 - p^{3-s})(1 - p^{4-5s})}\\ &= \zeta_{\mathbb{Z}^4}(s)\zeta(5s - 4)\end{aligned}$$

Now, we will show two methods of arriving at the rational function form of the local parts of the normal subgroup growth cotype zeta function of H_{2d+1}. The first method is illustrated for H_3 below. The second method will be used to derive rational function expressions for Heisenberg groups of any rank.

When $F = H_3$, we have $p^{bd}k^d_{\mathbf{a}}(p)h^{\mathbf{a}^\dagger}_b(p) = p^{2b}k^2_{(a_1,a_2)}(p)h^{a_2}_b(p)$ and the p-part of the *normal* *subgroup growth cotype zeta function* of the Heisenberg group is

$$\zeta^{\triangleleft}_{H_3,p}(s,z,w) = \sum_{n=0}^{\infty} \sum_{\substack{a_1\geq a_2\geq b\geq 0\\ a_1+a_2+b=n}} p^{2b} k^2_{(a_1,a_2)}(p) h^{a_2}_b(p) p^{-a_1 s - a_2 z - bw} \tag{2.3.2}$$

Thus, the normal subgroup growth cotype zeta function is then

$$\zeta^{\triangleleft}_{H_3}(s,z,w) = \prod_p \Big(\sum_{n=0}^{\infty} \sum_{\substack{a_1\geq a_2\geq b\geq 0\\ a_1+a_2+b=n}} p^{2b} k^2_{a_1,a_2}(p) h^{a_2}_b(p) p^{-a_1 s - a_2 z - bw}\Big) \tag{2.3.3}$$

The number of subgroups of index p^b in an abelian p-group of type (a_2), which is the cyclic group $\mathbb{Z}/p^{a_2}\mathbb{Z}$, is always 1. Therefore, $h^{a_2}_b = 1$. Also, $k^2_{(a_1,a_2)}(p)$ which is the number of subgroups K in $\mathbb{Z}^2$ such that $\mathbb{Z}^2/K$ is a p-group of type (a_1,a_2) is exactly $a_{(a_1,a_2)}(\mathbb{Z}^2) = F(2,(a_1,a_2))$ as given in equation (19) of [P07]. For the rest of this section, we will denote $a_\nu(\mathbb{Z}^d)$ by $F(d,\nu)$.

By definition of the p-part, 1.1.6 , we have

$$\zeta_{\mathbb{Z}^d,p}(s_1,\ldots,s_d) = \sum_{\nu_1\geq\ldots\geq\nu_d\geq 0} F(d,\nu) p^{-s_1\nu_1-\cdots-s_d\nu_d}, \tag{2.3.4}$$

where $F(d,\nu)$ is the number of sublattices in $\mathbb{Z}^d$ of cotype $\nu = (\nu_1,\ldots,\nu_d)$.

Also, by the formulas on page 1142 of [P07], we deduce that

$$\zeta_{\mathbb{Z}^2,p}(s_1,s_2) = \sum_{a_1\geq a_2\geq 0} F(2,(a_1,a_2)) p^{-a_1 s_1 - a_2 s_2} = \frac{1-p^{-2s_1}}{(1-p^{-s_1})(1-p^{1-s_1})(1-p^{-s_1-s_2})}. \tag{2.3.5}$$

Proposition 14. *The normal cotype zeta function of the discrete Heisenberg group H_3 is*

$$\zeta^{\triangleleft}_{H_3}(s,z,w) = \frac{\zeta(s)\zeta(s-1)\zeta(s+z)\zeta(s+z+w-2)}{\zeta(2s)} = \zeta_{\mathbb{Z}^2}(s,z)\zeta(s+z+w-2)$$

Proof. From the definition of $F(2,(a_1,a_2))$, it can easily be checked that

$$F(2,(a_1,a_2)) = \begin{cases} (1+p)p^{a_1-a_2-1} & \text{if} \quad a_1-a_2\neq 0 \\ 1 & \text{otherwise.} \end{cases} \tag{2.3.6}$$

So for any $n \in \mathbb{Z}_{\geq 0}, F(2,(b_1+n, b_2+n)) = F(2,(b_1,b_2))$. Using these, we evaluate $\zeta_{H_3,p}(s,z,w)$ as follows.

$$\begin{aligned}
\zeta^{\triangleleft}_{H_3,p}(s,z,w) &= \sum_{n=0}^{\infty} \sum_{\substack{a_1\geq a_2\geq b\geq 0\\ a_1+a_2+b=n}} p^{2b} k^2_{(a_1,a_2)}(p) h^{a_2}_b(p) p^{-a_1 s - a_2 z - bw}\\
&= \sum_{n=0}^{\infty} \sum_{\substack{a_1\geq a_2\geq b\geq 0\\ a_1+a_2+b=n}} F(2,(a_1,a_2)) p^{2b} p^{-a_1 s - a_2 z - bw}\\
&= \sum_{a_1\geq a_2\geq b\geq 0} F(2,(a_1,a_2)) p^{2b - a_1 s - a_2 z - bw}\\
&= \sum_{b\geq 0} p^{b(2-w)} \sum_{a_1\geq a_2\geq b} F(2,(a_1,a_2)) p^{-a_1 s - a_2 z}\\
&= \sum_{a_1\geq a_2\geq 0} F(2,(a_1,a_2)) p^{-a_1 s - a_2 z} + p^{2-w} \sum_{a_1\geq a_2\geq 1} F(2,(a_1,a_2)) p^{-a_1 s - a_2 z}\\
&+ p^{2(2-w)} \sum_{a_1\geq a_2\geq 2} F(2,(a_1,a_2)) p^{-a_1 s - a_2 z} + \dots\\
&= \zeta_{\mathbb{Z}^2,p}(s,z) + p^{2-w} \sum_{b_1\geq b_2\geq 0} F(2,(b_1+1,b_2+1)) p^{-(b_1+1)s - (b_2+1)z}\\
&+ p^{2(2-w)} \sum_{b_1\geq b_2\geq 0} F(2,(b_1+2,b_2+2)) p^{-(b_1+2)s - (b_2+2)z} + \dots\\
&+ p^{n(2-w)} \sum_{b_1\geq b_2\geq 0} F(2,(b_1+2,b_2+2)) p^{-(b_1+n)s - (b_2+n)z} + \dots\\
&= \zeta_{\mathbb{Z}^2,p}(s,z) + p^{2-w-s-z} \sum_{b_1\geq b_2\geq 0} F(2,(b_1,b_2)) p^{-b_1 s - b_2 z}\\
&+ p^{2(2-w)-2s-2z} \sum_{b_1\geq b_2\geq 0} F(2,(b_1,b_2)) p^{-b_1 s - b_2 z} + \dots\\
&= \zeta_{\mathbb{Z}^2,p}(s,z) + p^{2-w-s-z} \zeta_{\mathbb{Z}^2,p}(s,z) + p^{2(2-w)-2s-2z} \zeta_{\mathbb{Z}^2,p}(s,z) + \dots\\
&= \zeta_{\mathbb{Z}^2,p}(s,z)(1 + p^{2-w-s-z} + p^{2(2-w)-2s-2z} + \dots)\\
&= \zeta_{\mathbb{Z}^2,p}(s,z)\left(\frac{1}{1-p^{2-s-z-w}}\right)\\
&= \frac{1-p^{-2s}}{(1-p^{-s})(1-p^{1-s})(1-p^{-s-z})(1-p^{2-s-z-w})}
\end{aligned}$$

Then using the Euler decomposition, we get the claimed formula. □

A similar pattern will appear for the higher rank Heisenberg groups. However, it is difficult to generalize this proof for $H_{2n+1}, n \geq 2$ as we currently don't yet have a simple formula for general Hall polynomials or $F(d, \nu)$. So, using another method that will work for integral Heisenberg groups of every rank, we will derive below a closed formula for the normal cotype zeta function.

Definition 15. *The higher rank integral Heisenberg group on* $2n$ *generators* $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ *is the group* H_{2n+1} *given by the following presentation [S96, p. 96]*

$$H_{2n+1} =< x_1, \ldots, x_n, y_1, \ldots, y_n | [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = [z, x_i] = [z, y_j] = 1$$
$$\textit{for all } i \neq j, [x_i, y_i] = z >$$

This means while $[x_i, y_i] = z$ for all i, all the other commutators of the generators are trivial. In other words, H_{2n+1} is the central product of n copies of H_3 [S96, pp. 90, 96], that is, n copies of H_3 glued along their commutators.

By Lemma 5.2 in [S96], H_{2n+1} is 2-step nilpotent and $[H_{2n+1}, H_{2n+1}] = Z(H_{2n+1}) =< z >$. Let $H \triangleleft H_{2n+1}$ be a normal subgroup of p-power index for a prime p. Let $A = ZH$. Then H_{2n+1}/A is an abelian p-group of type $(d_1, \ldots, d_n, e_1, \ldots, e_n)$ where $d_1 \geq \ldots \geq d_n \geq e_1 \geq \ldots \geq e_n$. Also since $[A, H_{2n+1}]$ is a subgroup of Z, we know it is cyclic. We will prove that $Z/[A, H_{2n+1}]$ has type (e_n).

Proposition 16. *Let* H *be a normal subgroup of* p*-power index of* H_{2n+1} *for a prime* p*. Then the cotype of* H *in* H_{2n+1} *has the form* $(d_1, \ldots, d_n, e_1, \ldots, e_n; e_n)$*. In particular,*

$$[ZH, H_{2n+1}] =< z^{p^{e_n}} > .$$

Proof. By applying an automorphism of H_{2n+1} [GSS88, p. 208], we can put $A = ZH$ in the following form: $A = Z < x_1^{p^{d_1}}, \ldots, x_n^{p^{d_n}}, y_1^{p^{e_1}}, \ldots, y_n^{p^{e_n}} >$. So, without loss of generality, let $A = Z < x_1^{p^{d_1}}, \ldots, x_n^{p^{d_n}}, y_1^{p^{e_1}}, \ldots, y_n^{p^{e_n}} >$. A general element of H_{2n+1} is $z^c y_n^{b_n} \cdots y_1^{b_1} x_n^{a_n} \cdots x_1^{a_1}$.

Also, any element of A is of the form

$$z^f y_n^{h_n p^{e_n}} \cdots y_1^{h_1 p^{e_1}} x_n^{g_n p^{d_n}} \cdots x_1^{g_1 p^{d_1}}. \tag{2.3.7}$$

Now, $[A, H_{2n+1}] =< [x,y] | x \in A, y \in H_{2n+1} >$. So, let's compute the generators $[x,y]$.

$$\begin{aligned}
[x,y] &= xyx^{-1}y^{-1} \\
&= (z^f y_n^{h_n p^{e_n}} \cdots y_1^{h_1 p^{e_1}} x_n^{g_n p^{d_n}} x_1^{g_1 p^{d_1}})(z^c y_n^{b_n} \cdots y_1^{b_1} x_n^{a_n} \cdots x_1^{a_1})(z^f y_n^{h_n p^{e_n}} \cdots y_1^{h_1 p^{e_1}} x_n^{g_n p^{d_n}} x_1^{g_1 p^{d_1}})^{-1} \\
&(z^c y_n^{b_n} \cdots y_1^{b_1} x_n^{a_n} \cdots x_1^{a_1})^{-1} \\
&= (z^f y_n^{h_n p^{e_n}} \cdots y_1^{h_1 p^{e_1}} x_n^{g_n p^{d_n}} x_1^{g_1 p^{d_1}})(z^c y_n^{b_n} \cdots y_1^{b_1} x_n^{a_n} \cdots x_1^{a_1}) \\
&(x_1^{-g_1 p^{d_1}} \cdots x_n^{-g_n p^{d_n}} y_1^{-h_1 p^{e_1}} \cdots y_n^{-h_n p^{e_n}} z^{-f})(x_1^{-a_1} \cdots x_n^{-a_n} y_1^{-b_1} \cdots y_n^{-b_n} z^{-c}) \\
&= z^{(b_1+\cdots+b_n)(g_1 p^{d_1}+\cdots+g_n p^{d_n})+(a_1+\cdots+a_n)(h_1 p^{e_1}+\cdots+h_n p^{e_n})} \\
&= z^{(\text{integer})*p^{e_n}}
\end{aligned}$$

This is because $d_1 \geq \ldots \geq d_n \geq e_1 \geq \ldots \geq e_n$. So, $[A, H_{2n+1}] =< z^{p^{e_n}} >$. □

Therefore, adapting the counting argument for free nilpotent class two groups in [GSS88] to the Heisenberg groups H_{2d+1}, we deduce that the number of normal subgroups of cotype $(a_1, \ldots, a_d, b_1, \ldots, b_d; b_d)$ is $p^{b(2d)} k^{2d}_{(a_1,\ldots,a_d,b_1,\ldots,b_d)}(p) h_b^{(b_d)}(p)$. Then the p-part of the *normal subgroup growth cotype zeta function of the Heisenberg group* H_{2d+1} is

$$\begin{aligned}
&\zeta^{\triangleleft}_{H_{2d+1},p}(s_1, \ldots, s_d, z_1, \ldots, z_d, w) \\
&= \sum_{n=0}^{\infty} \sum_{\substack{a_1 \geq a_2 \geq \ldots \geq a_d \geq b_1 \geq b_2 \geq \ldots \geq b_d \geq b \geq 0 \\ a_1+\cdots+a_d+b_1+\cdots+b_d+b=n}} p^{b(2d)} k^{2d}_{(a_1,\ldots,a_d,b_1,\ldots,b_d)}(p) h_b^{(b_d)}(p) p^{-a_1 s_1 - \cdots - a_d s_d - b_1 z_1 - \cdots - b_d z_d - bw} \\
&= \sum_{a_1 \geq a_2 \geq \ldots \geq a_d \geq b_1 \geq b_2 \geq \ldots \geq b_d \geq b \geq 0} F(2d, (a_1, \ldots, b_d)) p^{2bd - a_1 s_1 - \cdots - a_d s_d - b_1 z_1 - \cdots - b_d z_d - bw} \qquad (2.3.8)
\end{aligned}$$

Thus, the normal subgroup growth cotype zeta function is then

$$\zeta^{\triangleleft}_{H_{2d+1}}(s_1, \ldots, s_d, z_1, \ldots, z_d, w)$$

$$= \prod_p \Big(\sum_{a_1 \geq a_2 \geq \ldots \geq a_d \geq b_1 \geq b_2 \geq \ldots \geq b_d \geq b \geq 0} F(2d, (a_1, \ldots, b_d)) p^{2bd - a_1 s_1 - \cdots - a_d s_d - b_1 z_1 - \cdots - b_d z_d - bw} \Big).$$

Let's now compute the normal subgroup growth cotype zeta function of H_{2d+1}.

Theorem 17. *The normal subgroup growth cotype zeta function of H_{2d+1} is*

$$\zeta^{\triangleleft}_{H_{2d+1}}(s_1, \ldots, s_{2d}, w) = \zeta_{\mathbb{Z}^{2d}}(s_1, \ldots, s_{2d}) \zeta(s_1 + \cdots + s_{2d} + w - 2d).$$

Equivalently, the p-part is

$$\zeta^{\triangleleft}_{H_{2d+1},p}(s_1, \ldots, s_{2d}, w) = \frac{1}{1 - p^{2d - s_1 - \cdots - s_{2d} - w}} \zeta^{\triangleleft}_{\mathbb{Z}^{2d},p}(s_1, \ldots, s_{2d})$$

Proof. Let's rename $b_1, \ldots, b_d$ as $a_{d+1}, \ldots, a_{2d}$ and $z_1, \ldots, z_d$ as $s_{d+1}, \ldots, s_{2d}$. Thus,

$$\begin{aligned}
&\zeta^{\triangleleft}_{H_{2d+1},p}(s_1, \ldots, s_{2d}, w) \\
&= \sum_{a_1 \geq \ldots \geq a_{2d} \geq b \geq 0} F(2d, (a_1, \ldots a_{2d})) p^{2bd - a_1 s_1 - \cdots - a_{2d} s_{2d} - bw} \\
&= \sum_{a_1 \geq \ldots \geq a_{2d} \geq 0} F(2d, (a_1, \ldots a_{2d})) p^{-a_1 s_1 - \cdots - a_{2d} s_{2d}} \Big(\sum_{a_{2d} \geq b \geq 0} p^{2bd - bw} \Big) \\
&= \sum_{a_1 \geq \ldots \geq a_{2d} \geq 0} F(2d, (a_1, \ldots a_{2d})) p^{-a_1 s_1 - \cdots - a_{2d} s_{2d}} \Big(\frac{1 - (p^{2d-w})^{a_{2d}+1}}{1 - p^{2d-w}} \Big) \\
&= \frac{1}{1 - p^{2d-w}} \zeta_{\mathbb{Z}^{2d},p}(s_1, \ldots, s_{2d}) - \frac{p^{2d-w}}{1 - p^{2d-w}} \zeta_{\mathbb{Z}^{2d},p}(s_1, \ldots, s_{2d-1}, s_{2d} + w - 2d) \\
&= \frac{\sum_{\lambda \subseteq \mathbb{N}_{2d-1}} w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-Z_j}}{(1 - p^{-Z_1})(1 - p^{-Z_2}) \cdots (1 - p^{-Z_{2d}})(1 - p^{2d-w})} - \frac{p^{2d-w}}{1 - p^{2d-w}} \frac{\sum_{\lambda \subseteq \mathbb{N}_{2d-1}} w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-W_j}}{(1 - p^{-W_1})(1 - p^{-W_2}) \cdots (1 - p^{-W_{2d}})}
\end{aligned}$$

where $Z_j = s_1 + \cdots + s_j - j(2d - j), W_j = Z_j$ for all $j < 2d$ and $W_{2d} = Z_{2d} + w - 2d$. Since $\sum_{\lambda \subseteq \mathbb{N}_{2d-1}} w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-Z_j}$ and $\sum_{\lambda \subseteq \mathbb{N}_{2d-1}} w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-W_j}$ are independent of Z_{2d} and W_{2d}, we have

$$\begin{aligned}
&\zeta^{\triangleleft}_{H_{2d+1},p}(s_1, \ldots, s_{2d}, w) \\
&= \frac{\sum_{\lambda \subseteq \mathbb{N}_{2d-1}} w_\lambda(p^{-1}) \prod_{j \in \lambda} p^{-Z_j}}{(1 - p^{-Z_1})(1 - p^{-Z_2}) \cdots (1 - p^{-Z_{2d-1}})} \Big(\frac{1}{(1 - p^{2d-w})(1 - p^{-Z_{2d}})} - \frac{p^{2d-w}}{(1 - p^{2d-w})(1 - p^{2d-w-Z_{2d}})} \Big) \\
&= \frac{1}{1 - p^{2d-w-Z_{2d}}} \zeta_{\mathbb{Z}^{2d},p}(s_1, \ldots, s_{2d})
\end{aligned}$$

$$= \frac{1}{1 - p^{2d-s_1 \cdots - s_{2d} - w}} \zeta_{\mathbb{Z}^{2d},p}(s_1, \ldots, s_{2d})$$

Then using the Euler decomposition, we get the claimed formula. □

2.3.1 Density results for the corank

Let H be a subgroup of finite index in G and let $(a_1, \ldots, a_m; b)$ be its cotype. We define the **corank** of H to be the largest index i for which $a_i \neq 1$. By convention, we let G have corank 0 and say H is **cocyclic** when $G/(ZH)$ is cyclic or equivalently H has cotype $(a_1, 1, 1, \ldots, 1, b)$.

As in [CKK], we define the **corank zeta function** to be the Dirichlet series counting normal subgroups of G of corank less than or equal to m as

$$\zeta_G^{\triangleleft,m}(s) = \sum_{\substack{H \triangleleft G \\ \mathrm{corank}(H) \leq m}} \frac{1}{[G:H]^s} \tag{2.3.9}$$

Using our computation of the normal subgroup growth cotype zeta function for H_{2d+1}, we deduce the following.

Corollary 18.

$$\zeta_{H_{2d+1}}^{\triangleleft,m}(s) = \zeta_{\mathbb{Z}^{2d}}^{(m)}(s).$$

Proof.

$$\begin{aligned}
\zeta_{H_{2d+1}}^{\triangleleft,m}(s) &= \lim_{s_{m+1}\to\infty} \ldots \lim_{s_{2d}\to\infty} \zeta_{H_{2d+1}}^{\triangleleft}(s, \ldots, s, s_{m+1}, \ldots, s_{2d}, s) \\
&= \lim_{s_{m+1}\to\infty} \ldots \lim_{s_{2d}\to\infty} \zeta_{\mathbb{Z}^{2d}}(s, \ldots, s, s_{m+1}, \ldots, s_{2d})\zeta(ms + s_{m+1} + \cdots + s_{2d} + s) \\
&= \lim_{s_{m+1}\to\infty} \ldots \lim_{s_{2d}\to\infty} \zeta_{\mathbb{Z}^{2d}}(s, \ldots, s, s_{m+1}, \ldots, s_{2d}) \\
&= \zeta_{\mathbb{Z}^{2d}}^{(m)}(s),
\end{aligned}$$

which is the corank zeta function of $\mathbb{Z}^{2d}$. □

In conclusion, the normal subgroups of finite index and corank at most m in the higher rank Heisenberg group of dimension $2d+1$ have the same densities as the sublattices of corank at most m in the integer lattice $\mathbb{Z}^{2d}$.

In [CDD+17], the authors initiate a study of random nilpotent groups formed from the free nilpotent group $F_{n,d}$ of class n and rank d by adding random relations. This is analogous to the study of random abelian groups as all nilpotent groups arise as quotients of free nilpotent groups $F_{n,d}$ while abelian groups are quotients of the free finitely generated abelian groups $\mathbb{Z}^d$. More generally, random groups are quotients of free groups by random relations for some model of randomness.

In particular, for balanced models (i.e. in which the number of relations equals the number of generators), they show that the probability that a random nilpotent group is abelian equals the probability that it is cyclic. Furthermore, it is shown that this probability is equal to the the probability that a random sublattice of $\mathbb{Z}^d$ is cocyclic as computed by Stanley and Wang [CKK; NS16; WS17]. Our results above show that this probability also appears when considering normal subgroups of finite index in the special family of Heisenberg groups.

2.4 Cotype zeta function of $F_{2,d}$

In Lemma 13, the p-part of the *normal cotype zeta function* of the free nilpotent group $F = F_{2,d}$ of class 2 on d generators is given as

$$\zeta^{\triangleleft}_{F,p}(s_1,\ldots,s_d,w) = \sum_{n=0}^{\infty} \sum_{\substack{a_1 \geq a_2 \geq \ldots \geq a_d \geq 0 \\ a_1+\cdots+a_d+b=n}} p^{bd} k^d_{\mathbf{a}}(p) h^{\mathbf{a}^\dagger}_b(p) p^{-a_1 s_1 - a_2 s_2 - \cdots - a_d s_d - bw}$$

This coincides with the normal subgroup growth zeta function of F when $s_1 = \cdots = s_d = w = s$. In addition, when $d = 3$, using the method of convolution, we prove in Corollary 29 that the p-parts of the normal cotype zeta function of $F_{2,3}$ are rational functions in powers of p. This is also true for $d \geq 4$ and the general proof along the same lines will be provided in the future.

For $F = F_{2,3}$, the coefficient of the series is $p^{3b} k^3_{(a_1,a_2,a_3)}(p) h_b^{(a_2,(a_3)^2)}(p)$. The p-part of the *normal subgroup growth cotype zeta function* of this step-2 free nilpotent group on 3 generators is then

$$\begin{aligned}\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1,s_2,s_3,w) &= \sum_{n=0}^{\infty} \sum_{\substack{a_1 \geq a_2 \geq a_3 \geq 0 \\ a_1+a_2+a_3+b=n}} p^{3b} k^3_{(a_1,a_2,a_3)}(p) h_b^{(a_2,(a_3)^2)}(p) p^{-a_1 s_1 - a_2 s_2 - a_3 s_3 - bw} \\ &= \sum_{\substack{a_1 \geq a_2 \geq a_3 \geq 0 \\ a_2+2a_3 \geq b \geq 0}} p^{3b} k^3_{(a_1,a_2,a_3)}(p) h_b^{(a_2,a_3,a_3)}(p) p^{-a_1 s_1 - a_2 s_2 - a_3 s_3 - bw} \qquad (2.4.1)\end{aligned}$$

Thus, the *normal subgroup growth multiple zeta function* of $F_{2,3}$ is

$$\zeta^{\triangleleft}_{F_{2,3}}(s_1,s_2,s_3,w) = \prod_p \Big(\sum_{\substack{a_1 \geq a_2 \geq a_3 \geq 0 \\ a_2+2a_3 \geq b \geq 0}} p^{3b} k^3_{(a_1,a_2,a_3)}(p) h_b^{(a_2,a_3,a_3)}(p) p^{-a_1 s_1 - a_2 s_2 - a_3 s_3 - bw} \Big) \qquad (2.4.2)$$

Now $k^3_{(a_1,a_2,a_3)}$ which is the number of subgroups K in $\mathbb{Z}^3$ such that $\mathbb{Z}^3/K$ is a p-group of type (a_1,a_2,a_3) is exactly $F(3,(a_1,a_2,a_3))$ as given in equation (19) of [P07]. Also, $h_b^{(a_2,(a_3)^2)}(p)$ is the number of subgroups of index p^b in an abelian p-group of type $(a_2,(a_3)^2) := (a_2,a_3,a_3)$. So, we will need to know how to count the number of subgroups of index p^b in the rank 3 abelian p-group $\mathbb{Z}/p^{a_2} \times \mathbb{Z}/p^{a_3} \times \mathbb{Z}/p^{a_3}$. This can be done by using a general recurrence relation in [S92] satisfied by $h_b^{\mathbf{a}}$.

In order to express the p-part $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1,s_2,s_3,w)$ as a rational function, we will use a convolution of rational functions formed from factors of the terms of $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1,s_2,s_3,w)$ above. This is explained in detail below.

The idea behind the method is the following. Let $F(x) = \sum a_n x^n$ and $G(x) = \sum b_n x^n$ be two convergent series. Then the series $H(x) = \sum a_n b_n x^n$ can be obtained from F and G as follows.

Definition 19. *The **convolution** of F and G denoted $F \star G(x)$ is the following contour integral around a small circle of radius ϵ going counterclockwise about 0 where x is a complex number sufficiently small in magnitude:*

$$F \star G(x) = \frac{1}{2\pi i} \oint_{y \in B_\epsilon(0)} F(y) G(\frac{x}{y}) \frac{dy}{y}$$

Lemma 20. *$H(x) = \sum a_n b_n x^n$ is the convolution of $F(x) = \sum a_n x^n$ and $G(x) = \sum b_n x^n$.*

Proof.

$$\begin{aligned} F \star G(x) &= \frac{1}{2\pi i} \oint_{y \in B_\epsilon(0)} F(y) G(\frac{x}{y}) \frac{dy}{y} \\ &= \frac{1}{2\pi i} \oint_{y \in B_\epsilon(0)} \sum a_n b_m y^n \frac{x^m}{y^m} \frac{dy}{y} \\ &= \sum a_n b_m x^m \frac{1}{2\pi i} \oint_{y \in B_\epsilon(0)} y^{n-m-1} dy \\ &= \sum a_n b_n x^n, \text{ as the integral is zero when } n \neq m. \end{aligned}$$

Thus $H(x) = F \star G(x)$. □

Before we apply this method to the computation of cotype zeta functions of $F_{2,d}$, let's demonstrate the main steps with an easier but similar computation. Consider the series

$$\sum_{a_1 \geq a_2 \geq r \geq 0} h_r^{(a_1,a_2)}(p) x_1^{a_1} x_2^{a_2} y^r. \tag{2.4.3}$$

It is the generating series for $h_r^{(a_1,a_2)}(p)$ that counts the number of subgroups of index p^r in the abelian p-group $\mathbb{Z}/p^{a_1} \times \mathbb{Z}/p^{a_2}$ with the restriction $0 \leq r \leq a_2$. We will show how to express this series as a repeated convolution of two rational functions and then evaluate the convolutions to arrive at the rational function expression of the series. Let

$$F_2(x_1, x_2, y) = \sum_{\substack{a=(a_1,a_2) \\ a_1 \geq a_2 \geq 0 \\ 0 \leq r \leq a_1+a_2}} h_r^{(a_1,a_2)}(p) x_1^{a_1} x_2^{a_2} y^r \tag{2.4.4}$$

and

$$G(x_1, x_2, y) = \sum_{a_1 \geq a_2 \geq r \geq 0} x_1^{a_1} x_2^{a_2} y^r. \tag{2.4.5}$$

Then by Lemma 20 we have

$$F_2 \star G(x_1, x_2, y) = \sum_{a_1 \geq a_2 \geq r \geq 0} h_r^{(a_1,a_2)}(p) x_1^{a_1} x_2^{a_2} y^r. \tag{2.4.6}$$

First, we find the rational function expressions of F_2, G and $\sum_{a_1 \geq a_2 \geq r \geq 0} h_r^{(a_1,a_2)}(p) x_1^{a_1} x_2^{a_2} y^r$. In general, let $F_d(x, y)$ be the generating series

$$F_d(x, y) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_d) \\ \alpha_1 \geq \cdots \alpha_d \geq 0 \\ 0 \leq r \leq \alpha_1 + \cdots + \alpha_d}} h_r^{(\alpha_1,\ldots,\alpha_d)}(p) x^\alpha y^r \tag{2.4.7}$$

where $x = (x_1, \ldots, x_d)$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$. Let's denote $h_r^\alpha(p)$ by $N_\alpha^{(d)}(r)$ or simply by $N_\alpha(r)$ when the rank is understood. By [S92, Corollary], we have the recurrence formula

$$N_\alpha(r) = N_{\tilde{\alpha}}(r-1) + p^r N_{\hat{\alpha}}(r) \tag{2.4.8}$$

where $\hat{\alpha} = (\alpha_2, \ldots, \alpha_d)$ and $\tilde{\alpha} = \alpha$ with -1 added to the k^{th} position where

$$k = \begin{cases} 1 & \text{if } \alpha_1 > \alpha_2 \\ 2 & \text{if } \alpha_1 = \alpha_2 > \alpha_3 \\ 3 & \text{if } \alpha_1 = \alpha_2 = \alpha_3 > \alpha_4 \\ \ldots & \ldots \end{cases}$$

Note $N_\alpha^{(1)}(r) = 1$ if $0 \leq r \leq \alpha$. So,

$$F_1(x, y) = \sum_{a \geq r \geq 0} h_r^{(a)}(p) x^a y^r = \sum_{a \geq r \geq 0} x^a y^r = \frac{1}{(1-x)(1-xy)}. \tag{2.4.9}$$

The last equality is obtained by first summing over a and then over r. For rank 2, let's break down $F_2(x_1, x_2, y)$ as follows:

$$F_2(x_1, x_2, y) = \sum_{\substack{\alpha_1 \geq \alpha_2 \geq 0 \\ 0 \leq r \leq \alpha_1 + \alpha_2}} N_{\alpha_1,\alpha_2}^{(2)}(r) x_1^{\alpha_1} x_2^{\alpha_2} y^r = F_2^{(0)} + F_2^{(1)} \tag{2.4.10}$$

where

$$F_2^{(0)}(x_1,x_2,y) = \sum_{\alpha_1=\alpha_2\geq 0} N^{(2)}_{\alpha_1,\alpha_2}(r)x_1^{\alpha_1}x_2^{\alpha_2}y^r \tag{2.4.11}$$

$$F_2^{(1)}(x_1,x_2,y) = \sum_{\alpha_1>\alpha_2\geq 0} N^{(2)}_{\alpha_1,\alpha_2}(r)x_1^{\alpha_1}x_2^{\alpha_2}y^r. \tag{2.4.12}$$

Let's first compute $F_2^{(0)}(x_1,x_2,y)$.

Lemma 21.

$$F_2^{(0)}(x_1,x_2,y) = \frac{1+x_2x_2y}{(1-x_1x_2)(1-px_1x_2y)(1-x_1x_2y^2)}.$$

Proof.

$$\begin{aligned}
F_2^{(0)}(x_1,x_2,y) &= \sum_{\alpha=0}^{\infty}\sum_r N^{(2)}_{\alpha,\alpha}(r)(x_1x_2)^\alpha y^r \\
&= \sum_\alpha\sum_r N^{(2)}_{\alpha,\alpha-1}(r-1)(x_1x_2)^\alpha y^r + \sum_\alpha p^r N^{(1)}_\alpha(r)(x_1x_2)^\alpha y^r \\
&= \sum_\alpha\sum_r N^{(2)}_{\alpha-1,\alpha-1}(r-2)(x_1x_2)^\alpha y^r + \sum_\alpha\sum_r p^{r-1}N^{(1)}_{\alpha-1}(r-1)(x_1x_2)^\alpha y^r \\
&+ \sum_\alpha p^r N^{(1)}_\alpha(r)(x_1x_2)^\alpha y^r \\
&= \sum_\alpha\sum_r N^{(2)}_{\alpha,\alpha}(r)(x_1x_2)^{\alpha+1}y^{r+2} + \sum_\alpha\sum_r N^{(1)}_\alpha(r)(x_1x_2)^{\alpha+1}p^r y^{r+1} \\
&+ \sum_\alpha N^{(1)}_\alpha(r)(x_1x_2)^\alpha p^r y^r \\
&= x_1x_2y^2F_2^{(0)}(x_1,x_2,y) + x_1x_2yF_1(x_1x_2,py) + F_1(x_1x_2,py).
\end{aligned}$$

Solving for $F_2^{(0)}$ results in the desired sum:

$$F_2^{(0)}(x_1,x_2,y) = \frac{1+x_2x_2y}{(1-x_1x_2)(1-px_1x_2y)(1-x_1x_2y^2)}.$$

□

Similarly, we sum $F_2^{(1)}(x_1,x_2,y)$.

Lemma 22.

$$F_2^{(1)}(x_1,x_2,y) = \frac{1+y-x_1y+x_1^2x_2y^2}{(1-x_1)(1-x_1y)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)}.$$

Proof.

$$\begin{aligned}
F_2^{(1)}(x_1,x_2,y) &= \sum_{\alpha_1>\alpha_2\geq 0}\sum_r N_\alpha^{(2)}(r)x_1^{\alpha_1}x_2^{\alpha_2}y^r \\
&= \sum_{\alpha_1>\alpha_2\geq 0}\left[N_{\tilde{\alpha}}^{(2)}(r-1)+p^rN_{\hat{\alpha}}^{(1)}(r)\right]x_1^{\alpha_1}x_2^{\alpha_2}y^r \\
&= \sum_{\alpha_1>\alpha_2\geq 0}\left[N_{\alpha_1-1,\alpha_2}^{(2)}(r-1)+p^rN_{\alpha_2}^{(1)}(r)\right]x_1^{\alpha_1}x_2^{\alpha_2}y^r \\
&= \sum_{\alpha_1\geq\alpha_2\geq 0}N_{\alpha_1,\alpha_2}^{(2)}(r-1)x_1^{\alpha_1+1}x_2^{\alpha_2}y^{r+1}+\sum_{\substack{\alpha_1>\alpha_2\geq 0\\ 0\leq r\leq\alpha_2}}x_1^{\alpha_1}x_2^{\alpha_2}(py)^r \\
&= x_1y\left[F_2^{(1)}(x_1,x_2,y)+F_2^{(0)}(x_1,x_2,y)\right]+\sum_{\alpha_1>\alpha_2\geq 0}x_1^{\alpha_1}x_2^{\alpha_2}\frac{1-(py)^{\alpha_2+1}}{1-py}.
\end{aligned}$$

Solving for $F_2^{(1)}(x_1,x_2,y)$, we arrive at

$$F_2^{(1)}(x_1,x_2,y)=\frac{1+y-x_1y+x_1^2x_2y^2}{(1-x_1)(1-x_1y)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)}.$$

□

Proposition 23.

$$F_2(x,y)=F_2(x_1,x_2,y)=\frac{x_1^2x_2y^2+x_1^2x_2y-x_1x_2y-1}{(1-x_1)(1-x_1y)(1-x_1x_2)(1-x_1x_2y^2)(px_1x_2y-1)}.$$

Proof. Combining the two sums in Lemmas 21 and 22, we deduce that

$$\begin{aligned}
F_2(x,y)=F_2(x_1,x_2,y) &= \sum_{\substack{\alpha_1\geq\alpha_2\geq 0\\ 0\leq r\leq\alpha_1+\alpha_2}}h_r^{(\alpha_1,\alpha_2)}(p)x_1^{\alpha_1}x_2^{\alpha_2}y^r \\
&= F_2^{(0)}+F_2^{(1)} \\
&= \frac{x_1^2x_2y^2+x_1^2x_2y-x_1x_2y-1}{(1-x_1)(1-x_1y)(1-x_1x_2)(1-x_1x_2y^2)(px_1x_2y-1)}.
\end{aligned}$$

□

Also, by successively summing over a_1, a_2 and r, respectively, we find

$$G(x_1,x_2,y)=\sum_{a_1\geq a_2\geq r\geq 0}x_1^{a_1}x_2^{a_2}y^r=\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2y)}. \tag{2.4.13}$$

The left hand side of 2.4.6 can also be summed similarly.

Proposition 24.

$$\sum_{a_1 \geq a_2 \geq r \geq 0} h_r^{(a_2,a_1)}(p) x_1^{a_1} x_2^{a_2} y^r = \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)}$$

Proof. We know from [T10] that when $0 \leq r \leq a_2$,

$$h_r^{(a_1,a_2)}(p) = \frac{p^{r+1}-1}{p-1}. \tag{2.4.14}$$

Using this, similar to the case of $G(x_1, x_2, y)$, we find that

$$\begin{aligned}\sum_{a_1 \geq a_2 \geq r \geq 0} h_r^{(a_2,a_1)}(p) x_1^{a_1} x_2^{a_2} y^r &= \sum_{a_1 \geq a_2 \geq r \geq 0} \frac{p^{r+1}-1}{p-1} x_1^{a_1} x_2^{a_2} y^r \\ &= \frac{p}{p-1} \sum_{a_1 \geq a_2 \geq r \geq 0} x_1^{a_1} x_2^{a_2} (py)^r - \frac{1}{p-1} \sum_{a_1 \geq a_2 \geq r \geq 0} x_1^{a_1} x_2^{a_2} y^r \\ &= \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)}\end{aligned}$$

□

Now it remains to verify that the evaluation of the convolution $F_2 \star G$ results in the same rational function as above which will confirm equation (2.4.6).

Theorem 25.

$$F_2 \star G(x_1, x_2, y) = \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)}$$

Proof. Fix small circles in the complex plane around 0 and take complex numbers x_1, x_2 and y with very small magnitude. Then by definition, we have

$$F_2 \star G(x_1, x_2, y) = \frac{1}{(2\pi i)^3} \oint_{u_3 \in B_{\epsilon_3}(0)} \oint_{u_2 \in B_{\epsilon_2}(0)} \oint_{u_1 \in B_{\epsilon_1}(0)} F_2(u_1, u_2, v) G(\frac{x_1}{u_1}, \frac{x_2}{u_2}, \frac{y}{v}) \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{dv}{v}$$

We evaluate the repeated contour integrals by applying Cauchy's residue theorem:

$$\begin{aligned}
F_2 \star G(x_1,x_2,y) &= \frac{1}{(2\pi i)^3}\oint_{u_3\in B_{\epsilon_3}(0)}\oint_{u_2\in B_{\epsilon_2}(0)}\oint_{u_1\in B_{\epsilon_1}(0)} F_2(u_1,u_2,v)G(\frac{x_1}{u_1},\frac{x_2}{u_2},\frac{y}{v})\frac{du_1}{u_1}\frac{du_2}{u_2}\frac{dv}{v}\\
&= \frac{1}{(2\pi i)^3}\oint\oint\oint \frac{(u_1^2u_2v^2+u_1^2u_2v-u_1u_2v-1)}{u_1u_2v(1-u_1)(1-u_1v)(1-u_1u_2)(1-u_1u_2v^2)(pu_1u_2v-1)}\\
&\qquad \frac{\times du_1du_2dv}{(1-\frac{x_1}{u_1})(1-\frac{x_1x_2}{u_1u_2})(1-\frac{x_1x_2y}{u_1u_2v})}\\
&= \text{(See SAGE and Mathematica codes for computing the integrals}\\
&\qquad \text{in Appendices A and B)}\\
&= \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2y)(1-px_1x_2y)},\ \text{as expected.}
\end{aligned}$$

□

We finish this section by giving an integral representation of $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1,s_2,s_3,w)$ whose evaluation will lead to a rational function expression. In order to do this, we will form two series whose convolution will be $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1,s_2,s_3,w)$. First let's fix the following notations: $x_1=p^{-s_1}, x_2=p^{-s_2}, x_3=p^{-s_3}$ and $y=p^{3-w}$. Let

$$\begin{aligned}
A(x_1,x_2,x_3,y) &= \sum_{\substack{a_1\geq a_2\geq a_3\geq 0\\ a_1+a_2+a_3\geq b\geq 0}} k^3_{(a_1,a_2,a_3)}(p)p^{-a_1s_1-a_2s_2-a_3s_3+b(3-w)}\\
&= \sum_{\substack{a_1\geq a_2\geq a_3\geq 0\\ a_1+a_2+a_3\geq r\geq 0}} k^3_{(a_1,a_2,a_3)}(p)x_1^{a_1}x_2^{a_2}x_3^{a_3}y^r \qquad (2.4.15)
\end{aligned}$$

and

$$\begin{aligned}
B(x_1,x_2,x_3,y) &= \sum_{\substack{a_1\geq a_2\geq a_3\geq 0\\ a_2+2a_3\geq r\geq 0}} h_r^{(a_2,a_3,a_3)}(p)p^{-a_1s_1-a_2s_2-a_3s_3+r(3-w)}\\
&= \sum_{\substack{a_1\geq a_2\geq a_3\geq 0\\ a_2+2a_3\geq r\geq 0}} h_r^{(a_2,a_3,a_3)}(p)x_1^{a_1}x_2^{a_2}x_3^{a_3}y^r \qquad (2.4.16)
\end{aligned}$$

We can deduce from [P07] that the p-part of the cotype zeta function of $\mathbb{Z}^3$ is

$$\zeta_{\mathbb{Z}^3,p}(x_1,x_2,x_3) = \frac{1+x_1+px_1+x_1x_2+px_1x_2+px_1^2x_2}{(1-p^2x_1)(1-p^2x_1x_2)(1-x_1x_2x_3)}. \qquad (2.4.17)$$

Moreover, applications of recurrence relations leads to rational function expressions for $A(x_1, x_2, x_3, y)$ and $B(x_1, x_2, x_3, y)$ described in the following lemmas.

Lemma 26.

$$A(x_1, x_2, x_3, y) = \frac{\zeta^{\triangleleft}_{\mathbb{Z}^3,p}(x_1, x_2, x_3) - y\zeta^{\triangleleft}_{\mathbb{Z}^3,p}(x_1 y, x_2 y, x_3 y)}{1-y}.$$

In the case of $B(x_1, x_2, x_3, y)$, we further break down the sum and deal with the parts seprately. Let

$$B_1(x_1, x_2, x_3, y) = \sum_{\substack{a_1 \geq a_2 > a_3 \geq 0 \\ 2a_3 \geq r \geq 0}} h_r^{(a_3,a_3)}(p) x_1^{a_1} x_2^{a_2} x_3^{a_3} (py)^r$$
$$= \frac{x_1 x_2 (1 + p x_1 x_2 x_3 y)}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-p^2x_1x_2x_3y)(1-p^2x_1x_2x_3y^2)}, \tag{2.4.18}$$

$$B_2(x_1, x_2, x_3, y) = \sum_{\substack{a_1 \geq a_2 = a_3 \geq 0 \\ 3a_3 \geq r \geq 0}} h_r^{(a_3,a_3,a_3)}(p) x_1^{a_1} (x_2x_3)^{a_3} y^r$$
$$= \frac{x_2 x_3 y}{(1-x_1)(1-x_1x_2x_3)(1-p^2x_1x_2x_3y)(1-x_2x_3y^3)}$$
$$+ \frac{(1 + p x_2 x_3 y^2 + x_2 x_3 y^2) B_3(x_1, x_2, x_3, y)}{1 - x_2 x_3 y^3} \tag{2.4.19}$$

and

$$B_3(x_1, x_2, x_3, y) = \sum_{\substack{a_1 \geq a_2 \geq 0 \\ 2a_2 \geq r \geq 0}} h_r^{(a_2,a_2)}(p) x_1^{a_1} (x_2x_3)^{a_2} (py)^r$$
$$= \frac{1 + p x_2 x_3 y}{(1-p^2x_1x_2x_3y)(1-p^2x_2x_3y^2)(1-x_1x_2x_3)(1-x_1)}. \tag{2.4.20}$$

Lemma 27.

$$B(x_1, x_2, x_3, y) = \frac{B_1(x_1, x_2, x_3, y) + B_2(x_1, x_2, x_3, y)}{1 - x_1 x_2 y}$$

Theorem 28. *The multiple integral representation of the p-part of the normal cotype zeta function of $F_{2,3}$ is*

$$\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1, s_2, s_3, w) = A \star B(x_1, x_2, x_3, y)$$

$$= \frac{1}{(2\pi i)^4} \oint_{v \in B_{\epsilon_4}(0)} \oint_{u_3 \in B_{\epsilon_3}(0)} \oint_{u_2 \in B_{\epsilon_2}(0)} \oint_{u_1 \in B_{\epsilon_1}(0)} A(u_1, u_2, u_3, v)$$
$$\times B(\frac{x_1}{u_1}, \frac{x_2}{u_2}, \frac{x_3}{u_3}, \frac{y}{v}) \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{du_3}{u_3} \frac{dv}{v}$$

Corollary 29. *The p-part of the normal cotype zeta function of $F_{2,3}$ is a rational function in powers of p.*

Proof. The preceding theorem shows that $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1, s_2, s_3, w)$ is the convolution of the rational functions $A(x_1, x_2, x_3, y)$ and $A(x_1, x_2, x_3, y)$. Since the convolution of two rational functions is again rational, $\zeta^{\triangleleft}_{F_{(2,3)},p}(s_1, s_2, s_3, w)$ is a rational function in x_1, x_2, x_3 and y, equivalently, a rational function in $p^{s_1}, p^{s_2}, p^{s_3}$ and p^w. □

When we set all variables equal, we have

$$\zeta^{\triangleleft}_{F_{2,3},p}(s) = \frac{1 + p^3x^3 + p^4x^3 + p^6x^5 + p^7x^5 + p^{10}x^8}{(1-x)(1-px)(1-p^2x)(1-p^8x^5)(1-p^6x^9)}, \tag{2.4.21}$$

where $x = p^{-s}$ [GSS88]. It will be interesting to compute this enormous integral and investigate the density of subgroups of certain cotypes similar to the Heisenberg groups treated earlier. This may lead to new type of distribution of subgroups unlike the same distribution observed in both $\mathbb{Z}^d$ and H_{2d+1} as $d \to \infty$.

Chapter 3

Counting finite class two nilpotent groups

In this chapter, we employ generating series in several variables to study finite nilpotent groups just as we used cotype zeta functions to study a family of infinite groups. Let's first recall a few well-known facts which can be found in [BNV07], [CMZ17] and [I08].

Counting finite groups of rank 2 and nilpotency class 2 can be reduced to counting finite indecomposable p-groups of rank 2 and class 2. Any d-generator p-group of class 2 is a central extension of a cyclic group by an abelian p-group of rank d. In general, all nilpotent groups of class 3 or less are metabelian. That is, their commutator subgroup is abelian or equivalently, there is an abelian normal subgroup such that the quotient of the group by this subgroup is abelian. Let G be a d-generator p-group of class 2 and order p^n. Then, G satisfies the following short exact sequence:

$$1 \longrightarrow \mathbb{Z}/p^{\gamma} \longrightarrow G \longrightarrow \mathbb{Z}/p^{\alpha_1} \times \ldots \times \mathbb{Z}/p^{\alpha_d} \longrightarrow 1,$$

where $n = \alpha_1 + \cdots + \alpha_d + \gamma$ and $\alpha_1 \geq \ldots \geq \alpha_d \geq \gamma \geq 1$.

Definition 30. *The **type** of a d-generated class 2 finite p-group of order p^n is defined as the tuple of numbers $(\alpha_1, \ldots, \alpha_d; \gamma)$ such that the abelianization of G, i.e., $G/[G,G]$, is isomorphic to $\mathbb{Z}/p^{\alpha_1} \times \ldots \times \mathbb{Z}/p^{\alpha_d}$ and the commutator subgroup $[G,G]$ is isomorphic to*

$\mathbb{Z}/p^\gamma$.

Let $G_p(\alpha_1, \ldots, \alpha_d; \gamma)$ denote the number of of nonisomorphic d-generated p-groups G of class 2 and type $(\alpha_1, \ldots, \alpha_d; \gamma)$. Equivalently, $G_p(\alpha_1, \ldots, \alpha_d; \gamma)$ is the number of nonisomorphic groups G that fit the short exact sequence above. This is a generalization of the 2-generators case in [AMM12]. We form the following generating function in $d+1$ variables to aid our counting.

Definition 31. *The* ***type series*** *of the class of d-generated class 2 finite p-groups is defined as the generating series*

$$Z_{2,d,p}(s_1, \ldots, s_d, s_{d+1}) = \sum_{\alpha_1 \geq \ldots \geq \alpha_d \geq \gamma > 0} G_p(\alpha_1, \ldots, \alpha_d; \gamma) p^{-\alpha_1 s_1 - \cdots - \alpha_d s_d - \gamma s_{d+1}}$$

When we set all variables equal to each other and sum the individual series for classes from 1 up to c, and number of generators from 1 up to d, the resulting sum becomes the local zeta function $\zeta_{c,d,p}(s)$ studied in [DS00] and [V09]. For fixed p, the zeta function $\zeta_{c,d,p}(s)$ is known to be a rational function in p^{-s} [DS00, Theorem 1.6]. While the computation of $\zeta_{1,d,p}(s)$ is a classical result reproduced below counting finite abelian groups, we know of other explicit rational functions only for $\zeta_{2,2,p}(s)$, i.e., for class at most two on at most two generators [KNV11, p. 138]. We will now consider classes 1 and 2.

3.1 Class 1

The groups of class 1 are abelian and the commutator subgroup is trivial. So, the type of a finite abelian group G on d generators becomes the usual tuple of elementary divisors $(\alpha_1, \ldots, \alpha_d)$. The type series then has only d variables as $\gamma = 0$. Moreover, $G_p(\alpha_1, \ldots, \alpha_d; 0) = 1$ as there is only one isomorphism class of a finite abelian group of a given type.

Proposition 32. *The series counting finite abelian p-groups by type is*

$$Z_{1,d,p}(s_1,\dots,s_d) = \frac{1}{(1-p^{-s_1})(1-p^{-s_1-s_2})\cdots(1-p^{-s_1-\cdots-s_d})}.$$

The series counting finite abelian groups by type is

$$\zeta(s_1)\zeta(s_1+s_2)\cdots\zeta(s_1+\cdots+s_d).$$

Proof. We compute as follows:

$$\begin{aligned} Z_{1,d,p}(s_1,\dots,s_d) &= \sum_{\substack{A \text{ f. ab. p-group of rk } \leq d \\ A\cong\mathbb{Z}/p^{\alpha_1}\times\ldots\times\mathbb{Z}/p^{\alpha_d} \\ \alpha_1\geq\ldots\geq\alpha_d\geq 0}} p^{-\alpha_1 s_1-\cdots-\alpha_d s_d} \\ &= \sum_{\alpha_1\geq\ldots\geq\alpha_d\geq\gamma=0} G_p(\alpha_1,\dots,\alpha_d;0)p^{-\alpha_1 s_1-\cdots-\alpha_d s_d} \\ &= \sum_{\alpha_1\geq\ldots\geq\alpha_d\geq 0} p^{-\alpha_1 s_1-\cdots-\alpha_d s_d} \\ &= \frac{1}{(1-p^{-s_1})(1-p^{-s_1-s_2})\cdots(1-p^{-s_1-\cdots-s_d})} \end{aligned}$$

Furthermore, taking the product over all primes gives $\zeta(s_1)\zeta(s_1+s_2)\cdots\zeta(s_1+\cdots+s_d)$. □

Remark 33. *Setting all variables equal leads one to the well-known generating series for counting finite abelian groups of rank $\leq d$, in increasing order:*

$$F(s) = \zeta(s)\zeta(2s)\cdots\zeta(ds). \tag{3.1.1}$$

3.2 Class 2

The type series we would like to compute is

$$Z_{2,2,p}(s_1,s_2,s_3) = \sum_{\alpha\geq\beta\geq\gamma>0} G_p(\alpha,\beta;\gamma)p^{-\alpha s_1-\beta s_2-\gamma s_3} \tag{3.2.1}$$

By theorem 4.1 in [AMM12], the number of nonisomorphic 2-generator p-groups of class 2 and type $(\alpha,\beta;\gamma)$ is

$$G_p(\alpha,\beta;\gamma) = \gamma+1+\frac{1}{2}\min(\gamma,\alpha-\beta)(2\gamma+1-\min(\gamma,\alpha-\beta)), \tag{3.2.2}$$

which is independent of p. This is also the order of the cohomology group $H^2(\mathbb{Z}/p^\alpha \times \mathbb{Z}/p^\beta, \mathbb{Z}/p^\gamma)$ where $\mathbb{Z}/p^\alpha \times \mathbb{Z}/p^\beta$ acts trivially on $\mathbb{Z}/p^\gamma$. As noted in [AMM12], this formula agrees with an independent computation of Voll's in [V09] via zeta functions. For instance, the first 12 coefficients of the Taylor series expansion in Corollary 1.2 of [V09] agree with the entries in Table 1 of [AMM12] enumerating the number of nonisomorphic finite p-groups of class at most 2 and generated by at most 2 elements.

Using this formula and breaking it down into several cases to remove the min function, we can directly sum the series (3.2.1). Another way is to extend a result of Voll in [V09] to several variables using $G_p(\alpha, \beta; \gamma)$ and take the difference between the count of finite p-groups of class at most 2 on at most 2 generators and those of class 1 on at most 2 generators. A third method is to first show that $G_p(\alpha, \beta; \gamma)$ satisfies a recurrence relation. Then we either sum the resulting series directly or separate the series into two series and then use the method of convolution to sum the series into a rational function in p, p^{-s_1}, p^{-s_2} and p^{-s_3}. We pursue the third method to compute the type series explicitly.

Lemma 34. *$G_p(\alpha, \beta; \gamma)$ satisfies the recurrence relations:*

$$G_p(\alpha, \beta; \gamma) = \begin{cases} G_p(\alpha - 1, \beta; \gamma) & \text{if } \quad \alpha - \beta > \gamma \\ G_p(\alpha - 1, \beta; \gamma) + \gamma + 1 - (\alpha - \beta) & \text{if } \quad 0 < \alpha - \beta \leq \gamma. \end{cases}$$

Moreover, we have $G_p(\beta, \beta; \gamma) = \gamma + 1$ and $G_p(\beta + \gamma, \beta; \gamma) = \frac{(\gamma+1)(\gamma+2)}{2}$.

Proof. In both cases of the recurrence above, we use equation (3.2.2). The first recurrence follows from $\min(\gamma, \alpha - \beta) = \min(\gamma, \alpha - 1 - \beta) = \gamma$ as $\alpha - \beta > \gamma$. When $\alpha - \beta \leq \gamma$, we know $\min(\gamma, \alpha - \beta) = \alpha - \beta$. Therefore, in order to arrive at the second recurrence, it suffices to write out the formulas for both $G_p(\alpha, \beta; \gamma)$ and $G_p(\alpha - 1, \beta; \gamma)$ and take their difference.

The formula $G_p(\beta, \beta; \gamma) = \gamma + 1$ follows by direct calculation. For $G_p(\beta + \gamma, \beta; \gamma)$, we apply the second recurrence and induction to deduce

$$G_p(\beta+\gamma,\beta;\gamma) = G_p(\beta,\beta;\gamma) + 1 + 2 + \cdots + \gamma.$$

Then we use $G_p(\beta,\beta;\gamma) = \gamma + 1$ to finish the computation. □

Definition 35. *Let $\zeta_{2,2,p}(s_1,s_2,s_3)$ denote the Dirichlet generating series counting finite p-groups of class at most 2 on at most 2 generators by type. That is,*

$$\zeta_{2,2,p}(s_1,s_2,s_3) = \sum_{\alpha\geq\beta\geq\gamma\geq 0} G_p(\alpha,\beta;\gamma)p^{-\alpha s_1-\beta s_2-\gamma s_3}$$

Remark 36. *When we set all variables equal, we get the one variable zeta function enumerating finite p-groups of class 2 on at most 2 generators [V09, see Theorem 1.1]. As a consequence of Theorem 4.2 in [AMM12], we have*

$$\begin{aligned}
\zeta_{2,2,p}(s,s,s) &= \sum_{\alpha\geq\beta\geq\gamma\geq 0} G_p(\alpha,\beta;\gamma)p^{-\alpha s-\beta s-\gamma s} \\
&= \sum_{\alpha\geq\beta\geq\gamma\geq 0} G_p(\alpha,\beta;\gamma)p^{-(\alpha+\beta+\gamma)s} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{\substack{n=\alpha+\beta+\gamma\\ \alpha\geq\beta\geq\gamma\geq 0}} G_p(\alpha,\beta;\gamma)\Big)p^{-ns} \\
&= \frac{1}{(1-p^{-s})(1-p^{-2s})(1-p^{-3s})^2(1-p^{-4s})}.
\end{aligned}$$

Now, we break down $\zeta_{2,2,p}(s_1,s_2,s_3)$ into a count of those of class 2 and those of class 1. That is, we have

$$\zeta_{2,2,p}(s_1,s_2,s_3) = Z_{1,2,p}(s_1,s_2) + Z_{2,2,p}(s_1,s_2,s_3). \tag{3.2.3}$$

Therefore, in order to compute $Z_{2,2,p}(s_1,s_2,s_3)$, it suffices to find the series $\zeta_{2,2,p}(s_1,s_2,s_3)$ and $Z_{1,2,p}(s_1,s_2,s_3)$. By Proposition 32, we know

$$Z_{1,2,p}(s_1,s_2) = \frac{1}{(1-p^{-s_1})(1-p^{-s_1-s_2})}. \tag{3.2.4}$$

It remains to compute $\zeta_{2,2,p}(s_1, s_2, s_3)$. To simplify notation, let's make the substitutions: $x = p^{-s_1}, y = p^{-s_2}$ and $z = p^{-s_3}$. Furthermore, we break down the series $\zeta_{2,2,p}(x, y, z)$ as follows.

$$\begin{aligned}\zeta_{2,2,p}(x,y,z) &= \sum_{\alpha\geq\beta\geq\gamma\geq 0} G_p(\alpha,\beta;\gamma)x^\alpha y^\beta z^\gamma \\ &= \sum_{\substack{\alpha-\beta>\gamma \\ \alpha\geq\beta\geq\gamma\geq 0}} G_p(\alpha,\beta;\gamma)x^\alpha y^\beta z^\gamma + \sum_{\substack{\alpha-\beta\leq\gamma \\ \alpha\geq\beta\geq\gamma\geq 0}} G_p(\alpha,\beta;\gamma)x^\alpha y^\beta z^\gamma \qquad (3.2.5)\end{aligned}$$

Let $\alpha = \beta + \gamma + r$ where $r \geq 1$ in the first sum and $\alpha = \beta + t$ where $0 \leq t \leq \gamma$ in the second sum. Then,

$$\begin{aligned}\zeta_{2,2,p}(x,y,z) &= \sum_{\substack{\beta\geq\gamma\geq 0 \\ r\geq 1}} G_p(\beta+\gamma+r,\beta;\gamma)x^{\beta+\gamma+r} y^\beta z^\gamma + \sum_{\beta\geq\gamma\geq t\geq 0} G_p(\beta+t,\beta;\gamma)x^{\beta+t} y^\beta z^\gamma \\ &:= \zeta^{(1)} + \zeta^{(2)}. \qquad (3.2.6)\end{aligned}$$

In the next two lemmas, we sum $\zeta^{(1)}$ and $\zeta^{(2)}$, separately.

Lemma 37.

$$\zeta^{(1)} = \frac{x}{(1-x)(1-xy)(1-x^2yz)^3}.$$

Proof. By Lemma 34, we know $G_p(\beta+\gamma+r, \beta; \gamma) = G_p(\beta+\gamma, \beta; \gamma) = \frac{(\gamma+1)(\gamma+2)}{2}$. Thus,

$$\begin{aligned}\zeta^{(1)} &= \sum_{\substack{\beta\geq\gamma\geq 0 \\ r\geq 1}} G_p(\beta+\gamma+r,\beta;\gamma)x^{\beta+\gamma+r} y^\beta z^\gamma \\ &= \sum_{\substack{\beta\geq\gamma\geq 0 \\ r\geq 1}} \frac{(\gamma+1)(\gamma+2)}{2}(xy)^\beta (xz)^\gamma \sum_{r\geq 1} x^r \\ &= \frac{x}{2(1-x)} \sum_{\gamma\geq 0} (\gamma^2+3\gamma+1)(xz)^\gamma \frac{(xy)^\gamma}{1-xy} \\ &= \frac{x}{2(1-x)(1-xy)} \sum_{\gamma\geq 0} (\gamma^2 w^\gamma + 3\gamma w^\gamma + w^\gamma), \qquad \text{where } w = x^2yz \\ &= \frac{1}{2(1-xy)} \left[\frac{w^2+w}{(1-w)^3} + \frac{3w}{(1-w)^2} + \frac{2}{(1-w)}\right]\end{aligned}$$

$$= \frac{x}{(1-x)(1-xy)(1-w)^3}$$
$$= \frac{x}{(1-x)(1-xy)(1-x^2yz)^3}.$$

In the fourth line, we applied the sums $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ and $\sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3}$. □

Lemma 38.

$$\zeta^{(2)} = \frac{1-x^3y^2z^2}{(1-xy)(1-xyz)^2(1-x^2yz)^3}.$$

Proof.

$$\begin{aligned}
\zeta^{(2)} &= \sum_{\beta\geq\gamma\geq t\geq 0} G_p(\beta+t,\beta;\gamma)x^{\beta+t}y^\beta z^\gamma \\
&= \sum_{\beta\geq\gamma\geq t\geq 0} (\gamma+1+\frac{1}{2}t(2\gamma+1-t))x^{\beta+t}y^\beta z^\gamma \\
&= \sum_{\gamma\geq t\geq 0} (\gamma+1+\frac{1}{2}t(2\gamma+1-t))x^t z^\gamma \sum_{\beta\geq\gamma}(xy)^\beta \\
&= \frac{1}{1-xy}\sum_{\gamma\geq t\geq 0} (\gamma+1+\frac{1}{2}t(2\gamma+1-t))x^t(xyz)^\gamma \\
&= \frac{1}{1-xy}\Big(\sum_{\gamma\geq t\geq 0}\gamma(1+t)u^\gamma x^t + \sum_{\gamma\geq t\geq 0}(1-\frac{t^2}{2})u^\gamma x^t + \sum_{\gamma\geq t\geq 0}\frac{t}{2}u^\gamma x^t\Big), \text{ where } u = xyz \\
&= \frac{1}{1-xy}(S_1+S_2+S_3) \qquad (3.2.7)
\end{aligned}$$

We sum S_1, S_2 and S_3 as follows.

$$\begin{aligned}
S_1 &= \sum_{\gamma\geq t\geq 0}\gamma(1+t)u^\gamma x^t \\
&= \sum_{t\geq 0}(1+t)x^t\sum_{\gamma\geq t}\gamma u^\gamma \\
&= \sum_{\gamma\geq 0}\gamma u^\gamma + 2x\sum_{\gamma\geq 1}\gamma u^\gamma + \sum_{t\geq 2}(1+t)x^t\sum_{\gamma\geq t}\gamma u^\gamma \\
&= \frac{u}{(1-u)^2} + \frac{2xu}{(1-u)^2} + \sum_{t\geq 2}(1+t)x^t\Big(\frac{u}{(1-u)^2} - \sum_{\gamma=1}^{t-1}\gamma u^\gamma\Big) \\
&= \frac{2xu+u}{(1-u)^2} + \frac{u}{(1-u)^2}\sum_{t\geq 2}(x^t+tx^t) - \sum_{t\geq 2}(1+t)x^t\Big(\frac{u-tu^t+(t-1)u^{t+1}}{(1-u)^2}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2xu+u}{(1-u)^2} + \frac{u}{(1-u)^2}\Big(\frac{1}{1-x} - 1 - x + \frac{x}{(1-x)^2} - x\Big) \\
&- \frac{u}{(1-u)^2}\Big(\frac{1}{1-x} - 1 - x + \frac{x}{(1-x)^2} - x\Big) \\
&+ \frac{1}{(1-u)^2}\Big(\frac{(ux)^2+ux}{(1-ux)^3} - ux + \frac{ux}{(1-ux)^2} - ux\Big) \\
&+ \frac{u}{(1-u)^2}\Big(\frac{(ux)^2+ux}{(1-ux)^3} - ux - \frac{(ux)^2}{1-ux}\Big) \\
&= \frac{2xu+u}{(1-u)^2} + \frac{1}{(1-u)^2}\Big(\frac{(ux)^2+ux}{(1-ux)^3} - ux + \frac{ux}{(1-ux)^2} - ux\Big) \\
&+ \frac{u}{(1-u)^2}\Big(\frac{(ux)^2+ux}{(1-ux)^3} - ux - \frac{(ux)^2}{1-ux}\Big) \qquad (3.2.8)
\end{aligned}
$$

In the fifth line above, we use the formula $\sum_{r=1}^{n} rx^r = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$. Similarly, we have

$$S_2 = \sum_{\gamma \geq t \geq 0} (1 - \frac{t^2}{2}) u^\gamma x^t = \frac{1}{1-u}\Big(\frac{1}{1-ux} - \frac{(ux)^2+ux}{2(1-ux)^3}\Big), \quad \text{and} \qquad (3.2.9)$$

$$S_3 = \sum_{\gamma \geq t \geq 0} \frac{t}{2} u^\gamma x^t = \frac{ux}{2(1-u)(1-ux)^2}. \qquad (3.2.10)$$

Therefore, summing S_1, S_2 and S_3 and simplifying, we get the required sum:

$$
\begin{aligned}
\zeta^{(2)} &= S_1 + S_2 + S_3 \\
&= \frac{1 - x^3y^2z^2}{(1-xy)(1-xyz)^2(1-x^2yz)^3}.
\end{aligned}
$$

□

Theorem 39. *The generating series counting finite p-groups of class at most 2 on at most 2 generators by type is*

$$
\begin{aligned}
\zeta_{2,2,p}(s_1, s_2, s_3) &= \sum_{\alpha \geq \beta \geq \gamma \geq 0} G_p(\alpha, \beta; \gamma) p^{-\alpha s_1 - \beta s_2 - \gamma s_3} \\
&= \frac{1}{(1-p^{-s_1})(1-p^{-s_1-s_2})(1-p^{-s_1-s_2-s_3})^2(1-p^{-2s_1-s_2-s_3})}.
\end{aligned}
$$

Equivalently,

$$\zeta_{2,2}(s_1, s_2, s_3) = \zeta(s_1)\zeta(s_1+s_2)\zeta^2(s_1+s_2+s_3)\zeta(2s_1+s_2+s_3).$$

Proof. We combine lemmas 37 and 38 for the proof. We have

$$\begin{aligned}\zeta_{2,2,p}(s_1,s_2,s_3) &= \zeta^{(1)}+\zeta^{(2)}\\ &= \frac{x}{(1-x)(1-xy)(1-x^2yz)^3}+\frac{1-x^3y^2z^2}{(1-xy)(1-xyz)^2(1-x^2yz)^3}\\ &= \frac{1}{(1-x)(1-xy)(1-xyz)^2(1-x^2yz)}\\ &= \zeta_p(s_1)\zeta_p(s_1+s_2)\zeta_p^2(s_1+s_2+s_3)\zeta_p(2s_1+s_2+s_3).\end{aligned}$$

□

Remark 40. *Indeed, when we set all variables equal, we get the corresponding one variable zeta function computed by Voll. See Remark 36.*

Corollary 41. *The type series counting the number of nonisomorphic 2-generator p-groups of class 2 is*

$$\begin{aligned}Z_{2,2,p}(s_1,s_2,s_3) &= \sum_{\alpha\geq\beta\geq\gamma>0} G_p(\alpha,\beta;\gamma)p^{-\alpha s_1-\beta s_2-\gamma s_3}\\ &= \frac{1-(1-p^{-s_1-s_3-s_3})^2(1-p^{-2s_1-s_2-s_3})}{(1-p^{-s_1})(1-p^{-s_1-s_2})(1-p^{-s_1-s_2-s_3})^2(1-p^{-s_1-s_2-s_3})}\end{aligned}$$

Proof. By equations 3.2.3, 3.2.4 and Theorem 39, we have

$$\begin{aligned}Z_{2,2,p}(s_1,s_2,s_3) &= \sum_{\alpha\geq\beta\geq\gamma>0} G_p(\alpha,\beta;\gamma)p^{-\alpha s_1-\beta s_2-\gamma s_3}\\ &= \zeta_{2,2,p}(s_1,s_2,s_3)-Z_{1,2,p}(s_1,s_2)\\ &= \frac{1}{(1-x)(1-xy)(1-xyz)^2(1-x^2yz)}-\frac{1}{(1-x)(1-xy)}\\ &= \frac{1-(1-xyz)^2(1-x^2yz)}{(1-x)(1-xy)(1-xyz)^2(1-x^2yz)}\\ &= \frac{1-(1-p^{-s_1-s_3-s_3})^2(1-p^{-2s_1-s_2-s_3})}{(1-p^{-s_1})(1-p^{-s_1-s_2})(1-p^{-s_1-s_2-s_3})^2(1-p^{-s_1-s_2-s_3})}\end{aligned}$$

□

Corollary 42. *Setting all variables equal, define*

$$Z_{2,2,p}(s) := Z_{2,2,p}(s,s,s) = \frac{1-(1-p^{-3s})(1-p^{-4s})}{(1-p^{-s})(1-p^{-2s})(1-p^{-3s})^2(1-p^{-4s})}$$

We have

$$Z_{2,2,p}(s) = 2t^3 + 3t^4 + 5t^5 + 9t^6 + 13t^7 + 18t^8 + 26t^9 + 34t^{10} + 44t^{11} + 58t^{12} + O(t^{13})$$

where $t = p^{-s}$. The coefficients of the series expansion above agree with the entries in Table 1 of [AMM12].

Remark 43. *In general, we can deduce that the type series $Z_{2,d,p}(s_1,\ldots,s_d,s_{d+1})$ is a rational function in p and $p^{-s_i}, 1 \leq i \leq d+1$. [V09]. However, the explicit rational function is currently unknown for $d \geq 3$. We were able to sum the series for $d = 2$, using recurrence relations satisfied by the coefficients of the series. When the number of generators d is 3 or more, finding the recurrence relations satisfied by $G_p(\alpha_1,\ldots,\alpha_d;\gamma)$ appears to be a substantially more technical task. Even if we don't know the recurrence relations, it would still be very interesting to look for recurrence relations, analogous to Lemma 34, experimentally and then make conjectures about the form of the rational functions. Applications of such formulas include finding the proportion of groups of class at most c among groups of order p^n as $n \to \infty$.*

Appendix A

Sage Code

A.1 Convolution Example

```
R.<q,p,u1,u2,u3,v,x1,x2,x3,y> = QQbar[]

C(x1,x2,x3) = 1/((1-x1)*(1-x1*x2)*(1-x1*x2*x3)) # an example rational
#function

CC(x1,u1,x2,u2,x3,u3) = C(x1/u1,x2/u2,x3/u3) # setting up for convolution
#around small circles about origin; x1,x2,x3 are sufficiently small numbers

def has_x1(ff): #function to check if a root has x1 as a factor
    ff0=ff.subs(x1=0)
    return(ff0==0)

def has_x2(ff): #function to check if a root has x2 as a factor
    ff0=ff.subs(x2=0)
    return(ff0==0)

def has_x3(ff): #function to check if a root has x3 as a factor
    ff0=ff.subs(x3=0)
    return(ff0==0)

#integrand of the convolution
f(x1,u1,x2,u2,x3,u3) = C(u1,u2,u3)*CC(x1,u1,x2,u2,x3,u3)/(u1*u2*u3)

fd = f.denominator() #need to extract denominator to get poles and then use
#Cauchy's residue theorem

ans1=0       #convolution wrt u1
for ff in fd.roots(u1):   #loop thru roots in denominator to compute residues
    if has_x1(ff[0].numerator()):
        tmp=f.residue(u1==ff[0])
        ans1+=tmp

ans1d = ans1.denominator() #use this to find poles wrt u1

#now do two more convolutions for u2 and u3
ans2=0       #convolution wrt u2
for ff in ans1d.roots(u2):   #loop thru roots in denominator of result of
#last convolution to compute residues
    if has_x2(ff[0].numerator()):
        tmp=ans1.residue(u2==ff[0])
        ans2+=tmp

```

```
ans2d = ans2.denominator() #use this to find poles wrt u3
ans3=0        #convolution wrt u3
for ff in ans2d.roots(u3):   #loop thru roots in denominator of result of
#last convolution to compute residues
    if has_x3(ff[0].numerator()):
        tmp=ans2.residue(u3==ff[0])
        ans3+=tmp

print(ans3) #is the required rational function which agrees with hand computation
"(x1, u1, x2, u2, x3, u3) |--> -1/((x1*x2*x3 - 1)*(x1*x2 - 1)*(x1 - 1))"
```

A.2 Cotype zeta function of $F_{2,3}$

```
R.<q,p,u1,u2,u3,v,x1,x2,x3,y> = QQbar[]

C(x1,x2,x3) = (1+x1+q*x1+x1*x2+q*x1*x2+q*x1^2*x2)/((1-q^2*x1)
*(1-q^2*x1*x2)*(1-x1*x2*x3)) #rational function for p-part of sgzf of Z^3

B3(x1,x2,x3,y) = (1+q*x2*x3*y)/((q^2*x1*x2*x3*y - 1)
*(q^2*x2*x3*y^2 - 1)*(x1*x2*x3 - 1)*(x1 - 1))
#rational function for sum over N_(a2,a2)(r)*x1^a1*(x2*x3)^a2*(py)^r
#with a1>=a2>=0, 2*a2>=r>=0

B1(x1,x2,x3,y) = x1*x2*(1+q*x1*x2*x3*y)/((1-x1)*(1-x1*x2)
*(1-x1*x2*x3)*(1-q^2*x1*x2*x3*y)*(1-q^2*x1*x2*x3*y^2))
#rational function for sum ove N_(a3,a3)(r)*x1^a1*x2^a2*x3^a3*(p*y)^r
#with a1>=a2>a3>=0, 2*a3>=r>=0

B2(x1,x2,x3,y) = ( (x2*x3*y)/((1-x1)*(1-x1*x2*x3)
*(1-q^2*x1*x2*x3*y)) + (1+q*x2*x3*y^2+x2*x3*y^2)
*B3(x1,x2,x3,y) )/(1-x2*x3*y^3)
#rational function for sum over N_(a3,a3,a3)(r)*x1^a1*(x2*x3)^a3*y^r
#with a1>=a2=a3>=0, 3*a3>=r>=0

#the integrand of the convolution
A(x1,x2,x3,y) = (C(x1,x2,x3) - y*C(x1*y,x2*y,x3*y))/(1-y) #rational function
#for sum over K_(a1,a2,a3)(p)*x^a*y^r with a1>=a2>=a3>=0, a1+a2+a3>=r>=0

B(x1,x2,x3,y) = (B1(x1,x2,x3,y) + B2(x1,x2,x3,y))/(1-x1*x2*y)
#rational function  for sum over N_(a2,a3,a3)(r)*x1^a2*x2^a3*x3^a3*y^r
#with a1>=a2>=a3>=0, a2+2*a3>=r>=0

BB(x1,u1,x2,u2,x3,u3,y,v) = B(x1/u1,x2/u2,x3/u3,y/v)

h(x1,u1,x2,u2,x3,u3,y,v) = A(u1,u2,u3,v)*BB(x1,u1,x2,u2,x3,u3,y,v)
/(u1*u2*u3*v)
#integrand of the convolution

def has_x1(ff):
    ff0=ff.subs(x1=0)
    return(ff0==0)

def has_x2(ff):
    ff0=ff.subs(x2=0)
    return(ff0==0)

def has_x3(ff):
    ff0=ff.subs(x3=0)
    return(ff0==0)

def has_y(ff):
    ff0=ff.subs(y=0)
    return(ff0==0)

#determine the poles from the factors of the denominator of the
#rational function
hd = h.denominator()

ans1=0
for ff in hd.roots(u1):
    if has_x1(ff[0].numerator()): #checks if the numerator of a pole has x1
        tmp=h.residue(u1==ff[0])
        ans1+=tmp

ans1d = ans1.denominator()
```

```

ans2=0
for ff in ans1d.roots(u2):
    if has_x2(ff[0].numerator()):
        tmp=ans1.residue(u2==ff[0])
        ans2+=tmp

ans2d = ans2.denominator()

ans3=0
for ff in ans2d.roots(u3):
    if has_x3(ff[0].numerator()):
        tmp=ans2.residue(u3==ff[0])
        ans3+=tmp

ans3d = ans3.denominator()

ans=0
for ff in ans3d.roots(v):
    if has_y(ff[0].numerator()):
        tmp=ans3.residue(v==ff[0])
        ans+=tmp

print(ans) #is the required local part of the subgroup growth zeta function
#of F23
```

Appendix B

Mathematica Code

The codes below are Mathematica versions of the Sage codes in appendix A.

B.1 Convolution Example

```
c[x1_, x2_, x3_] := 1/((1 - x1)*(1 - x1*x2)*(1 - x1*x2*x3))
cc[x1_, u1_, x2_, u2_, x3_, u3_] := c[x1/u1, x2/u2, x3/u3]
f[x1_, u1_, x2_, u2_, x3_, u3_] :=
 c[u1, u2, u3]*cc[x1, u1, x2, u2, x3, u3]/(u1*u2*u3)
hasx1[ff_] := If[(Numerator[ff] /. x1 -> 0) == 0, True, False]
hasx2[ff_] := If[(Numerator[ff] /. x2 -> 0) == 0, True, False]
hasx3[ff_] := If[(Numerator[ff] /. x3 -> 0) == 0, True, False]
fden[x1_, u1_, x2_, u2_, x3_, u3_] :=
 Denominator[Together[f[x1, u1, x2, u2, x3, u3]]]
ans1 = 0;
lst1 = u1 /. {ToRules[Roots[fden[x1, u1, x2, u2, x3, u3] == 0, u1]]} //
    Flatten;
ans1 = Total[
   Table[Boole[hasx1[lst1[[i]]]]*
     Residue[f[x1, u1, x2, u2, x3, u3], {u1, lst1[[i]]}], {i,
     Length[lst1]}]];
ans1den[x1_, x2_, u2_, x3_, u3_] := Denominator[Together[ans1]]
ans2 = 0;
lst2 = u2 /. {ToRules[Roots[ans1den[x1, x2, u2, x3, u3] == 0, u2]]} //
    Flatten;
ans2 = Total[
   Table[Boole[hasx2[lst2[[i]]]]*Residue[ans1, {u2, lst2[[i]]}], {i,
     Length[lst2]}]];
ans2den[x1_, x2_, x3_, u3_] := Denominator[Together[ans2]]
ans3 = 0;
lst3 = u3 /. {ToRules[Roots[ans2den[x1, x2, x3, u3] == 0, u3]]} //
   Flatten;
ans3 = Total[
  Table[Boole[hasx3[lst3[[i]]]]*Residue[ans2, {u3, lst3[[i]]}], {i,
    Length[lst3]}]]
```

B.2 Cotype zeta function of $F_{2,3}$

```
c[x1_, x2_, x3_,
  q_] := (1 + x1 + q*x1 + x1*x2 + q*x1*x2 +
    q*x1^2*x2)/((1 - q^2*x1)*(1 - q^2*x1*x2)*(1 - x1*x2*x3))
b3[x1_, x2_, x3_, y_,
  q_] := (1 +
    q*x2*x3*y)/((q^2*x1*x2*x3*y - 1)*(q^2*x2*x3*y^2 - 1)*(x1*x2*x3 -
      1)*(x1 - 1))
b1[x1_, x2_, x3_, y_, q_] :=
 x1*x2*(1 +
     q*x1*x2*x3*
      y)/((1 - x1)*(1 - x1*x2)*(1 - x1*x2*x3)*(1 -
       q^2*x1*x2*x3*y)*(1 - q^2*x1*x2*x3*y^2))
b2[x1_, x2_, x3_, y_,
  q_] := (x2*x3*
     y)/((1 - x1)*(1 - x1*x2*x3)*(1 - q^2*x1*x2*x3*y)*(1 -
       x2*x3*y^3)) + (1 + q*x2*x3*y^2 + x2*x3*y^2)*
   b3[x1, x2, x3, y, q]/(1 - x2*x3*y^3)
b[x1_, x2_, x3_, y_,
  q_] := (b1[x1, x2, x3, y, q] + b2[x1, x2, x3, y, q])/(1 - x1*x2*y)
a[x1_, x2_, x3_, y_,
  q_] := (c[x1, x2, x3, q] - y*c[x1*y, x2*y, x3*y, q])/(1 - y)
bb[x1_, u1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 b[x1/u1, x2/u2, x3/u3, y/v, q]
h[x1_, u1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 a[u1, u2, u3, v, q]*bb[x1, u1, x2, u2, x3, u3, y, v, q]/(u1*u2*u3*v)
hasx1[ff_] := If[(Numerator[ff] /. x1 -> 0) == 0, True, False]
hasx2[ff_] := If[(Numerator[ff] /. x2 -> 0) == 0, True, False]
hasx3[ff_] := If[(Numerator[ff] /. x3 -> 0) == 0, True, False]
hasy[ff_] := If[(Numerator[ff] /. y -> 0) == 0, True, False]
hden[x1_, u1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 Denominator[Together[h[x1, u1, x2, u2, x3, u3, y, v, q]]]
lst1 = u1 /. {ToRules[
     Roots[hden[x1, u1, x2, u2, x3, u3, y, v, q] == 0, u1]]} // Flatten
ans1[x1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 Total[Table[
   Boole[hasx1[lst1[[i]]]]*
    Residue[h[x1, u1, x2, u2, x3, u3, y, v, q], {u1, lst1[[i]]}], {i,
    Length[lst1]}]]
ans1den[x1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 Denominator[Together[ans1[x1, x2, u2, x3, u3, y, v, q]]]
ans2 = 0
lst2 = u2 /. {ToRules[
     Roots[ans1den[x1, u1, x2, u2, x3, u3, y, v, q] == 0, u2]]} //
  Flatten
ans2 = Total[
  Table[Boole[hasx2[lst2[[i]]]] * Residue[ans1, {u2, lst2[[i]]}], {i,
    Length[lst2]}]]
ans2den[x1_, u1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 Denominator[Together[ans2]]
ans3 = 0
lst3 = u3 /. {ToRules[
     Roots[ans2den[x1, u1, x2, u2, x3, u3, y, v, q] == 0, u3]]} //
  Flatten
ans3 = Total[
  Table[Boole[hasx3[lst3[[i]]]] * Residue[ans2, {u3, lst3[[i]]}], {i,
    Length[lst3]}]]
ans3den[x1_, u1_, x2_, u2_, x3_, u3_, y_, v_, q_] :=
 Denominator[Together[ans3]]
ans = 0
lst4 = v /. {ToRules[
     Roots[ans3den[x1, u1, x2, u2, x3, u3, y, v, q] == 0, v]]} //
  Flatten
```

```
ans =
 Total[Table[
   Boole[hasy[lst4[[i]]]] * Residue[ans3, {u3, lst3[[i]]}], {i,
    Length[lst3]}]]
```

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