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The Early Development of Quantitative Cognition: Correspondences and One-Way Functions

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THE EARLY DEVELOPMENT OF QUANTITATIVE COGNITION:

CORRESPONDENCES AND ONE-WAY FUNCTIONS

by

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Abstract

THE EARLY DEVELOPMENT OF QUANTITATIVE COGNITION:
CORRESPONDENCES AND ONE-WAY FUNCTIONS

by

Irvin Sam Schonfeld

Adviser: Dr. Harry Beilin

This investigation was designed to examine the development of the child's capacity to make numerical and quantitative comparisons. It was hypothesized that the nature of the child's understanding of correspondences and one-way functions informs his capacity to compare arrays of decals and quantities of liquid. The relation of Piagetian operative level to the child's capacity to use crystallized skills, or solution aids (Cattell, 1963; Horn, 1968), in comparing arrays was also investigated.

A total of 171 children who ranged in age from four to seven years were administered numerical and liquid comparison tasks. The numerical tasks included paired arrays of green and red decals represented as the candies of puppets named Bert and Ernie. Some paired arrays were related by injective and/or surjective correspondences. Other paired arrays were equal in length but different in density. In a different set of tasks, the green and red arrays each comprised two

subarrays. In these tasks, one puppet got more candy in one green-red subarray comparison and again in the second subarray comparison, or a different puppet got more candy in each of the two green-red subarray comparisons. In all subarray tasks the child was asked to determine the relative numerosity of the total amount of green and red decals. In the liquid tasks, paired green and red liquids were represented as Bert and Ernie's juice. Tasks included paired quantities of juice contained in transparent cylinders that were either the same or different in diameter. In the latter task, paired quantities had the same height. In other liquid tasks, each quantity of green and red liquid comprised two subquantities. In these tasks, one puppet got more juice in one green-red subquantity comparison and again in the second subquantity comparison, or a different puppet got more juice in each of the two green-red subquantity comparisons. In all subquantity tasks, the child was asked to determine the relative quantity of the total amount of green and red liquid.

Results indicate that: (1) Preoperational children possess comparison-making capabilities reflecting a rudimentary understanding of injective and surjective correspondences, one-way function based mappings of height on to quantity, and one-way compositions of same-directional subquantity comparisons. (2) Concrete operational children develop powerful comparison-making capabilities based on the capacity to coordinate countervailing subquantity and correspondence relations; they also begin to judge quantities on joint bases such as density and length, or diameter and height. (3) Comparison-making capabilities found in the preoperational subperiod become more accurate with the development of the concrete operations. (4) The child's capacity to use solution aids in making accurate numerical comparisons is structured by operative level.

The educational implications of correspondence and function based knowledge were discussed.

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CHAPTER I

INTRODUCTION

The study of the development of mathematical cognition owes much to Piaget. Piaget's books on number (with Szeminska, 1952), quantity (with Inhelder, 1974), space (with Inhelder, 1956), and geometry (with Inhelder & Szeminska, 1960) delineate structures of thought which are hypothesized to underlie an interrelated set of mathematical behaviors. There are, however, important forms of mathematical behavior that Piaget's theory does not systematically address. These forms of mathematical behavior include ordinary numerical and quantitative comparisons and enumerational strategies.

Researchers have recently begun to investigate the nature of the relationship of Piagetian ontogenetic status to the varieties of cognitive functioning Piaget did not address. Such studies have been undertaken in the domains of mathematical and nonmathematical cognition. For example, in the domain of nonmathematical cognition, researchers have linked operative level to the understanding of conjunctive relationships (Gallagher, Wright, & Noppe, 1974) and the utilization of problem solving strategies (Gholson & Beilin, 1978; Gholson, O'Connor, & Stern, 1976).

Many researchers have attempted to relate operative level to the domain of mathematical cognition. This is perhaps because a major theme in Piaget's writing has been the progressive arithmetization of thought. Furthermore, in his book on number, Piaget (with Szeminska, 1952) advanced the view that conservation, possibly his most studied concept, is a basic constituent of quantitative thought. It is, thus, understandable that a large number of research endeavors were stimulated by the arithmetical implications of Piaget's theory.

A considerable body of research indicates that measures of Piagetian operativity predict achievement in arithmetic (Almy, Chittenden, & Miller, 1966; Ayers, Rohr, & Ayers, 1974; Bearison, 1975; Dimitrovsky & Almy, 1975; Dudek, Goldberg, Lester, & Harris, 1969; Dudek, Lester, Goldberg, & Dyer, 1969; Freyberg, 1966; Goldschmid & Bentler, 1968; Kaminsky, 1970; Kaufman & Kaufman, 1972; Lunzer, Wilkenson, & Dolan, 1976; Melnick, Bernstein, & Lehrer, 1974; Nelson, 1970; Omotoso & Shapiro, 1976; Riggs & Nelson, 1976; Rohr, 1973; Wheatley, 1969). In studies in which the effects of psychometric intelligence were statistically controlled, operativity was still found to be significantly related to arithmetic achievement (Bearison, 1975; Dudek, Goldberg, Lester, & Harris, 1969; Dudek, Lester, Goldberg, & Dyer, 1969; Melnick, Bernstein, & Lehrer, 1974; Riggs & Nelson, 1976). Most of the studies which have addressed the issue of the relationship between operativity and arithmetic achievement employed standard tests of arithmetic achievement, the overall arithmetic achievement score constituting the dependent measure. As a consequence of the widespread use of global achievement scores as dependent measures, the relationship of operativity to each of the manifold arithmetical skills and concepts that young children typically master remains largely unexplored.

Some researchers, however, have undertaken finer grained studies of the relationship of operativity to arithmetical functioning. These researchers have generally investigated the relationship between operative level and specific varieties of arithmetical activity. Since

Piaget and Szeminska (1952) contend that arithmetical thought presupposes conservation, conservation has been widely used as an index of operative level.

Computational skill is one variety of arithmetical activity that has been studied in relation to operative level. Baker and Sullivan (1970) attempted to establish a relationship between conservation of number and mastery of addition and subtraction concepts. The conservation test they employed, however, was not a true test of conservation. Like Mehler and Bever's (1967) measure of conservation, Baker and Sullivan's (1970) measure did not employ the necessary transformational procedures that must characterize conservation tasks (Beilin, 1968; Piaget, 1968). Leblanc (1968) found that conservation of number is related to the ability to solve verbal subtraction problems. However, studies by Hood (1962) and Sohns (1975) indicate that conservation and mastery of elementary subtraction problems are independent of each other. Sohns (1975) also found mastery of subtraction problems to be unrelated to performance on seriation and class inclusion tasks. Evidence adduced by Steffe (in Lovell, 1972) indicates that conservation of number and the mastery of elementary addition facts are unrelated. Furthermore, Stahl (1973) found that performance on conservation of number and classification tasks did not predict achievement on written tests of elementary addition and subtraction. Although she found that performance on a conservation of mass task predicted achievement in subtraction, this finding must be taken cautiously since it was her only significant result in testing eight hypotheses and, thus, may have been a chance effect. The majority of the above cited studies suggest that the elementary arithmetic operations of addition and subtraction may not be structured by Piagetian operative level. This conclusion is understandable since the acquisition of elementary computational facts is more likely to be a product of informal (Brush, 1978) and rote learning experiences.

Moreover, Piaget and Szeminska (1952) assert that the sheer learning of arithmetical facts does not constitute arithmetical understanding. In order for arithmetical facts to become meaningful, that is to say, truly arithmetical, they must be assimilated into the concrete operational system of thought. Concrete operational thought structure enables the child to deepen his understanding of the quantitative features of his environment.

In contrast, Gelman (1972b) advanced the view that counting constitutes the major vehicle with which the child extends what he knows in the context of small quantities to the context of larger quantities. According to Gelman (1972a; 1972b; Gelman & Gallistel, 1978; Gelman & Tucker, 1975; Bullock & Gelman, 1977), children as young as three years (and sometimes younger) understand, at least in the context of the first three or four numbers, several logico-arithmetical properties. These include an understanding of number invariant displacements, addition, subtraction, and order relations. Through an increase in skill and confidence in counting these properties are extended to larger numbers (Gelman, 1972B).

There are three weaknesses in this view. First, Gelman (1972b; Gelman & Gallistel, 1978) does not present any hypotheses that pertain to the way in which children's initial understanding of the properties comes about. Although they eschew such an interpretation, Gelman and Gallistel (1978) provide no alternative to a nativist interpretation of the acquisition of an understanding of the properties.

Secondly, the behaviors Gelman describes are as susceptible to perceptually based explanations as they are to conceptually based explanations. Gelman and Gallistel (1978) do not convincingly dispel a subitizing (Jevons, 1971; Kaufman, Lord, Reese, & Volkman, 1941; Taves, 1941; Woodworth & Scholsberg, 1954) based explanation of children's ability to represent small quantities. A number of phenomena they attempt to explain are amenable to subitizing based explanations. For example, the question of the invariance of two or three counters in the context of configurational change is likely to be qualitatively different from that of the invariance of, say, eight counters. That Koehler's (1956) work indicates there is a functional parallel between human beings' capacity to subitize small quantities and animals' capacity to apprehend small quantities reinforces a perceptual interpretation of the phenomena Gelman and Gallistel report. An array of two or three counters is likely to remain within subitizing range even if its elements are displaced. An array of eight counters is beyond subitizing range before and after displacement. If a large quantity of counters is beyond subitizing range, conceptual processes must be called into play to enable the child to understand number invariant displacement.

Finally, evidence from studies of the relation of counting to conservation of number contradicts Gelman's (1972b) conceptualization of counting as a device whose function is to extend an understanding of number invariance from the context of small quantities to contexts which involve large quantities. Available evidence suggests that accurate counting, although an excellent means of determing the cardinal value of an array, does not guarantee the attainment of conservation of number (Carpenter, 1971; Greco, 1962; Piaget & Szeminska, 1952; Wallach & Sprott, 1964; Williams, 1971; Wohlwill, 1960; Wohlwill & Lowe, 1962; Zimiles, 1966). In addition, Saxe (1979b) found that a group of children who counted inaccurately because of learning disabilities mastered number conservation. Gelman, in response to the findings of Saxe (1979b) and others, revised their view to indicate that number conservation cannot be considered an extension of the ability to count (Gelman & Gallistel, 1978).

Werner (1957) provides an alternative conceptualization of counting. Counting, within the framework of Werner's theory, is a human construction that, with development, becomes, on one hand, less bound to the configuration of the objects being counted and, on the other hand, progressively governed by rules of order and, therefore, evolves into an information extracting tool. It is not an information extracting tool from the first. Saxe (1977) also regards counting as an important knowledge extracting tool that undergoes developmental change. He considers three interrelated functions of counting: "first, as a means to extract (or determine) number of an array of elements; second, as a means to compare two arrays numerically; and third, as a means to reproduce a model numerically" (p. 1512). He found an age developmental improvement in counting accuracy and adequacy of counting strategy.

Much of Gelman's research has been motivated by a problem that pervades the research on the cognitive capacities of preschool children, namely that preschool children have frequently been characterized by the cognitive capacities they lack rather than by those capacities they possess. The Genevans have also responded to this problem. Although the preoperational child has been found to lack many of the cognitive developmental capacities attributed to the concrete operational child, the Genevans have begun to investigate the nature of the cognitive capacities the preoperational child does possess. These have been characterized as one-way functions (Piaget, 1968, 1970a, 1970b, 1970c; Piaget, Grize, Szeminiska, & Vinh Bang, 1977).

A function expresses a particular relation. A relation is defined on sets X and Y if each ordered pair of elements (x,y), where x is an element of X and y is an element of Y, is meaningful. There is no requirement that an element of X correspond to one element of Y. A function is that relation in which an element of X is mapped on to exactly one element of Y. Thus a function involves a unique mapping in one direction, or, as the Genevans write, functions are "univocal to the right" (Piaget et al., 1977, p. 14).

Since these terms were developed in a mathematical context two simple mathematical examples will make the distinction between relations and functions clear. First consider the equation $X^2 + 4Y^2 = 36$. The relation expressed by this equation is not a function. There is no unique Y to be mapped on to each value of X (for example, if $X = 0$, $Y = \pm 3$). By contrast, consider the equation $Y = X^2 - 2X + 3$. The variable Y is a function of X. To each value of X there corresponds exactly one value of Y (for example, if $X = 0$, $Y = 3$). The inverse, however, is not true (if $Y = 3$, X can assume two values, 0 or 2).

There are functions where the uniqueness property is fulfilled in both directions. Consider the equation $Y = 2X + 3$. Each value of X maps on to exactly one value of Y. By rewriting the equation as $X = 1/2(Y - 3)$, it is evident that each value of Y maps on to exactly one value of X. The Genevans designate functions in which the uniqueness condition is fulfilled in either direction "biunivocal" or "one-to-one" (Piaget et al., 1977).

The research of Piaget and his coworkers (Piaget, Grize, Szeminska, & Vinh Bang, 1977) indicates that the preoperational child manifests some understanding of one-way order functions. The logic (the Genevans use the term "semilogic", cf. Inhelder, Sinclair, & Bovet, 1974) of these order functions is hypothesized to underlie the preoperational child's use of spatial extent to index and compare quantities. For example, the length of an array may be thought to covary with its numerosity. The longer of two arrays might thus be considered the more numerous. Such comparisons often constitute a somewhat fair yet imperfect (hence semi-logical) substitute for metric quantification. This logic is also thought to contribute to a rudimentary understanding of the regularities in the child's environment. Preoperational children were found, for example, to be capable of constructing regular sequences of objects of alternating color. In addition, preoperational children were found to be able to order pairs of objects. In one study, children were asked to locate contexts in which pairwise exchanges of cards depicting different kinds of flowers were appropriate. Preoperational children appeared to have mastered the ingredients of such pairwise exchanges and, with considerable difficulty, exchanges to obtain objects which themselves could be exchanged for target objects.

The Genevans attribute some understanding of correspondences, another category of functions, to preoperational children. Morphisms are correspondences that, because they are sustained by, although not coincident with, operative transformations, have become logically necessary (Piaget, 1976). Piaget (1977) writes that "correspondences and morphisms are essentially comparisons that do not transform objects to be compared but that extract common forms from them or analogies between them" (p. 351). They develop out of "primitive applications" of action schemes to objects in the environment (Piaget et al., 1977)

In one study of morphisms, Piaget and his coworkers asked children to identify those members of a series of movable red cut-outs that, when appropriately placed, covered a specified portion of each of four base cards. Each base card consisted of two areas, one red, the other white. The task amounted to finding those cut-outs which, when properly superimposed upon a particular base card, made the entire area appear red. Three cut-outs, differing slightly among themselves, corresponded to one and only one base card. Since there were four base cards, twelve cut-outs were employed.

Although there was much opportunity to relate base cards to cut-outs, the youngest children, five-year olds, were at best able to match each base card to one cut-out. This constitutes an example of the bijection, or term-by-term, morphism. At later ages, children could match each base card to more than one cut-out. This more advanced ability constitutes a many-to-one correspondence, a variety of the surjection morphism. Two levels of mastery of this morphism were in evidence. The more primitive level of mastery entails the discovery of those features of a base card which correspond to a single cut-out followed by trial-and-error extension of the correspondence to other cut-outs. The more advanced level entails the immediate assignment of base cards to all appropriate cut-outs.

Injection is a third type of morphism. That every element in a set B corresponds to at most one element in a set A constitutes the injection morphism (Piaget, 1977; Piaget et al., 1977). In other words, injection entails the condition in which every element in B corresponds to one or no elements in A. In contrast, surjection, in its most general form, entails the condition in which every element in set B corresponds to one or more elements in set A. Note that in both injection and surjection, the mapping of A on to B is univocal to the right.

Piaget et al. (1977) found that seven-year-olds could partition a collection of tokens into subsets such that one subset had more than the other. However, they were unable to quantify the relation between the subsets. When asked to create from a collection of 10 tokens two subsets which differ by four, they succeeded in creating unequal subsets, but were unable to quantify the difference between the subsets as asked. It is evident that the children created subset pairs that conformed to the injection mappings, at least in the qualitative sense. Piaget et al. emphasized that among preoperational children morphisms are not fully developed and heavily qualitative in character. They become quantitative and exact with operative development.

Piaget (1977) also stressed the role of morphisms in the attainment of conservation. This new interpretation of conservation can be made clear if one considers a uniformly dense row of counters, B, composed of subsections A and A' where A is detachable from a stationary A'. Suppose A is moved, in a density preserving manner, from one side of A' to another. B can be decomposed into a number of different A-A' subsections: A1-A1', A2-A2', etc. Clearly the unions of A1 and A1', A2 and A2', etc. are equal. The equivalence of the unions indicates that a vicariance relationship (Grouping II, see Flavell, 1973; Piaget, 1960, 1972) among the pairs A1-A1', A2-A2', etc. exists.

Two morphisms are implied in this vicariance relationship. First, the surjection morphism characterizes the relationship of each A-A' pair with respect to B. That is to say, both members of each A-A' pair correspond to a single whole, B. Secondly reciprocal injection morphisms characterize the relationship between the members of each of the following noncomplementary pairs: A1-A2' and A2-A1'. Reciprocal injection also characterizes the relationship between two pairs of subclasses where each pair consists of a subclass and the complement of an alternative subclass located at the same level of discourse as the first member of the pair (e.g., cats-non-dogs and dog-non-cats). Every element of one members of each pair of subclasses or subsections corresponds to at most one element of the other member of the pair.

That these morphisms are implied in the conservation of number raises two theoretical issues that are important to the Genevans. The first is that functions rooted in the preoperational subperiod provide a basis from which the child advances to the concrete operations. Piaget et al. (1977) argue that the progressive coordination of these functions leads to the "reversible mobility of operations." The second is that functions become progressively controlled by the concrete operations. That is to say, with the advent of the concrete operations, functions become progressively reversible and arithmetical (see Chapter 3, Regularities to

Proportion, in Piaget et al., 1977).

In the sections that follow the role of the child's understanding of correspondences and functions in comparing quantities will be treated. The types of comparisons to be treated differ from classic conservation comparisons. The quantities to be described are static. That is to say, the child does not witness the transformation of the quantities--decals pasted on to cardboard, liquid held in glasses--to be compared, as is the case in classic conservation.

The Genevans developed a psychology of correspondences and functions in connection with the study of the development of the understanding of physical causality and mathematical relations (Piaget et al., 1977). The mathematical relations investigated by the Genevans were studied within the context of children's performance on tasks in which sets were transformed. For example, Piaget et al. (177) investigated the relation between two collections of tokens--collections that were initially equal--where tokens in one collection were transferred one by one to the other collection. The Genevans and others have devoted much research effort to the study of the child's capacity to compare of quantities under conditions of transformation. However, the investigation of the child's understanding of relations between untransformed quantities is of equal importance. This is because many real life situations call for the comparison of quantities where no transformation of stimuli is involved (Beilin, 1969).

Although investigators have studied the child's capacity to compare untransformed quantities (e.g., Beilin, 1969; Pufall & Shaw, 1972; Pufall, Shaw, & Syrdal-Lasky, 1973; Saxe, 1977; Schwartz & Scholnick, 1970; Zimiles, 1966), little has been done within the framework of studies of the development of the child's capacity to compare untransformed quantities to elucidate the Genevan theory of correspondences and functions. A number of hypotheses that pertain to the relation of the child's understanding of correspondences and functions to his capacity to compare static quantities are developed in the sections that follow.

Static Numerical Comparisons

Static numerical and quantitative comparisons constitute categories of mathematical behaviors. Numerical and quantitative comparisons are materially distinguishable. The terms of numerical comparisons comprise arrays of countable items such as rows of decals. The child's task within the framework of the static numerical comparisons presented in this paper (see Figure 1) is to determine the relative numerosity of the members of a pair of arrays. The arrays are so constructed that the relative numerosity of the members of each pair can be accurately determined on a more or less deductive basis where the role of counting is minimized (e.g., the red array has more because it is more dense than the green array and both arrays are the same length). Alternatively, straightforward counting can be used to inform the comparisons. The terms of the quantitiative comparisons comprise paired amounts of colored liquid (see Figure 2). Such quantities lend themselves to comparison without the aid of measurement devices insofar as their dimensions are easily perceived to be different (see the Static Quantitative Comparisons section).

Beilin (1969) advanced the view that conservation of number, because one of its constituents is the capacity to make internal transformations, contributes to the attainment of the ability to compare two static arrays whose relative cardinal values to not correspond to the arrays' linear extent. His experimental results, which indicate that children tend to acquire conservation of number before they are capable of discovering that counters aligned in numerically equal rows of dissimilar length are in fact equal, support this view. He also obtained parallel results in the domain of area. Evidence adduced by Zimiles (1966), which indicates that performance on conservation of number tasks is predictive of success at making static numerical comparisons, supports Beilin's view.

It follows from the Genevans' treatment of functions that preoperational children ought to succeed at certain numerical comparison tasks. This is not to say that preoperational children succeed at comparison tasks which concrete operational children find challenging. However, the issue of discerning the types of comparisons at which preoperational children succeed requires theoretically informed hypotheses that discriminate comparison tasks whose solutions require the application of primitive functions from comparison tasks whose solutions require the application of concrete operational logic.

Evidence adduced by several researchers (Piaget, 1968; Pufall & Shaw, 1972; Pufall, Shaw, & Syrdal-Lasky, 1973) suggests that preoperational children master two types of static comparisons. One type involves arrays that are the same length and number. The other involves arrays in which the longer of two is the more numerous. Note that these comparisons require no more than one-way mappings of spatial extent schemata on to schemata of numerosity. The preoperational child in applying one-way spatial extent schemata to compare arrays in which relative numerosity and spatial extent conflict, as in Beilin's (1969) research, is bound to err.

It is expected that, by virtue of the role played by one-way functions in preoperational thought, preoperational should succeed at a number of static numerical comparisons. These comparisons require either injective or surjective mappings of one array on to the other. The comparisons that require injective and surjective mappings can be found in the Injective Preoperational (IP) and Surjective Preoperational (SP) series illustrated in Figure 1.

The comparisons depicted in the IP and SP series involve pairs of arrays in which terminal points are aligned. Some of the comparisons employed by Zimiles (1966) and Beilin (1969)--Beilin's static "conservation of inequality" (SCI in Figure 1) comparisons--involved pairs of arrays that are unequal in number but whose terminal points are aligned. The IP and SP series, because they embody injective and surjective mappings of one array on to another, to some degree employ one-to-one correspondence of interior elements. This is not the case in the SCI comparions. No attempt was made to match the interior elements one-to-one. In the context of injective mappings as many elements of one array as possible correspond one-to-one to the elements of the other array. The result is that one or more elements of the larger array go unmatched. In the context of surjective mappings, as many elements of one array as possible correspond one-to-one to the elements of the other array, but where one-to-one matching does not obtain, two-to-one matching does.

If spatial extent schemata play a role in evaluating the relative numerosity of two arrays, one would expect that there would be some tendency to to judge arrays that are of equal length to be equal in number. However, this tendency is likely to be diminished in the context of the IP and SP series because the array pairs composing those series are constructed in such a way as to engage the preoperational child's capacity to form judgments in accordance with injective and surjective mappings. On the other hand, the tendency to base judgments on spatial extent is more likely to inform the preoperational child's evaluation of SCI than IP and SP array pairs. This is because the SCI series is not constructed in such a way as to engage developing morphism based processes that at least provide some counterweight to the tendency to use spatial extent as an index of numerosity. It is, therefore, predicted that preoperational children will perform better on IP and SP comparisons than on comparable SCI comparisons.

As mentioned earlier, the Genevans hold that the concrete operations enrich the child's understanding of correspondences. Compensation, an important feature of concrete operational thought, appears to play a role in the development of the child's understanding of correspondences. In this context, compensation refers to the capacity to coordinate functional relations (Piaget et al., 1977).

The capacity to coordinate functional relations is relevant to the comparison of arrays that are organized such that each aggregate comprises two or more spatially distinct subarrays. Comparisons of such arrays call for some cross-referencing of comparisons between corresponding subarrays. However, in order to avoid making the discussion of the comparison of aggregates comprised of subarrays unnecessarily complex, two simplifying constraints, will, to some degree, be imposed. The first requires that each array comprise two colinear subarrays. This constraint holds in the TPO-I and TPR-I series but is relaxed slightly in the TPO-S and TPR-S series. The second constraint requires that each subarray of one aggregate visually correspond to a subarray of the other aggregate. The visual correspondence between subarrays involves either injective or surjective mappings.

It is expected that concrete operational children are more likely than preoperational children to succeed at comparison tasks in which two conditions hold: (1) visually corresponding subarrays are unequal; (2) the direction of the inequality which holds between the members of one pair of corresponding subarrays is the reverse of the direction of the inequality which holds between the members of the other pair (Two Part Reverse-Injective, or TPR-I, and Two Part Reverse-Surjective, or TPR-S, series). Thus, in the TPR conditions the comparison of a pair of arrays calls for the coordination of countervailing one-way subarray comparisons. Consider, for example, the two subarrays of the more numerous array in the first TPR-I comparison illustrated in Figure 1. Let us call the left subarrays, from above to below, R1 and G1 and the right subarrays, R2 and G2. The upper row, R, then, comprises R1 and R2; the lower row, G, comprises G1 and G2. Although R is greater than G, R1 is less than G1. R2 is greater than G2. The absolute difference between R2 and G2 exceeds that of R1 and G1. Therefore the R2-G2 subarray comparison informs the direction of the R-G comparison. If R and G were equal, the R1-G1 and R2-G2 differences would exactly compensate for each other (see the second TPR-S comparison in Figure 1).

Some comparisons which involve arrays comprising spatially distinct subgroups engage injective and surjective mappings in a simpler way. For example, each subarray of a more numerous array might also be more numerous than the subarray to which it corresponds (see the Two Part One-Way-Injective and-Surjective, TPO-I and -S, Series depicted in Figure 1). No more than a one-way composition of the two same-directional subarray comparisons is necessary for success: greater and greater yield greater. Also included within the TPO-I and -S series are other relatively uncomplicated types of comparisons of pairs of aggregates comprising visually corresponding subarrays. One subarray of the more numerous aggregate might equal the subarray to which it corresponds while the other subarray of the more numerous aggregate is greater than its correspondent (see the second TPO-S comparison in Figure 1). Here the comparison of the unequal pair informs the overall comparison: equal and greater yield greater.

Piaget et al. (1977) hypothesized decalage effects in performance on tasks reflecting the extent to which children understand functions and functions of functions. As mentioned earlier, the IP and SP comparisons embody elementary mappings and the TPO comparisons, one-way functional compositions of those mappings. Genevan theory implies that preoperational children should perform better on the IP and SP tasks than on corresponding TPO tasks, and that concrete operational children should perform better on the TPO tasks than preoperational children. Pilot data, however, suggest that the implied differences in performance are small. Perhaps the child, in order to succeed on the TPO tasks, needs only to center on cues that call for "greater than" responses. Unlike the TPR tasks, conflict deriving from the subarray relations is minimized, and, thus, preoperational children are expected to perform better on the TPO tasks than corresponding TPR tasks.

As in the TPR tasks, the capacity to coordinate functional relations is relevant to the child's performance in comparing pairs of arrays of the type depicted in the Injective Surjective (IS) series. Note that each of the examples of the IS pairs illustrated in Figure 1 does not comprise spatially distinct subgroups. However, in order to compare the arrays in the first IS pair depicted in Figure 1, the child must coordinate two-to-one and none-to-one mappings. Other IS pairs comprise arrays that are also irregularly matched one-to-one. For example, the second IS pair depicted in Figure 1 comprises two rows between which one or two none-to-one mappings obtain from the perspective of either row. Otherwise the arrays are matched one-to-one. Another IS pair not depicted comprises two rows between which one or two two-to-one mappings obtain from the perspective of either row, with the arrays otherwise matched one-to-one. In order to compare the latter two pairings, the child must coordinate countervailing none-to-one or two-to-one mappings.

Within the context of some comparison tasks, counting should constitute a more direct logical solution approach (cf. Saxe, 1979a) than the application of a logic in which the role of counting is minimized. Counting is an indexing operation. It entails the one-to-one correspondence of ordered number names to countable objects. The last number named serves as a summary representation of the cardinal value of the aggregate of counted objects. This representation of cardinal value can be compared with numbers representing the cardinal values of other aggregates. The comparison of cardinal numbers informs the comparison of the aggregates they represent. Because counting becomes routinized and systematic with development, it should constitute a cognitively efficient (Beilin, 1969) means of making numerical comparisons.

Pilot data indicate that the SCI, TPR, and IS series are the most difficult tasks. Their mastery is thought to require a concrete operational level of functioning. These tasks, therefore, constitute candidate conditions in which children who employ counting in comparing arrays may perform better than children who attempt to deduce the solution without the help of counting. It should be noted that in the TPR comparisons, the counter must iterate successive members of each array and, within the framework of the count, ignore the division of an array into subarrays. The need to compensate countervailing subarray comparisons is apparently minimized. In arriving at correct judgments in the SCI task, the counter needs to apply routinized counting procedures while ignoring conflicting length and density cues. Similarly, in the IS task the counter need not struggle with coordinating two- and none-to-one mappings but apply routinized counting skills across the peaks and valleys that make up the arrays.

Counting may, thus, serve as what Horn (1968) calls a "generalized solution aid." A generalized solution aid is a "technique which may be used to compensate for limitations in anlage capacities" (Horn, 1968, p. 244). The concept of a generalized solution aid was developed within the framework of Cattell and Horn's (Cattell, 1963; Horn, 1967, 1968; Horn & Cattell, 1966) theory of intelligence. Cattell and Horn hypothesized that a compensatory relationship may exist between fluid and crystallized abilities. Individuals whose limitations in fluid ability or anlage capacity (e.g., span of immediate memory) make certain problems potentially unsolvable might, through the application of appropriate crystallized skills (e.g., algebraic rules, mnemonic devices), find solutions. In contrast, the Genevans, as will be discussed later, hypothesize no such compensatory relationship between the achievements of learning in the strict sense and operativity.

In the context of the comparison tasks, counting and counting minimized logical approaches constitute alternative avenues to solutions. Preoperational children, while likely to have mastered counting, are, typically, unable to coordinate functional relations. Thus, there is some reason to believe that inducing preoperational children to count the elements of the arrays is more likely to improve performance on the SCI, IS, and TPR tasks than inducing them to approach the tasks from a functional-deductive standpoint.

Matching might also constitute a generalized solution aid. Inducing a child to consider pairwise correspondences between the appropriate members of two arrays might help him to determine the arrays' relative numerosity. If a child matches pairs of elements proceeding from left to right, as soon as he finds members of one array without correspondents in the other array he can infer that the array containing the unmatched elements is more numerous. If all the elements of the two arrays can be matched one-to-one, he can infer their equivalence. A principle underlying satisfactory performance is that the child persist in matching in one direction and ignore such configurational subtleties as conflicting length and density cues or the division of arrays into two parts.

Genevan theory, however, suggests that the degree to which counting or matching, relative to a functional-deductive approach, leads to improved performance on the comparison tasks is limited. This limitation must be considered within the framework of the Genevans' distinction between learning in the strict sense and operativity (Furth, 1969, 1974; Inhelder, Sinclair, & Bovet, 1974; Piaget, 1970b, 1971). Strict sense learning refers to information acquired through the impress of sheer physical experience. Operativity refers to

> the action aspect of intelligence at all periods, including sensory-motor intelligence. Operativity is the essential,

generalizable structuring aspect of intelligence. (Furth, 1969, p. 263)

The Genevans hold that strict sense learning is regulated by operative level. This regulation is, for example, manifest in the relation between counting and number conservation. Previously mentioned research indicates that the child's use of counting to read off the cardinal value of an array does not insure an understanding of number invariance. Once number conservation is attained, the child may use the cardinal values he abstracts from arrays in his explanation of the invariance property (e.g., "You put out eight then spread it out. You did not add or subtract any. It stays eight."). As for the static numerical comparisons, Genevan theory suggests that the configurational subtleties of the SCI, TPR, and IS tasks are likely to mislead the able counter whose thinking is preoperational. The preoperational child who employs matching in the context of the above tasks should similarly be misled. The concrete operational thinker who uses counting or matching is not expected to be misled and is predicted to be able to make accurate comparisons.

Static Quantitative Comparisons

It follows from the Genevans' treatment of functions that preoperational children should succeed at certain quantitative comparison tasks. For example, given two glasses of identical diameter, both containing liquid, judgments of relative quantity can be founded on a one-way mapping of height on to quantity. There is no need to enlist a compensatory understanding of the relation between dimensions. One dimension informs relative quantity.

By the same token, given that the heights of liquids contained in two glasses of different diameter are equal, judgments of relative quantity can be founded on the one-way mapping of diameter on to quantity. Again there is no need to enlist compensation. The diameter dimension informs relative quantity.

Height, however, may be a prepotent cue (P. H. Miller, 1973). Preoperational children may be more likely to succeed at comparison tasks which call for the one-way mapping of height on to quantity than at tasks which call for the one-way mapping of diameter on to quantity. Children who center on height might maintain that two quantities with the same height but different diameters are equal. Schwartz and Scholnick (1970) attempted to address this and a related issue.

Schwartz and Scholnick administered to children who ranged in age from 53 to 76 months a series of tasks which included nonverbal comparisons of discontinuous quantities and verbal and nonverbal tests of conservation of discontinuous quantity. In the nonverbal tasks the child indicated that a glass of "candies" belonging to a fictitious boy named Billy contained more than or the same amount as the interviewer's glass by pointing to a picture of a smiling face. The child indicated that Billy's glass contained fewer candies by pointing to a sad face. In one task, children were asked to compare glasses of candies of the same diameter. Ninety-five percent of the children succeeded at this task. Children tended to perform more poorly on a comparison task in which the heights of the candy contained in glasses of different diameter were equal. Both tasks were found to be easier than either the verbal or nonverbal tests of conservation. In another task, children were asked to compare equal quantities of candies that were contained in glasses of differing diameter. Results indicated that this comparison task was considerably more difficult than either of the conservation tasks.

There are three issues that seriously qualify these finding. First,

S. A. Miller (1976) reported a problem with Schwartz and Scholnick's nonverbal tasks which raises some doubt regarding the results. The tasks did not differentiate the judgment that Billy's glass has more than the interviewer's from the judgment that Billy's and the interviewer's glasses have the same amount. Consequently the child could have been credited with a correct judgment when his judgment was wrong. Secondly, half the trials of the "same diameter" task comprised comparisons in which the heights of the quantities to be compared were also equal. Although such comparisons merit status as "same height" trials, no trials in the "same height" task comprised comparisons in which corresponding heights and diameters were equal. This asymmetry probably made the "same diameter" task less difficult relative to the "same height" task. Finally, Schwartz and Scholnick did not differentiate important logical properties of the above mentioned comparison tasks. In the first two comparison tasks, the relevant cues logically inform the solution (taller = more or same height = same quantity; wider = more). In the third comparison task, there are no logical grounds that unambiguously determine that the quantities are equal. That one quantity is short and wide and the other, tall and thin does not logically guarantee the quantities' equality.

The one-way functional character of preoperational thought suggests that preoperational children will succeed at liquid comparison tasks that are solved through the one-way mapping of relative height ("same diameter" or SD task) and diameter ("same height" or SH task) on to quantity. In the SD task, the diameters of the containers holding two quantities of liquid are the same. The height of the liquids may or may not differ. Thus relative height maps on to relative quantity. It follows that the taller of two quantities of liquid is necessarily the greater. In the SH task, two quantities of liquid are equal in height. The diamaters of their containers may or may not differ. Relative diameter thus maps on to relative quantity. It follows that, of two quantities of liquid, the quantity in the fatter container is necessarily the greater of the two. However, because height appears to be strongly fixed as a cue which maps on to quantity (P. H. Miller, 1973) it is expected that some degree of decentering is necessary before diameter can be successfully mapped on to quantity. It is, therefore, expected that the decentering that attends the development of the concrete operations will enable the child to utilize diameter as an index of quantity.

A number of two-part quantitative comparison tasks will also be introduced. Two different colored liquids are used. Each liquid is contained in two glasses. That is to say, each quantity of liquid is divided into two subquantities. Each subquantity of a liquid of one color is placed near, and, hence, in visual correspondence with, a subquantity of the liquid of the other color.

The capacity to coordinate functional relations is relevant to the comparison of quantities that comprise discernibly different subquantities. Comparisons of such quantities may call for some cross-referencing of comparisons between corresponding subquantities. As in the case of the comparison of arrays comprising subarrays, the characteristic of concrete operational thought known as compensation may well enrich the child's capacity to coordinate liquid subquantity relations.

It is, therefore, expected that concrete operational children are more likely than preoperational children to succeed at two-part quantitative comparison tasks in which the following conditions hold: (1) visually corresponding subquantities are unequal; (2) the direction of the inequality between members of one pair of different colored subquantities is the reverse of the direction of the inequality between the members of the other pair (Two Part Reverse-Same Diameter or TPR-SD). Like the TPR numerical tasks, the TPR liquid task comprises comparisons of two quantities which require the coordination of countervailing one-way subquantity comparisons. Consider the subquantities that comprise the TPR-SD comparison illustrated in Figure 2. Let us call the left pairs G1 (light) and R1 (dark) and the right pairs G2 and R2. Note that, overall, G is greater than R. Also note that R1 is more than G1 and G2 is more than R2.

In order to determine which quantity, G or R, is greater, the child must coordinate the differences found in the subquantity comparisons. The child can do this by discerning a difference in how the corresponding pairs of subquantities differ. The absolute difference between the R1-G1 pair is discernibly less pronounced than that of the R2-G2 pair. Therefore, the direction of the difference between the R2-G2 pair informs the direction of the overall difference in the quantities. If R and G were equal the differences between the R1-G1 and R2-G2 pairs would exactly compensate each other. Moreover, noncorresponding subquantities of R and G would be equal: R1 = G2 and R2 = G1.

Other varieties of two-part quantitative comparison tasks engage one-way mappings in a simpler way. In the Two Part One-Way-Same Diameter (TPO-SD, see Figure 2) comparisons each subquantity of the greater total quantity is more than the corresponding subquantity of the lesser total quantity. As in the case of the TPO numerical comparisons, a one-way composition of two same-directional subquantity comparisons is needed for success at TPO-SD comparisons: greater and greater yield greater.

As mentioned in the section on static numerical comparisons, Piaget et al. (1977) hypothesized decalage effects in performance on tasks reflecting the extent to which children understand functions and functions of functions. While success at the SD task requires a one-way mapping of height on to quantity, success on the TPO-SD task requires one-way functional compositions of one-way mappings of height on to quantity. Genevan theory suggests that preoperational children ought to perform better on the SD task than on the TPO-SD task, and concrete operational children should perform better on the TPO-SD task than preoperational children. Genevan theory also suggests that preoperational children are better able to compose same-directional one-way subquantity comparisons than coordinate countervailing one-way subquantity comparisons. Therefore, preoperational children are expected to perform better on the TPO-SD task than on the TPR-SD task.

The TPO-SD comparison illustrated in Figure 2 typifies the TPO-SD comparisons investigated here. Note that R is greater than G. With regard to subquantity relations, R1 is greater than G1, and R2 is greater than G2; however, G2 is greater than R1, G2 and R1 being the interior subquantities. If a child, in comparing R and G, centered on the interior subquantities, and mapped the comparison of the interior subquantities on to the overall R-G comparison, he would compare the total quantities of R and G inaccurately. There is no feature in the TPO numerical tasks that parallels the problems of interior subquantities in the TPO liquid task. Insofar as some degree of cognitive decentering is required for adequate performance on the TPO-SD task, decalage effects are expected.

The effects of solution aids on performance on the static quantitative comparison tasks requires assessment. Although behavior relevant to comparing numerical quantity may not carry over to the comparison of liquid quantity, it is possible that a functional-deductive or matching orientation may be more useful than a counting orientation when comparing liquid quantity without the aid of a metric.

Instructional Set

Instructional set procedures were used in two studies. In Study 1 children were induced to inspect arrays without systematically counting elements. In Study 2 different instructional sets were used in order to pit the effects of inspection against those of counting and matching. Johnson wrote:

> The essence of set is a prepared adjustment. . . . In problem-solving experiments, where the problem situation is complex and several alternative response patterns are possible, establishment of the set means that the critical features of the situation have been identified (as a result of instructions and perhaps of a training series) and a response pattern integrated, so that the subject is prepared to respond quickly, without attending to other possibilities, when another problem is presented. (p. 162)

Woodworth (1937) described the influence of instructional set in terms of an "inner steer." The experimenter's instructions and the situational context of the subject's behaviors arouse within the subject an "inner steer" toward carrying out specific behaviors. Generally, verbal instructions, practice, and meaningful context have been thought to constitute major determinants of set (Gibson, 1941; Johnson, 1955, 1972; Woodworth, 1937). These three means of inducing set were used in concert in Studies 1 and 2.

In all set conditions children were introduced to Bert and Ernie finger puppets and three pairs of practice arrays. Each pair comprised green and red decals. The green decals were represented as Bert's candies and the red, Ernie's. Depending upon the condition to which they were assigned, children practiced inspecting arrays without counting elements, counting the elements in the arrays, or using a finger to match green and red decals. The children were asked to engage in these behaviors in order to "help Bert and Ernie" determine who had more candies or if both had the same amount. In the inspection condition the children were told that systematic counting would not help the puppets because neither Bert nor Ernie could count. The children were induced to use schemes other than counting. A popular scheme among children sampled in a pilot study was to locate those elements in one array to which no elements in the other array corresponded. The array which included the unmatched elements was then judged as more numerous.

Children in the counting set condition were told that Bert and Ernie like counting and that counting should be used to inform the puppets of the arrays' relative numerosity. In the matching set condition, as in the inspection set condition, children were told that counting would not be useful in helping the puppets learn of the arrays' relative numerosity. The children were explicitly told that Bert and Ernie like matching and that the systematic matching of elements, from left to right, should be employed.

The interviewer monitored the children's behavior during the administration of the static numerical comparison tasks. Children of the age levels sampled tend to be unable to engage in covert problem-solving behavior (Brainerd, 1973; Kohlberg, Yaeger, & Jhertholm, 1968; Vygotsky, 1962). If any child was observed engaging in set excluded behavior (e.g., counting to oneself in the inspection or matching set conditions) a standard prompt was administered. The prompt consisted of a reminder of Bert and

Ernie's needs (e.g., "Bert and Ernie don't know how to count. They compare candies by looking carefully.") and a request to engage in behavior that conformed to set instructions. The prompt, however, was rarely required.

Statement of the Problem

Ten hypotheses that pertain to the development of children's understanding of correspondences and one-way functions were tested in two studies. Study 1 addressed the first three hypotheses, and Study 2 addressed Hypotheses 4 to 10.

Study 1 was designed to investigate the child's emerging capacity to use corresondence based reasoning, unaided by counting, to compare arrays. Study 2 was designed for two principal purposes. First, it addressed the nature of the relationship between operative level and the child's capacity to use solution aids such as counting and matching in comparing arrays. Second it examined the development of the child's emerging capacity to use one-way function based reasoning in comparing quantities of liquid. An exploratory analysis of the relationship of operative level and solution aid effects on liquid task performance was also undertaken. Both Studies 1 and 2 provide data which are relevant to characterizing the cognitive capacities of preoperational children.

The following hypotheses were tested:

Study 1

1. It was expected that preoperational and concrete operational children would perform similarly on the IP and SP tasks since success at the tasks was hypothesized to require no more than a preoperational level of functioning. Moreover, preoperational children were expected to perform better on the IP and SP tasks than on the SCI task, reflecting an elementary understanding of injective and surjective correspondences.

2. It was expected that concrete operational children would perform better than preoperational children on the SCI, IS, TPR-I and TPR-S

tasks because success at these tasks was hypothesized to require a concrete operational level of functioning.

3. Since Genevan theory suggests that there should be decalage effects in performance on tasks reflecting an understanding of functions and functions of functions, preoperational children were expected to perform better on the IP and SP tasks than on corresponding TPO tasks, and concrete operational children were expected to perform better than preoperational children on the TPO tasks, although the differences in performance were expected to be mitigated somewhat by task structure. Genevan theory also suggests that preoperational children are better able to compose same-directional subarray comparisons than coordinate countervailing subarray comparisons; therefore, preoperational children were expected to perform better on the TPO tasks than on corresponding TPR tasks. A more adequate test of the decalage position can be found in Hypothesis 8.

Study 2

4. Genevan theory suggests that preoperational children who use counting or matching should not perform better on the static numerical comparison tasks than peers who inspect arrays. Cattell and Horn's theory suggests that counting and, perhaps, matching are solution aids that should enhance the preoperational child's performance on the static numerical comparison tasks. In order to help resolve these conflicting formulations, an exploratory investigation of the effects of three different set induced solution approaches--inspection, counting, and matching--on the performance of preoperational children on the most difficult static numerical comparison tasks was undertaken. It was, however, expected that concrete

operational children would perform better than preoperational children on each of the four static numerical comparison tasks administered in Study 2.

5. It was expected that some children who failed to conserve on a standard conservation of number test administered prior to the static numerical tasks would conserve on a standard test of number conservation administered just after the static numerical comparison tasks. The performance of these "improvers" would be contrasted with that of children whose performance on the conservation tests remained stable. Since the improvers are incipiently concrete operatonal, two results were expected: (a) Improvers would perform better on the static numerical comparison tasks than stable nonconservers; (b) Improvers would perform about as well on the tasks as stable conservers.

6. It was expected that concrete operatonal and preoperational children would perform similarly on the SD task since success at the task is thought to require no more than a preoperational level of functioning.

7. Because success on the SH task was thought to require some decentering away from the prepotent cue height, it was expected that concrete operatonal children would perform better than preoperational children on the SH task.

8. Since Genevan theory suggests that there should be decalage effects on performance on tasks reflecting an understanding of functions and functions of functions, preoperational children were expected to perform better on the SD task than on the TPO-SD task. Genevan theory also implies that preoperational children are better able to compose same-directional comparisons than coordinate countervailing comparisons;

therefore, preoperational children were expected to perform better on the TPO-SD task than on the TPR-SD task. Moreover, concrete operational children were believed to be better able to decenter away from the misleading interior subquantity comparisons, and were, therefore, expected to perform better than preoperational children on the TPO-SD task.

9. Concrete operational children were expected to perform better than preoperational children on the TPR-SD task because success at the task requires the coordination of countervailing subquantity comparisons, a hypothesized capacity of concrete operational functioning.

10. It was expected that some children who failed to conserve on a standard conservation of liquid test administered prior to the static quantitative comparison tasks would conserve on a standard conservation of liquid test administered just after the static quantitative comparison tasks. The performance of these improvers would be contrasted with that of the children whose performance on both liquid conservation tests remained stable. Since the improvers are incipiently concrete operational, two results were expected: (a) Improvers would perform better on the static quantitative comparison tasks than stable nonconservers; and (b) Improvers would perform about as well on the tasks as stable conservers.

Exploratory analysis. The effects of set condition on performance on the static quantitative comparison tasks was explored.

CHAPTER II

METHODOLOGY

STUDY 1

Method

Design

Each child in this study was initially administered the same instructional set, an inspection set. The purpose of the set was to induce the child to inspect pairs of arrays of "candies" in order to evaluate the relative numerosity of the members of each pair. Each child was then administered all the static numerical comparison tasks, counterbalanced for subgrouping (unitary arrays vs. arrays comprising subgroups) and hypothesized task difficulty. Four orders of administration were used: (1) IP, SP, SCI, IS, TPO-I, TPO-S, TPR-I, TPR-S; (2) SCI, IS, IP, SP, TPR-I, TPR-S, TPO-I, TPO-S; (3) TPO-I, TPO-S, TPR-I, TPR-S, IP, SP, SCI, IS; and (4) TPR-I, TPR-S, TPO-I, TPO-S, SCI, IS, IP, SP. After the series of comparison tasks was completed, each child was administered a conservation of number test. Performance on the conservation test was used as an index of operative level.

Subjects

A total of 64 children who ranged in age from 4 years, 0 months to 7 years, 5 months were included in the sample. The mean age was 5 years, 6 months. All children attended tuition-charging private schools. Approximately 90% of the children were white. An informal review of parents' occupations indicated that they were employed as physicians, college professors, computer programmers, and school teachers, among other middle-class occupations.

Materials

Sesame Street finger puppets named Bert and Ernie were used in both the static numerical comparison tests and the inspection set instructions. Each pair of arrays used in the comparison tasks and the instructional set condition consisted of a row of green and a row of red decals that had been pasted to a 15 x 4 inch white rectangular cardboard surface. The diameter of each decal was three-quarters of an inch. Diagrams depicting paired arrays can be found in Figure 1. Red and black checkers were used in the conservation of number test.

Procedure

Inspection Set

Each child was introduced to the Bert and Ernie puppets, and was told that each puppet received candy--Bert received green candy and Ernie, red candy--from his mother. Three pairs of arrays of red and green decals were represented as the candy. The rows of decals making up a pair of arrays were linear, equally dense, and, as far as possible, matched one-to-one from left to right. Upon the presentation of each pair of arrays, the child was asked to determine if Bert had more candy, if Ernie had more candy, or if both puppets had the same amount of candy. Each child was told that, since neither Bert nor Ernie knew how to count, he had to compare the members of each pair of arrays the way the puppets wanted, by careful inspection without using counting. The child viewed the following pairs of arrays: 5 red vs. 2 green; 3 red vs. 6 green;

4 red vs. 4 green.

Static Numerical Comparison Tasks

Every child was administered all eight static numerical comparison tasks. Each task consisted of seven comparisons (see Figure 1 for examples of task comparisons). In each comparison, an array of green decals, represented as Bert's candy, was to be compared with an array of red decals, represented as Ernie's candy. The cardinal values of the arrays ranged from 7 to 10. In every task but the SCI task, the child viewed two unequal arrays in each of five or six of the seven task comparisons. Two equal arrays were viewed in each of the one or two remaining comparisons. The child viewed two unequal arrays in every one of the seven SCI comparisons. The tasks involving the comparison of unitary arrays--that is, arrays that were not divided into subarrays (IP, SP, SCI, and IS)--were presented in a context in which the child was to help Bert and Ernie by determining if Bert got more candy, if Ernie got more candy, or if both puppets got the same amount of candy. All comparisons involving arrays that had been divided into subarrays (TPO-I and -S and TPR-I and -S) were introduced such that the left pair of red and green subarrays was represented as the candy the puppets received in the morning and the right pair of subarrays, as the candy they received in the afternoon. At the beginning of each subarray task, the child was asked to help Bert and Ernie by determining the relative numerosity of, first, the two subarrays that were represented as Bert and Ernie's morning candy and, then, the two subarrays that were represented as the puppets' afternoon candy. Data indicate that all children made within morning and within afternoon comparisons accurately. Next, each child was asked to determine the relative numerosity of the total amounts of green and red candy, i.e., the candy Bert and Ernie got for "the whole day." Any child who attempted to use counting to compare a pair of arrays was prompted to inspect the arrays.

Conservation of Number Test

The conservation of number test consisted of three trials. In the first trial eight red (black) checkers were placed in a row before the child. The interviewer then asked the child to remove from a bag "just as many black (red) checkers as there are red (black), one black (red) for each red (black)." Once a one-to-one correspondence between black and red checkers was established, the interviewer spread the row of red (black) checkers and then asked the child if there was still the same number of red as black checkers or if there were more red or more black. The child was then asked the reason for his judgment. The next trial began once the one-to-one correspondence was restored and the one-to-one correspondence between the rows, reestablished. During this trial the black (red) row was compressed, and the child was questioned. The order of the first two trials and the colors of the checkers that were expanded and compressed were counterbalanced. The third trial parallels the second except that the red (black) row was stacked to form a cylinder. Children who responded correctly and supplied adequate justification for their responses (e.g., reversibility, addition/subtraction, etc.) on at least two of the three trials were operationally defined as conservers of number. The criterion of two of three correct trials was employed in order to include within the conserver category children, who, due to unfamiliarity with the test, anxiety, etc., may have responded incorrectly on one trial but correctly on the other two. This criterion was employed in Study 2 as well as Study 1. The majority of children (85%) in Studies

1 and 2 who were classified as conservers responded correctly on all three conservation trials.

Children who failed to respond correctly on all trials and failed to place the red and black checkers in a one-to-one correspondence were operationally defined as level 1 nonconservers (level 1 NCs). Those children who responded incorrectly on every trial but spontaneously placed the red and black checkers in one-to-one correspondence were operationally defined as level 2 nonconservers (level 2 NCs). Four children who manifested mastery of one-to-one correspondence and responded correctly on one or more conservation trials without supplying adequate justification were classified as transitional but were too few to be included in the sample.

Consistent with the Genevan scheme for classifying children who range in age from 4 to 7 years, and for the purposes of testing Hypotheses 1-3, conservers were regarded as concrete operational, and nonconservers, as preoperational.

STUDY 2

Method

Design

The children in this study were interviewed twice no more than 4 days apart. At the beginning of the first interview session, each child was administered a counting task. Any child who counted inaccurately was not included in the sample. Approximately half of the children who were included in the sample were administered four static numerical comparison tasks during the first interview session and four static quantitative tasks during the second. For these children Session 1 consisted of, in order, the following: a number conservation test, one of three possible instructional sets, four static numerical tasks, and a re-presentation of the number conservation test; Session 2 consisted of the following: a liquid conservation test, a reminder of the instructional set administered the previous session, four static quantitative comparison tasks, and a re-presentation of the liquid conservation test. The order of administration was reversed for the remaining children but constrained such that the instructional set was always administered in Session 1, immediately after the liquid conservation test, and the reminder, in Session 2, immediately after the number conservation test. The children were randomly assigned to one of the three instructional set conditions: inspection, counting, and matching. The order in which the two groups of static comparison tasks were administered was counterbalanced for subgrouping (unitary arrays vs. arrays comprising subgroups; each of two colored liquids contained in one vs. two glasses). The four orders of task administration were: (1) number conservation (NC), SCI, IS, TPR-I, TPR-S, NC, liquid conservation (LC), SD, SH, TPO-SD, TPR-SD, LC; (2) NC, TPR-I, TPR-S, SCI, IS, NC, LC, TPO-SD, TPR-SD, SD, SH, LC; (3) LC, SD, SH, TPO-SD, TPR-SD, LC, NC, SCI, IS, TPO-I, TPR-I, NC; and (4) LC, TPO-SD, TPR-SD, SD, SH, LC, NC, TPR-I, TPR-S, SCI, IS, NC. Performance on the conservation tests was used as an index of operative level.

Subjects

A total of 107 children who ranged from 4 years, 0 months to 7 years, 11 months were interviewed in Study 2. The mean age was 5 years, 11 months. One child who was administered the numerical tasks in the first session was absent for the administration of the quantitative tasks in the second session. Five children who were administered the quantitative tasks in Session 1 were absent for the administration of the numerical tasks in Session 2. As in Study 1, the children attended tuition-charging private schools, and an informal review of parental occupations indicated that the parents were generally employed as professionals and business people. Approximately 90% of the children were white.

Materials

The materials used in the static numerical comparison tasks (see Figure 1), the instruction set conditions, and the conservation of number test were similar to those used in Study 1. A Sesame Street finger puppet named Grover and 10 black checkers were used in the counting task. A number of 5-inch-high clear plastic cylinders were used in the static quantitative comparison tasks. The cylinders had diameters of 2, 3 and 4 inches and held water that had been colored either red or green by food coloring. Diagrams depicting how the cylinders were used are provided in Figure 2. In the conservation of liquid task, two 4 x 3 inch clear plastic cylinders served a standard containers and one 4 x 4 inch cylinder, one 9 x 2 inch cylinder, and two 4 x 2 inch cylinders were employed as comparison containers.

Procedure

Counting Task

Each child was introduced to the Grover puppet. The child was then shown a row of 10 black checkers. The interviewer informed the child that the checkers were Grover's and asked the child to help Grover by counting the checkers. Two children counted inaccurately and were excluded fom the sample.

Instructional Set

Children assigned to all three instructional set conditions practiced comparing the same three pairs of arrays described in Study 1. Every child was told that the green decals were Bert's candies and the red, Ernie's. The inspection set was identical to the set administered in Study 1. Each child in the counting set condition was told that Bert and Ernie liked counting and the puppets wanted him to count the green and red candies in order to compare the practice arrays. Each child in the matching set condition was told that Bert and Ernie liked matching and that the puppets wanted him to match, as far as possible, the green and red candies one-to-one in order to compare the practice arrays. Any child who did not understand the set instructions was briefly shown how to perform in accordance with the set instructions. This was rare; however, some of the younger children in the matching condition required one demonstration.

Conservation of Number Test

The number conservation test that was administered before and after the static numerical comparison tasks was identical to the number conservation test employed in Study 1. Children who, on each number conservation test, responded correctly and supplied adequate justifications for their responses on at least two or three trials were operationally defined as conservers of number. Children who failed to respond correctly on all pre- and posttest trials and failed to place the red and black checkers in one-to-one correspondence on either test were operationally defined as level 1 nonconservers (level 1 NC). Children who responded incorrectly on every pre- and posttest trial but placed red and black checkers in one-to-one correspondence on both conservation of number tests were operationally defined as level 2 nonconservers (level 2 NC). If any child responded incorrectly on all pretest trials but correctly, and with justification, on at least two of three posttest trials, he would have been operationally defined as an improver. No child whose behavior conformed to this pattern was found. One child who evinced mastery of one-to-one correspondence on both conservation tests and responded correctly on all pre- and posttest trials without supplying an adequate justification for his responses was classified as transitional. Since there was only one transitional child, the category was excluded from the analysis.

Consistent with the Genevan scheme for classifying children who range in age from 4 to 7 years, and for the purpose of testing Hypothesis 4, conservers of number were regarded as concrete operational, and nonconservers, as preoperational.

Static Numerical Comparison Tasks

As in Study 1, each child who was administered the static numerical comparison tasks was asked to compare arrays of green and red decals. The numerical tasks selected for Study 2 were the four most difficult Study 1 tasks--the SCI, IS, and TPR-I and -S tasks. Each numerical task selected for Study 2 consisted of four comparisons (see Figure 1 for examples of task comparisons). The pairs of arrays that were used in the Study 2 numerical tasks were chosen on the basis of pilot and Study 1 data. These data indicated that the two arrays within each of the selected pairs were either moderately difficult or the most difficult arrays to compare. The child viewed two unequal arrays in each of three IS, two TPR-I, and two TPR-S comparisons. Two equal arrays were presented in each of the remaining IS and TPR-I and -S comparisons. The child viewed two unequal arrays in every one of the four SCI comparisons. Any child whose behavior failed to conform to set instructions was prompted to inspect, count, or match accordingly.

Conservation of Liquid Test

A conservation of liquid test was administered before and after the series of static quantitative comparison tasks. Each test consisted of three trials. In the first trial the child was shown the two 4 x 3 inch standard containers holding equal amounts of red and green "juice." The standard containers were half filled. Once the child was satisfied that there was as much red as green liquid, the 4 x 4 inch comparison container was presented and the interviewer poured all the red (green) liquid into it. The child was then asked, "Is there still the same amount of red as green juice or is there more red or is there more green?" and the reason for his judgment. The next trial began when, with the child's agreement, equal amounts of red and green liquid were placed in the two standard containers. During this trial the green (red) liquid was poured into the 9 x 2 inch container, and the child was questioned. The order of the first two trials and the color of the liquid that was poured were counterbalanced. The third trial paralleled the second except that the red (green) liquid was poured into the two 4 x 2 inch comparison containers. Children who on each liquid conservation test responded correctly and supplied adequate justification for their responses on at least two of the three trials were operationally defined as conservers of liquid. An improver was defined as a child who failed to respond correctly on all three pretest trials but who responded correctly and supplied adequate justification for his correct responses on at least two of the three posttest trials. Since the performance of only two children conformed to this pattern, the category of improver was excluded from the analysis. Children who responded incorrectly on all pre- and posttest trials were operationally defined as nonconservers.

Consistent with the Genevan scheme for classifying children who range in age from 4 to 7 years, and for the purpose of testing Hypotheses 6-9, conservers of liquid were regarded as concrete operational, and nonconservers, as preoperational.

Static Quantitative Comparison Tasks

Every child who was administered the tasks was told that each puppet got juice--that Bert got green juice and Ernie, red juice--from his mother. The child was then asked to help the puppets by comparing Bert and Ernie's juice and was administered the SD, SH, TPO-SD, and TPR-SD tasks. Each task consisted of four comparisons (see Figure 2 for examples of task comparisons). In every task the child viewed two unequal quantities of liquid in each of three of the four task comparisons and two equal quantities in the remaining comparison. In the context of the tasks in which liquid of each color was not divided into subquantities (SD and SH), the child was asked to determine if Bert got more juice, if Ernie got more juice, or if the puppets got the same amount of juice. In the tasks in which each liquid was divided into subquantities (TPO-SD and TPR-SD), the red and green liquid held in containers on the left was represented as the juice the puppets got in the morning. The juice held in the containers on the right was represented as the afternoon juice. For each comparison in the subquantity tasks, every child initially compared the two subquantities that were represented as the morning juice. Then the child compared the two subquantities that were represented as the afternoon juice. Data indicate that all children made these comparisons accurately. Next each child compared the total amount of green and red juice, i.e., the juice that Bert and Ernie got for "the whole day."

Reminder of the Instructional Set

On Session 2, just prior to the administration of the second group of comparison tasks, children were reminded of the set instructions administered at the beginning of Session 1. The interviewer showed the child the Bert and Ernie puppets, and asked the child how the puppets compared candies. If the child responded incorrectly, the interviewer repeated the set instructions. Every child then practiced comparing the same three pairs of arrays viewed in the instructional set phase of Session 1.

CHAPTER III

RESULTS

The results are presented in three sections. The first section reports findings relevant to general characteristics of the data. The second section reports findings relevant to each of the ten hypotheses presented in Chapter I. The third section provides anecdotal data relevant to the hypotheses.

General Characteristics of the Data

Correlations among Tasks

Correlations among Study 1 tasks are presented in Table 1. All correlations except the correlation between the IP and SCI tasks were statistically significant. The TPO-I and -S tasks were the most highly correlated tasks (r = .77). Most between task correlations, however, were moderate. The median between task correlation was .445. Study 1 tasks were, in general, moderately but significantly correlated with age. The SP task had the smallest correlation with age (r = .34) and the SCI task, the largest (r = .67). The median task-age correlation was .565.

Correlations between Study 2 tasks are presented in Table 2. The SD task was the only task that failed to correlate significantly with other tasks. It correlated significantly with only one other task, the TPR-SD task; however, the correlation was low (r = .20). The SCI and IS tasks were the most highly correlated tasks (r = .68) but most of the between task correlations were moderate. The median between task correlation was .455. Study 2 tasks were, in general, moderately but significantly correlated with age. The SD task had the smallest

Table 1

Zero Order Correlations among Study 1 Tasks

Tasks	1	2	3	4	5	6	7	8
1. IP								
2. SP	.45**							
3. SCI	.22+	.44**						
4. IS	.41**	.53**	.59**					
5. TPO-I	.43**	.55**	.47**	.57**				
6. TPO-S	.47**	.66**	.41**	.48**	.77**			
7. TPR-I	.33**	.33**	.45**	.46**	.49**	.40**		
8. TPR-S	.27*	.29*	.37**	.41**	.42**	.45**	.28*	
9. AGE	.34**	.56**	.67**	.63**	.60**	.52**	.57**	.42**

Note. Statistical tests were two-tailed.

$^{+}p < .10$

$^{*}p < .05$

$^{**}p < .01$

Table 2

Zero Order Correlations among Study 2 Tasks

Tasks	1	2	3	4	5	6	7	8
1. SCI								
2. IS	.68**							
3. TPR-I	.67**	.55**						
4. TPR-S	.54**	.51**	.64**					
5. SD	.08	.06	.07	.07				
6. SH	.46**	.39**	.46**	.46**	.14			
7. TPO-SD	.46**	.55**	.28**	.14	.11	.46**		
8. TPR-SD	.46**	.54**	.46**	.42**	.20*	.42**	.46**	
9. AGE	.73**	.63**	.54**	.48**	.22*	.50**	.54**	.62**

Note. Statistical tests were two-tailed.

*p < .05

**p < .01

correlation with age (r = .22) and the SCI task, the largest (r = .73). The median correlation with age was .54.

Order Effects

The effects of task order were analyzed for Studies 1 and 2. For each study a multivariate analysis of variance (MANOVA) was performed because the analysis of order effects was exploratory and no hypotheses regarding order effects were made. The test statistic obtained in each MANOVA was either the Hotelling T^2 statistic, or the generalized F statistic, F_o (Bock, 1975). The Hotelling T^2 statistic, along with its two distribution parameters (appearing in parenthesis), is reported when the hypothesis being tested places on restriction on the multivariate general linear model.[1] The generalized F statistic, along with its three distribution parameters (appearing in parenthesis), is reported when the hypothesis being tested places more than one restriction on the model.[2] The Hotelling T^2 statistic as well as the widely used univariate F statistic is a special case of the generalized F statistic.

The effects of the two major task orders used in Study 1 were analyzed. In the first major task order, the static numerical tasks in which arrays did not comprise spatially distinct subgroups (IP, SP,

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[1]The two distribution parameters of the Hotelling T^2 statistic are computed as follows: the first distribution parameter is equal to the number of dependent variables; the second distribution parameter is equal to the total number of subjects minus the number of dependent variables minus one (Bock, 1975).

[2]The three distribution parameters of the generalized F statistic--s, m, and n--are computed as follows: s = min(r - 1, NDV); m = (|r - NDV| - 1)/2; n = (NSUBJ - r - NDV - 1)/2. The symbol r represents the number of restrictions the hypothesis places on the model, NDV, the number of dependent variables, and NSUBJ, the number of subjects included in the sample. A full description of the distribution parameters is provided by Bock (1975).

SCI, and IS) were administered before the tasks in which arrays comprised spatially distinct subgroups (TPO-I and -S and TPR-I and -S). The sequence was reversed in the second major task order. A MANOVA was performed on each set of four tasks with subjects stratified by operative level. On the four tasks in which arrays did not comprise subgroups, the order main effect, $\underline{T}^2(4, 59) = 3.88$, and the operative level by order interaction, $\underline{F}_o$ (2.00, .50, 26.50) = 2.14, were not significant. On the four tasks in which arrays did comprise subgroups, the order main effect, $\underline{T}^2(4, 59) = 2.59$, and the operative level by order interaction, $\underline{F}_o$(2.00, .50, 26.50) = 2.41, were also not significant.

The effects of the two major task orders used in Study 2 were also analyzed. In the first major task order, the static numerical tasks (SCI, IS, and TPR-I and -S) were administered before the static quantitative tasks (SD, SH, TPO-SD, and TPR-SD). The sequence was reversed in the second major task order. A MANOVA was performed on each set of four tasks with subjects stratified by the categorical scheme created by the crossing of operative level and set. The order main effect, $\underline{T}^2(4, 100) = 1.99$, on performance on the four numerical tasks was not significant. In addition, none of the interaction effects was significant: operative level by order, $\underline{F}_o$(2.00, .50, 41.00) = 3.78; set by order, $\underline{F}_o$(2.00, .50, 41.00) = 4.17; operative level by set by order, $\underline{F}_o$(4.00, -.50, 41.00) = 2.83.

The order main effect on the four quantitative tasks, $\underline{T}^2(4, 95) = 5.11$, was not significant. Two of the three interaction effects were not significant: set by order, $\underline{F}_o$(2.00, .50, 41.50) = 2.17; operative level by set by order, $\underline{F}_o$(2.00, .50, 41.50) = 4.06. Only the operative level by order interaction was significant, $\underline{T}^2(4, 95) = 18.64$, $\underline{p} < .01$.

Inspection of the scaled discriminant function weights indicates that the SH task largely contributed to the interaction. For conservers of liquid, when the set of quantitative tasks was administered prior to the set of numerical tasks, the mean number of correct responses on the SH task was 3.67. When the order was reversed, conservers did not perform as well, $\underline{M} = 2.47$. Nonconservers, in contrast, performed about the same within both orders (quantitative tasks first: $\underline{M} = 1.27$; numerical tasks first: $\underline{M} = 1.40$).

Tests of Hypotheses

Hypothesis 1: Performance on the IP and SP Tasks

Preoperational children were expected to perform as well as concrete operational children on the IP and SP tasks. In addition, preoperational children were expected to perform better on the IP and SP tasks than on the SCI task. The mean number of correct responses on each of tasks is presented in Table 3. Throughout Study 1, performance on the number conservation test was used as an index of operative level, and the one-way analysis of variance (ANOVA) was used to test operative level effects. However, where within operative level task specific differences were investigated, the correlated $\underline{t}$-test was employed.

In order to determine if children's performance on the IP and SP tasks was related to operative level, a one-way ANOVA was performed for each of the tasks. Each ANOVA revealed a significant operative level effect: IP, $\underline{F}(2, 61) = 9.12$, $\underline{p} < .001$; SP, $\underline{F}(2, 61) = 11.08$, $\underline{p} < .001$. A Scheffe post hoc test (Winer, 1971) was performed after each of the analyses. This test is exact for unequal group size. Here and throughout the results section, whenever the Scheffe post hoc test was employed <u>only</u> statistically significant results are reported.

Table 3

Mean Number of Correct Judgments on Each of the Study 1 Tasks[a]

	IP	SP	SCI	IS	TPO-I	TPO-S	TPR-I	TPR-S	_n_
Level 1 NC	4.71	3.57	.43	2.29	3.86	3.36	2.71	2.79	14
Level 2 NC	6.12	5.19	.88	3.04	4.50	4.62	3.31	3.65	26
Conserver	6.88	6.29	3.17	4.38	6.29	6.25	3.83	4.33	24
Task Mean	6.09	5.25	1.64	3.38	5.03	4.95	3.38	3.72	

[a] Maximum number correct = 7.

Scheffe post hoc tests indicated that both conservers ($p < .01$) and level 2 NCs ($p < .05$) performed significantly better than level 1 NCs on the IP task. On the SP task, post hoc tests also revealed that conservers ($p < .01$) and level 2 NCs ($p < .05$) performed significantly better than level 1 NCs.

It was expected that preoperational children would not differ in IP and SP task performance from concrete operational children. Although this expectation was not supported, other analyses support the view that, to some extent, success on the IP and SP tasks is accessible to preoperational children. One-tailed correlated t-tests were performed to contrast the performance of preoperational children on the IP and SP tasks with their performance on the SCI task. Preoperational thinking is believed to be a greater impediment to success on the SCI task than on either the IP or SP tasks. The correlated t-tests revealed that level 1 NCs performed significantly better on the IP, $t(13) = 5.94$, $p < .001$, and SP, $t(13) = 4.48$, $p < .001$, tasks than on the SCI task. Correlated t-tests also revealed that level 2 NCs performed significantly better on the IP, $t(25) = 13.30$, $p < .001$, and SP, $t(25) = 12.16$, $p < .001$, tasks than on the SCI task.

Additional evidence indicated that nonconservers tended to judge two arrays that were equal in length but different in density (SCI comparisons) to have the same amount. This tendency was almost absent when nonconservers compared two arrays that had the same length but were related by either an injective (IP) or surjective (SP) correspondence. Eleven of 14 level 1 NCs (79%) and 15 of 26 level 2 NCs (58%) judged the two arrays making up an SCI comparison to have the same amount of candies in every one of the seven SCI comparisons. In contrast, only one of 14 level 1 NCs (7%) and no level 2 NC judged every one of the seven IP comparisons to involve the same amount. Two of the 14 level 1 NCs (14%) and no level 2 NC judged every one of the seven SP comparisons to involve the same amount.

Conservers' performance on the SCI task contrasts with that of nonconservers. Only four of 24 conservers (17%) judged every SCI comparison to involve the same amount. With regard to the IP and SP tasks, nonconservers and conservers, in an important respect, performed similarly. Like the nonconservers, the conservers manifested little tendency to judge two unequal arrays that were equal in length, but related by an injective or surjective correspondence, to have the same amount of candies. No conserver judged every one of the seven IP comparisons or every one of the seven SP comparisons to involve the same amount.

Hypothesis 2: Effect of Operative Level on Performance on the SCI, IS, and TPR-I and -S Tasks

It was expected that performance on the SCI, IS, and TPR-I and -S tasks would be directly related to the child's operative level. The mean number of correct responses on these tasks is presented in Table 3. A one-way ANOVA was performed on each of the tasks. The analyses revealed significant effects for operative level on each of the tasks: SCI, $F(2, 61) = 13.43$, $p < .001$, IS, $F(2, 61) = 10.26$, $p < .001$; TPR-I, $F(2, 61) = 4.54$, $p < .05$; TPR-S, $F(2, 61) = 7.12$, $p < .01$.

The results of Scheffe post hoc tests underlined the superiority of the conservers' performance. On the SCI and IS tasks conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs

($p < .01$). Conservers performed significantly better than level 1 NCs on the TPR-I ($p < .05$) and -S tasks ($p < .01$).

Hypothesis 3: Performance on the TPO Tasks

Genevan theory suggests that concrete operational children should perform better than preoperational children on the TPO-I and -S tasks. The mean number of correct responses on each of these tasks is provided in Table 3. A one-way ANOVA was performed for each of the tasks. Each ANOVA revealed a significant operative level effect: TPO-I, $F(2, 61) = 12.90$, $p < .001$, TPO-S, $F(2, 61) = 12.56$, $p < .001$. Scheffe post hoc tests revealed that conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$) on the TPO-I task. Post hoc tests also indicated that on the TPO-S task conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$).

Preoperational children were expected to perform better on the TPO tasks than on the TPR tasks. One-tailed correlated t-tests indicated that level 1 NCs performed significantly better on the TPO-I task than on the TPR-I task, $t(13) = 2.58$, $p < .05$, but that their performance on the TPO-S and TPR-S tasks did not differ significantly, $t(13) = .94$. Correlated t-tests also revealed that level 2 NCs performed significantly better on the TPO-I task than on the TPR-I task, $t(25) = 3.49$, $p < .01$, and that they performed better on the TPO-S task than on the TPR-S task $t(25) = 2.91$, $p < .01$.

Preoperational children were also expected to perform better on the IP and SP tasks than on corresponding TPO tasks. One-tailed correlated t-tests indicated that for level 1 NCs the difference in performance on the IP and TPO-I tasks, $t(13) = 1.17$, was nonsignificant, and the difference in the performance on the SP and TPO-S tasks, $t(13) = .42$, was also nonsignificant. Level 2 NCs, however, performed significantly better on the IP task than on the TPO-I task, $t(25) = 4.34$, $p < .001$. The difference in level 2 NCs' performance on the SP and TPO-S tests, $t(25) = 1.68$, was not significant.

Hypothesis 4: Operative Level Structures the Influence of Set on the Child's Performance on the Static Numerical Tasks

With performance on the number conservation tests as an index of operative level, it was expected that set effects would not influence the performance of preoperational children on the static numerical tasks administered in Study 2. It was, however, expected that set would affect the performance of concrete operational children. In order to test these hypotheses, the simple main effects of set were examined at each operative level. Testing for every simple main effect of set is tantamount to testing for the set main effect and the operative level by set interaction. The variation owed to the simple main effects of set at each of the two nonconserver levels and the conserver level is redundant with the variation owed to the set main effect and the operative level by set interaction (Winer, 1971). The mean number of correct responses on each of the numerical tasks administered in Study 2, cross-classified by operative level and set, is presented in Table 4.

Analyses of variance indicated that on each numerical task no simple main effect of set at level 1 NC was significant: SCI, $F(2, 96) = .41$; IS, $F(2, 96) = .79$; TPR-I, $F(2, 96) = .05$; TPR-S, $F(2, 96) = 1.98$. Significant simple main effects of set were evident at more advanced operative

Table 4

Mean Number of Correct Judgments on Each of the Study 2 Static Numerical Comparison Tasks[a]

	Number Conservation Level	Inspection	Counting	Matching	Operative Level Mean	n
SCI	Level 1 NC	.12	.50	.00	.22	23
	Level 2 NC	.33	2.77	1.64	1.62	39
	Conserver	2.93	4.00	3.44	3.40	43
IS	Level 1 NC	1.62	1.00	1.14	1.26	23
	Level 2 NC	1.50	2.46	1.79	1.92	39
	Conserver	3.20	3.60	3.22	3.30	43
TPR-I	Level 1 NC	1.12	1.25	1.29	1.22	23
	Level 2 NC	1.17	2.62	1.36	1.72	39
	Conserver	2.27	3.70	2.67	2.72	43
TPR-S	Level 1 NC	.50	1.50	1.43	1.13	23
	Level 2 NC	1.67	2.00	1.36	1.67	39
	Conserver	2.27	3.60	2.83	2.81	43

[a] Maximum number correct = 4

levels: SCI, level 2 NCs, $F(2, 96) = 14.63$, $p < .001$; TPR-I, level 2 NCs, $F(2, 96) = 7.07$, $p < .01$; TPR-I, conservers, $F(2, 96) = 5.57$, $p < .01$; TPR-S, conservers, $F(2, 96) = 4.28$, $p < .05$.

Scheffe post hoc tests were performed whenever significant simple main effects were found. In the context of the SCI task, level 2 NCs assigned to the counting condition performed significantly better than their counterparts in the matching ($p < .05$) and inspection ($p < .01$) conditions. Moreover, level 2 NCs assigned to the matching condition performed significantly better than their counterparts in the inspection condition ($p < .05$). On the TPR-I task, level 2 NCs assigned to the counting condition performed significantly better than their counterparts in either the inspection ($p < .01$) or the matching ($p < .05$) conditions. On the same task, conservers assigned to the counting condition performed significantly better than their counterparts in the inspection ($p < .01$) condition. In the context of the TPR-S task, conservers assigned to the counting condition performed significantly better than their counterparts in the inspection condition ($p < .05$).

Simple main effects of set were sometimes not significant at advanced operative levels: IS, conservers, $F(2, 96) = .55$; TPR-S, level 2 NCs, $F(2, 96) = 1.11$. The simple main effects of set for conservers performing on the SCI task, $F(2, 96) = 2.75$, $p < .10$, and level 2 NCs performing on the IS task, $F(2, 96) = 2.89$, $p < .10$, were marginally significant.

It was also expected that across all set conditions the performance on the four numerical tasks administered in Study 2 would be directly related to operative level. Two-way ANOVAs with subjects stratified by set condition revealed highly significant operative level effects on all four tasks: SCI, $F(2, 96) = 66.38$, $p < .001$; IS, $F(2, 96) = 35.31$,

$p < .001$; TPR-I, $F(2, 96) = 21.14$, $p < .001$; TPR-S, $F(2, 96) = 21.53$, $p < .001$; TPR-S, $F(2, 96) = 21.53$, $p < .001$.

The Scheffe post hoc test was performed after each ANOVA. On the SCI task, conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$), and level 2 NCs performed significantly better than level 1 NCs ($p < .01$). Similarly, on the IS task, conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$). Conservers performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$) on the TPR-I task. On the TPR-S task, conservers also performed significantly better than level 1 ($p < .01$) and level 2 NCs ($p < .01$).

Hypothesis 5: The Performance of Improvers on the Static Numerical Tasks

With regard to number conversation, an improver was defined as a child who, although failing to conserve on the number conservation test administered prior to the static numerical tasks, conserved on the post-test. The performance of improvers on the static numerical tasks was expected: (a) to be superior to the performance of stable nonconservers and (b) to resemble the performance of stable conservers. This hypothesis could not be tested because no child among the 105 children who were included in the sample could be classified as an improver.

Hypothesis 6: The Effect of Operative Level on Performance on the SD Task

It was expected that preoperational and concrete operational children would perform similarly on the SD task. The mean number of correct responses on the SD task is presented in Table 5. A two-way ANOVA with subjects stratified by the set condition to which they were assigned was

Table 5

Mean Number of Correct Judgments on Each of the Study 2 Static Quantitative Comparison Tasks[a]

	Liquid Conservation Level	Inspection	Counting	Matching	Operative Level Mean	n
SD	Nonconserver	3.90	3.70	3.85	3.82	66
	Conserver	3.85	4.00	4.00	3.94	34
SH	Nonconserver	1.50	1.10	1.35	1.29	66
	Conserver	2.54	3.60	3.00	3.00	34
TPO-SD	Nonconserver	2.30	2.40	2.42	2.38	66
	Conserver	3.85	3.60	3.64	3.71	34
TPR-SD	Nonconserver	1.60	1.75	1.96	1.79	66
	Conserver	3.46	3.10	3.09	3.24	34

[a] Maximum number correct = 4

performed to assess the effect of operative level. The analysis revealed no significant operative level effect: $F(1, 94) = 2.46$.

Hypothesis 7: Performance on the SH Task

It was expected that performance on the SH task would be directly related to the child's operative level. The mean number of correct responses on this task is presented in Table 5. A two-way ANOVA with subjects stratified by set condition revealed a highly significant operative level effect, $F(1, 94) = 66.14$, $p < .001$.

Additional evidence indicated that liquid conservers' performance on the SH task differed from that of nonconservers in an important respect. Nonconservers, to a much greater extent than conservers, tended to judge two quantities of liquid that were equal in height to be the same regardless of the relative diameters of the cylinders containing them. A total of 44 of 66 nonconservers (67%) judged every one of the four same-height comparisons to be the same amount. In contrast, only nine of a total of 34 conservers (26%) judged the SH comparisons in the same manner.

Hypothesis 8: Performance on the TPO-SD Task

It was expected that concrete operational children would perform better than preoperational children on the TPO-SD task. The mean number of correct responses on this task is presented in Table 5. A two-way ANOVA with subjects stratified by set condition revealed a highly significant operative level effect, $F(1, 94) = 29.73$, $p < .001$.

It was also expected that preoperational children would perform better on the TPO-SD task than on the TPR-SD task. In order to contrast the performance of preoperational children on the TPO-SD and TPR-SD tasks, a repeated measures ANOVA with nonconservers stratified by set condition was performed. The ANOVA indicated that nonconservers performed significantly better on the TPO-SD task, $F(1, 63) = 11.04$, $p < .01$.

Preoperational children were also expected to perform better on the SD task than on the TPO-SD task. A repeated measures ANOVA with nonconservers stratified by set condition indicated that nonconservers' SD task performance was significantly better than their TPO-SD task performance, $F(1, 63) = 73.72$, $p < .001$.

Hypothesis 9: Performance on the TPR-SD Task

It was expected that performance on the TPR-SD task is directly related to the child's operative level. The mean number of correct responses on this task is presented in Table 5. A two-way ANOVA with subjects stratified by set condition revealed a highly significant operative level effect, $F(1, 94) = 40.15$, $p < .001$.

Hypothesis 10: The Performance of Improvers on the Static Quantitative Tasks

With regard to conservation of liquid, an improver was defined as a child who, although failing to conserve on the liquid conservation test administered prior to the static quantitative tasks, conserved on the posttest. The performance of improvers on the static quantitative tasks was expected: (a) to be superior to the performance of stable nonconservers and (b) to resemble the performance of stable conservers. This hypothesis could not be tested because only two children of the 102 children who were administered the conservation of liquid pre- and posttests could be classified as improvers.

Exploratory Analysis: Operative Level and the Influence of Set on the Child's Performance on the Static Quantitative Tasks

Although no predictions were made regarding the relation of operative level to the influence of set on performance on the static quantitative tasks, tests of the simple main effects of set were carried out. The mean number of correct responses on each of the quantitative tasks administered in Study 2, cross-classified by operative level and set, is presented in Table 5. Analyses of variance indicated that on each quantitative task no simple main effect of set at the nonconserver level was significant: SD, $F(2, 94) = 1.37$; SH, $F(2, 94) = .81$; TPO-SD, $F(2, 94) = .07$; TPR-SD, $F(2, 94) = .65$. The simple main effect of set for conservers performing on the SH task was significant, $F(2, 94) = 3.18$, $p < .05$. Among conservers, children assigned to the counting condition manifested the highest mean SH score. Scheffe post hoc comparisons, however, did not indicate that counters performed significantly better than children assigned to the other conditions. Otherwise the simple main effects of set were not significant at the conserver level: SD, $F(2, 94) = .56$; TPO-SD, $F(2, 94) = .17$; TPR-SD, $F(2, 94) = .46$.

Anecdotal Record of Children's Behavior During Task Administration

The purpose of this section is to provide ancillary, anecdotal data that further elucidate the hypotheses. Although the anecdotal data do not purport to be complete, these data are, nevertheless, informative because they depict salient aspects of the children's performance.

During Studies 1 and 2 children were not encouraged to discuss their responses while performing the tasks. However, the few spontaneous comments that were uttered by children were recorded. Children in a pilot study performed prior to Studies 1 and 2 were sometimes asked to comment on how they compared arrays and these comments were recorded. Whenever children made their comparison strategies manifest the interviewer recorded those strategies.

During the administration of the IP and SP tasks in Study 1, a conserver who, in conformity with Study 1 inspection set instructions, looked at the arrays without counting decals spontaneously reported, in referring to each of several correct responses, "I can tell it's more." Another conserver spontaneously moved her finger in such a way as to establish, as far as could be established, a one-to-one correspondence between green and red decals before responding on each of the Study 1 IP and SCI comparisons. She never responded incorrectly on either of the two tasks. A level 2 nonconserver in the pilot study who responded correctly on most of the IP and TPO-I comparisons indicated that he knew which array had more because "candies were missing," i.e., there were gaps in the less numerous row. These are the result of the none-to-one correspondences that mark the injective tasks. He responded correctly on most SP and TPO-S comparisons and justified his responses by indicating that the second decals in the two-to-one correspondences were "extra candies." Two-to-one correspondences mark the surjective tasks.

A number of children spontaneously commented on the first IS comparison. In the first IS comparison, which is depicted in Figure 1, each child viewed two arrays that were numerically equal. Of the seven green and seven red decals that were viewed, five green and five red were matched one-to-one. No red decal corresponded to the centermost (fourth from the left) green decal. It appeared as a gap in the red array. Two red decals corresponded to one green decal in the second position from the left. A number of children in the pilot sample, the Study 1 sample, and the Study 2 matching and inspecting samples reported to have accurately compared the arrays by mentally moving ("in my mind," "I moved this one in my head") the upper one of the two red decals that corresponded to the single green decal into the gap in the red array. In that way a mental representation of a one-to-one correspondence could be established (cf. Beilin, 1969).

In Study 2 level 1 NCs assigned to the counting condition tended to employ counterproductive counting strategies on the numerical tasks. Many of these children included all the green and red decals on a card in the same count. They counted every decal of one color, then increased that total by counting every decal of the other color. Finally they arrived at the total number of decals on a card, a number that was not relevant to ascertaining the relative numerosity of the arrays. During the TPR-I and -S tasks, a small number of children counted the total number of decals by, first, counting the total number of green and red morning decals, then, continuing the count with the afternoon decals. It is interesting to note that counting accuracy was not a problem. All the children who attempted to count the total number of decals counted accurately or missed arriving at the total number of decals on a card by no more than one or two. The children's inappropriate counting strategies, not counting accuracy, led to poor performances (see Saxe, 1977).

Other level 1 NCs, as well as some young level 2 NCs, employed, in contrast to the above children, more adequate counting strategies, yet performed poorly on the numerical tasks. These children accurately counted the seven green decals and the nine red decals in the first SCI comparison depicted in Figure 1, yet indicated that there was the same amount of green as red, i.e., that Bert and Ernie got the same amount of candy. That is, they responded as if the relative length of the arrays--a pair of SCI arrays had the same length but different density--was the sole index of relative numerosity.

Level 1 NCs assigned to the matching condition committed comparable errors. On the SCI task, some of these children matched green and red decals one-to-one until there were no more decals in the less numerous array to match to the remaining decals in the more numerous array. For example, in the first SCI comparison depicted in Figure 1, the children matched green-red, green-red, . . . until the green decals were exhausted and two red decals remained unmatched. Instead of indicating that the array that contained unmatched decals was the more numerous array, the children indicated that there was the same amount of red and green decals. The children responded as if the relative length of the arrays was a more appropriate index of relative numerosity than the finding of unmatched decals. Study 2 children who conserved number tended to use counting and matching to compare SCI arrays accurately. Comparable children in the inspection condition also performed well on the SCI task. During an SCI comparison a conserver in the inspection condition commented that the decals in the less numerous array were "further away" from each other, i.e., less dense than the decals in the more numerous array.

Many children, including conservers, manifested some difficulty in applying matching to the IS and the TPR-I and -S tasks. In the IS task, children sometimes ignored two-to-one correspondences. They tended to match, as in the first IS comparison, the two red decals that faced a single green decal to that green decal. They failed to match the upper red decal to the next available green. Children sometimes ignored none-to-one correspondences during the IS comparisons. A none-to-one correspondence appeared as a gap in the flow of green-red mappings. A child would proceed in matching green and red decals one-to-one, then, where faced with a none-to-one correspondence, would continue to move his finger as if to match the decal which faced the gap to an imaginary decal opposite it. A few more advanced children were able to accommodate for none-to-one and two-to-one correspondences in matching green-red, green-red . . . until all possible matches were exhausted.

In the TPR-I task, level 1 and 2 NCs assigned to the matching condition tended to ignore gaps created by none-to-one correspondences. As in the IS task, a child would proceed to match green and red decals one-to-one, then where faced with a gap created by a none-to-one correspondence, continue moving his finger as if to match the decal that faced the gap to an imaginary decal opposite it. Some advanced children accommodated to the gaps by continuing to match decals one-to-one in a slantwise pattern. Other advanced children employed a different strategy. Recall that in the TPR-I task one puppet's candy was more numerous in the morning and the other puppet's, in the afternoon. In the first TPR-I comparison (see Figure 1), there were more green than red decals in the morning, and green was matched to red one-to-one and one-to-none. Red was more numerous in the afternoon, and red was matched to green one-to-one and one-to-none. That is to say, there were gaps in the red array in the morning portion and gaps in the green array in the afternoon portion. Some number conservers accurately compared arrays by matching the morning and afternoon gaps. In the first TPR-I comparison, these children discerned that there were fewer gaps in the morning portion of the red array than in the afternoon portion of the green array. From that standpoint they concluded that there was more red, i.e., that Ernie got more candy in the whole day.

Similar behaviors were observed among children assigned to the matching condition during the TPR-S comparisons. Many children ignored the two-to-one comparisons and matched green and red decals as if the two-to-one correspondences were one-to-one correspondences. Some number conservers, but not all, accommodated for the two-to-one correspondences found in the TPR-S comparisons. These children matched only what some children called the "extra candies." The extra candies were the seconds in the two-to-one correspondences. The extra candies stood out because they were either above or below the two main rows of decals (see Figure 1). In each comparison the children matched second greens to second reds but did not match the greens and reds that made up the two main rows of decals. The green and red decals that made up the two main rows on each card were in one-to-one correspondence, and the one-to-one correspondence was easily discerned without the aid of finger movements. If the child found that the red and green seconds were matched one-to-one, he responded that Bert and Ernie had the same amount. If the matching of seconds resulted in his finding that there were more seconds of one color than the other, the child indicated that the puppet with more seconds had more candy for the whole day.

Children exhibited interesting task relevant behavior during the administration of the Static Quantitative Comparison tasks in Study 2. During the administration of the SD task, several children while inspecting the green and red liquids responded that the liquid in one container was "more up" or "higher" than the liquid in the other container. Others pointed to the level of each liquid. As reported earlier, performance on the SD task was excellent. Almost every child made all four SD comparisons accurately.

Most children who failed to conserve liquid performed poorly on the SH task. Nonconservers of liquid generally responded incorrectly on each of the three comparisons in which the green and red liquids were unequal. Recall that in this task the heights of the two quantities of liquid that were presented in an SH comparison were the same. Thus in each of the inequality comparisons the diameters of the cylinders holding the liquids differed.

Nonconservers overwhelmingly indicated that the two quantities in any SH comparison were the same regardless of the diameters of the cylinders. One nonconserver justified his responses with "because I can see." This child and most other nonconservers reasoned as if relative height was the sole index of relative quantity and relative diameter was irrelevant to the comparison. Their pattern of responses paralleled the pattern of responses found among number nonconservers during the administration of the SCI task. Number nonconservers overwhelmingly responded that the puppets had the same amount of candies in each of the SCI comparisons. These children responded as if relative length was the sole index of relative numerosity and relative density was irrelevant to the comparison.

Occasionally liquid nonconservers responded correctly during the SH comparisons in which the liquids were unequal. One such child volunteered the explanation that one puppet had more because his juice was in a "bigger cup." Conservers of liquid who spontaneously justified their responses tended to volunteer more sophisticated explanations. They tended to mention that one cylinder was "fatter" than the other. One child said, "I can tell [green has more] because this one [red] is skinny."

Conservers of liquid tended to perform better than nonconservers on the TPO-SD and TPR-SD tasks. During the TPO-SD comparisons a number of conservers responded that a particular puppet had more in the whole day and justified the response by indicating that the puppet had more in the morning and the afternoon. During the one equality comparison within the TPO-SD task, a conserver responded that the puppets had the same amount of juice in the whole day because they had the same amount in the morning and the same amount in the afternoon.

Conservers exhibited a number of interesting behaviors during the TPR-SD task. In attempting to compare Bert and Ernie's juice, one child used his thumb and forefinger to measure the difference between the heights of the green and red morning juice. He then measured the difference between the heights of the green and red afternoon juice. In the TPR-SD comparison depicted in Figure 2 he concluded that Bert got more juice in the whole day because Bert got a lot more juice than Ernie in the morning and Ernie got a little more juice than Bert in the afternoon. This strategy parallels an explanation provided by a number conserver assigned to the inspection condition in Study 2. She justified her correct responses in the TPR-I and -S tasks on the basis of the inequality between the difference in the morning subarrays and the difference in the afternoon subarrays. For example, in one comparison the child indicated that Ernie got more candy in the whole day because Bert got a little more candy than Ernie in the morning and Ernie got a lot more candy than Bert in the afternoon.

Conservers of liquid also tended to make another variety of cross-referencing comparisons during the TPR-SD task. Several conservers compared Bert's morning juice with Ernie's afternoon juice and Ernie's morning juice with Bert's afternoon juice. In the comparison in which Bert got more in the morning and Ernie got more in the afternoon (depicted in Figure 2), these children found that Bert's morning juice was greater than Ernie's afternoon juice and Bert's afternoon juice was greater than Ernie's morning juice and concluded that Bert got more juice in whe whole day. Although several children justified their responses in this manner, only one child spontaneously moved the cylinders about to compare directly one puppet's morning juice with the other puppet's afternoon juice.

Other conservers took a different approach to the TPR-SD task. Before judging the relative quantity of the green and red juice, they predicted what would result if all the red juice in the morning container were poured into the container holding the red afternoon juice. They also made a comparable prediction concerning the green juice. They thus judged which quantity of juice would overfill the afternoon container. For example, in the TPR-SD comparison depicted in Figure 2, the children predicted that the morning green would overfill the cylinder containing the afternoon green and the morning red would <u>not</u> overfill the cylinder containing the afternoon red. They concluded that there was more green juice, i.e., that Bert got more for the whole day.

Many nonconservers responded to each TPR-SD comparison that Bert and Ernie got the same amount of juice in the whole day regardless of the actual quantities presented. To justify his response that Bert and Ernie got the same amount of juice in the whole day, one child indicated that "Bert had big and little [juice] and Ernie had big and little [juice]. The child quantified the morning and afternoon juice in the most global terms. Data supporting Hypothesis 8 indicate that nonconservers performed better on the TPO-SD task than on the TPR-SD task. However, nonconservers sometimes responded incorrectly on the TPO-SD comparisons involving unequal amounts of green and red liquid although they tended to respond accurately on the one equality comparison. On the inequality comparisons one puppet had more than the other in the morning and the afternoon. However, the puppet who had less juice in the morning had more morning juice than the other puppet had afternoon juice. For example, if Ernie had more juice in the morning and afternoon, Bert's morning juice was more than Ernie's afternoon juice. This situation is depicted in Figure 2. Some of the nonconservers responded as if the latter morning-afternoon comparison informed the comparison of the total quantity of morning and afternoon juice and, thus, mistakenly responded that Bert had more total juice. As depicted in Figure 2, this morning-afternoon comparison involved the green and red juice contained in the two interior cylinders. Thus it appeared as if the children centered on one salient aspect of the comparison to the detriment of the overall comparison.

CHAPTER IV

DISCUSSION

The discussion of the results comprises five sections. Each of the first four sections is devoted to a major research problem the hypotheses were designed to address. The research problems concern the development of the child's capacity to make numerical and quantitative comparisons. The first section integrates the results that pertain to the nature of the cognitive capacities of the preoperational child. The second section provides a discussion of knowledge of functional relations in connection with the acquisition of conservation. The third section characterizes the growth of correspondence and one-way function based reasoning in making numerical and quantitative comparisons. The fourth section addresses the relation of operative level to the child's capacity to utilize solution aids in making numerical comparisons. The investigator's conclusions are provided in the fifth, and final, section.

The Cognitive Capacities of Preoperational Children

While the results strongly indicate that the cognitive capacities of the concrete operational child are more adequate than those of the preoperational child, the cognitive capacities of the preoperational child can be characterized in a positive way. Study 1 nonconservers tended to perform better on some tasks in which relative numerosity and spatial extent cues conflicted than on others. They performed better on the IP and SP tasks than might have been expected given their poor performance on the SCI task. This is not to say that they performed as well on the IP and SP tasks as conservers. The data pertaining to Hypothesis 1 indicate that conservers performed significantly better on the IP and SP tasks than level 1 NCs. However, in some respects, nonconservers' performance on the IP and SP tasks resembled that of conservers. Like conservers, level 1 and 2 NCs manifested little tendency to judge IP and SP arrays on the basis of spatial extent although the two arrays making up an IP and SP comparison were equal in length. In contrast, spatial extent schemata appeared to govern nonconservers' judgment on the SCI comparisons. Level 1 and 2 NCs tended to indicate that a pair of same-length SCI arrays had the same amount of candies although the arrays were actually unequal in number.

Nonconservers' performance on the IP and SP tasks, however, was superior to their performance on the SCI task. It appears that an emergent understanding of the injective and surjective correspondences that characterize the IP and SP comparisons provided some counterweight to the preoperational child's tendency to judge quantity on the basis of spatial extent, as was manifest in the SCI comparisons.

The results pertaining to Hypothessis 6 indicate that in Study 2 liquid nonconservers, as expected, performed no differently on the SD task than liquid conservers. The two quantities of liquid presented in an SD comparison were contained in cylinders that were equal in diameter; therefore, the relative height of each column of liquid informed its relative quantity. Accurate comparisons of this type required a one-way mapping of height on to quantity. The almost perfect scores of the nonconservers ($\underline{M}$ = 3.82 where a score of 4.00 reflects perfect performance) indicate that preoperational children are accomplished at making accurate SD comparisons.

Number nonconservers manifested some capacity to judge TPO-I and

-S comparisons adequately, although data pertaining to Hypothesis 3 indicate that task performance is directly related to operative level. With regard to the two injective subarray tasks, level 1 and 2 NCs, as expected, performed significantly better on the TPO-I task than on the TPR-I task. Level 2, but not level 1, NCs performed better on the TPO-S task than on the TPR-S task. Level 1 and 2 NCs tended to perform as well on the TPO numerical tasks as they had on the IP and SP tasks.

To make accurate TPO comparisons, the child needed to compose one-way subarray relations such as greater and greater yield greater or equal and greater yield greater. For example, if Bert got more candy in the morning and the afternoon, then Bert got more the whole day. In contrast, the TPR comparisons required a certain amount of cross-referencing between subarray comparisons. For example, if Bert got more candy in the morning, and Ernie got more in the afternoon, the whole-day comparison is not as straightforward as in the case of the TPO comparisons.

Results, thus, suggest that the preoperational child is more capable of making one-way compositions of same-directional subarray comparisons (TPO) than coordinating countervailing subgroup comparisons (TPR). Moreover, this view is supported by the Study 2 results that pertain to Hypothesis 8. The TPO-SD and TPR-SD tasks used in Study 2 are, in the domain of liquid, cognate with the TPO and TPR numerical tasks used in Study 1. In the TPO-SD comparisons one puppet got more juice than the other in the morning and in the afternoon. The TPR-SD comparisons were presented in a context in which one puppet got more juice in the morning, and the other puppet got more in the afternoon. Study 2 liquid nonconservers performed significantly better on the TPO-SD task than on the TPR-SD task.

Thus it appears that the preoperational child's capacity to make accurate numerical and quantitative comparisons can be characterized in terms other than what the child lacks. Although his capacity to compare quantities is not as adequate as that of the concrete operational child, the preoperational child does not compare quantities unintelligently. For example, the preoperational child's propensity to base quantitative comparisons on a single dimension such as height --quantity often covaries with height, albeit imperfectly--is more adaptive than to respond haphazardly. In sum, preoperational children possess adaptive comparison making capabilities that are founded on a rudimentary understanding of injective and surjective correspondences, one-way function based mappings of height on to quantity, and one-way compositions of same-directional subquantity comparisons.

Piaget et al. (1977) posit that schemes of action constitute the source of the child's understanding of functions. They define a scheme as "that which makes [an action] repeatable, transposable, or generalizable, in others words, its structure or form as opposed to the objects which serve as its variable contents" (p. 171). The types of elementary actions that become schematized include acts of finding, displacing, and modifying objects.

Four modes of functioning characterize action schemes (Piaget et al., 1977). These include reproductory, recognitory, generalizing, and reciprocal assimilation. Each of these modes of functioning is linked to an understanding of functional relations among objects.

Reproductory assimilation refers to the capacity to reproduce an action and apply it to one or more objects. Reproductory assimilation, according to Piaget et al. (1977), forms the basis for the expectation that "phenomena or the behaviors of objects are expected to repeat themselves" (p. 176). Recognitory assimilation makes possible the discrimination and identification of objects. Recognitory assimilation, according to the Genevans, underlies the child's capacity to conserve the identity of objects (in a qualitative but not quantitative sense).

Generalizing assimilation refers to the application of a scheme to new objects. It constitutes the basis for the child's expectation that parallel actions on objects lead to parallel results. Reciprocal assimilation refers to the coordination of successive applications of the same scheme.

The above characterization of the functioning of action schemes provides a framework within which to understand the performance of preoperational children on the comparison tasks. Although such a framework is far from complete, the conceptualization of the functioning of schemes provides a starting point for the analysis of preoperational children's performance on the comparison tasks.

Consider, for example, the IP and TPO-I tasks. Recognitory assimilation may be implicated in the success preoperational children manifest on these tasks. Recall that as a result of the none-to-one correspondences that characterize these tasks, there appeared one or more gaps in one of the two arrays of decals presented for comparison. The array having gaps was the smaller of the two. Anecdotal data reveal that preoperational children were capable of recognizing the presence of gaps and justifying their numerosity judgments by identifying gaps.

In the SP and TPO-S tasks, what have been called "extra" candies (the seconds in the two-to-one correspondences) mark the larger of the two arrays presented in any comparison. Anecdotal data indicate that preoperational children are capable of recognizing the presence of extra decals and justifying their numerosity judgments by identifying the extras. Thus extra decals in the SP and TPO-S arrays as well as gaps in the IP and TPO-I arrays map on to judgments of relative numerosity.

Judgments in the TPO-I and -S tasks, while possibly sustained by recognitory functioning, may also require the support of reciprocal and generalizing assimilation. Recall that in most TPO numerical comparisons a puppet who received more candy in the morning also received more candy in the afternoon. Some coordination of successive same-directional subarray mappings appears to be needed. Moreover, the generalization of the subarray comparisons is manifest in the judgment that the puppet who received more in the morning and the afternoon also received more in the whole day.

Preoperational performance on the SH and SCI tasks may be characterized by what might be termed the misapplication of generalization. This view can be made clear by citing results from the liquid draining study of Piaget et al. (1977). In this study preoperational children tended to predict that liquid which drained out of one container into a differently shaped container would collect to a level that matched the level of the original container. These children mapped height on to height without an operative understanding of the relation between height and diameter dimensions. The performance of preoperational children on the SH tasks is characterized by the mapping of water level on to quantity without regard to differences in the containers' diameters. Similarly, in the SCI task, preoperational children tended to map length on to quantity without regard for density. Thus on both SCI and SH comparisons, preoperational children tended to overgeneralize the relation of one dimension to total quantity. On the SD comparisons, however, the mapping of one dimension, height, on to quantity provides the basis for accurate judgments.

Just as reciprocal and generalizing assimilation were used to account for the preoperational child's success on the TPO-I and -S tasks, these two modes of functioning may also underlie preoperational performance on the TPO-SD task. In the TPO-SD task, one puppet received more juice in the morning and the afternoon. Some coordination of same-directional subquantity mappings may be needed. In addition, the generalization of the subquantity comparisons is that the puppet who received more juice in the morning and the afternoon also received more in the whole day.

The nature of preoperational functioning is relevant to the presence of decalage effects in children's performance on the numerical and liquid TPO tasks and corresponding simple one-way tasks (IP, SP, and SD). Study 1 data that pertain to Hypothesis 3 indicate, at best, weak decalage effects. Among level 1 and 2 NCs, the difference in performance on the SP and TPO-S tasks, although in the expected direction, was not significant. Level 2, but not level 1, NCs performed significantly better on the IP task than on the TPO-I task. Study 2 results that pertain to Hypothesis 8 revealed that liquid nonconservers' performance on the SD task was significantly better than their performance on the TPO-SD task, although nonconservers' TPO-SD task performance was superior to their performance on the TPR-SD task. The pattern of results supports the expectation that task structure is relevant to the presence of decalage effects. The numerical and liquid TPO tasks constitute function-of-function tasks. Success on the TPO tasks calls for the same-directional composition of subquantity relations, which was expected to make these tasks more difficult than related correspondence (IP and SP) and one-way mapping (SD) tasks. As mentioned in Chapter I, a feature of the TPO liquid task was expected to engage the thinking of preoperational children and, thus, create a difficulty not present in the TPO numerical tasks. In the TPO-SD (liquid) task, the tendency to center on the comparison of the interior subquantities was expected to provoke inaccurate judgments. Anecdotal data indicate that some liquid nonconservers responded as if the interior subquantity comparison informed the comparison of the total amount of green and red liquid.

An alternative to a Genevan oriented explanation of the preoperational child's performance on the Study 1 and 2 tasks might be based on perceptual salience. Length, for example, is arguably a salient cue; therefore, performance on the SCI task is explained by the salience of the length cue. Such an explanation parallels a view espoused by Wallach (1969) which holds that performance on conservation tasks, both nonconserving and conserving, is the based on the criterion stimuli the child employs as indices of quantity.

A perceptual salience based explanation of children's task performance is subject to three interrelated problems. In addressing these problems, a major asset of the Genevan viewpoint, namely theoretical coherence, will be underlined. First consider performance on the IP, SP, and SCI tasks. In each of the three tasks the terminal points of any pair of arrays presented for comparison were aligned. If length were their only basis for comparing quantity, preoperational children would have performed on the IP and SP tasks as they had on the SCI task. That is to say, preoperational children would have repeatedly judged the two arrays making up an IP or SP pair to be equal. Such a result was not obtained. In contrast, preoperational children performed significantly better on the IP and SP tasks than on the SCI task. Moreover, hypotheses informed by a theory of the child's understanding of correspondence relations, i.e., the Genevan theory of functions, anticipated the differences in task performance.

Secondly, a perceptual salience based explanation is taxed by the problem of determining the source of a cue's salience. It is not enough to examine the cue. That a cue is salient indicates than an organism reponds to it in a consistent way. To determine if a cue is salient, then, one must look to the responding organism as well as to the cue. The Genevan theory of functions provides a general developmental formulation which constitutes a basis for generating hypotheses that are relevant to task performance. Length appears to be a singly dominant cue in the preoperational (but not concrete operational) child's performance on the SCI task. The Genevans (Inhelder, Sinclair, & Bovet, 1974; Piaget, Inhelder, & Szeminska, 1960) advanced the view that preoperational quantification is based on length because of the early elaboration of ordinal schemes of "going beyond." Genevan theory holds that among late preoperational children (aged approximately four years) "evaluation by length is actually based on an ordinal quantification which is already of a conceptual (emphasis mine) nature" (Piaget, 1968, p. 976). Thus, the preoperational child's response that two same-length SCI arrays are equal is conceptually, not perceptually, based.

Moreover, Genevan theory embeds the late preoperational child's tendency to evaluate quantity by length within a general developmental formulation. With operative development, evaluation by length is expected to yield to a more sophisticated evaluation of quantity, such as the joint evaluation of quantity by length and density. Prior to the late preoperational period, Genevan theory holds that topological notions of crowding govern the child's evaluation of quantity, only to be supplanted by evaluations by length (Piaget, 1968).

Third, perceptual salience based explanations lack parsimony. Wallach, Wall, and Anderson (1967) found that, in the standard number conservation paradigm, length constitutes a cue that frequently misleads young children. P. H. Miller (1973) found that, in the standard liquid conservation paradigm, height constitutes a cue that frequently misleads young children. However, neither study provides a basis for linking cue salience in one domain with cue salience in the other. By constrast, the Genevan theory of functions provides a coherent framework within which to view the salience of length and height cues. Genevan theory holds that the preoperational child's use of length and height as indices of quantity reflect a level of conceptual development. The level of conceptual development is governed by the semilogic made manifest by the Genevan theory of functions.

Knowledge of Functional Relations and the Acquisition of Conservation

Although this investigation does not purport to be a conservation training study, data relevant to Hypotheses 5 and 10 indicate that experience with the static numerical and quantitative comparison tasks does not promote the acquisition of conservation. Such findings do not, however, imply that experience with functional relations is unrelated to the acquisition of conservation. In view of the Genevan theory of functions, a reexamination of American conservation training research might indicate that instruction in functional relations attends conservation training. Since the conservation training literature is vast, the following discussion will be confined to two exemplary studies.

In a number conservation training study, Wohlwill and Lowe (1962) attempted to modify the preoperational child's tendency to use length as an index of numerosity. The training procedure began with the child counting the members of each of two equal rows of stars. The rows were initially in one-to-one correspondence. As in the standard number conservation test, one row was expanded or contracted during the training trials. However, during two-thirds of the training trials the interviewer, just prior to changing the length of a row, added a star to (or subtracted a star from) the row to be altered. Despite having features that were designed to affect the child's understanding of length-numerosity relations, the procedure was reported to have had marginal effects on conservation performance.

In contrast, Wallach, Wall, and Anderson (1967) developed a training procedure that was better suited to altering the young child's tendency to map length on to numerosity. They developed a "doll reversibility training" procedure which began with the child placing each of six dolls in one of six beds, thus, creating a one-to-one relation between dolls and beds. The interviewer removed the dolls from the beds and placed them closer together (further apart). The child was questioned about the possibility of restoring the original doll-bed correspondence.

Children in the reversibility training condition tended to perform better on a number conservation posttest than comparable children assigned to an addition-subtraction training group. Wallach et al. (1967) argued that reversibility training, by inducing children to rely less upon misleading length cues, provoked improved performance on number conservation tests.

From the standpoint of the theory of functions, Wallach et al. (1967) and Wohlwill and Lowe (1962) attempted to enrich the child's understanding of the length-numerosity relation. An important aspect of this enriched understanding is the quantity-informing character of the one-to-one correspondence. Doll reversibility training, in particular, emphasizes the relation between sets of objects within the framework of an attractive--to a child--thematic context (cf. the "provoked correspondence" of Piaget and Szeminska, 1952). The thematic context, a doll in every bed, anchored every change in the length of the set of dolls. After each change the child was questioned about the doll-bed relation and witnessed the dolls' return to their beds. In doll reversibility training the notion that the doll-bed correspondence relation nullifies the length-numerosity relation is emphasized. Piaget and Szeminska (1952) evolved the view that in the domain of number conservation, an understanding of the reversibility of action enables the child to proceed "exclusively by reference to the one-to-one correspondence" (p. 89).

One-to-one correspondence constitutes a basic functional relation, bijection. The acquisition of number conservation implies that the child's understanding of bijection has become "operatory" (Piaget, 1968). In the preoperational subperiod, one-to-one correspondence marks an equivalence relation just as injective and surjective correspondences may mark nonequivalence relations. However, these relations apply only to the domain of static comparisons. With operative development the equivalence-informing character of the one-to-one correspondence is preserved despite the destruction of the optical correspondence. It is in this way that number conservation constitutes a more developed understanding of bijection.

Suppose, in a slight variation of the number conservation paradigm, an injective or surjective arrangement (such as the IP or SP arrangements used in Study 1) of two rows of tokens was placed before a child. Study 1 data indicate that if the rows are left untransformed, preoperational children would be capable of indicating which row held more. However, the young child's understanding of injective and surjective relations is not yet operatory. If the optical correspondences were altered such that the terminal points of the less numerous array were extended beyond those of the more numerous array, it is likely that the preoperational child's judgment of the numerosity relation would reflect the new length relations. With operative development it is expected that the child would become capable of maintaining his initial judgment despite the destruction of the initial optical correspondence. That is to say, the child's understanding of injective and surjective correspondences would become operatory.

It is also expected that the child's judgment of height relations also undergoes developmental change. Consider two glasses having the same diameter but containing unequal quantities of liquid (as in the SD task). Study 2 data indicate that preoperational children are highly accurate at comparing liquids thus arranged. Suppose the lesser quantity of liquid is poured into a narrower glass and, after pouring, rises to a level which is higher than that of the other quantity.

It is expected that the preoperational child would alter his judgment to reflect the new height relations. However, with development the child's understanding of height relations should become operatory. The child should become capable of maintaining a judgment based on pre-transformation height relations despite their alteration. This advance in cognitive functioning is thought to be supported by a growing understanding of height-diameter relations and the reversibility of actions.

The Growth of Correspondence and One-Way Function Based Comparisons

The results of the investigation generally indicate that the child's capacity to make accurate numerical and quantitative comparisons becomes more adequate with operative development. Significant operative level effects were found in every instance of task performance except the SD task.

As anticipated in Hypothesis 2, significant operative level effects on the performance of the Study 1 SCI task were found. The results pertaining to the SCI task indicate that number conservers are significantly more accurate in judging SCI arrays than nonconservers. Conservers' performance appeared to be governed less by spatial extent schemata than the performance of nonconservers. Since Study 1 children were not permitted to count in comparing arrays, conservers appeared to be more sensitive to another quantity informing cue, relative density. In order to use relative density as an index of relative quantity, the child's judgments of SCI arrays must be decentered from the spatial extent schemata that appeared to dominate the judgements of the preoperational child. That is not to say that quantitative judgments become insensitive to spatial extent cues. Rather, density and spatial extent provide joint bases which, in concert, inform judgments of relative quantity. Relative density was an informative cue because paired SCI arrays were equal in length.

This interpretation is supported by the Study 2 results anticipated in Hypothesis 7. The results pertain to performance on the SH task, the liquid cognate of the SCI task. In each of the tasks, a prepotent cue, height of two liquids in the SH task and length of two arrays in the SCI task, was held constant. In the SH task, relative diameter had the same quantity informing function relative density had in the SCI task. The results obtained on SH performance paralleled the results obtained on Study 1 SCI performance. Just as spatial extent appeared to govern the judgments of number nonconservers on the SCI task, relative height appeared to govern the judgments of liquid nonconservers on the SH task. That is to say, liquid nonconservers tended to judge two quantities of liquid that were equal in height to be the same regardless of the relative diameters of the cylinders containing them.

It was, therefore, expected that some degree of decentering away from an overreliance on the height cue would attend operative development. Moreover, height and diameter, with development, would be expected to constitute joint bases which inform judgments of relative quantity. Relative diameter was an informative cue in the SH task because the heights of the quantities compared were equal. The results relevant to Hypothesis 7 support this view in that they reveal a highly significant relation between accurate judgment on the SH comparisons and level of operative development.

The performance of Study 1 children on the TPR tasks revealed significant operative level effects. It was thought that accurate judgment on the TPR comparisons required a capcity to coordinate countervailing subarray relations. The general capacity to coordinate one-way functional relations, i.e., compensation, is considered a characteristic of concrete operational thought (Piaget et al., 1977). Results pertinent to Hypothesis 2, which revealed a significant relation between TPR-I and -S task performance and operative level, support this view.

Moreover, the view is supported by Study 2 results that pertain to Hypothesis 9. Hypothesis 9 addresses performance on the TPR-SD task. The TPR-SD task is, in the domain of liquid, cognate with the TPR numerical tasks. In order to compare adequately the overall quantity of green and red liquid presented in the task, the child must coordinate subquantity comparisons. The results relevant to Hypothesis 9 revealed highly significant operative effects in the expected direction.

The view is further supported by the Study 1 results that pertain to the children's performance on the IS task. It was thought that accurate judgment on the IS task required some capacity to coordinate countervailing two-to-one and none-to-one correspondences. The results pertaining to Hypothesis 2 revealed the expected significant operative level effects on performance on the IS task.

Operative level, thus, appears to be reflected in performance on tasks in which a certain degree of cognitive decentering or coordination of countervailing relationships is required. Operative level effects have also been observed in the performance on tasks in which preoperational children manifest some degree of success.

Preoperational children manifested some success in comparing IP and SP arrays. This success appears to reflect some rudimentary understanding of injective and surjective correspondences. Preoperational children also manifested some success on the numerical and liquid TPO tasks. Success on these tasks appears to reflect an understanding of one-way compositions of same-directional comparisons. Results pertaining to Hypothesis 1 revealed significant operative level related improvements in performance on the IP and SP tasks. Significant results relevant to Hypothesis 3 indicated that performance on the TPO-I and -S tasks is related to operative level. Results relevant to Hypothesis 8 revealed operative level effects on the TPO-SD task.

With respect to the child's capacity to make numerical and quantitative comparisons, the concrete operations appear to support two kinds of functioning: the emergence of new comparison-making capabilities and the strengthening of capabilities in which preoperational children manifest some degree of competence. In regard to the first change in functioning, powerful new comparison-making capabilities emerge in the concrete operational period. Results pertinent to SCI and SH task performance revealed that with operative development children become capable of using two dimensions as joint bases for comparisons. SCI judgments become based on density and length cues, and SH judgments, on diameter and height cues. At the preoperational level, performance on these tasks appeared to be informed by only one dimension, length in the instance of the SCI task, and height in the instance of the SH task. Results pertaining to the TPR task revealed an emergent capacity to make cross-referencing types of comparisons. This was underlined by the anecdotal data on TPR-SD task performance. These data depicted liquid conservers who spontaneously reported making cross-comparisons of one puppet's morning juice and another puppet's afternoon juice or imagining the overfilling of a cylinder if morning juice were to be poured into the cylinder containing same colored afternoon juice. Nonconservers, on the other hand, manifested none of these tendencies.

The second type of change in functioning to occur with the development of the concrete operations involves the strengthening of comparison-making capacities that first appear in the preoperational subperiod. Children's performance on the IP, SP, and TPO-I, -S and -SD tasks, tasks on which preoperational children manifest some positive capability appears to improve further with operative development. That is to say, with operative development children become increasingly accurate in applying elementary correspondence and function based knowledge.

The investigation of the child's capacity to compare quantities on the basis of his developing understanding of correspondence and one-way function relations raises three issues that pertain to future research. The first issue involves a major implication of the Genevan theory of functions, namely, the "limitless" extension of the child's capacity to understand functional relations (Piaget et al., 1977). The Genevans contend that functions "are constituted without limit" as a result of an abstracting process arising from the concrete operations (p. 194). It follows from Genevan theory that the comparison tasks employed in Studies 1 and 2 constitute a mere sample of the comparisons concrete operational children eventually master. Such comparisons might include tasks in which the quantities to be compared consist of materials that differ from the materials used in Studies 1 and 2 (e.g., clay, rice, wood--the variety is great). More importantly, concrete operational children would be expected, on the basis of Genevan theory, to become capable of making quantitative comparisons within conceptual domains not explored in Studies 1 and 2. These conceptual domains might include length, area, and speed among others. Comparisons might also involve quantities that are "packaged" such that they are divided into three or more subquantities or arrays in which three-, four-, or more-to-one mappings are found. New research is needed to explore more fully the growth of the child's capacity to compare quantities that embody each of the many material and conceptual domains.

The second research issue involves the relation of school learning to a major component of development, that of the progressive arithmetization of thought. The present investigation underlines the emergence of new comparison-making capabilities and the increasing accuracy with which children compare quantities. However, success at the tasks employed in this investigation did not require an understanding of arithmetic [as opposed to logical] multiplication and proportion. It is only with further development that children acquire knowledge of multiplicative relations (e.g., $n + n + \ldots + n_m = mn$) and proportion and how to use that knowledge to inform comparisons. In a number of studies, Piaget et al. (1977) examined aspects of the child's late developing understanding of multiplicative relations (e.g., in the operation of a lever; in the problem of the "buckling square") and proportion (e.g., in double seriation problems in which a constant ratio marked the relation between the quantitative values--number, length, and area--of corresponding elements). Despite their interest in multiplicative relations and proportion, Piaget and his colleagues, surprisingly, do not characterize the role of school learning in the development of the child's understanding in these school relevant areas. Even in their work on the development of the child's understanding of geometry concepts (Piaget, Inhelder,

& Szeminska, 1960), the Genevans neglect the role of school learning--and this neglect is all the more conspicuous because they deal with school oriented topics such as measurement.

Future research on the relation between development and formal instruction can take two coordinate courses. The role of development in structuring the child's capacity to profit from mathematical instruction constitutes one course of research. However, since mathematics instruction constitutes a major source of mathematical knowledge, future research must also examine the products of mathematics instruction as they feed into the developmental process. Surely the children who manifested some knowledge of multiplicative relations and proportionality, as depicted by Piaget et al. (1977), attended school. It is almost inconceivable that these children could have reached the highest levels of problem solving in the area of, for example, proportion without school experiences. It is, therefore, important that future research investigate the role of school learning in the course of general developmental advance.

The third research issue involves the relation of correspondence and function based thought to problems that are not purely mathematical in content. Piaget et al. (1977) provide a starting point for such research by studying the child's understanding of functional relations in the context of simple machines (e.g., the operation of a lever; the force of a weight on a spring). Future research can extend this work by examining the relation between the child's understanding of functional relations in mathematical and causal-scientific contexts. Such research is important from the standpoint of elucidating the connection between the child's mathematical understanding and his understanding of physical phenomena.

The Relation of Operative Level to the Effective Use of Solution Aids

In Study 2 children were randomly assigned to three instructional set conditions in which they were induced to employ different solution aids in comparing arrays. The solution aids consisted of the following comparison procedures: inspection, counting, and matching. Across all set conditions, results pertinent to Hypothesis 4 revealed a highly significant relation between operative level and performance on each of the static numerical comparison tasks administered in Study 2. However, the principal purpose of the investigation of children's performance on the Study 2 numerical tasks was to examine the effectiveness of the different solution aid approaches within each operative level. Results pertaining to within operative level performance were expected to elucidate the formulations of Cattell and Horn, on one hand, and the Genevans, on the other.

Cattell and Horn's theory of fluid and crystallized intelligence posits that solution aids may "compensate for limitations in anlage capacities" (Horn, 1968, p. 244). Counting and matching are knowledge extracting skills that ought to inform the numerical comparisons. To the extent they are a subset of the learned noegenetic skills the child has acquired during his lifetime, counting and matching constitute crystallized skills, or solution aids (Horn, 1968; Horn & Cattell, 1966).

The Cattell-Horn view is relevant to the performance of the preoperational group on the Study 2 static numerical comparison tasks.

However, in the domain of number nonconservation, preoperational children were classified according to two developmental levels. Children in the more advanced preoperational group, level 2 NCs, although failing to conserve number, tended to establish spontaneously a one-to-one correspondence in creating two equal sets of checkers. Children in the less advanced preoperational group, level 1 NCs, failed to conserve and evidenced difficulty in spontaneously establishing a one-to-one correspondence for the purpose of creating equal sets. According to the Cattell-Horn view, all children, including those who were likely to perform poorest, i.e., level 1 NCs, when induced to employ counting or matching in comparing arrays, should manifest better performance relative to comparable children in the inspection condition. Moreover, counting was potentially a viable solution aid for all groups. All children, level 1 and 2 NCs and conservers, included in Study 2 were capable of counting a set of items whose cardinal value, ten, was equal to the greatest cardinal value a static numerical comparison array could assume. Passing a preliminary counting task where the child was asked to count a set of ten items was a criterion for inclusion in the study.

The Genevans, to some extent, differ with this formulation. The Genevans hold that operative level structures strict sense learning (Furth, 1969, 1974; Inhelder, Sinclair, & Bovet, 1974; Piaget, 1970b, 1971). This view implies that the child's operative level largely determines whether a solution aid would be used effectively in comparing arrays. Level 1 NCs, in contrast to children of more advanced operative levels, constitute the group least expected to manifest improved performance when induced to employ counting or matching.

The four static numerical comparison tasks included in Study 2 for the purpose of investigating set induced solution aid effects were abridged versions of the four most difficult Study 1 tasks. On three of the tasks--the IS, TPR-I, and TPR-S tasks--Table 3 indicates that in Study 1 level 1 NCs performed at approximately chance levels of responding, given that 2.33 correct responses constitutes chance level.[3] Level 1 NCs tended not to respond randomly on the SCI task. They consistently, and inaccurately, judged same-length SCI arrays to have the same amount of candies regardless of differences in density. Thus, the four static numerical comparison tasks that were selected to be administered in Study 2 constituted good candidate tasks in which to examine set induced solution aid effects among level 1 NCs.

Results pertaining to Hypothesis 4 support the Genevan view. Simple main effects for set on the four static numerical comparison tasks administered in Study 2 were nonsignificant among level 1 NCs. In contrast, simple main set effects were found among level 2 NCs and conservers. Study 2 results indicate that simple main set effects for SH task performance were found among conservers of liquid. Although the counting set condition appeared to induce the best performance, the reason for this effect to occur in the domain of liquid comparisons is unclear. Of note is that the simple main effects of set occurred among liquid conservers, and not among nonconservers.

Genevan theory, with its stress on universal features of cognitive development, sometimes neglects issues concerning localized functioning

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[3]In Study 1, 2.33 correct judgments was considered chance responding since there were seven comparisons in each task, and one of three possible responses was considered correct: Bert has more candy; Ernie has more candy; both puppets have the same amount of candy.

such as the question of the efficacy of rival solution approaches. The Cattell-Horn view, by contrast, implies that well-learned knowledge-extracting skills constitute potential solution aids, and that children differ in what solution aids they may adopt. That children assigned to the counting condition tended to make more accurate judgments than children assigned to other solution aid conditions implies that counting constitutes a solution aid that is more cognitively efficient (Beilin, 1969) than either inspection or matching.

A number of characteristics of counting are possible sources of efficiency. Counting is an indexing operation. Number names are ordered and assigned one-to-one to countable objects. By conforming to rules of order and one-to-one correspondence, the counter is guaranteed an accurate representation of the cardinal value of an aggregate. In addition, for many children, counting becomes highly routinized. Routinization makes counting easy to invoke. According to Werner (1957), with development counting becomes progressively less disturbed by the configuration of the countables in an aggregate.

A characteristic of matching suggests that it too constitutes a cognitively efficient solution aid. Matching and counting share a feature not shared with inspection. Both matching and counting require the establishment of one-to-one correspondences. In matching, a one-to-one correspondence between decals making up each of two rows is established. In counting, a one-to-one correspondence between an ordered list of number names and decals is established.

However, despite the common feature, counting and matching are not always equally superior to inspection. Study 2 results indicate that, in the IS, TPR-I, and TPR-S tasks, matching, unlike counting, does not constitute a solution aid that is superior to inspection. This is perhaps because of the features counting and matching do not share. Counting is a "tool" that has wide cultural-environmental support within the context of the child's everyday experiences (Saxe, 1979a; Vygotsky, 1978). An ordered list of number names is provided by the culture and can be invoked regardless of the configuration of the aggregate to be counted. It is perhaps because of wide cultural-environmental support that counting functions as a quantitative skill that is acquired relatively early in life. Saxe (1977, 1979b) adduced evidence indicating that children employ counting to compare and reproduce arrays even before they conserve number.

Study 2 anecdotal data indicate that children of all operative levels manifested difficulty in matching green and red decals in the IS, TPR-I, and TPR-S tasks. This is probably because children have less cultural-environmental support for matching sets of items, especially when the sets vary in configuration, than counting items. However, in the SCI task, matching tended to improve performance relative to inspection. The features of the IS, TPR-I, and TPR-S arrays--gaps, two-to-one mappings, the division of arrays into subarrays--that interfered with the children's attempts to match decals one-to-one, were absent in the SCI arrays. Anecdotal evidence suggests that the uninterrupted linear character of the SCI arrays facilitates matching.

Study 1 data indicate that on the SCI task, the performance of level 2 NCs was very inaccurate and resembled the performance of level 1 NCs. All Study 1 children were induced to inspect arrays. In study 2, when inspection was pitted against counting and matching, the performance of level 2 NCs assigned to the counting and matching set conditions was superior to that of peers assigned to the inspection set condition. When children counted or matched decals, they discovered which array had more, and, in effect, freed themselves from the tendency to base judgments on spatial extent that characterized the performance of level 2 NCs who were induced to inspect arrays. By contrast, counting and matching sets appeared to be insufficient to free level 1 NCs from the tendency to base judgments on spatial extent. Anecdotal data revealed that some level 1 NCs matched SCI decals one-to-one until finding unmatched decals in the more numerous row, yet these children always responded that the puppets had the same amount of candy. Analogous anecdotal findings bear on the performance of level 1 NCs assigned to the counting condition. Some level 1 NCs accurately counted each member of a pair of unequal SCI arrays but reported that the puppets had the same amount of candy. This finding reflects Saxe's (1977) result that among some children, "numerical comparisons based on counting and spatial extent compete with one another" (p. 1515). Other level 1 NCs assigned to the counting set condition employed a counting strategy that was inappropriate given that the SCI or any other numerical task called for the comparison of two arrays (cf. Saxe, 1977). These children tended to count out the total number of decals presented instead of the number of decals in each of the two arrays.

Operative level appears to provide a framework within which the competition between the results of counting or matching and spatial extent can be understood. Study 2 data indicate that level 1 NCs were influenced by the length of the SCI arrays in all solution set conditions. However, matching and, to a greater extent, counting tended to induce level 2 NCs to compare arrays more accurately. An important characteristic of the functioning of level 2 NCs may sustain these results. Level 2 NCs are capable of creating equal sets of items by establishing a one-to-one correspondence between the members of the sets. Level 1 NCs, by contrast, are incapable of this type of functioning. The child who uses matching to compare arrays establishes a one-to-one correspondence between green and red decals. The child who uses counting to compare arrays establishes a one-to-one correspondence between green decals and ordered number names, and again between red decals and ordered number names. Since level 2 NCs are capable of establishing one-to-one correspondences to inform quantity judgements, when induced to count or match in comparing arrays, the quantity informing capability inherent in establishing one-to-one correspondences is probably invoked. Thus, their SCI task performance is likely to be better than the performance of comparable children assigned to the inspection set condition, a condition in which the quantity informing capability inherent in establishing one-to-one correspondences is probably not invoked. Given level 1 NCs' limitations in functioning, no such performance differential among children at this level is likely.

On the Study 2 IS, TPR-I, and TPR-S tasks, level 1 NCs performed at approximately chance levels, given that 1.33 correct responses constitutes chance level.[4] Counting or matching did not lead to

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[4]In study 2, 1.33 correct judgments was considered chance responding since there were four comparisons in each task, and one of three possible responses was considered correct: Bert has more candy; Ernie has more candy; both puppets have the same amount of candy.

improved performance relative to inspection. However, at more advanced operative levels, i.e., among level 2 NCs and conservers, children in the counting condition tended to perform better than peers in the inspection and matching conditions.

Study 2 results, thus, support the view derived from the Genevan interpretation of the relation of learning in the strict sense and operative level. The Genevan view suggests that children at more advanced operative levels are more flexible than level 1 NCs in adjusting their counting to the configurational subtleties of the SCI, TPR, and IS numerical tasks. This flexibility is likely to derive from a growing capacity to coordinate each of the following: length and density relations, countervailing subgroup comparisons, and countervailing injective and surjective relations. This flexibility, largely absent in early preoperational thought, is characteristic of the emerging reversibility inherent in concrete operational thinking.

The relation of operative level to the child's capacity to utilize solution aids has practical applications in the area of instruction. A major implication of this relation is that with development children become increasingly capable of solving problems by a variety of means. The investigation revealed that counting and, sometimes, matching constitute quantity-informing solution approaches that are effective alternatives to inspection. Data indicated that among conservers and level 2, but not level 1, nonconservers counting and matching were effective solution approaches. Moreover, counting and, perhaps, matching are solution approaches that may be considered the result of informal instruction.

The above cited results should not be construed within the limits of time-worn "readiness" explanations. Vygotsky (1962) wrote that instruction "must be aimed not so much at the ripe as at the ripening [cognitive] functions" (p. 104). He further argued that a certain "minimal ripeness of functions is required" before instruction can be effective (p. 104). However, Vygotsky (1962) rejected the view that "instruction hobbles behind development" (p. 94). Vygotsky contended that instruction must be slightly in advance of development in that it is aimed at ripening cognitive functions. The results of the present investigation support Vygotsky's views.

Recall that in the present investigation a subgroup of the preoperational children tended to apply counting and matching effectively in responding to the static numerical comparison tasks. What distinguished the preoperational children who tended to apply counting and matching effectively from the preoperational children who failed to apply the solution aids effectively was the capacity to invoke the quantity-informing character of the one-to-one correspondence. The former, in contrast to the latter, group of children exhibited, within the context of the Piagetian conservation of number paradigm, a greater capacity to invoke the quantity-informing character of one-to-one correspondence. That is to say, the quantity-informing character of one-to-one correspondence constituted a "ripening" function for the former, but not the latter, group. Members of neither group, however, were able to conserve number. This finding is important because a considerable body of research (cited in Chapter 1) assumes that number conservation--number conservation entails a relatively advanced, or "ripe", understanding of one-to-one correspondences (see pp. 84 to 85) --is a basic constituent of quantitative thinking. Thus conserver status does not constitute the minimum prerequisite for applying counting or matching based enumerational solution strategies, underlining the Vygotskean notion that instruction is best linked to ripening rather than ripe cognitive functions.

The relation of operative level to learning should be construed in a broad sense. Development paves the way for a variety of approaches to a concept. One characteristic of concrete operational thinking is the emergence of the child's understanding of vicariance relations (Flavell, 1963; Piaget, 1960, 1972), i.e., the great variety of ways in which conceptual relations can be recast (e.g., mammals = dogs + non-dogs = cats + non-cats = ...). Furthermore, the Genevan theory of functions and corresondences stresses the virtually "limitless" capacity to the concrete operational child to master the ingredients of mathematical and physical-causal functional relations (Piaget et al., 1977). In addition, the Genevan investigations of ontogenesis open up, rather than exclude, areas of instruction. For example, geometry is a subject that has long been considered appropriate to high school. However, Genevan research (Piaget & Inhelder, 1967; Piaget, Inhelder, & Szeminska, 1960) revealed that children's concepts of space and geometry have a long developmental history. As a result of this research, it becomes possible to adjust instruction in geometry to levels of conceptual development found among children in lower grades.

The present investigation identifies certain content domains as objects of mathematics instruction for kindergarten and first grade. These content domains, estimating and comparing quantities, make use of children's informal learning in the areas of counting and matching.

Individuals routinely estimate quantities without counting, and with practice improve their accuracy in estimating quantities. In kindergarten and first grade classrooms, children can be induced to estimate arrays of discrete objects and contrast the results of inspection based estimation activities with the results of counting. Configurational subtleties such as length of an array or the division of the array into subarrays, which are expected affect the child's accuracy in estimating the array's numerosity, can be varied. Estimation activity can be centered around arrays of objects of interest such as arrays consisting of pictures of toys. Activity of this sort can be extended to include the estimation of the size of groups of classmates. For example, some children can be asked to stand in a group while remaining class members can estimate group size, and then check their estimates by counting the number of children in the group. The configuration of the group can also be altered.

The purpose of instruction in estimation is twofold. First, by receiving feedback regarding their estimates, children can be induced to become better estimators. Secondly, children should begin to become aware of the difficulties configurational subtleties impose upon inspection based estimation activity and how these difficulties might be systematically overcome. It is expected that children will vary in the strategies they use to overcome configurational subtleties. These strategies can be shared among class members and compared for effectiveness.

The second content domain, the comparison of quantities, is related to the first. Children can be induced to compare arrays of objects without counting, and then compare the same arrays using counting and/or matching. The different approaches to comparing arrays can thus be contrasted. Groups of children can also be compared. For example, the teacher might ask if there are more boys or girls. A variety of strategies ranging from one-to-one correspondence to subgrouping procedures are potentially elicitable. The children's strategies can be shared and contrasted for effectiveness. In addition to inducing children to become more accurate in making numerical comparisons, instruction in this content domain should help to make children aware of distorting configurational features and how to overcome them.

The present investigation indicates that children as young as four years can benefit from instruction in the estimation and comparison content domains. A child need not be a number conserver in order to participate in a classroom in which these content domains are addressed. The present investigation, however, does emphasize that the child should evidence some understanding of one-to-one correspondence if he is to benefit from instruction in these domains.

Conclusion

The present investigation was designed to examine three aspects of the development of the child's capacity to make numerical and quantitative comparisons. These include characterizing the cognitive capacities of the preoperational child, the achievements of the concrete operational child, and the relation of operative level to the child's capacity to utilize solution aids.

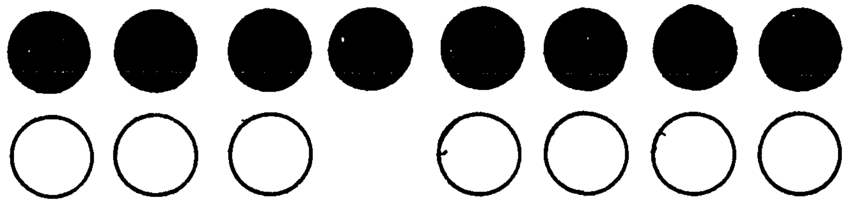
Preoperational children--defined by their nonconserving performance on conservation of number and liquid tests--were found to have positive comparison-making capabilities reflecting a rudimentary understanding of injective and surjective correspondences, one-way function based mappings of height on to quantity, and one-way compositions of same-directional subquantity comparisons (e.g., greater and greater yield greater). The concrete operations were found to sustain two types of changes in functioning. One type of change in functioning involves the emergence of powerful new comparison-making capabilities. Unlike preoperational children who based quantity judgments exclusively on a single dimension such as length or height, concrete operational children--conservers of number, conservers of liquid--tended to use joint bases such as length and density or height and diameter to inform quantity judgments. Concrete operational children were also found to be capable of comparing quantities in which some degree of cross-referencing of subquantity comparisons was required. The other change in functioning that characterizes the concrete operational period involves the strengthening of comparison-making capabilities that first appear in the preoperational subperiod.

The investigation also supports the Genevan view that operative level regulates the child's capacity to use solution aids effectively. Number nonconservers who evinced little mastery of one-to-one correspondence tended to perform poorly on numerical comparison tasks regardless of the solution aids the children were induced to adopt. In contrast, when nonconservers who evinced mastery of one-to-one correspondence and conservers were assigned to the counting condition, they tended to perform better than peers in the inspection and matching conditions. The capacity of children at more advanced operative levels to benefit from the use of solution aids is an expression of the flexibility that is characteristic of the emerging reversibility inherent in concrete operational thought.

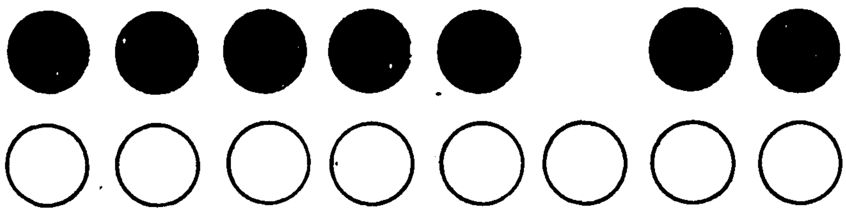
FIGURE 1

SAMPLE STATIC NUMERICAL COMPARISONS
(RED ROW ON TOP - GREEN ROW ON BOTTOM)

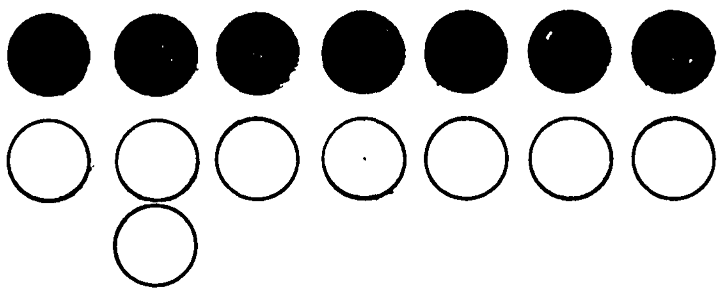
IP_1



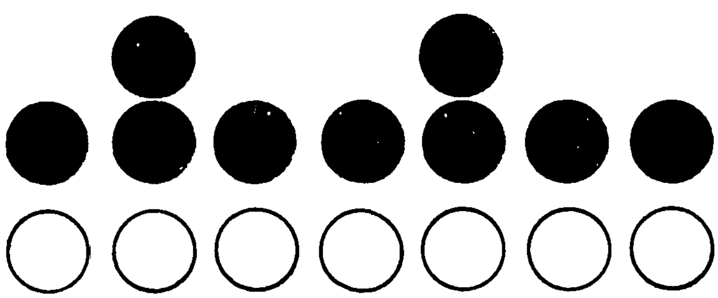
IP_2



SP_1



SP_2



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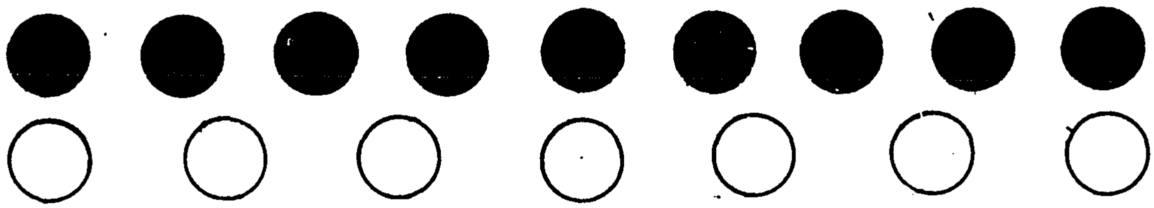

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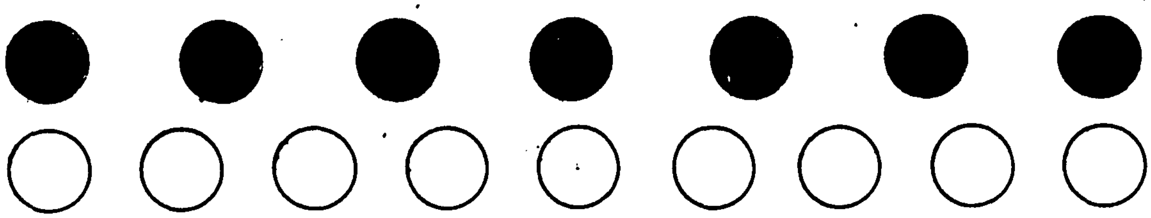

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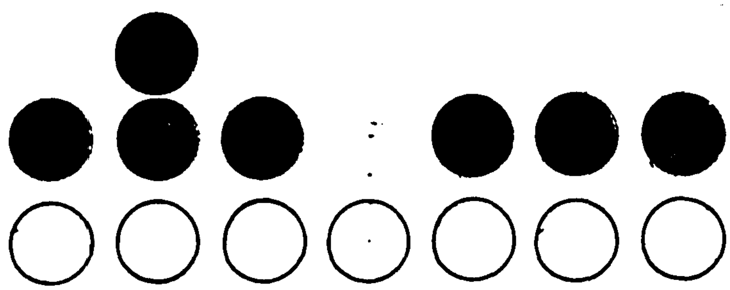

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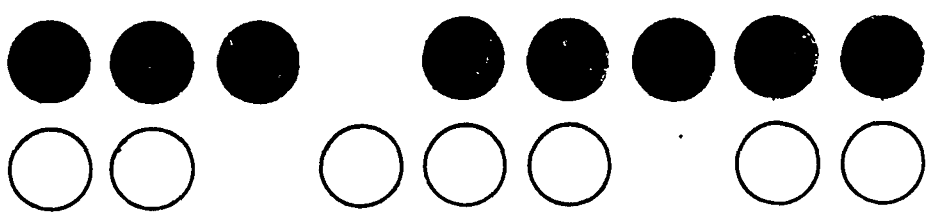

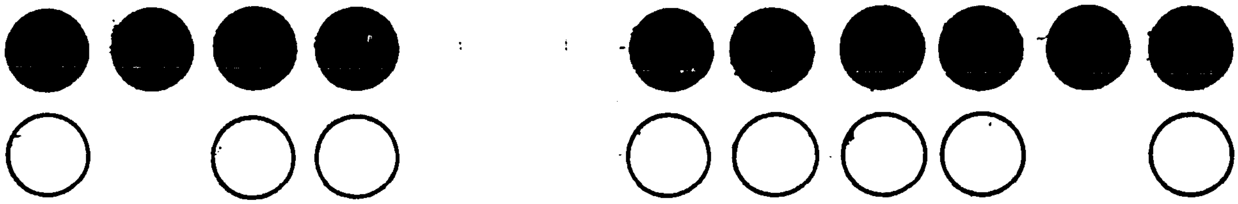

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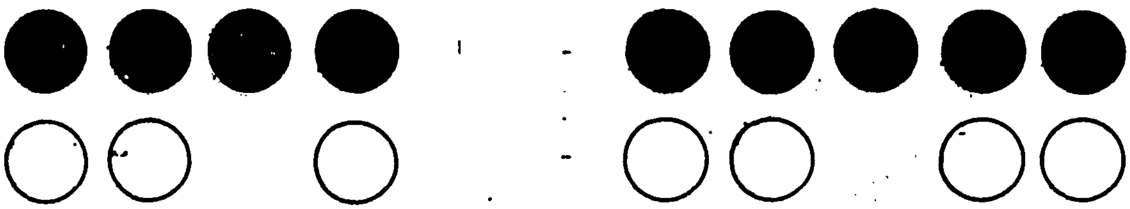

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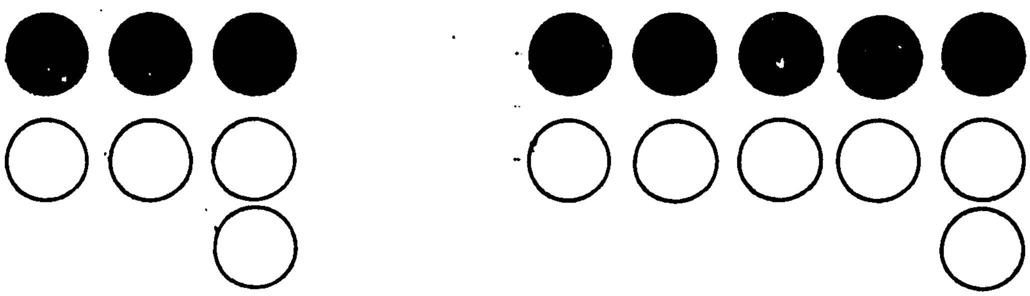

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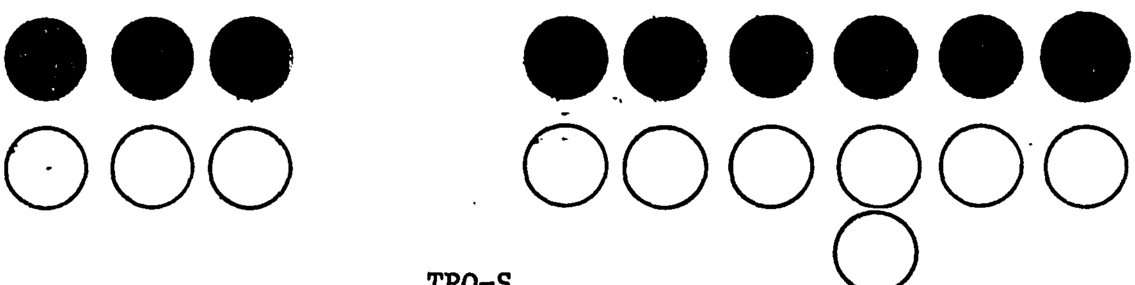

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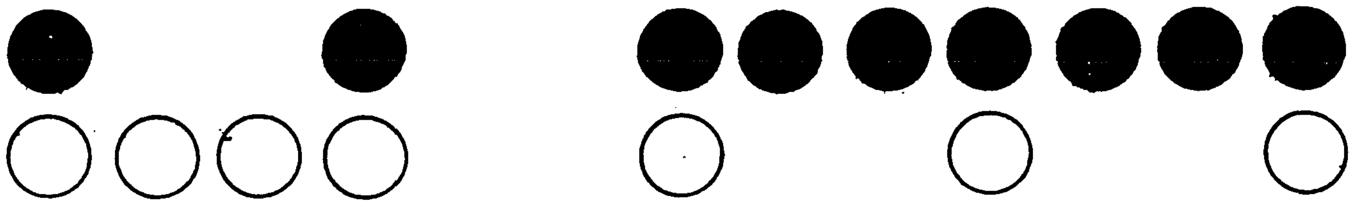

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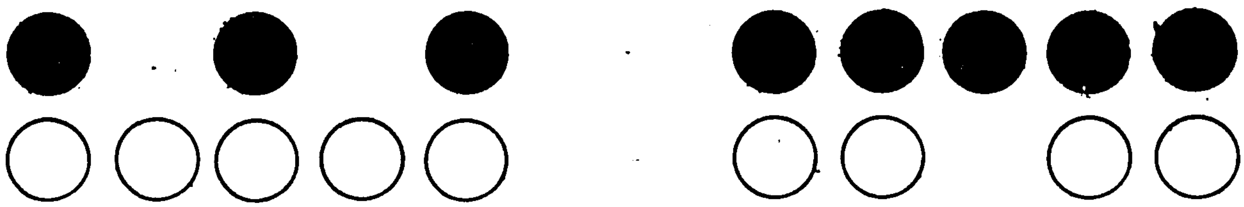

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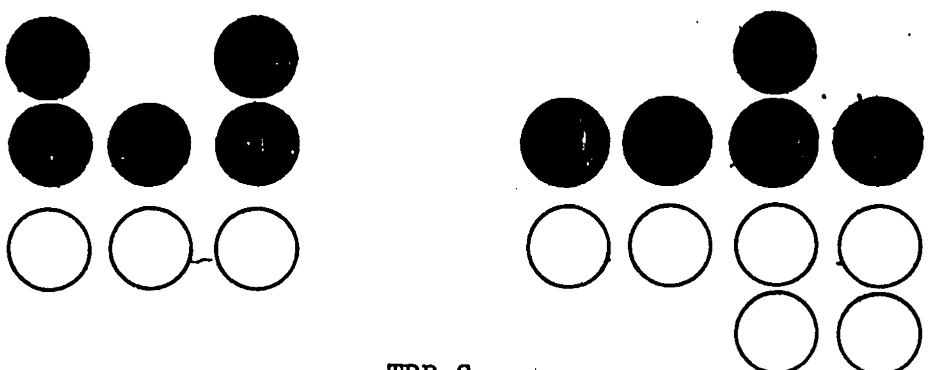

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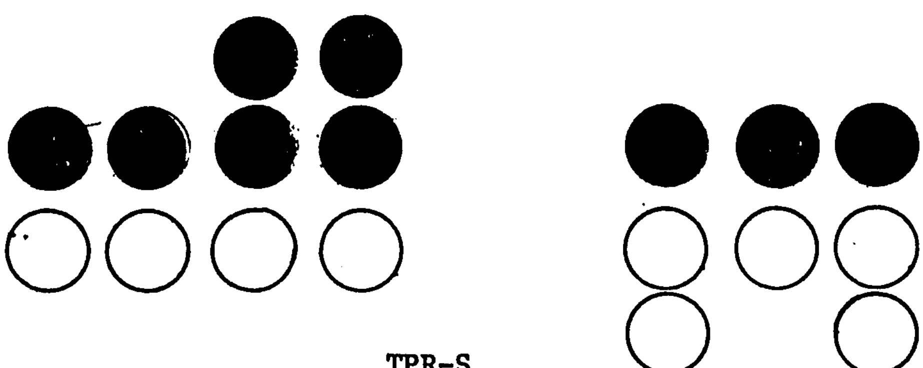
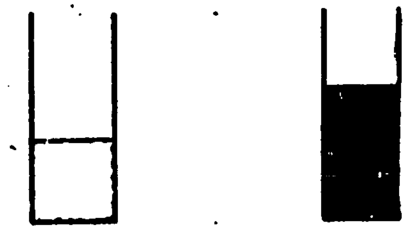


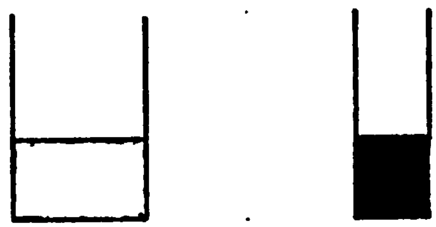
TPR-S_2

FIGURE 2

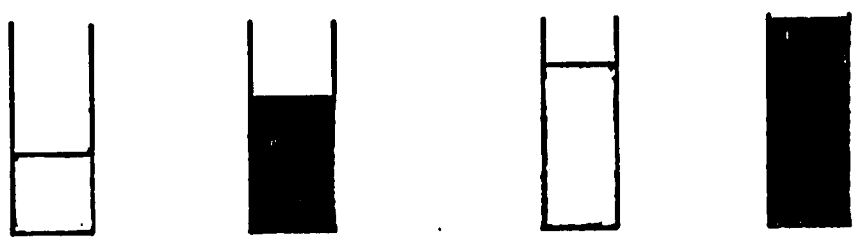
SAMPLE STATIC QUANTITATIVE COMPARISONS
(DARK COLOR REPRESENTS RED LIQUID AND
LIGHT COLOR REPRESENTS GREEN LIQUID)



SD



SH



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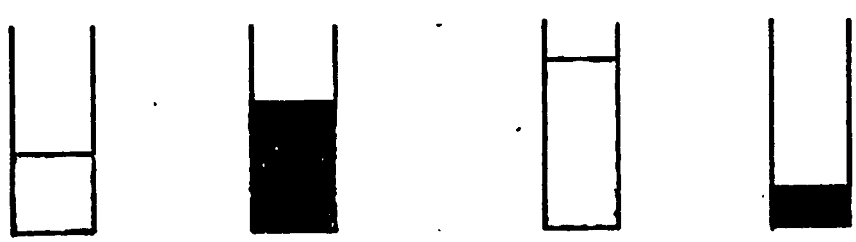

TPR-SD

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