6-2020

Quadratic Packing Polynomials on Sectors of R^2

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QUADRATIC PACKING POLYNOMIALS ON SECTORS OF $\mathbb{R}^2$

by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2020

Typesetting is done with LaTeX.
All figures are created with the TikZ package.

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Abstract

Quadratic Packing Polynomials on Sectors of $\mathbb{R}^2$

by

Kåre Schou Gjaldbæk

Advisor: Melvyn B. Nathanson

In this text, we provide a few general results concerning quadratic packing polynomials on sectors defined as the convex hull of the rays $\{(x, 0) : x \geq 0\}$ and $\{(x, \alpha x) : x \geq 0\}$ with $\alpha > 0$.

We procced to establish a formula for α under the assumption of a quadratic polynomial fulfilling requirements imposed by being a quadratic packing polynomial.

We then show that a quadratic polynomial with non-zero discriminant can not be injective on any subset of $\mathbb{R}^2$ containg an affine convex cone. We use this result to show that a classical proof of the Fueter-Pólya theorem can be employed without invoking the Lindemann-Weierstraß theorem. Furthermore, a consequence of the non-injectivity of non-zero discriminant polynomials is the non-existence of quadratic packing polynomials on sectors with irrational α.

We conclude by determining all quadratic packing packing polynomials on all sectors with rational α.

Acknowledgments

I would like to thank my advisor, Professor Melvyn B. Nathanson, for his continued support and guidance, his endless patience, and for being a lighthouse of inspiration. His profound knowledge and his tireless contributions to the mathematical society make him a role model which any student can only hope to aspire to.

Furthermore, special thanks to Professor Kevin O'Bryant for his involvement from the very beginning of my dissertation research. His interest and input has been invaluable.

I would also like to thank Professor Mark Sheingorn for taking the time to read my work and for being on the committee for my defense.

Finally, I would like thank my wife, Laura Gardner, for her unwavering support in this endeavor. Her emotional support and encouragement has been without bound and her help with the English language has, no doubt, made me look less silly than I otherwise would have. And to our three wonderful children, Freja, Erik and Valdemar: You are the reason I wake up every morning. Literally. Every morning.

Contents

List of Figures

Notation

List of Symbols

$\mathbb{N}$	The set of positive integers
$\mathbb{N}_0$	The set of non-negative integers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rationals
$\mathbb{R}$	The set of reals
$\mathbb{R}_{\geq 0}$	The set of non-negative reals
$S(\alpha)$	Sector of the first quadrant defined by *alpha*
$I(\alpha)$	Set of integral points of $S(\alpha)$
$S(\infty)$	$\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$
$I(\infty)$	$\mathbb{N}_0 \times \mathbb{N}_0$
$P_1(x, y)$	Homogenous linear part of polynomial $P(x, y)$
$P_2(x, y)$	Homogenous quadratic part of polynomial $P(x, y)$
Δ	Discriminant of polynomial $P(x, y)$
$L(p, q)$	Line $py = qx$
$L(p, q; i)/L_i$	Line $py = qx + i$
J_i	$L_i \cap I(\alpha)$
$\mathrm{d}(A, B)$	Natural density of set A in B

continues

List of Symbols (continued)

$\#A$	Number of elements in A
$\lvert I \rvert$	Length of interval I
$\bigsqcup$	Disjoint union
QPP	Quadratic packing polynomial
$\mathrm{GL}_n(A)$	General group of $n \times n$ matrices with entries in A
$\mathrm{LR}(P, I(\alpha))$	Lew-Rosenberg limit
R_n	$S(\alpha) \cap P^{-1}([0,n])$
$\mathrm{area}(\Omega)$	Area of region Ω
$\pi_x(\Omega)$	Projection of Ω onto the x-axis
$\mathbb{1}_A$	Indicator function of the set A
$\mathcal{C}(\boldsymbol{v}_1, \boldsymbol{v}_2)$	Closed convex cone in $\mathbb{R}^2$ spanned by vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$
$\mathcal{C}_{\boldsymbol{v}_0}(\boldsymbol{v}_1, \boldsymbol{v}_2)$	Affine closed convex cone $\mathcal{C}(\boldsymbol{v}_1, \boldsymbol{v}_2) + \boldsymbol{v}_0$
$[\![S]\!]$	Conditional on statement S: 1, if S is true, 0, if false
$\widehat{S}, \widehat{I}, \widehat{J}, \widehat{P}$	Equivalents of S, I, J, P on skewed sector
$\bar{y}_i$	First step on ith staircase

Chapter 1

Background

Published in 1878, the paper *Ein Beitrag zur Manningfaltigkeitslehre* by Georg Cantor [3] introduces the polynomial

$$f(x,y) = x + \frac{(x+y-1)(x+y-2)}{2}$$

which bijectively maps $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. The two *Cantor polynomials*

$$F(x,y) = \frac{1}{2}(x+y)(x+y+1) + x \tag{1.0.1}$$

$$G(x,y) = \frac{1}{2}(x+y)(x+y+1) + y \tag{1.0.2}$$

obtained from $F(x,y) = f(x+1,y+1) - 1$ and $G(x,y) = F(y,x)$ are quadratic polynomials which bijectively map $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$. In the 1923 article *Rationale Abzählung der Gitterpunkte* [6], Fueter and Pólya proved the following.

Theorem 1.1 (Fueter-Pólya)**.** *The polynomials* (1.0.1) *and* (1.0.2) *are the only two quadratic polynomials which are bijections* $\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$.

The proof uses analytic methods and rests on the Lindemann-Weierstraß theorem concerning the transcendence of algebraic combinations of e. In 2001, Vsemirnov [12] provided two proofs of the result using elementary methods. Fueter and Pólya further conjectured

Conjecture 1.2 (Fueter-Pólya)**.** *No polynomial of degree higher than two can be a bijection from* $\mathbb{N}_0 \times \mathbb{N}_0$ *onto* $\mathbb{N}_0$*.*

It is also mentioned that no linear polynomial can provide such a bijection (careful arguments are provided in [8] and [10]).

In two papers, [8] and [9], published in 1978, Lew and Rosenberg develop a more general theory. Here, they introduce the terms *storing polynomial* and *packing polynomial* for polynomials which are injections, resp. bijections from some lattice to $\mathbb{N}_0$. In the second paper, they prove that Fueter and Pólya's conjecture is true for polynomials of degree 3 and 4. The conjecture remains open for higher degrees.

In the 2014 paper *Cantor Polynomials for Semigroup Sectors* [10], Nathanson studies packing polynomials on the integer lattices of subsets of $\mathbb{R}^2_{\geq 0}$ bounded by the lines $y = 0$ and $y = \alpha x$ with $\alpha > 0$. Nathanson provided the following result.

Theorem 1.3 (Nathanson)**.** *For* $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$, *the polynomials*

$$F_{1/m}(x, y) = \frac{1}{2}(x - (m-1)y)(x - (m-1)y - 1) + x + (2 - m)y \tag{1.0.3}$$

$$G_{1/m}(x, y) = \frac{1}{2}(x - (m-1)y)(x - (m-1)y + 1) + x - my \tag{1.0.4}$$

are the only quadratic packing polynomials.

Furthermore, Nathanson finds two packing polynomials for integral α and certain rationals.

Theorem 1.4 (Nathanson)**.** *For* $\alpha = n \in \mathbb{N}$*, the polynomials*

$$F_n(x, y) = \frac{n}{2}x(x-1) + x + y \tag{1.0.5}$$

$$G_n(x, y) = \frac{n}{2}x(x+1) + x - y \tag{1.0.6}$$

are packing polynomials.

For $n, m \in \mathbb{N}$, $\gcd(m, n) = 1$, $1 \leq n \leq m$ *and provided* $n \mid (m-1)$*, the polynomials*

$$F_{n/m}(x, y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)\left(x - \frac{m-1}{n}y - 1\right) + x + \frac{n-(m-1)}{n}y \tag{1.0.7}$$

$$G_{n/m}(x, y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)\left(x - \frac{m-1}{n}y + 1\right) + x + \frac{-n-(m-1)}{n}y \tag{1.0.8}$$

are packing polynomials for $\alpha = n/m$.

Nathanson goes on to, for all rational α, construct *quasi-polynomials*, that is functions

$$H(x, y) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} c_{i,j}(x, y) x^i y^j,$$

where $c_{i,j}(x, y)$ depends only on the congruence classes of x and y modulo m. The paper concludes with a list of (at the time) open problems. Nathanson asks

(1) Is there a quadratic packing polynomial with $\alpha = 3/5$? Is there a quadratic packing polynomial for the sector $\alpha = 3/2$? For what rational numbers α do there exist quadratic packing polynomials?

(2) Are the two polynomials (1.0.5) and (1.0.6) the only quadratic packing polynomials for integral $\alpha \geq 2$?

(3) Can there be more than two quadratic packing polynomials for any rational α?

(4) Can there be more than two packing polynomials for any rational α?

(5) Can there be a packing polynomial of degree greater than two for rational α?

(6) Prove that there is no packing polynomial when α is irrational.

The answer to questions (2) and (3) (and partly (1)) was given by Stanton [11]. She classified quadratic packing polynomials for all integral α and in doing so found that $\alpha = 3$ and $\alpha = 4$ allows for four different solutions. She also provided a necessary form that the homogeneous quadratic part of a packing polynomial must take for arbitrary rational α. The necessary form immediately rules out the possibilities $\alpha = 3/5$ and $\alpha = 3/2$. Brandt [2] used Stanton's results to answer question (1) by providing a classification of all quadratic packing polynomials for arbitrary rational α. Brandt's work on the subject is not published and the cited paper contains a few errors. The strategy of the proof is sound and the conclusions of the paper are correct.

In this thesis, we will address problems (1), (2), (3) and (6) in Nathanson's list. The answer to Problem (4) is implied by (3) since the answer is yes. The question concerning higher degree polynomials, however, remains open at the time of writing. Chapter 2 provides the basic definitions and setup and a few preliminary results. In chapter 3, we show that quadratic polynomials that meet certain criteria (criteria necessary for a packing polynomial) force α to take a specific form dependant only on the homogeneous quadratic part of the polynomial. Chapter 4 provides a general results about quadratic polynomials, a corollary of which settles problem (6) on Nathanson's list. In chapter 5, we will revisit the results of Stanton and Brandt, providing a full stand-alone proof of the classification of quadratic packing polynomials for rational α.

Chapter 2

Packing Polynomials on Sectors of $\mathbb{R}^2$

In this chapter, we will provide the basic definitions and a few preliminary results. We will introduce the fundamental tool of sector transformations allowing us to group sectors into classes of equivalence.

2.1 Preliminaries

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a subset. A function $P : \Omega \to \mathbb{R}$ is called a *packing function* on Ω if it maps $\Omega \cap \mathbb{Z}^n$ bijectively onto $\mathbb{N}_0$. If P is a polynomial, we call it a *packing polynomial*.

Definition 2.2. Let $\alpha > 0$. We define the sector $S(\alpha)$ as the closed convex cone

$$S(\alpha) = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x, 0 \leq y \leq \alpha x\}$$

and the lattice point sector

$$I(\alpha) = S(\alpha) \cap \mathbb{Z}^2$$

as the set of integral lattice points in $S(\alpha)$. We set $S(\infty) = \mathbb{R}^2_{\geq 0}$ and $I(\infty) = S(\infty) \cap \mathbb{Z}^2 = \mathbb{N}_0^2$.

Example 2.3. The Cantor polynomials (1.0.1) and (1.0.2) are, of course, examples of packing functions on $S(\infty)$. A different classic example stems from the observation that every natural

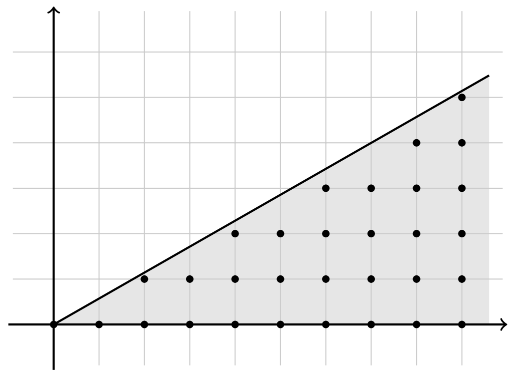

Figure 2.1: The sector $S(\alpha)$

number can be uniquely written as a product of a power of 2 and an odd number. Therefore, the function

$$F(x, y) = 2^x(2y + 1) - 1$$

is a packing function on $S(\alpha)$, but it is not a polynomial.

The focus of this thesis is on quadratic packing polynomials (QPPs for short) on arbitrary sectors. Our starting point is the following fact about polynomials which take integer values on integer points.

Proposition 2.4. *If a quadratic polynomial $P(x, y)$ takes integer values on $I(\alpha)$, then it must be of the form*

$$\begin{aligned} P(x, y) &= A\frac{x(x-1)}{2} + Bxy + C\frac{y(y-1)}{2} + Dx + Ey + F \\ &= \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 + \left(D - \frac{A}{2}\right)x + \left(E - \frac{C}{2}\right)y + F \qquad (2.1.1) \end{aligned}$$

with $A, B, C, D, E, F \in \mathbb{Z}$.

Proof. It is a standard result (see [7] Chp. X §6, Lem. 6.4) that if $P(x)$ is a polynomial of degree d and if $P(n) \in \mathbb{Z}$ for all sufficiently large integers n, then

$$P(x) = a_d\binom{x}{d} + a_{d-1}\binom{x}{d-1} + \cdots + a_0$$

with $a_i \in \mathbb{Z}$, where

$$\binom{x}{d} = \frac{x(x-1)\cdots(x-d+1)}{d!}.$$

Furthermore, any polynomial of the above form is integer-valued.

Let $P(x, y)$ be a quadratic polynomial, so we can write it as

$$P(x, y) = A\frac{x(x-1)}{2} + Bxy + C\frac{y(y-1)}{2} + Dx + Ey + F$$

with $A, B, C, D, E, F \in \mathbb{Q}$ and A, B, C not all zero. If $P(x, y)$ is integer-valued on $I(\alpha)$, then

$$P(n, 0) = A\frac{n(n-1)}{2} + Dx + F \in \mathbb{Z}$$

for all integral $n \geq 0$. By the above, we must have

$$A, D, F \in \mathbb{Z}.$$

For all sufficient large $n \in \mathbb{N}$, we have $(n, 1) \in I(\alpha)$, so

$$P(n, 1) = A\frac{n(n-1)}{2} + Bn + Dn + E + F \in \mathbb{Z}$$

for all large enough n. Since $An(n-1)/2$, Dn, F are all integers, we must have $Bn + E \in \mathbb{Z}$ for all sufficiently large n. We conclude that

$$B, E \in \mathbb{Z}.$$

Finally, since $(n, 2) \in I(\alpha)$ for all n large enough, we have

$$P(n, 2) = A\frac{n(n-1)}{2} + 2Bn + C + Dn + 2E + F \in \mathbb{Z}$$

for some n. All terms but C are known to be integral, so we can conlude the same for C. $\square$

We will throughout assume that $P(x, y)$ has this form and we will denote its discriminant by $\Delta = B^2 - AC$. By $P_2(x, y)$ we will denote the homogeneous quadratic part of $P(x, y)$. That is

$$P_2(x, y) = \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2.$$

Similarly, $P_1(x, y)$ will denote the homogeneous linear part,

$$P_1(x, y) = \left(D - \frac{A}{2}\right)x + \left(E - \frac{C}{2}\right)y.$$

Furthermore, whenever it is convenient, we will use the shorthand notation

$$D' = D - \frac{A}{2}, \quad E' = E - \frac{C}{2}.$$

Lemma 2.5. *Let $P(x,y)$ be a QPP on $S(\alpha)$. The homogeneous quadratic part, $P_2(x,y)$, can not vanish on any line $L(p,q) : y = \frac{q}{p}x$ with rational slope $0 \le \frac{q}{p} \le \alpha$.*

Proof. Let $\frac{q}{p} \in \mathbb{Q}$ be a fixed rational in lowest terms with $0 \le \frac{q}{p} \le \alpha$. Define for $i \in \mathbb{Z}$ the lines

$$L_i : py = qx + i.$$

Define the sets

$$J_i = L_i \cap I(\alpha).$$

Then $I(\alpha)$ is the disjoint union of the J_i's. That is

$$I(\alpha) = \bigsqcup_{i \in \mathbb{Z}} J_i.$$

Restricted to the line $L_0 : y = \frac{q}{p}x$, we have

$$\begin{aligned}
P(x,y) &= P(px, qx) \\
&= \frac{A}{2}(px)^2 + B(px)(qx) + \frac{C}{2}(qx)^2 + D'(px) + E'(qx) + F \\
&= \left(\frac{A}{2}p^2 + Bpq + \frac{C}{2}q^2\right)x^2 + (D'p + E'q)x + F
\end{aligned}$$

If $P_2(x,y)$ vanishes on $y = \frac{q}{p}x$, we have

$$\frac{A}{2}p^2 + Bpq + \frac{C}{2}q^2 = 0.$$

On any line, L_i, we then have

$$\begin{aligned}
P(x,y) &= P\left(px, qx + \frac{i}{p}\right) \\
&= \frac{A}{2}(px)^2 + Bpx\left(qx + \frac{i}{p}\right) + \frac{C}{2}\left(qx + \frac{i}{p}\right)^2 \\
&\qquad + D'px + E'\left(qx + \frac{i}{p}\right) + F \\
&= \left(\frac{A}{2}p^2 + Bpq + \frac{C}{2}q^2\right)x^2 \\
&\qquad + \left(Bp\frac{i}{p} + Cq\frac{i}{p} + D'p + E'q\right)x + \frac{C}{2}\left(\frac{i}{p}\right)^2 + E'\frac{i}{p} + F \\
&= \left((Bp + Cq)\frac{i}{p} + D'p + E'q\right)x + \frac{C}{2}\left(\frac{i}{p}\right)^2 + E'\frac{i}{p} + F.
\end{aligned}$$

If the inequalities are sharp, $0 < \frac{q}{p} < \alpha$, then an infinitely long segment of each L_i falls inside $S(\alpha)$. If we choose an i for which

$$(Bp + Cq)\frac{i}{p} + D'p + E'q < 0,$$

then there will be an x_0 such that

$$P\left(px, qx + \frac{i}{p}\right) < 0$$

for $x > x_0$. Since P is a packing polynomial, this can not happen, so the only possibilities are $\frac{q}{p} = 0$ or $\frac{q}{p} = \alpha$.

Assume $\frac{q}{p} = 0$. If $P_2(x, y)$ vanishes on $y = 0$, then

$$P(x, 0) = \frac{A}{2}x^2 + D'x + F = D'x + F,$$

so $A = 0$, and so,

$$\begin{aligned} P(x, i) &= Bxi + \frac{C}{2}i^2 + D'x + E'i + F \\ &= (Bi + D')x + \frac{C}{2}i^2 + E'i + F. \end{aligned}$$

Now, for $i \geq 0$, we have $L_i : y = i$ and so

$$J_i = L_i \cap I(\alpha) = \{(x, i) \ : \ x \in \mathbb{Z}, i \leq \alpha x\}$$

and

$$I(\alpha) = \bigsqcup_{i=0}^{\infty} J_i.$$

Let

$$\begin{aligned} P(J_i) &= \{P(x, y) \ : \ (x, y) \in J_i\} \\ &= \{(Bi + D')x + \frac{C}{2}i^2 + E'i + F \ : \ \alpha x \geq i\} \\ &= \{a(i)x + b(i) \ : \ \alpha x \geq i\}, \end{aligned}$$

where, for notational convenience,

$$a(i) = Bi + D', \quad b(i) = \frac{C}{2}i^2 + E'i + F.$$

Define the natural density

$$\mathrm{d}(P(J_i), \mathbb{N}_0) = \lim_{n\to\infty} \frac{\#(P(J_i) \cap [0,n])}{n}.$$

We have

$$\begin{aligned}\#(P(J_i) \cap [0,n]) &= \#\{(x,y) \in J_i \ : \ 0 \le P(x,y) \le n\} \\ &= \#\{x \in \mathbb{Z} \ : \ \alpha x \ge i, 0 \le a(i)x + b(i) \le n\}.\end{aligned}$$

The inequalities $\alpha x \ge i$ and $0 \le a(i)x + b(i) \le n$ implies that

$$x \in \left[-\frac{b(i)}{a(i)}, \frac{n-b(i)}{a(i)}\right] \cap \left[\frac{i}{p}, \infty\right) = \left[\max\left(-\frac{b(i)}{a(i)}, \frac{i}{p}\right), \frac{n-b(i)}{a(i)}\right].$$

The length of this interval is

$$\begin{aligned}l_{i,n} &= \left|\left[\max\left(-\frac{b(i)}{a(i)}, \frac{i}{p}\right), \frac{n-b(i)}{a(i)}\right]\right| \\ &= \max\left(\min\left(\frac{n}{a(i)}, \frac{n}{a(i)} - \frac{b(i)}{a(i)} - \frac{i}{p}\right), 0\right).\end{aligned}$$

The number of integers in the interval is

$$\#(P(J_i) \cap [0,n]) = \lfloor l_{i,n} \rfloor \pm 1$$

depending on how many endpoints are integers. So

$$\begin{aligned}\mathrm{d}(P(J_i), \mathbb{N}_0) &= \lim_{n\to\infty} \frac{1}{n} \#(P(J_i) \cap [0,n]) \\ &= \lim_{n\to\infty} \frac{1}{n} (\lfloor l_{i,n} \rfloor \pm 1) \\ &= \lim_{n\to\infty} \frac{1}{n} \left(\left\lfloor \max\left(\min\left(\frac{n}{a(i)}, \frac{n}{a(i)} - \frac{b(i)}{a(i)} - \frac{i}{p}\right), 0\right)\right\rfloor \pm 1\right) \\ &= \lim_{n\to\infty} \left(\left\lfloor \max\left(\min\left(\frac{1}{a(i)}, \frac{1}{a(i)} - \frac{b(i)}{na(i)} - \frac{i}{np}\right), 0\right)\right\rfloor \pm \frac{1}{n}\right) \\ &= \frac{1}{a(i)} \\ &= \frac{1}{Bi + D'}.\end{aligned}$$

For a packing polynomial, we have $\#(P(I(\alpha)) \cap [0,n]) = n+1$ for each n, so if $P(x,y)$ is a packing polynomial, we must have $\mathrm{d}(P(I(\alpha)), \mathbb{N}_0) = 1$. Now,

$$\begin{aligned}
\mathrm{d}(P(I(\alpha)), \mathbb{N}_0) &= \lim_{n\to\infty} \frac{\#(P(I(\alpha)) \cap [0,n])}{n} \\
&= \lim_{n\to\infty} \frac{\#(P(\bigsqcup_{i=0}^{\infty} J_i) \cap [0,n])}{n} \\
&= \lim_{n\to\infty} \frac{\#(\bigsqcup_{i=0}^{\infty}(P(J_i) \cap [0,n]))}{n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=0}^{\infty} \#(P(J_i) \cap [0,n])}{n} \\
&= \sum_{i=0}^{\infty} \lim_{n\to\infty} \frac{1}{n} \#(P(J_i) \cap [0,n]) \\
&= \sum_{i=0}^{\infty} \mathrm{d}(P(J_i), \mathbb{N}_0) \\
&= \sum_{i=0}^{\infty} \frac{1}{Bi + D'},
\end{aligned}$$

where changing union and summation is justified by the J_i's being disjoint and interchanging limit and summation is okay since the convergence of the limit inside the summation is absolute. The sum diverges, a contradiction.

If $\frac{q}{p} = \alpha$, the situation is similar. On L_i, we have

$$\begin{aligned}
P\left(x, \frac{q}{p}x + \frac{i}{p}\right) &= \left(\frac{A}{2} + B\frac{q}{p} + \frac{C}{2}\left(\frac{q}{p}\right)^2\right) x^2 \\
&\qquad + \left(\left(B + C\frac{q}{p}\right)\frac{i}{p} + D' + E'\frac{q}{p}\right) x \\
&\qquad + \frac{C}{2}\left(\frac{i}{p}\right)^2 + E'\frac{i}{p} + F \\
&= a(i)x + b(i),
\end{aligned}$$

where

$$a(i) = \left(B + C\frac{q}{p}\right)\frac{i}{p} + D' + E'\frac{q}{p}, \quad b(i) = \frac{C}{2}\left(\frac{i}{p}\right)^2 + E'\frac{i}{p} + F.$$

We have

$$\begin{aligned} J_i &= L_i \cap I(\alpha) \\ &= \left\{ \left(x, \frac{q}{p}x + \frac{i}{p} \right) \in \mathbb{Z}^2 \ : \ 0 \le \frac{q}{p}x + \frac{i}{p} \le \alpha x = \frac{q}{p}x \right\} \\ &= \left\{ \left(x, \frac{q}{p}x + \frac{i}{p} \right) \in \mathbb{Z}^2 \ : \ 0 \le -\frac{i}{q} \le x \right\} \end{aligned}$$

so $i \le 0$, $I(\alpha) = \bigsqcup_{i=0}^{\infty} J_{-i}$, and

$$\begin{aligned} \#(P(J_i) \cap [0,n]) &= \#\{(x,y) \in J_i \ : \ 0 \le P(x,y) \le n\} \\ &= \#\left\{ x \in \mathbb{Z} \ : \ x \ge -\frac{i}{q}, \ qx + i \equiv 0 \pmod{p}, \ 0 \le a(i)x + b(i) \le n \right\} \end{aligned}$$

This means

$$x \in \left[-\frac{b(i)}{a(i)}, \frac{n - b(i)}{a(i)} \right] \cap \left[-\frac{i}{q}, \infty \right),$$

an interval of length

$$l_{i,n} = \max\left(\min\left(\frac{n}{a(i)}, \frac{n}{a(i)} - \frac{b(i)}{a(i)} + \frac{i}{q} \right), 0 \right).$$

Depending on endpoints, this interval contains $\lfloor l_{i,n}/p \rfloor \pm 1$ integers x with $qx + i \equiv 0 \pmod{p}$.

This means that

$$\begin{aligned} \mathrm{d}(P(J_i), \mathbb{N}_0) &= \lim_{n \to \infty} \frac{1}{n} \#(P(J_i) \cap [0,n] \\ &= \lim_{n \to \infty} \frac{1}{n} \left(\lfloor l_{i,n}/p \rfloor \pm 1 \right) \\ &= \frac{1}{pa(i)} \\ &= \frac{1}{(Bp + Cq)\frac{i}{p} + D'p + E'q} \end{aligned}$$

Again,

$$\begin{aligned} \mathrm{d}(P(I(\alpha)), \mathbb{N}_0) &= \lim_{n\to\infty} \frac{1}{n} \# \left(P\Big(\bigsqcup_{i=0}^{\infty} J_{-i} \Big) \cap [0,n] \right) \\ &= \sum_{i=0}^{\infty} \lim_{n\to\infty} \frac{1}{n} \# P(J_{-i}) \cap [0,n]) \\ &= \sum_{i=0}^{\infty} \mathrm{d}(P(J_{-i}), \mathbb{N}_0) \\ &= \sum_{i=0}^{\infty} \frac{1}{-(Bp+Cq)\frac{i}{p} + D'p + E'q} \end{aligned}$$

which diverges. □

Lemma 2.6. *For P to be a packing polynomial on $S(\alpha)$, we must have $A > 0$.*

Proof. If A is negative, then $P(x, 0)$ will be negative for large enough x. This is not allowed for a packing polynomial.

If $A = 0$, then the quadratic part vanishes on $y = 0$. This can't happen by Lem. 2.5. □

Lemma 2.7. *For a QPP on $S(\alpha)$, we must have*

$$\frac{A}{2} + B\alpha + \frac{C}{2}\alpha^2 \geq 0.$$

Furthermore, if α is rational, then the inequality is sharp.

Proof. We have $P_2(x, \alpha x) = \left(\frac{A}{2} + B\alpha + \frac{C}{2}\alpha^2\right) x^2$. Since we can choose points $(q,p) \in I(\alpha)$ with $\frac{p}{q}$ arbitrarily close to α, if $\frac{A}{2} + B\alpha + \frac{C}{2}\alpha^2 < 0$, we can find $(q,p) \in I(\alpha)$ such that $\frac{A}{2} + B\frac{p}{q} + \frac{C}{2}\frac{p^2}{q^2} < 0$. But then $P(px, qx)$ would eventually become negative. We conclude that $\frac{A}{2} + B\alpha + \frac{C}{2}\alpha^2 \geq 0$.

If $\alpha = q/p$ is rational and $\frac{A}{2} + B\alpha + \frac{C}{2}\alpha^2 = 0$, then $P_2(x, y)$ vanishes on the line $y = \alpha x$. This is impossible by Lem 2.5. □

2.2 Tranformations of the Sector

It can be convenient to apply an invertible linear transformation, M, to the sector $S(\alpha)$. This will allow us to transfer results from a sector to one equivalent under such transformations.

In particular, if

$$M(I(\alpha)) = M(S(\alpha)) \cap \mathbb{Z}^2,$$

then packing polynomials on $S(\alpha)$ are in 1–1 correspondence with packing polynomials on $M(S(\alpha))$. This is the case when $M \in \mathrm{GL}_2(\mathbb{Z})$.

Consider transformations of the form

$$M_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ with inverse } (M_t)^{-1} = M_{-t} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \tag{2.2.1}$$

Nathanson [10] used transformations of this type (along with one other involutory transformation discussed later) to obtain the polynomials of Thm. 1.3. The effect of this is skewing the sector left if $t < 0$, and right if $t > 0$. If $t \in \mathbb{Z}$, then P is a packing polynomial on $S(\alpha)$,

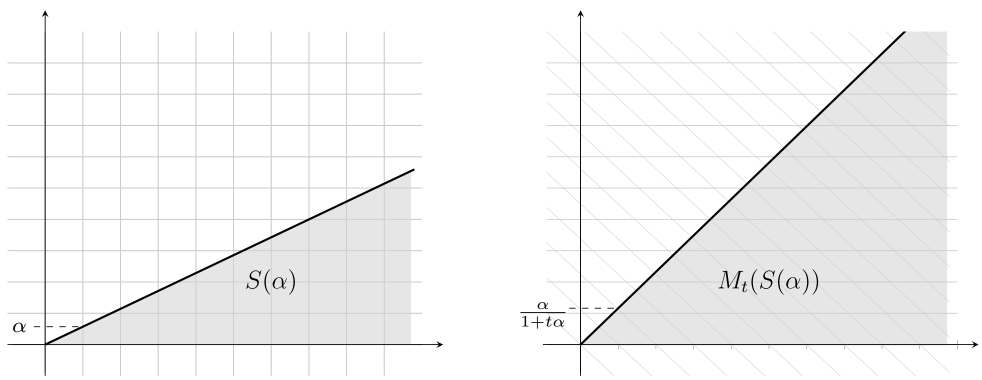

Figure 2.2: $S(\alpha)$ and $M_t(S(\alpha))$ with $-\alpha < t < 0$.

if and only if $P \circ M_{-t}$ is a packing polynomial on $M_t(S(\alpha))$ since $I(\alpha)$ is mapped bijectively onto $M_t(I(\alpha))$. To be concrete, since $M_t(1, \alpha) = (1 + \alpha t, \alpha)$, we have

$$M_t(S(\alpha)) = S\left(\frac{\alpha}{1 + \alpha t}\right),$$

and if $P(x, y)$ is a packing polynomial on $S(\alpha)$, then the equivalent packing polynomial, $\widehat{P}(x, y)$, on $M_t(S(\alpha))$ is

$$\begin{aligned}\widehat{P}(x, y) &= P(x - ty, y) \\ &= \frac{A}{2}x^2 + (B - At)xy + \frac{At^2 - 2Bt + C}{2}y^2 \\ &\quad + D'x + (E' - D't)y + F\end{aligned}$$

Remark 2.8. Since $A > 0$, by choosing t big enough, we can ensure that $B < 0$ and $C > 0$.

Example 2.9. The polynomials Nathanson found for integral sectors and sectors of the form $S(n/m)$ with $n \mid (m-1)$, Thm. 1.4, are equivalent under a transformation of the above type. On $S(n)$, the polynomial (1.0.5)

$$F_n(x,y) = \frac{n}{2}x(x-1) + x + y$$

is a QPP. Let $m \in \mathbb{N}$ such that $n \mid (m-1)$. Then

$$M_{\frac{m-1}{n}}(S(n)) = S\left(\frac{n}{m}\right)$$

and on $S(n/m)$,

$$\begin{aligned} P(x,y) = F_n\left(M_{-\frac{m-1}{n}}(x,y)\right) &= F_n\left(x - \frac{m-1}{n}y, y\right) \\ &= \frac{n}{2}\left(x - \frac{m-1}{n}y\right)\left(x - \frac{m-1}{n}y - 1\right) + x - \frac{m-1}{n}y + y \end{aligned}$$

is a QPP. It is the polynomial (1.0.7). Similarly, the polynomials (1.0.6) and (1.0.8) are equivalent.

Lemma 2.10. *Let P be a QPP on $S(\alpha)$ with $B < 0$ and $C > 0$. If $\Delta > 0$, then*

$$\alpha \le \frac{-B - \sqrt{\Delta}}{C} < \frac{-B + \sqrt{\Delta}}{C}.$$

If $\Delta = 0$, then

$$\alpha < -\frac{B}{C}.$$

Proof. The roots of the polynomial $\frac{A}{2} + Bt + \frac{C}{2}t^2$ are $\frac{-B \pm \sqrt{\Delta}}{C}$. Given $B < 0$, $A > 0$ and $C > 0$, these are both positive and the polynomial has negative values between the roots. If $\alpha > \frac{-B - \sqrt{\Delta}}{C}$, then there is a rational line $y = \frac{q}{p}x$ which falls inside $S(\alpha)$ on which $P_2(x,y)$ is negative. This is impossible, since it would imply that $P(px, qx) < 0$ for large enough x.

If $\Delta = 0$, then $B = -\sqrt{AC}$ and we can write

$$P_2(x,y) = \frac{1}{2}\left(\sqrt{A}x - \sqrt{C}y\right)^2.$$

So on the line $y = \sqrt{A/C}x$,

$$P_2\left(x, \sqrt{A/C}x\right) = 0.$$

But $\sqrt{A/C} = -B/C$, so the line is rational. By Lem. 2.7, $P_2(x,y)$ can not vanish on a rational line falling inside $S(\alpha)$. This means that we must have $\sqrt{A/C} = -B/C > \alpha$. □

We will apply other types of invertible transformations in the parts to come, including some not in $\mathrm{GL}_2(\mathbb{Z})$. In order to treat images under such transformations as equivalent, we need to remember that where a packing polynomial on $S(\alpha)$ maps $I(\alpha) = S(\alpha) \cap \mathbb{Z}^2$ onto $\mathbb{N}_0$, the equavalent polynomial on $M(S(\alpha))$ maps $M(I(\alpha))$ bijectively onto $\mathbb{N}_0$, and $M(I(\alpha)) \neq M(S(\alpha)) \cap \mathbb{Z}^2$ is a possibility. This will be explained in more detail when it appears. At the moment, we can apply transformations to settle a question regarding sectors between two arbitrary rational rays, i.e. not confining one to the x-axis.

Proposition 2.11. *Let $\frac{q}{p} > \frac{s}{r}$ be positive rationals in lowest terms and let*

$$S\left(\frac{q}{p}, \frac{s}{r}\right) = \left\{(x,y) \ : \ \frac{s}{r}x \leq y \leq \frac{q}{p}x\right\}.$$

There exists a positive rational, $\alpha \in \mathbb{Q}_+$, such that the sectors $S(\alpha)$ and $S\left(\frac{q}{p}, \frac{s}{r}\right)$ are equivalent under a linear transformation $M \in \mathrm{GL}_2(\mathbb{Z})$. In particular, every packing polynomial on $S\left(\frac{q}{p}, \frac{s}{r}\right)$ is in 1–1 correspondence with a packing polynomial on $S(\alpha)$.

Proof. Since $\gcd(r,s) = 1$, there exist integers, $a, b \in \mathbb{Z}$, such that $ar + bs = 1$. Also, $(a - ks)r + (b + kr)s = 1$ for all $k \in \mathbb{Z}$. The matrix

$$M = \begin{pmatrix} a - ks & b + kr \\ -s & r \end{pmatrix}$$

has $\det(M) = 1$, so $M \in \mathrm{GL}_2(\mathbb{Z})$. We have

$$M(r,s) = (1,0) \quad \text{and} \quad M(p,q) = (ap + bq + k(rq - ps), rq - sp).$$

Since $q/p > s/r$, we have $rq - sp > 0$. Choosing k big enough, we can ensure that also $ap + bq + k(rq - ps) > 0$. So, with

$$\alpha = \frac{rq - sp}{ap + bq + k(rq - ps)},$$

we have

$$M\left(S\left(\frac{q}{p}, \frac{s}{r}\right)\right) = S(\alpha).$$

□

Chapter 3

Slope Constraint

In this chapter, we will show that if $P(x, y)$ is a quadratic polynomial satisfying conditions necessary for a packing polynomial on the sector $S(\alpha)$, then α must follow a specific formula that depends only on the quadratic part of $P(x, y)$. The method used to obtain this result is similar to that used by Lew and Rosenberg [8].

3.1 Limit Considerations

Consider the following limit

$$\mathrm{LR}(P, I(\alpha)) = \lim_{n\to\infty} \left(\frac{1}{n} \# \big(I(\alpha) \cap P^{-1}([0, n]) \big) \right).$$

Note that this only coincides with the natural density if P is injective. The label LR is in honor of Lew and Rosenberg who introduced this in [8] where they call it a density as they assume injectivity for their study. For arbitrary polynomials there is no reason to expect this limit to exist, but for a polynomial P to be a packing polynomial on $S(\alpha)$, it is a necessary condition that $\mathrm{LR}(P, I(\alpha)) = 1$. This is true since, if P is a packing polynomial, we have $\#\big(I(\alpha) \cap P^{-1}([0, n])\big) = n + 1$ for all n. We want to calculate this limit and, assuming it is 1, obtain a restriction on α.

Define the regions

$$R_n = R_n(S(\alpha), P) = S(\alpha) \cap P^{-1}([0,n])$$
$$= \{(x,y) \in S(\alpha) \ : \ 0 \leq P(x,y) \leq n\}. \tag{3.1.1}$$

The limit can then be written as

$$\mathrm{LR}(P, I(\alpha)) = \lim_{n\to\infty} \frac{1}{n} \#(\mathbb{Z}^2 \cap R_n).$$

Assuming the limit exists, we will compute it by finding an estimate for the number of lattice points in these regions. We will invoke a theorem of Davenport. The following is the theorem as stated in [4, 5] (with slight notation modification).

Let Ω be a closed region of $\mathbb{R}^d$ and let $\mathrm{vol}(\Omega)$ denote its volume. Let $V_m(\Omega)$ denote the sum of the m-dimensional volumes of the projections of Ω onto the subspaces obtained by setting any $d-m$ coordinates to zero. We set $V_0(\Omega) = 1$ by convention. Suppose the following conditions are met:

I Any line parallel to one of the d coordinate axes intersects Ω in a point set consisting of at most h intervals, given it is non-empty.

II The same is true (with m in place of d) for any of the m dimensional regions obtained by projection Ω onto one of the coordinate spaces defined by setting $d-m$ of the coordinates to 0. This must hold for all $m = 1, ..., d-1$.

Theorem 3.1 (Davenport)**.** *If Ω satifies conditions (I) and (II), then*

$$|\#(\Omega \cap \mathbb{Z}^d) - \mathrm{vol}(\Omega)| \leq \sum_{m=0}^{d-1} h^{d-m} V_m(\Omega).$$

In our case, the regions of interest, the regions R_n, are the subsets of $S(\alpha)$ bounded by the level curves $P(x,y) = 0$ and $P(x,y) = n$. These regions, if they are bounded, fulfill the conditions of the theorem. $S(\alpha)$ is a convex region, so any straight line intersects $S(\alpha)$ in at most one line segment. The level curves, $P(x,y) = 0$ and $P(x,y) = n$, are given by quadratic equations, so a straight line can intersect each in at most two points. This means that any straight line can enter and leave the region R_n at most twice. In particular, the intersection with a line parallel to an axis can consist of at most two disjoint intervals. We can therefore set the coefficient h in the theorem to 2. The subspaces in the sum of the theorem are the projections onto the x-axis and y-axis respectively (along with V_0, the volume of which is 1). Since our only interest is the case $d = 2$, we will state and prove this special case. Davenport's proof similar, only using induction over d.

Theorem 3.2 (Davenport, case $d = 2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a region with the property that any line parallel to either the x- or y-axis intersects Ω in a set consisting of a finite number of disjoint intervals (here, a point is considered an interval). Let h be the highest number of intervals in any such intersection. Then*

$$|\operatorname{area}(\Omega) - \#(\Omega \cap \mathbb{Z}^2)| \leq h|\pi_x(\Omega)| + h|\pi_y(\Omega)| + h^2,$$

where $\pi_x(\Omega)$ (resp. $\pi_y(\Omega)$) denotes the projection of Ω onto the x-axis (resp. y-axis), and $|\pi_x(\Omega)|$ and $|\pi_y(\Omega)|$ their respective sizes, that is, the sum of the lengths of disjoint intervals.

Proof. Let

$$\mathbb{1}_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

be the indicator function. We can then write

$$|\pi_x(\Omega)| = \int_{\mathbb{R}} \mathbb{1}_{\pi_x(\Omega)}(x)dx, \quad |\pi_y(\Omega)| = \int_{\mathbb{R}} \mathbb{1}_{\pi_y(\Omega)}(y)dy,$$

and

$$\operatorname{area}(\Omega) = \iint_{\mathbb{R}^2} \mathbb{1}_\Omega(x,y)dxdy = \int_{\mathbb{R}} dx \int_{\mathbb{R}} \mathbb{1}_\Omega(x,y)dy$$

and

$$\#(\Omega \cap \mathbb{Z}^2) = \sum_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} \mathbb{1}_\Omega(x,y).$$

For the 1-dimensional projections, for each disjoint interval, the discrepancy between length and number of integral points is ± 1, dependent on whether the two endpoints of the interval include an integral point. The total is then at most the number of disjoint intervals, which is at most h, so

$$||\pi_x(\Omega)| - \#(\pi_x(\Omega) \cap \mathbb{Z})| \leq h, \quad ||\pi_y(\Omega)| - \#(\pi_y(\Omega) \cap \mathbb{Z})| \leq h,$$

or

$$\left| \int_{\mathbb{R}} \mathbb{1}_{\pi_x(\Omega)}(x) dx - \sum_{x \in \mathbb{Z}} \mathbb{1}_{\pi_x(\Omega)}(x) \right| \leq h, \tag{3.1.2}$$

and

$$\left| \int_{\mathbb{R}} \mathbb{1}_{\pi_y(\Omega)}(y) dy - \sum_{y \in \mathbb{Z}} \mathbb{1}_{\pi_y(\Omega)}(y) \right| \leq h. \tag{3.1.3}$$

Note that (3.1.2) can be reformulated as

$$\sum_{x \in \mathbb{Z}} \mathbb{1}_{\pi_x(\Omega)}(x) \leq |\pi_x(\Omega)| + h \tag{3.1.4}$$

Inequalites (3.1.2) and (3.1.3) can be extended, taking the other variable into account, as for fixed y

$$\left| \int_{\mathbb{R}} \mathbb{1}_{\Omega}(x, y) dx - \sum_{x \in \mathbb{Z}} \mathbb{1}_{\Omega}(x, y) \right| \leq h \mathbb{1}_{\pi_y(\Omega)}(y), \tag{3.1.5}$$

and for fixed x

$$\left| \int_{\mathbb{R}} \mathbb{1}_{\Omega}(x, y) dy - \sum_{y \in \mathbb{Z}} \mathbb{1}_{\Omega}(x, y) \right| \leq h \mathbb{1}_{\pi_x(\Omega)}(x), \tag{3.1.6}$$

Integrating (3.1.5) wrt. y, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{1}_{\Omega}(x, y) dx - \sum_{x \in \mathbb{Z}} \mathbb{1}_{\Omega}(x, y) \right) dy \right| \\ \leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathbb{1}_{\Omega}(x, y) dx - \sum_{x \in \mathbb{Z}} \mathbb{1}_{\Omega}(x, y) \right| dy \\ \leq \int_{\mathbb{R}} h \mathbb{1}_{\pi_y(\Omega)}(y) dy. \end{aligned} \tag{3.1.7}$$

We have

$$\begin{aligned}\left|\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dx-\sum_{x\in\mathbb{Z}}\mathbb{1}_\Omega(x,y)\right)dy\right| \\ &= \left|\iint_{\mathbb{R}^2}\mathbb{1}_\Omega(x,y)dxdy-\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy\right| \\ &= \left|\text{area}(\Omega)-\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy\right|,\end{aligned}$$

and

$$\int_{\mathbb{R}} h\mathbb{1}_{\pi_y(\Omega)}(y)dy = h|\pi_y(\Omega)|,$$

so (3.1.7) yields

$$\left|\text{area}(\Omega)-\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy\right| \le h|\pi_y(\Omega)|. \tag{3.1.8}$$

Summing (3.1.6) over integral x, we obtain

$$\begin{aligned}\left|\sum_{x\in\mathbb{Z}}\left(\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\sum_{y\in\mathbb{Z}}\mathbb{1}_\Omega(x,y)\right)\right| \\ &\le \sum_{x\in\mathbb{Z}}\left|\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\sum_{y\in\mathbb{Z}}\mathbb{1}_\Omega(x,y)\right| \\ &\le \sum_{x\in\mathbb{Z}} h\mathbb{1}_{\pi_x(\Omega)}(x).\end{aligned} \tag{3.1.9}$$

We have

$$\begin{aligned}\left|\sum_{x\in\mathbb{Z}}\left(\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\sum_{y\in\mathbb{Z}}\mathbb{1}_\Omega(x,y)\right)\right| \\ &= \left|\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}\mathbb{1}_\Omega(x,y)\right| \\ &= \left|\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\#(\Omega\cap\mathbb{Z}^2)\right|,\end{aligned}$$

and, by (3.1.4),

$$\sum_{x\in\mathbb{Z}} h\mathbb{1}_{\pi_x(\Omega)}(x) \le h(|\pi_x(\Omega)|+h).$$

From (3.1.9) we then get

$$\left|\sum_{x\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_\Omega(x,y)dy-\#(\Omega\cap\mathbb{Z}^2)\right| \le h|\pi_x(\Omega)|+h^2. \tag{3.1.10}$$

Adding inequalites (3.1.8) and (3.1.10) yields

$$
\begin{aligned}
&|\operatorname{area}(\Omega) - \#(\Omega \cap \mathbb{Z}^2)| \\
&\quad = \left| \operatorname{area}(\Omega) - \sum_{x\in\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1}_\Omega(x,y)dy + \sum_{x\in\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1}_\Omega(x,y)dy - \#(\Omega \cap \mathbb{Z}^2) \right| \\
&\quad \le \left| \operatorname{area}(\Omega) - \sum_{x\in\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1}_\Omega(x,y)dy \right| + \left| \sum_{x\in\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1}_\Omega(x,y)dy - \#(\Omega \cap \mathbb{Z}^2) \right| \\
&\quad \le h|\pi_x(\Omega)| + h|\pi_y(\Omega)| + h^2
\end{aligned}
$$

which is what we wanted to show. □

Applying the theorem to the regions R_n (as defined in (3.1.1) and assuming they are bounded), we obtain the estimate

$$|\#(R_n \cap \mathbb{Z}^2) - \operatorname{area}(R_n)| \le 2|\pi_x(R_n)| + 2|\pi_y(R_n)| + 4. \tag{3.1.11}$$

The coefficients 2 and constant 4 arise from setting $h = 2$ which is the worst case scenario. Any constant would do for the following analysis.

Example 3.3 (Gauß's circle problem)**.** Let $C(r)$ be the circle centered at the origin with radius r. Let $A(C(r))$ be the area of the circle and $N(C(r))$ the number of lattice points in its closed interior. Gauß showed that

$$|A(C(r)) - N(C(r))| \le 2\sqrt{2}\pi r.$$

Asymptotically, the best known error at the time of writing, due to Huxley, is $O(r^{46/73+\epsilon})$ and it is conjectured to be $O(r^{1/2+\epsilon})$.

The lengths of the projections onto the coordinate axes are

$$|\pi_x(C(r))| = |\pi_y(C(r))| = 2r,$$

All straight lines intersect the circle in at most one interval, so Davenport gives an error of

$$|A(C(r)) - N(C(r))| \le 4r + 1,$$

slightly better than Gauß, but asymptotically not optimal.

3.2 Restriction on α

In the following we will assume a polynomial of the standard form (2.1.1) with $A > 0$, $B < 0$, $C > 0$. In the case of a QPP, this is justified by Lem. 2.6 and Rem. 2.8. We will furthermore assume that if $\Delta \geq 0$, then

$$\alpha < \frac{-B - \sqrt{\Delta}}{C}.$$

For a QPP, this is justified by Lem. 2.10, except for the case $\alpha = \frac{-B-\sqrt{\Delta}}{C}$, $\Delta > 0$. This situation can be ruled out for a QPP also, as we shall see.[1]

Lemma 3.4. *Let R_n be the regions defined by (3.1.1). We have*

$$2|\pi_x(R_n)| + 2|\pi_y(R_n)| + 4 = O(\sqrt{n})$$

as n tends to infinity.

Proof. We will treat the cases $\Delta < 0$, $\Delta > 0$ and $\Delta = 0$, corresponding to the level curves being respecitively ellipses, hyperbolas and parabolas, individually.

If $\Delta \neq 0$, we can rewrite $P(x, y)$ as

$$P(x, y) = \frac{A}{2}(x - x_0)^2 + B(x - x_0)(y - y_0) + \frac{C}{2}(y - y_0)^2 + K \tag{3.2.1}$$

where a calculation shows that

$$x_0 = \frac{CD' - BE'}{\Delta}, \quad y_0 = \frac{AE' - BD'}{\Delta}, \quad K = F + \frac{D'}{2}x_0 + \frac{E'}{2}y_0 \tag{3.2.2}$$

The coordinate (x_0, y_0) is the center of the level curve $P(x, y) = n$, be it an ellipse or a hyperbola.

Case $\Delta < 0$: The level curves $P(x, y) = n$ are ellipses. The length of the projection of R_n onto the x-axis is bounded by the difference between the smallest and biggest x-coordinate of all points on the ellipse. Similar for the y-axis projection. Let

$$y - y_0 = a(x - x_0)$$

[1]We will eventually rule out any polynomial with non-zero discriminant, making the point a bit moot.

be a line through the center. The x-coordinate of where it crosses the ellipse is given by

$$\frac{A}{2}(x-x_0)^2 + Ba(x-x_0)^2 + \frac{C}{2}a^2(x-x_0)^2 + K$$
$$= \left(\frac{A}{2} + Ba + \frac{C}{2}a^2\right)(x-x_0)^2 + K = n,$$

$$x = \pm\sqrt{\frac{n-K}{\frac{A}{2} + Ba + \frac{C}{2}a^2}} + x_0.$$

The difference between the two is

$$2\sqrt{\frac{n-K}{\frac{A}{2} + Ba + \frac{C}{2}a^2}}.$$

Since $C > 0$ and $\Delta = B^2 - AC < 0$, we have $\frac{A}{2} + Ba + \frac{C}{2}a^2 > 0$, so the maximal distance between x-coordinates is where $\frac{A}{2} + Ba + \frac{C}{2}a^2$ is minimal, that is

$$a = -\frac{B}{C}, \quad \frac{A}{2} + Ba + \frac{C}{2}a^2 = -\frac{\Delta}{2C}.$$

So the $|\pi_x(R_n)|$ is bounded by

$$2\sqrt{\frac{n-K}{-\frac{\Delta}{2C}}} = O(\sqrt{n}).$$

The same reasoning leads to the same conclusion for the y-projection.

Case $\Delta > 0$*:* The level curves

$$P(x,y) = \frac{A}{2}(x-x_0)^2 + B(x-x_0)(y-y_0) + \frac{C}{2}(y-y_0)^2 + K = n \tag{3.2.3}$$

are hyperbolas. We are assuming $B < 0$ and $C > 0$. Solving

$$\frac{A}{2} + B\frac{y-y_0}{x-x_0} + \frac{C}{2}\left(\frac{y-y_0}{x-x_0}\right)^2 = 0$$

for $\frac{y-y_0}{x-x_0}$ shows that the slopes of the asymptotes of the level curves are

$$0 < m_1 = -\frac{B}{C} - \frac{\sqrt{\Delta}}{C} < m_2 = -\frac{B}{C} + \frac{\sqrt{\Delta}}{C}$$

By assumption, we have

$$0 < \alpha < m_1 < m_2.$$

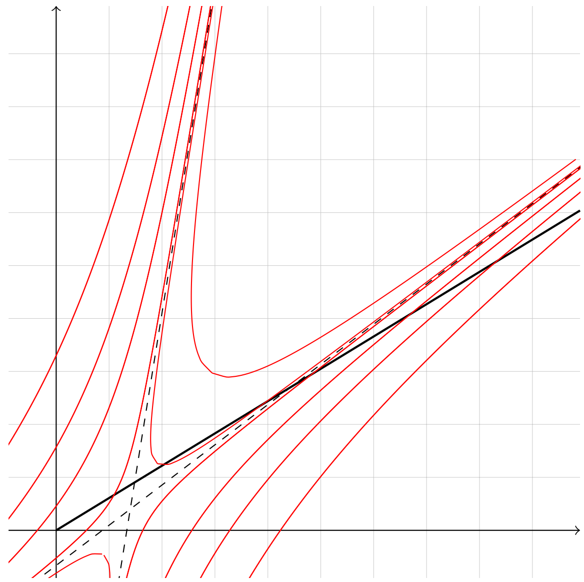

Figure 3.1: Hyperbolic level curves.

For n big enough, the projection of R_n onto the y-axis is the y-coordinate of the intersection of the hyperbola with the line $y = \alpha x$. The projection onto the x-axis the x-coordinate of the intersections of the hyperbola with the line $y = \alpha x$.

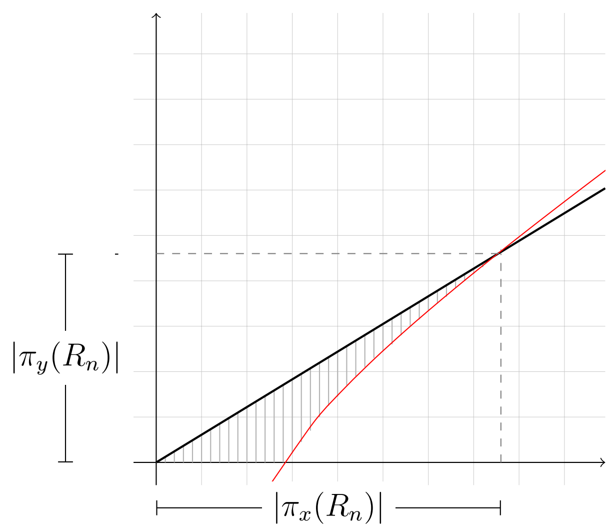

Figure 3.2: Projections of R_n onto the axes.

The equation $P(x, \alpha x) = n$ is of the form

$$ax^2 + bx + c - n = 0,$$

where $a, b, c \in \mathbb{R}$ are independent of n. The solutions are $O(\sqrt{n})$ as $n \to \infty$, so also in this case, we find that $4 + 2\pi_x(R_n) + 2\pi_y(R_n) = O(\sqrt{n})$ as $n \to \infty$.

Case $\Delta = 0$*:* By Rem. 2.8, we may assume that $B < 0$ and $C > 0$. $B^2 = AC$ then implies that $B = -\sqrt{AC}$, and we can write $P(x, y)$ as

$$P(x, y) = \frac{1}{2}\left(\sqrt{A}x - \sqrt{C}y\right)^2 + D'x + E'y + F.$$

The level curves are given by $P(x, y) = n$. The symmetry axis for the parabolas has slope $\sqrt{A/C} = -B/C > \alpha$, by Lem. 2.10. We conclude that the picture must look as Fig. 3.3. The lengths of the projections are bounded by whichever is larger of the intersection with

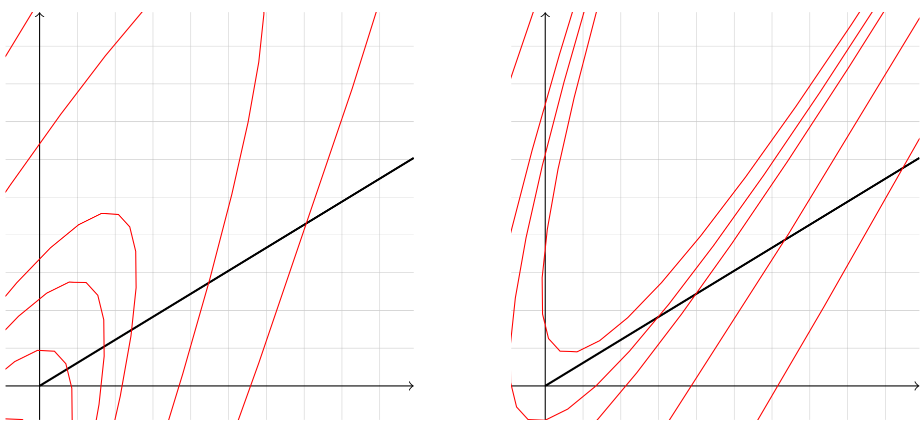

Figure 3.3: Parabolic level curves.

$y = \alpha x$, the x-axis, the topmost point on the level curve, the bottommost or the rightmost point. Taking the derivative of

$$\frac{1}{2}\left(\sqrt{A}x - \sqrt{C}y\right)^2 + D'x + E'y + F = n$$

wrt. x yields

$$\left(\sqrt{A}x - \sqrt{C}y\right)\left(\sqrt{A} - \sqrt{C}\frac{dy}{dx}\right) + D' + E'\frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = -\frac{Ax + D' - \sqrt{AC}y}{Cy + E' - \sqrt{AC}x}$$

The top or bottom point on the curve is found where the derivative wrt. x is zero, so

$$y = \sqrt{\frac{A}{C}}x + \frac{D'}{\sqrt{AC}}.$$

Since $\sqrt{A/C} > \alpha$, for n large enough, the projection onto the y-axis will be bounded by the intersection with the line $y = \alpha x$. Taking the derivative wrt. y yields

$$\left(\sqrt{A}x - \sqrt{C}y\right)\left(\sqrt{A}\frac{dx}{dy} - \sqrt{C}\right) + D'\frac{dx}{dy} + E' = 0,$$

so

$$\frac{dx}{dy} = -\frac{Cy + E' - \sqrt{AC}x}{Ax + D' - \sqrt{AC}y}.$$

The rightmost point on the level curves is where this vanishes, so

$$y = \sqrt{\frac{A}{C}}x - \frac{E'}{C}$$

and the conclusion is the same as above. The intersection of $P(x, y) = n$ and the x-axis or $y = \alpha x$ are given by quadratic equations

$$ax^2 + bx + c - n = 0,$$

where $a, b, c \in \mathbb{R}$ are independent of n. The solutions are $O(\sqrt{n})$ as $n \to \infty$. □

Remark 3.5. As mentioned above, all conditions of the lemma are justified by assuming a QPP with the exception of the case $\Delta > 0$ and

$$\alpha = \frac{-B - \sqrt{\Delta}}{C} = m_1,$$

m_1 being the lower asymptote of the hyperbolic level curves. By Lem. 2.7, α is irrational. Let (x_0, y_0) be the center, given by (3.2.2). If (x_0, y_0) lies above the line $y = \alpha x$, then all level curves cross this line, the regions R_n are bounded and the conclusions of the lemma hold. If (x_0, y_0) is below the line, then negative level curves will fall inside $S(\alpha)$ which can't happen if $P(x, y)$ is a packing polynomial. We may therefore assume that (x_0, y_0) falls on the line. Since x_0, y_0 are rational and α is irrational, this means that $(x_0, y_0) = (0, 0)$. The level curves thus take the form

$$\frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 = n - F \tag{3.2.4}$$

which can be rewritten as

$$(Ax + By)^2 - \Delta y^2 = 2A(n - F).$$

Since α is irrational, Δ is not a perfect square, so this is a Pell type equation. If it has one solution in integers, it has infinitely many. Specifically, if (x_1, y_1) is a solution to (3.2.4), any solution (s_i, t_i) to the Pell equation $s^2 - \Delta t^2 = 1$ generates a new solution

$$((x_1 s_i - (Bx_1 + Cy_1)t_i), (Ax_1 + By_1)t_i + y_1 s_i))$$

to (3.2.4), since

$$\begin{aligned}
(A(x_1 s_i - (Bx_1 + Cy_1)t_i) + B((Ax_1 + By_1)t_i + y_1 s_i))^2 \\
- \Delta((Ax_1 + By_1)t_i + y_1 s_i)^2 \\
= (s_i(Ax_1 + By_1) + \Delta t_i y_1)^2 \\
- \Delta((Ax_1 + By_1)^2 t_i^2) + 2(Ax_1 + By_1)t_i y_1 s_i + y_1^2 s_i^2) \\
= s_i^2(Ax_1 + By_1)^2 + 2s_i(Ax_1 + By_1)\Delta t_i y_1 + \Delta^2 t_i^2 y_1^2 \\
- \Delta(Ax_1 + By_1)^2 t_i^2 - 2\Delta(Ax_1 + By_1)t_i y_1 s_i - \Delta y_1^2 s_i^2 \\
= (s_i^2 - \Delta t_i^2)(Ax_1 + By_1)^2 - (s_i^2 - \Delta t_i^2)\Delta y_1^2 \\
= (Ax_1 + By_1)^2 - \Delta y_1^2.
\end{aligned}$$

Remark 3.6. Note that R_n isn't necessarily simply a region bounded by the level curve $P(x, y) = n$, the line $y = \alpha x$ and the x-axis. There could be a small area "missing" bounded by the level curve $P(x, y) = 0$. For the projections of concern here, this doesn't change anything. It will be worth a comment when the area is being calculated, but it won't cause issues.

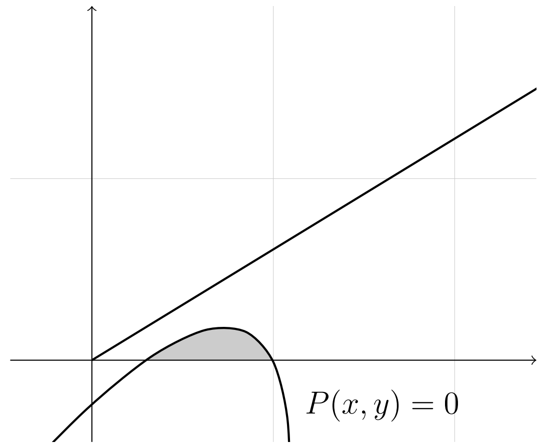

Figure 3.4: Area "missing" from R_n.

Lemma 3.7. *We have*

$$\text{area}(R_n) = \frac{n}{2}\int_0^{\arctan\alpha} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} + O(\sqrt{n}) + O(1)$$

Proof. To calculate the area of R_n, we switch to polar coordinates, putting $(x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$. The equation of the level curve then takes the form

$$P(r(\theta)\cos\theta, r(\theta)\sin\theta) - n = \\ r^2(\theta)P_2(\cos\theta, \sin\theta) + r(\theta)P_1(\cos\theta, \sin\theta) + F - n = 0.$$

Aside from a possible region bounded by the level curve $P(x, y) = 0$ (cf. Rem. 3.6), the region R_n is the simply-connected region bounded by the x-axis, the line $y = \alpha x$ and the the level curve $P(x, y) = n$. Supressing the variables in our notation for a moment, we find that

$$r(\theta) = -\frac{P_1}{2P_2} \pm \frac{\sqrt{P_1^2 - 4P_2(F - n)}}{2P_2}.$$

Note that the denominator is never zero, as

$$\begin{aligned} P_2(\cos\theta, \sin\theta) &= \frac{A}{2}\cos^2\theta + B\cos\theta\sin\theta + \frac{C}{2}\sin^2\theta \\ &= \frac{1}{\cos^2\theta}\left(\frac{A}{2} + B\tan\theta + \frac{C}{2}\tan^2\theta\right) = 0 \end{aligned}$$

implies that $\Delta \geq 0$ and $\tan\theta = \frac{-B\pm\sqrt{\Delta}}{C}$, but

$$0 \leq \theta \leq \arctan\alpha < \arctan\left(\frac{-B - \sqrt{\Delta}}{C}\right) \leq \frac{\pi}{2}.$$

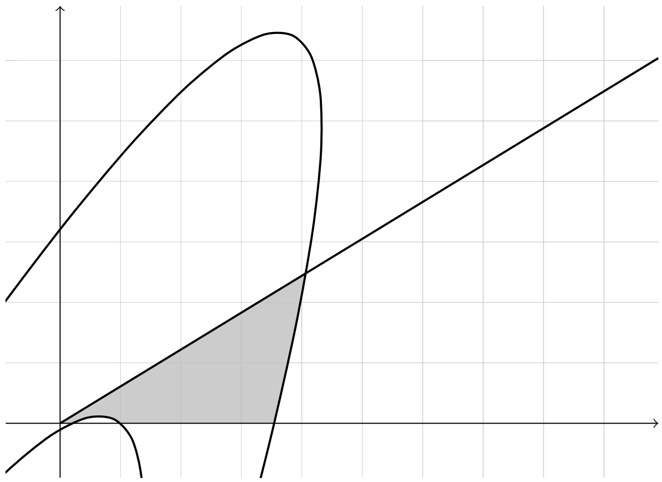

Figure 3.5: The region R_n.

Let A_0 be the area of the possibly missing region. The area of R_n is then given by

$$\text{area}(R_n) = \frac{1}{2}\int_0^{\arctan\alpha} r^2(\theta)d\theta - A_0$$

We have

$$r(\theta)^2 = \left(\frac{P_1}{2P_2}\right)^2 \pm \frac{P_1\sqrt{P_1^2 - 4P_2(F-n)}}{2P_2^2} + \frac{P_1^2 - 4P_2F}{4P_2^2} + \frac{n}{P_2}.$$

So the area is

$$\begin{aligned}\text{area}(R_n) = \frac{1}{2}\int_0^{\arctan\alpha} & \left(\left(\frac{P_1}{2P_2}\right)^2 + \frac{P_1^2 - 4P_2F}{4P_2^2}\right) d\theta - A_0 \\ & \pm \frac{1}{2}\int_0^{\arctan\alpha} \frac{P_1\sqrt{P_1^2 - 4P_2(F-n)}}{2P_2^2} d\theta \\ & + \frac{1}{2}\int_0^{\arctan\alpha} \frac{n}{P_2} d\theta \\ = O(1) + O(\sqrt{n}) + \frac{n}{2}\int_0^{\arctan\alpha} & \frac{d\theta}{P_2(\cos\theta, \sin\theta)}\end{aligned}$$

□

We can now compute the limit.

Theorem 3.8. *Let $P(x,y)$ be a polynomial of the form (2.1.1) with $A > 0$, $B < 0$, $C > 0$ and assume that if $\Delta \geq 0$, then $\alpha < \frac{-B-\sqrt{\Delta}}{C}$.*

If $\Delta = 0$*, then*

$$\mathrm{LR}(P, I(\alpha)) = \frac{1}{B} - \frac{1}{\alpha C + B}.$$

If $\Delta > 0$*, then*

$$\mathrm{LR}(P, I(\alpha)) = \frac{1}{2\sqrt{\Delta}} \log \frac{\alpha(B + \sqrt{\Delta}) + A}{\alpha(B - \sqrt{\Delta}) + A}$$

if $\Delta < 0$*, then*

$$\mathrm{LR}(P, I(\alpha)) = \frac{1}{\sqrt{-\Delta}} \left(\arctan \frac{\alpha\sqrt{-\Delta}}{\alpha B + A} \pmod{\pi} \right)$$

or, in case $\alpha = -A/B$*,*

$$\mathrm{LR}(P, I(\alpha)) = \frac{\pi}{2\sqrt{-\Delta}}$$

Note that the assumptions of the theorem can be assumed if $P(x, y)$ is a QPP on $S(\alpha)$ by Lem. 2.6, Rem. 2.8, Lem. 2.10 and Rem. 3.5.

Proof of Thm. 3.8. By Lem. 3.4 and Lem. 3.7,

$$\begin{aligned}
\mathrm{LR}(P, I(\alpha)) &= \lim_{n\to\infty} \frac{1}{n} \#(\mathbb{Z}^2 \cap R_n) \\
&= \lim_{n\to\infty} \frac{1}{n} (\mathrm{area}(R_n) + O(\sqrt{n})) \\
&= \lim_{n\to\infty} \frac{\mathrm{area}(R_n)}{n} \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\frac{n}{2} \int_0^{\arctan \alpha} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} + O(\sqrt{n}) + O(1) \right) \\
&= \frac{1}{2} \int_0^{\arctan \alpha} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} + \lim_{n\to\infty} \left(\frac{O(\sqrt{n})}{n} + \frac{O(1)}{n} \right) \\
&= \frac{1}{2} \int_0^{\arctan \alpha} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} \\
&= \frac{1}{2} \int_0^{\arctan \alpha} \frac{d\theta}{\frac{A}{2}\cos^2\theta + B\cos\theta\sin\theta + \frac{C}{2}\sin^2\theta} \\
&= \frac{1}{2} \int_0^{\arctan \alpha} \frac{d\theta}{\cos^2\theta(\frac{A}{2} + B\tan\theta + \frac{C}{2}\tan^2\theta)}
\end{aligned}$$

Apply the change of variable $t = \tan\theta$, so $dt = \frac{d\theta}{\cos^2\theta}$ and the new boundaries are $t = 0$ and $t = \alpha$. So the integral becomes

$$\int_0^{\alpha} \frac{dt}{A + 2Bt + Ct^2} = \frac{1}{C} \int_0^{\alpha} \frac{dt}{\frac{A}{C} + 2\frac{B}{C}t + t^2} = \frac{1}{C} \int_0^{\alpha} \frac{dt}{(t + \frac{B}{C})^2 - \frac{\Delta}{C^2}}$$

Case $\Delta = 0$*:*

$$\begin{aligned}\mathrm{LR}(P, I(\alpha)) &= \frac{1}{C}\int_0^\alpha \frac{dt}{(t+\frac{B}{C})^2} = \frac{1}{C}\left[-\frac{1}{t+\frac{B}{C}}\right]_0^\alpha = -\frac{1}{C}\left(\frac{1}{\alpha+\frac{B}{C}} - \frac{1}{\frac{B}{C}}\right)\\ &= \frac{1}{B} - \frac{1}{\alpha C + B}.\end{aligned}$$

Note that, by Rem. 2.10, we may assume $-B/C > \alpha$.

Case $\Delta > 0$*:* With Δ positive, we can write

$$\begin{aligned}\mathrm{LR}(P, I(\alpha)) &= \frac{1}{C}\int_0^\alpha \frac{dt}{(t+\frac{B}{C})^2 - \frac{\Delta}{C^2}} = \frac{1}{C}\int_0^\alpha \frac{dt}{\frac{\Delta}{C^2}((\frac{C}{\sqrt{\Delta}}t + \frac{B}{\sqrt{\Delta}})^2 - 1)}\\ &= \frac{C}{\Delta}\int_0^\alpha \frac{dt}{(\frac{Ct+B}{\sqrt{\Delta}})^2 - 1}\end{aligned}$$

Apply the change of variables

$$u = \frac{Ct+B}{\sqrt{\Delta}}, \quad du = \frac{C}{\sqrt{\Delta}}dt$$

yielding boundaries

$$u = \frac{B}{\sqrt{\Delta}}, \quad \text{and} \quad u = \frac{\alpha C + B}{\sqrt{\Delta}}.$$

So we have

$$\begin{aligned}\mathrm{LR}(P, I(\alpha)) &= \frac{1}{\sqrt{\Delta}}\int_{\frac{B}{\sqrt{\Delta}}}^{\frac{\alpha C+B}{\sqrt{\Delta}}} \frac{du}{(u+1)(u-1)}\\ &= \frac{1}{\sqrt{\Delta}}\int_{\frac{B}{\sqrt{\Delta}}}^{\frac{\alpha C+B}{\sqrt{\Delta}}} \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)}\right) du\\ &= \frac{1}{2\sqrt{\Delta}}\left[\log\frac{u-1}{u+1}\right]_{\frac{B}{\sqrt{\Delta}}}^{\frac{\alpha C+B}{\sqrt{\Delta}}}\\ &= \frac{1}{2\sqrt{\Delta}}\left(\log\frac{\frac{\alpha C+B}{\sqrt{\Delta}} - 1}{\frac{\alpha C+B}{\sqrt{\Delta}} + 1} - \log\frac{\frac{B}{\sqrt{\Delta}} - 1}{\frac{B}{\sqrt{\Delta}} + 1}\right) = \frac{1}{2\sqrt{\Delta}}\log\frac{\frac{\alpha C+B-\sqrt{\Delta}}{\alpha C+B+\sqrt{\Delta}}}{\frac{B-\sqrt{\Delta}}{B+\sqrt{\Delta}}}\\ &= \frac{1}{2\sqrt{\Delta}}\log\frac{\alpha C(B+\sqrt{\Delta}) + B^2 - \Delta}{\alpha C(B-\sqrt{\Delta}) + B^2 - \Delta} = \frac{1}{2\sqrt{\Delta}}\log\frac{\alpha(B+\sqrt{\Delta}) + A}{\alpha(B-\sqrt{\Delta}) + A}\end{aligned}$$

Note $u = \pm 1$ is of no concern, since that would imply $t = \frac{-B\pm\sqrt{\Delta}}{C}$, but those are the slopes of the asymptotes, both of which are greater than α.

Case $\Delta < 0$*:* With Δ negative, we can write

$$\mathrm{LR}(P, I(\alpha)) = \frac{C}{-\Delta} \int_0^\alpha \frac{dt}{(\frac{Ct+B}{\sqrt{-\Delta}})^2 + 1}$$

Apply the same change of variables as in the previous case, with $-\Delta$ instead of Δ. We then have

$$\begin{aligned}\mathrm{LR}(P, I(\alpha)) &= \frac{1}{\sqrt{-\Delta}} \int_{\frac{B}{\sqrt{-\Delta}}}^{\frac{\alpha C+B}{\sqrt{-\Delta}}} \frac{du}{u^2+1} = \frac{1}{\sqrt{-\Delta}} \Big[\arctan u \Big]_{\frac{B}{\sqrt{-\Delta}}}^{\frac{\alpha C+B}{\sqrt{-\Delta}}} \\ &= \frac{1}{\sqrt{-\Delta}} \left(\arctan \frac{\alpha C + B}{\sqrt{-\Delta}} - \arctan \frac{B}{\sqrt{-\Delta}} \right)\end{aligned}$$

If $\alpha = -A/B$, we have

$$\frac{\alpha C + B}{\sqrt{-\Delta}} = \frac{-AC + B^2}{B\sqrt{-\Delta}} = -\frac{\sqrt{-\Delta}}{B}.$$

Since, in general,

$$\arctan x + \arctan \frac{1}{x} = \pm \frac{\pi}{2},$$

we conclude that

$$\mathrm{LR}(P, I(\alpha)) = \frac{\pi}{2\sqrt{-\Delta}}$$

If $\alpha \neq -A/B$, we can use the difference formula for arctan:

$$\arctan u - \arctan v = \arctan \frac{u - v}{1 + uv} \pmod{\pi}, \quad uv \neq 1.$$

We find that

$$\begin{aligned}\mathrm{LR}(P, I(\alpha)) &= \frac{1}{\sqrt{-\Delta}} \left(\arctan \frac{\frac{\alpha C+B}{\sqrt{-\Delta}} - \frac{B}{\sqrt{-\Delta}}}{1 + \frac{(\alpha C+B)B}{-\Delta}} \pmod{\pi} \right) \\ &= \frac{1}{\sqrt{-\Delta}} \left(\arctan \frac{\alpha C\sqrt{-\Delta}}{\alpha BC + B^2 - \Delta} \pmod{\pi} \right) \\ &= \frac{1}{\sqrt{-\Delta}} \left(\arctan \frac{\alpha\sqrt{-\Delta}}{\alpha B + A} \pmod{\pi} \right).\end{aligned}$$

□

Theorem 3.9. *Let* $\alpha > 0$ *and*

$$P(x,y) = A\frac{x(x-1)}{2} + Bxy + C\frac{y(y-1)}{2} + Dx + Ey + F,$$

with $A, B, C, D, E, F \in \mathbb{Z}$ *and put* $\Delta = B^2 - AC$*. If* $A > 0$*,* $B < 0$*,* $C > 0$*, and if* $\alpha < \frac{-B-\sqrt{\Delta}}{C}$ *when* $\Delta \geq 0$*, then the condition that* $\mathrm{LR}(P, I(\alpha)) = 1$ *implies the following.*

If $\Delta = 0$*, then*

$$\alpha = \frac{A}{1-B}.$$

If $\Delta > 0$*, then*

$$\alpha = \frac{A}{\sqrt{\Delta}\coth\sqrt{\Delta} - B}.$$

If $\Delta < 0$*, then*

$$\alpha = \frac{A}{\sqrt{-\Delta}\cot\sqrt{-\Delta} - B}.$$

Proof. For each case, $\Delta = 0, \Delta > 0, \Delta < 0$, we assume that we have $\mathrm{LR}(P, S(\alpha)) = 1$, apply Thm. 3.8 and solve for α.

Case $\Delta = 0$*:* By Thm. 3.8, we have

$$\frac{1}{B} - \frac{1}{\alpha C + B} = 1.$$

This implies

$$\alpha = \frac{A}{1-B}.$$

Case $\Delta > 0$*:* By Thm. 3.8, we have

$$\frac{1}{2\sqrt{\Delta}}\log\frac{\alpha(B+\sqrt{\Delta}) + A}{\alpha(B-\sqrt{\Delta}) + A} = 1,$$

which implies

$$e^{2\sqrt{\Delta}} = \frac{\alpha(B+\sqrt{\Delta}) + A}{\alpha(B-\sqrt{\Delta}) + A},$$

so

$$\alpha = \frac{A(e^{2\sqrt{\Delta}} - 1)}{-B(e^{2\sqrt{\Delta}} - 1) + \sqrt{\Delta}(e^{2\sqrt{\Delta}} + 1)} = \frac{A}{\sqrt{\Delta}\frac{e^{2\sqrt{\Delta}}+1}{e^{2\sqrt{\Delta}}-1} - B}$$

$$= \frac{A}{\sqrt{\Delta}\coth\sqrt{\Delta} - B}.$$

Case $\Delta < 0$*:* By Thm. 3.8, we have

$$\frac{1}{\sqrt{-\Delta}}\left(\arctan\frac{\alpha\sqrt{-\Delta}}{\alpha B + A} \pmod{\pi}\right) = 1.$$

This implies

$$\frac{\alpha\sqrt{-\Delta}}{\alpha B + A} = \tan\sqrt{-\Delta},$$

and so

$$\alpha = \frac{A}{\sqrt{-\Delta}\cot\sqrt{-\Delta} - B}.$$

□

As noted after the statement of Thm. 3.8, the conditions of the theorem can be assumed if $P(x, y)$ is a packing polynomial on $S(\alpha)$. In the next chapter, we will prove a result that renders considering non-zero discriminant QPPs superfluous.

Chapter 4

Non-zero Discriminants and Non-injectivity

We will now show that no integer-valued quadratic polynomial can be injective on the lattice points of any sector of $\mathbb{R}^2$ if its discriminant is non-zero. Coupled with the result from the previous chapter on the necessary form of α given a QPP, we will conclude that there can be no quadratic packing polynomials on irrational sectors.

4.1 Polynomials with Non-zero Discriminant

Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$. We define the closed convex cone

$$\mathcal{C}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \{s\boldsymbol{v}_1 + t\boldsymbol{v}_2 \ : \ s, t \geq 0\}$$

and for $\boldsymbol{v}_0 \in \mathbb{R}^2$ the affine convex cone

$$\mathcal{C}_{\boldsymbol{v}_0}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \mathcal{C}(\boldsymbol{v}_1, \boldsymbol{v}_2) + \boldsymbol{v}_0.$$

Theorem 4.1. *Let $P(x, y)$ be an integer-valued polynomial with non-zero discriminant. $P(x, y)$ can not be injective on the lattice points of any subset of $\mathbb{R}^2$ containing an affine convex cone.*

Proof. By Prop. 2.4, $P(x,y)$ has the form

$$P(x,y) = \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 + D'x + E'y + F,$$

where $D' = D - A/2$, $E' = E - C/2$, and $A, B, C, D, E, F \in \mathbb{Z}$. As we saw in the proof of Lem. 3.4, when $\Delta = B^2 - AC \neq 0$, we can rewrite $P(x,y)$ as

$$P(x,y) = \frac{A}{2}(x-x_0)^2 + B(x-x_0)(y-y_0) + \frac{C}{2}(y-y_0)^2 + K,$$

where

$$x_0 = \frac{CD' - BE'}{\Delta}, \quad y_0 = \frac{AE' - BD'}{\Delta}, \quad K = F + \frac{D'}{2}x_0 + \frac{E'}{2}y_0.$$

The point (x_0, y_0) is the center of the level curve $P(x,y) = n$ for all n, whether it is an ellipse or a hyperbola.

Define, as in Lem. 2.5, for each $p, q, i \in \mathbb{Z}$ with $\gcd(p,q) = 1$, the line

$$L(p,q;i) : py = qx + i.$$

Then

$$\mathbb{Z}^2 \cap \left(\bigsqcup_{i \in \mathbb{Z}} L(p,q;i) \right) = \mathbb{Z}^2.$$

For fixed coprime p, q, let $Q_i(x)$ be $P(x,y)$ restricted to the line $L(p,q;i)$. That is,

$$\begin{aligned} Q_i(x) &= P\left(x, \frac{q}{p}x + \frac{i}{q}\right) \\ &= \frac{A}{2}x^2 + Bx\left(\frac{q}{p}x + \frac{i}{q}\right) + \frac{C}{2}\left(\frac{q}{p}x + \frac{i}{q}\right)^2 + D'x + E'\left(\frac{q}{p}x + \frac{i}{q}\right) + F \\ &= \left(\frac{A}{2} + B\frac{q}{p} + \frac{C}{2}\left(\frac{q}{p}\right)^2\right)x^2 + \left(\left(B\frac{1}{p} + C\frac{q}{p^2}\right)i + D' + E'\frac{q}{p}\right)x + K(i), \end{aligned}$$

where $K(i)$ is an irrelevant constant depending on i. If $\frac{A}{2} + B\frac{q}{p} + \frac{C}{2}\left(\frac{q}{p}\right)^2 \neq 0$, the graph of $Q_i(x)$ is a parabola with vertex whose x-coordinate is

$$x(i) = -\frac{\left(B\frac{1}{p} + C\frac{q}{p^2}\right)i + D' + E'\frac{q}{p}}{A + 2B\frac{q}{p} + C\left(\frac{q}{p}\right)^2} = \frac{-(Bp + Cq)i - (D'p + E'q)p}{Ap^2 + 2Bpq + Cq^2}.$$

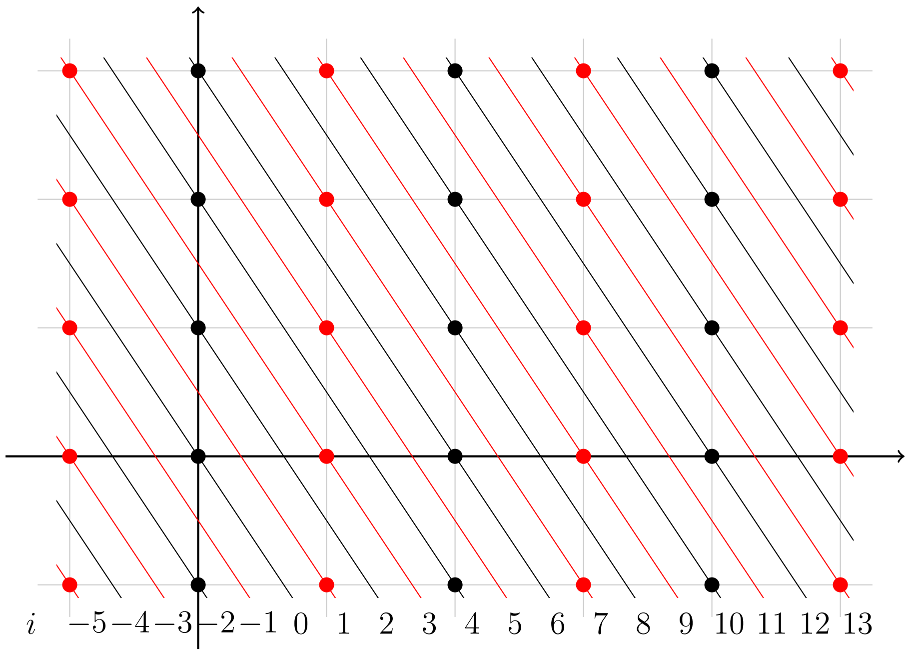

Figure 4.1: The lines $L(2, -3; i)$ covering $\mathbb{Z} \times \mathbb{Z}$.

The corresponding y-coordinate on $L(p, q; i)$ is

$$\begin{aligned} y(i) &= \frac{q}{p}\left(\frac{-(Bp + Cq)i - (D'p + E'q)p}{Ap^2 + 2Bpq + Cq^2}\right) + \frac{i}{p} \\ &= \frac{-Bpqi - Cq^2 i - (D'p + E'q)pq + Ap^2 i + 2Bpqi + Cq^2 i}{p(Ap^2 + 2Bpq + Cq^2)} \\ &= \frac{(Ap + Bq)i - (D'p + E'q)q}{Ap^2 + 2Bpq + Cq^2}. \end{aligned}$$

This means that the max/min value of $P(x, y)$ restricted to the line $L(p, q; i)$ falls on the point $(x(i), y(i))$. The symmetry of parabolas implies that $P(x(i)+p, y(i)+q) = P(x(i)-p, y(i)-q)$.

Claim: *Let $\mathcal{C} = \mathcal{C}_{(x_c, y_c)}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ be an affine convex cone. We can choose p and q such that the points $(x(i) + p, y(i) + q)$ and $(x(i) - p, y(i) - q)$ are both in $\mathcal{C} \cap \mathbb{Z}^2$ for some i.*

The points $(x(i), y(i))$ fall on the line $L_0(p, q)$ parametrized by

$$(x(t), y(t)) = \left(\frac{-(Bp+Cq)t - (D'p+E'q)p}{Ap^2 + 2Bpq + Cq^2}, \frac{(Ap+Bq)t - (D'p+E'q)q}{Ap^2 + 2Bpq + Cq^2}\right)$$

with t being a continuous variable. The slope of $L_0(p, q)$ is $-\frac{Ap+Bq}{Bp+Cq}$. At

$$t = \frac{(AE' - BD')p - (CD' - BE')q}{\Delta},$$

we have

$$\begin{aligned} x(t) &= \frac{-(Bp+Cq)\frac{(AE'-BD')p-(CD'-BE')q}{\Delta} - (D'p+E'q)p}{Ap^2+2Bpq+Cq^2} \\ &= \frac{-(Bp+Cq)((AE'-BD')p-(CD'-BE')q)-(B^2-AC)(D'p+E'q)p}{\Delta(Ap^2+2Bpq+Cq^2)} \\ &= \frac{Ap^2(CD'-BE')+2Bpq(CD'-BE')+Cq^2(CD'-BE')}{\Delta(Ap^2+2Bpq+Cq^2)} \\ &= \frac{CD'-BE'}{\Delta} = x_0 \end{aligned}$$

and

$$\begin{aligned} y(t) &= \frac{(Ap+Bq)\frac{(AE'-BD')p-(CD'-BE')q}{\Delta} - (D'p+E'q)q}{Ap^2+2Bpq+Cq^2} \\ &= \frac{(Ap+Bq)((AE'-BD')p-(CD'-BE')q)-(B^2-AC)(D'p+E'q)q}{\Delta(Ap^2+2Bpq+Cq^2)} \\ &= \frac{Ap^2(AE'-BD')+2Bpq(AE'-BD')+Cq^2(AE'-BD')}{\Delta(Ap^2+2Bpq+Cq^2)} \\ &= \frac{AE'-BD'}{\Delta} = y_0. \end{aligned}$$

What this shows is (perhaps to no surprise) that, regardless of the choice of p and q, the line $L_0(p,q)$ passes through the central point (x_0, y_0) of the level curves.

Now, let $\mathcal{C}_0 = \mathcal{C}_{(x_0,y_0)}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and choose a lattice point $(m,n) \in \mathcal{C} \cap \mathcal{C}_0$. The line passing through the points (m,n) and (x_0,y_0) has an infinite segment inside $\mathcal{C}$ (see Fig. 4.2). Let

$$\frac{p}{q} = -\frac{A(m-x_0)+B(n-y_0)}{B(m-x_0)+C(n-y_0)}$$

in lowest terms. As noted above, we don't want $\frac{A}{2} + B\frac{q}{p} + \frac{C}{2}(\frac{q}{p})^2 = 0$. Should that be the case, pick a different (m,n). Then $\gcd(p,q) = 1$ and

$$\frac{n-y_0}{m-x_0} = -\frac{Ap+Bq}{Bp+Cq}.$$

This is the slope of $L_0(p,q)$ which also passes through (x_0,y_0), so $L_0(p,q)$ passes through (m,n). This means that for some $i' \in \mathbb{Z}$, the lattice point $(m,n) \in \mathcal{C}$ is the point of symmetry for $P(x,y)$ restricted to $L(p,q;i')$. Let

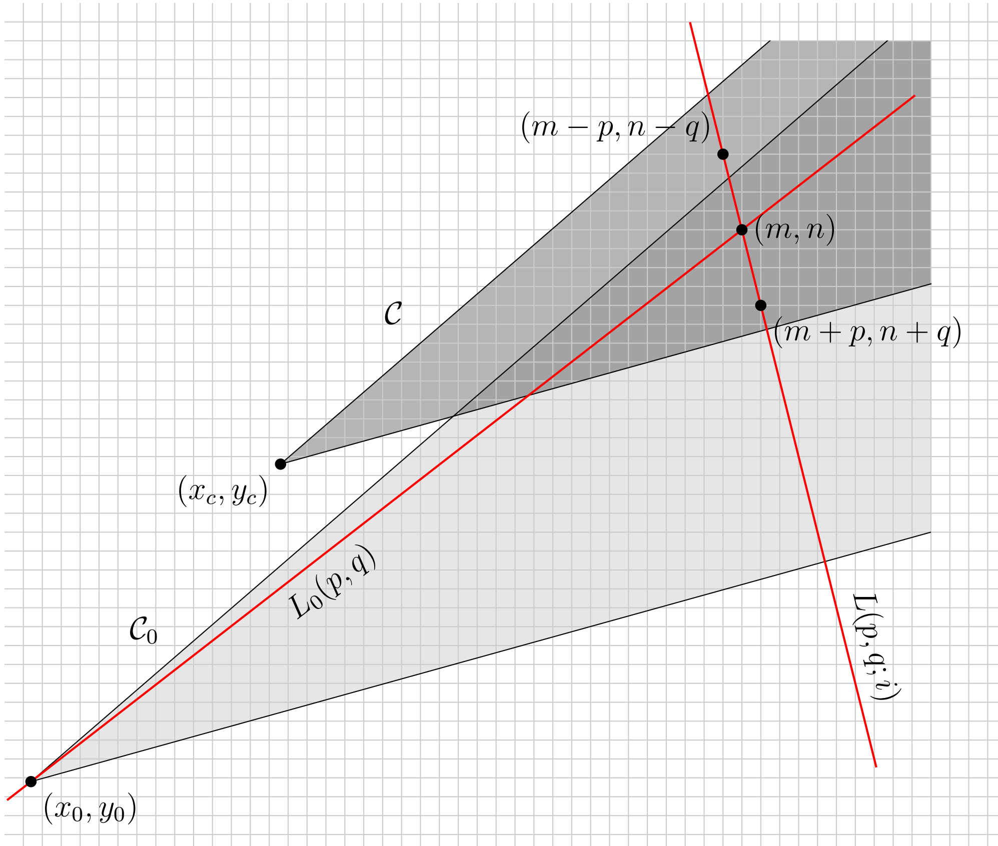

Figure 4.2: $L_0(p,q)$ with $\frac{q}{p} = -\frac{A(m-x_0)+B(n-y_0)}{B(m-x_0)+C(n-y_0)}$.

$$\frac{s}{r} = \frac{n-y_0}{m-x_0}$$

in lowest terms. Then $(m+jr, n+js)$ are lattice points on $L_0(p,q) \cap \mathcal{C}$ for all $j \in \mathbb{N}$. For j big enough, we have $(m+jr+p, n+js+q), (m+jr-p, n+js-q) \in \mathcal{C} \cap \mathbb{Z}^2$, and

$$P(m+jr+p, n+js+q) = P(m+jr-p, n+js-q).$$

This proves the claim and the lemma. □

4.2 Irrational Sectors

We can now prove the main result which addresses Nathanson's sixth problem.

Theorem 4.2. *There can be no quadratic packing polynomial on the sector $S(\alpha)$ if α is irrational.*

Proof. Let $P(x, y)$ be a QPP with discriminant Δ on $S(\alpha)$. By Thm. 3.9, if $\Delta = 0$, then α is rational. Therefore, if α is irrational, we must have $\Delta \neq 0$. Since $S(\alpha) = \mathcal{C}((1, 0), (1, \alpha))$ is a (affine) convex cone, by Thm. 4.1, $P(x, y)$ can not be injective and therefore not a packing polynomial. □

4.3 The Cantor Polynomials

Before Vsemirnov's 2001 article [12], all proofs of Fueter-Pólya's theorem relied on the Lindemann-Weierstraß theorem (further details are given in the next chapter). Vsemirnov's proof is combinatorial and completely different from that of Fueter-Pólya or Lew-Rosenberg whose methods are similar. With Thm. 4.1, we can provide a proof along the latter strategy but without the need for Lindemann-Weierstraß.

Theorem 4.3 (Fueter-Pólya)**.** *The Cantor polynomials*

$$F(x, y) = \frac{1}{2}(x + y)(x + y + 1) + x$$
$$G(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y$$

are the only quadratic packing polynomials on $S(\infty)$*.*

Proof. First, we observe that the Cantor polynomials are indeed packing polynomials. Enumerating $I(\infty)$ along the diagonals as in Fig. 4.3 produces a bijection $I(\infty) \to \mathbb{N}_0$. This enumeration is obtained by the Cantor polynomial $F(x, y)$. We note that the triagonal numbers fall on the y-axis, so we have

$$P(0, y) = \frac{1}{2}y(y + 1).$$

We also have

$$P(x, y - x) = P(0, y) + x,$$

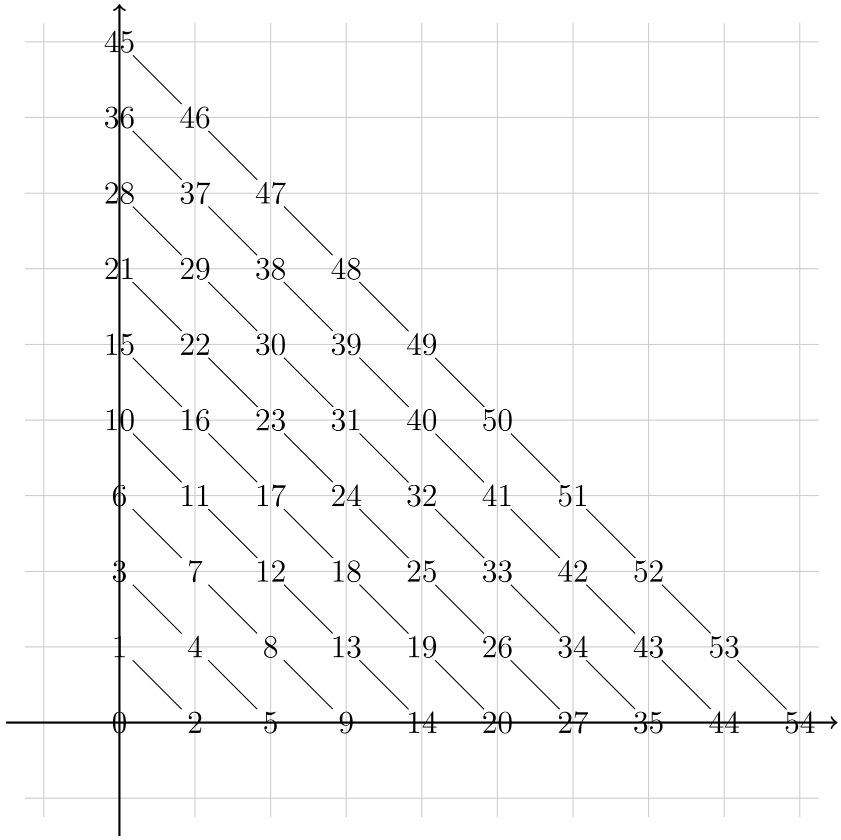

Figure 4.3: Enumerating the first quadrant lattice points

and so

$$\begin{aligned} P(x, y) &= P(x, (x+y) - x) \\ &= P(0, x+y) + x \\ &= \frac{1}{2}(x+y)(x+y+1) + x \\ &= F(x, y). \end{aligned}$$

$G(x, y)$ is obtained by swapping the x- and y-coordinates.

Let $P(x, y)$ be a QPP on $S(\infty)$. By Prop. 2.4, we must have

$$P(x, y) = \frac{A}{2}x^2 + Bxy + \frac{C}{2}y^2 + \left(D - \frac{A}{2}\right)x + \left(E - \frac{C}{2}\right)y + F$$

with $A, B, C, D, E, F \in \mathbb{Z}$. By Lem. 2.6, we must have $A > 0$. On $S(\infty)$, if $P(x, y)$ is a QPP, so is $P(y, x)$. We conclude that also $C > 0$. Since $S(\infty)$ is a closed convex cone, by Thm. 4.1, we must have $B^2 - AC = 0$, or $B = \pm\sqrt{AC}$, else $P(x, y)$ can not be injective. $P(x, y)$ can therefore be written as

$$P(x,y) = \frac{1}{2}\left(\sqrt{A}x \pm \sqrt{C}y\right)^2 + \left(D - \frac{A}{2}\right)x + \left(E - \frac{C}{2}\right)y + F.$$

If $B = -\sqrt{AC}$, then the quadratic part of $P(x, y)$ vanishes on the rational line $y = -\sqrt{A/C}x = -(B/C)x$. This is impossible by Lem. 2.5. We thus have

$$P(x,y) = \frac{1}{2}\left(\sqrt{A}x + \sqrt{C}y\right)^2 + \left(D - \frac{A}{2}\right)x + \left(E - \frac{C}{2}\right)y + F.$$

As in Chp. 3, define the regions

$$R_n = \{(x, y) \in S(\infty) \ : \ 0 \leq P(x, y) \leq n\}.$$

We can't apply Lem. 3.4 directly as it assumes $B < 0$ where in this situation we have $B > 0$. For the parabolic case, the only matter of importance is that the axis of symmetry for the level curves doesn't fall inside $S(\alpha)$ as this implies that the regions R_n are bounded and we can apply Davenport's lemma. That doesn't happen here either and the arguments transfer. To recap, we must have

$$\mathrm{LR}(P, I(\infty)) = \lim_{n\to\infty} \frac{1}{n}\#\left(\mathbb{Z}^2 \cap R_n\right) = 1.$$

By Davenport's lemma 3.2 and Lem. 3.7, we have

$$\begin{aligned}
\mathrm{LR}(P, I(\infty)) &= \lim_{n\to\infty} \frac{1}{n}\#\left(\mathbb{Z}^2 \cap R_n\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\mathrm{area}(R_n) + O(\sqrt{n})\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\mathrm{area}(R_n)\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} + O(\sqrt{n}) + O(1)\right) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{P_2(\cos\theta, \sin\theta)} \\
&= \frac{1}{C}\int_0^{\infty} \frac{dt}{\left(t + \frac{B}{C}\right)^2} \\
&= \frac{1}{B}.
\end{aligned}$$

The details are the same as in the proof of Thm. 3.8. The condition $\mathrm{LR}(P, I(\infty)) = 1$ implies $B = 1$ and, since $B^2 = AC$, $A = C = 1$. This means that we must have

$$\begin{aligned} P(x,y) &= \frac{1}{2}(x+y)^2 + \left(D - \frac{1}{2}\right)x + \left(E - \frac{1}{2}\right)y + F \\ &= \frac{1}{2}(x+y)(x+y+1) + (D-1)x + (E-1)y + F. \end{aligned}$$

Assume $D \geq E$ and write

$$P(x,y) = \frac{1}{2}(x+y)(x+y+1) + (D-E)x + (E-1)(x+y) + F.$$

We can't have $D = E$ as $P(x,y)$ then would be constant on lines $x + y = n$. We have

$$\begin{aligned} &P(x, (D-E-1)x - E) \\ &\qquad = \frac{1}{2}((D-E)x - E)((D-E)x - E + 1) \\ &\qquad\qquad + (D-E)x + (E-1)((D-E)x - E) + F \\ &\qquad = \frac{1}{2}((D-E)x - E)((D-E)x - E + 1) \\ &\qquad\qquad + (D-E)x - E + 1 + (E-1)((D-E)x - E + 1) + F \\ &\qquad = \frac{1}{2}((D-E)x - E + 2)((D-E)x - E + 1) \\ &\qquad\qquad + (E-1)((D-E)x - E + 1) + F \\ &\qquad = P(0, (D-E)x + 1 - E). \end{aligned}$$

This means that $P(x,y)$ can only be injective on $S(\infty)$ if $D - E = 1$. Having

$$P(x, -E) = P(0, x + 1 - E)$$

then implies that we must have $E > 0$. E and $D - E$ both being positive implies that $P(x,y)$ takes its smallest value on $(0,0)$, hence it is necessary that $F = 0$. We thus have

$$P(x,y) = \frac{1}{2}(x+y)(x+y+1) + x + (E-1)(x+y).$$

Restricted to a line $x + y = n$, the values of $P(x, n-x)$ are $n+1$ consecutive integers, increasing from $(0,n)$ to $(n,0)$. See Fig. 4.4. For P to be a packing polynomial, we must have $P(0,n+1)=P(n,0)+1$, so

$$\frac{1}{2}(n+1)(n+2)+(E-1)(n+1)=\frac{1}{2}n(n+1)+n+(E-1)n+1,$$
$$\frac{1}{2}(n+1)n+n+1+(E-1)n+E-1=\frac{1}{2}n(n+1)+n+(E-1)n+1,$$
$$E-1=0.$$

So if $D\geq E$, we must have

$$P(x,y)=\frac{1}{2}(x+y)(x+y+1)+x.$$

If $E\geq D$, we can repeat the argument. This leads to the other option

$$P(x,y)=\frac{1}{2}(x+y)(x+y+1)+y.$$

□

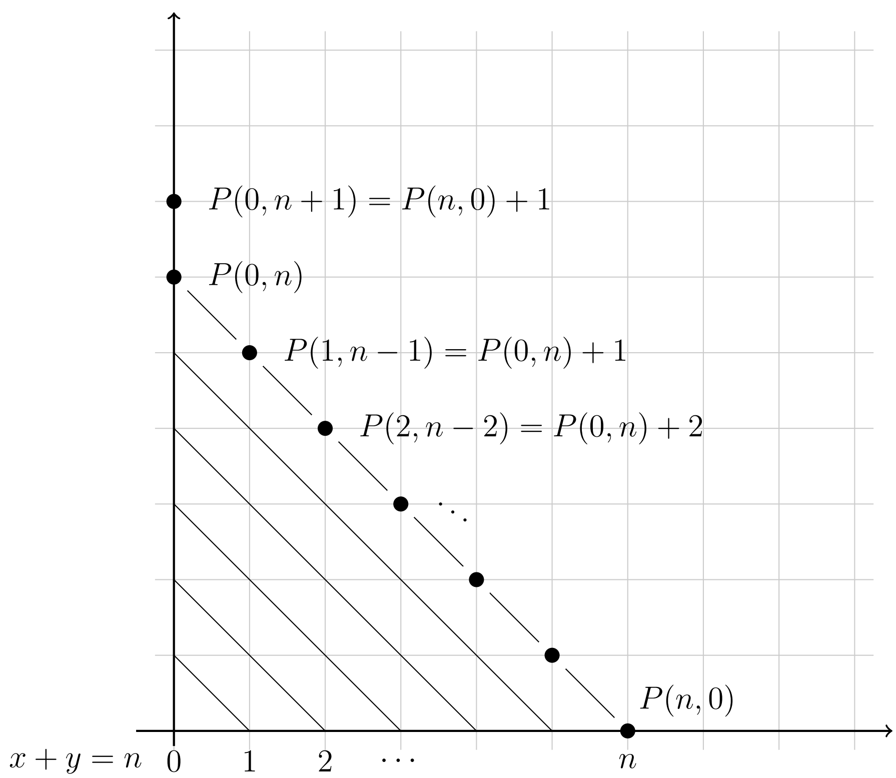

Figure 4.4: Lines $x+y=n$

Chapter 5

Rational Sectors

In this chapter, we determine all quadratic packing polynomials (QPPs) on all rational sectors $S(n/m)$. Nathanson's work [10] covered all sectors of form $S(1/m)$, $m \in \mathbb{N}$, cf. 1.3. Integral sectors were treated by Stanton [11]. Stanton further provided a necessary form a quadratic packing polynomial on arbitrary rational sectors $S(n/m)$. We will provide proofs for Stanton's results in first two sections of this chapter. In section 5.3, we will, heavily inspired by the work of Brandt [2], provide a necessary form that a QPP on the sector $S(n/m)$ must have and we will determine all rational sectors, $S(n/m)$, not equivalent to integral or Nathanson sectors, on which polynomials of the necessary form are packing polynomials. We conclude the chapter by collecting all results in a complete classification.

5.1 Necessary Form of $P_2(x, y)$

In Chp. 3, we established (Thm. 3.9) that if $P(x, y)$ of the form (2.1.1) is a QPP on sector $S(\alpha)$, then

$$\alpha = \frac{A}{\sqrt{\Delta}\coth(\Delta) - B},$$

Where $\Delta = B^2 - AC$ is the discriminant of P and $0 \cdot \coth 0$ is assigned the value 1. In the previous chapter, we used the fact that an irrational α implies non-zero discriminant.

Previous results on the subject of quadratic packing polynomials[1] have relied on a similar strategy, obtaining a formula for the density involving a trigonometric term and then applying the Lindemann-Weierstraß theorem to rule out certain cases.

Theorem 5.1 (Lindemann-Weierstraß)**.** *For any distinct algebraic numbers* $\alpha_1, \dots, \alpha_n$ *and non-zero algebraic* $\beta_1, \dots, \beta_n$*, we have*

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0.$$

For a proof, see e.g. [1]. The theorem implies that $\cot(x)$ and $\coth(x)$ are both transcendental for all algebraic $x \neq 0$, and our formula for α then rules out polynomials with non-zero discriminants when α is rational and vice versa. With Thm. 4.1, we don't need such heavy tools.

Corollary 5.2. *If* $P(x, y)$ *is a QPP on* $S(\alpha)$*, then* α *is rational if and only if the discriminant of* P *is zero.*

Proof. By Thm. 5.2, if $P(x, y)$ is a QPP on $S(\alpha)$, then it must have zero discriminant, as otherwise it would not be injective. If $P(x, y)$ has zero discriminant, then by Thm. 3.9, $\alpha = \frac{A}{1-B}$ is rational. □

Theorem 5.3 (Stanton)**.** *If* P *is a QPP on the rational sector* $S(\frac{n}{m})$ *where* $\gcd(n, m) = 1$*, then* $n \mid (m-1)^2$ *and* P *is of the form*

$$P(x, y) = \frac{n}{2}\left(x - \frac{m-1}{n}y\right)^2 + \left(D - \frac{n}{2}\right)x + \left(E - \frac{(m-1)^2/n}{2}\right)y + F, \tag{5.1.1}$$

with $D, E, F \in \mathbb{Z}$.

Proof. Stanton proved this result in [11] using results of Lew and Rosenberg [8]. It follows from the above statements without much work, so I am including a proof here.

[1] Fueter-Pólya [6] as well as Lew-Rosenberg [8], and in turn Stanton [11] (Stanton cites Lew-Rosenberg's paper).

If $\alpha = \frac{n}{m}$ with $\gcd(n,m) = 1$ is rational, then, by Thm 3.9 and Cor. 5.2, we have

$$\frac{n}{m} = \frac{A}{1-B}, \text{ so } 1 - B = \frac{Am}{n}.$$

Since $(n,m) = 1$, this implies $n|A$. By Cor. 5.2 again, we have

$$AC = B^2 = \left(1 - \frac{Am}{n}\right)^2 = \frac{(n-Am)^2}{n^2} = \frac{n^2 - 2Anm + A^2m^2}{n^2},$$

so

$$nC = \frac{n^2 - 2Anm + A^2m^2}{nA} = \frac{n}{A} - 2m + \frac{A}{n}m^2.$$

Since nC, $2m$, $\frac{A}{n}m^2$ are all integers, we conclude that $A \mid n$. By Lem. 2.6, we have $A > 0$. Hence

$$A = n,$$
$$B = 1 - \frac{Am}{n} = 1 - m,$$
$$C = \frac{B^2}{A} = \frac{(1-m)^2}{n}.$$

Substituting in these values to (2.1.1) leads to (5.1.1). □

Example 5.4. Nathanson's first open problem asks if a QPP exists for $I(3/5)$ and $I(3/2)$. Since $3 \nmid 4^2$ and $3 \nmid 1$, the answer in both cases is no.

5.2 Integral Sectors

Stanton dealt with integral sectors in [11]. I am including a proof for completeness and for clarity. The non-integral rational case will be handled using a strategy similar to the one used in this section, only that case gets a bit more technical.

If $\alpha = n/m$ is an integer, then $m = 1$ and, by Thm. 5.3, we have $A = n$, $B = C = 0$. Hence, a QPP on the sector $S(n)$ must have the form

$$P(x,y) = \frac{n}{2}x^2 + \left(D - \frac{n}{2}\right)x + Ey + F$$
$$= \frac{n}{2}x(x-1) + Dx + Ey + F, \tag{5.2.1}$$

with $D, E, F \in \mathbb{Z}$. So P is linear in y, in particular,

$$P(x, y+1) - P(x, y) = E.$$

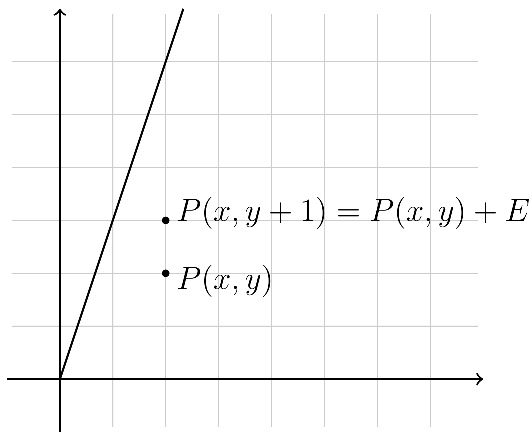

Figure 5.1: Consecutive points in $S(n)$ for a fixed x.

The involutory transformation

$$L = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}$$

maps (x, y) to $(x, nx - y)$. We have $L(1, 0) = (1, n)$ and $L(1, n) = (1, 0)$, so $L(S(n)) = S(n)$. The effect on the corresponding packing polynomial is changing the sign of E.

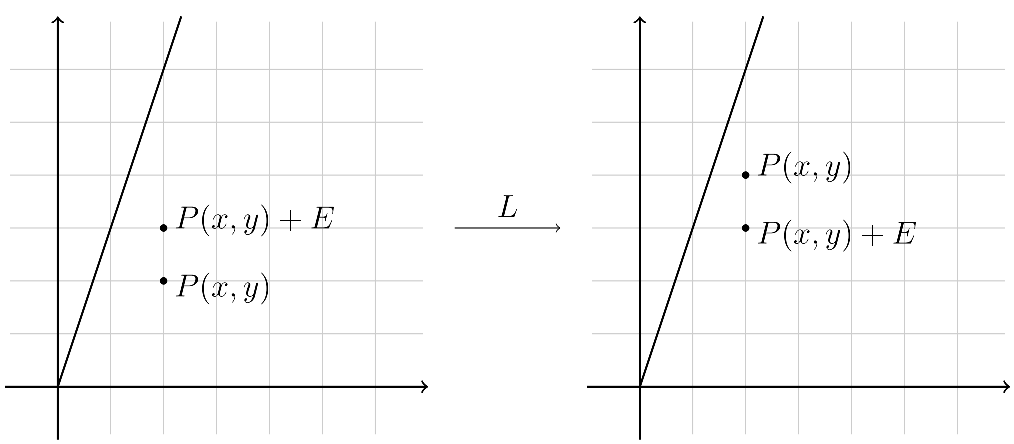

Figure 5.2: Effect of the involution L.

By applying this transformation, we can assume that $E > 0$.

For fixed x, all values of $P(x, y)$ are congruent modulo E. Furthermore, $P(x, 0)$ is an increasing function for large enough x, so for P to be a packing polynomial, from a certain value of x, we must have

$$P(x, nx) + E = P(x', 0),$$

where $x' > x$ is the smallest x-value for which $P(x, y) \equiv P(x', y) \pmod{E}$.

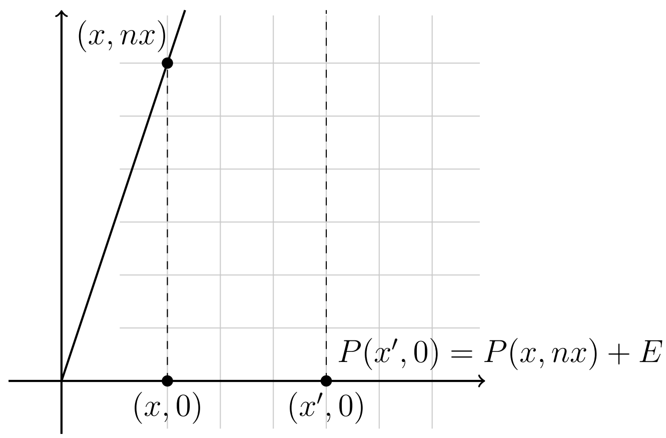

Figure 5.3: Smallest x' with $P(x, y) \equiv P(x', y) \pmod{E}$

Proposition 5.5. *We have*

$$D = 1 - \frac{n}{2}(E - 1),$$

so the necessary form of P becomes

$$P(x, y) = \frac{n}{2}x(x - E) + x + Ey + F.$$

Proof. For any x, let i_x be the smallest positive integer such that

$$P(x, y) \equiv P(x + i_x, y) \pmod{E}.$$

Claim: *From a certain $x = x_0$, $i_x = E$ for all $x > x_0$.*

Assume the contrary, that we can choose arbitrarily large $x = x_0$ such that $i_{x_0} \neq E$. If $i_{x_0} < E$, the consecutive

$$x_0, x_0 + 1, \ldots, x_0 + i_{x_0}, \ldots, x_0 + E - 1$$

can represent at most $E-1$ congruence classes. This means that there must then be an x_1 with $i_{x_1} > E$. Let that be the assumption, and furthermore, choose x_1 large enough that

(i) $P(x_1, nx_1) + E = P(x_1 + i_{x_1}, 0)$.

(ii) $nx_1 > E + F$.

We have

$$\begin{aligned} P(x_1 + i_{x_1}, 0) &= \frac{n}{2}(x_1 + i_{x_1})(x_1 + i_{x_1} - 1) + D(x_1 + i_{x_1}) + F \\ &= \frac{n}{2}x_1(x_1 - 1) + \frac{n}{2}(2x_1 + i_{x_1} - 1)i_{x_1} + Dx_1 + Di_{x_1} + F \\ &= \frac{n}{2}x_1(x_1 - 1) + Dx_1 + Enx_1 + F \\ &\qquad + \frac{n}{2}i_{x_1}(i_{x_1} - 1) + Di_{x_1} + F \\ &\qquad + i_{x_1}nx_1 - Enx_1 - F \\ &= P(x_1, nx_1) + P(i_{x_1}, 0) + nx_1(i_{x_1} - E) - F, \end{aligned}$$

which implies that

$$P(x_1 + i_{x_1}, 0) - P(x_1, nx_1) = E = P(i_{x_1}, 0) + nx_1(i_{x_1} - E) - F,$$

so

$$P(i_{x_1}, 0) = E + F - nx_1(i_{x_1} - E). \tag{5.2.2}$$

This is negative by the assumptions which is impossible.

The conclusion is that $i_{x_1} = E$, so by (5.2.1) and (5.2.2), we find that

$$P(E, 0) = \frac{n}{2}E(E-1) + DE + F = E + F,$$

which yields

$$D = 1 - \frac{n}{2}(E - 1).$$

Inserting into (5.2.1), we get

$$P(x, y) = \frac{n}{2}x(x - E) + x + Ey + F.$$

□

Lemma 5.6. *If $E > 0$ and*

$$P(x,y) = \frac{n}{2}x(x-E) + x + Ey + F,$$

then P is a packing polynomial on $I(n)$, if and only if

$$\{P(0,0), P(1,0), \dots, P(E-1,0)\} = \{0,1,\dots,E-1\}.$$

Proof. Let P be of the necessary form. Then, for all x, we have

$$\begin{aligned} P(x+E,0) - P(x,nx) &= \frac{n}{2}(x+E)x + x + E + F \\ &\qquad - \left(\frac{n}{2}x(x-E) + x + Enx + F\right) \\ &= \frac{n}{2}x(x+E-(x-E)) - Enx + E \\ &= E. \end{aligned}$$

So it is necessary and sufficient for P to be a bijection on to $\mathbb{N}_0$ that $P(0,0), P(1,0), \dots, P(E-1)$ take on the values $0, 1, \dots, E-1$ in any order. ☐

Lemma 5.7. *We have*

$$F = \frac{n(E+1)(E-1)}{12}.$$

Proof. By Lem. 5.6, we have

$$\sum_{i=0}^{E-1} P(i,0) = \sum_{i=0}^{E-1} i. \tag{5.2.3}$$

By Lem. 5.5, the left-hand side expands to

$$\begin{aligned} \sum_{i=0}^{E-1} P(i,0) &= \sum_{i=0}^{E-1}\left(\frac{n}{2}i(i-E) + i + F\right) \\ &= \frac{n}{2}\sum_{i=0}^{E-1} i^2 - \frac{nE}{2}\sum_{i=0}^{E-1} i + \sum_{i=0}^{E-1} i + \sum_{i=0}^{E-1} F \\ &= \frac{n}{2}\left(\frac{(E-1)E(2E-1)}{6} - E\frac{E(E-1)}{2}\right) + EF + \sum_{i=0}^{E-1} i. \end{aligned}$$

Combined with (5.2.3), we have

$$E\frac{n}{2}\left(\frac{(E-1)(2E-1)}{6} - \frac{E(E-1)}{2}\right) + EF = 0$$

yielding

$$F = \frac{n(E-1)(E+1)}{12}.$$

□

We can now determine all quadratic packing polynomials on integral sectors.

Theorem 5.8 (Stanton)**.** *For all n, the polynomials*

$$P(x,y) = \frac{n}{2}x(x-1) + x + y$$
$$P(x,y) = \frac{n}{2}x(x+1) + x - y$$

are packing polynomials on $S(n)$.

The polynomials

$$P(x,y) = 2x(x-2) + x + 2y + 1$$
$$P(x,y) = 2x(x+2) + x - 2y + 1$$

are packing polynomials on $S(4)$.

The polynomials

$$P(x,y) = \frac{3}{2}x(x-3) + x + 3y + 2$$
$$P(x,y) = \frac{3}{2}x(x+3) + x - 3y + 2$$

are packing polynomials on $S(3)$.

There are no other quadratic packing polynomials on integral sectors.

Proof. We'll do a case by case study, assuming $E > 0$. By Lem. 5.7 and Lem. 5.6, we have

$$P(0,0) = F = \frac{n(E-1)(E+1)}{12} \leq E - 1,$$

so $E = 1$ or $n(E+1) \leq 12$. The latter imposes immediate restrictions on the possibilities of E and n. We have $2 \leq E \leq 11$. Remembering that F must be an integer, these are the options:

Case $E = 2$: $F = \frac{n \cdot 3}{12} \leq 1$, so $n = 4$.

Case $E = 3$: $n \leq 3$. $F = \frac{n \cdot 2 \cdot 4}{12} \leq 2$, so $n = 3$.

Case $E = 4$: $n \leq 2$. $F = \frac{n \cdot 3 \cdot 5}{12}$. This is impossible.

Case $E = 5$: $n \leq 2$. $F = \frac{n \cdot 4 \cdot 6}{12} \leq 4$. So $n = 2, F = 4$, or $n = 1, F = 2$.

Case $E = 6$: $n = 1$. $F = \frac{5 \cdot 7}{12}$. This is impossible.

Case $E = 7$: $n = 1$. $F = \frac{6 \cdot 8}{12} = 4$.

Case $E = 8, 9, 10$: $n = 1$. $F = \frac{7 \cdot 9}{12}$, $F = \frac{8 \cdot 10}{12}$ or $F = \frac{9 \cdot 11}{12}$. All are impossible.

Case $E = 11$: $n = 1$. $F = \frac{10 \cdot 12}{12} = 10$.

So the candidates are boiled down to

- $E = 1$, $n > 0$, $F = 0$.
- $E = 2$, $n = 4$, $F = 1$.
- $E = 3$, $n = 3$, $F = 2$.
- $E = 5$, $n = 1$, $F = 2$.
- $E = 5$, $n = 2$, $F = 4$.
- $E = 7$, $n = 1$, $F = 4$.
- $E = 11$, $n = 1$, $F = 10$.

If the resulting polynomials satisfy the necessary conditions given in Lem. 5.6, we have a winner.

Case $E = 1, n > 0, F = 0$: The polynomial must have the form

$$P(x, y) = \frac{n}{2}x(x - 1) + x + y. \tag{5.2.4}$$

Since $P(0, 0) = 0$ for all n, it is a packing polynomial, by Lem. 5.6.

Case $E = 2$, $n = 4, F = 1$: The polynomial must have the form

$$P(x, y) = 2x(x - 2) + x + 2y + 1. \tag{5.2.5}$$

We have $P(0,0)=1,\ P(1,0)=0$, so it is a packing polynomial.

Case $E=3, n=3, F=2$: The polynomial must have the form

$$P(x,y)=\frac{3}{2}x(x-3)+x+3y+2. \tag{5.2.6}$$

We have $P(0,0)=2,\ P(1,0)=0,\ P(2,0)=1$, so it is a packing polynomial.

Case $E=5, n=1, F=2$: The polynomial must have the form

$$P(x,y)=\frac{1}{2}x(x-5)+x+5y+2.$$

We have $P(0,0)=P(3,0)=2$, so P is not a packing polynomial.

Case $E=5, n=2, F=4$: The polynomial must have the form

$$P(x,y)=x(x-5)+x+5y+4.$$

We have $P(0,0)=P(4,0)=4$, so P is not a packing polynomial.

Case $E=7, n=1, F=4$: The polynomial must have the form

$$P(x,y)=\frac{1}{2}x(x-7)+x+7y+4.$$

We have $P(0,0)=P(5,0)=4$, so P is not a packing polynomial.

Case $E=11, n=1$: The polynomial must have the form

$$P(x,y)=\frac{1}{2}x(x-11)+x+11y+F.$$

We have $P(0,0)=P(9,0)=F$, so P is not a packing polynomial.

The corresponding polynomials for $-E$ arise from applying the transformation $L=\begin{pmatrix}1 & 0\\ n & -1\end{pmatrix}$. Note that L is its own inverse. For all $S(n)$, (5.2.4) is a packing polynomial, so

$$\begin{aligned}P(L(x,y)) &= P(x,nx-y)\\ &= \frac{n}{2}x(x-1)+x+nx-y\\ &= \frac{n}{2}x(x+1)+x-y.\end{aligned}$$

For $S(4)$, (5.2.5) is a packing polynomial, so

$$\begin{aligned} P(L(x,y)) &= P(x, 4x-y) \\ &= 2x(x-2) + x + 2(4x-y) + 1 \\ &= 2x(x+2) + x - 2y + 1. \end{aligned}$$

For $S(3)$, (5.2.6) is a packing polynomial, so

$$\begin{aligned} P(L(x,y)) &= P(x, 3x-y) \\ &= \frac{3}{2}x(x-3) + x + 3(3x-y) + 2 \\ &= \frac{3}{2}x(x+3) + x - 3y + 2. \end{aligned}$$

$\square$

Example 5.9. Stanton's results settled Nathanson's problem (2) and (3). The packing polynomials found for all sectors $S(n)$ are the same Nathanson found (1.0.5) and (1.0.6). Nathanson asked if these were the only QPPs on integral sectors. With the exception of the sectors $S(3)$ and $S(4)$, the answer is yes. On $S(3)$, the polynomials

$$\begin{aligned} P(x,y) &= \frac{3}{2}x(x-1) + x + y \\ P(x,y) &= \frac{3}{2}x(x+1) + x - y \\ P(x,y) &= \frac{3}{2}x(x-3) + x + 3y + 2 \\ P(x,y) &= \frac{3}{2}x(x+3) + x - 3y + 2 \end{aligned}$$

are all QPPs. On $S(4)$, the polynomials

$$\begin{aligned} P(x,y) &= 2x(x-1) + x + y \\ P(x,y) &= 2x(x+1) + x - y \\ P(x,y) &= 2x(x-2) + x + 2y + 1 \\ P(x,y) &= 2x(x+2) + x - 2y + 1 \end{aligned}$$

are all QPPs. This also answered problem (3) of whether any rational sector allows for more than two QPPs. In the following, we will be able to determine all rational sectors on which there are more than two QPPs.

5.3 General Rational Sectors

We will now focus on general rational sectors $S(\frac{n}{m})$ with $n, m \in \mathbb{N}$, $\gcd(m, n) = 1$. The procedure follows closely that laid out by Brandt in [2].

By applying the transformation

$$M_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ with inverse } M_t^{-1} = M_{-t} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$

with $t = -\left\lfloor \frac{m}{n} \right\rfloor$, thereby changing the sector to $S\left(\frac{n}{m - \left\lfloor \frac{m}{n} \right\rfloor n}\right)$, we can assume that $n > m$. Note that the resulting sector is the unique representative of the class of sectors equivalent under M_t with $n > m$. There are three cases:

- $n = 1$. The sector is equivalent to $S(\infty)$. Nathanson [10] dealt with this case, cf. Thm. 1.3.
- $m \equiv 1 \pmod{n}$. The sector is equivalent to $S(n)$, dealt with by Stanton [11], treated in section 5.2.
- $m \not\equiv 1 \pmod{n}$. The sector is equivalent to the unique sector $S(n/m)$ with $n > m > 1$, $\gcd(m, n) = 1$.

Throughout this section, we will put

$$l = \gcd(n, m - 1), \quad \frac{n}{l} = s, \quad \frac{m - 1}{l} = r.$$

So $s > r \geq 1$, $\gcd(s, r) = 1$. Also note that, because of the necessary condition

$$sl = n \mid (m - 1)^2 = (rl)^2,$$

and the fact that r and s are relatively prime, we immediately have $s \mid l$. So there is a $d \in \mathbb{N}$ such that $ds = l$ and we can write

$$n = ds^2, \quad m - 1 = drs.$$

Under this notation, the necessary form of P in (5.1.1) can be rewritten as

$$\begin{aligned}
P(x,y) &= \frac{n}{2}\left(x - \frac{m-1}{n}y\right)^2 + \left(D - \frac{n}{2}\right)x + \left(E - \frac{(m-1)^2/n}{2}\right)y + F \\
&= \frac{ds^2}{2}\left(x - \frac{r}{s}y\right)^2 + \left(D - \frac{ds^2}{2}\right)x + \left(E - \frac{dr^2}{2}\right)y + F \\
&= \frac{d}{2}(sx - ry)^2 + Dx - \frac{ds^2}{2}x + Ey - \frac{dr^2}{2}y + F \\
&= \frac{d}{2}(sx - ry)^2 - \frac{ds^2}{2}x + \frac{drs}{2}y - \frac{drs}{2}y + Dx + Ey - \frac{dr^2}{2}y + F \\
&= \frac{d}{2}(sx - ry)^2 - \frac{d}{2}s(sx - ry) + Dx + Ey - \frac{dr}{2}(s + r)y + F \\
&= \frac{d}{2}(sx - ry)(sx - ry - s) + Dx + \left(E - \frac{dr}{2}(s + r)\right)y + F
\end{aligned} \tag{5.3.1}$$

The role of the vertical lines in the case of integral sectors, on which the polynomials were linear in y, is now taken by lines given by $sx - ry = i$. The difference between P evaluated at two consecutive integral points on such a line is constant. We have

$$\begin{aligned}
P(x + r, y + s) &= \frac{d}{2}(s(x + r) - r(y + s))(s(x + r) - r(y + s) - s) \\
&\qquad + D(x + r) + \left(E - \frac{dr}{2}(s + r)\right)(y + s) + F \\
&= \frac{d}{2}(sx - ry)(sx - ry - s) + Dx + \left(E - \frac{dr}{2}(s + r)\right)y + F \\
&\qquad + Dr + \left(E - \frac{dr}{2}(s + r)\right)s \\
&= P(x,y) + Dr + Es - \frac{drs}{2}(s + r).
\end{aligned}$$

We put

$$k = P(x + r, y + s) - P(x,y) = Dr + Es - \frac{drs}{2}(s + r) \tag{5.3.2}$$

Adopting the terminology of Brandt [2], we refer to P as a *k-stair polynomial*. We call

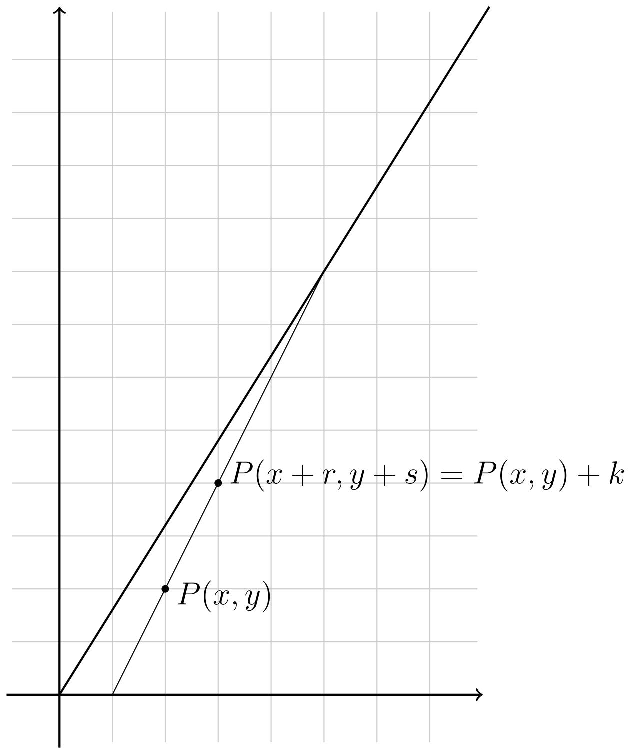

Figure 5.4: P evaluated on consecutive steps of a fixed staircase.

$$J_i = \{(x,y) \in I(n/m) : ry = sx - i\}$$

the *ith staircase* and refer to the points on each staircase as *steps* with the step having the lowest y-coordinate being the first step.[2]

Transforming the Sector

To facilitate computation, we apply to the sector $S(n/m)$ the linear transformation $M_{-r/s} = \begin{pmatrix} 1 & -\frac{r}{s} \\ 0 & 1 \end{pmatrix}$ with inverse $M_{r/s} = \begin{pmatrix} 1 & \frac{r}{s} \\ 0 & 1 \end{pmatrix}$. Note that

$$M_{-r/s}(m,n) = \begin{pmatrix} 1 & -\frac{r}{s} \\ 0 & 1 \end{pmatrix} (drs+1, ds^2) = (1, ds^2) = (1, n)$$

[2] Brandt calls a polynomial which differs by absolute value k on consecutive steps a k-stair polynomial, differentiating between positive and negative k by the label *ascending* or *descending*.

We put

$$\widehat{S}\left(\frac{n}{m}\right) = M_{-\frac{r}{s}}\left(S\left(\frac{n}{m}\right)\right) = S(n)$$
$$\widehat{I}\left(\frac{n}{m}\right) = M_{-\frac{r}{s}}\left(I\left(\frac{n}{m}\right)\right)$$
$$= \left\{\left(\frac{i}{s}, y\right) \ : \ i, y \in \mathbb{N}_0,\ 0 \le y \le dsi,\ y \equiv -\frac{i}{r} \pmod{s}\right\},$$

$$\widehat{J}_i = M_{-\frac{r}{s}}(J_i) = \left\{(x,y) \in \widehat{I}\left(\frac{n}{m}\right) \ : \ x = \frac{i}{s}\right\}.$$

To avoid confusion and appease the architecturally inclined, we will refer to $\widehat{J}_i$ as the *ith ladder*. Note that

$$\#\widehat{J}_i = di + [\![s \mid i]\!], \tag{5.3.3}$$

where

$$[\![s \mid i]\!] = \begin{cases} 1 & \text{if } s \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

We will use the notation $\bar{y}_i$ to mean the y-coordinate of the first step on the ith ladder.

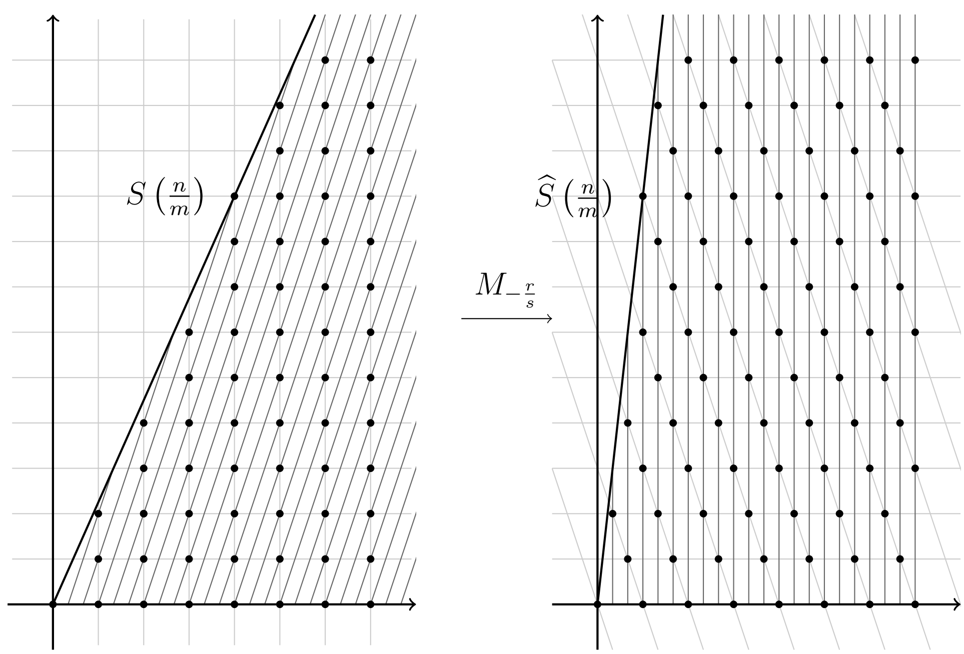

Figure 5.5: $S(n/m)$ skewed, transformaing staircases to vertical lines.

The transformed lattice $\widehat{I}(n/m)$ is no longer integral. Still, under the above transformation, packing polynomials P on $I(n/m)$ correspond bijectively to packing polynomials $\widehat{P}$ on $\widehat{I}(n/m)$

by

$$P = \widehat{P} \circ M_{-\frac{r}{s}}, \quad \widehat{P} = P \circ M_{\frac{r}{s}}$$

The necessary form (5.3.1) of $P(x, y)$ on $\widehat{S}\left(\frac{n}{m}\right)$ thus takes the shape

$$\begin{aligned}\widehat{P}(x, y) &= P\left(x + \frac{r}{s}y, y\right) \\ &= \frac{d}{2}sx(sx - s) + Dx + \frac{1}{s}\left(Dr + Es - \frac{dr}{2}(s + r)\right)y + F \\ &= \frac{d}{2}sx(sx - s) + Dx + \frac{k}{s}y + F. \end{aligned} \tag{5.3.4}$$

It is immediately evident from this form that the difference between the values of $\widehat{P}$ at two consecutive steps on any ladder is k, as the y-coordinates of consecutive steps differ by s.

As in the integral case, we can apply the involution $\begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}$ to $\widehat{S}\left(\frac{n}{m}\right)$ to ensure that consecutive steps on each ladder differ by a positive k. For ladders with non-integral x-coordinates, this might cause the steps to change position. When transforming back to a sector with integral lattice points, we can therefore not simply apply the inverse $M_{r/s}$. To recover an integer lattice, we apply the transformation $M_{1-r/s} = M_{1-(m-1)/n}$. On $S(n/m)$, the total transformation is

$$L = \begin{pmatrix} 1 & 1 - \frac{m-1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{m-1}{n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n - m + 2 & \frac{(m-1)^2}{n} - m \\ n & -m \end{pmatrix}$$

with inverse

$$L^{-1} = \begin{pmatrix} m & \frac{(m-1)^2}{n} - m \\ n & -(n - m + 2) \end{pmatrix}.$$

The effect is

$$L\left(S\left(\frac{n}{m}\right)\right) = S\left(\frac{n}{n - m + 2}\right).$$

So, if P is a k-stairs polynomial on $S\left(\frac{n}{m}\right)$, then $P^{-1} = P \circ L^{-1}$ is a $(-k)$-stairs polynomial on $S\left(\frac{n}{n-m+2}\right)$. This allows us to assume $k > 0$.

Example 5.10. We have (see Fig. 5.3)

$$L\left(S\left(\frac{9}{7}\right)\right) = S\left(\frac{9}{9 - 7 + 2}\right) = S\left(\frac{9}{4}\right),$$

so there is a k-stairs QPP on $S(9/7)$, if and only if there is a $(-k)$-stairs QPP on $S(9/4)$.

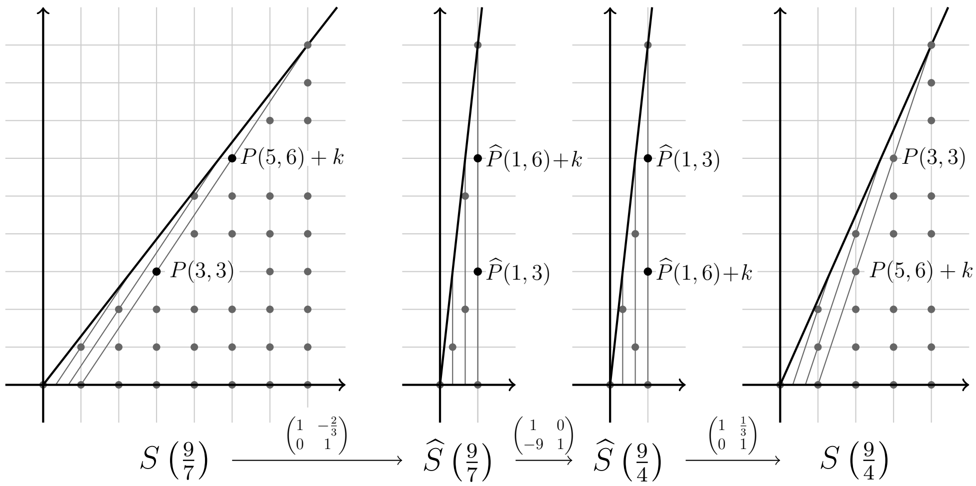

Figure 5.6: $S(9/7)$ and $S(9/4)$. k has opposite sign.

Restrictions on D and E

We will now see that the coefficient D (and thus E, by (5.3.2)) can be expressed in terms of s, d and k. The procedure is similar to the case of integral sectors, but the the general rational sectors require a bit more work. Assume $k > 0$.

Lemma 5.11. *We have*

$$\widehat{P}\left(\frac{k}{s}, \bar{y}_k\right) = F + k$$

and

$$D = s - \frac{ds}{2}(k - s) - \bar{y}_k.$$

Proof. $\widehat{J}_{si}$ are the ladders with integral x-coordinates. On these, $\bar{y}_{si} = 0$. For each si, let i' be the smallest positive integer such that the values of $\widehat{P}$ are in the same congruence class modulo k on $\widehat{J}_{is}$ and $\widehat{J}_{is+i'}$.

Claim: *For i large enough, we have $i' = k$.*

Assume this is not the case so that we can choose arbitrarily large i, such that $i' \neq k$. As argued in Prop. 5.5, this means that there must be arbitrarily large i for which $i' > k$. For $\widehat{P}$

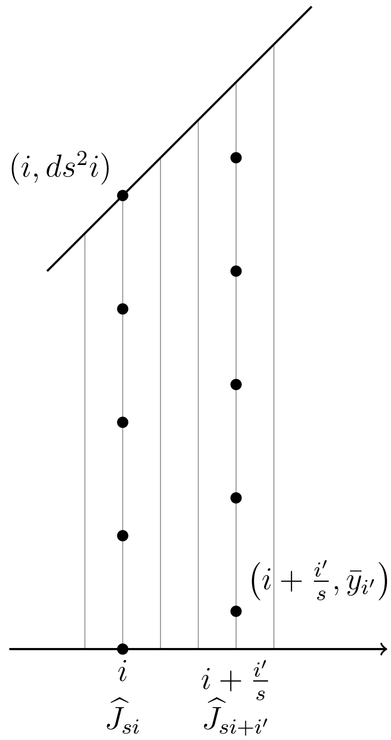

Figure 5.7: Next ladder with same congruence class mod k.

to be a packing polynomial, for i large enough, the first step on $\widehat{J}_{si+i'}$ must be k more than the last step on $\widehat{J}_{si}$. Otherwise, either values would be missing from the congruence class, since $\widehat{P}(x, 0)$ is eventually increasing, or values would be repeated. Specifically,

$$\widehat{P}(i, ds^2 i) + k = \widehat{P}(i + i'/s, \bar{y}_{i'}) \tag{5.3.5}$$

for i large enough. Also note that, by (5.3.3),

$$\widehat{P}(i, ds^2 i) = \widehat{P}(i, 0) + (\#\widehat{J_{si}} - 1)k = \widehat{P}(i, 0) + dsik. \tag{5.3.6}$$

We have

$$\begin{aligned}\widehat{P}\left(\frac{si+i'}{s},\bar{y}_{i'}\right) &= \frac{d}{2}(si+i')(si+i'-s)+D\left(i+\frac{i'}{s}\right)+\frac{k}{s}\bar{y}_{i'}+F\\ &= \frac{d}{2}si(si-s)+\frac{d}{2}(i'(si-s)+sii'+i'^2)\\ &\qquad +Di+D\frac{i'}{s}+\frac{k}{s}\bar{y}_{i'}+F\\ &= \frac{d}{2}si(si-s)+Di+F\\ &\qquad +\frac{d}{2}s\frac{i'}{s}\left(s\frac{i'}{s}-s\right)+D\frac{i'}{s}+\frac{k}{s}\bar{y}_{i'}+F\\ &\qquad +dsii'-F\\ &= \widehat{P}(i,0)+\widehat{P}\left(\frac{i'}{s},\bar{y}_{i'}\right)+dsii'-F. \end{aligned} \tag{5.3.7}$$

Along with (5.3.5) and (5.3.6), this implies that

$$\widehat{P}\left(\frac{i'}{s},\bar{y}_{i'}\right)=F+k+dsi(k-i'). \tag{5.3.8}$$

Since we assumed that we could choose arbitrarily large i with $i'>k$, this means that for some i, we have $\widehat{P}\left(\frac{i'}{s},\bar{y}_{i'}\right)<0$, a contradiction. This proves the claim that, from a certain point, $i'=k$.

From (5.3.4) and (5.3.8), we then have

$$\widehat{P}\left(\frac{k}{s},\bar{y}_k\right)=\frac{d}{2}k(k-s)+D\frac{k}{s}+\frac{k}{s}\bar{y}_k+F=F+k,$$

which implies

$$D=s-\frac{ds}{2}(k-s)-\bar{y}_k.$$

□

Lemma 5.12. *For all $i\in\mathbb{N}_0$, we have*

$$\widehat{P}\left(\frac{i}{s},\bar{y}_i\right)\equiv\widehat{P}\left(\frac{i+k}{s},\bar{y}_{i+k}\right)\pmod{k}.$$

Proof. In the previous lemma, we proved that this is true for si for i large enough. For any i, we find that

$$\begin{aligned}
\widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) &= \frac{d}{2}(i+k)(i+k-s) + D\frac{i+k}{s} + \frac{k}{s}\bar{y}_{i+k} + F \\
&= \frac{d}{2}i(i-s) + \frac{d}{2}(k(i-s) + ik + k^2) \\
&\qquad + D\frac{i}{s} + D\frac{k}{s} + \frac{k}{s}\bar{y}_{i+k} + F \\
&= \frac{d}{2}i(i-s) + D\frac{i}{s} + \frac{k}{s}\bar{y}_i + F \\
&\qquad + \frac{d}{2}k(k-s) + D\frac{k}{s} + \frac{k}{s}\bar{y}_k + F \\
&\qquad + dki + \frac{k}{s}\bar{y}_{i+k} - \frac{k}{s}\bar{y}_i - \frac{k}{s}\bar{y}_k - F \\
&= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + \widehat{P}\left(\frac{k}{s}, \bar{y}_k\right) + dki + \frac{k}{s}\bar{y}_{i+k} - \frac{k}{s}\bar{y}_i - \frac{k}{s}\bar{y}_k - F
\end{aligned}$$

By the previous lemma, we have $\widehat{P}\left(\frac{k}{s}, \bar{y}_k\right) = F + k$, so

$$\begin{aligned}
\widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) &= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + F + k + dki + \frac{k}{s}\bar{y}_{i+k} - \frac{k}{s}\bar{y}_i - \frac{k}{s}\bar{y}_k - F \\
&= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + k\left(1 + di + \frac{\bar{y}_{i+k} - (\bar{y}_i + \bar{y}_k)}{s}\right)
\end{aligned}$$

Since

$$\bar{y}_{i+k} \equiv -\frac{i+k}{r} \equiv -\frac{i}{r} - \frac{k}{r} \equiv \bar{y}_i + \bar{y}_k \pmod{s},$$

we have $\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) \equiv \widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) \pmod{k}$. □

Lemma 5.13. *We have*

$$\bar{y}_k = s - 1.$$

Proof. By (5.3.4) and Lem 5.11, we have

$$\begin{aligned}
\widehat{P}(x,y) &= \frac{d}{2}sx(sx-s) + \left(s - \frac{ds}{2}(k-s) - \bar{y}_k\right)x + \frac{k}{s}y + F \\
&= \frac{d}{2}sx(sx-s) - \frac{d}{2}sx(k-s) + sx - \bar{y}_k x + \frac{k}{s}y + F \\
&= \frac{d}{2}sx(sx-k) + (s - \bar{y}_k)x + \frac{k}{s}y + F
\end{aligned} \tag{5.3.9}$$

Since, by Lem 5.12, $\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) \equiv \widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) \pmod{k}$ for all i, from a certain i, we must have that the value of $\widehat{P}$ on the last step on $\widehat{J}_{i-k}$ is k less than the value on the first step of $\widehat{J}_i$. $\widehat{J}_{i-k}$ intersects the line $y = ds^2x$ in the point $(\frac{i-k}{s}, ds(i-k))$. So we have

$$\widehat{P}\left(\frac{i-k}{s}, ds(i-k)\right) \geq \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) - k. \tag{5.3.10}$$

Evaluating $\widehat{P}\left(\frac{i-k}{s}, ds(i-k)\right)$ using (5.3.9), we find

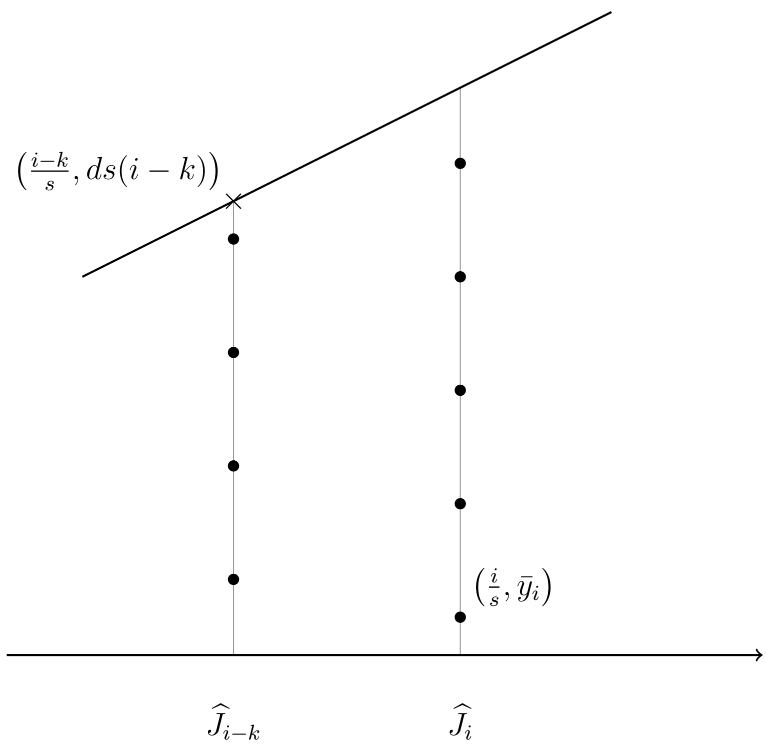

Figure 5.8: The point where $\widehat{J}_{i-k}$ intersects $y = ds^2x$.

$$\begin{aligned}
\widehat{P}\left(\frac{i-k}{s}, ds(i-k)\right) &= \frac{d}{2}(i-k)(i-2k) + (s-\bar{y}_k)\frac{i-k}{s} + \frac{k}{s}ds(i-k) + F \\
&= \frac{d}{2}i(i-k) - dk(i-k) + (s-\bar{y}_k)\frac{i}{s} - (s-\bar{y}_k)\frac{k}{s} \\
&\qquad + dk(i-k) + F \\
&= \frac{d}{2}i(i-k) + (s-\bar{y}_k)\frac{i}{s} + \frac{k}{s}\bar{y}_i + F - \frac{k}{s}\bar{y}_i - (s-\bar{y}_k)\frac{k}{s} \\
&= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) - k + \frac{k}{s}(\bar{y}_k - \bar{y}_i).
\end{aligned}$$

Together with (5.3.10), this means that

$$\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) - k + \frac{k}{s}(\bar{y}_k - \bar{y}_i) \geq \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) - k,$$

which implies $\bar{y}_k \geq \bar{y}_i$. Assuming i was large enough, it was arbitrary, so we conclude that $\bar{y}_k = s - 1$, since $s - 1$ is the highest possible value for a first step on a ladder. □

We can now express the necessary form of $P(x, y)$ in terms of d, r, s, k and the constant term F.

Proposition 5.14. *We have*

$$D = 1 - \frac{d}{2}s(k - s),$$

so P takes the necessary form

$$P(x, y) = \frac{d}{2}(sx - ry)(sx - ry - k) + x + \frac{k - r}{s}y + F,$$

or correspondingly on $\widehat{S}\left(\frac{n}{m}\right)$

$$\widehat{P}(x, y) = \frac{d}{2}sx(sx - k) + x + \frac{k}{s}y + F. \tag{5.3.11}$$

Proof. By Lem. 5.11, we have $D = s - \frac{ds}{2}(k - s) - \bar{y}_k$. By Lem. 5.13, we have $\bar{y}_k = s - 1$. Combined, this means that

$$D = 1 - \frac{d}{2}s(k - s).$$

By (5.3.2), we have

$$E - \frac{d}{2}r(s + r) = \frac{k}{s} - D\frac{r}{s},$$

so (5.3.1) can be rewritten as

$$\begin{aligned}
P(x,y) &= \frac{d}{2}(sx-ry)(sx-ry-s) + Dx + \left(E - \frac{dr}{2}(s+r)\right) y + F \\
&= \frac{d}{2}(sx-ry)(sx-ry-s) + Dx + \left(\frac{k}{s} - D\frac{r}{s}\right) y + F \\
&= \frac{d}{2}(sx-ry)(sx-ry-s) + \left(1 - \frac{d}{2}s(k-s)\right) x \\
&\qquad\qquad\qquad + \frac{k}{s}y - \left(1 - \frac{d}{2}s(k-s)\right)\frac{r}{s}y + F \\
&= \frac{d}{2}(sx-ry)(sx-ry-s) - \frac{d}{2}s(k-s)x + \frac{d}{2}r(k-s)y \\
&\qquad\qquad\qquad + x + \frac{k-r}{s}y + F \\
&= \frac{d}{2}(sx-ry)(sx-ry-s) + \frac{d}{2}(sx-ry)(s-k) + x + \frac{k-r}{s}y + F \\
&= \frac{d}{2}(sx-ry)(sx-ry-k) + x + \frac{k-r}{s}y + F.
\end{aligned}$$

The corresponding necessary form (5.3.11) of $\widehat{P}$ follows immediately from (5.3.9) and Lem. 5.13. □

Sufficient Conditions for a Packing Polynomial

Having found that for $\widehat{P}$ to be a QPP on $\widehat{I}(n/m)$, it must have the form (5.3.11), we can now supply a sufficient condition for a polynomial of that form to be a packing polynomial.

Lemma 5.15. *$\widehat{P}$ is a QPP on $\widehat{S}(n/m)$ wrt. $\widehat{I}(n/m)$, if and only if it is of the necessary form (5.3.11) and*

$$\left\{\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) \;:\; 0 \le i \le k-1\right\} = \{0, 1, \ldots, k-1\}$$

Proof. We want to show that if $\widehat{P}$ has the form (5.3.11), then its value on the last step of $\widehat{J}_i$ is k less than the first step on $\widehat{J}_{i+k}$ for all $i \in \mathbb{N}_0$. Equivalently, the value on the first step of $\widehat{J}_i$ is $(\#\widehat{J}_i)k$ less than the first step on $\widehat{J}_{i+k}$. If that is the case, then $\widehat{P}$ taking the values $0, \ldots, k-1$ on the first steps of $\widehat{J}_0, \ldots, \widehat{J}_{k-1}$ will make the polynomial bijective onto $\mathbb{N}_0$, i.e. a packing polynomial.

We have

$$\begin{aligned}\widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) &= \frac{d}{2}(i+k)i + \frac{i+k}{s} + \frac{k}{s}\bar{y}_{i+k} + F \\ &= \frac{d}{2}i(i-k) + \frac{i}{s} + \frac{k}{s}\bar{y}_i + F + dki + \frac{k}{s} + \frac{k}{s}\bar{y}_{i+k} - \frac{k}{s}\bar{y}_i \\ &= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + k\left(di + \frac{1+\bar{y}_{i+k}-\bar{y}_i}{s}\right). \end{aligned} \tag{5.3.12}$$

From Lem. 5.13, we have

$$\bar{y}_k = s-1 \equiv -1 \equiv -\frac{k}{r} \pmod{s},$$

so $k \equiv r \pmod{s}$. Hence

$$\bar{y}_{i+k} - \bar{y}_i \equiv -\frac{i}{r} - \frac{k}{r} + \frac{i}{r} \equiv -1 \pmod{s}.$$

We have $0 \le \bar{y}_i \le s-1$ for all i, so $\bar{y}_{i+k} - \bar{y}_i = -1$, except when $\bar{y}_{i+k} = s-1$ and $\bar{y}_i = 0$ which happens exactly when $s \mid i$. So (5.3.12) can be rewritten as

$$\begin{aligned}\widehat{P}\left(\frac{i+k}{s}, \bar{y}_{i+k}\right) &= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + k\left(di + \frac{1+s[\![s \mid i]\!]-1}{s}\right) \\ &= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + k(di + [\![s \mid i]\!]) \\ &= \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) + k(\#\widehat{J}_i),\end{aligned}$$

by (5.3.3), which is what we wanted to show. □

What is left for us is to pinpoint which values of k allow for quadratic packing polynomials on a sector $S(n/m)$ and find the constant terms F to achieve such.

Lemma 5.16.

$$\sum_{i=0}^{k-1} \bar{y}_i = \frac{(k-1)(s-1)}{2}.$$

Proof. Note that, since $k \equiv r \pmod{s}$,

$$\bar{y}_i + \bar{y}_{k-i} \equiv -\frac{i}{r} - \frac{k-i}{r} \equiv -1 \pmod{s}.$$

$0 \le \bar{y}_i \le s-1$ for all i, so $\bar{y}_i + \bar{y}_{k-i} = s-1$. Also, if k is even, then

$$2\bar{y}_{k/2} \equiv 2\left(-\frac{k/2}{r}\right) \equiv -1 \pmod{s},$$

in which case $\bar{y}_{k/2} = \frac{s-1}{2}$.

We now find, for odd k, remembering $\bar{y}_0 = 0$,

$$\sum_{i=0}^{k-1} \bar{y}_i = \sum_{i=1}^{k-1} \bar{y}_i = \sum_{i=1}^{\frac{k-1}{2}} (\bar{y}_i + \bar{y}_{k-i}) = \sum_{i=1}^{\frac{k-1}{2}} (s-1) = \frac{(k-1)(s-1)}{2}.$$

For even k,

$$\sum_{i=0}^{k-1} \bar{y}_i = \sum_{i=1}^{\frac{k}{2}-1} (\bar{y}_i + \bar{y}_{k-i}) + \bar{y}_{k/2} = \left(\frac{k}{2} - 1\right)(s-1) + \frac{s-1}{2} = \frac{(k-1)(s-1)}{2}.$$

□

Lemma 5.17. *Let F be the constant term in (5.3.11). We have*

$$F = \frac{d}{12}(k+1)(k-1)$$

Proof. By Lemma 5.15, we have

$$\sum_{i=0}^{k-1} \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) = \sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$$

Evaluating $\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right)$ by (5.3.11), and applying Lem. 5.16 along the way, we find

$$\begin{aligned}
\frac{k(k-1)}{2} &= \sum_{i=0}^{k-1} \widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) \\
&= \sum_{i=0}^{k-1} \left(\frac{d}{2} i(i-k) + \frac{i}{s} + \frac{k}{s}\bar{y}_i + F\right) \\
&= \frac{d}{2}\sum_{i=0}^{k-1} i^2 - \frac{dk}{2}\sum_{i=0}^{k-1} i + \frac{1}{s}\sum_{i=0}^{k-1} i + \frac{k}{s}\sum_{i=0}^{k-1} \bar{y}_i + \sum_{i=0}^{k-1} F \\
&= \frac{d}{2}\frac{(k-1)k(2k-1)}{6} + \left(\frac{1}{s} - \frac{dk}{2}\right)\frac{k(k-1)}{2} \\
&\qquad + \frac{k}{s}\frac{(k-1)(s-1)}{2} + kF \\
&= k\left((k-1)\left(\frac{ds(2k-1)}{12s} + \frac{3(2-dsk)}{12s} + \frac{6(s-1)}{12s}\right) + F\right) \\
&= k\left((k-1)\frac{6-d(k+1)}{12} + F\right),
\end{aligned}$$

so

$$F = \frac{k-1}{2} - (k-1)\frac{6-d(k+1)}{12} = \frac{d(k-1)(k+1)}{12}.$$

□

Classification of QPPs on Rational Sectors

We can now find all non-integral rational sectors on which there are QPPs and we can state a specific formula for any QPP on such sectors.

Still, assume $k > 0$ and choose as the representative of equivalent sectors the unique sector with $n/m > 1$.

Proposition 5.18. *Let $n > m > 1$, $\gcd(m, n) = 1$ with $n \mid (m-1)^2$.*

Let $l = \gcd(m-1, n)$ and put $n = ls$, $m-1 = lr$ with $\gcd(r, s) = 1$. So we have $s > r \geq 1$. The conditions imply $s \mid l$, so there is and integer $d \geq 1$ such that $l = ds$. This means we can write $n = ds^2$, $m = drs + 1$ with $d, r, s \in \mathbb{Z}$, $d > 1$, $s > r \geq 1$, $\gcd(r, s) = 1$.

If $r = 1$, so

$$\frac{n}{m} = \frac{ds^2}{ds+1},$$

then

$$P(x, y) = \frac{d}{2}(sx - y)(sx - y - 1) + x$$

is a 1-stairs packing polynomial on $S(n/m)$ for all d, s.

If $r = 2$, $d = 4$, so

$$\frac{n}{m} = \frac{4s^2}{8s+1},$$

then

$$P(x, y) = 2(sx - 2y)(sx - 2y - 2) + x + 1$$

is a 2-stairs packing polynomial for $S(n/m)$ for all odd *s.*

If $r = 3$, $d = 3$, so

$$\frac{n}{m} = \frac{3s^2}{9s+1},$$

then

$$P(x,y) = \frac{3}{2}(sx-3y)(sx-3y-3)+x+2,$$

is a 3*-stairs packing polynomial on* $S(n/m)$ *for all* $s \not\equiv 0 \pmod 3$*.*

If $r=1,\ d=3,\ s=2,$ *so*

$$\frac{n}{m} = \frac{12}{7},$$

then

$$P(x,y) = \frac{3}{2}(2x-y)(2x-y-3)+x+2y+2$$

is a 3*-stairs packing polynomial on* $S(12/7)$ *with* $k=3$*.*

There are no other sectors with $n/m>1$ *which admit quadratic packing polynomials with* $k>0$*.*

Proof. We will do case study on all values of $k \geq 1$.

Case $k=1$*:* $F=0$, by Lem. 5.17. $k \equiv r \pmod s$ implies that $r=1$, so $n=ds^2$, $m=ds+1$.

The necessary form of the polynomial is

$$\begin{aligned} P(x,y) &= \frac{d}{2}(sx-ry)(sx-ry-k)+x+\frac{k-r}{s}y+F \\ &= \frac{d}{2}(sx-y)(sx-y-1)+x. \end{aligned}$$

By Lem. 5.15, this is a packing polynomial for all values of $s>1, d>0$, since $P(0,0)=0$.

Case $k=2$*:*

$$F = \frac{d(k-1)(k+1)}{12} = \frac{3d}{12} = \begin{cases} 0 \\ 1 \end{cases}$$

Since $d>0$, we must have $d=4$. On $\widehat{S}(n/m)$, the necessary form of the polynomial is

$$\begin{aligned} \widehat{P}(x,y) &= \frac{d}{2}sx(sx-k)+x+\frac{k}{s}y+F \\ &= 2sx(sx-2)+x+\frac{2}{s}y+1. \end{aligned}$$

The first steps on the first two stairs are $(0,0)$ and $(1/s,\bar{y}_1)$. $k \equiv r \pmod{s}$ implies that $r = 2$, and hence that s is odd. We have $\bar{y}_1 \equiv -\frac{1}{2} \pmod{s}$, so $\bar{y}_1 = \frac{s-1}{2}$. We find that

$$\begin{aligned} \widehat{P}(0,0) &= 1 \\ \widehat{P}\left(\frac{1}{s},\bar{y}_1\right) &= \widehat{P}\left(\frac{1}{s},\frac{s-1}{2}\right) \\ &= 2(1-2) + \frac{1+2\frac{s-1}{2}}{s} + 1 \\ &= -2+1+1 = 0. \end{aligned}$$

So this is a packing polynomial for all odd s. The corresponding packing polynomial on $S(n/m)$ is

$$P(x,y) = 2(sx-2y)(sx-2y-2) + x + 1.$$

Case $k = 3$:

$$F = \frac{d(k-1)(k+1)}{12} = \frac{8d}{12} = \frac{2d}{3} = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$$

The only option is $d = 3$, $F = 2$. On $\widehat{S}(n/m)$, the necessary form of the polynomial is

$$\begin{aligned} \widehat{P}(x,y) &= \frac{d}{2}sx(sx-k) + x + \frac{k}{s}y + F \\ &= \frac{3}{2}sx(sx-3) + x + \frac{3}{s}y + 2. \end{aligned}$$

$k = 3 \equiv r \pmod{s}$ leaves two options: Either $r = 3$, or $r = 1$ and $s = 2$. In the latter case, we have

$$\begin{aligned} \bar{y}_1 &\equiv -\frac{1}{1} \mod 2, \text{ so } \bar{y}_1 = 1, \\ \bar{y}_2 &\equiv -\frac{2}{1} \mod 2, \text{ so } \bar{y}_2 = 0. \end{aligned}$$

We have

$$\begin{aligned}
\widehat{P}(0,0) &= 2 \\
\widehat{P}\left(\frac{1}{s}, \bar{y}_1\right) &= \widehat{P}\left(\frac{1}{2}, 1\right) \\
&= \frac{3}{2}(1-3) + \frac{1+3\cdot 1}{2} + 2 \\
&= -3+2+2 = 1. \\
\widehat{P}\left(\frac{2}{s}, \bar{y}_2\right) &= \widehat{P}(1,2) \\
&= \frac{3}{2}\cdot 2(2-3) + \frac{2+3\cdot 0}{2} + 2 \\
&= -3+1+2 = 0.
\end{aligned}$$

This checks. The corresponding packing polynomial on $S(n/m)$ is

$$P(x,y) = \frac{3}{2}(2x-y)(2x-y-3) + x + 2y + 2.$$

In the case $r = 3$, we have $s \equiv 1$ or $2 \pmod 3$.

$$\bar{y}_1 \equiv -\frac{1}{3} \pmod s, \text{ so } \bar{y}_1 = \begin{cases} \frac{s-1}{3} & \text{if } s \equiv 1 \pmod 3 \\ \frac{2s-1}{3} & \text{if } s \equiv 2 \pmod 3 \end{cases}$$

$$\bar{y}_2 \equiv -\frac{2}{3} \pmod s, \text{ so } \bar{y}_2 = \begin{cases} \frac{2s-2}{3} & \text{if } s \equiv 1 \pmod 3 \\ \frac{s-2}{3} & \text{if } s \equiv 2 \pmod 3 \end{cases}$$

If $s \equiv 1 \pmod 3$, we have

$$\begin{aligned}
\widehat{P}(0,0) &= 2 \\
\widehat{P}\left(\frac{1}{s}, \bar{y}_1\right) &= \widehat{P}\left(\frac{1}{s}, \frac{s-1}{3}\right) \\
&= \frac{3}{2}(1-3) + \frac{1}{s} + \frac{s-1}{s} + 2 \\
&= -3 + 1 + 2 = 0. \\
\widehat{P}\left(\frac{2}{s}, \bar{y}_2\right) &= \widehat{P}\left(\frac{2}{s}, \frac{2s-2}{3}\right) \\
&= \frac{3}{2} \cdot 2(2-3) + \frac{2}{s} + \frac{2s-2}{s} + 2 \\
&= -3 + 2 + 2 = 1.
\end{aligned}$$

If $s \equiv 2 \pmod 3$, we have

$$\begin{aligned}
\widehat{P}(0,0) &= 2 \\
\widehat{P}\left(\frac{1}{s}, \bar{y}_1\right) &= \widehat{P}\left(\frac{1}{s}, \frac{2s-1}{3}\right) \\
&= \frac{3}{2}(1-3) + \frac{1}{s} + \frac{2s-1}{s} + 2 \\
&= -3 + 2 + 2 = 1. \\
\widehat{P}\left(\frac{2}{s}, \bar{y}_2\right) &= \widehat{P}\left(\frac{2}{s}, \frac{s-2}{3}\right) \\
&= \frac{3}{2} \cdot 2(2-3) + \frac{2}{s} + \frac{s-2}{s} + 2 \\
&= -3 + 1 + 2 = 0.
\end{aligned}$$

Both check. The corresponding packing polynomial on $S(n/m)$ is

$$P(x,y) = \frac{3}{2}(sx-3y)(sx-3y-3) + x + 2.$$

Case $k > 3$: Again, using the fact that F is an integer and

$$0 \leq F = \frac{d(k+1)(k-1)}{12} \leq k-1,$$

we find that $k+1 \leq \frac{12}{d}$, so $k \leq 11$. Also $d \leq \frac{12}{k+1}$ implies that $d = 1$ or 2 (2 only being a possiblity in the case $k = 4$ or 5). We test each option.

k	4	5	6	7	8	9	10	11
F	$\frac{3\cdot5}{12}$ or $\frac{2\cdot3\cdot5}{12}$	$\frac{4\cdot6}{12}$ or $\frac{2\cdot4\cdot6}{12}$	$\frac{5\cdot7}{12}$	$\frac{6\cdot8}{12}$	$\frac{7\cdot9}{12}$	$\frac{8\cdot10}{12}$	$\frac{9\cdot11}{12}$	$\frac{10\cdot12}{12}$
Integer?		✓		✓				✓

In particular, k must be odd. Since $k \equiv r \pmod{s}$ and $\gcd(s, r) = 1$, we have $\gcd(s, k) = 1$. This means that both 2, d and s have inverses modulo k. We find

$$\begin{aligned}\widehat{P}\left(\frac{i}{s}, \bar{y}_i\right) &= \frac{d}{2}i(i-k) + \frac{i}{s} + \frac{k}{s}\bar{y}_i + F \\ &\equiv \frac{d}{2}i^2 + \frac{i}{s} + F \pmod{k} \\ &\equiv \frac{d}{2}\left(i + \frac{1}{ds}\right)^2 - \frac{1}{2ds^2} + F \pmod{k}\end{aligned}$$

Let $1 \leq j \leq k-1$ such that $j \equiv -\frac{1}{ds} \pmod{k}$. Then we see that

$$\widehat{P}\left(\frac{j+1}{s}, \bar{y}_{j+1}\right) \equiv \widehat{P}\left(\frac{j-1}{s}, \bar{y}_{j-1}\right) \pmod{k},$$

which is impossible by Lem. 5.12, since $k > 2$. □

All Rational Sectors

To wrap things up, let us investigate what happens when applying the transformations L and M to the sectors representing their equivalence class.

Negative k

We begin by determining which sectors $S(n/m)$ with $n > m > 1$, $\gcd(n, m) = 1$ admit k-stairs packing polynomials with $k < 0$. Under the transformation

$$L = \begin{pmatrix} n-m+2 & \frac{(m-1)^2}{n} - m \\ n & -m \end{pmatrix}, \quad L : S\left(\frac{n}{m}\right) \mapsto S\left(\frac{n}{n-m+2}\right),$$

k-stair packing polynomials on $S(\frac{n}{m})$ is in 1-1 correspondance with $(-k)$-stair packing polynomials on $S(\frac{n}{n-m+2})$.

Proposition 5.19. *Let* $n > m > 1$, $\gcd(m, n) = 1$ *with* $n \mid (m-1)^2$.

We can write $n = ds^2$, $m = drs + 1$. *where* $s > r \geq 1$, $\gcd(r, s) = 1$, $d \geq 1$ *are integers.*

If $r = s - 1$, *so*

$$\frac{n}{m} = \frac{ds^2}{d(s-1)s+1},$$

then

$$P(x, y) = \frac{d}{2}(sx - (s-1)y)(sx - (s-1)y + 1) + x - y$$

is a (-1)*-stairs packing polynomial on* $S(n/m)$ *for all* d, s.

If $r = s - 2$ *and* $d = 4$, *so*

$$\frac{n}{m} = \frac{4s^2}{4(s-2)s+1},$$

then

$$P(x, y) = 2(sx - (s-2)y)(sx - (s-2)y + 2) + x - y + 1$$

is a (-2)*-stairs packing polynomial on* $S(n/m)$ *for all* odd s.

If $r = s - 3$ *and* $d = 3$, *so*

$$\frac{n}{m} = \frac{3s^2}{3(s-3)s+1},$$

then

$$P(x, y) = \frac{3}{2}(sx - (s-3)y)(sx - (s-3)y + 3) + x - y + 2$$

is a (-3)*-stairs packing polynomial on* $S(n/m)$ *for all* $s \not\equiv 3 \pmod 3$.

If $r = 1$, $d = 3$, $s = 2$, *so*

$$\frac{n}{m} = \frac{12}{7},$$

then

$$\frac{3}{2}(2x - y)(2x - y + 3) + x - 2y + 2$$

is a (-3)*-stairs packing polynomial.*

There are no other sectors $S(n/m)$ *with* $n > m > 1$ *which admit* k*-stairs QPPs with* $k < 0$.

Proof. A $(-k)$-stairs QPP on $S(n/m)$ corresponds to a k-stairs QPP on $S(n/(n-m+2))$.

By Prop. 5.18, there is a k-stairs QPP on $S(n/(n-m+2))$ with $k>0$, only if

$$n = ds^2, \quad n - m + 2 = drs + 1,$$

where $s>r>0$, $d>1$ are integers with $\gcd(r,s)=1$ and at least one of the following is met.

(i) $r = 1$

(ii) $r = 2$, $d = 4$

(iii) $r = 3$, $d = 3$

(iv) $r = 1$, $d = 3$, $s = 2$

Then

$$Q(x,y) = \frac{d}{2}(sx - ry)(sx - ry - k) + x + \frac{k-r}{s}y + \frac{d(k^2-1)}{12},$$

where $k = r$ or, in case (iv), $k = 3$.

This means that on $S(n/m)$, for $k>0$, there is a $(-k)$-stairs polynomial only if

$$n = ds^2, \quad m = d(s-r)s + 1,$$

with $s > s - r \geq 1$, $\gcd(s, s-r)$ and at least one of the following is met.

(i) $s - r = 1$

(ii) $s - r = 2$, $d = 4$

(iii) $s - r = 3$, $d = 3$

(iv) $r = 1$, $d = 3$, $s = 2$

The packing polynomial is

$$P(x,y) = Q(L(x,y))$$

$$= Q\left(\begin{pmatrix} n-m+2 & \frac{(m-1)^2}{n}-m \\ n & -m \end{pmatrix}(x,y)\right)$$

$$= Q\left(\begin{pmatrix} drs+1 & -(dr(s-r)+1) \\ ds^2 & -(ds(s-r)+1) \end{pmatrix}(x,y)\right)$$

$$= Q((drs+1)x-(dr(s-r)+1)y, ds^2x-(ds(s-r)+1)y)$$

$$= \frac{d}{2}(s((drs+1)x-(dr(s-r)+1)y)-r(ds^2x-(ds(s-r)+1)y))$$
$$(s((drs+1)x-(dr(s-r)+1)y)-r(ds^2x-(ds(s-r)+1)y)-k)$$
$$+(drs+1)x-(dr(s-r)+1)y+\frac{k-r}{s}(ds^2x-(ds(s-r)+1)y)$$
$$+\frac{d(k^2-1)}{12}$$

$$= \frac{d}{2}(sx-(s-r)y)(sx-(s-r)y-k)+x+(drs+(k-r)ds)x$$
$$-\left(dr(s-r)+1+(k-r)d(s-r)+\frac{k-r}{s}\right)y+\frac{d(k^2-1)}{12}$$

$$= \frac{d}{2}(sx-(s-r)y)(sx-(s-r)y-k)+x+dskx$$
$$-\left(1+d(s-r)k+\frac{k-r}{s}\right)y+\frac{d(k^2-1)}{12}$$

$$= \frac{d}{2}(sx-(s-r)y)(sx-(s-r)y-k)+\frac{d}{2}(sx-(s-r)y)(2k)$$
$$+x+\frac{-k-(s-r)}{s}y+\frac{d(k^2-1)}{12}$$

$$= \frac{d}{2}(sx-(s-r)y)(sx-(s-r)y+k)+x+\frac{-k-(s-r)}{s}y+\frac{d(k^2-1)}{12},$$

with $k=r$ or, in case (iv), $k=3$. Substituting in the values for k, d, r and s corresponding to each case provides the result. □

Example 5.20 (Sectors with multiple QPPs)**.** As we have seen, all integral sectors and all sectors $S(1/m)$ allow for two different QPPs. Also, the sectors $S(3)$ and $S(4)$ allow for two more QPPs.

Prop. 5.18 and Prop. 5.19 reveal exactly which sectors $S(n/m)$ with $n>m$ allow for multiple QPPs.

	$r=1$	$r=2$, $d=4$	$r=3$, $d=3$
$r=s-1$	$s=2$	$s=3$	$s=4$
$r=s-2$, $d=4$	$s=3$	$s=4$, $\gcd(2,4)\neq 1$	$3\neq 4$
$r=s-3$, $d=3$	$s=4$	$3\neq 4$	$s=6$, $\gcd(3,6)\neq 1$

The table identifies where there can be an overlap in the various cases.

- On $S\left(\frac{4d}{2d+1}\right)$, there are QPPs with $k=\pm 1$ for all $d\geq 1$. If $d=3$, i.e. the sector is $S\left(\frac{12}{7}\right)$, then there are also QPPs with $k=\pm 3$.
- On $S\left(\frac{36}{13}\right)$, there are QPPs with $k=1$ and $k=-2$.
- On $S\left(\frac{36}{25}\right)$, there are QPPs with $k=-1$ and $k=2$.
- On $S\left(\frac{48}{13}\right)$, there are QPPs with $k=1$ and $k=-3$.
- On $S\left(\frac{48}{25}\right)$, there are QPPs with $k=-1$ and $k=3$.

On all other sectors which allow for a QPP, there is only one.

General Rational Sectors

We will now collect all results to formulate one theorem capturing all rational sectors.

Theorem 5.21. *Let $m,n\in\mathbb{Z}$ with $m\geq 0$, $n\geq 1$, $\gcd(m,n)=1$. Put $l=\gcd(m-1,n)$. The polynomial*

$$P(x,y)=\frac{n}{2}\left(x-\frac{m-1}{n}y\right)\left(x-\frac{m-1}{n}y-\frac{kl}{n}\right)+x+\frac{kl-(m-1)}{n}y+|k|-1$$

is a packing polynomial on $S(n/m)$ if and only if any of the following is true.

(i) $k = 1$ *and either*

(a) $m = n\left\lfloor \frac{m}{n} \right\rfloor + 1 + l$

(b) $m = nt + 1,\ t \geq 0,\ n > 1$

(c) $n = 1$

(ii) $k = 2$ *and either*

(a) $m = n\left\lfloor \frac{m}{n} \right\rfloor + 1 + 2l,\ n = \frac{l^2}{4}$

(b) $m = 4t + 1,\ n = 4,\ t \geq 0$

(iii) $k = 3$ *and either*

(a) $m = n\left\lfloor \frac{m}{n} \right\rfloor + 1 + 3l,\ n = \frac{l^2}{3}$

(b) $m = 3t + 1,\ n = 3,\ t \geq 0$

(c) $m = 7 + 12t,\ n = 12,\ t \geq 0$

(iv) $k = -1$ *and either*

(a) $m = n\left(\left\lfloor \frac{m}{n} \right\rfloor + 1\right) + 1 - l$

(b) $m = tn + 1,\ t \geq 0,\ n > 1$

(c) $n = 1$

(v) $k = -2$ *and either*

(a) $m = n\left(\left\lfloor \frac{m}{n} \right\rfloor + 1\right) + 1 - 2l,\ n = \frac{l^2}{4}$

(b) $m = 4t + 1,\ n = 4,\ t \geq 0$

(vi) $k = -3$ *and either*

(a) $m = n\left(\left\lfloor \frac{m}{n} \right\rfloor + 1\right) + 1 - 3l,\ n = \frac{l^2}{3}$

(b) $m = 3t + 1,\ n = 3,\ t \geq 0$

(c) $m = 7 + 12t,\ n = 12,\ t \geq 0$

Proof. First, let $n \geq 2$, $m \geq 2$, $\gcd(m, n) = 1$, $m \not\equiv 1 \pmod{n}$. Under the transformation $M_{-\lfloor m/n \rfloor} = \begin{pmatrix} 1 & -\lfloor m/n \rfloor \\ 0 & 1 \end{pmatrix}$, the sector $S(n/m)$ is equivalent to $S(n/(m - n\lfloor m/n \rfloor))$, and $n > m - n\lfloor m/n \rfloor > 1$, the condition $m \not\equiv 1 \pmod{n}$ ensuring that $S(n/m)$ is not equivalent to an integral sector.

Any QPP on $S(n/m)$ is equivalent to a QPP on $S(n/(m - n\lfloor m/n \rfloor))$. By Prop. 5.18 and Prop. 5.19, there is a k-stairs QPP on $S(n/(m - n\lfloor m/n \rfloor))$, k positive or negative, only if

$$n = ds^2, \quad m - n\lfloor m/n \rfloor = drs + 1, \tag{5.3.13}$$

where $s > r > 0$, $d > 1$ are integers with $\gcd(r, s) = 1$ and at least one of the following is met.

(i) $r = 1$ or $r = s - 1$

(ii) $r = 2$ or $r = s - 2$ and $d = 4$

(iii) $r = 3$ or $r = s - 3$ and $d = 3$

(iv) $r = 1$, $d = 3$, $s = 2$

Then

$$Q(x, y) = \frac{d}{2}(sx - ry)(sx - ry - k) + x + \frac{k - r}{s}y + \frac{d(k^2 - 1)}{12},$$

where $k = r$, $k = s - r$ or, in case (iv), $k = \pm 3$, is a QPP on $S(n/(m - n\lfloor m/n \rfloor))$. Note that in all cases, we may write

$$\frac{d(k^2 - 1)}{12} = |k| - 1$$

The equivalent packing polynomial on $S(n/m)$ is

$$\begin{aligned} P(x, y) &= Q\left(M_{-\lfloor m/n \rfloor}(x, y)\right) \\ &= Q\left(x - \lfloor m/n \rfloor y, y\right) \\ &= \frac{d}{2}(s(x - \lfloor m/n \rfloor y) - ry)(s(x - \lfloor m/n \rfloor y) - ry - k) \\ &\quad + x - \lfloor m/n \rfloor y + \frac{k - r}{s}y + |k| - 1 \\ &= \frac{d}{2}(sx - (r + \lfloor m/n \rfloor s)y)(sx - (r + \lfloor m/n \rfloor s)y - k) \\ &\quad + x + \frac{k - (r + \lfloor m/n \rfloor s)}{s}y + |k| - 1. \end{aligned} \tag{5.3.14}$$

Now, to bring back m and n, by (5.3.13),

$$n = ds^2, \quad m = d(r + \lfloor m/n \rfloor s)s + 1$$

and, with $l = \gcd(n, m - 1) = ds$, we have

$$s = \frac{n}{l}, \quad r + \lfloor m/n \rfloor s = \frac{m - 1}{l}, \quad d = \frac{l^2}{n}.$$

Rewriting (5.3.14) in these terms, we get

$$P(x,y) = \frac{l^2/n}{2}\left(\frac{n}{l}x - \frac{m-1}{l}y\right)\left(\frac{n}{l}x - \frac{m-1}{l}y - k\right)$$
$$+ x + \frac{k-(m-1)/l}{n/l}y + |k| - 1$$
$$= \frac{n}{2}\left(x - \frac{m-1}{n}y\right)\left(x - \frac{m-1}{n}y - \frac{kl}{n}\right) + x + \frac{kl-(m-1)}{n}y + |k| + 1$$

The remaining rational sectors are those equivalent to $S(\infty)$, that is sectors of the form $S(1/m)$, $m \geq 0$, and those equivalent to integral sectors, i.e. sectors of the form $S(n/(nt+1)$, $n \geq 1$, $t \geq 0$.

By Fueter-Pólya, Thm. 1.1, the only QPPs on $S(\infty)$ are the two Cantor polynomials

$$P_\infty(x,y) = \frac{1}{2}(x+y)(x+y-k) + x + (k+1)y$$

with $k = \pm 1$. By Nathanson, Thm. 1.3, the only QPPs on $S(1/m)$ are the two

$$\begin{aligned} P_{\frac{1}{m}}(x,y) &= P_\infty\left(M_{-m}(x,y)\right) \\ &= P_\infty(x-my,y) \\ &= \frac{1}{2}(x-my+y)(x-my+y-k) + x - my + (k+1)y \\ &= \frac{1}{2}(x-(m-1)y)(x-(m-1)y-k) + x(k-(m-1))y \end{aligned} \tag{5.3.15}$$

with $k = \pm 1$. For $S(\infty)$, we set $n = 1$, $m = 0$ and have $l = 1$. On $S(1/m)$, $m \geq 1$, we also have $n = l = 1$. Inserting into the formula for $P(x,y)$ given in the theorem leads to the same formula as (5.3.15).

By Stanton, Prop. 5.8, on $S(n)$,

$$P_n(x,y) = \frac{n}{2}x(x-k) + x + ky + |k| - 1,$$

is a QPP only if

(i) $k = \pm 1$

(ii) $k = \pm 2$ and $n = 4$

(iii) $k = \pm 3$ and $n = 3$

and there no other QPPs on integral sectors. Under the transformation $M_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, with $t \geq 0$, $S(n)$ is equivalent to $S(n/(nt+1))$. The corresponding packing polynomial is

$$\begin{aligned} P_{\frac{n}{nt+t}}(x,y) &= P_n(M_{-t}(x,y)) \\ &= P_n(x-ty,y) \\ &= \frac{n}{2}(x-ty)(x-ty-k) + x - ty + ky + |k| - 1 \\ &= \frac{n}{2}(x-ty)(x-ty-k) + x + (k-t)y + |k| - 1 \end{aligned} \tag{5.3.16}$$

For the sector $S(n/(nt+1))$, we have $m = nt+1$ and $l = \gcd(n, m-1) = n$, so $\frac{m-1}{n} = t$. Inserting into the formula for $P(x,y)$ again leads to the same formula as (5.3.16). □

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