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by

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Abstract

AVERAGES AND NONVANISHING OF CENTRAL VALUES OF TRIPLE PRODUCT L-FUNCTIONS VIA THE RELATIVE TRACE FORMULA

by

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Advisor: Professor Brooke Feigon

Harris and Kudla [HK04] proved a conjecture of Jacquet, that the central value of a triple product L-function does not vanish if and only if there exists a quaternion algebra over which a period integral of three corresponding automorphic forms does not vanish. Moreover, Gross and Kudla [GK92] established an explicit identity relating central L-values and period integrals (which are finite sums in their case), when the cusp forms are of prime levels and weight 2. Böcherer, Schulze–Pillot [BSP96] and Watson [Wat02] generalized this identity to more general levels and weights, and Ichino [Ich08] proved an adelic period formula which would work for all the cases. In this thesis we use Ichino's period formula combined with a relative trace formula to show exact averages of certain families of triple product L-functions. We also present some applications of the average formulas to the nonvanishing problem.

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Contents

Chapter 1

Introduction

1.1 Main result

Let $N, k \geq 1$ be integers and N be square-free. Let $\mathcal{F}_{2k}(N)$ be the set of normalized cusp newforms of weight $2k$ on $\Gamma_0(N)$ which are eigenforms of Hecke operators. Normalizing $f(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inz}, g, h \in \mathcal{F}_{2k}(N)$ such that $a_1 = 1$, we can define the triple product L-function as the Euler product

$$L_{\text{fin}}(s, f \times g \times h) := \prod_p L_p(s, f \times g \times h).$$

The local factors are defined as follows. When $p \nmid N$,

$$L_p(s, f \times g \times h) := \prod_{i_1,i_2,i_3\in\{1,2\}} (1 - \alpha_p^{(i_1)}(f)\alpha_p^{(i_2)}(g)\alpha_p^{(i_3)}(h)p^{-s})^{-1}$$

where $\alpha_p^{(1)}(f)$, $\alpha_p^{(2)}(f)$ are defined to be the roots of $X^2 - a_p(f)X + p^{2k-1} = 0$; when $p \mid N$, noticing that $a_p(f)p^{-(k-1)} = \pm 1$, we define

$$L_p(s, f \times g \times h) := (1 + \varepsilon_p p^{3(k-1)}p^{-s})^{-1}(1 + \varepsilon_p p^{3(k-1)}p^{1-s})^{-2}$$

where $\varepsilon_p := -a_p(f)a_p(g)a_p(h)p^{-3(k-1)}$. Then $L_{\text{fin}}(s, f \times g \times h)$ is absolutely convergent in the half plane $\text{Re}(s) > 3k - \frac{1}{2}$. Moreover, Böcherer and Schulze–Pillot proved that

Theorem 1.1.1 ([BSP96] Theorem 4.3)**.** *The function*

$$\Lambda(s, f \times g \times h) := (2\pi)^{6k-3-4s}\Gamma(s)\Gamma(s+1-2k)^3 L_{\text{fin}}(s, f \times g \times h)$$

has an analytic continuation to the entire s-plane and satisfies the functional equation

$$\Lambda(s, f \times g \times h) = \varepsilon N^{5(3k-1-s)}\Lambda(6k-2-s, f \times g \times h), \quad \textit{where } \varepsilon = -\prod_{p|N} \varepsilon_p = \pm 1.$$

Remark 1.1.2. *One can observe that the central value is at $s = 3k - 1$. But after a translation $s \mapsto s + 3k - \frac{3}{2}$, the functional equation may be written in the form*

$$\Lambda(s, f \times g \times h) = \varepsilon N^{5(\frac{1}{2}-s)}\Lambda(1-s, f \times g \times h)$$

so that the central value is $\Lambda(\frac{1}{2}, f \times g \times h)$. See Lemma 3.2.2.

In [FW10] Theorem 1.1, Feigon and Whitehouse gave an exact average formula of the central values of triple product L-functions associated to three newforms of weight 2 and of the same prime level p: for $h \in \mathcal{F}_2(p)$,

$$\frac{p}{2^7\pi^5}\sum_{f,g\in\mathcal{F}_2(p)}\frac{L_{\text{fin}}(2, f\times g\times h)}{(f,f)(g,g)(h,h)} = 1 - \frac{24}{p-1} + \frac{1}{8\pi^2(h,h)}\times \begin{cases} 0, & p \equiv 1 \pmod{12}; \\ 6\sqrt{3}L_{\text{fin}}(1,h)L_{\text{fin}}(1,h,\chi_{-3}), & p \equiv 5 \pmod{12}; \\ 4L_{\text{fin}}(1,h)L_{\text{fin}}(1,h,\chi_{-4}), & p \equiv 7 \pmod{12}; \\ (4L_{\text{fin}}(1,h)L_{\text{fin}}(1,h,\chi_{-4}) + 6\sqrt{3}L_{\text{fin}}(1,h)L_{\text{fin}}(1,h,\chi_{-3})), & p \equiv 11 \pmod{12}. \end{cases} \tag{1.1}$$

Here $(\cdot,\cdot)$ is the Petersson inner product defined in Lemma 3.2.2. Their approach does not use the relative trace formula. Rather, using the classical period formula of Gross–Kudla [GK92], they write $L_{\mathrm{fin}}(2, f \times g \times h)$ as a finite sum of functions defined on a finite set.

In this paper, we will briefly present the relative trace formula (RTF), originally introduced by Jacquet to study periods integrals, and apply it to automorphic forms on a specific quaternion algebra, so that we can generalize the above result to the case of general weight and level.

Theorem 1.1.3 (Main Theorem)**.** *Let N be a square-free integer with an odd number of prime factors. For any $h \in \mathcal{F}_2(N)$,*

$$\frac{N}{2^8\pi^5}\sum_{\substack{f,g\in\mathcal{F}_2(N)\\ \varepsilon_p=-1,\forall p|N}}\frac{L_{\mathrm{fin}}(2,f\times g\times h)}{(f,f)(g,g)(h,h)}=\frac{1-24/\varphi(N)}{2^{\omega(N)}}+\frac{A_0+\frac{3\sqrt{3}}{2}A_1}{4\pi^2(h,h)} \tag{1.2}$$

where

$$A_0 = L_{\mathrm{fin}}(1,h)_{\mathrm{fin}}L(1,h,\chi_{-4})\prod_{p|N}\frac{1-\chi_{-4}(p)}{2}\cdot\begin{cases}1, & 2\nmid N,\\ (1+a_2(h)), & 2\mid N;\end{cases}$$

$$A_1 = L_{\mathrm{fin}}(1,h)L_{\mathrm{fin}}(1,h,\chi_{-3})\prod_{p|N}\frac{1-\chi_{-3}(p)}{2}\cdot\begin{cases}1, & 3\nmid N,\\ (1+a_3(h)), & 3\mid N.\end{cases}$$

If N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$ (or if N has a prime factor $\equiv 1 \pmod{12}$), for any $h \in \mathcal{F}_{2k}(N)$,

$$\frac{N}{2^{12k-4}\pi^{6k-1}}\sum_{\substack{f,g\in\mathcal{F}_{2k}(N)\\ \varepsilon_p=-1,\ \forall p|N}}\frac{L_{\mathrm{fin}}(3k-1,f\times g\times h)}{(f,f)(g,g)(h,h)}=\frac{1-24\delta(k)/\varphi(N)}{2^{\omega(N)}\Gamma(2k-1)^2\Gamma(2k)}. \tag{1.3}$$

Here $\varphi(N)$ is the Euler's totient function, $\omega(N) := \sum_{p|N} 1$ is the number of distinct prime *factors of N, and*

$$\delta(k) := \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.1.4. *1) Since $\varepsilon_\infty = -1$ and N has an odd number of prime factors, the global ε-factor (the global root number) for the triple product L-function is 1 in the case of the main theorem.*

2) Here χ_d is the Dirichlet character defined by the Kronecker symbol $(\frac{d}{\cdot})$, where $d \equiv 0, 1 \pmod 4$ is a fundamental discriminant. The product over $p \mid N$ can be seen as a congruency condition. For example, when N is square-free,

$$\prod_{p|N} \frac{1 - \chi_{-4}(p)}{2} = \begin{cases} 0, & \text{if } N \text{ has a prime factor} \equiv 1 \pmod 4; \\ \frac{1}{2}, & \text{if } N \text{ has no prime factor} \equiv 1 \pmod 4 \text{ and } 2 \mid N; \\ 1, & \text{if all prime factors of } N \text{ are} \equiv 3 \pmod 4. \end{cases}$$

3) When N is a prime with $N = 11$ or $N > 13$, this reproves (1.1), that is Theorem 1.1 in [FW10].

4) When f, g, h all have weight 4, a similar result as (1.2) can be obtained by the same method. See Theorem 5.6.4.

In Section 6.2 we apply the above theorem to the nonvanishing problem.

Corollary 1.1.5. *Let N be a square-free integer with an odd number of prime factors. Then*

$$\#\{(f, g) \in \mathcal{F}_{2k}(N) \times \mathcal{F}_{2k}(N) : L_{\text{fin}}(3k - 1, f \times g \times h) \neq 0\} \gg_{k,\epsilon} N^{3/4-\epsilon}$$

holds in the following cases:

- $2k = 2$ *and* $\varphi(N) > 24$*;*

- *$2k = 4$;*
- *$2k > 2$, N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$.*

The adelic RTF, along with Ichino's period formula [Ich08], plays an important role in the proof of the adelic version of the Main Theorem (Theorem 3.2.3) as well as the case of general weight and level. This method is more flexible since it could also be applied to the case of triple product L-functions attached to Hilbert modular forms over a totally real number field.

1.2 Ichino's period formula

From an adelic point of view, one can consider the triple product L-function $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ (defined in Section 3.2) associated to three irreducible unitary cuspidal automorphic representations of $\mathrm{PGL}(2, \mathbb{A})$, where $\mathbb{A}$ is the adele ring over $\mathbb{Q}$. Harris and Kudla [HK04] proved a conjecture of Jacquet, that the central value $L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \neq 0$ if and only if there exists a quaternion algebra D over $\mathbb{Q}$ such that the period integral

$$\int_{\mathbb{A}^\times D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A})} \phi_1(x)\phi_2(x)\phi_3(x)\ d^\times x \neq 0$$

for some $\phi_i \in \pi_i'$, where $\mathbb{A}^\times$ is diagonally embedded in $D^\times(\mathbb{A})$ as its center, and π' is the irreducible unitary automorphic representation of $D^\times(\mathbb{A})$ associated to π by the Jacquet–Langlands correspondence.

Moreover, Gross and Kudla [GK92] established an explicit identity relating central L-values and period integrals (which are finite sums in their case), when the cusp forms are of prime levels and weight 2. This Gross–Kudla period formula is the key ingredient when [FW10] proves (1.1). Böcherer, Schulze–Pillot [BSP96] and Watson [Wat02] generalized this identity to more general levels and weights. At last, Ichino [Ich08] proved an adelic version of this period formula which would work for all the cases:

$$\frac{\left|\int_{[D^\times]} \phi_1(h)\phi_2(h)\phi_3(h)\ dh\right|^2}{\prod_{i=1}^3 \int_{[D^\times]} \phi_i(h)\overline{\phi_i(h)}\ dh} \sim \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})}.$$

Here $[D^\times] := \mathbb{A}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})$. The exact formula can be found in Theorem 4.5.1.

1.3 Jacquet's relative trace formula

Here we consider a general version of the RTF. Let G be an anisotropic algebraic group defined over a global field F and H_1, H_2 be closed subgroups of G. Let $f \in C_c^\infty(G(\mathbb{A}_F))$. Integrating f against the action of $G(\mathbb{A}_F)$ gives a linear map

$$R(f) : L^2(G(F)\backslash G(\mathbb{A}_F)) \to L^2(G(F)\backslash G(\mathbb{A}_F))$$

defined by

$$(R(f)\phi)(x) := \int_{G(\mathbb{A}_F)} f(g)\phi(xg)\ dg.$$

One sees that $R(f)$ is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y), \quad x, y \in G(\mathbb{A}_F).$$

Let $\mathcal{A}(G)$ denote the set of automorphic representations on $G(\mathbb{A}_F)$. Fixing automorphic forms ϕ_{H_1}, ϕ_{H_2} in $\pi_1 \in \mathcal{A}(H_1)$ and $\pi_2 \in \mathcal{A}(H_2)$ respectively, we define a distribution

$$I(f) = \int_{H_1(F)\backslash H_1(\mathbb{A}_F)} \int_{H_2(F)\backslash H_2(\mathbb{A}_F)} K_f(h_1, h_2)\phi_{H_1}(h_1)\overline{\phi_{H_2}(h_2)}\ dh_1\ dh_2.$$

The RTF for the case $H_1\backslash G/H_2$ gives two expressions of $I(f)$.

From the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}_F))$,

$$K_f(x,y) = \sum_{\pi\in\mathcal{A}(G)} \sum_{\phi\in\mathcal{ONB}(\pi)} (\pi(f)\phi)(x)\overline{\phi(y)},$$

where for each $\pi \in \mathcal{A}(G)$, $\mathcal{ONB}(\pi)$ denotes an orthonormal basis of V_π. The spectral expansion for the kernel $K_f(x,y)$ gives

$$I(f) = \sum_{\pi\in\mathcal{A}(G)} I_\pi(f),$$

where for each $\pi \in \mathcal{A}(G)$

$$I_\pi(f) := \sum_{\phi\in\mathcal{ONB}(\pi)} \int_{H_1(F)\backslash H_1(\mathbb{A}_F)} (\pi(f)\phi)(h_1)\phi_{H_1}(h_1)\ dh_1 \overline{\int_{H_2(F)\backslash H_2(\mathbb{A}_F)} \phi(h_2)\phi_{H_2}(h_2)\ dh_2}.$$

From the geometric expansion of the RTF,

$$I(f) = \int_{H_1(F)\backslash H_1(\mathbb{A}_F)} \int_{H_2(F)\backslash H_2(\mathbb{A}_F)} \sum_{[\gamma]} \sum_{(\theta_1,\theta_2)} f(h_1^{-1}\theta_1^{-1}\gamma\theta_2 h_2)\phi_{H_1}(h_1)\overline{\phi_{H_2}(h_2)}\ dh_1\ dh_2.$$

Here $[\gamma]$ runs through representatives of $H_1(F)\backslash G(F)/H_2(F)$; and (θ_1,θ_2) runs through $H_1(F)\times H_2(F)/(H_1(F)\times H_2(F))_\gamma$, where we define

$$(H_1(F)\times H_2(F))_\gamma := \{(\theta_1,\theta_2) \in H_1(F)\times H_2(F) : \theta_1^{-1}\gamma\theta_2 = \gamma\}.$$

Let $\theta_i h_i$ be the new h_i $(i = 1, 2)$. Then we have

$$I(f) = \sum_{[\gamma]\in H_1(F)\backslash G(F)/H_2(F)} I_{[\gamma]}(f)$$

where

$$I_{[\gamma]}(f) = \int_{(H_1(F)\times H_2(F))_\gamma\backslash H_1(\mathbb{A}_F)\times H_2(\mathbb{A}_F)} f(h_1^{-1}\gamma h_2)\phi_{H_1}(h_1)\overline{\phi_{H_2}(h_2)}\ d(h_1,h_2).$$

As a generalization of the Arthur–Selberg trace formula, Jacquet's relative trace formula (RTF) is a powerful tool in the study of period integrals. With a period formula like Ichino's, the average of central values of L-functions appears in the spectral decomposition of a certain distribution. In the compact quotient case one can get an explicit orbital decomposition of the same distribution. For example, Feigon and Whitehouse [FW09] considered the RTF for the case $E^\times\backslash D^\times/E^\times$ (where D is a quaternion algebra over a totally real number field F and E/F is a quadratic extension embedded in D) and, using a period formula of Waldspurger, obtained an exact formula for averages of central values of twisted quadratic base change L-functions associated to Hilbert modular forms. In this paper an analogous method is applied to the case $D^\times\backslash(D^\times \times D^\times)/D^\times$ to obtain exact formulas for averages of central values of triple product L-functions.

Chapter 2

Quaternion Algebras

For any field F of characteristic $\neq 2$, and $a, b \in F^\times$, let

$$D = \left(\frac{a, b}{F}\right) := F\{i, j\}/(i^2 - a,\ j^2 - b,\ ij + ji),$$

denote the quaternion algebra with F-basis $1, i, j, k$ such that $i^2 = a$, $j^2 = b$ and $ij = -ji = k$ (so $k^2 = -ab$). We know that either $D \cong M_2(F)$ (called the **split** quaternion algebra) or D is a division algebra.

In this chapter we recall and prove some facts about quaternion algebras that will be needed later in this paper.

2.1 How to represent certain quaternion algebras

We recall the main theorem on the classification of quaternion algebras over a global number field.

Theorem 2.1.1 ([Voi20] Theorem 14.6.1)**.** *Let F be a number field and $S(F)$ be the set of its places.*

- *If D is a quaternion algebra over F, the set* $\mathrm{Ram}(D) \subset S(F)$ *of places v such that D is* ***ramified*** *at v, i.e. such that $D_v := D \otimes_F F_v$ is not split, is a finite set with an even number of elements.*

- *For any finite subset $S \subset S(F)$ of non-complex places such that $\#(S)$ is even, there is a unique quaternion algebra D over F such that* $\mathrm{Ram}(D) = S$.

When $F = \mathbb{Q}$, a quaternion algebra over $\mathbb{Q}$ is called **definite** if $D_\infty = D \otimes_{\mathbb{Q}} \mathbb{R}$ is not split (i.e. isomorphic to the algebra $\mathbb{H}$ of Hamilton quaternions), **indefinite** otherwise. Of course $(\frac{a,b}{\mathbb{Q}})$ is definite if and only if $a, b < 0$. We define $\mathrm{disc}(D)$, the **discriminant** of D, as the (square-free) product of the finite primes at which D is ramified. The quaternion algebra D corresponding to a fixed subset of $S(\mathbb{Q})$ (i.e. to a fixed square-free discriminant) can be constructed explicitly. In this paper we only consider the following two kinds of discriminants.

Lemma 2.1.2. *Let N be a square-free integer with an odd number of prime divisors.*

(1) If N has no prime divisor of the form $4n+1$, $(\frac{-1,-N}{\mathbb{Q}})$ is the definite quaternion algebra over $\mathbb{Q}$ with discriminant N;

(2) If N has no prime divisor of the form $3n+1$, $(\frac{-3,-N}{\mathbb{Q}})$ is the definite quaternion algebra over $\mathbb{Q}$ with discriminant N.

Proof. It is easy to prove this lemma for $p \nmid N$ and $p \neq 2, 3$, since $(\frac{a,b}{\mathbb{Q}_p})$ is split if p is unramified in $\mathbb{Q}(\sqrt{a})$ and $v_p(b) = 0$ (see [Voi20] Corollary 13.4.1). For $p \mid N$ and $p \neq 2, 3$ the lemma also holds, noticing that $(\frac{a,\varpi}{\mathbb{Q}_p})$ is the only non-split quaternion algebra over $\mathbb{Q}_p$ (up to isomorphism) if ϖ is a uniformizer of $\mathbb{Q}_p$ and $a \in \mathbb{Z}_p^\times$ is an element such that $\mathbb{Q}_p(\sqrt{a})$ is the unramified quadratic extension of $\mathbb{Q}_p$ (see [Voi20] Theorem 13.3.10).

A more detailed proof of this lemma using the Hilbert symbol can be found in Appendix A.1. $\square$

2.2 The centralizer of a quaternion

We define the reduced **trace** and **norm** in D as

$$\mathrm{Tr}_D(\alpha + \beta i + \gamma j + \delta k) = 2\alpha,$$
$$N_D(\alpha + \beta i + \gamma j + \delta k) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2.$$

We have a lemma about the norm group $N_D(D^\times)$.

Lemma 2.2.1 ([Voi20] Lemma 13.4.9)**.** *For any quaternion algebra D over a local field F,*

$$N_D(D^\times) = \begin{cases} \mathbb{R}^\times_{>0} & \text{if } D \cong \mathbb{H} = (\frac{-1,-1}{\mathbb{R}}), \\ F^\times & \text{otherwise.} \end{cases}$$

It is obvious that elements conjugate to each other have the same trace and norm, in either a matrix algebra or a division algebra. The following lemma gives a stronger result for conjugacy classes in a division quaternion algebra.

Lemma 2.2.2 ([Voi20] Corollary 7.1.7)**.** *Let D be a non-split quaternion algebra over F. Then a quaternion x is $D^\times$-conjugate to x' (that is, $x' = \gamma x \gamma^{-1}$ for some $\gamma \in D^\times$) if and only if they have the same trace and norm.*

One can also say that conjugacy classes in $D^\times$ can be parametrized by traces and norms.

Corollary 2.2.3. *Fix a set Σ of representatives in $F^\times/(F^\times)^2$. (For example, when $F = \mathbb{Q}$, Σ can be the set of square-free integers.) Then $[\bar{x}] \mapsto (\mathrm{Tr}_D(x), N_D(x))$ is a well-defined injection from the set of conjugacy classes of $G'(F) := F^\times \backslash D^\times$ to $(\{\pm 1\}\backslash F) \times \Sigma$.*

Proof. For any two representatives $x_1, x_2 \in D^\times$ of $\bar{x} \in G'(F)$, there exists $\lambda \in F^\times$ so that $x_2 = \lambda x_1$, and we have $\mathrm{Tr}_D(x_2) = \lambda \,\mathrm{Tr}_D(x_1)$, $N_D(x_2) = \lambda^2 N_D(x_1)$. Fixing Σ, moreover, with $N_D(x)$ fixed, we can only take $\lambda^2 = 1$. So the traces of representatives in $\bar{x}$ might differ by a sign. With Lemma 2.2.2 the statement is proved. $\square$

For a fixed quaternion, one can check the following result about its centralizer by direct calculation.

Lemma 2.2.4. *Suppose $D = (\frac{a,b}{F})$ is a division algebra. Consider the centralizer of $x \in G'(F) := F^\times \backslash D^\times$ given by $G'_x(F) := \{g \in G'(F) : gx = xg\}$. Then*

- $G'_x(F) = G'(F)$ *when* $x = 1$;
- *when* $x = 1 + \beta i + \gamma j + \delta k \neq 1$, $G'_x(F)$ *is the image of* $\{\lambda + \mu x \in D^\times : \lambda, \mu \in F\}$;
- *and when* $x = \beta i + \gamma j + \delta k$, $G'_x(F)$ *is the image of*

$$\{\lambda+\mu x \in D^\times : \lambda, \mu \in F\} \cup \{x_1 i + x_2 j + x_3 k \in D^\times : x_1, x_2, x_3 \in F;\ a\beta x_1 + b\gamma x_2 = ab\delta x_3\}.$$

2.3 Maximal orders of quaternion algebras

Let D be a quaternion algebra over $F = \mathbb{Q}$ or $\mathbb{Q}_p$. An **order** of D is an $\mathcal{O}_F$-subalgebra $\mathcal{O} \subset D$ which is an $\mathcal{O}_F$-lattice of the underlying F-vector space of D. It is well known that any $\mathcal{O}$ is contained in a maximal order. The maximal orders of quaternion algebras over $\mathbb{Q}_p$ can be described as following.

Proposition 2.3.1 ([Voi20] Sections 10.1, 14.1)**.**

- *When* $D = M_2(\mathbb{Q}_p)$, *the maximal orders of* D *are the* $\mathrm{GL}_2(\mathbb{Q}_p)$-*conjugates of* $M_2(\mathbb{Z}_p)$;
- *When* D *is the non-split quaternion algebra over* $\mathbb{Q}_p$, *there is a unique maximal order*

$$\mathcal{O} = \{x \in D : N_D(x) \in \mathbb{Z}_p\}$$

which contains all $\mathbb{Z}_p$-*integral elements of* D.

2.4 Normalization of measures

Let F be a number field. For a finite place v of F, the ring of integers in F_v is denoted by $\mathcal{O}_{F_v}$. Let ϖ_v denote a uniformizer in F_v and $q_v := \#(\mathcal{O}_{F_v}/(\varpi_v))$.

Fix an additive character ψ of $F\backslash\mathbb{A}_F$. For a place v of F we take the additive Haar measure dx_v on F_v which is self-dual with respect to ψ_v. On $F_v^\times$ we take the measure

$$d^\times x_v := \zeta_{F_v}(1)\frac{dx_v}{|x|_v} = \begin{cases} \frac{dx}{|x|}, & F_v = \mathbb{R}; \\ \pi^{-1}\frac{2dx_0dx_1}{x\bar{x}}, & F_v = \mathbb{C},\ x = x_0 + x_1 i; \\ (1-q_v^{-1})^{-1}\frac{dx_v}{|x|_v}, & v < \infty. \end{cases}$$

Let E be a quadratic extension over F. We define measures on $E_v = F_v \otimes_F E$ and $E_v^\times$ similarly with respect to the additive character $\psi \circ \mathrm{Tr}_{E/F}$.

We note that with these choices of measures we have, for $v < \infty$,

$$\mathrm{vol}(\mathcal{O}_{F_v}^\times; d^\times x_v) = \mathrm{vol}(\mathcal{O}_{F_v}; dx_v) = |\mathfrak{d}_v|^{1/2}$$

with $\mathfrak{d}_v \in F_v$ such that $\mathfrak{d}_v\mathcal{O}_{F_v}$ is the different of F_v over $\mathbb{Q}_p$; and for a quadratic field extension E_v/F_v, we have

$$\mathrm{vol}(F_v^\times\backslash E_v^\times) = \begin{cases} 2, & \text{if } F_v = \mathbb{R},\ E_v = \mathbb{C}; \\ |\mathfrak{d}_v|^{1/2}, & \text{if } E_v/F_v \text{ is the unramified field extension;} \\ 2|\mathfrak{D}_v\mathfrak{d}_v|^{1/2}, & \text{if } E_v/F_v \text{ is ramified,} \end{cases}$$

with $\mathfrak{D}_v \in \mathcal{O}_{F_v}$ such that $\mathfrak{D}_v\mathcal{O}_{F_v}$ is the relative discriminant of E_v/F_v.

For a quaternion algebra D defined over a number field F, fix a maximal order $\mathcal{O} \subset D$.

For a finite place v of F we take the Haar measure dg_v on $D_v^\times$ as

$$dg_v := \zeta_{F_v}(1)|N_{D_v}(g_v)|^{-2}\ d\mu_v(g_v)$$

where μ_v is the additive Haar measure on D_v which is self-dual with respect to ψ_v. Then, with the quotient measure defined on $G'_v := Z(F_v)\backslash D_v^\times$, for $v < \infty$, we have (see [Voi20] 29.7.23)

$$\mathrm{vol}(K_v) = |\mathfrak{d}_v|^{\frac{3}{2}}\zeta_{F_v}(2)^{-1} \cdot \begin{cases} (q_v-1)^{-1}, & \text{if } v \in \mathrm{Ram}(D), \\ 1, & \text{if } v \notin \mathrm{Ram}(D), \end{cases}$$

where K_v is the image of $Z(F_v)\mathcal{O}_v^\times$ in G'_v.

For $v \mid \infty$ and a definite quaternion algebra D, $D_v \cong \mathbb{H}$ and $G'_v = Z(\mathbb{R})\backslash\mathbb{H}^\times \cong \{\pm 1\}\backslash \mathrm{SU}(2)$. We parametrize $h = \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2)$ by setting

$$\alpha = re^{i\theta}\cos\gamma, \quad \beta = re^{i\varphi}\sin\gamma, \qquad r>0,\ 0 \le \gamma \le \frac{\pi}{2},\ 0 \le \theta, \varphi < 2\pi.$$

So $d\alpha d\beta = 2r^3 \sin 2\gamma\ dr d\gamma d\theta d\varphi$ (notice that $d\alpha d\beta$ is the self-dual additive measure on $\mathbb{H}$), and for a function $\Phi \in L^1(\mathrm{SU}(2))$,

$$\int_{\mathrm{SU}(2)} \Phi(h)\ dh = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Phi(\gamma,\theta,\varphi)\cdot 2\sin 2\gamma\ d\gamma\ d\theta\ d\varphi.$$

This choice of Haar measure on SU(2) implies $\mathrm{vol}(G'_v) = \mathrm{vol}(G'_\infty) = 4\pi^2$ (See [CC19] (5.9)).

Globally we take the product of these local measures and give discrete subgroups the counting measures, and define the Tamagawa measure on

$$[E^\times] := \mathbb{A}_F^\times E^\times \backslash \mathbb{A}_E^\times \quad \text{and} \quad [D^\times] := Z(\mathbb{A}_F)D^\times(F)\backslash D^\times(\mathbb{A}_F)$$

as the quotient measure. In this way we get

$$\mathrm{vol}([E^\times]) = 2L(1,\eta) \quad \text{and} \quad \mathrm{vol}([D^\times]) = 2. \tag{2.1}$$

Here η is the quadratic character of $F^\times\backslash\mathbb{A}_F^\times$ associated to E/F by class field theory. For example, when $E = \mathbb{Q}(\sqrt{d})$ such that d is a fundamental discriminant, η is the Hecke character corresponding to the Dirichlet character χ_d such that

$$\chi_d(p) := \left(\frac{d}{p}\right) = \begin{cases} 1, & p \text{ splits in } E; \\ -1, & p \text{ remains prime in } E; \\ 0, & p \text{ ramifies in } E. \end{cases}$$

We recall a useful lemma about quotient measure.

Lemma 2.4.1 ([KL06] Corollary 7.14)**.** *Let G be a unimodular group and suppose $G = KH$ for closed unimodular subgroups H and K. Suppose further that $K \cap H$ is unimodular. Let dh denote a right H-invariant measure on $(K\cap H)\backslash H$. Then*

$$\int_{(K\cap H)\backslash H}\int_K f(kh)\ dk\ dh \quad (f \in C_c(KH))$$

defines a Haar measure on $G = KH$. Moreover, with this measure on G,

$$\int_{K\backslash G} f(g)\ dg = \int_{(K\cap H)\backslash H} f(h)\ dh$$

for all $f \in C_c(K\backslash G)$.

Chapter 3

Automorphic Forms and L-Functions

Let D be a quaternion algebra over $\mathbb{Q}$. Define an algebraic group $D^\times$ over $\mathbb{Q}$ by $D^\times(A) = (A \otimes_{\mathbb{Q}} D)^\times$ for a $\mathbb{Q}$-algebra A. Thus $D^\times$ is a reductive algebraic group and we therefore have a theory of automorphic forms and representations of $D^\times$. We will be more interested in the forms that correspond to some automorphic forms on $\mathrm{GL}(2, \mathbb{Q})$ via the Jacquet–Langlands correspondence.

3.1 Jacquet–Langlands correspondence

Let N be a square-free integer with an odd number of prime factors. Fix a positive integer k. Denote by $\mathcal{F}(N, 2k)$ the set of cuspidal automorphic representations of $\mathrm{PGL}(2, \mathbb{A})$ of level N and weight $2k$. The following theorem shows the 1-1 correspondence between $\mathcal{F}_{2k}(N)$ and $\mathcal{F}(N, 2k)$.

Theorem 3.1.1 ([Gel75] Theorem 5.21, [LW12])**.** *Suppose $N = \prod p_i$ is a product of distinct primes. If $f(z) = \sum_{n \geq 1} a_n(f) e^{2\pi i n z} \in \mathcal{F}_{2k}(N)$ (normalized such that $a_1 = 1$), then its corresponding cuspidal automorphic representation $\pi_f = \otimes_v \pi_v$ of* $\mathrm{PGL}(2, \mathbb{A})$ *can be described as follows:*

- $\pi_\infty \cong \pi_{\mathrm{dis}}^{2k} = \sigma(|\cdot|^{k-1/2}, |\cdot|^{-(k-1/2)})$ *is the discrete series representation of weight* $2k$*;*

- *if* $p \nmid N$, π_p *is the spherical representation* $\pi(\mu_1, \mu_2)$ *such that* $\mu_1\mu_2$ *is trivial and* $a_p(f) = p^{\frac{2k-1}{2}}(\mu_1(p) + \mu_2(p))$*; and*

- *if* $p \mid N$, π_p *is the special representation* σ_δ *of* $\mathrm{GL}(2, \mathbb{Q}_p)$ *with trivial central character, where* δ *is the unramified character of* $\mathbb{Q}_p^\times$ *with* $\delta(p) = a_p(f)p^{-(k-1)} = \pm 1$.

Let D be the definite quaternion algebra with discriminant N (i.e. the quaternion algebra defined over $\mathbb{Q}$ which is ramified precisely at the infinite place of $\mathbb{Q}$ and the primes dividing N). We have taken G' to be the algebraic group defined over $\mathbb{Q}$ with $G'(\mathbb{Q}) = Z(\mathbb{Q})\backslash D^\times(\mathbb{Q})$. Denote by $\mathcal{A}(G')$ the set of irreducible automorphic representations of $G'(\mathbb{A})$. Since the quotient $G'(\mathbb{Q})\backslash G'(\mathbb{A})$ is compact, we have the decomposition

$$L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A})) = \widehat{\bigoplus}_{\pi' \in \mathcal{A}(G')} V_{\pi'},$$

where $V_{\pi'}$ denotes the space of π'.

Clearly $\mathcal{A}(G')$ contains all the characters of $G'(\mathbb{Q})\backslash G'(\mathbb{A})$, which are of the form $\delta \circ N_D$ where $\delta : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \{\pm 1\}$ is a Hecke character. Let $\mathcal{A}_{res}(G')$ be the set of these characters. Then its complement, denoted by $\mathcal{A}_{cusp}(G')$, contains all the infinite-dimensional irreducible automorphic representations of $G'(\mathbb{A})$. Every representation $\pi' \in \mathcal{A}_{cusp}(G')$, according to Jacquet–Langlands [JL70], can correspond to a cuspidal automorphic representation of $\mathrm{PGL}(2, \mathbb{A})$.

Let $\mathcal{F}'(N, 2k)$ be the set of representations $\pi' \in \mathcal{A}_{cusp}(G')$ which map to representations in $\mathcal{F}(N, 2k)$ under the Jacquet–Langlands correspondence. The compatibility between the local and global Jacquet–Langlands correspondence gives the following theorem, which describes explicitly $\pi' = \otimes\pi'_v \in \mathcal{F}'(N, 2k)$.

Theorem 3.1.2 (Jacquet–Langlands correspondence, [JL70] [FW09])**.** *Under the Jacquet–Langlands correspondence* $\mathrm{JL} : \mathcal{A}(G') \hookrightarrow \mathcal{A}(\mathrm{PGL}(2))$*, the image of* $\mathcal{A}_{cusp}(G')$ *is equal to the set of cuspidal automorphic representations* $\pi = \otimes_v \pi_v$ *of* $\mathrm{PGL}(2, \mathbb{A})$ *such that* π_v *is a discrete series representation of* $\mathrm{PGL}(2, \mathbb{Q}_v)$ *at all places* v *where* D *is ramified. In particular, when* D *is definite and has discriminant* N*, for* $\pi \in \mathcal{F}(N, 2k)$*, there exists* $\pi' = \otimes \pi'_v \in \mathcal{A}_{cusp}(G')$ *such that* $\mathrm{JL}(\pi') = \pi$ *and*

1. $\pi_\infty \cong \pi^{2k}_{\mathrm{dis}}$*,* $\pi'_\infty \cong \pi'_{2k}$ *is a* $(2k-1)$*-dimensional irreducible representation of* $G'_\infty = Z(\mathbb{R})\backslash\mathbb{H}^\times$ *(which is described in Appendix B);*

2. *for* $v = p \mid N$*,* π_p *is the special representation* σ_{δ_p} *where* $\delta_p : \mathbb{Q}_p^\times \to \{\pm 1\}$ *is an unramified character,* $\pi'_p \cong \delta_p \circ N_{D_p}$ *is a character of* G'_p*; and*

3. *for all the other* v*,* π_v *is unramified,* $\pi'_v \cong \pi_v$*.*

For $\pi' \in \mathcal{F}'(N, 2k)$, the following lemma defines a new-line vector $\phi \in V_{\pi'}$.

Lemma 3.1.3. *Fix a maximal order* $\mathcal{O} \subset D$ *such that* $\mathcal{O}_p = M_2(\mathbb{Z}_p)$ *whenever* D *splits at* p*. For any* $p < \infty$*, let* K_p *be the image of* $Z(\mathbb{Q}_p)\mathcal{O}_p^\times$ *in* $G'_p = G'(\mathbb{Q}_p)$*, and* $K_{\mathrm{fin}} := \prod_p K_p$ *be an open subgroup of* $G'_{\mathrm{fin}} := \prod_p G'_p$*. Then (with* $X^{2k-2} \in V_{\pi'_{2k}}$ *being the unit highest weight vector defined in Appendix B)*

$$\mathbb{C}X^{2k-2} \otimes (\pi'_{\mathrm{fin}})^{K_{\mathrm{fin}}}$$

is a one-dimensional subspace of $V_{\pi'}$ *for* $\pi' \in \mathcal{F}'(N, 2k)$*. We call any nonzero vector* ϕ *in this subspace a* ***new-line vector*** *in* π'*, and write it as*

$$\phi = \otimes_v \phi_v \quad \textit{with} \quad \phi_\infty := \|\phi\| X^{2k-2}$$

and ϕ_p *being the unit spherical vector in* π'_p *(we fix a* G'_p*-invariant bilinear form on* $\pi'_p \otimes \tilde{\pi}'_p$*).*

Proof. When $p \mid N$, the non-ramification of δ_p implies that $\delta_p \circ N_{D_p}$ is K_p-invariant. Then this Lemma is a direct result from Theorem 3.1.2. □

Remark 3.1.4. *In particular, when $2k = 2$, $\pi'_\infty \cong \mathrm{Sym}^0 V \otimes \det^0$ is trivial (see Appendix B). So every automorphic form in $\pi' \in \mathcal{F}'(N, 2)$, in particular the new-line vector, can be seen as a function defined on $G'(\mathbb{A}_{\mathrm{fin}})$.*

3.2 L-functions

Let F be a local field. According to the local Langlands correspondence, for every irreducible admissible representation π of $\mathrm{GL}(2, F)$, there is a representation $\rho : W_F \to \mathrm{GL}(2, \mathbb{C})$ of the Weil group such that $L(s, \rho) = L(s, \pi)$. The triple product local L-factor can be defined by

$$L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = L(s, \rho_1 \otimes \rho_2 \otimes \rho_3),$$
$$L(s, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad}) = L(s, \oplus_i \mathrm{Ad}(\rho_i)) = \prod_i L(s, \mathrm{Ad}(\rho_i)) = \prod_i L(s, \pi_i, \mathrm{Ad}),$$

where $\mathrm{Ad}(\rho_i) : W_F \to \mathrm{GL}(3, \mathbb{C})$ is the adjoint representation. For the cases at hand, we can define the local L-factors more explicitly.

Let $F = \mathbb{R}$ or $\mathbb{C}$ be an Archimedean local field. Recall that, for $s \in \mathbb{C}$,

$$\zeta_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2), \quad \zeta_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s),$$

where $\Gamma(s)$ is the standard Γ-function. For a character $\mu : F^\times \to \mathbb{C}^\times$, define

$$L(s, \mu) = \begin{cases} \zeta_{\mathbb{R}}(s + r + m), & \text{when } F = \mathbb{R},\ \mu(x) = |x|_{\mathbb{R}}^r \operatorname{sgn}^m(x),\ r \in \mathbb{C},\ m \in \{0, 1\}; \\ \zeta_{\mathbb{C}}(s + r + |m|), & \text{when } F = \mathbb{C},\ \mu(z) = |z|_{\mathbb{C}}^r (z/\bar{z})^m,\ r \in \mathbb{C},\ m \in \frac{1}{2}\mathbb{Z}. \end{cases}$$

For a discrete series representation π_{dis}^{2k} of $\mathrm{GL}(2,\mathbb{R})$ with weight $2k$, one can define

$$L(s,\pi_{\mathrm{dis}}^{2k}) = \zeta_{\mathbb{C}}(s+k-\frac{1}{2}), \quad L(s,\pi_{\mathrm{dis}}^{2k},\mathrm{Ad}) = \zeta_{\mathbb{R}}(s+1)\zeta_{\mathbb{C}}(s+2k-1);$$
$$L(s,\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}) = \zeta_{\mathbb{C}}(s+3k-\frac{3}{2})\zeta_{\mathbb{C}}(s+k-\frac{1}{2})^3.$$

For a quadratic extension $E/\mathbb{Q}$ we have that $(\eta_{E/\mathbb{Q}})_\infty$ is trivial or sgn. Notice that $\pi_{\mathrm{dis}}^{2k}\otimes\mathrm{sgn}\cong\pi_{\mathrm{dis}}^{2k}$. Then

$$L(s,(\pi_{\mathrm{dis}}^{2k})_E) = L(s,\pi_{\mathrm{dis}}^{2k})L(s,\pi_{\mathrm{dis}}^{2k}\otimes(\eta_{E/\mathbb{Q}})_\infty) = L(s,\pi_{\mathrm{dis}}^{2k})^2.$$

Remark 3.2.1. *According to the Legendre duplication formula*

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}),$$

we can write all these Archimedean L-factors in terms of $\zeta_{\mathbb{R}}(s)$. For example,

$$\zeta_{\mathbb{C}}(s) = \zeta_{\mathbb{R}}(s)\zeta_{\mathbb{R}}(s+1),$$
$$L(s,\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}) = \zeta_{\mathbb{R}}(s+3k-\frac{3}{2})\zeta_{\mathbb{R}}(s+3k-\frac{1}{2})\zeta_{\mathbb{R}}(s+k-\frac{1}{2})^3\zeta_{\mathbb{R}}(s+k+\frac{1}{2})^3.$$

Notice that the Euler factor $L(s,\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k})$ at infinity for the triple product L-function is of degree 8, just like the non-Archimedean L-factor (when the three representations are all spherical).

Now let F be a non-Archimedean local field with uniformizer ϖ, and let $q = \#(\mathcal{O}_F/(\varpi))$ be the order of the residue field. For an unramified character μ (perhaps with superscripts and subscripts),

$$L(s,\mu) = (1-\mu(\varpi)q^{-s})^{-1}, \quad \zeta_F(s) = L(s,\mathbf{1}_F) = (1-q^{-s})^{-1}.$$

For a spherical representation $\pi(\mu_1, \mu_2)$ with μ_1, μ_2 unramified,

$$L(s, \pi(\mu_1, \mu_2)) = L(s, \mu_1)L(s, \mu_2) = (1 - \mu_1(\varpi)q^{-s})^{-1}(1 - \mu_2(\varpi)q^{-s})^{-1};$$

$$L(s, \pi(\mu_1, \mu_2), \mathrm{Ad}) = \zeta_F(s)L(s, \mu_1\mu_2^{-1})L(s, \mu_1^{-1}\mu_2);$$

$$L(s, \pi(\mu_1^{(1)}, \mu_2^{(1)}) \otimes \pi(\mu_1^{(2)}, \mu_2^{(2)}) \otimes \pi(\mu_1^{(3)}, \mu_2^{(3)})) = \prod_{i_1, i_2, i_3 \in \{1,2\}} L(s, \mu_{i_1}^{(1)}\mu_{i_2}^{(2)}\mu_{i_3}^{(3)}).$$

For a special representation σ_μ with μ unramified,

$$L(s, \sigma_\mu) = L(s + \frac{1}{2}, \mu) = (1 - \mu(\varpi)q^{-s-1/2})^{-1};$$

$$L(s, \sigma_\mu, \mathrm{Ad}) = \zeta_F(s + 1) = (1 - q^{-s-1})^{-1};$$

$$L(s, \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \sigma_{\mu_3}) = L(s + \frac{3}{2}, \mu_1\mu_2\mu_3)L(s + \frac{1}{2}, \mu_1\mu_2\mu_3)^2.$$

In this case the local root numbers are

$$\varepsilon(\frac{1}{2}, \sigma_\mu) = -\mu(\varpi), \quad \varepsilon(\frac{1}{2}, \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \sigma_{\mu_3}) = -\mu_1\mu_2\mu_3(\varpi).$$

For a quadratic extension E/F, the base change L-factors can be defined in the same way as above, noticing that ([GG12] Appendix E.6)

$$(\pi(\mu_1, \mu_2))_E = \pi(\mu_1 \circ N_{E/F}, \mu_2 \circ N_{E/F}), \quad (\sigma_\mu)_E = \sigma_{\mu \circ N_{E/F}}.$$

Globally, for a number field F, a Hecke character μ on $\mathbb{A}_F^\times$, and automorphic representations π, π_1, π_2, π_3 of $\mathrm{GL}(2, \mathbb{A}_F)$, we define the completed L-functions

$$\zeta_F^*(s), \quad L(s, \mu), \quad L(s, \pi), \quad L(s, \pi_1 \otimes \pi_2 \otimes \pi_3), \quad L(s, \pi, \mathrm{Ad}), \quad L(s, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})$$

as Euler products of corresponding local L-factors over all places of F.

Lemma 3.2.2. *Let $f, g, h \in \mathcal{F}_{2k}(N)$ be cusp forms of the same level and weight, and π_f, π_g, π_h be the cuspidal automorphic representations corresponding (via Theorem 3.1.1) to f, g, h respectively. Then*

$$L_{\text{fin}}(\frac{1}{2}, \pi_f \otimes \pi_g \otimes \pi_h) = L_{\text{fin}}(3k-1, f \times g \times h),$$

and

$$L(1, \pi_f, \text{Ad}) = \frac{2^{2k}}{N}(f, f) \quad \textit{([CST14] Proposition 1.11).}$$

Here $(\cdot,\cdot)$ is the Petersson inner product on $\mathcal{F}_{2k}(N)$ defined by

$$(f_1, f_2) := \int_{\Gamma_0(N)\backslash\mathcal{H}} f_1(z)\overline{f_2(z)} y^{2k} \, \frac{dx \; dy}{y^2}.$$

With the above lemma and the definition of Archimedean L-factors, one can easily check that the following theorem is equivalent to Theorem 1.1.3. Recall that for $h \in \mathcal{F}_{2k}(N)$, when $p \mid N$, $(\pi_h)_p \cong \sigma_{\delta_p}$ is a special representation with

$$\delta_p(p) = a_p(h)p^{-(k-1)} = -\varepsilon_p(\frac{1}{2}, \pi_h) = \pm 1.$$

Theorem 3.2.3 (Main Theorem, adelic version)**.** *Let N be a square-free integer with an odd number of prime factors, and $\mathcal{F}(N, 2k)$ be the set of cuspidal automorphic representations of* $\mathrm{PGL}(2, \mathbb{A})$ *of level N and weight $2k$. For any $\pi_3 \in \mathcal{F}(N, 2)$,*

$$\frac{1}{2N^2} \sum_{\substack{\pi_1, \pi_2 \in \mathcal{F}(N,2) \\ \varepsilon_p = -1, \forall p \mid N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \text{Ad})} = \frac{1 - 24/\varphi(N)}{2^{\omega(N)}} + \frac{I_0 + \frac{3\sqrt{3}}{2} I_1}{N \cdot L(1, \pi_3, \text{Ad})}, \tag{3.1}$$

where

$$I_0 = L(\frac{1}{2}, (\pi_3)_{\chi_{-4}}) \prod_{p|N} \frac{1-\chi_{-4}(p)}{2} \cdot \begin{cases} 1, & 2 \nmid N, \\ (1-\varepsilon_2(\frac{1}{2}, \pi_3)), & 2 \mid N; \end{cases}$$

$$I_1 = L(\frac{1}{2}, (\pi_3)_{\chi_{-3}}) \prod_{p|N} \frac{1-\chi_{-3}(p)}{2} \cdot \begin{cases} 1, & 3 \nmid N, \\ (1-\varepsilon_3(\frac{1}{2}, \pi_3)), & 3 \mid N. \end{cases}$$

In addition, if N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$, for any $\pi_3 \in \mathcal{F}(N, 2k)$,

$$\frac{2^{\omega(N)}}{2N^2} \sum_{\substack{\pi_1, \pi_2 \in \mathcal{F}(N,2k) \\ \varepsilon_p = -1,\ \forall p|N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})} = \begin{cases} 1 - \frac{24}{\varphi(N)}, & \textit{if } 2k = 2; \\ \frac{\Gamma(k)^3 \Gamma(3k-1)}{\Gamma(2k-1)^2 \Gamma(2k)}, & \textit{otherwise}. \end{cases} \tag{3.2}$$

In the following two chapters we are going to prove Theorem 3.2.3 using the relative trace formula (RTF).

Chapter 4

Spectral Side of the RTF

Let D be the definite quaternion algebra over $\mathbb{Q}$ with discriminant N. Let G' be the algebraic group defined over $\mathbb{Q}$ with $G'(\mathbb{Q}) = Z(\mathbb{Q})\backslash D^\times(\mathbb{Q})$. We consider the RTF introduced in Section 1.3 for the case $G'\backslash(G' \times G')/G'$.

Let $f \in C_c^\infty(G'(\mathbb{A}) \times G'(\mathbb{A}))$. Integrating f against the action of $G'(\mathbb{A}) \times G'(\mathbb{A})$ gives a linear map

$$R(f) : L^2(G'(\mathbb{Q}) \times G'(\mathbb{Q})\backslash G'(\mathbb{A}) \times G'(\mathbb{A})) \to L^2(G'(\mathbb{Q}) \times G'(\mathbb{Q})\backslash G'(\mathbb{A}) \times G'(\mathbb{A}))$$

defined by

$$(R(f)\Phi)(x_1, x_2) = \int_{G'(\mathbb{A})} \int_{G'(\mathbb{A})} f(g_1, g_2)\Phi(x_1 g_1, x_2 g_2)\ dg_1\ dg_2.$$

From the spectral decomposition of $L^2(G'(\mathbb{Q}) \times G'(\mathbb{Q})\backslash G'(\mathbb{A}) \times G'(\mathbb{A}))$ one sees that $R(f)$ is an integral operator with kernel

$$K_f(x_1, x_2; y_1, y_2) = \sum_{\pi_1' \otimes \pi_2' \in \mathcal{A}(G' \times G')} \sum_{\Phi \in \mathcal{ONB}(\pi_1' \otimes \pi_2')} ((\pi_1' \otimes \pi_2')(f)\Phi)(x_1, x_2)\overline{\Phi(y_1, y_2)},$$

where $\mathcal{ONB}(\pi)$ denotes an orthonormal basis of V_π.

Having fixed the diagonal embedding $G' \hookrightarrow G' \times G'$ we get an injection $G'(\mathbb{A}) \hookrightarrow G'(\mathbb{A}) \times G'(\mathbb{A})$. Let $\pi'_3 \in \mathcal{F}'(N, 2k)$ be an automorphic representation of G'. Fixing an automorphic form $\phi_3 \in \mathbb{C}X_3^{2k-2} \otimes (\pi'_{3,\mathrm{fin}})^{K_{\mathrm{fin}}}$ on G' (a new-line vector defined in Lemma 3.1.3), we define a distribution

$$I(f) := \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} K_f(h_1, h_1; h_2, h_2)\phi_3(h_1)\overline{\phi_3(h_2)}\ dh_1\ dh_2. \tag{4.1}$$

The spectral expansion for the kernel $K_f(x_1, x_2; y_1, y_2)$ gives

$$I(f) = \sum_{\pi'_1 \otimes \pi'_2} I_{\pi'_1,\pi'_2}(f)$$

where for each $\pi'_1, \pi'_2 \in \mathcal{A}(G')$ we define

$$I_{\pi'_1,\pi'_2}(f) := \sum_{\Phi \in \mathcal{ONB}(\pi'_1 \otimes \pi'_2)} \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} ((\pi'_1 \otimes \pi'_2)(f)\Phi)(h_1, h_1)\phi_3(h_1)\ dh_1 \cdot \overline{\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \Phi(h_2, h_2)\phi_3(h_2)\ dh_2}.$$

Recall that $\mathcal{A}(G') = \mathcal{A}_{cusp}(G') \sqcup \mathcal{A}_{res}(G')$ where $\mathcal{A}_{res}(G')$ is the set of characters of $G'(\mathbb{Q})\backslash G'(\mathbb{A})$. The automorphic representations of $G' \times G'$ are of the form $\pi'_1 \otimes \pi'_2$ where

- $\pi'_1, \pi'_2 \in \mathcal{A}_{cusp}(G')$;
- one in $\mathcal{A}_{cusp}(G')$ and another in $\mathcal{A}_{res}(G')$; or
- $\pi'_1, \pi'_2 \in \mathcal{A}_{res}(G')$.

Correspondingly we can decompose $I(f)$ as

$$I(f) = \sum_{\pi'_1,\pi'_2 \in \mathcal{A}_{cusp}(G')} I_{\pi'_1,\pi'_2}(f) + 2 \sum_{\pi'_1 \in \mathcal{A}_{res}(G')} \sum_{\pi'_2 \in \mathcal{A}_{cusp}(G')} I_{\pi'_1,\pi'_2}(f) + \sum_{\pi'_1,\pi'_2 \in \mathcal{A}_{res}(G')} I_{\pi'_1,\pi'_2}(f). \tag{4.2}$$

Our goal is to choose a suitable test function $f \in C_c^\infty(G'(\mathbb{A}) \times G'(\mathbb{A}))$ such that $R(f)$ kills all $\pi_1' \otimes \pi_2' \in \mathcal{A}_{cusp}(G') \otimes \mathcal{A}_{cusp}(G')$ unless $\pi_1', \pi_2' \in \mathcal{F}'(N, 2k)$.

4.1 Test function

First we give two lemmas in the representation theory of compact groups. The first lemma can be obtained through direct calculation.

Lemma 4.1.1. *Let K be a compact topological group with Haar measure dk, Π be a unitary representation (might not be irreducible) of K. Define*

$$P_\Pi(v) := \frac{1}{\mathrm{vol}(K; dk)} \int_K \Pi(k)v \; dk, \quad v \in V_\Pi.$$

Then P_Π is the projection map from V_Π to its K-invariant subspace V_Π^K, and

$$\int_K \langle \Pi(k)u, v\rangle \; dk = \mathrm{vol}(K; dk) \langle P_\Pi(u), P_\Pi(v)\rangle$$

where $\langle\ ,\ \rangle$ is a K-invariant inner product defined on V_Π.

Lemma 4.1.2 ([Kna01] Schur Orthogonality Relations)**.** *Let K be a compact Lie group, π, π' be two finite-dimensional irreducible unitary representations of K, $\langle\ ,\ \rangle$ be a K-invariant inner product of π or π'. Then, for $u, v \in V_\pi$, $u', v' \in V_{\pi'}$,*

$$\int_K \langle \pi(k)u, v\rangle \overline{\langle \pi'(k)u', v'\rangle} \; dk = \begin{cases} 0, & \text{if } \pi \not\cong \pi'; \\ \mathrm{vol}(K; dk) \frac{\langle u, u'\rangle \overline{\langle v, v'\rangle}}{\deg \pi}, & \text{if } \pi = \pi'. \end{cases}$$

For $v = \infty$, recall that π'_{2k} (defined in Appendix B) corresponds to π_{dis}^{2k}, via the local Jacquet–Langlands correspondence. Let $\langle\ ,\ \rangle$ be a G'_∞-invariant inner product of $\pi'_{2k} \otimes \pi'_{2k}$. By Lemma 3.1.3 we fix $\phi_{3,\infty} = \|\phi_3\| X_3^{2k-2}$, and Lemma B.0.4 shows that there exists a vector

$w_{2k}^{\circ} \in \pi'_{2k} \otimes \pi'_{2k}$ such that

$$\int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w_{2k}^{\circ}, w_{2k}^{\circ} \rangle \pi'_{2k}(g) X_3^{2k-2} \, dg$$

is a constant multiple of X_3^{2k-2}. We fix such a nonzero vector $w_{2k}^{\circ} \in \pi'_{2k} \otimes \pi'_{2k}$ as in Lemma B.0.2 and define $f_\infty \in C_c^\infty(G'_\infty \times G'_\infty)$ by

$$f_\infty(g_1, g_2) := \overline{\langle \pi'_{2k} \otimes \pi'_{2k}(g_1, g_2) w_{2k}^{\circ}, w_{2k}^{\circ} \rangle} / \langle w_{2k}^{\circ}, w_{2k}^{\circ} \rangle.$$

For $v = p < \infty$, fix a maximal order $\mathcal{O}_p$ of $D_p = D(\mathbb{Q}_p)$. In particular, fix $\mathcal{O}_p = M(2, \mathbb{Z}_p)$ and $\mathcal{O}_p^\times = \mathrm{GL}(2, \mathbb{Z}_p)$ when $p \nmid N$. Let K_p be the image of $Z(\mathbb{Q}_p)\mathcal{O}_p^\times$ in G'_p. Clearly

$$f_0 := \prod_{p<\infty} f_p \quad \text{where } f_p = \mathbf{1}_{K_p}$$

is a function in $C_c^\infty(G'(\mathbb{A}_{\mathrm{fin}}))$. We define the test function on $G'(\mathbb{A}) \times G'(\mathbb{A})$ by $f := f_\infty \times (f_0 \otimes f_0)$, i.e.

$$f(g_1, g_2) := f_\infty(g_{1,\infty}, g_{2,\infty}) \prod_{p<\infty} \mathbf{1}_{K_p}(g_{1,p}) \mathbf{1}_{K_p}(g_{2,p}). \tag{4.3}$$

In particular when $2k = 2$, π'_{2k} is trivial and so is f_∞. In this case the test function is simply

$$f(g_1, g_2) = f_0 \otimes f_0 = \prod_{p<\infty} \mathbf{1}_{K_p}(g_{1,p}) \mathbf{1}_{K_p}(g_{2,p}).$$

When choosing $\Phi \in \mathcal{ONB}(\pi'_1 \otimes \pi'_2)$ we take $\Phi = \Phi_\infty \cdot \phi_{1,\mathrm{fin}} \otimes \phi_{2,\mathrm{fin}}$, where Φ_∞ is a unit vector in $\pi'_{1,\infty} \otimes \pi'_{2,\infty}$, and $\phi_{1,\mathrm{fin}}, \phi_{2,\mathrm{fin}}$ are functions in $\pi'_{1,\mathrm{fin}}, \pi'_{2,\mathrm{fin}}$ respectively. With this test function we have

$$\begin{aligned}(R(f)\Phi)(x_1, x_2) =& ((\pi'_1 \otimes \pi'_2)(f_\infty \cdot f_0 \otimes f_0)\Phi)(x_1, x_2) \\ =& (\pi'_{1,\infty} \otimes \pi'_{2,\infty}(f_\infty)\Phi_\infty) \cdot (R(f_0 \otimes f_0)(\phi_{1,\mathrm{fin}} \otimes \phi_{2,\mathrm{fin}}))(x_1, x_2)\end{aligned}$$

and

$$\begin{aligned}
&(R(f_0 \otimes f_0)(\phi_{1,\mathrm{fin}} \otimes \phi_{2,\mathrm{fin}}))(x_1, x_2)\\
=&\int_{G'(\mathbb{A}_{\mathrm{fin}})}\int_{G'(\mathbb{A}_{\mathrm{fin}})} f_0(g_1)f_0(g_2)\phi_{1,\mathrm{fin}}(x_1g_1)\phi_{2,\mathrm{fin}}(x_2g_2)\ dg_1\ dg_2\\
=&(R(f_0)\phi_{1,\mathrm{fin}})(x_1)\cdot(R(f_0)\phi_{2,\mathrm{fin}})(x_2) = (\pi'_{1,\mathrm{fin}}(f_0)\phi_{1,\mathrm{fin}})(x_1)\cdot(\pi'_{2,\mathrm{fin}}(f_0)\phi_{2,\mathrm{fin}})(x_2).
\end{aligned}$$

Then the distribution $I_{\pi'_1,\pi'_2}$ becomes

$$\begin{aligned}
&I_{\pi'_1,\pi'_2}(f_\infty\cdot f_0\otimes f_0)\\
=&\sum_{\substack{\Phi\in\mathcal{ONB}(\pi'_1\otimes\pi'_2)\\ \Phi=\Phi_\infty\cdot\phi_{1,\mathrm{fin}}\otimes\phi_{2,\mathrm{fin}}}}\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\Big(\pi'_{1,\infty}\otimes\pi'_{2,\infty}(f_\infty)\Phi_\infty\Big)\Big(\pi'_{1,\mathrm{fin}}(f_0)\phi_{1,\mathrm{fin}}\Big)\Big(\pi'_{2,\mathrm{fin}}(f_0)\phi_{2,\mathrm{fin}}\Big)\phi_3(h_1)\ dh_1\\
&\cdot\overline{\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\Phi(h_2,h_2)\phi_3(h_2)\ dh_2}.
\end{aligned}$$

In the next three sections we will answer the question that, with the test function defined by (4.3), which representations would contribute to the spectral decomposition (4.2).

4.2 Cusp $\otimes$ Cusp

For an irreducible admissible representation σ of G'_v acting on the space V_σ and for $f_v \in C_c^\infty(G'_v)$, we define $\sigma(f_v): V_\sigma \to V_\sigma$ by

$$\sigma(f_v)w := \int_{G'_v} f_v(g_v)\sigma(g_v)w\ dg_v.$$

Lemma 4.2.1 ([FW09] Lemma 3.2, 3.3)**.** *For $v = p < \infty$, let $f_p = \mathbf{1}_{K_p}$ as above. Let σ be an irreducible unitary representation of G'_p. Then $\sigma(f_p)$ kills the orthogonal complement of σ^{K_p} in V_σ, and $\sigma(f_p)w = \mathrm{vol}(K_p)w$ for $w \in \sigma^{K_p}$.*

Proof. By definition,

$$\sigma(f_p)w = \int_{G'_p} \mathbf{1}_{K_p}(g)\sigma(g)w\ dg = \int_{K_p} \sigma(g)w\ dg \quad \text{for } w \in V_\sigma.$$

The rest can be shown by Lemma 4.1.1. □

On the Archimedean place we can prove a similar result as [FW09] Lemma 3.4.

Lemma 4.2.2. *Let $\sigma = \sigma_1 \otimes \sigma_2$ be an irreducible unitary representation of $G'_\infty \times G'_\infty$. Then, with the definition of f_∞ as above, $\sigma(f_\infty)$ kills the space V_σ unless $\sigma \cong \pi'_{2k} \otimes \pi'_{2k}$. Furthermore for $\sigma = \pi'_{2k} \otimes \pi'_{2k}$, $\sigma(f)$ kills the orthogonal complement of $\mathbb{C}w^\circ_{2k}$ in V_σ, and*

$$\pi'_{2k} \otimes \pi'_{2k}(f_\infty)w^\circ_{2k} = \left(\frac{\mathrm{vol}(G'_\infty)}{2k-1}\right)^2 w^\circ_{2k}.$$

Proof. Since f_∞ is a matrix coefficient, Schur Orthogonality Relations (Lemma 4.1.2) show that $\sigma(f_\infty)$ kills the space V_σ unless $\sigma \cong \pi'_{2k} \otimes \pi'_{2k}$, and

$$\begin{aligned}
&\langle \pi'_{2k} \otimes \pi'_{2k}(f_\infty)w_1, w_2\rangle = \int_{G'_\infty \times G'_\infty} f_\infty(g)\langle \pi'_{2k} \otimes \pi'_{2k}(g)w_1, w_2\rangle\ dg \\
&= \int_{G'_\infty \times G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g)w_1, w_2\rangle \frac{\overline{\langle \pi'_{2k} \otimes \pi'_{2k}(g)w^\circ_{2k}, w^\circ_{2k}\rangle}}{\langle w^\circ_{2k}, w^\circ_{2k}\rangle}\ dg \\
&= \frac{\mathrm{vol}(G'_\infty \times G'_\infty)}{\dim \pi'_{2k} \otimes \pi'_{2k}} \frac{\langle w_1, w^\circ_{2k}\rangle \overline{\langle w_2, w^\circ_{2k}\rangle}}{\langle w^\circ_{2k}, w^\circ_{2k}\rangle} = \begin{cases} \left(\frac{\mathrm{vol}(G'_\infty)}{2k-1}\right)^2 \|w^\circ_{2k}\|^2 & \text{when } w_1 = w_2 = w^\circ_{2k}; \\ 0 & \text{when } w_1 \text{ or } w_2 \in (\mathbb{C}w^\circ_{2k})^\perp. \end{cases}
\end{aligned}$$

□

Now we apply the lemmas for $\pi'_{1,v} \otimes \pi'_{2,v}$ and work on $\pi'_1 \otimes \pi'_2(f)\Phi$ for $\Phi \in \mathcal{ONB}(\pi'_1 \otimes \pi'_2)$.

Lemma 4.2.3. *For cuspidal representations π_1', π_2' of G',*

$$\sum_{\Phi \in \mathcal{ONB}(\pi_1' \otimes \pi_2')} \Big(\pi_1' \otimes \pi_2'(f)\Phi\Big)(x) = \begin{cases} \Phi_{\pi_1'\otimes\pi_2'}(x) \left(\frac{\mathrm{vol}(K')}{2k-1}\right)^2, & \pi_1', \pi_2' \in \mathcal{F}'(N, 2k); \\ 0, & \textit{otherwise.} \end{cases}$$

Here $K' := G'_\infty \prod_{p<\infty} K_p$ is an open subgroup of $G'(\mathbb{A})$, $\Phi_{\pi_1'\otimes\pi_2'}$ is the orthonormal basis of the 1-dimensional subspace $W_{\pi_1'\otimes\pi_2'} := \mathbb{C}w_{2k}^\circ \otimes (\pi'_{1,\mathrm{fin}})^{K_{\mathrm{fin}}} (\pi'_{2,\mathrm{fin}})^{K_{\mathrm{fin}}}$.

Proof. It follows from the previous two lemmas that $R(f)$ kills the orthogonal complement of $\Phi_{\pi_1'\otimes\pi_2'}$ in $V_{\pi_1'\otimes\pi_2'}$ and

$$\Big(\pi_1' \otimes \pi_2'(f)\Phi_{\pi_1'\otimes\pi_2'}\Big)(x_1, x_2) = \Phi_{\pi_1'\otimes\pi_2'}(x_1, x_2) \left(\frac{\mathrm{vol}(G'_\infty)}{2k-1} \prod_{p<\infty} \mathrm{vol}(K_p)\right)^2.$$

□

This lemma implies that, for $\pi_3' \in \mathcal{F}'(N, 2k)$,

$$\sum_{\pi_1', \pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1', \pi_2'}(f) = \left(\frac{\mathrm{vol}(K')}{2k-1}\right)^2 \sum_{\pi_1', \pi_2' \in \mathcal{F}'(N,2k)} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \Phi_{\pi_1'\otimes\pi_2'}(h)\phi_3(h)\, dh \right|^2.$$

4.3 Res ⊗ Res

Every $\pi' \in \mathcal{A}_{res}(G')$, is a character $\delta \circ N_D$ for some $\delta : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \{\pm 1\}$. We can take $\phi \in \mathcal{ONB}(\pi')$ to be the normalization of $\delta \circ N_D$.

Lemma 4.3.1. *$R(f)(\delta_1 \circ N_D \otimes \delta_2 \circ N_D) \equiv 0$ unless $2k = 2$ and δ_1, δ_2 are unramified everywhere, in which case (with $K' := G'_\infty \prod_{p<\infty} K_p$)*

$$R(f)(\delta_1 \circ N_D \otimes \delta_2 \circ N_D)(x_1, x_2) = \delta_1(N_D(x_1))\delta_2(N_D(x_2))\, \mathrm{vol}(K')^2.$$

Proof. By definition we have

$$\begin{aligned}
&R(f)(\delta_1 \circ N_D \otimes \delta_2 \circ N_D)(x_1, x_2) \\
&= \int_{G'(\mathbb{A}) \times G'(\mathbb{A})} f(g_1, g_2)\delta_1(N_D(x_1 g_1))\delta_2(N_D(x_2 g_2))\ dg_1\ dg_2 \\
&= \int_{G'_\infty \times G'_\infty} f_\infty(g_1, g_2)\delta_{1,\infty}(N_{D_\infty}(x_{1,\infty} g_1))\delta_{2,\infty}(N_{D_\infty}(x_{2,\infty} g_2))\ dg_1\ dg_2 \\
&\qquad \cdot \prod_{p<\infty} \int_{G'_p} \mathbf{1}_{K_p}(g)\delta_v(N_{D_p}(x_{1,p} g))\ dg \int_{G'_p} \mathbf{1}_{K_p}(g)\delta_p(N_{D_p}(x_{2,p} g))\ dg.
\end{aligned}$$

When $p < \infty$, since the norm map $N_{D_p} : \mathcal{O}_p^\times \to \mathbb{Z}_p^\times$ is surjective (whether D_p is split or not), we have

$$\begin{aligned}
\int_{G'_p} \mathbf{1}_{K_p}(g)\delta_p(N_{D_p}(xg))\ dg &= \delta_p(N_{D_p}(x)) \int_{K_p} \delta_p(N_{D_p}(g))\ dg \\
&= \begin{cases} 0, & \delta_p \text{ ramified;} \\ \delta_p(N_{D_p}(x)) \operatorname{vol}(K_p), & \delta_p \text{ unramified.} \end{cases}
\end{aligned}$$

When $v = \infty$, $N_{D_\infty}(D_\infty^\times) = \mathbb{R}_{>0}^\times$ by Lemma 2.2.1. Then $\delta_\infty(N_{D_\infty}(g)) = 1$ for all $g \in D_\infty^\times$ since δ_∞ is quadratic. Thus

$$\begin{aligned}
&\int_{G'_\infty \times G'_\infty} f_\infty(g_1, g_2)\delta_{1,\infty}(N_{D_\infty}(x_1 g_1))\delta_{2,\infty}(N_{D_\infty}(x_2 g_2))\ dg_1\ dg_2 \\
&= \delta_{1,\infty}(N_{D_\infty}(x_1))\delta_{2,\infty}(N_{D_\infty}(x_2)) \overline{\frac{\int_{G'_\infty \times G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g_1, g_2) w_{2k}^\circ, w_{2k}^\circ \rangle\ d(g_1, g_2)}{\langle w_{2k}^\circ, w_{2k}^\circ \rangle}} \\
&= \begin{cases} 0, & \text{if } 2k > 2; \\ \delta_{1,\infty}(N_D(x_1))\delta_{2,\infty}(N_D(x_2)) \operatorname{vol}(G'_\infty)^2, & \text{if } \pi'_{2k} \otimes \pi'_{2k} \cong \operatorname{id}., \text{ i.e. } 2k = 2. \end{cases}
\end{aligned}$$

Putting these local calculation together shows the statement. □

For any global field F, we define $X^{\mathrm{un}}(F)$ to be the set of Hecke characters $\delta : F^\times \backslash \mathbb{A}_F^\times \to$

$\{\pm 1\}$ that are unramified everywhere. Notice that $\|\delta \circ N_D\| = \mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A}))^{1/2}$. Then

$$\frac{\sum\limits_{\pi_1',\pi_2' \in \mathcal{A}_{res}(G')} I_{\pi_1',\pi_2'}(f)}{\mathrm{vol}(K')^2} = \begin{cases} \sum\limits_{\delta_1,\delta_2 \in X^{\mathrm{un}}(\mathbb{Q})} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \frac{\delta_1\delta_2(N_D(h))\phi_3(h)}{\mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A}))} \, dh \right|^2, & \text{if } 2k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma shows that any character $\delta : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \{\pm 1\}$ that is unramified everywhere can only be trivial, i.e. $X^{\mathrm{un}}(\mathbb{Q}) = \{\mathbf{1}\}$. Then

$$\frac{\sum\limits_{\pi_1',\pi_2' \in \mathcal{A}_{res}(G')} I_{\pi_1',\pi_2'}(f_0 \otimes f_0)}{\mathrm{vol}(K')^2} = \frac{1}{4} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \phi_3(h) \, dh \right|^2 = \frac{1}{4} |\langle \phi_3, \mathbf{1} \rangle|^2.$$

By orthogonality, $\langle \phi_3, \mathbf{1} \rangle = 0$. So

$$\sum_{\pi_1',\pi_2' \in \mathcal{A}_{res}(G')} I_{\pi_1',\pi_2'}(f) = 0.$$

Lemma 4.3.2. *Let F be $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ with class number 1. Then the set $X^{un}(F)$ of all characters $\delta : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ which are unramified everywhere is parameterized by $\widehat{F_\infty^\times/\mathcal{O}_F^\times}$, the set of characters of $F_\infty^\times$ which are invariant under $\mathcal{O}_F^\times$. Here $\mathcal{O}_F^\times$ is the group of units in the ring $\mathcal{O}_F$ of algebraic integers, which is actually the set of roots of unity in F.*

Proof. This is a direct corollary of the strong approximation theorem:

$$\mathcal{O}_F^\times\backslash F_\infty^\times \times \prod_{v<\infty} \mathcal{O}_{F_v}^\times \cong F^\times\backslash\mathbb{A}_F^\times;$$

and the Dirichlet's unit theorem. □

4.4 Res $\otimes$ Cusp

In the same way as in the previous section we can show that

Lemma 4.4.1. *For $\pi_2' \in \mathcal{A}_{cusp}(G')$, $R(f)(\delta \circ N_D \otimes \phi_2) \equiv 0$ unless δ is unramified everywhere, $2k = 2$, $\pi_2' \in \mathcal{F}'(N,2)$ and $\phi_2 \in (\pi_2')^{K_{\mathrm{fin}}}$, in which case (with $K' := G_\infty' \prod_{p<\infty} K_p$)*

$$R(f)(\delta \circ N_D \otimes \phi_2)(x_1, x_2) = \delta(N_D(x_1))\phi_2(x_2)\,\mathrm{vol}(K')^2.$$

Proof. The proof on the non-Archimedean places has been done in the proof of Lemma 4.2.1 and Lemma 4.3.1. On the Archimedean place, one can apply a similar proof as in Lemma 4.3.1 to show that, for $\pi_{1,\infty}' = \delta_\infty \circ N_{D_\infty}$, $\pi_{2,\infty}' = \pi_{2k'}'$,

$$\pi_{1,\infty}' \otimes \pi_{2,\infty}'(f_\infty)(\delta_\infty \circ N_{D_\infty} \otimes \phi_{2,\infty}) = 0$$

unless $2k = 2k' = 2$, in which case $\phi_{2,\infty} = \mathbf{1}$ and

$$\pi_{1,\infty}' \otimes \pi_{2,\infty}'(f_\infty)(\delta_\infty \circ N_{D_\infty} \otimes \phi_{2,\infty}) = (\delta_\infty \circ N_{D_\infty} \otimes \phi_{2,\infty})\,\mathrm{vol}(G_\infty')^2.$$

$\square$

For $\pi' \in \mathcal{A}_{cusp}(G')$, let $\phi_{\pi'} \in \mathbb{C}X^{2k-2} \otimes (\pi_{\mathrm{fin}}')^{K_{\mathrm{fin}}}$ be the normalized new-line vector as defined in Lemma 3.1.3. Then we can write

$$\frac{\sum_{\pi_1' \in \mathcal{A}_{res}(G')} \sum_{\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f)}{\mathrm{vol}(K')^2}$$
$$= \begin{cases} \sum_{\delta_1 \in X^{\mathrm{un}}(\mathbb{Q})} \sum_{\pi_2' \in \mathcal{F}'(N,2)} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \frac{\delta_1(N_D(h))}{\mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A}))^{1/2}} \phi_{\pi_2'}(h)\phi_3(h)\, dh \right|^2, & \text{if } 2k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

We already know that $X^{\mathrm{un}}(\mathbb{Q}) = \{\mathbf{1}\}$. So when $2k = 2$,

$$\begin{aligned}\frac{\sum_{\pi_1' \in \mathcal{A}_{res}(G')} \sum_{\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f_0 \otimes f_0)}{\mathrm{vol}(K')^2} &= \frac{1}{2} \sum_{\pi_2' \in \mathcal{F}'(N,2)} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \phi_{\pi_2'}(h)\phi_3(h)\ dh \right|^2 \\ &= \frac{1}{2} \sum_{\pi_2' \in \mathcal{F}'(N,2)} \left| \langle \phi_3, \overline{\phi_{\pi_2'}} \rangle \right|^2 .\end{aligned}$$

By orthogonality, $\langle \phi_3, \overline{\phi_2} \rangle \neq 0$ only when $\phi_{\pi_2'} = \overline{\phi_3}/\|\phi_3\|$ up to a constant multiple. So

$$\frac{\sum_{\pi_1' \in \mathcal{A}_{res}(G')} \sum_{\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f_0 \otimes f_0)}{\mathrm{vol}(K')^2} = \frac{1}{2} \left| \langle \phi_3, \frac{\phi_3}{\|\phi_3\|} \rangle \right|^2 = \frac{1}{2} \langle \phi_3, \phi_3 \rangle.$$

4.5 Application of Ichino's formula

In summary, for $\pi_3' \in \mathcal{F}'(N, 2k)$,

$$\begin{cases} \dfrac{\sum_{\pi_1',\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f)}{\mathrm{vol}(K')^2} = \left(\dfrac{1}{2k-1}\right)^2 \sum_{\pi_1',\pi_2' \in \mathcal{F}'(N,2k)} \left| \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \Phi_{\pi_1' \otimes \pi_2'}(h)\phi_3(h)\ dh \right|^2 . \\ \dfrac{\sum_{\pi_1' \in \mathcal{A}_{res}(G')} \sum_{\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f)}{\mathrm{vol}(K')^2} = \begin{cases} \frac{1}{2}\langle \phi_3, \phi_3 \rangle, & \text{if } 2k = 2; \\ 0, & \text{otherwise.} \end{cases} \\ \dfrac{\sum_{\pi_1',\pi_2' \in \mathcal{A}_{res}(G')} I_{\pi_1',\pi_2'}(f)}{\mathrm{vol}(K')^2} = 0. \end{cases} \tag{4.4}$$

Now we can apply Ichino's triple product formula to the sum $\sum_{\pi_1',\pi_2' \in \mathcal{A}_{cusp}(G')} I_{\pi_1',\pi_2'}(f)$.

Theorem 4.5.1 (Ichino's period formula, [Ich08] Theorem 1.1)**.** *Let F be a number field, $E = F \times F \times F$, π_i be an cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A}_F)$ *for $i = 1, 2, 3$. Assume that the product of the central characters of π_i is trivial. D is a quaternion algebra over F so that there exists an irreducible unitary automorphic representation* $\Pi' = \pi_1' \otimes \pi_2' \otimes \pi_3'$

of $(D^\times(\mathbb{A}_F))^3$ *associated to* $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ *by the Jacquet–Langlands correspondence. Then, for* $\Phi = \otimes_v \Phi_v \in \Pi'$,

$$\frac{\left|\int_{G'(F)\backslash G'(\mathbb{A}_F)} \Phi(g)\ dg\right|^2}{\langle \Phi, \Phi\rangle} = \frac{\zeta_F^*(2)^2}{2^3} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})} \prod_v I_v,$$

where $dg = \prod_v dg_v$ *is the Tamagawa measure on* $G'(\mathbb{A}_F)$,

$$I_v = \frac{1}{\zeta_{F_v}(2)^2} \frac{L_v(1, \Pi, \mathrm{Ad})}{L_v(1/2, \Pi)} \int_{G'_v} \frac{\mathcal{B}_v(\Pi'_v(g_v)\phi_v, \overline{\phi_v})}{\mathcal{B}_v(\phi_v, \overline{\phi_v})}\ dg_v,$$

the $\mathcal{B}_v$*'s are* $(D^\times(\mathbb{A}_F))^3$*-invariant pairings between* Π'_v *and its contragredient* $\tilde{\Pi}'_v$ *so that*

$$\prod_v \mathcal{B}_v(\Phi_v, \overline{\Phi'_v}) = \langle \Phi, \Phi'\rangle := \iiint_{(G'(F)\backslash G'(\mathbb{A}_F))^3} \Phi(g_1, g_2, g_3)\overline{\Phi'(g_1, g_2, g_3)}\ dg_1\ dg_2\ dg_3$$

and $\mathcal{B}_v(\Phi_v, \overline{\Phi'_v}) = 1$ *for almost all* v.

We calculate the local factors I_v for the cases at hand. Assume $F = \mathbb{Q}$, D is definite, and $N = \mathrm{disc}(D)$ is, as before, square-free with an odd number of prime factors. Let Φ be of the form $\Phi_{\pi'_1 \otimes \pi'_2} \otimes \phi_3$ which contributes to the sum $\sum_{\pi'_1, \pi'_2 \in \mathcal{A}_{cusp}(G')} I_{\pi'_1, \pi'_2}(f)$. Then

$$\Phi_\infty = \frac{w_{2k}^\circ}{\|w_{2k}^\circ\|} \otimes \|\phi_3\| X_3^{2k-2} \tag{4.5}$$

and on the non-Archimedean places Φ_p are tensor products of three K_p-invariant unit vectors.

When $p \nmid N$, according to [Ich08] Lemma 2.2,

$$I_p = \mathrm{vol}(K_p; dg_p) = \zeta_p(2)^{-1}, \quad p \nmid N.$$

When $p \mid N$, $(\pi_i)_p$ is the special representation σ_{δ_i} for some unramified character δ_i : $\mathbb{Q}_p^\times \to \{\pm 1\}$. The corresponding ϕ_i, up to a constant multiple, is $\delta_i \circ N_{D_p}$. [Woo11] Proposition 5.5 shows that

$$I_p = (1-\varepsilon_p)\frac{1}{p}(1-\frac{1}{p})\zeta_p(2)^{-1}, \quad p \mid N$$

with $\varepsilon_p = -(\delta_1\delta_2\delta_3)(p) = \varepsilon_p(\frac{1}{2}, \Pi)$. (Note that the measure in [Woo11] differs by a factor $\zeta_p(2)$.)

On the Archimedean place we have

$$I_\infty = \frac{1}{\zeta_{\mathbb{R}}(2)^2}\frac{(L(1,\pi_{\mathrm{dis}}^{2k},\mathrm{Ad}))^3}{L(1/2,\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k})}\int_{G'_\infty}\frac{\mathcal{B}_\infty(\Pi'_\infty(g_\infty)\Phi_\infty,\overline{\Phi_\infty})}{\mathcal{B}_\infty(\Phi_\infty,\overline{\Phi_\infty})}\,dg_\infty.$$

Here $\mathcal{B}_\infty(\cdot,\bar{\cdot}) = \langle\cdot,\cdot\rangle$ is the inner product defined in (B.1). By (4.5) $\langle\Phi_\infty,\overline{\Phi_\infty}\rangle_\infty = \|\phi_3\|^2$, and, applying Lemma 4.1.1, we have

$$\begin{aligned}
&\int_{G'_\infty}\mathcal{B}_\infty(\Pi'_\infty(g)\Phi_\infty,\overline{\Phi_\infty})\,dg\\
&= \frac{\|\phi_3\|^2}{\|w_{2k}^\circ\|^2}\int_{G'_\infty}\langle\pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}\circ\Delta_3(g)w_{2k}^\circ\otimes X_3^{2k-2}, w_{2k}^\circ\otimes X_3^{2k-2}\rangle\,dg\\
&= \frac{\|\phi_3\|^2}{\|w_{2k}^\circ\|^2}\,\mathrm{vol}(G'_\infty)\langle w_{2k}^\circ\otimes X_3^{2k-2},\frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle\overline{\langle w_{2k}^\circ\otimes X_3^{2k-2},\frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle}\\
&= \frac{\|\phi_3\|^2}{\|w_{2k}^\circ\|^2}4\pi^2\frac{|\langle w_{2k}^\circ\otimes X_3^{2k-2},\mathbb{P}_{2k}\rangle|^2}{\|\mathbb{P}_{2k}\|^2}.
\end{aligned}$$

Here $\mathbb{P}_{2k}$, as defined in Lemma B.0.2, is the only $D^\times(\mathbb{R})$-invariant vector in $\pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}\circ\Delta_3$ up to a constant multiple. In Lemma B.0.4 we show that $\langle w_{2k}^\circ\otimes X_3^{2k-2},\mathbb{P}_{2k}\rangle = \|w_{2k}^\circ\|^2$. Moreover,

$$\zeta_{\mathbb{R}}(2) = \pi^{-1}, \quad L(1,\pi_{\mathrm{dis}}^{2k},\mathrm{Ad}) = \zeta_{\mathbb{C}}(2k)\zeta_{\mathbb{R}}(2) = 2^2(2\pi)^{-1-2k}\Gamma(2k),$$

$$L(1/2,\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}\otimes\pi_{\mathrm{dis}}^{2k}) = \zeta_{\mathbb{C}}(k)^3\zeta_{\mathbb{C}}(3k-1) = 2^4(2\pi)^{1-6k}\Gamma(k)^3\Gamma(3k-1).$$

Therefore, with Lemma B.0.2 one can check that

$$I_\infty = \frac{\Gamma(2k)^2\Gamma(2k-1)}{\Gamma(k)^3\Gamma(3k-1)}.$$

Now we see that $\prod_v I_v = 0$ unless $\varepsilon_p = -1$ for any $p \mid N$, in which case

$$\prod_v I_v = \zeta_{\mathbb{Q}}^*(2)\frac{\Gamma(2k)^2\Gamma(2k-1)}{\Gamma(k)^3\Gamma(3k-1)}\frac{2^{\omega(N)}\varphi(N)}{\pi N^2}.$$

Theorem 4.5.1 explicitly becomes

Lemma 4.5.2.

$$\left|\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \Phi_{\pi_1'\otimes\pi_2'}(h)\phi_3(h)\ dh\right|^2 = \|\phi_3\|^2\frac{\Gamma(2k)^2\Gamma(2k-1)}{\Gamma(k)^3\Gamma(3k-1)}\frac{2^{\omega(N)}\varphi(N)}{48N^2}\frac{L(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)}{L(1,\pi_1\otimes\pi_2\otimes\pi_3,\mathrm{Ad})}$$

if $\varepsilon_p = -1$ for every $p \mid N$; otherwise it vanishes.

Recall that

$$\mathrm{vol}(K') = \mathrm{vol}(G'_\infty;\ dg_\infty)\prod_p \mathrm{vol}(K_p;\ dg_p) = \frac{24}{\varphi(N)}.$$

Combining (4.2) (4.4) and the above lemma, we get that:

Theorem 4.5.3 (Main Theorem, spectral side)**.** *Let N be a square-free product of an odd number of primes. For $\pi_3' \in \mathcal{F}'(N,2k)$,*

$$\frac{I(f)}{\mathrm{vol}(K')} = \langle\phi_3,\phi_3\rangle\frac{\Gamma(2k-1)^3}{\Gamma(k)^3\Gamma(3k-1)}\frac{2^{\omega(N)}}{2N^2}\sum_{\substack{\pi_1,\pi_2\in\mathcal{F}(N,2k)\\ \varepsilon_p=-1,\ \forall p\mid N}}\frac{L(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)}{L(1,\pi_1\otimes\pi_2\otimes\pi_3,\mathrm{Ad})}$$
$$+\begin{cases}\langle\phi_3,\phi_3\rangle\frac{24}{\varphi(N)}, & \text{if } 2k=2;\\ 0, & \text{otherwise.}\end{cases}$$

Chapter 5

Geometric Side of the RTF

Recall that $G' = Z\backslash D^\times$, K_p is the image of $Z_p\mathcal{O}_p^\times$ in G'_p;

$$f(g_1, g_2) = f_\infty(g_1, g_2) \prod_{p<\infty} \mathbf{1}_{K_p}(g_1)\mathbf{1}_{K_p}(g_2),$$

$$f_\infty(g_1, g_2) = \overline{\langle \pi'_{2k} \otimes \pi'_{2k}(g_1, g_2) w^\circ_{2k}, w^\circ_{2k}\rangle}/\langle w^\circ_{2k}, w^\circ_{2k}\rangle, \quad w^\circ_{2k} = \left(-Y_1 Y_2 \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}\right)^{k-1};$$

$$\mathbb{P}_{2k} = \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} \otimes \begin{vmatrix} X_2 & X_3 \\ Y_2 & Y_3 \end{vmatrix}^{k-1} \otimes \begin{vmatrix} X_3 & X_1 \\ Y_3 & Y_1 \end{vmatrix}^{k-1};$$

ϕ_3 is a new-line vector in $\pi'_3 \in \mathcal{F}'(N, 2k)$ with $\phi_{3,\infty}$ a highest weight vector of π'_{2k}, as defined in Lemma 3.1.3. In Section 5.1 we will proof the following theorem which gives the orbital expansion of the distribution

$$I(f) = \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} K_f(h_1, h_1; h_2, h_2)\phi_3(h_1)\overline{\phi_3(h_2)}\, dh_1\, dh_2.$$

Theorem 5.0.1. *Let N be a square-free integer which has an odd number of prime factors. Let D be the quaternion algebra $\mathbb{Q}$ which is ramified precisely at ∞ and the primes dividing*

N. Then

$$I(f) = I_{[1]} + I_{[\gamma_0]} + I_{[\gamma_1]}$$

where γ_0, $\gamma_1 \in D^\times(\mathbb{Q})$ such that

$$\mathrm{Tr}_D(\gamma_0) = 0, \quad N_D(\gamma_0) = 1; \quad \mathrm{Tr}_D(\gamma_1) = N_D(\gamma_1) = 1,$$

with

$$I_{[1]} = \langle \phi_3, \phi_3 \rangle \frac{\mathrm{vol}(K')}{2k-1},$$
$$I_{[\gamma_0]} = \frac{1}{2}\,\mathrm{vol}(K') \int_{\mathbb{A}_{E_0}^\times \backslash D^\times(\mathbb{A})} \varphi_{\gamma_0}(h) \int_{\mathbb{A}^\times E_0^\times \backslash \mathbb{A}_{E_0}^\times} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt\ dh,$$
$$I_{[\gamma_1]} = \mathrm{vol}(K') \int_{\mathbb{A}_{E_1}^\times \backslash D^\times(\mathbb{A})} \varphi_{\gamma_1}(h) \int_{\mathbb{A}^\times E_1^\times \backslash \mathbb{A}_{E_1}^\times} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt\ dh.$$

Here $K' = G'_\infty \prod_{p<\infty} K_p$; $E = \mathbb{Q}(\gamma)$ is the quadratic extension of $\mathbb{Q}$ which can be embedded in D when γ exists (in particular $E_0 = \mathbb{Q}(\gamma_0) = \mathbb{Q}(\sqrt{-1})$, $E_1 = \mathbb{Q}(\gamma_1) = \mathbb{Q}(\sqrt{-3})$); ϕ^*, $\phi^{**} \in \pi'_3$ such that

$$\phi^* = \phi_3; \quad \phi^{**} = \otimes \phi_v^{**}, \quad \phi_\infty^{**} = \frac{\|\phi_3\|}{\|\mathbb{P}_{2k}\|^2} e_\gamma, \quad \phi_p^{**} = \phi_{3,p}; \tag{5.1}$$
$$e_\gamma = \sum_{i=0}^{2k-2} \binom{2k-2}{i} \langle \pi'_{2k} \otimes \pi'_{2k} \otimes \pi'_{2k}(h^{-1}\gamma h, 1, 1)\mathbb{P}_{2k}, w_{2k}^\circ \otimes X_3^{2k-2-i} Y_3^i \rangle X_3^{2k-2-i} Y_3^i; \tag{5.2}$$

φ_γ is the characteristic function of the set

$$\{h \in G'_\gamma(\mathbb{A}) \backslash G'(\mathbb{A}) : h_p^{-1}\gamma h_p \in K_p \textit{ for all primes } p\}.$$

And

- *$I_{[\gamma_0]} = 0$ if N has a prime factor of the form $4n+1$ (in this case γ_0 does not exist);*

- $I_{[\gamma_1]} = 0$ *if N has a prime factor of the form $3n+1$ (in this case γ_1 does not exist).*

(These two primes do not have to be distinct.)

For some particular N only the trivial orbit appears in the orbital decomposition. In fact we have:

Corollary 5.0.2. *With assumptions and notations as before, if N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$, we have*

$$\frac{I(f)}{\mathrm{vol}(K')} = \frac{\langle \phi_3, \phi_3 \rangle}{2k-1}.$$

With Theorem 4.5.3 we have that

$$\frac{2^{\omega(N)}}{2N^2} \sum_{\substack{\pi_1,\pi_2 \in \mathcal{F}(N,2k) \\ \varepsilon_p = -1,\ \forall p | N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})} = \begin{cases} 1 - \frac{24}{\varphi(N)}, & \text{if } 2k = 2; \\ \frac{\Gamma(k)^3 \Gamma(3k-1)}{\Gamma(2k-1)^2 \Gamma(2k)} = \binom{2k-2}{k-1}^{-2} \binom{3k-2}{k-1}, & \text{otherwise.} \end{cases}$$

This corollary and Lemma 3.2.2 together give a proof of (1.3).

After proving Theorem 5.0.1 we will use Waldspurger's formula (Theorem 5.2.1) to compute $I_{[\gamma_0]}$ and $I_{[\gamma_1]}$ for weight $2k = 2$ in Section 5.2. (Calculation of these terms for weight 4 can be found in Section 5.6.) This combined with the calculations in Sections 5.3 and 5.4 gives a proof of the following theorem.

Theorem 5.0.3 (Main Theorem, geometric side)**.** *Let N be a square-free product of an odd number of primes. For $\pi_3' \in \mathcal{F}'(N, 2)$,*

$$I(f) = I_{[1]} + I_{[\gamma_0]} + I_{[\gamma_1]}$$

with

$$\frac{I_{[1]}}{\langle \phi_3, \phi_3 \rangle \,\mathrm{vol}(K')} = 1;$$

$$\frac{I_{[\gamma_0]}}{\langle\phi_3,\phi_3\rangle\operatorname{vol}(K')} = \frac{2^{\omega(N)}L(\frac{1}{2},(\pi_3)_{E_0})}{N\cdot L(1,\pi_3,\mathrm{Ad})}\prod_{p|N}\frac{1-\chi_{-4}(p)}{2}\cdot\begin{cases}1, & 2\nmid N,\\ (1-\varepsilon_2(\frac{1}{2},\pi_3)), & 2\mid N;\end{cases}$$

$$\frac{I_{[\gamma_1]}}{\langle\phi_3,\phi_3\rangle\operatorname{vol}(K')} = \frac{3\sqrt{3}}{2}\frac{2^{\omega(N)}L(\frac{1}{2},(\pi_3)_{E_1})}{N\cdot L(1,\pi_3,\mathrm{Ad})}\prod_{p|N}\frac{1-\chi_{-3}(p)}{2}\cdot\begin{cases}1, & 3\nmid N,\\ (1-\varepsilon_3(\frac{1}{2},\pi_3)), & 3\mid N.\end{cases}$$

Theorem 4.5.3, 5.0.3 and Corollary 5.0.2 together imply the Main Theorem 3.2.3.

5.1 Orbital decomposition

We apply the geometric side of the RTF to the distribution $I(f)$. When $G = G'\times G'$ and $H_1 = H_2 = G'$ ($H_1, H_2 \hookrightarrow G$ diagonally), the representatives $[(\gamma_1,\gamma_2)]$ in $G'(\mathbb{Q})\backslash G'\times G'(\mathbb{Q})/G'(\mathbb{Q})$ can be chosen such that $\gamma_2 = 1$ and $[\gamma_1]$ runs through all conjugacy classes of $G'(\mathbb{Q})$. For $\theta_1,\theta_2\in G'(\mathbb{Q})$, $\theta_1^{-1}(\gamma,1)\theta_2 = (\gamma,1)$ if and only if $\theta_1=\theta_2\in G'_\gamma(\mathbb{Q})$, the centralizer of γ in $G'(\mathbb{Q})$. By the orbital expansion of the RTF, we have

$$\begin{aligned} I(f) &= \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})}\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} K_f(h_1,h_1;h_2,h_2)\phi_3(h_1)\overline{\phi_3(h_2)}\,dh_1\,dh_2\\ &= \iint_{(G'(\mathbb{Q})\backslash G'(\mathbb{A}))^2}\sum_{\gamma_1,\gamma_2\in G'(\mathbb{Q})} f(h_1^{-1}\gamma_1h_2,h_1^{-1}\gamma_2h_2)\phi_3(h_1)\overline{\phi_3(h_2)}\,dh_1\,dh_2\\ &= \iint_{(G'(\mathbb{Q})\backslash G'(\mathbb{A}))^2}\sum_{\substack{[\gamma_1,\gamma_2]=[\gamma,1]\\ \gamma\in[G'(\mathbb{Q})]}}\sum_{(\theta_1,\theta_2)} f(h_1^{-1}\theta_1^{-1}\gamma\theta_2h_2,h_1^{-1}\theta_1^{-1}\theta_2h_2)\phi_3(h_1)\overline{\phi_3(h_2)}\,dh_1\,dh_2\end{aligned}$$

Here (θ_1,θ_2) runs through

$$G'(\mathbb{Q})\times G'(\mathbb{Q})/\{(\theta_1,\theta_2):\theta_1=\theta_2\in G'_\gamma(\mathbb{Q})\}\cong (G'_\gamma(\mathbb{Q})\backslash G'(\mathbb{Q}))\times G'(\mathbb{Q}),$$

i.e. for a fixed $\theta_1 \in G'_\gamma(\mathbb{Q})\backslash G'(\mathbb{Q})$, θ_2 runs through $G'(\mathbb{Q})$. So we have

$$I(f) = \sum_{[\gamma]} \sum_{\theta_1,\theta_2} \iint_{(G'(\mathbb{Q})\backslash G'(\mathbb{A}))^2} f\big((\theta_1 h_1)^{-1}\gamma(\theta_2 h_2), (\theta_1 h_1)^{-1}(\theta_2 h_2)\big)\phi_3(\theta_1 h_1)\overline{\phi_3(\theta_2 h_2)}\, d\theta_1 h_1\, d\theta_2 h_2$$

where $[\gamma]$ runs through conjugacy classes of $G'(\mathbb{Q})$. Let $\theta_i h_i$ be the new h_i $(i = 1, 2)$. Then

$$I(f) = \sum_{[\gamma]} \int_{G'_\gamma(\mathbb{Q})\backslash G'(\mathbb{A})} \left(\int_{G'(\mathbb{A})} f(h_1^{-1}\gamma h_2, h_1^{-1}h_2)\overline{\phi_3(h_2)}\ dh_2 \right) \phi_3(h_1)\ dh_1.$$

We split $h_1 = th$ with $h \in G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})$ and $t \in G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})$, and let $g = t^{-1}h_2$. Then

$$\begin{aligned} I(f) &= \sum_{[\gamma]} \int_{G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})} \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} \left(\int_{G'(\mathbb{A})} f((th)^{-1}\gamma(tg), (th)^{-1}tg)\overline{\phi_3(tg)}\ d(tg) \right) \phi_3(th)\ dh\ dt \\ &= \sum_{[\gamma]} \int_{G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})} \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} \left(\int_{G'(\mathbb{A})} f(h^{-1}\gamma g, h^{-1}g)\overline{\phi_3(tg)}\ dg \right) \phi_3(th)\ dh\ dt. \end{aligned}$$

Denote the summand as $I_{[\gamma]}(f)$.

Recall that the new-line vector $\phi_3 = \otimes\phi_{3,v} \in \mathbb{C}X_3^{2k-2} \otimes \pi'^{K_{\mathrm{fin}}}_{3,\mathrm{fin}}$ can be written such that $\phi_{3,\infty} = \|\phi_3\|X_3^{2k-2}$ and $\phi_{3,p}$'s are K_p-invariant unit vectors for $p < \infty$ (see Lemma 3.1.3). Let $f = f_\infty \cdot (f_0 \otimes f_0)$ be the test function defined in (4.3). Then

Lemma 5.1.1. $\int_{G'(\mathbb{A})} \overline{f(h^{-1}\gamma g, h^{-1}g)}(R(g)\phi_3)\ dg$ *is a pure tensor in* π'_3.

Proof. For $f = f_\infty \cdot (f_0 \otimes f_0)$,

$$\begin{aligned} \int_{G'(\mathbb{A})} \overline{f(h^{-1}\gamma g, h^{-1}g)}(R(g)\phi_3)\ dg = \|\phi_3\| &\int_{G'_\infty} \overline{f_\infty(h_\infty^{-1}\gamma g_\infty, h_\infty^{-1}g_\infty)}(\pi'_{2k}(g_\infty)X_3^{2k-2})\ dg_\infty \\ &\cdot \prod_{p<\infty} \int_{G'_p} \overline{\mathbf{1}_{K_p}(h_p^{-1}\gamma g_p)\ \mathbf{1}_{K_p}(h_p^{-1}g_p)}R(g_p)\phi_{3,p}\ dg_p. \end{aligned}$$

For $v = p < \infty$ the local test function $\mathbf{1}_{K_p}(h_p^{-1}\gamma h_{2,p})\ \mathbf{1}_{K_p}(h_p^{-1}h_{2,p})$ is nonzero only if both

$h_p^{-1}\gamma g_p,\ h_p^{-1}g_p \in K_p$, and hence

$$\int_{G'_p} \overline{\mathbf{1}_{K_p}(h_p^{-1}\gamma g_p)\ \mathbf{1}_{K_p}(h_p^{-1}g_p)} R(g_p)\phi_{3,p}\ dg_p = \int_{h_pK_p\cap\gamma^{-1}h_pK_p} R(g_p)\phi_{3,p}\ dg_p.$$

These two left cosets either coincide or are disjoint, and $h_pK_p = \gamma^{-1}h_pK_p$ if and only if $h_p^{-1}\gamma h_p \in K_p$. Let $\varphi_\gamma = \prod \varphi_{\gamma,p}$ and $\varphi_{\gamma,p} : G'_\gamma(\mathbb{Q}_p)\backslash G'_p \to \mathbb{C}$ be the characteristic function of the set $\{h_p : h_p^{-1}\gamma h_p \in K_p\}$. Since $\phi_{3,p}$ is K_p-invariant, we have

$$\int_{G'_p} \overline{\mathbf{1}_{K_p}(h_p^{-1}\gamma g_p)\ \mathbf{1}_{K_p}(h_p^{-1}g_p)} R(g_p)\phi_{3,p}\ dg_p$$
$$= \varphi_{\gamma,p}(h_p)\int_{h_pK_p} R(g_p)\phi_{3,p}\ dg_p = \mathrm{vol}(K_p)\varphi_{\gamma,p}(h_p)R(h_p)\phi_{3,p}.$$

For $v=\infty$, by the following lemma we have

$$\int_{G'_\infty} \overline{f_\infty(h_\infty^{-1}\gamma g_\infty, h_\infty^{-1}g_\infty)}\pi'_{2k}(g_\infty)X_3^{2k-2}\ dg_\infty = \frac{\mathrm{vol}(G'_\infty)}{\|\mathbb{P}_{2k}\|^2}\pi'_{2k}(h_\infty)e_\gamma$$

where e_γ is defined in (5.1). □

Lemma 5.1.2. *For $\gamma \in G'(\mathbb{Q})$, $h \in G'_\gamma(\mathbb{R})\backslash G'_\infty$, with e_γ defined in (5.1),*

$$\int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g, h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle \pi'_{2k}(g)X_3^{2k-2}\ dg = \mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\pi'_{2k}(h)e_\gamma.$$

Proof. First we consider the inner product

$$\left\langle \int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g, h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle \pi'_{2k}(g)X_3^{2k-2}\ dg, \pi'_{2k}(h)X_3^{2k-2-i}Y_3^i \right\rangle$$
$$= \int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(g,g)w^\circ_{2k}, \pi'_{2k}\otimes\pi'_{2k}(\gamma^{-1}h,h)w^\circ_{2k}\rangle \langle \pi'_{2k}(g)X_3^{2k-2}, \pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\rangle\ dg$$
$$= \int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(g,g,g)w^\circ_{2k}\otimes X_3^{2k-2}, \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(\gamma^{-1}h,h,h)w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle\ dg.$$

By Lemma 4.1.1 the above integral is equal to

$$\mathrm{vol}(G'_\infty)\langle w^\circ_{2k}\otimes X_3^{2k-2}, \frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle\overline{\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(\gamma^{-1}h,h,h)w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i, \frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle}.$$

Recall that $\mathbb{P}_{2k}$ is G'_∞-invariant, and we have $\langle w^\circ_{2k}\otimes X_3^{2k-2}, \mathbb{P}_{2k}\rangle = \|w^\circ_{2k}\|^2$ by Lemma B.0.4. Now the above integral is equal to

$$\begin{aligned}
&\mathrm{vol}(G'_\infty)\|w^\circ_{2k}\|^2\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(\gamma,1,1)\mathbb{P}_{2k}, \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(h,h,h)w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle\\
=&\mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(\gamma h,h,h)\mathbb{P}_{2k}, \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(h,h,h)w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle\\
=&\mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma h,1,1)\mathbb{P}_{2k}, w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle.
\end{aligned}$$

Lemma B.0.1 shows that, for a fixed h, $\{\pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\}$ forms an orthogonal basis of $V_{\pi'_{2k}}$. So we have

$$\begin{aligned}
&\int_{G'_\infty}\langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g,h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle\pi'_{2k}(g)X_3^{2k-2}\,dg\\
=&\sum_{i=0}^{2k-2}\frac{\left\langle\int_{G'_\infty}\langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g,h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle\pi'_{2k}(g)X_3^{2k-2}\,dg, \pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\right\rangle}{\langle\pi'_{2k}(h)X_3^{2k-2-i}Y_3^i, \pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\rangle}\pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\\
=&\sum_{i=0}^{2k-2}\frac{\mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma h,1,1)\mathbb{P}_{2k}, w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle}{\langle X_3^{2k-2-i}Y_3^i, X_3^{2k-2-i}Y_3^i\rangle}\pi'_{2k}(h)X_3^{2k-2-i}Y_3^i\\
=&\mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\sum_{i=0}^{2k-2}\binom{2k-2}{i}\langle \pi'_{2k}\otimes\pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma h,1,1)\mathbb{P}_{2k}, w^\circ_{2k}\otimes X_3^{2k-2-i}Y_3^i\rangle\pi'_{2k}(h)X_3^{2k-2-i}Y_3^i.
\end{aligned}$$

$\square$

Take $\phi^*, \phi^{**}\in\pi'_3$ such that

$$\phi^* = \phi_3;\quad \phi^{**} = \otimes\phi^{**}_v,\quad \phi^{**}_v = \begin{cases}\frac{\|\phi_3\|}{\|\mathbb{P}_{2k}\|^2}e_\gamma, & v=\infty;\\ \phi_{3,p}, & v=p<\infty.\end{cases}$$

We can write

$$I_{[\gamma]}(f) = \int_{G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})} \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} \overline{\mathrm{vol}(K')(R(h)\phi^{**})(t) \prod_{p<\infty} \varphi_{\gamma,p}(h_p)}(R(h)\phi^*)(t)\; dh\; dt$$
$$= \mathrm{vol}(K') \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_\gamma(h) \int_{G'_\gamma(\mathbb{Q})\backslash G'_\gamma(\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\; dt\; dh,$$

with K', e_γ, φ_γ defined in Theorem 5.0.1.

When $\gamma = 1$, the centralizer $G'_\gamma = G'$. According to Lemma B.0.4,

$$e_\gamma = \|w^\circ_{2k}\|^2 X_3^{2k-2}.$$

Then by Lemma B.0.2

$$\frac{I_{[1]}(f)}{\mathrm{vol}(K')} = \int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \phi^*(t)\overline{\phi^{**}(t)}\; dt = \langle \phi^*, \phi^{**}\rangle = \prod_v \langle \phi^*_v, \phi^{**}_v\rangle_v$$
$$= \|\phi_3\|^2 \langle X_3^{2k-2}, \frac{1}{\|\mathbb{P}_{2k}\|^2} e_\gamma\rangle_\infty = \|\phi_3\|^2 \frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2} = \frac{\|\phi_3\|^2}{2k-1}.$$

Now we study the property for the other $[\gamma]$'s such that φ_γ is not identically 0. Instead of $\mathbb{Q}$, we consider it over an arbitrary number field F.

Lemma 5.1.3. *$\varphi_\gamma = 0$ unless $\mathrm{Tr}_D(\gamma) \in \{\pm 1\}\backslash \mathcal{O}_F$ and $N_D(\gamma) \in \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$. In particular, when $F = \mathbb{Q}$, $\varphi_\gamma = 0$ unless $\mathrm{Tr}_D(\gamma) \in \{\pm 1\}\backslash \mathbb{Z} \cong \mathbb{Z}_{\geq 0}$ and $N_D(\gamma) = \pm 1$.*

Proof. Fix a set $\Sigma \subset \mathcal{O}_F - \{0\}$ of representatives in $F^\times/(F^\times)^2$. We can choose Σ to be the set of "square-free" integers. More precisely, the factorization of the principal ideal of $\mathcal{O}_F$ generated by any number in Σ has exactly one factor for each prime ideal that appears in it. Then as in Lemma 2.2.3 we can fix a representative of γ in $D^\times(F)$ (also denoted γ) so that $N_D(\gamma) \in \Sigma$. Under this assumption we see that $\mathrm{ord}_v(N_D(\gamma))$ is either 0 or 1.

Suppose that $\varphi_\gamma(h) \neq 0$. This means $h_v^{-1}\gamma h_v \in K_v$ for all $v < \infty$.

When $v \notin \mathrm{Ram}(D)$, $K_v = \mathrm{PGL}(2, \mathcal{O}_{F_v})$. We can say that, fixing a representative of γ in

$\mathrm{GL}(2, F_v)$ there is an $h_v \in \mathrm{GL}(2, F_v)$ such that $\lambda_v h_v^{-1}\gamma h_v \in \mathrm{GL}_2(\mathcal{O}_{F_v})$ for some $\lambda_v \in F_v^\times$. As a matrix in $\mathrm{GL}_2(\mathcal{O}_{F_v})$, we have that

$$\mathrm{Tr}_{D_v}(\lambda_v h_v^{-1}\gamma h_v) \in \mathcal{O}_{F_v}, \quad N_{D_v}(\lambda_v h_v^{-1}\gamma h_v) \in \mathcal{O}_{F_v}^\times.$$

Conjugate matrices have the same norm and trace, so

$$\mathrm{ord}_v(\mathrm{Tr}_{D_v}(\lambda_v\gamma)) = \mathrm{ord}_v(\lambda_v \,\mathrm{Tr}_D(\gamma)) \geq 0, \quad \mathrm{ord}_v(N_{D_v}(\lambda_v\gamma)) = \mathrm{ord}_v(\lambda_v^2 N_D(\gamma)) = 0.$$

With the assumption of Σ, one can imply that $\mathrm{ord}_v(N_D(\gamma)) = 0$, $\lambda_v \in \mathcal{O}_{F_v}^\times$ is a unit, and then $\mathrm{ord}_v(\mathrm{Tr}_D(\gamma)) \geq 0$.

When $v \in \mathrm{Ram}(D)$ and $v < \infty$, D_v is a division algebra and it has only one maximal order

$$\mathcal{O}_v = \{x \in D_v : N_{D_v}(x) \in \mathcal{O}_{F_v}\}.$$

We still know that, there is an $h_v \in D_v^\times$ such that $\lambda_v h_v^{-1}\gamma h_v \in \mathcal{O}_v^\times$ for some $\lambda_v \in F_v^\times$. With the assumption of Σ, as in the previous case, one can imply that $\mathrm{ord}_v(N_D(\gamma)) = 0$, i.e. $\gamma \in \mathcal{O}_v^\times$. Moreover, Proposition 2.3.1 shows that γ is $\mathcal{O}_{F_v}$-integral and then $\mathrm{ord}_v(\mathrm{Tr}_D(\gamma)) \geq 0$.

Globally we see that, fixing γ such that $N_D(\gamma) \in \Sigma$, for any place v of F, $N_D(\gamma)$ cannot be divisible by v, and the order of v in the ideal decomposition of $\mathrm{Tr}_D(\gamma) \in F$ has to be nonnegative. In other words $N_D(\gamma)$ has to be a unit and $\mathrm{Tr}_D(\gamma)$ in $\mathcal{O}_F$. With Lemma 2.2.3 we get the conclusion. $\square$

With this lemma, we only need to consider the summand $I_{[\gamma]}(f)$ with $N_D(\gamma)$ a "square-free" unit in $\mathcal{O}_F^\times$ and $\mathrm{Tr}_D(\gamma) \in \{\pm 1\}\backslash\mathcal{O}_F$. Going back to the case when $F = \mathbb{Q}$, we only need to consider the summand $I_{[\gamma]}(f)$ with $N_D(\gamma) = 1$ and $\mathrm{Tr}_D(\gamma) \in \mathbb{Z}_{\geq 0}$ (obviously $N_D(\gamma)$ cannot be -1 when D is definite). Now we consider the infinite place $\mathbb{Q}_\infty = \mathbb{R}$. When D is definite, $D(\mathbb{R}) \cong \mathbb{H}$ is non-split. That means the characteristic polynomial

$$X^2 - \mathrm{Tr}_D(x)X + N_D(x)$$

of any $x \in D^\times(\mathbb{Q})$ is irreducible over $\mathbb{R}$ if and only if $x \notin \mathbb{R} \cap D^\times(\mathbb{Q}) = \mathbb{Q}^\times$. For $\gamma \in G'(\mathbb{Q}) = Z(\mathbb{Q})\backslash D^\times(\mathbb{Q})$, if $\gamma \neq 1$, $X^2 - \mathrm{Tr}_D(\gamma)X + N_D(\gamma)$ is irreducible over $\mathbb{R}$, which implies that $\mathrm{Tr}_D(\gamma)^2 < 4N_D(\gamma)$. Now $N_D(\gamma) = 1$, we only need $\mathrm{Tr}_D(\gamma) = 0$ or 1.

Proposition 5.1.4. *Denote N and D as in Theorem 5.0.1.*

(1) There is an $x \in D$ with $\mathrm{Tr}_D(x) = 0$, $N_D(x) = 1$ if and only if N has no prime factor of the form $4n + 1$.

(2) There is an $x \in D$ with $\mathrm{Tr}_D(x) = 1$, $N_D(x) = 1$ if and only if N has no prime factor of the form $3n + 1$.

Proof. The sufficiency can be seen directly from Lemma 2.1.2. If N has no prime factor of the form $4n + 1$, D has a presentation $D = (\frac{-1,-N}{\mathbb{Q}})$ and clearly $\mathrm{Tr}_D(i) = 0$, $N_D(i) = 1$. If N has no prime factor of the form $3n + 1$, D has a presentation $D = (\frac{-3,-N}{\mathbb{Q}})$ and one can check $\mathrm{Tr}_D(\frac{1+i}{2}) = 1$, $N_D(\frac{1+i}{2}) = 1$.

See Appendix A.2 for the proof of the necessity. $\square$

Lemma 5.1.3, Proposition 5.1.4, and the following lemma together imply Theorem 5.0.1.

Lemma 5.1.5. *With notations in Theorem 5.0.1,*

$$\frac{I_{[\gamma_0]}(f)}{\mathrm{vol}(K')} = \frac{1}{2} \int_{\mathbb{A}_{E_0}^\times \backslash D^\times(\mathbb{A})} \varphi_i(h) \left(\int_{\mathbb{A}^\times E_0^\times \backslash \mathbb{A}_{E_0}^\times} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt \right) dh,$$

$$\frac{I_{[\gamma_1]}(f)}{\mathrm{vol}(K')} = \int_{\mathbb{A}_{E_1}^\times \backslash D^\times(\mathbb{A})} \varphi_{1+i}(h) \left(\int_{\mathbb{A}^\times E_1^\times \backslash \mathbb{A}_{E_1}^\times} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt \right) dh.$$

Here $E_0 = \mathbb{Q}[X]/(X^2+1) = \mathbb{Q}(\sqrt{-1})$, $E_1 = \mathbb{Q}[X]/(X^2+3) = \mathbb{Q}(\sqrt{-3})$ *are quadratic extensions of* $\mathbb{Q}$ *which can be embedded in* D *when* γ_0, γ_1 *exist respectively.*

Proof. (1) When N has no prime factor of the form $4n+1$, we can write $D = (\frac{-1,-N}{\mathbb{Q}})$ by Lemma 2.1.2 and take $\gamma_0 = i_D$ (the i in the $\mathbb{Q}$-basis $\{1, i, j, k\}$ of D). Then

$$I_{[\gamma_0]} = I_{[i]} = \mathrm{vol}(K') \int_{G'_i(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_i(h) \left(\int_{G'_i(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt \right)\ dh,$$

where φ_i is the characteristic function of the set

$$\{h \in G'_i(\mathbb{A})\backslash G'(\mathbb{A}) : h_p^{-1} i_D h_p \in K_p \text{ for all primes } p\}.$$

Lemma 2.2.4 shows that $G'_i(\mathbb{Q})$ is the image in $G'(\mathbb{Q})$ of

$$\{\lambda + \mu i \in D^\times(\mathbb{Q}) : \lambda, \mu \in \mathbb{Q}\} \cup \{\lambda j + \mu k \in D^\times(\mathbb{Q}) : \lambda, \mu \in \mathbb{Q}\} = \mathbb{Q}(i)^\times \cdot \{1, j\} = \{1, j\} \cdot \mathbb{Q}(i)^\times.$$

Instead of using G'_i we can simply consider a subgroup T in G' such that $T(\mathbb{Q})$ is the image in $G'(\mathbb{Q})$ of $\mathbb{Q}(i)^\times$. By Lemma 2.4.1, for $f \in C_c^\infty(G'_i(\mathbb{Q})\backslash G'_i(\mathbb{A}))$,

$$\int_{G'_i(\mathbb{Q})\backslash G'_i(\mathbb{A})} f(g)\ dg = \int_{G'_i(\mathbb{Q})\backslash G'_i(\mathbb{Q})T(\mathbb{A})} f(g)\ dg = \int_{(G'_i(\mathbb{Q})\cap T(\mathbb{A}))\backslash T(\mathbb{A})} f(t)\ dt = \int_{T(\mathbb{Q})\backslash T(\mathbb{A})} f(t)\ dt$$

and hence

$$\int_{G'_i(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt = \int_{T(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt,$$

$$\begin{aligned}\frac{I_{[i]}}{\mathrm{vol}(K')} &= \int_{G'_i(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_i(h) \left(\int_{T(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt \right)\ dh \\ &= \frac{1}{2} \int_{T(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_i(h) \left(\int_{T(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\ dt \right)\ dh.\end{aligned}$$

(2) When N has no prime factor of the form $3n+1$, we can write $D = (\frac{-3,-N}{\mathbb{Q}})$ by Lemma 2.1.2 and take $\gamma_1 = \frac{1}{2}(1+i_D)$. Then

$$I_{[\gamma_1]} = I_{[1+i]} = \mathrm{vol}(K') \int_{G'_{1+i}(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_{1+i}(h) \left(\int_{G'_{1+i}(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt \right) dh,$$

where φ_{1+i} is the characteristic function of the set

$$\{h \in G'_{1+i}(\mathbb{A})\backslash G'(\mathbb{A}) : h_p^{-1}(1+i)h_p \in K_p \text{ for all primes } p\}.$$

Lemma 2.2.4 shows that $G'_{1+i}(\mathbb{Q})$ is the image in $G'(\mathbb{Q})$ of

$$\{\lambda + \mu(1+i) \in D^\times(\mathbb{Q}) : \lambda, \mu \in \mathbb{Q}\} = \mathbb{Q}(i)^\times = T(\mathbb{Q})$$

(notice that this $i = \sqrt{-3}$ is different with the $i = \sqrt{-1}$ in the previous case). So

$$\frac{I_{[1+i]}}{\mathrm{vol}(K')} = \int_{T(\mathbb{A})\backslash G'(\mathbb{A})} \varphi_{1+i}(h) \left(\int_{T(\mathbb{Q}\backslash\mathbb{A})} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt \right) dh.$$

□

5.2 Nontrivial orbits and Waldspurger's formula

Now we calculate $I_{[\gamma_0]}$ and $I_{[\gamma_1]}$ when the weight $2k = 2$. In this case $(\pi'_3)_\infty \cong \pi'_{2k}$ is a trivial representation, and $\phi^* = \phi^{**} = \phi_3$ since their Archimedean factors are all trivial. Lemma 5.1.5 says

$$\frac{I_{[\gamma]}(f)}{\mathrm{vol}(K')} = c_\gamma \int_{\mathbb{A}_E^\times\backslash D^\times(\mathbb{A})} \varphi_\gamma(h) \left(\int_{\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times} |(R(h)\phi_3)(t)|^2\, dt \right) dh.$$

Here $c_\gamma = \frac{1}{2}$ when $\gamma = \gamma_0$, and $c_\gamma = 1$ when $\gamma = \gamma_1$.

For any two forms $\phi', \phi'' \in \pi'_3$, as functions in $L^2([D^\times])$, we already have a Petersson inner product $\langle \cdot, \cdot \rangle$ defined as

$$\langle \phi', \phi'' \rangle := \int_{[D^\times]} \phi'(g)\overline{\phi''(g)}\, dg. \tag{5.3}$$

We can also consider them as functions in $L^2([E^\times])$, in which we have an inner product

$$\langle \phi', \phi'' \rangle_E := \int_{[E^\times]} \phi'(t)\overline{\phi''(t)}\, dt.$$

Here $E/\mathbb{Q}$ is a quadratic field extension as in the above lemma, embedded in D by $E = \mathbb{Q}(i_D)$. We can choose the set of all characters on $E^\times \backslash \mathbb{A}_E^\times$ (whose restrictions on $\mathbb{A}^\times$ are trivial) as a basis of $L^2([E^\times])$ and decompose $\phi|_{[E^\times]}$ by

$$\phi|_{[E^\times]} = \sum_{\Omega \in \widehat{[E^\times]}} \frac{\langle \phi|_{[E^\times]}, \Omega \rangle_E}{\langle \Omega, \Omega \rangle_E} \Omega = \sum_{\Omega \in \widehat{[E^\times]}} \frac{P_{\Omega^{-1}}(\phi)}{\langle \Omega, \Omega \rangle_E} \Omega.$$

Here $P_\Omega : \pi' \to \mathbb{C}$ is a period integral defined by

$$P_\Omega(\phi) := \int_{[E^\times]} \phi(t)\Omega(t)\, dt, \quad \phi \in \pi',$$

where the Haar measure gives total volume $2L(1, \eta)$ on $[E^\times]$, with $\eta : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \{\pm 1\}$ the quadratic character associated to $E/\mathbb{Q}$ by class field theory. Then

$$\begin{aligned} &\int_{[E^\times]} \phi'(t)\overline{\phi''(t)}\, dt = \langle \phi'|_{[E^\times]}, \phi''|_{[E^\times]} \rangle_E \\ = &\left\langle \sum_{\Omega \in \widehat{[E^\times]}} \frac{P_{\Omega^{-1}}(\phi')}{\langle \Omega, \Omega \rangle_E} \Omega, \sum_{\Omega \in \widehat{[E^\times]}} \frac{P_{\Omega^{-1}}(\phi'')}{\langle \Omega, \Omega \rangle_E} \Omega \right\rangle_E = \sum_{\Omega \in \widehat{[E^\times]}} \frac{P_{\Omega^{-1}}(\phi')}{\langle \Omega, \Omega \rangle_E} \overline{\left(\frac{P_{\Omega^{-1}}(\phi'')}{\langle \Omega, \Omega \rangle_E} \right)} \langle \Omega, \Omega \rangle_E. \end{aligned}$$

Noticing $\langle \Omega, \Omega \rangle_E = \mathrm{vol}([E^\times])$, we have

$$\int_{[E^\times]} \phi'(t)\overline{\phi''(t)}\ dt = \frac{1}{\mathrm{vol}([E^\times])} \sum_{\Omega \in \widehat{[E^\times]}} P_{\Omega^{-1}}(\phi') P_\Omega(\overline{\phi''}). \tag{5.4}$$

The product of two period integrals is related via a period formula of Waldspurger to the central value of a base change L-function.

Theorem 5.2.1 (Waldspurger's formula, [Wal85], [YZZ13] Section 1.4.1 and Theorem 1.4.2)**.** *For any $\phi', \phi'' \in \pi'$, we have*

$$\frac{P_\Omega(\phi') \cdot P_{\Omega^{-1}}(\overline{\phi''})}{\langle \phi', \phi'' \rangle} = \frac{\zeta_F^*(2) L(\frac{1}{2}, \pi_E \otimes \Omega)}{2L(1, \pi, \mathrm{Ad})} \prod_v \alpha_v(\phi'_v, \overline{\phi''_v}; \Omega_v),$$

with

$$\alpha_v(\phi'_v, \overline{\phi''_v}; \Omega_v) := \frac{L_v(1,\eta) L_v(1, \pi, \mathrm{Ad})}{\zeta_{F_v}(2) L_v(\frac{1}{2}, \pi_E \otimes \Omega)} \int_{E_v^\times / F_v^\times} \frac{B_v(\pi'_v(t_v)\phi'_v, \overline{\phi''_v})}{B_v(\phi'_v, \overline{\phi''_v})} \Omega_v(t_v)\ dt_v.$$

Here the B_v's are $D_v^\times$-invariant bilinear forms on $\pi'_v \otimes \tilde{\pi}'_v$ such that $\prod_v B_v(\cdot, \bar{\cdot}) = \langle \cdot, \cdot \rangle$ as defined in (5.3). The integral converges absolutely if both π'_v and Ω_v are unitary. Moreover, for $v < \infty$,

$$\alpha_v(\phi'_v, \phi''_v; \Omega_v) = \mathrm{vol}(\mathcal{O}_{E_v}^\times / \mathcal{O}_{F_v}^\times; dt_v)$$

when $D_v \cong M(2, F_v)$, E_v/F_v is unramified, $\pi'_v \cong \pi_v$ and Ω_v are both unramified, and $\phi'_v \in \pi_v$, $\phi''_v \in \tilde{\pi}_v$ are unit spherical vectors.

For weight 2, we apply Waldspurger's formula to the case when $\phi' = \phi'' = R(h)\phi_3$. Let $F = \mathbb{Q}$, $E = E_0$ or E_1 be a quadratic extension of $\mathbb{Q}$ defined as in Lemma 5.1.5. We have

$$\frac{\left| \int_{[E^\times]} (R(h)\phi_3)(t)\Omega(t)\ dt \right|^2}{\langle R(h)\phi_3, R(h)\phi_3 \rangle} = \frac{\zeta_{\mathbb{Q}}^*(2) L(\frac{1}{2}, (\pi_3)_E \otimes \Omega)}{2L(1, \pi_3, \mathrm{Ad})} \prod_v \alpha_v(\pi'_{3,v}(h_v)\phi_{3,v}, \overline{\pi'_{3,v}(h_v)\phi_{3,v}}; \Omega_v),$$

and by (5.4) we get

$$\frac{\int_{[E^\times]} |(R(h)\phi_3)(t)|^2\, dt}{\langle \phi_3, \phi_3\rangle} = \frac{1}{\mathrm{vol}([E^\times])} \sum_{\Omega \in \widehat{[E^\times]}} \frac{\left|\int_{[E^\times]} (R(h)\phi_3)(t)\Omega(t)\, dt\right|^2}{\langle R(h)\phi_3, R(h)\phi_3\rangle}$$
$$= \frac{1}{\mathrm{vol}([E^\times])} \frac{\zeta_{\mathbb{Q}}^*(2)}{2L(1,\pi_3,\mathrm{Ad})} \sum_{\Omega \in \widehat{[E^\times]}} L(\frac{1}{2}, (\pi_3)_E \otimes \Omega) \prod_v \alpha_v(\pi'_{3,v}(h_v)\phi_{3,v}, \overline{\pi'_{3,v}(h_v)\phi_{3,v}}; \Omega_v).$$

Actually, most characters Ω do not contribute to this sum. For example:

Lemma 5.2.2. *For weight $2k = 2$ and $E = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, $\alpha_\infty = 0$ unless Ω_∞ is trivial, in which case*

$$\alpha_\infty(\phi'_\infty, \overline{\phi''_\infty}, \mathbf{1}) = \frac{L(1,\mathrm{sgn})L(1,\pi^2_{\mathrm{dis}},\mathrm{Ad})}{\zeta_{\mathbb{R}}(2)L(\frac{1}{2},(\pi^2_{\mathrm{dis}})_E)} \mathrm{vol}(\mathbb{C}^\times/\mathbb{R}^\times) = (2\pi)^{-1}\,\mathrm{vol}(\mathbb{C}^\times/\mathbb{R}^\times).$$

Proof. Now π'_{2k} is trivial. So the integral in the definition of $\alpha_\infty(\phi'_\infty, \overline{\phi''_\infty}, \Omega_\infty)$ becomes $\int_{\mathbb{C}^\times/\mathbb{R}^\times} \Omega_\infty(t_\infty)\, dt_\infty$, which vanishes unless $\Omega_\infty = \mathbf{1}$.

Also recall that η is the quadratic Hecke character of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ corresponding to the Dirichlet character χ_{-d} where $d = 4$ or 3. Then the rest can be verified by noticing that $\eta_\infty = \mathrm{sgn}$. □

Moreover, Lemma 5.3.1 and 5.4.3 show that $\alpha_p = 0$ unless Ω_p is unramified for any finite p. With Lemma 4.3.2, now we can glue all these conditions together and get that

$$\prod_v \alpha_v(\pi'_v(h_v)\phi_{3,v}, \overline{\pi'_v(h_v)\phi_{3,v}}; \Omega_v) = 0 \quad \text{unless } \Omega = \mathbf{1}.$$

Now we have

$$\frac{\int_{[E^\times]} |(R(h)\phi_3)(t)|^2\, dt}{\langle \phi_3, \phi_3\rangle} = \frac{\zeta_{\mathbb{Q}}^*(2)L(\frac{1}{2},(\pi_3)_E)}{2L(1,\pi_3,\mathrm{Ad})\,\mathrm{vol}([E^\times])} \prod_v \alpha_v(\pi'_{3,v}(h_v)\phi_{3,v}, \overline{\pi'_{3,v}(h_v)\phi_{3,v}}; \mathbf{1}),$$

and therefore

$$\frac{I_{[\gamma]}(f)}{\langle\phi_3,\phi_3\rangle\operatorname{vol}(K')} = \frac{c_\gamma\zeta_{\mathbb{Q}}^*(2)L(\frac{1}{2},(\pi_3)_E)}{2L(1,\pi_3,\mathrm{Ad})\operatorname{vol}([E^\times])}\prod_v\int_{E_v^\times\backslash D_v^\times}\alpha_v(\pi'_{3,v}(h_v)\phi_{3,v},\overline{\pi'_{3,v}(h_v)\phi_{3,v}};\mathbf{1})\varphi_{\gamma_v}(h_v)\,dh_v. \tag{5.5}$$

With the above lemma we have that

$$\int_{\mathbb{C}^\times\backslash D_\infty^\times}\alpha_\infty\varphi_{\gamma,\infty}\,dh_\infty = (2\pi)^{-1}\operatorname{vol}(G'_\infty);$$

by Lemma 5.4.2 and 5.4.3 we have

$$\int_{E_p^\times\backslash D_p^\times}\alpha_p\varphi_{\gamma,p}(h_p)\,dh_p = \operatorname{vol}(K_p),\quad p\nmid N;$$

and by (5.6), when $p\mid N$,

$$\int_{E_p^\times\backslash D_p^\times}\alpha_p\varphi_{\gamma,p}\,dh_p = (1-\chi_{-d}(p))(1-p^{-1})\operatorname{vol}(K_p)\cdot\begin{cases}1, & E_p/\mathbb{Q}_p \text{ unramified;}\\ (1-\varepsilon_p(\frac{1}{2},\pi_3)), & E_p/\mathbb{Q}_p \text{ ramified.}\end{cases}$$

Here χ_{-d} is the Dirichlet character corresponding to the Hecke character η, and $d=4$ or 3. We will show in the next two sections the calculation of the above local integrals at finite places.

Recall that $\operatorname{vol}([E^\times]) = 2L(1,\eta)$. The special values of completed L-functions are given by Dirichlet's class number formula.

Theorem 5.2.3 (Dirichlet class number formula, [Dav00] (6.15))**.** *Let $d<0$ be the fundamental discriminant (so $d\equiv 0,1 \pmod 4$). Then*

$$L(1,\chi_d) = \frac{2h}{w\sqrt{|d|}}$$

where h is the class number of $\mathbb{Q}(\sqrt{d})$ and w is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$. In particular, $w = 6$ when $d = -3$, $w = 4$ when $d = -4$, and hence

$$L(1, \chi_{-4}) = \frac{1}{4}, \quad L(1, \chi_{-3}) = \frac{1}{3\sqrt{3}}.$$

With Theorem 5.0.1 and all the above results, we can get the orbital decomposition of $I(f)$ (Theorem 5.0.3) for the case when weight $2k = 2$.

5.3 Local calculation: Non-Archimedean, $p \mid \mathrm{disc}(D)$

In the following two sections we will explicitly calculate the local integrals in (5.5).

When $p \mid \mathrm{disc}(D)$, with Proposition 5.1.4, we can assume $\chi_{-d}(p) \neq 1$, i.e. $E_p/\mathbb{Q}_p$ is non-split. Recall that $E = \mathbb{Q}(\sqrt{-d})$ (where $-d$ is a fundamental discriminant) corresponds to a Dirichlet character χ_{-d}. When $E_p/\mathbb{Q}_p$ is unramified and non-split, $\chi_{-d}(p) = -1$, the ramification index is $e = 1$, and the inertia degree is $f = 2$; when $E_p/\mathbb{Q}_p$ is ramified, $\chi_{-d}(p) = 0$, $e = 2$, $f = 1$.

In this case, $\pi_p = \sigma_{\delta_p}$ is the special representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ and $\pi'_p = \delta_p \circ N_{D_p}$ is a character of G'_p, where δ_p is an unramified character of $\mathbb{Q}_p^\times$ of order at most 2. We can take

$$\phi'_p = \phi''_p = \pi'_p(h_p)\phi_p, \quad \phi_p := \delta_p \circ N_{D_p}$$

with $B_p(\phi_p, \overline{\phi_p})$ defined to be 1.

Lemma 5.3.1. *When $p \mid \mathrm{disc}(D)$, $\pi_p = \sigma_{\delta_p}$, we have $\alpha_p(\phi'_p, \overline{\phi''_p}; \Omega_p) = 0$ unless $\Omega_p = \overline{\delta_p \circ N_{D_p}}$, in which case*

$$\alpha_p(\phi'_p, \overline{\phi''_p}; \overline{\delta_p \circ N_{D_p}}) = (1 - p^{-1}) \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times).$$

In particular,

$$\alpha_p(\phi'_p, \overline{\phi''_p}; \mathbf{1}) = (1-p^{-1}) \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times) \cdot \begin{cases} 1, & \chi_{-d}(p) = -1; \\ \frac{1}{2}(1-\varepsilon(\frac{1}{2}, (\pi_3)_p)), & \chi_{-d}(p) = 0. \end{cases}$$

Proof. We have that

$$\begin{aligned} &\int_{E_p^\times/\mathbb{Q}_p^\times} \frac{B_p(\pi'_p(t_p)\pi'_p(h_p)\phi_p, \overline{\pi'_p(h_p)\phi_p})}{B_p(\pi'_p(h_p)\phi_p, \overline{\pi'_p(h_p)\phi_p})} \Omega_p(t_p)\ dt_p \\ = &\int_{E_p^\times/\mathbb{Q}_p^\times} \frac{B_p(\delta_p(N_{D_p}(t_p))\pi'_p(h_p)\phi_p, \overline{\pi'_p(h_p)\phi_p})}{B_p(\pi'_p(h_p)\phi_p, \overline{\pi'_p(h_p)\phi_p})} \Omega_p(t_p)\ dt_p \\ = &\int_{E_p^\times/\mathbb{Q}_p^\times} \delta_p(N_{D_p}(t_p))\Omega_p(t_p)\ dt_p = \begin{cases} \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times), & \text{if } \Omega_p = \overline{\delta_p \circ N_{D_p}}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

So $\alpha_p(\phi'_p, \overline{\phi''_p}; \Omega_p) = 0$ unless $\Omega_p = \overline{\delta_p \circ N_{D_p}}$, in which case

$$\begin{aligned} &\alpha_p(\phi'_p, \overline{\phi''_p}; \overline{\delta_p \circ N_{D_p}}) := \frac{L_p(1,\eta)L_p(1,\pi,\mathrm{Ad})}{\zeta_{\mathbb{Q}_p}(2)L_p(\frac{1}{2}, \pi_E\otimes\Omega)} \int_{E_p^\times/\mathbb{Q}_p^\times} \frac{B_p(\pi'_p(t_p)\phi'_p, \overline{\phi''_p})}{B_p(\phi'_p, \overline{\phi''_p})} \overline{\delta_p(N_{D_p}(t_p)))}\ dt_p \\ = &\begin{cases} \frac{(1-(-1)p^{-1})^{-1}}{(1-p^{-2})^{-1}} \frac{L_p(1,\pi,\mathrm{Ad})}{L_p(\frac{1}{2},\pi_E\otimes\Omega)} \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times), & \text{if } E_p/\mathbb{Q}_p \text{ is unramified and non-split} \\ \frac{1}{(1-p^{-2})^{-1}} \frac{L_p(1,\pi,\mathrm{Ad})}{L_p(\frac{1}{2},\pi_E\otimes\Omega)} \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times), & \text{if } E_p/\mathbb{Q}_p \text{ is ramified} \end{cases} \\ = &\operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times)(1-p^{-e}) \frac{L_p(1,\pi,\mathrm{Ad})}{L_p(\frac{1}{2},\pi_E\otimes\Omega)} = \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times)(1-p^{-e}) \frac{1-p^{-f}}{1-p^{-2}} = \operatorname{vol}(E_p^\times/\mathbb{Q}_p^\times)(1-p^{-1}). \end{aligned}$$

Recall that for $\pi_p = \sigma_{\delta_p}$, one has $(\pi_E)_p = \sigma_{\delta_p \circ N_{E_p/\mathbb{Q}_p}}$ and

$$L_p(s,\pi,\xi) = (1-\delta_p(\varpi_p)\xi_p(\varpi_p)|\varpi_p|^{s+1/2})^{-1},$$

$$L_p(\frac{1}{2}, \pi_E\otimes\Omega) = (1-\delta_p(N_{E_p/\mathbb{Q}_p}(\varpi_{E_p}))\Omega_p(\varpi_{E_p})|\varpi_{E_p}|)^{-1} = (1-p^{-f})^{-1};$$

$$L_p(s,\pi,\mathrm{Ad}) = \zeta_p(s+1), \quad L_p(1,\pi,\mathrm{Ad}) = (1-p^{-2})^{-1}.$$

For the rest of the lemma, notice that δ_p is unramified and of order at most 2. So when $E_p/\mathbb{Q}_p$ is unramified and non-split, $N_{D_p}(\varpi_{E_p}) = N_{E_p/\mathbb{Q}_p}(p) = p^2$ and therefore $\delta_p \circ N_{D_p} = \mathbf{1}_{E_p^\times}$ always holds. But if $E_p/\mathbb{Q}_p$ is ramified we know $N_{D_p}(\varpi_{E_p}) = N_{E_p/\mathbb{Q}_p}(\varpi_{E_p}) = p$; so

$$\overline{\delta_p \circ N_{D_p}}(\varpi_{E_p}) = \delta_p(p),$$

and then $\alpha_p(\phi'_p, \overline{\phi''_p}; \mathbf{1}) = 0$ unless $\delta_p(p) = 1$. Notice that we have $\varepsilon(\frac{1}{2}, \pi_p) = -\delta_p(p)$ in this case. □

Next we study $\int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p)\, dh_p$.

Lemma 5.3.2. *For $p \mid \mathrm{disc}(D)$,*

$$\int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p)\, dh_p = \mathrm{vol}(E_p^\times \backslash D_p^\times).$$

Proof. When $p \mid \mathrm{disc}(D)$, $D_p = (\frac{-q,-N}{\mathbb{Q}_p})$ is a division algebra ($q = 1$ or 3). Proposition 2.3.1 shows $\mathcal{O}_p^\times = \{x \in D_p^\times : v_p(N_{D_p}(x)) = 0\}$. Clearly for any $h_p \in D_p$, $N_{D_p}(h_p^{-1}\gamma h_p) = N_{D_p}(\gamma) = 1$ for $\gamma = \gamma_0$ or γ_1. So the condition $h_p^{-1}\gamma h_p \in \mathcal{O}_p^\times$ is trivial, and therefore

$$\int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p)\, dh_p = \mathrm{vol}(E_p^\times \backslash D_p^\times).$$

□

The above two lemmas show that, when $\alpha_p \neq 0$,

$$\int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p)\alpha_p\, dh_p = (1 - p^{-1})\,\mathrm{vol}(\mathbb{Q}_p^\times \backslash E_p^\times)\,\mathrm{vol}(E_p^\times \backslash D_p^\times) = (1 - p^{-1})\,\mathrm{vol}(G'_p).$$

Notice that

$$N_{D_p}(\mathbb{Q}_p^\times) = (\mathbb{Q}_p^\times)^2 \quad \text{while } N_{D_p}(D_p^\times) = \mathbb{Q}_p^\times \quad \text{by Lemma 2.2.1.}$$

Since in this case $\mathcal{O}_p^\times = \{g_p \in D_p^\times : v_p(N_{D_p}(g_p)) = 0\}$, we can write

$$\begin{aligned} G'_p &= \mathbb{Q}_p^\times \backslash D_p^\times = \mathbb{Q}_p^\times \backslash \Big(\{g_p : 2 \mid N_{D_p}(g_p)\} \sqcup \{g_p : 2 \nmid N_{D_p}(g_p)\} \Big) \\ &= \mathbb{Q}_p^\times \backslash \mathbb{Q}_p^\times \mathcal{O}_p^\times \sqcup \mathbb{Q}_p^\times \backslash j \mathbb{Q}_p^\times \mathcal{O}_p^\times = K_p \sqcup j K_p. \end{aligned}$$

(Here $v_p(N_{D_p}(j)) = v_p(N) = 1$ for $j \in D_p = (\frac{-q,-N}{\mathbb{Q}_p})$.) And therefore

$$\mathrm{vol}(G'_p) = \mathrm{vol}(\mathbb{Q}_p^\times \backslash D_p^\times) = 2\,\mathrm{vol}(K_p).$$

In conclusion, for $p \mid N$ and $\chi_{-d}(p) \neq 1$,

$$\alpha_p \int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p)\, dh_p = (1-\chi_{-d}(p))(1-p^{-1})\,\mathrm{vol}(K_p) \cdot \begin{cases} 1, & \chi_{-d}(p) = -1; \\ (1 - \varepsilon(\frac{1}{2}, (\pi_3)_p)), & \chi_{-d}(p) = 0. \end{cases} \tag{5.6}$$

5.4 Local calculation: Non-Archimedean, $p \nmid \mathrm{disc}(D)$

In this case we fix an isomorphism $D_p \cong M(2, \mathbb{Q}_p)$ and take the maximal order $\mathcal{O}_p = M(2, \mathbb{Z}_p)$.

Lemma 5.4.1. *Under the natural embedding*

$$E_p = \mathbb{Q}_p(\gamma) \hookrightarrow D_p \cong M(2, \mathbb{Q}_p),$$

we have $\mathcal{O}_{E_p} = E_p \cap \mathcal{O}_p$.

The isomorphism $D_p \cong M(2, \mathbb{Q}_p)$ and the proof of the above lemma can be found later case by case in this section.

When $p \nmid \mathrm{disc}(D)$, $\pi'_p \cong \pi_p$ and they are spherical. We can write

$$\pi_p = \pi(\mu_p, \mu_p^{-1}) = \mathrm{Ind}_{B_p}^{G_p}(\mu_p \otimes \mu_p^{-1}),$$

where μ_p is an unramified quasicharacter of $\mathbb{Q}_p^\times$ such that $\mu_p^2 \neq |\cdot|$. A spherical vector in the principal series representation can be given as a constant multiple of

$$\phi_p(\begin{pmatrix} a & b \\ & d \end{pmatrix} k_p) = \mu_p\left(\frac{a}{d}\right)\left|\frac{a}{d}\right|^{1/2} \qquad \text{for any } k_p \in \mathrm{GL}(2, \mathbb{Z}_p).$$

We can fix $\phi'_p = \phi''_p = \pi_p(h_p)\phi_p$.

Notice that here h_p satisfies $h_p^{-1}\gamma h_p \in K_p$ for all p. (Our choice of γ has $N_D(\gamma) = 1$, so $h_p^{-1}\gamma h_p \in \mathrm{GL}(2, \mathbb{Z}_p)$.) For $\phi'_p = \phi''_p = \pi_p(h_p)\phi_p$ we care about

- $\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi'_p, \overline{\phi''_p})\Omega_p(t_p)\ dt_p$ for $h_p^{-1}\gamma h_p \in K_p$; and
- $\int_{E_p^\times\backslash D_p^\times} \mathbf{1}_{K_p}(h_p^{-1}\gamma h_p)\ dh_p$.

For the second integral we have the following lemma.

Lemma 5.4.2. *Assume that $p \nmid \mathrm{disc}(D)$, $\gamma = \gamma_0 = \sqrt{-1}$ or $= \gamma_1 = \frac{1+\sqrt{-3}}{2}$.*

(1) For $h \in D_p^\times$,

$$\textit{(i)} \quad h \in E_p^\times \mathrm{GL}(2, \mathbb{Z}_p) \quad \textit{if and only if} \quad \textit{(ii)} \quad h^{-1}\gamma h \in \mathrm{GL}(2, \mathbb{Z}_p).$$

(2)

$$\int_{E_p^\times\backslash D_p^\times} \mathbf{1}_{K_p}(h_p^{-1}\gamma h_p)\ dh_p = \frac{\mathrm{vol}(\mathcal{O}_p^\times)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times)}.$$

Proof. The second statement can be shown with Lemma 2.4.1:

$$\int_{E_p^\times\backslash D_p^\times} \mathbf{1}_{K_p}(h_p^{-1}\gamma h_p)\ dh_p = \int_{E_p^\times\backslash E_p^\times\mathcal{O}_p^\times} dh_p = \mathrm{vol}((\mathcal{O}_p^\times\cap E_p^\times)\backslash\mathcal{O}_p^\times) = \frac{\mathrm{vol}(\mathcal{O}_p^\times)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times)}.$$

For the first statement, (i)⇒(ii) is obvious since $\gamma\in\mathrm{GL}(2,\mathbb{Z}_p)$ and γ commutes with $E_p^\times$, and (ii)⇒(i) will be proved case by case later in this section. □

For the first integral we have

Lemma 5.4.3. *For $h_p\in E_p^\times\,\mathrm{GL}(2,\mathbb{Z}_p)$, $\phi_p'=\phi_p''=\pi_p(h_p)\phi_p$, ϕ_p a unit spherical vector in π_p, we have*

$$\alpha(\phi_p',\overline{\phi_p''};\Omega_p) = \begin{cases}\frac{\mathrm{vol}(\mathcal{O}_{E_p}^\times)}{\mathrm{vol}(\mathbb{Z}_p^\times)}, & \Omega_p \text{ unramified;}\\ 0, & \Omega_p \text{ ramified.}\end{cases}$$

Proof. Write $h=t_hu_h$ with $t_h\in E_p^\times$, $u_h\in\mathcal{O}_p^\times$. Then $h^{-1}th=u_h^{-1}t_h^{-1}tt_hu_h=u_h^{-1}tu_h$. Recall that the $\mathrm{GL}(2,\mathbb{Q}_p)$-invariant bilinear form on $\pi_p\otimes\tilde\pi_p$ can be defined by

$$B_p(\phi,\phi') := \int_{\mathrm{GL}(2,\mathbb{Z}_p)} \phi(k)\phi'(k)\ dk, \quad \phi\in\pi_p, \phi'\in\tilde\pi_p.$$

Then

$$\begin{aligned} B_p(\pi_p(t_p)\phi_p',\overline{\phi_p''}) &= B_p(\pi_p(h^{-1}th)\phi_p,\overline{\phi_p}) = \int_{\mathcal{O}_p^\times}\pi_p(h^{-1}th)\phi_p(k)\overline{\phi_p(k)}\ dk\\ &= \int_{\mathcal{O}_p^\times}\phi_p(kh^{-1}th)\overline{\phi_p(k)}\ dk = \int_{\mathcal{O}_p^\times}\phi_p(ku_h^{-1}tu_h)\overline{\phi_p(k)}\ dk\\ &= \int_{\mathcal{O}_p^\times}\phi_p(ku_h^{-1}t)\overline{\phi_p(ku_h^{-1})}\ d(ku_h^{-1}) = \int_{\mathcal{O}_p^\times}\phi_p(kt)\overline{\phi_p(k)}\ dk = B_p(\pi_p(t)\phi_p,\overline{\phi_p}). \end{aligned}$$

So

$$\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p',\overline{\phi_p''})\Omega_p(t_p)\ dt_p = \int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p,\overline{\phi_p})\Omega_p(t_p)\ dt_p$$

and therefore, by Theorem 5.2.1,

$$\alpha(\phi'_p, \overline{\phi''_p}; \Omega_p) = \alpha(\phi_p, \overline{\phi_p}; \Omega_p) = \mathrm{vol}(\mathcal{O}_{E_p}^{\times}) / \mathrm{vol}(\mathbb{Z}_p^{\times})$$

when Ω_p is unramified and $E_p/\mathbb{Q}_p$ is unramified.

The rest of this lemma (when Ω_p is ramified or $E_p/\mathbb{Q}_p$ is ramified) is proved later in this section. □

When $E_p/\mathbb{Q}_p$ is split

For $D(\mathbb{Q}_p) = \left(\frac{-1,-N}{\mathbb{Q}_p}\right)$ or $\left(\frac{-3,-N}{\mathbb{Q}_p}\right)$ we fix the isomorphism $D(\mathbb{Q}_p) \cong M(2, \mathbb{Q}_p)$ with

$$1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} \alpha & N\beta \\ \beta & -\alpha \end{pmatrix}, \quad j \mapsto \begin{pmatrix} & -N \\ 1 & \end{pmatrix}, \quad k \mapsto \begin{pmatrix} N\beta & -N\alpha \\ -\alpha & -N\beta \end{pmatrix}$$

where (α, β) is a solution of $\alpha^2 + N\beta^2 = -q$ in $\mathbb{Q}_p$, $q = 1$ or 3. When p splits in E, $-q$ is a square in $\mathbb{Q}_p$ and we can take

$$i \mapsto \begin{pmatrix} \sqrt{-q} & \\ & -\sqrt{-q} \end{pmatrix}$$

where $\sqrt{-q}$ is a fixed solution of $x^2 = -q$ in $\mathbb{Q}_p$. Then $E_p = \mathbb{Q}_p \oplus \mathbb{Q}_p$ is embedded in $M(2, \mathbb{Q}_p)$ as the set of all the diagonal matrices. With this embedding Lemma 5.4.1 is obvious.

Proof of Lemma 5.4.2 when $E_p/\mathbb{Q}_p$ is split.

The Iwasawa decomposition gives that

$$h = c_h \begin{pmatrix} p^r & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x_h \\ & 1 \end{pmatrix} k_h, \quad c_h \in \mathbb{Q}_p^\times,\ r \in \mathbb{Z},\ x_h \in \mathbb{Q}_p,\ k_h \in \mathrm{GL}(2, \mathbb{Z}_p).$$

Then its inverse is

$$h^{-1} = (c_h p^r)^{-1} k_h^{-1} \begin{pmatrix} 1 & -x_h p^r \\ & p^r \end{pmatrix}.$$

Let $\bar{t} = x - y i_D$ for $t = x + y i_D \in E_p = \mathbb{Q}_p(i_D)$. Then, for $\gamma = i_D = \sqrt{-1}$ (or $= \frac{1+i_D}{2} = \frac{1+\sqrt{-3}}{2}$), γ can be embedded in $\mathrm{GL}(2, \mathbb{Q}_p)$ as $\begin{pmatrix} \gamma & \\ & \bar{\gamma} \end{pmatrix}$, and

$$h^{-1}\gamma h = (c_h p^r)^{-1} k_h^{-1} \begin{pmatrix} 1 & -x_h p^r \\ & p^r \end{pmatrix} \begin{pmatrix} \gamma & \\ & \bar{\gamma} \end{pmatrix} c_h \begin{pmatrix} p^r & x_h p^r \\ & 1 \end{pmatrix} k_h = k_h^{-1} \begin{pmatrix} \gamma & (\gamma - \bar{\gamma})\, x_h \\ & \bar{\gamma} \end{pmatrix} k_h.$$

Therefore $h^{-1}\gamma h \in \mathrm{GL}(2, \mathbb{Z}_p)$ if and only if $(\gamma - \bar{\gamma})\, x_h \in \mathbb{Z}_p$. This implies $v_p(x_h) \geq 0$ when $p \neq 2$ (or $p \neq 3$ respectively). Then $h \in t_h \begin{pmatrix} p^r & \\ & 1 \end{pmatrix} \mathrm{GL}(2, \mathbb{Z}_p)$. This proves the statement when $E_p/\mathbb{Q}_p$ is split. □

Proof of Lemma 5.4.3 when $E_p/\mathbb{Q}_p$ is split.

Now $E_p^\times = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Notice that $E_p^\times/\mathbb{Q}_p^\times \cong \{(x, 1) : x \in \mathbb{Q}_p^\times\}$, We can write

$$\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p, \overline{\phi_p})\Omega_p(t_p)\ dt_p = \int_{\mathbb{Q}_p^\times} B_p(\pi_p \begin{pmatrix} x & \\ & 1 \end{pmatrix} \phi_p, \overline{\phi_p})\Omega_p(x, 1)\ d^\times x.$$

For $\Omega_p(t_1,t_2)=\nu_p(t_1)\nu_p^{-1}(t_2)$ we have

$$\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p,\overline{\phi_p})\Omega_p(t_p)\ dt_p = \sum_{r\in\mathbb{Z}}\int_{v_p(x)=r} B_p(\pi_p\begin{pmatrix} x & \\ & 1\end{pmatrix}\phi_p,\overline{\phi_p})\Omega_p(x,1)\ d^\times x$$

$$=\sum_{r\in\mathbb{Z}}\int_{u\in\mathbb{Z}_p^\times} B_p(\pi_p\begin{pmatrix} p^r & \\ & 1\end{pmatrix}\begin{pmatrix} u & \\ & 1\end{pmatrix}\phi_p,\overline{\phi_p})\nu_p(p^r u)\ du$$

$$=\sum_{r\in\mathbb{Z}} B_p(\pi_p\begin{pmatrix} p^r & \\ & 1\end{pmatrix}\phi_p,\overline{\phi_p})\nu_p(p)^r\int_{\mathbb{Z}_p^\times}\nu_p(u)\ du = 0 \quad \text{when } \nu_p \text{ is ramified.}$$

$\square$

When $E_p/\mathbb{Q}_p$ is not split

As before, we fix the isomorphism $D(\mathbb{Q}_p)\cong M(2,\mathbb{Q}_p)$ with

$$i\mapsto\begin{pmatrix}\alpha & N\beta\\ \beta & -\alpha\end{pmatrix},\quad j\mapsto\begin{pmatrix} & -N\\ 1 & \end{pmatrix}$$

where (α,β) is a solution of $\alpha^2+N\beta^2=-q$ in $\mathbb{Q}_p$, $q=1$ or 3. In particular we can choose $\alpha,\beta\in\mathbb{Z}_p$:

When $p\neq 2$, we can find a solution satisfying $\alpha,\beta\in\mathbb{Z}_p^\times\cup\{0\}$. With Hensel's Lemma we only need to prove $\alpha^2+N\beta^2\equiv -q \pmod p$ has a solution in $\mathbb{Z}/p\mathbb{Z}$, that means, there is at least one quadratic residue (including 0) mod p in

$$\{-q,-q-1^2N,-q-2^2N,\ldots,-q-((p-1)/2)^2N\},$$

which is a set of $\frac{p+1}{2}$ congruency classes mod p (here p,q,N are relatively prime to each other). If not, the $\frac{p+1}{2}$ quadratic residues can only be found in the complement of the above set, which has $\frac{p-1}{2}$ numbers, a contradiction.

When $p = q = 3 \nmid N$, N is the product of an odd number of distinct primes of the form $3n+2$. Then $-3-N \equiv 1 \pmod 3$ so $-3-N$ is a square in $\mathbb{Q}_3^\times$. We have a solution

$$\alpha^2 = -3-N, \quad \beta = 1.$$

When $p = 2 \nmid N$, actually $\alpha^2 + N\beta^2 = -q$ has no solution in $\mathbb{Z}_2^\times \cup \{0\}$, But one can get a solution such that one of α and β is in $\mathbb{Z}_2^\times$. For example, when $q = 1$, according to Corollary 5.0.1, N is the product of an odd number of distinct primes of the form $4n+3$ and we have the following solution:

$$\begin{aligned} \alpha &= 2, \quad \beta^2 = 1 - 8N^{-1}(\frac{N-3}{8}+1) \quad \text{for } N \equiv 3 \pmod 8, \\ \alpha &= 4, \quad \beta^2 = 1 - 8N^{-1}(\frac{N-7}{8}+3) \quad \text{for } N \equiv 7 \pmod 8; \end{aligned}$$

when $q = 3$, N is the product of an odd number of distinct primes of the form $6n+5$:

$$\alpha^2 = 1 - 8\frac{N+1}{2}, \quad \beta = 2.$$

Recall that a unit $u \in \mathbb{Z}_2^\times$ is a square if and only if $u \equiv 1 \pmod 8$.

Proof of Lemma 5.4.1. When $E_p/\mathbb{Q}_p$ is non-split, the isomorphism $D(\mathbb{Q}_p) \cong M(2, \mathbb{Q}_p)$ maps $\lambda + \mu\sqrt{-q}$ to

$$\begin{pmatrix} \lambda + \alpha\mu & N\beta\mu \\ \beta\mu & \lambda - \alpha\mu \end{pmatrix}.$$

Then $\mathcal{O}_{E_p} \supseteq E_p \cap \mathcal{O}_p$ is obvious since the determinant of a matrix in $M(2, \mathbb{Z}_p)$ is always in $\mathbb{Z}_p$ and $\det \begin{pmatrix} \lambda + \alpha\mu & N\beta\mu \\ \beta\mu & \lambda - \alpha\mu \end{pmatrix} = \lambda^2 + q\mu^2$ is exactly the norm over $E_p/\mathbb{Q}_p$ of $\lambda + \mu\sqrt{-q}$.

Now assume $\lambda + \mu\sqrt{-q} \in \mathcal{O}_{E_p}^\times$. Then with our choice of α, β and the next lemma, both λ

and μ are in $\mathbb{Z}_p$ for most cases, and therefore $\begin{pmatrix} \lambda+\alpha\mu & N\beta\mu \\ \beta\mu & \lambda-\alpha\mu \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p)$. This also holds in the exceptional case ($q=3$, $p=2$) since

$$\frac{1}{2}+\frac{1}{2}\sqrt{-3} \mapsto \begin{pmatrix} \frac{1+\alpha}{2} & \frac{N\beta}{2} \\ \frac{\beta}{2} & \frac{1-\alpha}{2} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_2) \quad \text{(recall that } \alpha\in\mathbb{Z}_2^\times,\ \beta=2).$$

□

Lemma 5.4.4. *For $\gamma=\sqrt{-1}$ or $\frac{1}{2}(1+\sqrt{-3})$, $\mathcal{O}_{\mathbb{Q}_p(\gamma)}=\mathbb{Z}_p[\gamma]$. Equivalently one can say, when $E_p=\mathbb{Q}_p(\sqrt{-q})$ ($q=1$ or 3) is non-split, $\lambda+\mu\sqrt{-q}\in\mathcal{O}_{E_p}^\times$ if and only if*

- *$\lambda,\mu\in\mathbb{Z}_p$ and $\lambda^2+q\mu^2\in\mathbb{Z}_p^\times$; or*
- *$\lambda,\mu\in\frac{1}{2}+\mathbb{Z}_2$ for $q=3$, $p=2$.*

Proof. With the normalized valuation on E_p, $\lambda+\mu\sqrt{-q}\in\mathcal{O}_{E_p}$ if and only if $\lambda^2+q\mu^2\in\mathbb{Z}_p$. When $p\neq 2$, both λ and μ are in $\mathbb{Z}_p$ because $-q$ is not a quadratic residue mod p. When $p=2$, assume that λ and μ are not together in $\mathbb{Z}_p$. Then $\lambda^2+q\mu^2\in\mathbb{Z}_2$ implies $v_2(\lambda)=v_2(\mu)=-r<0$ for some r. Say $\lambda=2^{-r}u$, $\mu=2^{-r}v$ for some $u,v\in\mathbb{Z}_2^\times$. Then

$$\lambda^2+q\mu^2=2^{-2r}(u^2+qv^2)\in\mathbb{Z}_2.$$

Recall that $u^2,v^2\equiv 1 \pmod 8$. Then $u^2+qv^2\equiv 1+q \pmod 8$, i.e.

$$v_2(\lambda^2+q\mu^2)=-2r+v_2(u^2+qv^2)=\begin{cases}-2r+1, & q=1;\\ -2r+2, & q=3.\end{cases}$$

The only possibility such that $v_2(\lambda^2+q\mu^2)\geq 0$ holds is that $q=3$ and $r=1$, which implies $\lambda,\mu\in\frac{1}{2}+\mathbb{Z}_2$. □

Now we prove Lemma 5.4.2 and 5.4.3.

Proof of Lemma 5.4.2 when $E_p/\mathbb{Q}_p$ is non-split (i.e. when $\chi_{-d}(p) \neq 1$).

We claim that, $h^{-1}\gamma h \in \mathcal{O}_p^\times$ if and only if

$$h \in Z_p\mathcal{O}_p^\times \sqcup \begin{cases} \emptyset, & p \text{ inert in } E = \mathbb{Q}(\gamma); \\ \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix} Z_p\mathcal{O}_p^\times, & \gamma = \sqrt{-1},\ p = 2; \\ \begin{pmatrix} 3 & \alpha\beta^{-1} \\ & 1 \end{pmatrix} Z_p\mathcal{O}_p^\times, & \gamma = \frac{1+\sqrt{-3}}{2},\ p = 3. \end{cases}$$

Notice that

$$E_p^\times = \begin{cases} \sqcup_{r\in\mathbb{Z}}\, p^r\mathcal{O}_{E_p}^\times = \mathbb{Q}_p^\times\mathcal{O}_{E_p}^\times, & p \text{ inert}, \\ \mathbb{Q}_p^\times\mathcal{O}_{E_p}^\times \sqcup \varpi_{E_p}\mathbb{Q}_p^\times\mathcal{O}_{E_p}^\times, & p \text{ ramified}, \end{cases}$$

and $\mathcal{O}_{E_p}^\times \subset \mathcal{O}_p^\times$ by Lemma 5.4.1. If the above statement holds, we only need to show

$$\varpi_{E_2} \in \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix} \mathrm{GL}(2,\mathbb{Z}_2), \quad \varpi_{E_3} \in \begin{pmatrix} 3 & \alpha\beta^{-1} \\ & 1 \end{pmatrix} \mathrm{GL}(2,\mathbb{Z}_3)$$

just for the ramified case, which can be easily checked by the following calculation, taking $\varpi_{E_2} = 1+\sqrt{-1}$, $\varpi_{E_3} = \sqrt{-3}$:

$$\begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix}^{-1} \varpi_{E_2} = \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+\alpha & N\beta \\ \beta & 1-\alpha \end{pmatrix} = \begin{pmatrix} \frac{1-\beta+\alpha}{2} & \frac{N\beta-1+\alpha}{2} \\ \beta & 1-\alpha \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_2);$$

$$\begin{pmatrix} 3 & \alpha\beta^{-1} \\ & 1 \end{pmatrix}^{-1} \varpi_{E_3} = \begin{pmatrix} 3 & \alpha\beta^{-1} \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & N\beta \\ \beta & -\alpha \end{pmatrix} = \begin{pmatrix} & -\beta^{-1} \\ \beta & -\alpha \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_3).$$

Now we prove the claim. The sufficiency is easy to check. To show the necessity, assume that $h^{-1}\gamma h \in \mathrm{GL}(2,\mathbb{Z}_p)$ with $h = \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_h$ for some $k_h \in Z_p\,\mathrm{GL}(2,\mathbb{Z}_p)$ using Iwasawa decomposition.

When $\gamma = \sqrt{-1}$, $h^{-1}\gamma h \in \mathrm{GL}(2,\mathbb{Z}_p)$ implies

$$\begin{pmatrix} y & x \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & N\beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} = \begin{pmatrix} X & -(X^2+1)Y^{-1} \\ Y & -X \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p),$$

where $X = \alpha - \beta x$, $Y = \beta y$. Our assumption on α and β says $\alpha \in \mathbb{Z}_p$, $\beta \in \mathbb{Z}_p^\times$, in which case

$$X = \alpha - \beta x \in \mathbb{Z}_p \Leftrightarrow x \in \mathbb{Z}_p; \quad Y = \beta y \in \mathbb{Z}_p \Leftrightarrow y \in \mathbb{Z}_p.$$

We will only consider the case when $h \notin \mathrm{GL}(2,\mathbb{Z}_p)$, i.e. when

$$v_p(X^2+1) \geq v_p(Y) > 0.$$

This cannot happen when $\chi_{-4}(p) = -1$ since -1 is not a quadratic residue mod p and $v_p(X^2+1) = 0$. But if $\chi_{-4}(p) = 0$ i.e. $p = 2$, we have

$$v_2(X^2+1) > 0 \Leftrightarrow X \in \mathbb{Z}_2^\times \Leftrightarrow X^2 \in 1 + 8\mathbb{Z}_2 \Leftrightarrow v_2(X^2+1) = 1$$

and then $v_2(Y) = 1$. So (recall that in this case $v_2(\alpha) \geq 1$) $x \in \mathbb{Z}_2^\times$ and $y \in 2\mathbb{Z}_2^\times$, which is equivalent to say that $\begin{pmatrix} y & x \\ & 1 \end{pmatrix} \in \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix} \mathrm{GL}(2,\mathbb{Z}_2)$.

When $\gamma = \frac{1+\sqrt{-3}}{2}$, $h^{-1}\gamma h \in \mathrm{GL}(2,\mathbb{Z}_p)$ implies

$$\begin{pmatrix} y & x \\ & 1 \end{pmatrix}^{-1} \frac{1}{2}\begin{pmatrix} 1+\alpha & N\beta \\ \beta & 1-\alpha \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+X & -(X^2+3)Y^{-1} \\ Y & 1-X \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p)$$

where $X = \alpha - \beta x$, $Y = \beta y$. When $p \neq 2$, we can do similar deduction as above, considering only the case when $h \notin \mathrm{GL}(2,\mathbb{Z}_p)$, and get $v_p(X^2+3) \geq v_p(Y) > 0$ and $p = 3$. But we know that

$$v_3(X^2+3) > 0 \Leftrightarrow X \in 3\mathbb{Z}_3 \Leftrightarrow v_3(X^2+3) = 1$$

and then $v_3(Y) = 1$. Recall that we choose $\alpha, \beta \in \mathbb{Z}_3^\times$. Therefore we have $x \in \alpha\beta^{-1} + 3\mathbb{Z}_3$, $y \in 3\mathbb{Z}_3^\times$, which is equivalent to say that $\begin{pmatrix} y & x \\ & 1 \end{pmatrix} \in \begin{pmatrix} 3 & \alpha\beta^{-1} \\ & 1 \end{pmatrix}\mathrm{GL}(2,\mathbb{Z}_3)$.

When $\gamma = \frac{1+\sqrt{-3}}{2}$ and $p = 2$, we can choose $\alpha \in \mathbb{Z}_2^\times$ and $\beta = 2$. Then the condition $h^{-1}\gamma h \in \mathrm{GL}(2,\mathbb{Z}_2)$ becomes

$$\begin{pmatrix} y & x \\ & 1 \end{pmatrix}^{-1} \frac{1}{2}\begin{pmatrix} 1+\alpha & 2N \\ 2 & 1-\alpha \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\alpha}{2} - x & -\frac{(\alpha-2x)^2+3}{4y} \\ y & \frac{1-\alpha}{2} + x \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p).$$

We still can get $x, y \in \mathbb{Z}_2$, but $\frac{(\alpha-2x)^2+3}{4y} \in \mathbb{Z}_2$ implies $y \in \mathbb{Z}_2^\times$, noticing that $(\alpha-2x)^2 \in 1+8\mathbb{Z}_2$ for $\alpha \in \mathbb{Z}_2^\times$. So in this case we can only have $\begin{pmatrix} y & x \\ & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_2)$. □

Proof of Lemma 5.4.3 when $E_p/\mathbb{Q}_p$ is unramified and non-split.

Assume that Ω_p has conductor c. In this case $E_p^\times = \sqcup_{r\in\mathbb{Z}} p^r \mathcal{O}_{E_p}^\times = \mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times$. Lemma 2.4.1

implies, for any integrable function f defined on $E_p^\times/\mathbb{Q}_p^\times$,

$$\int_{E_p^\times/\mathbb{Q}_p^\times} f(t)\ dt = \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} f(t)\ dt.$$

Recall that $\mathcal{O}_{E_p} = E_p \cap \mathcal{O}_p$ and ϕ_p is spherical. So we have

$$\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p, \overline{\phi_p})\Omega_p(t_p)\ dt_p = \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} B_p(\pi_p(t_p)\phi_p, \overline{\phi_p})\Omega_p(t_p)\ dt_p$$

$$= B_p(\phi_p, \overline{\phi_p}) \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} \Omega_p(t_p)\ dt_p = \begin{cases} B_p(\phi_p, \overline{\phi_p}) \operatorname{vol}(\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times), & c = 0; \\ 0, & c > 0. \end{cases}$$

□

Proof of Lemma 5.4.3 when $E_p/\mathbb{Q}_p$ is ramified.

Now $E_p^\times = \mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times \sqcup \varpi_{E_p} \mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times$. Then for any integrable function f on $E_p^\times/\mathbb{Q}_p^\times$,

$$\int_{E_p^\times/\mathbb{Q}_p^\times} f(t)\ dt = \int_{\mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times/\mathbb{Q}_p^\times} f(t)\ dt + \int_{\mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times/\mathbb{Q}_p^\times} f(\varpi_{E_p} t)\ dt.$$

By Lemma 2.4.1

$$\int_{\mathbb{Q}_p^\times \mathcal{O}_{E_p}^\times/\mathbb{Q}_p^\times} f(t)\ dt = \int_{\mathcal{O}_{E_p}^\times/(\mathbb{Q}_p^\times \cap \mathcal{O}_{E_p}^\times)} f(t)\ dt = \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} f(t)\ dt,$$

and hence we can write

$$\int_{E_p^\times/\mathbb{Q}_p^\times} f(t)\ dt = \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} (f(t) + f(\varpi_{E_p} t))\ dt.$$

Recall that $\mathcal{O}_{E_p}^\times \subset \mathcal{O}_p^\times$ by Lemma 5.4.1. Now we have

$$\begin{aligned}
&\int_{E_p^\times/\mathbb{Q}_p^\times} B_p(\pi_p(t_p)\phi_p, \overline{\phi_p})\Omega_p(t_p)\ dt_p \\
=&\int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} B_p(\pi_p(t_p)\phi_p, \overline{\phi_p})\Omega_p(t_p)\ dt_p + \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} B_p(\pi_p(\varpi_{E_p}t_p)\phi_p, \overline{\phi_p})\Omega_p(\varpi_{E_p}t_p)\ dt_p \\
=&B_p(\phi_p, \overline{\phi_p}) \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} \Omega_p(t_p)\ dt_p + B_p(\pi_p(\varpi_{E_p})\phi_p, \overline{\phi_p})\Omega_p(\varpi_{E_p}) \int_{\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times} \Omega_p(t_p)\ dt_p \\
=&\begin{cases} \mathrm{vol}(\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times)\Big(B_p(\phi_p, \overline{\phi_p}) + B_p(\pi_p(\varpi_{E_p})\phi_p, \overline{\phi_p})\Omega_p(\varpi_{E_p})\Big), & \Omega_p \text{ unramified;} \\ 0, & \Omega_p \text{ ramified.} \end{cases}
\end{aligned}$$

By the proof of Lemma 5.4.2, in this case, $\varpi_{E_p} \in \begin{pmatrix} p & * \\ & 1 \end{pmatrix} \mathcal{O}_p^\times$. The Macdonald formula (see [Bum97] Theorem 4.6.6) implies that

$$\begin{aligned}
\frac{B_p\left(\pi_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} \phi_p, \overline{\phi_p}\right)}{B_p(\phi_p, \overline{\phi_p})} =& \frac{1}{1+p^{-1}} p^{-1/2} \left(\mu_p(p) \frac{1 - p^{-1}\mu_p(p)^{-2}}{1-\mu_p(p)^{-2}} + \mu_p(p)^{-1} \frac{1-p^{-1}\mu_p(p)^2}{1-\mu_p(p)^2} \right) \\
=& \frac{p^{-1/2}}{1+p^{-1}} (\mu_p(p) + \mu_p(p)^{-1}).
\end{aligned}$$

For unramified Ω_p,

$$\begin{aligned}
\alpha_p(\phi'_p, \overline{\phi''_p}; \Omega_p) :=& \frac{L_p(1,\eta)L_p(1,\pi,\mathrm{Ad})}{\zeta_{\mathbb{Q}_p}(2)L_p(\frac{1}{2}, \pi_E \otimes \Omega)} \int_{E_p^\times/\mathbb{Q}_p^\times} \frac{B_p(\pi'_p(t_p)\phi'_p, \overline{\phi''_p})}{B_p(\phi'_p, \overline{\phi''_p})} \Omega_p(t_p)\ dt_p \\
=& \frac{L_p(1,\eta)L_p(1,\pi,\mathrm{Ad})}{\zeta_{\mathbb{Q}_p}(2)L_p(\frac{1}{2}, \pi_E \otimes \Omega)} \mathrm{vol}(\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times) \left(1 + \frac{p^{-1/2}}{1+p^{-1}} (\mu_p(p) + \mu_p(p)^{-1}) \Omega_p(\varpi_{E_p}) \right).
\end{aligned}$$

Here $L_p(1,\eta)=1$, $\zeta_{\mathbb{Q}_p}(2)=(1-p^{-2})^{-1}$,

$$\begin{aligned}L_p(1,\pi(\mu_1,\mu_2),\mathrm{Ad}) &= \frac{L_p(1,\mathbf{1})}{(1-\mu_1(p)\mu_2^{-1}(p)p^{-1})(1-\mu_1^{-1}(p)\mu_2(p)p^{-1})}\\ &= \frac{(1-p^{-1})^{-1}}{(1-\mu^2(p)p^{-1})(1-\mu^{-2}(p)p^{-1})}.\end{aligned}$$

For $\pi_p=\pi(\mu_p,\mu_p^{-1})$, one has $e=2$, $f=1$, $(\pi_E)_p=\pi(\mu_p\circ N_{E_p/F_p},\mu_p^{-1}\circ N_{E_p/F_p})$ and

$$\begin{aligned}L_p(\frac{1}{2},\pi_E\otimes\Omega) =&(1-\mu_p(N_{E_p/F_p}(\varpi))\Omega_p(\varpi)q^{-1/2})^{-1}(1-\mu_p^{-1}(N_{E_p/F_p}(\varpi))\Omega_p(\varpi)q^{-1/2})^{-1}\\ =&(1-\mu_p(p)\Omega_p(\varpi)p^{-1/2})^{-1}(1-\mu_p^{-1}(p)\Omega_p(\varpi)p^{-1/2})^{-1}\quad (\text{here } \varpi=\varpi_{E_p}).\end{aligned}$$

Then

$$\begin{aligned}\frac{\alpha_p(\phi_p',\overline{\phi_p''};\Omega_p)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times)} = (1+p^{-1})&\frac{(1-\mu^2(p)p^{-1})^{-1}(1-\mu^{-2}(p)p^{-1})^{-1}}{(1-\mu_p(p)\Omega_p(\varpi)p^{-1/2})^{-1}(1-\mu_p^{-1}(p)\Omega_p(\varpi)p^{-1/2})^{-1}}\\ &\cdot\left(1+\frac{p^{-1/2}}{1+p^{-1}}(\mu_p(p)+\mu_p(p)^{-1})\Omega_p(\varpi)\right).\end{aligned}$$

When Ω is unramified everywhere, $\Omega_p(\varpi)^2=1$ and

$$\begin{aligned}&\frac{\alpha_p(\phi_p',\overline{\phi_p''};\Omega_p)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times/\mathbb{Z}_p^\times)}\\ =&\frac{1+p^{-1}}{(1+\mu_p(p)\Omega_p(\varpi)p^{-1/2})(1+\mu_p^{-1}(p)\Omega_p(\varpi)p^{-1/2})}\left(1+\frac{p^{-1/2}}{1+p^{-1}}(\mu_p(p)+\mu_p(p)^{-1})\Omega_p(\varpi)\right)\\ =&\frac{1+p^{-1}+p^{-1/2}(\mu_p(p)+\mu_p(p)^{-1})\Omega_p(\varpi)}{(1+\mu_p(p)\Omega_p(\varpi)p^{-1/2})(1+\mu_p^{-1}(p)\Omega_p(\varpi)p^{-1/2})}=1.\end{aligned}$$

□

5.5 Compatibility of two nontrivial orbits

When N has no prime factor $\equiv 1 \pmod 4$ and none $\equiv 1 \pmod 3$ (i.e. all prime factors of N are either 2, 3, or $\equiv 11 \pmod{12}$), γ_0 and γ_1 both appear in the quaternion algebra D. We applied different presentations of D in this case, which, when $p \nmid N$, lead to local maximal orders that differ by conjugation by an element in $\mathrm{GL}(2, \mathbb{Q}_p)$. But the spherical vector $\phi_{3,p}$ in the definition of the distribution $I(f)$ should be determined by a fixed maximal order of D, independent of the different ways to represent D.

Say $D(F) = \left(\frac{-1,-N}{F}\right)$ and $D'(F) = \left(\frac{-3,-N}{F}\right)$. For the case at hand, by Lemma 2.1.2, both $D(\mathbb{Q})$ and $D'(\mathbb{Q})$ have discriminant N; and by Theorem 2.1.1 they are isomorphic to each other. More explicitly the isomorphism $D' \xrightarrow{\sim} D$ can be given by $i' \mapsto xi + yk$, $j' \mapsto j$ with $x^2 + Ny^2 = 3$ for some $x, y \in \mathbb{Q}$.

We consider the choice of ϕ_3 in the definition of the distribution $I(f)$ (see (4.1)):

$$\phi_3 \in \mathbb{C}X_3^{2k-2} \otimes (\pi'_{3,\mathrm{fin}})^{K_{\mathrm{fin}}}$$

is a new-line vector in π'_3 as defined in Lemma 3.1.3. It depends on the definition of X_3^{2k-2} (i.e. the way we construct $\pi'_{3,\infty}$, which is trivial when $2k = 2$) and of $K_{\mathrm{fin}} = \prod_{p<\infty} K_p$ (i.e. the choice of maximal orders $\mathcal{O}_p$ at every finite place). For $p \mid \mathrm{disc}(D)$ there is a unique maximal order $\mathcal{O}_p$ of D_p and $\phi_{3,p}$ is a constant multiple of $\delta_p \circ N_{D_p}$ for both cases.

For each $p \nmid \mathrm{disc}(D)$, in the previous section we have fixed an isomorphism $D_p = \left(\frac{-1,-N}{\mathbb{Q}_p}\right) \xrightarrow{\sim} M(2, \mathbb{Q}_p)$, under which the preimage of $M(2, \mathbb{Z}_p)$ gives a maximal order $\mathcal{O}_p$. We choose $\phi_{3,p}$ to be the normalized $\mathcal{O}_p^\times$-invariant vector in the spherical representation $\pi_{3,p}$. Lemma 5.4.2 and 5.4.3 show that

$$\int_{E_p^\times \backslash D_p^\times} \alpha_p \varphi_{\gamma,p}(h_p)\, dh_p = \mathrm{vol}(K_p), \quad p \nmid N$$

holds for $\gamma = \gamma_0$. We will show it still holds for $\gamma = \gamma_1$, i.e. for D'_p.

Analogously we fix an isomorphism $D'_p = \left(\frac{-3,-N}{\mathbb{Q}_p}\right) \xrightarrow{\sim} M(2, \mathbb{Q}_p)$. Let $\mathcal{O}'_p$ be the preimage of $M(2, \mathbb{Z}_p)$, which is the a maximal order we fixed in the previous section. The endomorphism

$$M(2, \mathbb{Q}_p) \xrightarrow{\sim} \left(\frac{-1, -N}{\mathbb{Q}_p}\right) = D_p \cong D'_p = \left(\frac{-3, -N}{\mathbb{Q}_p}\right) \xrightarrow{\sim} M(2, \mathbb{Q}_p)$$

is a conjugation $A \mapsto TAT^{-1}$ for some $T \in \mathrm{GL}(2, \mathbb{Q}_p)$. This endomorphism gives another maximal order: the image of $\mathcal{O}_p$ under the isomorphism $D_p \cong D'_p$. Its image under the isomorphism $D'_p \xrightarrow{\sim} M(2, \mathbb{Q}_p)$ becomes $T \cdot M(2, \mathbb{Z}_p) \cdot T^{-1}$ and we still denote it by $\mathcal{O}_p$. Then the condition for $\phi_{3,p}$ becomes that it is $T \cdot \mathrm{GL}(2, \mathbb{Z}_p) \cdot T^{-1}$-invariant. It's easy to check that $\pi_p(T^{-1})\phi_p$ is $\mathrm{GL}(2, \mathbb{Z}_p)$-invariant i.e. $(\mathcal{O}'_p)^\times$-invariant.

With the new notations, Lemma 5.4.2 shows that $h \in E_p^\times(\mathcal{O}'_p)^\times$ if and only if $h^{-1}\gamma h \in (\mathcal{O}'_p)^\times$ and therefore

$$\int_{E_p^\times \backslash D_p^\times} \mathbf{1}_{K'_p}(h_p^{-1}\gamma h_p)\, dh_p = \frac{\mathrm{vol}((\mathcal{O}'_p)^\times)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times)};$$

Lemma 5.4.3 shows that for $h \in E_p^\times(\mathcal{O}'_p)^\times$,

$$\alpha(\pi_p(h)\pi_p(T^{-1})\phi_p, \overline{\pi_p(h)\pi_p(T^{-1})\phi_p}; \Omega_p) = \begin{cases} \frac{\mathrm{vol}(\mathcal{O}_{E_p}^\times)}{\mathrm{vol}(\mathbb{Z}_p^\times)}, & \Omega_p \text{ unramified;} \\ 0, & \Omega_p \text{ ramified.} \end{cases}$$

We work on another maximal order $\mathcal{O}_p = T\mathcal{O}'_pT^{-1}$. Now we have

$$h^{-1}\gamma h \in \mathcal{O}_p^\times \Leftrightarrow T^{-1}h^{-1}\gamma hT \in \mathcal{O}'_p \Leftrightarrow hT \in E_p^\times(\mathcal{O}'_p)^\times \Leftrightarrow h \in E_p^\times(\mathcal{O}'_p)^\times T^{-1};$$

and then

$$\int_{E_p^\times \backslash D_p^\times} \mathbf{1}_{K_p}(h_p^{-1}\gamma h_p)\, dh_p = \int_{E_p^\times \backslash D_p^\times} \mathbf{1}_{E_p^\times(\mathcal{O}'_p)^\times T^{-1}}(h_p)\, dh_p = \frac{\mathrm{vol}((\mathcal{O}'_p)^\times T^{-1})}{\mathrm{vol}(\mathcal{O}_{E_p}^\times)} = \frac{\mathrm{vol}((\mathcal{O}'_p)^\times)}{\mathrm{vol}(\mathcal{O}_{E_p}^\times)};$$

under the condition that $h^{-1}\gamma h \in \mathcal{O}_p^\times$, by Lemma 5.4.3,

$$\alpha(\pi_p(hT)(\pi_p(T^{-1})\phi_p), \overline{\pi_p(hT)(\pi_p(T^{-1})\phi_p)}; \Omega_p) = \begin{cases} \frac{\mathrm{vol}(\mathcal{O}_{E_p}^\times)}{\mathrm{vol}(\mathbb{Z}_p^\times)}, & \Omega_p \text{ unramified}, \\ 0, & \Omega_p \text{ ramified}, \end{cases}$$

because that $hT \in E_p^\times(\mathcal{O}'_p)^\times$. Thus we have

$$\int_{E_p^\times\backslash D_p^\times} \alpha_p \varphi_{\gamma,p}(h_p)\, dh_p = \mathrm{vol}(K_p), \quad p \nmid N$$

also holds for $\gamma = \gamma_1$.

5.6 When weight > 2

Recall that

$$\frac{I_{[\gamma]}(f)}{\mathrm{vol}(K')} = c_\gamma \int_{\mathbb{A}_E^\times\backslash D^\times(\mathbb{A})} \varphi_\gamma(h) \left(\int_{\mathbb{A}^\times E^\times\backslash \mathbb{A}_E^\times} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt \right) dh$$

by Lemma 5.1.5. Here $c_\gamma = \frac{1}{2}$ when $\gamma = \gamma_0$, and $c_\gamma = 1$ when $\gamma = \gamma_1$. Moreover with (5.4) and Theorem 5.2.1

$$\frac{\int_{[E^\times]} (R(h)\phi^*)(t)\overline{(R(h)\phi^{**})(t)}\, dt}{\langle R(h)\phi^*, R(h)\phi^{**}\rangle} = \frac{1}{\mathrm{vol}([E^\times])} \frac{\zeta_{\mathbb{Q}}^*(2)}{2L(1,\pi_3,\mathrm{Ad})} \sum_{\Omega\in\widehat{[E^\times]}} L(\frac{1}{2}, (\pi_3)_E\otimes\Omega) \prod_v \alpha_v(\pi'_{3,v}(h_v)\phi_v^*, \overline{\pi'_{3,v}(h_v)\phi_v^{**}}; \Omega_v).$$

The calculation for the finite places still holds, that is, by Lemma 5.3.1 and 5.4.3 we still have that $\alpha_p = 0$ unless Ω_p is unramified for any finite p. The difficulty of the case when $2k > 2$ is that, some terms with nontrivial Hecke characters of $E = \mathbb{Q}(\gamma)$ do not vanish.

Lemma 4.3.2 shows that, if $\Omega : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ is unramified everywhere, it is determined by its Archimedean factor, i.e. a character of $\mathbb{C}^\times$ which is invariant under $\mathcal{O}_E^\times$. Here

$$\mathcal{O}_{E_0}^\times = \{\pm 1, \pm i\}, \quad E_0 = \mathbb{Q}(\gamma_0) = \mathbb{Q}(\sqrt{-1}), \quad i = \sqrt{-1};$$
$$\mathcal{O}_{E_1}^\times = \{\zeta_6^r\}_{r=0}^5, \qquad E_1 = \mathbb{Q}(\gamma_1) = \mathbb{Q}(\sqrt{-3}), \quad \zeta_6 = \tfrac{1}{2}(1+\sqrt{-3}).$$

Then the sum over $\Omega \in \widehat{[E^\times]}$ becomes that over $\Omega_\infty \in \mathcal{O}_E^\times \widehat{\backslash \mathbb{C}^\times / \mathbb{R}^\times}$. Recall that every character of $\mathbb{C}^\times/\mathbb{R}^\times$ is of the form $z \mapsto (z/\bar{z})^m$. Denote it by sgn^{2m} where $\mathrm{sgn}(z) := z/|z|$ is the "sign" of $z \in \mathbb{C}^\times$ on the complex unit circle (and hence $\mathrm{sgn}(z)^2 := z/\bar{z}$). Obviously

$$\begin{aligned} &\mathrm{sgn}^{2m} \text{ is } \mathcal{O}_{E_0}^\times\text{-invariant} \Leftrightarrow m \in \mathbb{Z},\ 2 \mid m; \\ &\mathrm{sgn}^{2m} \text{ is } \mathcal{O}_{E_1}^\times\text{-invariant} \Leftrightarrow m \in \mathbb{Z},\ 3 \mid m. \end{aligned} \tag{5.7}$$

These are the Ω_∞ that may appear in the sum.

Recall that by Lemma 5.3.1, for $p \mid \mathrm{disc}(D)$, $\alpha_p(\phi_p', \overline{\phi_p''}; \Omega_p) = 0$ unless $\Omega_p = \overline{\delta_p \circ N_{D_p}}$, where δ_p is the character such that $\pi_p = \sigma_{\delta_p}$. This is to say, besides the unramification of Ω_p, we also need that

$$\Omega_p(\varpi_{E_p}) = \overline{\delta_p(N_{D_p}(\varpi_{E_p}))} = \overline{\delta_p(N_{E_p/\mathbb{Q}_p}(\varpi_{E_p}))}.$$

This gives no extra information when $E_p/\mathbb{Q}_p$ is unramified. But when $E_p/\mathbb{Q}_p$ is ramified, this leads to more restriction for Ω_∞, noticing that Ω is trivial on $E^\times$:

- For $E = \mathbb{Q}(\sqrt{-1})$ and $p = 2$, $1+i$ is a uniformizer of E_p, and then

$$\Omega_\infty(1+i)\Omega_2(1+i) = 1 \Rightarrow \Omega_\infty(1+i) = \Omega_2(1+i)^{-1} = \delta_2(N_{E_2/\mathbb{Q}_2}(1+i)) = \delta_2(2).$$

For $\Omega_\infty = \mathrm{sgn}^{2m}$ with $2 \mid m$, this shows that

$$\mathrm{sgn}^{2m}(1+i) = \delta_2(2) = \begin{cases} 1 & \Rightarrow m \equiv 0 \pmod 4, \\ -1 & \Rightarrow m \equiv 2 \pmod 4. \end{cases}$$

- For $E = \mathbb{Q}(\sqrt{-3})$ and $p = 3$, $\sqrt{-3}$ is a uniformizer of E_p. And similarly for $\Omega_\infty = \mathrm{sgn}^{2m}$ with $3 \mid m$, this shows that

$$\mathrm{sgn}^{2m}(\sqrt{-3}) = \delta_3(3) = \begin{cases} 1 & \Rightarrow m \equiv 0 \pmod 6, \\ -1 & \Rightarrow m \equiv 3 \pmod 6. \end{cases}$$

Moreover, analogous to Lemma 5.2.2, we can find a condition when α_∞ vanishes. Recall that, when $v = \infty$, $\pi'_v \cong \pi'_{2k} = \mathrm{Sym}^{2k-2} V \otimes \det^{-k+1}$ can be realized on the space of homogeneous polynomials in X, Y of degree $2k-2$, i.e.

$$V_{\pi'_{2k}} = \bigoplus_{n=0}^{2k-2} \mathbb{C} X^{2k-2-n} Y^n$$

with

$$\pi'_{2k}(g) P(X,Y) = P((X,Y)g) \det(g)^{1-k} \quad \text{for } g \in \left\{ \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) \right\} \cong D^\times(\mathbb{R}).$$

(See Appendix B.) This representation is determined by the last isomorphism. To make the action on $E_0^\times$ and $E_1^\times$ consistent (so that the two nontrivial orbits are compatible as we mentioned in the previous section), we fix the following isomorphism for $q = 1$ and 3:

$$\begin{aligned} D(\mathbb{R}) = \left(\frac{-q,-N}{\mathbb{R}}\right) &\xrightarrow{\sim} \left\{ \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in M(2,\mathbb{C}) \right\}, \\ i_D \mapsto \begin{pmatrix} \sqrt{-q} & \\ & -\sqrt{-q} \end{pmatrix}&, \quad j_D \mapsto \begin{pmatrix} & -\sqrt{N} \\ \sqrt{N} & \end{pmatrix}. \end{aligned} \tag{5.8}$$

Then the image of $E_\infty = \mathbb{R}(\gamma)$ are the same if $\left(\frac{-1,-N}{\mathbb{Q}}\right)$ and $\left(\frac{-3,-N}{\mathbb{Q}}\right)$ express the same quaternion algebra; and for $t \in E_\infty^\times = \mathbb{C}^\times$, we have

$$t \mapsto \begin{pmatrix} t & \\ & \bar{t} \end{pmatrix} \quad \text{and} \quad \pi'_{2k}(t) X^{2k-2-n} Y^n = \left(t/\bar{t}\right)^{k-1-n} X^{2k-2-n} Y^n.$$

In general

$$\pi'_{2k}(t) \left(\sum_{n=0}^{2k-2} c_n X^{2k-2-n} Y^n \right) = \sum_{n=0}^{2k-2} \operatorname{sgn}^{2(k-1-n)}(t) c_n X^{2k-2-n} Y^n, \tag{5.9}$$

where $\operatorname{sgn}^{2m}(t) := \left(t/\bar{t}\right)^m$. We fix the bilinear form B_∞ such that $B_\infty(\cdot, \bar{\cdot}) = \langle \cdot, \cdot \rangle_{2k}$ is the inner product on π'_{2k} defined as Lemma B.0.1.

Lemma 5.6.1. *For any $\phi'_v, \phi''_v \in V_{\pi'_{2k}}$,*

$$\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_v, \overline{\phi''_v}) \operatorname{sgn}^{2m}(t)\, dt = 0 \quad \textit{unless } m \in \mathbb{Z},\ -(k-1) \leq m \leq k-1.$$

Proof. Write

$$\phi'_v = \sum_{r=0}^{2k-2} c'_r X^{2k-2-r} Y^r, \quad \phi''_v = \sum_{r=0}^{2k-2} c''_r X^{2k-2-r} Y^r.$$

Recall that $\{X^{2k-2-r} Y^r\}$ forms an orthogonal basis of $V_{\pi'_{2k}}$ which are eigenvectors under the action of $\mathbb{C}^\times$. Then

$$\begin{aligned}
&\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_v, \overline{\phi''_v}) \operatorname{sgn}^{2m}(t)\ dt \\
=& \int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty\left(\sum_{r=0}^{2k-2} \operatorname{sgn}^{2(k-1-r)}(t) c'_r X^{2k-2-r} Y^r, \overline{\phi''_v}\right) \operatorname{sgn}^{2m}(t)\ dt \\
=& \sum_{r=0}^{2k-2} c'_r \overline{c''_r} B_\infty(X^{2k-2-r}Y^r, \overline{X^{2k-2-r}Y^r}) \int_{\mathbb{C}^\times/\mathbb{R}^\times} \operatorname{sgn}^{2(k-1-r)}(t) \operatorname{sgn}^{2m}(t)\ dt \\
=& \begin{cases} c'_{k-1+m}\overline{c''_{k-1+m}} \|X^{k-1-m}Y^{k-1+m}\|^2 \operatorname{vol}(\mathbb{C}^\times/\mathbb{R}^\times), & \text{if } m \in \mathbb{Z},\ -(k-1) \le m \le k-1; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

□

With the above lemma and (5.7) we can give a result for weight $2k = 4$. In this case

$$\prod_v \alpha_v(\pi'_v(h_v)\phi^*_v, \overline{\pi'_v(h_v)\phi^{**}_v}; \Omega_v) = 0 \quad \text{unless } \Omega = \mathbf{1}$$

still holds as in weight 2 case. So similarly we have

$$\frac{\int_{[E^\times]} R(h)\phi^*(t)\overline{R(h)\phi^{**}(t)}\ dt}{\langle \phi^*, \phi^{**}\rangle} = \frac{\zeta^*_{\mathbb{Q}}(2) L(\frac{1}{2}, (\pi_3)_E)}{2L(1, \pi_3, \operatorname{Ad}) \operatorname{vol}([E^\times])} \prod_v \alpha_v(\pi'_{3,v}(h_v)\phi^*_v, \overline{\pi'_{3,v}(h_v)\phi^{**}_v}; \mathbf{1}).$$

Denote $\phi' = R(h)\phi^*$, $\phi'' = R(h)\phi^{**}$. Notice that $B_\infty(\phi'_\infty, \overline{\phi''_\infty}) = \langle \phi^*, \phi^{**}\rangle$. The above calculation shows

$$\begin{aligned}
\frac{I_{[\gamma]}(f)}{\operatorname{vol}(K')} &= c_\gamma \int_{\mathbb{A}_E^\times \backslash D^\times(\mathbb{A})} \varphi_\gamma(h) \left(\frac{\zeta^*_{\mathbb{Q}}(2) L(\frac{1}{2}, (\pi_3)_E) B_\infty(\phi'_\infty, \overline{\phi''_\infty})}{2L(1, \pi_3, \operatorname{Ad}) \operatorname{vol}([E^\times])} \prod_v \alpha_v(\phi'_v, \overline{\phi''_v}; \mathbf{1}) \right) dh \\
&= \frac{c_\gamma \zeta^*_{\mathbb{Q}}(2) L(\frac{1}{2}, (\pi_3)_E)}{2L(1, \pi_3, \operatorname{Ad}) \operatorname{vol}([E^\times])} \int_{\mathbb{C}^\times \backslash D^\times_\infty} B_\infty(\phi'_\infty, \overline{\phi''_\infty}) \alpha_\infty\ dh_\infty \prod_p \int_{E_p^\times \backslash D_p^\times} \varphi_{\gamma,p}(h_p) \alpha_p\ dh_p,
\end{aligned} \tag{5.10}$$

with

$$\begin{aligned} B_\infty(\phi'_\infty, \overline{\phi''_\infty})\alpha_\infty &= \frac{L(1,\mathrm{sgn})L(1,\pi^{2k}_{\mathrm{dis}},\mathrm{Ad})}{\zeta_{\mathbb{R}}(2)L(\frac{1}{2},(\pi^{2k}_{\mathrm{dis}})_E)} \int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_\infty, \overline{\phi''_\infty})\,\mathrm{sgn}^0(t)\ dt \\ &= (2\pi)^{-1}\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_\infty, \overline{\phi''_\infty})\ dt. \end{aligned} \tag{5.11}$$

The following lemma gives a formula to calculate $\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_\infty, \overline{\phi''_\infty})\ dt$.

Lemma 5.6.2. *For $\phi'_\infty = \|\phi_3\|\pi'_{2k}(h)X^{2k-2}$, $\phi''_\infty = \frac{\|\phi_3\|}{\|\mathbb{P}_{2k}\|^2}\pi'_{2k}(h)e_\gamma \in \pi'_{2k}$, and $0 \le r \le 2k-2$,*

$$\begin{aligned} &\int_{\mathbb{C}^\times\backslash D^\times_\infty}\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_\infty, \overline{\phi''_\infty})\,\mathrm{sgn}^{-2(k-1-r)}(t)\ dt\ dh \\ = &\ \mathrm{vol}(G'_\infty)\frac{\|\phi_3\|^2}{2k-1}\frac{\Gamma(2k-1)^3}{\Gamma(k)^3\Gamma(3k-1)}\binom{2k-2}{r}^{-1} \\ &\cdot \sum_{\substack{0\le i,j\le 2k-2\\ i+j=3(k-1)-r}} \gamma^{2(k-1-i)}\binom{2k-2}{i}^{-1}\binom{2k-2}{j}^{-1}|C_{i,j,r}|^2, \end{aligned}$$

where $C_{i,j,r}$ is the coefficient in $\mathbb{P}_{2k}$ of $X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j \otimes X_3^{2k-2-r}Y_3^r$. In particular

$$\begin{aligned} &\int_{\mathbb{C}^\times\backslash D^\times_\infty}\int_{\mathbb{C}^\times/\mathbb{R}^\times} B_\infty(\pi'_{2k}(t)\phi'_\infty, \overline{\phi''_\infty})\ dt\ dh \\ &\qquad = \mathrm{vol}(G'_\infty)\frac{\|\phi_3\|^2}{2k-1}\frac{\Gamma(2k-1)^2}{\Gamma(k)\Gamma(3k-1)}\sum_{i=0}^{2k-2}\gamma^{2(k-1-i)}\binom{2k-2}{i}^{-2}|C_{i,2k-2-i,k-1}|^2. \end{aligned}$$

Proof. By the proof of Lemma 5.6.1 we have

$$\int_{\mathbb{C}^\times/\mathbb{R}^\times}\langle\pi'_{2k}(t)\phi'_v, \phi''_v\rangle_{2k}\,\mathrm{sgn}^{-2(k-1-r)}(t)\ dt = c'_r\overline{c''_r}\|X^{2k-2-r}Y^r\|^2\,\mathrm{vol}(\mathbb{C}^\times/\mathbb{R}^\times) \tag{5.12}$$

with

$$c'_r\overline{c''_r} = \frac{\langle\phi'_\infty, X^{2k-2-r}Y^r\rangle_{2k}\overline{\langle\phi''_\infty, X^{2k-2-r}Y^r\rangle_{2k}}}{\|X^{2k-2-r}Y^r\|^4}. \tag{5.13}$$

We first deal with $\langle \phi''_\infty, X^{2k-2-r}Y^r\rangle$. Recall that in Lemma 5.1.2, e_γ is defined by

$$\int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g, h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle \pi'_{2k}(g)X^{2k-2}\, dg = \mathrm{vol}(G'_\infty)\frac{\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^2}\pi'_{2k}(h)e_\gamma.$$

We write ϕ''_∞ back as an integral:

$$\phi''_\infty = \frac{\|\phi_3\|}{\mathrm{vol}(G'_\infty)\|w^\circ_{2k}\|^2}\int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g, h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle \pi'_{2k}(g)X^{2k-2}\, dg.$$

Then we have

$$\begin{aligned}
&\langle \phi''_\infty, X^{2k-2-r}Y^r\rangle \frac{\mathrm{vol}(G'_\infty)\|w^\circ_{2k}\|^2}{\|\phi_3\|}\\
=&\langle \int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(h^{-1}\gamma g, h^{-1}g)w^\circ_{2k}, w^\circ_{2k}\rangle \pi'_{2k}(g)X^{2k-2}\, dg, X^{2k-2-r}Y^r\rangle \qquad (5.14)\\
=&\int_{G'_\infty} \langle \pi'_{2k}\otimes\pi'_{2k}(\gamma g, g)w^\circ_{2k}, \pi'_{2k}\otimes\pi'_{2k}(h,h)w^\circ_{2k}\rangle\langle \pi'_{2k}(g)X^{2k-2}, X^{2k-2-r}Y^r\rangle\, dg.
\end{aligned}$$

Noticing that $\{X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\}$ forms an orthogonal basis of $\pi'_{2k}\otimes\pi'_{2k}$, the first matrix coefficient $\langle \pi'_{2k}\otimes\pi'_{2k}(\gamma g, g)w^\circ_{2k}, \pi'_{2k}\otimes\pi'_{2k}(h,h)w^\circ_{2k}\rangle$ is equal to

$$\begin{aligned}
&\sum_{0\le i,j\le 2k-2} \langle \pi'_{2k}\otimes\pi'_{2k}(\gamma g, g)w^\circ_{2k}, \frac{X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j}{\|X_1^{2k-2-i}Y_1^i\|\|X_2^{2k-2-j}Y_2^j\|}\rangle\\
&\qquad\cdot\overline{\langle \pi'_{2k}\otimes\pi'_{2k}(h,h)w^\circ_{2k}, \frac{X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j}{\|X_1^{2k-2-i}Y_1^i\|\|X_2^{2k-2-j}Y_2^j\|}\rangle}\\
=&\sum_{0\le i,j\le 2k-2} \|X_1^{2k-2-i}Y_1^i\|^{-2}\|X_2^{2k-2-j}Y_2^j\|^{-2}\\
&\qquad\cdot\langle \pi'_{2k}\otimes\pi'_{2k}(g,g)w^\circ_{2k}, \pi'_{2k}\otimes\pi'_{2k}(\gamma^{-1},1)(X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j)\rangle\\
&\qquad\cdot\overline{\langle \pi'_{2k}\otimes\pi'_{2k}(h,h)w^\circ_{2k}, X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\rangle}.
\end{aligned}$$

By (5.9), when $|\gamma| = 1$,

$$\pi'_{2k}(\gamma^{-1})X_1^{2k-2-i}Y_1^i = \gamma^{-2(k-1-i)}X_1^{2k-2-i}Y_1^i.$$

Therefore

$$\begin{aligned}
&\langle \pi'_{2k} \otimes \pi'_{2k}(\gamma g, g)w^\circ_{2k}, \pi'_{2k} \otimes \pi'_{2k}(h,h)w^\circ_{2k}\rangle \\
&= \sum_{0 \le i,j \le 2k-2} \|X_1^{2k-2-i}Y_1^i\|^{-2}\|X_2^{2k-2-j}Y_2^j\|^{-2}\gamma^{-2(k-1-i)} \\
&\qquad \cdot \langle \pi'_{2k} \otimes \pi'_{2k}(g,g)w^\circ_{2k}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j\rangle \\
&\qquad \cdot \overline{\langle \pi'_{2k} \otimes \pi'_{2k}(h,h)w^\circ_{2k}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j\rangle}.
\end{aligned} \tag{5.15}$$

Combining (5.12) (5.13) (5.14) (5.15) we have

$$\begin{aligned}
&\int_{\mathbb{C}^\times/\mathbb{R}^\times} \langle \pi'_{2k}(t)\phi'_\infty, \phi''_\infty\rangle_{2k} \,\mathrm{sgn}^{-2(k-1-r)}(t)\; dt \\
=&\frac{\mathrm{vol}(\mathbb{C}^\times/\mathbb{R}^\times)}{\mathrm{vol}(G'_\infty)} \frac{\|\phi_3\|^2}{\|w^\circ_{2k}\|^2} \|X^{2k-2-r}Y^r\|^{-2} \sum_{0\le i,j\le 2k-2} \gamma^{2(k-1-i)}\|X^{2k-2-i}Y^i\|^{-2}\|X^{2k-2-j}Y^j\|^{-2} \\
&\cdot \langle \pi'_{2k}(h)X^{2k-2}, X^{2k-2-r}Y^r\rangle_{2k} \langle \pi'_{2k} \otimes \pi'_{2k}(h,h)w^\circ_{2k}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j\rangle \\
&\cdot \int_{G'_\infty} \overline{\langle \pi'_{2k} \otimes \pi'_{2k}(g,g)w^\circ_{2k}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j\rangle \langle \pi'_{2k}(g)X^{2k-2}, X^{2k-2-r}Y^r\rangle_{2k}}\; dg \\
=&\frac{\mathrm{vol}(\mathbb{C}^\times/\mathbb{R}^\times)}{\mathrm{vol}(G'_\infty)} \frac{\|\phi_3\|^2}{\|w^\circ_{2k}\|^2} \|X^{2k-2-r}Y^r\|^{-2} \sum_{0\le i,j\le 2k-2} \gamma^{2(k-1-i)}\|X^{2k-2-i}Y^i\|^{-2}\|X^{2k-2-j}Y^j\|^{-2} \\
&\cdot \langle \big({\pi'_{2k}}^{\otimes 3} \circ \Delta_3(h)\big) w^\circ_{2k} \otimes X_3^{2k-2}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j \otimes X_3^{2k-2-r}Y_3^r\rangle \\
&\cdot \int_{G'_\infty} \overline{\langle \big({\pi'_{2k}}^{\otimes 3} \circ \Delta_3(g)\big) w^\circ_{2k} \otimes X_3^{2k-2}, X_1^{2k-2-i}Y_1^i \otimes X_2^{2k-2-j}Y_2^j \otimes X_3^{2k-2-r}Y_3^r\rangle}\; dg.
\end{aligned}$$

Here we denote by Δ_3 the diagonal embedding from G'_∞ to three copies of it. By the definition of quotient measure on $\mathbb{C}^\times\backslash D_\infty^\times \cong (\mathbb{R}^\times\backslash\mathbb{C}^\times)\backslash G'_\infty$, we have

$$\int_{\mathbb{C}^\times\backslash D_\infty^\times}\int_{\mathbb{C}^\times/\mathbb{R}^\times}\langle\pi'_{2k}(t)\phi'_\infty,\phi''_\infty\rangle_{2k}\operatorname{sgn}^{2m}(t)\ dt\ dh = \frac{\int_{G'_\infty}\int_{\mathbb{C}^\times/\mathbb{R}^\times}\langle\pi'_{2k}(t)\phi'_\infty,\phi''_\infty\rangle_{2k}\operatorname{sgn}^{2m}(t)\ dt\ dh}{\operatorname{vol}(\mathbb{C}^\times/\mathbb{R}^\times)},$$

and then

$$\begin{aligned}&\int_{\mathbb{C}^\times\backslash D_\infty^\times}\int_{\mathbb{C}^\times/\mathbb{R}^\times}\langle\pi'_{2k}(t)\phi'_\infty,\phi''_\infty\rangle_{2k}\operatorname{sgn}^{-2(k-1-r)}(t)\ dt\ dh\\ =&\frac{1}{\operatorname{vol}(G'_\infty)}\frac{\|\phi_3\|^2}{\|w^\circ_{2k}\|^2}\|X^{2k-2-r}Y^r\|^{-2}\sum_{0\le i,j\le 2k-2}\gamma^{2(k-1-i)}\|X^{2k-2-i}Y^i\|^{-2}\|X^{2k-2-j}Y^j\|^{-2}\\ &\quad\cdot\left|\int_{G'_\infty}\langle\big({\pi'_{2k}}^{\otimes 3}\circ\Delta_3(g)\big)w^\circ_{2k}\otimes X_3^{2k-2},X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\otimes X_3^{2k-2-r}Y_3^r\rangle\ dg\right|^2.\end{aligned}$$

Recall that, up to a constant multiple, $\mathbb{P}_{2k}$ is the only G'_∞-invariant vector in $(\pi'_{2k})^{\otimes 3}\cdot\Delta_3$. Then Lemma 4.1.1 gives that

$$\begin{aligned}&\int_{G'_\infty}\langle\big({\pi'_{2k}}^{\otimes 3}\circ\Delta_3(g)\big)w^\circ_{2k}\otimes X_3^{2k-2},X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\otimes X_3^{2k-2-r}Y_3^r\rangle\ dg\\ &\qquad=\operatorname{vol}(G'_\infty)\langle w^\circ_{2k}\otimes X_3^{2k-2},\frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle\overline{\langle X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\otimes X_3^{2k-2-r}Y_3^r,\frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|}\rangle}.\end{aligned}$$

Lemma B.0.4 shows that $\langle w^\circ_{2k}\otimes X_3^{2k-2},\mathbb{P}_{2k}\rangle = \|w^\circ_{2k}\|^2$. So

$$\begin{aligned}&\int_{\mathbb{C}^\times\backslash D_\infty^\times}\int_{\mathbb{C}^\times/\mathbb{R}^\times}\langle\pi'_{2k}(t)\phi'_\infty,\phi''_\infty\rangle_{2k}\operatorname{sgn}^{-2(k-1-r)}(t)\ dt\ dh\\ =&\operatorname{vol}(G'_\infty)\frac{\|\phi_3\|^2\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^4}\|X^{2k-2-r}Y^r\|^{-2}\sum_{0\le i,j\le 2k-2}\gamma^{2(k-1-i)}\|X^{2k-2-i}Y^i\|^{-2}\|X^{2k-2-j}Y^j\|^{-2}\\ &\qquad\cdot\left|\langle\mathbb{P}_{2k},X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\otimes X_3^{2k-2-r}Y_3^r\rangle\right|^2\\ =&\operatorname{vol}(G'_\infty)\frac{\|\phi_3\|^2\|w^\circ_{2k}\|^2}{\|\mathbb{P}_{2k}\|^4}\binom{2k-2}{r}^{-1}\sum_{\substack{0\le i,j\le 2k-2\\ i+j=3(k-1)-r}}\gamma^{2(k-1-i)}\binom{2k-2}{i}^{-1}\binom{2k-2}{j}^{-1}\\ &\qquad\cdot\left|\Big(\text{the coefficient in }\mathbb{P}_{2k}\text{ of }X_1^{2k-2-i}Y_1^i\otimes X_2^{2k-2-j}Y_2^j\otimes X_3^{2k-2-r}Y_3^r\Big)\right|^2.\end{aligned}$$

By Lemma B.0.2 we have

$$\frac{\|w_{2k}^{\circ}\|^2}{\|\mathbb{P}_{2k}\|^4} = \frac{1}{2k-1}\frac{\Gamma(2k-1)^3}{\Gamma(k)^3\Gamma(3k-1)},$$

and this completes the proof. □

We go back to the case when weight $2k = 4$. Here

$$\begin{aligned}\mathbb{P}_4 &= (X_1Y_2 - X_2Y_1)(X_2Y_3 - X_3Y_2)(X_3Y_1 - X_1Y_3)\\ &= (-Y_1^2X_2^2 + X_1^2Y_2^2)X_3Y_3 + \text{other terms.}\end{aligned}$$

The coefficient $C_{i,2-i,1}$ in $\mathbb{P}_4$ of $X_1^{2-i}Y_1^i \otimes X_2^iY_2^{2-i} \otimes X_3Y_3$ is given by

$$C_{i,2-i,1} = \begin{cases} 1, & i = 0; \\ 0, & i = 1; \\ -1, & i = 2. \end{cases}$$

Applying (5.11) and the above lemma, we have

$$\int_{\mathbb{C}^\times \backslash D_\infty^\times} B_\infty(\phi'_\infty, \overline{\phi''_\infty})\alpha_\infty \, dh_\infty = (2\pi)^{-1} \operatorname{vol}(G'_\infty)\|\phi_3\|^2 \frac{1}{18}(\gamma^2 + \gamma^{-2}).$$

For the non-Archimedean factors, by Lemma 5.4.2, 5.4.3 and (5.6), we still have

$$\int_{E_p^\times \backslash D_p^\times} \alpha_p \varphi_{\gamma,p}(h_p)\, dh_p = \begin{cases} \operatorname{vol}(K_p), & p \nmid N; \\ \operatorname{vol}(K_p)(1-\chi_{-d}(p))(1-p^{-1}), & p \mid N,\ \chi_{-d}(p) = -1; \\ \operatorname{vol}(K_p)(1-\chi_{-d}(p))(1-p^{-1})(1-\varepsilon_p(\tfrac{1}{2}, \pi_3)), & p \mid N,\ \chi_{-d}(p) = 0. \end{cases}$$

Therefore when $2, 3 \nmid N$, with (5.10)

$$\frac{I_{[\gamma]}(f)}{\mathrm{vol}(K')} = c_\gamma \int_{\mathbb{A}_E^\times \backslash D^\times(\mathbb{A})} \varphi_\gamma(h) \left(\int_{[E^\times]} R(h)\phi^*(t)\overline{R(h)\phi^{**}(t)}\, dt \right) dh$$
$$= c_\gamma \|\phi_3\|^2 \,\mathrm{vol}(K') \frac{\zeta_{\mathbb{Q}}^*(2) L(\frac{1}{2}, (\pi_3)_E)}{2L(1, \pi_3, \mathrm{Ad}) 2L(1, \eta)} (2\pi)^{-1} \frac{1}{18} (\gamma^2 + \gamma^{-2}) \frac{\varphi(N)}{N} \prod_{p|N} (1 - \chi_{-d}(p)).$$

In particular, $I_{[\gamma_0]} = 0$ if $2 \mid N$ and $(\delta_{\pi_3})_p(p) = -1$ for $p = 2$, otherwise

$$\frac{I_{[\gamma_0]}(f)}{\|\phi_3\|^2 \,\mathrm{vol}(K')} = \frac{-1}{9N} \frac{L(\frac{1}{2}, (\pi_3)_{\chi_{-4}})}{L(1, \pi_3, \mathrm{Ad})} 2^{\mathrm{ord}_2(N)} \prod_{p|N} (1 - \chi_{-4}(p));$$

and $I_{[\gamma_1]} = 0$ if $3 \mid N$ and $(\delta_{\pi_3})_p(p) = -1$ for $p = 3$, otherwise

$$\frac{I_{[\gamma_1]}(f)}{\|\phi_3\|^2 \,\mathrm{vol}(K')} = \frac{-\sqrt{3}}{12N} \frac{L(\frac{1}{2}, (\pi_3)_{\chi_{-3}})}{L(1, \pi_3, \mathrm{Ad})} 2^{\mathrm{ord}_3(N)} \prod_{p|N} (1 - \chi_{-3}(p)).$$

Recall that

$$\frac{I_{[1]}(f)}{\|\phi_3\|^2 \,\mathrm{vol}(K')} = \frac{1}{2k-1} = \frac{1}{3}$$

and from the spectral side (Theorem 4.5.3)

$$\frac{I(f)}{\langle \phi_3, \phi_3 \rangle \,\mathrm{vol}(K')} = \frac{\Gamma(2k-1)^3}{\Gamma(k)^3 \Gamma(3k-1)} \frac{2^{\omega(N)}}{2N^2} \sum_{\substack{\pi_1, \pi_2 \in \mathcal{F}(N, 2k) \\ \varepsilon_p = -1,\ \forall p|N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})}$$
$$= \frac{1}{3} \frac{2^{\omega(N)}}{2N^2} \sum_{\substack{\pi_1, \pi_2 \in \mathcal{F}(N, 4) \\ \varepsilon_p = -1,\ \forall p|N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})}.$$

Theorem 5.6.3 (Weight 4 Case). *Let N be a square-free integer with an odd number of prime factors. For any $\pi_3 \in \mathcal{F}(N, 4)$,*

$$\frac{1}{2N^2} \sum_{\substack{\pi_1, \pi_2 \in \mathcal{F}(N, 4) \\ \varepsilon_p = -1, \forall p|N}} \frac{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \pi_1 \otimes \pi_2 \otimes \pi_3, \mathrm{Ad})} = \frac{1}{2^{\omega(N)}} - \frac{\frac{1}{3} I_0 + \frac{\sqrt{3}}{4} I_1}{N \cdot L(1, \pi_3, \mathrm{Ad})}, \tag{5.16}$$

where

$$I_0 = L(\frac{1}{2}, (\pi_3)_{\chi_{-4}}) \prod_{p|N} \frac{1-\chi_{-4}(p)}{2} \cdot \begin{cases} 1, & 2 \nmid N, \\ (1-\varepsilon_2(\frac{1}{2},\pi_3)), & 2 \mid N; \end{cases}$$

$$I_1 = L(\frac{1}{2}, (\pi_3)_{\chi_{-3}}) \prod_{p|N} \frac{1-\chi_{-3}(p)}{2} \cdot \begin{cases} 1, & 3 \nmid N, \\ (1-\varepsilon_3(\frac{1}{2},\pi_3)), & 3 \mid N. \end{cases}$$

We can get a classical version of this result. Notice that, when $2k = 4$, for $f \in \mathcal{F}_4(N)$,

$$\begin{gathered} \varepsilon_p(\frac{1}{2}, \pi_f) = -a_p(f)p^{-1}, \quad p \mid N, \\ L(1, \pi_f, \mathrm{Ad}) = \frac{2^4}{N}(f,f), \quad L(\frac{1}{2}, \pi^4_{\mathrm{dis}}) = \zeta_{\mathbb{C}}(2) = 2^{-1}\pi^{-2}, \\ L(\frac{1}{2}, \pi^4_{\mathrm{dis}} \otimes \pi^4_{\mathrm{dis}} \otimes \pi^4_{\mathrm{dis}}) = \zeta_{\mathbb{C}}(5)\zeta_{\mathbb{C}}(2)^3 = 3 \cdot 2^{-4}\pi^{-11}, \\ L(\frac{1}{2}, (\pi^4_{\mathrm{dis}})_E) = L(\frac{1}{2}, \pi^4_{\mathrm{dis}})^2 = 2^{-2}\pi^{-4}. \end{gathered}$$

Theorem 5.6.4 (Weight 4 Case)**.** *Let N be a square-free integer with an odd number of prime factors. For any $h \in \mathcal{F}_4(N)$,*

$$\frac{3N}{2^{17}\pi^{11}} \sum_{\substack{f,g \in \mathcal{F}_4(N) \\ \varepsilon_p = -1, \forall p|N}} \frac{L_{\mathrm{fin}}(5, f \times g \times h)}{(f,f)(g,g)(h,h)} = \frac{1}{2^{\omega(N)}} - \frac{\frac{1}{3}A_0 + \frac{\sqrt{3}}{4}A_1}{2^6\pi^4(h,h)}, \tag{5.17}$$

where

$$A_0 = L_{\mathrm{fin}}(2,h)L_{\mathrm{fin}}(2,h,\chi_{-4}) \prod_{p|N} \frac{1-\chi_{-4}(p)}{2} \cdot \begin{cases} 1, & 2 \nmid N, \\ (1+\frac{a_2(h)}{2}), & 2 \mid N; \end{cases}$$

$$A_1 = L_{\mathrm{fin}}(2,h)L_{\mathrm{fin}}(2,h,\chi_{-3}) \prod_{p|N} \frac{1-\chi_{-3}(p)}{2} \cdot \begin{cases} 1, & 3 \nmid N, \\ (1+\frac{a_3(h)}{3}), & 3 \mid N. \end{cases}$$

Chapter 6

Applications

6.1 Sum over three forms

Feigon–Whitehouse [FW09] shows that

Lemma 6.1.1 ([FW09] Theorem 6.10)**.** *Let E be an imaginary quadratic field of fundamental discriminant $-d < 0$, with associated quadratic character $\chi_{-d} = (\frac{-d}{\cdot})$. Let $h = h_E$ be the class number of E, $u := \#\mathcal{O}_E^\times/\{\pm 1\}$. Let N be a square-free integer which is the product of an odd number of primes p satisfying $\chi_{-d}(p) = -1$ and $N > d$, and let $2k \geq 2$ be an even integer. Then*

$$\frac{(2k-2)!d^{1/2}u}{2\pi(4\pi)^{2k-1}} \sum_{f \in \mathcal{F}_{2k}(N)} \frac{L_{\mathrm{fin}}(k,f)L_{\mathrm{fin}}(k,f,\chi_{-d})}{(f,f)} = \begin{cases} h(1 - \frac{12h}{u\varphi(N)}), & \text{if } 2k = 2; \\ h, & \text{otherwise.} \end{cases}$$

In particular, for $d = 4$ and N being the product of an odd number of distinct primes of the form $4n+3$, we have $u = 2$ and

$$\frac{(2k-2)!4}{2\pi(4\pi)^{2k-1}} \sum_{f \in \mathcal{F}_{2k}(N)} \frac{L_{\text{fin}}(k,f)L_{\text{fin}}(k,f,\chi_{-4})}{(f,f)} = \begin{cases} 1 - \frac{6}{\varphi(N)}, & \text{if } 2k = 2; \\ 1, & \text{otherwise.} \end{cases}$$

For $d = 3$ and N being the product of an odd number of distinct primes of the form $3n+2$, we have $u = 3$ and

$$\frac{(2k-2)!3\sqrt{3}}{2\pi(4\pi)^{2k-1}} \sum_{f \in \mathcal{F}_{2k}(N)} \frac{L_{\text{fin}}(k,f)L_{\text{fin}}(k,f,\chi_{-3})}{(f,f)} = \begin{cases} 1 - \frac{4}{\varphi(N)}, & \text{if } 2k = 2; \\ 1, & \text{otherwise.} \end{cases}$$

With Theorem 1.1.3 we can get average formulas of central L-values over all three forms.

Corollary 6.1.2. *Let N be a square-free integer with an odd number of prime factors. When $2, 3 \nmid N$,*

$$\begin{aligned} \frac{N}{2^8\pi^5} \sum_{\substack{f,g,h \in \mathcal{F}_2(N) \\ \varepsilon_p = -1, \forall p \mid N}} \frac{L_{\text{fin}}(2, f \times g \times h)}{(f,f)(g,g)(h,h)} &= \frac{1 - 24/\varphi(N)}{2^{\omega(N)}} \#\mathcal{F}_2(N) \\ &+ \left(\frac{1 - 6/\varphi(N)}{2} \prod_{p \mid N} \frac{1 - \chi_{-4}(p)}{2} + (1 - 4/\varphi(N)) \prod_{p \mid N} \frac{1 - \chi_{-3}(p)}{2} \right); \end{aligned}$$

$$\begin{aligned} \frac{3N}{2^{17}\pi^{11}} \sum_{\substack{f,g,h \in \mathcal{F}_4(N) \\ \varepsilon_p = -1, \forall p \mid N}} \frac{L_{\text{fin}}(5, f \times g \times h)}{(f,f)(g,g)(h,h)} &= \frac{1}{2^{\omega(N)}} \#\mathcal{F}_4(N) \\ &- \frac{1}{12} \left(\prod_{p \mid N} \frac{1 - \chi_{-4}(p)}{2} + \prod_{p \mid N} \frac{1 - \chi_{-3}(p)}{2} \right). \end{aligned}$$

If N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$, we have

$$\frac{N}{2^{12k-4}\pi^{6k-1}} \sum_{\substack{f,g,h\in\mathcal{F}_{2k}(N) \\ \varepsilon_p=-1,\ \forall p|N}} \frac{L_{\mathrm{fin}}(3k-1, f\times g\times h)}{(f,f)(g,g)(h,h)} = \frac{1-24\delta(k)/\varphi(N)}{2^{\omega(N)}\Gamma(2k-1)^2\Gamma(2k)} \#\mathcal{F}_{2k}(N),$$

where $\delta(k) = \begin{cases} 1, & \textit{if } k=1, \\ 0, & \textit{otherwise.} \end{cases}$

The size of $\mathcal{F}_{2k}(N)$ is given by the following lemma.

Lemma 6.1.3 ([Mar05] Theorem 1)**.** *For any integer $k \geq 1$ and N a square-free integer with an odd number of prime factors, the dimension of the space of weight $2k$ newforms on $\Gamma_0(N)$ is*

$$\#\mathcal{F}_{2k}(N) = \frac{2k-1}{12}\varphi(N) - c_2(2k)\prod_{p|N}(1-\chi_{-4}(p)) - c_3(2k)\prod_{p|N}(1-\chi_{-3}(p)) - \delta(k),$$

where $\delta(k)$ is defined in the above corollary, and c_2, c_3 are defined by

$$c_2(n) = \frac{1}{4} + \lfloor\frac{n}{4}\rfloor - \frac{n}{4} = \begin{cases} 1/4, & n\equiv 0 \pmod 4, \\ -1/4, & n \equiv 2 \pmod 4; \end{cases}$$

$$c_3(n) = \frac{1}{3} + \lfloor\frac{n}{3}\rfloor - \frac{n}{3} = \begin{cases} 1/3, & n\equiv 0 \pmod 3, \\ 0, & n\equiv 1 \pmod 3, \\ -1/3, & n \equiv 2 \pmod 3. \end{cases}$$

In particular,

$$\#\mathcal{F}_2(N) = \frac{\varphi(N)}{12} + \frac{1}{4}\prod_{p|N}(1-\chi_{-4}(p)) + \frac{1}{3}\prod_{p|N}(1-\chi_{-3}(p)) - 1;$$
$$\#\mathcal{F}_4(N) = \frac{\varphi(N)}{4} - \frac{1}{4}\prod_{p|N}(1-\chi_{-4}(p));$$

if N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$,

$$\#\mathcal{F}_{2k}(N) = \frac{2k-1}{12}\varphi(N) - \delta(k).$$

With the above lemma we can have an explicit result.

Corollary 6.1.4. *Let N be a square-free integer with an odd number of prime factors. If N has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$, we have*

$$\frac{N}{2^8\pi^5}\sum_{\substack{f,g,h\in\mathcal{F}_2(N)\\ \varepsilon_p=-1,\forall p|N}}\frac{L_{\mathrm{fin}}(2,f\times g\times h)}{(f,f)(g,g)(h,h)} = \frac{(\varphi(N)-24)(\varphi(N)-12)}{2^{\omega(N)}12\varphi(N)};$$

$$\frac{N}{2^{12k-4}\pi^{6k-1}}\sum_{\substack{f,g,h\in\mathcal{F}_{2k}(N)\\ \varepsilon_p=-1,\ \forall p|N}}\frac{L_{\mathrm{fin}}(3k-1,f\times g\times h)}{(f,f)(g,g)(h,h)} = \frac{\varphi(N)}{2^{\omega(N)}12\Gamma(2k-1)^3}, \quad \text{if } 2k>2.$$

6.2 The nonvanishing problems

When the weight $2k=2$, we can get a similar result of the nonvanishing problem as [FW10] Corollary 5.2.

Corollary 6.2.1. *Let N be the product of an odd number of distinct primes such that $\varphi(N) > 24$. For each $h\in\mathcal{F}_2(N)$, there exist $f,g\in\mathcal{F}_2(N)$ such that $L_{\mathrm{fin}}(2,f\times g\times h)\neq 0$; moreover,*

$$\#\{(f,g)\in\mathcal{F}_2(N)\times\mathcal{F}_2(N) : L_{\mathrm{fin}}(2,f\times g\times h)\neq 0\} \gg_\epsilon N^{3/4-\epsilon}.$$

Proof. The first statement comes from (1.2) and the non-negativity of $L(1,h)L(1,h,\chi_{-d})$.

For $f \in \mathcal{F}_{2k}(N)$ we know from Hoffstein–Lockhart [HL94] that $(f,f) \gg N(\log N)^{-2}$. Applying this to (1.2) together with the non-negativity of $L(1,h)L(1,h,\chi_{-d})$, we have

$$\sum_{f,g\in\mathcal{F}_2(N)} L(2, f\times g\times h) \gg \sum_{\substack{f,g\in\mathcal{F}_2(N)\\ \varepsilon_p=-1,\forall p|N}} L_{\mathrm{fin}}(2, f\times g\times h) \gg \frac{1-24/\varphi(N)}{2^{\omega(N)}} N^2(\log N)^{-6}.$$

Robin [Rob83] shows that, for $N > 2$,

$$\omega(N) := \sum_{p|N} 1 \ll \log N/\log\log N. \tag{6.1}$$

So $2^{\omega(N)} \ll N^{1/\log\log N}$. When $\varphi(N) > 24$ we have

$$\sum_{f,g\in\mathcal{F}_2(N)} L_{\mathrm{fin}}(2, f\times g\times h) \gg N^{2-1/\log\log N}(\log N)^{-6}.$$

Moreover, for any weight $2k$ we have the convexity bound $L_{\mathrm{fin}}(3k-1, f\times g\times h) \ll_{k,\epsilon} N^{5/4+\epsilon}$ ([IK04]). Therefore

$$\#\{(f,g)\in\mathcal{F}_2(N)\times\mathcal{F}_2(N) : L_{\mathrm{fin}}(2, f\times g\times h)\neq 0\} \gg_\epsilon \frac{N^{3/4-\epsilon}}{N^{1/\log\log N}(\log N)^6} \gg_\epsilon N^{3/4-\epsilon}.$$

□

Analogously we can get a nonvanishing result for weight 4. The lower bound of (f,f) and the convexity bound of $L_{\mathrm{fin}}(5, f\times g\times h)$ do not change, but we notice that in (5.17), negative signs appear with the base change L-functions. Recall that, for $f \in \mathcal{F}_4(N)$, the convexity bound of central value of base change L-function is

$$L_{\mathrm{fin}}(2,f)L_{\mathrm{fin}}(2,f,\chi_{-d}) \ll_\epsilon N^{1/2+\epsilon}.$$

Corollary 6.2.2. *Let N be the product of an odd number of distinct primes. For $h \in \mathcal{F}_4(N)$,*

$$\#\{(f,g) \in \mathcal{F}_4(N) \times \mathcal{F}_4(N) : L_{\text{fin}}(5, f \times g \times h) \neq 0\} \gg_\epsilon N^{3/4-\epsilon}.$$

Proof. With the notations in Theorem 5.6.4 and the convexity bound of $L_{\text{fin}}(2,f)L_{\text{fin}}(2,f,\chi_{-d})$, we have

$$\frac{\frac{1}{3}A_0 + \frac{\sqrt{3}}{4}A_1}{2^6\pi^4(h,h)} \ll_\epsilon \frac{N^{1/2+\epsilon}}{N(\log N)^{-2}} \ll N^{-1/2+\epsilon};$$

by (5.17) and (6.1)

$$\frac{N}{2^{20}\pi^{11}} \sum_{\substack{f,g \in \mathcal{F}_4(N) \\ \varepsilon_p = -1, \forall p | N}} \frac{L_{\text{fin}}(5, f \times g \times h)}{(f,f)(g,g)(h,h)} = \frac{1}{2^{\omega(N)}} - \frac{\frac{1}{3}A_0 + \frac{\sqrt{3}}{4}A_1}{2^6\pi^4(h,h)} \gg N^{-1/\log\log N};$$

and therefore

$$\sum_{f,g \in \mathcal{F}_4(N)} L(5, f \times g \times h) \gg \sum_{\substack{f,g \in \mathcal{F}_4(N) \\ \varepsilon_p = -1, \forall p | N}} L_{\text{fin}}(5, f \times g \times h) \gg N^{2-1/\log\log N}(\log N)^{-6}.$$

The rest of the proof is the same as the case of weight 2. $\square$

The above work still holds for any weight $2k > 2$, if we only consider certain levels N such that only the trivial orbit appears in the geometric side of the RTF.

Corollary 6.2.3. *Let $2k > 2$, N be the product of an odd number of distinct primes which has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$. For each $h \in \mathcal{F}_{2k}(N)$, there exist $f, g \in \mathcal{F}_{2k}(N)$ such that $L_{\text{fin}}(3k-1, f \times g \times h) \neq 0$; moreover,*

$$\#\{(f,g) \in \mathcal{F}_{2k}(N) \times \mathcal{F}_{2k}(N) : L_{\text{fin}}(3k-1, f \times g \times h) \neq 0\} \gg_{k,\epsilon} N^{3/4-\epsilon}.$$

Now we study the nonvanishing modulo suitable primes p of the algebraic part of triple product L-values. Given $f, g, h \in \mathcal{F}_{2k}(N)$, we define

$$L^{\mathrm{alg}}(3k-1, f\times g\times h) := \Gamma(3k-1)\Gamma(k)^3 \frac{2^{\omega(N)}N}{2^{8k-3}\pi^{6k-1}} \frac{L_{\mathrm{fin}}(3k-1, f\times g\times h)}{(f,f)(g,g)(h,h)}.$$

According to [BSP96] Theorem 5.7 (a revised version of this theorem can be found in [BSSP03], in the proof of Proposition 2.1), $L^{\mathrm{alg}}(3k-1, f\times g\times h)$ lies in the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, g and h and hence is algebraic.

Corollary 6.2.4. *Let p be a prime such that $p \geq 3k-1$ and $p \neq 2$, and $\mathfrak{p}$ be a place in $\overline{\mathbb{Q}}$ above p. Let N be a square-free integer with an odd number of prime factors which has a prime factor $\equiv 1 \pmod 4$ and one $\equiv 1 \pmod 3$. Then, when $2k > 2$, for any $h \in \mathcal{F}_{2k}(N)$, there exist $f, g \in \mathcal{F}_{2k}(N)$ such that*

$$L^{\mathrm{alg}}(3k-1, f\times g\times h) \not\equiv 0 \pmod{\mathfrak{p}}.$$

This holds for $2k = 2$ too if in addition $p \nmid \varphi(N) - 24$.

Proof. The corollary is obvious, noting from (1.3) that, for any $h \in \mathcal{F}_{2k}(N)$,

$$\sum_{\substack{f,g\in\mathcal{F}_{2k}(N)\\ \varepsilon_p=-1,\ \forall p|N}} L^{alg}(3k-1, f\times g\times h) = \Big(1 - \frac{24}{\varphi(N)}\delta(k)\Big)2^{4k-1}\frac{\Gamma(3k-1)\Gamma(k)^3}{\Gamma(2k-1)^2\Gamma(2k)}$$

$$= \begin{cases} 2^3\,\frac{\varphi(N)-24}{\varphi(N)}, & 2k=2; \\ 2^{4k-1}\,\frac{\Gamma(3k-1)\Gamma(k)^3}{\Gamma(2k-1)^2\Gamma(2k)}, & 2k>2. \end{cases}$$

□

Appendix A

Hilbert Symbol

For a local field F, the **Hilbert symbol** $(\cdot,\cdot)_F : F^\times/(F^\times)^2 \times F^\times/(F^\times)^2 \to \{\pm 1\}$ can be defined by that $(a,b)_F = 1$ if $(\frac{a,b}{F}) \cong M_2(F)$ is split, and $= -1$ if $(\frac{a,b}{F})$ is a division algebra. It can be calculated by the following lemma.

Lemma A.0.1 ([Ser73] §III.1.2 Theorem 1)**.** *If $F = \mathbb{R}$, we have $(a,b)_{\mathbb{R}} = 1$ if a or b is > 0, and $= -1$ if a and b are < 0.*

If $F = \mathbb{Q}_p$ and if we write a, b in the form $p^\alpha u, p^\beta v$ where $u, v \in \mathbb{Z}_p^\times$, we have

$$(a,b)_{\mathbb{Q}_p} = \begin{cases} (-1)^{\alpha\beta\varepsilon(p)}(\frac{u}{p})^\beta(\frac{v}{p})^\alpha & \text{if } p \neq 2, \\ (-1)^{\varepsilon(u)\varepsilon(v)+\alpha\omega(v)+\beta\omega(u)} & \text{if } p = 2. \end{cases}$$

Here $(\frac{u}{p})$ denotes the Legendre symbol, and ε, ω are defined by

$$\varepsilon(z) \equiv \frac{z-1}{2} \pmod 2 = \begin{cases} 0 & \text{if } z \equiv 1 \pmod 4, \\ 1 & \text{if } z \equiv 3 \pmod 4; \end{cases}$$

$$\omega(z) \equiv \frac{z^2-1}{8} (\text{mod } 2) = \begin{cases} 0 & \textit{if } z \equiv \pm 1 \pmod 8, \\ 1 & \textit{if } z \equiv \pm 5 \pmod 8. \end{cases}$$

A.1 Application to quaternion algebras

We can use the Hilbert symbol to prove Lemma 2.1.2.

Proof of Lemma 2.1.2. (1) When $p \nmid 2N$, by Lemma A.0.1,

$$(-N,-1)_{\mathbb{Q}_p} = (-1)^0 \Big(\frac{-N}{p}\Big)^0 \Big(\frac{-1}{p}\Big)^0 = 1.$$

When $p \mid N$ but $p \neq 2$ (then $p \equiv 3 \pmod 4$),

$$(-N,-1)_{\mathbb{Q}_p} = (-1)^0 \Big(\frac{-N/p}{p}\Big)^0 \Big(\frac{-1}{p}\Big)^1 = -1.$$

When $p = 2$, if $2 \nmid N$, N is the product of an odd number of primes of the form $4n+3$, and therefore $-N \equiv 1 \pmod 4$,

$$(-N,-1)_{\mathbb{Q}_2} = (-1)^{\varepsilon(-N)\varepsilon(-1)} = 1;$$

if $2 \mid N$, $\frac{N}{2}$ is the product of an even number of primes of the form $4n+3$, and then $-\frac{N}{2} \equiv -1 \pmod 4$,

$$(-N,-1)_{\mathbb{Q}_2} = (-1)^{\varepsilon(-N/2)\varepsilon(-1)+\omega(-1)} = -1.$$

(2) First we notice that, for a prime $p \neq 2, 3$, by the Quadratic Reciprocity Law,

$$\Big(\frac{-3}{p}\Big) = \Big(\frac{3}{p}\Big)\Big(\frac{-1}{p}\Big) = (-1)^{\varepsilon(p)}\Big(\frac{p}{3}\Big)\Big(\frac{-1}{p}\Big) = \Big(\frac{p}{3}\Big) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3 \\ -1 & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{A.1}$$

When $p \nmid 6N$, $(-N,-3)_{\mathbb{Q}_p} = 1$. When $p \mid N$ but $p \nmid 6$ (then $p \equiv 2 \pmod 3$),

$$(-N,-3)_{\mathbb{Q}_p} = (-1)^0 \Big(\frac{-N/p}{p}\Big)^0 \Big(\frac{-3}{p}\Big)^1 = -1.$$

When $p = 2$, if $2 \nmid N$,

$$(-N,-3)_{\mathbb{Q}_2} = (-1)^{\varepsilon(-N)\varepsilon(-3)} = 1;$$

if $2 \mid N$,

$$(-N,-3)_{\mathbb{Q}_2} = (-1)^{\varepsilon(-N/2)\varepsilon(-3)+\omega(-3)} = -1.$$

When $p = 3$, if $3 \nmid N$, N is the product of an odd number of primes of the form $3n+2$ (2 might be included), and therefore $-N \equiv 1 \pmod 3$,

$$(-N,-3)_{\mathbb{Q}_3} = (-1)^0 \Big(\frac{-N}{3}\Big)^1 \Big(\frac{-3/3}{3}\Big)^0 = 1;$$

if $3 \mid N$, $\frac{N}{3}$ is the product of an even number of primes of the form $3n+2$, and then $-\frac{N}{3} \equiv -1 \pmod 3$,

$$(-N,-3)_{\mathbb{Q}_3} = (-1)^{\varepsilon(3)} \Big(\frac{-N/3}{3}\Big)^1 \Big(\frac{-3/3}{3}\Big)^1 = (-1)(-1)(-1) = -1.$$

□

A.2 Application to quadratic forms

Another application of the Hilbert symbol is to study what numbers can be represented in a given quadratic form.

Lemma A.2.1 ([Ser73] §IV.2.2 Corollary of Theorem 6)**.** *Let $f \sim a_1X_1^2 + \cdots + a_nX_n^2$ be a quadratic form of rank n and $a \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$. When $n = 3$, in order that f represents a in*

$\mathbb{Q}_p$, it is necessary and sufficient that $a \neq -a_1a_2a_3$ in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$, or

$$a = -a_1a_2a_3 \quad \textit{and} \quad (-1,a)_{\mathbb{Q}_p} = (a_1,a_2)_{\mathbb{Q}_p}(a_2,a_3)_{\mathbb{Q}_p}(a_1,a_3)_{\mathbb{Q}_p}.$$

With this lemma we give a proof of Proposition 5.1.4.

Proof of Proposition 5.1.4. To prove the necessity, by the definition of the Hilbert symbol, we write $D(\mathbb{Q}) = (\frac{a,b}{\mathbb{Q}})$ with $(a,b)_{\mathbb{Q}_p} = -1$ for any $p \mid N$. Recall that

$$N_D(X_0 + X_1 i + X_2 j + X_3 k) = X_0^2 - aX_1^2 - bX_2^2 + abX_3^2.$$

(1) When $p \equiv 1 \pmod 4$ is a prime factor of N, we claim that $-aX_1^2 - bX_2^2 + abX_3^2$ cannot represent 1 in $\mathbb{Q}_p$. Then it cannot represent 1 in $\mathbb{Q}$, i.e. $D(\mathbb{Q})$ has no element with trace 0 and norm 1.

To prove the claim, we only need to show

$$1 = -(-a)(-b)(ab) \text{ in } \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2, \quad \text{and} \quad (-1,1) \neq (-a,-b)(-b,ab)(-a,ab)$$

(we omit the subscript $\mathbb{Q}_p$ when there is no confusion). Notice that when $p \equiv 1 \pmod 4$, -1 is a square in $(\mathbb{Z}/p\mathbb{Z})^\times$ and therefore a square in $\mathbb{Q}_p^\times$ by Hensel's Lemma. This proves the first statement. For the second one, by the properties of Hilbert symbols (see [Voi20] Lemma 5.6.3 and Lemma 12.4.1),

$$(-a,-b) = (-a, -(-a)(-b)) = (-a,-ab);$$

$$(-b,ab)(-a,ab) = (ab,ab) = (-1,ab) = (-1,-1)(-1,-ab);$$

$$(-a,-b)(-b,ab)(-a,ab) = (-a,-ab)(-1,-1)(-1,-ab) = (-1,-1)(a,-ab).$$

With the assumption $(a, b) = -1$ (and therefore $(a, -ab) = -1$) we have

$$(-a, -b)(-b, ab)(-a, ab) = -(-1, -1).$$

(This result does not depend on the congruence condition of p.) When $p \equiv 1 \pmod 4$, -1 is a square in $\mathbb{Q}_p^\times$, and then $(-1, -1) = 1 = (-1, 1)$. This completes the proof of the claim.

(2) One can check that

$$\mathrm{Tr}_D(x) = 1,\ N_D(x) = 1 \Leftrightarrow \mathrm{Tr}_D(2x - 1) = 0,\ N_D(2x - 1) = 3.$$

So for this case we will show that, when $p \equiv 1 \pmod 3$ is a prime factor of N, $-aX_1^2 - bX_2^2 + abX_3^2$ cannot represent 3 in $\mathbb{Q}_p$; i.e.

$$-3 \in (\mathbb{Q}_p^\times)^2, \quad \text{and} \quad (-1, 3) \neq (-a, -b)(-b, ab)(-a, ab).$$

When $p \equiv 1 \pmod 3$, -3 is a square in $(\mathbb{Z}/p\mathbb{Z})^\times$ according to (A.1), and therefore a square in $\mathbb{Q}_p^\times$ by Hensel's Lemma. By the deduction of the previous case we still have

$$(-a, -b)(-b, ab)(-a, ab) = -(-1, -1);$$

and $-3 \in (\mathbb{Q}_p^\times)^2$ implies

$$(-1, 3) = (-1, -3)(-1, -1) = 1 \cdot (-1, -1).$$

This completes the proof of the proposition. $\square$

Appendix B

Representation Theory of $\mathrm{SU}(2)$

Notice that $G'_\infty = \mathbb{R}^\times \backslash D^\times(\mathbb{R}) \cong \mathrm{SU}(2)/\{\pm 1\}$. Set

$$\pi'_{2k} := \mathrm{Sym}^{2k-2} V \otimes \det{}^{-k+1},$$

where V denotes the irreducible 2-dimensional representation of G'_∞ coming from the isomorphism $D^\times(\overline{\mathbb{R}}) \xrightarrow{\sim} \mathrm{GL}(2,\mathbb{C})$ (see (5.8)). We have

$$\dim \pi'_{2k} = \dim(\mathrm{Sym}^{2k-2} V) = \binom{\dim V + (2k-2) - 1}{2k-2} = 2k-1.$$

More explicitly, π'_{2k} can be realized on the space of homogeneous polynomials in X, Y of degree $2k-2$, i.e.

$$V_{\pi'_{2k}} := \bigoplus_{n=0}^{2k-2} \mathbb{C} X^n Y^{2k-2-n}$$

with

$$\pi'_{2k}(g) P(X,Y) := P((X,Y)g) \det(g)^{1-k} \quad \text{for } g \in \left\{ \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \right\} \cong D^\times(\mathbb{R}).$$

Lemma B.0.1. *Define*

$$\langle X^iY^{2k-2-i}, X^jY^{2k-2-j}\rangle_{2k} = \begin{cases} \binom{2k-2}{i}^{-1}, & \text{if } i=j; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\langle\cdot,\cdot\rangle_{2k}$ *defines a* G'_∞*-invariant inner product on* $V_{\pi'_{2k}}$.

Proof. First of all, we have the following identity

$$\binom{d}{m}\binom{m}{k}\binom{d-m}{n-k} = \binom{d}{n}\binom{n}{k}\binom{d-n}{m-k}$$

because of the symmetry of m and n in

$$\binom{d}{m}\binom{m}{k}\binom{d-m}{n-k} = \frac{d!}{k!(m-k)!(n-k)!(d-m-n+k)!}.$$

For $g = \begin{pmatrix} \alpha & -\beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in \mathrm{SU}(2)$,

$$\begin{aligned}
&\langle \pi'_{2k}(g)X^iY^{2k-2-i}, \pi'_{2k}(g)X^jY^{2k-2-j}\rangle_{2k} \\
=&\langle (\alpha X+\bar\beta Y)^i(-\beta X+\bar\alpha Y)^{2k-2-i}, (\alpha X+\bar\beta Y)^j(-\beta X+\bar\alpha Y)^{2k-2-j}\rangle_{2k} \\
=&\sum_{r=0}^{2k-2}\Bigg\langle \sum_{\substack{0\le s\le i \\ 0\le t\le 2k-2-i \\ s+t=r}} \binom{i}{s}\alpha^s\bar\beta^{i-s}\binom{2k-2-i}{t}(-\beta)^t\bar\alpha^{2k-2-i-t}X^rY^{2k-2-r}, \\
&\qquad \sum_{\substack{0\le s'\le j \\ 0\le t'\le 2k-2-j \\ s'+t'=r}} \binom{j}{s'}\alpha^{s'}\bar\beta^{j-s'}\binom{2k-2-j}{t'}(-\beta)^{t'}\bar\alpha^{2k-2-j-t'}X^rY^{2k-2-r}\Bigg\rangle_{2k} \\
=&\sum_{r=0}^{2k-2}\alpha^{2k-2-j-r}\bar\alpha^{2k-2-i-r}(-\beta)^{r-i}(-\bar\beta)^{r-j}\langle X^rY^{2k-2-r}, X^rY^{2k-2-r}\rangle_{2k} \\
&\qquad\cdot\left(\sum_s\binom{i}{s}\binom{2k-2-i}{r-s}(\alpha\bar\alpha)^s(-\beta\bar\beta)^{i-s}\overline{\sum_{s'}\binom{j}{s'}\binom{2k-2-j}{r-s'}(\alpha\bar\alpha)^{s'}(-\beta\bar\beta)^{j-s'}}\right)
\end{aligned}$$

Notice that

$$\binom{i}{s}\binom{2k-2-i}{r-s} = \binom{2k-2}{i}^{-1}\binom{2k-2}{r}\binom{r}{s}\binom{2k-2-r}{i-s}.$$

We have

$$\begin{aligned}
&\langle \pi'_{2k}(g)X^iY^{2k-2-i}, \pi'_{2k}(g)X^jY^{2k-2-j}\rangle_{2k} \\
=&\sum_{r=0}^{2k-2} \alpha^{2k-2-j-r}\bar{\alpha}^{2k-2-i-r}(-\beta)^{r-i}(-\bar{\beta})^{r-j}\binom{2k-2}{r}^{-1}\binom{2k-2}{i}^{-1}\binom{2k-2}{j}^{-1}\binom{2k-2}{r}^{2} \\
&\cdot\left(\sum_s \binom{r}{s}\binom{2k-2-r}{i-s}(\alpha\bar{\alpha})^s(-\beta\bar{\beta})^{i-s}\sum_{s'}\binom{r}{s'}\binom{2k-2-r}{j-s'}(\alpha\bar{\alpha})^{s'}(-\beta\bar{\beta})^{j-s'}\right) \\
=&\binom{2k-2}{i}^{-1}\binom{2k-2}{j}^{-1}\alpha^{-j}\bar{\alpha}^{-i}(-\beta)^{-i}(-\bar{\beta})^{-j}\sum_{r=0}^{2k-2}\binom{2k-2}{r}(\alpha\bar{\alpha})^{2k-2-r}(\beta\bar{\beta})^r A_i B_j \\
=&\binom{2k-2}{i}^{-1}\binom{2k-2}{j}^{-1}\alpha^{-j}\bar{\alpha}^{-i}(-\beta)^{-i}(-\bar{\beta})^{-j}C_{ij},
\end{aligned}$$

where

$$A_i = \text{the coefficient of } x^i \text{ in } (\alpha\bar{\alpha}x+1)^r(-\beta\bar{\beta}x+1)^{2k-2-r},$$

$$B_j = \text{the coefficient of } y^i \text{ in } (\alpha\bar{\alpha}y+1)^r(-\beta\bar{\beta}y+1)^{2k-2-r},$$

and C_{ij} is the coefficient of x^iy^j in

$$\begin{aligned}
&\sum_{r=0}^{2k-2}\binom{2k-2}{r}(\alpha\bar{\alpha})^{2k-2-r}(\beta\bar{\beta})^r(\alpha\bar{\alpha}x+1)^r(-\beta\bar{\beta}x+1)^{2k-2-r}(\alpha\bar{\alpha}y+1)^r(-\beta\bar{\beta}y+1)^{2k-2-r} \\
=&\Big(\beta\bar{\beta}(\alpha\bar{\alpha}x+1)(\alpha\bar{\alpha}y+1)+\alpha\bar{\alpha}(-\beta\bar{\beta}x+1)(-\beta\bar{\beta}y+1)\Big)^{2k-2} \\
=&\Big(\big((\alpha\bar{\alpha})^2\beta\bar{\beta}+\alpha\bar{\alpha}(\beta\bar{\beta})^2\big)xy+\alpha\bar{\alpha}+\beta\bar{\beta}\Big)^{2k-2}.
\end{aligned}$$

Observe that it is a polynomial of xy. So the coefficient C_{ij} of x^iy^j is 0 unless $i=j$. Also notice that $\alpha\bar{\alpha}+\beta\bar{\beta}=1$. So

$$C_{ii} = \big(\text{the coefficient of } (xy)^i \text{ in } (\alpha\bar{\alpha}\beta\bar{\beta}xy+1)^{2k-2}\big) = \binom{2k-2}{i}(\alpha\bar{\alpha}\beta\bar{\beta})^i$$

and then

$$\langle \pi'_{2k}(g)X^iY^{2k-2-i}, \pi'_{2k}(g)X^iY^{2k-2-i}\rangle_{2k} = \binom{2k-2}{i}^{-1} = \langle X^iY^{2k-2-i}, X^iY^{2k-2-i}\rangle_{2k}.$$

□

One can check that π'_{2k} is an irreducible representation with highest weight $2k-2$, and X^{2k-2} is a highest weight vector. As in Lemma 3.1.3, let ϕ_∞ be the highest weight vector $\|\phi\|X^{2k-2}$. We have $\langle\phi,\phi\rangle = \prod_v \langle\phi_v,\phi_v\rangle_v$ since the length of ϕ_v is assumed to be 1 for any $v<\infty$.

Denote by Δ_2 (or Δ_3) the diagonal embedding from G'_∞ to two (or three, respectively) copies of it. One can view ${\pi'_{2k}}^{\otimes 2}\circ\Delta_2$ and ${\pi'_{2k}}^{\otimes 3}\circ\Delta_3$ as representations of G'_∞. Denote by $\{X_1^iY_1^{2k-2-i}\otimes X_2^jY_2^{2k-2-j}\}$ (or $\{X_1^iY_1^{2k-2-i}\otimes X_2^jY_2^{2k-2-j}\otimes X_3^rY_3^{2k-2-r}\}$) a basis of ${\pi'_{2k}}^{\otimes 2} = \pi'_{2k}\otimes\pi'_{2k}$ (or ${\pi'_{2k}}^{\otimes 3}$). [CC19] shows that,

$$\mathbb{P}_{2k} = \det\begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2\end{pmatrix}^{k-1} \otimes \det\begin{pmatrix} X_2 & X_3 \\ Y_2 & Y_3\end{pmatrix}^{k-1} \otimes \det\begin{pmatrix} X_3 & X_1 \\ Y_3 & Y_1\end{pmatrix}^{k-1}$$

is the only G'_∞-invariant vector in ${\pi'_{2k}}^{\otimes 3}\circ\Delta_3$ up to a constant multiple.

Let $\langle\cdot,\cdot\rangle$ be the $D^\times(\mathbb{Q})^2$ or $D^\times(\mathbb{Q})^3$-invariant pairing on ${\pi'_{2k}}^{\otimes 2}$ or ${\pi'_{2k}}^{\otimes 3}$ given by

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{2k}\otimes\langle\cdot,\cdot\rangle_{2k} \quad \text{or} \quad \langle\cdot,\cdot\rangle_{2k}\otimes\langle\cdot,\cdot\rangle_{2k}\otimes\langle\cdot,\cdot\rangle_{2k}. \tag{B.1}$$

We can calculate the lengths of some particular vectors.

Lemma B.0.2. *Let*

$$w_{2k}^{\circ} := \left(-Y_1Y_2\begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}\right)^{k-1}; \qquad \mathbb{P}_{2k} := \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} \otimes \begin{vmatrix} X_2 & X_3 \\ Y_2 & Y_3 \end{vmatrix}^{k-1} \otimes \begin{vmatrix} X_3 & X_1 \\ Y_3 & Y_1 \end{vmatrix}^{k-1}.$$

Then

$$\langle w_{2k}^{\circ}, w_{2k}^{\circ}\rangle = \frac{\Gamma(k)^3\Gamma(3k-1)}{\Gamma(2k-1)^2\Gamma(2k)}, \quad \langle \mathbb{P}_{2k}, \mathbb{P}_{2k}\rangle = \frac{\Gamma(k)^3\Gamma(3k-1)}{\Gamma(2k-1)^3}.$$

In particular $\langle w_{2k}^{\circ}, w_{2k}^{\circ}\rangle = \langle \mathbb{P}_{2k}, \mathbb{P}_{2k}\rangle/(2k-1)$.

Proof. The length of $\mathbb{P}_{2k}$ is shown in [CC19] Proposition 5.1.

By definition

$$\begin{aligned}\langle w_{2k}^{\circ}, w_{2k}^{\circ}\rangle &= \Bigg\langle \sum_{r=0}^{k-1}\binom{k-1}{r} X_1^r(-Y_1)^{2k-2-r}X_2^{k-1-r}Y_2^{k-1+r}, \\ &\qquad\qquad \sum_{r'=0}^{k-1}\binom{k-1}{r'} X_1^{r'}(-Y_1)^{2k-2-r'}X_2^{k-1-r'}Y_2^{k-1+r'}\Bigg\rangle \\ &= \sum_{r=0}^{k-1}\binom{k-1}{r}^2 \langle X_1^rY_1^{2k-2-r}, X_1^rY_1^{2k-2-r}\rangle_{2k}\langle X_2^{k-1-r}Y_2^{k-1+r}, X_2^{k-1-r}Y_2^{k-1+r}\rangle_{2k} \\ &= \sum_{r=0}^{k-1}\binom{k-1}{r}^2\binom{2k-2}{r}^{-1}\binom{2k-2}{k-1-r}^{-1}.\end{aligned}$$

One can verify that

$$\binom{k-1}{r}^2\binom{2k-2}{r}^{-1}\binom{2k-2}{k-1-r}^{-1} = \binom{2k-2}{k-1}^{-2}\binom{k-1+r}{k-1}\binom{2k-2-r}{k-1}.$$

Therefore, with Lemma B.0.3 we have

$$\begin{aligned}\langle w_{2k}^{\circ}, w_{2k}^{\circ}\rangle &= \binom{2k-2}{k-1}^{-2}\sum_{r=0}^{k-1}\binom{k-1+r}{k-1}\binom{2k-2-r}{k-1} \\ &= \binom{2k-2}{k-1}^{-2}\binom{3k-2}{k-1} = \frac{\Gamma(k)^3\Gamma(3k-1)}{\Gamma(2k-1)^2\Gamma(2k)}.\end{aligned}$$

□

Lemma B.0.3. *For any $m, n \geq 1$*

$$\sum_{r=0}^{n} \binom{n+r}{n} \binom{m+n-r}{n} = \binom{2n+m+1}{m}.$$

Proof. Assume that a point moves from $(0,0)$ to $(2n+1, m)$ by moving up or to the right by one unit each time. Then the point has $\binom{2n+m+1}{m}$ possible paths. But the path can intersect the vertical line $x = n + 1/2$ only once, say it passes (n, r) and $(n+1, r)$ where $0 \leq r \leq m$. Then $\binom{n+r}{n}\binom{n+m-r}{n}$ is the number of all possible paths that the point moves from $(0,0)$ to (n, r) and then from $(n+1, r)$ to $(2n+1, m)$. While r varies from 0 to m, all possible paths are counted. □

At last we prove a lemma which we use to motivate our choice of the test function in Section 4.1.

Lemma B.0.4. *With w_{2k}° and $\mathbb{P}_{2k}$ defined in Lemma B.0.2, we have*

(1)

$$\langle w_{2k}^{\circ} \otimes X_3^{2k-2-i} Y_3^i, \mathbb{P}_{2k} \rangle = \begin{cases} \langle w_{2k}^{\circ}, w_{2k}^{\circ} \rangle, & i = 0; \\ 0, & i \neq 0. \end{cases}$$

(2)

$$\left\langle \int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w_{2k}^{\circ}, w_{2k}^{\circ} \rangle \pi'_{2k}(g) X^{2k-2} \, dg, X^{2k-2-i} Y^i \right\rangle = \begin{cases} \mathrm{vol}(G'_\infty) \frac{\|w_{2k}^{\circ}\|^4}{\|\mathbb{P}_{2k}\|^2}, & i = 0; \\ 0, & i \neq 0. \end{cases}$$

Proof. (1) Here w_{2k}° has nothing to do with X_3, Y_3. So only the terms in $\mathbb{P}_{2k}$ with $X_3^{2k-2-i} Y_3^i$

contribute to the inner product $\langle w_{2k}^\circ \otimes X_3^{2k-2-i} Y_3^i, \mathbb{P}_{2k} \rangle$, i.e. the inner product is equal to

$$\begin{aligned}
&\Bigg\langle \left(-Y_1 Y_2 \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \right)^{k-1} X_3^{2k-2-i} Y_3^i, \\
&\qquad \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} X_3^{2k-2-i} Y_3^i \sum_{r_1+r_2=i} \binom{k-1}{r_1} (X_2)^{r_1} (-Y_2)^{k-1-r_1} \binom{k-1}{r_2} (Y_1)^{k-1-r_2} (-X_1)^{r_2} \Bigg\rangle \\
&= \Bigg\langle \left(-Y_1 Y_2 \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \right)^{k-1}, \\
&\qquad \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} \sum_{r_1+r_2=i} \binom{k-1}{r_1} (X_2)^{r_1} (-Y_2)^{k-1-r_1} \binom{k-1}{r_2} (Y_1)^{k-1-r_2} (-X_1)^{r_2} \Bigg\rangle \\
&\cdot \langle X_3^{2k-2-i} Y_3^i, X_3^{2k-2-i} Y_3^i \rangle.
\end{aligned}$$

Notice that the sum of exponents of X_1, X_2 in the first term $w_{2k}^\circ = \big(-Y_1 Y_2 (X_1 Y_2 - X_2 Y_1)\big)^{k-1}$ is always $k-1$, while that in the second term is always $k-1+i$. So the inner product is 0 unless $i = 0$.

When $i = 0$, we see that

$$\begin{aligned}
&\langle w_{2k}^\circ \otimes X_3^{2k-2}, \mathbb{P}_{2k} \rangle \\
&= \Bigg\langle \left(-Y_1 Y_2 \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \right)^{k-1}, \\
&\qquad \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} \sum_{r_1+r_2=0} \binom{k-1}{r_1} (X_2)^{r_1} (-Y_2)^{k-1-r_1} \binom{k-1}{r_2} (Y_1)^{k-1-r_2} (-X_1)^{r_2} \Bigg\rangle \\
&\cdot \langle X_3^{2k-2}, X_3^{2k-2} \rangle \\
&= \Bigg\langle \left(-Y_1 Y_2 \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix} \right)^{k-1}, \begin{vmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{vmatrix}^{k-1} (-Y_2)^{k-1} (Y_1)^{k-1} \Bigg\rangle \cdot 1 = \langle w_{2k}^\circ, w_{2k}^\circ \rangle.
\end{aligned}$$

(2) We have that

$$\begin{aligned}
&\left\langle \int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w^\circ_{2k}, w^\circ_{2k} \rangle \pi'_{2k}(g) X^{2k-2} \; dg, X^{2k-2-i} Y^i \right\rangle \\
&= \int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w^\circ_{2k}, w^\circ_{2k} \rangle \langle \pi'_{2k}(g) X^{2k-2}, X^{2k-2-i} Y^i \rangle \; dg \\
&= \int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k} \otimes \pi'_{2k}(g,g,g) w^\circ_{2k} \otimes X_3^{2k-2}, w^\circ_{2k} \otimes X_3^{2k-2-i} Y_3^i \rangle \; dg \\
&= \int_{G'_\infty} \langle {\pi'_{2k}}^{\otimes 3} \circ \Delta_3(g) w^\circ_{2k} \otimes X_3^{2k-2}, w^\circ_{2k} \otimes X_3^{2k-2-i} Y_3^i \rangle \; dg.
\end{aligned}$$

Recall that $\mathbb{P}_{2k}$ is the only G'_∞-invariant vector in ${\pi'_{2k}}^{\otimes 3} \circ \Delta_3$ up to a constant multiple. Hence, by Lemma 4.1.1, the above integral is equal to

$$\mathrm{vol}(G'_\infty) \langle w^\circ_{2k} \otimes X_3^{2k-2}, \frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|} \rangle \overline{\langle w^\circ_{2k} \otimes X_3^{2k-2-i} Y_3^i, \frac{\mathbb{P}_{2k}}{\|\mathbb{P}_{2k}\|} \rangle}.$$

The value of $\langle w^\circ_{2k} \otimes X_3^{2k-2-i} Y_3^i, \mathbb{P}_{2k} \rangle$ completes the proof. $\square$

Remark B.0.5. *This lemma implies that*

$$\int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w^\circ_{2k}, w^\circ_{2k} \rangle \pi'_{2k}(g) X^{2k-2} \; dg$$

is a constant multiple of X^{2k-2}*; more explicitly,*

$$\int_{G'_\infty} \langle \pi'_{2k} \otimes \pi'_{2k}(g,g) w^\circ_{2k}, w^\circ_{2k} \rangle \pi'_{2k}(g) X^{2k-2} \; dg = \mathrm{vol}(G'_\infty) \frac{\|w^\circ_{2k}\|^4}{\|\mathbb{P}_{2k}\|^2} X^{2k-2}.$$

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