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Growth of Conjugacy Classes of Reciprocal Words in Triangle Groups

by

Blanca Teresa Marmolejo

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

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This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

Dr. Ara Basmajian

Date | Chair of Examining Committee

Dr. Ara Basmajian

Date | Executive Officer

Dr. Melvyn Nathanson

Dr. Dragomir Saric

Dr. Robert Suzzi Valli

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Growth of Conjugacy Classes of Reciprocal Words in Triangle Groups

by

Blanca Teresa Marmolejo

Adviser: Professor Ara Basmajian

In this thesis we obtain the growth rates for conjugacy classes of reciprocal words for triangle groups of the form $G = \mathbb{Z}_2 * H$ where H is finitely generated and does not contain an order 2 element. We explore cases where H is infinite cyclic and finite cyclic. The quotient $\mathcal{O} = \mathbb{H}/G$ is an orbifold and contains a cone point of order 2, due to the first factor $\mathbb{Z}_2$ in the free product G. The reciprocal words in G correspond to geodesics on $\mathcal{O}$ which pass through the order 2 cone point on $\mathcal{O}$. We use methods from analytic combinatorics to count these words and to analyze the asymptotic behavior of their conjugacy classes with respect to the word length in the group for chosen generators of minimal length in G. We specifically use recurrence relations, and techniques for obtaining the closed forms of these relations, in order to study the growth of these geometric objects. With these methods we are able to describe the asymptotic behavior of the conjugacy classes of the primitive reciprocal geodesics in the groups as well. We find exponential asymptotics in all of the cases studied here and can compare the different bases between them.

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This thesis is dedicated to the memory of both Teresa C. Aguila who set me on my life's path and to Prof. John Loustau whose guidance helped me change it for the better.

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Notation

$\mathbb{C}$: complex plane

$\mathbb{U}$: upper half-plane model of the hyperbolic plane

$d_{\mathbb{U}}$: hyperbolic metric in $\mathbb{U}$

$\ell(\gamma)$: hyperbolic length of the path γ

$\mathbb{R}$: set of real numbers/real axis of the complex plane

$\mathbb{Z}$: set of integers

$\mathbb{Z}_k$: multiplicative group of integers modulo k

$\mathcal{A}_g$: axis of the hyperbolic isometry g

$\mathcal{T}_g$: translation length of the hyperbolic isometry g

$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$

$tr(A)$: trace of the square matrix A

$|tr|$: absolute value of the trace function

$\mathbb{D}$: unit disk model of the hyperbolic plane

$d_{\mathbb{D}}$: hyperbolic metric in $\mathbb{D}$

S^1: unit circle

ρ: hyperbolic metric

$\mathbb{H}$: hyperbolic plane

$\partial\mathbb{H}$: boundary of the hyperbolic plane

$\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$: closed hyperbolic plane

$\mathrm{Isom}(\mathbb{H})$: group of isometries of $\mathbb{H}$

$\mathrm{Isom}^+(\mathbb{H})$: group of orientation-preserving isometries of $\mathbb{H}$

Λ_G: limit set of G

Ω_G: ordinary set of G

$\mathcal{O} = \mathbb{H}/G$: 2-dimensional orientable hyperbolic orbifold for a group G with torsion

$\mathcal{W} = \{$reduced words in the generators of $G\}$

$\|g\|$: the word length of element $g \in G$

$[g]$: the conjugacy class of $g \in G$

$\|[g]\| = min\, \{\|h\| : h \in [g]\}$: length of conjugacy class

W: the set of conjugacy classes of reduced words in G

W^p: the set of primitive conjugacy classes of reduced words in G

W^{np}: the set of non-primitive conjugacy classes of reduced words in G

$\mathcal{R}$: the set of reciprocal words

$\mathcal{N}$: the normal form for reciprocal words

$\phi_n(x)$: the number of compositions of x with n components

$\phi_n^m(x)$: the number of compositions of x with n components $\leq m$

$|\mathcal{N}_{2t}|$: the number of reciprocal words of length $2t$ in normal form

$|R_{2t}|$: the number of conjugacy classes of reciprocal words of length $2t$

$P(z)$: the characteristic polynomial of a recurrence relation

ρ_k: the dominant real positive root of the characteristic polynomial for the recurrence in $\mathbb{Z}_2 * \mathbb{Z}_k$

Chapter 1

Introduction

In this thesis we study the growth of curves which pass through an order two cone point on specific orbifolds. This is achieved with respect to the length of the words in the group which represent these curves on the given orbifold. The orbifolds $\mathcal{O}$ of interest here arise from the quotient $\mathcal{O} = \mathbb{H}/G$ where G is a free product of the form $G = \mathbb{Z}_2 * H$ and H is finitely generated, not containing an order 2 element. Thus the order two fixed point comes from the first factor $\mathbb{Z}_2$ in G and produces its corresponding cone point on the quotient. We call these words of interest in G *reciprocal words* and exploit their *normal form*, in our work below. We borrow the word reciprocal from the literature, in our case the work of Sarnak in [13], a paper which served as a motivation for the investigation we present here. Sarnak studies these curves, which he names reciprocal geodesics, in the setting of the modular group but in terms of the trace of their corresponding matrices. Through our approach using word length, the study of this growth becomes a question of combinatorics. We are taking geometric objects and using non-geometric techniques to count them making our work computational in nature. In analytic combinatorics generating functions are useful tools and appear in the process of developing algorithmic approaches to well known counting problems. In some cases the generating functions have simple recurrence relations associated with them which can give insight into the structure of the growth of these objects. These recurrence relations can simplify the work by providing us with polynomials to study in our analyses. This is indeed the case for the objects we study in the groups we are working with here.

In chapter 2, we present the basic hyperbolic geometry definitions and theorems which are the foundation for this work. We present the general ideas to understand how curves on surfaces arise and how this extends to orbifolds. We use the typical notions of discrete groups and quotients of the hyperbolic plane by these groups, in particular the projection of axes of hyperbolic isometries, to produce curves on the orbifold.

In chapter 3, we continue with some fundamentals of geometric group theory. The notion of words in the groups, word length and theorems about normal forms in groups are crucial here. Furthermore, we develop the specific word forms for the reciprocal words which correspond to the curves on our orbifolds of interest which pass through the order two cone point on the orbifold. We show lemmas which will come into play in the counting of the conjugacy classes of these curves in our results in chapter 5.

The results of this thesis rely heavily on techniques from analytic combinatorics. Since these methods are outside of the realm of traditional hyperbolic geometry and geometric group theory, we use chapter 4 to outline the computational tools necessary. This field is vast and so we focus on the specific types of solutions used in our development. We also use this chapter to highlight some purely algebraic theorems which were implemented in order to use these combinatorial techniques for our results.

Finally, we present our results in chapter 5. Here is where we demonstrate the marriage of hyperbolic geometry and analytic combinatorics through the use of recurrence relations and their closed forms. From our work establishing the normal form for reciprocal words, we remark that these words are of even length. In section 5.1 we study $\mathbb{Z}_2 * \mathbb{Z}$ through the recurrence relation for the Jacobsthal sequence. We find in theorem 5.1.4 that the conjugacy classes of both the reciprocal and primitive reciprocal words of length $2t$ grow like $\frac{1}{6}2^t$ as $t \to \infty$ with respect to word length. In 5.2 the count is a bit easier for $\mathbb{Z}_2 * \mathbb{Z}_3$ and we have in 5.2.3 a growth rate of 2^{t-1} as $t \to \infty$ for reciprocal words of length $4t$. And in 5.3 we study the Hecke groups $\mathbb{Z}_2 * \mathbb{Z}_k$ and get a result for k odd by constructing a recurrence relation which builds the reciprocal words, and using it to get a closed form. From this closed form we have that in theorem 5.3.6 we are able to get a growth rate of $c_k \rho_k^t$ as $t \to \infty$ with respect to a word length of $2t$ where the values of the constant c_k and ρ_k depend on k. We may summarize as follows.

Theorem 1.0.1. *(1) Let $P(z) = z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$ where $m = \dfrac{k-1}{2}$. The conjugacy classes of primitive reciprocal words of length $2t$ in $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd, are asymptotic to $c_k \rho_k^t$ where ρ_k is the unique positive root of $P(z)$ in the interval $[\sqrt{2}, 2]$. Moreover, the ρ_k are strictly increasing and $\rho_k \to 2$ as $k \to \infty$.*

(2) Let $P(z) = z^2 - z - 2$. The conjugacy classes of primitive reciprocal words of length $2t$ in $\mathbb{Z}_2 * \mathbb{Z}$, are asymptotic to $\frac{1}{6}2^t$ where 2 is the unique positive root of $P(z)$.

Chapter 2

The Hyperbolic Plane, Fuchsian Groups and Orbifolds

2.1 The Hyperbolic Plane

There are several models used to describe the hyperbolic plane. Below are two standard models which are useful in terms of computations and visualizations. They are the *Upper Half-Plane Model* and the *Poincaré disk model*. We note that many of these definitions and basics for hyperbolic geometry are well known and can be found in ([1]), ([5]) and ([3]).

2.1.1 The Upper Half-Plane Model

Begin with the set $\mathbb{U} = \{z \in \mathbb{C} \mid Im(z) > 0\}$. Then construct the metric $d_{\mathbb{U}}$ which is derived from the differential $\frac{|dz|}{Im(z)}$ where $|dz| = \sqrt{dRe(z)^2 + dIm(z)^2}$. First, we need a way to express hyperbolic length. To each piecewise continuously differentiable curve in $\mathbb{U}$, say $\gamma : [a, b] \to \mathbb{U}$ we define length to be

$$\|\gamma\| = \int_a^b \frac{|\gamma'(t)|}{Im(\gamma(t))} dt$$

We now define $d_{\mathbb{U}}$ to be

$$d_{\mathbb{U}}(z, w) = inf\|\gamma\|$$

where the infimum is taken over all γ which join z to w in $\mathbb{U}$. As this is non-negative, symmetric and satisfies the triangle inequality, this $d_{\mathbb{U}}$ is a metric on $\mathbb{U}$. The metric space $(\mathbb{U}, d_{\mathbb{U}})$ is called the *upper half-plane model* of the hyperbolic plane. The *boundary at infinity* of $\mathbb{U}$ is $\mathbb{R} \cup \infty$ and is denoted $\partial\mathbb{U}$. The set $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$ represents the closed hyperbolic plane.

The infimum above is realized by certain paths for any $z, w \in \mathbb{U}$. These paths, the *geodesics* of $\mathbb{U}$, are Euclidean vertical lines and Euclidean semi-circles orthogonal to $\mathbb{R}$. We now summarize some important facts about geodesics in the hyperbolic plane.

(1) There is a unique geodesic through any two distinct points of the hyperbolic plane.

(2) A geodesic is uniquely determined by its endpoints on $\partial\mathbb{U}$.

(3) Two distinct geodesics intersect in at most one point in the closed hyperbolic plane.

(4) Given any two geodesics L_1 and L_2 which are disjoint in the closed hyperbolic plane, there is a unique geodesic which is orthogonal to both L_1 and L_2.

2.1.2 Isometries

An isometry of $\mathbb{U}$ is a bijective transformation $f : \mathbb{U} \to \mathbb{U}$ such that $d_{\mathbb{U}}(z, w) = d_{\mathbb{U}}(f(z), f(w))$ for all $z, w \in \mathbb{U}$. Let f be the map defined by

$$f(z) = \frac{az + b}{cz + d} \tag{2.1}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. f maps $\mathbb{U}$ onto $\mathbb{U}$ and using our definition of length along with the following fact,

$$\frac{|f'(z)|}{Im(f(z))} = \frac{1}{Im(z)}$$

we then have

$$\begin{aligned} l(f\gamma) &= \int_a^b \frac{|f'(\gamma(t))||\gamma'(t)|}{Im(f(\gamma(t)))} dt \\ &= l(\gamma) \end{aligned}$$

Thus it follows that the map (2.1) above is an isometry of $\mathbb{U}$. The notion of *angles* in $\mathbb{U}$ is the same as in $\mathbb{C}$ as we define an angle between two paths to be the angle between the tangent vectors to the paths at their point of intersection in $\mathbb{U}$. The map f above also preserves angles between paths in $\mathbb{U}$.

The set of all isometries of $\mathbb{U}$, Isom($\mathbb{U}$), form a group under composition and is generated by the set of maps of the form (2.1) together with the map $z \to -\overline{z}$. Below are some properties of Isom($\mathbb{U}$).

(1) Given any two points $z, w \in \mathbb{U}$, there exists some $f \in$ Isom($\mathbb{U}$) such that $f(z) = w$. (Acts transitively)

(2) For two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of distinct points on $\partial\mathbb{U}$, there exists some $f \in$ Isom($\mathbb{U}$) such that $f(z_1) = w_1$, $f(z_2) = w_2$, $f(z_3) = w_3$. (Acts triply transitively on $\partial\mathbb{U}$)

(3) Given any two geodesics L_1 and L_2, there exists some $f \in \mathrm{Isom}(\mathbb{U})$ such that $f(L_1) = L_2$. (Acts transitively on the set of geodesics in $\mathbb{U}$)

$\mathrm{Isom}^+(\mathbb{U})$ is the subgroup of orientation preserving isometries of $\mathrm{Isom}(\mathbb{U})$ and consists of the maps having the form (2.1). The non-trivial elements of $\mathrm{Isom}^+(\mathbb{U})$ can be classified into three distinct types as follows.

Theorem 2.1.1. ***Classification of isometries:*** *Let f be a non-trivial orientation preserving isometry of the upper half-plane model of the hyperbolic plane. Then exactly one of the following holds:*

*(i) f is **elliptic**: In this case f has exactly one fixed point in $\mathbb{U}$ and is conjugate to the map $z \mapsto e^{i\theta} z$ for some $\theta \in \mathbb{R}$;*

*(ii) f is **parabolic**: In this case f has exactly one fixed point on $\partial\mathbb{U}$ and is conjugate to the map $z \mapsto z + 1$;*

*(iii) f is **hyperbolic**: In this case f has exactly two fixed points on $\partial\mathbb{U}$ and is conjugate to the map $z \mapsto \lambda z$ for some $\lambda > 0$, $\lambda \neq 1$.*

Recall $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. Thus an element of $\mathrm{PSL}(2,\mathbb{R})$ is an equivalence class of matrices, in particular 2×2 matrices with entries in $\mathbb{R}$ and determinant 1. This class can be written as

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\}$$

Define the following map ϕ, such that $\phi : \mathrm{Isom}^+(\mathbb{U}) \mapsto \mathrm{PSL}(2,\mathbb{R})$ by

$$\left\{ f(z) = \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \mapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$

The map ϕ is an isomorphism and it allows us to switch from the setting of the composition of rational maps to matrix multiplication. As a result, the composition of two such rational maps say $f(z)$ and $g(z)$ where $g(z) = \dfrac{a'z + b'}{c'z + d'}$ can be found by using the entries obtained from the matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

This is simpler from a computational point of view but also allows us to characterize the classification of isometries according to the trace of the isometry's corresponding matrix. Recall that the trace of a matrix is the sum of that matrix's diagonal entries. Thus, the function $|tr| : \mathrm{PSL}(2, \mathbb{R}) \mapsto \mathbb{R}_{\geq 0}$ takes

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \mapsto |a + d|$$

For $f \in \mathrm{Isom}^+(\mathbb{U})$ let M_f denote the corresponding matrix for $\phi(f) \in \mathrm{PSL}(2, \mathbb{R})$. Then theorem 2.1.1 can be restated as follows.

Theorem 2.1.2. *Let f be a non-trivial element of Isom$^+(\mathbb{U})$. Then*

*(i) f is **elliptic** $\iff |tr(M_f)| < 2$;*

*(ii) f is **parabolic** $\iff |tr(M_f)| = 2$;*

*(iii) f is **hyperbolic** $\iff |tr(M_f)| > 2$.*

2.1.3 The Unit Disk Model

We define the unit disk to be the set $\mathbb{D} = \{z \in \mathbb{C} \quad | \quad |z| < 1\}$. The map $f(z) = \dfrac{z - i}{z + i}$ is a bijective, conformal map which sends $\mathbb{U}$ to $\mathbb{D}$. It is an isometry of $\mathbb{C}$ and so the map induces a metric, $d_{\mathbb{D}}$ in the unit disk with an arc-length element of $\frac{2|dz|}{1-|z|^2}$. The unit disk model of the hyperbolic plane is then the metric space $(\mathbb{D}, d_{\mathbb{D}})$. Since $f(z)$ is conformal, the boundary at ∞ of the upper half-plane, $\partial\mathbb{U}$ maps to the unit circle and so the boundary of the disk model, $\partial\mathbb{D}$ is S^1. The geodesics of $\mathbb{U}$, i.e. the vertical lines and the semi-circles perpendicular to $\partial\mathbb{U}$, respectively map to the diameters of S^1 and to the arcs of Euclidean circles in $\mathbb{C}$ which intersect S^1 orthogonally. Thus the facts observed for geodesics in the upper half-plane model have an equivalent in the disk model. The orientation preserving isometries of the disk model are as follows:

$$Isom^+(\mathbb{D}) = \left\{ f(z) = \frac{az+\overline{c}}{cz+\overline{a}} \quad \middle| \quad a, c \in \mathbb{C} \quad and \quad |a|^2 - |c|^2 = 1 \right\}$$

In general, if the model is clear or it is not necessary to specify, we refer to the hyperbolic plane as $\mathbb{H}$, the closed hyperbolic plane as $\overline{\mathbb{H}}$, and the hyperbolic metric as ρ.

As a result of the classification of isometries, we have that parabolic and hyperbolic isometries have infinite order while elliptic isometries may have finite or infinite order. Upon examination of the action of these elements in the group, we see that each of the three types leaves specific subsets of the hyperbolic plane invariant. This is easy to visualize in the disc model and works as follows.

For an elliptic isometry f with fixed point $z \in \mathbb{D}$ the action of f is a non-Euclidean rotation about z by 2θ as f is conjugate to $e^{i\theta}z$. More specifically, for any hyperbolic circle $C \in \mathbb{D}$ which has center z, where such a hyperbolic circle is defined as $C = \{\ w \in \mathbb{D} \quad | \quad d_{\mathbb{D}}(z,w) = r, \quad r > 0, \quad r \quad fixed\}$, $f(C) = C$. Thus all such circles will be kept invariant by the action of f.

For a parabolic isometry f the picture differs from the elliptic case as the fixed point z is no longer inside the disc but lies on the boundary at infinity $\partial\mathbb{D}$, or S^1. Circles which are tangent to $\partial\mathbb{D}$ at the fixed point z are now of interest. These tangent circles are called horocircles, which enclose regions of $\mathbb{D}$ known as horodiscs. These horodiscs can be mapped to the region $\{z \in \mathbb{U} \quad | \quad Im(z) > 1\}$ with its corresponding horocircle mapped to the horizontal line in $\mathbb{U}$ where $Im(z) = 1$ and fixed point mapped to ∞ as we can always conjugate to $z \mapsto z + 1$. This isometry translates the region and the line horizontally, and fixes infinity. And so we can see how for any horocircle C we have $f(C) = C$ thus f keeps these horocircles invariant.

For a hyperbolic isometry f we instead have two distinct fixed points on the boundary. From the facts listed previously for the hyperbolic plane there exists a unique geodesic in $\mathbb{D}$ which connects these two points. This geodesic is called the *axis* of the isometry f, denoted by $\mathcal{A}_f$ and is invariant under f. The action of f yields a translation along its axis $\mathcal{A}_f$. If $w \in \mathcal{A}_f$ then $\mathcal{T}_f = d_{\mathbb{D}}(w, f(w))$ is called the *translation length* of f and this quantity is invariant under conjugation. Since the trace of an isometry is also invariant under conjugation, we can state the following formula which relates these two quantities.

$$|tr(M_f)| = 2\cosh\left(\frac{\mathcal{T}_f}{2}\right)$$

We note the special form of the distance formula in both models according to their respective version of the metric. In the upper half plane model for $z, w \in \mathbb{U}$ we have

$$\cosh d_{\mathbb{U}}(z, w) = 1 + \frac{|z - w|^2}{2Im(z)Im(w)}$$

and equivalently in the disc model for $z, w \in \mathbb{D}$ we have

$$\cosh^2\left(\frac{1}{2}d_{\mathbb{D}}(z, w)\right) = \frac{|1 - z\bar{w}|^2}{(1 - |z|^2)(1 - |w|^2)}$$

where $|\cdot|$ is the Euclidean norm in either version of the formula.

2.2 Fuchsian Groups, Surfaces and Orbifolds

2.2.1 Fuchsian Groups

We begin by studying some groups and their properties in order to understand objects that arise as a result of group actions on the hyperbolic plane. We note many of these concepts and definitions are standard and can be found in works such as ([3]) and ([11]). A *topological group* G is both a group and a topological space, with the conditions that the maps $x \mapsto x^{-1}$ of G onto G and $(x, y) \mapsto xy$ of $G \times G$ onto G are continuous. Note, for each $y \in G$ the map $x \mapsto xy$ is a homeomorphism of G onto itself (as is the map $x \mapsto yx$). A topological group is *discrete* if the topology on G is the discrete topology. G being discrete is equivalent to saying that one point of G is isolated. We can take this point to be the identity for simplicity or in terms of sequences of $g_i \in G$, G is discrete if and only if $g_n \mapsto I$ and $g_n \in G$ imply that $g_n = I$ for all but finitely many n. A subgroup S of a discrete group G is discrete as well. Also, if G is discrete then so is any conjugate group gGg^{-1}.

Definition 2.2.1. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ equivalently, $\mathrm{Isom}^+(\mathbb{H})$, is called a Fuchsian group.

Let X be a topological space and G a group of homeomorphisms of X onto itself. G acts *properly discontinuously* on X if and only if for every compact subset K of X $g(K) \cap K = \varnothing$ except for finitely many $g \in G$. From the characterization above we have discrete groups if and only if the group acts properly discontinuously. Thus Fuchsian groups act properly discontinuously as they are discrete. If $Y \subseteq X$ then Y is a G-invariant subset of X if $G(Y) = Y$.

Definition 2.2.2. Let G be a subgroup of $\mathrm{Isom}^+(\mathbb{H})$ and $\Gamma \leq G$ and $Y \subseteq \mathbb{H}$. Then we say that Y is *precisely invariant* under Γ in G if

(1) $\gamma Y = Y, \forall \gamma \in \Gamma$

(2) $gY \cap Y = \emptyset, \forall g \in G - \Gamma$

Definition 2.2.3. Let G be a Fuchsian group. The *limit set* of G denoted Λ_G is a subset of the boundary of the hyperbolic plane, $\partial\mathbb{H}$, where $\Lambda_G = \{x \in \partial\mathbb{H} \mid g_i(y) \mapsto x$ for some $y \in \mathbb{H}$ and a sequence of $g_{i's} \in G\}$

Definition 2.2.4. The *set of discontinuity* for G, denoted Ω_G is the complement of Λ_G.

The limit set is a closed set in $\partial\mathbb{H}$ and the set of discontinuity is an open set in $\partial\mathbb{H}$. If Λ_G is finite then G is *elementary*, otherwise it is a *non-elementary* group. If the limit set is the entire boundary, i.e. if $\Lambda_G = \partial\mathbb{H}$, then we say G is of the *first kind*, otherwise G is of the *second kind.* In the case where G is of the second kind, then $\Omega_G \cap \partial\mathbb{H}$ is a union of open intervals called *intervals of discontinuity.*

Definition 2.2.5. A *fundamental set* F for a Fuchsian group G acting on the hyperbolic plane is a subset F of $\mathbb{H}$ which contains exactly one point from each orbit in $\mathbb{H}$.

Definition 2.2.6. A *fundamental domain* D is a domain which with part of its boundary forms a fundamental set for G. There must be a fundamental set F such that $D \subset F \subset \tilde{D}$ where $\tilde{D}$ is the closure of D in $\mathbb{H}$. The boundary of D, ∂D has area of 0 and for all $g \in G$, $g \neq$ identity in G, $g(D) \cap D = \emptyset$ and $\cup_{g \in G} \quad g(\tilde{D}) = \mathbb{H}$.

2.2.2 Surfaces and Orbifolds

Let $G \leq \mathrm{Isom}^+(\mathbb{H})$ where G is a discrete torsion free subgroup, then $\mathbb{H}/G$ is a two dimensional hyperbolic manifold, a surface, which is a complete Riemannian manifold of constant curvature -1. There exists a covering map p which is the projection map from $\mathbb{H}$ to $\mathbb{H}/G$ which is a local isometry. A *geodesic* in $\mathbb{H}/G$ is the projection under p of a geodesic in $\mathbb{H}$.

Definition 2.2.7. A *closed geodesic* on $\mathbb{H}/G$ is the projection under p of the axis of a hyperbolic element in G.

We recall that a path in a topological space X is a continuous map $\alpha : [0,1] \to X$ and that a *homotopy* of paths between two points in the space is a continuous deformation from one path to another which fixes the endpoints. For a point x in X a loop at x is a path that starts and ends at x. The *fundamental group* of X with base point x denoted $\pi_1(X, x)$ is defined to be the set of equivalence classes of loops at x where equivalence is by homotopy. We can look at classes of loops that do not depend on a chosen base point and thus look at *free homotopy classes.* If X and Y are two topological spaces and $f : X \to Y$ and $g : X \to Y$ are two continuous maps then f and g are freely homotopic if and only if there exists a mapping $H : X \times [0,1] \to Y$ such that $\forall x \in X$ $H(x,0) = f$ and $H(x,1) = g$.

Proposition 2.2.8. *Let $\mathbb{H}/G$ be a hyperbolic surface. For any free homotopy class in $\mathbb{H}/G$ either*

(1) There exists a unique geodesic in the free homotopy class, or

(2) *No closed geodesic in the free homotopy class. Furthermore, there is a closed curve in the free homotopy class that bounds a cusp (i.e. a punctured disk).*

We say a closed curve is *essential* if it is not homotopic to a point, a puncture, or a boundary component. Note there is a direct correspondence between the conjugacy classes on the fundamental group of a surface and the free homotopy classes of oriented closed curves on the surface. An element of a group is *primitive* if it is not a non-trivial power of another element. A curve is primitive if it does not trace over another curve multiple times.

Now we consider the case for $G \leq \mathrm{Isom}^+(\mathbb{H})$ where G is discrete but G has torsion. In this case $\mathbb{H}/G$ is a two dimensional hyperbolic orbifold, and not a surface. So for G discrete with torsion, we have finite order elliptic elements in the group G. Here the projection map $p : \mathbb{H} \to \mathbb{H}/G$ sets up a correspondence between the elliptic fixed points of the isometries in $\mathbb{H}$ and *cone points* in the quotient. These cone points are the places where we lack smoothness on the quotient, i.e. what keeps it from being a smooth manifold and thus from being a surface. Points on $\mathbb{H}/G$ which are not cone points are called *regular points*.

There is a way of describing homotopy for orbifolds, which can be found in more detail in ([14]). If a loop based at a regular point on $\mathbb{H}/G$ does not pass through a cone point, then the homotopy is the same as described above for a surface as the lifts from the quotient to $\mathbb{H}$ are unique. If a loop does pass through the cone point, we can still have an equivalent notion of homotopy as we again pick a regular point as a base point and although there are different lifts, the lifts are homotopic since the base point is a regular point. We have the following theorem for the fundamental group of an orbifold.

Theorem 2.2.9. *Suppose G is a Fuchsian group and $p : \mathbb{H} \to \mathbb{H}/G$ is the projection map. Take a regular point $x \in \mathbb{H}/G$ and a lift $\tilde{p} \in p^{-1}(x)$ then $\pi_1(\mathbb{H}/G, x)$ is isomorphic to G.*

This allows us to work directly with the group G when convenient. As for closed geodesics on orbifolds, these are still the projections of axes of hyperbolic elements in the group G. The difference is that if a projection of the axis of a hyperbolic isometry passes through a cone point on the orbifold, it is a piecewise closed geodesic and it is not smooth at the cone point itself.

Our previous theorem for surfaces changes in the orbifold context in the following manner.

Proposition 2.2.10. *Let $\mathbb{H}/G$ be a hyperbolic orbifold. For any free homotopy class in $\mathbb{H}/G$ either*

(1) There exists a unique geodesic in the free homotopy class, or

(2) No closed geodesic in the free homotopy class. Furthermore, there is a closed curve in the free homotopy class that bounds a cusp. Or

(3) No closed geodesic in the free homotopy class. Furthermore, there is a closed curve in the free homotopy class that bounds a cone point.

We will be focusing on closed curves on orbifolds which contain order two cone points. In particular, counting conjugacy classes of closed curves which actually pass through this special point and studying their growth with respect to word length. In the next chapter we describe the groups we work with and the elements with which we are most concerned.

Chapter 3

Free products, conjugacy classes and geometric group theory

3.1 Free products

A group can be described in terms of a set of *generators* and a set of defining relations known as *relators*. This is known as *presenting* a group. A *word* in the group is a finite sequence of symbols $g_1, g_2, ..., g_{n-1}, g_n$ such that each of the g_i's comes from the set $\{a, b, c, ..., a^{-1}, b^{-1}, c^{-1}, ...\}$ of generators in the group and their inverses. A *relator* is a word which is equivalent to the identity in the group. A presentation for a group is generally written as

$$G = \langle a, b, c, ...; P, Q, R, ...\rangle$$

where $\{a, b, c, ...\}$ are the generators in the group and $\{P, Q, R, ...\}$ are relator words in terms of the generators. A group is *finitely generated* if it has a presentation with finitely many generators. Similarly, a group is *finitely related* if it has a presentation with finitely many relators. We will be studying finitely generated and finitely related groups here. The *empty* or trivial word is the word 1 in the group. We note that a word of the form aa^{-1} is equivalent to the empty word, as are relators, and that words such as $aaaaaaa$ can be written as a^7 i.e. as a generator raised to a power. A *reduced word* is a non-trivial word which does not contain any subwords which are equivalent to the empty word. The word $abb^{-1}abcc^{-3}aaa$ is not a reduced word and is equivalent to $a^2bc^{-2}a^3$ which is in reduced form. For a finitely generated, finitely presented group $G = \langle a_1, ..., a_n; R_1, ..., R_p \rangle$, a reduced word $w = g_1^{k_1}, g_2^{k_2}, ..., g_{n-1}^{k_{n-1}}, g_n^{k_n}$ for $k_i \in \mathbb{Z} - \{0\}, n \in \mathbb{N}$ and the $g_i's \in \{a_1, ..., a_n\}$ has a *word length* of $\sum_{i=1}^{n} |k_i|$ thus counting the number of symbols in the sequence which makes up the word w. The *empty word* 1, or the identity, has a word length of zero. A reduced word such as w is *cyclically reduced* if we also have the condition that $g_1 \neq g_n$ if $n \neq 1$. A *free group* is a group with no defining relators. For example the group $< a, b >$ is the free group on two generators and is usually denoted by F_2. Similarly, the group F_m is the free group on m generators.

We can also construct groups out of other groups. The *free product* $G = G_1 * G_2$ of two groups G_1 and G_2 where $G_1 = \langle a_1, ..., a_n; R_1, ..., R_p \rangle$ and $G_2 = \langle b_1, ..., b_m; S_1, ..., S_q \rangle$ is the group $G_1 * G_2 = \langle a_1, ..., a_n, b_1, ..., b_m; R_1, ..., R_p, S_1, ..., S_q \rangle$. G_1 and G_2 are called the *free factors* of the group G. For free products a sequence of elements $g_1, g_2, \ldots, g_n$ from $G_1 * G_2$ is called *reduced* if $g_i \neq 1$, g_i is in G_1 or G_2, and g_i, g_{i+1} are not in the same free factor. A word $w \in G_1 * G_2$ with reduced form $w = g_1 \cdots g_n$ is called *cyclically reduced* if g_n and g_1 are in different factors or if $n \leq 1$. We have the following theorems about normal form and for conjugacy in free products.

Theorem 3.1.1. *([10]) The Normal Form Theorem for Free Products: Consider the free product $G = G_1 * G_2$. Each element of $g \in G$ can be uniquely expressed as a product $g = g_1 g_2 \cdots g_n$, where $g_1, g_2, \ldots, g_n$ is a reduced sequence.*

Theorem 3.1.2. *([9]) The Conjugacy Theorem for Free Products: Each element of $G_1 * G_2$ is conjugate to a cyclically reduced word. If $u = g_1 \cdots g_n$ and $v = h_1 \cdots h_m$ are cyclically reduced words which are conjugate in $G_1 * G_2$ and $n > 1$, then $m = n$ and the sequences $g_1, \ldots, g_n$ and $h_1, \ldots, h_m$ are cyclic permutations of each other. If $n \leq 1$, then $m = n$, and u and v are in the same factor and are conjugate in that factor.*

Definition 3.1.3. Let G be a group with generating set S. The *Cayley graph* $\mathcal{C}(G, S)$ associated to G is a graph where the vertices are the elements of G and two vertices g_1 and g_2 are connected by an edge if and only if there is some $s \in S$ such that $g_1 \cdot s = g_2$.

3.2 Examples of free products

For the first two examples, we have *free products of cyclic groups.* These groups have a general presentation of the form $\langle x_1, ..., x_m; x_1^{r_1}, x_2^{r_2} ......, x_n^{r_n} \rangle$ where $r_i \in \mathbb{Z} - \{0\}$.

Example 3.2.1. The Modular Group: $\mathrm{PSL}(2, \mathbb{Z})$ with presentation

$$\langle a, b \mid a^2 = b^3 = I \rangle$$

under the discrete faithful representation ρ mapping $a \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

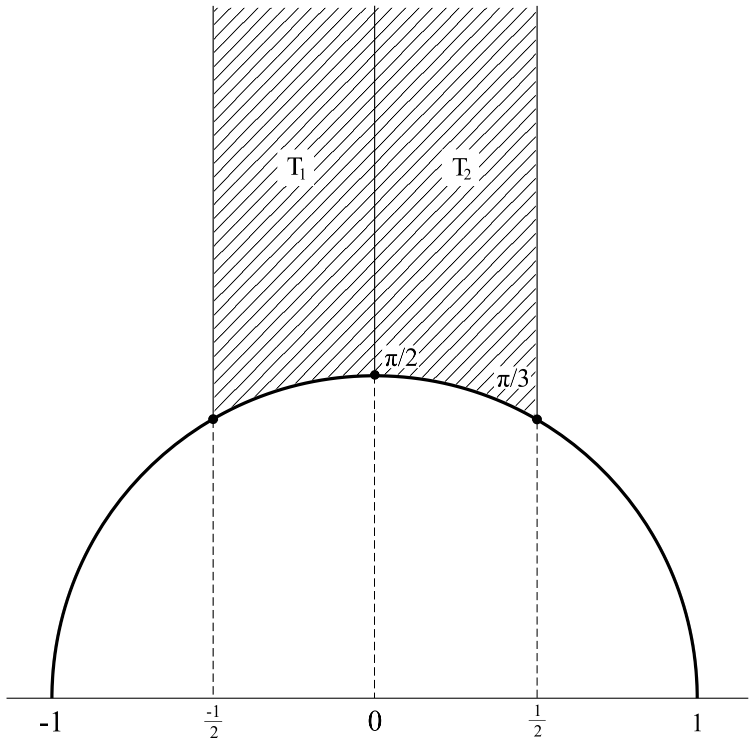

Figure 3.1: Fundamental domain of modular group

$b \to \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Note, this is the $(2, 3, \infty)$ triangle group. Thus we see from the presentation above that $\mathrm{PSL}(2, \mathbb{Z})$ can be thought of as a free product $\langle a; a^2 \rangle * \langle b; b^3 \rangle$ or $\mathbb{Z}_2 * \mathbb{Z}_3$.

In figure (3.1) we see a fundamental domain T_2 for the modular group with our chosen generators. In figure (3.2), we see the orbifold we get from the quotient $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$.

Example 3.2.2. The Hecke Groups, H_q: These are a type of generalization of the Modular group and are a class of triangle groups of the $(2, q, \infty)$ form. In this setting we still have an order two elliptic element but we also have

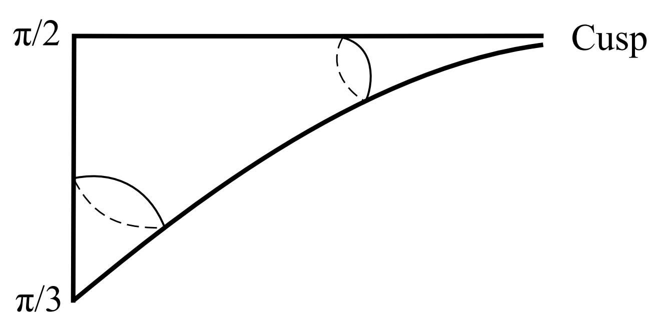

Figure 3.2: Orbifold $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$

an order q elliptic element in the group, i.e. $\langle a; a^2\rangle * \langle b; b^q\rangle$ as a presentation and so $H_q \simeq \mathbb{Z}_2 * \mathbb{Z}_q$. Thus, the presentation can also be written as

$$H_q = \langle a, b \mid a^2 = b^q = I\rangle$$

under the discrete faithful representation ρ taking

$$a \to \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix}, \quad b \to \begin{pmatrix} \cos(\frac{\pi}{q}) & -\lambda_{2,q}^{-1}\sin(\frac{\pi}{q}) \\ \lambda_{2,q}\sin(\frac{\pi}{q}) & \cos(\frac{\pi}{q}) \end{pmatrix},$$

$$\lambda_{2,q} = \frac{K_{2,q} + \left(K_{2,q}^2 - \sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{q}\right)\right)^{\frac{1}{2}}}{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{q}\right)}$$

where $K_{2,q} = 1 + \cos(\frac{\pi}{2})\cos(\frac{\pi}{q})$ by following the known general forms. Upon simplification we have $K_{2,q} = 1$ and get the following matrices

$$a \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b \to \begin{pmatrix} \cos(\frac{\pi}{q}) & -\lambda_{2,q}^{-1}\sin(\frac{\pi}{q}) \\ \lambda_{2,q}\sin(\frac{\pi}{q}) & \cos(\frac{\pi}{q}) \end{pmatrix},$$

with

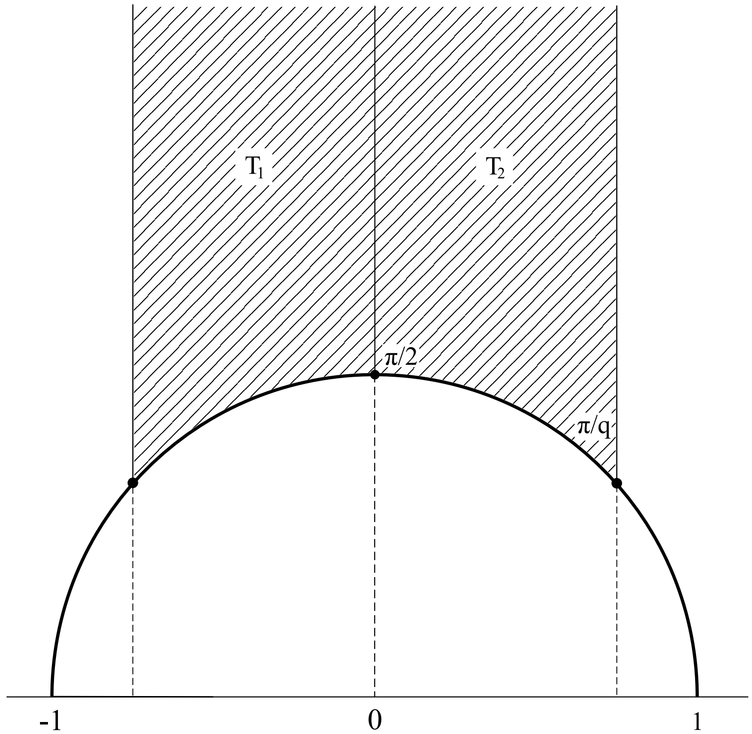

Figure 3.3: Fundamental domain of the Hecke groups

$$\lambda_{2,q} = \frac{1 + \left(1 - \sin^2\left(\frac{\pi}{q}\right)\right)^{\frac{1}{2}}}{\sin\left(\frac{\pi}{q}\right)}$$

In figure (3.3) we see a fundamental domain T_2 for the Hecke groups with our chosen generators. In figure (3.4), we see the orbifold we get from the quotient $\mathbb{H}/\rho(\mathbb{Z}_2 * \mathbb{Z}_q)$.

We remark that as discrete groups which act properly discontinuously on $\mathbb{H}$, our examples above as well as $\mathbb{Z}_2 * \mathbb{Z}$ are hyperbolic groups. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ is not studied here as it has a Cayley graph which is isomorphic to $\mathbb{Z}$, or alternatively, when looking at words in the group we see they grow

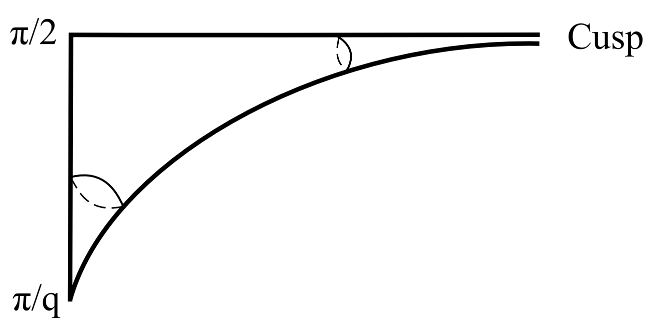

Figure 3.4: Orbifold $\mathbb{H}/\rho(\mathbb{Z}_2 * \mathbb{Z}_q)$

linearly.

Example 3.2.3. $\mathbb{Z}_2 * \mathbb{Z}$: The $(2, \infty, \infty)$ triangle group has the presentation

$$\langle a, b \mid a^2 = I \rangle$$

under the discrete faithful representation ρ taking $a \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b \to \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

In figure (3.5) we see a fundamental domain T_2 for the group $\mathbb{Z}_2 * \mathbb{Z}$ with our chosen generators. In figure (3.6), we see the orbifold we get from the quotient $\mathbb{H}/\rho(\mathbb{Z}_2 * \mathbb{Z})$. Observe, as $q \longrightarrow \infty$ in the Hecke groups, our limiting case is the group $\mathbb{Z}_2 * \mathbb{Z}$ as the π/q angle goes to zero and the vertex of the fundamental triangle goes to the boundary of $\mathbb{H}$.

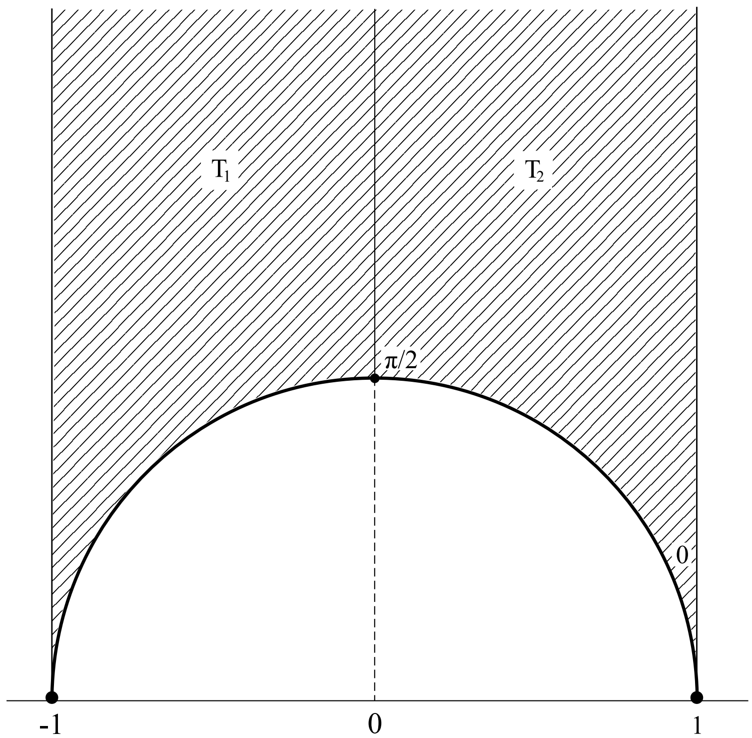

Figure 3.5: Fundamental domain of $\mathbb{Z}_2 * \mathbb{Z}$

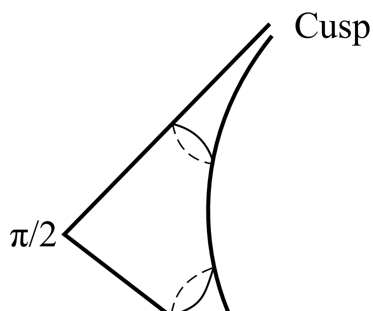

Figure 3.6: Orbifold $\mathbb{H}/\rho(\mathbb{Z}_2 * \mathbb{Z})$

3.3 Conjugacy classes

We now focus on groups $G = \mathbb{Z}_2 * H$ with H a (finite or infinite) cyclic group. Assume the generator of $\mathbb{Z}_2$ is a and the generator of H is b. We choose these generators to minimize word length. When b is infinite cyclic G is isomorphic to a subgroup of the modular group, namely $\mathbb{Z}_2 * \mathbb{Z}$. We recall that an element $g \in G$ is *primitive* if it is not a non-trivial power of another element of G. Set $\mathcal{W} = \{\text{reduced words in the generators of } G\}$. The *word length* of the element $g = g_1^{k_1}, g_2^{k_2}, ..., g_{n-1}^{k_{n-1}}, g_n^{k_n}$ for $k_i \in \mathbb{Z} - \{0\}, n \in \mathbb{N}$ and the $g_i' s \in \{a, b\}$ is $\sum_{i=1}^{n} |k_i|$ denoted by $\|g\|$ and is the minimum length among all words representing g using the symmetric set of generators $\left\{a, b, a^{-1}, b^{-1}\right\}$. The conjugacy class of $g \in G$ is denoted, $[g]$. For a positive integer s, since conjugation commutes with taking powers, we may define $[g]^s := [g^s]$. The *length of a conjugacy class* $[g]$ is given by $\|[g]\| = min\,\{\|h\| : h \in [g]\}$. From our previous definition, it is clear that a word in $\mathcal{W}$ is cyclically reduced if any cyclic permutation of it is a reduced word. Though cyclically reduced words in a conjugacy class are not unique they do realize the minimum length in the conjugacy class. These minimal length words are cyclic permutations of each other within the conjugacy class due to our previous theorem 3.1.2.

Definition 3.3.1. Each element $g \in G$ can be expressed uniquely in one of the following reduced forms, where each $k_i \in \mathbb{Z} - \{0\}, n \in \mathbb{N}$.

(1) $ab^{k_1}ab^{k_2} \cdots ab^{k_n}$, (($ab$)-words, as they start with a and end with b)

(2) $b^{k_1}ab^{k_2} \cdots ab^{k_n}a$, (($ba$)-words, start with b and end with a)

(3) $ab^{k_1}ab^{k_2} \cdots ab^{k_n}a$, (($aa$)-words, start and end with a)

(4) $b^{k_1}ab^{k_2} \cdots ab^{k_n}$, (($bb$)-words, start and end with b)

We remark the important fact that an (ab) or (ba) word above is cyclically reduced.

Lemma 3.3.2. *Let $x \in \mathcal{W}$ where x is not conjugate to a or b. Then*

(1) x is conjugate to an (ab)-word y with $\|x\| \geq \|y\|$.

(2) The only conjugates of the word $ab^{k_1}...ab^{k_n}$ that are (ab)-words are its cyclic permutations. That is, $ab^{k_n}ab^{k_1}...ab^{k_{n-1}}$ and so on.

(3) If y is an (ab)-word and $x^s = y$ for s a positive integer, then x is an (ab)-word and $s\|x\| = \|y\|$.

(4) If $[x]^s = [y]$ then $s\|[x]\| = \|[y]\|$.

Proof. Proof of Item (1): We proceed according to cases. First, if x is an (ab)-word then we are done and $\|x\|$ is already minimal. Suppose instead that x is a (ba)-word, thus we can write $x = b^{k_1}...ab^{k_n}a$. It follows that $\|x\| = \sum_{i=1}^{n} |k_i| + n$ since there are n many a's which contribute to the overall word length of w. We now conjugate w by a and since a is its own inverse, we get $y = ab^{k_1}...ab^{k_n}$ which is an (ab)-word of length $\|y\| = \sum_{i=1}^{n} |k_i| + n$ and so $\|x\| = \|y\|$. If x is a (bb)-word, say $x = b^{k_1}...ab^{k_n}$ then we can conjugate by b^{k_1} to get $y = ab^{k_2}...ab^{k_n}b^{k_1}$ or $y = ab^{k_2}...ab^{k_n+k_1}$ which is an (ab)-word. Now, $\|x\| = \sum_{i=1}^{n} |k_i| + n - 1$ here and upon inspection we find $\|y\| = \sum_{i=2}^{n-1} |k_i| + |k_n + k_1| + n - 1$. Using the triangle inequality we have $\|y\| \leq \sum_{i=2}^{n-1} |k_i| + |k_n| + |k_1| + n - 1$. But $\sum_{i=2}^{n-1} |k_i| + |k_n| + |k_1| + n - 1 = \sum_{i=1}^{n} |k_i| + n - 1$, thus $\|x\| \geq \|y\|$ as desired. Finally, we consider when x is an (aa)-word. So then we can write $x = ab^{k_1}...ab^{k_n}a$ and $\|x\| = \sum_{i=1}^{n} |k_i| + n + 1$. We can then conjugate x by ab^{k_1} to get $y = ab^{k_2}...ab^{k_n+k_1}$ with $\|y\| = \sum_{i=2}^{n-1} |k_i| + |k_n + k_1| + n - 1$. As before,

$\|y\| \leq \sum_{i=2}^{n-1} |k_i|+|k_n|+|k_1|+n-1$ or $\sum_{i=1}^{n} |k_i|+n-1$. Note this is strictly less than $\|x\| = \sum_{i=1}^{n} |k_i| + n + 1$ and so again $\|x\| \geq \|y\|$ as desired. We remark that by the work above, any conjugation of a word already in (ab)-form by a non-trivial element in the group, can only produce a word of greater length than that original (ab)-word. Item (2): Follows from theorem 3.1.2. Item (3): x must be an (ab)-word by using the definition of free product and writing out all possible cases as there are few. Since $x^s = y$ and we are in a free product, meaning there will not be any reductions from raising an (ab)-word to a power, then the number of a's in x^s must be the same as the number of a's in $y = ab^{k_1}...ab^{k_n}$ and the number of b's must match up as well. So for $x = ab^{l_1}...ab^{l_m}$, x^s will contain sm many a's and $s\sum_{i=1}^{m} |l_i|$ many b's. Then $\|y\| = \sum_{i=1}^{n} |k_i| + n = s\sum_{i=1}^{m} |l_i| + sm = s(\sum_{i=1}^{m} |l_i| + m) = s\|x\|$. Item (4): We may assume, by conjugating if necessary, that y is an (ab)-word. Now, by assumption there exists an x so that $x^s = y$, and hence x is an (ab)-word and we have $\|[y]\| = \|y\| = s\|x\| = s\|\,[x]\,\|$ where the second and third equalities follow from item (3). $\square$

3.4 Reciprocal words

We now describe the specific elements in the free products of the form $G = \mathbb{Z}_2 * H$ with H a (finite or infinite) cyclic, that are of interest to us. The subset of these elements will be words in the group of a specific type which essentially correspond to the set of what we will call reciprocal geodesics on the orbifold surface. So for a free group $G = \mathbb{Z}_2 * H$, we will have an orbifold $\mathbb{H}/G$ where the *reciprocal geodesics* are the closed geodesics on the orbifold which pass through the order two cone point on the quotient. To study this set and understand its properties, we must develop some more basic concepts and notation.

Definition 3.4.1. A subset of infinite order elements $\mathcal{A}$ of an abstract group G is said to satisfy condition (*) if it is closed under conjugation, taking powers, and each element $x \in \mathcal{A}$ is contained in a maximal infinite cyclic subgroup H of G such that $H \subset \mathcal{A}$. In particular, if $x \in \mathcal{A}$ and there is a $y \in G$, so that $y^n = x$ then $y \in \mathcal{A}$. Thus, every element in $\mathcal{A}$ has a unique primitive element in $\mathcal{A}$ associated to it.

Setting $\mathcal{A}^p = \{\text{primitive elements of } \mathcal{A}\}$ we have, $\mathcal{A}^p \subseteq \mathcal{A} \subseteq \mathcal{W}$. Thus the set $\mathcal{W}$ of all reduced words in the group G contains a subset $\mathcal{A}$ of reduced words with a specific property, and within those $\mathcal{A}$, there is a subset of words with the desired property which are also primitive, the $\mathcal{A}^p$. Since each of these subsets is closed under conjugation by elements of G we define the conjugacy classes of these subsets by W, A, and A^p respectively. W is the full set of conjugacy classes in G and we denote the *non-primitive conjugacy classes* in A by A^{np}.

We will choose particular $\mathcal{A}$'s and study the growth rate of their conjugacy and primitive conjugacy classes in $\mathcal{A}$. Our *growth* is measured according to word length in terms of the generators. Let t be a positive integer. A_t denotes the conjugacy classes in A of length t and $|\mathrm{A}_t|$ is the number of conjugacy classes in A of length t. A *proper divisor* of t is a positive integer that divides t but is not 1 or t. We next define a map from primitive conjugacy classes to non-primitive conjugacy classes given by a power map.

Lemma 3.4.2. *The map* $\iota : \bigcup_{s|t} \mathrm{A}_s^p \to \mathrm{A}_t$ *given by* $[x] \mapsto [x^{t/s}]$ *is well defined and a bijection onto the set of non-primitive conjugacy classes in* A_t. *That is, the non-primitive conjugacy classes in* A_t *are in one to one* *correspondence with elements of* $\dot{\bigcup}\iota(\mathrm{A}_s^p)$*, where the union is over all proper divisors,* s*, of* t*.*

Proof. ι is well defined since powers commute with conjugation. In other words, mapping an element in $[x]$ will produce an element in $[x^{t/s}]$. We choose an element in $[x]$, say gxg^{-1} and now take $\iota(gxg^{-1})$, we get $(gxg^{-1})^{t/s} = gxg^{-1}gxg^{-1}\ldots gxg^{-1}$ where there are t/s many copies of gxg^{-1} on the right hand side. This simplifies upon reduction to $gx^{t/s}g^{-1}$ which is an element of $[x^{t/s}]$ as desired. To prove surjectivity, suppose $[y] \in \mathrm{A}_t$ is non-primitive and that there exists $[x] \in \mathrm{A}_s^p$ so that $[x]^s = [y]$. By item (4) of Lemma 3.3.2, $s\|[x]\| = \|[y]\|$ and hence s divides t. If $[y]$ was a proper power of $[x]$, then s is a proper divisor of t.

Next to prove injectivity we need to establish the following.

(1) if $[x_1] \neq [x_2]$ in A_s^p, then $\iota(x_1)$ is not conjugate to $\iota(x_2)$.

(2) if s_1 and s_2 divide t , $s_1 \neq s_2$, then $\iota(\mathrm{A}_{s_1}^p) \bigcap \iota(\mathrm{A}_{s_2}^p) = \emptyset$.

For item (1), suppose that $[x_1] \neq [x_2]$ and $\iota(x_1)$ is conjugate to $\iota(x_2)$. Thus we have $[x_1^{t/s}] = [x_2^{t/s}]$ that is, $x_1^{t/s} = gx_2^{t/s}g^{-1}$ for some g. Then $x_1^{t/s} = (gx_2g^{-1})^{t/s}$ and so $x_1 = gx_2g^{-1}$ i.e. $[x_1] = [x_2]$ which is a contradiction. For item (2) If $s_1 \mid t$ and $s_2 \mid t$ then $s_1 n = t$ and $s_2 m = t$ for some $n, m \in \mathbb{Z}$. Since $s_1 \neq s_2$, then $n \neq m$. So $\|[\iota(\mathrm{A}_{s_1}^p)]\| = n$ and $\|[\iota(\mathrm{A}_{s_2}^p)]\| = m$. They are not the same length, and so not in the same conjugacy class and so $\iota(\mathrm{A}_{s_1}^p) \bigcap \iota(\mathrm{A}_{s_2}^p) = \emptyset$.

□

We have established,

Theorem 3.4.3. *Suppose* $\mathcal{A} \subset G$ *satisfies condition (*). Then*

$$\mathrm{A}_t^p = \mathrm{A}_t - \dot{\bigcup}\iota(\mathrm{A}_s^p) \text{ and } |\mathrm{A}_t^p| = |\mathrm{A}_t| - \sum |\mathrm{A}_s^p|$$

where the union and sum are over all proper divisors, s, of t.

We next recall the following definition from abstract algebra.

Definition 3.4.4. For a group G the *commutator* of two elements $g, h \in G$ is $ghg^{-1}h^{-1}$ which is denoted by $[g, h]$.

We return to the setting of free products, in particular, $G = \mathbb{Z}_2 * H$ where a is the generator in $\mathbb{Z}_2$ and b the generator in a cyclic group H. Set $\mathcal{R} = \{xy : x \text{ and } y \text{ distinct order two elements } \in G\}$ which is equivalent to writing

$$\mathcal{R} = \left\{x[a, y]x^{-1} : x, y \in G, y \neq a\right\} \tag{3.1}$$

$$= \left\{[xax^{-1}, xyx^{-1}] : x, y \in G, y \neq a\right\} \tag{3.2}$$

Definition 3.4.5. A *primitive reciprocal word* is a primitive word that can be expressed as the product of two conjugates of a. Thus a primitive reciprocal word is conjugate to $[a, \gamma]$ where γ can be taken to be a (bb)-word.

Definition 3.4.6. A *reciprocal word* is the power of a primitive reciprocal word.

Reciprocal words have the following form

$$ab^{k_1}ab^{k_2} \cdots ab^{k_n}ab^{-k_n} \cdots ab^{-k_2}ab^{-k_1} \tag{3.3}$$

We remark that the set of reciprocal words is closed under taking powers and moreover if an element of the set of reciprocal words is a power of say $y \in G$, then y is also in the set of reciprocal words. Thus the set of reciprocal words satisfies condition $(*)$. The reciprocal words correspond to reciprocal

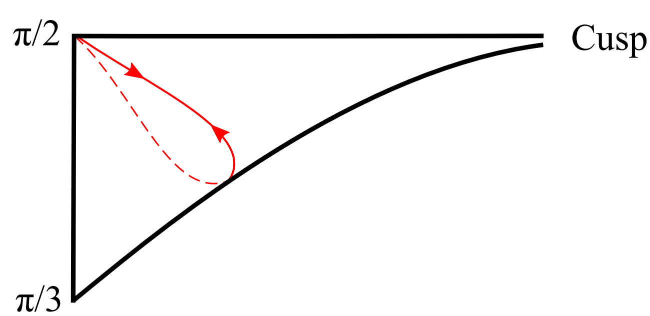

Figure 3.7: A reciprocal geodesic on $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$

geodesics on the orbifold $\mathbb{H}/G$. Figure (3.7) illustrates an example of a reciprocal geodesic on $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$.

Let $w = ab^{\epsilon_0}ab^{\epsilon_1}\cdots ab^{\epsilon_{2t-1}}$ represent a word in $G = \mathbb{Z}_2 * H$ in (ab)-form as described in 3.3.1 for a fixed positive integer t. We've switched notation here to have subscripts starting at zero to simplify computations modulo $2t$ in our subsequent proofs. The ϵ_i's are integer values from the set Y_m which depends on the second factor H of the free product. When H is $\mathbb{Z}$, then the allowable integer values Y_m for the ϵ_i's are $\mathbb{Z}$. When H is $\mathbb{Z}_k$, where k is odd and $k \geq 3$, set $m = \dfrac{k-1}{2}$, then ϵ_i's come from the set

$$Y_m = \{-m, -m+1, \ldots, -2, -1, 1, 2, \ldots, m-1, m\}$$

We identify the words of the form $w = ab^{\epsilon_0}ab^{\epsilon_1}\cdots ab^{\epsilon_{2t-1}}$ with the set of $2t$-tuples, $X_{2t} = \{(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) : \epsilon_i \in Y_m\}$ as in ([2]) and define α to be the action of taking a cyclic permutation of the coordinates of a tuple where the last coordinate is moved to the first position. Thus,

$$\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) = (\epsilon'_0, \epsilon'_1, ..., \epsilon'_{2t-1})$$

where $\epsilon'_j = \epsilon_{j-k}$ for all $j = 0, 1, ..., 2t-1$. We remind the reader of 3.3.2

and remark that the orbit of the action of α produces all elements in the conjugacy class of w.

Definition 3.4.7. The *diagonal* of the orbit of α consists of the tuples in reciprocal word form. We denote this subset of X_{2t} by D_{2t} and specify its form by $D_{2t} = \{(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) : \epsilon_j = -\epsilon_{2t-j-1} \quad \forall j\}$.

Lemma 3.4.8. *Fix an integer k. For any $(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in X_{2t}$,*

(1) $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) = (\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) =$ iff $\epsilon_j = \epsilon_{j-k}$ for all j.

(2) $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ iff $\epsilon_{j-k} = -\epsilon_{2t-j-1-k}$ for all j.

Proof. Item (1) follows from the definition of the cyclic action. For item (2), using our condition on diagonal elements, we have that if $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$, then

$$\epsilon'_j = -\epsilon'_{2t-j-1}$$

Now using the action of α^k we have

$$\epsilon_{j-k} = -\epsilon_{2t-j-1-k}$$

Each step in this proof is reversible.

□

Lemma 3.4.9. *If $(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ then $\alpha^t(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$*

Proof. Assume $(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$. By definition of the diagonal, we have that

$$\epsilon_{j-t} = -\epsilon_{2t-(j-t)-1}$$

$$= -\epsilon_{3t-j-1}$$

$$= -\epsilon_{t-j-1}$$

which is equal to item (2) of lemma 3.4.8 for $k = t$.

□

Lemma 3.4.10. *Fix an integer k and an $(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$. $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ iff for all $j = 0, 1, ..., 2t-1$*

$$\epsilon_j = \epsilon_{j-2k}$$

Proof. Since $(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$, we have that $\epsilon_j = -\epsilon_{2t-j-1}$ for all j. Also, assuming $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$, we have item (2) of lemma 3.4.8 in which we replace j with $2t - j - 1 + k$ to get

$$\epsilon_{2t-j-1} = -\epsilon_{2t-(2t-j-1+k)-1-k}$$

$$= -\epsilon_{j-2k}$$

And so using our condition to be on the diagonal above, we have

$$\epsilon_j = -\epsilon_{2t-j-1}$$

$$= -(-\epsilon_{j-2k})$$

$$= \epsilon_{j-2k}$$

For the reverse direction of the proof, assume $\epsilon_j = \epsilon_{j-2k}$ and replace ϵ_j with ϵ_{j-2k} in the condition for a tuple to be in the diagonal to get

$$\epsilon_{j-2k} = -\epsilon_{2t-j-1}$$

Replacing j with $j+k$ yields item (2) of lemma 3.4.8 and so $\alpha^k(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ as desired.

□

Let O be the orbit under the action of α of an element in $D_{2t} \subset X_{2t}$. Set k_0 to be the smallest power of α for which an element of $O \cap D_{2t}$ maps back into $O \cap D_{2t}$. k_0 is an invariant of O and $1 \leq k_0 \leq t$ as k_0 is the minimal value for which we return to the diagonal.

Lemma 3.4.11. *(1) For any integer n, $\alpha^{nk_0}(O \cap D_{2t}) = O \cap D_{2t}$.*

(2) $\alpha^l(O \cap D_{2t}) \cap (O \cap D_{2t}) = \emptyset$ for l not a multiple of k_0.

(3) $\alpha^{k_0}(x) \neq x$ for any $x \in O \cap D_{2t}$.

Proof. Set $x = (\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ for the entire proof. For the proof of item (1), only one containment direction is necessary, we need only show $\alpha^{nk_0}(O \cap D_{2t}) \subset O \cap D_{2t}$. Since k_0 is the smallest value for which an element of $O \cap D_{2t}$ maps back into $O \cap D_{2t}$ we know $\alpha^{k_0}(x) \in O \cap D_{2t}$. By lemma 3.4.10, we have $\epsilon_j = \epsilon_{j-2k_0} = \cdots = \epsilon_{j-2nk_0}$ for all j and so $\alpha^{nk_0}(x) \subset O \cap D_{2t}$. For item (2), we proceed by contradiction assuming $\alpha^l(x) \in O \cap D_{2t}$ and that $nk_0 \leq l \leq (n+1)k_0$. $\alpha^{l-nk_0}(\alpha^{nk_0}(x)) = \alpha^l \in O \cap D_{2t}$. By item (1), $\alpha^{nk_0}(x) \in O \cap D_{2t}$ and so $\alpha^{l-nk_0}(O \cap D_{2t}) \cap (O \cap D_{2t}) \neq \emptyset$. But this contradicts the minimality if k_0.

For item (3), Assume $\alpha^{k_0}(x) = x$. Thus there is a repeating subtuple y of length k_0 which fills out the tuple representation of x. Moreover, since the elements of $\mathcal{R}$ satisfy condition $(*)$, this subtuple must be in D_{k_0}. Thus, k_0 is even as elements of D_{k_0} are of even length. However, if k_0 is even then then item (1) of 3.4.8 can be written as $\epsilon_j = \epsilon_{j-2\frac{k_0}{2}}$. Comparing this to 3.4.10, we see that $\frac{k_0}{2}$ would be an integer value for which $\alpha^{\frac{k_0}{2}}(\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in D_{2t}$ which contradicts the minimality of k_0. And so $\alpha^{k_0}(x) \neq x$.

□

Proposition 3.4.12. *Let $x \in O \cap D_{2t}$. $\alpha^{nk_0}(x) = x$ for n even and $\alpha^{nk_0}(x) = \alpha^{k_0}(x)$ for n odd. Namely, $O \cap D_{2t}$ consists of two distinct elements, $\{x, \alpha^{k_0}(x)\}$*

Proof. Set $x = (\epsilon_0, \epsilon_1, ..., \epsilon_{2t-1}) \in O \cap D_{2t}$. By items (1) and (2) of 3.4.11, we know all the points in $O \cap D_{2t}$ are of the form $\alpha^{nk_0}(x)$. Since $\alpha^{k_0}(x) \in O \cap D_{2t}$, we have $\epsilon_j = \epsilon_{j-2k_0}$ by lemma 3.4.10, thus $\alpha^{nk_0}(x) = x$ for n even. For n odd, we can write $n = 2m + 1$ and so we have that $\alpha^{nk_0}(x) = \alpha^{(2m+1)k_0}(x) = \alpha^{k_0}(\alpha^{2mk_0}(x)) = \alpha^{k_0}(x)$. Finally, recall $\alpha^{k_0}(x) \neq x$ by item (3) of 3.4.11. □

From the discussion above we see that the $\langle\alpha\rangle$-orbit of an element in X_{2t} contains $2k_0 - 1$ many points. Of these points, there are exactly two which are distinct and in the diagonal D_{2t}.

Definition 3.4.13. Let t be a positive integer and $x \in X_t$. x is called *nonprimitive* if there exists an s which properly divides t and a $z \in X_{t/s}$ such that $x = z^s$. Thus, if x is nonprimitive, its tuple is comprised of a subtuple z which is repeated s many times. Otherwise, we say x is primitive.

Every element of X_{2t} is an integer power of a primitive subtuple. If $x \in D_{2t} \in X_{2t}$ is nonprimitive, then $x = z^s$ where $z \in D_{2t/s}$.

Proposition 3.4.14. *Let $x \in D_{2t}$. x is primitive if and only if $k_0 = t$.*

Proof. Suppose x is nonprimitive. Then $x = z^s$ where $z \in D_{2t/s}$, i.e. x is a subtuple repeated s many times. The length of the subtuple z is $2t/s$ and $k_0 < 2t/s$. Since $2t/s < t$, then $k_0 < t$. Now suppose x is primitive, then there is no such proper s and so $s = t$. □

$$\mathcal{N} = \{[a, \gamma] : \gamma \text{ is a (bb)-word}\} = \{[a, \beta]^n : [a, \beta] \text{ is primitive}\} \qquad (3.4)$$

We remark that when $[a, \beta]$ is primitive, then β cannot be order two. From our work on tuples and using the notation above we have the following lemma.

Lemma 3.4.15. *Each conjugacy class of an element of $\mathcal{R}$ has exactly two representatives in the normal form $\mathcal{N}$. The two conjugates in $\mathcal{N}$ are $[a, \beta]^n$ and $[a, \beta^{-1}]^n$ where $[a, \beta]$ is primitive, n a unique positive integer and β is a unique (bb)-word which is not of order two.*

Chapter 4

Analytic combinatorics, computational and algebraic tools

In order to prove the results in our next chapter, we will need to borrow techniques from other areas of mathematics than the traditional methods of hyperbolic geometry. Our work here relies heavily on recurrence relations, combinatorics and some results from algebra to justify the asymptotic analysis of reciprocal words. We are using more applied methods and counting techniques to make statements about the growth of very specific geometric objects. This chapter is intended to bridge the gap between the geometry and computations by presenting the basics about these techniques that we will be using for our main results.

4.1 Analytic combinatorics

The area of analytic combinatorics is a vast subject with many beautiful applications. Our reference ([7]) alone provides over 800 pages in this field with many interesting problems to consider. We can also find these standard methods in ([8]) and ([15]). We will of course focus here on the specific methods used for our results and note that this is barely scratching the surface in terms of the computational tools available in this area.

We first recall some basic combinatoric definitions.

Definition 4.1.1. The number of possible combinations of n objects taken r at a time is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Definition 4.1.2. Pascal's identity for binomial coefficients is given by

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Definition 4.1.3. The number of compositions, that is, distinct ways to get n many positive integer valued numbers to add up to x is given by

$$\Phi_n(x) = \binom{x-1}{n-1}$$

Definition 4.1.4. The number of distinct ways to get n many positive integer valued numbers, where the integer values must all be $\leq m$, to add up to x is denoted by the function $\Phi_n^m(x)$. This function does not have a simple combinatoric form as the function in definition 4.1.3 above. At the time of writing this thesis there is no generalized form for these bounded compositions, only a few formulas for very specific cases.

A *recurrence relation* is an equation which describes a recursive sequence. The recursive way of producing terms in such a sequence involves initial conditions and rules for obtaining terms based on previous terms in the sequence. A *linear recurrence relation* is one where the recurrence is a linear function of the previous terms and can be of two types, homogeneous and non-homogeneous. A linear *homogeneous* recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_1 + c_2 a_2 + \cdots + c_k a_{n-k}$$

where the $c_1, c_2 ...., c_k$ are non-zero real coefficients. A linear recurrence relation which is *non-homogeneous* of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_1 + c_2 a_2 + \cdots + c_k a_{n-k} + f(n)$$

where the $c_1, c_2 ...., c_k$ are non-zero real coefficients and f(n) is a function which depends only on n. For the purposes of our work in this thesis, we will only need to consider linear homogeneous recurrence relations.

While these types of sequences can be written in terms of generating functions which are useful for other types of analysis, we will be using the linear recursive form as described above to get closed forms for our sequences by way of the characteristic root method. We begin by taking a recurrence such as

$$a_n = c_1 a_1 + c_2 a_2 + \cdots + c_k a_{n-k}$$

and rewrite it in the form

$$a_n - c_1a_1 - c_2a_2 - \cdots - c_ka_{n-k} = 0$$

and use the coefficients and the form above to get the characteristic equation

$$x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_k = 0$$

We then find the roots of the polynomial

$$P(x) = x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_k \tag{4.1}$$

which we will refer to as $r_1, ..., r_k$. The closed form solution to the recurrence will depend on these roots. In the case where these roots are distinct, the closed form will be

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_n r_n^n$$

where the α_i's are constants. In the case where we have multiple roots, the coefficients are not all constants, but in fact are polynomials in n for the terms in the solution corresponding to the multiple roots of the equation. The groups we will be working with in the next chapter we produce characteristic equations with distinct roots and so we will not discuss the multiple root case any further as we will only need the form above for our computations.

Example 4.1.5. A well known example of a linear homogeneous recurrence relation is the Fibonacci sequence given by

$$a_n = a_{n-1} + a_{n-2}$$

a degree 2 recurrence where both coefficient values are 1 and with initial conditions of $a_0 = 0$ and $a_1 = 1$. Using

$$a_n - a_{n-1} - a_{n-2} = 0$$

we obtain the characteristic equation

$$x^2 - x - 1 = 0$$

The roots of which are $x = (1+\sqrt{5})/2$ and $x = (1-\sqrt{5})/2$. Since the roots are distinct, we can use the method above to obtain the form

$$a_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$$

Using the initial conditions we can find the full form by solving for the coefficients. Upon substitution we have

$$\alpha_1 + \alpha_2 = 0 \text{ and } \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 1$$

and so our coefficients are $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$ giving us a final closed form of

$$a_n = 1/\sqrt{5}((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$$

4.2 Computational and algebraic tools

Definition 4.2.1. Given functions $f(x)$ and $g(x)$, we say $f(x)$ is *asymptotic* to $g(x)$, denoted by $f(x) \sim g(x)$, if and only if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$

Definition 4.2.2. For $r \neq 1$ the sum of the first n terms of a geometric series with a as its first term and r as the common ratio is

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right)$$

Theorem 4.2.3. ***Bolzano's theorem*** *(special case of intermediate value theorem). Let f be a continuous function on a closed interval $[a, b]$ of the real line such that $f(a)$ and $f(b)$ have opposite signs, then there exists a $c \in [a, b]$ such that $f(c) = 0$.*

Theorem 4.2.4. ***Descarte's rule of signs****. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients:*

1. *The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.*

2. *The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.*

Theorem 4.2.5. ***Rational root theorem****. If a polynomial with integer coefficients $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has a rational root of the form $r = \pm\frac{a}{b}$ with $gcd(a, b) = 1$ then $a|a_0$ and $b|a_n$.*

Theorem 4.2.6. ***Eisenstein Criterion*** *([4]). Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function with integer coefficients. Suppose there exists a prime p such that*

1. $p \nmid a_n$

2. $p | a_{n-1}, a_{n-2}, ..., a_1, a_0$

3. $p^2 \nmid a_0$

Then $f(x)$ is irreducible over $\mathbb{Z}$.

Theorem 4.2.7. *([4]) Let $f(x)$ be an irreducible polynomial over $\mathbb{Q}$. Then $f(x)$ cannot have repeated roots in $\mathbb{C}$.*

Theorem 4.2.8. ***Gauss lemma for monic polynomials*** *([4]). Every real root of a monic polynomial with integer coefficients is either an integer or irrational.*

Definition 4.2.9. ([12]) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be an n-th degree monic polynomial with complex coefficients. Then the *Cauchy bound* of p denoted ρ is the unique positive root of the polynomial $z^n - \sum_{k=0}^{n-1} |a_k| z^k$.

Theorem 4.2.10. *([12]) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be an n-th degree monic polynomial with complex coefficients. Then all zeros of $p(z)$ have modulus less than or equal to ρ.*

Chapter 5

Counting reciprocal words and their growth

5.1 $\mathbb{Z}_2 * \mathbb{Z}$

Consider reciprocal words in $G = \mathbb{Z}_2 * \mathbb{Z}$. Let a be a generator in $\mathbb{Z}_2$ and we let b be a generator in $\mathbb{Z}$ as described in example (3.2.3). Since reciprocal words have even length we use the parameter $2t$ for ease of computation. We begin with a computational lemma which will be used in counting the conjugacy classes $|R_{2t}|$. Recall the function $\Phi_n(x) = \binom{x-1}{n-1}$ from definition 4.1.3 in the previous chapter.

Lemma 5.1.1. *The expression* $\sum_{n=1}^{\lfloor t/2 \rfloor} \Phi_n(t-n)2^n$ *yields the closed form* $\frac{1}{3}(2^t + 2(-1)^t)$.

Proof. Starting with $\sum_{n=1}^{\lfloor t/2 \rfloor} \Phi_n(t-n)2^n$, we work our way to a summation form for what is known in number theory as the Jacobsthal sequence ([6]). Using $\lfloor x/2 \rfloor = \left\lfloor \dfrac{x}{2} - 1 + 1 \right\rfloor = \left\lfloor \dfrac{x}{2} - 1 \right\rfloor + 1 = \left\lfloor \dfrac{x-2}{2} \right\rfloor + 1$, we rewrite our sum as

$$\sum_{n=1}^{\lfloor t/2 \rfloor} \Phi_n(t-n)2^n \quad = \quad \sum_{n=1}^{\left\lfloor \frac{t-2}{2} \right\rfloor + 1} \binom{t-n-1}{n-1} 2^n$$

$$= \quad \sum_{n=1}^{\left\lfloor \frac{(t-1)-1}{2} \right\rfloor + 1} \binom{(t-1)-n)}{n-1} 2^n$$

we replace $m = t - 1$ and get

$$\sum_{n=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor + 1} \binom{m-n}{n-1} 2^n \quad = \quad \sum_{n=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor + 1} \binom{m-1-(n-1)}{n-1} 2^n$$

then replace $r = n - 1$ to get

$$\sum_{r=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-1-r}{r} 2^{r+1} \quad = \quad 2 \sum_{r=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-1-r}{r} 2^r \quad = 2J_m$$

Thus we have twice the Jacobsthal sequence, J_m, known in number theory as one of the Fibonacci type sequences ([6]). As such, this sequence can be written in the form of a linear recurrence relation with constant coefficients as follows,

$$J_m = J_{m-1} + 2J_{m-2} \quad \text{or} \quad J_m - J_{m-1} - 2J_{m-2} = 0$$

Following the standard method for recurrence relations ([7]) which was outlined in chapter four, we will use this equation to get the characteristic polynomial.

$$x^2 - x - 2 = 0$$

which we factor to get the roots $x = -1$ and $x = 2$. These roots are then used as the bases in the following form

$$J_m = a(-1)^m + b(2)^m$$

We then use initial values of the sequence to solve for a and b and we get

$$J_m = -\frac{1}{3}(-1)^m + \frac{1}{3}(2)^m \quad = \quad \frac{1}{3}((2)^m - (-1)^m)$$

□

Lemma 5.1.2. *For $G = \mathbb{Z}_2 * \mathbb{Z}$,*

$$|R_{2t}| = \frac{1}{6}(2^t + 2(-1)^t). \tag{5.1}$$

Proof. We start with a reciprocal word of the form

$$ab^{k_1}ab^{k_2}\cdots ab^{k_n}ab^{-k_n}\cdots ab^{-k_2}ab^{-k_1} \tag{5.2}$$

The exponents for the generator b satisfy, $\sum_{i=1}^{n}|k_i| = t - n$. Denoting the number of ordered choices of positive integers $(|k_1|, ..., |k_n|)$ which sum to $t - n$ by $\Phi_n(t-n)$ and noting that there are two possible signs for each exponent k_i, we have that the total number of words of the form (5.2) is equal to

$$|\mathcal{N}_{2t}| = \sum_{n=1}^{\lfloor t/2 \rfloor} \Phi_n(t-n)2^n.$$

We get the expression (5.1) equals $\frac{1}{3}(2^t + 2(-1)^t)$ by lemma 5.1.1 and we divide by 2 using lemma 3.4.15 to get

$$|R_{2t}| = \frac{1}{6}(2^t + 2(-1)^t).$$

as a closed form for counting the conjugacy classes as desired.

□

Lemma 5.1.3. *For $G = \mathbb{Z}_2 * \mathbb{Z}$,*

$$|R_{2t}^{np}| \leq \frac{1}{6}t2^{t/2} \tag{5.3}$$

Proof. Fix t a positive integer. All sums in this proof are over the proper divisors of t. Note that by Lemma 3.4.2 we have,

$$|R_{2t}^{np}| = \sum |R_{2s}^{p}| \leq \sum |R_{2s}| = \sum \frac{1}{6}(2^s + 2(-1)^s) \leq \frac{2}{6}\sum 2^s \tag{5.4}$$

Using the fact that the largest possible proper divisor of t is $s = t/2$, we have that the right hand side of equation (5.4) is,

$$\leq \frac{2}{6}2^{t/2}\sum 1 \leq \frac{1}{6}t2^{t/2}, \tag{5.5}$$

where the last inequality follows from the fact that the most number of proper divisors an integer t can have is $t/2$.

□

Applying Lemma 5.1.3 to Theorem 3.4.3 we have the inequality,

$$|R_{2t}| - \frac{1}{6}t2^{t/2} \leq |R_{2t}^{p}| \leq |R_{2t}|, \tag{5.6}$$

and hence,

$$1 - \frac{\frac{1}{6}t2^{t/2}}{|R_{2t}|} \leq \frac{|R_{2t}^{p}|}{|R_{2t}|} \leq 1. \tag{5.7}$$

Using this inequality and that $\frac{\frac{1}{6}t2^{t/2}}{|R_{2t}|} \longrightarrow 0$ as $t \longrightarrow \infty$ we prove the following theorem.

Theorem 5.1.4. *Let $G = \mathbb{Z}_2 * \mathbb{Z}$, R the reciprocal word conjugacy classes in G, and R^p the primitive classes in R.*

$$|R^p_{2t}| \sim |R_{2t}| \sim \frac{1}{6}2^t, \text{ as } t \to \infty. \tag{5.8}$$

5.2 $\mathbb{Z}_2 * \mathbb{Z}_3$

We next compute the growth rate of conjugacy classes of reciprocal words in $\mathbb{Z}_2 * \mathbb{Z}_3$. The reciprocal words in this group have word length a multiple of 4 and so we use the parameter $4t$ for ease of computation.

Lemma 5.2.1. *For $G = \mathbb{Z}_2 * \mathbb{Z}_3$,*

$$|R_{4t}| = \frac{1}{2}2^t. \tag{5.9}$$

Proof. Let $[w] \in R_{4t}$. We pick as representative a cyclically reduced word of the form

$$w = ab^{x_1}...ab^{x_n}ab^{-x_n}...ab^{-x_1}$$

where $x_i = \pm 1$. The result follows since there are exactly two conjugates of this form as per Lemma 3.4.15 and there are 2^t many words of this form. $\square$

Lemma 5.2.2. *For $G = \mathbb{Z}_2 * \mathbb{Z}_3$,*

$$|R^{np}_{4t}| \leq \frac{1}{4}t2^{t/2}$$

Proof. Fix t a positive integer. All sums in this proof are over the proper divisors of t. We have

$$|R_{4t}^{np}| \leq \sum_{s|t} |R_{4s}^{p}| \leq \sum_{s|t} |R_{4s}|$$

$$= \frac{1}{2} \sum_{s|t} 2^{s} \leq \frac{1}{4} t 2^{t/2}$$

where the last inequality follows from the fact that the largest proper divisor of t is $t/2$ and there are at most $t/2$ many divisors.

$\square$

Theorem 5.2.3. *Let $G = \mathbb{Z}_2 * \mathbb{Z}_3$, R the reciprocal word conjugacy classes in G, and R^p the primitive conjugacy classes in R. Then*

$$|\mathrm{R}_{4t}^{p}| \sim |\mathrm{R}_{4t}| \sim 2^{t-1}, \ \textit{as } t \to \infty.$$

Proof. Applying Lemma 5.2.2 and Theorem 3.4.3 we have

$$|R_{4t}| - \frac{1}{4} t 2^{t/2} \leq |R_{4t}^{p}| \leq |R_{4t}|$$

$$1 - \frac{\frac{1}{4} t 2^{t/2}}{|R_{4t}|} \leq \frac{|R_{4t}^{p}|}{|R_{4t}|} \leq 1$$

and thus the desired asymptotic since

$$\frac{\frac{1}{4} t 2^{t/2}}{|R_{4t}|} \longrightarrow 0 \quad as \quad t \longrightarrow \infty$$

$\square$

5.3 $\mathbb{Z}_2 * \mathbb{Z}_k$

In this section $G = \mathbb{Z}_2 * \mathbb{Z}_k$, $k \geq 4$, and we let a represent a generator for the first factor $\mathbb{Z}_2$, and b a generator in the second factor $\mathbb{Z}_k$. These generators are chosen as they achieve minimal length. We set $m = \dfrac{k-1}{2}$, so the choices for exponents on the generator b that will yield words with minimal length and that will stay within the constraints of the second factor come from the set

$$Y_m = \{-m, -m+1, \ldots, -2, -1, 1, 2, \ldots, m-1, m\}$$

As in our previous groups, we are able to conjugate to (ab) type words in the free product. We specifically focus on reciprocal words in normal form shown below where the $x_i's$ are in Y_m.

$$w = ab^{x_1}ab^{x_2} \cdots ab^{x_n}ab^{-x_n} \cdots ab^{-x_2}ab^{-x_1} \tag{5.10}$$

5.3.1 $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd

For the following theorem recall $\Phi_n^m(t-n)$ as defined in 4.1.4 in chapter four. Due to the restriction of the part sizes by m, there will be less compositions than for the general $\mathbb{Z}_2 * \mathbb{Z}$ situation.

Theorem 5.3.1. *Let $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd, and $m = \dfrac{k-1}{2}$. Then the number of reciprocal words in $\mathbb{Z}_2 * \mathbb{Z}_k$ is given by*

$$|\mathcal{N}_{2t}| = \sum_{n=\lceil t/(m+1) \rceil}^{\lfloor t/2 \rfloor} \Phi_n^m(t-n)2^n \tag{5.11}$$

Proof. Since k is odd here, our set for possible exponents on the generator b of $\mathbb{Z}_k$ is $Y_m = \{-m, -m+1, \ldots, -2, -1, 1, 2, \ldots, m-1, m\}$. This yields two possible words with length 1 in $\mathbb{Z}_k$, two of length 2, all the way up to and including length m hence 2^n. The upper bound on the sum is due to the overall word length of $2t$. When $t - n \leq nm$ we will have nonzero terms for the sum as $\Phi_n^m(t-n)$ is non-zero and so we have the lower integer bound of $n = \lceil t/(m+1) \rceil$ for the sum.

□

The theorem above describes conceptually a counting scheme similar to the one used in the $\mathbb{Z}_2 * \mathbb{Z}$ case. We cannot however proceed in the same fashion as there is no simple general formula for the restricted Φ_n^m function here. Instead, we must produce the reciprocal words by way of constructing a recurrence relation which in turn can be used to get a closed form.

Theorem 5.3.2. *For $\mathbb{Z}_2 * \mathbb{Z}_k$ and $m = \dfrac{k-1}{2}$ with k odd, the number of reciprocal words of length $2t$, $|\mathcal{N}_{2t}|$, satisfies the following:*

(i) for $t \leq m+1$, $|\mathcal{N}_{2t}| = \frac{1}{3}(2^t + 2(-1)^t)$

(ii) for $t = m+2$, $|\mathcal{N}_{2t}| = \frac{1}{3}(2^t + 2(-1)^t) - 2$

(iii) for $t > m+2$, $|\mathcal{N}_{2t}| = 2|\mathcal{N}_{2(t-2)}| + 2|\mathcal{N}_{2(t-3)}| + \cdots + 2|\mathcal{N}_{2(t-(m+1))}|$

Proof. We recall work from our previous section for the group $\mathbb{Z}_2 * \mathbb{Z}$ that the number of words in reciprocal form in $\mathbb{Z}_2 * \mathbb{Z}$ is given by

$$\sum_{n=1}^{\lfloor t/2 \rfloor} \Phi_n(t-n)2^n \tag{5.12}$$

which we found to have an equivalent closed form of

$$\frac{1}{3}(2^t + 2(-1)^t) \tag{5.13}$$

(i) For $t \leq m+1$, the lower bound on the sum in (5.11) from 5.3.1 will be equal to 1 matching the lower bound for (5.12).

Fix $t = m+1$. Then we have $\Phi_n(m+1-n)$ in (5.12) and $\Phi_n^m(m+1-n)$ in (5.11). When $n = 1$, $\Phi_1(m) = \Phi_1^m(m) = 1$ as only one exponent of value m is possible. As n increases and thus the number of parts in the composition increase, the argument $t - n$ decreases in both functions. The size of the parts must decrease as we compute the rest of the terms in the sum from $n = 2$ on and in turn all exponents must be less than m for both functions. As a result, both Φ_n and Φ_n^m match for the rest of the values of n up to and including $\lfloor t/2 \rfloor$.

For fixed values $t < m+1$, the parts for all values of n must be strictly less than m, and we again have the same matching behavior as described above.

(ii) Fix $t = m+2$. Then we have $\Phi_n(m+2-n)$ in (5.12) and $\Phi_n^m(m+2-n)$ in (5.11). When $n = 1$, $\Phi_1(m+1) = 1$ while $\Phi_1^m(m+1) = 0$. As n increases and thus the number of parts in the composition increase, the argument $t - n$ decreases in both functions. The size of the parts must decrease as we compute the rest of the terms in the sum from $n = 2$ on and in turn all exponents must be less than or equal to m for both functions. As a result, both Φ_n and Φ_n^m match for the rest of the values of n up to and including $\lfloor t/2 \rfloor$.

(iii) We take w to be a reciprocal word in normal form as in (5.10). Take $w \in \mathcal{N}_{2t}$ with word length $|w| = 2t = 2\left(n + \sum_{i=1}^{n} |x_i|\right)$. Recall that within a set $\mathcal{N}_{2t}$ the n values may vary. For $j \in Y_m$ set

$$\mathcal{N}_{2t:j} = \{w \in \mathcal{N}_{2t} \quad | \quad w = ab^j ab^{x_2} \cdots ab^{x_n} ab^{-x_n} \cdots ab^{-x_2} ab^{-j}\} \tag{5.14}$$

$$\mathcal{N}_{2t} = \dot{\cup}_{j \in Y_m} \mathcal{N}_{2t:j} \tag{5.15}$$

We claim that

$$\dot{\cup}_{j \in Y_m} \mathcal{N}_{2t:j} \longleftrightarrow \dot{\cup}_{j \in Y_m} \mathcal{N}_{2(t-|j|-1)} \tag{5.16}$$

We map an element from $\mathcal{N}_{2t:j}$ to $\mathcal{N}_{2(t-|j|-1)}$ by

$$ab^j ab^{x_2} \cdots ab^{x_n} ab^{-x_n} \cdots ab^{-x_2} ab^{-j} \longmapsto ab^{x_2} \cdots ab^{x_n} ab^{-x_n} \cdots ab^{-x_2} \tag{5.17}$$

Observe $t - |j| = n + \sum_{i=2}^{n} |x_i|$ for our word in $\mathcal{N}_{2t:j}$. The length of the new word under our map is $t - |j| - 1 = (n-1) + \sum_{i=2}^{n} |x_i|$ thus we are in fact in $\mathcal{N}_{2(t-|j|-1)}$.

Going from $\mathcal{N}_{2(t-|j|-1)}$ to $\mathcal{N}_{2t:j}$,

$$ab^{x_1} \cdots ab^{x_n} ab^{-x_n} \cdots ab^{-x_1} \longmapsto ab^j ab^{x_1} \cdots ab^{x_n} ab^{-x_n} \cdots ab^{-x_1} ab^{-j} \tag{5.18}$$

Now $t - |j| - 1 = n + \sum_{i=1}^{n} |x_i|$ for our word in $\mathcal{N}_{2t(t-|j|-1)}$. The length of the new word under our map in the opposite direction is $t - |j| - 1 + 1 = n + 1 + \sum_{i=1}^{n} |x_i|$ or $t - |j| = (n+1) + \sum_{i=2}^{n+1} |x_i|$ thus it is in $\mathcal{N}_{2t:j}$.

Next, since

$$\mathcal{N}_{2t} = \dot{\cup}_{j \in Y_m} \mathcal{N}_{2t:j} \tag{5.19}$$

and

$$\dot{\cup}_{j \in Y_m} \mathcal{N}_{2t:j} \longleftrightarrow \dot{\cup}_{j \in Y_m} \mathcal{N}_{2(t-|j|-1)} \tag{5.20}$$

then for Y_m^+ the positive values of Y_m

$$|\mathcal{N}_{2t}| = \sum_{j \in Y_m} |\mathcal{N}_{2(t-|j|-1)}| = 2 \sum_{j \in Y_m^+} |\mathcal{N}_{2(t-|j|-1)}| \tag{5.21}$$

□

Theorem 5.3.3. *The recurrence relation* $|\mathcal{N}_{2t}| = 2|\mathcal{N}_{2(t-2)}| + 2|\mathcal{N}_{2(t-3)}| + \cdots + 2|\mathcal{N}_{2(t-(m+1))}|$ *has a Binet closed form which grows like* $c\rho^t$ *as* $t \longrightarrow \infty$ *for* c *a constant and* ρ *a value between* $\sqrt{2}$ *and* 2.

Proof. We begin with the recurrence relation

$$|\mathcal{N}_{2t}| = 2|\mathcal{N}_{2(t-2)}| + 2|\mathcal{N}_{2(t-3)}| + \cdots + 2|\mathcal{N}_{2(t-(m+1))}|$$

and rewrite it as

$$|\mathcal{N}_{2t}| - 2|\mathcal{N}_{2(t-2)}| - 2|\mathcal{N}_{2(t-3)}| - \cdots - 2|\mathcal{N}_{2(t-(m+1))}| = 0$$

In this form we are able to get the characteristic equation

$$z^{m+1} - 2z^{m-1} - 2z^{m-2} - \cdots - 2z^2 - 2z - 2 = 0 \tag{5.22}$$

We observe that at $z = 2$, $z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$ becomes

$$2^{m+1} - \left(2 \cdot 2^{m-1} + \cdots + 2 \cdot 2 + 2\right)$$

$$= 2^{m+1} - \bigl(2^m + 2^{m-1} + \cdots + 4 + 2\bigr)$$

$$= 2^{m+1} - \left(\left(\frac{1 - 2^{m+1}}{1 - 2}\right) - 1\right) =$$

$$= 2^{m+1} - \bigl(2^{m+1} - 2\bigr) = 2$$

Thus at $z = 2$, $z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$ has a value of 2, a positive value. Also, at $z = \sqrt{2}$ we have

$$(\sqrt{2})^{m+1} - 2(\sqrt{2})^{m-1} - \cdots - 2(\sqrt{2}) - 2$$

$$2^{(m+1)/2} - 2 \cdot 2^{(m-1)/2} - \cdots - 2 \cdot 2^{1/2} - 2$$

$$2^{(m+1)/2} - 2^{((m-1)/2)+1} - \cdots - 2^{3/2} - 2$$

$$2^{(m+1)/2} - 2^{(m+1)/2} - \cdots - 2^{3/2} - 2$$

Since the first two terms cancel, the overall value of the sum of the remaining terms above is negative. So by the intermediate value theorem, there exists at least one real root between $\sqrt{2}$ and 2 for our characteristic polynomial.

Upon further inspection of the characteristic polynomial, observe that

$$P(z) = z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$$

has only one sign change. By Descartes rule of signs, there is only one positive real root for this polynomial, thus the only one real positive root for this characteristic polynomial is the root that exists between $\sqrt{2}$ and 2. By Gauss' lemma for monic polynomials with integer coefficients 4.2.8, this root is irrational. We call this root ρ_k.

Next, justify the dominance of this value for the roots. Applying theorem 4.2.10 to our monic polynomial $P(z) = z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$, we see our ρ_k between $\sqrt{2}$ and 2 is in fact our dominant root.

As discussed in the analytic combinatorics section of chapter four, typical computations to find the closed Binet form for any linear recurrence relation with constant coefficients of the form

$$A_n = c_1 A_{n-1} + c_2 A_{n-2} + \cdots + c_k A_{n-k} \tag{5.23}$$

and with characteristic equation

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_{k-1} x - c_k = 0 \tag{5.24}$$

we get a closed form expression of

$$A_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_q r_q^n \tag{5.25}$$

where the α_i's are constants and the r_i's are the roots that satisfy the corresponding characteristic equation. We claim the roots here are unique and point out that A_n would have a slightly different form were they not. We justify that there are no multiple roots for our current case by considering the following. The characteristic polynomial in this case has a leading coefficient of 1 followed by coefficients of -2 for all the remaining terms. Thus there exists a prime number, 2, which does not divide the leading coefficient of 1, divides all the other coefficients, and whose square, $2^2 = 4$ does not divide the constant term of 2. And so we can use the Eisenstein Criterion 4.2.6

here to conclude that this polynomial is irreducible over the rationals $\mathbb{Q}$. We can then conclude that this polynomial does not have multiple roots in $\mathbb{C}$ as it is irreducible in $\mathbb{Q}$ from theorem 4.2.7. Thus we satisfy the criteria for this type of solution form and we get that the actual closed form for $|\mathcal{N}_{2t}|$ is

$$|\mathcal{N}_{2t}| = \alpha_1 r_1^t + \alpha_2 r_2^t + \cdots + \alpha_{(m+1)} r_{(m+1)}^t \tag{5.26}$$

However, for our purposes it is not important to use the entire Binet form but to recognize that the largest root ρ_k, between $\sqrt{2}$ and 2, will dominate this expression as t gets large. Thus as $t \longrightarrow \infty$, $|\mathcal{N}_{2t}| \sim c_k \rho_k^t$ where c_k is the corresponding α_i coefficient for root ρ_k.

□

Theorem 5.3.4. *For $\mathbb{Z}_2 * \mathbb{Z}_k$ and $m = \dfrac{k-1}{2}$ with k odd, the number of conjugacy classes of reciprocal words with respect to word length $2t$ is*

$$|R_{2t}| = \frac{1}{2} \sum_{n=\lceil t/(m+1) \rceil}^{\lfloor t/2 \rfloor} \Phi_n^m (t-n) 2^n \tag{5.27}$$

Proof. Observe 5.3.1 above provides us with a way to count the number of reciprocal words in normal form in the group $G = \mathbb{Z}_2 * \mathbb{Z}_k$ for k odd. We recall 3.4.15, and apply it in this context. This allows us to divide the total count by two, yielding

$$\frac{1}{2} \sum_{n=\lceil t/(m+1) \rceil}^{\lfloor t/2 \rfloor} \Phi_n^m (t-n) 2^n$$

as desired.

□

For t large we have the Binet form found in 5.3.3 in place of the summation above and so take half of the $c_k\rho_k^t$ approximation we found in that proof. We abuse notation and absorb the 1/2 into the constant and write $c_k\rho_k^t$ to simplify later computations for the asymptotics. Thus $|R_{2t}| \sim c_k\rho_k^t$ as $t \longrightarrow \infty$.

5.3.2 Asymptotic behavior for primitive reciprocal words in $G = \mathbb{Z}_2 * \mathbb{Z}_k$ for k odd

We have that the conjugacy classes for reciprocal words, in $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd, is asymptotic to an exponential function. Specifically, $|R_{2t}| \sim c_k\rho_k^t$ as $t \longrightarrow \infty$, where c_k is a constant and ρ_k is a value between $\sqrt{2}$ and 2 which depends on the value of k for the group. We can now use this for an asymptotic analysis of the primitive reciprocal words in $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd as we did in $\mathbb{Z}_2 * \mathbb{Z}$. Denote the non primitive reciprocal words by $|R_{2t}^{np}|$ and the primitive reciprocal words by $|R_{2t}^{p}|$.

Lemma 5.3.5. *For $\mathbb{Z}_2 * \mathbb{Z}_k$, and for γ_1 a universal constant,*

$$|R_{2t}^{np}| \leq \gamma_1 t \rho_k^{t/2}$$

Proof. Fix $t \in \mathbb{Z}_+$ and note that the sums in this proof are over the proper divisors of t. By Lemma 3.4.2 we have,

$$|R_{2t}^{np}| = \sum |R_{2s}^{p}| \leq \sum |R_{2s}| \leq c_k\rho_k \sum \rho_k^s \tag{5.28}$$

since $|R_{2s}|$ grows approximately like $c_k\rho_k^s$. The largest proper divisor of t possible is $s = t/2$ and so

$$|R_{2t}^{np}| \leq c_k \rho_k \sum \rho_k^s \leq c_k \rho_k \rho_k^{t/2} \sum 1 \leq \frac{c_k \rho_k}{2} t \rho_k^{t/2} \tag{5.29}$$

since there are at most $t/2$ many proper divisors possible for integer t. If we replace $\frac{c_k \rho_k}{2}$ by γ_1, we get the desired inequality.

□

Applying Lemma 5.3.5 to Theorem 3.4.3 we have,

$$|R_{2t}| - \gamma_1 t \rho_k^{t/2} \leq |R_{2t}^p| \leq |R_{2t}| \tag{5.30}$$

$$1 - \frac{\gamma_1 t \rho_k^{t/2}}{|R_{2t}|} \leq \frac{|R_{2t}^p|}{|R_{2t}|} \leq 1 \tag{5.31}$$

and so since $\frac{\gamma_1 t \rho_k^{t/2}}{|R_{2t}|} \longrightarrow 0$ as $t \longrightarrow \infty$ we have shown the following.

Theorem 5.3.6. *Let $\mathbb{Z}_2 * \mathbb{Z}_k$, c_k constant*

$$|R_{2t}^p| \sim |R_{2t}| \sim c_k \rho_k^t, \text{ as } t \to \infty.$$

5.4 Summary of asymptotic results

Tables 5.1 and 5.2 summarize the asymptotic results obtained in this work.

From a geometric standpoint, we see that for $\mathbb{Z}_2 * \mathbb{Z}_k$ the limiting case as $k \to \infty$ is $\mathbb{Z}_2 * \mathbb{Z}$. As $k \to \infty$ the angle $\frac{\pi}{k}$ in the fundamental domain of $\mathbb{Z}_2 * \mathbb{Z}_k$ goes to 0, thus approaching the fundamental domain picture for

Table 5.1: Asymptotics of reciprocal words

	$\lvert R_{2t}\rvert$	$\lvert R^p_{2t}\rvert$
$\mathbb{Z}_2 * \mathbb{Z}_k$ k odd	$c_k \rho_k^t$	$c_k \rho_k^t$
$\mathbb{Z}_2 * \mathbb{Z}$	$\frac{1}{6} 2^t$	$\frac{1}{6} 2^t$

Table 5.2: Asymptotics of reciprocal words on $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$

	$\lvert R_{4t}\rvert$	$\lvert R^p_{4t}\rvert$
$\mathbb{Z}_2 * \mathbb{Z}_3$	2^{t-1}	2^{t-1}

$\mathbb{Z}_2 * \mathbb{Z}$. Thus $\rho_k \to 2$ as $k \to \infty$. We can also show that these ρ_k are strictly increasing by the following.

Recall the characteristic polynomial $P(z) = z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$ where $m = \dfrac{k-1}{2}$, which has a dominant unique positive root of ρ_k. Let $P_{k+2}(z) = z^{(m+1)+1} - 2z^{(m+1)-1} - \cdots - 2z - 2$ be the characteristic polynomial for the next odd k value. We work with $P_{k+2}(z)$ in a similar manner as in ([16]) to get

$$P_{k+2}(z) = z^{(m+1)+1} - 2z^{(m+1)-1} - \cdots - 2z - 2 \tag{5.32}$$

$$z^{m+2} - 2z^m - 2z^{m-1} - \cdots - 2z - 2 \tag{5.33}$$

$$z(z^{m+1} - 2z^{m-1} - \cdots - 2z - 2) - 2 \tag{5.34}$$

Notice that inside the parentheses we have $P(z)$. If we substitute ρ_k into $P_{k+2}(z)$, we get a value of -2 and we know $P(2) = 2$ for any k by our previous work. Thus, by the intermediate value theorem, the unique positive dominant root ρ_{k+2} must be greater than ρ_k and so the sequence of ρ_k's are strictly increasing as $k \to \infty$.

In light of the methods used, we can rewrite the results from table 5.1 as

Theorem 5.4.1. *(1) Let $P(z) = z^{m+1} - 2z^{m-1} - \cdots - 2z - 2$ where $m = \dfrac{k-1}{2}$. The conjugacy classes of primitive reciprocal words of length $2t$ in $\mathbb{Z}_2 * \mathbb{Z}_k$ for k odd, are asymptotic to $c_k\rho_k^t$ where ρ_k is the unique positive root of $P(z)$ in the interval $[\sqrt{2}, 2]$. Moreover, the ρ_k are strictly increasing and $\rho_k \to 2$ as $k \to \infty$.*

(2) Let $P(z) = z^2 - z - 2$. The conjugacy classes of primitive reciprocal words of length $2t$ in $\mathbb{Z}_2 * \mathbb{Z}$, are asymptotic to $\frac{1}{6}2^t$ where 2 is the unique positive root of $P(z)$.

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