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Some Model Theory of Free Groups

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Abstract

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There are two main sets of results, both pertaining to the model theory of free groups. In the first set of results, we prove that non-abelian free groups of finite rank at least 3 or of countable rank are not $\forall$-homogeneous. We then build on the proof of this result to show that two classes of groups, namely finitely generated free groups and finitely generated elementary free groups, fail to form $\forall$-Fraïssé classes and that the class of non-abelian limit groups fails to form a strong $\forall$-Fraïssé class.

The second main result is that if a countable group is elementarily equivalent to a non-abelian free group and all of its finitely generated abelian subgroups are cyclic, then the group is a union of a chain of regular NTQ groups (i.e., hyperbolic towers).

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Chapter 1

Introduction

In 1945, Tarski conjectured that non-abelian free groups are elementarily equivalent and that their common (first-order) theory is decidable. Proofs of the both conjectures were given at the turn of the century by Kharlampovich and Myasnikov [KM06] and, independently, of the first conjecture by Sela [Sel06]. In answering these questions, they developed existing results about equations over free groups into fuller theories, deemed algebraic geometry over groups or, alternatively, Diophantine geometry over groups. This machinery became the foundation for further investigation in the model theory of free groups and torsion-free hyperbolic groups undertaken in the twenty-first century.

Many model-theoretic aspects of free groups were studied and proven, including stability, homogeneity, forking, and embeddings. The field also widened to study groups that are universally equivalent to free groups and elementarily equivalent to free groups. In the setting of finitely generated groups, both are well understood: the former are limit groups and the latter are hyperbolic towers, both of which are major pieces in the machinery of algebraic geometry over groups. Less is understood about non-finitely generated groups that are elementarily equivalent to free groups. Progress in this direction could be helpful to model theorists studying the theory of non-abelian free groups, their monster models, and other non-standard models.

1.1 Background

We recall a couple foundational results in the model theory of free groups. The solutions to Tarski's problems actually proved a stronger result:

Theorem 1.1.1. *[KM06] [Sel06] Let F_n denote the free group on n generators. If $2 \leq n < m$, then the canonical embedding $F_n \to F_m$ is elementary.*

Perin proved the converse, thus characterizing elementary substructures for free groups:

Theorem 1.1.2. *[Per11, Theorem 1.3] Let H be a proper subgroup of a finitely generated free group F. Then H is an elementary substructure of F if and only if H is a non-abelian free factor of F.*

Another result to recall is about quantifier elimination:

Theorem 1.1.3. *[KM06] [Sel06] The theory of non-abelian free groups has quantifier elimination to boolean combinations of $\forall\exists$-formulas.*

Note that there is no quantifier elimination to $\forall$-formulas because the theory of a non-abelian free group is not model complete [Per11].

One avenue of interest in the model theory of free groups is concerned with the model-theoretic property called homogeneity. Given a structure M and a tuple $\bar{a} \in M$, the *type* of $\bar{a}$ in M is $\mathrm{tp}^M(\bar{a}) = \{\phi(\bar{x}) : M \models \phi(\bar{a})\}$. We say that M is *homogeneous* if for any $\bar{a}, \bar{b} \in M$ with $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b})$, there is an automorphism of M sending $\bar{a}$ to $\bar{b}$. (This is often called $\aleph_0$-homogeneity or sometimes strong $\aleph_0$-homogeneity. The cardinality $\aleph_0$ indicates that the tuples $\bar{a}, \bar{b}$ are finite, i.e., have length less than $\aleph_0$, but the definition can be extended analogously to tuples of larger cardinalities.) Perin and Sklinos [PS12] and, independently, Ould Houcine [Oul11] showed that non-abelian free groups are homogeneous.

One can tweak this question to one-quantifier types. Let the $\forall$-*type* of a tuple $\bar{a}$ in M be $\mathrm{tp}^M_\forall(\bar{a}) = \{\phi(\bar{x}) : M \models \phi(\bar{a})$ and $\phi(\bar{x})$ is a universal formula$\}$. In complete analogy with homogeneity, we say that M is $\forall$-*homogeneous* if for any $\bar{a}, \bar{b} \in M$ with $\mathrm{tp}^M_\forall(\bar{a}) = \mathrm{tp}^M_\forall(\bar{b})$, there is an automorphism of M sending $\bar{a}$ to $\bar{b}$. (Note that Ould Houcine calls this property $\exists$-homogeneity.) In some of the first results on homogeneity in free groups, Nies [Nie03] showed that the free group on two generators is not only homogeneous but $\forall$-homogeneous. It was an open question whether other non-abelian free groups are $\forall$-homogeneous, although partial results were proven by Ould Houcine. In particular, he gave the following characterization of $\forall$-homogeneity for finite-rank non-abelian free groups:

Proposition 1.1.4. *[Oul11, Proposition 4.10] Let F be a finitely generated free group. Then F is $\forall$-homogeneous if and only if F satisfies the following two conditions:*

1. *If a tuple $\bar{a} \in F$ is a power of a primitive element (i.e., there is a single primitive element $x \in F$ such that $\bar{a} \in \langle x \rangle$) and if $\mathrm{tp}_\forall^F(\bar{a}) = \mathrm{tp}_\forall^F(\bar{b})$, then $\bar{b}$ is a power of a primitive element.*

2. *Every existentially closed subgroup of F is a free factor.*

Note that the second condition would require strengthening Theorem 1.1.2 for F.

Another avenue of interest in the model theory of free groups is the study of universally free groups, i.e., groups that have the same universal theory as a non-abelian free group, and *elementary free* groups, i.e., groups that are elementarily equivalent to a non-abelian free group. Finitely generated elementary free groups were completely characterized by [KM06] and, independently, [Sel06] as hyperbolic towers or, equivalently, regular NTQ groups, to be discussed in Section 2.2. Finitely generated universally free groups are equivalent to limit groups (see Section 2.3).

Elementary free groups and universally free groups that are not finitely generated are not as well-studied. One approach taken by Kharlampovich, Myasnikov, and Sklinos in [KMS20] is modifying the model-theoretic constructions known as Fraïssé classes and Fraïssé limits to use stronger kinds of embeddings, the details of which are discussed in Section 2.1. Certain classes of groups, in particular, non-abelian limit groups and finitely generated elementary free groups were shown to satisfy the properties of these modified Fraïssé classes. In the latter case, the resulting group, namely the Fraïssé limit of finitely generated non-abelian free groups, provides a new example of an elementary free group. They posit a few open questions on whether some other classes of groups satisfy the criteria necessary for their Fraïssé limits and whether all countable elementary free groups can be obtained as unions of chains of finitely generated elementary free groups, some of which we answer below.

1.2 Main results

The first collection of main results are from [KN20b] and pertain to $\forall$-homogeneity and $\forall$-Fraïssé classes:

Theorem (3.1.2). *Free groups of finite rank at least 3 or of countable rank are not $\forall$-homogeneous.*

We obtain the following as an immediate corollary:

Theorem (3.1.3). *The first-order theory of a non-abelian free group does not have quantifier elimination to boolean combinations of $\forall$-formulas.*

We build upon the proofs that went into these results to show that certain classes of groups do not form $\forall$-Fraïssé classes or, in the third case, a strong $\forall$-Fraïssé class:

Theorem (3.2.1). *The class of finitely generated free groups is not a $\forall$-Fraïssé class.*

Theorem (3.2.8). *The class of finitely generated elementary free groups is not a $\forall$-Fraïssé class.*

Theorem (3.3.2). *The class of non-abelian limit groups is not a strong $\forall$-Fraïssé class.*

The second main result, from [KN20a], pertains to the structure of certain countable elementary free groups:

Theorem (4.0.1)**.** *Let M be a countable elementary free group in which all finitely generated abelian subgroups are cyclic. Then M is a union of a chain of regular NTQ groups (i.e., hyperbolic towers).*

1.3 Outline

In Chapter 2 we define some notions from the model theory of free groups, in particular, $\forall$-Fraïssé limits, strong $\forall$-Fraïssé limits, hyperbolic towers, JSJ decompositions, and limit groups.

In Chapter 3, we provide an example (Example 3.1.1) of a non-abelian limit group that is not $\forall$-homogeneous, and then use it to prove that all non-abelian free groups of finite rank at least 3 or rank ω are not $\forall$-homogeneous (Theorem 3.1.2). We then build on this example to show that finitely generated non-abelian free groups do not form a $\forall$-Fraïssé class (Theorem 3.2.1), using some results from combinatorial group theory. We then show two corollaries: finitely generated elementary free groups do not form a $\forall$-Fraïssé class (Theorem 3.2.8), and non-abelian limit groups do not form a strong $\forall$-Fraïssé class (Theorem 3.3.2).

In Chapter 4, we give an example (Theorem 4.1.2) of a countable elementary free group that is not a union of a chain of finitely generated elementary free groups. We then show that any countable elementary free group with an extra condition, namely that all finitely generated abelian subgroups are cyclic, is indeed a union of a chain of finitely generated elementary free groups (Theorem 4.0.1).

Chapter 2

Preliminaries

2.1 $\forall$-Fraïssé limits and strong $\forall$-Fraïssé limits

Classical model-theoretic Fraïssé limits arise from classes of finite or finitely generated structures that embed and amalgamate nicely. Given such a class, the resulting limit structure is countable, homogenous, and "universal" in the sense that it contains all the structures in the class, and it is also the unique structure with these properties. The classic example of a Fraïssé limit is the random graph in graph theory. Algebraic examples are less common, although Phillip Hall's universal locally finite group is the Fraïssé limit of all finite groups. Kharlampovich, Myasnikov, and Sklinos [KMS20] noticed that strengthening the morphisms from embeddings to $\forall$-embeddings or elementary embeddings yields new kinds of Fraïssé classes and Fraïssé limits, which can be applied to some classes of finitely generated groups of concern to combinatorial group theorists.

Given a language $\mathcal{L}$ and $\mathcal{L}$-structures A, B, we say that an embedding $f : A \to B$ is a $\forall$-*embedding*, denoted $\to_\forall$, if it preserves universal formulas, i.e., if for every universal formula $\phi(\bar{x})$ and for every $\bar{a} \in A$, $A \models \phi(\bar{a})$ implies $B \models \phi(f(\bar{a}))$. Note that if $f : A \to B$ is the inclusion map, then f being a $\forall$-embedding is equivalent to A being *existentially closed* in B, i.e., for any quantifier-free formula $\phi(\bar{x}, \bar{y})$ and for any $\bar{a} \in A$, if $B \models \exists x \phi(\bar{x}, \bar{a})$ then $A \models \exists x \phi(\bar{x}, \bar{a})$. If A is a substucture of B, we say A is a $\forall$-substructure if A is existentitally closed in B.

Given a $\mathcal{L}$-structure M, the *universal age* of M is the class of all finitely generated $\forall$-substructures of M up to isomorphism.

Definition 2.1.1. Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- ($\forall$-HP) the class $\mathcal{K}$ is closed under finitely generated $\forall$-substructures;
- ($\forall$-JEP) if A_1, A_2 are in $\mathcal{K}$, then there are B in $\mathcal{K}$ and $\forall$-embeddings $f_i : A_i \to_\forall B$ for $i \leq 2$;

- ($\forall$-AP) if A_0, A_1, A_2 are in $\mathcal{K}$ and $f_i : A_0 \to_\forall A_i$ for $i \leq 2$, then there are B in $\mathcal{K}$ and $\forall$-embeddings $g_i : A_i \to_\forall B$ for $i \leq 2$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Then $\mathcal{K}$ is a *universal Fraïssé class* or for short a $\forall$-*Fraïssé class.*

Theorem 2.1.2. *[KMS20, Theorem 2.4] Let $\mathcal{K}$ be a $\forall$-Fraïssé class. Then there exists a countable $\mathcal{L}$-structure M such that*

- *the $\forall$-age of M is exactly $\mathcal{K}$;*
- *the $\mathcal{L}$-structure M is weakly $\forall$-homogeneous, i.e., every isomorphism between finitely generated $\forall$-substructures of M extends to an automorphism of M;*
- *the $\mathcal{L}$-structure M is the union of a $\forall$-chain of $\mathcal{L}$-structures in $\mathcal{K}$.*

Moreover, any other countable $\mathcal{L}$-structure with the above properties is isomorphic to M.

M is called the $\forall$-*Fraïssé limit* of $\mathcal{K}$.

Strengthening the $\forall$-AP yields a stronger homogeneity property in the Fraïssé limit. We first define a partial $\forall$-embedding. Given two $\mathcal{L}$-structures A, B and a subset A_0 of A, a set map $f : A_0 \to B$ is a *partial* $\forall$-*embedding* if for any quantifier-free formula $\phi(\bar{x}, \bar{a})$ over A_0, if $A \models \forall \bar{x} \phi(\bar{x}, \bar{a})$ then $B \models \forall \bar{x} \phi(\bar{x}, f(\bar{a}))$.

Definition 2.1.3. Let $\mathcal{K}$ be a countable (with respect to isomorphism types) non-empty class of finitely generated $\mathcal{L}$-structures with the following properties:

- (IP) the class $\mathcal{K}$ is closed under isomorphisms;
- ($\forall$-HP) the class $\mathcal{K}$ is closed under finitely generated $\forall$-substructures;
- ($\forall$-JEP) if $A_1 r A_2$ are in $\mathcal{K}$, then there are B in $\mathcal{K}$ and $\forall$-embeddings $f_i : A_i \to_\forall B$ for $i \leq 2$;
- (strong $\forall$-AP) if A_0, A_1, A_2 are in $\mathcal{K}$, $\bar{a} \in A_0$, and $f_i : \bar{a} \to_\forall A_i$ for $i \leq 2$ are partial $\forall$-embeddings, then there are B in $\mathcal{K}$ and $\forall$-embeddings $g_i : A_i \to_\forall B$ for $i \leq 2$ with $g_1 \circ f_1(\bar{a}) = g_2 \circ f_2(\bar{a})$.

Then $\mathcal{K}$ is a *strong universal Fraïssé class* or for short a *strong* $\forall$*-Fraïssé class*.

Theorem 2.1.4. *[KMS20, Theorem 2.6] Let $\mathcal{K}$ be a strong $\forall$-Fraïssé class. Then there exists a countable $\mathcal{L}$-structure M such that*

- *the $\forall$-age of M is exactly $\mathcal{K}$;*
- *the $\mathcal{L}$-structure M is $\forall$-homogeneous;*
- *the $\mathcal{L}$-structure M is the union of a $\forall$-chain of $\mathcal{L}$-structures in $\mathcal{K}$.*

Moreover, any other countable $\mathcal{L}$-structure with the above properties is isomorphic to M.

M is called the *strong* $\forall$*-Fraïssé limit* of $\mathcal{K}$.

[KMS20, Theorem 3.3] showed that the class of abelian limit groups, i.e., finitely generated free abelian groups, is a strong $\forall$-Fraïssé class and its strong $\forall$-Fraïssé limit is the direct sum of countably many copies of $\mathbb{Z}$. They also showed [KMS20, Theorem 3.11] that the class of non-abelian limit groups is a $\forall$-Fraïssé class.

Among other open questions, they asked whether the class of non-abelian limit groups is a strong $\forall$-Fraïssé class, whether the class of finitely generated free groups is a $\forall$-Fraïssé class, and whether the class of finitely generated elementary free groups is a $\forall$-Fraïssé class. We give negative answers to these three questions below (Theorem 3.3.2, Theorem 3.2.1, and Theorem 3.2.8, respectively), by showing that they fail to satisfy the relevant $\forall$-APs.

2.2 Hyperbolic towers and JSJ decompositions

We use two special graphs of groups constructions, both involving surfaces: hyperbolic towers and JSJ decompositions. In the discussions below, note that a *maximal boundary subgroup* of a compact connected surface Σ with non-empty boundary is the cyclic fundamental group of a boundary component of Σ.

Definition 2.2.1. Let Γ be a graph of groups with fundamental group G. Then a vertex $v \in \Gamma$ is called a *surface type vertex* if the following conditions hold:

- the vertex group $G_v = \pi_1(\Sigma)$ for a connected compact surface Σ with non-empty boundary such that either the Euler characteristic of Σ is at most -2 or Σ is a once punctured torus;
- For every edge $e \in \Gamma$ adjacent to v, the edge group G_e embeds onto a maximal boundary subgroup of $\pi_1(\Sigma)$, and this induces a one-to-one correspondence between the set of edges adjacent to v and the set of boundary components of Σ.

Definition 2.2.2. Let G be a group and H be a subgroup of G. Then G is a *hyperbolic floor* over H if G admits a graph of groups decomposition Γ where the set of vertices can be partitioned in two subsets V_S, V_R such that

- each vertex in V_S is a surface type vertex;
- Γ is a bipartite graph between V_S and V_R;
- the subgroup H of G is the free product of the vertex groups in V_R;

- either there exists a retraction $r : G \to H$ that, for each $v \in V_S$, sends G_v to a non-abelian image or H is cyclic and there exists a retraction $r' : G * \mathbb{Z} \to H * \mathbb{Z}$ that, for each $v \in V_S$, sends G_v to a non-abelian image.

A hyperbolic tower is a sequence of hyperbolic floors and free products with finite-rank free groups and closed surface groups of Euler characteristic at most -2.

Definition 2.2.3. A group G is a *hyperbolic tower* over a subgroup H if there exists a sequence $G = G^m > G^{m-1} > \ldots > G^0 = H$ such that for each i, $0 \leq i < m$, one of the following holds:

- the group G^{i+1} has the structure of a hyperbolic floor over G^i, in which H is contained in one of the vertex groups that generate G_i in the floor decomposition of G^{i+1} over G^i;

- the group G^{i+1} is a free product of G^i with a finite-rank free group or the fundamental group of a compact surface without boundary of Euler characteristic at most -2.

Note that a hyperbolic tower over a free group is equivalent to the class of groups known as *regular NTQ groups* [KM06]. Hyperbolic towers completely characterize finitely generated elementary free groups as well as elementary embeddings among torsion-free hyperbolic groups.

Theorem 2.2.4. *[Kharlampovich-Myasnikov, Sela] A finitely generated group G is elementarily equivalent to a nonabelian free group F if and only if G is a nonabelian hyperbolic tower over F.*

Theorem 2.2.5. *[Kharlampovich-Myasnikov, Sela] Let G be torsion-free hyperbolic. If G is a hyperbolic tower over a nonabelian subgroup H, then H is an elementary subgroup of G.*

Theorem 2.2.6. *[Per11] Let G be a torsion-free hyperbolic group. If H is an elementary subgroup of G, then G is a hyperbolic tower over H.*

We now turn to JSJ decompositions, recalling definitions of Dehn twists and canonical automorphisms as well.

A *one-edge cyclic splitting* of a group G is a graph of groups with fundamental group G that is either a segment (i.e., amalgamated product) or a loop (i.e., HNN extension), with the single edge group being infinite cyclic. Given a graph Λ of groups with fundamental group G, $H \leq G$ is *elliptic* if it is contained in a conjugate of a vertex group in Λ; analogously for an element of G.

Definition 2.2.7. Let Λ be a one-edge cyclic splitting of G into either $A*_C B$ or $A*_C$, and let $\gamma \in C_G(C)$. The *Dehn twist* of Λ by γ is the automorphism that restricts to identity on A and to conjugation on B or, respectively, restricts to identity on A and sends the stable letter t to $t\gamma$. A *canonical automorphism* of G is an automorphism generated by Dehn twists and inner automorphisms. In the case of H-automorphisms for some subgroup $H \leq G$, canonical H-automorphisms must fix H pointwise.

Let Λ be a graph of groups with fundamental group G. A vertex of Λ is *quadratically hanging* or *QH* if the corresponding vertex group is a fundamental group of a closed hyperbolic surface with boundary, each incident edge group is a finite-index subgroup of a maximal boundary subgroup, and, conversely, every maximal boundary subgroup contains an incident edge group as a finite-index subgroup. Note that this is weaker than Definition 2.2.1 of surface type vertices.

Definition 2.2.8. A *(cyclic) JSJ decomposition* of a group G is a graph of groups Λ with fundamental group G that is maximal in the sense that all possible one-edge cyclic splittings of G are either splittings along edges in Λ or can be found by splitting a QH vertex along a simple closed curve. A graph of groups Λ with fundamental group G is a *(cyclic) JSJ decomposition of G* *relative to* $H \leq G$ if H is elliptic in a non-QH vertex of Λ and Λ is maximal only with respect to one-edge cyclic splittings in which H is elliptic.

Note that any pair of noncompatible (intersecting) splittings of G comes from a pair of intersecting simple closed curves on one of the corresponding surfaces in the JSJ decomposition of G.

2.3 Equations over groups and limit groups

We review some basic notions from algebraic geometry over groups from [BMR99]. Suppose G is a finitely generated group and denote by $F(X)$ the free group on $X = \{x_2, \ldots, x_n\}$. Let $G[X] = G * F(X)$. Given $S \subset G[X]$, the formal expression $S(X) = 1$ is called a *system of equations over* G. Any elements of G in the words in S are called *coefficients*. A *solution* to $S(X) = 1$ over G is tuple $(g_1, \ldots, g_n) \in G$ such that for every word in S, replacing x_i by g_i for all i gives the trivial element. There is a natural bijection between solutions $(g_1, \ldots, g_n) \in G$ and G-homomorphisms $\phi : G[X] \to G$ (i.e., homomorphisms that fix G pointwise) such that $\phi(S) = 1$. We will sometimes use such maps ϕ interchangeably with the solutions.

Let $\operatorname{ncl}(S)$ be the normal closure of S in $G[X]$, and let $G_S = G[X]/\operatorname{ncl}(S)$. Then every solution $\phi : G[X] \to G$ factors through the quotient, i.e., gives rise to a G-homomorphism $G_S \to G$, and conversely, every G-homomorphism

$G_S \to G$ induces a solution $G[X] \to G$. The *radical* of S is the set of all consequences of $S(X) = 1$, i.e.,

$$R(S) = \{T(X) \in G[X] : \forall A \in G^n(S(A) = 1 \to T(A) = 1)\},$$

a normal subgroup of $G[X]$ containing S. The quotient group $G_{R(S)} = G[X]/R(S)$ is called the *coordinate group* for S. Analogously, there is a bijection between solutions and G-homomorphisms $G_{R(S)} \to G$.

The set $V_G(S)$ of all solutions in G to $S(X) = 1$ is the *variety* or *algebraic set* corresponding to $S(X) = 1$. A Zariski topology can be defined on G^n by taking varieties as the sub-basis of closed sets. Notably, the Zariski topology is Noetherian, i.e., every proper descending chain of closed sets in is finite. Therefore every variety is a finite union of *irreducible* subsets, i.e., subsets that are not unions of two closed proper subsets. Coordinate groups of irreducible varieties over free groups are equivalent to limit groups, to be discussed below.

A group G is *fully residually free* or *freely discriminated* if given finitely many non-trivial elements $g_1, \ldots, g_n \in G$, there exists a homomorphism ϕ from G to a free group such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$.

We now review some equivalent definitions of limit groups that we need for our purposes.

Definition 2.3.1. Suppose G is a finitely generated group. Then G is a *limit group* if it satisfies one (i.e., all) of the following equivalent properties:

- G is freely discriminated;
- G is universally equivalent to a non-abelian free group in the language without constants (i.e., $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(F)$);
- G is the coordinate group of an irreducible variety over a free group.

Definition 2.3.2. Suppose G is a finitely generated group containing a non-abelian free group $F \leq G$. Then G is a *restricted limit group* if it satisfies one (i.e., all) of the following equivalent properties:

- G is F-discriminated by F (i.e., given finitely many non-trivial elements $g_1, \ldots, g_n \in G$, there exists an F-homomorphism $\phi : G \to F$ such that $\phi(g_i) \neq 1$ for $i = 1, \ldots, n$);
- G is universally equivalent to F in the language with F as constants;
- G is the coordinate group of an irreducible variety over F.

We refer the reader to [KM10] for more equivalent definitions of limit groups and a proof of their equivalence.

[KMS20] give a characterization of existentially closed subgroups of limit groups in terms a construction called extensions of centralizers.

Definition 2.3.3. An *extension of a centralizer* of a group G is a group $\langle G, t \mid [C_G(u), t] = 1\rangle$ where $u \neq 1$ is some fixed element in G and t is a new letter. If $G = G_0 < \cdots < G_n$ and each G_{k+1} is an extension of a centralizer of G_k, i.e., $G_{k+1} = \langle G_k, t_k \mid [C_{G_k}(u_k), t_k] = 1\rangle$ where $u_k \in G_k$ is nontrivial, we call G_n a *finite iterated extension of centralizers* over G.

Lemma 2.3.4. *[KMS20, Theorem 3.6 and Lemma 3.7] Let L, M be limit groups with $L \leq M$. Then L is existentially closed in M if and only if there is a finite iterated centralizer extension L_n of L such that $M \leq L_n$.*

Remark 2.3.5. For embeddings into a sequence of centralizer extensions, we can without loss consider the sequence to be a mixture of centralizer extensions and free products with free groups. I.e., suppose L is a non-abelian limit group and M embeds in a sequence $L = L_0 < \cdots < L_n$ where for all k, L_{k+1} is a centralizer extension of L_k or a free product of L_k with a countable free group. Then M can be obtained as a subgroup of a finite iterated centralizer extension over L. Indeed, if $L_{k+1} = L_k * \langle x_1, \ldots, x_m\rangle$ where x_i are new letters, then L_{k+1} embeds in the centralizer extension $\langle L_k, t \mid [C_{L_k}(u), t] = 1\rangle$ by mapping x_i to $t^i g t^i g t^i$ where $g \in L_k - C_{L_k}(u)$.

Chapter 3

Non-$\forall$-homogeneity in free groups

3.1 Main example

We first describe a free group M of rank 4 that is not $\forall$-homogeneous and then a free group M_3 of rank 3 that is not $\forall$-homogeneous.

Example 3.1.1. Let

$$L = \langle a, b \rangle \quad \leq \quad L_1 = L * \langle x \rangle \quad \leq \quad L_2 = \langle L_1, t \mid [u, t] = 1 \rangle \, ,$$

where x is a new letter and $u = x^2(bx^{2n})^m$. We construct M as an amalgamated product $M_1 *_{u=u^t} M_2$ living in L_2, where $M_1 = \langle a, b, x^2 \rangle$ and $M_2 = \langle b^t, x^t \rangle$. Nielsen transformations show that $\{a, bx^{2n}, x^2(bx^{2n})^m\}$ is a basis for M_1. Then

$$M = M_1 \underset{u=u^t}{*} M_2 = \left\langle a, bx^{2n}, u \right\rangle \underset{u=u^t}{*} \left\langle b^t, x^t \right\rangle = \left\langle a, bx^{2n} \right\rangle * \left\langle b^t, x^t \right\rangle .$$

So M is a free subgroup of L_2 containing L, and by Lemma 2.3.4, L is existentially closed in M.

Note that b embeds into M as

$$b = h\left(h^m\left(b^t(x^t)^{2n}\right)^{-m}(x^t)^{-2}\right)^n$$

where $h = bx^{2n}$. If we take the Nielsen transformation $b^t \mapsto b^t(x^t)^{2n}$, then in the basis $\{a, h, b^t(x^t)^{2n}, x^t\}$, the Whitehead graph for b contains the cycle pictured in Figure 3.1. In particular, it does not contain a cut vertex. [Whi36] showed that if the Whitehead graph of a cyclically reduced word does not contain a cut vertex, then that word is not primitive (see also [HW19]). So b is not primitive, hence L is not a free factor of M.

$$\begin{tikzcd}
h \arrow[r, dash] \arrow[d, dash] & b^t(x^t)^{2n} \arrow[r, dash] & \left(b^t(x^t)^{2n}\right)^{-1} \arrow[d, dash] \\
h^{-1} \arrow[r, dash] & (x^t)^{-1} \arrow[r, dash] & x^t
\end{tikzcd}$$

Figure 3.1: In the basis $\{a, h, b^t(x^t)^2n, x^t\}$, the Whitehead graph for the word $h\left(h^m\left(b^t(x^t)^{2n}\right)^{-m}(x^t)^{-2}\right)^n$ contains the above cycle.

Since L is existentially closed in M, we have $\mathrm{tp}_\forall^L(a,b) = \mathrm{tp}_\forall^M(a,b)$. Also, the tuple (a, bx^{2n}) has the same ∀-type in M as it does in $\langle a, bx^{2n}\rangle$, because this group is a free factor in M and therefore, by Theorem 2.2.5, is elementarily embedded in M. Moreover, L and $\langle a, bx^{2n}\rangle$ are isomorphic, so $\mathrm{tp}_\forall^M(a,b) = \mathrm{tp}_\forall^M(a, bx^{2n})$. But there is no automorphism of M sending (a,b) to (a, bx^{2n}) because L is not a free factor, so M is not $\forall$-homogeneous.

Now let M_3 be the subgroup of M generated by bx^{2n}, b^t, x^t. Since $\mathrm{tp}_\forall^M(a,b) = \mathrm{tp}_\forall^M(a, bx^{2n})$, we have $\mathrm{tp}_\forall^M(b) = \mathrm{tp}_\forall^M(bx^{2n})$. Since M_3 is a non-abelian free factor of M containing both b and bx^{2n}, we have $\mathrm{tp}_\forall^{M_3}(b) = \mathrm{tp}_\forall^{M_3}(bx^{2n})$. But bx^{2n} is primitive in M_3 and b is not, so they cannot be in the same automorphic orbit.

An alternative proof that M is not $\forall$-homogeneous follows from Proposition 1.1.4. L is an example of an existentially closed subgroup of M that is not a free factor, hence M is not $\forall$-homogeneous.

Theorem 3.1.2. *Free groups of finite rank at least 3 or of countable rank are not $\forall$-homogeneous.*

Proof. Let L, M, M_3 be as in Example 3.1.1. We have shown that M_3 and M are free groups of ranks 3 and 4, respectively, and neither are $\forall$-homogeneous. Suppose F is a finitely generated free group of rank greater than 4, and canonically embed M into F. Since this embedding is elementary, L is existentially closed in F. Suppose by way of contradiction that F is $\forall$-homogeneous. Then by Proposition 1.1.4, L is a free factor of F, say $F = L * K$. By Bass-Serre theory, we can write

$$M = L * (L^{x_1} \cap M) * \cdots * (L^{x_p} \cap M) * (K^{y_1} \cap M) * \cdots * (K^{y_q} \cap M) * F',$$

where $x_i, y_i \in F$ and F' is some free group. But then L is a free factor of M.

Let F_ω be a free group of rank ω, and embed M in F_ω canonically. Choose tuples $\bar{a}, \bar{b} \in M$ such that $\mathrm{tp}_\forall^M(\bar{a}) = \mathrm{tp}_\forall^M(\bar{b})$ but there is no automorphism of M sending $\bar{a}$ to $\bar{b}$. It is a result from model theory (see for example [Mar02, Proposition 2.3.11]) that any structure in an elementary chain is an elementary substructure of the union of the chain. In particular, if we let F_i denote the free group on i generators, then $F_2 \prec F_3 \prec M \prec F_5 \prec \cdots$ form an elementary chain, so $M \prec \bigcup_{i<\omega} F_i = F_\omega$. Then $\mathrm{tp}_\forall^{F_\omega}(\bar{a}) = \mathrm{tp}_\forall^{F_\omega}(\bar{b})$. If F_ω were ∀-homogeneous, then there would be an automorphism of F_ω sending $\bar{a}$ to $\bar{b}$. Then $\mathrm{tp}^{F_\omega}(\bar{a}) = \mathrm{tp}^{F_\omega}(\bar{b})$, so $\mathrm{tp}^M(\bar{a}) = \mathrm{tp}^M(\bar{b})$. Since M is homogeneous, there is an automorphism of M carrying $\bar{a}$ to $\bar{b}$. $\square$

The lack of ∀-homogeneity in countable free groups gives us an intermediate result on quantifier elimination:

Corollary 3.1.3. *The first-order theory of a non-abelian free group does not have quantifier elimination to boolean combinations of ∀-formulas.*

3.2 ∀-AP

In this section we will prove the following:

Theorem 3.2.1. *The class of finitely generated free groups is not a ∀-Fraïssé class.*

In the next example we double Example 3.1.1 to obtain two finitely generated free groups that cannot be amalgamated to satisfy ∀-AP.

Example 3.2.2. Following Example 3.1.1, we let $L = \langle a, b \rangle$ be a common subgroup of H and K, where

$$H = \left\langle a, h, \tilde{b}, \tilde{x} \right\rangle \qquad K = \left\langle a, k, \hat{b}, \hat{y} \right\rangle$$
$$b \mapsto h\left(\tilde{x}^8(\tilde{b}\tilde{x}^{8n})^m h^{-m}\right)^{-n} \qquad b \mapsto k\left(\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}\right)^{-p},$$

n, m, p, q are sufficiently large, and m, q are even. Here, we change the exponents, $\tilde{b}$ plays the role of b^t from Example 3.1.1, $\tilde{x}$ plays the role of x^t, h plays the role of bx^{8n}, and the embedding of b into H is based on the equation $u = u^t$ from M. Similarly for K, with a new letter y replacing x.

We require a few lemmas to show that H and K cannot be amalgamated over L into a free group to satisfy the ∀-AP. The first follows from [LS62, Lemma 4].

Lemma 3.2.3. *Suppose F is a finitely generated free group. Fix a basis of F, and let u and v be cyclically reduced words in that basis. Suppose w is a subword of u^{n_1} and a subword of v^{n_2} where $n_1, n_2 \in \mathbb{Z}$, and suppose that $|w| \geq |u| + |v|$. Then there exist a_1, a_2 cyclic shifts of one another such that $u = a_1^{k_1}$ and $v = a_2^{k_2}$ for some $k_1, k_2 \in \mathbb{Z}$. In particular, u commutes with a conjugate of v.*

Definition 3.2.4. An equation E is *quadratic* if each variable in E occurs exactly twice. Let $F(A)$ denote the free group with finite basis A, and let $(A \cup A^{-1})^*$ be the set of all words (not necessarily reduced) with alphabet A. A quadratic equation E with variables $\{x_i, y_i, z_j\}$ and non-trivial coefficients $\{C_j, C\} \in F(A)$ is said to be in *standard form* if its coefficients are expressed as freely and cyclically reduced words in $(A \cup A^{-1})^*$ and E has either the form

$$\left(\prod_{i=1}^{g}[x_i, y_i]\right)\left(\prod_{j=1}^{m_{\mathrm{coef}}-1} z_j^{-1} C_j z_j\right) C = 1 \quad \text{or} \quad \left(\prod_{i=1}^{g}[x_i, y_i]\right) C = 1 \tag{3.1}$$

where $[x, y] = x^{-1}y^{-1}xy$, in which case we say it is *orientable*, or it has the form

$$\left(\prod_{i=1}^{g} x_i^2\right)\left(\prod_{j=1}^{m_{\mathrm{coef}}-1} z_j^{-1} C_j z_j\right) C = 1 \quad \text{or} \quad \left(\prod_{i=1}^{g} x_i^2\right) C = 1 \tag{3.2}$$

in which case we say it is non-orientable. The *genus* of a quadratic equation is the number g in Equations (3.1) and (3.2) and m_{coef} is the number of coefficients. If $g = 0$ then we will define E to be orientable. If E is a quadratic equation we define its *reduced Euler characteristic*, $\bar{\chi}$ as follows:

$$\bar{\chi}(E) = \begin{cases} 2 - 2g & \text{if } E \text{ is orientable} \\ 2 - g & \text{if } E \text{ is not orientable.} \end{cases}$$

For example, $C_1 u^{-1} C_2 u v^{-1} C_3 v = 1$ is an orientable quadratic equation in standard form with variables u, v, coefficients C_1, C_2, C_3, genus $g = 0$, and $m_{\text{coef}} = 3$. Similarly, $C_1 v^{-1} C_2 v = 1$ is an orientable quadratic equation in standard form in a single variable v, with coefficients C_1, C_2, genus $g = 0$, and $m_{\text{coef}} = 2$.

Lemma 3.2.5. *[Ols89] or [KV12, Theorem 4] Let E be a quadratic equation in standard form over $F(A)$. If either $g = 0$ and $m_{\text{coef}} = 2$, or E is non-orientable and $g = m_{\text{coef}} = 1$, then we set $N = 1$. Otherwise we set $N = 3(m_{\text{coef}} - \bar{\chi}(E))$. If E has a solution, then for some $n \leq N$,*

(i) there is a set $P = \{p_1, \dots p_n\}$ of variables and a collection of discs $D_1, \dots, D_{m_{\text{coef}}}$ such that

(ii) the boundaries of these discs are circular 1-complexes with directed and labelled edges such that each edge has a label in P and each $p_j \in P$ occurs exactly twice in the union of boundaries;

(iii) if we glue the discs together by edges with the same label, respecting the

edge orientations, then we will have a collection $\Sigma_0, \ldots, \Sigma_l$ of closed surfaces and the following inequalities: if E is orientable then each Σ_i is orientable and

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \bar{\chi}(E);$$

if E is non-orientable either at least one Σ_i is non-orientable and

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \bar{\chi}(E)$$

or, each Σ_i is orientable and

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \bar{\chi}(E) + 2;$$

(iv) there is a mapping $\bar{\psi} : P \to (A \cup A^{-1})^*$ such that upon substitution, the coefficients $C_1, \ldots, C_{m_{\mathrm{coef}}}$ can be read without cancellations around the boundaries of $D_1, \ldots, D_{m_{\mathrm{coef}}}$, respectively.

Lemma 3.2.6. *Suppose a, b, c are reduced words in a finitely generated free group F and a is cyclically reduced. Let $m \geq 9$ and $j \in \{7, 8\}$. If $|a^m b^m c^j| < |a|$, then one of a, b, c commutes with a conjugate of another or an inverse of another.*

Proof. Let $d = a^m b^m c^j$, and let $b = u^{-1} b_0 u$ and $c = v^{-1} c_0 v$, where b_0 and c_0 are cyclically reduced. Then we have

$$d^{-1} a^m u^{-1} b_0^m u v^{-1} c_0^j v = 1,$$

which must have a solution in F. We apply Lemma 3.2.5 to this equation in variables u, v. Here, $N = m_{\text{coef}} = 3$. So there are three discs D_1, D_2, D_3 with the words $d^{-1}a^m, b^m, c^j$ on the boundaries, and there are two possibilities for P, namely, $P = \{p_1, p_2\}$ or $P = \{p_1, p_2, p_3\}$.

Suppose $P = \{p_1, p_2\}$. If any disc is labeled by $p_i p_i$ for some i, then that disc is nonorientable, contradicting Lemma 3.2.5(iii). So the only possibility for labeling the boundaries of the discs (up to reordering the discs) is that p_1 labels the entirety of ∂D_1, p_2 labels the entirety of ∂D_2, and $p_1 p_2$ labels ∂D_2. Suppose $\bar{\psi}$ in Lemma 3.2.5(iv) sends p_1 to a cyclic permutation of $d^{-1}a^m$, p_2 to a cyclic permutation of b_0^m, and $p_1 p_2$ to a cyclic permutation of c_0^j.

If there is no cancellation in the $ad^{-1}a$ segment of the cyclic word $d^{-1}a^m$, then $p_1 = p_{11} d^{-1} p_{12}$, where $p_{11} = a_1 a^{m_1}$, $p_{12} = a^{m_2} a_2$, $m_1 + m_2 = m - 1$, and $a_2 a_1 = a$. If there is cancellation in the $ad^{-1}a$ segment, then we can rewrite the cyclic word $d^{-1}a^m$ as $d' a_0^{m-1} a_0'$ where d' is a subword of d^{-1}, a_0 is a cyclic permutation of a, and a_0' is an initial segment of a_0. Then $p_1 = p_{11} d' p_{12}$, where $p_{11} = a_1 a_0^{m_1} a_0'$, $p_{12} = a_0^{m_2} a_2$, $m_1 + m_2 = m - 2$, and $a_2 a_1 = a_0$. In either case, p_{11} and p_{12} are subwords of a^m, as well as subwords of c_0^j. Also, $|p_1| < |p_{11}| + |p_{12}| + |a|$.

By Lemma 3.2.3, either a commutes with a conjugate of c_0, in which case we're done, or both $|p_{11}| < |a| + |c_0|$ and $|p_{12}| < |a| + |c_0|$. Similarly, either

b_0 commutes with a conjugate of c_0 or $|p_2| < |b_0| + |c_0|$. If neither b_0 nor a commutes with a conjugate of c_0, then

$$|p_{11}| + |p_{12}| + |p_2| < 2|a| + |b_0| + 3|c_0|$$

$$|p_1| + |p_2| < 3|a| + |b_0| + 3|c_0|.$$

But this contradicts

$$(m-1)|a| + m|b_0| + j|c_0| < |\partial D_1| + |\partial D_2| + |\partial D_3| = 2|p_1| + 2|p_2|,$$

since $|\partial D_1| + |\partial D_2| + |\partial D_3| = 2|p_1| + 2|p_2|$. Any other choices of $\bar{\psi}$ are analogous.

Suppose $P = \{p_1, p_2, p_3\}$ and ∂D_1 is labeled by $p_2 p_3$, ∂D_2 by $p_1 p_3$, and ∂D_3 by $p_1 p_2$. Suppose $\bar{\psi}$ sends $p_1 p_2$ to a cyclic permutation of $d^{-1} a^m$, $p_2 p_3$ to a cyclic permutation of b_0^m, and $p_1 p_3$ to a cyclic permutation of c_0^j. If p_1 covers $a d^{-1} a$, then let $p_1 = p_{11} d_1 p_{12}$ where p_{11} and p_{12} are subwords of a^m and $|p_1| < |p_{11}| + |p_{12}| + |a|$. Again, by Lemma 3.2.3, we have either

- a commutes with a conjugate of b_0, b_0 commutes with a conjugate of c_0, or a commutes with a conjugate of c_0, in which case we're done; or
- $|p_{11}|, |p_{12}| < |a| + |c_0|$ and $|p_2| < |a| + |b_0|$ and $|p_3| < |b_0| + |c_0|$.

If the latter, then

$$|p_1| + |p_2| + |p_3| < |p_{11}| + |p_{12}| + |a| + |p_2| + |p_3| < 4|a| + 2|b| + 3|c_0|,$$

which contradicts

$$(m-1)|a| + m|b| + j|c_0| < |\partial D_1| + |\partial D_2| + |\partial D_3| = 2|p_1| + 2|p_2| + 2|p_3|.$$

Any other labelings of the boundaries or choices of $\bar{\psi}$ are similar. □

Proposition 3.2.7. *There is no finitely generated free group satisfying the $\forall$-AP with respect to L, H, and K.*

Proof. Suppose there is such a free group F, i.e., suppose H and K $\forall$-embed into F such that the embedding of L along $H \hookrightarrow F$ equals the embedding of L along $K \hookrightarrow F$. Then F must be a quotient of

$$\begin{aligned} G &= H \underset{L}{*} K \\ &= \left\langle a_G, h_G, \tilde{b}_G, \tilde{x}_G, k_G, \hat{b}_G, \hat{y}_G \;\middle|\; h_G \left(\tilde{x}_G^8 (\tilde{b}_G \tilde{x}_G^{8n})^m h_G^{-m}\right)^{-n} = k_G \left(\hat{y}_G^7 (\hat{b}_G \hat{y}_G^{7p})^q k_G^{-q}\right)^{-p} \right\rangle, \end{aligned}$$

where the copies of generators of H and K in G are denoted with subscripts. Denote the images in F of the generators in G without the subscripts, e.g., the image of $h_G \in G$ under the quotient map is a word $h \in F$. The embeddings of H and K into F must commute with their embeddings into G composed with the quotient map, e.g., $h \in H$ is mapped to $h \in F$. So we treat H and K as subgroups of F.

Then in F we have the equation

$$h \left(h^m (\tilde{x}^{-8n} \tilde{b}^{-1})^m \tilde{x}^{-8}\right)^n \left(\hat{y}^7 (\hat{b} \hat{y}^{7p})^q k^{-q}\right)^p k^{-1} = 1.$$

Considering $\tilde{b}, \tilde{x}, h$ as coefficients (i.e., parameters) in H, we have for all $s > 1$,

$$H \models \forall z \left(\tilde{x}^8 (\tilde{b}\tilde{x}^{8n})^m h^{-m} \neq z^s \right).$$

Since H is existentially closed in F, the same sentence holds in F, i.e., $\tilde{x}^8(\tilde{b}\tilde{x}^{8n})^m h^{-m}$ is not a proper power in F. Similarly $\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}, a, h, k, \tilde{b}, \tilde{x}, \hat{b}, \hat{y}$ are not proper powers in F.

We can assume (changing the values of $\tilde{x}, \tilde{b}, h, \hat{b}, \hat{y}, k$ by conjugation, if necessary) that the reduced word in F obtained from $h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m \tilde{x}^{-8}$ is cyclically reduced. We can also assume that the reduced form of $(\hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q})^p k^{-1}$ is $v^{-1}(\bar{y}^7(\bar{b}\bar{y}^{7p})^q \bar{k}^{-q})^p \bar{k}^{-1} v$ for some v, where $\bar{y}, \bar{b}, \bar{k}$ are conjugates of $\hat{y}, \hat{b}, \hat{k}$ by v and the reduced word obtained from $\bar{y}^7(\bar{b}\bar{y}^{7p})^q \bar{k}^{-q}$ is cyclically reduced. Finally, we can assume that h and $\bar{k}$ are cyclically reduced, by changing $\tilde{x}, \tilde{b}, \bar{y}, \bar{b}, v$ if necessary.

Now we have

$$h \left(h^m (\tilde{x}^{-8n}\tilde{b}^{-1})^m \tilde{x}^{-8} \right)^n v^{-1} \left(\bar{y}^7 (\bar{b}\bar{y}^{7p})^q \bar{k}^{-q} \right)^p \bar{k}^{-1} v = 1. \tag{3.3}$$

We apply Lemma 3.2.5 to this equation in variable v. Here, $N = 1$ and $m_{\text{coef}} = 2$, so we have a single variable p_1 that is the label of both discs D_1 and D_2. So p_1 equals a cyclic permutation of $h\big(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\big)^n$ and also equals a cyclic permutation of $\big(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\big)^p\,\bar{k}^{-1}$ or its inverse. So, without loss of generality, $h\big(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\big)^n$ is a cyclic permutation of

$\left(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\right)^p \bar{k}^{-1}$. Then we can write $hw_1 = w_2\bar{k}^{-1}w_3$, where $w_1 = \left(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\right)^n$ and w_2, w_3 are subwords of $\left(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\right)^p$.

Note that if $|h| > |h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}|$, then by Lemma 3.2.6, one of $h, \tilde{x}^{-8n}\tilde{b}^{-1}, \tilde{x}$ commutes with a conjugate of another or an inverse of another. But if $F \models \exists z[h_1, h_2^z] = 1$ for $h_1, h_2 \in \{h^{\pm 1}, (\tilde{x}^{-8n}\tilde{b}^{-1})^{\pm 1}, \tilde{x}^{\pm 1}\}$, then $H \models \exists z[h_1, h_2^z] = 1$, contradicting the fact that H is freely generated by $a, \tilde{x}, \tilde{b}, h$. So $|h| \leq |h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}|$ and, similarly, $|\bar{k}| \leq |\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}|$.

If $w_3 \neq 1$, then w_3 is a common subword of $\left(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\right)^n$ and $\left(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\right)^p$. Otherwise, we have $hw_1 = w_2\bar{k}^{-1}$. Since $|h| \leq |h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}|$ and $|\bar{k}| \leq |\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}|$, we can choose w_1', w_2' such that $hw_1 = hw_1'\bar{k}^{-1}$ and $w_2\bar{k}^{-1} = hw_2'\bar{k}^{-1}$. Then $w_1' = w_2'$, i.e., $\left(h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}\right)^n$ and $\left(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\right)^p$ have a common subword.

Therefore by Lemma 3.2.3, $h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}$ must equal a conjugate of $\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}$. Then

$$K \models \exists \check{x}, \check{b}, \check{h}\left(\check{x}^8(\check{b}\check{x}^{8n})^m\check{h}^{-m} = \hat{y}^7(\hat{b}\hat{y}^{7p})^q k^{-q}\right).$$

But this equation does not have a solution in K, since in general, a free group $F(e_1, e_2, e_3)$ cannot have a solution to the equation $x^8y^mz^{-m} = e_1^7e_2^qe_3^{-q}$ where m, q are even, because it is impossible for the exponential sum of e_1 in the left-hand side to be 7. □

Now Theorem 3.2.1 follows from Proposition 3.2.7.

The proof of Proposition 3.2.7 can also be extended to finitely generated elementary free groups, i.e., groups that model the common theory of non-abelian free groups.

Theorem 3.2.8. *The class of finitely generated elementary free groups is not a $\forall$-Fraïssé class.*

Proof. Let L, H, K be as in the proof of Proposition 3.2.7. Suppose an elementary free group E satisfies the $\forall$-AP with respect to L, H, K. The proof of Proposition 3.2.7 shows that any free group F models the following $\forall\exists$-sentence without parameters: For any values of $h, \tilde{x}, \tilde{b}, \bar{k}, \bar{y}, \bar{b}, v$ that solve Equation (3.3), there exists $u \in F$ such that $[x^u, y] = 1$ for some $x, y \in S_1$ or $x, y \in S_2$ or $x, y \in S_3$, where

$$S_1 = \left\{h, \tilde{x}, \tilde{b}\tilde{x}^{8n}\right\}, \quad S_2 = \left\{\bar{k}, \bar{y}, \bar{b}\bar{y}^{7p}\right\}, \quad S_3 = \left\{h^m(\tilde{x}^{-8n}\tilde{b}^{-1})^m\tilde{x}^{-8}, \left(\bar{y}^7(\bar{b}\bar{y}^{7p})^q\bar{k}^{-q}\right)\right\}$$

and $x \neq y$. Then E models the same sentence. However, plugging in the words $h, \tilde{x}, \tilde{b}, \bar{k}, \bar{y}, \bar{b} \in E$ given by the $\forall$-embeddings of H and K into E results in the same contradictions as in Proposition 3.2.7. $\square$

3.3 Strong ∀-AP

We again modify Example 3.1.1, this time to show that non-abelian limit groups do not form a strong ∀-Fraïssé class.

Example 3.3.1. Let $L_0 = \langle b, x\rangle$ and $u_1 = x^2(bx^{2n})^m$. Define a single centralizer extension $L_1 = \langle L_0, t_1 \mid [u_1, t_1] = 1\rangle$ with a subgroup $H = \langle h, \tilde{b}, \tilde{x}\rangle$ where $\tilde{b} = b^{t_1}$, $\tilde{x} = x^{t_1}$, and $h = bx^{2n}$. Note H contains both b and x^4 as

$$b = h\left(\tilde{x}^2(\tilde{b}\tilde{x}^{2n})^m h^{-m}\right)^{-n} \qquad x^4 = \left(\tilde{x}^2(\tilde{b}\tilde{x}^{2n})^m h^{-m}\right)^2.$$

Analogously, let $u_2 = x^4(bx^{4p})^q$ and define another centralizer extension of L_0 as $L_2 = \langle L_0, t_2 \mid [u_2, t_2] = 1\rangle$. Let $K = \langle k, \hat{b}, \hat{x}\rangle$ where $\hat{b} = b^{t_2}$, $\hat{x} = x^{t_2}$, and $k = bx^{4p}$. K contains both b and x^4 as

$$b = k\left(\hat{x}^4(\hat{b}\hat{x}^{4p})^q k^{-q}\right)^{-p} \qquad x^4 = \hat{x}^4(\hat{b}\hat{x}^{4p})^q k^{-q}.$$

Note that the inclusions of the tuple (b, x^4) into H and K are partial ∀-embeddings. Indeed, if ϕ is a quantifier-free formula and $L_0 \models \forall\bar{y}\,\phi(\bar{y}, b, x^4)$, then by Lemma 2.3.4, we have $L_1 \models \forall\bar{y}\,\phi(\bar{y}, b, x^4)$. Since $H \leq L_1$, we have $H \models \forall\bar{y}\,\phi(\bar{y}, b, x^4)$. Similarly for K.

Let

$$\begin{aligned} G &= H \underset{\langle b, x^4\rangle}{*} K \\ &= \left\langle h, \tilde{b}, \tilde{x}, k, \hat{b}, \hat{x} \;\middle|\; \begin{array}{rl} h\big(\tilde{x}^2(\tilde{b}\tilde{x}^{2n})^m h^{-m}\big)^{-n} & = k\big(\hat{x}^4(\hat{b}\hat{x}^{4p})^q k^{-q}\big)^{-p} \\ \big(\tilde{x}^2(\tilde{b}\tilde{x}^{2n})^m h^{-m}\big)^2 & = \hat{x}^4(\hat{b}\hat{x}^{4p})^q k^{-q} \end{array} \right\rangle. \end{aligned}$$

Suppose M is a limit group satisfying the strong ∀-AP with respect to L_0, H, K and the tuple (b, x^4). Then M must be a quotient of G. From the second relation in G, we have $M \models \exists u \left(u^2 = \hat{x}^4(\hat{b}\hat{x}^{4p})^q k^{-q}\right)$. So K models the same sentence, which is a contradiction.

Theorem 3.3.2. *The class of non-abelian limit groups is not a strong ∀-Fraïssé class.*

3.4 Finite iterated centralizer extensions and free factors

We prove a result of independent interest, characterizing free factors of free groups in terms of a restricted kind of finite iterated centralizer extensions. We will use a theorem from Wilton [Wil12, Theorem 18], for which we slightly correct the formulation:

Lemma 3.4.1. *Let Γ be a graph of groups with infinite cyclic edge groups and a finitely generated fundamental group L. Suppose every vertex group has rank at least 2 or, if it is cyclic, then the vertex has exactly one incident edge and* *the inclusion map of the edge group into that vertex is an isomorphism. Then L is one-ended if and only if every vertex group in Γ is freely indecomposable relative to the incident edge groups.*

Note that free groups are not one-ended (they have infinitely many ends).

Proposition 3.4.2. *Let $L < M$ be free groups. Then L is a free factor of M if and only if M embeds in $L_n = \langle L, t_1, \ldots, t_n \mid [c_i, t_i] = 1 \rangle$, where $c_1, \ldots, c_n \in L$ are primitive and distinct, such that the embedding of L along $M \hookrightarrow L_n$ equals L.*

Proof. ($\Rightarrow$) This follows immediately from Remark 2.3.5.

($\Leftarrow$) Consider L_n as the fundamental group of a graph of groups with a single vertex group L and n loops. M acts on the corresponding Bass-Serre tree, inducing a graph of groups Γ with fundamental group M. We will use induction on the rank of M. Consequently, we need only consider the connected component of Γ containing L.

By Lemma 3.4.1, there is a vertex group G_v in Γ that is freely decomposable relative to its incident edge groups. If $G_v \neq L$ and v is a cut-point of Γ, then we apply the induction hypothesis to the connected component containing L. If $G_v \neq L$ and v is not a cut-point, replace v with two vertices, one for each factor of G_v, and an edge with trivial edge group G_e between them.

Then M is a free product of $\pi_1(\Gamma - e)$ and the stable letter corresponding to G_e, so we apply the induction hypothesis to $\pi_1(\Gamma - e)$, which has lower rank.

Suppose $G_v = L$ and no other vertex group is freely decomposable relative to its edge groups. Let $L = A * B$ and suppose $c_1, \ldots, c_k$ are conjugate into A and $c_{k+1}, \ldots, c_n$ are conjugate into B. Consider another vertex group $L^x \cap M$, where $x \in L_n$. By Bass-Serre theory, we have

$$L^x \cap M = A_0^x * \cdots * A_p^x * B_0^x * \cdots * B_q^x * F,$$

where A_j is conjugate into A for all j, B_j is conjugate into B for all j, and F is free. Let $A' = A_0^x * \cdots * A_p^x$ and let $B' = B_0^x * \cdots * B_q^x$. We claim that either $L^x \cap M = A'$ or $L^x \cap M = B'$. If F is nontrivial then $L^x \cap M$ is freely decomposable with all incident edge groups conjugate into $A' * B'$, contradicting our assumption. The situation is similar if there are no incident edge groups conjugate into A' or no edge groups conjugate into B'. Suppose both A' and B' contain conjugates of edge groups. Every edge group is of the form $\langle c_i^{x_i} \rangle$ for some $x_i \in L_n$. If $i \leq k$, c_i is conjugate into A, so $c_i^{x_i}$ must be conjugate into A'. Otherwise $c_i^{x_i}$ is conjugate into B'. So $L^x \cap M$ is freely decomposable relative to its edge groups, contradicting our assumption.

Therefore, every vertex group other than L is either of the form $A_0^x * \cdots * A_p^x$ or of the form $B_0^x * \cdots * B_q^x$. Furthermore, since c_i is conjugate into A if and only if $i \leq k$, there is no edge connecting a vertex of the form $A_0^x * \cdots * A_p^x$ to one of the form $B_0^x * \cdots * B_q^x$. So removing the trivial edge between A and B disconnects Γ into two components. Then M is the free product of freely indecomposable groups and hence not free. $\square$

Chapter 4

On countable elementary free groups

Groups that are elementarily equivalent to non-abelian free groups called *elementary free groups*. Finitely generated elementary free groups were fully characterized as hyperbolic towers or, equivalently, regular NTQ groups (see 2.2.4 or [KM06] and [Sel06]). Elementary free groups that are not finitely generated are not as well understood.

Ultraproducts are natural examples of elementary free groups, but they are uncountable and less tractable from a group-theoretic point of view. Some progress has been made in the direction of understanding countable elementary free groups by Kharlampovich, Myasnikov, and Sklinos [KMS20] by another modification of Fraïssé limits, by strengthening the morphisms to elementary embeddings, which yielded a countable elementary free group that is the union of a chain of finitely generated elementary free groups, in addition to other nice model-theoretic properties among which it is unique. They ask whether every countable elementary free group can be obtained as the union of a chain of finitely generated elementary free groups. (Note that by a chain, we mean a chain with order type ω, i.e., a sequence of groups $G_0 \leq G_1 \leq \cdots \leq G_n \leq \cdots$ where $n < \omega$.) Compare this to the fact that every countable *universally* free group is a union of a chain of finitely generated universally free groups, i.e., limit groups.

In this chapter we give $\mathbb{Z} * (\mathbb{Z} \oplus \mathbb{Q})$ as a counter-example, employing the fact that finitely generated elementary free groups cannot contain non-cyclic abelian subgroups. We then positively answer a modified version of the question from [KMS20], in which we add the extra condition that finitely generated abelian subgroups must be cyclic. Our main theorem for this chapter is as follows:

Theorem 4.0.1. *Let M be a countable elementary free group in which all finitely generated abelian subgroups are cyclic. Then M is a union of a chain of regular NTQ groups (i.e., hyperbolic towers).*

We prove this theorem in two cases: when the language is that of pure groups and when the language is expanded by adding generators for a fixed non-abelian free group $F \leq M$.

First, we recall some theory of torsion-free abelian groups (see for example [Fuc73, Section 85]), to show that in a countable elementary free group, assuming just finitely generated abelian subgroups are cyclic is enough to show that all abelian subgroups are cyclic.

The *rank* of a torsion-free abelian group is the cardinality of a maximal linearly independent subset. Given a torsion-free abelian group A of rank 1, $a \in A$, and a prime number p, the *p-height* $h_p(a)$ of a is the largest integer k such that there exists a p^kth root of a, i.e., some $r \in A$ with $r^{p^k} = a$, if such a root exists and $h_p(a) = \infty$ otherwise. The sequence $(h_2(a), h_3(a), h_5(a), \ldots)$ is called the *height-sequence* of a. Two height-sequences are *equivalent* if they differ at only a finite number of indices and by a finite difference at those indices. In torsion-free abelian groups of rank 1, the height-sequence of every non-trivial element is equivalent. The height-sequence equivalence class of a torsion-free abelian group of rank 1 is called its *type* (not to be confused with types from model theory). Baer [Bae37] showed that two torsion-free abelian groups of rank 1 are isomorphic if they have the same type.

Lemma 4.0.2. *Let M be a countable elementary free group in which all finitely generated abelian subgroups are cyclic. Then any abelian subgroup of M is also cyclic.*

Proof. Let $A \leq M$ be abelian. We can suppose without loss of generality that A is maximal abelian. Note that the rank of A is 1, since otherwise A would contain a non-cyclic but finitely generated abelian group. Since M is elementary free, we have for every prime p,

$$M \models \forall x_1 \forall x_2 \exists y \left([x_1, x_2] = 1 \to \bigvee_{(m_1, m_2) \in S} x_1^{m_1} x_2^{m_2} = y^p \right),$$

where S is the set of all non-trivial pairs (m_1, m_2) with $0 \leq m_i < p$. In particular, either $A/pA = \mathbb{Z}/p\mathbb{Z}$ or $A/pA = 1$. Moreover, for each p we have

$$M \models \forall x \exists y \forall z \left([x, y] = 1 \to y \neq z^p \right).$$

In particular, $A/pA = \mathbb{Z}/p\mathbb{Z}$ for each p. For any $a \in A$ and any p, $h_p(a)$ is finite, so the type of A is equivalent to $(0, 0, 0, \ldots)$. By Baer, A is cyclic. □

4.1 Example with non-cyclic abelian subgroups

Theorem 4.1.1. *A free product of abelian groups that are each elementarily equivalent to $\mathbb{Z}$ is an elementary free group.*

This follows from [Sel10, Theorem 7.1], which states that for groups A_1, B_1, A_2, B_2, if A_1 is elementarily equivalent to A_2 and B_1 is elementarily equivalent to B_2, then $A_1 * B_1$ is elementarily equivalent to $A_2 * B_2$.

We now recall some of Szmielew's results on torsion-free abelian groups (see [EF72]). Let A be a torsion-free abelian group. Define $\alpha_p(A) = \dim(A/pA)$

over the field of p elements, if it is finite, and $\alpha_p = \infty$ otherwise. For example, for any prime p, we have $\alpha_p(\mathbb{Z}) = 1$. The Szmielew characteristic of A is $\psi(A) = (\alpha_2(A), \alpha_3(A), \alpha_5(A), \ldots)$. Then for a torsion-free abelian group B, $\mathrm{Th}(A) = \mathrm{Th}(B)$ if and only if $\psi(A) = \psi(B)$. In particular, if C is divisible, then $\mathrm{Th}(A) = \mathrm{Th}(A \oplus C)$, e.g., $\mathrm{Th}(\mathbb{Z}) = \mathrm{Th}(\mathbb{Z} \oplus \mathbb{Q})$.

Given a group G, $\dim(A/pA) < 2$ for any abelian subgroup $A \leq G$ if and only if

$$G \models \forall x_1, x_2 \exists y \left([x_1, x_2] = 1 \to \bigvee_{(m_1,m_2) \in S} x_1^{m_1} x_2^{m_2} = y^p \right),$$

where S is the set of all non-trivial tuples (m_1, m_2) where $0 \leq m_i < p$.

Theorem 4.1.2. *The elementary free group $T = \mathbb{Z} * (\mathbb{Z} \oplus \mathbb{Q})$ cannot be represented as a union of a chain of finitely generated elementary free groups.*

Proof. Any chain of finitely generated groups whose union is $\mathbb{Z} \oplus \mathbb{Q}$ must at some step be isomorphic to the free abelian group of rank 2. So a chain whose union is T must at some step include a non-cyclic abelian subgroup. But no finitely generated elementary free group can contain a non-cyclic abelian subgroup. $\square$

4.2 JSJ decompositions of subgroups of an elementary free group without non-cyclic abelian subgroups

Let F be a finitely generated free group. We can consider the theory $\mathrm{Th}(F)$ with or without coefficients, i.e., with or without the generators of F included as constants in the language. In the case where we have constants, all the groups elementarily equivalent to F contain a designated copy of F. Below, in the case of a coefficient-free system $S(X) = 1$ we put $F_{R(S)} = F(X)/R(S)$ and in the case when there are coefficients $F_{R(S)} = (F * F(X))/R(S)$.

Let M be a countably generated elementary free group in which all abelian subgroups are cyclic. Suppose the theory has coefficients. Then $F \leq M$. Let G be a subgroup of M generated by F and finitely many elements. We will show (Corollary 4.2.3) that for every free factor in the free decomposition of G relative to F, the JSJ decomposition of such a factor is either trivial or has a QH subgroup.

Note first that in the language with constants, we have $\mathrm{Th}_\forall(F) \supseteq \mathrm{Th}_\forall(G) \supseteq \mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(F)$, so by Definition 2.3.2, G is a limit group and so there exists an irreducible system of equations $S(X) = 1$ over F such that $G = F_{R(S)}$. In the case without constants, $\mathrm{Th}_\forall(G) \supseteq \mathrm{Th}_\forall(M)$. In particular, for any reduced word $w(x, y)$, $M \models \forall x \forall y([x, y] \neq 1 \to w(x, y) \neq 1)$, hence so does

G. So for any non-commuting $a, b \in G$, $\langle a, b\rangle$ is free, hence $\mathrm{Th}_\forall(\langle a, b\rangle) \supseteq \mathrm{Th}_\forall(G) \supseteq \mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(\langle a, b\rangle)$, so again we can write G as the coordinate group of an irreducible variety. The group G does not have non-cyclic abelian subgroups.

In the free decomposition of G relative to F, one free factor contains F and the others have trivial intersection with F. The factors that are isomorphic to closed surface groups and cyclic groups have trivial JSJ decompositions. We assume that at least one factor is not isomorphic to a surface group. We first wish to understand the JSJ decomposition of such a factor. Since the same argument gives a proof for both cases, when $F \leq G$ and when their intersection is trivial, we may assume that G is freely indecomposable relative to F and consider only one case, say $F \leq G$.

Then G has a non-degenerate cyclic JSJ decomposition [KM05a] relative to F; denote it by D. Let B be a basis of F. Then $G = \langle B, X \mid S\rangle$ gives the canonical finite presentation of G as the fundamental group of D. Let E_r be the set of edges between rigid subgroups. We will assume that the edge groups corresponding to edges in E_r are maximal cyclic in M. If not, we add the roots of the maximal cyclic subgroups in M to G, denote the new group by $\bar{G}$ and replace D by the cyclic JSJ decomposition of $\bar{G}$. Note that by Lemma 4.0.2, all abelian subgroups of M are cyclic, which guarantees that there is a deepest root to add to G. Moreover, since G inherits the splitting from the JSJ decomposition of $\bar{G}$, we do not have to add more roots to $\bar{G}$.

Let A_E be the group of F-automorphisms (or simply automorphisms, in the case where $F \cap G = 1$) of G generated by Dehn twists along the edges of D. The group A_E is abelian by [Lev04]. Recall that two solutions ϕ_1 and ϕ_2 of the equation $S(X) = 1$ are A_E-*equivalent* if there is an automorphism $\sigma \in A_E$ such that $\phi_1 \sigma = \phi_2$.

Recall that if A is a group of canonical automorphisms of G, then the maximal standard quotient of G with respect to A is defined as the quotient G/R_A of G by the intersection R_A of the kernels of all solutions of $S(X) = 1$ that are minimal with respect to A (see [KM05a] for details). Minimality is taken with respect to the length of a solution, where the length of a solution ψ is defined as $|\psi| = \min_{h \in F} \max_{x \in X} |h\psi(x)h^{-1}|$.

By [KM05a, Theorem 9.1] the maximal standard quotient G/R_{A_D} of G with respect to the whole group of canonical automorphisms A_D is a proper quotient of G, i.e., there exists a disjunction of systems of equations (it is equivalent to one equation if we consider the theory with coefficients) $V(X) = 1$ such that $V \not\in R(S)$ and all minimal solutions of $S(X) = 1$ with respect to the canonical group of automorphisms A_D satisfy $V(X) = 1$. Notice that there is a finite family of maximal limit groups $L_1, \dots, L_k$ such that every homomorphism $G/R_{A_D} \to F$ factors through one of them. Now, compare this with the following result.

Lemma 4.2.1. *The maximal standard quotient of G with respect to the group A_E is equal to G, i.e., the set of minimal solutions with respect to A_E discriminates G.*

Proof. Suppose, to the contrary, that the maximal standard quotient G/R_{A_E} is a proper quotient of G, i.e., there exists $V \in G$ such that $V \neq 1$ and $V^\phi = 1$ for any solution ϕ of S minimal with respect to A_E. Recall that the group A_E is generated by Dehn twists along the edges of D. If c_e is a given generator of the cyclic subgroup associated with the edge e, then we know how the Dehn twist σ associated with e acts on the generators from the set X. Namely, if $x \in X$ is a generator of a vertex group, then either $x^\sigma = x$ or $x^\sigma = c_e^{-1} x c_e$. Similarly, if $x \in X$ is a stable letter then either $x^\sigma = x$ or $x^\sigma = x c_e$. It follows that for $x \in X$ one has $x^{\sigma^n} = x$ or $x^{\sigma^n} = c_e^{-n} x c_e^n$ or $x^{\sigma^n} = x c_e^n$ for every $n \in \mathbb{Z}$. Now, since the centralizer of c_e in G is cyclic the following equivalence holds:

$$\exists n \in \mathbb{Z}(x^{\sigma^n} = z) \Longleftrightarrow \begin{cases} x = z & \text{if } x^\sigma = x \\ \exists y([y, c_e] = 1 \wedge y^{-1} x y = z) & \text{if } x^\sigma = c_e^{-1} x c_e \\ \exists y([y, c_e] = 1 \wedge x y = z) & \text{if } x^\sigma = x c_e \end{cases}.$$

Similarly, since the group A_E is finitely generated abelian one can write down a formula which describes the relation

$$\exists \alpha \in A_E (x^\alpha = z).$$

One can write the elements c_e as words in the generators X, say $c_e = c_e(X)$. Now the sentence

$$\forall X \exists Y \exists Z \left(S(X) = 1 \to \left(\bigwedge_{i=1}^{|Y|} [y_i, c_i(X)] = 1 \wedge Z = X^{\sigma_Y} \wedge V(Z) = 1 \right) \right)$$

holds in the group F. Indeed, this formula tells one that each solution of $S(X) = 1$ is A_E-equivalent to a (minimal) solution Z that satisfies the equation $V(Z) = 1$. Hence this formula is true in M; in particular, it is true for $X \subset G$. The subgroup generated by Z is isomorphic to the subgroup generated by X in M, and X and Z satisfy the same relations. But $V(X) \neq 1$, so the formula cannot be true in M: contradiction. □

Lemma 4.2.2. *There exist QH subgroups in D.*

Proof. By [KM05a, Theorem 9.1], G/R_{A_D} is a proper quotient of G, but by Lemma 4.2.1, G/R_{A_E} is not proper, so we have $A_D \neq A_E$, hence D has QH subgroups. □

Corollary 4.2.3. *A JSJ decomposition of a finitely generated freely indecomposable subgroup of M is either trivial or has a QH subgroup.*

4.3 Proof of the main theorem

We continue the notation (F, M, G, D, etc.) from Section 4.2. Let K be the fundamental group of the graph of groups obtained from D by removing all QH subgroups.

Lemma 4.3.1. *There is a K-homomorphism from G into M with a non-trivial kernel containing R_{A_D}. The quotient is also a quotient of one of the limit groups $L_1, \dots, L_k$ that are maximal limit quotients of G/R_{A_D}.*

Proof. The generators X in G corresponding to the decomposition D can be partitioned as $X = X_1 \cup X_2$ where $B \cup X_2$ generate K. Since G/R_{A_D} is a proper quotient, the system of equations $V(X_1, X_2) = 1$ defining R_{A_D} is not in $R(S)$.

Any solution $\psi : G \to F$ can be precomposed by an automorphism $\sigma \in A_D$ such that $\psi\sigma$ solves V. Since $A_E \trianglelefteq A_D$, we can factor $\sigma = \gamma\beta$ where $\beta \in A_E$ and $\gamma \in A_Q$. Note A_E is abelian and $\gamma|_K$ only conjugates the free factors of K. The existence of such an automorphism σ for any solution ψ is expressed by the following sentence:

$$\begin{aligned} &\forall X_1 \forall X_2 \exists Y_1 \exists Y_2 \exists T \; S(X_1, X_2) = 1 \\ &\to \left(\bigwedge_{i=1}^{|T|} [t_i, c_i(X_1, X_2)] = 1 \wedge Y_2 = X_2^{\sigma_T} \wedge S(Y_1, Y_2) = 1 \wedge V(Y_1, Y_2) = 1 \right). \end{aligned}$$

It is true in F, and therefore also in M. If we take X_1, X_2 to be the generators of G and denote the witnesses to Y_1, Y_2, T in M by Y_1, Y_2, T, then M models

$$\bigwedge_{i=1}^{|T|}[t_i, c_i(X_1, X_2)] = 1 \wedge Y_2 = X_2^{\sigma_T} \wedge S(Y_1, Y_2) = 1 \wedge V(Y_1, Y_2) = 1.$$

Let $G_1 = \langle G, Y_1, Y_2 \rangle$ and define a homomorphism $\phi : G \to G_1$ by $\phi(X_1) = Y_1$ and $\phi(X_2) = Y_2$, i.e., $\phi|_K = \gamma\beta|_K$. The subgroup of M generated by Y_2 is isomorphic to K. Let $\phi_1 = \phi\beta^{-1}$. Then $\phi_1|_K = \gamma|_K$ acts by conjugation on the free factors of K, i.e., if $K = K_1 * \cdots * K_s$ is a free decomposition of K, then for each i, there exists $g_i \in G$ such that for all $k \in K_i$, $\phi_1(k) = k^{g_i}$.

Let $\phi_1(G) = P_1 * \cdots * P_m$ be a free decomposition of $\phi_1(G)$. Define a map ν on $\phi_1(G)$ as follows: if P_j contains $K_i^{g_i}$ for some i, let $\nu(\ell) = \ell^{g_i^{-1}}$ for all $\ell \in P_j$; otherwise, ν acts identically on P_j. Also, define η on $\nu\phi_1(G)$ by $\eta\nu(P_j) = 1$ if P_j does not contain any factor K_i and $\eta\nu(P_j) = P_j$ otherwise. Then $\phi_2 = \eta\nu\phi_1$ is the desired homomorphism. □

Remark 4.3.2. We can also view ϕ_2 from a geometric perspective as follows. For each QH subgroup in D we associate a possibly trivial collection of simple closed curves on the corresponding surfaces that are mapped to the identity by ϕ_2. We construct another graph of groups Γ obtained from D by cutting the surfaces corresponding to QH vertex groups along this collection of curves, filling each curve with a disk and removing the surfaces that are not connected to any boundary components of the original QH vertex subgroups. Let $\bar{G}$ be the fundamental group of Γ. There is a natural epimorphism $\mu : G \to \bar{G}$. We define $\tau : \bar{G} \to G_1$ on each free factor $\bar{G}_i$ of $\bar{G}$ that is associated with a connected component of Γ by $\tau(\bar{G}_i) = \phi_2\mu^{-1}(\bar{G}_i)$. Then $\phi_2 = \tau\mu$. Notice that no non-trivial simple closed curve on a QH vertex group of Γ is mapped to the trivial element in G_1.

Lemma 4.3.3. *There exists a fundamental sequence*

$$G \xrightarrow{\pi_1} G_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} G_n = F$$

in the canonical Hom-diagram for G (see [KM12]) such that for each $i = 1, \ldots, n$ there is a subgroup $H_i < M$ such that

1. *H_i is a quotient of G_i;*
2. *the JSJ decomposition Δ_i of H_i (if H_i is a free product, then Δ_i is a Grushko decomposition followed by JSJ decompositions of the free factors) contains a QH subgroup;*
3. *if K_i is the fundamental group of the subgraph of groups of Δ_i containing all rigid vertex groups, then $K_i \leq H_{i+1}$;*
4. *there is a proper K_i-homomorphism $\beta_{i+1} : H_i \to H_{i+1}$ such that its kernel contains $R_{A_{\Delta_i}}$;*

5. *H_i is freely indecomposable relative to the isomorphic image of K_{i-1} in H_i. Each free factor of H_i contains a conjugate of a free factor of K_{i-1}.*

If

$$G \xrightarrow{\beta_1} H_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} H_n = F \tag{4.1}$$

is the corresponding fundamental sequence describing some homomorphisms from G to a free group, then the NTQ group N for this fundamental sequence is a regular NTQ group. Moreover, M contains a quotient of N containing G.

Proof. We set $\beta_1 = \phi_2$ and $H_1 = \phi_2(G)$. Since every chain of proper limit quotients of a limit group is finite, the statement of the lemma can be proved using Lemma 4.3.1 inductively. □

The above lemma is used to prove the following theorem.

Theorem 4.3.4. *For every finitely generated subgroup G of M (in the case with coefficients, G must contain F) there exists a regular NTQ group that is a subgroup of M and contains G.*

The proof of this theorem is given in Section 4.4. To give an idea, notice that by Lemma 4.3.3, M contains a quotient of the regular NTQ group N such that this quotient contains G. Denote this quotient by G_1. If G_1 is isomorphic to N the theorem is proved. Otherwise we will construct a regular NTQ group $\bar{N}_1$ containing G_1 and such that M contains a quotient G_2 of $\bar{N}_1$, moreover $G_1 \leq G_2$. If $G_2 = \bar{N}_1$, the theorem is proved. Otherwise, we construct a regular NTQ group $\bar{N}_2$ such that $G_2 \leq \bar{N}_2$ and M contains a quotient G_3 of $\bar{N}_2$. We will show that the construction can be designed such a way that it stops, namely on some step i, $G_{i+1} = \bar{N}_i$.

Proof of Theorem 4.0.1. Let G be the subgroup of M generated by F and the first n generators of M. By Theorem 4.3.4 there is a regular NTQ group that is a subgroup of M and contains G. Denote this NTQ group by F_1. It is the NTQ group over the base group F. Take now the subgroup Γ of M generated by F_1 and next n generators of M. There is a regular NTQ group F_2 over F_1 that is a subgroup of M and contains Γ.

Suppose, by induction, that a regular NTQ group $F_i \leq M$ containing the first ni generators of M has been already constructed. Let Γ_i be the subgroup of M containing F_i and the first $n(i+1)$ generators of M. We can construct a regular NTQ group F_{i+1} over F_i containing Γ_i. Then M is the union of the chain $F \prec F_1 < \ldots < F_i < \ldots$. $\square$

Similarly to Theorem 4.0.1, one can show the following using a completely analogous proof:

Theorem 4.3.5. *Let H be a non-cyclic torsion-free hyperbolic group. Let $M \geq H$ be a countable group that is elementarily equivalent to H in the language with constants for the generators of H, and suppose all finitely generated abelian subgroups of M are cyclic. Then M is a union of a chain of hyperbolic towers over H.*

4.4 Proof of Theorem 4.3.4

In this section we give the proof of Theorem 4.3.4 from [KN20a], written by the first author. It is included in this dissertation only for completeness of the presentation. The proof is, basically, the adaptation for our needs of the construction of the $\forall\exists$-tree from [KM06] or [KM12].

The strategy of the proof.

In the free decomposition of G relative to F, one free factor contains F and the others have trivial intersection with F. If all factors are isomorphic to closed surface groups and cyclic groups, we are done because in both cases G is a regular NTQ group as required. Therefore we assume that at least one factor is not isomorphic to a surface group.

Let $G = F_{R(U_0)}$ be a limit group generated by elements X that is a subgroup of the elementary free group M. We will use the following strategy. Let N be the regular NTQ group from Lemma 4.3.3. If $N \leq M$ the theorem is proved. Suppose that $N \not\leq M$. Then M contains the proper quotient of N corresponding to the system of equations $U_1 = 1$, and $G_1 = F_{R(U_1)}$ contains G. Denote $N = N_0 = \bar{N}_0$. We say that $N_0, \bar{N}_0$ are constructed on the **initial step**.

Definition 4.4.1. We call fundamental sequences satisfying the first and second restrictions introduced in [KM06] (see also [KM12]) *well-aligned* fundamental sequences.

We perform the **first step** and construct the so called block-NTQ groups N_1 describing all homomorphisms $G \to F$ that factor through $\bar{N}_0$, satisfy $U_1 = 1$ and belong to well-aligned fundamental sequences. Such block-NTQ group N_1 contains as a subgroup a regular NTQ group $\bar{N}_1$ containing G. Since this can be described by first order sentences, we conclude the same way as we did in Lemma 4.3.3 that M contains a quotient of at least one of the groups $\bar{N}_1$ containing G. If this quotient is isomorphic to $\bar{N}_1$ we stop, and the theorem is proved.

Otherwise $G_2 = F_{R(U_2)}$ is a proper quotient of $\bar{N}_1$, we go to the **second step** and construct refined block-NTQ systems N_2 describing all homomorphisms $G \to F$ that factor through N_1 and $\bar{N}_1$, satisfy $U_2 = 1$ and belong to well-aligned fundamental sequences.

We will prove that this construction stops. The construction of the "refined" NTQ systems is quite complicated but it is very similar to the construction of a branch of the $\exists\forall$-tree in the proof of the decidability of the theory of a free group [KM06], [KM12].

We will represent the construction as a path. We assign to each vertex of the path some set of homomorphisms $G \to F$ and a regular NTQ group (hyperbolic tower) containing G as a subgroup. We assign the set of all homomorphisms $G \to F$ to the initial vertex w_0.

If Q is a QH subgroup, let $size(Q) = (genus(Q), |\chi(Q)|)$. Pairs are ordered left lexicographically. Let complexity of the JSJ decomposition be the tuple of complexities

$$(size(Q_1), \ldots, size(Q_k)),$$

where $size(Q_1) \geq \ldots \geq size(Q_k)$. We order tuples left lexicographically. Similarly the *regular size* of an NTQ group is defined.

It is convenient to define the notion of complexity of a fundamental sequence ($Cmplx(Var_{fund})$) at follows:

$$Cmplx(Var_{\text{fund}}) =$$

$$(dim(Var_{\text{fund}}) + factors(Var_{\text{fund}}), (size(Q_1), \ldots, size(Q_m))),$$

where $factors(Var_{\text{fund}})$ is the number of freely indecomposable, non-cyclic terminal factors (this component only appears in fundamental sequences relative to subgroups), and $(size(Q_1), \ldots, size(Q_m))$ is the regular size of the system. The complexity is a tuple of numbers which we compare in the left lexicographic order. The number $dim(Var_{\text{fund}}) + factors(Var_{\text{fund}})$ is called the Kurosh rank of the fundamental sequence.

Let K be a finitely generated group. Recall that any family of homomorphisms $\Psi = \{\psi_i : K \to F\}$ factors through a finite set of maximal fully residually free groups $H_1, \ldots, H_k$ that all are quotients of K. We first take a quotient K_1 of K by the intersection of the kernels of all homomorphisms from Ψ, and then construct maximal fully residually free quotients $H_1, \ldots, H_k$ of K_1. We say that Ψ *discriminates* groups $H_1, \ldots, H_k$, and that each H_i is *a fully residually free group discriminated by* Ψ.

Let S be a compact surface. Given a homomorphism $\phi : \pi_1(S) \to H$ a family of pinched curves is a collection $\mathcal{C}$ of disjoint, non-parallel, two-sided simple closed curves, none of which is null-homotopic, such that the fundamental group of each curve is contained in ker ϕ (the curves may be parallel to a boundary component). The map ϕ is *non-pinching* if there is no pinched curve: ϕ is injective in restriction to the fundamental group of any simple closed curve which is not null-homotopic

4.4.1 Initial step

We can construct algorithmically a finite number of NTQ systems corresponding to branches b of the canonical Hom-diagram described in [KM12, Section 2.2]. For at least one of them we can construct the well aligned fundamental sequence (4.1) and the regular NTQ group N, denote this branch by b and the system corresponding to N by $S(b) = 1$. $(S(b)$: $S_1(X_1, \ldots, X_n) = 1, \ldots, S_n(X_n) = 1)$. Then $N = F_{R(S(b))}$ For this fundamental sequence $Var_{\text{fund}}(S(b))$ and regular NTQ group $N = N_0 = \bar{N}_0 = F_{R(S(b))}$ we assign a vertex w_1 of the path. We draw an edge from vertex w_0 to each vertex w_1.

We say that a fundamental sequence is constructed modulo some subgroups of the coordinate group if these subgroups are elliptic in the JSJ decompositions on all levels in the construction of this fundamental sequence.

Definition 4.4.2. Let $S = 1$ be a canonical NTQ system corresponding to a well-aligned fundamental sequence for a group L relative to limit groups $L_1, \ldots, L_k$:

$$S_1(X_1, X_2, \ldots, X_n) = 1,$$

$$S_2(X_2, \ldots, X_n) = 1,$$

$$\ldots$$

$S_n(X_n) = 1$

We say that values of $X_2, \ldots, X_n$ are *minimal with respect to the images of* L if for each $i > 1$ values of $X_i, \ldots, X_n$ are minimal in fundamental sequences for the groups $F_{R(S_i,\ldots,S_n)}$ modulo rigid subgroups and edge groups of the decomposition of the image of L on level $i-1$ from the top. In particular, values of $X_2, \ldots, X_n$ are minimal in fundamental sequences for $F_{R(X_2,\ldots,X_n)}$ modulo rigid subgroups and edge groups of the image of L in $F_{R(X_1,\ldots,X_n)}$.

4.4.2 First step

Let $U_1(X_1, \ldots, X_n) = 1$ be the system of equations satisfied by the images of $X_1, \ldots, X_n$ in M. If $F_{R(U_1)}$ is not a proper quotient of $N_0 = F_{S(b)}$ we stop. Suppose it is a proper quotient. Let $Var_{\text{fund}}(U_1)$ be the subset of homomorphisms from the set $Var_{\text{fund}}(S(b))$ minimal with respect to the canonical automorphisms modulo the images of G and satisfying the additional equation $U_1(X_1, \ldots, X_n) = 1$.

Let G_1 be a fully residually free group discriminated by the set of homomorphisms $Var_{\text{fund}}(U_1)$. $G_1 = F_{R(U_1)}$ and we can suppose that the system $U_1(X_1, \ldots, X_n) = 1$ is irreducible. Consider the family of well-aligned fundamental sequences for G_1 modulo the images $R_1, \ldots, R_s$ of the factors in the free decomposition of the subgroup $H_1 = \langle X_2, \ldots, X_n \rangle$. Since we only consider well aligned canonical fundamental sequences c for G_1 modulo $R_1, \dots, R_s$ they have, in particular, the following property: c is consistent with the decompositions of surfaces corresponding to quadratic equations of S_1 by a collection of simple closed curves mapped to the identity by π_1. Namely, if we refine the JSJ decomposition of G by adding splittings corresponding to the simple closed curves that are mapped to the identity when G is mapped to the free product in [KM12, Section 3.1] then the boundary elements of QH subgroups in this new decomposition are mapped to elliptic elements on all the levels of c. Let k_1 be the sum of the ranks of the maximal free groups that can be the images of the closed surfaces after we refined the JSJ decomposition.

The group G is embedded in the NTQ group corresponding to the fundamental sequence c.

Suppose the fundamental sequence c has the top dimension component k_1. If the NTQ system corresponding to the top level of the sequence c is the same as $S_1 = 1$, we extend the fundamental sequences modulo $R_1, \dots, R_s$ by canonical fundamental sequences for H_1 modulo the factors in the free decomposition of the subgroup $\langle X_3, \dots, X_n \rangle$. Since the fundamental sequence is well alligned it has dimension less or equal to k_2. We continue this way to construct fundamental sequences $Var_{\text{fund}}(S_1(b))$.

Suppose now that the fundamental sequence c for G_1 modulo $R_1, \dots, R_s$ has dimension strictly less than k_1 or has dimension k_1, but the NTQ system corresponding to the top level of c is not the same as $S_1 = 1$. Then we use the following lemma (in which we suppose that $R_1, \dots, R_s$ are non-trivial).

Lemma 4.4.3. *The image G_t of G in the group H_t appearing on the terminal level t of the sequence c is a proper quotient of G unless G is a free group.*

Proof. Consider the terminal group of c; denote it H_t. Suppose G_t is isomorphic to G. Denote the abelian JSJ decomposition of H_t by D_t. Then there is an abelian decomposition of G induced by D_t. Therefore rigid (non-abelian and non-QH) subgroups and edge groups of G are elliptic in this decomposition. But this is impossible because this means that the homomorphisms we are considering can be shortened by applying canonical automorphisms of H_t modulo those subgroups $\{R_1, \dots, R_s\}$ but $V_{fund}(U_1)$ contains only homomorphisms minimal in fundamental sequences modulo $R_1, \dots, R_s$ (see Def. 4.4.2).

□

Therefore the image of G on all the levels of the fundamental sequence c above some level p is isomorphic to G and on level p is a proper quotient of G. Denote the complete set of fundamental sequences that encode all the homomorphisms $G_p \to F$ by $\mathcal{F}$. One can extract from c modulo level p the induced well aligned fundamental sequence for G, see the definition of the induced fundamental sequence in [KM06], [KM12]. Denote this induced fundamental sequence up to level p by c_2. Consider a fundamental sequence c_3 that consists of homomorphisms obtained by the composition of a homomorphism from c_2 and from a fundamental sequence $b_2 \in \mathcal{F}$. Let $\bar{N}_1$ be the NTQ group corresponding to c_3. Then $\bar{N}_1$ is a regular NTQ group. The existence of the groups occurring in the fundamental sequence c_3 can be described by first order formulas and, therefore, there is a quotient of $\bar{N}_1$ inside M.

Denote by $M(X, Z_1)$ (where Z_1 is the set of generators, $X \subset Z_1$) the group generated by the top p levels of the NTQ group corresponding to the fundamental sequence c. Consider the *block-NTQ group* N_1 that is a fully residually free quotient of the amalgamated product of $M(X, Z_1)$ and $\bar{N}_1$ amalgamated along the top p levels of $\bar{N}_1$. To obtain a fully residually free quotient we add relations to make it commutative transitive. Assign the sequence c_3, regular NTQ group $\bar{N}_1$, block-NTQ group N_1, and $M(X, Z_1)$ to the vertex w_2 of the path. We draw an edge from the vertex w_1 to w_2.

If there are no additional equations on generators of $\bar{N}_1$ in M we stop because in this case $\bar{N}_1$ is the regular NTQ system that is contained in M and contains G. Suppose there are additional equations on generators of $\bar{N}_1$.

Let $U_2 = 1$ be a system of all equations on generators of $\bar{N}_1$. Moreover, we suppose that $G_2 = F_{R(U_2)}$ is discriminated by homomorphisms that are minimal for $\bar{N}_1$ with respect to images of G and minimal for $M(X, Z_1)$ with respect to images of $F_{R(U_1)}$.

4.4.3 Second step

We will describe the next step in the construction which basically is general. The fundamental sequence and the block-NTQ group obtained on the second step will be assigned to vertex w_3.

Suppose the JSJ decomposition for the NTQ system corresponding to the top level of c corresponds to the equation $S_{11}(X_{11}, X_{12}, \ldots) = 1$; some of the variables X_{11} are quadratic, the others correspond to extensions of centralizers. Construct a canonical fundamental sequence $c^{(2)}$ for G_2 modulo the factors in the free decomposition of the subgroup generated by $X_{12}, \ldots$.

Denote by N_0^1 the image of the subgroup generated by $X_1, \ldots, X_n$ in the group $M(X, Z_1)$ discriminated by c, we will write $N_0^1 = \langle X_1, \ldots, X_n \rangle_c$. Denote by $N_0^2 = \langle X_1, \ldots, X_n \rangle_{c^{(2)}}$ the image of $\langle X_1, \ldots, X_n \rangle$ in the group discriminated by $c^{(2)}$. N_0^1 must be isomorphic to N_0^2 because they correspond to the same subgroup of M.

Case 1. If the top levels of c and $c^{(2)}$ are the same, then we go to the second level of c and consider it the same way as the first level.

Case 2. If the top levels of the NTQ system for c and S_1 are the same (therefore c has only one level). We work with $c^{(2)}$ the same way as we did for c. Suppose $c^{(2)}$ is not the same as c. Then the image of G on some level p of $c^{(2)}$ is a proper quotient of G by Lemma 4.4.3. Let $M(X, Z_2)$ be the NTQ group corresponding to the top p levels of $c^{(2)}$. We consider fundamental sequences constructed as follows: the top part is the fundamental sequence induced by the top part of $c^{(2)}$ above level p for G, and the bottom part is a fundamental sequence for this quotient of G (solutions will go along the first fundamental sequence from the top level to level p and then continue along one of the fundamental sequences for the image of G on level k). We assign each of these fundamental sequences to a vertex w_3, then take the corresponding canonical NTQ group $\bar{N}_2$. Then we construct the block-NTQ group as we did on the first step, denote it by N_2. We also assign N_2 and $\bar{N}_2$ to the vertex w_3.

Case 3. If the top levels of the NTQ system for c and S_1 are not the same and the top levels of c and $c^{(2)}$ are not the same, then we look at $N_0^2 = N_0^1$. It follows from the minimality restriction made for the solutions used to construct $F_{R(U_2)}$ that there is some level k from the top of $c^{(2)}$ such that we can suppose that the image of either G or N_0^1 on this level is a proper quotient of it (and on the levels above k is isomorphic to G (resp. to N_0^1).

If the image of G that we denote G_t is a proper quotient of G we do what we did on step 1. Namely, we construct the fundamental sequence for G induced from $c^{(2)}$ and continue it with a canonical fundamental sequence for G, construct the NTQ group $\bar{N}_2$ for this fundamental sequence and the block NTQ group N_2 and assign them to a vertex w_3.

Suppose now that G_t is isomorphic to G and $N_{0,t}^1$ is a proper quotient of N_0^1. Consider fundamental sequences for $N_{0,t}^1$ modulo the images of subgroups $R_1, \dots, R_s$, and apply to them step 1. Denote the obtained fundamental sequences by f_i. Construct fundamental sequences for the subgroup generated by the images of $X_1, \dots, X_n$ with the top part being induced from the top part of $c^{(2)}$ (above level k) and bottom part being some f_i, but not the sequence with the same top part as c. We construct the block-NTQ group amalgamating the top k levels of $c^{(2)}$ and the block-NTQ group constructed for f_i as on the first step. We denote this block-NTQ group (consisting of three blocks) by N_2 and its subgroup that is a regular NTQ group by $\bar{N}_2$. These groups are assigned then to w_3.

Since we started this step assuming that $U_2 = 1$ is a proper equation on the generators of $\bar{N}_1$ It must be some level s of c such that we do not have cases 2 or 3 on the levels above s but have one of these cases on level s.

4.4.4 General step

We now describe the n'th step of the construction. Denote by N_i the block-NTQ group constructed on the i'th step and assigned to vertex w_{i+1} and by $N_i^j, j > i$ its image on the j'th step. Denote by $\bar{N}_i$ the NTQ groups constructed on i'th step.

Let $\{j_k,\ k = 1, \dots, s\}$ be all the indices for which the top level of N_{j_k+1} is different from the top level of N_{j_k}. Let $M(X, Z_i)$ be the group corresponding to the top block on step i and $M^j(X, Z_i)$ its image in N_{j-1}^j.

On each step i we consider fundamental sequences for the proper quotient of N_{i-1} modulo the images of freely indecomposable free factors of the second level block NTQ group N_{i-1} (images of rigid subgroups in the relative decomposition of $M(X, Z_{i-1})$). Let $R^{r(i)}$ be the family of these rigid subgroups on step i. Every time when we increase parameter subgroups $R^{r(i)}$, $r(i)$ increases by 1. Let $g(t)$ be the last step when we constructed fundamental sequences modulo R^t. On step $g(t)+1$ parameters subgroups are increased to become R^{t+1} and fundamental sequences for the top block are constructed modulo R^{t+1}, so $r(g(t)+1) = t+1$.

Notice that G and all the groups $M^{g(r)}(X, Z_{g(r)})$, $g(r) < n$, are embedded into N_{n-1}^n.

Case 1. The top levels of $c^{(n)}$ and $c^{(n-1)}$ are the same. In this case we go to the second level and consider it the same way as the first level.

Case 2. The top levels of $c^{(n-1)}, c^{(n-2)}, \ldots, c^{(n-i)}$ are the same, and the top levels of $c^{(n-1)}$ and $c^{(n)}$ are not the same. Then on some level p of the NTQ group for $c^{(n)}$ we can suppose that the image of $M^n(X, Z_{n-1})$ is a proper quotient of it (and on the levels above p it is isomorphic to $M^n(X, Z_{n-1})$). Let r be the minimal such index that $M^n(X, Z_{g(r)})_t$ is a proper quotient of $M^n(X, Z_{g(r)}) = M^{g(r)}(X, Z_{g(r)})$. Then the levels below the first level (level 2 and below) of the block NTQ group N_n that we are constructing on step n will correspond to the block NTQ group $\tilde{N}_{g(r)+1}$ that we would construct on step $g(t) + 1$ for $M^n(X, Z_{g(r)})_t$. Moreover, we consider only fundamental sequences for $\tilde{N}_{g(r)+1}$ with the top level different from $c^{(g(r))}$ (not of maximal complexity).

Case 3. The top levels of $c^{(n-2)}$ and $c^{(n-1)}$ are not the same and the top levels of $c^{(n-1)}$ and $c^{(n)}$ are not the same. Then on some level p of $c^{(n)}$ the image of $M^n(X, Z_{n-1})$ is a proper quotient of $M^n(X, Z_{n-1})$. Let r be the minimal such index that $M^n(X, Z_{g(r)})_t$ is a proper quotient of $M^n(X, Z_{g(r)}) = M^{g(r)}(X, Z_{g(r)})$. Then the levels below the first level (level 2 and below) of the block NTQ group N_n that we are constructing on step n will correspond to the block NTQ group $\tilde{N}_{g(r)+1}$ that we would construct on step $g(t) + 1$ for

$M^n(X, Z_{g(r)})_t$. Moreover, we consider only fundamental sequences for $\tilde{N}_{g(r)+1}$ with the top level different from $c^{(g(r))}$ (not of maximal complexity).

If going from the top to the bottom of the block-NTQ system, on some level of the block-NTQ group we can apply Case 2 or 3 to this level we apply it. There will be always such a level.

4.4.5 Some auxiliary results

We recall several results that we will need.

Let G be a freely indecomposable limit group with the JSJ decomposition D. Let K be the fundamental group of the subgraph of groups generated by the rigid subgroups of D. Let ϕ be an epimorphism from G to a limit group H, and suppose that H is freely indecomposable relative to $\phi(K)$ and Δ is the JSJ decomposition of H relative to $\phi(K)$. Suppose that ϕ is non-pinching on each QH subgroup.

Let Q be a QH subgroup in D and let S be the corresponding punctured surface. Since the boundary elements of Q are mapped by ϕ to elliptic elements in Δ, the (maybe trivial) cyclic decomposition induced on $\phi(Q)$ by Δ can be lifted to (maybe trivial) maximal cyclic decomposition of Q, in which every cyclic edge group is mapped by ϕ to an elliptic element in Δ which corresponds to some decomposition of the punctured surface S along a maximal collection of disjoint non-homotopic simple closed curves. Let $\Delta(Q)$ be the corresponding cyclic decomposition of Q and let $\Delta(S)$ be the associated collection of simple closed curves.

Lemma 4.4.4. *Let $\bar{Q}$ be a QH subgroup in Δ, and let $\bar{S}$ be the associated punctured surface. If ϕ maps non-trivially a connected subsurface S_1 of $S - \Delta(S)$ into $\bar{S}$, then $genus(\bar{S}) \leq genus(S)$ and $|\chi(\bar{S})| \leq |\chi(S)|$. Moreover, in this case ϕ maps the fundamental group of the subsurface S_1 into a finite index subgroup of $\bar{Q}$.*

Proof. Since the boundary components of S_1 are mapped to non-trivial elliptic elements of in Δ, we have $genus(\bar{S}) \leq genus(S)$ and $|\chi(\bar{S})| \leq |\chi(S)|$. The last statement of the lemma follows from [KM06, Lemma 7]. $\square$

In the case when there is a pinching corresponding to ϕ we construct another graph of groups Γ obtained from D by cutting the surfaces corresponding to QH vertex groups along the pinching and filling each curve with a disk. Then $\phi = \phi_1 \circ \theta$, where θ maps the pinching to the identity and ϕ_1 is non-pinching.

We call a QH subgroup of D *stable* if there exists a QH subgroup $\bar{Q}$ in Δ so that ϕ_2 maps Q into $\bar{Q}$ and the sizes of Q and $\bar{Q}$ are the same.

We now give the setting for the next lemma. Let a regular NTQ system

$S(X_1,\dots,X_n)=1$ have the form

$$S_1(X_1,\dots,X_n)=1,$$

$$\dots$$

$$S_n(X_n)=1,$$

where $S_1=1$ corresponds to the top level of the JSJ decomposition for $\Gamma_{R(S)}$, variables from X_1 are quadratic. Consider this system together with the fundamental sequence $V_{\text{fund}}(S)$ defining it. Let $V_{\text{fund}}(U_1)$ be the subset of $V_{\text{fund}}(S)$ satisfying some additional equation $U_1=1$, and G_1 a group discriminated by this subset. Consider the family of those canonical fundamental sequences for G_1 modulo the images $R_1,\dots,R_s$ of the factors $P_1,\dots,P_s$ in the free decomposition of the subgroup $\langle X_2,\dots,X_m\rangle$, which have the same Kurosh rank modulo them as $S_1=1$ and are compatible up to an automorphism with the pinching of QH subgroups of $F_{R(S)}$. Denote this free decomposition by H_1*. The terminal group of such a fundamental sequence does not have a sufficient splitting (see [KM06]) relative to H_1*.

Denote such a fundamental sequence by c, and the corresponding NTQ system $\bar{S}=1(mod\ H_1*)$, where $\bar{S}=1$ has the form

$$\bar{S}_1(X_{11},\dots,X_{1m})=1$$

$$\ldots$$

$$\bar{S}_m(X_{1m}) = 1.$$

Denote by D_S a canonical decomposition corresponding to the group $F_{R(S)}$. Non-QH subgroups in this decomposition are $P_1, \ldots, P_s$, QH subgroups correspond to the system $S_1(X_1, \ldots, X_n) = 1$. For each i there exists a canonical homomorphism

$$\eta_i : F_{R(S)} \to F_{R(\bar{S}_i, \ldots, \bar{S}_m)}$$

such that $P_1, \ldots, P_s$ are mapped into rigid subgroups in the canonical decomposition of $\eta_i(F_{R(S)})$.

Each QH subgroup in the decomposition of $F_{R(\bar{S}_i, \ldots, \bar{S}_m)}$ as an NTQ group is a QH subgroup of $\eta_i(F_{R(S)})$. Since η_i can be represented as a composition of the map killing the pinching on QH subgroups of D_S and a non-pinching map, by Lemma 4.4.4, for each QH subgroup Q_1 of $\eta_i(F_{R(S)})$ there exists a QH subgroup of the image of $F_{R(S)}$ after killing the pinching (not necessarily a maximal QH subgroup) that is mapped into a subgroup of finite index in Q_1. The size of this QH subgroup is, obviously, greater or equal to the size of Q_1. Those QH subgroups of $F_{R(S)}$ that are mapped into QH subgroups of the same size by some η_i are stable.

Lemma 4.4.5. *In the conditions above there are the following possibilities:*

(i) size $(\bar{S}) =$ *size* (S_1)*, in this case* c *has only one level identical to* S_1*;*

(ii) It is possible to modify the system $\bar{S} = 1$ *in such a way that size* $(\bar{S}) <$ *size* (S_1)*;*

Proof. The fundamental sequence c modulo the decomposition H_1* has the same Kurosh rank as $S_1 = 1$. The Kurosh rank of $Q_1 = 1$ is the sum of the following numbers:

1) the dimension of a free factor F_1 in the free decomposition of $F_{R(S)}$ corresponding to an empty equation in $S_1 = 1$;

2) the sum of dimensions of surface group factors;

3) the number of free variables of quadratic equations with coefficients in $S_1 = 1$ corresponding to the fundamental sequence $Var_{\text{fund}}(S)$,

4) $factors(Var_{\text{fund}})$.

Because c has the same Kurosh rank, the free factor F_1 is unchanged. Surface factors are sent into different free factors.

If *size* $(\bar{S}) =$ *size* (S_1), then the generic family of solutions for $S_1 = 1$ factors through $\bar{S} = 1$ and in this case c has only one level identical to S_1.

Otherwise, by the analog of [KM05b, Theorem 9] we move all the stable QH subgroups corresponding to the system $S_1 = 1$ to the bottom level m of $\bar{S}$. If all QH subgroups corresponding to the system $S_1 = 1$ were stable we would have $size\ (\bar{S}) = size\ (S_1)$. Therefore there are non-stable QH subgroups. Then $size\ (\bar{S}) < size\ (S_1)$ by Lemma 4.4.4 and the paragraph after it. The lemma is proved. □

4.4.6 The procedure is finite

In this subsection we will prove the following result.

Proposition 4.4.6. *The path $w_0, w_1, ...,$ is finite.*

We will use induction on the complexity of $S_1(X_1, \ldots, X_n) = 1$. The induction assumption is that for any block-NTQ system for which the complexity of the first level is less than the complexity of $S_1(X_1, \ldots, X_n) = 1$ the procedure is finite. Suppose the procedure is infinite.

By Lemma 4.4.5, every time we apply the transformation of Case 3 (we refer to the cases from Section 4.4.4) in the construction we either (i) decrease the dimension in the top block, therefore decrease the Kurosh rank, or (ii) replace the NTQ system in the top block by another NTQ system of the same dimension but of a smaller size. Hence the complexity defined in the beginning of this section decreases. Notice that the complexity of the top block is bounded by the complexity of $S_1(X_1, \ldots, S_n) = 1$. Hence, Case 3 cannot be applied infinitely many times to the top block.

Therefore starting at some step n_1 for any $n > n_1$, $r(n) = r(n_1)$, the top blocks of all N_n are the same and fundamental sequences modulo $R^{r(n)} = R^{r(n_1)}$ for the top block are of maximal complexity.

Therefore eventually beginning at some step after step n_1 we are applying the procedure to the second block, more precisely to the terminal level of first block fundamental sequence.

If at some step $n > n_1$ $G_t^{(n)}$ is a proper quotient of G then the procedure for the second level is eventually applied to $G_t^{(n)}$ and is finite by induction. Otherwise, if we apply Case 2, we consider the second block for proper quotients of a finite number of groups. Let $r < r(n_1)$ be the minimal index for which $N^n(X, Z_{g(r)})_t$ is a proper quotient of $N^{g(r)}(X, Z_{g(r)})$ for some $n > n_1$. Moreover we only consider fundamental sequences for $N^n(X, Z_{g(r)})_t$ modulo R^{r+1} that are not of maximal complexity, in particular, their complexity is less than the complexity of $S_1(X_1, \ldots, X_n) = 1$. By induction, the procedure for $N^n(X, Z_{g(r)})_t$ is finite.

This proves the proposition. Theorem 4.3.4 follows from the proposition.

Bibliography

[Bae37] R. Baer. "Abelian groups without elements of finite order". In: *Duke Mathematical Journal* 3.1 (1937), pp. 68–122.

[BMR99] G. Baumslag, A. Myasnikov, and V. Remeslennikov. "Algebraic geometry over groups I. Algebraic sets and ideal theory". In: *Journal of Algebra* 219.1 (1999), pp. 16–79.

[EF72] P. C. Eklof and E. R. Fischer. "The elementary theory of abelian groups". In: *Annals of Mathematical Logic* 4.2 (1972), pp. 115–171.

[Fuc73] L. Fuchs. *Infinite Abelian Groups*. Vol. 36-II. Academic Press, 1973.

[HW19] M. Heusener and R. Weidmann. "A remark on Whitehead's cut-vertex lemma". In: *Journal of Group Theory* 22.1 (2019), pp. 15–21.

[KM05a] O. Kharlampovich and A. Myasnikov. "Effective JSJ decompositions". In: *Groups, languages, algorithms*. Ed. by A. Borovik. Vol. 378. Contemporary Mathematics, 2005.

[KM05b] O. Kharlampovich and A. Myasnikov. "Implicit function theorem over free groups". In: *Journal of Algebra* 290.1 (2005), pp. 1–203.

[KM06] O. Kharlampovich and A. Myasnikov. "Elementary theory of free non-abelian groups". In: *Journal of Algebra* 302.2 (2006), pp. 451–552.

[KM10] O. Kharlampovich and A. Myasnikov. "Equations and fully residually free groups". In: *Combinatorial and geometric group theory.* Springer, 2010, pp. 203–242.

[KM12] O. Kharlampovich and A. Myasnikov. *Quantifier elimination algorithm to boolean combination of ∃∀-formulas in the theory of a free group*. 2012. arXiv: `1207.1900 [math.GR]`.

[KMS20] O. Kharlampovich, A. Myasnikov, and R. Sklinos. "Fraïssé limits of limit groups". In: *Journal of Algebra* 545 (2020), pp. 300–323.

[KN20a] O. Kharlampovich and C. Natoli. *On countable elementary free groups*. 2020. arXiv: `2006.02414 [math.LO]`.

[KN20b] Olga Kharlampovich and Christopher Natoli. *Non-∀-homogeneity in free groups*. 2020. arXiv: `2001.09904 [math.LO]`.

[KV12] O. Kharlampovich and A. Vdovina. "Linear estimates for solutions of quadratic equations in free groups". In: *International Journal of Algebra and Computation* 22.01 (2012), p. 1250004.

[Lev04] G. Levitt. "Automorphisms of Hyperbolic Groups and Graphs of Groups". In: *Geometriae Dedicata* 114 (2004), pp. 49–70.

[LS62] R. C. Lyndon and M. P. Schützenberger. "The equation $a^M = b^N c^P$ in a free group". In: *The Michigan Mathematical Journal* 9.4 (1962), pp. 289–298.

[Mar02] D. Marker. *Model theory: An introduction*. Vol. 217. Springer, 2002.

[Nie03] A. Nies. "Aspects of free groups". In: *Journal of Algebra* 263.1 (2003), pp. 119–125.

[Ols89] A. Y. Olshanskii. "Diagrams of homomorphisms of surface groups". In: *Sibirsk. Mat. Zh.* 30.6 (1989), pp. 150–171.

[Oul11] A. Ould Houcine. "Homogeneity and prime models in torsion-free hyperbolic groups". In: *Confluentes Mathematici* 3.01 (2011), pp. 121–155.

[Per11] C. Perin. "Elementary embeddings in torsion-free hyperbolic groups". In: *Annales scientifiques de l'École Normale Supérieure* 44.4 (2011), pp. 631–681.

[PS12] C. Perin and R. Sklinos. "Homogeneity in the free group". In: *Duke Mathematical Journal* 161.13 (2012), pp. 2635–2668.

[Sel06] Z. Sela. "Diophantine geometry over groups VI: The elementary theory of a free group". In: *Geometric & Functional Analysis GAFA* 16.3 (2006), pp. 707–730.

[Sel10] Z. Sela. “Diophantine geometry over groups X: The elementary theory of free products of groups”. In: *arXiv preprint arXiv:1012.0044* (2010).

[Whi36] J. H. C. Whitehead. “On certain sets of elements in a free group”. In: *Proceedings of the London Mathematical Society* 2.1 (1936), pp. 48–56.

[Wil12] H. Wilton. “One-ended subgroups of graphs of free groups with cyclic edge groups”. In: *Geometry & Topology* 16.2 (2012), pp. 665–683.