9-2021

Representing the Derivative of Trace of Holonomy

Jeffrey Peter Kroll
The Graduate Center, City University of New York

Representing the Derivative of Trace of Holonomy

by

Jeffrey Peter Kroll

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York.

2021

This manuscript has been read and accepted for the Graduate Faculty in Mathematics in satisfaction of the dissertation requirements for the degree of Doctor of Philosophy.

(required signature)

May 18, 2021 Mahmoud Zeinalian

Date Chair of Examining Committee

(required signature)

May 18, 2021 Ara Basmajian

Date Executive Officer

Mahmoud Zeinalian

Scott Wilson

Martin Bendersky

Supervisory Committee

THE CITY UNIVERSITY OF NEW YORK

Abstract

Representing the Derivative of Trace of Holonomy

by

Jeffrey Peter Kroll

Advisor: Mahmoud Zeinalian

Trace of holonomy around a fixed loop γ defines a function on the space of unitary connections on a hermitian vector bundle over a Riemannian manifold. Using the derivative of trace of holonomy, the loop γ, and a flat unitary connection ∇, a functional is defined on the vector space $H^1_\nabla(M;\mathfrak{u}E)$ of twisted degree 1 cohomology classes with coefficients in skew-hermitian endomorphisms. It is shown that this functional is obtained by pairing elements of $H^1_\nabla(M;\mathfrak{u}E)$ with a degree 1 *homology* class built directly from the 1-cycle γ and equipped with a flat section obtained from the variation of holonomy around γ. When the base manifold is closed Kähler, hard Lefschetz duality implies that $H^1_\nabla(M;\mathfrak{u}E)$ is a symplectic vector space and, coupled with Poincaré duality, can be identified with $H_1^\nabla(M;\mathfrak{u}E)$. In this case, the functional is obtained by contracting the symplectic pairing on $H^1_\nabla(M;\mathfrak{u}E)$ with the hard Lefschetz dual of the Poincaré dual of the twisted homology class built using γ.

Acknowledgements

My doctoral advisor **Mahmoud Zeinalian** – it is impossible to express all you've done for and mean to me. Once we started working together it felt like I was learning for the first time. I learned more our first year together than I had from all my previous education. I've been to your home, I've met your family, you've fed me, you've walked with me, you've talked with me about mathematics and about life. You provided me forums to sort through new mathematical concepts and ideas. Your knowledge is overwhelming; your patience, abundant; your kindness, boundless. Thank you for everything.

My masters advisor **Vincent Bouchard** – you brought me to Canada for the first time, to my first mathematics workshop, and we were together the first time I summited a mountain in the snow. Thank you for demonstrating balance of intellect, creativity, activity.

My college thesis advisor **Curtis Greene** – even though I was a physics student at the time, you took me under your guidance as summer research assistant before my senior year. Thank you for your benevolence, for sharing your immense knowledge of combinatorics, for bringing me into your research team, and for motivating me to obtain my degree in mathematics.

My first college mathematics professor **Lynne Butler** – first semester college mathematics opened my eyes to how little I knew. While this was originally intimidating and discouraging, it quickly inspired me to appreciate the precision and challenges of working in mathematics. You were the first person to tell me that I have a mathematical mind. Thank you for believing in and encouraging me from the very beginning.

Thank you to the my thesis committee members **Scott Wilson** and **Martin Bendersky** – both of you taught me and exposed me to new areas of mathematics. **Stephen Wang**, **Charles Doran**, **Linda Keen**, **Abhijit Champanerkar**, **Joseph Maher**, **Dennis Sullivan**, **Brooke Feigon**, **Krzysztof Klosin**, **Thomas Tradler**, **Ilya Kofman**, **Jeremy Kahn**, **Radoslaw Wojciechowski** – thank you for teaching me. Thank you to *everyone* – faculty, students, staff and peers alike – who has contributed to my education both inside and outside of mathematics.

Thank you to my family – you have always been supportive of my endeavors. I should give special acknowledgement to Krollfam – my loving parents Michael and Penny, my awesome siblings Samantha and Caleb, and my amazing wife Lyubava. Thank you to the Greenhouse family for all your help especially when I moved to New York. Thank you to the Fartushenko family for your faith in me. Thank you to all my friends – those from my life in Texas, in Pennsylvania, in Canada, in New York – who are scattered across the globe.

Much love to everyone.

Dedicated to Sylvia

Contents

Chapter 1

Introduction

Fix a smooth complex vector bundle $\pi : E \to M$ equipped with a (hermitian) bundle metric $h : E \otimes \bar{E} \to M \times \mathbb{C}$. Given any connection $\nabla : \Gamma(E) \xrightarrow{\mathbb{C}-\text{linear}} \Omega^1(M, \mathbb{C}) \otimes_{\Omega^0(M,\mathbb{C})} \Gamma(E)$ that is compatible with the metric and any (piece-wise) smooth path $\gamma : [0, 1] \to M$, parallel transport along γ defines a linear isometry $P_\gamma(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)}$ between the fibers. When γ is closed, parallel transport is referred to as holonomy $\mathrm{Hol}_\gamma(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)} = E_{\gamma(0)}$ and defines a unitary automorphism of the fiber over $\gamma(0) = \gamma(1)$.

The set $\mathcal{A}(E, h)$ of all connections that are compatible with the metric is an affine space modeled on $\Omega^1(M; \mathfrak{u}E)$, where $\mathfrak{u}E \subset \mathrm{End}E$ is the sub-bundle of h-skew-hermitian bundle endomorphisms. Thus when the base M is Riemannian, the set of all connections is a metric space. Parallel transport along a fixed loop γ defines the holonomy function $\mathrm{Hol}_\gamma : \mathcal{A}(E, h) \to UE_{\gamma(0)} \subset \mathrm{Aut}(E_{\gamma(0)})$ and $\mathbf{Tr}\mathrm{Hol}_\gamma$ defines a scalar-valued function on the space of all connections.

If the base M is a closed symplectic manifold with symplectic form $\omega \in \Omega^2(M, \mathbb{R})$, the space $\mathcal{A}(E, h)$ of all metric connections has a symplectic structure described as follows. The tangent space to every $\nabla \in \mathcal{A}(E,h)$ is naturally identified with $\Omega^1(M;\mathfrak{u}E)$ and the symplectic form is defined at $\nabla \in \mathcal{A}(E,h)$ by the pairing

$$\Omega^1(M;\mathfrak{u}E) \times \Omega^1(M;\mathfrak{u}E) \ni (\Phi,\Psi) \mapsto \int_M h(\Phi \wedge \Psi) \wedge \omega^{\dim_{\mathbb{R}} M - 1}.$$

This dissertation is motivated by a consideration of the Hamiltonian vector field associated to the trace of holonomy function. When $M = \Sigma$ is a closed oriented (real) surface, Goldman [6] discovered a *symplectic structure on the moduli space of gauge equivalence classes of flat metric connections*. Goldman proceeded to analyze the trace of holonomy functions on the moduli space [5] and realized that the subspace of functions consisting of the trace of holonomy functions is closed under the Poisson bracket. In doing so, Goldman discovered a Lie algebra structure on free homotopy classes of oriented loops within a closed oriented surface.

Karshon [10] provided a purely algebraic proof of Goldman's symplectic result and observed that the moduli space of flat unitary connections should be symplectic provided that the base is a closed Kähler manifold. The Riemannian metric on the base, together with the bundle metric, defines an inner product on the space of bundle-valued differential forms which allows for formal adjoints of linear operators on differential forms and the use of Hodge theory. The linear operator $L \in \mathrm{End}^2\big(\Omega^\bullet(M;E)\big)$ is defined by taking the wedge product with the Kähler form ω. The operator L and its formal adjoint $\Lambda \in \mathrm{End}^{-2}\big(\Omega^\bullet(M;E)\big)$ turn out to be the images of two of three generators of $\mathfrak{sl}(2,\mathbb{C})$ under a representation on $\Omega^\bullet(M;E)$. Next, each flat unitary connection ∇ induces a unique holomorphic structure on the bundle so that the connection becomes the Chern connection for the corresponding holomorphic hermitian vector bundle and, in this context, there is a twisted version of the Kähler identities. The Kähler identities are used to show that the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\Omega^\bullet(M;E)$ restricts to a representation on the space $\mathcal{H}^\bullet_\nabla(M;E) = \ker\nabla \cap \ker\nabla^*$ of ∇-harmonic forms, which by the Hodge theorem is both finite dimensional and isomorphic to $H^\bullet_\nabla(M;E)$. Therefore one has a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^\bullet_\nabla(M;E)$. In particular $H^\bullet_\nabla(M;E)$ satisfies hard Lefschetz duality which says that cup product with the Kähler class $[\omega]$ induces isomorphisms $H^k_\nabla(M;E) \cong H^{\dim_{\mathbb{R}} M-k}(M;E)$ for all $k \leq \dim_{\mathbb{C}} M$.

The symplectic form on the moduli space is given at the point $[\nabla]$ by the pairing

$$H^1_\nabla(M;\mathfrak{u}E)\times H^1_\nabla(M;\mathfrak{u}E) \cong H^1_\nabla(M;\mathfrak{u}E)\times H^{\dim_{\mathbb{R}} M-1}(M;\mathfrak{u}E) \xrightarrow{\cup_h} H^{\dim_{\mathbb{R}} M}(M) \xrightarrow{\int} \mathbb{R}$$

which uses the cup-product along with the bundle metric on $\mathfrak{u}E$ induced from the metric h on E.

The main result of this dissertation will now be described. The context is a fixed smooth hermitian vector bundle $\pi : E \to M$ over a closed Kähler manifold equipped with a flat metric connection ∇. A functional on $H^1_\nabla(M;\mathfrak{u}E)$ is constructed from a given (homotopy class of a) loop $\gamma : [0,1] \to M$ using trace of holonomy. The (directional) derivatives of $\mathbf{Tr}\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to \mathbb{C}$ at a connection $\nabla \in \mathcal{A}(E,h)$ define the function

$$d\mathbf{Tr}\mathrm{Hol}_\gamma(\nabla) : \Omega^1(M;\mathfrak{u}E) \to \mathbb{C}.$$

The functional on cohomology $H^1_\nabla(M;\mathfrak{u}E)$ is obtained by restricting the domain of $d\mathbf{Tr}\mathrm{Hol}_\gamma(\nabla)$

to $Z^1_\nabla(M;\mathfrak{u}E) := \Omega^1(M;\mathfrak{u}E)\cap\ker\nabla$ and observing this descends to cohomology. The resulting functional is computed as

$$H^1_\nabla(M;\mathfrak{u}E) \ni [\Phi] \mapsto -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

where $\dot\gamma(t)\in T_{\gamma(t)}M$ is the velocity vector of γ, and $\gamma^t(s)=\gamma(s+t \mod 1)$ is the t-shifted loop of γ. The functional is referred to as the *(derivative of) trace of holonomy functional* $\mathbb{T}$. It is shown that $\mathbb{T}$ is represented by a homology class

$$[\gamma, F\mathrm{Hol}_\gamma(\nabla)] \in H_1^\nabla(M;\mathfrak{u}E)$$

using the natural pairing between degree 1 homology and cohomology. This is the homology class that is represented by the twisted singular 1-cycle γ equipped with the flat section of $\gamma^*\mathfrak{u}E$ obtained by applying the variation map $F: UE\to\mathfrak{u}E$ to holonomy.

Let $\Upsilon^{-1}(\gamma)\in H^1_\nabla(M;\mathfrak{u}E)$ be hard Lefschetz dual to the Poinaré dual of $[\gamma, F\mathrm{Hol}_\gamma(\nabla)]\in H_1^\nabla(M;\mathfrak{u}E)$. The main result is that for all $\Phi\in H^1_\nabla(M;\mathfrak{u}E)$

$$\int_M h(\Phi\wedge\Upsilon^{-1}(\gamma))\wedge[\omega]^{n-1} = -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

which equates contraction of the symplectic pairing on $H^1_\nabla(M;\mathfrak{u}E)$ with the trace of holonomy functional $\mathbb{T}$.

Chapter 2

Derivative of parallel transport

2.1 Vector bundles and connections

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. A smooth $\mathbb{F}$-*vector bundle of rank* r is a smooth surjection $\pi : E \to M$ of manifolds such that every point $x \in M$ is contained in an open neighborhood $U \ni x$ equipped with a diffeomorphism $\phi_U : \pi^{-1}(U) \cong U \times \mathbb{F}^r$ which satisfies that, for all $y \in U$, the restriction $\phi_U|_y : \pi^{-1}(y) \to \{y\} \times \mathbb{F}^k$ is a linear isomorphism. The space E is the *total space*, the space M is the *base space* and π is the *projection* of the vector bundle. The *fiber over* $x \in M$ is the preimage $E_x := \pi^{-1}(x) \subset E$.

Remark. One may refer to *the vector bundle* E (without explicit reference to the projection π and the base M) whenever $\pi : E \to M$ is understood.

Definition 2.1.2. A smooth *section* of a smooth vector bundle $\pi : E \to M$ is a smooth right inverse of π, i.e. a smooth morphism $s : M \to E$ such that $\pi \circ s = \mathbf{1}_M$. The collection of smooth sections of the vector bundle is written $\Gamma(E) \subset \mathrm{Hom}(M, E)$.

Definition 2.1.3. Let M be a manifold and V a $\mathbb{F}$-vector space. The cochain complex of *V-valued differential forms on M*, denoted $\Big(\Omega^\bullet(M, V), d\Big)$, is the graded module $\Omega^\bullet(M, V) := \Omega^\bullet(M) \otimes_{\mathbb{F}} V$ equipped with de Rham's exterior differential $d(\phi \otimes v) := d\phi \otimes v$.

Remark. Note that $\Omega^0(M, \mathbb{F})$ consists of smooth $\mathbb{F}$-valued functions on M and $\Gamma(E)$ is a module over $\Omega^0(M, \mathbb{F})$.

Example 2.1.4. Let V be a $\mathbb{F}$-vector space. Sections $\Gamma(M \times V)$ of the trivial vector bundle $M \times V \xrightarrow{(m,v)\mapsto m} M$ are precisely V-valued functions on M,

$$\Gamma(M \times V) \ni \big(M \xrightarrow{m\mapsto(m,f(m))} M \times V\big) \leftrightarrow f \in \mathrm{Hom}(M, V).$$

Definition 2.1.5. The *differential forms on M with values in a $\mathbb{F}$-vector bundle $\pi : E \to M$* are elements of the graded module

$$\Omega^\bullet(M; E) := \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Gamma(E). \tag{2.1}$$

Example 2.1.6. Consider the trivial vector bundle $\pi : M \times V \to M$ where V is a $\mathbb{F}$-vector space. Then $\Omega^\bullet(M; M\times V) = \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Gamma(M\times V) = \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Omega^0(M, V) = \Omega^\bullet(M, V)$.

Remark. One may write $\Omega^\bullet(M)$ when $\mathbb{F}$ is understood.

Definition 2.1.7. A *connection on a $\mathbb{F}$-vector bundle $\pi : E \to M$* is a $\mathbb{F}$-linear map

$$\nabla : \Omega^0(M; E) := \Gamma(E) \to \Omega^1(M) \otimes_{\Omega^0(M)} \Gamma(E) =: \Omega^1(M; E) \tag{2.2}$$

such that $\nabla(fs) = df \otimes s + f\nabla(s)$ for all $(f, s) \in \Omega^0(M) \times \Gamma(E)$. A *connection* ∇ *extends to a unique degree* $+1$ *linear operator* ∇ *on* $\Omega^\bullet(M; E)$ by imposing the Leibnitz condition

$$\nabla(\phi \otimes s) = d\phi \otimes s + (-1)^{|\phi|}\phi \wedge \nabla(s). \tag{2.3}$$

Write $\mathcal{A}(E)$ for *the set of all connections on a vector bundle* $\pi : E \to M$. If $\nabla \in \mathcal{A}(E)$ we refer to the pair (E, ∇) as a *vector bundle with connection.*

Proposition 2.1.8. The set $\mathcal{A}(E)$ of all connections on a given vector bundle $\pi : E \to M$ is an affine space modeled on $\Omega^1(M; \mathrm{End}E)$.

Proof. Given $\nabla_1, \nabla_2 \in \mathcal{A}(E)$, the a priori $\mathbb{F}$-linear map $\nabla_1 - \nabla_2$ is readily seen to be $\Omega^0(M)$-linear: for all $(f, s) \in \Omega^0(M) \times \Gamma(E)$,

$$(\nabla_1 - \nabla_2)(fs) = \nabla_1(fs) - \nabla_2(fs) = df \otimes s + f\nabla_1(s) - df \otimes s - f\nabla_2(s) = f(\nabla_1 - \nabla_2)s.$$

Therefore $(\nabla_1 - \nabla_2) \in \mathrm{Hom}_{\Omega^0(M)-Mod}\big(\Gamma(E), \Omega^1(M; E)\big) \cong \Omega^1(M; \mathrm{End}E)$. Conversely, given any $\phi \in \Omega^1(M; \mathrm{End}E)$ we have $(\nabla_1 + \phi)(fs) = df \otimes s + f\nabla_1 s + f\phi s = df \otimes s + f(\nabla_1 + \phi)s$ and thus $\nabla_1 + \phi \in \mathcal{A}(E)$. $\square$

Remark. The above proof used the canonical (linear algebra) identification

$$\mathrm{Hom}_{\Omega^0(M)-Mod}(\Gamma E, \Omega^k(M; E)) \cong \Omega^k(M; \mathrm{End}E). \tag{2.4}$$

Corollary 2.1.9. For $\nabla \in \mathcal{A}(E)$, there is a natural identification

$$T_\nabla \mathcal{A}(E) \equiv \Omega^1(M; \mathrm{End}E) \tag{2.5}$$

of the tangent vector space to the affine space $\mathcal{A}(E)$ at the point ∇. $\square$

2.2 Flat sections, parallel translation, and holonomy

Definition 2.2.1. Let $\pi : E \to M$ be a vector bundle with connection $\nabla \in \mathcal{A}(E)$. A section $s \in \Gamma(E)$ is called *parallel* (or *flat*) *with respect to* ∇ if and only if $\nabla(s) = 0 \in \Omega^1(M; E)$.

Definition 2.2.2. Let $\pi : E \to M$ be a vector bundle with connection ∇ and let $\gamma : [0,1] \to M$ be a path in the base. A lift $\widetilde{\gamma}$ of γ is called *parallel* (or *flat*) *with respect to* ∇ if and only if $\widetilde{\gamma}$ is a parallel section of the pullback bundle $\gamma^*E \to [0,1]$ with respect to the pulled-back connection $\gamma^*\nabla \in \mathcal{A}(\gamma^*E)$.

Example 2.2.3. A parallel section is – tautologically – a parallel lift of its projection. More precisely, a path $\alpha : [0,1] \to E$ in the total space of a vector bundle $\pi : E \to M$ is ∇-flat if and only if α is a $(\pi\alpha)^*\nabla$-flat section of $(\pi\alpha)^*E$.

Remark. A lift $\widetilde{\gamma}$ of $\gamma : [0,1] \to M$ is the same thing as a section of $\gamma^*E \to [0,1]$. It is convenient to work with the bundle $\gamma^*E \to [0,1]$ because it is trivializable. Then $\widetilde{\gamma}$ is parallel if and only if $(\gamma^*\nabla)(\widetilde{\gamma}) = 0$ which holds if and only if $\nabla(\widetilde{\gamma})(\dot{\gamma}(t)) = 0$ for every $t \in [0,1]$ where $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is the velocity vector of γ at $\gamma(t) \in M$.

Proposition 2.2.4. Let $\gamma : [0,1] \to M$ be given. For each $v \in E_{\gamma(0)}$, there is a unique ∇-parallel lift $\widetilde{\gamma}_v$ of γ such that $\widetilde{\gamma}_v(\gamma(0)) = v$. We say that $\widetilde{\gamma}_v$ defines *parallel translation of* v *along* γ *with respect to the connection* ∇. Furthermore, the path γ induces a linear isomorphism $\widetilde{\gamma} : E_{\gamma(0)} \to E_{\gamma(1)}$ defined by $\widetilde{\gamma}(v) = \widetilde{\gamma}_v(1)$.

Proof. The lifted path $\widetilde{\gamma}_v$ would have velocity field $\frac{d\widetilde{\gamma}_v(t)}{dt}$ satisfying $\nabla(\widetilde{\gamma}_v)(\frac{d\widetilde{\gamma}_v(t)}{dt}) = 0$ and

$\widetilde{\gamma}_v(0) = v$, which is a first order ordinary differential equation with initial value. Thus $\widetilde{\gamma}_v$ exists as the unique solution to the initial value problem and we can define the linear map $\widetilde{\gamma} : v \mapsto \widetilde{\gamma}_v(1)$. The inverse of $\widetilde{\gamma}(t)$ is $\widetilde{\gamma}(1-t)$. See 2.22 below for an explicit construction of the solution for the parallel lift. □

Definition 2.2.5. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle and $\gamma : [0,1] \to M$ a path. The *parallel translation function*

$$P_\gamma : \mathcal{A}(E) \to \mathrm{Hom}_{\mathbb{F}-Mod}(E_{\gamma(0)}, E_{\gamma(1)}) \tag{2.6}$$

sends a connection ∇ to the parallel translation isomorphism $P_\gamma(\nabla) = \widetilde{\gamma} : v \mapsto \widetilde{\gamma}_v(1)$.

Definition 2.2.6. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle. If $\gamma : [0,1] \to M$ is a closed path (i.e. $\gamma(0) = \gamma(1)$) then parallel translation along γ is called *holonomy along* γ and is denoted by

$$\mathrm{Hol}_\gamma := P_\gamma : \mathcal{A}(E) \to \mathrm{Aut}(E_{\gamma(0)}). \tag{2.7}$$

2.3 Flat connections and parallel translation

Proposition 2.3.1. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle with connection $\nabla \in \mathcal{A}(E)$. The operator

$$\nabla \circ \nabla : \Omega^0(M; E) \to \Omega^2(M; E)$$

is $\Omega^0(M)$-linear and therefore corresponds to a unique $\mathrm{End}E$-valued 2-form $R(\nabla) \in \Omega^2(M; \mathrm{End}E)$.

Proof. For all $(f, s) \in \Omega^0(M) \times \Gamma(E)$ one sees that

$$(\nabla \circ \nabla)(fs) = \nabla(df \otimes s + f\nabla s) = d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\nabla(\nabla s) = f(\nabla \circ \nabla)s.$$

$\square$

Definition 2.3.2. Let $\pi : E \to M$ be a vector bundle with connection $\nabla \in \mathcal{A}(E)$. The *curvature of* ∇ is the element $R(\nabla) \in \Omega^2(M; \mathrm{End}E)$ corresponding to $\nabla \circ \nabla$.

Definition 2.3.3. A connection ∇ is *flat* if and only if $R(\nabla) = 0$. The *subset of all flat connections on* E will be denoted by $\mathcal{F}(E) \subset \mathcal{A}(E)$. A *flat vector bundle* is a pair (E, ∇) where $E \to M$ is a vector bundle and $\nabla \in \mathcal{F}(E)$ is a flat connection on E.

Definition 2.3.4. The *de Rham cohomology of* M *with values in a flat vector bundle* $(E, \nabla) \to M$ is the cohomology $H^\bullet_\nabla(M; E)$ of the differential graded module $\left(\Omega^\bullet(M; E), \nabla\right)$.

Proposition 2.3.5. Let $\pi : (E, \nabla) \to M$ be a flat vector bundle. If $\gamma \sim \eta : [0,1] \to M$ are homotopy equivalent paths with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$, then $P_\gamma(\nabla) = P_\eta(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)}$.

Proof. $\widetilde{\gamma}_v$ denotes the unique parallel lift of γ with $\widetilde{\gamma}_v(0) = v$ so that $P_\gamma(\nabla) : v \mapsto \widetilde{\gamma}_v(1)$.

Write $x = \gamma(0) = \eta(0)$ and $y = \gamma(1) = \eta(1)$. Let $H_s(t) = H(s,t) : [0,1] \times [0,1] \to M$ be a homotopy with $H(0,t) = \gamma(t)$ and $H(1,t) = \eta(t)$ that fixes the endpoints, i.e. $H(s,0) = x$ and $H(s,1) = y$ for all $s \in [0,1]$. Pick $v \in \pi^{-1}(x)$ and define $\sigma \in \Gamma(H_s(t)^*E)$ by requiring that (i) $\sigma(s,0) = v$ is a constant function of s and (ii) for fixed s, $\sigma(s,t) = \widetilde{(H_s)}_v(t)$. Condition (ii) says that $\sigma(s,t)$, as a function of t, assigns to t the element of $\pi^{-1}(H(s,t))$ obtained by parallel translating $H(s,0) = v$ along the path $[0,1] \ni t \mapsto (s,t) \in [0,1] \times [0,1]$.

Write $D = H^*\nabla$ and let $X_s, X_t \in \Gamma(T([0,1] \times [0,1]))$ be the standard unit vector fields. By condition (ii) $(D\sigma)(X_t) = 0$ since σ is parallel in the t-direction. Since $\sigma(s,0) = v$ for all s, we have $D\sigma(X_s)|_{t=0} = 0$. By assumption ∇ is flat so that $D^2 = 0$ and therefore

$$0 = (D^2\sigma)(X_s, X_t) = X_s(D\sigma)(X_t) - X_t(D\sigma)(X_s) - (D\sigma)[X_s, X_t] = -X_t(D\sigma)(X_s)$$

which shows that $D\sigma(X_s)$ is constant in the t-direction, and thus $D\sigma(X_s)|_{t=1} = D\sigma(X_s)|_{t=0} = 0$. Hence $\sigma(s,1)$ is a constant path of s and $P_\gamma(\nabla)(v) = \widetilde{\gamma}_v(1) = \sigma(0,1) = \sigma(1,1) = \widetilde{\eta}_v(1) = P_\eta(\nabla)(v)$. $\square$

Corollary 2.3.6. If $(E, \nabla) \to M$ is a flat vector bundle, then for all $[\gamma] \in \pi_1(M, x)$ the holonomy

$$\mathrm{Hol}_{[\gamma]}(\nabla) \in \mathrm{Aut}(E_x) \tag{2.8}$$

is well-defined by $\mathrm{Hol}_{[\gamma]}(\nabla) := \mathrm{Hol}_\gamma(\nabla)$ for any representative closed path γ. In particular, there is the *holonomy representation of the fundamental group of M*

$$\mathrm{Hol}_{(-)}(\nabla) : \pi_1(M, x) \to \mathrm{Aut}(E_x).$$

$\square$

Remark. The significance of the holonomy representation is discussed in 3.2.

Corollary 2.3.7. One may restrict the domain of $\mathrm{Hol} : \mathcal{A}(E) \times \mathrm{Map}\big(([0,1], 0, 1), (M, x, x)\big) \to \mathrm{Aut}(E_x)$ to flat connections $\mathrm{Hol} : \mathcal{F}(E) \times \mathrm{Map}\big(([0,1], 0, 1), (M, x, x)\big) \to \mathrm{Aut}(E_x)$ and then quotient by homotopy classes of loops to obtain the holonomy function

$$\mathrm{Hol} : \mathcal{F}(E) \times \pi_1(M, x) \to \mathrm{Aut}(E_x). \tag{2.9}$$

□

2.4 Metrics on vector bundles

Definition 2.4.1. An *inner product* on a $\mathbb{F}$-vector space V is sesquilinear pairing – i.e. a linear map $V \otimes \bar{V} \to \mathbb{F}$ – that is conjugate-symmetric and positive-definite. An inner product on a complex vector space is called a *hermitian inner product*.

Remark. $\bar{V}$ is the $\mathbb{F}$-module V equipped with the $\mathbb{F}$ action $a \cdot v = \bar{a}v$. In particular, if V is real then $\bar{V} = V$. Thus the convention adopted here is that a hermitian inner product is complex linear in the first argument and anti-linear (i.e. conjugate linear) in the second argument.

Definition 2.4.2. A *bundle metric on a $\mathbb{F}$-vector bundle* $\pi : E \to M$ is a smooth section $h \in \Gamma\big(\mathrm{Hom}(E \otimes \bar{E}, M \times \mathbb{F})\big)$ such that, for all $x \in M$, h_x is an inner product on the fiber E_x.

Example 2.4.3. A *Riemannian manifold* is a (smooth) manifold whose tangent bundle has a bundle metric. Similarly, a *hermitian manifold* is (complex) manifold whose holomorphic tangent bundle has a hermitian metric.

Proposition 2.4.4. Let h be a bundle metric on $\pi : E \to M$. Then $\wedge_h := \wedge \otimes h$ defines a sesquilinear pairing

$$\wedge_h : \Omega^\bullet(M; E) \times \Omega^\bullet(M; E) \ni \big(\phi \otimes s, \psi \otimes t\big) \mapsto \phi \wedge \bar{\psi} h(s, t) \in \Omega^\bullet(M, \mathbb{F}). \tag{2.10}$$

Remark. In the above proposition sesquilinear means *sesquilinear over* $\Omega^0(M, \mathbb{F})$:

$$(f\Phi) \wedge_h (g\Psi) = f\bar{g}(\Phi \wedge_h \Psi)$$

for all $f, g \in \Omega^0(M, \mathbb{F})$ and $\Phi, \Psi \in \Omega^\bullet(M; E)$.

Proof. $\wedge_h$ is a well-defined pairing on $\Omega^\bullet(M; E) = \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Gamma(E)$ since for all $f, g \in \Omega^0(M)$

$$(\phi \otimes fs) \wedge_h (\psi \otimes gt) = \phi \wedge \bar{\psi} h(fs, gt) = \phi \wedge \bar{\psi} f \bar{g} h(s,t) = \phi f \wedge \bar{\psi} \bar{g} h(s,t) = (\phi f \otimes s) \wedge_h (\psi g \otimes t).$$

This computation, coupled with the fact that both $\wedge$ and h are bilinear over $\Omega^0(M, \mathbb{R})$, also shows that $\wedge_h$ is sesquilinear over $\Omega^0(M, \mathbb{F})$. $\square$

Definition 2.4.5. Given a vector bundle $\pi : E \to M$ with bundle metric h, a connection $\nabla \in \mathcal{A}(E)$ is said to be *compatible with the bundle metric* if and only if for all $s, t \in \Gamma(E)$, the function $h(s,t) : M \ni x \mapsto h(s(x), t(x)) \in \mathbb{F}$ satisfies

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t) \in \Omega^1(M, \mathbb{F}). \tag{2.11}$$

Definition 2.4.6. A *hermitian vector bundle* (E, h) is a complex vector bundle $\pi : E \to M$ equipped with a hermitian bundle metric h. A connection ∇ that is compatible with a hermitian bundle (E, h) is called a *unitary connection* and the triple (E, h, ∇) a *unitary bundle.* A connection that is both flat and unitary is called a *flat unitary connection* and a bundle with such a connection is a *flat unitary bundle.* Let $\mathcal{A}(E, h) \supset \mathcal{F}(E, h)$ denote the *set of all unitary connections* and the *subset of flat unitary connections.*

Definition 2.4.7. Let $(\pi : E \to M, h)$ be a hermitian vector bundle. The $\mathbb{R}$-vector bundle

$$\mathfrak{u}E \to M \tag{2.12}$$

is defined as the sub-bundle of $\mathrm{End}E \to M$ consisting of h-skew-hermitian endomorphisms.

Proposition 2.4.8. The set $\mathcal{A}(E,h)$ of all unitary connections is an affine space modeled on $\Omega^1(M;\mathfrak{u}E)$.

Proof. For $\nabla_1, \nabla_2 \in \mathcal{A}(E,h) \subset \mathcal{A}(E)$, we know that $\nabla_1 - \nabla_2$ corresponds to a unique element of $\Omega^1(M;\mathrm{End}E)$. Furthermore, since both connections are unitary we have

$$h(\nabla_1 s,t) + h(s,\nabla_1 t) = dh(s,t) = h(\nabla_2 s,t) + h(s,\nabla_2 t)$$

$$\Rightarrow h((\nabla_1 - \nabla_2)s,t) + h(s,(\nabla_1 - \nabla_2)t) = 0$$

and therefore $\nabla_1 - \nabla_2$ is h-skew-hermitian. □

Corollary 2.4.9. The tangent space $T_\nabla \mathcal{A}(E,h)$ to a unitary connection ∇ is canonically identified with $\Omega^1(M;\mathfrak{u}E)$. □

Proposition 2.4.10. If $(\pi : E \to M, h)$ is a hermitian bundle over a closed Riemannian manifold, then $\mathcal{A}(E,h)$ is a Banach manifold.

Proof. $\mathcal{A}(E,h)$ is an affine space modeled on $\Omega^1(M;\mathfrak{u}E)$. The Riemann metric on M combined with the bundle metric h induce an inner product on $\wedge^k T^\vee M \otimes \mathfrak{u}E$ and on $\Omega^k(M;\mathfrak{u}E)$. See 4.5. One has the L^p space of functions $L^p(M, \wedge^k T^\vee M \otimes \mathfrak{u}E)$ whose elements are smooth functions $f : M \to \wedge^k T^\vee M \otimes \mathfrak{u}E$ such that

$$||f||_p := \Big(\int_M ||f||^p_{\mathfrak{u}E} dM \Big)^{1/p} < \infty.$$

Then the space $L^p\Omega^k(M;\mathfrak{u}E) := \Omega^k(M;\mathfrak{u}E) \cap L^p(M, \wedge^k T^\vee_{\mathbb{C}} M \otimes \mathfrak{u}E)$ is a Banach manifold and a Hilbert manifold when $p = 2$.

To get a better approximation of $\Omega^1(M;\mathfrak{u}E)$, one may use the Sobolev spaces $W^{j,2}(M,\wedge^k T^\vee M\otimes \mathfrak{u}E)\subset L^2(M,\wedge^k T^\vee M\otimes \mathfrak{u}E)$ consisting of functions f such that f and its first j-derivatives are in L^2. $W^{j,2}(M,\wedge^k T^\vee M\otimes \mathfrak{u}E)\cap\Omega^k(M;\mathfrak{u}E)$ is a Hilbert space and $\Omega^k(M;\mathfrak{u}E)$ is dense in the space. Formally one may treat $\mathcal{A}(E,h)$ as a Sobolev space of sections in order to get a Hilbert manifold. However, since $\Omega^1(M;\mathfrak{u}E)$ is dense in the Sobolev space, it is safe to work within $\Omega^1(M;\mathfrak{u}E)$ equipped with its inner product. Issues of convergence are dealt with in the larger Sobolev space but are not relevant to this dissertation. See [3] for basics on Sobolev functors in a similar context. $\square$

2.5 Constructions with metrics and connections

Example 2.5.1. Let $\pi : E\to M$ be a $\mathbb{F}$-vector bundle. Every smooth map $\phi : N\to M$ induces a map of connections $\phi^* : \mathcal{A}(E)\to\mathcal{A}(\phi^*E)$ via pullback to the pullback bundle $\phi^*E\to N$. By definition $\phi^*(\mathcal{F}(E))\subset\mathcal{F}(\phi^*E)$. As a special case, given a path $\gamma : [0,1]\to M$, every connection $\nabla\in\mathcal{A}(E)$ gives a connection $\gamma^*\nabla$ on the vector bundle $\gamma^*E\to[0,1]$.

Example 2.5.2. Let ∇ be a connection on $\pi : E\to M$. There is an induced connection $\widetilde{\nabla}\in\mathcal{A}(\mathrm{End}E)$ on the vector bundle $\mathrm{End}E\to M$. First recall that every $\Phi\in\Omega^k(M;\mathrm{End}E)$ corresponds to a unique $\Omega^0(M)$-linear map $\Phi : \Gamma(E)\to\Omega^k(M;E)$. See 2.4. Secondly, observe that $[\nabla, \Phi] : \Gamma(E) \to \Omega^{k+1}(M; E)$ is also $\Omega^0(M)$-linear:

$$\begin{aligned}
[\nabla, \Phi](fs) &= \big(\nabla\Phi - (-1)^k \Phi\nabla\big) fs = \nabla\Phi(fs) - (-1)^k \Phi\nabla(fs) \\
&= \nabla(f\Phi s) - (-1)^k \Phi(df \otimes s + f\nabla s) \\
&= df \wedge \Phi s + f\nabla(\Phi s) - (-1)^k \Phi(df \otimes s) - (-1)^k \Phi(f\nabla s) \\
&= df \wedge \Phi s + f\nabla(\Phi s) - df \wedge \Phi s - (-1)^k f\Phi(\nabla s) \\
&= f\big(\nabla\Phi - (-1)^k \Phi\nabla\big)s.
\end{aligned}$$

Therefore $[\nabla, \Phi]$ corresponds to a unique element of $\Omega^{k+1}(M; \mathrm{End}E)$. Implicitly using the identifications $\Omega^\bullet(M; \mathrm{End}E) \cong \mathrm{Hom}_{\Omega^0(M)-Mod}(\Gamma(E), \Omega^\bullet(M; E))$, one defines

$$\widetilde{\nabla} := [\nabla, -] : \Omega^\bullet(M; \mathrm{End}E) \to \Omega^\bullet(M; \mathrm{End}E) \tag{2.13}$$

and verifies that $\widetilde{\nabla}$ is indeed a connection on $\mathrm{End}E$. For all $(f, \sigma) \in \Omega^0(M) \times \Gamma(\mathrm{End}E)$

$$\begin{aligned}
\widetilde{\nabla}(f\sigma) &= [\nabla, f\sigma] = \nabla(f\sigma) - f\sigma(\nabla) = df \otimes \sigma + f\nabla\sigma - f\sigma\nabla \\
&= df \otimes \sigma + f(\nabla\sigma - \sigma\nabla) = df \otimes \sigma + f[\nabla, \sigma] = df \otimes \sigma + f\widetilde{\nabla}\sigma
\end{aligned}$$

so $\widetilde{\nabla} \in \mathcal{A}(\mathrm{End}E)$. Furthermore for all $\sigma \in \Gamma(\mathrm{End}E)$

$$\widetilde{\nabla}(\widetilde{\nabla}\sigma) = \widetilde{\nabla}(\nabla\sigma - \sigma\nabla) = \nabla(\nabla\sigma - \sigma\nabla) + (\nabla\sigma - \sigma\nabla)\nabla = \nabla^2\sigma - \sigma\nabla^2$$

hence $\widetilde{\nabla}$ is flat whenever ∇ is flat.

Example 2.5.3. A connection ∇ on E induces a connection $\nabla^\vee$ on the dual bundle $E^\vee := \mathrm{Hom}_{VB}(E, \mathbb{F})$ by requiring that for all $(\phi, s) \in \Gamma(E^\vee) \times \Gamma(E)$, the *function* $\phi(s) : M \ni x \mapsto$

$\phi(x)(s(x)) \in \mathbb{F}$ satisfies

$$d(\phi(s)) = (\nabla^\vee \phi)s + \phi(\nabla s). \tag{2.14}$$

Indeed $\phi \mapsto f\phi$ gives

$$\nabla^\vee f\phi(s) + f\phi(\nabla s) = d(f\phi(s)) = df \wedge \phi(s) + f\nabla^\vee \phi(s) + f\phi(\nabla s)$$

$$\Rightarrow \nabla^\vee f\phi = df \wedge \phi + f\nabla^\vee \phi \Rightarrow \nabla^\vee \in \mathcal{A}(E^\vee).$$

Observe that $\nabla^\vee$ is flat if and only if ∇ is flat:

$$\begin{aligned} 0 = d^2(\phi(s)) &= d\big((\nabla^\vee \phi)s + \phi(\nabla s)\big) \\ &= (\nabla^\vee\nabla^\vee\phi)s - (\nabla^\vee\phi)\nabla s + (\nabla^\vee\phi)\nabla s - \phi\nabla\nabla s \\ &= (\nabla^\vee\nabla^\vee\phi)s - \phi(\nabla\nabla s). \end{aligned}$$

Example 2.5.4. If ∇ is a connection on $E \to M$, and if $E_1 \to M$ is another vector bundle with connection ∇_1 then $\nabla \otimes 1 + 1 \otimes \nabla_1 \in \mathcal{A}(E \otimes E_1)$. Under the identification $E^\vee \otimes E \cong \mathrm{End}E$, the connection $\nabla^\vee \otimes 1 + 1 \otimes \nabla$ corresponds to $\widetilde{\nabla}$. Explicitly, the identification is given by $\phi \otimes s : v \mapsto \phi(v)s$ and thus

$$\begin{aligned} (\nabla^\vee \otimes 1 + 1 \otimes \nabla)(\phi \otimes s)v &= (\nabla^\vee\phi \otimes s + \phi \otimes \nabla s)v = (\nabla^\vee\phi)v \otimes s + \phi v \otimes \nabla s \\ &= d(\phi(v)) \otimes s - \phi(\nabla v) \otimes s + \phi v \otimes \nabla s \\ &= \nabla(\phi(v)s) - \phi(\nabla v) \otimes s = (\nabla \circ \phi \otimes s)v - (\phi \otimes s \circ \nabla)v \\ &= [\nabla, \phi \otimes s](v) = \widetilde{\nabla}(\phi \otimes s)(v) \Rightarrow \nabla^\vee \otimes 1 + 1 \otimes \nabla = \widetilde{\nabla}. \end{aligned}$$

Example 2.5.5. If h is a metric on E then there is an induced metric $h^\vee$ on the dual bundle $E^\vee$. Simply use the metric h to induce the canonical (anti-linear) isomorphism $\flat : E \to E^\vee$ defined by $v^\flat(w) := h(w, v)$ whose inverse is denoted $\sharp : E^\vee \to E$. Define the metric $h^\vee$ on $E^\vee$ by $h^\vee(f, g) = h(g^\sharp, f^\sharp)$.

$\nabla^\vee$ is compatible with $h^\vee$ whenever ∇ is compatible with h. To see this consider the natural extension $\flat : \Omega^1(M; E) \leftrightarrows \Omega^1(M; E^\sharp) : \sharp$ given by $\flat(\phi \otimes s) = \bar{\phi} \otimes s^\flat$, and similarly $\sharp(\psi \otimes f) = \bar{\psi} \otimes f^\sharp$. For all $(\phi \otimes f, g) \in \Omega^1(M; E^\vee) \times \Omega^0(M; E^\vee)$

$$h^\vee(\phi \otimes f, g) = \phi h^\vee(f, g) = \phi h(g^\sharp, f^\sharp) = h(g^\sharp, \bar{\phi} \otimes f^\sharp) = h\big(g^\sharp, \sharp(\phi \otimes f)\big)$$

$$h^\vee(g, \phi \otimes f) = \bar{\phi} h^\vee(g, f) = \bar{\phi} h(f^\sharp, g^\sharp) = h(\bar{\phi} \otimes f^\sharp, g^\sharp) = h\big(\sharp(\phi \otimes f), g\big).$$

Using that h is compatible with ∇ and that $h(s, t) = t^\flat(s)$ for all $s, t \in \Gamma(E)$,

$$dh(s, t) = h(\nabla s, t) + h(s, \nabla t) = t^\flat(\nabla s) + h(s, \nabla t)$$

$$= d(t^\flat(s)) = (\nabla^\vee t^\flat)s + t^\flat(\nabla s)$$

$$\Leftrightarrow \nabla^\vee t^\flat = (\nabla t)^\flat$$

$$\Leftrightarrow \nabla^\vee(f) = (\nabla(f^\sharp))^\flat \text{ for all } f \in \Gamma(E^\vee)$$

which gives another way to characterize the dual connection whenever there is a compatible bundle metric. At this point the compatibility between $\nabla^\vee$ and $h^\vee$ is immediate:

$$dh^\vee(f, g) = dh(g^\sharp, f^\sharp) = h(\nabla g^\sharp, f^\sharp) + h(g^\sharp, \nabla f^\sharp)$$

$$= h^\vee(f, \nabla^\vee g) + h^\vee(\nabla^\vee f, g).$$

Example 2.5.6. End$E$ inherits a metric $\widetilde{h}$ from a metric h on E: give $E^\vee \otimes E \cong \mathrm{End}E$ the metric $\widetilde{h}(\phi \otimes s, \psi \otimes t) := h^\vee(\phi, \psi)h(s,t)$. Again, if ∇ is compatible with h then $\widetilde{\nabla}$ is compatible with $\widetilde{h}$:

$$\begin{aligned}
d\widetilde{h}(\phi \otimes s, \psi \otimes t) &= d\big(h^\vee(\phi,\psi)h(s,t)\big) = dh^\vee(\phi,\psi)h(s,t) + h^\vee(\phi,\psi)dh(s,t) \\
&= h^\vee(\nabla^\vee\phi,\psi)h(s,t) + h^\vee(\phi,\nabla^\vee\psi)h(s,t) + h^\vee(\phi,\psi)h(\nabla s,t) + h^\vee(\phi,\psi)h(s,\nabla t) \\
&= \widetilde{h}(\nabla^\vee\phi \otimes s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \nabla^\vee\psi \otimes t) + \widetilde{h}(\phi \otimes \nabla s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \psi \otimes \nabla t) \\
&= \widetilde{h}(\nabla^\vee\phi \otimes s, \psi \otimes t) + \widetilde{h}(\phi \otimes \nabla s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \nabla^\vee\psi \otimes t) + \widetilde{h}(\phi \otimes s, \psi \otimes \nabla t) \\
&= \widetilde{h}(\widetilde{\nabla}(\phi \otimes s), \psi \otimes t) + \widetilde{h}(\phi \otimes s, \widetilde{\nabla}(\psi \otimes t)).
\end{aligned}$$

Example 2.5.7. If (E,h) is a hermitian bundle, the vector bundle $\mathfrak{u}E \subset \mathrm{End}E$ is the sub-bundle consisting of h-skew-hermitian endomorphisms. Note that $\mathfrak{u}E$ is a *real* vector bundle and that the connection $\widetilde{\nabla} \in \mathcal{A}(\mathrm{End}E)$ restricts to a connection on $\mathfrak{u}E$ which will also be written $\widetilde{\nabla}$. Similarly, the (hermitian) metric $\widetilde{h}$ on End$E$ restricts to give a (real) metric on $\mathfrak{u}E$ which is also written $\widetilde{h}$.

The metric $\widetilde{h}$ on End$E$ and on $\mathfrak{u}E$ is the familiar metric $\widetilde{h}(A,B) = \mathbf{Tr}(B^*A)$ where B^* denotes the adjoint of B. Indeed, under the identification $\mathrm{End}E \cong E^\vee \otimes E$, conjugate transpose is $(\phi \otimes v)^* = v^\flat \otimes \phi^\sharp$, $\mathbf{Tr} = \mathrm{ev} : E^\vee \otimes E \to \mathbb{C}$ is $\mathbf{Tr}(\phi \otimes v) = \phi(v)$ and $(\psi \otimes w) \circ (\phi \otimes v) = \phi \otimes w\psi(v) \in E^\vee \otimes E \cong \mathrm{End}E$ so that

$$\begin{aligned}
\mathbf{Tr}\Big((\psi \otimes w)^* \circ (\phi \otimes v)\Big) &= \mathbf{Tr}\Big(w^\flat \otimes \psi^\sharp) \circ (\phi \otimes v)\Big) = \mathbf{Tr}\Big(\phi \otimes \psi^\sharp w^\flat(v)\Big) = \phi\big(\psi^\sharp w^\flat(v)\big) \\
&= \phi(\psi^\sharp)w^\flat(v) = h(\psi^\sharp, \phi^\sharp)h(v,w) = h^\vee(\phi,\psi)h(v,w) = \widetilde{h}(\phi \otimes v, \psi \otimes w).
\end{aligned}$$

Remark. We will use the notation **Tr** only when applied to linear operators. The notation ev will be used more generally to denote the pairing of a dual object with an object.

Remark. Restating the definition of a unitary connection, a connection ∇ on a hermitian bundle $(E,h) \to M$ is h-unitary if and only if

$$h : \Big(\Omega^\bullet(M;E) \otimes \Omega^\bullet(M;E), \nabla \otimes 1 + 1 \otimes \nabla\Big) \to \Big(\Omega^\bullet(M), d\Big)$$

is a chain map.

Proposition 2.5.8. If $\nabla \in \mathcal{F}(E,h)$ is a flat unitary connection, the tangent space $T_\nabla \mathcal{F}(E,h)$ is naturally identified with $Z^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) := \ker \widetilde{\nabla} \cap \Omega^1(M;\mathfrak{u}E)$.

Proof. $T_\nabla \mathcal{F}(E,h) \subset T_\nabla \mathcal{A}(E,h) = \Omega^1(M;\mathfrak{u}E)$. Let $\Phi \in \Omega^1(M;\mathfrak{u}E) = T_\nabla \mathcal{A}(E,h)$. Then $\Phi \in T_\nabla \mathcal{F}(E,h)$ if and only if $(\nabla + \Phi) \circ (\nabla + \Phi) \equiv 0 \mod \Phi^2$. This holds if and only if $0 = \nabla\Phi + \Phi\nabla = [\nabla, \Phi] = \widetilde{\nabla}(\Phi)$. □

2.6 Connections locally and explicit formula for parallel translation

Definition 2.6.1. Let $\pi : E \to M$ be a vector bundle and $U \subset M$ an open set. A *frame* on $E|_U \to U \subset M$ is a set $\{s_i\}_{i=1}^r \subset \Gamma(E|_U)$ of (local) sections such that, for every $x \in U$, the set of vectors $\{s_i(x)\}_{i=1}^r \subset E_x$ is a basis.

Remark. Given a frame, we write $s = (s_1, \ldots, s_r)^T$ for the column vector of sections. Then an arbitrary local section can be written $as = a^i s_i = \sum_i a^i s_i$ where each $a^i : U \to \mathbb{F}$ is a (local) function. We will view coefficients (with respect to a frame s) as a row vector $a = (a^1, \dots, a^r)$ of functions or, equivalently, as a vector-valued function $a : U \to \mathbb{F}^r$.

Example 2.6.2. A vector bundle $\pi : E \to M$ is locally trivial by definition and thus for every $m \in M$ there is an open neighborhood $U \ni m$ such that $\phi_U : \pi^{-1}(U) \cong U \times \mathbb{F}^r$. One can define a frame by $\{\phi_U^{-1}(x, e_i)\}$ using the local trivialization ϕ_U and any basis $\{e_i\}$ of $\mathbb{F}^r$.

Definition 2.6.3. Let ∇ be a connection on a vector bundle $\pi : E \to M$ of rank r. The *local connection forms of* ∇ *with respect to a local frame* $\{s_i\}_{i=1}^r$ on $E|_U \to U$ are the differential forms $\Theta_i^j \in \Omega^1(U)$ defined by

$$\nabla(s_i) = \sum_j \Theta_i^j \otimes s_j \in \Omega^1(U) \otimes \Gamma(E|_U). \tag{2.15}$$

Remark. For an arbitrary local section one has

$$\nabla(as) = da \otimes s + a\Theta \otimes s. \tag{2.16}$$

This is abbreviated by writing $\nabla|_U = d + \Theta$ (with respect to the frame s). According to the notation $\Theta(a \otimes s) = a\Theta \otimes s$ so the matrix Θ can be viewed either as an element of $\Omega^1(U, gl(\mathbb{F}^r))$ which acts on the coefficient $a : U \to \mathbb{F}^r$ from the right or as an element of $\Omega^1(U; \mathrm{End}(E|_U))$ that acts on s from the left. Of course, using the local trivialization $\Omega^1(U; \mathrm{End}E|_U) \cong \Omega^1(U; gl(\mathbb{F}^r))$ so this is consistent.

Proposition 2.6.4. Let $\{\sigma_i\}$ be a local frame over $U \subset M$ with local connection form Θ. Then $\{\tau_i\}$ is another local frame over U with connection form Φ if and only if there is some

$g : U \to \mathrm{Aut}(E|_U)$ such that

$$\Phi = dgg^{-1} + g\Theta g^{-1}. \tag{2.17}$$

Proof. Since σ is a frame, τ is another frame if and only if there is a unique $g : U \to \mathrm{Aut}(E|_U)$ such that $\tau = g\sigma$ and therefore

$$\Phi\tau = \nabla(\tau) = \nabla(g\sigma) = dg \otimes \sigma + g\Theta \otimes \sigma = dg \otimes g^{-1}\tau + g\Theta \otimes g^{-1}\tau = dgg^{-1} \otimes \tau + g\Theta g^{-1} \otimes \tau$$

$$\Leftrightarrow \Phi = dgg^{-1} + g\Theta g^{-1}.$$

$\square$

Remark. One uses local trivializations to work with a connection: give M an open cover $\{U_i\}$ of local trivializations with local connections $d + \Theta(i)$ on $E|_{U_i} \to U_i$ which combine by the condition $\Theta(i) = dg_{ij}g_{ij}^{-1} + g_{ij}\Theta(j)g_{ij}^{-1}$ on $E|_{U_i \cap U_j}$ where $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(E|_{U_i \cap U_j})$ is the transition function on $U_i \cap U_j$.

Example 2.6.5. Let $\pi : E \to M$ be a rank r vector bundle with connection ∇ and let $\gamma : [0,1] \to M$ be given. Since $\gamma^*E \to [0,1]$ is a trivial bundle it admits a global frame $s = (s, \ldots, s_r)^T \subset \Gamma(\gamma^*E)$ and a corresponding connection form Θ where $\gamma^*\nabla(s) = \Theta \otimes s$. An arbitrary lift $as = \sum_i a^i s_i \in \Gamma(\gamma^*E)$ of γ is flat if and only if $0 = \gamma^*\nabla(a\sigma) = da \otimes s + a\Theta \otimes s$. This gives a condition on the vector valued function a

$$da + a\Theta = 0 \tag{2.18}$$

which is an equation of vector-valued 1-forms on $[0,1]$. The right side vanishes if and only if it vanishes on the canonical unit vector field $X_t := \frac{d}{dt} \in \Gamma(T[0,1])$. Hence the coefficient sections a of a flat lift are determined by the differential equation

$$\frac{da(t)}{dt}a(t) + a(t)\Theta(X_t) = 0. \tag{2.19}$$

By abuse of notation we write $\Theta(t) = \Theta(X_t) = \nabla(\dot{\gamma}(t))$ for convenience.

$$\begin{aligned}
a(t) &= a(0) - \int_0^t a(t_1)\Theta(t_1)dt_1 = a(0) - \int_0^t \Big(a(0) - \int_0^{t_1} a(t_2)\Theta(t_2)dt_2\Big)\Theta(t_1)dt_1 \\
&= a(0) - \int_0^t a(0)\Theta(t_1)dt_1 + \int_0^t \int_0^{t_1 \le t} a(0)a(t_2)\Theta(t_2)\Theta(t_1)dt_2dt_1 \\
&= a(0) - \int_0^t a(0)\Theta(t_1)dt_1 + \int_0^t \int_0^{t_1 \le t} a(0)\Theta(t_2)\Theta(t_1)dt_2dt_1 \\
&- \int_0^t \int_0^{t_1 \le t} \int_0^{t_2 \le t_1} a(0)a(t_3)\Theta(t_3)\Theta(t_2)\Theta(t_1)dt_3dt_2dt_1 \\
\Rightarrow a(t) &= \sum_{n=0}^{\infty} (-1)^n a(0) \int_{0 \le t_n \le \cdots \le t_1 \le t} \Theta(t_n)\dots\Theta(t_1)dt_n\dots dt_1.
\end{aligned}$$

Hence the solution to the differential equation $da + a\Theta = 0$ is given by

$$a(t) = \sum_{n=0}^{\infty} (-1)^n a(0) \int_{0 \le t_n \le \cdots \le t_1 \le t} \Theta(t_n)\dots\Theta(t_1)dt_n\dots dt_1. \tag{2.20}$$

This computes the coefficients a required so that as is a flat section.

Proposition 2.6.6. The series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \int_{0 \le t_n \le \cdots \le t_1 \le 1} \Theta(t_n)\dots\Theta(t_1)dt_n\dots dt_1$ is absolutely convergent.

Proof. Since $\Theta \in \Omega^1([0,1], \gamma^*\mathrm{End}E)$ is a smooth form on a compact object, there exists $B < \infty$

such that $||\Theta(t)|| \leq B$ for all $t \in [0,1]$. Hence the absolute series is

$$\sum_{n=0}^{\infty} \int_{0\leq t_n \leq \cdots \leq t_1 \leq 1} \Big|\Big|\Theta(t_n)\ldots\Theta(t_1)dt_n\ldots dt_1\Big|\Big| \leq \sum_{n=0}^{\infty} \int_{0\leq t_n \leq \cdots \leq t_1 \leq 1} B^n dt_n \ldots dt_1$$
$$= \sum_{n=0}^{\infty} \int_{[0,1]^n} \frac{B^n}{n!} dt_n \ldots dt_1 \leq \exp(B).$$

□

Definition 2.6.7. For $a \leq b \in [0,1]$ we define

$$p_\gamma(\nabla;\Theta^s)_a^b := \sum_{n\geq 0}(-1)^n \int_{a\leq t_n\leq\cdots\leq t_1\leq b} \Theta(t_n)\ldots\Theta(t_1)dt_n\ldots dt_1 \in gl(\mathbb{F}^k). \tag{2.21}$$

The notation Θ^s serves as a reminder that the formula involves the connection form Θ which is defined by a choice of frame s.

Definition 2.6.8. Parallel translation along γ is given by

$$P_\gamma(\nabla;\Theta^s)_a^b : E_{\gamma(a)} \to E_{\gamma(b)}$$
$$c(a)s(a) \mapsto \big(c(a)p_\alpha(\nabla;\Theta^s)_a^b\big)s(b) \tag{2.22}$$

which is the linear isomorphism obtained by parallel lifts of γ that connect the fibers $E_{\gamma(a)}$ and $E_{\gamma(b)}$.

Proposition 2.6.9. $P_\alpha(\nabla;\Theta^s)_a^b : E_{\alpha(a)} \to E_{\alpha(b)}$ is independent of the choice of the frame s.

Proof. Let $\tau = (\tau_1,\ldots,\tau_r)^T$, $\tau_i \in \Gamma(\alpha^*E)$, be another frame. Write Φ for the corresponding connection matrix 1-form. Since τ and s are both frames, there exists some $g : [0,1] \to \mathrm{Aut}(\alpha^*E)$ such that $\tau = gs$ and $\Phi = dgg^{-1} + g\Theta g^{-1}$. The coefficients a of s are determined by $da + a\Theta = 0$;

the coefficients b of τ are determined by $db + b\Phi = 0$. Fixing $a(0)s(0) = b(0)\tau(0)$, a and b are the unique solutions to their initial value problem. Now $0 = db + b\Phi = db + bdgg^{-1} + bg\Theta g^{-1}$ if and only if $0 = dbg + bdg + bg\Theta = d(bg) + (bg)\Theta$. By uniqueness, we must have $a = bg$, and hence $as = bgs = b\tau$. Thus $p_\alpha(\nabla; \Theta^s) = p_\alpha(\nabla; \Phi^\tau)$. □

Definition 2.6.10. The parallel translation with respect to ∇ along $\gamma : [0, 1] \to M$ is denoted

$$P_\gamma(\nabla)_a^b : E_{\gamma(a)} \to E_{\gamma(b)}. \tag{2.23}$$

$P_\gamma(\nabla)$ without mention of a and b means parallel transport from $E_{\gamma(0)}$ to $E_{\gamma(1)}$.

2.7 Holonomy in unitary bundles

Proposition 2.7.1. Let $(E, \nabla, h) \to M$ be a unitary vector bundle. For every $\gamma : [0, 1] \to M$ the linear isomorphism $P_\gamma(\nabla)$ is unitary.

Proof. If $v, w \in E_{\gamma(0)}$, then $h\big(P_\gamma(\nabla)v, P_\gamma(\nabla)w\big) = h\big(\widetilde{\gamma}_v(1), \widetilde{\gamma}_w(1)\big)$ where $\widetilde{\gamma}_v$ and $\widetilde{\gamma}_w$ are ∇-flat lifts. But since ∇ and h are compatible

$$dh(\widetilde{\gamma}_v, \widetilde{\gamma}_w) = h(\nabla\widetilde{\gamma}_v, \widetilde{\gamma}_w) + h(\widetilde{\gamma}_v, \nabla\widetilde{\gamma}_w) = 0$$

so that $h\big(P_\gamma(\nabla)v, P_\gamma(\nabla)w\big) = h(v, w)$ and thus $P_\gamma(\nabla)$ preserves the metric h. □

Proposition 2.7.2. The linear transformation $p_\alpha(\nabla, \Theta^s)_0^1$, which acts on the coefficients determined by a frame s, is unitary.

Proof.

$$
\begin{aligned}
\Big(p_\alpha(\nabla,\Theta^s)_0^1\Big)^* &= \Big(\sum_{n=0}^{\infty}(-1)^n \int_{0\le t_n\le\cdots\le t_1\le 1}\Theta(t_n)\dots\Theta(t_1)dt_n\dots dt_1\Big)^* \\
&= \sum_{n=0}^{\infty}(-1)^n \int_{0\le t_n\le\cdots\le t_1\le 1}\Big(\Theta(t_n)\dots\Theta(t_1)\Big)^* dt_n\dots dt_1 \\
&= \sum_{n=0}^{\infty}(-1)^n \int_{0\le t_n\le\cdots\le t_1\le 1}\Theta^*(t_1)\dots\Theta^*(t_n)dt_n\dots dt_1 \\
&= \sum_{n=0}^{\infty} \int_{0\le t_n\le\cdots\le t_1\le 1}\Theta(t_1)\dots\Theta(t_n)dt_n\dots dt_1.
\end{aligned}
$$

Performing the change of variable $t'_k := 1 - t_k$, hence $dt'_k = -dt_k$, the above becomes

$$
\sum_{n=0}^{\infty}(-1)^n \int_{0\le t'_n\le\cdots\le t'_1\le 1}\Theta(t'_n)\dots\Theta(t'_1)dt'_n\dots dt'_1
$$

which is $p_{\gamma^{-1}}(\nabla,\Theta^S): E_{\gamma(1)} \to E_{\gamma(0)}$ the linear operator that computes the coefficients of parallel translation along the reverse of γ. Thus $\big(p_\gamma(\nabla,\Theta^s)\big)^* = p_{\gamma^{-1}}(\nabla,\Theta^s) = \big(p_\gamma(\nabla,\Theta^s)\big)^{-1}$. □

Corollary 2.7.3. If $(E,h,\nabla) \to M$ is a *flat* unitary bundle, the holonomy representation is unitary

$$
\mathrm{Hol}_{(-)}(\nabla) : \pi_1(M,x) \to U(E_x). \tag{2.24}
$$

Proof. By 2.3.5 parallel translation along a path with respect to a flat connection is independent of the homotopy class of the path. □

Corollary 2.7.4. If (E,h) is a hermitian vector bundle, then the domain of the holonomy function can be restricted to give the function

$$
\mathrm{Hol} : \mathcal{F}(E,h) \times \pi_1(M,x) \to U(E_x). \tag{2.25}
$$

□

2.8 Derivative of trace of holonomy functional

Let $\gamma : [0,1] \to M$ be a path in the base of a hermitian vector bundle $(\pi : E \to M, h)$. Parallel transport defines a function

$$P_\gamma : \mathcal{A}(E,h) \to \mathrm{Hom}_{\mathbb{C}-Mod}(E_{\gamma(0)}, E_{\gamma(1)})$$

whose range sits within the space of unitary transformations between the fibers over the endpoints.

Theorem 2.8.1. The derivative of parallel transport $P_\gamma : \mathcal{A}(E,h) \to \mathrm{Hom}_{\mathbb{C}-Mod}(E_{\gamma(0)}, E_{\gamma(1)})$ at $\nabla \in \mathcal{A}(E,h)$ in the direction of $\Phi \in \Omega^1(M; \mathfrak{u}E) = T_\nabla \mathcal{A}(E,h)$ is given by

$$dP_\gamma(\nabla, \Phi) := \frac{dP_\gamma(\nabla + \epsilon\Phi)}{d\epsilon}|_{\epsilon=0} = -\int_0^1 P_\gamma(\nabla)_t^1 \circ \Phi(\dot\gamma(t)) \circ P_\gamma(\nabla)_0^t dt.$$

Proof. Every $\Phi \in \Omega^1(M; \mathrm{End}E)$ defines the connections $\nabla + \Phi \in \mathcal{A}(E)$ and $\gamma^*\nabla + \gamma^*\Phi \in \mathcal{A}(\gamma^*E)$. Using a frame s for γ^*E, write $\Theta = \Theta^s$ for the connection 1-forms corresponding to $\gamma^*\nabla$ and write $B = \gamma^*\Phi$. Note that Θ and B can be viewed as elements of $\Omega^1([0,1], gl(\mathbb{F}^k))$. For example, given $X \in \Gamma(T[0,1])$, $B(X) : [0,1] \to gl(\mathbb{F}^k)$ is defined by $\gamma^*\Phi(c^i s_i)(X) : t \mapsto c^i(t)B(X_t)s_i(t)$, i.e. $B(X)$ acts on the coefficients from the right. (Of course, one can also view B as an element of $\Omega^1([0,1]; \mathrm{End}(\alpha^*E))$ which acts on *sections* from the left.) Recall that $p_\gamma(\nabla, \Theta^s)$ computes parallel translation of the coefficients of an arbitrary section $as \in \Gamma(\gamma^*E)$ where the coefficients are viewed as sections of the trivial bundle using the trivialization defined by the frame s.

Lemma 2.8.2.

$$\frac{dp_\gamma(\nabla+\epsilon\Phi;(\Theta+\epsilon B)^s)}{d\epsilon}|_{\epsilon=0} = -\int_0^1 \Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)dt. \tag{2.26}$$

Proof. This amounts to looking at

$$\sum_{n=0}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}(\Theta(t_n)+\epsilon B(t_n))\dots(\Theta(t_1)+\epsilon B(t_1))dt_n\dots dt_1$$

and extracting the coefficient of ϵ modulo ϵ^2. We will use the fact 2.6.6 that the series is absolutely convergent in order to rearrange terms. The order one terms are

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}&\Big(\sum_{m=0}^{\infty}(-1)^m\int_{0\le r_m\le\cdots\le r_1\le t_n}\Theta(r_m)\dots\Theta(r_1)dr_m\dots dr_1\Big)\\
&\times B(t_n)\Theta(t_{n-1})\dots\Theta(t_1)dt_n\dots dt_1\\
&=\sum_{n=1}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}\Big(p_\gamma(\nabla,\Theta^s)_0^{t_n}\Big)B(t_n)\Theta(t_{n-1})\dots\Theta(t_1)dt_n\dots dt_1\\
&=-\int_{t=0}^1\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\sum_{n-1=0}^{\infty}(-1)^{n-1}\int_{t\le t_{n-1}\le\cdots\le t_1\le 1}\Theta(t_{n-1})\dots\Theta(t_1)dt_{n-1}\dots dt_1 dt\\
&=-\int_{t=0}^1\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)dt.
\end{aligned}$$

$\square$

Applying $-\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)$ to the coefficients $a=(a^1,\dots,a^r)$ *from the right* results in first parallel transporting $-v=-a(0)s(0)\in E_{\gamma(0)}$ to $-P_\gamma(\nabla)_0^t v\in E_{\gamma(t)}$, applying $\Phi(\dot\gamma(t))\in \mathrm{End}(E_{\gamma(t)})$, and proceeding to parallel transport until arriving at $-P_\gamma(\nabla)_t^1\circ\Phi(\dot\gamma(t))\circ P_\gamma(\nabla)_0^t\in E_{\gamma(1)}$.

$\square$

Corollary 2.8.3. If γ is a closed path, the derivative of holonomy

$$\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to U(E_{\gamma(0)})$$

at $\nabla \in \mathcal{A}(E,h)$ in the direction of $\Phi \in \Omega^1(M;\mathfrak{u}E)$ is given by

$$d\mathrm{Hol}_\gamma(\nabla,\Phi) = -\int_0^1 P_\gamma(\nabla)_t^1 \circ \Phi(\dot\gamma(t)) \circ P_\gamma(\nabla)_0^t dt. \tag{2.27}$$

Proof. If $\gamma : [0,1] \to M$ is a closed path then $\mathrm{Hol}_\gamma = P_\gamma : \mathcal{A}(E,h) \to U(E_{\gamma(0)})$. □

Corollary 2.8.4. The derivative of (the real) trace of holonomy $\Re\mathbf{Tr}\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to \mathbb{R}$ is given by

$$d(\Re\mathbf{Tr}\mathrm{Hol}_\gamma)(\nabla,\Phi) = -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

where $\gamma^t : [0,1] \ni s \mapsto \gamma(s+t \mod 1) \in M$ denotes γ shifted up by t.

Proof.

$$\begin{aligned}
d(\Re\mathbf{Tr}\mathrm{Hol}_\gamma)(\nabla,\Phi) &= d(\Re\mathbf{Tr})_{\mathrm{Hol}_\gamma(\nabla)} \circ d\mathrm{Hol}_\gamma(\nabla,\Phi) = -\Re\mathbf{Tr}\Big(\int_0^1 P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))P_\gamma(\nabla)_0^t dt\Big) \\
&= -\int_0^1 \Re\mathbf{Tr}\Big(P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))P_\gamma(\nabla)_0^t\Big)dt = -\int_0^1 \Re\mathbf{Tr}\Big(P_\gamma(\nabla)_0^t P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))\Big)dt \\
&= -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.
\end{aligned}$$

□

Definition 2.8.5. For a given loop γ and flat unitary connection ∇, define the functional

$$\begin{aligned}
\mathbb{T} = \mathbb{T}_\nabla^\gamma &: Z^1_{\nabla}(M;\mathfrak{u}E) \to \mathbb{R} \\
&\Phi \mapsto -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.
\end{aligned} \tag{2.28}$$

Proposition 2.8.6. The functional $\mathbb{T}$ vanishes on $B^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ and therefore defines a functional

$$\mathbb{T} : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to \mathbb{R}. \tag{2.29}$$

Proof. If $\Phi \in B^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \subset Z^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$, then $\Phi = \widetilde{\nabla} f$ for some $f \in \Omega^0(M;\mathfrak{u}E)$ and therefore $\mathrm{Hol}_{\gamma^t}(\nabla)\widetilde{\nabla} f(\dot\gamma(t)) = \widetilde{\nabla}\Big(\mathrm{Hol}_{\gamma^t} f(\dot\gamma(t))\Big)$ since Hol_{γ^t} is flat by construction. As γ is closed, the result follows from Stokes' theorem. $\square$

Remark. Let $x = \gamma(0)$ be the basepoint of a closed path γ. It will be convenient to describe $d\mathrm{Hol}_\gamma(\nabla)$ and $\mathbb{T}$ by left-translating the images to the tangent space at the identity $\mathbf{1} \in UE_x$ in order to obtain an element of the Lie algebra $\mathfrak{u}E_x \cong \mathfrak{u}(r)$ where r is the rank of E. The method of doing this results in a *variation function* $F : UE_x \to \mathfrak{u}E_x$ as described by Goldman in [5].

Definition 2.8.7. Given a conjugate invariant function $f : UE_x \to \mathbb{R}$ (e.g. $f = \Re\mathbf{Tr}$) the *covariation function* $\hat F : UE_x \to (\mathfrak{u}E_x)^\vee$ is defined by the commutative diagram

$$\begin{tikzcd} T_u UE_x \arrow[r, "df_u"] & \mathbb{R} \\ T_{\mathbf{1}} UE_x = \mathfrak{u}E_x \arrow[u, "dLu"] \arrow[ur, dotted, "\hat F(u)"'] & \end{tikzcd}$$

where Lu is left multiplication by u.

Remark. By assumption h is a hermitian metric on E and $\widetilde{h}$ is the induced metric on (the *real* vector bundle) $\mathfrak{u}E$. In particular, $\widetilde{h}_x : \mathfrak{u}E_x \otimes \mathfrak{u}E_x \to \mathbb{R}$ is the metric $\widetilde{h}_x(A,B) = \mathbf{Tr}(B^*A) = -\mathbf{Tr}(BA) = -\Re\mathbf{Tr}(BA)$. The metric induces the linear isomorphisms $h^\flat : \mathfrak{u}E_x \leftrightarrows (\mathfrak{u}E_x)^\vee : \widetilde{h}^\sharp$.

Definition 2.8.8. The *variation function of a conjugate invariant function* $f : UE_x \to \mathbb{R}$ *with* *respect to the metric* $\widetilde{h}$ is the function

$$F = \widetilde{h}^\sharp \circ \hat{F} : UE_x \to \mathfrak{u}E_x.$$

Equivalently, the variation function is characterized by

$$\widetilde{h}(F(u), X) = \hat{F}(u)(X) = df_u(dLuX) = \frac{d}{ds}|_{s=0} f(ue^{sX}) \tag{2.30}$$

for all $u \in UE_x$ and for all $X \in \mathfrak{u}E_x$.

Remark. These definitions can be applied globally to a bundle. For example, if $f : UE \to M \times \mathbb{R}$ is a bundle map with each $f_x : UE_x \to \mathbb{R}$ conjugate invariant, then take the variation function $F_x : UE_x \to \mathfrak{u}_x$ at each fiber in order to obtain a global variation bundle map $F : UE \to \mathfrak{u}E$. This global context provides an optimal backdrop if one needs to consider different (homotopy classes of) closed curves with different basepoints.

Proposition 2.8.9. Let $(\pi : E \to M, h)$ be a hermitian vector bundle. The variation bundle map for $\Re\mathbf{Tr} : UE \to M \times \mathbb{R}$ with respect to the metric $\widetilde{h}$ on $\mathfrak{u}E$ is given by

$$F : UE \to \mathfrak{u}E \text{ where for all } x \in M \tag{2.31}$$

$$F_x : UE_x \ni A \mapsto \frac{-1}{2}\left(A - A^{-1}\right) \in \mathfrak{u}E_x.$$

Proof. For $A \in UE_x$ and $X \in \mathfrak{u}E$, $AA^* = \mathbf{1} \in UE_x$ and $X^* = -X$. Recall that $\widetilde{h}$ is defined on $\mathfrak{u}E$ via its restriction on the larger bundle $\mathrm{End}E$, and that both $A, X \in \mathrm{End}E_x$. Thus

$$\begin{aligned}\widetilde{h}\big(\frac{-1}{2}(A-A^{-1}),X\big) &= -\frac{1}{2}\widetilde{h}(A,X)+\frac{1}{2}\widetilde{h}(A^{-1},X) = -\frac{1}{2}\mathbf{Tr}(X^*A)+\frac{1}{2}\mathbf{Tr}(X^*A^{-1})\\ &= \frac{1}{2}\mathbf{Tr}(XA)+\frac{1}{2}\mathbf{Tr}(X^*A^*) = \frac{1}{2}\mathbf{Tr}(AX)+\frac{1}{2}\mathbf{Tr}((AX)^*)\\ &= \frac{1}{2}\mathbf{Tr}(AX)+\frac{1}{2}\overline{\mathbf{Tr}(AX)} = \Re\mathbf{Tr}(AX) = \frac{d}{ds}|_{s=0}\Re\mathbf{Tr}(Ae^{sX}).\end{aligned}$$

□

Proposition 2.8.10. For a given closed curve γ and flat unitary connection ∇, the functional $\mathbb{T}=\mathbb{T}^{\gamma}_{\nabla}$ can be expressed using the variation map $F: UE \to \mathfrak{u}E$:

$$\mathbb{T}(\Phi) = -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.$$

Proof. Starting with 2.8.3

$$\begin{aligned}d(\mathbf{Tr}\mathrm{Hol}_{\gamma})(\nabla,\Phi) &= -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt = -\int_0^1 \Re\mathbf{Tr}\Big(\frac{d}{ds}|_{s=0}\mathrm{Hol}_{\gamma^t}(\nabla)\exp(s\Phi\dot\gamma(t))\Big)dt\\ &= -\int_0^1 \frac{d}{ds}|_{s=0}\Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\exp(s\Phi\dot\gamma(t))\Big)dt\\ &= -\int_0^1 \widetilde{h}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla),\Phi(\dot\gamma(t))\Big)dt = -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt\end{aligned}$$

where the penultimate step is given by 2.30. □

2.9 Twisted singular (co)homology

Let $\pi: E\to M$ be a smooth vector bundle equipped with a flat connection ∇. In this section we define the de Rham map

$$\mathbf{I}: \Big(\Omega^{\bullet}(M;E),\nabla\Big) \to \Big(S^{\bullet}_{\nabla}(M;E),\delta\Big)$$

from the differential graded module of differential forms on M with values in (E, ∇) to singular cochains on M with values in (E, ∇) and show that it is a quasi-isomorphism.

Definition 2.9.1. A *singular k-chain on M with values in a flat vector bundle* $(E, \nabla) \to M$ is a finite summation of pairs

$$(\sigma, s) \in S_k(M) \times \Gamma(\sigma^* E)$$

where $\sigma : \Delta^k \to M$ is an ordinary (smooth) elementary singular k-chain and $s \in \Gamma(\sigma^* E)$ is a $\sigma^*\nabla$-flat section of $\sigma^* E \to \Delta^k$. The *vector space of all (E, ∇)-valued singular k-chains* will be denoted by $S_k^{\nabla}(M; E)$ and $S_\bullet^{\nabla}(M; E) := \bigoplus_{k \geq 0} S_k^{\nabla}(M; E)$ is the *graded vector space of all (E, ∇)-valued singular chains.* $S_\bullet^{\nabla}(M; E)$ is equipped with the *boundary operator* ∂ defined using the usual boundary operator for ordinary singular homology

$$\partial(\sigma, s) := (\partial\sigma, s|_{\partial\sigma})$$

where $\partial\sigma$ is the alternating summation over the codimension 1-faces of $\sigma : \Delta^k \to M$ and $s|_{\partial\sigma}$ denotes the restriction of the section $s : \Delta^k \to \sigma^* E$ to the the boundary $\partial\Delta^k \subset \Delta^k$. By definition ∂ is linear and $\partial^2 = 0$ so that

$$\left(S_\bullet^{\nabla}(M; E), \partial\right)$$

is a chain complex. The homology of this chain complex is the *singular homology of M with values in* (E, ∇) and is denoted by $H_\bullet^{\nabla}(M; E)$.

Remark. "*Homology* with local coefficients" refers to singular homology. The space of singular cycles is $Z^{\nabla}_{\bullet}(M;E) := \ker\partial \cap S^{\nabla}_{\bullet}(M;E)$ and the space of boundaries is $B^{\nabla}_{\bullet}(M;E) = \partial\big(S^{\nabla}_{\bullet}(M;E)\big)$.

Definition 2.9.2. A *singular k-cochain with values in a flat vector bundle* (E,∇) is an element ϕ that assigns to every singular k-chain $\sigma : \Delta^k \to M$ a $\sigma^*\nabla$-flat section $\phi(\sigma) \in \Gamma(\sigma^* E)$.

Write $S^k_{\nabla}(M;E)$ for *the vector space of all (E,∇)-valued singular k-cochains* and $S^{\bullet}_{\nabla}(M;E) = \bigoplus_k S^k_{\nabla}(M;E)$ for the *graded vector space of all singular cochains with values in* (E,∇). $S^{\bullet}_{\nabla}(M;E)$ is equipped with the *coboundary operator* δ defined for all $(\phi,\sigma) \in S^k_{\nabla}(M;E) \times S_{k+1}(M)$ by the requirement that $(\delta\phi)(\sigma)$ is the (summation over the) unique flat sections that extend the flat sections $\phi(\partial\sigma) : \partial\Delta^{k+1} \to \sigma^* E$ over all of $\Delta^{k+1} \supset \partial\Delta^k$. The cohomology of the cochain complex $\left(S^{\bullet}_{\nabla}(M;E), \delta\right)$ is the *singular cohomology of M with values in* (E,∇) and is written $H^{\bullet}(M;E^{\nabla})$.

Definition 2.9.3. Given $\phi \in S^k_{\nabla}(M;E)$ and $\sigma \in S_j(M)$, their cap product $\sigma \cap_1 \phi \in S^{\nabla}_{j-k}(M;E)$ is constructed by taking the flat section $\phi(\sigma|_{\text{front } k-\text{face}})$ over the front k-face of σ, extending it by parallel transport to a flat section over all of σ, and then restricting to the back $j-k$ face of σ. The cap product $\sigma \cap_1 \phi$ is defined as the back $j-k$ face of σ equipped with the flat section just described.

Definition 2.9.4. If E is equipped with a metric h, then $\cap_h : S^{\nabla}_j(M;E) \times S^k_{\nabla}(M;E) \to S_{j-k}(M; M \times \mathbb{F}) = S_{j-k}(M,\mathbb{F})$ is given as follows. Let $\phi \in S^k_{\nabla}(M;E)$ and $(\sigma, s) \in S^{\nabla}_j(M;E)$. Then $\sigma \cap_1 \phi \in S^{\nabla}_{j-k}(M;E)$ is a flat section over the back face of σ. Restricting the flat section

$s : \Delta^j \to E$ gives another flat section over the back $j - k$ face of σ.

$$(\sigma, s) \cap_h \phi := h(\sigma \cap_1 \phi, \sigma|_{\text{back } j-k \text{ face}})$$

Definition 2.9.5. $\cup_1 : S^j_\nabla(M;E) \times S^k(M) \to S^{j+k}_\nabla(M;E)$ is defined in the usual manner. Given $\phi \in S^j_\nabla(M;E)$, $\psi \in S^k(M)$ and $\sigma : \Delta^{j+k} \to M$, then $(\phi \cup_1 \psi)(\sigma)$ assigns the flat section over Δ^{j+k} formed by taking the flat section $\phi(\sigma_{\text{front } j-\text{face}})$, extending over Δ^{j+k} using parallel transport, and scaling that section by $\psi(\sigma_{\text{back } k-\text{face}})$. We also write $\cup_1 : S^j(M) \times S^k_\nabla(M;E) \to S^{j+k}_\nabla(M;E)$ when the second argument is twisted. In particular $\cup_1 : S^j_\nabla(M;E) \times S^k_\nabla(M;E) \to S^{j+k}_\nabla(M;E \otimes E)$ is defined.

Definition 2.9.6. If E is equipped with a metric h, then $\cup_h : S^j_\nabla(M;E) \times S^j_\nabla(M;E) \to S^{j+k}(M;M \times \mathbb{F})$ is defined following the above. Given $\phi \in S^j_\nabla(M;E)$, $\psi \in S^k_\nabla(M;E)$ and $\sigma : \Delta^{j+k} \to M$, extend $\phi(\sigma_{\text{front } j-\text{face}})$ to a flat section over Δ^{j+k} and extend $\psi(\sigma_{\text{back } k-\text{face}})$ to a flat section over Δ^{j+k}. Use the metric h to pair the two flat sections into the flat section $(\phi \cup_h \psi)(\sigma)$ of the trivial line bundle.

Remark. The cap and cup product can be used to obtain twisted Poincaré duality.

Theorem 2.9.7 (twisted Poincaré duality)**.** If M is a closed orientable manifold of dimension n and (E, ∇) is a flat vector bundle over M then $[M]\cap_1 : H^k(M;E^\nabla) \to H^\nabla_{n-k}(M;E)$ is an isomorphism.

Proof. The proof of (ordinary) Poincaré duality [8] carries through in this twisted version. For a complete overview of twisted (co)homology theory including a proof of Poinaré duality see [17]. ☐

Definition 2.9.8. Given $\phi \in \Omega^k_\nabla(M; E)$ and $\sigma : \Delta^k \to M$, define the integral

$$\int_{\Delta^k} \sigma^*\phi := (\int_{\Delta^k} \alpha)s : \Delta^k \to E$$

where $\sigma^*\phi = \alpha \otimes s \in \Omega^k(\Delta^k) \otimes \Gamma(E)$ and s is a $\sigma^*\nabla$-flat section of $\sigma^*E \to \Delta^k$.

Remark. To see that the integral is well-defined, assume $\sigma^*\phi = \alpha \otimes s = \beta \otimes \tau$ where both s and τ are $\sigma^*\nabla$-flat so that there exists some $g \in \mathrm{Aut}(\sigma^*E)$ such that $\tau = gs$ and necessarily $dg = 0$. Then $\alpha \otimes s = \beta \otimes gs = \beta g \otimes s \Rightarrow \alpha = \beta g$ and therefore

$$(\int_{\Delta^k} \beta)\tau = (\int_{\Delta^k} \beta)gs = (\int_{\Delta^k} \beta g)s = (\int_{\Delta^k} \alpha)s.$$

Lemma 2.9.9. If $E \to U$ is a smooth vector bundle equipped with a flat connection ∇ over a contractible base U, then there exists a ∇-flat frame for E.

Proof. Pick any $v \in E$ and use parallel translation to extend v. This is well-defined because ∇ is flat and U is contractible. ☐

Corollary 2.9.10. If $E \to \Delta^n$ is a smooth vector bundle with flat connection ∇, every bundle-valued differential form $\Omega^k(\Delta^n; E)$ can be written as $\phi \otimes s \in \Omega^k(\Delta^n) \otimes_{\Omega^0(\Delta^n)} \Gamma(E)$ where $\nabla(s) = 0$ and therefore $\nabla(\phi \otimes s) = d\phi \otimes s$. Hence any ∇-closed differential form can be written $\phi \otimes s \in \Omega^k(\Delta^n) \otimes_{\Omega^0(\Delta^n)} \Gamma(E)$ where $\nabla(s) = 0$ and $d\phi = 0$. ☐

Definition 2.9.11. The de Rham map from differential forms to singular cochains is given by

$$\mathbf{I} : \Omega^\bullet_\nabla(M;E) \to S^\bullet_\nabla(M;E) \text{ where for } \phi \in \Omega^k_\nabla(M;E)$$
$$\mathbf{I}\phi(\sigma : \Delta^k \to M) = \int_{\Delta^k} \sigma^*\phi \in \Gamma(\sigma^*E \to \Delta^k)$$

and extended linearly over finite singular chains.

Lemma 2.9.12. $\mathbf{I} : \Omega^\bullet_\nabla(M;E) \to S^\bullet_\nabla(M;E)$ is a chain map.

Proof. Take any $\phi \in \Omega^k(M;E)$ and $\sigma : \Delta^{k+1} \to M$. Write $\sigma^*\phi = \alpha \otimes s \in \Omega^k(\Delta^{k+1}) \otimes \Gamma(\sigma^*E)$ where $\sigma^*\nabla(s) = 0$. Note that by definition of $\mathbf{I}$ and the singular boundary map ∂ one has

$$(\delta\mathbf{I}\phi)(\sigma) = (\mathbf{I}\phi)(\partial\sigma) = \int_{\partial\Delta^{k+1}} (\partial\sigma)^*\phi$$

following the established abuse of notation where the section over $\partial\Delta^{k+1}$ is extended to a flat section over all of Δ^{k+1}. On the other hand

$$\begin{aligned}(\mathbf{I}\nabla\phi)(\sigma) &= \int_{\Delta^{k+1}} \sigma^*(\nabla\phi) = \int_{\Delta^{k+1}} (\sigma^*\nabla)(\sigma^*\phi) = \int_{\Delta^{k+1}} (\sigma^*\nabla)(\alpha \otimes s) = \int_{\Delta^{k+1}} (d\alpha \otimes s) \\ &= (\int_{\Delta^{k+1}} d\alpha)s = (\int_{\partial\Delta^{k+1}} \alpha)s = \int_{\partial\Delta^{k+1}} \sigma^*\phi.\end{aligned}$$

Both $(\partial\sigma)^*\phi$ and $\sigma^*\phi$ agree on $\partial\Delta^{k+1}$.

□

Example 2.9.13. Let $\phi \in Z^1_\nabla(M;E)$ and $\sigma : [0,1] \to M$. Then $\sigma^*\phi \in Z^1_{\sigma^*\nabla}([0,1];\sigma^*E)$ so that one may choose a flat section $s \in \Gamma(\sigma^*E)$ and write $\sigma^*\phi = \alpha \otimes s \in \Omega^1([0,1]) \otimes_{\Omega^0([0,1])} \Gamma(\sigma^*E)$. It follows that $d\alpha = 0$, and thus $\alpha = f(t)dt$ where $f(t) = c$ is a constant function. Write ∂_t for the basis of $T_t[0,1]$ which is dual to the basis dt of $T_t^\vee[0,1]$. Then

$$\phi(\dot\sigma(t)) = (\sigma^*\phi)(\partial_t) = (\alpha \otimes s)(\partial_t) = f(t)s(t) = cs(t)$$

and since $(\mathbf{I}\phi)(\sigma) = (\int_{[0,1]} \alpha)s = cs$, the section $(\mathbf{I}\phi)(\sigma) \in \Gamma(\sigma^*E \to [0,1])$ is given by $t \mapsto \phi(\dot\sigma(t))$.

Remark. If $\mathfrak{U}$ is a cover of M, then $S_\bullet^{\mathfrak{U}}(M) \subset S_\bullet(M)$ is the subspace of *singular chains subordinate to the cover* $\mathfrak{U}$ which consists of finite sums of elementary chains $\sigma : \Delta^k \to M$ such that there is some $U_\sigma \in \mathfrak{U}$ with $\sigma(\Delta^k) \subset U_\sigma$. The inclusion $S_\bullet^{\mathfrak{U}}(M) \hookrightarrow S_\bullet(M)$ is a chain-equivalence [2]. Thus, when dealing with singular homology of a smooth manifold M it is sufficient to work with smooth singular chains subordinate to some good cover $\mathfrak{U}$ of M.

Theorem 2.9.14 (twisted de Rham)**.** The de Rham map is a quasi-isomorphism.

Proof. Since M is a manifold it admits a good cover $\mathfrak{U}$. Singular cohomology with values in (E, ∇) satisfies the Mayer-Vietoris sequence [19]. This can be seen by working with singular chains subordinate to the cover $\mathfrak{U}$. For differential forms

$$0 \to \Omega^\bullet(U \cup V) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$

is a short exact sequence of graded $\Omega^0(U \cup V)$-modules. Hence

$$0 \to \Omega^\bullet(U \cup V) \otimes \Gamma(E) \to \Omega^\bullet(U) \otimes \Gamma(E) \oplus \Omega^\bullet(V) \otimes \Gamma(E) \to \Omega^\bullet(U \cap V) \otimes \Gamma(E) \to 0$$

is also a short exact sequence and we obtain the Mayer-Vietoris sequence for $H^\bullet_\nabla(U \cup V; E)$. If $U \subset M$ is contractible, then $H^\bullet_\nabla(U; E) = H^\bullet_\delta(U; E^\nabla) \cong \left(\mathbb{F}^r \to 0 \to \dots\right)$ and, since $\mathbf{I}$:

$H^0_\nabla(U;E) \to H^0_\delta(U;E^\nabla)$ is injective, $\mathbf{I} : H^\bullet_\nabla(U;E) \cong H^\bullet_\delta(U;E^\nabla)$. Using the good cover $\mathfrak{U}$ of M, the Mayer-Vietoris sequences and the five-lemma, the result follows by induction on the number of sets in $\mathfrak{U}$. □

Definition 2.9.15. Let $(\pi : E \to M, \nabla)$ be a flat bundle equipped with a compatible metric h. Given $\Phi \in \Omega^k_\nabla(M;E)$ and $(\sigma, s) \in S^\nabla_k(M;E)$, define the section

$$\langle \Phi, (\sigma, s)\rangle_h := h(\Phi(\sigma), s) : \Delta^k \to \mathbb{F}$$

which uses the metric h to pair $\Phi(\sigma) \in \Gamma(\sigma^* E)$ with $s \in \Gamma(\sigma^* E)$. Integrating over the domain of σ gives the following pairing

$$\int \langle -, -\rangle_h : \Omega^k_\nabla(M;E) \times S^\nabla_k(M;E) \to \mathbb{F} \tag{2.32}$$

$$\int \langle \Phi, (\sigma, s)\rangle_h = \int_{\Delta^k} \langle \Phi, (\sigma, s)\rangle_h.$$

Theorem 2.9.16. When M is a closed orientable manifold, the pairing above induces a non-degenerate pairing

$$H^k_\nabla(M;E) \times H^\nabla_k(M;E) \to \mathbb{F}. \tag{2.33}$$

Proof. This follows immediately from Poincaré duality and de Rham's theorem. □

2.10 Representing the functional $\mathbb{T}$

Proposition 2.10.1. Given a loop $\gamma : [0,1] \to M$ and flat unitary connection $\nabla \in \mathcal{F}(E,h)$, there is a twisted 1-cycle

$$\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$$

where $F\mathrm{Hol}_\gamma(\nabla) : t \mapsto F\mathrm{Hol}_{\gamma^t}(\nabla) \in \mathfrak{u}E_{\gamma(t)}$ uses the variation map $F : UE \to \mathfrak{u}E$.

Proof. The loop γ is an ordinary singular 1-cycle. $F\mathrm{Hol}_\gamma(\nabla) \in \Gamma(\gamma^*\mathfrak{u}E)$ is $\widetilde{\nabla}$-flat, and therefore the pair $(\gamma, F\mathrm{Hol}_\gamma(\nabla)) \in S_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$ is a twisted singular 1-chain. Furthermore, it is a twisted *cycle* because

$$\partial\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) = \big(\gamma(1), F\mathrm{Hol}_{\gamma^1}(\nabla)\big) - \big(\gamma(0), F\mathrm{Hol}_{\gamma^0}(\nabla)\big)$$

and $F\mathrm{Hol}_{\gamma^0}(\nabla) = F\mathrm{Hol}_{\gamma^1}(\nabla)$ as both are parallel translation around γ (in the same direction) starting at $\gamma(0) = \gamma(1)$. $\square$

Definition 2.10.2. The *holonomy cycle of* γ *with respect to* ∇ is $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$.

Theorem 2.10.3. Given $\nabla \in \mathcal{F}(E,h)$ and a closed curve γ, the functional $\mathbb{T} = \mathbb{T}_\nabla^\gamma : Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}$ is represented by the holonomy cycle $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$ in the sense that

$$\mathbb{T}(\Phi) = \int \Big\langle \Phi, \big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \Big\rangle_h$$

for every $\Phi \in Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$.

Proof. By de Rham's Theorem every $\Phi \in Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$ represents a *singular cocycle* with values in $\mathfrak{u}E$. Since γ is a(n ordinary) singular 1-chain, $\Phi(\gamma) = \mathbf{I}\Phi(\gamma)$ is the $\widetilde{\nabla}$-flat section of $\gamma^*\mathfrak{u}E$

given by $t \mapsto \Phi(\dot\gamma(t))$. More generally, if (γ, s) is a *twisted* singular 1-chain, then $\langle \Phi, (\gamma, s)\rangle : t \mapsto \Phi(\dot\gamma(t)) \otimes s(t)$ is a flat section of $\gamma^*(\mathfrak{u}E \otimes \mathfrak{u}E)$. Post-composing with $\widetilde{h} : \mathfrak{u}E \otimes \mathfrak{u}E \to \mathbb{R}$ yields a section of the trivial bundle over the $[0,1]$ (the domain of γ):

$$\langle \Phi, \gamma \otimes s\rangle_h : [0,1] \ni t \mapsto \widetilde{h}(\Phi(\dot\gamma(t)), s(t)) \in \mathbb{R}.$$

Integrating the above function gives a number and defines the pairing

$$\int \langle -, -\rangle_h : \Omega^1(M; \mathfrak{u}E) \times S_1^{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}$$

which, when applied to Φ and $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big)$, gives

$$\begin{aligned}\int \Big\langle \Phi, \big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big)\Big\rangle_h &= \int_0^1 \widetilde{h}\Big(\Phi(\dot\gamma(t)), F\mathrm{Hol}_\gamma(\nabla)\Big)dt \\ &= -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_\gamma(\nabla)\Phi(\dot\gamma(t))\Big)dt = \mathbb{T}(\Phi).\end{aligned}$$

$\square$

Chapter 3

A motivational perspective

This chapter provides an overview of the *moduli space of flat unitary (irreducible) connections.*

There is a bijection between equivalence classes of flat hermitian vector bundles (which can be identified with flat principal $U(r)$-bundles) and conjugacy classes of representations $\rho : \pi_1(M) \to U(r)$. For compact manifolds $\pi_1(M)$ is finitely generated and therefore $\mathrm{Hom}(\pi_1(M), U(r)) \subset U(r)^{\#\text{of generators}}$. The quotient of this representation variety is one model for the moduli space of flat unitary connections. Given a flat unitary vector bundle $(E, \nabla) \to M$ of rank r, the corresponding representation is given by holonomy $\mathrm{Hol}(\nabla) : \pi_1(M, x) \to U(E_x) \cong U(r)$. In particular, the *trace of holonomy functions are the characters of the representation that determines* (E, ∇) *up to gauge equivalence* and this motivates our interest in the trace of holonomy functions.

Furthermore we give an informal explanation of how $H^1_{\bar{\nabla}}(M; \mathfrak{u}E)$ may be viewed as (a model for) the tangent space to the class $[\nabla]$ within the moduli space.

3.1 Gauge equivalence

Definition 3.1.1. Let $\pi : E \to M$ be a vector bundle. The *automorphism bundle of* E is the group bundle $\mathrm{Aut}E \to M$ with fibers $(\mathrm{Aut}E)_x = \mathrm{Aut}(E_x) = GL(E_x)$. The *gauge group* $G(E)$ *of the bundle* E is the the group of sections $\Gamma(\mathrm{Aut}E)$. Two connections $\nabla_1, \nabla_2 \in \mathcal{A}(E)$ are *gauge equivalent* $\nabla_1 \sim \nabla_2$ if and only if there is an element $g \in G(E)$ such that $g \circ \nabla_1 = \nabla_2 \circ g$.

Remark. Elements of $\Gamma(\mathrm{Aut}E)$ are the same as vector bundle automorphisms of $\pi : E \to M$ that project to the identity on M.

Definition 3.1.2. The *gauge group acts on the space of connections* by

$$\mathcal{A}(E) \times G(E) \ni (\nabla, g) \mapsto g^{-1} \circ \nabla \circ g \in \mathcal{A}(E).$$

Proposition 3.1.3. $\nabla_1 \sim \nabla_2$ if and only if they are in the same orbit of the $G(E)$-action. □

Proposition 3.1.4. The gauge action preserves flat connections.

Proof. $(g^{-1} \circ \nabla \circ g) \circ (g^{-1} \circ \nabla \circ g) = g^{-1} \circ (\nabla \circ \nabla) \circ g$, hence $\nabla^2 = 0 \Leftrightarrow (\nabla \cdot g)^2 = 0$. □

Definition 3.1.5. The *(naïve) moduli space* $\mathcal{M}(E)$ *of flat connections on* $\pi : E \to M$ is the set of gauge equivalence classes of flat connections.

Definition 3.1.6. If $(E, h) \to M$ is a hermitian vector bundle, the *unitary gauge group* $G(E, h)$ of the hermitian bundle is the subgroup of the gauge group consisting of automorphisms that preserve the metric h.

Proposition 3.1.7. The subgroup $G(E,h)$ acts on the space $\mathcal{A}(E,h)$ of all unitary connections and also preserves the subspace $\mathcal{F}(E,h)$ of flat unitary connections.

Proof. Let $\nabla \in \mathcal{A}(E,h)$ and $\phi \in G(E,h)$. Then for all $s,t \in \Gamma(E)$

$$\begin{aligned} h\big((\nabla \cdot \phi)s,t\big) + h\big(s,(\nabla \cdot \phi)t\big) &= h\big(\phi^{-1}\nabla(\phi s),t\big) + h\big(s,\phi^{-1}\nabla(\phi t)\big) \\ &= h\big(\nabla(\phi s),\phi t\big) + h\big(\phi s,\nabla(\phi t)\big) = dh(\phi s,\phi t) = dh(s,t) \end{aligned}$$

using that both ∇ and ϕ are h-unitary. Therefore $\nabla \cdot \phi$ is a unitary connection. As $G(E,h)$ preserves both $\mathcal{A}(E,h)$ and $\mathcal{F}(E)$, it preserves $\mathcal{A}(E,h) \cap \mathcal{F}(E) = \mathcal{F}(E,h)$. □

Definition 3.1.8. If (E,h) is a hermitian vector bundle, the *(naïve) moduli space* $\mathcal{M}(E,h)$ *of flat unitary connections* is the set of unitary gauge equivalence classes of flat unitary connections.

3.2 Holonomy representation and the moduli space

Definition 3.2.1. The *holonomy representation of a flat connection* $\nabla \in \mathcal{F}(E)$ is the representation

$$\mathrm{Hol}(\nabla) : \pi_1(M,x) \to \mathrm{Aut}(E_x) \tag{3.1}$$

of the fundamental group on E_x. The assignment $\nabla \mapsto \mathrm{Hol}(\nabla)$ is the *holonomy (representation) function*

$$\mathrm{Hol} : \mathcal{F}(E) \to \mathrm{Hom}\Big(\pi_1(M,x), \mathrm{Aut}(E_x)\Big). \tag{3.2}$$

Remark. If (E, h) is a hermitian vector bundle and $\nabla \in \mathcal{F}(E, h)$ is a flat unitary connection, holonomy takes values in unitary transformations and therefore the holonomy representation function restricts to

$$\mathrm{Hol} : \mathcal{F}(E, h) \to \mathrm{Hom}\Big(\pi_1(M, x), U(E_x)\Big). \tag{3.3}$$

Proposition 3.2.2. If g is an element of the (unitary) gauge group then

$$\mathrm{Hol}_\gamma(\nabla \cdot g) = g(\gamma(0))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0)).$$

Proof. Let $\widetilde{\gamma}_\bullet$ denote the family of ∇-parallel lifts of γ and let $\eta_\bullet$ denote the family of $\nabla \cdot g$-parallel lifts of γ. For example, $\widetilde{\gamma}_v$ is the ∇-parallel lift with $\widetilde{\gamma}_v(0) = v$. By definition $\mathrm{Hol}_\gamma(\nabla)(v) = \widetilde{\gamma}_v(1)$ and $\mathrm{Hol}_\gamma(\nabla \cdot g)(v) = \eta_v(1)$. Consider the action $g^{-1} \cdot \widetilde{\gamma}_\bullet := g^{-1} \circ \widetilde{\gamma}_\bullet : [0, 1] \to E \to E$. The family of lifts $g^{-1} \cdot \widetilde{\gamma}_\bullet$ is $\nabla \cdot g$ parallel if and only if $\gamma_\bullet$ is a ∇-parallel family of lifts:

$$\begin{aligned} 0 &= \gamma^*\nabla(\widetilde{\gamma}_\bullet) = (\gamma^*\nabla \circ g)(g^{-1} \circ \widetilde{\gamma}_\bullet) \\ \Leftrightarrow 0 &= (g^{-1} \circ \gamma^*\nabla \circ g)(g^{-1} \circ \widetilde{\gamma}_\bullet) = (\gamma^*\nabla \cdot g)(g^{-1} \cdot \widetilde{\gamma}_\bullet). \end{aligned}$$

By uniqueness, $\eta_\bullet = g^{-1} \cdot \widetilde{\gamma}_\bullet$ as families. Note that $v = \eta_v(0) = g^{-1}(\widetilde{\gamma}_\bullet(0)) \Leftrightarrow \widetilde{\gamma}_\bullet(0) = g(v)$, i.e. $\eta_v = g^{-1}\widetilde{\gamma}_{gv}$. Then

$$v \mapsto g(v) \mapsto \mathrm{Hol}_\gamma(\nabla)(g(v)) = \widetilde{\gamma}_{g(v)}(1) \mapsto g^{-1}\widetilde{\gamma}_{g(v)}(1) = \eta_v(1) = \mathrm{Hol}_\gamma(\nabla \cdot g)(v)$$

hence $\mathrm{Hol}_\gamma(\nabla \cdot g) = g(\gamma(1))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0)) = g(\gamma(0))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0))$. $\square$

Corollary 3.2.3. The holonomy representation $\mathrm{Hol} : \mathcal{F}(E) \to \mathrm{Hom}\big(\pi_1(M, x), \mathrm{Aut}(E_x)\big)$ descends to a map

$$\mathrm{Hol}: \frac{F(E)}{G(E)} \to \frac{\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)}{\mathrm{Aut}(E_x)} \tag{3.4}$$

where $\mathrm{Aut}(E_x)$ acts on $\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)$ by post-composition with inner automorphisms. Similarly, if (E,h) is a hermitian bundle, then the map

$$\mathrm{Hol}: \frac{F(E,h)}{G(E,h)} \to \frac{\mathrm{Hom}\big(\pi_1(M,x), U(E_x)\big)}{U(E_x)} \tag{3.5}$$

is well-defined. □

Theorem 3.2.4. The holonomy representation gives bijections

$$\mathrm{Hol}: \frac{\mathcal{F}(E)}{G(E)} \leftrightarrow \frac{\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)}{\mathrm{Aut}(E_x)} \tag{3.6}$$

$$\mathrm{Hol}: \frac{\mathcal{F}(E,h)}{G(E,h)} \leftrightarrow \frac{\mathrm{Hom}\big(\pi_1(M,x), U(E_x)\big)}{U(E_x)}. \tag{3.7}$$

Proof. The most efficient manner of proof uses principal bundles and connections which have not been introduced in this dissertation and thus only a sketch will be provided. The idea is to construct an essential inverse which builds a flat vector bundle from a representation $\pi_1(M,x) \to \mathrm{Aut}(E_x)$. For complete details see [12, 14].

Start with a universal cover $\widetilde{M} \to M$. We can use the path model where

$$\widetilde{M} = \{\gamma \in M^{[0,1]} : \gamma(0) = x\} \ni \gamma \mapsto \gamma(1) \in M.$$

Equip $\widetilde{M} \times E_x \to \widetilde{M}$ with the trivial connection. Note that $\widetilde{M} \times E_x \to \widetilde{M} \to M$ has a natural flat connection using the trivial connection on $\widetilde{M} \times E_x \to \widetilde{M}$ along with the homotopy lifting property of the universal cover $\widetilde{M} \to M$.

Given $\rho : \pi_1(M, x) \to \mathrm{Aut}(E_x)$, construct the associated flat vector bundle

$$\widetilde{M} \times_\rho E_x \to M$$

where $\widetilde{M} \times_\rho E_x$ denotes equivalence classes of pairs $(\gamma, v) \in \widetilde{M} \times E_x$ under the relation $(\gamma \cdot \alpha, v) \sim (\gamma, \rho(\alpha)v)$. The result follows from two observations.

The first observation is that a flat vector bundle (E, ∇) is isomorphic to the flat vector bundle $\widetilde{M} \times_{\mathrm{Hol}(\nabla)} E_x$. An isomorphism is given by sending $v \in E_y$ to $(\gamma, P_{\gamma^{-1}}(\nabla)(v))$ where $\gamma \in \widetilde{M}_y$. This is well defined since any other path from x to y can be written, up to homotopy, as $\gamma \cdot \alpha$ where α is a loop based at x. Hence a different choice in $\widetilde{M}_y$ gives $v \mapsto (\gamma \cdot \alpha, P_{(\gamma\cdot\alpha)^{-1}}(\nabla)(v)) = (\gamma \cdot \alpha, P_{\alpha^{-1}}(\nabla) \circ P_{\gamma^{-1}}(\nabla)(v)) \sim (\gamma, \mathrm{Hol}_\alpha(\nabla) \circ P_{\alpha^{-1}}(\nabla) \circ P_{\gamma^{-1}}(\nabla)(v)) = (\gamma, P_{\gamma^{-1}}(\nabla)(v))$.

Now consider $\rho : \pi_1(M, x) \to \mathrm{Aut}(E_x)$. Given a loop α representing an element of $\pi_1(M, x)$, take a parallel lift (γ_t, v) of α in $\widetilde{M} \times_\rho E_x$ that starts at $(\gamma_0, v) \in (\widetilde{M} \times_\rho E_x)$ and ends at (γ_1, v). The second observation is that for all $t \in [0, 1]$, $\gamma_t : [0, 1] \to M$ is a path from x to $\gamma_t(1) = \alpha(t)$ and in particular γ_0 and γ_1 are both loops based at x. In fact, $\gamma_1 \sim \gamma_0 \cdot \alpha$ and parallel transport sends $(\gamma_0, v) \mapsto (\gamma_1, v) \sim (\gamma_0 \cdot \alpha, v) \sim (\gamma_0, \rho(\alpha)(v))$. Thus holonomy in $\widetilde{M} \times_\rho E_x$ is given by ρ. □

Remark. The purpose of the above result is to construct the moduli space as a quotient of the algebraic variety $\mathrm{Hom}(\pi_1(M, x), U(E_x))$. This is the approach taken in [6,10]. The quotient will have singularities at any point $\rho : \pi_1(M, x) \to U(E_x)$ whose stabilizer subgroup is strictly larger than the center $ZU(E_x)$ i.e. non-singular points are precisely the irreducible representations of

$\pi_1(M, x)$.

Definition 3.2.5. A flat unitary connection ∇ on $\pi : E \to M$ is *irreducible* if $\mathrm{Hol}(\nabla) : \pi_1(M, x) \to U(E_x)$ is an irreducible representation.

Remark. An alternative construction of the moduli space begins with the observation 2.4.8 that $\mathcal{A}(E, h)$ is an affine space modeled on $\Omega^1(M; \mathfrak{u}E) \subset \mathrm{Hom}(M, T^\vee M \otimes \mathfrak{u}E)$. One uses Sobolev spaces of connections of class $k > \frac{\dim_{\mathbb{R}} M}{2}$, with respect to the L^2 norm, in order to obtain Hilbert spaces of connections $\mathcal{A}^k(E, h)$ of Sobolev class k. Similarly, $\Gamma(UE) \subset \mathrm{Hom}(M, UE)$ can be completed into Sobolev spaces of appropriate class k resulting in Hilbert Lie groups $G^k(E, h)$ whose corresponding Lie algebras are Sobolev completions of $\Gamma(\mathfrak{u}E) = \Omega^1(M; \mathfrak{u}E)$.

Theorem 3.2.6 (see [13])**.** Let M be a Riemannian manifold of dimension n and $k > \frac{n}{2} + 1$. If $\widetilde{\mathcal{A}}^k(E, h)$ is the class k Sobolev space of *irreducible* unitary connections, and $\widetilde{G}^{k+1}(E, h)$ is the class $k + 1$ Sobolev space completion of $G(E, h)/Z(E, h)$, then the induced action $\widetilde{\mathcal{A}}^k(E, h) \times \widetilde{G}^{k+1}(E, h) \to \widetilde{\mathcal{A}}^k(E, h)$ is free. □

It is helpful to keep the following informal narrative in mind. One considers the space $\widetilde{\mathcal{F}}(E, h)$ of irreducible flat unitary connections. Then $\widetilde{\mathcal{G}}(E, h) \to \widetilde{\mathcal{F}}(E, h) \to \widetilde{\mathcal{M}}(E, h)$ is to be thought of as a principal bundle where $\widetilde{\mathcal{M}}(E, h)$ is the moduli space of irreducible flat unitary connections. From this perspective, vertical tangent vectors are identified with $\Omega^0(M; \mathfrak{u}E)$. If $\nabla \in \widetilde{\mathcal{F}}(E, h)$ is an irreducible flat unitary connection and $\Phi \in \Omega^1(M; \mathfrak{u}E) = T_\nabla \mathcal{A}(E, h)$, then Φ is a tangent vector in the space $\widetilde{\mathcal{F}}(E,h)$ if and only if $[\nabla,\phi] = 0$. Thus $T_{\nabla}\widetilde{\mathcal{F}}(E,h)$ is identified with (a Sobolev completion of) $Z^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$. Since the $\widetilde{\mathcal{G}}(E,h)$ action is free, there is an identification of $T_{[\nabla]}\widetilde{\mathcal{M}}(E,h)$ with $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ using any representative ∇ of the class $[\nabla]$. In other words, $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ is viewed as the tangent space to $[\nabla]$ in the moduli space.

Chapter 4

Unitary bundles over Kähler manifolds

4.1 Pre-symplectic manifolds and Hamiltonian vector fields

Definition 4.1.1. A *symplectic vector space* is a $\mathbb{R}$-vector space V equipped with a non-degenerate 2-form $\omega : V \wedge V \to \mathbb{R}$.

Proposition 4.1.2. A finite symplectic vector space (V, ω) has even dimension.

Proof. If $V \neq \mathbf{0}$ then there is some $x_1 \in V - \{0\}$ and since ω is non-degenerate and alternating there is some $y_1 \notin \mathbb{R}\{x_1\}$ such that $\omega(x_1, y_1) = 1$. Set $W_1 = \mathbb{R}\{x_1, y_1\}$ and define $C_1 = \ker\big(\omega(x_1, -)\big) \cap \ker\big(\omega(y_1, -)\big)$ so that $W_1 \cap C_1 = \{0\}$. For every $z \in V$, it is easily verified that $\big(z - \omega(z, y_1)x_1 + \omega(z, x_1)y_1\big) \in C_1$, and therefore $z = \big(\omega(z, y_1)x_1 - \omega(z, x_1)y_1\big) + \big(z - \omega(z, y_1)x_1 + \omega(z, x_1)y_1\big)$ provides a decomposition $V = W_1 \oplus C_1$. If $C_1 \neq \mathbf{0}$ then there exist $x_2, y_2 \in C_1$ such that $\omega(x_2, y_2) = 1$. Set $W_2 = \mathbb{R}\{x_2, y_2\}$ and define $C_2 = \ker\big(\omega(x_2, -)\big) \cap \ker\big(\omega(y_2, -)\big)$ which ultimately yields $V = W_1 \oplus W_2 \oplus C_2$. Continue by induction and, since V is finite, eventually

$C_k = \{0\}$ hence $V = W_1 \oplus \cdots \oplus W_k$ is even dimensional. $\square$

Remark. By following the above inductive construction, every symplectic vector space of dimension $2n$ can be equipped with a basis $\{x_i, y_i\}_{i=1}^n$ where $\omega(x_i, y_i) = 1$ and all other ω-pairings yield zero.

Definition 4.1.3. A *pre-symplectic manifold* (M, ω) is a real manifold equipped with a closed differential 2-form $\omega \in \Omega^2(M)$. If ω is also non-degenerate then (M, ω) is a *symplectic manifold.*

Remark. If M is symplectic, then by definition each tangent space T_xM is a symplectic vector space, hence is of even dimension. Thus the dimension of a symplectic manifold is necessarily even, say $\dim M = 2n$. Every symplectic manifold M has a canonical top form $\frac{\omega^n}{n!} \neq 0 \in \Omega^{2n}(M)$ so is orientable with a natural orientation.

Proposition 4.1.4. If (M, ω) is a symplectic manifold then $\omega^\flat : TM \to T^\vee M$ is a linear injection. If M is a finite symplectic manifold then $\omega^\flat : TM \cong T^\vee M$.

Proof. $\omega^\flat(X) = \omega(X, -)$ but ω is non-degenerate so that $\ker \omega^\flat = \{0\}$ and therefore $\omega^\flat : TM \to T^\vee M$ is injective. Hence $\omega^\flat : TM \cong T^\vee M$ when M is finite. $\square$

Remark. Note that $\omega^\flat(X) = \iota_X \omega$. When (M, ω) is finite symplectic the inverse of $\omega^\flat$ is denoted $\omega^\sharp : T^\vee M \to TM$.

Definition 4.1.5. Let (M, ω) be a pre-symplectic manifold. A vector field $X \in \Gamma(TM)$ is *Hamiltonian* if $\omega^\flat(X) \in B^1(M)$. In other words, $X \in \Gamma(TM)$ is Hamiltonian if and only if there exists some function $f \in \Omega^0(M)$ such that $df = \omega^\flat(X) = \iota_X(\omega) = \omega(X, -)$. Similarly, a *Hamiltonian function* is a function $f \in \Omega^0(M)$ such that $df \in \omega^\flat\big(\Gamma(TM)\big)$.

Remark. If $X \in \Gamma(TM)$ is a Hamiltonian vector field on a pre-symplectic manifold, we write $f_X \in \Omega^0(M)$ to denote an arbitrary Hamiltonian function corresponding to X.

Lemma 4.1.6. When M is finite dimensional symplectic, every function $f \in \Omega^0(M)$ has a corresponding Hamiltonian vector field $H_f \in \Gamma(TM)$.

Proof. Given $f \in \Omega^0(M)$, define $H_f \in \Gamma(TM)$ by $H_f := \omega^\sharp(df)$. □

Remark. Thus for symplectic manifolds there is a one-to-one correspondence between Hamiltonian vector fields and non-constant functions.

Lemma 4.1.7. Hamiltonian vector fields on a pre-symplectic manifold preserve the pre-symplectic form ω.

Proof. Let $X \in \Gamma(TM)$ be Hamiltonian. Then there exists some $f \in \Omega^0(M)$ such that $\iota_X\omega = df$. Thus $\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega) = d(\iota_X\omega) = d(df) = 0$. □

Proposition 4.1.8. If (M, ω) is pre-symplectic, the subspace $\mathbf{Ham}(M) \subset \Gamma(TM)$ of Hamiltonian vector fields is closed under the Lie bracket of vector fields.

Proof. If $X, Y \in \Gamma(TM)$ are Hamiltonian vector fields then

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega = \mathcal{L}_X\iota_Y\omega = d\iota_X\iota_Y\omega + \iota_X d\iota_Y\omega = d\iota_X\iota_Y\omega$$

which shows that $[X, Y]$ is also Hamiltonian with corresponding Hamiltonian function $\iota_X \iota_Y \omega \in \Omega^0(M)$. $\square$

Proposition 4.1.9. The commutative algebra $\Omega^0(M)$ of functions on a symplectic manifold (M, ω) is a Poisson algebra when endowed with the Poisson bracket defined by

$$\{f, g\} := \omega(H_f, H_g) \tag{4.1}$$

for $f, g \in \Omega^0(M)$.

Remark. Note that $\{f, g\} = \omega(H_f, H_g) = df(H_g) = H_g(f) = \mathcal{L}_{H_g} f$.

Proof. It is immediate that $\{-,-\}$ is skew-symmetric since ω is alternating. For $f, g \in \Omega^0(M)$

$$\begin{aligned} d\{f, g\} &= d\big(\omega(H_f, H_g)\big) = d\big(\iota_{H_g} \iota_{H_f} \omega\big) = d\iota_{H_g}\big(\iota_{H_f}\omega\big) = \mathcal{L}_{H_g}(\iota_{H_f}\omega) \\ &= \mathcal{L}_{H_g}(\iota_{H_f}\omega) + \iota_{H_g}(\mathcal{L}_{H_f}\omega) = \iota_{[H_g, H_f]}\omega = \omega([H_g, H_f], -) \end{aligned}$$

where the first equality on the second line uses that $\mathcal{L}_{H_f}\omega = 0$. Therefore $H_{\{f,g\}} = [H_g, H_f]$ and

$$\begin{aligned} \big\{f, \{g, h\}\big\} &= df(H_{\{g,h\}}) = df[H_h, H_g] = \mathcal{L}_{[H_h, H_g]} f = [\mathcal{L}_{H_h}, \mathcal{L}_{H_g}] f \\ &= \mathcal{L}_{H_h}\mathcal{L}_{H_g} f - \mathcal{L}_{H_g}\mathcal{L}_{H_h} f = \mathcal{L}_{H_h} df(H_g) - \mathcal{L}_{H_g} df(H_h) = \mathcal{L}_{H_h}\{f, g\} - \mathcal{L}_{H_g}\{f, h\} \\ &= \big\{\{f, g\}, h\big\} - \big\{\{f, h\}, g\big\} = \big\{\{f, g\}, h\big\} + \big\{g, \{f, h\}\big\} \end{aligned}$$

which shows that $\{-,-\}$ satisfies the Jacobi identity. Thus $\{-,-\}$ is a Lie bracket on $\Omega^0(M)$.

Finally, since $\omega(H_{gh}, -) = d(gh) = (dg)h + g(dh) = \omega(H_g, -)h + g\omega(H_h, -)$ we see that

$$\{f, gh\} = \omega(H_f, H_{gh}) = \omega(H_f, H_g)h + g\omega(H_f, H_g) = \{f, g\}h + g\{f, h\}$$

hence the bracket is a derivation of the product of functions. □

4.2 Complex manifolds and holomorphic vector bundles

Definition 4.2.1. An *almost complex structure on a smooth manifold* M is a linear operator $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathbf{1}$. An *almost complex manifold* (M, J) is a smooth manifold M equipped with an almost complex strucutre J.

Remark. Since $J^2 = -\mathbf{1}_{TM}$, a (finite) almost complex manifold has even dimension.

Definition 4.2.2. The *complexified tangent bundle of a smooth manifold* M is the $\mathbb{C}$-vector bundle $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C} \to M$.

Definition 4.2.3. Let (M, J) be an almost complex manifold. Define $J^{\mathbb{C}} := J \otimes \mathbf{1}_{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}M)$ as the $\mathbb{C}$-linear extension of J.

Remark. Since $(J^{\mathbb{C}})^2 = -\mathbf{1}_{T^{\mathbb{C}}M}$ the operator $J^{\mathbb{C}}$ has eigenvalues $\pm i := \pm\sqrt{-1}$. The eigenspace (or "eigenbundle") corresponding to eigenvalue $+i$ is denoted by $T^{(1,0)}M \subset T^{\mathbb{C}}M$ and the eigenspace corresponding to eigenvalue $-i$ is $T^{(0,1)}M \subset T^{\mathbb{C}}M$.

Proposition 4.2.4. If (M, J) is an almost complex manifold then $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ and $\overline{T^{(1,0)}M} = T^{(0,1)}M$.

Proof. Every $X \in T^{\mathbb{C}}M$ can be written $X = \frac{1}{2}(X - iJ^{\mathbb{C}}X) + \frac{1}{2}(X + iJ^{\mathbb{C}}X)$ and observe that

$$J^{\mathbb{C}}\big(\frac{1}{2}(X \mp iJ^{\mathbb{C}}X)\big) = \frac{1}{2}\big(J^{\mathbb{C}}X \pm iX\big) = \pm i\frac{1}{2}(X \mp iJ^{\mathbb{C}}X)$$

so that $\frac{1}{2}(X - iJ^{\mathbb{C}}X) \in T^{(1,0)}M$ and $\frac{1}{2}(X + iJ^{\mathbb{C}}X) \in T^{(0,1)}M$. Furthermore $T^{(1,0)}M \cap T^{(0,1)}M = \{0\} \in T^{\mathbb{C}}M$ and therefore $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. This also shows that $Y \in T^{(1,0)}M$ if and only if $Y = \frac{1}{2}(X - iJ^{\mathbb{C}}X)$ for some $X \in T^{\mathbb{C}}M$. Hence $Y \in T^{(1,0)}M$ if and only if $\overline{Y} \in T^{(0,1)}M$. □

Remark. Since $J^{\mathbb{C}}$ is just the $\mathbb{C}$-(bi)linear extension of J, henceforth both operators will be denoted by J. In context it will always be clear whether one is working with TM or with $T^{\mathbb{C}}M$.

Corollary 4.2.5. If (M, J) is an almost complex manifold then the complexified cotangent bundle has a decomposition $(T^{\mathbb{C}}M)^\vee = (T^{(1,0)}M \oplus T^{(0,1)}M) = (T^{(1,0)}M)^\vee \oplus (T^{(0,1)}M)^\vee$. Thus

$$\bigwedge(T^{\mathbb{C}}M)^\vee = \bigoplus_n \wedge^n(T^{\mathbb{C}}M)^\vee = \bigoplus_n \bigoplus_{p+q=n} \wedge^p(T^{(1,0)}M)^\vee \otimes \wedge^q(T^{(0,1)}M)^\vee. \tag{4.2}$$

□

Definition 4.2.6. Let (M, J) be an almost complex manifold. *Differential forms of type* (p, q) are the $\mathbb{C}$-valued differential forms

$$\Omega^{(p,q)}(M) := \Gamma\big(\wedge^p\,(T^{(1,0)}M)^\vee \otimes \wedge^q(T^{(0,1)}M)^\vee\big).$$

Corollary 4.2.7. If (M, J) is an almost complex manifold then the differential graded module of $\mathbb{C}$-valued differential forms has the following (p, q)-decomposition

$$\Omega^\bullet(M, \mathbb{C}) = \bigoplus_n \Omega^n(M, \mathbb{C}) = \bigoplus_n \bigoplus_{p+q=n} \Omega^{(p,q)}(M).$$

□

Definition 4.2.8. A *complex manifold of dimension* n is a smooth manifold M of real dimension $2n$ whose coordinate charts $\phi_U : U \to \mathbb{R}^2 \cong \mathbb{C}^n$ have holomorphic transition functions.

Example 4.2.9. Every complex manifold M is an almost complex manifold. Using local coordinates $\{z_k = x_k + iy_k\}$ for $U \subset M$, $\Gamma(TM|_U)$ has a local frame $\{\partial_{x_k}, \partial_{y_k}\}$. Define J with respect to this coordinate system by $J(\partial_{x_k}) = \partial_{y_k}$ and $J(\partial_{y_k}) = -\partial_{x_k}$ so that $J^2 = -1$.

Definition 4.2.10. The *Nijenhuis tensor* $N_J \in \Omega^2(M, TM)$ *of an almost complex structure* J is defined by

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].$$

Theorem 4.2.11 (Newlander-Nirenberg)**.** An almost complex manifold (M, J) is complex if and only if $N_J = 0$. □

Proof. The proof of this classic result is an extensive detour into analysis beyond the scope of this dissertation. See [15, 18] for a proof when M is assumed analytic or [16] for the original proof.

□

Lemma 4.2.12. $N_J = 0$ if and only if $T^{(1,0)}M$ is integrable.

Proof. Vector fields are in $T^{(1,0)}M$ if and only if they can be written $X - iJX$ where $X \in$

$\Gamma(T^{\mathbb{C}}M)$. Computing

$$\begin{aligned}[X - iJX, Y - iJY] &= [X,Y] - [X, iJY] - [iJX, Y] + [iJX, iJY] \\ &= \Big([X,Y] - [JX, JY]\Big) - i\Big([X, JY] + [JX, Y]\Big)\end{aligned}$$

we see that $[X - iJX, Y - iJY] \in T^{(1,0)}$ if and only if

$$\begin{aligned}[X, JY] + [JX, Y] &= J\Big([X,Y] - [JX, JY]\Big) \\ \Leftrightarrow 0 &= [X, JY] + [JX, Y] - J[X,Y] + J[JX, JY] \\ \Leftrightarrow 0 &= J[X, JY] + J[JX, Y] + [X,Y] - [JX, JY] = N_J(X,Y).\end{aligned}$$

□

Corollary 4.2.13. An almost complex manifold (M, J) is a complex manifold if and only if $T^{(1,0)}M$ is an integrable distribution if and only if $T^{(0,1)}M$ is an integrable distribution. □

Remark. An almost complex manifold is called *integrable* if its almost complex structure comes from a complex structure, i.e. if the almost complex manifold is actually a complex manifold. The terminology is justified by the preceding corollary.

Corollary 4.2.14. An almost complex manifold (M, J) is integrable if and only if $d(\Omega^{(1,0)}(M)) \subset \Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M)$.

Proof. Let $\phi \in \Omega^{(1,0)}M$ and consider $d\phi \in \Omega^2(M, \mathbb{C})$. For arbitrary $X, Y \in \Gamma(T^{(0,1)}M)$ we have

$$d\phi(X,Y) = X\phi(Y) - Y\phi(X) - \phi([X,Y]) = -\phi([X,Y])$$

and therefore $d(\Omega^{(1,0)}(M)) \subset \Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M)$ if and only if $T^{(0,1)}M$ is an integrable distribution. $\square$

Remark. Equivalently, (M, J) is complex if and only if $d(\Omega^{(0,1)}(M)) \subset \Omega^{(1,1)}(M) \oplus \Omega^{(0,2)}(M)$. It follows that (M, J) is complex if and only if $d(\Omega^{(p,q)}(M)) \subset \Omega^{(p+1,q)}(M) \oplus \Omega^{(p,q+1)}(M)$. This implies the following.

Proposition 4.2.15. (M, J) is a complex manifold if and only if the differential $d \in \mathrm{End}^1\big(\Omega^\bullet(M, \mathbb{C})\big)$ on $\mathbb{C}$-valued differential forms can be written $d = \partial + \bar{\partial}$ where $\partial \in \mathrm{End}^{(1,0)}\big(\Omega^\bullet(M, \mathbb{C})\big)$ and $\bar{\partial} \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(M, \mathbb{C})\big)$.

Proof. Use the natural projections $\pi_{(j,k)} : \Omega^n(M, \mathbb{C}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(M) \to \Omega^{(j,k)}(M)$ to define

$$\partial := \pi_{(p+1,q)} \circ d : \Omega^{(p,q)}(M) \to \bigoplus_{a+b=p+q+1} \Omega^{(a,b)}(M) \to \Omega^{(p+1,q)}(M)$$
$$\bar{\partial} := \pi_{(p,q+1)} \circ d : \Omega^{(p,q)}(M) \to \bigoplus_{a+b=p+q+1} \Omega^{(a,b)}(M) \to \Omega^{(p,q+1)}(M).$$

(M, J) is complex if and only if $d(\Omega^{(p,q)}(M)) \subset \Omega^{(p+1,q)}(M) \oplus \Omega^{(p,q+1)}(M)$ which holds if and only if $d = \partial + \bar{\partial}$. $\square$

Remark. If M is a complex manifold, local coordinates $\{z_k = x_k + iy_k\}$ in some neighborhood are holomorphic by definition. Then $dz_k = dx_k + idy_k$ and $d\bar{z}_k := \overline{dz_k} = dx_k - idy_k$ are elements of $\Omega^1(M, \mathbb{C})$. Dual to dz_k and $d\bar{z}_k$ are the elements $\frac{1}{2}\big(\partial_{x_k} - i\partial_{y_k}\big) = \frac{1}{2}\big(\partial_{x_k} - iJ\partial_{x_k}\big) \in T^{(1,0)}M$ and $\frac{1}{2}\big(\partial_{x_k} + iJ\partial_{x_k}\big) \in T^{(0,1)}M$ using that $J(\partial_{x_k}) = \partial_{y_k}$ and $J(\partial_{y_k}) = -\partial_{x_k}$. Thus $\{dz_k\} \subset \Omega^{(1,0)}(M)$

and $\{d\bar{z}_k\} \subset \Omega^{(0,1)}(M)$. For $f \in \Omega^0(M,\mathbb{C})$, $\partial f = \sum_k \partial_{z_k} f$ and $\bar{\partial} f = \sum_k \partial_{\bar{z}_k} f$. Note that

$$\partial_{\bar{z}_k} f = (\partial_{x_k} + i\partial_{y_k})(\Re f + i\Im f) = (\partial_{x_k}\Re f - \partial_{y_k}\Im f) + i(\partial_{x_k}\Im f + \partial_{y_k}\Re f)$$

so that a function f is holomorphic (i.e. satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial} f = 0$. More generally

Definition 4.2.16. If (M, J) is a complex manifold the *holomorphic differential forms* are $\{\phi \in \Omega^{(\bullet,0)}(M) : \bar{\partial}\phi = 0\} \subset \Omega^{(\bullet,0)}(M) \subset \Omega^\bullet(M,\mathbb{C})$.

Definition 4.2.17. A *holomorphic vector bundle* $\pi : E \to M$ is a $\mathbb{C}$-vector bundle over a complex manifold such that the total space E is a complex manifold and π is holomorphic.

Remark. A complex vector bundle over a complex manifold is holomorphic if and only if the transition functions between local trivializations are bi-holomorphic.

Proposition 4.2.18. Let $\pi : E \to M$ be a holomorphic vector bundle. The operator $\bar{\partial} \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(M,\mathbb{C})\big)$ extends to an operator on $\Omega^\bullet(M;E)$.

Proof. In a frame σ for E over $U \subset M$ we set $\bar{\partial}(\phi \otimes \sigma) := (\bar{\partial}\phi) \otimes \sigma$. A different frame τ is given by $\tau = g\sigma$ where g_{ij} are holomorphic functions. Then for $s = \phi \otimes \sigma = \phi \otimes g^{-1}\tau$ we find that

$$\bar{\partial}(\phi \otimes g^{-1}\tau) = \bar{\partial}(\phi g^{-1} \otimes \tau) := \bar{\partial}(\phi g^{-1}) \otimes \tau = (\bar{\partial}\phi)g^{-1} \otimes \tau = \bar{\partial}\phi \otimes g^{-1}\tau = \bar{\partial}\phi \otimes \sigma$$

so $\bar{\partial}$ is well-defined on $\Omega^\bullet(M;E)$. ∎

Definition 4.2.19. If $\pi : E \to M$ is holomorphic, the cochain complex $\left(\Omega^\bullet(M;E), \bar{\partial}\right)$ is the

Dolbeault complex for the holomorphic bundle E and its cohomology yields *Dolbeault cohomology with values in the holomorphic bundle E.*

Definition 4.2.20. A *holomorphic structure* on a complex vector bundle $\pi : E \to M$ over a complex manifold is a differential operator $D'' \in \mathrm{End}^{(0,1)}(\Omega^\bullet(M; E))$ such that (i) $D'' \circ D'' = 0$ and (ii) $D''(fs) = \bar{\partial} f \otimes s + f D''(s)$ for all $(f, s) \in \Omega^0(M, \mathbb{C}) \times \Gamma(E)$.

Proposition 4.2.21. If $\pi : E \to M$ is a holomorphic vector bundle then there exists a connection $\nabla \in \mathcal{A}(E)$ such that $\nabla^{(0,1)} = \bar{\partial} \in \mathrm{End}^{(0,1)}(\Omega^\bullet(M; E))$.

Proof. Give M an open cover and a partition of unity subordinate to this cover. In an open set U, endow $\Gamma(E|_U)$ with a holomorphic frame $\{\sigma\}$ and define $\nabla|_U$ by $\nabla|_U(\sigma) = 0 = \bar{\partial}\sigma$. Use the partition of unity to combine $\nabla|_U$ to a global connection. □

Proposition 4.2.22. A complex vector bundle over a complex manifold is a holomorphic vector bundle if and only if it has a holomorphic structure.

Proof. We have already observed that every holomorphic vector bundle has a holomorphic sturcture given by $\bar{\partial}$. For the converse we follow [11, 15].

Let D'' be a holomorphic structure on $\pi : E \to M$. As usual let $\sigma = (\sigma_1, \ldots, \sigma_r)^T$ be a local frame for $E|_U$ and define $\theta_{ij} \in \Omega^{(0,1)}(U)$ by $D''\sigma_i = \sum_j \theta_{ij} \otimes \sigma_j \in \Omega^{(0,1)}(U; E|_U)$. Thus $D''\sigma = \theta \otimes \sigma$ and

$$0 = D''(D''\sigma) = D''(\theta \otimes \sigma) = \bar{\partial}\theta \otimes \sigma - \theta \wedge D''\theta = \bar{\partial}\theta \otimes \sigma - \theta \wedge \theta \otimes \sigma = \big(\bar{\partial}\theta - \theta \wedge \theta\big) \otimes \sigma$$

and therefore $\bar{\partial}\theta = \theta \wedge \theta$. Endow $\pi^{-1}(U) \to U$ with an almost complex structure $\widetilde{J}$ by declaring $\{dz\} \cup \{d\sigma - \theta\sigma\}$ to be a basis for the holomorphic differential forms $\Omega^{(1,0)}(\pi^{-1}U)$. Here σ is the local frame and z denotes the the local holomorphic coordinates of $U \subset M$. $\widetilde{J}$ is a complex structure if and only if $d\Omega^{(1,0)}(\pi^{-1}U) \subset \Omega^{(2,0)}(\pi^{-1}U) \oplus \Omega^{(1,1)}(\pi^{-1}U)$. Certainly $d(dz) = d(d\sigma) = 0 \in \Omega^{(2,0)}(\pi^{-1}U) \oplus \Omega^{(1,1)}(\pi^{-1}U)$ and

$$
\begin{aligned}
d(d\sigma - \theta\sigma) = -d(\theta\sigma) &= -(d\theta)\sigma + \theta \wedge d\sigma = -(\partial\theta + \bar{\partial}\theta)\sigma + \theta \wedge d\sigma \\
&= -(\partial\theta)\sigma - (\bar{\partial}\theta)\sigma + \theta \wedge d\sigma = -(\partial\theta)\sigma - (\theta \wedge \theta)\sigma + \theta \wedge d\sigma \\
&= -(\partial\theta)\sigma - \theta \wedge (d\sigma - \theta\sigma) \in \Omega^{(1,1)}(\pi^{-1}U)
\end{aligned}
$$

where in the second line we used $\bar{\partial}\theta = \theta \wedge \theta$. So $\widetilde{J}$ is a complex structure and therefore there exists a $\widetilde{J}$-holomorphic frame $\tau = (\tau_1, \ldots, \tau_r)$ for the bundle $\pi^{-1}(U)$. Note that τ gives an explicit trivialization $\pi^{-1}(U) \ni v \leftrightarrow \big(\pi(v), \tau(v)\big) \in U \times \mathbb{C}^r$. Since $d(\tau) = d\tau$ is a vector of holomorphic 1-forms, we can write $d\tau = Adz + B(d\sigma - \theta\sigma)$ for some $A : U \to GL_n(\mathbb{C})$ and $B \in \Gamma(\mathrm{Aut}(E|_U)$. Now

$$
\begin{aligned}
0 = d^2\tau = d\Big(Adz + B(d\sigma - \theta\sigma)\Big) &= dA \wedge dz + dB \wedge (d\sigma - \theta\sigma) - B(d\theta)\sigma + B\theta \wedge d\sigma \in \Omega^2(\pi^{-1}U) \\
= \partial A \wedge dz + \bar{\partial}A \wedge dz &+ \partial B \wedge (d\sigma - \theta\sigma) + \bar{\partial}B \wedge (d\sigma - \theta\sigma) - B(\partial\theta)\sigma - B(\bar{\partial}\theta)\sigma + B\theta \wedge d\sigma
\end{aligned}
$$

can be projected onto forms corresponding under the trivialization to type $\Omega^{(0,1)}(U) \otimes \Omega^1(\mathbb{C}^r)$:

$$0 = \bar{\partial}B \wedge (d\sigma - \theta\sigma) + B\theta \wedge d\sigma.$$

Evaluating this when $\sigma = 0$ gives $0 = \bar{\partial}B \wedge d\sigma + B\theta \wedge d\sigma \Rightarrow \bar{\partial}B + B\theta = 0$. Using the original frame σ, we obtain a new frame $B\sigma$ which is readily seen to be holomorphic since

$$D''(B\sigma) = \bar{\partial}B \otimes \sigma + BD''\sigma = \bar{\partial}B \otimes \sigma + B\theta \otimes \sigma = 0.$$

This allows one to construct local holomorphic frames that are compatible with the differential operator D'' which can be patched together to make E into a holomorphic vector bundle. $\square$

Corollary 4.2.23. If $\pi : E \to M$ is a complex vector bundle over a complex manifold equipped with a connection ∇ that satisfies $\nabla^{(0,1)} \circ \nabla^{(0,1)} = 0$, then there is a unique holomorphic structure such that $\nabla^{(0,1)} = \bar{\partial}$. In particular, every flat connection on E induces a unique holomorphic structure. $\square$

4.3 Chern connections on holomorphic hermitian vector bundles

Definition 4.3.1. Let $(\pi : E \to M, h)$ be a holomorphic hermitian vector bundle. A *Chern connection* is a connection ∇ that is compatible with the metric and that satisfies $\nabla^{(0,1)} = \bar{\partial}$.

Proposition 4.3.2. A Chern connection is unique.

Proof. Assume ∇_1, ∇_2 are Chern connections so that $\nabla_i = \nabla_i^{(1,0)} + \bar{\partial}$. Compatibility with the metric means

$$dh(s,t) = h(\nabla_i s, t) + h(s, \nabla_i t) = h(\nabla_i^{(1,0)} s, t) + h(\bar{\partial} s, t) + h(s, \nabla_i^{(1,0)} t) + h(s, \bar{\partial} t).$$

Since h is Hermitian, $\partial h(s,t) = h(\nabla_i^{(1,0)} s, t) + h(s, \bar{\partial} t)$ and therefore for all $s, t \in \Gamma(E)$

$$0 = h(\nabla_1^{(1,0)} s, t) + h(s, \bar{\partial} t) - h(\nabla_2^{(1,0)} s, t) - h(s, \bar{\partial} t) = h((\nabla_1^{(1,0)} - \nabla_2^{(1,0)}) s, t)$$

hence $\nabla_1^{(1,0)} - \nabla_2^{(1,0)} = 0$. Therefore $\nabla_1 = \nabla_2$. $\square$

Proposition 4.3.3. Every holomorphic hermitian vector bundle $(\pi : E \to M, h)$ has a Chern connection which is necessarily unique.

Proof. Only existence needs to be shown. Working in a local holomorphic frame $\sigma = \{s_i\}$ over U, the metric is given by a section of hermitian matrices $h_{ij} := h(s_i, s_j)$. Write $\nabla^{(1,0)}(s_i) = \sum_j \Theta_{ij} \otimes s_j$ where $\Theta_{ij} \in \Omega^{(1,0)}(U)$. Using that σ is a holomorphic frame,

$$dh(s_i, s_j) = h(\nabla^{(1,0)} s_i, s_j) + h(s_i, \nabla^{(1,0)} s_j) \Rightarrow$$

$$\partial h_{ij} = h(\nabla^{(1,0)} s_i, s_j) = h(\sum_k \Theta_{ik} s_k, s_j) = \sum_k \Theta_{ik} h_{kj}$$

$$\bar{\partial} h_{ij} = h(s_i, \nabla^{(1,0)} s_j) = h(s_i, \sum_k \Theta_{jk} s_k) = \sum_k h_{ik} \bar{\Theta}_{jk} = h_{ik} \bar{\Theta}^T_{kj}.$$

Therefore $\partial h = \Theta h$ and $\bar{\partial} h = h \bar{\Theta}^T$. The first equation implies $\Theta = \partial h h^{-1}$. Therefore

$$\bar{\Theta}^T = (\overline{\partial h h^{-1}})^T = (\overline{h^{-1}})^T (\overline{\partial h})^T = h^{-1} \bar{\partial} h$$

which solves the second equation. So locally the Chern connection is given by $\Theta = \partial h h^{-1}$ which is well-defined by uniqueness: given a different holomorphic frame over U one would get a different representation for the local Chern connection but by uniqueness the two representations yield the same Chern connection. For the same reason, one obtains a global Chern connection by patching together these locally defined Chern connections which necessarily agree on overlaps. $\square$

Proposition 4.3.4. If $(E, h, \nabla) \to M$ is a flat unitary bundle over a complex manifold, then there is a *unique* holomorphic structure on E such that ∇ is the Chern connection.

Proof. Since (E, ∇) is flat, there is a unique holomorphic structure on E such that $\nabla^{(0,1)} = \bar{\partial}$. Since ∇ is compatible with the metric, ∇ is the Chern connection with respect to the holomorphic structure. □

Corollary 4.3.5. For a fixed hermitian vector bundle over a complex manifold, there is a bijection between *the space of flat unitary connections* and *the space of holomorphic structures with flat Chern connections.* □

4.4 Hermitian and Kähler manifolds

Definition 4.4.1. A *hermitian metric on a complex manifold* (M, J) is a hermitian bundle metric h on the holomorphic tangent bundle $T^{(1,0)}M \to M$. A *hermitian manifold* (M, h) is a complex manifold $M = (M, J)$ equipped with a hermitian metric h.

Remark. Hermitian metrics are compatible with the complex structure J by definition since, for all $X, Y \in T^{(1,0)}M$, $h(JX, JY) = h(iX, iY) = -i^2 h(X, Y) = h(X, Y)$ using that h is sesquilinear.

Remark. Since $\overline{T^{(1,0)}M} = T^{(0,1)}M$, every hermitian metric is equivalent to a bundle map $h : T^{(1,0)}M \otimes T^{(0,1)}M \to M \times \mathbb{C}$. It follows that in local holomorphic coordinates $\{z_k = x_k + iy_k\}$ one can write the metric as $h = h_{jk} dz_j \otimes d\bar{z}_k$.

Proposition 4.4.2. Let (M, J, h) be a Hermitian manifold. Then $\Re h$ is a Riemannian metric on M and $\Im h \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$ is non-degenerate.

Proof. It follows from $\Re h(X,Y) + i\Im h(X,Y) = h(X,Y) = \overline{h(Y,X)} = \Re h(Y,X) - i\Im h(Y,X)$ that $\Re h$ is symmetric positive definite and that $\Im h \in \Omega^2(M)$ is non-degenerate. Thus $\Re h$ is a Riemannian metric by definition. Denote the $\mathbb{C}$-extension of $\Im h$ by $\omega \in \Omega^2(M,\mathbb{C})$. If $X, Y \in T^{(1,0)}M \subset T^{\mathbb{C}}M$ then $\omega(X,Y) = \omega(JX,JY) = \omega(iX,iY) = -\omega(X,Y) = 0$, and therefore $\omega \notin \Omega^{(2,0)}(M)$. A similar calculation shows that $\omega \notin \Omega^{(0,2)}(M)$ and therefore $\omega \in \Omega^{(1,1)}(M)$. □

Definition 4.4.3. Let (M,J,h) be a hermitian manifold. The *Riemannian metric of M* is the metric $g := \Re h$ on TM. The *fundamental form of (M,J,h)* is the non-degenerate form $\omega := -\Im h \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$.

Remark. If (M,J,h) is hermitian then

$$2g(X,Y) = 2\Re h(X,Y) = h(X,Y) + \overline{h(X,Y)} = h(X,Y) + h(Y,X)$$

$$2\omega(X,Y) = -2\Im h(X,Y) = i\overline{h(X,Y)} - ih(X,Y) = h(iY,X) + h(X,iY)$$

hence $\omega(X,Y) = g(X,JY)$ and $g(X,Y) = \omega(JX,Y)$. It follows that a hermitian manifold can be defined as a complex manifold (M,J) equipped with a Riemannian metric g such that $g(-,-) = g(J-,J-)$. Then $h = g(-,-) + ig(-,J-)$ defines the corresponding hermitian metric.

Definition 4.4.4. A *Kähler manifold* is a hermitian manifold whose fundamental 2-form is closed. The fundamental form of a Kähler manifold is called the *Kähler form*.

Remark. By definition every Kähler manifold is a symplectic manifold.

Example 4.4.5. Let $\mathbb{C}^n$ be equipped with the hermitian metric $h = \sum_j dz_j \otimes d\bar{z}_j$ where $\{z_j\}$ are standard holomorphic coordinates. Then

$$\omega = -\Im h = \frac{i}{2}(h - \bar{h}) = \frac{i}{2}\Big(\sum_j dz_j \otimes d\bar{z}_j - \sum_j d\bar{z}_j \otimes dz_j\Big) = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$$

and it is immediate that $d\omega = 0$.

Remark. More generally, if (M, J, h) is a hermitian manifold, in any local holomorphic coordinate system $\{z_j\}$ the metric is given by the matrix $h_{jk} := h(\partial_{z_j}, \partial_{z_k})$ and the fundamental form, with respect to the local coordinate system, is $\omega = \frac{i}{2}\sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$. The following fact shows that the Kähler condition $d\omega = 0$ allows for a more refined local picture.

Proposition 4.4.6. A hermitian manifold (M, J, h) is Kähler if and only if every point has a local holomorphic coordinate system such that the metric h agrees with the standard metric on $\mathbb{C}^n$ to order 2.

Proof. Given $x \in M$ let $\{z_j\}$ be local holomorphic coordinates of $U \ni x$ so that $\omega = \frac{i}{2}\sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$ and therefore

$$d\omega = \frac{i}{2}\sum_{jk}\Big(\sum_{p \neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k + \sum_{q \neq k}(\partial_{\bar{z}_q} h_{jk}) d\bar{z}_q \wedge dz_j \wedge d\bar{z}_k\Big).$$

$d\omega = 0$ if and only if both $\sum_{jk}\sum_{p\neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k = 0 \in \Omega^{(2,1)}(U)$ and $\sum_{jk}\sum_{q\neq k}(\partial_{\bar{z}_q} h_{jk}) d\bar{z}_q \wedge dz_j \wedge d\bar{z}_k = 0 \in \Omega^{(1,2)}(U)$. Furthermore, since $\{dz_p \wedge dz_j \wedge d\bar{z}_k\}$ is a basis of $\Omega^{(2,1)}(U)$, $0 = \sum_{jk}\sum_{p\neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k$ if and only if $\partial_{z_p} h_{jk} = 0$. Similarly, looking at $\Omega^{(1,2)}(U)$ one finds that$\partial_{\bar{z}_p} h_{jk} = 0$. Thus $d\omega = 0$ if and only if $\partial_{z_p} h_{jk} = 0$ for all $p \neq j$ and $\partial_{\bar{z}_p} h_{jk} = 0$ for all $q \neq k$. □

4.5 Inner products on differential forms

If V is a $\mathbb{R}$ vector space with $\dim V = n < \infty$ and equipped with a metric $\langle -, -, \rangle$ and a volume form dV, the star operator $* : \wedge^k V \to \wedge^{n-k} V$ is the linear map defined by $a \wedge *b = \langle a, b \rangle dV$. Working with a basis one readily verifies that $*^2 = (-1)^{k(n-k)} : \wedge^k V \to \wedge^k V$.

More generally, let M be a *closed* manifold with Riemannian metric $g : TM \otimes TM \to M \times \mathbb{R}$, volume form dM, and $\dim_{\mathbb{R}} M = n$. The metric g induces a metric, also written g, on $\wedge^k (TM)^\vee$ for all k. If $\phi, \psi \in \Omega^k(M, \mathbb{R}) = \Gamma(\wedge^k (TM)^\vee)$ then $g(\phi, \psi) \in \Omega^0(M)$.

Definition 4.5.1. The metric on $\Omega^k(M, \mathbb{R})$ is given by $\langle \phi, \psi \rangle_{\mathbb{R}} = \int_M g(\phi, \psi) dM$.

Definition 4.5.2. The *Hodge star operator on $\mathbb{R}$-valued differential forms (with respect to the metric g)* is the linear operator $* : \Omega^k(M, \mathbb{R}) \to \Omega^{n-k}(M, \mathbb{R})$ defined by

$$\phi \wedge *\psi = g(\phi, \psi) dM \in \Omega^n(M, \mathbb{R}). \tag{4.3}$$

Lemma 4.5.3. $*^{-1} = (-1)^{k(n-k)} * : \Omega^{n-k}(M) \to \Omega^k(M)$.

Proof. $*^2 = (-1)^{k(n-k)} : \Omega^k(M) \to \Omega^k(M)$. □

Definition 4.5.4. The *formal adjoint of* $A \in \mathrm{End}\big(\Omega^\bullet(M, \mathbb{R})\big)$ is the operator A^* defined by $\langle A\Phi, \Psi \rangle_{\mathbb{R}} = \langle \Phi, A^* \Psi \rangle_{\mathbb{R}}$ with respect to the inner product on $\Omega^\bullet(M, \mathbb{R})$.

Remark. A formal adjoint is unique since $\langle A\Phi, \Psi \rangle = \langle \Phi, A^*\Psi \rangle_{\mathbb{R}} = \langle \Phi, A'\Psi \rangle_{\mathbb{R}} \Rightarrow \big\langle \Phi, (A^* - A')\Psi \big\rangle_{\mathbb{R}} = 0 \Rightarrow A^* = A'$.

Proposition 4.5.5. The formal adjoint to $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is

$$d^* = (-1)^{nk+1} * d * .$$

Proof. For $\phi, \psi \in \Omega^k(M)$ using Stokes' theorem

$$\begin{aligned}\langle d\phi, \psi\rangle &= \int_M d\phi \wedge *\psi = -(-1)^k \int_M \phi \wedge d(*\psi) = -(-1)^{k+k(n-k)} \int_M \phi \wedge *^2(d * \psi) \\ &= -(-1)^{k+k(n-k)} \langle \phi, *d * \psi\rangle = (-1)^{nk+1} \langle \phi, *d * \psi\rangle.\end{aligned}$$

□

Definition 4.5.6. The metric g on $\wedge^k(TM)^\vee$ can be extended to give a hermitian metric $g_{\mathbb{C}}$ on $\wedge^k(T^{\mathbb{C}}M)^\vee$ by $\mathbb{C}$-(bi)linearly extending the metric g and defining $g_{\mathbb{C}}(\phi, \psi) := g(\phi, \bar{\psi})$. The Hodge star operator can be $\mathbb{C}$-linearly extended to yield the *Hodge star operator for* $\Omega^k(M, \mathbb{C})$ and is characterized by

$$\phi \wedge *\bar{\psi} = g_{\mathbb{C}}(\phi, \psi) dM \in \Omega^{\dim_{\mathbb{R}} M}(M, \mathbb{C}).$$

The inner product on $\Omega^\bullet(M, \mathbb{C})$ is $\langle \phi, \psi\rangle = \displaystyle\int_M g_{\mathbb{C}}(\phi, \psi) dM = \int_M \phi \wedge *\bar{\psi}$.

Remark. Note if (M, J) is a complex manifold with $\dim_{\mathbb{R}} M = 2n$, then $* : \Omega^k(M, \mathbb{C}) \to \Omega^{2n-k}(M, \mathbb{C})$ must satisfy $* : \Omega^{(p,q)}(M) \to \Omega^{(n-q,n-n)}(M)$.

Corollary 4.5.7. If (M, J) is a complex manifold equipped with a Riemannian metric g so that both $\Omega^k(M, \mathbb{R})$ and $\Omega^k(M, \mathbb{C})$ are metric spaces as described above, then

$$\partial^* = - * \bar{\partial} *$$

$$\bar{\partial}^* = - * \partial *$$

where $d = \partial + \bar{\partial}$.

Proof. The differential d on $\Omega^\bullet(M, \mathbb{C})$ is the $\mathbb{C}$-linear extension of the differential on $\Omega^\bullet(M, \mathbb{R})$. Therefore $d^* = -{*}d{*}$ on both real and complex forms. Hence $(\partial+\bar{\partial})^* = -{*}(\partial+\bar{\partial}){*}$. Comparing bi-degrees, $\partial^* = -*\bar{\partial}*$ and $\bar{\partial}^* = -*\partial*$. $\square$

Definition 4.5.8. Let $(E, h) \to M$ be a hermitian vector bundle over a closed complex manifold with underlying Riemannian metric g and $\dim_{\mathbb{C}} M = n$. Then

$$g_h(\phi \otimes s, \psi \otimes t) := g_{\mathbb{C}}(\phi, \psi) h(s, t) \tag{4.4}$$

is a hermitian inner product on $\wedge^k (T^{\mathbb{C}} M)^\vee \otimes E \to M$. The *hermitian inner product on* $\Omega^k(M; E)$ is

$$\langle \phi \otimes s, \psi \otimes t \rangle_E = \int_M g_{\mathbb{C}}(\phi, \psi) h(s, t) dM. \tag{4.5}$$

Recall the metric h on E indues a conjugate linear bundle isomorphism $\flat = h^\flat : E \to E^\vee$ by $v^\flat := h(-, v)$ since h is $\mathbb{C}$-conjugate linear in the second factor. The *Hodge star* operator for $\Omega^k(M; E)$ is the $\mathbb{C}$-conjugate linear map $\star : \Omega^k(M; E) \to \Omega^{2n-k}(M; E^\vee)$ defined by

$$\mathrm{ev}(\Phi \wedge \star\Psi) = g_h(\Phi, \Psi) dM \in \Omega^{2n}(M, \mathbb{C}) \tag{4.6}$$

where $\mathrm{ev}(\alpha \otimes s \wedge \beta \otimes f) := \alpha \wedge \beta f(s) \in \Omega^\bullet(M, \mathbb{C})$.

Proposition 4.5.9. $\star(\psi \otimes t) = *\bar{\psi} \otimes t^\flat$.

Proof. For the moment write $\star(\psi \otimes t) = \star\psi \otimes \star t$ and then deduce what these symbols mean.

$$(\phi \wedge *\bar{\psi}) t^\flat(s) = g_{\mathbb{C}}(\phi, \psi) h(s, t) dM = g_h(\phi \otimes s, \psi \otimes t) dM = \mathrm{ev}\big(\phi \otimes s \wedge \star(\psi \otimes t)\big) = (\phi \wedge \star\psi)(\star t) s$$

shows that $\star\psi = *\bar{\psi}$ and $(\star t)s = t^\flat(s)$. $\square$

Remark. One should write $\langle -, - \rangle_E$ and $\star_E$ when there is ambiguity about the hermitian vector bundle E. It follows from above that $\star_{E^\vee}(\psi \otimes f) = *\bar{\psi} \otimes f^\sharp$.

Corollary 4.5.10. $\star_{E^\vee} \circ \star_E = (-1)^k : \Omega^k(M; E) \to \Omega^k(M; E)$ and $\star_E \circ \star_{E^\vee} = (-1)^k : \Omega^k(M; E^\vee) \to \Omega^k(M; E^\vee)$. $\square$

Proposition 4.5.11. The formal adjoint to a unitary connection $\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$ is

$$\nabla^* = -1 \star_{E^\vee} \nabla^\vee \star_E \tag{4.7}$$

where $\nabla^\vee$ is the dual unitary connection on the hermitian bundle $(E^\vee, h^\vee) \to M$.

Proof. Let $\Phi \in \Omega^k(M; E)$ and $\Psi \in \Omega^{k+1}(M; E)$. By definition of $h^\vee$ [see 2.5],

$$d\mathrm{ev}(\Phi \wedge \star_E \Psi) = \mathrm{ev}\big(\nabla\Phi \wedge \star_E \Psi\big) + (-1)^k \mathrm{ev}\big(\Phi \wedge \nabla^\vee(\star_E \Psi)\big)$$

and since M is without boundary Stokes' theorem implies

$$\mathrm{ev}\big(\nabla\Phi \wedge \star_E \Psi\big) = -(-1)^k \mathrm{ev}\big(\Phi \wedge \nabla^\vee(\star_E \Psi)\big) = -\mathrm{ev}\big(\Phi \wedge \star_E \star_{E^\vee} \nabla^\vee(\star\Psi)\big)$$

$$\Rightarrow \langle \nabla\Phi, \Psi \rangle = \langle \Phi, -1 \star_{E^\vee} \nabla^\vee \star_E \Psi \rangle.$$

$\square$

Corollary 4.5.12. If $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ is unitary, then

$$\nabla^{(1,0)*} = - \star_{E^\vee} \nabla^{(0,1)} \star_E$$

$$\nabla^{(0,1)*} = - \star_{E^\vee} \nabla^{(1,0)} \star_E .$$

Proof. Use that $-1 \star_{E^\vee} \nabla \star_E = (\nabla)^* = \nabla^{(1,0)*} + \nabla^{(0,1)*}$ and compare bi-degrees. $\square$

Remark. For ease of notation, for the remainder of this chapter we will write $\nabla^{10} := \nabla^{(1,0)}$ and $\nabla^{01} := \nabla^{(0,1)}$.

4.6 Hermitian bundles over Kähler manifolds

Let $(E, h) \to M$ be a hermitian vector bundle over a closed Kähler manifold. Then $g := \Re h$ is a Riemannian metric and $\langle -, - \rangle_E : \Omega^\bullet(M; E) \otimes \overline{\Omega^\bullet(M; E)} \to \mathbb{C}$ is the hermitian inner product as described in 4.5.

Definition 4.6.1. Assume $\dim_{\mathbb{R}} M = 2n$. Define the following operators on $\Omega^\bullet(M; E)$

$$\mathrm{End}^2\big(\Omega^\bullet(M; E)\big) \ni L : \Phi \mapsto \omega \wedge \Phi$$

$$\mathrm{End}^{-2}\big(\Omega^\bullet(M; E)\big) \ni \Lambda := L^*$$

$$\mathrm{End}\big(\Omega^\bullet(M; E)\big) \ni \Pi := \sum_k (n-k)\pi_k$$

where π_k is projection $\Omega^\bullet(M; E) \to \Omega^k(M; E)$. Thus $\Pi|_{\Omega^k(M;E)} = (n-k)\mathbf{1}|_{\Omega^k(M;E)}$. $\Lambda = L^*$ is the formal adjoint of L with respect to $\langle -, - \rangle_E$.

Proposition 4.6.2. For every unitary connection $\nabla \in \mathcal{A}(E, h)$

$$[L, \nabla] = [L, \nabla^{10}] = [L, \nabla^{01}] = 0$$

$$[\Lambda, \nabla^*] = [\Lambda, \nabla^{10*}] = [\Lambda, \nabla^{01*}] = 0.$$

Proof. $[L, \nabla]\Phi = \omega \wedge \nabla(\Phi) - \nabla(\omega \wedge \Phi) = \omega \wedge \nabla(\Phi) - d\omega \wedge \Phi - \omega \wedge \nabla(\Phi) = 0$ since $d\omega = 0$. Therefore $[L, \nabla^{10}] + [L, \nabla^{01}] = 0$. But $[L, \nabla^{10}]$ is an operator of bi-degree $(2, 1)$ whereas $[L, \nabla^{01}]$ has bi-degree $(1, 2)$ so it must be that $[L, \nabla^{10}] = [L, \nabla^{01}] = 0$. Passing to formal adjoints gives $[\Lambda, \nabla^*] = [\Lambda, \nabla^{10*}] = [\Lambda, \nabla^{01*}] = 0$. □

Lemma 4.6.3. $[\Lambda, L] = \Pi$.

Proof. We proceed following [7] by working in a local holomorphic coordinate system $\{z_k\}$. First note that $||dz_k|| = \langle dx_k + idy_k, dx_k + idy_k\rangle = ||dx_k|| + ||dy_k|| = 2$. Define $e_k \in \mathrm{End}^{(1,0)}\big(\Omega^\bullet(U; E)\big)$ by $e_k\phi = dz_k \wedge \phi$ and define $\bar{e}_k \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(U; E)\big)$ similarly. Let f_k and $\bar{f}_k$ be the formal adjoints to e_k and and $\bar{e}_k$ respectively. Then $\langle f_k dz_A \wedge d\bar{z}_B, \phi\rangle = \langle dz_A \wedge d\bar{z}_B, dz_k \wedge \phi\rangle$ is zero for $k \notin A$ and thus $f_k dz_A \wedge d\bar{z}_B = 0$ for $k \notin A$. Also $\langle f_k dz_k \wedge dz_A \wedge d\bar{z}_B, \phi\rangle = \langle dz_k \wedge dz_A \wedge d\bar{z}_B, dz_k \wedge \phi\rangle = ||dz_k||\langle dz_A \wedge d\bar{z}_B, \phi\rangle$ and therefore $f_k dz_k \wedge dz_A \wedge d\bar{z}_B = 2dz_A \wedge d\bar{z}_B$. Similarly we find that $\bar{f}_k dz_A \wedge d\bar{z}_B = 0$ for $k \notin B$ and that $\bar{f}_k d\bar{z}_k \wedge dz_A \wedge d\bar{z}_B = 2dz_A \wedge d\bar{z}_B$. It follows that

$$f_k e_k + e_k f_k = 2$$

$$e_j f_k + f_k e_j = 0 \text{ for } j \neq k$$

$$\bar{e}_j f_k + f_k \bar{e}_j = 0 \text{ for } j \neq k$$

and similar expressions obtained by taking conjugates.

Locally $\omega \equiv \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$ modulo order 2 terms so $L \equiv \frac{i}{2}\sum_j e_j\bar{e}_j$ and $\Lambda = L^* \equiv -\frac{i}{2}\sum_k \bar{f}_k f_k$ and therefore

$$4[\Lambda, L] \equiv \sum_{j\neq k} \bar{f}_k f_k e_j \bar{e}_j - e_j\bar{e}_j\bar{f}_k f_k + \sum_j \bar{f}_j f_j e_j\bar{e}_j - e_j\bar{e}_j\bar{f}_j f_j.$$

Note that $\bar{f}_k(f_k e_j)\bar{e}_j = -(\bar{f}_k e_j)(f_k\bar{e}_j) = -e_j(\bar{f}_k\bar{e}_j)f_k = e_j\bar{e}_j\bar{f}_k f_k$. Also

$$\bar{f}_j f_j e_j\bar{e}_j = \bar{f}_j(2 - e_j f_j)\bar{e}_j = 2\bar{f}_j\bar{e}_j - \bar{f}_j e_j f_j\bar{e}_j$$

$$e_j\bar{e}_j\bar{f}_j f_j = e_j(2 - \bar{f}_j\bar{e}_j)f_j = 2e_j f_j - e_j\bar{f}_j\bar{e}_j f_j = 2e_j f_j - \bar{f}_j e_j f_j\bar{e}_j$$

$$\Rightarrow \bar{f}_j f_j e_j\bar{e}_j - e_j\bar{e}_j\bar{f}_j f_j = 2\bar{f}_j\bar{e}_j - 2e_j f_j$$

and therefore

$$[\Lambda, L] = \frac{1}{2}\sum_j \bar{f}_j\bar{e}_j - e_j f_j = \frac{1}{2}\sum_j \Big(2 - \bar{e}_j\bar{f}_j - e_j f_j\Big) = n - \frac{1}{2}\sum_j \left(\bar{e}_j\bar{f}_j + e_j f_j\right).$$

Now $e_j f_j dz_A \wedge d\bar{z}_B = \begin{cases} 0 \text{ if } j \notin A \\ 2dz_A \wedge d\bar{z}_B \text{ else} \end{cases}$ and similarly $\bar{e}_j\bar{f}_j dz_A \wedge d\bar{z}_B = \begin{cases} 0 \text{ if } j \notin B \\ 2dz_A \wedge d\bar{z}_B \text{ else} \end{cases}$. Therefore for $\phi \in \Omega^{(p,q)}(U)$ we have $[\Lambda, L]\phi = \big(n - (p+q)\big)\phi = \Pi\phi$.

It follows that $[\Lambda, L] = \Pi$ on $\Omega^\bullet(M; M \times \mathbb{C})$. The result holds for an arbitrary vector bundle over $E \to M$ since the operator L, and therefore the operator Λ, acts on $\Omega^\bullet(M; E) = \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Gamma(E)$ solely through its action on $\Omega^\bullet(M) = \Omega^\bullet(M; M \times \mathbb{C})$. See also [1].

$\square$

4.7 Representations of $\mathfrak{s}l(2,\mathbb{C})$ and hard Lefschetz property

The special linear group is $SL(2,\mathbb{F}) := \{A \in GL(2,\mathbb{F}) : \det(A) = 1\}$ and its Lie algebra is $\mathfrak{s}l(2,\mathbb{F}) = T_{\mathbf{1}}SL(2,\mathbb{F}) = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbb{F})\} = \{X \in \mathrm{Mat}_{2\times 2} : \mathbf{Tr}(X) = 0\}$. In particular, $\mathfrak{s}l(2,\mathbb{F})$ is generated over $\mathbb{F}$ by the matricies $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which satisfy the relations $[D,A] = 2A$, $[D,B] = -2B$ and $[A,B] = D$.

Lemma 4.7.1. Let $\rho : \mathfrak{s}l(2,\mathbb{C}) \to \mathrm{End}(V)$ be a Lie algebra representation. If (v,λ) is an eigenpair for D (i.e. $Dv := \rho(D)v = \lambda v$) then $(Av, \lambda+2)$ and $(Bv, \lambda-2)$ are also eigenpairs for D.

Proof.

$$D(Av) = DAv - ADv + ADv = [D,A]v + ADv = 2Av + A(\lambda v) = (2+\lambda)Av$$

$$D(Bv) = DBv - BDv + BDv = [D,B]v + BDv = -2Bv + B(\lambda v) = (-2+\lambda)Bv.$$

□

Definition 4.7.2. Let $\rho : \mathfrak{s}l(2,\mathbb{C}) \to \mathrm{End}(V)$ be a representation. An eigenvector $v \in V$ for $D = \rho(D)$ is called *primitive* if $Av := \rho(A)v = 0$.

Lemma 4.7.3. If v is primitive with $Dv = \lambda v$, then $AB^k v = (k\lambda - k^2 + k)B^{k-1}v$.

Proof. Notice that

$$ABv = ABv - BAv + BAv = Dv + BAv = \lambda v$$

$$AB^2v = (AB - BA)Bv + BABv = D(Bv) + B(ABv) = (\lambda - 2)Bv + B(\lambda v) = \big((\lambda - 2) + \lambda\big)Bv$$

$$AB^3v = (AB - BA)B^2v + BAB^2v = D(B^2v) + B(AB^2v) = (\lambda - 4)B^2v + B^2\big((\lambda - 2) + \lambda\big)v$$

$$= \big((\lambda - 4) + (\lambda - 2) + \lambda\big)B^2v.$$

Assume that $AB^{k-1}v = \big((\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-2}v$ (which holds for $k \leq 4$) and proceed inductively:

$$\begin{aligned} AB^kv &= (AB - BA)B^{k-1}v + BAB^{k-1}v = D(B^{k-1}v) + B(AB^{k-1}v) \\ &= (\lambda - 2(k-1))B^{k-1}v + B\big((\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-2}v \\ &= \big((\lambda - 2k + 2) + (\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-1}v \\ &= \frac{k\big(\lambda - 2(k-1) + \lambda\big)}{2}B^{k-1}v = (k\lambda - k^2 + k)B^{k-1}v. \end{aligned}$$

□

Proposition 4.7.4. Let $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ be an *irreducible* representation and let $v \in V$ be primitive. Then V is generated by $\{v, Bv, B^2v, \dots\}$.

Proof. Let S be the span of $\{v, Bv, B^2v, \dots\}$ so $BS \subset S$ by construction. Each B^kv is an eigenvector of D and therefore $DS \subset S$. Lastly, $A(B^kv) = (k\lambda - k^2 + k)B^{k-1}v \subset S$, hence $AS \subset S$. Thus ρ preserves the subspace $S \subset V$ but since ρ is assumed irreducible, it must be that $S = V$. □

Remark. If v is primitive with eigenvalue λ of D, then each non-zero $B^k v \in V$ is also an eigenvector of D with eigenvalue $\lambda - 2k$. Thus $\{B^k v\}$ is a linearly independent set of vectors in, hence a basis of, V. This gives $V = \oplus_\lambda V_\lambda$ where each $V_\lambda \simeq \mathbb{C}$ is an 1-dimensional D-eigenspace. Furthermore $D(V_\lambda) = V_\lambda$, $A(V_\lambda) = V_{\lambda+2}$ and $B(V_\lambda) = V_{\lambda-2}$.

Proposition 4.7.5. Let $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ be a finite irreducible representation with v primitive. Then the eigen-decomposition is $V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$ for some $n \in \mathbb{N}$.

Remark. This says that the eigenvalues are all integers, distributed symmetrically about zero.

Proof. Let λ be the D-eigenvalue of v. $\{B^k v\} \subset V$ has only finitely many non-zero terms since V is finite. Let n be such that $B^n v \neq 0$ and $B^{n+1} v = 0$. Then

$$0 = AB^{n+1}v = ((n+1)\lambda - (n+1)^2 + n + 1)B^n v = (n+1)(\lambda - n)B^n v \Rightarrow \lambda = n \in \mathbb{N}.$$

The largest D-eigenvalue is n corresponding to v and the smallest eigenvalue is $n - 2n = -n$ corresponding to $B^n v$ and therefore $V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$. □

Theorem 4.7.6. If $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ is a finite representation, then $\rho(B)^\lambda : V_\lambda \to V_{-\lambda}$ is an isomorphism for all positive eigenvalues λ of $\rho(D)$.

Proof. $\mathfrak{sl}(2,\mathbb{C})$ is a complex simple Lie algebra thus every finite representation of $\mathfrak{sl}(2,\mathbb{C})$ is a direct sum of irreducible representations [4]. Say $V = \oplus_{i=1}^d V^i$ with corresponding $\rho_i : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V^i)$ is a decomposition of ρ into irreducible components. For each V^i there exists a $v_i \in V^i$ such that $Dv_i = n_i v_i$, $Av_i = 0$ and $V^i = V^i_{-n_i} \oplus \cdots \oplus V^i_{n_i}$ where $V^i_{n_i - 2k} = \mathbb{C}\{B^k v_i\}_{k=1}^{d_i}$ is the eigenspace of V^i with eigenvalue $n_i - 2k$. Note that $A : V^i_{n_i-2k} \leftrightarrows V^i_{n_i-2k+2} : B$ gives an isomorphism between one dimensional vector spaces. Thus we can eigen-decompose along the irreducible components

$$V = \bigoplus_{i=1}^{d} V^i = \bigoplus_{i=1}^{d} \bigoplus_{k=0}^{n_i} V^i_{-n_i+2k}.$$

Of course we no longer have $A : V_m = \oplus_{i=1}^{d} V^i_m \to \oplus_{i=1}^{d} V^i_{m+2}$ is an isomorphism because some eigenspaces may be trivial, i.e. the dimensions of neighboring eigenspaces can differ. But since each V^i has an eigen-decomposition symmetric about zero, we necessarily have $\dim V_{-m} = \dim \oplus_{i=1}^{d} V^i_{-m} = \dim \oplus_{i=1}^{d} V^i_m = \dim V_m$.

For $m > 0$, $v \in V_m$ if and only if $v = B^k w$ for some $k \in \mathbb{N}$ and some $w \in V$ such that $Dw = (m + 2k)w$. Then $B^m v = B^{m+k} w$ so that $D(B^m v) = -mv$. Thus if $v \neq 0$, then $D(B^m v) \neq 0$ which implies that $B^m v \neq 0$. Therefore $B^m : V_m \to V_{-m}$ is injective and thus bijective. □

Proposition 4.7.7. There is a representation $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}\big(\Omega^\bullet(M; E)\big)$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \Pi \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Lambda \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto L.$$

Proof. $[\Lambda, L] = \Pi$ by 4.6.3. For all $\Phi \in \Omega^k(M; E)$

$$[\Pi, L]\Phi = \Pi(L\Phi) - L(\Pi\Phi) = (n - k - 2)L(\Phi) - L\big((n - k)\Phi\big) = -2L(\Phi)$$

$$[\Pi, \Lambda]\Phi = \Pi(\Lambda\Phi) - \Lambda(\Pi\Phi) = (n - k + 2)\Lambda(\Phi) - \Lambda\big((n - k)\Phi\big) = 2\Lambda(\Phi).$$

□

4.8 Kähler identities

Lemma 4.8.1. Let M be a Kähler manifold. Then $[\Lambda, \partial] = i\bar{\partial}^* \in \operatorname{End}\big(\Omega^\bullet(M, \mathbb{C})\big)$.

Proof. Following [7] let $e_j, \bar{e}_j, f_j, \bar{f}_j$ be as in 4.6.3 with respect to some local holomorphic coordinate system for M so that locally we have $L = \frac{i}{2}\sum_j e_j\bar{e}_j$ and $\Lambda = -\frac{i}{2}\sum_j \bar{f}_j f_j$. Write $\partial_k := \frac{\partial}{\partial z_k}$ and $\bar{\partial}_k := \frac{\partial}{\partial \bar{z}_k}$ which are defined to act on functions so that $\partial = \sum_k \partial_k e_k$ and $\bar{\partial} = \sum_k \bar{\partial}_k \bar{e}_k$. Note that ∂_k and $\bar{\partial}_k$ commute with all of the elementary operators $e_j, f_j, \bar{e}_j, \bar{f}_j$.

For all functions s, t on M we have $\partial\langle s, t\rangle = \langle \partial s, t\rangle + \langle s, \bar{\partial} t\rangle$ hence by Stokes' theorem $\partial_k^* = -\bar{\partial}_k$ and $\bar{\partial}_k^* = -\partial_k$. This gives $\bar{\partial}^* = \sum_k \bar{f}_k \bar{\partial}_k^* = \sum_k \bar{\partial}_k^* \bar{f}_k = -\sum_k \partial_k \bar{f}_k$ and therefore

$$\begin{aligned}
\Lambda\partial &= \frac{i}{2}\sum_{jk} \bar{f}_k f_k \partial_j e_j = \frac{i}{2}\sum_{jk} \partial_j \bar{f}_k f_k e_j = \frac{i}{2}\Big(\sum_k \partial_k \bar{f}_k f_k e_k + \sum_{j\neq k} \partial_j \bar{f}_k f_k e_j\Big) \\
&= \frac{i}{2}\Big(\sum_k \partial_k \bar{f}_k (2 - e_k f_k) + \sum_{j\neq k} \partial_j e_j \bar{f}_k f_k\Big) = i\sum_k \partial_k \bar{f}_k + \frac{i}{2}\Big(\sum_k \partial_k e_k \bar{f}_k f_k) + \sum_{j\neq k} \partial_j e_j \bar{f}_k f_k\Big) \\
&= i\sum_k \partial_k \bar{f}_k + \frac{i}{2}\sum_{jk} \partial_j e_j \bar{f}_k f_k = \Big(i\sum_k \partial_k \bar{f}_k\Big) + \partial\Lambda = i\bar{\partial}^* + \partial\Lambda \Rightarrow [\Lambda, \partial] = i\bar{\partial}^*.
\end{aligned}$$

See also [1]. □

Corollary 4.8.2. If M is Kähler then $[\Lambda, \bar{\partial}] = -i\partial^*$, $[L, \partial^*] = i\bar{\partial}$, and $[L, \bar{\partial}^*] = -i\partial$.

Proof. Recall that $\omega \in \Omega^{(1,1)}(M) \cap \Omega^2(M, \mathbb{R})$ hence $\bar{L} = L$ and $\bar{\Lambda} = \Lambda$. Taking the conjugate of $[\Lambda, \partial] = i\bar{\partial}^*$ gives $[\Lambda, \bar{\partial}] = -i\partial^*$; taking the adjoint gives $[L, \partial^*] = i\bar{\partial}$ the conjugate of which gives $[L, \bar{\partial}^*] = -i\partial$. □

Proposition 4.8.3. Let $(E, h) \to M$ be a holomorphic hermitian vector bundle over a closed Kähler manifold with flat Chern connection $\nabla = \nabla^{10} + \nabla^{01} = \nabla^{10} + \bar{\partial}$. Then the *Kähler* *identities* hold for operators on $\Omega^\bullet(M;E)$:

$$[L, \nabla^{10*}] = i\nabla^{01}$$

$$[L, \nabla^{01*}] = -i\nabla^{10}$$

$$[\Lambda, \nabla^{10}] = i\nabla^{01*}$$

$$[\Lambda, \nabla^{01}] = -i\nabla^{10*}.$$

Proof. Locally we can give flat holomorphic frames; the result follows from the previous result for operators on $\Omega^\bullet(M,\mathbb{C})$. See also [1]. □

4.9 Hard Lefschetz for unitary bundles over Kähler manifolds

As usual $(E,h) \to M$ is a hermitian vector bundle over a closed Kähler manifold with $\dim_{\mathbb{C}} M = n$. $\Omega^\bullet(M;E)$ has the hermitian inner product $\langle -,-\rangle_E$ as defined in 4.5 which allows us to define the formal adjoint of a linear operators on $\Omega^\bullet(M;E)$.

Definition 4.9.1. If ∇ is a flat unitary connection on (E,h), the ∇-*Laplacian* is the self-adjoint degree 0 operator $\Delta := \nabla\nabla^* + \nabla^*\nabla$. The ∇-*harmonic* k-*forms with values in the bundle* E are $\mathcal{H}^k_\nabla(M;E) := \ker\Delta \cap \Omega^k(M;E)$. The harmonic forms are a graded sub-space $\mathcal{H}^\bullet_\nabla(M;E)$ of $\Omega^\bullet_\nabla(M;E)$.

Lemma 4.9.2. $\ker\Delta = \ker\nabla \cap \ker\nabla^*$.

Proof. Certainly $\ker\nabla \cap \ker\nabla^* \subset \ker\Delta$. If $\Phi \in \ker\Delta$ then $0 = \langle\Delta\Phi,\Phi\rangle_E = \langle\nabla^*\Phi,\nabla^*\Phi\rangle_E +$

$\langle \nabla\Phi, \nabla\Phi\rangle_E \Rightarrow \Phi \in \ker\nabla \cap \ker\nabla^*$. □

Lemma 4.9.3. Every harmonic form is the minimal element of its cohomology class with respect to $||-||_E^2$.

Proof. Let $\Phi \in \ker\Delta$. Thus $\nabla^*\Phi = \nabla\Phi = 0$ hence $[\Phi] \in H^\bullet_\nabla(M;E)$. Elements in the cohomology class of $[\Phi]$ are necessarily of the form $\Phi + \nabla\Psi$.

$$||\Phi+\nabla\Psi||_E^2 = \langle \Phi+\nabla\Psi, \Phi+\nabla\Psi\rangle_E = ||\Phi||_E^2 + ||\nabla\Psi||_E^2 + 2\Re\langle\Phi,\nabla\Psi\rangle_E = ||\Phi||_E^2 + ||\nabla\Psi||_E^2$$

so that Φ is minimal within its cohomology class.

Conversely, assume that $\Phi \in \ker\nabla$ is minimal within its cohomology class. Without loss of generality assume $\Phi \in \Omega^k(M;E) \cap \ker\nabla$. Every $\Psi \in \Omega^{k-1}(M;E)$ defines the functions $c_1(\epsilon) = ||\Phi+\epsilon\nabla\Psi||_E^2$ and $c_2(\epsilon) = ||\Phi+i\epsilon\nabla\Psi||_E^2$. By assumption $0 = c_1'(0) = c_2'(0)$ which implies that $0 = \Re\langle\Phi,\nabla\Psi\rangle_E = \Im\langle\Phi,\nabla\Psi\rangle_E$. Thus $0 = \langle\Phi,\nabla\Psi\rangle_E = \langle\nabla^*\Phi,\Psi\rangle_E$. Since this holds for all for all $\Psi \in \Omega^{k-1}(M;E)$ we must have $\nabla^*\Phi = 0$ and therefore $\Phi \in \ker\nabla^* \cap \nabla = \ker\Delta$. □

Proposition 4.9.4. Let $(E,h,\nabla) \to M$ be a flat unitary bundle over a closed Kähler manifold and assume that E is endowed with the unique holomorphic structure such that $\nabla^{01} = \bar\partial$. Then the ∇-Laplacian Δ commutes with L , Λ, and Π.

Proof. It is immediate that $[\Delta,\Pi] = 0$ and according to 4.6.2 $[L,\nabla^{01}] = [L,\nabla^{10}] = [\Lambda,\nabla^{10*}] =$

$[\Lambda, \nabla^{01*}] = 0$. Consider the Laplacian operators

$$\Delta_{10} := \nabla^{10}\nabla^{10*} + \nabla^{10*}\nabla^{10}$$

$$\Delta_{01} := \nabla^{01}\nabla^{01*} + \nabla^{01*}\nabla^{01}.$$

Using the Kähler identites 4.8.3 we find

$$\begin{aligned}
[L, \Delta_{10}] &= L\nabla^{10}\nabla^{10*} + L\nabla^{10*}\nabla^{10} - \nabla^{10}\nabla^{10*}L - \nabla^{10*}\nabla^{10}L \\
&= \nabla^{10}L\nabla^{10*} + L\nabla^{10*}\nabla^{10} - \nabla^{10}\nabla^{10*}L - \nabla^{10*}L\nabla^{10} \\
&= \nabla^{10}L\nabla^{10*} + \big(\nabla^{10*}L + i\nabla^{01}\big)\nabla^{10} - \nabla^{10}\big(L\nabla^{10*} - i\nabla^{01}\big) - \nabla^{10*}L\nabla^{10} \\
&= i\nabla^{01}\nabla^{10} + i\nabla^{10}\nabla^{01} = 0
\end{aligned}$$

since $\nabla^2 = 0 \leftrightarrow \nabla^{10}\nabla^{01} + \nabla^{01}\nabla^{10} = 0$. A similar computation shows that $[L, \Delta_{01}] = 0$.

Finally notice that

$$\begin{aligned}
\Delta &= \nabla\nabla^* + \nabla^*\nabla = (\nabla^{10} + \nabla^{01})(\nabla^{10*} + \nabla^{01*}) + (\nabla^{10*} + \nabla^{01*})(\nabla^{10} + \nabla^{01}) \\
&= \nabla^{10}\nabla^{10*} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{01}\nabla^{01*} + \nabla^{10*}\nabla^{10} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} + \nabla^{01*}\nabla^{01} \\
&= \nabla^{10}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01}\nabla^{01*} + \nabla^{01*}\nabla^{01} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} \\
&= \Delta_{10} + \Delta_{01} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} \\
&= \Delta_{10} + \Delta_{01} + i[L, \nabla^{01*}]\nabla^{01*} - i[L, \nabla^{10*}]\nabla^{10*} - i\nabla^{10*}[L, \nabla^{10*}] + i\nabla^{01*}[L, \nabla^{01*}] \\
&= \Delta_{10} + \Delta_{01} - i\nabla^{01*}L\nabla^{01*} + i\nabla^{10*}L\nabla^{10*} - i\nabla^{10*}L\nabla^{10*} + i\nabla^{01*}L\nabla^{01*} \\
&= \Delta_{10} + \Delta_{01}
\end{aligned}$$

and therefore $[L, \Delta] = [L, \Delta_{10}] + [L, \Delta_{01}] = 0$. As Δ is self-adjoint, passing to the adjoint gives $[\Lambda, \Delta] = 0$. □

Corollary 4.9.5. There is a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathcal{H}^{\bullet}_{\nabla}(M; E)$.

Proof. Since Δ commutes with L, Λ, and Π, the representation $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}\big(\Omega^{\bullet}(M; E)\big)$ restricts to a representation on $\mathcal{H}^{\bullet}_{\nabla}(M; E)$. □

Proposition 4.9.6. $\mathcal{H}^{\bullet}_{\nabla}(M; E) \cong H^{\bullet}_{\nabla}(M; E)$.

Proof. $\Delta : \Omega^k(M; E) \to \Omega^k(M; E)$ is a an elliptic operator for all k and therefore $\ker \Delta \cap \Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E)$ is finite and there is an orthogonal (with respect to $\langle -, - \rangle_E$) decomposition [18]

$$\Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E) \oplus \Delta(\Omega^k(M; E)).$$

Since $\nabla^2 = 0 = (\nabla^*)^2$ observe that $\langle \nabla\Phi, (\nabla\nabla^* + \nabla^*\nabla)\Psi \rangle_E = \langle \Phi, (\nabla^*)^2 \nabla\Psi \rangle_E + \langle \nabla^2\Phi, \nabla^*\Psi \rangle_E = 0$ and therefore $\nabla(\Omega^{k-1}(M; E)) \perp \mathcal{H}^k_{\nabla}(M; E)$. Similarly $\nabla^*(\Omega^{k+1}(M; E)) \perp \mathcal{H}^k_{\nabla}(M; E)$. Therefore $\nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E)) \subset \Delta(\Omega^k(M; E))$. Also observe that $\nabla(\Omega^{k-1}(M; E)) \perp \nabla^*(\Omega^{k+1}(M; E))$ and therefore $\Delta(\Omega^k(M; E)) \subset \nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E))$. Hence one has the *Hodge decomposition*

$$\Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E) \oplus \nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E)).$$

Now consider the natural map $\mathcal{H}^k_{\nabla}(M; E) \to H^k_{\nabla}(M; E)$ that sends a harmonic form to the cohomology class it represents. The kernel of this map is, by definition, $\nabla(\Omega^{k-1}(M; E)) \cap$

$\mathcal{H}^k_\nabla(M;E) = \{0\}$ hence the map is injective. Let $\Phi \in \Omega^k(M;E) \cap \ker\nabla$ represent an arbitrary element of $H^k_\nabla(M;E)$. Use the decomposition to write $\Phi = \phi + \nabla a + \nabla^* b$ where $(\phi, a, b) \in \mathcal{H}^k_\nabla(M;E) \times \Omega^{k-1}(M;E) \times \Omega^{k+1}(M;E)$. Notice that $\nabla^* b \in \ker\nabla$ and, furthermore, for all $\nabla^* c \in \nabla^*(\Omega^{k+1}(M;E)$, $\langle \nabla^* b, \nabla^* c\rangle_E = \langle \nabla\nabla^* b, c\rangle_E = 0$. Thus $\nabla^* b = 0$ so $\Phi = \phi + \nabla a$ and $\mathcal{H}^k_\nabla(M;E) \ni \phi \mapsto [\phi] = [\phi + \nabla a] = [\Phi] \in H^k_\nabla(M;E)$ is surjective. □

Theorem 4.9.7. If $(E, h, \nabla) \to (M, \omega)$ is a flat unitary bundle over a closed Kähler manifold then for all $k \leq \dim_{\mathbb{C}} M = n$, the maps $L^{n-k} : \Omega^k_\nabla(M;E) \ni \Phi \mapsto \omega^{n-k} \wedge \Phi \in \Omega^{2n-k}_\nabla(M;E)$ induce isomorphisms

$$L^{n-k} : H^k_\nabla(M;E) \xrightarrow{\sim} H^{2n-k}_\nabla(M;E). \tag{4.8}$$

Proof. Using $\mathcal{H}^\bullet_\nabla(M;E) \cong H^\bullet_\nabla(M;E)$, $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}\big(H^\bullet_\nabla(M;E)\big)$ is a finite representation and therefore $\rho(B)^{n-k} = L^{n-k}$ is an isomorphism between the eigenspaces $H^k_\nabla(M;E)$ and $H^{2n-k}_\nabla(M;E)$. □

Corollary 4.9.8. If (E, h, ∇) is a flat unitary bundle then $(\mathrm{End}E, \widetilde{h}, \widetilde{\nabla})$ is also a flat unitary bundle and therefore for all $k \leq \dim_{\mathbb{C}} M$

$$L^{n-k} : H^k_{\widetilde{\nabla}}(M;\mathrm{End}E) \xrightarrow{\sim} H^{2n-k}_{\widetilde{\nabla}}(M;\mathrm{End}E).$$

□

Corollary 4.9.9. $L^{n-k} : H^k_{\widetilde{\nabla}}(M;\mathfrak{u}E) \xrightarrow{\sim} H^{2n-k}_{\widetilde{\nabla}}(M;\mathfrak{u}E)$.

Proof. $\mathfrak{u}E \subset \mathrm{End}E$ is a real sub-bundle and the symplectic form is $\omega \in \Omega^2(M,\mathbb{R}) \cap \Omega^{(1,1)}(M)$. Hence $L^j(H^\bullet_{\widetilde{\nabla}}(M;\mathfrak{u}E)) \subset H^\bullet_{\widetilde{\nabla}}(M;\mathfrak{u}E)$. Since $H^\bullet_{\widetilde{\nabla}}(M;\mathrm{End}E)$ is finite dimensional, $L^{n-k}|_{H^k_{\widetilde{\nabla}}(M;\mathfrak{u}E)} : H^k_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to H^{2n-k}_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ is injective and thus an isomorphism.

□

4.10 Functional on the symplectic space $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$

Now assume that the hermitian vector bundle $(\pi : E \to M, h)$ has a closed Kähler manifold M as its base. Let $\omega \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$ denote the Kähler form.

For all $\nabla \in \mathcal{F}(E,h)$, the module $H^\bullet_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ satisfies hard Lefschetz duality [4.9.7]. In particular, the pairing

$$\omega_\nabla : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \times H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \ni (\Phi,\Psi) \mapsto \int_M h(\Phi \wedge \Psi) \wedge [\omega]^{n-1} \in \mathbb{R} \tag{4.9}$$

makes the vector space $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ into a symplectic vector space. Hard Lefschetz duality and its composition with Poincaré duality gives the isomorphism

$$\Upsilon := PD \circ \mathcal{L} : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to H_1^{\widetilde{\nabla}}(M;\mathfrak{u}E). \tag{4.10}$$

For $\Psi \in H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$, consider the map induced from the symplectic structure on $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$

$$\omega_\nabla(-,\Psi) = \int_M \widetilde{h}(- \wedge \Psi) \wedge [\omega]^{n-1} : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to \mathbb{R}.$$

Using the pairing introduced in 2.10.3, every homology class $N \in H_1^{\widetilde{\nabla}}(M;\mathfrak{u}E)$ defines the function

$$\int \langle -, N \rangle_h : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to \mathbb{R}.$$

Notice that

$$\Psi \mapsto \int_M \widetilde{h}(- \wedge \Psi) \wedge [\omega]^{n-1} = \int_M \widetilde{h}\big(- \wedge(\Psi \wedge [\omega^{n-1}])\big)$$

corresponds to the Poincaré dual of the hard Lefschetz dual of Ψ. Therefore, for all $\Phi, \Psi \in H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$,

$$\int \langle \Phi, \Upsilon(\Psi) \rangle_h = \int_M \widetilde{h}(\Phi \wedge \Psi) \wedge [\omega]^{n-1}.$$

This establishes the following

Theorem 4.10.1. Let $(\pi : E \to M, h)$ be a hermitian vector bundle over a closed Kähler manifold with Kähler form ω. Let $\nabla \in \mathcal{F}(E, h)$ be a flat unitary connection and let γ be a closed curve in M. Then the trace of holonomy functional satisfies

$$\mathbb{T} = \omega_{\nabla}\big(-, \Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)]\big) : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R} \tag{4.11}$$

and thus for all $\Phi \in H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$

$$\int_M h(\Phi \wedge \Upsilon(\gamma)) \wedge [\omega]^{n-1} = -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot{\gamma}(t))\Big) dt. \tag{4.12}$$

Proof. Using 2.10.3

$$\begin{aligned}\mathbb{T}(\Phi) &= \int \Big\langle \Phi, [\gamma, F\mathrm{Hol}_\gamma(\nabla)] \Big\rangle_h = \int \Big\langle \Phi, \Upsilon\Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)] \Big\rangle_h \\ &= \int_M \widetilde{h}(\Phi \wedge \Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)]) \wedge [\omega]^{n-1}.\end{aligned}$$

□

Bibliography

[1] Werner Ballmann, *Lectures on Kähler manifolds*, Vol. 2, European Mathematical Society, (2006).

[2] Raoul Bott and Loring W Tu, *Differential forms in algebraic topology*, Vol. 82, Springer Science & Business Media, (2013).

[3] Daniel S Freed and Karen K Uhlenbeck, *Instantons and four-manifolds*, Vol. 1, Springer Science & Business Media, 2012.

[4] William Fulton and Joe Harris, *Representation theory: a first course*, Vol. 129, Springer Science & Business Media, (2013).

[5] William M Goldman, *Invariant functions on Lie groups and Hamiltonion flows on surface groups representations*, Inventions mathematicae **85** (1986), 263 - 302.

[6] William M Goldman, *Symplectic Nature of Fundamental Groups of Surfaces*, Advances in Mathematics **54** (1984), 200-225.

[7] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, John Wiley & Sons, (1978).

[8] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[9] Daniel Huybrechts, *Complex geometry: an introduction*, Springer Science & Business Media, 2005.

[10] Yael Karshon, *An algebraic proof for the symplectic structure of moduli space*, Proceedings of the American Mathematical Society **116** (1992), no. 3.

[11] Shoshichi Kobayashi, *Differential geometry of complex vector bundles*, Vol. 793, Princeton University Press, (2014).

[12] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry*, Vol. 1, New York, London, 1963.

[13] P K Mitter and Claude-Michel Viallet, *On the bundle of connections and the gauge orbit manifold in Yang-Mills theory*, Communications in Mathematical Physics **79** (1981), no. 4, 457–472.

[14] Shigeyuki Morita, *Geometry of characteristic classes*, American Mathematical Soc., 2001.

[15] Andrei Moroianu, *Lectures on Kähler geometry*, Vol. 69, Cambridge University Press, (2007).

[16] August Newlander and Louis Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Annals of Mathematics (1957), 391–404.

[17] Edwin Spanier, *Singular homology and cohomology with local coefficients and duality for manifolds*, Pacific journal of mathematics **160** (1993), no. 1, 165–200.

[18] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, Vol. 76, Cambridge University Press, Cambridge, (2002).

[19] George W Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, (1978).