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Representing the Derivative of Trace of Holonomy

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by

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Abstract

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Trace of holonomy around a fixed loop γ defines a function on the space of unitary connections on a hermitian vector bundle over a Riemannian manifold. Using the derivative of trace of holonomy, the loop γ, and a flat unitary connection ∇, a functional is defined on the vector space $H^1_\nabla(M;\mathfrak{u}E)$ of twisted degree 1 cohomology classes with coefficients in skew-hermitian endomorphisms. It is shown that this functional is obtained by pairing elements of $H^1_\nabla(M;\mathfrak{u}E)$ with a degree 1 *homology* class built directly from the 1-cycle γ and equipped with a flat section obtained from the variation of holonomy around γ. When the base manifold is closed Kähler, hard Lefschetz duality implies that $H^1_\nabla(M;\mathfrak{u}E)$ is a symplectic vector space and, coupled with Poincaré duality, can be identified with $H_1^\nabla(M;\mathfrak{u}E)$. In this case, the functional is obtained by contracting the symplectic pairing on $H^1_\nabla(M;\mathfrak{u}E)$ with the hard Lefschetz dual of the Poincaré dual of the twisted homology class built using γ.

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Much love to everyone.

Dedicated to Sylvia

Contents

Chapter 1

Introduction

Fix a smooth complex vector bundle $\pi : E \to M$ equipped with a (hermitian) bundle metric $h : E \otimes \bar{E} \to M \times \mathbb{C}$. Given any connection $\nabla : \Gamma(E) \xrightarrow{\mathbb{C}-\text{linear}} \Omega^1(M, \mathbb{C}) \otimes_{\Omega^0(M,\mathbb{C})} \Gamma(E)$ that is compatible with the metric and any (piece-wise) smooth path $\gamma : [0, 1] \to M$, parallel transport along γ defines a linear isometry $P_\gamma(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)}$ between the fibers. When γ is closed, parallel transport is referred to as holonomy $\mathrm{Hol}_\gamma(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)} = E_{\gamma(0)}$ and defines a unitary automorphism of the fiber over $\gamma(0) = \gamma(1)$.

The set $\mathcal{A}(E, h)$ of all connections that are compatible with the metric is an affine space modeled on $\Omega^1(M; \mathfrak{u}E)$, where $\mathfrak{u}E \subset \mathrm{End}E$ is the sub-bundle of h-skew-hermitian bundle endomorphisms. Thus when the base M is Riemannian, the set of all connections is a metric space. Parallel transport along a fixed loop γ defines the holonomy function $\mathrm{Hol}_\gamma : \mathcal{A}(E, h) \to UE_{\gamma(0)} \subset \mathrm{Aut}(E_{\gamma(0)})$ and $\mathbf{Tr}\mathrm{Hol}_\gamma$ defines a scalar-valued function on the space of all connections.

If the base M is a closed symplectic manifold with symplectic form $\omega \in \Omega^2(M, \mathbb{R})$, the space $\mathcal{A}(E, h)$ of all metric connections has a symplectic structure described as follows. The tangent space to every $\nabla \in \mathcal{A}(E,h)$ is naturally identified with $\Omega^1(M;\mathfrak{u}E)$ and the symplectic form is defined at $\nabla \in \mathcal{A}(E,h)$ by the pairing

$$\Omega^1(M;\mathfrak{u}E) \times \Omega^1(M;\mathfrak{u}E) \ni (\Phi,\Psi) \mapsto \int_M h(\Phi \wedge \Psi) \wedge \omega^{\dim_{\mathbb{R}} M-1}.$$

This dissertation is motivated by a consideration of the Hamiltonian vector field associated to the trace of holonomy function. When $M = \Sigma$ is a closed oriented (real) surface, Goldman [6] discovered a *symplectic structure on the moduli space of gauge equivalence classes of flat metric connections*. Goldman proceeded to analyze the trace of holonomy functions on the moduli space [5] and realized that the subspace of functions consisting of the trace of holonomy functions is closed under the Poisson bracket. In doing so, Goldman discovered a Lie algebra structure on free homotopy classes of oriented loops within a closed oriented surface.

Karshon [10] provided a purely algebraic proof of Goldman's symplectic result and observed that the moduli space of flat unitary connections should be symplectic provided that the base is a closed Kähler manifold. The Riemannian metric on the base, together with the bundle metric, defines an inner product on the space of bundle-valued differential forms which allows for formal adjoints of linear operators on differential forms and the use of Hodge theory. The linear operator $L \in \mathrm{End}^2\big(\Omega^\bullet(M;E)\big)$ is defined by taking the wedge product with the Kähler form ω. The operator L and its formal adjoint $\Lambda \in \mathrm{End}^{-2}\big(\Omega^\bullet(M;E)\big)$ turn out to be the images of two of three generators of $\mathfrak{sl}(2,\mathbb{C})$ under a representation on $\Omega^\bullet(M;E)$. Next, each flat unitary connection ∇ induces a unique holomorphic structure on the bundle so that the connection becomes the Chern connection for the corresponding holomorphic hermitian vector bundle and, in this context, there is a twisted version of the Kähler identities. The Kähler identities are used to show that the representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\Omega^\bullet(M;E)$ restricts to a representation on the space $\mathcal{H}^\bullet_\nabla(M;E) = \ker\nabla \cap \ker\nabla^*$ of ∇-harmonic forms, which by the Hodge theorem is both finite dimensional and isomorphic to $H^\bullet_\nabla(M;E)$. Therefore one has a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^\bullet_\nabla(M;E)$. In particular $H^\bullet_\nabla(M;E)$ satisfies hard Lefschetz duality which says that cup product with the Kähler class $[\omega]$ induces isomorphisms $H^k_\nabla(M;E) \cong H^{\dim_\mathbb{R} M-k}(M;E)$ for all $k \leq \dim_\mathbb{C} M$.

The symplectic form on the moduli space is given at the point $[\nabla]$ by the pairing

$$H^1_\nabla(M;\mathfrak{u}E) \times H^1_\nabla(M;\mathfrak{u}E) \cong H^1_\nabla(M;\mathfrak{u}E) \times H^{\dim_\mathbb{R} M-1}(M;\mathfrak{u}E) \xrightarrow{\cup_h} H^{\dim_\mathbb{R} M}(M) \xrightarrow{\int} \mathbb{R}$$

which uses the cup-product along with the bundle metric on $\mathfrak{u}E$ induced from the metric h on E.

The main result of this dissertation will now be described. The context is a fixed smooth hermitian vector bundle $\pi : E \to M$ over a closed Kähler manifold equipped with a flat metric connection ∇. A functional on $H^1_\nabla(M;\mathfrak{u}E)$ is constructed from a given (homotopy class of a) loop $\gamma : [0,1] \to M$ using trace of holonomy. The (directional) derivatives of $\mathbf{Tr}\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to \mathbb{C}$ at a connection $\nabla \in \mathcal{A}(E,h)$ define the function

$$d\mathbf{Tr}\mathrm{Hol}_\gamma(\nabla) : \Omega^1(M;\mathfrak{u}E) \to \mathbb{C}.$$

The functional on cohomology $H^1_\nabla(M;\mathfrak{u}E)$ is obtained by restricting the domain of $d\mathbf{Tr}\mathrm{Hol}_\gamma(\nabla)$

to $Z^1_\nabla(M;\mathfrak{u}E) := \Omega^1(M;\mathfrak{u}E) \cap \ker\nabla$ and observing this descends to cohomology. The resulting functional is computed as

$$H^1_\nabla(M;\mathfrak{u}E) \ni [\Phi] \mapsto -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

where $\dot\gamma(t) \in T_{\gamma(t)}M$ is the velocity vector of γ, and $\gamma^t(s) = \gamma(s+t \mod 1)$ is the t-shifted loop of γ. The functional is referred to as the *(derivative of) trace of holonomy functional* $\mathbb{T}$. It is shown that $\mathbb{T}$ is represented by a homology class

$$[\gamma, F\mathrm{Hol}_\gamma(\nabla)] \in H_1^\nabla(M;\mathfrak{u}E)$$

using the natural pairing between degree 1 homology and cohomology. This is the homology class that is represented by the twisted singular 1-cycle γ equipped with the flat section of $\gamma^*\mathfrak{u}E$ obtained by applying the variation map $F : UE \to \mathfrak{u}E$ to holonomy.

Let $\Upsilon^{-1}(\gamma) \in H^1_\nabla(M;\mathfrak{u}E)$ be hard Lefschetz dual to the Poinaré dual of $[\gamma, F\mathrm{Hol}_\gamma(\nabla)] \in H_1^\nabla(M;\mathfrak{u}E)$. The main result is that for all $\Phi \in H^1_\nabla(M;\mathfrak{u}E)$

$$\int_M h(\Phi \wedge \Upsilon^{-1}(\gamma)) \wedge [\omega]^{n-1} = -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

which equates contraction of the symplectic pairing on $H^1_\nabla(M;\mathfrak{u}E)$ with the trace of holonomy functional $\mathbb{T}$.

Chapter 2

Derivative of parallel transport

2.1 Vector bundles and connections

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. A smooth $\mathbb{F}$-*vector bundle of rank* r is a smooth surjection $\pi : E \to M$ of manifolds such that every point $x \in M$ is contained in an open neighborhood $U \ni x$ equipped with a diffeomorphism $\phi_U : \pi^{-1}(U) \cong U \times \mathbb{F}^r$ which satisfies that, for all $y \in U$, the restriction $\phi_U|_y : \pi^{-1}(y) \to \{y\} \times \mathbb{F}^k$ is a linear isomorphism. The space E is the *total space*, the space M is the *base space* and π is the *projection* of the vector bundle. The *fiber over* $x \in M$ is the preimage $E_x := \pi^{-1}(x) \subset E$.

Remark. One may refer to *the vector bundle* E (without explicit reference to the projection π and the base M) whenever $\pi : E \to M$ is understood.

Definition 2.1.2. A smooth *section* of a smooth vector bundle $\pi : E \to M$ is a smooth right inverse of π, i.e. a smooth morphism $s : M \to E$ such that $\pi \circ s = \mathbf{1}_M$. The collection of smooth sections of the vector bundle is written $\Gamma(E) \subset \mathrm{Hom}(M, E)$.

Definition 2.1.3. Let M be a manifold and V a $\mathbb{F}$-vector space. The cochain complex of *V-valued differential forms on M*, denoted $\Big(\Omega^\bullet(M,V), d\Big)$, is the graded module $\Omega^\bullet(M,V) := \Omega^\bullet(M) \otimes_{\mathbb{F}} V$ equipped with de Rham's exterior differential $d(\phi \otimes v) := d\phi \otimes v$.

Remark. Note that $\Omega^0(M, \mathbb{F})$ consists of smooth $\mathbb{F}$-valued functions on M and $\Gamma(E)$ is a module over $\Omega^0(M, \mathbb{F})$.

Example 2.1.4. Let V be a $\mathbb{F}$-vector space. Sections $\Gamma(M \times V)$ of the trivial vector bundle $M \times V \xrightarrow{(m,v)\mapsto m} M$ are precisely V-valued functions on M,

$$\Gamma(M \times V) \ni \big(M \xrightarrow{m \mapsto (m, f(m))} M \times V\big) \leftrightarrow f \in \mathrm{Hom}(M, V).$$

Definition 2.1.5. The *differential forms on M with values in a $\mathbb{F}$-vector bundle $\pi : E \to M$* are elements of the graded module

$$\Omega^\bullet(M; E) := \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Gamma(E). \tag{2.1}$$

Example 2.1.6. Consider the trivial vector bundle $\pi : M \times V \to M$ where V is a $\mathbb{F}$-vector space. Then $\Omega^\bullet(M; M \times V) = \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Gamma(M \times V) = \Omega^\bullet(M, \mathbb{F}) \otimes_{\Omega^0(M,\mathbb{F})} \Omega^0(M, V) = \Omega^\bullet(M, V)$.

Remark. One may write $\Omega^\bullet(M)$ when $\mathbb{F}$ is understood.

Definition 2.1.7. A *connection on a $\mathbb{F}$-vector bundle $\pi : E \to M$* is a $\mathbb{F}$-linear map

$$\nabla : \Omega^0(M; E) := \Gamma(E) \to \Omega^1(M) \otimes_{\Omega^0(M)} \Gamma(E) =: \Omega^1(M; E) \tag{2.2}$$

such that $\nabla(fs) = df \otimes s + f\nabla(s)$ for all $(f,s) \in \Omega^0(M) \times \Gamma(E)$. A *connection* ∇ *extends to a unique degree* $+1$ *linear operator* ∇ *on* $\Omega^\bullet(M;E)$ by imposing the Leibnitz condition

$$\nabla(\phi \otimes s) = d\phi \otimes s + (-1)^{|\phi|}\phi \wedge \nabla(s). \tag{2.3}$$

Write $\mathcal{A}(E)$ for *the set of all connections on a vector bundle* $\pi : E \to M$. If $\nabla \in \mathcal{A}(E)$ we refer to the pair (E, ∇) as a *vector bundle with connection.*

Proposition 2.1.8. The set $\mathcal{A}(E)$ of all connections on a given vector bundle $\pi : E \to M$ is an affine space modeled on $\Omega^1(M; \mathrm{End}E)$.

Proof. Given $\nabla_1, \nabla_2 \in \mathcal{A}(E)$, the a priori $\mathbb{F}$-linear map $\nabla_1 - \nabla_2$ is readily seen to be $\Omega^0(M)$-linear: for all $(f,s) \in \Omega^0(M) \times \Gamma(E)$,

$$(\nabla_1 - \nabla_2)(fs) = \nabla_1(fs) - \nabla_2(fs) = df \otimes s + f\nabla_1(s) - df \otimes s - f\nabla_2(s) = f(\nabla_1 - \nabla_2)s.$$

Therefore $(\nabla_1 - \nabla_2) \in \mathrm{Hom}_{\Omega^0(M)-Mod}\big(\Gamma(E), \Omega^1(M;E)\big) \cong \Omega^1(M; \mathrm{End}E)$. Conversely, given any $\phi \in \Omega^1(M; \mathrm{End}E)$ we have $(\nabla_1 + \phi)(fs) = df \otimes s + f\nabla_1 s + f\phi s = df \otimes s + f(\nabla_1 + \phi)s$ and thus $\nabla_1 + \phi \in \mathcal{A}(E)$. □

Remark. The above proof used the canonical (linear algebra) identification

$$\mathrm{Hom}_{\Omega^0(M)-Mod}(\Gamma E, \Omega^k(M;E)) \cong \Omega^k(M; \mathrm{End}E). \tag{2.4}$$

Corollary 2.1.9. For $\nabla \in \mathcal{A}(E)$, there is a natural identification

$$T_\nabla \mathcal{A}(E) \equiv \Omega^1(M; \mathrm{End}E) \tag{2.5}$$

of the tangent vector space to the affine space $\mathcal{A}(E)$ at the point ∇. □

2.2 Flat sections, parallel translation, and holonomy

Definition 2.2.1. Let $\pi : E \to M$ be a vector bundle with connection $\nabla \in \mathcal{A}(E)$. A section $s \in \Gamma(E)$ is called *parallel* (or *flat*) *with respect to* ∇ if and only if $\nabla(s) = 0 \in \Omega^1(M; E)$.

Definition 2.2.2. Let $\pi : E \to M$ be a vector bundle with connection ∇ and let $\gamma : [0,1] \to M$ be a path in the base. A lift $\widetilde{\gamma}$ of γ is called *parallel* (or *flat*) *with respect to* ∇ if and only if $\widetilde{\gamma}$ is a parallel section of the pullback bundle $\gamma^* E \to [0,1]$ with respect to the pulled-back connection $\gamma^*\nabla \in \mathcal{A}(\gamma^* E)$.

Example 2.2.3. A parallel section is – tautologically – a parallel lift of its projection. More precisely, a path $\alpha : [0,1] \to E$ in the total space of a vector bundle $\pi : E \to M$ is ∇-flat if and only if α is a $(\pi\alpha)^*\nabla$-flat section of $(\pi\alpha)^* E$.

Remark. A lift $\widetilde{\gamma}$ of $\gamma : [0,1] \to M$ is the same thing as a section of $\gamma^* E \to [0,1]$. It is convenient to work with the bundle $\gamma^* E \to [0,1]$ because it is trivializable. Then $\widetilde{\gamma}$ is parallel if and only if $(\gamma^*\nabla)(\widetilde{\gamma}) = 0$ which holds if and only if $\nabla(\widetilde{\gamma})(\dot{\gamma}(t)) = 0$ for every $t \in [0,1]$ where $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is the velocity vector of γ at $\gamma(t) \in M$.

Proposition 2.2.4. Let $\gamma : [0,1] \to M$ be given. For each $v \in E_{\gamma(0)}$, there is a unique ∇-parallel lift $\widetilde{\gamma}_v$ of γ such that $\widetilde{\gamma}_v(\gamma(0)) = v$. We say that $\widetilde{\gamma}_v$ defines *parallel translation of* v *along* γ *with respect to the connection* ∇. Furthermore, the path γ induces a linear isomorphism $\widetilde{\gamma} : E_{\gamma(0)} \to E_{\gamma(1)}$ defined by $\widetilde{\gamma}(v) = \widetilde{\gamma}_v(1)$.

Proof. The lifted path $\widetilde{\gamma}_v$ would have velocity field $\frac{d\widetilde{\gamma}_v(t)}{dt}$ satisfying $\nabla(\widetilde{\gamma}_v)(\frac{d\widetilde{\gamma}_v(t)}{dt}) = 0$ and

$\widetilde{\gamma}_v(0) = v$, which is a first order ordinary differential equation with initial value. Thus $\widetilde{\gamma}_v$ exists as the unique solution to the initial value problem and we can define the linear map $\widetilde{\gamma} : v \mapsto \widetilde{\gamma}_v(1)$. The inverse of $\widetilde{\gamma}(t)$ is $\widetilde{\gamma}(1-t)$. See 2.22 below for an explicit construction of the solution for the parallel lift. □

Definition 2.2.5. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle and $\gamma : [0,1] \to M$ a path. The *parallel translation function*

$$P_\gamma : \mathcal{A}(E) \to \mathrm{Hom}_{\mathbb{F}-Mod}(E_{\gamma(0)}, E_{\gamma(1)}) \tag{2.6}$$

sends a connection ∇ to the parallel translation isomorphism $P_\gamma(\nabla) = \widetilde{\gamma} : v \mapsto \widetilde{\gamma}_v(1)$.

Definition 2.2.6. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle. If $\gamma : [0,1] \to M$ is a closed path (i.e. $\gamma(0) = \gamma(1)$) then parallel translation along γ is called *holonomy along* γ and is denoted by

$$\mathrm{Hol}_\gamma := P_\gamma : \mathcal{A}(E) \to \mathrm{Aut}(E_{\gamma(0)}). \tag{2.7}$$

2.3 Flat connections and parallel translation

Proposition 2.3.1. Let $\pi : E \to M$ be a $\mathbb{F}$-vector bundle with connection $\nabla \in \mathcal{A}(E)$. The operator

$$\nabla \circ \nabla : \Omega^0(M; E) \to \Omega^2(M; E)$$

is $\Omega^0(M)$-linear and therefore corresponds to a unique End$E$-valued 2-form $R(\nabla) \in \Omega^2(M; \mathrm{End}E)$.

Proof. For all $(f, s) \in \Omega^0(M) \times \Gamma(E)$ one sees that

$$(\nabla \circ \nabla)(fs) = \nabla(df \otimes s + f\nabla s) = d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\nabla(\nabla s) = f(\nabla \circ \nabla)s.$$

$\square$

Definition 2.3.2. Let $\pi : E \to M$ be a vector bundle with connection $\nabla \in \mathcal{A}(E)$. The *curvature of* ∇ is the element $R(\nabla) \in \Omega^2(M; \mathrm{End}E)$ corresponding to $\nabla \circ \nabla$.

Definition 2.3.3. A connection ∇ is *flat* if and only if $R(\nabla) = 0$. The *subset of all flat connections on* E will be denoted by $\mathcal{F}(E) \subset \mathcal{A}(E)$. A *flat vector bundle* is a pair (E, ∇) where $E \to M$ is a vector bundle and $\nabla \in \mathcal{F}(E)$ is a flat connection on E.

Definition 2.3.4. The *de Rham cohomology of* M *with values in a flat vector bundle* $(E, \nabla) \to M$ is the cohomology $H^\bullet_\nabla(M; E)$ of the differential graded module $\Big(\Omega^\bullet(M; E), \nabla\Big)$.

Proposition 2.3.5. Let $\pi : (E, \nabla) \to M$ be a flat vector bundle. If $\gamma \sim \eta : [0,1] \to M$ are homotopy equivalent paths with $\gamma(0) = \eta(0)$ and $\gamma(1) = \eta(1)$, then $P_\gamma(\nabla) = P_\eta(\nabla) : E_{\gamma(0)} \to E_{\gamma(1)}$.

Proof. $\widetilde{\gamma}_v$ denotes the unique parallel lift of γ with $\widetilde{\gamma}_v(0) = v$ so that $P_\gamma(\nabla) : v \mapsto \widetilde{\gamma}_v(1)$.

Write $x = \gamma(0) = \eta(0)$ and $y = \gamma(1) = \eta(1)$. Let $H_s(t) = H(s,t) : [0,1] \times [0,1] \to M$ be a homotopy with $H(0,t) = \gamma(t)$ and $H(1,t) = \eta(t)$ that fixes the endpoints, i.e. $H(s,0) = x$ and $H(s,1) = y$ for all $s \in [0,1]$. Pick $v \in \pi^{-1}(x)$ and define $\sigma \in \Gamma(H_s(t)^*E)$ by requiring that (i) $\sigma(s,0) = v$ is a constant function of s and (ii) for fixed s, $\sigma(s,t) = \widetilde{(H_s)}_v(t)$. Condition (ii) says that $\sigma(s,t)$, as a function of t, assigns to t the element of $\pi^{-1}(H(s,t))$ obtained by parallel translating $H(s,0) = v$ along the path $[0,1] \ni t \mapsto (s,t) \in [0,1] \times [0,1]$.

Write $D = H^*\nabla$ and let $X_s, X_t \in \Gamma(T([0,1] \times [0,1]))$ be the standard unit vector fields. By condition (ii) $(D\sigma)(X_t) = 0$ since σ is parallel in the t-direction. Since $\sigma(s,0) = v$ for all s, we have $D\sigma(X_s)|_{t=0} = 0$. By assumption ∇ is flat so that $D^2 = 0$ and therefore

$$0 = (D^2\sigma)(X_s, X_t) = X_s(D\sigma)(X_t) - X_t(D\sigma)(X_s) - (D\sigma)[X_s, X_t] = -X_t(D\sigma)(X_s)$$

which shows that $D\sigma(X_s)$ is constant in the t-direction, and thus $D\sigma(X_s)|_{t=1} = D\sigma(X_s)|_{t=0} = 0$. Hence $\sigma(s,1)$ is a constant path of s and $P_\gamma(\nabla)(v) = \widetilde{\gamma}_v(1) = \sigma(0,1) = \sigma(1,1) = \widetilde{\eta}_v(1) = P_\eta(\nabla)(v)$. □

Corollary 2.3.6. If $(E, \nabla) \to M$ is a flat vector bundle, then for all $[\gamma] \in \pi_1(M, x)$ the holonomy

$$\mathrm{Hol}_{[\gamma]}(\nabla) \in \mathrm{Aut}(E_x) \tag{2.8}$$

is well-defined by $\mathrm{Hol}_{[\gamma]}(\nabla) := \mathrm{Hol}_\gamma(\nabla)$ for any representative closed path γ. In particular, there is the *holonomy representation of the fundamental group of M*

$$\mathrm{Hol}_{(-)}(\nabla) : \pi_1(M, x) \to \mathrm{Aut}(E_x).$$

□

Remark. The significance of the holonomy representation is discussed in 3.2.

Corollary 2.3.7. One may restrict the domain of $\mathrm{Hol} : \mathcal{A}(E) \times \mathrm{Map}\big(([0,1], 0, 1), (M, x, x)\big) \to \mathrm{Aut}(E_x)$ to flat connections $\mathrm{Hol} : \mathcal{F}(E) \times \mathrm{Map}\big(([0,1], 0, 1), (M, x, x)\big) \to \mathrm{Aut}(E_x)$ and then quotient by homotopy classes of loops to obtain the holonomy function

$$\mathrm{Hol} : \mathcal{F}(E) \times \pi_1(M, x) \to \mathrm{Aut}(E_x). \tag{2.9}$$

□

2.4 Metrics on vector bundles

Definition 2.4.1. An *inner product* on a $\mathbb{F}$-vector space V is sesquilinear pairing – i.e. a linear map $V \otimes \bar{V} \to \mathbb{F}$ – that is conjugate-symmetric and positive-definite. An inner product on a complex vector space is called a *hermitian inner product*.

Remark. $\bar{V}$ is the $\mathbb{F}$-module V equipped with the $\mathbb{F}$ action $a \cdot v = \bar{a}v$. In particular, if V is real then $\bar{V} = V$. Thus the convention adopted here is that a hermitian inner product is complex linear in the first argument and anti-linear (i.e. conjugate linear) in the second argument.

Definition 2.4.2. A *bundle metric on a $\mathbb{F}$-vector bundle* $\pi : E \to M$ is a smooth section $h \in \Gamma\big(\mathrm{Hom}(E \otimes \bar{E}, M \times \mathbb{F})\big)$ such that, for all $x \in M$, h_x is an inner product on the fiber E_x.

Example 2.4.3. A *Riemannian manifold* is a (smooth) manifold whose tangent bundle has a bundle metric. Similarly, a *hermitian manifold* is (complex) manifold whose holomorphic tangent bundle has a hermitian metric.

Proposition 2.4.4. Let h be a bundle metric on $\pi : E \to M$. Then $\wedge_h := \wedge \otimes h$ defines a sesquilinear pairing

$$\wedge_h : \Omega^\bullet(M;E) \times \Omega^\bullet(M;E) \ni \big(\phi \otimes s, \psi \otimes t\big) \mapsto \phi \wedge \bar{\psi} h(s,t) \in \Omega^\bullet(M,\mathbb{F}). \tag{2.10}$$

Remark. In the above proposition sesquilinear means *sesquilinear over* $\Omega^0(M,\mathbb{F})$:

$$(f\Phi) \wedge_h (g\Psi) = f\bar{g}(\Phi \wedge_h \Psi)$$

for all $f, g \in \Omega^0(M, \mathbb{F})$ and $\Phi, \Psi \in \Omega^\bullet(M; E)$.

Proof. $\wedge_h$ is a well-defined pairing on $\Omega^\bullet(M; E) = \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Gamma(E)$ since for all $f, g \in \Omega^0(M)$

$$(\phi \otimes fs) \wedge_h (\psi \otimes gt) = \phi \wedge \bar{\psi} h(fs, gt) = \phi \wedge \bar{\psi} f \bar{g} h(s,t) = \phi f \wedge \bar{\psi} \bar{g} h(s,t) = (\phi f \otimes s) \wedge_h (\psi g \otimes t).$$

This computation, coupled with the fact that both $\wedge$ and h are bilinear over $\Omega^0(M, \mathbb{R})$, also shows that $\wedge_h$ is sesquilinear over $\Omega^0(M, \mathbb{F})$. □

Definition 2.4.5. Given a vector bundle $\pi : E \to M$ with bundle metric h, a connection $\nabla \in \mathcal{A}(E)$ is said to be *compatible with the bundle metric* if and only if for all $s, t \in \Gamma(E)$, the function $h(s,t) : M \ni x \mapsto h(s(x), t(x)) \in \mathbb{F}$ satisfies

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t) \in \Omega^1(M, \mathbb{F}). \tag{2.11}$$

Definition 2.4.6. A *hermitian vector bundle* (E, h) is a complex vector bundle $\pi : E \to M$ equipped with a hermitian bundle metric h. A connection ∇ that is compatible with a hermitian bundle (E, h) is called a *unitary connection* and the triple (E, h, ∇) a *unitary bundle.* A connection that is both flat and unitary is called a *flat unitary connection* and a bundle with such a connection is a *flat unitary bundle.* Let $\mathcal{A}(E, h) \supset \mathcal{F}(E, h)$ denote the *set of all unitary connections* and the *subset of flat unitary connections.*

Definition 2.4.7. Let $(\pi : E \to M, h)$ be a hermitian vector bundle. The $\mathbb{R}$-vector bundle

$$\mathfrak{u}E \to M \tag{2.12}$$

is defined as the sub-bundle of $\mathrm{End}E \to M$ consisting of h-skew-hermitian endomorphisms.

Proposition 2.4.8. The set $\mathcal{A}(E,h)$ of all unitary connections is an affine space modeled on $\Omega^1(M;\mathfrak{u}E)$.

Proof. For $\nabla_1, \nabla_2 \in \mathcal{A}(E,h) \subset \mathcal{A}(E)$, we know that $\nabla_1 - \nabla_2$ corresponds to a unique element of $\Omega^1(M;\mathrm{End}E)$. Furthermore, since both connections are unitary we have

$$h(\nabla_1 s,t) + h(s,\nabla_1 t) = dh(s,t) = h(\nabla_2 s,t) + h(s,\nabla_2 t)$$

$$\Rightarrow h((\nabla_1 - \nabla_2)s,t) + h(s,(\nabla_1 - \nabla_2)t) = 0$$

and therefore $\nabla_1 - \nabla_2$ is h-skew-hermitian. □

Corollary 2.4.9. The tangent space $T_\nabla\mathcal{A}(E,h)$ to a unitary connection ∇ is canonically identified with $\Omega^1(M;\mathfrak{u}E)$. □

Proposition 2.4.10. If $(\pi : E \to M, h)$ is a hermitian bundle over a closed Riemannian manifold, then $\mathcal{A}(E,h)$ is a Banach manifold.

Proof. $\mathcal{A}(E,h)$ is an affine space modeled on $\Omega^1(M;\mathfrak{u}E)$. The Riemann metric on M combined with the bundle metric h induce an inner product on $\wedge^k T^\vee M \otimes \mathfrak{u}E$ and on $\Omega^k(M;\mathfrak{u}E)$. See 4.5. One has the L^p space of functions $L^p(M,\wedge^k T^\vee M \otimes \mathfrak{u}E)$ whose elements are smooth functions $f : M \to \wedge^k T^\vee M \otimes \mathfrak{u}E$ such that

$$||f||_p := \Big(\int_M ||f||^p_{\mathfrak{u}E} dM\Big)^{1/p} < \infty.$$

Then the space $L^p\Omega^k(M;\mathfrak{u}E) := \Omega^k(M;\mathfrak{u}E) \cap L^p(M,\wedge^k T^\vee_{\mathbb{C}} M \otimes \mathfrak{u}E)$ is a Banach manifold and a Hilbert manifold when $p = 2$.

To get a better approximation of $\Omega^1(M;\mathfrak{u}E)$, one may use the Sobolev spaces $W^{j,2}(M,\wedge^k T^\vee M\otimes \mathfrak{u}E)\subset L^2(M,\wedge^k T^\vee M\otimes\mathfrak{u}E)$ consisting of functions f such that f and its first j-derivatives are in L^2. $W^{j,2}(M,\wedge^k T^\vee M\otimes\mathfrak{u}E)\cap\Omega^k(M;\mathfrak{u}E)$ is a Hilbert space and $\Omega^k(M;\mathfrak{u}E)$ is dense in the space. Formally one may treat $\mathcal{A}(E,h)$ as a Sobolev space of sections in order to get a Hilbert manifold. However, since $\Omega^1(M;\mathfrak{u}E)$ is dense in the Sobolev space, it is safe to work within $\Omega^1(M;\mathfrak{u}E)$ equipped with its inner product. Issues of convergence are dealt with in the larger Sobolev space but are not relevant to this dissertation. See [3] for basics on Sobolev functors in a similar context. $\square$

2.5 Constructions with metrics and connections

Example 2.5.1. Let $\pi : E\to M$ be a $\mathbb{F}$-vector bundle. Every smooth map $\phi : N\to M$ induces a map of connections $\phi^* : \mathcal{A}(E)\to\mathcal{A}(\phi^*E)$ via pullback to the pullback bundle $\phi^*E\to N$. By definition $\phi^*(\mathcal{F}(E))\subset\mathcal{F}(\phi^*E)$. As a special case, given a path $\gamma : [0,1]\to M$, every connection $\nabla\in\mathcal{A}(E)$ gives a connection $\gamma^*\nabla$ on the vector bundle $\gamma^*E\to[0,1]$.

Example 2.5.2. Let ∇ be a connection on $\pi : E\to M$. There is an induced connection $\widetilde{\nabla}\in\mathcal{A}(\mathrm{End}E)$ on the vector bundle $\mathrm{End}E\to M$. First recall that every $\Phi\in\Omega^k(M;\mathrm{End}E)$ corresponds to a unique $\Omega^0(M)$-linear map $\Phi : \Gamma(E)\to\Omega^k(M;E)$. See 2.4. Secondly, observe that $[\nabla, \Phi] : \Gamma(E) \to \Omega^{k+1}(M; E)$ is also $\Omega^0(M)$-linear:

$$\begin{aligned}
[\nabla, \Phi](fs) &= \big(\nabla\Phi - (-1)^k \Phi\nabla\big) fs = \nabla\Phi(fs) - (-1)^k \Phi\nabla(fs) \\
&= \nabla(f\Phi s) - (-1)^k \Phi(df \otimes s + f\nabla s) \\
&= df \wedge \Phi s + f\nabla(\Phi s) - (-1)^k \Phi(df \otimes s) - (-1)^k \Phi(f\nabla s) \\
&= df \wedge \Phi s + f\nabla(\Phi s) - df \wedge \Phi s - (-1)^k f\Phi(\nabla s) \\
&= f\big(\nabla\Phi - (-1)^k \Phi\nabla\big)s.
\end{aligned}$$

Therefore $[\nabla, \Phi]$ corresponds to a unique element of $\Omega^{k+1}(M; \mathrm{End}E)$. Implicitly using the identifications $\Omega^\bullet(M; \mathrm{End}E) \cong \mathrm{Hom}_{\Omega^0(M)-Mod}(\Gamma(E), \Omega^\bullet(M; E))$, one defines

$$\widetilde{\nabla} := [\nabla, -] : \Omega^\bullet(M; \mathrm{End}E) \to \Omega^\bullet(M; \mathrm{End}E) \tag{2.13}$$

and verifies that $\widetilde{\nabla}$ is indeed a connection on $\mathrm{End}E$. For all $(f, \sigma) \in \Omega^0(M) \times \Gamma(\mathrm{End}E)$

$$\begin{aligned}
\widetilde{\nabla}(f\sigma) &= [\nabla, f\sigma] = \nabla(f\sigma) - f\sigma(\nabla) = df \otimes \sigma + f\nabla\sigma - f\sigma\nabla \\
&= df \otimes \sigma + f(\nabla\sigma - \sigma\nabla) = df \otimes \sigma + f[\nabla, \sigma] = df \otimes \sigma + f\widetilde{\nabla}\sigma
\end{aligned}$$

so $\widetilde{\nabla} \in \mathcal{A}(\mathrm{End}E)$. Furthermore for all $\sigma \in \Gamma(\mathrm{End}E)$

$$\widetilde{\nabla}(\widetilde{\nabla}\sigma) = \widetilde{\nabla}(\nabla\sigma - \sigma\nabla) = \nabla(\nabla\sigma - \sigma\nabla) + (\nabla\sigma - \sigma\nabla)\nabla = \nabla^2\sigma - \sigma\nabla^2$$

hence $\widetilde{\nabla}$ is flat whenever ∇ is flat.

Example 2.5.3. A connection ∇ on E induces a connection $\nabla^\vee$ on the dual bundle $E^\vee := \mathrm{Hom}_{VB}(E, \mathbb{F})$ by requiring that for all $(\phi, s) \in \Gamma(E^\vee) \times \Gamma(E)$, the *function* $\phi(s) : M \ni x \mapsto$

$\phi(x)(s(x)) \in \mathbb{F}$ satisfies

$$d(\phi(s)) = (\nabla^\vee \phi)s + \phi(\nabla s). \tag{2.14}$$

Indeed $\phi \mapsto f\phi$ gives

$$\nabla^\vee f\phi(s) + f\phi(\nabla s) = d(f\phi(s)) = df \wedge \phi(s) + f\nabla^\vee \phi(s) + f\phi(\nabla s)$$

$$\Rightarrow \nabla^\vee f\phi = df \wedge \phi + f\nabla^\vee \phi \Rightarrow \nabla^\vee \in \mathcal{A}(E^\vee).$$

Observe that $\nabla^\vee$ is flat if and only if ∇ is flat:

$$\begin{aligned} 0 = d^2(\phi(s)) &= d\big((\nabla^\vee \phi)s + \phi(\nabla s)\big) \\ &= (\nabla^\vee \nabla^\vee \phi)s - (\nabla^\vee \phi)\nabla s + (\nabla^\vee \phi)\nabla s - \phi\nabla\nabla s \\ &= (\nabla^\vee \nabla^\vee \phi)s - \phi(\nabla\nabla s). \end{aligned}$$

Example 2.5.4. If ∇ is a connection on $E \to M$, and if $E_1 \to M$ is another vector bundle with connection ∇_1 then $\nabla \otimes 1 + 1 \otimes \nabla_1 \in \mathcal{A}(E \otimes E_1)$. Under the identification $E^\vee \otimes E \cong \mathrm{End}E$, the connection $\nabla^\vee \otimes 1 + 1 \otimes \nabla$ corresponds to $\widetilde{\nabla}$. Explicitly, the identification is given by $\phi \otimes s : v \mapsto \phi(v)s$ and thus

$$\begin{aligned} (\nabla^\vee \otimes 1 + 1 \otimes \nabla)(\phi \otimes s)v &= (\nabla^\vee \phi \otimes s + \phi \otimes \nabla s)v = (\nabla^\vee \phi)v \otimes s + \phi v \otimes \nabla s \\ &= d(\phi(v)) \otimes s - \phi(\nabla v) \otimes s + \phi v \otimes \nabla s \\ &= \nabla(\phi(v)s) - \phi(\nabla v) \otimes s = (\nabla \circ \phi \otimes s)v - (\phi \otimes s \circ \nabla)v \\ &= [\nabla, \phi \otimes s](v) = \widetilde{\nabla}(\phi \otimes s)(v) \Rightarrow \nabla^\vee \otimes 1 + 1 \otimes \nabla = \widetilde{\nabla}. \end{aligned}$$

Example 2.5.5. If h is a metric on E then there is an induced metric $h^\vee$ on the dual bundle $E^\vee$. Simply use the metric h to induce the canonical (anti-linear) isomorphism $\flat : E \to E^\vee$ defined by $v^\flat(w) := h(w, v)$ whose inverse is denoted $\sharp : E^\vee \to E$. Define the metric $h^\vee$ on $E^\vee$ by $h^\vee(f, g) = h(g^\sharp, f^\sharp)$.

$\nabla^\vee$ is compatible with $h^\vee$ whenever ∇ is compatible with h. To see this consider the natural extension $\flat : \Omega^1(M; E) \leftrightarrows \Omega^1(M; E^\sharp) : \sharp$ given by $\flat(\phi \otimes s) = \bar{\phi} \otimes s^\flat$, and similarly $\sharp(\psi \otimes f) = \bar{\psi} \otimes f^\sharp$. For all $(\phi \otimes f, g) \in \Omega^1(M; E^\vee) \times \Omega^0(M; E^\vee)$

$$h^\vee(\phi \otimes f, g) = \phi h^\vee(f, g) = \phi h(g^\sharp, f^\sharp) = h(g^\sharp, \bar{\phi} \otimes f^\sharp) = h\big(g^\sharp, \sharp(\phi \otimes f)\big)$$

$$h^\vee(g, \phi \otimes f) = \bar{\phi} h^\vee(g, f) = \bar{\phi} h(f^\sharp, g^\sharp) = h(\bar{\phi} \otimes f^\sharp, g^\sharp) = h\big(\sharp(\phi \otimes f), g\big).$$

Using that h is compatible with ∇ and that $h(s, t) = t^\flat(s)$ for all $s, t \in \Gamma(E)$,

$$dh(s, t) = h(\nabla s, t) + h(s, \nabla t) = t^\flat(\nabla s) + h(s, \nabla t)$$

$$= d(t^\flat(s)) = (\nabla^\vee t^\flat)s + t^\flat(\nabla s)$$

$$\Leftrightarrow \nabla^\vee t^\flat = (\nabla t)^\flat$$

$$\Leftrightarrow \nabla^\vee(f) = (\nabla(f^\sharp))^\flat \text{ for all } f \in \Gamma(E^\vee)$$

which gives another way to characterize the dual connection whenever there is a compatible bundle metric. At this point the compatibility between $\nabla^\vee$ and $h^\vee$ is immediate:

$$dh^\vee(f, g) = dh(g^\sharp, f^\sharp) = h(\nabla g^\sharp, f^\sharp) + h(g^\sharp, \nabla f^\sharp)$$

$$= h^\vee(f, \nabla^\vee g) + h^\vee(\nabla^\vee f, g).$$

Example 2.5.6. End$E$ inherits a metric $\widetilde{h}$ from a metric h on E: give $E^\vee \otimes E \cong \mathrm{End}E$ the metric $\widetilde{h}(\phi \otimes s, \psi \otimes t) := h^\vee(\phi, \psi)h(s,t)$. Again, if ∇ is compatible with h then $\widetilde{\nabla}$ is compatible with $\widetilde{h}$:

$$\begin{aligned} d\widetilde{h}(\phi \otimes s, \psi \otimes t) &= d\big(h^\vee(\phi,\psi)h(s,t)\big) = dh^\vee(\phi,\psi)h(s,t) + h^\vee(\phi,\psi)dh(s,t) \\ &= h^\vee(\nabla^\vee\phi,\psi)h(s,t) + h^\vee(\phi,\nabla^\vee\psi)h(s,t) + h^\vee(\phi,\psi)h(\nabla s,t) + h^\vee(\phi,\psi)h(s,\nabla t) \\ &= \widetilde{h}(\nabla^\vee\phi \otimes s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \nabla^\vee\psi \otimes t) + \widetilde{h}(\phi \otimes \nabla s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \psi \otimes \nabla t) \\ &= \widetilde{h}(\nabla^\vee\phi \otimes s, \psi \otimes t) + \widetilde{h}(\phi \otimes \nabla s, \psi \otimes t) + \widetilde{h}(\phi \otimes s, \nabla^\vee\psi \otimes t) + \widetilde{h}(\phi \otimes s, \psi \otimes \nabla t) \\ &= \widetilde{h}(\widetilde{\nabla}(\phi \otimes s), \psi \otimes t) + \widetilde{h}(\phi \otimes s, \widetilde{\nabla}(\psi \otimes t)). \end{aligned}$$

Example 2.5.7. If (E,h) is a hermitian bundle, the vector bundle $\mathfrak{u}E \subset \mathrm{End}E$ is the sub-bundle consisting of h-skew-hermitian endomorphisms. Note that $\mathfrak{u}E$ is a *real* vector bundle and that the connection $\widetilde{\nabla} \in \mathcal{A}(\mathrm{End}E)$ restricts to a connection on $\mathfrak{u}E$ which will also be written $\widetilde{\nabla}$. Similarly, the (hermitian) metric $\widetilde{h}$ on End$E$ restricts to give a (real) metric on $\mathfrak{u}E$ which is also written $\widetilde{h}$.

The metric $\widetilde{h}$ on End$E$ and on $\mathfrak{u}E$ is the familiar metric $\widetilde{h}(A,B) = \mathbf{Tr}(B^*A)$ where B^* denotes the adjoint of B. Indeed, under the identification $\mathrm{End}E \cong E^\vee \otimes E$, conjugate transpose is $(\phi \otimes v)^* = v^\flat \otimes \phi^\sharp$, $\mathbf{Tr} = \mathrm{ev} : E^\vee \otimes E \to \mathbb{C}$ is $\mathbf{Tr}(\phi \otimes v) = \phi(v)$ and $(\psi \otimes w) \circ (\phi \otimes v) = \phi \otimes w\psi(v) \in E^\vee \otimes E \cong \mathrm{End}E$ so that

$$\begin{aligned} \mathbf{Tr}\Big((\psi \otimes w)^* \circ (\phi \otimes v)\Big) &= \mathbf{Tr}\Big(w^\flat \otimes \psi^\sharp) \circ (\phi \otimes v)\Big) = \mathbf{Tr}\Big(\phi \otimes \psi^\sharp w^\flat(v)\Big) = \phi\big(\psi^\sharp w^\flat(v)\big) \\ &= \phi(\psi^\sharp)w^\flat(v) = h(\psi^\sharp,\phi^\sharp)h(v,w) = h^\vee(\phi,\psi)h(v,w) = \widetilde{h}(\phi \otimes v, \psi \otimes w). \end{aligned}$$

Remark. We will use the notation **Tr** only when applied to linear operators. The notation ev will be used more generally to denote the pairing of a dual object with an object.

Remark. Restating the definition of a unitary connection, a connection ∇ on a hermitian bundle $(E, h) \to M$ is h-unitary if and only if

$$h : \Big(\Omega^\bullet(M; E) \otimes \Omega^\bullet(M; E), \nabla \otimes 1 + 1 \otimes \nabla\Big) \to \Big(\Omega^\bullet(M), d\Big)$$

is a chain map.

Proposition 2.5.8. If $\nabla \in \mathcal{F}(E, h)$ is a flat unitary connection, the tangent space $T_\nabla \mathcal{F}(E, h)$ is naturally identified with $Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) := \ker \widetilde{\nabla} \cap \Omega^1(M; \mathfrak{u}E)$.

Proof. $T_\nabla \mathcal{F}(E, h) \subset T_\nabla \mathcal{A}(E, h) = \Omega^1(M; \mathfrak{u}E)$. Let $\Phi \in \Omega^1(M; \mathfrak{u}E) = T_\nabla \mathcal{A}(E, h)$. Then $\Phi \in T_\nabla \mathcal{F}(E, h)$ if and only if $(\nabla + \Phi) \circ (\nabla + \Phi) \equiv 0 \mod \Phi^2$. This holds if and only if $0 = \nabla\Phi + \Phi\nabla = [\nabla, \Phi] = \widetilde{\nabla}(\Phi)$. □

2.6 Connections locally and explicit formula for parallel translation

Definition 2.6.1. Let $\pi : E \to M$ be a vector bundle and $U \subset M$ an open set. A *frame* on $E|_U \to U \subset M$ is a set $\{s_i\}_{i=1}^r \subset \Gamma(E|_U)$ of (local) sections such that, for every $x \in U$, the set of vectors $\{s_i(x)\}_{i=1}^r \subset E_x$ is a basis.

Remark. Given a frame, we write $s = (s_1, \ldots, s_r)^T$ for the column vector of sections. Then an arbitrary local section can be written $as = a^i s_i = \sum_i a^i s_i$ where each $a^i : U \to \mathbb{F}$ is a (local) function. We will view coefficients (with respect to a frame s) as a row vector $a = (a^1, \dots, a^r)$ of functions or, equivalently, as a vector-valued function $a : U \to \mathbb{F}^r$.

Example 2.6.2. A vector bundle $\pi : E \to M$ is locally trivial by definition and thus for every $m \in M$ there is an open neighborhood $U \ni m$ such that $\phi_U : \pi^{-1}(U) \cong U \times \mathbb{F}^r$. One can define a frame by $\{\phi_U^{-1}(x, e_i)\}$ using the local trivialization ϕ_U and any basis $\{e_i\}$ of $\mathbb{F}^r$.

Definition 2.6.3. Let ∇ be a connection on a vector bundle $\pi : E \to M$ of rank r. The *local connection forms of* ∇ *with respect to a local frame* $\{s_i\}_{i=1}^r$ on $E|_U \to U$ are the differential forms $\Theta_i^j \in \Omega^1(U)$ defined by

$$\nabla(s_i) = \sum_j \Theta_i^j \otimes s_j \in \Omega^1(U) \otimes \Gamma(E|_U). \tag{2.15}$$

Remark. For an arbitrary local section one has

$$\nabla(as) = da \otimes s + a\Theta \otimes s. \tag{2.16}$$

This is abbreviated by writing $\nabla|_U = d + \Theta$ (with respect to the frame s). According to the notation $\Theta(a \otimes s) = a\Theta \otimes s$ so the matrix Θ can be viewed either as an element of $\Omega^1(U, gl(\mathbb{F}^r))$ which acts on the coefficient $a : U \to \mathbb{F}^r$ from the right or as an element of $\Omega^1(U; \mathrm{End}(E|_U))$ that acts on s from the left. Of course, using the local trivialization $\Omega^1(U; \mathrm{End}E|_U) \cong \Omega^1(U; gl(\mathbb{F}^r))$ so this is consistent.

Proposition 2.6.4. Let $\{\sigma_i\}$ be a local frame over $U \subset M$ with local connection form Θ. Then $\{\tau_i\}$ is another local frame over U with connection form Φ if and only if there is some

$g : U \to \mathrm{Aut}(E|_U)$ such that

$$\Phi = dgg^{-1} + g\Theta g^{-1}. \tag{2.17}$$

Proof. Since σ is a frame, τ is another frame if and only if there is a unique $g : U \to \mathrm{Aut}(E|_U)$ such that $\tau = g\sigma$ and therefore

$$\Phi\tau = \nabla(\tau) = \nabla(g\sigma) = dg \otimes \sigma + g\Theta \otimes \sigma = dg \otimes g^{-1}\tau + g\Theta \otimes g^{-1}\tau = dgg^{-1} \otimes \tau + g\Theta g^{-1} \otimes \tau$$

$$\Leftrightarrow \Phi = dgg^{-1} + g\Theta g^{-1}.$$

□

Remark. One uses local trivializations to work with a connection: give M an open cover $\{U_i\}$ of local trivializations with local connections $d + \Theta(i)$ on $E|_{U_i} \to U_i$ which combine by the condition $\Theta(i) = dg_{ij}g_{ij}^{-1} + g_{ij}\Theta(j)g_{ij}^{-1}$ on $E|_{U_i \cap U_j}$ where $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(E|_{U_i \cap U_j})$ is the transition function on $U_i \cap U_j$.

Example 2.6.5. Let $\pi : E \to M$ be a rank r vector bundle with connection ∇ and let $\gamma : [0,1] \to M$ be given. Since $\gamma^*E \to [0,1]$ is a trivial bundle it admits a global frame $s = (s, \ldots, s_r)^T \subset \Gamma(\gamma^*E)$ and a corresponding connection form Θ where $\gamma^*\nabla(s) = \Theta \otimes s$. An arbitrary lift $as = \sum_i a^i s_i \in \Gamma(\gamma^*E)$ of γ is flat if and only if $0 = \gamma^*\nabla(a\sigma) = da \otimes s + a\Theta \otimes s$. This gives a condition on the vector valued function a

$$da + a\Theta = 0 \tag{2.18}$$

which is an equation of vector-valued 1-forms on $[0,1]$. The right side vanishes if and only if it vanishes on the canonical unit vector field $X_t := \frac{d}{dt} \in \Gamma(T[0,1])$. Hence the coefficient sections a of a flat lift are determined by the differential equation

$$\frac{da(t)}{dt}a(t) + a(t)\Theta(X_t) = 0. \tag{2.19}$$

By abuse of notation we write $\Theta(t) = \Theta(X_t) = \nabla(\dot{\gamma}(t))$ for convenience.

$$\begin{aligned}
a(t) &= a(0) - \int_0^t a(t_1)\Theta(t_1)dt_1 = a(0) - \int_0^t \Big(a(0) - \int_0^{t_1} a(t_2)\Theta(t_2)dt_2\Big)\Theta(t_1)dt_1 \\
&= a(0) - \int_0^t a(0)\Theta(t_1)dt_1 + \int_0^t \int_0^{t_1 \le t} a(0)a(t_2)\Theta(t_2)\Theta(t_1)dt_2dt_1 \\
&= a(0) - \int_0^t a(0)\Theta(t_1)dt_1 + \int_0^t \int_0^{t_1 \le t} a(0)\Theta(t_2)\Theta(t_1)dt_2dt_1 \\
&- \int_0^t \int_0^{t_1 \le t} \int_0^{t_2 \le t_1} a(0)a(t_3)\Theta(t_3)\Theta(t_2)\Theta(t_1)dt_3dt_2dt_1 \\
\Rightarrow a(t) &= \sum_{n=0}^{\infty} (-1)^n a(0) \int_{0 \le t_n \le \cdots \le t_1 \le t} \Theta(t_n)\ldots\Theta(t_1)dt_n \ldots dt_1.
\end{aligned}$$

Hence the solution to the differential equation $da + a\Theta = 0$ is given by

$$a(t) = \sum_{n=0}^{\infty} (-1)^n a(0) \int_{0 \le t_n \le \cdots \le t_1 \le t} \Theta(t_n)\ldots\Theta(t_1)dt_n \ldots dt_1. \tag{2.20}$$

This computes the coefficients a required so that as is a flat section.

Proposition 2.6.6. The series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \int_{0 \le t_n \le \cdots \le t_1 \le 1} \Theta(t_n)\ldots\Theta(t_1)dt_n \ldots dt_1$ is absolutely convergent.

Proof. Since $\Theta \in \Omega^1([0,1], \gamma^*\mathrm{End}E)$ is a smooth form on a compact object, there exists $B < \infty$

such that $||\Theta(t)|| \leq B$ for all $t \in [0,1]$. Hence the absolute series is

$$\sum_{n=0}^{\infty} \int_{0 \leq t_n \leq \cdots \leq t_1 \leq 1} \Big|\Big|\Theta(t_n) \ldots \Theta(t_1) dt_n \ldots dt_1\Big|\Big| \leq \sum_{n=0}^{\infty} \int_{0 \leq t_n \leq \cdots \leq t_1 \leq 1} B^n dt_n \ldots dt_1$$
$$= \sum_{n=0}^{\infty} \int_{[0,1]^n} \frac{B^n}{n!} dt_n \ldots dt_1 \leq \exp(B).$$

□

Definition 2.6.7. For $a \leq b \in [0,1]$ we define

$$p_\gamma(\nabla; \Theta^s)_a^b := \sum_{n \geq 0} (-1)^n \int_{a \leq t_n \leq \cdots \leq t_1 \leq b} \Theta(t_n) \ldots \Theta(t_1) dt_n \ldots dt_1 \in gl(\mathbb{F}^k). \tag{2.21}$$

The notation Θ^s serves as a reminder that the formula involves the connection form Θ which is defined by a choice of frame s.

Definition 2.6.8. Parallel translation along γ is given by

$$\begin{aligned} P_\gamma(\nabla; \Theta^s)_a^b : E_{\gamma(a)} &\to E_{\gamma(b)} \\ c(a)s(a) &\mapsto \big(c(a) p_\alpha(\nabla; \Theta^s)_a^b\big) s(b) \end{aligned} \tag{2.22}$$

which is the linear isomorphism obtained by parallel lifts of γ that connect the fibers $E_{\gamma(a)}$ and $E_{\gamma(b)}$.

Proposition 2.6.9. *$P_\alpha(\nabla; \Theta^s)_a^b : E_{\alpha(a)} \to E_{\alpha(b)}$ is independent of the choice of the frame s.*

Proof. Let $\tau = (\tau_1, \ldots, \tau_r)^T$, $\tau_i \in \Gamma(\alpha^* E)$, be another frame. Write Φ for the corresponding connection matrix 1-form. Since τ and s are both frames, there exists some $g : [0,1] \to \mathrm{Aut}(\alpha^* E)$ such that $\tau = gs$ and $\Phi = dgg^{-1} + g\Theta g^{-1}$. The coefficients a of s are determined by $da + a\Theta = 0$;

the coefficients b of τ are determined by $db + b\Phi = 0$. Fixing $a(0)s(0) = b(0)\tau(0)$, a and b are the unique solutions to their initial value problem. Now $0 = db + b\Phi = db + bdgg^{-1} + bg\Theta g^{-1}$ if and only if $0 = dbg + bdg + bg\Theta = d(bg) + (bg)\Theta$. By uniqueness, we must have $a = bg$, and hence $as = bgs = b\tau$. Thus $p_\alpha(\nabla; \Theta^s) = p_\alpha(\nabla; \Phi^\tau)$. □

Definition 2.6.10. The parallel translation with respect to ∇ along $\gamma : [0,1] \to M$ is denoted

$$P_\gamma(\nabla)_a^b : E_{\gamma(a)} \to E_{\gamma(b)}. \tag{2.23}$$

$P_\gamma(\nabla)$ without mention of a and b means parallel transport from $E_{\gamma(0)}$ to $E_{\gamma(1)}$.

2.7 Holonomy in unitary bundles

Proposition 2.7.1. Let $(E, \nabla, h) \to M$ be a unitary vector bundle. For every $\gamma : [0,1] \to M$ the linear isomorphism $P_\gamma(\nabla)$ is unitary.

Proof. If $v, w \in E_{\gamma(0)}$, then $h\big(P_\gamma(\nabla)v, P_\gamma(\nabla)w\big) = h\big(\widetilde{\gamma}_v(1), \widetilde{\gamma}_w(1)\big)$ where $\widetilde{\gamma}_v$ and $\widetilde{\gamma}_w$ are ∇-flat lifts. But since ∇ and h are compatible

$$dh(\widetilde{\gamma}_v, \widetilde{\gamma}_w) = h(\nabla\widetilde{\gamma}_v, \widetilde{\gamma}_w) + h(\widetilde{\gamma}_v, \nabla\widetilde{\gamma}_w) = 0$$

so that $h\big(P_\gamma(\nabla)v, P_\gamma(\nabla)w\big) = h(v, w)$ and thus $P_\gamma(\nabla)$ preserves the metric h. □

Proposition 2.7.2. The linear transformation $p_\alpha(\nabla, \Theta^s)_0^1$, which acts on the coefficients determined by a frame s, is unitary.

Proof.

$$\begin{aligned}
\Big(p_\alpha(\nabla,\Theta^s)_0^1\Big)^* &= \Big(\sum_{n=0}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}\Theta(t_n)\ldots\Theta(t_1)dt_n\ldots dt_1\Big)^*\\
&=\sum_{n=0}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}\Big(\Theta(t_n)\ldots\Theta(t_1)\Big)^* dt_n\ldots dt_1\\
&=\sum_{n=0}^{\infty}(-1)^n\int_{0\le t_n\le\cdots\le t_1\le 1}\Theta^*(t_1)\ldots\Theta^*(t_n)dt_n\ldots dt_1\\
&=\sum_{n=0}^{\infty}\int_{0\le t_n\le\cdots\le t_1\le 1}\Theta(t_1)\ldots\Theta(t_n)dt_n\ldots dt_1.
\end{aligned}$$

Performing the change of variable $t'_k := 1 - t_k$, hence $dt'_k = -dt_k$, the above becomes

$$\sum_{n=0}^{\infty}(-1)^n\int_{0\le t'_n\le\cdots\le t'_1\le 1}\Theta(t'_n)\ldots\Theta(t'_1)dt'_n\ldots dt'_1$$

which is $p_{\gamma^{-1}}(\nabla,\Theta^S) : E_{\gamma(1)} \to E_{\gamma(0)}$ the linear operator that computes the coefficients of parallel translation along the reverse of γ. Thus $\big(p_\gamma(\nabla,\Theta^s)\big)^* = p_{\gamma^{-1}}(\nabla,\Theta^s) = \big(p_\gamma(\nabla,\Theta^s)\big)^{-1}$. □

Corollary 2.7.3. If $(E, h, \nabla) \to M$ is a *flat* unitary bundle, the holonomy representation is unitary

$$\mathrm{Hol}_{(-)}(\nabla) : \pi_1(M, x) \to U(E_x). \tag{2.24}$$

Proof. By 2.3.5 parallel translation along a path with respect to a flat connection is independent of the homotopy class of the path. □

Corollary 2.7.4. If (E, h) is a hermitian vector bundle, then the domain of the holonomy function can be restricted to give the function

$$\mathrm{Hol} : \mathcal{F}(E, h) \times \pi_1(M, x) \to U(E_x). \tag{2.25}$$

$\square$

2.8 Derivative of trace of holonomy functional

Let $\gamma : [0,1] \to M$ be a path in the base of a hermitian vector bundle $(\pi : E \to M, h)$. Parallel transport defines a function

$$P_\gamma : \mathcal{A}(E,h) \to \mathrm{Hom}_{\mathbb{C}-Mod}(E_{\gamma(0)}, E_{\gamma(1)})$$

whose range sits within the space of unitary transformations between the fibers over the end-points.

Theorem 2.8.1. The derivative of parallel transport $P_\gamma : \mathcal{A}(E,h) \to \mathrm{Hom}_{\mathbb{C}-Mod}(E_{\gamma(0)}, E_{\gamma(1)})$ at $\nabla \in \mathcal{A}(E,h)$ in the direction of $\Phi \in \Omega^1(M; \mathfrak{u}E) = T_\nabla \mathcal{A}(E,h)$ is given by

$$dP_\gamma(\nabla, \Phi) := \frac{dP_\gamma(\nabla + \epsilon\Phi)}{d\epsilon}|_{\epsilon=0} = -\int_0^1 P_\gamma(\nabla)_t^1 \circ \Phi(\dot\gamma(t)) \circ P_\gamma(\nabla)_0^t dt.$$

Proof. Every $\Phi \in \Omega^1(M; \mathrm{End}E)$ defines the connections $\nabla + \Phi \in \mathcal{A}(E)$ and $\gamma^*\nabla + \gamma^*\Phi \in \mathcal{A}(\gamma^*E)$. Using a frame s for γ^*E, write $\Theta = \Theta^s$ for the connection 1-forms corresponding to $\gamma^*\nabla$ and write $B = \gamma^*\Phi$. Note that Θ and B can be viewed as elements of $\Omega^1([0,1], gl(\mathbb{F}^k))$. For example, given $X \in \Gamma(T[0,1])$, $B(X) : [0,1] \to gl(\mathbb{F}^k)$ is defined by $\gamma^*\Phi(c^i s_i)(X) : t \mapsto c^i(t)B(X_t)s_i(t)$, i.e. $B(X)$ acts on the coefficients from the right. (Of course, one can also view B as an element of $\Omega^1([0,1]; \mathrm{End}(\alpha^*E))$ which acts on *sections* from the left.) Recall that $p_\gamma(\nabla, \Theta^s)$ computes parallel translation of the coefficients of an arbitrary section $as \in \Gamma(\gamma^*E)$ where the coefficients are viewed as sections of the trivial bundle using the trivialization defined by the frame s.

Lemma 2.8.2.

$$\frac{dp_\gamma(\nabla+\epsilon\Phi;(\Theta+\epsilon B)^s)}{d\epsilon}|_{\epsilon=0} = -\int_0^1 \Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)dt. \tag{2.26}$$

Proof. This amounts to looking at

$$\sum_{n=0}^{\infty}(-1)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq 1}(\Theta(t_n)+\epsilon B(t_n))\dots(\Theta(t_1)+\epsilon B(t_1))dt_n\dots dt_1$$

and extracting the coefficient of ϵ modulo ϵ^2. We will use the fact 2.6.6 that the series is absolutely convergent in order to rearrange terms. The order one terms are

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq 1}\Big(\sum_{m=0}^{\infty}(-1)^m\int_{0\leq r_m\leq\cdots\leq r_1\leq t_n}\Theta(r_m)\dots\Theta(r_1)dr_m\dots dr_1\Big)\\
&\qquad\times B(t_n)\Theta(t_{n-1})\dots\Theta(t_1)dt_n\dots dt_1\\
&\qquad=\sum_{n=1}^{\infty}(-1)^n\int_{0\leq t_n\leq\cdots\leq t_1\leq 1}\Big(p_\gamma(\nabla,\Theta^s)_0^{t_n}\Big)B(t_n)\Theta(t_{n-1})\dots\Theta(t_1)dt_n\dots dt_1\\
&\qquad=-\int_{t=0}^1\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\sum_{n-1=0}^{\infty}(-1)^{n-1}\int_{t\leq t_{n-1}\leq\cdots\leq t_1\leq 1}\Theta(t_{n-1})\dots\Theta(t_1)dt_{n-1}\dots dt_1dt\\
&\qquad=-\int_{t=0}^1\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)dt.
\end{aligned}$$

□

Applying $-\Big(p_\gamma(\nabla,\Theta^s)_0^t\Big)B(t)\Big(p_\gamma(\nabla,\Theta^s)_t^1\Big)$ to the coefficients $a=(a^1,\dots,a^r)$ *from the right* results in first parallel transporting $-v=-a(0)s(0)\in E_{\gamma(0)}$ to $-P_\gamma(\nabla)_0^t v\in E_{\gamma(t)}$, applying $\Phi(\dot\gamma(t))\in \mathrm{End}(E_{\gamma(t)})$, and proceeding to parallel transport until arriving at $-P_\gamma(\nabla)_t^1\circ\Phi(\dot\gamma(t))\circ P_\gamma(\nabla)_0^t\in E_{\gamma(1)}$.

□

Corollary 2.8.3. If γ is a closed path, the derivative of holonomy

$$\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to U(E_{\gamma(0)})$$

at $\nabla \in \mathcal{A}(E,h)$ in the direction of $\Phi \in \Omega^1(M;\mathfrak{u}E)$ is given by

$$d\mathrm{Hol}_\gamma(\nabla,\Phi) = -\int_0^1 P_\gamma(\nabla)_t^1 \circ \Phi(\dot\gamma(t)) \circ P_\gamma(\nabla)_0^t dt. \tag{2.27}$$

Proof. If $\gamma : [0,1] \to M$ is a closed path then $\mathrm{Hol}_\gamma = P_\gamma : \mathcal{A}(E,h) \to U(E_{\gamma(0)})$. □

Corollary 2.8.4. The derivative of (the real) trace of holonomy $\Re\mathbf{Tr}\mathrm{Hol}_\gamma : \mathcal{A}(E,h) \to \mathbb{R}$ is given by

$$d(\Re\mathbf{Tr}\mathrm{Hol}_\gamma)(\nabla,\Phi) = -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt$$

where $\gamma^t : [0,1] \ni s \mapsto \gamma(s+t \mod 1) \in M$ denotes γ shifted up by t.

Proof.

$$\begin{aligned}
d(\Re\mathbf{Tr}\mathrm{Hol}_\gamma)(\nabla,\Phi) &= d(\Re\mathbf{Tr})_{\mathrm{Hol}_\gamma(\nabla)} \circ d\mathrm{Hol}_\gamma(\nabla,\Phi) = -\Re\mathbf{Tr}\Big(\int_0^1 P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))P_\gamma(\nabla)_0^t dt\Big) \\
&= -\int_0^1 \Re\mathbf{Tr}\Big(P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))P_\gamma(\nabla)_0^t\Big)dt = -\int_0^1 \Re\mathbf{Tr}\Big(P_\gamma(\nabla)_0^t P_\gamma(\nabla)_t^1\Phi(\dot\gamma(t))\Big)dt \\
&= -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.
\end{aligned}$$

□

Definition 2.8.5. For a given loop γ and flat unitary connection ∇, define the functional

$$\begin{aligned}
\mathbb{T} = \mathbb{T}_\nabla^\gamma &: Z_\nabla^1(M;\mathfrak{u}E) \to \mathbb{R} \\
&\Phi \mapsto -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.
\end{aligned} \tag{2.28}$$

Proposition 2.8.6. The functional $\mathbb{T}$ vanishes on $B^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ and therefore defines a functional

$$\mathbb{T} : H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \to \mathbb{R}. \tag{2.29}$$

Proof. If $\Phi \in B^1_{\widetilde{\nabla}}(M;\mathfrak{u}E) \subset Z^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$, then $\Phi = \widetilde{\nabla} f$ for some $f \in \Omega^0(M;\mathfrak{u}E)$ and therefore $\mathrm{Hol}_{\gamma^t}(\nabla)\widetilde{\nabla} f(\dot{\gamma}(t)) = \widetilde{\nabla}\Big(\mathrm{Hol}_{\gamma^t} f(\dot{\gamma}(t))\Big)$ since Hol_{γ^t} is flat by construction. As γ is closed, the result follows from Stokes' theorem. $\square$

Remark. Let $x = \gamma(0)$ be the basepoint of a closed path γ. It will be convenient to describe $d\mathrm{Hol}_\gamma(\nabla)$ and $\mathbb{T}$ by left-translating the images to the tangent space at the identity $\mathbf{1} \in UE_x$ in order to obtain an element of the Lie algebra $\mathfrak{u}E_x \cong \mathfrak{u}(r)$ where r is the rank of E. The method of doing this results in a *variation function* $F : UE_x \to \mathfrak{u}E_x$ as described by Goldman in [5].

Definition 2.8.7. Given a conjugate invariant function $f : UE_x \to \mathbb{R}$ (e.g. $f = \Re\mathbf{Tr}$) the *covariation function* $\hat{F} : UE_x \to (\mathfrak{u}E_x)^\vee$ is defined by the commutative diagram

$$\begin{tikzcd} T_u UE_x \arrow[r, "df_u"] & \mathbb{R} \\ T_{\mathbf{1}} UE_x = \mathfrak{u}E_x \arrow[u, "dLu"] \arrow[ur, dotted, "\hat{F}(u)"'] & \end{tikzcd}$$

where Lu is left multiplication by u.

Remark. By assumption h is a hermitian metric on E and $\widetilde{h}$ is the induced metric on (the *real* vector bundle) $\mathfrak{u}E$. In particular, $\widetilde{h}_x : \mathfrak{u}E_x \otimes \mathfrak{u}E_x \to \mathbb{R}$ is the metric $\widetilde{h}_x(A,B) = \mathbf{Tr}(B^*A) = -\mathbf{Tr}(BA) = -\Re\mathbf{Tr}(BA)$. The metric induces the linear isomorphisms $h^\flat : \mathfrak{u}E_x \leftrightarrows (\mathfrak{u}E_x)^\vee : \widetilde{h}^\sharp$.

Definition 2.8.8. The *variation function of a conjugate invariant function* $f : UE_x \to \mathbb{R}$ *with* *respect to the metric* $\widetilde{h}$ is the function

$$F = \widetilde{h}^{\sharp} \circ \hat{F} : UE_x \to \mathfrak{u}E_x.$$

Equivalently, the variation function is characterized by

$$\widetilde{h}(F(u), X) = \hat{F}(u)(X) = df_u(dLuX) = \frac{d}{ds}|_{s=0} f(ue^{sX}) \tag{2.30}$$

for all $u \in UE_x$ and for all $X \in \mathfrak{u}E_x$.

Remark. These definitions can be applied globally to a bundle. For example, if $f : UE \to M \times \mathbb{R}$ is a bundle map with each $f_x : UE_x \to \mathbb{R}$ conjugate invariant, then take the variation function $F_x : UE_x \to \mathfrak{u}_x$ at each fiber in order to obtain a global variation bundle map $F : UE \to \mathfrak{u}E$. This global context provides an optimal backdrop if one needs to consider different (homotopy classes of) closed curves with different basepoints.

Proposition 2.8.9. Let $(\pi : E \to M, h)$ be a hermitian vector bundle. The variation bundle map for $\Re\mathbf{Tr} : UE \to M \times \mathbb{R}$ with respect to the metric $\widetilde{h}$ on $\mathfrak{u}E$ is given by

$$F : UE \to \mathfrak{u}E \text{ where for all } x \in M \tag{2.31}$$

$$F_x : UE_x \ni A \mapsto \frac{-1}{2}\left(A - A^{-1}\right) \in \mathfrak{u}E_x.$$

Proof. For $A \in UE_x$ and $X \in \mathfrak{u}E$, $AA^* = \mathbf{1} \in UE_x$ and $X^* = -X$. Recall that $\widetilde{h}$ is defined on $\mathfrak{u}E$ via its restriction on the larger bundle $\mathrm{End}E$, and that both $A, X \in \mathrm{End}E_x$. Thus

$$\begin{aligned}\widetilde{h}\Big(\frac{-1}{2}(A-A^{-1}),X\Big) &= -\frac{1}{2}\widetilde{h}(A,X)+\frac{1}{2}\widetilde{h}(A^{-1},X) = -\frac{1}{2}\mathbf{Tr}(X^*A)+\frac{1}{2}\mathbf{Tr}(X^*A^{-1})\\ &= \frac{1}{2}\mathbf{Tr}(XA)+\frac{1}{2}\mathbf{Tr}(X^*A^*) = \frac{1}{2}\mathbf{Tr}(AX)+\frac{1}{2}\mathbf{Tr}((AX)^*)\\ &= \frac{1}{2}\mathbf{Tr}(AX)+\frac{1}{2}\overline{\mathbf{Tr}(AX)} = \Re\mathbf{Tr}(AX) = \frac{d}{ds}|_{s=0}\Re\mathbf{Tr}(Ae^{sX}).\end{aligned}$$

□

Proposition 2.8.10. For a given closed curve γ and flat unitary connection ∇, the functional $\mathbb{T}=\mathbb{T}^{\gamma}_{\nabla}$ can be expressed using the variation map $F: UE \to \mathfrak{u}E$:

$$\mathbb{T}(\Phi) = -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt.$$

Proof. Starting with 2.8.3

$$\begin{aligned}d(\mathbf{Tr}\mathrm{Hol}_\gamma)(\nabla,\Phi) &= -\int_0^1 \Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt = -\int_0^1 \Re\mathbf{Tr}\Big(\frac{d}{ds}|_{s=0}\mathrm{Hol}_{\gamma^t}(\nabla)\exp(s\Phi\dot\gamma(t))\Big)dt\\ &= -\int_0^1 \frac{d}{ds}|_{s=0}\Re\mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\exp(s\Phi\dot\gamma(t))\Big)dt\\ &= -\int_0^1 \widetilde{h}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla),\Phi(\dot\gamma(t))\Big)dt = -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot\gamma(t))\Big)dt\end{aligned}$$

where the penultimate step is given by 2.30. □

2.9 Twisted singular (co)homology

Let $\pi: E \to M$ be a smooth vector bundle equipped with a flat connection ∇. In this section we define the de Rham map

$$\mathbf{I}: \Big(\Omega^\bullet(M;E),\nabla\Big) \to \Big(S^\bullet_\nabla(M;E),\delta\Big)$$

from the differential graded module of differential forms on M with values in (E, ∇) to singular cochains on M with values in (E, ∇) and show that it is a quasi-isomorphism.

Definition 2.9.1. A *singular k-chain on M with values in a flat vector bundle* $(E, \nabla) \to M$ is a finite summation of pairs

$$(\sigma, s) \in S_k(M) \times \Gamma(\sigma^* E)$$

where $\sigma : \Delta^k \to M$ is an ordinary (smooth) elementary singular k-chain and $s \in \Gamma(\sigma^* E)$ is a $\sigma^*\nabla$-flat section of $\sigma^* E \to \Delta^k$. The *vector space of all (E, ∇)-valued singular k-chains* will be denoted by $S_k^{\nabla}(M; E)$ and $S_{\bullet}^{\nabla}(M; E) := \bigoplus_{k \geq 0} S_k^{\nabla}(M; E)$ is the *graded vector space of all (E, ∇)-valued singular chains.* $S_{\bullet}^{\nabla}(M; E)$ is equipped with the *boundary operator* ∂ defined using the usual boundary operator for ordinary singular homology

$$\partial(\sigma, s) := (\partial\sigma, s|_{\partial\sigma})$$

where $\partial\sigma$ is the alternating summation over the codimension 1-faces of $\sigma : \Delta^k \to M$ and $s|_{\partial\sigma}$ denotes the restriction of the section $s : \Delta^k \to \sigma^* E$ to the the boundary $\partial\Delta^k \subset \Delta^k$. By definition ∂ is linear and $\partial^2 = 0$ so that

$$\left(S_{\bullet}^{\nabla}(M; E), \partial\right)$$

is a chain complex. The homology of this chain complex is the *singular homology of M with values in* (E, ∇) and is denoted by $H_{\bullet}^{\nabla}(M; E)$.

Remark. "*Homology* with local coefficients" refers to singular homology. The space of singular cycles is $Z^{\nabla}_{\bullet}(M;E) := \ker\partial \cap S^{\nabla}_{\bullet}(M;E)$ and the space of boundaries is $B^{\nabla}_{\bullet}(M;E) = \partial\big(S^{\nabla}_{\bullet}(M;E)\big)$.

Definition 2.9.2. A *singular k-cochain with values in a flat vector bundle* (E,∇) is an element ϕ that assigns to every singular k-chain $\sigma : \Delta^k \to M$ a $\sigma^*\nabla$-flat section $\phi(\sigma) \in \Gamma(\sigma^* E)$.

Write $S^k_{\nabla}(M;E)$ for *the vector space of all (E,∇)-valued singular k-cochains* and $S^{\bullet}_{\nabla}(M;E) = \bigoplus_k S^k_{\nabla}(M;E)$ for the *graded vector space of all singular cochains with values in* (E,∇). $S^{\bullet}_{\nabla}(M;E)$ is equipped with the *coboundary operator* δ defined for all $(\phi,\sigma) \in S^k_{\nabla}(M;E) \times S_{k+1}(M)$ by the requirement that $(\delta\phi)(\sigma)$ is the (summation over the) unique flat sections that extend the flat sections $\phi(\partial\sigma) : \partial\Delta^{k+1} \to \sigma^* E$ over all of $\Delta^{k+1} \supset \partial\Delta^k$. The cohomology of the cochain complex $\left(S^{\bullet}_{\nabla}(M;E), \delta\right)$ is the *singular cohomology of M with values in* (E,∇) and is written $H^{\bullet}(M;E^{\nabla})$.

Definition 2.9.3. Given $\phi \in S^k_{\nabla}(M;E)$ and $\sigma \in S_j(M)$, their cap product $\sigma \cap_1 \phi \in S^{\nabla}_{j-k}(M;E)$ is constructed by taking the flat section $\phi(\sigma|_{\text{front } k-\text{face}})$ over the front k-face of σ, extending it by parallel transport to a flat section over all of σ, and then restricting to the back $j-k$ face of σ. The cap product $\sigma \cap_1 \phi$ is defined as the back $j-k$ face of σ equipped with the flat section just described.

Definition 2.9.4. If E is equipped with a metric h, then $\cap_h : S^{\nabla}_j(M;E) \times S^k_{\nabla}(M;E) \to S_{j-k}(M; M \times \mathbb{F}) = S_{j-k}(M,\mathbb{F})$ is given as follows. Let $\phi \in S^k_{\nabla}(M;E)$ and $(\sigma, s) \in S^{\nabla}_j(M;E)$. Then $\sigma \cap_1 \phi \in S^{\nabla}_{j-k}(M;E)$ is a flat section over the back face of σ. Restricting the flat section

$s : \Delta^j \to E$ gives another flat section over the back $j-k$ face of σ.

$$(\sigma, s) \cap_h \phi := h(\sigma \cap_1 \phi, \sigma|_{\text{back } j-k \text{ face}})$$

Definition 2.9.5. $\cup_1 : S^j_\nabla(M; E) \times S^k(M) \to S^{j+k}_\nabla(M; E)$ is defined in the usual manner. Given $\phi \in S^j_\nabla(M; E)$, $\psi \in S^k(M)$ and $\sigma : \Delta^{j+k} \to M$, then $(\phi \cup_1 \psi)(\sigma)$ assigns the flat section over Δ^{j+k} formed by taking the flat section $\phi(\sigma_{\text{front } j-\text{face}})$, extending over Δ^{j+k} using parallel transport, and scaling that section by $\psi(\sigma_{\text{back } k-\text{face}})$. We also write $\cup_1 : S^j(M) \times S^k_\nabla(M; E) \to S^{j+k}_\nabla(M; E)$ when the second argument is twisted. In particular $\cup_1 : S^j_\nabla(M; E) \times S^k_\nabla(M; E) \to S^{j+k}_\nabla(M; E \otimes E)$ is defined.

Definition 2.9.6. If E is equipped with a metric h, then $\cup_h : S^j_\nabla(M; E) \times S^j_\nabla(M; E) \to S^{j+k}(M; M \times \mathbb{F})$ is defined following the above. Given $\phi \in S^j_\nabla(M; E)$, $\psi \in S^k_\nabla(M; E)$ and $\sigma : \Delta^{j+k} \to M$, extend $\phi(\sigma_{\text{front } j-\text{face}})$ to a flat section over Δ^{j+k} and extend $\psi(\sigma_{\text{back } k-\text{face}})$ to a flat section over Δ^{j+k}. Use the metric h to pair the two flat sections into the flat section $(\phi \cup_h \psi)(\sigma)$ of the trivial line bundle.

Remark. The cap and cup product can be used to obtain twisted Poincaré duality.

Theorem 2.9.7 (twisted Poincaré duality)**.** If M is a closed orientable manifold of dimension n and (E, ∇) is a flat vector bundle over M then $[M]\cap_1 : H^k(M; E^\nabla) \to H^\nabla_{n-k}(M; E)$ is an isomorphism.

Proof. The proof of (ordinary) Poincaré duality [8] carries through in this twisted version. For a complete overview of twisted (co)homology theory including a proof of Poinaré duality see [17]. ☐

Definition 2.9.8. Given $\phi \in \Omega^k_\nabla(M;E)$ and $\sigma : \Delta^k \to M$, define the integral

$$\int_{\Delta^k} \sigma^*\phi := (\int_{\Delta^k} \alpha)s : \Delta^k \to E$$

where $\sigma^*\phi = \alpha \otimes s \in \Omega^k(\Delta^k) \otimes \Gamma(E)$ and s is a $\sigma^*\nabla$-flat section of $\sigma^*E \to \Delta^k$.

Remark. To see that the integral is well-defined, assume $\sigma^*\phi = \alpha \otimes s = \beta \otimes \tau$ where both s and τ are $\sigma^*\nabla$-flat so that there exists some $g \in \mathrm{Aut}(\sigma^*E)$ such that $\tau = gs$ and necessarily $dg = 0$. Then $\alpha \otimes s = \beta \otimes gs = \beta g \otimes s \Rightarrow \alpha = \beta g$ and therefore

$$(\int_{\Delta^k} \beta)\tau = (\int_{\Delta^k} \beta)gs = (\int_{\Delta^k} \beta g)s = (\int_{\Delta^k} \alpha)s.$$

Lemma 2.9.9. If $E \to U$ is a smooth vector bundle equipped with a flat connection ∇ over a contractible base U, then there exists a ∇-flat frame for E.

Proof. Pick any $v \in E$ and use parallel translation to extend v. This is well-defined because ∇ is flat and U is contractible. ☐

Corollary 2.9.10. If $E \to \Delta^n$ is a smooth vector bundle with flat connection ∇, every bundle-valued differential form $\Omega^k(\Delta^n;E)$ can be written as $\phi \otimes s \in \Omega^k(\Delta^n) \otimes_{\Omega^0(\Delta^n)} \Gamma(E)$ where $\nabla(s) = 0$ and therefore $\nabla(\phi \otimes s) = d\phi \otimes s$. Hence any ∇-closed differential form can be written $\phi \otimes s \in \Omega^k(\Delta^n) \otimes_{\Omega^0(\Delta^n)} \Gamma(E)$ where $\nabla(s) = 0$ and $d\phi = 0$. ☐

Definition 2.9.11. The de Rham map from differential forms to singular cochains is given by

$$\mathbf{I} : \Omega^\bullet_\nabla(M; E) \to S^\bullet_\nabla(M; E) \text{ where for } \phi \in \Omega^k_\nabla(M; E)$$
$$\mathbf{I}\phi(\sigma : \Delta^k \to M) = \int_{\Delta^k} \sigma^*\phi \in \Gamma(\sigma^* E \to \Delta^k)$$

and extended linearly over finite singular chains.

Lemma 2.9.12. $\mathbf{I} : \Omega^\bullet_\nabla(M; E) \to S^\bullet_\nabla(M; E)$ is a chain map.

Proof. Take any $\phi \in \Omega^k(M; E)$ and $\sigma : \Delta^{k+1} \to M$. Write $\sigma^*\phi = \alpha \otimes s \in \Omega^k(\Delta^{k+1}) \otimes \Gamma(\sigma^* E)$ where $\sigma^*\nabla(s) = 0$. Note that by definition of $\mathbf{I}$ and the singular boundary map ∂ one has

$$(\delta \mathbf{I}\phi)(\sigma) = (\mathbf{I}\phi)(\partial\sigma) = \int_{\partial\Delta^{k+1}} (\partial\sigma)^*\phi$$

following the established abuse of notation where the section over $\partial\Delta^{k+1}$ is extended to a flat section over all of Δ^{k+1}. On the other hand

$$(\mathbf{I}\nabla\phi)(\sigma) = \int_{\Delta^{k+1}} \sigma^*(\nabla\phi) = \int_{\Delta^{k+1}} (\sigma^*\nabla)(\sigma^*\phi) = \int_{\Delta^{k+1}} (\sigma^*\nabla)(\alpha \otimes s) = \int_{\Delta^{k+1}} (d\alpha \otimes s)$$
$$= (\int_{\Delta^{k+1}} d\alpha)s = (\int_{\partial\Delta^{k+1}} \alpha)s = \int_{\partial\Delta^{k+1}} \sigma^*\phi.$$

Both $(\partial\sigma)^*\phi$ and $\sigma^*\phi$ agree on $\partial\Delta^{k+1}$.

□

Example 2.9.13. Let $\phi \in Z^1_\nabla(M; E)$ and $\sigma : [0, 1] \to M$. Then $\sigma^*\phi \in Z^1_{\sigma^*\nabla}([0, 1]; \sigma^* E)$ so that one may choose a flat section $s \in \Gamma(\sigma^* E)$ and write $\sigma^*\phi = \alpha \otimes s \in \Omega^1([0, 1]) \otimes_{\Omega^0([0,1])} \Gamma(\sigma^* E)$. It follows that $d\alpha = 0$, and thus $\alpha = f(t)dt$ where $f(t) = c$ is a constant function. Write ∂_t for the basis of $T_t[0,1]$ which is dual to the basis dt of $T_t^\vee[0,1]$. Then

$$\phi(\dot{\sigma}(t)) = (\sigma^*\phi)(\partial_t) = (\alpha \otimes s)(\partial_t) = f(t)s(t) = cs(t)$$

and since $(\mathbf{I}\phi)(\sigma) = (\int_{[0,1]} \alpha)s = cs$, the section $(\mathbf{I}\phi)(\sigma) \in \Gamma(\sigma^*E \to [0,1])$ is given by $t \mapsto \phi(\dot{\sigma}(t))$.

Remark. If $\mathfrak{U}$ is a cover of M, then $S_\bullet^{\mathfrak{U}}(M) \subset S_\bullet(M)$ is the subspace of *singular chains subordinate to the cover* $\mathfrak{U}$ which consists of finite sums of elementary chains $\sigma : \Delta^k \to M$ such that there is some $U_\sigma \in \mathfrak{U}$ with $\sigma(\Delta^k) \subset U_\sigma$. The inclusion $S_\bullet^{\mathfrak{U}}(M) \hookrightarrow S_\bullet(M)$ is a chain-equivalence [2]. Thus, when dealing with singular homology of a smooth manifold M it is sufficient to work with smooth singular chains subordinate to some good cover $\mathfrak{U}$ of M.

Theorem 2.9.14 (twisted de Rham)**.** The de Rham map is a quasi-isomorphism.

Proof. Since M is a manifold it admits a good cover $\mathfrak{U}$. Singular cohomology with values in (E, ∇) satisfies the Mayer-Vietoris sequence [19]. This can be seen by working with singular chains subordinate to the cover $\mathfrak{U}$. For differential forms

$$0 \to \Omega^\bullet(U \cup V) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$

is a short exact sequence of graded $\Omega^0(U \cup V)$-modules. Hence

$$0 \to \Omega^\bullet(U \cup V) \otimes \Gamma(E) \to \Omega^\bullet(U) \otimes \Gamma(E) \oplus \Omega^\bullet(V) \otimes \Gamma(E) \to \Omega^\bullet(U \cap V) \otimes \Gamma(E) \to 0$$

is also a short exact sequence and we obtain the Mayer-Vietoris sequence for $H^\bullet_\nabla(U \cup V; E)$. If $U \subset M$ is contractible, then $H^\bullet_\nabla(U; E) = H^\bullet_\delta(U; E^\nabla) \cong \left(\mathbb{F}^r \to 0 \to \dots\right)$ and, since $\mathbf{I}$:

$H^0_\nabla(U;E) \to H^0_\delta(U;E^\nabla)$ is injective, $\mathbf{I} : H^\bullet_\nabla(U;E) \cong H^\bullet_\delta(U;E^\nabla)$. Using the good cover $\mathfrak{U}$ of M, the Mayer-Vietoris sequences and the five-lemma, the result follows by induction on the number of sets in $\mathfrak{U}$. □

Definition 2.9.15. Let $(\pi : E \to M, \nabla)$ be a flat bundle equipped with a compatible metric h. Given $\Phi \in \Omega^k_\nabla(M;E)$ and $(\sigma, s) \in S^\nabla_k(M;E)$, define the section

$$\langle \Phi, (\sigma, s)\rangle_h := h(\Phi(\sigma), s) : \Delta^k \to \mathbb{F}$$

which uses the metric h to pair $\Phi(\sigma) \in \Gamma(\sigma^* E)$ with $s \in \Gamma(\sigma^* E)$. Integrating over the domain of σ gives the following pairing

$$\int \langle -, - \rangle_h : \Omega^k_\nabla(M;E) \times S^\nabla_k(M;E) \to \mathbb{F} \tag{2.32}$$
$$\int \langle \Phi, (\sigma, s)\rangle_h = \int_{\Delta^k} \langle \Phi, (\sigma, s)\rangle_h.$$

Theorem 2.9.16. When M is a closed orientable manifold, the pairing above induces a non-degenerate pairing

$$H^k_\nabla(M;E) \times H^\nabla_k(M;E) \to \mathbb{F}. \tag{2.33}$$

Proof. This follows immediately from Poincaré duality and de Rham's theorem. □

2.10 Representing the functional $\mathbb{T}$

Proposition 2.10.1. Given a loop $\gamma : [0,1] \to M$ and flat unitary connection $\nabla \in \mathcal{F}(E,h)$, there is a twisted 1-cycle

$$\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$$

where $F\mathrm{Hol}_\gamma(\nabla) : t \mapsto F\mathrm{Hol}_{\gamma^t}(\nabla) \in \mathfrak{u}E_{\gamma(t)}$ uses the variation map $F : UE \to \mathfrak{u}E$.

Proof. The loop γ is an ordinary singular 1-cycle. $F\mathrm{Hol}_\gamma(\nabla) \in \Gamma(\gamma^*\mathfrak{u}E)$ is $\widetilde{\nabla}$-flat, and therefore the pair $(\gamma, F\mathrm{Hol}_\gamma(\nabla)) \in S_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$ is a twisted singular 1-chain. Furthermore, it is a twisted *cycle* because

$$\partial\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) = \big(\gamma(1), F\mathrm{Hol}_{\gamma^1}(\nabla)\big) - \big(\gamma(0), F\mathrm{Hol}_{\gamma^0}(\nabla)\big)$$

and $F\mathrm{Hol}_{\gamma^0}(\nabla) = F\mathrm{Hol}_{\gamma^1}(\nabla)$ as both are parallel translation around γ (in the same direction) starting at $\gamma(0) = \gamma(1)$. $\square$

Definition 2.10.2. The *holonomy cycle of* γ *with respect to* ∇ is $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$.

Theorem 2.10.3. Given $\nabla \in \mathcal{F}(E,h)$ and a closed curve γ, the functional $\mathbb{T} = \mathbb{T}_\nabla^\gamma : Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}$ is represented by the holonomy cycle $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \in Z_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$ in the sense that

$$\mathbb{T}(\Phi) = \int \Big\langle \Phi, \big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big) \Big\rangle_h$$

for every $\Phi \in Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$.

Proof. By de Rham's Theorem every $\Phi \in Z^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$ represents a *singular cocycle* with values in $\mathfrak{u}E$. Since γ is a(n ordinary) singular 1-chain, $\Phi(\gamma) = \mathbf{I}\Phi(\gamma)$ is the $\widetilde{\nabla}$-flat section of $\gamma^*\mathfrak{u}E$

given by $t \mapsto \Phi(\dot{\gamma}(t))$. More generally, if (γ, s) is a *twisted* singular 1-chain, then $\langle \Phi, (\gamma, s)\rangle : t \mapsto \Phi(\dot{\gamma}(t)) \otimes s(t)$ is a flat section of $\gamma^*(\mathfrak{u}E \otimes \mathfrak{u}E)$. Post-composing with $\widetilde{h} : \mathfrak{u}E \otimes \mathfrak{u}E \to \mathbb{R}$ yields a section of the trivial bundle over the $[0,1]$ (the domain of γ):

$$\langle \Phi, \gamma \otimes s\rangle_h : [0,1] \ni t \mapsto \widetilde{h}(\Phi(\dot{\gamma}(t)), s(t)) \in \mathbb{R}.$$

Integrating the above function gives a number and defines the pairing

$$\int \langle -, -\rangle_h : \Omega^1(M; \mathfrak{u}E) \times S_1^{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}$$

which, when applied to Φ and $\big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big)$, gives

$$\begin{aligned}\int \Big\langle \Phi, \big(\gamma, F\mathrm{Hol}_\gamma(\nabla)\big)\Big\rangle_h &= \int_0^1 \widetilde{h}\Big(\Phi(\dot{\gamma}(t)), F\mathrm{Hol}_\gamma(\nabla)\Big)dt \\ &= -\int_0^1 \mathbf{Tr}\Big(F\mathrm{Hol}_\gamma(\nabla)\Phi(\dot{\gamma}(t))\Big)dt = \mathbb{T}(\Phi).\end{aligned}$$

$\square$

Chapter 3

A motivational perspective

This chapter provides an overview of the *moduli space of flat unitary (irreducible) connections.*

There is a bijection between equivalence classes of flat hermitian vector bundles (which can be identified with flat principal $U(r)$-bundles) and conjugacy classes of representations $\rho : \pi_1(M) \to U(r)$. For compact manifolds $\pi_1(M)$ is finitely generated and therefore $\mathrm{Hom}(\pi_1(M), U(r)) \subset U(r)^{\#\text{of generators}}$. The quotient of this representation variety is one model for the moduli space of flat unitary connections. Given a flat unitary vector bundle $(E, \nabla) \to M$ of rank r, the corresponding representation is given by holonomy $\mathrm{Hol}(\nabla) : \pi_1(M, x) \to U(E_x) \cong U(r)$. In particular, the *trace of holonomy functions are the characters of the representation that determines* (E, ∇) *up to gauge equivalence* and this motivates our interest in the trace of holonomy functions.

Furthermore we give an informal explanation of how $H^1_{\nabla}(M; \mathfrak{u}E)$ may be viewed as (a model for) the tangent space to the class $[\nabla]$ within the moduli space.

3.1 Gauge equivalence

Definition 3.1.1. Let $\pi : E \to M$ be a vector bundle. The *automorphism bundle of* E is the group bundle $\mathrm{Aut}E \to M$ with fibers $(\mathrm{Aut}E)_x = \mathrm{Aut}(E_x) = GL(E_x)$. The *gauge group* $G(E)$ *of the bundle* E is the the group of sections $\Gamma(\mathrm{Aut}E)$. Two connections $\nabla_1, \nabla_2 \in \mathcal{A}(E)$ are *gauge equivalent* $\nabla_1 \sim \nabla_2$ if and only if there is an element $g \in G(E)$ such that $g \circ \nabla_1 = \nabla_2 \circ g$.

Remark. Elements of $\Gamma(\mathrm{Aut}E)$ are the same as vector bundle automorphisms of $\pi : E \to M$ that project to the identity on M.

Definition 3.1.2. The *gauge group acts on the space of connections* by

$$\mathcal{A}(E) \times G(E) \ni (\nabla, g) \mapsto g^{-1} \circ \nabla \circ g \in \mathcal{A}(E).$$

Proposition 3.1.3. $\nabla_1 \sim \nabla_2$ if and only if they are in the same orbit of the $G(E)$-action. □

Proposition 3.1.4. The gauge action preserves flat connections.

Proof. $(g^{-1} \circ \nabla \circ g) \circ (g^{-1} \circ \nabla \circ g) = g^{-1} \circ (\nabla \circ \nabla) \circ g$, hence $\nabla^2 = 0 \Leftrightarrow (\nabla \cdot g)^2 = 0$. □

Definition 3.1.5. The *(naïve) moduli space* $\mathcal{M}(E)$ *of flat connections on* $\pi : E \to M$ is the set of gauge equivalence classes of flat connections.

Definition 3.1.6. If $(E, h) \to M$ is a hermitian vector bundle, the *unitary gauge group* $G(E, h)$ of the hermitian bundle is the subgroup of the gauge group consisting of automorphisms that preserve the metric h.

Proposition 3.1.7. The subgroup $G(E,h)$ acts on the space $\mathcal{A}(E,h)$ of all unitary connections and also preserves the subspace $\mathcal{F}(E,h)$ of flat unitary connections.

Proof. Let $\nabla \in \mathcal{A}(E,h)$ and $\phi \in G(E,h)$. Then for all $s,t \in \Gamma(E)$

$$\begin{aligned} h\big((\nabla\cdot\phi)s,t\big)+h\big(s,(\nabla\cdot\phi)t\big) &= h\big(\phi^{-1}\nabla(\phi s),t\big)+h\big(s,\phi^{-1}\nabla(\phi t)\big) \\ &= h\big(\nabla(\phi s),\phi t\big)+h\big(\phi s,\nabla(\phi t)\big) = dh(\phi s,\phi t) = dh(s,t) \end{aligned}$$

using that both ∇ and ϕ are h-unitary. Therefore $\nabla\cdot\phi$ is a unitary connection. As $G(E,h)$ preserves both $\mathcal{A}(E,h)$ and $\mathcal{F}(E)$, it preserves $\mathcal{A}(E,h)\cap\mathcal{F}(E) = \mathcal{F}(E,h)$. $\square$

Definition 3.1.8. If (E,h) is a hermitian vector bundle, the *(naïve) moduli space* $\mathcal{M}(E,h)$ *of flat unitary connections* is the set of unitary gauge equivalence classes of flat unitary connections.

3.2 Holonomy representation and the moduli space

Definition 3.2.1. The *holonomy representation of a flat connection* $\nabla \in \mathcal{F}(E)$ is the representation

$$\mathrm{Hol}(\nabla) : \pi_1(M,x) \to \mathrm{Aut}(E_x) \tag{3.1}$$

of the fundamental group on E_x. The assignment $\nabla \mapsto \mathrm{Hol}(\nabla)$ is the *holonomy (representation) function*

$$\mathrm{Hol} : \mathcal{F}(E) \to \mathrm{Hom}\Big(\pi_1(M,x),\mathrm{Aut}(E_x)\Big). \tag{3.2}$$

Remark. If (E, h) is a hermitian vector bundle and $\nabla \in \mathcal{F}(E, h)$ is a flat unitary connection, holonomy takes values in unitary transformations and therefore the holonomy representation function restricts to

$$\mathrm{Hol} : \mathcal{F}(E, h) \to \mathrm{Hom}\Big(\pi_1(M, x), U(E_x)\Big). \tag{3.3}$$

Proposition 3.2.2. If g is an element of the (unitary) gauge group then

$$\mathrm{Hol}_\gamma(\nabla \cdot g) = g(\gamma(0))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0)).$$

Proof. Let $\widetilde{\gamma}_\bullet$ denote the family of ∇-parallel lifts of γ and let $\eta_\bullet$ denote the family of $\nabla{\cdot}g$-parallel lifts of γ. For example, $\widetilde{\gamma}_v$ is the ∇-parallel lift with $\widetilde{\gamma}_v(0) = v$. By definition $\mathrm{Hol}_\gamma(\nabla)(v) = \widetilde{\gamma}_v(1)$ and $\mathrm{Hol}_\gamma(\nabla \cdot g)(v) = \eta_v(1)$. Consider the action $g^{-1} \cdot \widetilde{\gamma}_\bullet := g^{-1} \circ \widetilde{\gamma}_\bullet : [0, 1] \to E \to E$. The family of lifts $g^{-1} \cdot \widetilde{\gamma}_\bullet$ is $\nabla \cdot g$ parallel if and only if $\gamma_\bullet$ is a ∇-parallel family of lifts:

$$\begin{aligned} 0 &= \gamma^*\nabla(\widetilde{\gamma}_\bullet) = (\gamma^*\nabla \circ g)(g^{-1} \circ \widetilde{\gamma}_\bullet) \\ \Leftrightarrow 0 &= (g^{-1} \circ \gamma^*\nabla \circ g)(g^{-1} \circ \widetilde{\gamma}_\bullet) = (\gamma^*\nabla \cdot g)(g^{-1} \cdot \widetilde{\gamma}_\bullet). \end{aligned}$$

By uniqueness, $\eta_\bullet = g^{-1} \cdot \widetilde{\gamma}_\bullet$ as families. Note that $v = \eta_v(0) = g^{-1}(\widetilde{\gamma}_\bullet(0)) \Leftrightarrow \widetilde{\gamma}_\bullet(0) = g(v)$, i.e. $\eta_v = g^{-1}\widetilde{\gamma}_{gv}$. Then

$$v \mapsto g(v) \mapsto \mathrm{Hol}_\gamma(\nabla)(g(v)) = \widetilde{\gamma}_{g(v)}(1) \mapsto g^{-1}\widetilde{\gamma}_{g(v)}(1) = \eta_v(1) = \mathrm{Hol}_\gamma(\nabla \cdot g)(v)$$

hence $\mathrm{Hol}_\gamma(\nabla \cdot g) = g(\gamma(1))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0)) = g(\gamma(0))^{-1}\mathrm{Hol}_\gamma(\nabla)g(\gamma(0))$. □

Corollary 3.2.3. The holonomy representation $\mathrm{Hol} : \mathcal{F}(E) \to \mathrm{Hom}\big(\pi_1(M, x), \mathrm{Aut}(E_x)\big)$ descends to a map

$$\mathrm{Hol} : \frac{F(E)}{G(E)} \to \frac{\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)}{\mathrm{Aut}(E_x)} \tag{3.4}$$

where $\mathrm{Aut}(E_x)$ acts on $\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)$ by post-composition with inner automorphisms. Similarly, if (E,h) is a hermitian bundle, then the map

$$\mathrm{Hol} : \frac{F(E,h)}{G(E,h)} \to \frac{\mathrm{Hom}\big(\pi_1(M,x), U(E_x)\big)}{U(E_x)} \tag{3.5}$$

is well-defined. $\square$

Theorem 3.2.4. The holonomy representation gives bijections

$$\mathrm{Hol} : \frac{\mathcal{F}(E)}{G(E)} \leftrightarrow \frac{\mathrm{Hom}\big(\pi_1(M,x), \mathrm{Aut}(E_x)\big)}{\mathrm{Aut}(E_x)} \tag{3.6}$$

$$\mathrm{Hol} : \frac{\mathcal{F}(E,h)}{G(E,h)} \leftrightarrow \frac{\mathrm{Hom}\big(\pi_1(M,x), U(E_x)\big)}{U(E_x)}. \tag{3.7}$$

Proof. The most efficient manner of proof uses principal bundles and connections which have not been introduced in this dissertation and thus only a sketch will be provided. The idea is to construct an essential inverse which builds a flat vector bundle from a representation $\pi_1(M,x) \to \mathrm{Aut}(E_x)$. For complete details see [12, 14].

Start with a universal cover $\widetilde{M} \to M$. We can use the path model where

$$\widetilde{M} = \{\gamma \in M^{[0,1]} : \gamma(0) = x\} \ni \gamma \mapsto \gamma(1) \in M.$$

Equip $\widetilde{M} \times E_x \to \widetilde{M}$ with the trivial connection. Note that $\widetilde{M} \times E_x \to \widetilde{M} \to M$ has a natural flat connection using the trivial connection on $\widetilde{M} \times E_x \to \widetilde{M}$ along with the homotopy lifting property of the universal cover $\widetilde{M} \to M$.

Given $\rho : \pi_1(M, x) \to \mathrm{Aut}(E_x)$, construct the associated flat vector bundle

$$\widetilde{M} \times_\rho E_x \to M$$

where $\widetilde{M} \times_\rho E_x$ denotes equivalence classes of pairs $(\gamma, v) \in \widetilde{M} \times E_x$ under the relation $(\gamma \cdot \alpha, v) \sim (\gamma, \rho(\alpha)v)$. The result follows from two observations.

The first observation is that a flat vector bundle (E, ∇) is isomorphic to the flat vector bundle $\widetilde{M} \times_{\mathrm{Hol}(\nabla)} E_x$. An isomorphism is given by sending $v \in E_y$ to $(\gamma, P_{\gamma^{-1}}(\nabla)(v))$ where $\gamma \in \widetilde{M}_y$. This is well defined since any other path from x to y can be written, up to homotopy, as $\gamma \cdot \alpha$ where α is a loop based at x. Hence a different choice in $\widetilde{M}_y$ gives $v \mapsto (\gamma \cdot \alpha, P_{(\gamma\cdot\alpha)^{-1}}(\nabla)(v)) = (\gamma \cdot \alpha, P_{\alpha^{-1}}(\nabla) \circ P_{\gamma^{-1}}(\nabla)(v)) \sim (\gamma, \mathrm{Hol}_\alpha(\nabla) \circ P_{\alpha^{-1}}(\nabla) \circ P_{\gamma^{-1}}(\nabla)(v)) = (\gamma, P_{\gamma^{-1}}(\nabla)(v))$.

Now consider $\rho : \pi_1(M, x) \to \mathrm{Aut}(E_x)$. Given a loop α representing an element of $\pi_1(M, x)$, take a parallel lift (γ_t, v) of α in $\widetilde{M} \times_\rho E_x$ that starts at $(\gamma_0, v) \in (\widetilde{M} \times_\rho E_x)$ and ends at (γ_1, v). The second observation is that for all $t \in [0, 1]$, $\gamma_t : [0, 1] \to M$ is a path from x to $\gamma_t(1) = \alpha(t)$ and in particular γ_0 and γ_1 are both loops based at x. In fact, $\gamma_1 \sim \gamma_0 \cdot \alpha$ and parallel transport sends $(\gamma_0, v) \mapsto (\gamma_1, v) \sim (\gamma_0 \cdot \alpha, v) \sim (\gamma_0, \rho(\alpha)(v))$. Thus holonomy in $\widetilde{M} \times_\rho E_x$ is given by ρ. □

Remark. The purpose of the above result is to construct the moduli space as a quotient of the algebraic variety $\mathrm{Hom}(\pi_1(M, x), U(E_x))$. This is the approach taken in [6,10]. The quotient will have singularities at any point $\rho : \pi_1(M, x) \to U(E_x)$ whose stabilizer subgroup is strictly larger than the center $ZU(E_x)$ i.e. non-singular points are precisely the irreducible representations of

$\pi_1(M, x)$.

Definition 3.2.5. A flat unitary connection ∇ on $\pi : E \to M$ is *irreducible* if $\mathrm{Hol}(\nabla) : \pi_1(M, x) \to U(E_x)$ is an irreducible representation.

Remark. An alternative construction of the moduli space begins with the observation 2.4.8 that $\mathcal{A}(E, h)$ is an affine space modeled on $\Omega^1(M; \mathfrak{u}E) \subset \mathrm{Hom}(M, T^\vee M \otimes \mathfrak{u}E)$. One uses Sobolev spaces of connections of class $k > \frac{\dim_{\mathbb{R}} M}{2}$, with respect to the L^2 norm, in order to obtain Hilbert spaces of connections $\mathcal{A}^k(E, h)$ of Sobolev class k. Similarly, $\Gamma(UE) \subset \mathrm{Hom}(M, UE)$ can be completed into Sobolev spaces of appropriate class k resulting in Hilbert Lie groups $G^k(E, h)$ whose corresponding Lie algebras are Sobolev completions of $\Gamma(\mathfrak{u}E) = \Omega^1(M; \mathfrak{u}E)$.

Theorem 3.2.6 (see [13])**.** Let M be a Riemannian manifold of dimension n and $k > \frac{n}{2} + 1$. If $\widetilde{\mathcal{A}}^k(E, h)$ is the class k Sobolev space of *irreducible* unitary connections, and $\widetilde{G}^{k+1}(E, h)$ is the class $k + 1$ Sobolev space completion of $G(E, h)/Z(E, h)$, then the induced action $\widetilde{\mathcal{A}}^k(E, h) \times \widetilde{G}^{k+1}(E, h) \to \widetilde{\mathcal{A}}^k(E, h)$ is free. □

It is helpful to keep the following informal narrative in mind. One considers the space $\widetilde{\mathcal{F}}(E, h)$ of irreducible flat unitary connections. Then $\widetilde{\mathcal{G}}(E, h) \to \widetilde{\mathcal{F}}(E, h) \to \widetilde{\mathcal{M}}(E, h)$ is to be thought of as a principal bundle where $\widetilde{\mathcal{M}}(E, h)$ is the moduli space of irreducible flat unitary connections. From this perspective, vertical tangent vectors are identified with $\Omega^0(M; \mathfrak{u}E)$. If $\nabla \in \widetilde{\mathcal{F}}(E, h)$ is an irreducible flat unitary connection and $\Phi \in \Omega^1(M; \mathfrak{u}E) = T_\nabla \mathcal{A}(E, h)$, then Φ is a tangent vector in the space $\widetilde{\mathcal{F}}(E,h)$ if and only if $[\nabla,\phi]=0$. Thus $T_\nabla\widetilde{\mathcal{F}}(E,h)$ is identified with (a Sobolev completion of) $Z^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$. Since the $\widetilde{\mathcal{G}}(E,h)$ action is free, there is an identification of $T_{[\nabla]}\widetilde{\mathcal{M}}(E,h)$ with $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ using any representative ∇ of the class $[\nabla]$. In other words, $H^1_{\widetilde{\nabla}}(M;\mathfrak{u}E)$ is viewed as the tangent space to $[\nabla]$ in the moduli space.

Chapter 4

Unitary bundles over Kähler manifolds

4.1 Pre-symplectic manifolds and Hamiltonian vector fields

Definition 4.1.1. A *symplectic vector space* is a $\mathbb{R}$-vector space V equipped with a non-degenerate 2-form $\omega : V \wedge V \to \mathbb{R}$.

Proposition 4.1.2. A finite symplectic vector space (V, ω) has even dimension.

Proof. If $V \neq \mathbf{0}$ then there is some $x_1 \in V - \{0\}$ and since ω is non-degenerate and alternating there is some $y_1 \notin \mathbb{R}\{x_1\}$ such that $\omega(x_1, y_1) = 1$. Set $W_1 = \mathbb{R}\{x_1, y_1\}$ and define $C_1 = \ker\big(\omega(x_1, -)\big) \cap \ker\big(\omega(y_1, -)\big)$ so that $W_1 \cap C_1 = \{0\}$. For every $z \in V$, it is easily verified that $\big(z - \omega(z, y_1)x_1 + \omega(z, x_1)y_1\big) \in C_1$, and therefore $z = \big(\omega(z, y_1)x_1 - \omega(z, x_1)y_1\big) + \big(z - \omega(z, y_1)x_1 + \omega(z, x_1)y_1\big)$ provides a decomposition $V = W_1 \oplus C_1$. If $C_1 \neq \mathbf{0}$ then there exist $x_2, y_2 \in C_1$ such that $\omega(x_2, y_2) = 1$. Set $W_2 = \mathbb{R}\{x_2, y_2\}$ and define $C_2 = \ker\big(\omega(x_2, -)\big) \cap \ker\big(\omega(y_2, -)\big)$ which ultimately yields $V = W_1 \oplus W_2 \oplus C_2$. Continue by induction and, since V is finite, eventually

$C_k = \{0\}$ hence $V = W_1 \oplus \cdots \oplus W_k$ is even dimensional. $\square$

Remark. By following the above inductive construction, every symplectic vector space of dimension $2n$ can be equipped with a basis $\{x_i, y_i\}_{i=1}^{n}$ where $\omega(x_i, y_i) = 1$ and all other ω-pairings yield zero.

Definition 4.1.3. A *pre-symplectic manifold* (M, ω) is a real manifold equipped with a closed differential 2-form $\omega \in \Omega^2(M)$. If ω is also non-degenerate then (M, ω) is a *symplectic manifold.*

Remark. If M is symplectic, then by definition each tangent space T_xM is a symplectic vector space, hence is of even dimension. Thus the dimension of a symplectic manifold is necessarily even, say $\dim M = 2n$. Every symplectic manifold M has a canonical top form $\frac{\omega^n}{n!} \neq 0 \in \Omega^{2n}(M)$ so is orientable with a natural orientation.

Proposition 4.1.4. If (M, ω) is a symplectic manifold then $\omega^\flat : TM \to T^\vee M$ is a linear injection. If M is a finite symplectic manifold then $\omega^\flat : TM \cong T^\vee M$.

Proof. $\omega^\flat(X) = \omega(X, -)$ but ω is non-degenerate so that $\ker \omega^\flat = \{0\}$ and therefore $\omega^\flat : TM \to T^\vee M$ is injective. Hence $\omega^\flat : TM \cong T^\vee M$ when M is finite. $\square$

Remark. Note that $\omega^\flat(X) = \iota_X \omega$. When (M, ω) is finite symplectic the inverse of $\omega^\flat$ is denoted $\omega^\sharp : T^\vee M \to TM$.

Definition 4.1.5. Let (M, ω) be a pre-symplectic manifold. A vector field $X \in \Gamma(TM)$ is *Hamiltonian* if $\omega^\flat(X) \in B^1(M)$. In other words, $X \in \Gamma(TM)$ is Hamiltonian if and only if there exists some function $f \in \Omega^0(M)$ such that $df = \omega^\flat(X) = \iota_X(\omega) = \omega(X, -)$. Similarly, a *Hamiltonian function* is a function $f \in \Omega^0(M)$ such that $df \in \omega^\flat\big(\Gamma(TM)\big)$.

Remark. If $X \in \Gamma(TM)$ is a Hamiltonian vector field on a pre-symplectic manifold, we write $f_X \in \Omega^0(M)$ to denote an arbitrary Hamiltonian function corresponding to X.

Lemma 4.1.6. When M is finite dimensional symplectic, every function $f \in \Omega^0(M)$ has a corresponding Hamiltonian vector field $H_f \in \Gamma(TM)$.

Proof. Given $f \in \Omega^0(M)$, define $H_f \in \Gamma(TM)$ by $H_f := \omega^\sharp(df)$. □

Remark. Thus for symplectic manifolds there is a one-to-one correspondence between Hamiltonian vector fields and non-constant functions.

Lemma 4.1.7. Hamiltonian vector fields on a pre-symplectic manifold preserve the pre-symplectic form ω.

Proof. Let $X \in \Gamma(TM)$ be Hamiltonian. Then there exists some $f \in \Omega^0(M)$ such that $\iota_X\omega = df$. Thus $\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X(d\omega) = d(\iota_X\omega) = d(df) = 0$. □

Proposition 4.1.8. If (M, ω) is pre-symplectic, the subspace $\mathbf{Ham}(M) \subset \Gamma(TM)$ of Hamiltonian vector fields is closed under the Lie bracket of vector fields.

Proof. If $X, Y \in \Gamma(TM)$ are Hamiltonian vector fields then

$$\iota_{[X,Y]}\omega = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega = \mathcal{L}_X\iota_Y\omega = d\iota_X\iota_Y\omega + \iota_X d\iota_Y\omega = d\iota_X\iota_Y\omega$$

which shows that $[X, Y]$ is also Hamiltonian with corresponding Hamiltonian function $\iota_X \iota_Y \omega \in \Omega^0(M)$. $\square$

Proposition 4.1.9. The commutative algebra $\Omega^0(M)$ of functions on a symplectic manifold (M, ω) is a Poisson algebra when endowed with the Poisson bracket defined by

$$\{f, g\} := \omega(H_f, H_g) \tag{4.1}$$

for $f, g \in \Omega^0(M)$.

Remark. Note that $\{f, g\} = \omega(H_f, H_g) = df(H_g) = H_g(f) = \mathcal{L}_{H_g} f$.

Proof. It is immediate that $\{-, -\}$ is skew-symmetric since ω is alternating. For $f, g \in \Omega^0(M)$

$$\begin{aligned} d\{f, g\} &= d\big(\omega(H_f, H_g)\big) = d\big(\iota_{H_g} \iota_{H_f} \omega\big) = d\iota_{H_g}\big(\iota_{H_f}\omega\big) = \mathcal{L}_{H_g}(\iota_{H_f}\omega) \\ &= \mathcal{L}_{H_g}(\iota_{H_f}\omega) + \iota_{H_g}(\mathcal{L}_{H_f}\omega) = \iota_{[H_g, H_f]}\omega = \omega([H_g, H_f], -) \end{aligned}$$

where the first equality on the second line uses that $\mathcal{L}_{H_f}\omega = 0$. Therefore $H_{\{f,g\}} = [H_g, H_f]$ and

$$\begin{aligned} \big\{f, \{g, h\}\big\} &= df(H_{\{g,h\}}) = df[H_h, H_g] = \mathcal{L}_{[H_h, H_g]} f = [\mathcal{L}_{H_h}, \mathcal{L}_{H_g}] f \\ &= \mathcal{L}_{H_h}\mathcal{L}_{H_g} f - \mathcal{L}_{H_g}\mathcal{L}_{H_h} f = \mathcal{L}_{H_h} df(H_g) - \mathcal{L}_{H_g} df(H_h) = \mathcal{L}_{H_h}\{f, g\} - \mathcal{L}_{H_g}\{f, h\} \\ &= \big\{\{f, g\}, h\big\} - \big\{\{f, h\}, g\big\} = \big\{\{f, g\}, h\big\} + \big\{g, \{f, h\}\big\} \end{aligned}$$

which shows that $\{-, -\}$ satisfies the Jacobi identity. Thus $\{-, -\}$ is a Lie bracket on $\Omega^0(M)$.

Finally, since $\omega(H_{gh}, -) = d(gh) = (dg)h + g(dh) = \omega(H_g, -)h + g\omega(H_h, -)$ we see that

$$\{f, gh\} = \omega(H_f, H_{gh}) = \omega(H_f, H_g)h + g\omega(H_f, H_g) = \{f, g\}h + g\{f, h\}$$

hence the bracket is a derivation of the product of functions. $\square$

4.2 Complex manifolds and holomorphic vector bundles

Definition 4.2.1. An *almost complex structure on a smooth manifold* M is a linear operator $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathbf{1}$. An *almost complex manifold* (M, J) is a smooth manifold M equipped with an almost complex strucutre J.

Remark. Since $J^2 = -\mathbf{1}_{TM}$, a (finite) almost complex manifold has even dimension.

Definition 4.2.2. The *complexified tangent bundle of a smooth manifold* M is the $\mathbb{C}$-vector bundle $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C} \to M$.

Definition 4.2.3. Let (M, J) be an almost complex manifold. Define $J^{\mathbb{C}} := J \otimes \mathbf{1}_{\mathbb{C}} \in \mathrm{End}(T^{\mathbb{C}}M)$ as the $\mathbb{C}$-linear extension of J.

Remark. Since $(J^{\mathbb{C}})^2 = -\mathbf{1}_{T^{\mathbb{C}}M}$ the operator $J^{\mathbb{C}}$ has eigenvalues $\pm i := \pm\sqrt{-1}$. The eigenspace (or "eigenbundle") corresponding to eigenvalue $+i$ is denoted by $T^{(1,0)}M \subset T^{\mathbb{C}}M$ and the eigenspace corresponding to eigenvalue $-i$ is $T^{(0,1)}M \subset T^{\mathbb{C}}M$.

Proposition 4.2.4. If (M, J) is an almost complex manifold then $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ and $\overline{T^{(1,0)}M} = T^{(0,1)}M$.

Proof. Every $X \in T^{\mathbb{C}}M$ can be written $X = \frac{1}{2}(X - iJ^{\mathbb{C}}X) + \frac{1}{2}(X + iJ^{\mathbb{C}}X)$ and observe that

$$J^{\mathbb{C}}\big(\frac{1}{2}(X \mp iJ^{\mathbb{C}}X)\big) = \frac{1}{2}\big(J^{\mathbb{C}}X \pm iX\big) = \pm i\frac{1}{2}(X \mp iJ^{\mathbb{C}}X)$$

so that $\frac{1}{2}(X - iJ^{\mathbb{C}}X) \in T^{(1,0)}M$ and $\frac{1}{2}(X + iJ^{\mathbb{C}}X) \in T^{(0,1)}M$. Furthermore $T^{(1,0)}M \cap T^{(0,1)}M = \{0\} \in T^{\mathbb{C}}M$ and therefore $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. This also shows that $Y \in T^{(1,0)}M$ if and only if $Y = \frac{1}{2}(X - iJ^{\mathbb{C}}X)$ for some $X \in T^{\mathbb{C}}M$. Hence $Y \in T^{(1,0)}M$ if and only if $\overline{Y} \in T^{(0,1)}M$. $\square$

Remark. Since $J^{\mathbb{C}}$ is just the $\mathbb{C}$-(bi)linear extension of J, henceforth both operators will be denoted by J. In context it will always be clear whether one is working with TM or with $T^{\mathbb{C}}M$.

Corollary 4.2.5. If (M, J) is an almost complex manifold then the complexified cotangent bundle has a decomposition $(T^{\mathbb{C}}M)^{\vee} = (T^{(1,0)}M \oplus T^{(0,1)}M) = (T^{(1,0)}M)^{\vee} \oplus (T^{(0,1)}M)^{\vee}$. Thus

$$\bigwedge(T^{\mathbb{C}}M)^{\vee} = \bigoplus_n \wedge^n(T^{\mathbb{C}}M)^{\vee} = \bigoplus_n \bigoplus_{p+q=n} \wedge^p(T^{(1,0)}M)^{\vee} \otimes \wedge^q(T^{(0,1)}M)^{\vee}. \tag{4.2}$$

$\square$

Definition 4.2.6. Let (M, J) be an almost complex manifold. *Differential forms of type* (p, q) are the $\mathbb{C}$-valued differential forms

$$\Omega^{(p,q)}(M) := \Gamma\big(\wedge^p (T^{(1,0)}M)^{\vee} \otimes \wedge^q(T^{(0,1)}M)^{\vee}\big).$$

Corollary 4.2.7. If (M, J) is an almost complex manifold then the differential graded module of $\mathbb{C}$-valued differential forms has the following (p, q)-decomposition

$$\Omega^{\bullet}(M, \mathbb{C}) = \bigoplus_n \Omega^n(M, \mathbb{C}) = \bigoplus_n \bigoplus_{p+q=n} \Omega^{(p,q)}(M).$$

$\square$

Definition 4.2.8. A *complex manifold of dimension* n is a smooth manifold M of real dimension $2n$ whose coordinate charts $\phi_U : U \to \mathbb{R}^2 \cong \mathbb{C}^n$ have holomorphic transition functions.

Example 4.2.9. Every complex manifold M is an almost complex manifold. Using local coordinates $\{z_k = x_k + iy_k\}$ for $U \subset M$, $\Gamma(TM|_U)$ has a local frame $\{\partial_{x_k}, \partial_{y_k}\}$. Define J with respect to this coordinate system by $J(\partial_{x_k}) = \partial_{y_k}$ and $J(\partial_{y_k}) = -\partial_{x_k}$ so that $J^2 = -1$.

Definition 4.2.10. The *Nijenhuis tensor* $N_J \in \Omega^2(M, TM)$ *of an almost complex structure* J is defined by

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY].$$

Theorem 4.2.11 (Newlander-Nirenberg)**.** An almost complex manifold (M, J) is complex if and only if $N_J = 0$. □

Proof. The proof of this classic result is an extensive detour into analysis beyond the scope of this dissertation. See [15, 18] for a proof when M is assumed analytic or [16] for the original proof.

□

Lemma 4.2.12. $N_J = 0$ if and only if $T^{(1,0)}M$ is integrable.

Proof. Vector fields are in $T^{(1,0)}M$ if and only if they can be written $X - iJX$ where $X \in$

$\Gamma(T^{\mathbb{C}}M)$. Computing

$$\begin{aligned}[X - iJX, Y - iJY] &= [X, Y] - [X, iJY] - [iJX, Y] + [iJX, iJY] \\ &= \Big([X, Y] - [JX, JY]\Big) - i\Big([X, JY] + [JX, Y]\Big)\end{aligned}$$

we see that $[X - iJX, Y - iJY] \in T^{(1,0)}$ if and only if

$$\begin{aligned}[X, JY] + [JX, Y] &= J\Big([X, Y] - [JX, JY]\Big) \\ \Leftrightarrow 0 &= [X, JY] + [JX, Y] - J[X, Y] + J[JX, JY] \\ \Leftrightarrow 0 &= J[X, JY] + J[JX, Y] + [X, Y] - [JX, JY] = N_J(X, Y).\end{aligned}$$

□

Corollary 4.2.13. An almost complex manifold (M, J) is a complex manifold if and only if $T^{(1,0)}M$ is an integrable distribution if and only if $T^{(0,1)}M$ is an integrable distribution. □

Remark. An almost complex manifold is called *integrable* if its almost complex structure comes from a complex structure, i.e. if the almost complex manifold is actually a complex manifold. The terminology is justified by the preceding corollary.

Corollary 4.2.14. An almost complex manifold (M, J) is integrable if and only if $d(\Omega^{(1,0)}(M)) \subset \Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M)$.

Proof. Let $\phi \in \Omega^{(1,0)}M$ and consider $d\phi \in \Omega^2(M, \mathbb{C})$. For arbitrary $X, Y \in \Gamma(T^{(0,1)}M)$ we have

$$d\phi(X, Y) = X\phi(Y) - Y\phi(X) - \phi([X, Y]) = -\phi([X, Y])$$

and therefore $d(\Omega^{(1,0)}(M)) \subset \Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M)$ if and only if $T^{(0,1)}M$ is an integrable distribution. $\square$

Remark. Equivalently, (M, J) is complex if and only if $d(\Omega^{(0,1)}(M)) \subset \Omega^{(1,1)}(M) \oplus \Omega^{(0,2)}(M)$. It follows that (M, J) is complex if and only if $d(\Omega^{(p,q)}(M)) \subset \Omega^{(p+1,q)}(M) \oplus \Omega^{(p,q+1)}(M)$. This implies the following.

Proposition 4.2.15. (M, J) is a complex manifold if and only if the differential $d \in \mathrm{End}^1\big(\Omega^\bullet(M, \mathbb{C})\big)$ on $\mathbb{C}$-valued differential forms can be written $d = \partial + \bar\partial$ where $\partial \in \mathrm{End}^{(1,0)}\big(\Omega^\bullet(M, \mathbb{C})\big)$ and $\bar\partial \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(M, \mathbb{C})\big)$.

Proof. Use the natural projections $\pi_{(j,k)} : \Omega^n(M, \mathbb{C}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(M) \to \Omega^{(j,k)}(M)$ to define

$$\partial := \pi_{(p+1,q)} \circ d : \Omega^{(p,q)}(M) \to \bigoplus_{a+b=p+q+1} \Omega^{(a,b)}(M) \to \Omega^{(p+1,q)}(M)$$
$$\bar\partial := \pi_{(p,q+1)} \circ d : \Omega^{(p,q)}(M) \to \bigoplus_{a+b=p+q+1} \Omega^{(a,b)}(M) \to \Omega^{(p,q+1)}(M).$$

(M, J) is complex if and only if $d(\Omega^{(p,q)}(M)) \subset \Omega^{(p+1,q)}(M) \oplus \Omega^{(p,q+1)}(M)$ which holds if and only if $d = \partial + \bar\partial$. $\square$

Remark. If M is a complex manifold, local coordinates $\{z_k = x_k + iy_k\}$ in some neighborhood are holomorphic by definition. Then $dz_k = dx_k + i dy_k$ and $d\bar z_k := \overline{dz_k} = dx_k - i dy_k$ are elements of $\Omega^1(M, \mathbb{C})$. Dual to dz_k and $d\bar z_k$ are the elements $\frac{1}{2}\big(\partial_{x_k} - i\partial_{y_k}\big) = \frac{1}{2}\big(\partial_{x_k} - iJ\partial_{x_k}\big) \in T^{(1,0)}M$ and $\frac{1}{2}\big(\partial_{x_k} + iJ\partial_{x_k}\big) \in T^{(0,1)}M$ using that $J(\partial_{x_k}) = \partial_{y_k}$ and $J(\partial_{y_k}) = -\partial_{x_k}$. Thus $\{dz_k\} \subset \Omega^{(1,0)}(M)$

and $\{d\bar{z}_k\} \subset \Omega^{(0,1)}(M)$. For $f \in \Omega^0(M, \mathbb{C})$, $\partial f = \sum_k \partial_{z_k} f$ and $\bar{\partial} f = \sum_k \partial_{\bar{z}_k} f$. Note that

$$\partial_{\bar{z}_k} f = (\partial_{x_k} + i\partial_{y_k})(\Re f + i\Im f) = (\partial_{x_k}\Re f - \partial_{y_k}\Im f) + i(\partial_{x_k}\Im f + \partial_{y_k}\Re f)$$

so that a function f is holomorphic (i.e. satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial} f = 0$. More generally

Definition 4.2.16. If (M, J) is a complex manifold the *holomorphic differential forms* are $\{\phi \in \Omega^{(\bullet,0)}(M) : \bar{\partial}\phi = 0\} \subset \Omega^{(\bullet,0)}(M) \subset \Omega^\bullet(M, \mathbb{C})$.

Definition 4.2.17. A *holomorphic vector bundle* $\pi : E \to M$ is a $\mathbb{C}$-vector bundle over a complex manifold such that the total space E is a complex manifold and π is holomorphic.

Remark. A complex vector bundle over a complex manifold is holomorphic if and only if the transition functions between local trivializations are bi-holomorphic.

Proposition 4.2.18. Let $\pi : E \to M$ be a holomorphic vector bundle. The operator $\bar{\partial} \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(M, \mathbb{C})\big)$ extends to an operator on $\Omega^\bullet(M; E)$.

Proof. In a frame σ for E over $U \subset M$ we set $\bar{\partial}(\phi \otimes \sigma) := (\bar{\partial}\phi) \otimes \sigma$. A different frame τ is given by $\tau = g\sigma$ where g_{ij} are holomorphic functions. Then for $s = \phi \otimes \sigma = \phi \otimes g^{-1}\tau$ we find that

$$\bar{\partial}(\phi \otimes g^{-1}\tau) = \bar{\partial}(\phi g^{-1} \otimes \tau) := \bar{\partial}(\phi g^{-1}) \otimes \tau = (\bar{\partial}\phi)g^{-1} \otimes \tau = \bar{\partial}\phi \otimes g^{-1}\tau = \bar{\partial}\phi \otimes \sigma$$

so $\bar{\partial}$ is well-defined on $\Omega^\bullet(M; E)$. □

Definition 4.2.19. If $\pi : E \to M$ is holomorphic, the cochain complex $\left(\Omega^\bullet(M; E), \bar{\partial}\right)$ is the

Dolbeault complex for the holomorphic bundle E and its cohomology yields *Dolbeault cohomology with values in the holomorphic bundle E.*

Definition 4.2.20. A *holomorphic structure* on a complex vector bundle $\pi : E \to M$ over a complex manifold is a differential operator $D'' \in \mathrm{End}^{(0,1)}(\Omega^\bullet(M;E))$ such that (i) $D'' \circ D'' = 0$ and (ii) $D''(fs) = \bar{\partial} f \otimes s + f D''(s)$ for all $(f, s) \in \Omega^0(M, \mathbb{C}) \times \Gamma(E)$.

Proposition 4.2.21. If $\pi : E \to M$ is a holomorphic vector bundle then there exists a connection $\nabla \in \mathcal{A}(E)$ such that $\nabla^{(0,1)} = \bar{\partial} \in \mathrm{End}^{(0,1)}(\Omega^\bullet(M;E))$.

Proof. Give M an open cover and a partition of unity subordinate to this cover. In an open set U, endow $\Gamma(E|_U)$ with a holomorphic frame $\{\sigma\}$ and define $\nabla|_U$ by $\nabla|_U(\sigma) = 0 = \bar{\partial}\sigma$. Use the partition of unity to combine $\nabla|_U$ to a global connection. □

Proposition 4.2.22. A complex vector bundle over a complex manifold is a holomorphic vector bundle if and only if it has a holomorphic structure.

Proof. We have already observed that every holomorphic vector bundle has a holomorphic sturcture given by $\bar{\partial}$. For the converse we follow [11, 15].

Let D'' be a holomorphic structure on $\pi : E \to M$. As usual let $\sigma = (\sigma_1, \ldots, \sigma_r)^T$ be a local frame for $E|_U$ and define $\theta_{ij} \in \Omega^{(0,1)}(U)$ by $D''\sigma_i = \sum_j \theta_{ij} \otimes \sigma_j \in \Omega^{(0,1)}(U; E|_U)$. Thus $D''\sigma = \theta \otimes \sigma$ and

$$0 = D''(D''\sigma) = D''(\theta \otimes \sigma) = \bar{\partial}\theta \otimes \sigma - \theta \wedge D''\theta = \bar{\partial}\theta \otimes \sigma - \theta \wedge \theta \otimes \sigma = \big(\bar{\partial}\theta - \theta \wedge \theta\big) \otimes \sigma$$

and therefore $\bar{\partial}\theta = \theta \wedge \theta$. Endow $\pi^{-1}(U) \to U$ with an almost complex structure $\widetilde{J}$ by declaring $\{dz\} \cup \{d\sigma - \theta\sigma\}$ to be a basis for the holomorphic differential forms $\Omega^{(1,0)}(\pi^{-1}U)$. Here σ is the local frame and z denotes the the local holomorphic coordinates of $U \subset M$. $\widetilde{J}$ is a complex structure if and only if $d\Omega^{(1,0)}(\pi^{-1}U) \subset \Omega^{(2,0)}(\pi^{-1}U) \oplus \Omega^{(1,1)}(\pi^{-1}U)$. Certainly $d(dz) = d(d\sigma) = 0 \in \Omega^{(2,0)}(\pi^{-1}U) \oplus \Omega^{(1,1)}(\pi^{-1}U)$ and

$$
\begin{aligned}
d(d\sigma - \theta\sigma) = -d(\theta\sigma) &= -(d\theta)\sigma + \theta \wedge d\sigma = -(\partial\theta + \bar{\partial}\theta)\sigma + \theta \wedge d\sigma \\
&= -(\partial\theta)\sigma - (\bar{\partial}\theta)\sigma + \theta \wedge d\sigma = -(\partial\theta)\sigma - (\theta \wedge \theta)\sigma + \theta \wedge d\sigma \\
&= -(\partial\theta)\sigma - \theta \wedge (d\sigma - \theta\sigma) \in \Omega^{(1,1)}(\pi^{-1}U)
\end{aligned}
$$

where in the second line we used $\bar{\partial}\theta = \theta \wedge \theta$. So $\widetilde{J}$ is a complex structure and therefore there exists a $\widetilde{J}$-holomorphic frame $\tau = (\tau_1, \ldots, \tau_r)$ for the bundle $\pi^{-1}(U)$. Note that τ gives an explicit trivialization $\pi^{-1}(U) \ni v \leftrightarrow \big(\pi(v), \tau(v)\big) \in U \times \mathbb{C}^r$. Since $d(\tau) = d\tau$ is a vector of holomorphic 1-forms, we can write $d\tau = Adz + B(d\sigma - \theta\sigma)$ for some $A : U \to GL_n(\mathbb{C})$ and $B \in \Gamma(\mathrm{Aut}(E|_U)$. Now

$$
\begin{aligned}
0 = d^2\tau = d\Big(Adz + B(d\sigma - \theta\sigma)\Big) &= dA \wedge dz + dB \wedge (d\sigma - \theta\sigma) - B(d\theta)\sigma + B\theta \wedge d\sigma \in \Omega^2(\pi^{-1}U) \\
= \partial A \wedge dz + \bar{\partial} A \wedge dz &+ \partial B \wedge (d\sigma - \theta\sigma) + \bar{\partial} B \wedge (d\sigma - \theta\sigma) - B(\partial\theta)\sigma - B(\bar{\partial}\theta)\sigma + B\theta \wedge d\sigma
\end{aligned}
$$

can be projected onto forms corresponding under the trivialization to type $\Omega^{(0,1)}(U) \otimes \Omega^1(\mathbb{C}^r)$:

$$0 = \bar{\partial} B \wedge (d\sigma - \theta\sigma) + B\theta \wedge d\sigma.$$

Evaluating this when $\sigma = 0$ gives $0 = \bar{\partial} B \wedge d\sigma + B\theta \wedge d\sigma \Rightarrow \bar{\partial} B + B\theta = 0$. Using the original frame σ, we obtain a new frame $B\sigma$ which is readily seen to be holomorphic since

$$D''(B\sigma) = \bar{\partial}B \otimes \sigma + BD''\sigma = \bar{\partial}B \otimes \sigma + B\theta \otimes \sigma = 0.$$

This allows one to construct local holomorphic frames that are compatible with the differential operator D'' which can be patched together to make E into a holomorphic vector bundle. □

Corollary 4.2.23. If $\pi : E \to M$ is a complex vector bundle over a complex manifold equipped with a connection ∇ that satisfies $\nabla^{(0,1)} \circ \nabla^{(0,1)} = 0$, then there is a unique holomorphic structure such that $\nabla^{(0,1)} = \bar{\partial}$. In particular, every flat connection on E induces a unique holomorphic structure. □

4.3 Chern connections on holomorphic hermitian vector bundles

Definition 4.3.1. Let $(\pi : E \to M, h)$ be a holomorphic hermitian vector bundle. A *Chern connection* is a connection ∇ that is compatible with the metric and that satisfies $\nabla^{(0,1)} = \bar{\partial}$.

Proposition 4.3.2. A Chern connection is unique.

Proof. Assume ∇_1, ∇_2 are Chern connections so that $\nabla_i = \nabla_i^{(1,0)} + \bar{\partial}$. Compatibility with the metric means

$$dh(s,t) = h(\nabla_i s, t) + h(s, \nabla_i t) = h(\nabla_i^{(1,0)} s, t) + h(\bar{\partial} s, t) + h(s, \nabla_i^{(1,0)} t) + h(s, \bar{\partial} t).$$

Since h is Hermitian, $\partial h(s,t) = h(\nabla_i^{(1,0)} s, t) + h(s, \bar{\partial} t)$ and therefore for all $s, t \in \Gamma(E)$

$$0 = h(\nabla_1^{(1,0)} s, t) + h(s, \bar{\partial} t) - h(\nabla_2^{(1,0)} s, t) - h(s, \bar{\partial} t) = h((\nabla_1^{(1,0)} - \nabla_2^{(1,0)}) s, t)$$

hence $\nabla_1^{(1,0)} - \nabla_2^{(1,0)} = 0$. Therefore $\nabla_1 = \nabla_2$. $\square$

Proposition 4.3.3. Every holomorphic hermitian vector bundle $(\pi : E \to M, h)$ has a Chern connection which is necessarily unique.

Proof. Only existence needs to be shown. Working in a local holomorphic frame $\sigma = \{s_i\}$ over U, the metric is given by a section of hermitian matrices $h_{ij} := h(s_i, s_j)$. Write $\nabla^{(1,0)}(s_i) = \sum_j \Theta_{ij} \otimes s_j$ where $\Theta_{ij} \in \Omega^{(1,0)}(U)$. Using that σ is a holomorphic frame,

$$dh(s_i, s_j) = h(\nabla^{(1,0)} s_i, s_j) + h(s_i, \nabla^{(1,0)} s_j) \Rightarrow$$

$$\partial h_{ij} = h(\nabla^{(1,0)} s_i, s_j) = h(\sum_k \Theta_{ik} s_k, s_j) = \sum_k \Theta_{ik} h_{kj}$$

$$\bar{\partial} h_{ij} = h(s_i, \nabla^{(1,0)} s_j) = h(s_i, \sum_k \Theta_{jk} s_k) = \sum_k h_{ik} \bar{\Theta}_{jk} = h_{ik} \bar{\Theta}^T_{kj}.$$

Therefore $\partial h = \Theta h$ and $\bar{\partial} h = h \bar{\Theta}^T$. The first equation implies $\Theta = \partial h h^{-1}$. Therefore

$$\bar{\Theta}^T = (\overline{\partial h h^{-1}})^T = (\overline{h^{-1}})^T (\overline{\partial h})^T = h^{-1} \bar{\partial} h$$

which solves the second equation. So locally the Chern connection is given by $\Theta = \partial h h^{-1}$ which is well-defined by uniqueness: given a different holomorphic frame over U one would get a different representation for the local Chern connection but by uniqueness the two representations yield the same Chern connection. For the same reason, one obtains a global Chern connection by patching together these locally defined Chern connections which necessarily agree on overlaps. $\square$

Proposition 4.3.4. If $(E, h, \nabla) \to M$ is a flat unitary bundle over a complex manifold, then there is a *unique* holomorphic structure on E such that ∇ is the Chern connection.

Proof. Since (E, ∇) is flat, there is a unique holomorphic structure on E such that $\nabla^{(0,1)} = \bar{\partial}$. Since ∇ is compatible with the metric, ∇ is the Chern connection with respect to the holomorphic structure. □

Corollary 4.3.5. For a fixed hermitian vector bundle over a complex manifold, there is a bijection between *the space of flat unitary connections* and *the space of holomorphic structures with flat Chern connections.* □

4.4 Hermitian and Kähler manifolds

Definition 4.4.1. A *hermitian metric on a complex manifold* (M, J) is a hermitian bundle metric h on the holomorphic tangent bundle $T^{(1,0)}M \to M$. A *hermitian manifold* (M, h) is a complex manifold $M = (M, J)$ equipped with a hermitian metric h.

Remark. Hermitian metrics are compatible with the complex structure J by definition since, for all $X, Y \in T^{(1,0)}M$, $h(JX, JY) = h(iX, iY) = -i^2 h(X, Y) = h(X, Y)$ using that h is sesquilinear.

Remark. Since $\overline{T^{(1,0)}M} = T^{(0,1)}M$, every hermitian metric is equivalent to a bundle map $h : T^{(1,0)}M \otimes T^{(0,1)}M \to M \times \mathbb{C}$. It follows that in local holomorphic coordinates $\{z_k = x_k + iy_k\}$ one can write the metric as $h = h_{jk} dz_j \otimes d\bar{z}_k$.

Proposition 4.4.2. Let (M, J, h) be a Hermitian manifold. Then $\Re h$ is a Riemannian metric on M and $\Im h \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$ is non-degenerate.

Proof. It follows from $\Re h(X,Y) + i\Im h(X,Y) = h(X,Y) = \overline{h(Y,X)} = \Re h(Y,X) - i\Im h(Y,X)$ that $\Re h$ is symmetric positive definite and that $\Im h \in \Omega^2(M)$ is non-degenerate. Thus $\Re h$ is a Riemannian metric by definition. Denote the $\mathbb{C}$-extension of $\Im h$ by $\omega \in \Omega^2(M,\mathbb{C})$. If $X, Y \in T^{(1,0)}M \subset T^{\mathbb{C}}M$ then $\omega(X,Y) = \omega(JX,JY) = \omega(iX,iY) = -\omega(X,Y) = 0$, and therefore $\omega \notin \Omega^{(2,0)}(M)$. A similar calculation shows that $\omega \notin \Omega^{(0,2)}(M)$ and therefore $\omega \in \Omega^{(1,1)}(M)$. $\square$

Definition 4.4.3. Let (M,J,h) be a hermitian manifold. The *Riemannian metric of* M is the metric $g := \Re h$ on TM. The *fundamental form of* (M,J,h) is the non-degenerate form $\omega := -\Im h \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$.

Remark. If (M,J,h) is hermitian then

$$2g(X,Y) = 2\Re h(X,Y) = h(X,Y) + \overline{h(X,Y)} = h(X,Y) + h(Y,X)$$

$$2\omega(X,Y) = -2\Im h(X,Y) = i\overline{h(X,Y)} - ih(X,Y) = h(iY,X) + h(X,iY)$$

hence $\omega(X,Y) = g(X,JY)$ and $g(X,Y) = \omega(JX,Y)$. It follows that a hermitian manifold can be defined as a complex manifold (M,J) equipped with a Riemannian metric g such that $g(-,-) = g(J-,J-)$. Then $h = g(-,-) + ig(-,J-)$ defines the corresponding hermitian metric.

Definition 4.4.4. A *Kähler manifold* is a hermitian manifold whose fundamental 2-form is closed. The fundamental form of a Kähler manifold is called the *Kähler form*.

Remark. By definition every Kähler manifold is a symplectic manifold.

Example 4.4.5. Let $\mathbb{C}^n$ be equipped with the hermitian metric $h = \sum_j dz_j \otimes d\bar{z}_j$ where $\{z_j\}$ are standard holomorphic coordinates. Then

$$\omega = -\Im h = \frac{i}{2}(h - \bar{h}) = \frac{i}{2}\Big(\sum_j dz_j \otimes d\bar{z}_j - \sum_j d\bar{z}_j \otimes dz_j\Big) = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$$

and it is immediate that $d\omega = 0$.

Remark. More generally, if (M, J, h) is a hermitian manifold, in any local holomorphic coordinate system $\{z_j\}$ the metric is given by the matrix $h_{jk} := h(\partial_{z_j}, \partial_{z_k})$ and the fundamental form, with respect to the local coordinate system, is $\omega = \frac{i}{2}\sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$. The following fact shows that the Kähler condition $d\omega = 0$ allows for a more refined local picture.

Proposition 4.4.6. A hermitian manifold (M, J, h) is Kähler if and only if every point has a local holomorphic coordinate system such that the metric h agrees with the standard metric on $\mathbb{C}^n$ to order 2.

Proof. Given $x \in M$ let $\{z_j\}$ be local holomorphic coordinates of $U \ni x$ so that $\omega = \frac{i}{2}\sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$ and therefore

$$d\omega = \frac{i}{2}\sum_{jk}\Big(\sum_{p\neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k + \sum_{q\neq k}(\partial_{\bar{z}_q} h_{jk}) d\bar{z}_q \wedge dz_j \wedge d\bar{z}_k\Big).$$

$d\omega = 0$ if and only if both $\sum_{jk}\sum_{p\neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k = 0 \in \Omega^{(2,1)}(U)$ and $\sum_{jk}\sum_{q\neq k}(\partial_{\bar{z}_q} h_{jk}) d\bar{z}_q \wedge dz_j \wedge d\bar{z}_k = 0 \in \Omega^{(1,2)}(U)$. Furthermore, since $\{dz_p \wedge dz_j \wedge d\bar{z}_k\}$ is a basis of $\Omega^{(2,1)}(U)$, $0 = \sum_{jk}\sum_{p\neq j}(\partial_{z_p} h_{jk}) dz_p \wedge dz_j \wedge d\bar{z}_k$ if and only if $\partial_{z_p} h_{jk} = 0$. Similarly, looking at $\Omega^{(1,2)}(U)$ one finds that$\partial_{\bar{z}_p} h_{jk} = 0$. Thus $d\omega = 0$ if and only if $\partial_{z_p} h_{jk} = 0$ for all $p \neq j$ and $\partial_{\bar{z}_p} h_{jk} = 0$ for all $q \neq k$. $\square$

4.5 Inner products on differential forms

If V is a $\mathbb{R}$ vector space with $\dim V = n < \infty$ and equipped with a metric $\langle -, -, \rangle$ and a volume form dV, the star operator $* : \wedge^k V \to \wedge^{n-k} V$ is the linear map defined by $a \wedge *b = \langle a, b \rangle dV$. Working with a basis one readily verifies that $*^2 = (-1)^{k(n-k)} : \wedge^k V \to \wedge^k V$.

More generally, let M be a *closed* manifold with Riemannian metric $g : TM \otimes TM \to M \times \mathbb{R}$, volume form dM, and $\dim_{\mathbb{R}} M = n$. The metric g induces a metric, also written g, on $\wedge^k (TM)^\vee$ for all k. If $\phi, \psi \in \Omega^k(M, \mathbb{R}) = \Gamma(\wedge^k (TM)^\vee)$ then $g(\phi, \psi) \in \Omega^0(M)$.

Definition 4.5.1. The metric on $\Omega^k(M, \mathbb{R})$ is given by $\langle \phi, \psi \rangle_{\mathbb{R}} = \int_M g(\phi, \psi) dM$.

Definition 4.5.2. The *Hodge star operator on $\mathbb{R}$-valued differential forms (with respect to the metric g)* is the linear operator $* : \Omega^k(M, \mathbb{R}) \to \Omega^{n-k}(M, \mathbb{R})$ defined by

$$\phi \wedge *\psi = g(\phi, \psi) dM \in \Omega^n(M, \mathbb{R}). \tag{4.3}$$

Lemma 4.5.3. $*^{-1} = (-1)^{k(n-k)} * : \Omega^{n-k}(M) \to \Omega^k(M)$.

Proof. $*^2 = (-1)^{k(n-k)} : \Omega^k(M) \to \Omega^k(M)$. $\square$

Definition 4.5.4. The *formal adjoint of* $A \in \operatorname{End}\big(\Omega^\bullet(M, \mathbb{R})\big)$ is the operator A^* defined by $\langle A\Phi, \Psi \rangle_{\mathbb{R}} = \langle \Phi, A^* \Psi \rangle_{\mathbb{R}}$ with respect to the inner product on $\Omega^\bullet(M, \mathbb{R})$.

Remark. A formal adjoint is unique since $\langle A\Phi, \Psi \rangle = \langle \Phi, A^*\Psi \rangle_{\mathbb{R}} = \langle \Phi, A'\Psi \rangle_{\mathbb{R}} \Rightarrow \big\langle \Phi, (A^* - A')\Psi \big\rangle_{\mathbb{R}} = 0 \Rightarrow A^* = A'$.

Proposition 4.5.5. The formal adjoint to $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is

$$d^* = (-1)^{nk+1} * d * .$$

Proof. For $\phi, \psi \in \Omega^k(M)$ using Stokes' theorem

$$\begin{aligned}\langle d\phi, \psi\rangle &= \int_M d\phi \wedge *\psi = -(-1)^k \int_M \phi \wedge d(*\psi) = -(-1)^{k+k(n-k)} \int_M \phi \wedge *^2(d * \psi) \\ &= -(-1)^{k+k(n-k)} \langle \phi, *d * \psi\rangle = (-1)^{nk+1} \langle \phi, *d * \psi\rangle.\end{aligned}$$

$\square$

Definition 4.5.6. The metric g on $\wedge^k(TM)^\vee$ can be extended to give a hermitian metric $g_\mathbb{C}$ on $\wedge^k(T^\mathbb{C}M)^\vee$ by $\mathbb{C}$-(bi)linearly extending the metric g and defining $g_\mathbb{C}(\phi, \psi) := g(\phi, \bar{\psi})$. The Hodge star operator can be $\mathbb{C}$-linearly extended to yield the *Hodge star operator for* $\Omega^k(M, \mathbb{C})$ and is characterized by

$$\phi \wedge *\bar{\psi} = g_\mathbb{C}(\phi, \psi) dM \in \Omega^{\dim_\mathbb{R} M}(M, \mathbb{C}).$$

The inner product on $\Omega^\bullet(M, \mathbb{C})$ is $\langle \phi, \psi\rangle = \displaystyle\int_M g_\mathbb{C}(\phi, \psi) dM = \int_M \phi \wedge *\bar{\psi}$.

Remark. Note if (M, J) is a complex manifold with $\dim_\mathbb{R} M = 2n$, then $* : \Omega^k(M, \mathbb{C}) \to \Omega^{2n-k}(M, \mathbb{C})$ must satisfy $* : \Omega^{(p,q)}(M) \to \Omega^{(n-q,n-n)}(M)$.

Corollary 4.5.7. If (M, J) is a complex manifold equipped with a Riemannian metric g so that both $\Omega^k(M, \mathbb{R})$ and $\Omega^k(M, \mathbb{C})$ are metric spaces as described above, then

$$\partial^* = - * \bar{\partial} *$$

$$\bar{\partial}^* = - * \partial *$$

where $d = \partial + \bar{\partial}$.

Proof. The differential d on $\Omega^\bullet(M, \mathbb{C})$ is the $\mathbb{C}$-linear extension of the differential on $\Omega^\bullet(M, \mathbb{R})$. Therefore $d^* = -*d*$ on both real and complex forms. Hence $(\partial + \bar{\partial})^* = -*(\partial + \bar{\partial})*$. Comparing bi-degrees, $\partial^* = -*\bar{\partial}*$ and $\bar{\partial}^* = -*\partial*$. $\square$

Definition 4.5.8. Let $(E, h) \to M$ be a hermitian vector bundle over a closed complex manifold with underlying Riemannian metric g and $\dim_{\mathbb{C}} M = n$. Then

$$g_h(\phi \otimes s, \psi \otimes t) := g_{\mathbb{C}}(\phi, \psi) h(s, t) \tag{4.4}$$

is a hermitian inner product on $\wedge^k (T^{\mathbb{C}} M)^\vee \otimes E \to M$. The *hermitian inner product on* $\Omega^k(M; E)$ is

$$\langle \phi \otimes s, \psi \otimes t \rangle_E = \int_M g_{\mathbb{C}}(\phi, \psi) h(s, t) dM. \tag{4.5}$$

Recall the metric h on E indues a conjugate linear bundle isomorphism $\flat = h^\flat : E \to E^\vee$ by $v^\flat := h(-, v)$ since h is $\mathbb{C}$-conjugate linear in the second factor. The *Hodge star* operator for $\Omega^k(M; E)$ is the $\mathbb{C}$-conjugate linear map $\star : \Omega^k(M; E) \to \Omega^{2n-k}(M; E^\vee)$ defined by

$$\mathrm{ev}(\Phi \wedge \star \Psi) = g_h(\Phi, \Psi) dM \in \Omega^{2n}(M, \mathbb{C}) \tag{4.6}$$

where $\mathrm{ev}(\alpha \otimes s \wedge \beta \otimes f) := \alpha \wedge \beta f(s) \in \Omega^\bullet(M, \mathbb{C})$.

Proposition 4.5.9. $\star(\psi \otimes t) = *\bar{\psi} \otimes t^\flat$.

Proof. For the moment write $\star(\psi \otimes t) = \star\psi \otimes \star t$ and then deduce what these symbols mean.

$$(\phi \wedge *\bar{\psi}) t^\flat(s) = g_{\mathbb{C}}(\phi, \psi) h(s, t) dM = g_h(\phi \otimes s, \psi \otimes t) dM = \mathrm{ev}\big(\phi \otimes s \wedge \star(\psi \otimes t)\big) = (\phi \wedge \star\psi)(\star t) s$$

shows that $\star\psi = *\bar{\psi}$ and $(\star t)s = t^\flat(s)$. $\square$

Remark. One should write $\langle -, - \rangle_E$ and $\star_E$ when there is ambiguity about the hermitian vector bundle E. It follows from above that $\star_{E^\vee}(\psi \otimes f) = *\bar{\psi} \otimes f^\sharp$.

Corollary 4.5.10. $\star_{E^\vee} \circ \star_E = (-1)^k : \Omega^k(M;E) \to \Omega^k(M;E)$ and $\star_E \circ \star_{E^\vee} = (-1)^k : \Omega^k(M;E^\vee) \to \Omega^k(M;E^\vee)$. $\square$

Proposition 4.5.11. The formal adjoint to a unitary connection $\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ is

$$\nabla^* = -1 \star_{E^\vee} \nabla^\vee \star_E \tag{4.7}$$

where $\nabla^\vee$ is the dual unitary connection on the hermitian bundle $(E^\vee, h^\vee) \to M$.

Proof. Let $\Phi \in \Omega^k(M;E)$ and $\Psi \in \Omega^{k+1}(M;E)$. By definition of $h^\vee$ [see 2.5],

$$d\mathrm{ev}(\Phi \wedge \star_E \Psi) = \mathrm{ev}\big(\nabla\Phi \wedge \star_E \Psi\big) + (-1)^k \mathrm{ev}\big(\Phi \wedge \nabla^\vee(\star_E \Psi)\big)$$

and since M is without boundary Stokes' theorem implies

$$\mathrm{ev}\big(\nabla\Phi \wedge \star_E \Psi\big) = -(-1)^k \mathrm{ev}\big(\Phi \wedge \nabla^\vee(\star_E \Psi)\big) = -\mathrm{ev}\big(\Phi \wedge \star_E \star_{E^\vee} \nabla^\vee(\star\Psi)\big)$$

$$\Rightarrow \langle \nabla\Phi, \Psi \rangle = \langle \Phi, -1 \star_{E^\vee} \nabla^\vee \star_E \Psi \rangle.$$

$\square$

Corollary 4.5.12. If $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$ is unitary, then

$$\nabla^{(1,0)*} = - \star_{E^\vee} \nabla^{(0,1)} \star_E$$

$$\nabla^{(0,1)*} = - \star_{E^\vee} \nabla^{(1,0)} \star_E .$$

Proof. Use that $-1 \star_{E^\vee} \nabla \star_E = (\nabla)^* = \nabla^{(1,0)*} + \nabla^{(0,1)*}$ and compare bi-degrees. $\square$

Remark. For ease of notation, for the remainder of this chapter we will write $\nabla^{10} := \nabla^{(1,0)}$ and $\nabla^{01} := \nabla^{(0,1)}$.

4.6 Hermitian bundles over Kähler manifolds

Let $(E, h) \to M$ be a hermitian vector bundle over a closed Kähler manifold. Then $g := \Re h$ is a Riemannian metric and $\langle -, - \rangle_E : \Omega^\bullet(M; E) \otimes \overline{\Omega^\bullet(M; E)} \to \mathbb{C}$ is the hermitian inner product as described in 4.5.

Definition 4.6.1. Assume $\dim_{\mathbb{R}} M = 2n$. Define the following operators on $\Omega^\bullet(M; E)$

$$\mathrm{End}^2\big(\Omega^\bullet(M; E)\big) \ni L : \Phi \mapsto \omega \wedge \Phi$$

$$\mathrm{End}^{-2}\big(\Omega^\bullet(M; E)\big) \ni \Lambda := L^*$$

$$\mathrm{End}\big(\Omega^\bullet(M; E)\big) \ni \Pi := \sum_k (n-k)\pi_k$$

where π_k is projection $\Omega^\bullet(M; E) \to \Omega^k(M; E)$. Thus $\Pi|_{\Omega^k(M;E)} = (n-k)\mathbf{1}|_{\Omega^k(M;E)}$. $\Lambda = L^*$ is the formal adjoint of L with respect to $\langle -, - \rangle_E$.

Proposition 4.6.2. For every unitary connection $\nabla \in \mathcal{A}(E, h)$

$$[L, \nabla] = [L, \nabla^{10}] = [L, \nabla^{01}] = 0$$

$$[\Lambda, \nabla^*] = [\Lambda, \nabla^{10*}] = [\Lambda, \nabla^{01*}] = 0.$$

Proof. $[L, \nabla]\Phi = \omega \wedge \nabla(\Phi) - \nabla(\omega \wedge \Phi) = \omega \wedge \nabla(\Phi) - d\omega \wedge \Phi - \omega \wedge \nabla(\Phi) = 0$ since $d\omega = 0$. Therefore $[L, \nabla^{10}] + [L, \nabla^{01}] = 0$. But $[L, \nabla^{10}]$ is an operator of bi-degree $(2, 1)$ whereas $[L, \nabla^{01}]$ has bi-degree $(1, 2)$ so it must be that $[L, \nabla^{10}] = [L, \nabla^{01}] = 0$. Passing to formal adjoints gives $[\Lambda, \nabla^*] = [\Lambda, \nabla^{10*}] = [\Lambda, \nabla^{01*}] = 0$. $\square$

Lemma 4.6.3. $[\Lambda, L] = \Pi$.

Proof. We proceed following [7] by working in a local holomorphic coordinate system $\{z_k\}$. First note that $||dz_k|| = \langle dx_k + idy_k, dx_k + idy_k\rangle = ||dx_k|| + ||dy_k|| = 2$. Define $e_k \in \mathrm{End}^{(1,0)}\big(\Omega^\bullet(U; E)\big)$ by $e_k\phi = dz_k \wedge \phi$ and define $\bar{e}_k \in \mathrm{End}^{(0,1)}\big(\Omega^\bullet(U; E)\big)$ similarly. Let f_k and $\bar{f}_k$ be the formal adjoints to e_k and and $\bar{e}_k$ respectively. Then $\langle f_k dz_A \wedge d\bar{z}_B, \phi\rangle = \langle dz_A \wedge d\bar{z}_B, dz_k \wedge \phi\rangle$ is zero for $k \notin A$ and thus $f_k dz_A \wedge d\bar{z}_B = 0$ for $k \notin A$. Also $\langle f_k dz_k \wedge dz_A \wedge d\bar{z}_B, \phi\rangle = \langle dz_k \wedge dz_A \wedge d\bar{z}_B, dz_k \wedge \phi\rangle = ||dz_k||\langle dz_A \wedge d\bar{z}_B, \phi\rangle$ and therefore $f_k dz_k \wedge dz_A \wedge d\bar{z}_B = 2dz_A \wedge d\bar{z}_B$. Similarly we find that $\bar{f}_k dz_A \wedge d\bar{z}_B = 0$ for $k \notin B$ and that $\bar{f}_k d\bar{z}_k \wedge dz_A \wedge d\bar{z}_B = 2dz_A \wedge d\bar{z}_B$. It follows that

$$f_k e_k + e_k f_k = 2$$

$$e_j f_k + f_k e_j = 0 \text{ for } j \neq k$$

$$\bar{e}_j f_k + f_k \bar{e}_j = 0 \text{ for } j \neq k$$

and similar expressions obtained by taking conjugates.

Locally $\omega \equiv \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$ modulo order 2 terms so $L \equiv \frac{i}{2}\sum_j e_j \bar{e}_j$ and $\Lambda = L^* \equiv -\frac{i}{2}\sum_k \bar{f}_k f_k$ and therefore

$$4[\Lambda, L] \equiv \sum_{j \neq k} \bar{f}_k f_k e_j \bar{e}_j - e_j \bar{e}_j \bar{f}_k f_k + \sum_j \bar{f}_j f_j e_j \bar{e}_j - e_j \bar{e}_j \bar{f}_j f_j.$$

Note that $\bar{f}_k(f_k e_j)\bar{e}_j = -(\bar{f}_k e_j)(f_k \bar{e}_j) = -e_j(\bar{f}_k \bar{e}_j) f_k = e_j \bar{e}_j \bar{f}_k f_k$. Also

$$\bar{f}_j f_j e_j \bar{e}_j = \bar{f}_j(2 - e_j f_j)\bar{e}_j = 2\bar{f}_j \bar{e}_j - \bar{f}_j e_j f_j \bar{e}_j$$

$$e_j \bar{e}_j \bar{f}_j f_j = e_j(2 - \bar{f}_j \bar{e}_j) f_j = 2 e_j f_j - e_j \bar{f}_j \bar{e}_j f_j = 2 e_j f_j - \bar{f}_j e_j f_j \bar{e}_j$$

$$\Rightarrow \bar{f}_j f_j e_j \bar{e}_j - e_j \bar{e}_j \bar{f}_j f_j = 2\bar{f}_j \bar{e}_j - 2 e_j f_j$$

and therefore

$$[\Lambda, L] = \frac{1}{2}\sum_j \bar{f}_j \bar{e}_j - e_j f_j = \frac{1}{2}\sum_j \Big(2 - \bar{e}_j \bar{f}_j - e_j f_j\Big) = n - \frac{1}{2}\sum_j \left(\bar{e}_j \bar{f}_j + e_j f_j\right).$$

Now $e_j f_j dz_A \wedge d\bar{z}_B = \begin{cases} 0 \text{ if } j \notin A \\ 2 dz_A \wedge d\bar{z}_B \text{ else} \end{cases}$ and similarly $\bar{e}_j \bar{f}_j dz_A \wedge d\bar{z}_B = \begin{cases} 0 \text{ if } j \notin B \\ 2 dz_A \wedge d\bar{z}_B \text{ else} \end{cases}$. Therefore for $\phi \in \Omega^{(p,q)}(U)$ we have $[\Lambda, L]\phi = \big(n - (p+q)\big)\phi = \Pi\phi$.

It follows that $[\Lambda, L] = \Pi$ on $\Omega^\bullet(M; M \times \mathbb{C})$. The result holds for an arbitrary vector bundle over $E \to M$ since the operator L, and therefore the operator Λ, acts on $\Omega^\bullet(M; E) = \Omega^\bullet(M) \otimes_{\Omega^0(M)} \Gamma(E)$ solely through its action on $\Omega^\bullet(M) = \Omega^\bullet(M; M \times \mathbb{C})$. See also [1].

$\square$

4.7 Representations of $\mathfrak{s}l(2,\mathbb{C})$ and hard Lefschetz property

The special linear group is $SL(2,\mathbb{F}) := \{A \in GL(2,\mathbb{F}) : \det(A) = 1\}$ and its Lie algebra is $\mathfrak{s}l(2,\mathbb{F}) = T_{\mathbf{1}}SL(2,\mathbb{F}) = \{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbb{F})\} = \{X \in \mathrm{Mat}_{2\times 2} : \mathbf{Tr}(X) = 0\}$. In particular, $\mathfrak{s}l(2,\mathbb{F})$ is generated over $\mathbb{F}$ by the matricies $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which satisfy the relations $[D,A] = 2A$, $[D,B] = -2B$ and $[A,B] = D$.

Lemma 4.7.1. Let $\rho : \mathfrak{s}l(2,\mathbb{C}) \to \mathrm{End}(V)$ be a Lie algebra representation. If (v,λ) is an eigenpair for D (i.e. $Dv := \rho(D)v = \lambda v$) then $(Av, \lambda+2)$ and $(Bv, \lambda-2)$ are also eigenpairs for D.

Proof.

$$D(Av) = DAv - ADv + ADv = [D,A]v + ADv = 2Av + A(\lambda v) = (2+\lambda)Av$$

$$D(Bv) = DBv - BDv + BDv = [D,B]v + BDv = -2Bv + B(\lambda v) = (-2+\lambda)Bv.$$

$\square$

Definition 4.7.2. Let $\rho : \mathfrak{s}l(2,\mathbb{C}) \to \mathrm{End}(V)$ be a representation. An eigenvector $v \in V$ for $D = \rho(D)$ is called *primitive* if $Av := \rho(A)v = 0$.

Lemma 4.7.3. If v is primitive with $Dv = \lambda v$, then $AB^k v = (k\lambda - k^2 + k)B^{k-1}v$.

Proof. Notice that

$$ABv = ABv - BAv + BAv = Dv + BAv = \lambda v$$

$$AB^2v = (AB - BA)Bv + BABv = D(Bv) + B(ABv) = (\lambda - 2)Bv + B(\lambda v) = \big((\lambda - 2) + \lambda\big)Bv$$

$$AB^3v = (AB - BA)B^2v + BAB^2v = D(B^2v) + B(AB^2v) = (\lambda - 4)B^2v + B^2\big((\lambda - 2) + \lambda\big)v$$

$$= \big((\lambda - 4) + (\lambda - 2) + \lambda\big)B^2v.$$

Assume that $AB^{k-1}v = \big((\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-2}v$ (which holds for $k \leq 4$) and proceed inductively:

$$\begin{aligned} AB^kv &= (AB - BA)B^{k-1}v + BAB^{k-1}v = D(B^{k-1}v) + B(AB^{k-1}v) \\ &= (\lambda - 2(k-1))B^{k-1}v + B\big((\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-2}v \\ &= \big((\lambda - 2k + 2) + (\lambda - 2k + 4) + \cdots + \lambda\big)B^{k-1}v \\ &= \frac{k\big(\lambda - 2(k-1) + \lambda\big)}{2}B^{k-1}v = (k\lambda - k^2 + k)B^{k-1}v. \end{aligned}$$

□

Proposition 4.7.4. Let $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ be an *irreducible* representation and let $v \in V$ be primitive. Then V is generated by $\{v, Bv, B^2v, \dots\}$.

Proof. Let S be the span of $\{v, Bv, B^2v, \dots\}$ so $BS \subset S$ by construction. Each B^kv is an eigenvector of D and therefore $DS \subset S$. Lastly, $A(B^kv) = (k\lambda - k^2 + k)B^{k-1}v \subset S$, hence $AS \subset S$. Thus ρ preserves the subspace $S \subset V$ but since ρ is assumed irreducible, it must be that $S = V$. □

Remark. If v is primitive with eigenvalue λ of D, then each non-zero $B^k v \in V$ is also an eigenvector of D with eigenvalue $\lambda - 2k$. Thus $\{B^k v\}$ is a linearly independent set of vectors in, hence a basis of, V. This gives $V = \oplus_\lambda V_\lambda$ where each $V_\lambda \simeq \mathbb{C}$ is an 1-dimensional D-eigenspace. Furthermore $D(V_\lambda) = V_\lambda$, $A(V_\lambda) = V_{\lambda+2}$ and $B(V_\lambda) = V_{\lambda-2}$.

Proposition 4.7.5. Let $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ be a finite irreducible representation with v primitive. Then the eigen-decomposition is $V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$ for some $n \in \mathbb{N}$.

Remark. This says that the eigenvalues are all integers, distributed symmetrically about zero.

Proof. Let λ be the D-eigenvalue of v. $\{B^k v\} \subset V$ has only finitely many non-zero terms since V is finite. Let n be such that $B^n v \neq 0$ and $B^{n+1} v = 0$. Then

$$0 = AB^{n+1}v = ((n+1)\lambda - (n+1)^2 + n + 1)B^n v = (n+1)(\lambda - n)B^n v \Rightarrow \lambda = n \in \mathbb{N}.$$

The largest D-eigenvalue is n corresponding to v and the smallest eigenvalue is $n - 2n = -n$ corresponding to $B^n v$ and therefore $V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$. □

Theorem 4.7.6. If $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V)$ is a finite representation, then $\rho(B)^\lambda : V_\lambda \to V_{-\lambda}$ is an isomorphism for all positive eigenvalues λ of $\rho(D)$.

Proof. $\mathfrak{sl}(2, \mathbb{C})$ is a complex simple Lie algebra thus every finite representation of $\mathfrak{sl}(2, \mathbb{C})$ is a direct sum of irreducible representations [4]. Say $V = \oplus_{i=1}^d V^i$ with corresponding $\rho_i : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}(V^i)$ is a decomposition of ρ into irreducible components. For each V^i there exists a $v_i \in V^i$ such that $Dv_i = n_i v_i$, $Av_i = 0$ and $V^i = V^i_{-n_i} \oplus \cdots \oplus V^i_{n_i}$ where $V^i_{n_i - 2k} = \mathbb{C}\{B^k v_i\}_{k=1}^{d_i}$ is the eigenspace of V^i with eigenvalue $n_i - 2k$. Note that $A : V^i_{n_i-2k} \leftrightarrows V^i_{n_i-2k+2} : B$ gives an isomorphism between one dimensional vector spaces. Thus we can eigen-decompose along the irreducible components

$$V = \bigoplus_{i=1}^{d} V^i = \bigoplus_{i=1}^{d} \bigoplus_{k=0}^{n_i} V^i_{-n_i+2k}.$$

Of course we no longer have $A : V_m = \oplus_{i=1}^d V^i_m \to \oplus_{i=1}^d V^i_{m+2}$ is an isomorphism because some eigenspaces may be trivial, i.e. the dimensions of neighboring eigenspaces can differ. But since each V^i has an eigen-decomposition symmetric about zero, we necessarily have $\dim V_{-m} = \dim \oplus_{i=1}^d V^i_{-m} = \dim \oplus_{i=1}^d V^i_m = \dim V_m$.

For $m > 0$, $v \in V_m$ if and only if $v = B^k w$ for some $k \in \mathbb{N}$ and some $w \in V$ such that $Dw = (m+2k)w$. Then $B^m v = B^{m+k} w$ so that $D(B^m v) = -mv$. Thus if $v \neq 0$, then $D(B^m v) \neq 0$ which implies that $B^m v \neq 0$. Therefore $B^m : V_m \to V_{-m}$ is injective and thus bijective. □

Proposition 4.7.7. There is a representation $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}\big(\Omega^\bullet(M;E)\big)$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \Pi \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Lambda \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto L.$$

Proof. $[\Lambda, L] = \Pi$ by 4.6.3. For all $\Phi \in \Omega^k(M;E)$

$$[\Pi, L]\Phi = \Pi(L\Phi) - L(\Pi\Phi) = (n-k-2)L(\Phi) - L\big((n-k)\Phi\big) = -2L(\Phi)$$

$$[\Pi, \Lambda]\Phi = \Pi(\Lambda\Phi) - \Lambda(\Pi\Phi) = (n-k+2)\Lambda(\Phi) - \Lambda\big((n-k)\Phi\big) = 2\Lambda(\Phi).$$

□

4.8 Kähler identities

Lemma 4.8.1. Let M be a Kähler manifold. Then $[\Lambda, \partial] = i\bar{\partial}^* \in \mathrm{End}\big(\Omega^\bullet(M, \mathbb{C})\big)$.

Proof. Following [7] let $e_j, \bar{e}_j, f_j, \bar{f}_j$ be as in 4.6.3 with respect to some local holomorphic coordinate system for M so that locally we have $L = \frac{i}{2}\sum_j e_j\bar{e}_j$ and $\Lambda = -\frac{i}{2}\sum_j \bar{f}_j f_j$. Write $\partial_k := \frac{\partial}{\partial z_k}$ and $\bar{\partial}_k := \frac{\partial}{\partial \bar{z}_k}$ which are defined to act on functions so that $\partial = \sum_k \partial_k e_k$ and $\bar{\partial} = \sum_k \bar{\partial}_k \bar{e}_k$. Note that ∂_k and $\bar{\partial}_k$ commute with all of the elementary operators $e_j, f_j, \bar{e}_j, \bar{f}_j$.

For all functions s, t on M we have $\partial\langle s, t\rangle = \langle \partial s, t\rangle + \langle s, \bar{\partial} t\rangle$ hence by Stokes' theorem $\partial_k^* = -\bar{\partial}_k$ and $\bar{\partial}_k^* = -\partial_k$. This gives $\bar{\partial}^* = \sum_k \bar{f}_k \bar{\partial}_k^* = \sum_k \bar{\partial}_k^* \bar{f}_k = -\sum_k \partial_k \bar{f}_k$ and therefore

$$\begin{aligned}
\Lambda\partial &= \frac{i}{2}\sum_{jk} \bar{f}_k f_k \partial_j e_j = \frac{i}{2}\sum_{jk} \partial_j \bar{f}_k f_k e_j = \frac{i}{2}\Big(\sum_k \partial_k \bar{f}_k f_k e_k + \sum_{j\neq k} \partial_j \bar{f}_k f_k e_j\Big) \\
&= \frac{i}{2}\Big(\sum_k \partial_k \bar{f}_k (2 - e_k f_k) + \sum_{j\neq k} \partial_j e_j \bar{f}_k f_k\Big) = i\sum_k \partial_k \bar{f}_k + \frac{i}{2}\Big(\sum_k \partial_k e_k \bar{f}_k f_k) + \sum_{j\neq k} \partial_j e_j \bar{f}_k f_k\Big) \\
&= i\sum_k \partial_k \bar{f}_k + \frac{i}{2}\sum_{jk} \partial_j e_j \bar{f}_k f_k = \Big(i\sum_k \partial_k \bar{f}_k\Big) + \partial\Lambda = i\bar{\partial}^* + \partial\Lambda \Rightarrow [\Lambda, \partial] = i\bar{\partial}^*.
\end{aligned}$$

See also [1]. ∎

Corollary 4.8.2. If M is Kähler then $[\Lambda, \bar{\partial}] = -i\partial^*$, $[L, \partial^*] = i\bar{\partial}$, and $[L, \bar{\partial}^*] = -i\partial$.

Proof. Recall that $\omega \in \Omega^{(1,1)}(M) \cap \Omega^2(M, \mathbb{R})$ hence $\bar{L} = L$ and $\bar{\Lambda} = \Lambda$. Taking the conjugate of $[\Lambda, \partial] = i\bar{\partial}^*$ gives $[\Lambda, \bar{\partial}] = -i\partial^*$; taking the adjoint gives $[L, \partial^*] = i\bar{\partial}$ the conjugate of which gives $[L, \bar{\partial}^*] = -i\partial$. ∎

Proposition 4.8.3. Let $(E, h) \to M$ be a holomorphic hermitian vector bundle over a closed Kähler manifold with flat Chern connection $\nabla = \nabla^{10} + \nabla^{01} = \nabla^{10} + \bar{\partial}$. Then the *Kähler* *identities* hold for operators on $\Omega^\bullet(M;E)$:

$$[L, \nabla^{10*}] = i\nabla^{01}$$

$$[L, \nabla^{01*}] = -i\nabla^{10}$$

$$[\Lambda, \nabla^{10}] = i\nabla^{01*}$$

$$[\Lambda, \nabla^{01}] = -i\nabla^{10*}.$$

Proof. Locally we can give flat holomorphic frames; the result follows from the previous result for operators on $\Omega^\bullet(M,\mathbb{C})$. See also [1]. □

4.9 Hard Lefschetz for unitary bundles over Kähler manifolds

As usual $(E,h) \to M$ is a hermitian vector bundle over a closed Kähler manifold with $\dim_{\mathbb{C}} M = n$. $\Omega^\bullet(M;E)$ has the hermitian inner product $\langle -, - \rangle_E$ as defined in 4.5 which allows us to define the formal adjoint of a linear operators on $\Omega^\bullet(M;E)$.

Definition 4.9.1. If ∇ is a flat unitary connection on (E,h), the ∇-*Laplacian* is the self-adjoint degree 0 operator $\Delta := \nabla\nabla^* + \nabla^*\nabla$. The ∇-*harmonic k-forms with values in the bundle E* are $\mathcal{H}^k_\nabla(M;E) := \ker\Delta \cap \Omega^k(M;E)$. The harmonic forms are a graded sub-space $\mathcal{H}^\bullet_\nabla(M;E)$ of $\Omega^\bullet_\nabla(M;E)$.

Lemma 4.9.2. $\ker\Delta = \ker\nabla \cap \ker\nabla^*$.

Proof. Certainly $\ker\nabla \cap \ker\nabla^* \subset \ker\Delta$. If $\Phi \in \ker\Delta$ then $0 = \langle \Delta\Phi, \Phi\rangle_E = \langle \nabla^*\Phi, \nabla^*\Phi\rangle_E +$

$\langle \nabla\Phi, \nabla\Phi\rangle_E \Rightarrow \Phi \in \ker\nabla \cap \ker\nabla^*$. $\square$

Lemma 4.9.3. Every harmonic form is the minimal element of its cohomology class with respect to $||-||_E^2$.

Proof. Let $\Phi \in \ker\Delta$. Thus $\nabla^*\Phi = \nabla\Phi = 0$ hence $[\Phi] \in H^\bullet_\nabla(M;E)$. Elements in the cohomology class of $[\Phi]$ are necessarily of the form $\Phi + \nabla\Psi$.

$$||\Phi + \nabla\Psi||_E^2 = \langle \Phi + \nabla\Psi, \Phi + \nabla\Psi\rangle_E = ||\Phi||_E^2 + ||\nabla\Psi||_E^2 + 2\Re\langle\Phi, \nabla\Psi\rangle_E = ||\Phi||_E^2 + ||\nabla\Psi||_E^2$$

so that Φ is minimal within its cohomology class.

Conversely, assume that $\Phi \in \ker\nabla$ is minimal within its cohomology class. Without loss of generality assume $\Phi \in \Omega^k(M;E) \cap \ker\nabla$. Every $\Psi \in \Omega^{k-1}(M;E)$ defines the functions $c_1(\epsilon) = ||\Phi + \epsilon\nabla\Psi||_E^2$ and $c_2(\epsilon) = ||\Phi + i\epsilon\nabla\Psi||_E^2$. By assumption $0 = c_1'(0) = c_2'(0)$ which implies that $0 = \Re\langle\Phi, \nabla\Psi\rangle_E = \Im\langle\Phi, \nabla\Psi\rangle_E$. Thus $0 = \langle\Phi, \nabla\Psi\rangle_E = \langle\nabla^*\Phi, \Psi\rangle_E$. Since this holds for all for all $\Psi \in \Omega^{k-1}(M;E)$ we must have $\nabla^*\Phi = 0$ and therefore $\Phi \in \ker\nabla^* \cap \nabla = \ker\Delta$. $\square$

Proposition 4.9.4. Let $(E, h, \nabla) \to M$ be a flat unitary bundle over a closed Kähler manifold and assume that E is endowed with the unique holomorphic structure such that $\nabla^{01} = \bar\partial$. Then the ∇-Laplacian Δ commutes with L , Λ, and Π.

Proof. It is immediate that $[\Delta, \Pi] = 0$ and according to 4.6.2 $[L, \nabla^{01}] = [L, \nabla^{10}] = [\Lambda, \nabla^{10*}] =$

$[\Lambda, \nabla^{01*}] = 0$. Consider the Laplacian operators

$$\Delta_{10} := \nabla^{10}\nabla^{10*} + \nabla^{10*}\nabla^{10}$$

$$\Delta_{01} := \nabla^{01}\nabla^{01*} + \nabla^{01*}\nabla^{01}.$$

Using the Kähler identites 4.8.3 we find

$$\begin{aligned}
[L, \Delta_{10}] &= L\nabla^{10}\nabla^{10*} + L\nabla^{10*}\nabla^{10} - \nabla^{10}\nabla^{10*}L - \nabla^{10*}\nabla^{10}L \\
&= \nabla^{10}L\nabla^{10*} + L\nabla^{10*}\nabla^{10} - \nabla^{10}\nabla^{10*}L - \nabla^{10*}L\nabla^{10} \\
&= \nabla^{10}L\nabla^{10*} + \big(\nabla^{10*}L + i\nabla^{01}\big)\nabla^{10} - \nabla^{10}\big(L\nabla^{10*} - i\nabla^{01}\big) - \nabla^{10*}L\nabla^{10} \\
&= i\nabla^{01}\nabla^{10} + i\nabla^{10}\nabla^{01} = 0
\end{aligned}$$

since $\nabla^2 = 0 \leftrightarrow \nabla^{10}\nabla^{01} + \nabla^{01}\nabla^{10} = 0$. A similar computation shows that $[L, \Delta_{01}] = 0$.

Finally notice that

$$\begin{aligned}
\Delta &= \nabla\nabla^* + \nabla^*\nabla = (\nabla^{10} + \nabla^{01})(\nabla^{10*} + \nabla^{01*}) + (\nabla^{10*} + \nabla^{01*})(\nabla^{10} + \nabla^{01}) \\
&= \nabla^{10}\nabla^{10*} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{01}\nabla^{01*} + \nabla^{10*}\nabla^{10} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} + \nabla^{01*}\nabla^{01} \\
&= \nabla^{10}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01}\nabla^{01*} + \nabla^{01*}\nabla^{01} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} \\
&= \Delta_{10} + \Delta_{01} + \nabla^{10}\nabla^{01*} + \nabla^{01}\nabla^{10*} + \nabla^{10*}\nabla^{01} + \nabla^{01*}\nabla^{10} \\
&= \Delta_{10} + \Delta_{01} + i[L, \nabla^{01*}]\nabla^{01*} - i[L, \nabla^{10*}]\nabla^{10*} - i\nabla^{10*}[L, \nabla^{10*}] + i\nabla^{01*}[L, \nabla^{01*}] \\
&= \Delta_{10} + \Delta_{01} - i\nabla^{01*}L\nabla^{01*} + i\nabla^{10*}L\nabla^{10*} - i\nabla^{10*}L\nabla^{10*} + i\nabla^{01*}L\nabla^{01*} \\
&= \Delta_{10} + \Delta_{01}
\end{aligned}$$

and therefore $[L, \Delta] = [L, \Delta_{10}] + [L, \Delta_{01}] = 0$. As Δ is self-adjoint, passing to the adjoint gives $[\Lambda, \Delta] = 0$. ☐

Corollary 4.9.5. There is a representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathcal{H}^{\bullet}_{\nabla}(M; E)$.

Proof. Since Δ commutes with L, Λ, and Π, the representation $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}\big(\Omega^{\bullet}(M; E)\big)$ restricts to a representation on $\mathcal{H}^{\bullet}_{\nabla}(M; E)$. ☐

Proposition 4.9.6. $\mathcal{H}^{\bullet}_{\nabla}(M; E) \cong H^{\bullet}_{\nabla}(M; E)$.

Proof. $\Delta : \Omega^k(M; E) \to \Omega^k(M; E)$ is a an elliptic operator for all k and therefore $\ker \Delta \cap \Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E)$ is finite and there is an orthogonal (with respect to $\langle -, - \rangle_E$) decomposition [18]

$$\Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E) \oplus \Delta(\Omega^k(M; E)).$$

Since $\nabla^2 = 0 = (\nabla^*)^2$ observe that $\langle \nabla\Phi, (\nabla\nabla^* + \nabla^*\nabla)\Psi \rangle_E = \langle \Phi, (\nabla^*)^2 \nabla\Psi \rangle_E + \langle \nabla^2\Phi, \nabla^*\Psi \rangle_E = 0$ and therefore $\nabla(\Omega^{k-1}(M; E)) \perp \mathcal{H}^k_{\nabla}(M; E)$. Similarly $\nabla^*(\Omega^{k+1}(M; E)) \perp \mathcal{H}^k_{\nabla}(M; E)$. Therefore $\nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E)) \subset \Delta(\Omega^k(M; E))$. Also observe that $\nabla(\Omega^{k-1}(M; E)) \perp \nabla^*(\Omega^{k+1}(M; E))$ and therefore $\Delta(\Omega^k(M; E)) \subset \nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E))$. Hence one has the *Hodge decomposition*

$$\Omega^k(M; E) = \mathcal{H}^k_{\nabla}(M; E) \oplus \nabla(\Omega^{k-1}(M; E)) \oplus \nabla^*(\Omega^{k+1}(M; E)).$$

Now consider the natural map $\mathcal{H}^k_{\nabla}(M; E) \to H^k_{\nabla}(M; E)$ that sends a harmonic form to the cohomology class it represents. The kernel of this map is, by definition, $\nabla(\Omega^{k-1}(M; E)) \cap$

$\mathcal{H}^k_\nabla(M;E) = \{0\}$ hence the map is injective. Let $\Phi \in \Omega^k(M;E) \cap \ker\nabla$ represent an arbitrary element of $H^k_\nabla(M;E)$. Use the decomposition to write $\Phi = \phi + \nabla a + \nabla^* b$ where $(\phi, a, b) \in \mathcal{H}^k_\nabla(M;E) \times \Omega^{k-1}(M;E) \times \Omega^{k+1}(M;E)$. Notice that $\nabla^* b \in \ker\nabla$ and, furthermore, for all $\nabla^* c \in \nabla^*(\Omega^{k+1}(M;E)$, $\langle \nabla^* b, \nabla^* c\rangle_E = \langle \nabla\nabla^* b, c\rangle_E = 0$. Thus $\nabla^* b = 0$ so $\Phi = \phi + \nabla a$ and $\mathcal{H}^k_\nabla(M;E) \ni \phi \mapsto [\phi] = [\phi + \nabla a] = [\Phi] \in H^k_\nabla(M;E)$ is surjective. $\square$

Theorem 4.9.7. If $(E, h, \nabla) \to (M, \omega)$ is a flat unitary bundle over a closed Kähler manifold then for all $k \leq \dim_{\mathbb{C}} M = n$, the maps $L^{n-k} : \Omega^k_\nabla(M;E) \ni \Phi \mapsto \omega^{n-k} \wedge \Phi \in \Omega^{2n-k}_\nabla(M;E)$ induce isomorphisms

$$L^{n-k} : H^k_\nabla(M;E) \xrightarrow{\sim} H^{2n-k}_\nabla(M;E). \tag{4.8}$$

Proof. Using $\mathcal{H}^\bullet_\nabla(M;E) \cong H^\bullet_\nabla(M;E)$, $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}\big(H^\bullet_\nabla(M;E)\big)$ is a finite representation and therefore $\rho(B)^{n-k} = L^{n-k}$ is an isomorphism between the eigenspaces $H^k_\nabla(M;E)$ and $H^{2n-k}_\nabla(M;E)$. $\square$

Corollary 4.9.8. If (E, h, ∇) is a flat unitary bundle then $(\mathrm{End}E, \widetilde{h}, \widetilde{\nabla})$ is also a flat unitary bundle and therefore for all $k \leq \dim_{\mathbb{C}} M$

$$L^{n-k} : H^k_{\widetilde{\nabla}}(M;\mathrm{End}E) \xrightarrow{\sim} H^{2n-k}_{\widetilde{\nabla}}(M;\mathrm{End}E).$$

$\square$

Corollary 4.9.9. $L^{n-k} : H^k_{\widetilde{\nabla}}(M;\mathfrak{u}E) \xrightarrow{\sim} H^{2n-k}_{\widetilde{\nabla}}(M;\mathfrak{u}E)$.

Proof. $\mathfrak{u}E \subset \mathrm{End}E$ is a real sub-bundle and the symplectic form is $\omega \in \Omega^2(M, \mathbb{R}) \cap \Omega^{(1,1)}(M)$. Hence $L^j(H^\bullet_{\widetilde{\nabla}}(M; \mathfrak{u}E)) \subset H^\bullet_{\widetilde{\nabla}}(M; \mathfrak{u}E)$. Since $H^\bullet_{\widetilde{\nabla}}(M; \mathrm{End}E)$ is finite dimensional, $L^{n-k}|_{H^k_{\widetilde{\nabla}}(M;\mathfrak{u}E)}$: $H^k_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to H^{2n-k}_{\widetilde{\nabla}}(M; \mathfrak{u}E)$ is injective and thus an isomorphism.

□

4.10 Functional on the symplectic space $H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$

Now assume that the hermitian vector bundle $(\pi : E \to M, h)$ has a closed Kähler manifold M as its base. Let $\omega \in \Omega^2(M) \cap \Omega^{(1,1)}(M)$ denote the Kähler form.

For all $\nabla \in \mathcal{F}(E, h)$, the module $H^\bullet_{\widetilde{\nabla}}(M; \mathfrak{u}E)$ satisfies hard Lefschetz duality [4.9.7]. In particular, the pairing

$$\omega_\nabla : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \times H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \ni (\Phi, \Psi) \mapsto \int_M h(\Phi \wedge \Psi) \wedge [\omega]^{n-1} \in \mathbb{R} \tag{4.9}$$

makes the vector space $H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$ into a symplectic vector space. Hard Lefschetz duality and its composition with Poincaré duality gives the isomorphism

$$\Upsilon := PD \circ \mathcal{L} : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to H_1^{\widetilde{\nabla}}(M; \mathfrak{u}E). \tag{4.10}$$

For $\Psi \in H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$, consider the map induced from the symplectic structure on $H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$

$$\omega_\nabla(-, \Psi) = \int_M \widetilde{h}(- \wedge \Psi) \wedge [\omega]^{n-1} : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}.$$

Using the pairing introduced in 2.10.3, every homology class $N \in H_1^{\widetilde{\nabla}}(M; \mathfrak{u}E)$ defines the function

$$\int \langle -, N \rangle_h : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R}.$$

Notice that

$$\Psi \mapsto \int_M \widetilde{h}(- \wedge \Psi) \wedge [\omega]^{n-1} = \int_M \widetilde{h}\big(- \wedge(\Psi \wedge [\omega^{n-1}])\big)$$

corresponds to the Poincaré dual of the hard Lefschetz dual of Ψ. Therefore, for all $\Phi, \Psi \in H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$,

$$\int \langle \Phi, \Upsilon(\Psi) \rangle_h = \int_M \widetilde{h}(\Phi \wedge \Psi) \wedge [\omega]^{n-1}.$$

This establishes the following

Theorem 4.10.1. Let $(\pi : E \to M, h)$ be a hermitian vector bundle over a closed Kähler manifold with Kähler form ω. Let $\nabla \in \mathcal{F}(E, h)$ be a flat unitary connection and let γ be a closed curve in M. Then the trace of holonomy functional satisfies

$$\mathbb{T} = \omega_{\nabla}\big(-, \Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)]\big) : H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E) \to \mathbb{R} \tag{4.11}$$

and thus for all $\Phi \in H^1_{\widetilde{\nabla}}(M; \mathfrak{u}E)$

$$\int_M h(\Phi \wedge \Upsilon(\gamma)) \wedge [\omega]^{n-1} = -\int_0^1 \mathbf{Tr}\Big(\mathrm{Hol}_{\gamma^t}(\nabla)\Phi(\dot{\gamma}(t))\Big) dt. \tag{4.12}$$

Proof. Using 2.10.3

$$\begin{aligned}\mathbb{T}(\Phi) &= \int \Big\langle \Phi, [\gamma, F\mathrm{Hol}_\gamma(\nabla)] \Big\rangle_h = \int \Big\langle \Phi, \Upsilon\Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)] \Big\rangle_h \\ &= \int_M \widetilde{h}(\Phi \wedge \Upsilon^{-1}[\gamma, F\mathrm{Hol}_\gamma(\nabla)]) \wedge [\omega]^{n-1}.\end{aligned}$$

∎

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