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Differentiability of the Liouville Map via Geodesic Currents

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by

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Abstract

DIFFERENTIABILITY OF THE LIOUVILLE MAP VIA GEODESIC CURRENTS

by

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For a conformally hyperbolic Riemann surface, the Teichmüller space is the space of quasiconformal maps factored by an equivalence relation, and it is a complex Banach manifold. The space of geodesic currents endowed with the uniform weak* topology is a subset of a Fréchet space of Hölder geodesic distributions. We introduce an appropriate topology on the space of Hölder geodesic distributions and this new topology coincides with the uniform weak* topology on the space of geodesic currents. The Liouville map of the Teichmüller space becomes differentiable in the Fréchet sense. In particular, the derivative of Liouville currents exists and belongs to the space of Hölder geodesic distributions, and the tangent map of the Liouville map is continuous and linear. The elements of the Teichmüller space can be represented by earthquake maps. Since an earthquake path is a differentiable path in the Teichmüller space, then the image of an earthquake path under the Liouville map is a differentiable path in the space of Hölder geodesic distributions. We compute the image of the tangent vector to an earthquake path in the space of Hölder geodesic distributions.

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Chapter 1

Introduction

A Riemann surface is said to be conformally hyperbolic, if the universal cover of the surface is biholomorphically equivalent to the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Teichmüller space of a Riemann surface is the space of quasiconformal maps factored by an equivalence relation. It is a contractible, simply connected complex Banach manifold; it has a natural metric called the Teichmüller metric.

From now on, we assume that X_0 is a conformally hyperbolic Riemann surface. Geodesic currents are defined as those measures on the space $G(\widetilde{X}_0)$ of complete geodesics of the universal cover $\widetilde{X}_0$, which are invariant under the action of the fundamental group $\pi_1(X_0)$. The space $\mathcal{C}(X_0)$ of geodesic currents as a space of measures, is endowed with the usual weak* topology. However, this topology fails to take into account the many symmetries of the universal cover $\widetilde{X}_0$ coming from the group $PSL(2, \mathbb{R})$ of all orientation preserving isometries of $\widetilde{X}_0$. With the consideration of uniformity in [8], Bonahon and Šarić defined the Liouville map of the Teichmüller space into the space of bounded geodesic currents endowed with the uniform weak* toplogy, to give a natural description of the Thurston boundary of the Teichmüller space of a noncompact Riemann surface. The Liouville map $L : \mathcal{T}(X_0) \to \mathcal{C}_{\text{bd}}(X_0)$ was proved to be an embedding. For an element $[f] \in \mathcal{T}(X_0)$,

$L([f]) = L_f$ is the Liouville current which belongs to $\mathcal{C}_{\mathrm{bd}}(X_0)$. This embedding is only topological and has no smoothness properties.

We prove that the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents can be enlarged into a Fréchet space of bounded Hölder geodesic distributions, denoted by $\mathcal{H}_{\mathrm{bd}}(X_0)$. The Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$ is proved to be differentiable in the Fréchet sense. Explicitly, we are looking into the derivatives of Liouville currents at each point $[f] \in \mathcal{T}(X_0)$. It turns out that those derivatives belong to $\mathcal{H}_{\mathrm{bd}}(X_0)$. This dissertation is arranged as follows:

In Chapter 2, we give the necessary background on Hyperbolic geometry and Teichmüller theory. In Chapter 3, we discuss the geodesic currents and the Liouville embedding. In Chapters 4, 5, and 6, we prove the main results of our dissertation. We end with a conjecture in Chapter 6.

In Chapter 4, we define the space of bounded Hölder geodesic distributions $\mathcal{H}_{\mathrm{bd}}(X_0)$ that is an enlargement of $\mathcal{C}_{\mathrm{bd}}(X_0)$. We explain why the extended map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$ is again an embedding. To do this, we introduce the uniform Hölder topology on $\mathcal{H}_{\mathrm{bd}}(X_0)$ and we prove the following result:

THEOREM A. *The uniform Hölder topology coincides with the uniform weak* topology on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents.*

In Chapter 5, we prove the differentiability of L in the Fréchet sense. Here is the precise statement of our result.

THEOREM B. *The Liouville map $L : \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0) \subset \mathcal{H}_{\mathrm{bd}}(X_0)$ is differentiable at each $m \in \mathcal{T}(X_0)$, in the Fréchet sense. In particular, there is a continuous linear map*

$$T_m L : T_m\mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$$

such that if $t \to m_t$, $t \in (-\varepsilon, \varepsilon)$, is a differentiable path in $\mathcal{T}(X_0)$ with the tangent vector $v_0 \in T_{m_0}\mathcal{T}(X_0)$ at $t = 0$, if $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ is a Hölder continuous function with compact *support, and if φ is an orientation preserving isometry of $\widetilde{X}_0$, then the derivative*

$$\frac{d}{dt}L_{m_t}(\xi\circ\varphi)|_{t=0}=\frac{d}{dt}\int_{G(\widetilde{X}_0)}\xi\circ\varphi dL_{m_t}|_{t=0}$$

exists and it is equal to $T_{m_0}L(v_0)$.

In addition, the tangent map T_mL varies continuously with $m\in\mathcal{T}(X_0)$.

In Chapter 6, we study the tangent vectors to earthquake paths. For a bounded geodesic lamination $\widehat{\Lambda}=(\Lambda,\mu)$, the earthquake path $t\mapsto E^{t\mu}$ is a differentiable path in $\mathcal{T}(X_0)$. By Theorem B, $L_{E^{t\mu}}$ is a differentiable path in $\mathcal{H}_{\mathrm{bd}}(X_0)$. We compute the derivative $\frac{d}{dt}L_{E^{t\mu}}|_{t=0}$ for the cases of elementary earthquake and simple earthquake maps.

THEOREM C. *Let (Λ_δ,δ) be a bounded measured lamination with finite support $\{g_1,\cdots,g_n\}$ where $\delta=\sum_i^n d_i\mathbb{1}_{g_i}$. Let $\xi:G(\widetilde{X}_0)\to\mathbb{R}$ be a Hölder continuous function with compact support. Then*

$$\frac{d}{dt}L_{E^{t\delta}}(\xi)|_{t=0}=\frac{d}{dt}\int_{G(\widetilde{X}_0)}\xi dL_{E^{t\delta}}|_{t=0}=\sum_{i=1}^n d_i\int_{G(\mathbb{D})}\xi(h)\cos(h,g_i)dL_{\mathbb{D}}(h)$$

where $h\in G(\mathbb{D})$ and $g_i\in\Lambda_\delta$.

We conjecture a formula for the derivative of a general earthquake path from the above simple earthquake case.

CONJECTURE. *Let (Λ,μ) be a bounded geodesic lamination. Let $\xi:G(\widetilde{X}_0)\to\mathbb{R}$ be a Hölder continuous function with compact support. Then*

$$\frac{d}{dt}L_{E^{t\mu}}(\xi)|_{t=0}=\frac{d}{dt}\int_{G(\widetilde{X}_0)}\xi dL_{E^{t\mu}}|_{t=0}\overset{?}{=}\int_\Lambda\int_{G(\mathbb{D})}\xi(h)\cos(h,g)dL_{\mathbb{D}}(h)d\mu$$

where $h\in G(\mathbb{D})$ and $g\in\Lambda$.

Chapter 2

Hyperbolic Geometry and Teichmüller Theory

2.1 Introduction

In this chapter, we provide necessary background on Hyperbolic geometry and Teichmüller theory. First, we recall some definitions including but not limited to different models of the hyperbolic plane, Fuchsian groups and Riemann Surfaces. Next we will focus on Hyperbolic Riemann surfaces and study its Teichmüller space, including quasiconformal mappings, quadratic differentials and Bers embedding.

2.2 Hyperbolic geometry

We begin with models of the hyperbolic plane.

(i) *upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$

with the metric derived from the infinitesimal form $ds = \dfrac{|dz|}{Im(z)}$;

(ii) *unit disk* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

with the metric derived from the infinitesimal form $ds = \frac{2|dz|}{1-|z|^2}$.

The expression for ds gives us a way to define the hyperbolic distance. To each piecewise continuously differentiable curve $\gamma : [a, b] \to \mathbb{H}$, we define the hyperbolic length of γ by the formula $l(\gamma) = \int_a^b \frac{|\gamma'(t)|}{Im(\gamma(t))} dt$. The distance between two points $p, q \in \mathbb{H}$ is defined to be $d_{\mathbb{H}}(p, q) = \inf_{\gamma} l(\gamma)$, where the infimum is taken over all paths γ joining p to q. Similarly for $d_{\mathbb{D}}(p, q)$. The Cayley transformation $f(z) = \frac{z-i}{z+i}$ is an isometry of $\mathbb{H}$ and $\mathbb{D}$. Thus $d_{\mathbb{D}}(p, q) = d_{\mathbb{H}}(f^{-1}(p), f^{-1}(q))$ for any two points $p, q \in \mathbb{D}$. A path γ from z_0 to z_1 is a *geodesic* if it locally minimizes distances. The *boundary at infinity* is: $\mathbb{R} \cup \{\infty\}$ for $\mathbb{H}$ denoted by $\partial\mathbb{H}$ and $\{z : |z| = 1\}$ for $\mathbb{D}$ denoted by $\partial\mathbb{D}$.

The *group of orientation preserving isometries* of $\mathbb{H}$ consists of maps of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, denoted by Möb($\mathbb{H}$). Similarly, the group of orientation preserving isometries of $\mathbb{D}$ consists of maps the form $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$ with $|a|^2 - |b|^2 = 1$, denoted by Möb($\mathbb{D}$). In particular, Möb($\mathbb{H}$) $= PSL(2, \mathbb{R})$. The Cayley transform conjugates Möb($\mathbb{D}$) to Möb($\mathbb{H}$).

Next, we look at discrete groups of isometries of the hyperbolic plane. In general, if the model is not clear or it is not necessary to specify, we refer the hyperbolic plane as $\mathbb{H}$. A *topological group* G is both a group and a topological space, with the properties that the maps $x \mapsto x^{-1}$ of G onto G and $(x, y) \mapsto xy$ of $G \times G$ onto G are continuous. A topological group G is *discrete* if the topology on G is the discrete topology. We say that G *acts properly discontinuously* on $\mathbb{H}$ if and only if for all compact subsets $K \subset \mathbb{H}$, $g(K) \cap K = \varnothing$ for all but finitely many $g \in G$. Note that a subgroup $G \subset PSL(2, \mathbb{R})$ is discrete if and only if its action on $\mathbb{H}$ is properly discontinuous, see [5]. A *Fuchsian group* Γ is a discrete subgroup of $PSL(2, \mathbb{R})$.

A Riemann surface X is a one-dimensional complex manifold. This means, X is a connected Hausdorff space that is endowed with an atlas of charts $\{(\phi_i, U_i) : i \in I\}$ such that

$\{U_i : i \in I\}$ is an open cover of X, each ϕ_i is a homeomorphism of U_i onto an open subset of the complex plane; and if $U = U_i \cap U_j \neq \varnothing$, then $\phi_i(\phi_j)^{-1} : \phi_j(U) \to \phi_i(U)$ is an holomorphic map. One way of constructing Riemann surfaces is by forming the quotient space with respect to a properly discontinuous group action. We have the following Theorem:

Theorem 2.2.1. *Let D be a subdomain of $\widehat{\mathbb{C}}$ and let G be a group of Möbius transformations which leaves D invariant and which acts freely and properly discontinuously in D. Then D/G is a Riemann surface.*

See Theorem 6.2.1 in [5] for a proof. □

Let $\widetilde{X}$ denote the universal cover of X and $p : \widetilde{X} \to X$ be a covering map. We can put a Riemann surface structure on $\widetilde{X}$ so that p becomes holomorphic, see [23]. The Uniformization Theorem states that any simply connected Riemann surface is conformally equivalent to one of the following surfaces: the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}$. Recall that a map $g : \widetilde{X} \to \widetilde{X}$ which satisfies $p \circ g = p$ is called *covering transformation.* The set of covering transformations form a group, the *covering group* of X. It is a standard result that the covering group of X can be identified with the fundamental group of $\pi_1(X)$. A Riemann surface can be written as $\widetilde{X}/\Gamma$, where $\widetilde{X}$ is a subset of $\widehat{\mathbb{C}}$ and Γ is the covering group of X. A Riemann surface X is said to be *conformally hyperbolic* if it is of the form $\mathbb{H}/\Gamma$, we write $X \cong \mathbb{H}/\Gamma$ where Γ is a Fuchsian group. In addition, we say that a Riemann surface is of finite topological type if it has a finitely generated fundamental group. Otherwise, it is of infinite topological type. From now on, all Riemann surfaces in the upcoming chapters will be implicitly assumed to be conformally hyperbolic and of infinite topological type.

2.3 Teichmüller theory

A complex-valued function f defined on a region $\Omega \subset \mathbb{C}$ is called a *quasiconformal map* if it is a sense preserving homeomorphism of Ω onto its image and its complex distributional derivative $f_z, f_{\bar{z}}$ are locally square integrable on Ω that satisfy the inequality $|f_{\bar{z}}| \leqslant k|f_z|$ almost everywhere on Ω for some real number $0 \leqslant k < 1$. If f is a quasiconformal map defined on a region Ω, then the function f_z is known to be nonzero almost everywhere on Ω. Therefore the function $\mu_f = \dfrac{f_{\bar{z}}}{f_z}$ is a well-defined L^∞ function on Ω, called the *complex dilatation* or the *Beltrami coefficient* of f. The L^∞ norm of every Beltrami coefficient is less than one. The positive number $K(f) = \dfrac{1 + ||\mu_f||_\infty}{1 - ||\mu_f||_\infty}$ is called the *dilatation* of f. We say that f is K-*quasiconformal* if f is a quasiconformal map and $K(f) \leqslant K$. Note that a map is 1-quasiconformal if and only if it is conformal. If f and g are quasiconformal and the image of f is contained in the domain of g, then $K(f \circ g) \leqslant K(f)K(g)$. In particular, if f is K-quasiconformal and g and h are conformal, then $g \circ f \circ h$ is K-quasiconformal. A sense preserving homeomorphism $f : X_1 \to X_2$ between two Riemann surfaces is called K-quasiconformal if its lift to the universal cover $\tilde{f} : \mathbb{H} \to \mathbb{H}$ is K-quasiconformal. We define the dilatiation $K(f)$ of f to be $K(\tilde{f})$.

Let X_0 be a Riemann surface which is conformally hyperbolic. A *Beltrami differential* μ on X_0 is a type $(-1, 1)$ tensor of the form $\mu(z)\dfrac{\overline{dz}}{dz}$, more precisely an assignment of an L^∞ function $\mu_1(z_1)$ to each local coordinate z_1 such that if z_2 is another local coordinate, then $\mu_1(z_1) = \mu_2(z_2)\dfrac{\overline{dz_2/dz_1}}{dz_2/dz_1}$. Note that the essential norm $||\mu||_\infty$ is bounded. We denote the space of such essentially bounded Beltrami differential μ on X_0 by $L^\infty(X_0)$. By classical functional analysis, $L^\infty(X_0)$ is a complex Banach space.

Any quasiconformal map $f : X_0 \to X$ from X_0 onto a Riemann surface X extends uniquely to a homeomorphism of $X_0 \cup \partial X_0$ onto $X \cup \partial X$. For two such quasiconformal maps f_1 and f_2 from X_0 onto the Riemann surfaces X_1 and X_2 respectively, we say that f_1 and f_2 are *Teichmüller equivalent* if there exists a biholomorphic isomorphism g of X_1 onto X_2 such that the homeomorphism $f_2^{-1} \circ g \circ f_1$ of $X_0 \cup \partial X_0$ onto itself is homotopic to the identity by a homotopy that maps X_0 onto itself and fixes ∂X_0 pointwise. The *Teichmüller space* $\mathcal{T}(X_0)$ of X_0 is the set of all quasiconformal maps with domain X_0 factored by this equivalence relation. We denote the equivalence class of the quasiconformal map $f : X_0 \to X$ by $[f]$. The basepoint of $\mathcal{T}(X_0)$ is the equivalence class of the identity map $id : X_0 \to X_0$. In addition, given $[f] \in \mathcal{T}(X_0)$ and if f' is a quasiconformal map from X_0 to X', the map $[f] \mapsto [f \circ (f')^{-1}]$ induces an isomorphism from $\mathcal{T}(X_0) \to \mathcal{T}(X')$, which is an isometry, see [10].

The following theorem of Ahlfors and Bers is essential in determining a complex structure for Teichmüller space.

Theorem 2.3.1. *Let $M_1(\mathbb{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Then for each Beltrami differential $\mu \in M_1(\mathbb{C})$, there exists a unique quasiconformal map f^μ of $\widehat{\mathbb{C}}$ onto itself normalized to fix $0, 1$ and ∞ whose Beltrami coefficient is $\mu_{f^\mu} = \mu$. Furthermore, for every fixed $z \in \mathbb{C}$, the map $\mu \mapsto f^\mu(z)$ of $M_1(\mathbb{C})$ onto $\mathbb{C}$ is holomorphic.*

See [3] for a proof. □

Recall that $X_0 \cong \mathbb{H}/\Gamma$, where Γ is a Fuchsian group. By definition, any quasiconformal map $f : X_0 \to X$ is associated with a Beltrami coefficient μ_f. Then a Beltrami differential μ on X_0 can be identified with an L^∞ function $\tilde{\mu}_f$ on $\mathbb{H}$ satisfying the equation

$$\tilde{\mu}_f(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \tilde{\mu}_f(z) \tag{2.1}$$

for all $z \in \mathbb{H}$ and for all $\gamma \in \Gamma$. Furthermore, every quasiconformal map $f : X_0 \to X$ lifts to a quasiconformal map $\tilde{f} : \mathbb{H} \to \mathbb{H}$. If the Beltrami coefficient $\mu_{\tilde{f}}$ satisfies equation (2.1), then $\tilde{f} \circ \Gamma \circ \tilde{f}^{-1} = \Gamma_{\tilde{f}}$ is also a Fuchsian group. Similarly, a Beltrami differential μ on $f(X_0)$ can be identified with an L^∞ function $\tilde{\mu}_f$ on $\mathbb{H}$ satisfying the equation

$$\tilde{\mu}_f(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \tilde{\mu}_f(z)$$

for all $z \in \mathbb{H}$ and for all $\gamma \in \Gamma_{\tilde{f}}$.

Let $B(\Gamma)$ denote the space of all Beltrami coefficients on $\mathbb{H}$ which satisfies equation (2.1); denote the lower half-plane by $\mathbb{L} = \{z \in \mathbb{C} : Im(z) < 0\}$. For a fixed $\mu \in B(\Gamma)$, we will use the following notations: f^μ is the quasiconformal self-map of $\mathbb{H}$ with Beltrami coefficient μ and normalized to fix 0, 1 and ∞. Note that f^μ extends continuously to $\mathbb{R}$. Define f_μ the quasiconformal self-map of $\bar{\mathbb{C}}$ with Beltrami coefficient equal to μ in the upper half-plane and identically equal to zero in the lower half-plane and normalized to fix 0, 1 and ∞, then f_μ is conformal in $\mathbb{L}$ because the Beltrami coefficient is 0 in $\mathbb{L}$.

Theorem 2.3.2. *Suppose that $\mu, \nu \in B(\Gamma)$. Then the following are equivalent:*

(i) $f^\mu(x) = f^\nu(x)$ for all $x \in \mathbb{R}$; and

(ii) $f_\mu(x) = f_\nu(x)$ for all $x \in \mathbb{R}$;

(iii) $f_\mu(z) = f_\nu(z)$ for all $z \in \mathbb{L}$.

See page 133 in [10] for a proof. □

Let $[f_1], [f_2] \in \mathcal{T}(X_0)$. The Teichmüller space $\mathcal{T}(X_0)$ is endowed with the *Teichmüller metric* defined by

$$d_T([f_1], [f_2]) = \frac{1}{2} \inf_g \log K(g)$$

where the infimum is taken over all quasiconformal maps $g : X_1 \to X_2$ with g homotopic to $f_2 \circ f_1^{-1}$. The Teichmüller space $\mathcal{T}(X_0)$ is complete with respect to this metric d_T. Now we are going to see that $\mathcal{T}(X_0)$ with the Teichmüller metric is a complex Banach manifold. To do this, we first need to construct the Banach spaces we will model $\mathcal{T}(X_0)$ on.

A *quadratic differential* q on X_0 is a type $(2, 0)$ tensor of the form $q(z)dz^2$, more precisely an assignment of a function $q_1(z_1)$ to each local coordinate z_1 such that if z_2 is another local coordinate, then $q_1(z_1) = q_2(z_2)(dz_2/dz_1)^2$. We identify q with holomorphic functions $\tilde{q}$ defined in $\mathbb{L}$ such that

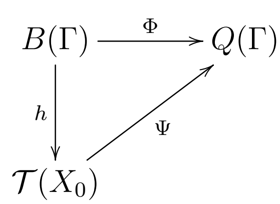
$$\tilde{q}(\gamma(z))\gamma'(z)^2 = \tilde{q}(z) \tag{2.2}$$

for all $\gamma \in \Gamma$. Let $Q(\Gamma)$ denote the Banach space of all bounded holomorphic quadratic differentials $\tilde{q}$ satisfying (2.2) with norm $||\tilde{q}||_{Q(\Gamma)} = ||\tilde{q}(z)y^2||_\infty < \infty$, where $y = Im(z)$. The complex structure on $T(X_0)$ is based on the following construction. For $\mu \in B(\Gamma)$, let $\tilde{\mu}$ be the Beltrami differential on $\mathbb{C}$ such that $\tilde{\mu}(z) = \mu(z)$ for $z \in \mathbb{H}$ and $\tilde{\mu}(z) = 0$ for $z \in \mathbb{L}$. Solve the Beltrami equation for $\tilde{\mu} \in \mathbb{C}$. The solution $f_{\tilde{\mu}}$ is conformal when restricted to $\mathbb{L}$ and fixes $0, 1$ and ∞, in particular it is injective on $\mathbb{L}$. We have the Schwarzian derivative of $f_{\tilde{\mu}}$ by

$$S(f_{\tilde{\mu}}) = \frac{f_{\tilde{\mu}}'''}{f_{\tilde{\mu}}'} - \frac{3}{2}\left(\frac{f_{\tilde{\mu}}''}{f_{\tilde{\mu}}'}\right)^2$$

and Nehari proved that for $\mu \in B(\Gamma)$, we have $S(f_{\tilde{\mu}}) \in Q(\Gamma)$, see Lemma 5 on page 133 in [10].

We can therefore define the Schwarzian derivative map $\Phi : B(\Gamma) \to Q(\Gamma)$ by $\Phi(\mu) = S(f_{\tilde{\mu}})$. Bers showed that $\Phi(\mu) = \Phi(\upsilon)$ if and only if $[f^\mu] = [f^\upsilon]$ in $\mathcal{T}(X_0)$. See the commutative diagram below, where h is a continuous map obtained by solving the Beltrami equation of each $\mu \in B(\Gamma)$ and taking the equivalence class of the solution. The map Φ induces a one-to-one map $\Psi : \mathcal{T}(X_0) \to Q(\Gamma)$ called *Bers embedding*; it maps onto an open bounded set in $Q(\Gamma)$. The map Ψ is homeomorphic onto its image and defines a global holomorphic chart for $\mathcal{T}(X_0)$.



Let $q \in Q(\Gamma)$. A Beltrami differential λ of the form $-2y^2q(\bar{z})$ for $z \in \mathbb{H}$ and $-2y^2\overline{q(z)}$ for $z \in \mathbb{L}$ is called a *harmonic Beltrami differential*. Note that $||q||_{Q(\Gamma)} = ||q(z)y^2||_\infty = \dfrac{1}{2}||\lambda||_\infty$. If $||q||_{Q(\Gamma)} < 1/2$, then $\lambda \in B(\Gamma)$. Ahlfors and Weill defined a map from $Q(\Gamma)$ to $B(\Gamma)$ by $\mathcal{AW}(q) = -2y^2q(\bar{z}) = \lambda(z)$ where $||q||_{Q(\Gamma)} < 1/2$ and proved that if $\tilde{\lambda}(z) = \lambda(z)$ for $z \in \mathbb{H}$ and $\tilde{\lambda}(z) = 0$ for $z \in \mathbb{L}$ then the Schwarzian derivative $S(f_{\tilde{\lambda}}) = q$, see [4] for more details. This means $\Psi([f^\lambda]) = q$, in particuluar $\Psi(\mathcal{T}(X_0))$ contains the neighborhood $\{q : ||q||_{Q(\Gamma)} < \frac{1}{2}\}$. Thus the inverse map $\Psi^{-1}(q) = [f^\lambda]$ from $\{q : ||q||_{Q(\Gamma)} < \frac{1}{2}\}$ of $[id] \in \mathcal{T}(X_0)$ to $B(\Gamma)$ is a harmonic chart for $\mathcal{T}(X_0)$ around $[id]$. The same property holds for all other points using the change of base point map, see page 136 in [10] for more details. Thus $\mathcal{T}(X_0)$ get a complex Banach manifold structure.

Next we describe the tangent space to the Teichmüller space. Since Bers embedding provides a global holomorphic chart for $\mathcal{T}(X_0)$, thus the tangent space at the basepoint of $\mathcal{T}(X_0)$ is identified with $Q(\Gamma)$. Let $[f] \in \mathcal{T}(X_0)$ be represented by a quasiconformal map $f : X_0 \to X$ and let μ_f be the Beltrami coefficient of f satisfying expression (2.1). It follows that $\tilde{f}$ conjugates Γ onto another Fuchsian group $\Gamma_{\tilde{f}}$. Note that $X \cong \mathbb{H}/\Gamma_{\tilde{f}}$. There is a natural bijection

$$T(\mu_f) : \mathcal{T}(X) \to \mathcal{T}(X_0), \quad [g] \mapsto [g \circ f]$$

which is an isometry for the Teichmüller metric. This map $T(\mu_f)$ is called the translation map. In particular, $T(\mu_f)$ is biholomorphic and maps the basepoint in $\mathcal{T}(X)$ to $[f] \in \mathcal{T}(X_0)$. Thus the tangent space at $[f] \in \mathcal{T}(X_0)$ is isomorphic to the tangent space at the basepoint of $\mathcal{T}(X)$, which can then be identified with $Q(\Gamma_{\tilde{f}})$. Since the Schwarzian derivative map $\Phi : B(\Gamma) \to Q(\Gamma)$ is holomorphic, a differentiable path $t \mapsto \mu_t$ in $B(\Gamma)$ projects to a differentiable path $t \mapsto \Phi(\mu_t)$ in $Q(\Gamma)$. Conversely, Ahlfors-Weill section shows a differentiable path in a neighborhood of $0 \in Q(\Gamma)$ lifts to a differentiable path in $B(\Gamma)$ through 0. Since the derivative of a differentiable path in $B(\Gamma)$ gives an element of $L^\infty(X_0)$, we conclude that each Beltrami differential $\mu \in L^\infty(X_0)$ represents a tangent vector at the basepoint in $\mathcal{T}(X_0)$, and conversely each tangent vector at the basepoint of $\mathcal{T}(X_0)$ is represented by some $\mu \in L^\infty(X_0)$. A single tangent vector is represented by many Beltrami differentials. We denote by $[\mu]_{tan}$ the class of all Beltrami differentials representing the same tangent vector as $\mu \in L^\infty(X_0)$.

Chapter 3

Liouville Embedding

3.1 Introduction

In this chapter, we assume our work in the unit disk model. Let X_0 be a Riemann surface which is conformally hyperbolic. This means that $\widetilde{X}_0$ is biholomorphically equivalent to the open unit disk $\mathbb{D}$, denote $\widetilde{X}_0 \cong \mathbb{D}$. We define the *geodesic currents* and investigate their properties. Then we will define the *Liouville map* of the Teichmüller space. The main feature of this map is that it is a topological embedding.

3.2 Geodesic currents

Let us consider Möb($\mathbb{D}$) the group of isometries of $\mathbb{D}$ of the form $z \mapsto \dfrac{az+b}{\bar{b}z+\bar{a}}$ where $a, b \in \mathbb{C}$ such that $|a|^2 - |b|^2 = 1$, and in particular these isometries of $\mathbb{D}$ extend to homeomorphisms of $\mathbb{D} \cup \partial\mathbb{D}$. This enables us to define Möb($\widetilde{X}_0$) using the biholomorphic map $\widetilde{X}_0 \to \mathbb{D}$. The map $\widetilde{X}_0 \to \mathbb{D}$ extends to a homeomorphism $\widetilde{X}_0 \cup \partial\widetilde{X}_0 \to \mathbb{D} \cup \partial\mathbb{D}$, where $\partial\widetilde{X}_0$ is topologically equivalent S^1.

Define $G(\mathbb{D}) = \partial\mathbb{D} \times \partial\mathbb{D} - \Delta$ to be the space of complete geodesics of $\mathbb{D}$, where Δ is the diagonal of $\partial\mathbb{D} \times \partial\mathbb{D}$. It is topologically equivalent to an open annulus. The map $\widetilde{X}_0 \to \mathbb{D}$ induces a homeomorphism $\partial\widetilde{X}_0 \to \partial\mathbb{D}$, which provides a homeomorphism from the space $G(\widetilde{X}_0) = \partial\widetilde{X}_0 \times \partial\widetilde{X}_0 - \Delta$ of complete geodesics of $\widetilde{X}_0$ to $G(\mathbb{D})$, where Δ is the diagonal of $\partial\widetilde{X}_0 \times \partial\widetilde{X}_0$. A Borel measure is called a Radon measure if it is inner regular, outer regular and locally finite, see [14]. A *geodesic current* for the Riemann surface X_0 is a Radon measure α on $G(\widetilde{X}_0)$ that is invariant under the action of $\pi_1(X_0)$. The Radon property here means that the integral $\alpha(K) = \int_K 1 \, d\alpha$ is finite and nonnegative for every compact subset $K \subset G(\widetilde{X}_0)$.

Let $\mathcal{C}(X_0)$ denote the space of geodesic currents and $C(\widetilde{X}_0)$ denote the space of continuous functions $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support. For each $\xi \in C(\widetilde{X}_0)$, we define a seminorm $|| \cdot ||_\xi$ on $\mathcal{C}(X_0)$ by

$$||\alpha||_\xi = \sup_{\varphi \in \text{Möb}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right|$$

for $\alpha \in \mathcal{C}(X_0)$. A *bounded geodesic current* is a geodesic current $\alpha \in \mathcal{C}(X_0)$ for which all seminorms $||\alpha||_\xi$ are finite, denote the space of bounded geodesic currents by $\mathcal{C}_{\text{bd}}(X_0)$. Moreover, the topology defined by the seminorms $||\alpha||_\xi$ is the *uniform weak* topology* of $\mathcal{C}_{\text{bd}}(X_0)$. It is known that uniform weak* topology of $\mathcal{C}_{\text{bd}}(X_0)$ is metrizable, a detailed proof is provided in Lemma 4 in [8].

3.3 Liouville map

Define $[a, b]$ to be a closed arc on S^1 between a and b for the counter clockwise orientation. Given disjoint arcs $[a, b]$ and $[c, d]$ in S^1, the set $[a, b] \times [c, d] \subset G(\mathbb{D})$ is called the *box of geodesics*. The *Liouville measure* for $\mathbb{D}$ defined on the box of geodesics is the logarithm of the cross ratio:

$$L_{\mathbb{D}}([a, b] \times [c, d]) = \log \frac{(a - c)(b - d)}{(a - d)(b - c)}$$

and we refer to $L_{\mathbb{D}}$ as the Liouville current for $\mathbb{D}$. If we parametrize the unit circle $\partial\mathbb{D} \subset \mathbb{C}$ by $t \mapsto e^{it}$, then

$$L_{\mathbb{D}}(A) = \int_A \frac{dtds}{|e^{it} - e^{is}|^2}$$

for any Borel subset $A \subset G(\mathbb{D}) = \partial\mathbb{D} \times \partial\mathbb{D} - \Delta$ as defined in [7]. If $\widetilde{X}$ is a Riemann surface conformally equivalent to $\mathbb{D}$, there is a conformal map $\tilde{g} : \widetilde{X} \to \mathbb{D}$ which extends to a homeomorphism $\partial\widetilde{X} \to \partial\mathbb{D}$. This extension provides a homeomorphism $G(\widetilde{X}) \to G(\mathbb{D})$, which we also denote by $\tilde{g}$. We can then pull back the Liouville measure $L_{\mathbb{D}}$ to a measure $L_{\widetilde{X}}$ on $G(\widetilde{X})$. The measure $L_{\widetilde{X}}$ is the Liouville measure of the Riemann surface $\widetilde{X} \cong \mathbb{D}$. Consider an element $[f] \in \mathcal{T}(X_0)$ represented by a quasiconformal map $f : X_0 \to X$. Let $\tilde{f} : \widetilde{X}_0 \to \widetilde{X}$ be the lift of f, where $\widetilde{X}_0$ and $\widetilde{X}$ are the respective universal covers. By Beurling-Ahlfors, see [6], there is a continuous extension $\tilde{f} : \widetilde{X}_0 \cup \partial\widetilde{X}_0 \to \widetilde{X} \cup \partial\widetilde{X}$ which provides a homeomorphism from $G(\widetilde{X}_0) \to G(\widetilde{X})$. We can then pull back the Liouville measure $L_{\widetilde{X}}$ by $\tilde{f}$ to a measure L_f on $G(\widetilde{X}_0)$ defined by $L_f(A) = L_{\widetilde{X}}(\tilde{f}(A))$ for every measurable subset $A \subset G(\widetilde{X}_0)$, while

$$\int_{G(\widetilde{X}_0)} \xi dL_f = \int_{G(\widetilde{X})} \xi \circ \tilde{f}^{-1} dL_{\widetilde{X}}$$

for every continuous function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support. In particular, the measure L_f is invariant under the action of $\pi_1(X_0)$ on $G(\widetilde{X}_0)$, L_f is a geodesic current in X_0.

We state as a lemma on a well-known property of the Liouville measure and quasisymmetric map. Recall that a homeomorphism $\tilde{f} : \partial\widetilde{X}_1 \to \partial\widetilde{X}_2$ is quasisymmetric if

$$\kappa(\tilde{f}) = \sup_{Q \text{ symmetric}} \frac{L_{\widetilde{X}_2}(\tilde{f}(Q))}{\log 2} < \infty,$$

as Q ranges over all symmetric boxes $Q \subset G(\widetilde{X}_1)$ where Q symmetric means $L_{\widetilde{X}_1}(Q) = \log 2$. By definition, $\kappa(\tilde{f})$ is the quasisymmetric constant of $\tilde{f}$.

Lemma 3.3.1. *Let $\tilde{f} : \partial\widetilde{X}_1 \to \partial\widetilde{X}_2$ be a quasisymmetric map, there exists a homeomorphism*

$\eta : [0, \infty) \to [0, \infty)$ *depending only on the quasisymmetric constant* $\kappa(\tilde{f})$ *such that*

$$L_{\widetilde{X}_2}(\tilde{f}(Q)) \leqslant \eta(L_{\widetilde{X}_1}(Q))$$

for every box $Q \subset G(\widetilde{X}_1)$.

In addition, the homeomorphism η *can be chosen so that it converges to the identity uniformly on compact subsets of the open interval* $(0, \infty)$, *as the quasisymmetric constant* $\kappa(\tilde{f})$ *tends to 1.*

See [21] for a proof. □

Proposition 3.3.2. *Let* $f : X_0 \to X$ *be a quasiconformal map between two Riemann surfaces and let* $\tilde{f} : \widetilde{X}_0 \to \widetilde{X}$ *be the lift of* f. *Then, the Liouville current* L_f *is bounded, and therefore belongs to* $\mathcal{C}_{\mathrm{bd}}(X_0)$.

Proof of Prop 3.3.2.

We want to show that for every continuous function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support,

$$\sup_{\varphi \in \mathrm{M\ddot{o}b}(\widetilde{X}_0)} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, dL_f \right| < \infty,$$

where the supremum is taken over all Möbius maps $\varphi : \widetilde{X}_0 \to \widetilde{X}_0$ and where L_f is the pullback under $\tilde{f}$ of the Liouville measure $L_{\widetilde{X}}$ of $\widetilde{X}$.

Cover the *Supp*(ξ) with finitely many symmetric boxes $Q_1, ..., Q_k$. Note that boxes

$\varphi^{-1}(Q_i) = Q_i', i = 1, ..., k$ are also symmetric. Then for every $\varphi \in \text{Möb}(\widetilde{X}_0)$,

$$\begin{aligned}\left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \; dL_f \right| &\leqslant \sum_{i=1}^{k} \int_{\varphi^{-1}(Q_i)} |\xi \circ \varphi| \; dL_f \\ &\leqslant ||\xi||_\infty \sum_{i=1}^{k} L_f(\varphi^{-1}(Q_i)) \\ &= ||\xi||_\infty \sum_{i=1}^{k} L_{\widetilde{X}}(\tilde{f}(Q_i')) \\ &\leqslant ||\xi||_\infty \sum_{i=1}^{k} \kappa(\tilde{f}) \log 2 \quad \text{this estimate is independent of } \varphi\end{aligned}$$

Hence we obtain the uniform bound for L_f.

This completes the proof of Proposition 3.3.2. □

If two quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ represents the same element $[f_1] = [f_2]$ in the Teichmüller space $\mathcal{T}(X_0)$, there exists a biholomorphic map $g : X_1 \to X_2$ such that $f_2^{-1} \circ g \circ f_1$ is homotopic to the identity in X_0 and fixes ∂X_0. We can then choose lifts $\tilde{f}_1 : \widetilde{X}_0 \to \widetilde{X}_1$, $\tilde{f}_2 : \widetilde{X}_0 \to \widetilde{X}_2$ and $\tilde{g} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{f}_2^{-1} \circ \tilde{g} \circ \tilde{f}_1$ is homotopic to the identity in $\widetilde{X}_0$ and fixes $\partial\widetilde{X}_0$. Hence, the restriction of $\tilde{f}_2$ and $\tilde{g} \circ \tilde{f}_1$ to maps $\partial\widetilde{X}_0 \to \widetilde{X}_2$ coincide. Since $\tilde{g}$ sends the Liouville measure $L_{\widetilde{X}_1}$ to $L_{\widetilde{X}_2}$, it follows that L_{f_1} and L_{f_2} coincide on $G(\widetilde{X}_0)$ since the Liouville measure is invariant under $\tilde{g}$.

The map

$$L : \mathcal{T}(X_0) \to \mathcal{C}_{\text{bd}}(X_0)$$

defined by $L([f]) = L_f$ is the *Liouville embedding.*

We will state a property of L and seminorms $|| \cdot ||_\xi$ due to Bonahon-Šarić.

Theorem 3.3.3. *Let X_0 be a conformally hyperbolic Riemann surface, let the Teichmüller space $\mathcal{T}(X_0)$ be equipped with the Teichmüller distance d_T, and let the space $\mathcal{C}_{\text{bd}}(X_0)$ of bounded geodesic currents be endowed with the uniform weak* topology. Then, the Liouville* *embedding* $L : \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ *is a homeomorphism onto its image, it is a proper map, and its image* $L(\mathcal{T}(X_0))$ *is closed in* $\mathcal{C}_{\mathrm{bd}}(X_0)$*.*

See Theorem 8 and Propositions 19, 21, 24 and 25 in [8]. □

Chapter 4

Hölder Distributions

4.1 Introduction

In this chapter, we assume our work in the unit disk model. Let X_0 be a Riemann surface which is conformally hyperbolic, meaning that $\widetilde{X}_0 \cong \mathbb{D}$. We introduce a topological vector space which contains the space $\mathcal{C}_{\text{bd}}(X_0)$ of bounded geodesic currents for which the Liouville map will be differentiable in a certain sense. To do that, we need to fix an identification of $\widetilde{X}_0$ with $\mathbb{D}$ in the following way. First, let $\varphi : \widetilde{X}_0 \to \mathbb{D}$ be a biholomorphic map and fix $\widetilde{x}_0 \in \widetilde{X}_0$ such that $\varphi(\widetilde{x}_0) = 0 \in \mathbb{D}$ is the origin. Next, we define an *Angle metric* on the boundary $\partial\widetilde{X}_0 \cong \partial\mathbb{D} = S^1$ of the universal cover $\widetilde{X}_0 \cong \mathbb{D}$. Fix a base point $\widetilde{x}_0 \in \widetilde{X}_0$. Then angle distance between $\widetilde{x}_1, \widetilde{x}_2 \in \partial\widetilde{X}_0$ is the angle at $\widetilde{x}_0$ between the hyperbolic geodesic rays connecting $\widetilde{x}_0$ to $\widetilde{x}_1$ and $\widetilde{x}_2$ respectively. Finally, It follows that the Angle metric defined by φ on $\partial\mathbb{D} = S^1$ coincides with the Angle metric in the Euclidean sense. This gives us the desired identification. Any other identifications of $\widetilde{X}_0$ with $\mathbb{D}$ differ by postcomposition with a Möbius map of Möb($\mathbb{D}$), the group of orientation preserving isometries of $\mathbb{D}$.

From Chapter 2, we know that the Teichmüller space $\mathcal{T}(X_0)$ has its structure as a complex Banach manifold. In Chapter 3, we introduced geodesic currents defined on the space $G(\widetilde{X}_0)$

of complete geodesics and the space $C(\widetilde{X}_0)$ of continuous functions $G(\widetilde{X}_0) \to \mathbb{R}$ with compact support. Also recall that $X_0 \cong \mathbb{D}/\Gamma$, where Γ is a Fuchsian group identified with $\pi_1(X_0)$. The action of Γ on $\partial\widetilde{X}_0 \cong \partial\mathbb{D}$ induces an action on $G(\widetilde{X}_0)$. Let $[f] \in \mathcal{T}(X_0)$ be represented by a quasiconformal map $f : X_0 \to X$, then the lift $\tilde{f} : \widetilde{X}_0 \to \widetilde{X}$ of f satisfying $\tilde{f} \circ \Gamma \circ \tilde{f}^{-1} = \Gamma_{\tilde{f}}$ is not smooth on $\partial\widetilde{X}_0$ in general and the same holds for the induced homeomorphism $F : G(\widetilde{X}_0) \to G(\widetilde{X})$. Notice that if $\xi \in C(\widetilde{X}_0)$, then F induces a map $\tilde{F} : C(\widetilde{X}_0) \to C(\widetilde{X})$ by $\xi \mapsto \xi \circ F^{-1}$. In particular, $C(\widetilde{X}_0)$ is isomorphic to $C(\widetilde{X})$. It follows that although $G(\widetilde{X}_0)$ is a smooth manifold, the map $\tilde{F} : C(\widetilde{X}_0) \to C(\widetilde{X})$ does not preserve the space of differentiable functions. Hence the Liouville map $L : \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ has no smoothness properties. For $[f] \in T(X_0)$, it is known that f is Hölder continuous with respect to the spherical metric. Then the induced homeomorphism $F : G(\widetilde{X}_0) \to G(\widetilde{X})$ is Hölder continuous. In particular, F induces an isomorphism between the spaces of Hölder continuous functions with compact support defined on $G(\widetilde{X}_0)$ and $G(\widetilde{X})$ respectively. It further induces an isomorphism between the spaces of Hölder geodesic distributions which will be formally defined later. It turns out that the space of Hölder geodesic distributions is a Fréchet space. Thus we expect to show that the Liouville map L is differentiable in the Fréchet sense. This leads us to consider a topological vector space equipped with a family of seminorms defined by the Hölder continuous functions on $G(\widetilde{X}_0)$, the space of Hölder geodesic distributions.

4.2 Space of Hölder geodesic distributions

Let us recall the definition of Hölder continuity. A function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ is Hölder continuous with respect to metric d if there exists constant $C > 0$ and ν, $0 < \nu \leqslant 1$ such that

$$|\xi(g_1) - \xi(g_2)| \leqslant C \cdot d(g_1, g_2)^\nu$$

for all $g_1, g_2 \in G(\widetilde{X}_0)$.

Let $H(\widetilde{X}_0)$ denote the space of Hölder continuous function $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ with compact support. The algebraic dual of $H(\widetilde{X}_0)$ is called the space of Hölder geodesic distributions, denote by $\mathcal{H}(\widetilde{X}_0)$. By previous identification of $\widetilde{X}_0$ with $\mathbb{D}$, for each $\xi \in H(\widetilde{X}_0)$ we define the seminorm $||\cdot||_\xi$ on $\mathcal{H}(\widetilde{X}_0)$ by

$$||\mathbf{W}||_\xi = \sup_{\varphi \in \text{Möb}(\mathbb{D})} |\mathbf{W}(\xi \circ \varphi)|$$

where $\mathbf{W} : H(\widetilde{X}_0) \to \mathbb{R}$ is a real linear functional. We denote $\mathcal{H}(X_0)$ to be the space of all real linear functionals in $\mathcal{H}(\widetilde{X}_0)$ which are invariant under the fundamental group $\pi_1(X_0)$.

Define

$$\mathcal{H}_{\text{bd}}(X_0) = \left\{ \begin{array}{l} \mathbf{W} \mid \mathbf{W} : H(\widetilde{X}_0) \to \mathbb{R} \text{ is a real linear functional} \\ \qquad \text{which is invariant under } \pi_1(X_0) \\ \qquad \text{with } ||\mathbf{W}||_\xi < \infty \text{ for all } \xi \in H(\widetilde{X}_0) \end{array} \right\}$$

to be the space of bounded Hölder geodesic distributions. Notice that $\mathcal{C}_{\text{bd}}(X_0)$ is a space of bounded positive measures, it is not a vector space. On the other hand, $\mathcal{H}_{\text{bd}}(X_0)$ is a vector space, a Fréchet space to be specific. The completeness of $\mathcal{H}_{\text{bd}}(X_0)$ follows from a standard result of functional analysis. We have $\mathcal{C}_{\text{bd}}(X_0) \subset \mathcal{H}_{\text{bd}}(X_0)$ by the definition of $\mathcal{C}_{\text{bd}}(X_0)$. We will call the topology defined by these seminorms $||\mathbf{W}||_\xi$ for all $\xi \in H(\widetilde{X}_0)$, the *uniform Hölder topology* of $\mathcal{C}_{\text{bd}}(X_0)$. In particular, if α is a geodesic current on $G(\widetilde{X}_0)$ we then set

$$||\alpha||_\xi = \sup_{\varphi \in \text{Möb}(\mathbb{D})} \left| \int_{G(\widetilde{X}_0)} \xi \circ \varphi \, d\alpha \right|$$

as ξ ranges over $H(\widetilde{X}_0)$.

By above inclusion, the Liouville map can therefore be viewed as $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$. We claim that it is a proper topological embedding: namely, it is a homeomorphism onto its image in $\mathcal{H}_{\text{bd}}(X_0)$ endowed with the uniform Hölder topology, a proper map and its image

$L(\mathcal{T}(X_0))$ is closed in $\mathcal{H}_{\mathrm{bd}}(X_0)$. Recall that the topology on $\mathcal{C}_{\mathrm{bd}}(X_0)$ is the uniform weak* topology and $L : \mathcal{T}(X_0) \to \mathcal{C}_{\mathrm{bd}}(X_0)$ is a proper topological embedding by Theorem 3.3.3. To prove our claim, we only need to show the following which we stated as THEOREM A in Chapter 1.

Theorem 4.2.1. *The uniform Hölder topology coincides with the uniform weak* topology on the space $\mathcal{C}_{\mathrm{bd}}(X_0)$ of bounded geodesic currents.*

Theorem 4.2.1 will be proved in several steps. We will need the following results.

Lemma 4.2.2. *Let $\psi : \mathbb{H} \to \mathbb{D}$ be a Möbius map. Then, the extension $\psi : \partial\mathbb{H} \to \partial\mathbb{D}$ is bi-Lipschitz with respect to the Euclidean metric on a compact subset of $\mathbb{R}$ and Angle metric respectively.*

Proof of Lemma 4.2.2.
Since ψ' is continuous on compact interval, it is bounded and we have $m < |\psi'| < M$. By the mean value theorem, ψ is Lipschitz. From the Inverse function theorem, we also have $|(\psi^{-1})'| = \left|\dfrac{1}{\psi'(\psi^{-1})}\right| < \dfrac{1}{m}$, again by the mean value theorem, ψ^{-1} is Lipschitz. Hence, ψ is bi-Lipschitz.

This proves Lemma 4.2.2. □

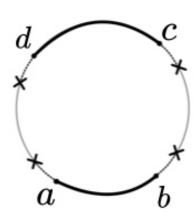


Figure 4.1: Q and $\overline{B_\delta(Q)}$

As we will see frequently for the rest of Chapter 4, for a box $Q = [a, b] \times [c, d]$ we introduce the notation $\overline{B_\delta(Q)}$ the closure of *δ-neighborhood* of Q, which we define it to satisfy the property such that each endpoint of $\overline{B_\delta(Q)}$ is δ-distance away from the each endpoint of Q respectively in the angle metric. To avoid ambiguity we assume that $\overline{B_\delta(Q)}$ is a proper box of geodesics. This is possible as long as we have control on δ. To do this, suppose that $Q = [a, b] \times [c, d] \subset G(\widetilde{X}_0)$ and let $d_\angle$ denote the angle distance for the Angle metric. If we assume $\delta < \frac{1}{2}\min\{d_\angle(a, d), d_\angle(b, c)\}$, then $\overline{B_\delta(Q)}$ is proper box.

Lemma 4.2.3. *Let $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ be a continuous function with support in a box Q. Fix $\delta > 0$. Then for a given $\varepsilon > 0$, there exists a differentiable function ξ'_δ such that $|\xi'_\delta - \xi| < \varepsilon$ and $Supp(\xi'_\delta) \subset \overline{B_\delta(Q)}$.*

Proof of Lemma 4.2.3.
Suppose that g is differentiable on $\mathbb{R}^2$ with compact support satisfying $\displaystyle\int_{\mathbb{R}^2} g(x)dA(x) = 1$, then $g_\delta(x) = \delta^{-2}g(x/\delta)$ is a mollifier. In particular, $Supp(g_\delta) \subset B_\delta(0)$.

Using change of variable $\delta y = x$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} g_\delta(x)dA(x) &= \int_{\mathbb{R}^2} \delta^{-2}g(x/\delta)dA(x)\\ &= \int_{\mathbb{R}^2} g(y)dA(y)\\ &= 1\end{aligned}$$

Note that mollifier is defined on $\mathbb{R}^2$. Consider a Möbius map $\psi : \mathbb{H} \to \mathbb{D}$ such that $\xi \circ \psi$ is continuous on $G(\mathbb{H})$ with $Supp(\xi \circ \psi) = \psi^{-1}(Q) = Q'$ which is compactly contained in $\mathbb{R} \times \mathbb{R}$. Then $\xi \circ \psi$ is in fact uniformly continuous, since every continuous function on a compact set is uniformly continuous.

Consider the convolution $(\xi \circ \psi) * g_\delta(x) = \displaystyle\int_{\mathbb{R}^2} (\xi \circ \psi)(y)g_\delta(x - y)dA(y)$, which is differentiable. Namely, $\dfrac{d}{dx}(\xi \circ \psi) * g_\delta(x) = \displaystyle\int_{\mathbb{R}^2} (\xi \circ \psi)(y)\frac{d}{dx}g_\delta(x - y)dA(y)$. We would like to have that $(\xi \circ \psi) * g_\delta(x)$ approximates $(\xi \circ \psi)$ in the sense of uniform convergence. If it is true, then we can set $\xi'_\delta = ((\xi \circ \psi) * g_\delta) \circ \psi^{-1}$ which will prove the first part of the Lemma 4.2.3.

To see this, by uniform continuity of $\xi \circ \psi$ and given ε, there exists $0 < \delta' < \delta$ such that we have the following:

$$\begin{aligned}
|(\xi \circ \psi) * g_{\delta'}(x) - (\xi \circ \psi)(x)| &= \left| \int_{\mathbb{R}^2} (\xi \circ \psi)(y) g_{\delta'}(x-y) dA(y) - \int_{\mathbb{R}^2} (\xi \circ \psi)(x) g_{\delta'}(x-y) dA(y) \right| \\
&= \left| \int_{\mathbb{R}^2} ((\xi \circ \psi)(y) - (\xi \circ \psi)(x)) g_{\delta'}(x-y) dA(y) \right| \\
&\leqslant \int_{\mathbb{R}^2} \left| (\xi \circ \psi)(y) - (\xi \circ \psi)(x) \right| g_{\delta'}(x-y) dA(y) \\
&= \int_{B_{\delta'}(x)} \left| (\xi \circ \psi)(y) - (\xi \circ \psi)(x) \right| g_{\delta'}(x-y) dA(y) \\
&\leqslant \max_{y \in B_{\delta'}(x)} \left| (\xi \circ \psi)(y) - (\xi \circ \psi)(x) \right| \int_{B_{\delta'}(x)} g_{\delta}(x-y) dA(y) \\
&= \max_{y \in B_{\delta'}(x)} \left| (\xi \circ \psi)(y) - (\xi \circ \psi)(x) \right| < \varepsilon
\end{aligned}$$

This implies that $|((\xi \circ \psi) * g_{\delta'}) \circ \psi^{-1} - (\xi \circ \psi) \circ \psi^{-1}| < \varepsilon$. Let $\xi'_\delta = ((\xi \circ \psi) * g_{\delta'}) \circ \psi^{-1}$, then ξ'_δ is differentiable.

Next suppose that $z \notin \overline{B_\delta(\psi^{-1}(Q))}$, then

$$(\xi \circ \psi) * g_{\delta'}(z) = \int_{B_{\delta'}(z)} (\xi \circ \psi)(y) g_{\delta'}(z-y) dA(y) = 0$$

since $\delta' < \delta$ and this implies that $(\xi \circ \psi)(y) = 0$ on $B_{\delta'}(z)$. Thus we have

$$Supp(\xi'_\delta) = Supp(((\xi \circ \psi) * g_{\delta'}) \circ \psi^{-1}) \subset \overline{B_\delta(Q)}$$

This proves Lemma 4.2.3. □

Proposition 4.2.4. *Let $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ be a continuous function with support in a compact set $K \subset \bigcup_{i=1}^{n} Q_i$ for finitely many boxes Q_i. Fix $\delta > 0$. Then, for a given $\varepsilon > 0$, there exists a differentiable function ξ'_δ such that $|\xi'_\delta - \xi| < \varepsilon$ and $Supp(\xi'_\delta) \subset \bigcup_{i=1}^{n} \overline{B_\delta(Q_i)}$.*

Proof of Proposition 4.2.4.

A partition of unity on K subordinated to $Q_1, ..., Q_n$ is a collection $\{\rho_i\}_{i=1,...,n}$ of continuous functions $\rho_i : K \to [0,1]$ such that

1) $Supp(\rho_i) \subset Q_i$ for each i
2) $\sum_{i=1}^{n} \rho_i(x) = 1$ for each $x \in K$.

Note that $Supp(\rho_i\xi) \subset Q_i$ for each i. Let $\xi = \sum_{i=1}^{n} \rho_i\xi$, by Lemma 4.2.3 there exists differentiable functions $\xi'_{\delta,i}$ such that $|\xi'_{\delta,i} - \rho_i\xi| < \varepsilon$ and $Supp(\xi'_{\delta,i}) \subset \overline{B_\delta(Q_i)}$. Then we have that $\xi'_\delta = \sum_{i=1}^{n} \xi'_{\delta,i}$ is differentiable such that $|\xi'_\delta - \xi| < \varepsilon$. Moreover,

$$Supp(\xi'_\delta) = Supp(\sum_{i=1}^{n} \xi'_{\delta,i}) \subset \bigcup_{i=1}^{n} Supp(\xi'_{\delta,i}) \subset \bigcup_{i=1}^{n} \overline{B_\delta(Q_i)}$$

which is a compact set, since the union of finitely many compact sets is compact.

This completes the proof of Proposition 4.2.4. □

Lemma 4.2.5. *Fix a single box Q. Then, for a given $\delta > 0$, there exists a differentiable function $\xi^{(*)}_\delta$ such that*

$$\xi^{(*)}_\delta = \begin{cases} 1 \textit{ inside } Q \\ 0 \textit{ outside } \overline{B_\delta(Q)} \end{cases}$$

Proof of Lemma 4.2.5.

We have $Q \subset \overline{B_{\frac{\delta}{3}}(Q)} \subset \overline{B_{\frac{2\delta}{3}}(Q)} \subset \overline{B_\delta(Q)}$.

We can find a continuous function $\xi^{(c)}_\delta$ with compact support such that

$$\xi^{(c)}_\delta = \begin{cases} 1 \text{ inside } \overline{B_{\frac{\delta}{3}}(Q)} \\ 0 \text{ outside } \overline{B_{\frac{2\delta}{3}}(Q)}. \end{cases}$$

Following the proof of Lemma 4.2.3, Let $\psi : \mathbb{H} \to \mathbb{D}$ be a Möbius map and g_δ be a mollifier. By uniform continuity of $\xi_\delta^{(c)} \circ \psi$ and given $\varepsilon > 0$, there exists $0 < \frac{\delta}{3} < \delta$ such that

$$|(\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}(x) - (\xi_\delta^{(c)} \circ \psi)(x)| \leqslant \max_{y \in B_{\frac{\delta}{3}}(x)} \left| (\xi_\delta^{(c)} \circ \psi)(y) - (\xi_\delta^{(c)} \circ \psi)(x) \right| < \varepsilon$$

which implies $|((\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}) \circ \psi^{-1} - (\xi_\delta^{(c)} \circ \psi) \circ \psi^{-1}| < \varepsilon$.

Let $\xi_\delta^{(*)} = ((\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}) \circ \psi^{-1}$. Since $(\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}$ is differentiable, then so is $\xi_\delta^{(*)}$.

If $z \notin \overline{B_\delta(\psi^{-1}(Q))}$, then we have

$$(\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}(z) = \int_{B_{\frac{\delta}{3}}(z)} (\xi_\delta^{(c)} \circ \psi)(y) g_{\frac{\delta}{3}}(z-y) dA(y) = 0$$

since $(\xi_\delta^{(c)} \circ \psi)(y) = 0$ on $B_{\frac{\delta}{3}}(z)$.

If $z \in \psi^{-1}(Q)$, then we have

$$(\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}(z) = \int_{B_{\frac{\delta}{3}}(z)} (\xi_\delta^{(c)} \circ \psi)(y) g_{\frac{\delta}{3}}(z-y) dA(y) = 1$$

since $(\xi_\delta^{(c)} \circ \psi)(y) = 1$ on $B_{\frac{\delta}{3}}(z)$.

This implies $\xi_\delta^{(*)} = ((\xi_\delta^{(c)} \circ \psi) * g_{\frac{\delta}{3}}) \circ \psi^{-1} = \begin{cases} 1 \text{ inside } Q \\ 0 \text{ outside } \overline{B_\delta(Q)}. \end{cases}$

This proves Lemma 4.2.5. □

Proposition 4.2.6. *Fix a compact set $K \subset \bigcup_{i=1}^{n} Q_i$ for finitely many boxes Q_i. For a given $\delta > 0$, there exists a differentiable function $\xi_\delta^{(*)}$ such that*

$$\xi_\delta^{(*)} = \begin{cases} 1 \ \textit{inside } K \\ 0 \ \textit{outside } \bigcup_{i=1}^{n} \overline{B_\delta(Q_i)} \end{cases}$$

Proof of Proposition 4.2.6.

We apply partition of unity on compact set K.

For each Q_i, it is possible to find continuous function $\xi_{\delta,i}^{(c)}$ such that

$$\xi_{\delta,i}^{(c)} = \begin{cases} 1 \text{ inside } \overline{B_{\frac{\delta}{3}}(Q_i)} \\ 0 \text{ outside } \overline{B_{\frac{2\delta}{3}}(Q_i)}. \end{cases}$$

Note that $Supp(\rho_i \xi_{\delta,i}^{(c)}) \subset Q_i$. Consider $\rho\xi_{\delta}^{(c)} = \sum_{i=1}^{n} \rho_i \xi_{\delta,i}^{(c)}$.

Then $Supp(\rho\xi_{\delta}^{(c)}) = Supp(\sum_{i=1}^{n} \rho_i \xi_{\delta,i}^{(c)}) \subset \bigcup_{i=1}^{n} Supp(\rho_i \xi_{\delta,i}^{(c)}) \subset \bigcup_{i=1}^{n} Q_i$ and

$$\rho\xi_{\delta}^{(c)} = \begin{cases} 1 \text{ inside } \bigcup_{i=1}^{n} \overline{B_{\frac{\delta}{3}}(Q_i)} \\ 0 \text{ outside } \bigcup_{i=1}^{n} \overline{B_{\frac{2\delta}{3}}(Q_i)}. \end{cases}$$

Let $\psi : \mathbb{H} \to \mathbb{D}$ be a Möbius map and g_δ be a mollifier. Set $\xi_{\delta}^{(*)} = ((\rho\xi_{\delta}^{(c)} \circ \psi) * g_{\frac{\delta}{3}}) \circ \psi^{-1}$.

We have found the desired differentiable function and the proof follows from Lemma 4.2.5.

This completes the proof of Proposition 4.2.6. □

Proof of Theorem 4.2.1.

The uniform weak* topology is defined by the basis consisting of balls

$$B_{\xi_1,\ldots,\xi_k}(\alpha; r) = \{\beta \in \mathcal{C}_{\mathrm{bd}}(X_0) : ||\alpha - \beta||_{\xi_i} < r \text{ for } i = 1, \ldots, k\}$$

where $\xi_i \in C(\widetilde{X}_0), i = 1, \ldots, k$.

We define the uniform Hölder topology by the basis consisting of balls

$$B_{\xi'_1,\ldots,\xi'_{k'}}(\alpha; r') = \{\beta \in \mathcal{C}_{\mathrm{bd}}(X_0) : ||\alpha - \beta||_{\xi'_i} < r' \text{ for } i = 1, \ldots, k'\}$$

where $\xi_i' \in H(\widetilde{X}_0), i = 1, ..., k'$.

We want to show the following:

1. Suppose that for fixed α and fixed r', $\forall \xi_1', ..., \xi_{k'}' \in H(\widetilde{X}_0)$, there exists $\xi_1, ..., \xi_k \in C(\widetilde{X}_0)$ and $\exists r > 0$ such that $B_{\xi_1,...,\xi_k}(\alpha; r) \subset B_{\xi_1',...,\xi_{k'}'}(\alpha; r')$.

   To see this, choose $\xi_1 = \xi_1', ..., \xi_k = \xi_{k'}'$ and $r = r'$.

2. Suppose that for fixed α and fixed r, $\forall \xi_1, ..., \xi_k \in C(\widetilde{X}_0)$, there exists $\xi_1', ..., \xi_{k'}' \in H(\widetilde{X}_0)$ and $\exists r' > 0$ such that $B_{\xi_1',...,\xi_{k'}'}(\alpha; r') \subset B_{\xi_1,...,\xi_k}(\alpha; r)$.

To prove 2, for fixed ξ. By Proposition 4.2.4 there exists a differentiable function ξ_δ' with support contained in a compact set K' such that $|\xi_\delta' - \xi| < \varepsilon$. By Proposition 4.2.6 there exists a differentiable function $\xi_\delta^{(*)}$ equals 1 inside K' and 0 outside some compact set K'' containing K'. Then we have $|\xi_\delta'(g) - \xi(g)| \leqslant \varepsilon \xi_\delta^{(*)}(g)$ for all $g \in G(\widetilde{X}_0)$. Therefore

$$\left| \int_{G(\widetilde{X}_0)} \xi_\delta' \circ \varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi \circ \varphi d\alpha \right| \leqslant \varepsilon \int_{G(\widetilde{X}_0)} \xi_\delta^{(*)} \circ \varphi d\alpha$$

and

$$\left| \int_{G(\widetilde{X}_0)} \xi_\delta' \circ \varphi d\beta - \int_{G(\widetilde{X}_0)} \xi \circ \varphi d\beta \right| \leqslant \varepsilon \int_{G(\widetilde{X}_0)} \xi_\delta^{(*)} \circ \varphi d\beta$$

for every $\alpha, \beta \in \mathcal{C}_{\mathrm{bd}}(X_0)$ and $\varphi \in \text{Möb}(\mathbb{D})$.

Using triangle inequality, we get

$$\begin{aligned}
\left|\int_{G(\widetilde{X}_0)} \xi\circ\varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi\circ\varphi d\beta\right| = & \left|\int_{G(\widetilde{X}_0)} \xi\circ\varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\alpha\right. \\
& + \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\beta \\
& \left. + \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\beta - \int_{G(\widetilde{X}_0)} \xi\circ\varphi d\beta\right| \\
\leqslant & \left|\int_{G(\widetilde{X}_0)} \xi\circ\varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\alpha\right| \\
+ & \left|\int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\alpha - \int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\beta\right| \\
+ & \left|\int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\beta - \int_{G(\widetilde{X}_0)} \xi\circ\varphi d\beta\right|
\end{aligned}$$

This implies that $||\alpha-\beta||_\xi \leqslant ||\alpha-\beta||_{\xi'_\delta} + \varepsilon||\alpha||_{\xi^{(*)}_\delta} + \varepsilon||\beta||_{\xi^{(*)}_\delta}$.

Recall that $B_\xi(\alpha; r)$ is a ball associated with a single function ξ.

Lemma 4.2.7. *Let $\alpha \in \mathcal{C}_{\mathrm{bd}}(x_0)$ and $r > 0$. There exists $\varepsilon > 0$ and differentiable functions ξ'_δ and $\xi^{(*)}_\delta$ such that $\varepsilon||\alpha||_{\xi^{(*)}_\delta} < \frac{r}{4}$ and*

$$B_{\xi'_\delta}(\alpha; \frac{r}{4}) \cap B_{\xi^{(*)}_\delta}(\alpha; \frac{r}{4\varepsilon}) \subset B_\xi(\alpha; r).$$

Proof of Lemma 4.2.7.

For a fixed δ we have $Supp(\xi'_\delta) \subset K' \subset K''$ and $Supp(\xi^{(*)}_\delta) \subset K''$.

For a fixed α we have,

$$||\alpha||_{\xi'_\delta} = \sup_{\varphi\in\text{Möb}(\mathbb{D})} \left|\int_{G(\widetilde{X}_0)} \xi'_\delta\circ\varphi d\alpha\right| \leqslant \sup_{\varphi\in\text{Möb}(\mathbb{D})} \alpha(\varphi^{-1}(K'')) < M < \infty$$

and similarly

$$||\alpha||_{\xi_\delta^{(*)}} = \sup_{\varphi\in \text{Möb}(\mathbb{D})} \left| \int_{G(\widetilde{X}_0)} \xi_\delta^{(*)} \circ \varphi d\alpha \right| \leqslant \sup_{\varphi\in \text{Möb}(\mathbb{D})} \alpha(\varphi^{-1}(K'')) < M < \infty.$$

This implies that $||\alpha||_{\xi'_\delta}$ and $||\alpha||_{\xi_\delta^{(*)}}$ are bounded by a fixed constant. Choose $\varepsilon = \frac{r}{4M}$, then $||\alpha||_{\xi_\delta^{(*)}} < \frac{r}{4\varepsilon}$ and $||\beta||_{\xi_\delta^{(*)}} \leqslant ||\beta-\alpha||_{\xi_\delta^{(*)}} + ||\alpha||_{\xi_\delta^{(*)}} < \frac{r}{4\varepsilon} + \frac{r}{4\varepsilon} = \frac{2r}{4\varepsilon}$. We then have

$$||\alpha-\beta||_\xi \leqslant ||\alpha-\beta||_{\xi'_\delta} + \varepsilon||\alpha||_{\xi_\delta^{(*)}} + \varepsilon||\beta||_{\xi_\delta^{(*)}} < \frac{r}{4} + \frac{r}{4} + \frac{2r}{4} = r.$$

This proves Lemma 4.2.7. □

Note that, by definition we have $B_{\xi_i}(\alpha; r) = \{\beta \in \mathcal{C}_{\text{bd}}(X_0) : ||\alpha-\beta||_{\xi_i} < r\}$ for each $\xi_i \in C(\widetilde{X}_0)$. Taking intersections of $B_{\xi_i}(\alpha; r)$ for $i = 1, ..., k$ we obtain exactly $B_{\xi_1,...,\xi_k}(\alpha; r)$ since it contains all bounded geodesic currents β with the property $||\alpha-\beta||_{\xi_i} < r$ for $i = 1, ..., k$. Thus

$$B_{\xi_1,...,\xi_k}(\alpha; r) = B_{\xi_1}(\alpha; r) \cap ... \cap B_{\xi_k}(\alpha; r)$$

and similarly

$$B_{\xi'_1,...,\xi'_{k'}}(\alpha; r') = B_{\xi'_1}(\alpha; r') \cap ... \cap B_{\xi'_{k'}}(\alpha; r').$$

Since for each ξ_i, we have chosen $\xi'_{i\delta}$ and $\xi_{i\delta}^{(*)}$ and r'_i with

$$B_{\xi'_{i\delta}}(\alpha, r'_i) \cap B_{\xi_{i\delta}^{(*)}}(\alpha, \frac{r}{4\varepsilon}) \subset B_{\xi_i}(\alpha; r).$$

By taking multiple intersections, there exist differentiable functions $\xi'_1, ..., \xi'_{2k}$ and $r' > 0$ such that

$$B_{\xi'_1,...,\xi'_{2k}}(\alpha; r') \subset B_{\xi_1,...,\xi_k}(\alpha; r)$$

This completes the proof of Theorem 4.2.1. □

Chapter 5

Differentiability

5.1 Introduction

In this chapter, we assume our work in the unit disk model. Let X_0 be a Riemann surface which is conformally hyperbolic, meaning that $\widetilde{X}_0 \cong \mathbb{D}$. We investigate the differentiability of the Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$ which maps the complex Banach manifold $\mathcal{T}(X_0)$ to the Fréchet space $\mathcal{H}_{\text{bd}}(X_0)$, equipped with family of semi-norms defined in Chapter 4. To describe the tangent space of $\mathcal{T}(X_0)$, we consider a one parameter family of Beltrami coefficients $\mu + t\nu \in B(\Gamma)$. The equivalence classes $[f^{\mu+t\nu}]$ of the one parameter family of solutions $f^{\mu+t\nu}$ to the Beltrami equations with coefficients $\mu + t\nu$ give a path in $\mathcal{T}(X_0)$. For fixed ν and $z \in \mathbb{C}$, the map $f^{\mu+t\nu}$ is differentiable in t which has been proved by Ahlfors and Bers in [3]. The derivative in the direction of ν is defined by

$$\dot{f}^{\mu}[\nu] := \frac{d}{dt} f^{\mu+t\nu}|_{t=0}$$

represents a tangent vector at the point $[f^{\mu}] \in \mathcal{T}(X_0)$. As described in Chapter 2, the space of tangent vectors at $[f^{\mu}] \in \mathcal{T}(X_0)$ is isomorphic to the space of tangent vectors at the basepoint of $\mathcal{T}(f^{\mu}(X_0))$, which is identified with $Q(\Gamma_{\tilde{f}^{\mu}})$ the space of bounded holomorphic quadratic differentials and then can be further identified with the space of equivalence classes of Beltrami differentials on $f^{\mu}(X_0)$. We define an equivalence relation as follows: two Beltrami differentials ν_1 and ν_2 on $f^{\mu}(X_0)$ are equivalent, denote $\nu_1 \sim \nu_2$ if

$$\dot{f}^{\mu}[\nu_1](z) = \dot{f}^{\mu}[\nu_2](z)$$

for all $z \in \partial\mathbb{D}$. We denote $[\nu]_{tan}$ the equivalence class of a Beltrami differential ν on $f^{\mu}(X_0)$. By Bers embedding, the image of the path $[f^{\mu+t\nu}]$ is a differentiable path through the point $\Psi([f^{\mu}]) = q_{\mu}$. The tangent vector at q_{μ} to the path $\Psi([f^{\mu+t\nu}])$ is a bounded holomorphic quadratic differential $q_{\mu}^{\nu} \in Q(\Gamma)$ and it maps to the tangent vector at $[f^{\mu}]$ identified with $[\nu]_{tan}$ under the tangent map $T_{q_{\mu}}\Psi^{-1}$. Now suppose that $\lambda = \mathcal{AW}(q)$ is a harmonic Beltrami differential, then we have $\Psi^{-1}(tq) = [f^{t\lambda}]$ due to Ahlfors and Weill. Thus the tangent map $T_0\Psi^{-1}$ at $0 \in Q(\Gamma)$ is defined by $T_0\Psi^{-1}(q) = [\mathcal{AW}(q)]_{tan} = [\lambda]_{tan}$ for $q \in Q(\Gamma)$.

Besides the Euclidean metric and Angle metric, we will also use the *Spherical metric* on $\widehat{\mathbb{C}}$. Recall that two finite points z_1 and z_2 have the spherical distance

$$k(z_1, z_2) = \arctan\left|\frac{z_1 - z_2}{1 + \bar{z}_1 z_2}\right|$$

where $0 \leqslant k(z_1, z_2) \leqslant \pi/2$. It is known that any K-quasiconformal map f of $\widehat{C}$ is $\frac{1}{K}$-Hölder continuous in the spherical metric, ie.

$$k(f(z_1), f(z_2)) \leqslant C(k(z_1, z_2))^{\frac{1}{K}}$$

for every $z_1, z_2 \in \widehat{\mathbb{C}}$, see [13]. We say that a family of K-quasiconformal maps is uniformly Hölder continuous if the above inequality holds for all maps in the family with fixed constant $C > 0$. A family of K-quasiconformal maps is uniformly Hölder continuous if and only if there exists three fixed points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ and a constant $d > 0$ such that $k(f(z_i), f(z_j)) > d$ for all maps f in the family, see Chapter 2 Theorem 4.2 in [13].

Recall the definition of Fréchet derivative on Banach spaces. Let V and W be normed vector spaces and U a non-empty open subset of V. A map $f : U \to W$ is said to be *Fréchet differentiable* at $x \in U$ if there exists a continuous linear map $A : V \to W$ such that

$$\lim_{y \to x} \frac{||f(y) - f(x) - A(y - x)||_W}{||y - x||_V} = 0.$$

We apply this definition to the Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$ in the following way: again by Bers embedding the map $\Psi : \mathcal{T}(X_0) \to Q(\Gamma)$ defines a global holomorphic chart for $\mathcal{T}(X_0)$ and its image $\Psi(\mathcal{T}(X_0)) = Q_b$ is an open bounded subset of $Q(\Gamma)$. Recall that if V is a vector space, then the tangent space $T_v(V) = V$ for all $v \in V$. We want to construct a linear map $T_{[f]}L : T_{[f]}\mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$ such that if $q = \Psi([f]) \in Q_b$, the map $T_{[f]}L \circ T_q\Psi^{-1} : Q(\Gamma) \to \mathcal{H}_{\text{bd}}(X_0)$ is continuous and

$$\lim_{q' \to q} \frac{||L \circ \Psi^{-1}(q') - L \circ \Psi^{-1}(q) - T_{[f]}L \circ T_q\Psi^{-1}(q' - q)||_{\mathcal{H}}}{||q' - q||_{Q(\Gamma)}} = 0.$$

Consider the basepoint, ie. $[f] = [id]$. Let $[f^{t\lambda}]$ be a path through $[id]$ in $\mathcal{T}(X_0)$ where λ is a harmonic Beltrami differential. Let $\alpha_t = L([f^{t\lambda}])$. It suffices to show that $T_{[id]}L$ is linear and $T_{[id]}L \circ \mathcal{T}_0\Psi^{-1}$ is continuous and

$$\lim_{q \to 0} \frac{||L \circ \Psi^{-1}(q) - L \circ \Psi^{-1}(0) - \mathcal{T}_{[id]}L \circ \mathcal{T}_0\Psi^{-1}(q)||_{\mathcal{H}}}{||q||_{Q(\Gamma)}} = 0.$$

Note that a linear map between two normed spaces is bounded if and only if it is continuous. Recall that $||q||_{Q(\Gamma)} = \frac{1}{2}||\lambda||_\infty$. Thus to show that $\mathcal{T}_{[id]}L \circ \mathcal{T}_0\Psi^{-1}$ is continuous, it is enough to show that

$$||\mathcal{T}_{[id]}L([\lambda]_{tan})||_{\mathcal{H}} \leqslant C \cdot ||\lambda||_\infty$$

for fixed constant C and for any harmonic Beltrami differential λ. Let $tq = \Psi([f^{t\lambda}])$. Since λ is a harmonic Beltrami differential, so is $t\lambda$ and we have $\mathcal{AW}(tq) = t\lambda$. Then due to Ahlfors and Weill, $\mathcal{T}_0\Psi^{-1}(tq) = [\mathcal{AW}(tq)]_{tan} = [t\lambda]_{tan}$. In addition, $||t\lambda||_\infty = |t| \cdot ||\lambda||_\infty = 2||tq||_{Q(\Gamma)}$ since $||q||_{Q(\Gamma)} = \frac{1}{2}||\lambda||_\infty$. Under the assumption of $\mathcal{T}_{[id]}L$ being linear and by the definition of the Liouville map L, ie. $L([f^{t\lambda}]) = \int \xi \circ \varphi \; d\alpha_t$; to show the above limit is equivalent to show

$$\lim_{t\to 0} \sup_{\varphi\in \text{Möb}(\mathbb{D})} \left| \frac{\int \xi \circ \varphi \; d\alpha_t - \int \xi \circ \varphi \; d\alpha_0 - t \cdot \mathcal{T}_{[id]}L([\lambda]_{tan})(\xi \circ \varphi)}{t \cdot ||\lambda||_\infty} \right| = 0$$

uniformly in λ as long as $||\lambda||_\infty$ is bounded.

5.2 Differentiability at the base point

THEOREM B of Chapter 1 consists of the following: Theorem 5.2.1, Theorem 5.2.7, Theorem 5.3.2, and Theorem 5.3.6. We begin by showing that the Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$ is differentiable at the basepoint of $\mathcal{T}(X_0)$.

Theorem 5.2.1. *Let $[f^{t\lambda}]$ be a path in $\mathcal{T}(X_0)$ where λ is a harmonic Beltrami differential. Let $\alpha_t = L([f^{t\lambda}])$. Then there is a $\mathbf{W} \in \mathcal{H}_{\text{bd}}(X_0)$ such that*

$$\frac{d}{dt} \int_{G(\mathbb{D})} \xi \circ \varphi \; d\alpha_t \Big|_{t=0} = \mathbf{W}(\xi \circ \varphi)$$

for all $\xi \in H(\widetilde{X}_0)$ and $\varphi \in$ Möb$(\mathbb{D})$.

We will prove Theorem 5.2.1 in several steps. Assume working in $\mathbb{D}$ and fix $\xi \in H(\widetilde{X}_0)$. For our convenience, we also denote $f^{t\lambda}$ for its lift between the universal covers.

Lemma 5.2.2. *Consider $f^{t\lambda} : \mathbb{D} \to \mathbb{D}$ with Betrami coefficient $t\lambda$, then for $\varphi \in$ Möb$(\mathbb{D})$, $f^{t\lambda} \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ has Beltrami coefficient $t\lambda'$ where $\lambda' = \lambda \circ \varphi^{-1}(z) \frac{\overline{(\varphi^{-1})'(z)}}{(\varphi^{-1})'(z)}$, and in particular $||\lambda||_\infty = ||\lambda'||_\infty$.*

Proof of Lemma 5.2.2.

Note that $(\varphi^{-1})_{\bar{z}}$ and $\overline{(\varphi^{-1})}_z$ are identically zero.

$$\begin{aligned}\frac{(f^{t\lambda}\circ\varphi^{-1})_{\bar{z}}}{(f^{t\lambda}\circ\varphi^{-1})_z} &= \frac{(f_z^{t\lambda}\circ\varphi^{-1})(\varphi^{-1})_{\bar{z}} + (f_{\bar{z}}^{t\lambda}\circ\varphi^{-1})\overline{(\varphi^{-1})}_{\bar{z}}}{(f_z^{t\lambda}\circ\varphi^{-1})(\varphi^{-1})_z + (f_{\bar{z}}^{t\lambda}\circ\varphi^{-1})\overline{(\varphi^{-1})}_z} \\ &= \frac{(f_{\bar{z}}^{t\lambda}\circ\varphi^{-1})\overline{(\varphi^{-1})'(z)}}{(f_z^{t\lambda}\circ\varphi^{-1})(\varphi^{-1})'(z)} \\ &= t\lambda\circ\varphi^{-1}(z)\frac{\overline{(\varphi^{-1})'(z)}}{(\varphi^{-1})'(z)}\end{aligned}$$

□

Consider a box $Q = [a,b]\times[c,d] \subset \mathbb{D}$ with $L_{\mathbb{D}}(Q) = M < \infty$ for some M. Choose Möbius maps $\psi_t : \mathbb{D} \to \mathbb{H}$ and $\psi_Q : \mathbb{H} \to \mathbb{D}$ such that $\psi_t \circ f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q : \mathbb{H} \to \mathbb{H}$ fixes 0, 1 and ∞.

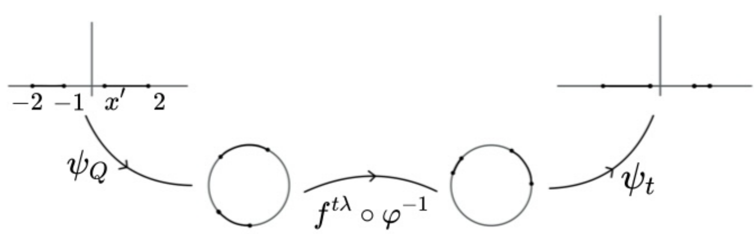


Figure 5.1: Choices of Möbius map ψ_Q and ψ_t

There exists one such choice of ψ_Q, say $\psi_Q(-2) = a$, $\psi_Q(-1) = b$ and $\psi_Q(2) = d$. Then we have $\psi_Q(x') = c$ for some x' lies in a compact subset of the closed interval $[-1, 2]$. To see this, by definition, we have

$$\begin{aligned}
L_{\mathbb{D}}(Q) &= \log \frac{(a-c)(b-d)}{(a-d)(b-c)} \\
&= \log \frac{(\psi_Q^{-1}(a) - \psi_Q^{-1}(b))(\psi_Q^{-1}(c) - \psi_Q^{-1}(d))}{(\psi_Q^{-1}(a) - \psi_Q^{-1}(d))(\psi_Q^{-1}(b) - \psi_Q^{-1}(c))} \\
&= \log \frac{(-2-x')(-1-2)}{(-2-2)(-1-x')} \\
&= M
\end{aligned}$$

Simplifying we get $\frac{(-2-x')}{(-1-x')} = \frac{4}{3}e^M$, we then have $x' = \frac{6-4e^M}{4e^M-3}$.

Note that M is bounded away from 0 and ∞ by some m_1, m_2, ie. $0 < m_1 < M < m_2 < \infty$. It implies that x' is bounded away from -1 and 2, ie. there exists n_1, n_2 such that $-1 < n_1 < x' < n_2 < 2$. Since Möbius map sends triples to triples, there is a unique choice of ψ_t. By Lemma 5.2.2, the Beltrami coefficient of $\psi_t \circ f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q$ is $t\lambda''$ where $||\lambda||_\infty = ||\lambda'||_\infty = ||\lambda''||_\infty$. Let $f^{t\lambda''} = \psi_t \circ f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q$ be the quasiconformal map with Beltrami coefficient $t\lambda''$. We need to show that $\frac{d}{dt} \int_{G(\mathbb{D})} \xi \circ \varphi \, d\alpha_t \big|_{t=0}$ exists; which is equivalent to show that

$$\begin{aligned}
\frac{d}{dt} \int_{G(\mathbb{D})} \xi \circ \psi_Q \circ \psi_Q^{-1} \circ \varphi \, d\alpha_t \Big|_{t=0} &= \frac{d}{dt} \int_{G(\mathbb{H})} \xi \circ \psi_Q \, d((\psi_Q^{-1} \circ \varphi)_* \alpha_t) \Big|_{t=0} \\
&= \frac{d}{dt} \int_{G(\mathbb{H})} \xi \circ \psi_Q \, d\beta_t \Big|_{t=0} \text{ exists}
\end{aligned} \tag{5.1}$$

where $\beta_t = (\psi_Q^{-1} \circ \varphi)_* \alpha_t$ is the pushforward of α_t by $\psi_Q^{-1} \circ \varphi : \mathbb{D} \to \mathbb{H}$.

Lemma 5.2.3. *Suppose that ξ is ν-Hölder continuous and $\psi_Q : \mathbb{H} \to \mathbb{D}$ is a Möbius map, then $\xi \circ \psi_Q$ is ν-Hölder continuous.*

Proof of Lemma 5.2.3.

To see this, notice that ψ'_Q is continuous and bounded on compact intervals of $\mathbb{R}$. Then by Lemma 4.2.2 ψ_Q is Lipschitz, which implies that ψ_Q is Hölder continuous. Namely,

$$\begin{aligned} |\xi \circ \psi_Q(x) - \xi \circ \psi_Q(y)| &\leqslant C \cdot |\psi_Q(x) - \psi_Q(y)|^\nu \\ &\leqslant C \cdot ||\psi'_Q||_\infty^\nu \cdot |x-y|^\nu \end{aligned}$$

Thus $\xi \circ \psi_Q$ is Hölder continuous with exponent ν, in fact $\xi \circ \psi_Q$ is uniformly continuous. □

Let $(p,q,r,s) = \dfrac{(p-r)(q-s)}{(p-s)(q-r)}$ denote the cross-ratio of $p,q,r,s \in \widehat{\mathbb{R}}$.

Lemma 5.2.4. *(Wolpert's Lemma)*

Let $\mu \in B(\Gamma)$ and $w^{\varepsilon\mu}$, $||\varepsilon\mu||_\infty < 1$ be the solution of Beltrami equation

$$\begin{cases} w_{\bar z} = \mu w_z, & z \in \mathbb{H} \\ w_{\bar z} = \bar\mu(\bar z) w_z, & z \in \mathbb{L}. \end{cases}$$

Given p, q, r, and $s \in \widehat{\mathbb{R}}$, distinct, then

$$\begin{aligned} &\frac{d}{d\varepsilon}(w^{\varepsilon\mu}(p), w^{\varepsilon\mu}(q), w^{\varepsilon\mu}(r), w^{\varepsilon\mu}(s))\Big|_{\varepsilon=0} \\ &= -\frac{2}{\pi}(p,q,r,s) Re \int_{\mathbb{H}} \mu(\zeta) \frac{pr+qs-ps-qr}{(\zeta-p)(\zeta-q)(\zeta-r)(\zeta-s)} d\sigma(\zeta). \end{aligned}$$

Refer to Lemma 1.1 in [22]. □

Proposition 5.2.5. *Suppose that $Supp(\xi) \subset Q = [a,b] \times [c,d]$ for a single box. Then*

$$\frac{d}{dt} \int_{G(\mathbb{D})} \xi \circ \varphi \; d\alpha_t \Big|_{t=0}$$

exists.

Proof of Proposition 5.2.5.

By expression (5.1), we need to show $\frac{d}{dt} \int_{G(\mathbb{H})} \xi \circ \psi_Q \; d\beta_t \Big|_{t=0}$ exists.

Note that $Supp(\xi \circ \psi_Q) \subset \psi_Q^{-1}(Q) = Q'$ for some box $Q' \subset G(\mathbb{H})$. We define a step function that approximates $\xi \circ \psi_Q$ as follows. Subdivide Q' into Q^*_{ij} satisfying $\cup Q^*_{ij} \subset Q'$. Explicitly, we divide the interval $[-2,-1]$ into 2^n equal size intervals $[a_{i-1}, a_i]$ for $i = 1, ..., 2^n$ where $a_0 = -2, a_{2^n} = -1$ and $a_i = -2 + \frac{i}{2^n}$. Similarly, we divide the interval $[x', 2]$, $-1 < x' < 2$ into 2^n equal size intervals $[c_{j-1}, c_j]$ for $j = 1, ..., 2^n$ where $c_0 = x', c_{2^n} = 2$ and $c_j = x' + (2 - x')\frac{j}{2^n}$. This defines 4^n boxes $Q^*_{ij} = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ for $i, j = 1, ..., 2^n$. Set

$$\xi_n \circ \psi_Q = \sum_{i,j=1}^{2^n} \xi \circ \psi_Q(g^*_{ij}) \mathbb{1}_{Q^*_{ij}} \quad g^*_{ij} = (x_i, y_j) \in Q^*_{ij} \text{ is an arbitrary geodesic.}$$

Recall that the derivative $\frac{d}{dt} f^{t\lambda''}(z)$ exists for t such that $||t\lambda''||_\infty < 1$ and for each fixed $z \in \mathbb{C}$, see Chapter 5 in [1]. By the definition of β_t and since the cross-ratio is invariant under Möbius map, $\frac{d}{dt}\beta_t(Q^*_{ij}) = \frac{d}{dt}\alpha_t((\psi_Q^{-1} \circ \varphi)^{-1}(Q^*_{ij}))$ exists because $\alpha_t((\psi_Q^{-1} \circ \varphi)^{-1})$ is the composition of differentiable functions:

$$\begin{aligned}
\beta_t(Q^*_{ij}) &= \alpha_t((\psi_Q^{-1} \circ \varphi)^{-1}(Q^*_{ij})) \\
&= L([f^{t\lambda}])((\psi_Q^{-1} \circ \varphi)^{-1}(Q^*_{ij})) \\
&= L([f^{t\lambda}])(\varphi^{-1} \circ \psi_Q(Q^*_{ij})) \\
&= L(f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q(Q^*_{ij})) \\
&= L(\psi_t \circ f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q(Q^*_{ij})), \quad \text{where } \psi_t : \mathbb{D} \to \mathbb{H} \text{ is Möbius} \\
&= L([f^{t\lambda''}])(Q^*_{ij})
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}\int \xi_n \circ \varphi \; d\alpha_t &= \frac{d}{dt}\int \xi_n \circ \psi_Q \; d((\psi_Q^{-1}\circ\varphi)_*\alpha_t) \\
&= \frac{d}{dt}\int \xi_n \circ \psi_Q \; d\beta_t \\
&= \sum_{i,j=1}^{2^n} \xi \circ \psi_Q(g_{ij}^*) \frac{d}{dt}\beta_t(Q_{ij}^*).
\end{aligned} \tag{5.2}$$

Also note that

$$\begin{aligned}
\left|\int \xi_n \circ \psi_Q \; d\beta_t - \int \xi \circ \psi_Q \; d\beta_t\right| &= \left|\int (\xi_n \circ \psi_Q - \xi \circ \psi_Q) \; d\beta_t\right| \\
&\leqslant \int |\xi_n \circ \psi_Q - \xi \circ \psi_Q| \; d\beta_t \\
&\leqslant \max_{i,j} \Big| \max_{g\in Q_{ij}^*} \xi \circ \psi_Q(g) - \min_{g\in Q_{ij}^*} \xi \circ \psi_Q(g) \Big| \sum_{i,j}^{2^n} \beta_t(Q_{ij}^*) \\
&\longrightarrow 0.
\end{aligned}$$

Thus $\displaystyle\int \xi_n \circ \psi_Q \; d\beta_t$ converges to $\displaystyle\int \xi \circ \psi_Q \; d\beta_t$ as $n \to \infty$.

By Theorem 7.17 in [15], to show that $\displaystyle\frac{d}{dt}\int \xi \circ \psi_Q \; d\beta_t\Big|_{t=0}$ exists, it is enough to show that $\frac{d}{dt}\int \xi_n \circ \psi_Q \; d\beta_t$ converges uniformly to some function g in an open interval containing $t = 0$. To show the convergence, we form the series

$$\frac{d}{dt}\int \xi_1 \circ \psi_Q \; d\beta_t + \sum_{n=1}^{\infty}\left[\frac{d}{dt}\int \xi_{n+1} \circ \psi_Q \; d\beta_t - \frac{d}{dt}\int \xi_n \circ \psi_Q \; d\beta_t\right]$$

whose n-th partial sum is $\displaystyle\frac{d}{dt}\int \xi_{n+1} \circ \psi_Q \; d\beta_t$. Then it is equivalent to show that the above series converges uniformly for small $|t|$.

Define $\dot{f}^{\mu}[\lambda](z) = \frac{d}{dt} f^{\mu+t\lambda}(z)\big|_{t=0}$ and $\dot{f}[\lambda](z) = \frac{d}{dt} f^{t\lambda}(z)\big|_{t=0}$.

By formula (1.12) of Ahlfors in [2],

$$\dot{f}^{\mu}[\lambda] = \dot{f}[L^{\mu}\lambda] \circ f^{\mu} \tag{5.3}$$

where

$$L^{\mu}\lambda = \left\{ \lambda \frac{(f_z^{\mu})^2}{|f_z^{\mu}|^2 - |f_{\bar{z}}^{\mu}|^2} \right\} \circ (f^{\mu})^{-1}.$$

Also by Lemma 5.2.4, for $t = 0$ we have

$$\frac{d}{dt}\beta_t(Q_{ij}^*)\Big|_{t=0} = -\frac{2}{\pi} Re \int_{\mathbb{H}} \lambda''(z) \frac{(a_{i-1} - a_i)(c_{j-1} - c_j)}{(z - a_{i-1})(z - a_i)(z - c_{j-1})(z - c_j)} dxdy \tag{5.4}$$

where $z = x + iy$.

Let $a_i^t = f^{t\lambda''}(a_i)$ and $c_j^t = f^{t\lambda''}(c_j)$.

From (5.3) and (5.4), for $t \neq 0$ we have

$$\frac{d}{dt}\beta_t(Q_{ij}^*) = -\frac{2}{\pi} Re \int_{\mathbb{H}} L^{t\lambda''}\lambda''(z) \frac{(a_{i-1}^t - a_i^t)(c_{j-1}^t - c_j^t)}{(z - a_{i-1}^t)(z - a_i^t)(z - c_{j-1}^t)(z - c_j^t)} dxdy \tag{5.5}$$

where $z = x + iy$.

By (5.2) and (5.5), we have

$$\frac{d}{dt}\int \xi_n \circ \psi_Q \; d\beta_t = \tag{5.6}$$

$$-\frac{2}{\pi} Re \int_{\mathbb{H}} L^{t\lambda''}\lambda''(z) \sum_{i,j=1}^{2^n} \frac{\xi \circ \psi_Q(g_{ij}^*)(a_{i-1}^t - a_i^t)(c_{j-1}^t - c_j^t)}{(z - a_{i-1}^t)(z - a_i^t)(z - c_{j-1}^t)(z - c_j^t)} dxdy$$

where $z = x + iy$ and $g_{ij}^* = (x_i, y_j) \in Q_{ij}^*$ is an arbitrary geodesic.

It follows that

$$\left|\frac{d}{dt}\int \xi_1 \circ \psi_Q \ d\beta_t\right| \leqslant \frac{2}{\pi}\sum_{i,j=1}^{2} |\xi \circ \psi_Q(g^*_{ij})| \ ||L^{t\lambda''}\lambda''||_\infty \tag{5.7}$$

$$\times \max_{i,j}\{|(a^t_{i-1} - a^t_i)(c^t_{j-1} - c^t_j)|\}$$

$$\times \max_{i,j}\left\{\int_{\mathbb{H}} \frac{dxdy}{|z - a^t_{i-1}||z - a^t_i||z - c^t_{j-1}||z - c^t_j|}\right\}.$$

To estimate $\left|\frac{d}{dt}\int \xi_{n+1} \circ \psi_Q \ d\beta_t - \frac{d}{dt}\int \xi_n \circ \psi_Q \ d\beta_t\right|$, we divide $[a_{i-1}, a_i]$ and $[c_{j-1}, c_j]$ into two equal sub-intervals.

Write $Q^*_{ij} = \bigcup_{k=1}^{4} Q^*_{ik,jk}$ where $Q^*_{ik,jk} = [a_{(i-1)k}, a_{ik}] \times [c_{(j-1)k}, c_{jk}]$ and $a_{ik} = a_{i-1}$ or a_i or midpoint of $[a_{i-1}, a_i]$. Similarly for c_{jk}.

Set $\xi_{n+1} \circ \psi_Q = \sum_{i,j=1}^{2^n}\sum_{k=1}^{4} \xi \circ \psi_Q(g^*_{ik,jk})\mathbb{1}_{Q^*_{ik,jk}}$, where $g^*_{ik,jk} = (x_{ik}, y_{jk}) \in Q^*_{ik,jk}$ is an arbitrary geodesic.

And let $\delta = \max\{1, 2 - x'\}$, the maximum length of two disjoint closed intervals of Q'. We already showed that $\xi \circ \psi_Q$ is Hölder continuous with exponent ν in Lemma 5.2.3. We have

$$\begin{aligned} |\xi \circ \psi_Q(g^*_{ik,jk}) - \xi \circ \psi_Q(g^*_{ij})| &\leqslant C_1|g^*_{ik,jk} - g^*_{ij}|^\nu \\ &\leqslant \frac{C_1\delta^\nu}{2^{n\nu}} = \frac{C_1\delta^\nu}{4^{\frac{n\nu}{2}}} \end{aligned}$$

for some fixed constant C_1 only depending on ξ and ψ_Q.

By (5.6) and definition of $\xi_{n+1} \circ \psi_Q$, we get

$$
\begin{aligned}
(5.8) \qquad & \left| \frac{d}{dt} \int \xi_{n+1} \circ \psi_Q \; d\beta_t - \frac{d}{dt} \int \xi_n \circ \psi_Q \; d\beta_t \right| \\
= & \left| \frac{d}{dt} \left(\int \xi_{n+1} \circ \psi_Q \; d\beta_t - \int \xi_n \circ \psi_Q \; d\beta_t \right) \right| \\
= & \left| \frac{d}{dt} \int (\xi_{n+1} \circ \psi_Q - \xi_n \circ \psi_Q) \; d\beta_t \right| \\
\leqslant & \quad \frac{2}{\pi} \frac{C_2 \delta^\nu 4^n}{4^{\frac{n\nu}{2}}} ||L^{t\lambda''} \lambda''||_\infty \\
& \times \max_{i,j,k} \{ |(a^t_{(i-1)k} - a^t_{ik})(c^t_{(j-1)k} - c^t_{jk})| \} \\
& \times \max_{i,j,k} \left\{ \int_{\mathbb{H}} \frac{dxdy}{|z - a^t_{(i-1)k}||z - a^t_{ik}||z - c^t_{(j-1)k}||z - c^t_{jk}|} \right\}.
\end{aligned}
$$

for some fixed constant C_2 only depending on C_1.

Recall that λ'' depends φ, we showed that $||\lambda''||_\infty = ||\lambda||_\infty$. And

$$L^{t\lambda''} \lambda'' = \left\{ \lambda'' \frac{(f_z^{t\lambda''})^2}{|f_z^{t\lambda''}|^2 - |f_{\bar{z}}^{t\lambda''}|^2} \right\} \circ (f^{t\lambda''})^{-1}$$

implies that

$$||L^{t\lambda''} \lambda''||_\infty = ||\lambda''||_\infty \frac{1}{1 - ||t\lambda''||_\infty} < C_3$$

for some fixed constant C_3, since $||\lambda''||_\infty$ is bounded and $||t\lambda''||_\infty < 1$ for small $|t|$.

For t small, the family $f^{t\lambda''}$ has quasiconformal constant $\dfrac{1 + ||t\lambda''||_\infty}{1 - ||t\lambda''||_\infty}$ close to 1 and also fixes 0, 1 and ∞. Then this family $f^{t\lambda''}$ is uniformly Hölder continuous with exponent ω close to 1 in the spherical metric. Using the fact that $\arctan(x) < x$ for all $x > 0$, this implies that

$$|(a^t_{(i-1)k} - a^t_{ik})(c^t_{(j-1)k} - c^t_{jk})| \leqslant \frac{C_4 \delta^{2\omega}}{4^{n\omega}}$$

for some fixed constant C_4 only depending on the uniform Hölder continuity of $f^{t\lambda''}$.

Next, we want to show that

$$\int_{\mathbb{H}} \frac{dxdy}{|z-a^t_{(i-1)k}||z-a^t_{ik}||z-c^t_{(j-1)k}||z-c^t_{jk}|} \leqslant C_5 + C_6 n \tag{5.9}$$

for some fixed constants C_5 and C_6 again only depending on the uniform Hölder continuity of $f^{t\lambda''}$. The computation uses the ideas from Section 3.4 in [10].

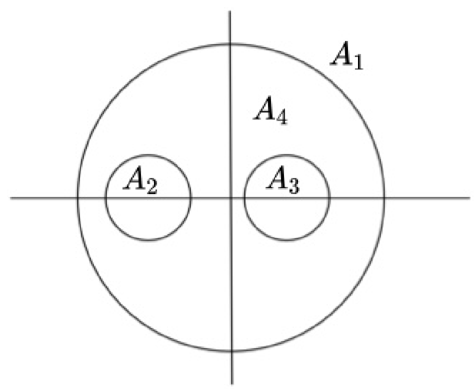


Figure 5.2: Regions for integrating expression (5.9)

To see this, divide the domain of the integral over $\mathbb{H}$ into four sets as follows. Let

$$A_1 = \{z \in \mathbb{H} : |z| > R\}, \text{ for large } R.$$

Note that since $f^{t\lambda''}$ is Hölder continuous with exponent ω close to 1 when t is small. Then $|f^{t\lambda''}(x') - f^{t\lambda''}(-1)| \leqslant C|x' - (-1)|^{\omega}$ is bounded by some number. Let

$$A_2 = \{z \in \mathbb{H} : |z - a^t_{ik}| \leqslant r\}, \text{ where } r = \frac{f^{t\lambda''}(x') - f^{t\lambda''}(-1)}{4}, \text{ depend on } t$$

and

$$A_3 = \{z \in \mathbb{H} : |z - c^t_{jk}| \leqslant r\}, \text{ where } r = \frac{f^{t\lambda''}(x') - f^{t\lambda''}(-1)}{4}, \text{ depend on } t.$$

Note that as $t \to 0$, we have $f^{t\lambda''}(x')$ and $f^{t\lambda''}(-1)$ which are close to x' and -1 respectively. Then A_2 and A_3 do not intersect.

Finally, we let

$$A_4 = \mathbb{H} \setminus (A_1 \cup A_2 \cup A_3).$$

For the integral in (5.9) over A_1, since $-2 < a_{ik} < -1$ and $x' < c_{jk} < 2$ for small t, by Hölder continuity of $f^{t\lambda''}$, we have that a^t_{ik} and c^t_{jk} are bounded. And for large R, we have $|z - a^t_{ik}| \geqslant \frac{1}{2}|z|$ and $|z - c^t_{jk}| \geqslant \frac{1}{2}|z|$. Then

$$\begin{aligned}\int_{|z|>R} \frac{dxdy}{|z-a^t_{(i-1)k}||z-a^t_{ik}||z-c^t_{(j-1)k}||z-c^t_{jk}|} &\leqslant \int_{|z|>R} \frac{dxdy}{16|z|^4}, \text{ write } z = \rho e^{i\theta} \\ &= \int_0^{\pi}\int_R^{\infty} \frac{\rho\, d\rho d\theta}{16\rho^4} = \int_0^{\pi} \frac{1}{32} R^{-2}\, d\theta \\ &= \frac{\pi}{32} R^{-2}.\end{aligned}$$

For the integral in (5.9) over A_4, we have

$$|a^t_{(i-1)k} - a^t_{ik}| \leqslant C|a_{(i-1)k} - a_{ik}|^{\omega} = \frac{C}{2^{n\omega}}$$

and

$$|c^t_{(j-1)k} - c^t_{jk}| \leqslant C|c_{(j-1)k} - c_{jk}|^{\omega} = \frac{C(x'-2)^{\omega}}{2^{n\omega}}.$$

There is a choice of r such that for $z \in A_4$, we have

$$|z - a^t_{(i-1)k}| \geqslant |z - a^t_{ik}| - |a^t_{(i-1)k} - a^t_{ik}| \geqslant r - \frac{C}{2^{n\omega}} > \frac{r}{2}$$

since $\frac{C}{2^{n\omega}}$ is small. Similarly

$$|z - c^t_{(j-1)k}| \geqslant |z - c^t_{jk}| - |c^t_{(j-1)k} - c^t_{jk}| \geqslant r - \frac{C(2-x')^{\omega}}{2^{n\omega}} > \frac{r}{2}$$

since $\dfrac{C(2-x')^{\omega}}{2^{n\omega}}$ is small. Then

$$\begin{aligned}\int_{A_4} \frac{dxdy}{|z-a^t_{(i-1)k}||z-a^t_{ik}||z-c^t_{(j-1)k}||z-c^t_{jk}|} &\leqslant \int_{A_4} \frac{dxdy}{r^4/4} \\ &= \frac{4}{r^4}\int_{A_4} dxdy \\ &\leqslant \frac{4}{r^4}\cdot\frac{1}{2}\pi R^2 \\ &= \frac{2\pi}{r^4}R^2\end{aligned}$$

For the integral in (5.9) over A_2, using substitution $(a^t_{ik}-a^t_{(i-1)k})w = z-a^t_{ik}$, then we have

$$\begin{aligned} z-a^t_{ik}+a^t_{ik}-a^t_{(i-1)k} &= (a^t_{ik}-a^t_{(i-1)k})w + (a^t_{ik}-a^t_{(i-1)k}) \\ &= (a^t_{ik}-a^t_{(i-1)k})(w+1).\end{aligned}$$

Note that $|z-c^t_{jk}||z-c^t_{(j-1)k}| > M$ for some M. Using change of variables where the Jacobian $= (a^t_{ik}-a^t_{(i-1)k})^2$, we have that

$$\int_{|w|\leqslant \frac{r}{|a^t_{ik}-a^t_{(i-1)k}|}} \frac{dudv}{M|w(w+1)|}.$$

Write $w=\rho e^{i\theta}$, $|w+1| = \sqrt{\rho^2+2\rho\cos\theta+1} \geqslant \sqrt{\rho^2-2\rho+1} = \rho-1$. Then the above integral is less than or equal to:

$$\frac{1}{M}\int_0^{\pi}\int_0^{\frac{r}{|a^t_{ik}-a^t_{(i-1)k}|}} \frac{\rho\, d\rho d\theta}{\rho(\rho-1)} = \frac{1}{M}\int_0^{\pi}\int_0^{\frac{r}{|a^t_{ik}-a^t_{(i-1)k}|}} \frac{d\rho d\theta}{\rho-1}.$$

Note that $(f^{t\lambda''})^{-1}$ is quasiconformal with the same quasiconformal constant as $f^{t\lambda''}$. Thus, it is also Hölder continuous with the same Hölder exponent as $f^{t\lambda''}$.

By definition, $a_{ik} = (f^{t\lambda''})^{-1}(a^t_{ik})$ and $a_{(i-1)k} = (f^{t\lambda''})^{-1}(a^t_{(i-1)k})$. Then we have

$$\begin{aligned} |a_{ik} - a_{(i-1)k}| &= |(f^{t\lambda''})^{-1}(a^t_{ik}) - (f^{t\lambda''})^{-1}(a^t_{(i-1)k})| \\ &\leqslant C|a^t_{ik} - a^t_{(i-1)k}|^{\omega} \end{aligned}$$

for some constant C.

This implies that $|a^t_{ik} - a^t_{(i-1)k}| \geqslant C|a_{ik} - a_{(i-1)k}|^{\frac{1}{\omega}} = \dfrac{C}{2^{\frac{n}{\omega}}}$ for some constant C. And we have

$$\begin{aligned} \frac{1}{M}\int_0^{\pi}\int_0^{\frac{r}{|a^t_{ik}-a^t_{(i-1)k}|}} \frac{d\rho d\theta}{\rho - 1} &= \frac{1}{M}\int_0^{\pi} \ln\left|\frac{r}{|a^t_{ik} - a^t_{(i-1)k}|} - 1\right| d\theta \\ &= \frac{\pi}{M}\ln\left|\frac{r}{|a^t_{ik} - a^t_{(i-1)k}|} - 1\right| \\ &\leqslant \frac{\pi}{M}\ln\left(\frac{r}{|a^t_{ik} - a^t_{(i-1)k}|} + 1\right) \\ &\leqslant \frac{\pi}{M}\ln\left(\frac{r\cdot 2^{\frac{n}{\omega}}}{C} + 1\right) \\ &\leqslant \frac{\pi}{M}\ln\left(\frac{(r+C)\cdot 2^{\frac{n}{\omega}}}{C}\right) \quad \text{since } 2^{\frac{n}{\omega}} > 1 \\ &= \frac{\pi}{M}(\ln(r+C) - \ln C) + \frac{\pi \ln 2}{M\omega}n \end{aligned}$$

Similar computation holds for the integral in (5.9) over A_3. And combining the above results, proves (5.9).

Next, we choose t small enough with ω close to 1 such that $\frac{\nu}{2} + \omega - 1 > 0$. Then we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\left|\frac{d}{dt}\xi_{n+1}\circ\psi_Q\ d\beta_t - \frac{d}{dt}\xi_n\circ\psi_Q\ d\beta_t\right| \\ &\leqslant C\cdot\sum_{n=1}^{\infty}\frac{n}{4^{n(\frac{\nu}{2}+\omega-1)}}\cdot||L^{t\lambda''}\lambda''||_{\infty} < \infty \end{aligned} \tag{5.10}$$

for some fixed constant C which does not depend on φ. Explicitly, the above series $\sum_{n=1}^{\infty} \frac{n}{4^{n(\frac{\nu}{2}+\omega-1)}}$ converges by ratio test since $\frac{\nu}{2}+\omega-1>0$.

Combining with the inequalities proved in (5.7), It follows that

$$\frac{d}{dt}\int \xi_1 \circ \psi_Q \ d\beta_t + \sum_{n=1}^{\infty}\left[\frac{d}{dt}\int \xi_{n+1}\circ\psi_Q \ d\beta_t - \frac{d}{dt}\int \xi_n \circ \psi_Q \ d\beta_t\right]$$

converges uniformly for small $|t|$ and this sum is bounded by a constant independent of φ.

In particular, $\sup_{\varphi\in \text{Möb}(\mathbb{D})} \frac{d}{dt}\int \xi\circ\varphi \ d\alpha_t$ is bounded by a constant.

Thus

$$\frac{d}{dt}\int \xi\circ\varphi \ d\alpha_t\big|_{t=0} = \frac{d}{dt}\int \xi\circ\psi_Q \ d\beta_t\big|_{t=0} = \mathbf{W}(\xi\circ\varphi)$$

exists where $\mathbf{W}\in\mathcal{H}_{\text{bd}}(X_0)$.

This completes the proof of Proposition 5.2.5. □

Notice that the Hölder continuity of ξ is crucial here, and we will discuss more by the end of this chapter. There is an immediate corollary which will be used later.

Corollary 5.2.6. *Let $[f^{t\lambda}]$ be a path in $\mathcal{T}(X_0)$ where λ is a harmonic Beltrami differential. Let $\alpha_t = L([f^{t\lambda}])$. Then for all $\xi\in H(\widetilde{X}_0)$ and $\varphi\in$ Möb$(\mathbb{D})$ we have*

$$\begin{aligned}\frac{d}{dt}\int_{G(\mathbb{D})} \xi\circ\varphi \ d\alpha_t\big|_{t=0} &= \frac{d}{dt}\int_{G(\mathbb{D})} \xi \ d(\varphi_*\alpha_t)\big|_{t=0} \\ &= \frac{d}{dt}\int_{G(\mathbb{D})} \xi \ d\alpha'_t\big|_{t=0} \\ &\leqslant C\cdot||\lambda||_\infty\end{aligned}$$

where $\alpha'_t = \varphi_*\alpha_t$ is the pushforward of α_t by $\varphi:\mathbb{D}\to\mathbb{D}$ and C only depending on the support, Hölder exponent and Hölder constant of ξ.

Proof of Corollary 5.2.6.

The proof follows directly from estimations in (5.7), (5.8) and (5.10). □

Proof of Theorem 5.2.1.

Consider $Supp(\xi) = K$ which is compact. Cover K by finitely many boxes $Q_1, ..., Q_n$. Using partition of unity defined in Chapter 4, $Supp(\rho_i \xi) \subset Q_i$ for each i. Applying Proposition 5.4, we get

$$\begin{aligned}\frac{d}{dt}\int \xi \circ \varphi \; d\alpha_t\Big|_{t=0} &= \frac{d}{dt}\int \xi \circ \psi_Q \; d\beta_t\Big|_{t=0} \\ &= \sum_{i=1}^{n} \frac{d}{dt}\int (\rho_i \xi) \circ \psi_Q \; d\beta_t\Big|_{t=0} \\ &= \sum_{i=1}^{n} \mathbf{W}((\rho_i \xi) \circ \varphi)\end{aligned}$$

exists and $\mathbf{W} \in \mathcal{H}_{\mathrm{bd}}(X_0)$ since $\sup_{\varphi \in \text{Möb}(\mathbb{D})} \mathbf{W}((\rho_i \xi) \circ \varphi) < \infty$ for each i is bounded by a constant independent of φ. Thus $\sup_{\varphi \in \text{Möb}(\mathbb{D})} \sum_{i=1}^{n} \mathbf{W}((\rho_i \xi) \circ \varphi) < \infty$.

This completes the proof of Theorem 5.2.1. □

From now on, we keep the notation $\mathbf{W}(\xi \circ \varphi) = \frac{d}{dt}\int \xi \circ \varphi \; d\alpha_t\Big|_{t=0}$ and λ denotes a harmonic Beltrami differential.

Theorem 5.2.7. *The map $T_{[id]}L : T_{[id]}\mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$ defined by $T_{[id]}L([\lambda]_{tan}) = \mathbf{W}$ is linear and bounded. And*

$$\lim_{t \to 0} \sup_{\varphi \in M\ddot{o}b(\mathbb{D})} \left| \frac{\int \xi \circ \varphi \; d\alpha_t - \int \xi \circ \varphi \; d\alpha_0 - t\mathbf{W}(\xi \circ \varphi)}{t||\lambda||_\infty} \right| = 0$$

uniformly in λ for all $\xi \in H(\widetilde{X}_0)$ and as long as $||\lambda||_\infty$ is bounded.

Proof of Theorem 5.2.7.

From the proof Theorem 5.2.1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left|\frac{d}{dt}\int \xi_{n+1}\circ\varphi\ d\alpha_t - \frac{d}{dt}\int \xi_n\circ\varphi\ d\alpha_t\right| \\
&= \sum_{n=1}^{\infty}\left|\frac{d}{dt}\int \xi_{n+1}\circ\psi_Q\ d\beta - \frac{d}{dt}\int \xi_n\circ\psi_Q\ d\beta\right| \\
&\leqslant C\cdot\sum_{n=1}^{\infty}\frac{n}{4^{n(\frac{\nu}{2}+\omega-1)}}\cdot ||L^{t\lambda''}\lambda''||_\infty
\end{aligned}$$

where $\beta = (\psi_Q^{-1}\circ\varphi)_*\alpha_t$ and $||L^{t\lambda''}\lambda''||_\infty = ||\lambda''||_\infty \cdot \dfrac{1}{1-||t\lambda''||_\infty}$.

When $t=0$, $|T_{[id]}L([\lambda]_{tan})(\xi\circ\varphi)| = |\mathbf{W}(\xi\circ\varphi)| \leqslant C||\lambda||_\infty$ for a fixed constant C and for all $\varphi \in \text{Möb}(\mathbb{D})$. Thus $T_{[id]}L$ is bounded.

Next we show the linearity, namely $T_{[id]}L([c_1\lambda_1+c_2\lambda_2]_{tan}) = c_1T_{[id]}L([\lambda_1]_{tan})+c_2T_{[id]}L([\lambda_2]_{tan})$. Using the Mapping Theorem from [10], we have $f^{t\lambda}(z) = z + tV_\lambda(z) + o(t)$ where

$$V_\lambda(z) = -\frac{z(z-1)}{\pi}\iint_{\mathbb{C}}\frac{\lambda(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\sigma d\eta, \quad \zeta = \sigma + i\eta.$$

Similarly, $f^{t(c_1\lambda_1+c_2\lambda_2)}(z) = z + tV_{c_1\lambda_1+c_2\lambda_2}(z) + o(t)$ where

$$\begin{aligned}
V_{c_1\lambda_1+c_2\lambda_2}(z) &= -\frac{z(z-1)}{\pi}\iint_{\mathbb{C}}\frac{(c_1\lambda_1+c_2\lambda_2)(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\sigma d\eta, \quad \zeta = \sigma + i\eta \\
&= -\frac{z(z-1)}{\pi}\iint_{\mathbb{C}}\frac{(c_1\lambda_1)(\zeta)+(c_2\lambda_2)(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\sigma d\eta \\
&= c_1V_{\lambda_1}(z) + c_2V_{\lambda_2}(z).
\end{aligned}$$

This shows that $\dfrac{d}{dt}f^{t(c_1\lambda_1+c_2\lambda_2)}\Big|_{t=0} = c_1\dfrac{d}{dt}f^{t\lambda_1}\Big|_{t=0} + c_2\dfrac{d}{dt}f^{t\lambda_2}\Big|_{t=0}$.

Note that the Liouville map L can be approximated by a sequence of maps L_n given by:

$$\begin{aligned} L_n([f^{t\lambda}])(\xi \circ \varphi) &= \int \xi_n \circ \varphi \, d\alpha_t \quad \text{where } \alpha_t = L([f^{t\lambda}]) \\ &= \int \xi_n \circ \psi_Q \, d((\psi_Q^{-1} \circ \varphi)_* \alpha_t) \\ &= \int \xi_n \circ \psi_Q \, d\beta_t \end{aligned}$$

where $\beta_t = (\psi_Q^{-1} \circ \varphi)_* \alpha_t$ is the pushforward of α_t by $\psi_Q^{-1} \circ \varphi : \mathbb{D} \to \mathbb{H}$.

Follow the proof of Proposition 5.2.5 and by (5.2), we have

$$\begin{aligned} \frac{d}{dt} L_n([f^{t\lambda}])(\xi \circ \varphi) &= \frac{d}{dt} \int \xi_n \circ \psi_Q \, d\beta_t \\ &= \sum_{i,j=1}^{2^n} \xi \circ \psi_Q(g_{ij}^*) \frac{d}{dt} \beta_t(Q_{ij}^*) \end{aligned}$$

Notice that $\frac{d}{dt} L_n([f^{t(c_1\lambda_1 + c_2\lambda_2)}])\Big|_{t=0}$ is linear.

Indeed, for a box $Q_{ij}^* = [a,b] \times [c,d]$, we have

$$\begin{aligned} \beta_t(Q_{ij}^*) &= L([f^{t(c_1\lambda_1'' + c_2\lambda_2'')}])(Q_{ij}^*) \\ &= \log \frac{(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(a) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(c))(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(b) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(d))}{(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(a) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(d))(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(b) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(c))} \\ &= \quad \log(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(a) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(c)) \\ &\quad + \log(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(b) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(d)) \\ &\quad - \log(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(a) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(d)) \\ &\quad - \log(f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(b) - f^{t(c_1\lambda_1'' + c_2\lambda_2'')}(c)). \end{aligned}$$

For simplicity, we give the computation for one piece, and the computations for remaining pieces are exactly the same. Recall that $f^{t\lambda}$ is differentiable in t, then so is $L([f^{t\lambda}])$.

As $t \to 0$, we have

$$\begin{aligned}
\frac{d}{dt}\beta_t(Q^*_{ij}) &= \frac{d}{dt}L([f^{t(c_1\lambda''_1+c_2\lambda''_2)}])(Q^*_{ij}) \\
&= \frac{\frac{d}{dt}(f^{t(c_1\lambda''_1+c_2\lambda''_2)}(a) - f^{t(c_1\lambda''_1+c_2\lambda''_2)}(c))}{f^{t(c_1\lambda''_1+c_2\lambda''_2)}(a) - f^{t(c_1\lambda''_1+c_2\lambda''_2)}(c)} \cdots \\
&= \frac{V_{c_1\lambda''_1+c_2\lambda''_2}(a) - V_{c_1\lambda''_1+c_2\lambda''_2}(c)}{(a-c) + t(V_{c_1\lambda''_1+c_2\lambda''_2}(a) - V_{c_1\lambda''_1+c_2\lambda''_2}(c)) + o(t)} \cdots \\
&= \frac{c_1(V_{\lambda''_1}(a) - V_{\lambda''_1}(c)) - c_2(V_{\lambda''_2}(a) - V_{\lambda''_2}(c))}{(a-c) + t(V_{c_1\lambda''_1+c_2\lambda''_2}(a) - V_{c_1\lambda''_1+c_2\lambda''_2}(c)) + o(t)} \cdots \\
&= \frac{c_1(V_{\lambda''_1}(a) - V_{\lambda''_1}(c))}{(a-c)} + \frac{c_2(V_{\lambda''_2}(a) - V_{\lambda''_2}(c))}{(a-c)} \cdots \\
&= c_1\frac{d}{dt}L([f^{t\lambda''_1}])(Q^*_{ij}) + c_2\frac{d}{dt}L([f^{t\lambda''_2}])(Q^*_{ij}).
\end{aligned}$$

Thus $T_{[id]}L_n$ is linear. Consequently, $T_{[id]}L$ is also linear since it is the limit of $T_{[id]}L_n$.

Finally, we show the limit goes to 0 as $t \to 0$. To simplify notation, we use $T_{[id]}L$ instead of $T_{[id]}L([\lambda]_{tan})$. Then we have

$$\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\int \xi\circ\varphi\ d\alpha_t - \int \xi\circ\varphi\ d\alpha_0 - t\mathbf{W}(\xi\circ\varphi)}{t||\lambda||_\infty}\right|$$

$$= \sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{L([f^{t\lambda}])(\xi\circ\varphi) - L(\xi\circ\varphi) - tT_{[id]}L([\lambda]_{tan})(\xi\circ\varphi)}{t||\lambda||_\infty}\right|$$

$$\leqslant \sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{L_n([f^{t\lambda}])(\xi\circ\varphi) - L_n(\xi\circ\varphi) - tT_{[id]}L_n([\lambda]_{tan})(\xi\circ\varphi)}{t||\lambda||_\infty}\right| \tag{5.11}$$

$$+ \sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{(L([f^{t\lambda}]) - L_n([f^{t\lambda}]))(\xi\circ\varphi) - (L-L_n)(\xi\circ\varphi) - t(T_{[id]}L - T_{[id]}L_n)(\xi\circ\varphi)}{t||\lambda||_\infty}\right| \tag{5.12}$$

We want to show that for given $\varepsilon > 0$, we can find n large such that expressions (5.11) $< \frac{\varepsilon}{2}$

and (5.12) $< \frac{\varepsilon}{2}$ for t small. We will first show that expression (5.12) $\to 0$ as $t \to 0$.

To see this, consider

$$\begin{aligned}
&\sup_{\varphi \in \text{Möb}(\mathbb{D})} \left| \frac{(L([f^{t\lambda}]) - L_n([f^{t\lambda}]))(\xi \circ \varphi) - (L - L_n)(\xi \circ \varphi) - t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)}{t||\lambda||_\infty} \right| \\
&\leqslant \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{|(L([f^{t\lambda}]) - L_n([f^{t\lambda}]))(\xi \circ \varphi) - (L - L_n)(\xi \circ \varphi))|}{|t| \cdot ||\lambda||_\infty} + \frac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty} \\
&= \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{|\int \xi \circ \varphi - \xi_n \circ \varphi \; d\alpha_t - \int \xi \circ \varphi - \xi_n \circ \varphi \; d\alpha_0|}{|t| \cdot ||\lambda||_\infty} + \frac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty} \\
&= \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{|\int (\xi \circ \varphi - \xi_n \circ \varphi) \; d(\alpha_t - \alpha_0)|}{|t| \cdot ||\lambda||_\infty} + \frac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty} \\
&= \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{|\int (\xi \circ \psi_Q - \xi_n \circ \psi_Q) \; d(\beta_t - \beta_0)|}{|t| \cdot ||\lambda||_\infty} + \frac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty} \\
&\leqslant \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{\int |\xi \circ \psi_Q - \xi_n \circ \psi_Q| \; d|\beta_t - \beta_0|}{|t| \cdot ||\lambda||_\infty} + \frac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty} \\
&\leqslant \sup_{\varphi \in \text{Möb}(\mathbb{D})} A + B \\
&\leqslant \sup_{\varphi \in \text{Möb}(\mathbb{D})} A + \sup_{\varphi \in \text{Möb}(\mathbb{D})} B
\end{aligned}$$

where $A = \dfrac{\int |\xi \circ \psi_Q - \xi_n \circ \psi_Q| \; d|\beta_t - \beta_0|}{|t| \cdot ||\lambda||_\infty}$ and $B = \dfrac{|t(T_{[id]}L - T_{[id]}L_n)(\xi \circ \varphi)|}{|t| \cdot ||\lambda||_\infty}$.

By Theorem 5.2.1, and the above inequality we have

$$\sup_{\varphi \in \text{Möb}(\mathbb{D})} A = \sup_{\varphi \in \text{Möb}(\mathbb{D})} \frac{\max_{i,j} |\max_{g \in Q^*_{ij}} \xi \circ \psi_Q(g) - \min_{g \in Q^*_{ij}} \xi \circ \psi_Q(g)| \cdot \sum_{i,j=1}^{2^n} |\beta_t - \beta_0|(Q^*_{ij})}{|t| \cdot ||\lambda||_\infty}.$$

We want to show that $\sup_{\varphi \in \text{Möb}(\mathbb{D})} A \to 0$ as $t \to 0$. Since ξ is ν-Hölder continuous, by

Lemma 5.2.3 and proof of Theorem 5.2.1, we have

$$\begin{aligned}
&\sup_{\varphi\in \text{Möb}(\mathbb{D})} \frac{\max_{i,j} |\max_{g\in Q^*_{ij}} \xi\circ\psi_Q(g) - \min_{g\in Q^*_{ij}} \xi\circ\psi_Q(g)| \cdot \sum_{i,j=1}^{2^n} |\beta_t-\beta_0|(Q^*_{ij})}{|t|\cdot||\lambda||_\infty} \\
&\leqslant \frac{C\cdot\delta}{2^{n\nu}} \cdot \frac{\sum_{i,j=1}^{2^n} |\beta_t-\beta_0|(Q^*_{ij})}{|t|\cdot||\lambda||_\infty} \qquad \text{write } \frac{d}{dt}\beta_t(Q^*_{ij}) = \beta'_t(Q^*_{ij}) \\
&= \frac{C\cdot\delta}{2^{n\nu}} \cdot \frac{\sum_{i,j=1}^{2^n} |\int_0^t \beta'_s(Q^*_{ij}))ds|}{|t|\cdot||\lambda||_\infty} \\
&< \frac{C\cdot\delta}{2^{n\nu}} \cdot \frac{\sum_{i,j=1}^{2^n} |t|\cdot\max\beta'_s(Q^*_{ij})}{|t|\cdot||\lambda||_\infty} \qquad \text{for } 0<s<t \\
&< \frac{C\cdot\delta}{2^{n\nu}} \cdot \frac{4^n\cdot n}{4^{n\omega}} \qquad \text{by (5.5) and estimates for (5.8), where } \omega\to 1 \text{ as } t\to 0 \\
&= \frac{C\cdot\delta n}{4^{n(\frac{\nu}{2}+\omega-1)}} \longrightarrow 0 \qquad \text{as } n\to\infty, \text{ since } \frac{\nu}{2}+\omega-1>0
\end{aligned}$$

Thus, we can find n_1 large enough and $\varepsilon_1 > 0$ such that $\sup_{\varphi\in\text{Möb}(\mathbb{D})} A < \frac{\varepsilon}{4}$ for all $|t| < \varepsilon_1$.

Next, we show that $\sup_{\varphi\in\text{Möb}(\mathbb{D})} B \longrightarrow 0$ as $t\to 0$.

To see this, by definition of $\frac{d}{dt}\int \xi_n\circ\varphi\, d\alpha_t$, we have

$$\sup_{\varphi\in\text{Möb}(\mathbb{D})} \frac{|t(\frac{d}{dt}\int\xi\circ\varphi\, d\alpha_t - \frac{d}{dt}\int\xi_n\circ\varphi\, d\alpha_t)|}{|t|\cdot||\lambda||_\infty} \longrightarrow 0$$

as $n\to\infty$. Similarly, we can find n_2 large enough and $\varepsilon_2>0$ such that $\sup_{\varphi\in\text{Möb}(\mathbb{D})} B < \frac{\varepsilon}{4}$ for all $|t|<\varepsilon_2$.

And in turn we have expression (5.12) $< \frac{\varepsilon}{2}$.

Now we fix $n = \max\{n_1, n_2\}$, we show that expression (5.11) $\to 0$ as $t\to 0$. To see this, by definition of L_n we have

$$\begin{aligned}
&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\int \xi_n\circ\varphi\ d\alpha_t - \int \xi_n\circ\varphi\ d\alpha_0 - t\frac{d}{dt}\int \xi_n\circ\varphi\ d\alpha_t}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\int \xi_n\circ\psi\ d\beta_t - \int \xi_n\circ\psi\ d\beta_0 - t\frac{d}{dt}\int \xi_n\circ\psi\ d\beta_t}{t||\lambda||_\infty}\right|
\end{aligned}$$

By Theorem 5.2.1 and Taylor polynomial of $\log(1+x)$, we have

$$\begin{aligned}
&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)\cdot\beta_t(Q_{ij}^*) - \sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)\cdot\beta_0(Q_{ij}^*) - t\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)\cdot\frac{d}{dt}\beta_t(Q_{ij}^*)}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)(\beta_t(Q_{ij}^*) - \beta_0(Q_{ij}^*) - t\frac{d}{dt}\beta_t(Q_{ij}^*)}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)(\log(f^{t\lambda}(a) - f^{t\lambda}(c)) - \log(a-c) - t\cdot\frac{V(a)-V(c)}{a-c}\cdots}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)(\log((a-c) + t(V(a)-V(c)) + o(t)) - \log(a-c) - t\cdot\frac{V(a)-V(c)}{a-c}\cdots}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)(\log(1+\frac{t(V(a)-V(c))}{a-c} + o(t)) - t\cdot\frac{V(a)-V(c)}{a-c}\cdots}{t||\lambda||_\infty}\right| \\
=&\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left|\frac{\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*)\cdot o(t)}{t||\lambda||_\infty}\right| \longrightarrow 0
\end{aligned}$$

since $\displaystyle\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g_{ij}^*) < \infty$ and $\left|\dfrac{o(t)}{t||\lambda||_\infty}\right| \to 0$ as $t\to 0$.

This completes the proof of Theorem 5.2.7. □

5.3 Differentiability at arbitrary point

Now we show the general case: the Liouville map $L:\mathcal{T}(X_0)\to\mathcal{H}_{\text{bd}}(X_0)$ is differentiable at any point in $\mathcal{T}(X_0)$.

Remark 5.3.1. To prove the general case, by formula (1.10) of Ahlfors in [2],

$$f^{\mu+t\lambda} = f^{p(\mu+t\lambda)} \circ f^{\mu} \tag{5.13}$$

where

$$p(\mu+t\lambda) = \left\{ \frac{t\lambda}{1-\bar{\mu}(\mu+t\lambda)} \left(\frac{f_z^{\mu}}{|f_z^{\mu}|} \right)^2 \right\} \circ (f^{\mu})^{-1}.$$

Let $\alpha_t^{\mu} = L([f^{\mu+t\lambda}])$ and let $\alpha_t' = L([f^{p(\mu+t\lambda)}])$. We look at the derivative

$$\frac{d}{dt}\int \xi \circ \varphi \, d\alpha_t^{\mu}\big|_{t=0} = \frac{d}{dt}\int \xi \circ \varphi \circ (f^{\mu})^{-1} \, d\alpha_t'\big|_{t=0}.$$

Notice that when $t = 0$, $f^{p(\mu+t\lambda)}$ is the identity. Therefore, the general case is reduced to the case for the basepoint proved in Theorem 5.2.1. In order to show that $\frac{d}{dt}\int \xi \circ \varphi \circ (f^{\mu})^{-1} \, d\alpha_t'$ converges uniformly for t small, we would like to have that for t small, the family of quasiconformal maps $f^{p(\mu+t\lambda)} \circ f^{\mu} \circ \varphi^{-1} \circ \psi_Q$ is uniformly Hölder continuous with exponent close to 1. However, this is clearly not the case here. We will take a different approach and choose a Möbius map $\varphi_1 : \mathbb{D} \to \mathbb{D}$ depending on φ such that $\varphi_1 \circ f^{\mu} \circ \varphi^{-1}$ is uniformly Hölder continuous.

Theorem 5.3.2. *Let $\alpha_t^{\mu} = L([f^{\mu+t\lambda}])$, where $[f^{\mu+t\lambda}]$ is a path in $\mathcal{T}(X_0)$ and λ is a harmonic Beltrami differential. Then L is differentiable at any $[f^{\mu}]$.*

Proof of Theorem 5.3.2.

Proceed similarly as in the proof of Theorem 5.2.1.

Assuming working in $\mathbb{D}$. By (5.13), $\alpha_t^{\mu} = (f^{\mu})_*^{-1}\alpha_t'$ is the pushforward of α_t' by $(f^{\mu})^{-1}$ satisfying $\alpha_t^{\mu}(A) = \alpha_t'(f^{\mu}(A))$ for any Borel set $A \subset G(\mathbb{D})$.

Suppose that $Supp(\xi) \subset Q = [a,b] \times [c,d]$, then $Supp(\xi \circ \varphi \circ (f^{\mu})^{-1}) = f^{\mu} \circ \varphi^{-1}(Q) = Q^{\mu}$. By definition of Möbius map, we can choose $\varphi_1 : \mathbb{D} \to \mathbb{D}$ such that $\varphi_1 \circ f^{\mu} \circ \varphi^{-1}$ fixes three vertices of Q, say it fixes b, c and d. To simplify the notation, let $f^{\mu_1} = \varphi_1 \circ f^{\mu} \circ \varphi^{-1}$. By

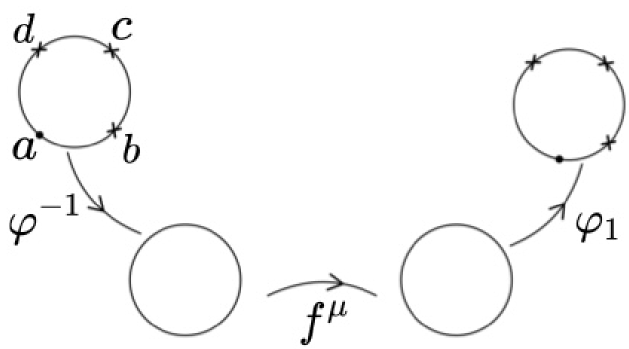


Figure 5.3: Choice of Möbius map φ_1

the choice of φ_1 and for fixed μ, the family of quasiconformal maps f^{μ_1} has the following properties:

i) Every map in the family fixes three vertices of box Q. ie. b, c and d.

ii) There exists a number $\delta > 0$ such that

$$k(f^{\mu_1}(i), f^{\mu_1}(j)) = k(i, j) > \delta$$

where k is the spherical metric and $i, j \in \{b, c, d\}$, $i \neq j$.

The family f^{μ_1} is equicontinuous and hence uniformly Hölder continuous as desired, see [13] for details. Let $f^{p_1(\mu+t\lambda)} = f^{p(\mu+t\lambda)} \circ \varphi_1^{-1}$. When $t = 0$, $f^{p_1(\mu+t\lambda)}$ is the identity. Set

$$\alpha_t'' = \varphi_{1*}\alpha_t' = L([f^{p(\mu+t\lambda)} \circ \varphi_1^{-1}]) = L([f^{p_1(\mu+t\lambda)}]). \tag{5.14}$$

Then we have

$$\begin{aligned}\int_{G(\mathbb{D})} \xi \circ \varphi \, d\alpha_t^{\mu} &= \int_{G(\mathbb{D})} \xi \circ \varphi \, d((f^{\mu})_*^{-1}\alpha_t') \\ &= \int_{G(\mathbb{D})} \xi \circ \varphi \circ (f^{\mu})^{-1} \, d\alpha_t' \\ &= \int_{G(\mathbb{D})} \xi \circ \varphi \circ (f^{\mu})^{-1} \circ \varphi_1^{-1} \circ \varphi_1 \, d\alpha_t' \\ &= \int_{G(\mathbb{D})} \xi \circ \varphi \circ (f^{\mu})^{-1} \circ \varphi_1^{-1} \, d(\varphi_{1*}\alpha_t') \\ &= \int_{G(\mathbb{D})} \xi \circ (f^{\mu_1})^{-1} \, d(\varphi_{1*}\alpha_t') \\ &= \int_{G(\mathbb{D})} \xi \circ (f^{\mu_1})^{-1} \, d\alpha_t''. \end{aligned} \tag{5.15}$$

where $\alpha_t'' = \varphi_{1*}\alpha_t'$ is the pushforward of α_t' by $\varphi_1 : \mathbb{D} \to \mathbb{D}$.

Consider $\xi' = \xi \circ (f^{\mu_1})^{-1}$. By Corollary 5.2.6, we have the base case

$$\begin{aligned}\frac{d}{dt}\int_{G(\mathbb{D})} \xi \circ \varphi \, d\alpha_t^{\mu}\big|_{t=0} &= \frac{d}{dt}\int_{G(\mathbb{D})} \xi' \, d\alpha_t''\big|_{t=0} \\ &\leqslant D \cdot ||\lambda||_{\infty}\end{aligned} \tag{5.16}$$

where D only depending on the Hölder exponent and constant of ξ', which only depending on the support, Hölder exponent and Hölder constant of ξ and the uniform Hölder exponent and uniform Hölder constant of f^{μ_1}.

The rest follows directly from Theorem 5.2.1. Hence, the Liouville map L is differentiable at any $[f^{\mu}]$ if it is differentiable at the base point $[id]$ of $\mathcal{T}[X_0]$.

This completes the proof of Theorem 5.3.2. □

We have shown the differentiability of the Liouville map L at any point in $\mathcal{T}(X_0)$. Now let us provide an explicit formula for the tangent map $\mathcal{T}_{[id]}L$.

Proposition 5.3.3. *The map $T_{[id]}L : T_{[id]}\mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$ at the base point $[id]$ is given by the formula*

$$T_{[id]}L([\lambda]_{tan})(\xi\circ\varphi) = \\ -\frac{2}{\pi}Re\int_{\mathbb{D}}\lambda(\varphi^{-1}(z))\frac{\overline{(\varphi^{-1})'(z)}}{(\varphi^{-1})'(z)}\left[\int_{G(\widetilde{X}_0)}\frac{\xi(\sigma,\tau)}{(z-\sigma)^2(z-\tau)^2}\,d\sigma d\tau\right]dxdy$$

where λ is a Beltrami differential representing a tangent vector at $[id]$, $\xi \in H(\widetilde{X}_0)$ and $\varphi \in Möb(\mathbb{D})$; where $z = x+iy \in \mathbb{D}$ and $g = (\sigma,\tau) \in G(\widetilde{X}_0)$.

Proof of Proposition 5.3.3.

Suppose $Supp(\xi) \subset Q = [a,b]\times[c,d]$ as in Proposition 5.2.5. Let $\psi_Q : \mathbb{H} \to \mathbb{D}$ be the Möbius map which maps the box $Q' = [-2,-1]\times[x',2]$ to Q where x' lies in a compact subset of the open interval $(-1,2)$. Let $Q^*_{ij} = [a_{i-1},a_i]\times[c_{j-1},c_j]$ be the subdivision of Q'.

Notice that

$$\int_{Q^*_{ij}}\frac{1}{(z-\sigma)^2(z-\tau)^2}\,d\sigma d\tau = \frac{(a_{i-1}-a_i)(c_{j-1}-c_j)}{(z-a_{i-1})(z-a_i)(z-c_{j-1})(z-c_j)}.$$

Then the integrand of expression (5.6) for t=0 can be written as

$$f_n(z) = \lambda''(z)\int_{Q'}\frac{\xi_n\circ\psi_Q(g')}{(z-\sigma)^2(z-\tau)^2}\,d\sigma d\tau = \lambda''(z)\int_{Q'}\frac{\xi_n\circ\psi_Q(\sigma,\tau)}{(z-\sigma)^2(z-\tau)^2}\,d\sigma d\tau$$

where $g' = (\sigma,\tau) \in Q'$ is an arbitrary geodesic. Recall that by the proof of Proposition 5.2.5, the series

$$\int_{\mathbb{H}}|f_1(z)|\,dxdy + \sum_{n=1}^{\infty}\int_{\mathbb{H}}|f_{n+1}(z)-f_n(z)|\,dxdy$$

converges. This implies that the series

$$|f_1(z)| + \sum_{n=1}^{\infty}|f_{n+1}(z)-f_n(z)| \tag{5.17}$$

converges almost everywhere, see Theorem 1.38 in [16] for details.

Let $g(z)$ denote the sum of the series (5.17).

Then it is an integrable function and $|f_n(z)| \leqslant g(z)$ for all n. Also note that $f_n(z)$ converges to

$$f(z) = \lambda''(z) \int_{Q'} \frac{\xi \circ \psi_Q(\sigma, \tau)}{(z-\sigma)^2 (z-\tau)^2} \, d\sigma d\tau$$

almost everywhere. Thus by Lebesgue's dominated convergence theorem, we have

$$\int_{\mathbb{H}} f_n(z) \, dxdy \longrightarrow \int_{\mathbb{H}} f(z) \, dxdy$$

as $n \to \infty$.

Define $K(z, a_{i-1}, a_i, c_{j-1}, c_j) = \dfrac{(a_{i-1} - a_i)(c_{j-1} - c_j)}{(z - a_{i-1})(z - a_i)(z - c_{j-1})(z - c_j)}$. Wolpert proved that K satisfies the following transformation law:

$$K(\psi(z), \psi(a_{i-1}), \psi(a_i), \psi(c_{j-1}), \psi(c_j))\psi'(z)^2 = K(z, a_{i-1}, a_i, c_{j-1}, c_j) \tag{5.18}$$

for any Möbius map ψ. Since by Lemma 5.2.2 $\lambda'' = \lambda' \circ \psi_Q(z) \frac{\overline{(\psi_Q)'(z)}}{(\psi_Q')(z)}$ and by expression (5.18), we have

$$f_n(z) = \lambda'(\psi_Q(z)) |\psi_Q'(z)|^2 \int_{Q'} \frac{\xi_n \circ \psi_Q(\sigma, \tau)}{(\psi_Q(z) - \psi_Q(\sigma))^2 (\psi_Q(z) - \psi_Q(\tau))^2} \, d\sigma d\tau.$$

Let $(\sigma', \tau') = \psi_Q(\sigma, \tau)$. And by a change of variable on $z' = \psi_Q(z)$, the above expression becomes

$$f_n(z) = \lambda'(\psi_Q(z))|\psi'_Q(z)|^2 \int_{Q'} \frac{\xi_n(\sigma',\tau')}{(\psi_Q(z)-\sigma')^2(\psi_Q(z)-\tau')^2}\, d\sigma' d\tau' \tag{5.19}$$
$$= \lambda'(z') \int_{Q} \frac{\xi_n(\sigma',\tau')}{(z'-\sigma')^2(z'-\tau')^2}\, d\sigma' d\tau'$$
$$= \lambda'(z') \int_{G(\widetilde{X}_0)} \frac{\xi_n(\sigma',\tau')}{(z'-\sigma')^2(z'-\tau')^2}\, d\sigma' d\tau'$$

where $|\psi'_Q(z)|^2$ is the Jacobian of $z' = \psi_Q(z)$. Note that the integral in (5.19) does not change if we integrate over $G(\widetilde{X}_0)$ because $Supp(\xi_n) \subset Q$.

Finally, by Lemma 5.2.2 we get

$$f_n(z) = \lambda(\varphi^{-1}(z'))\frac{\overline{(\varphi^{-1})'(z')}}{(\varphi^{-1})'(z')} \int_{G(\widetilde{X}_0)} \frac{\xi_n(\sigma',\tau')}{(z'-\sigma')^2(z'-\tau')^2}\, d\sigma' d\tau'.$$

Sending $n \to \infty$, we obtain the desired integral.

This completes the proof of Proposition 5.3.3. □

Remark 5.3.4. For the integral formula proved in Proposition 5.3.3, its convergence strongly depends on the fact that ξ is Hölder continuous. By the proof of the Proposition 5.2.5, we expect that if we replace ξ with a continuous function with compact support then the series

$$\int_{\mathbb{H}} |f_1(z)|\, dxdy + \sum_{n=1}^{\infty} \int_{\mathbb{H}} |f_{n+1}(z) - f_n(z)|\, dxdy$$

does not converge uniformly. Using the change of base point of $T(X_0)$ as described in Remark 5.3.1 and recall that $L^\mu\lambda$ is a Beltrami differential on $f^\mu(X_0)$ from expression (5.3), a similar formula can be obtained for the tangent map $\mathcal{T}_{[f^\mu]}L$. We state as an immediate corollary below.

Corollary 5.3.5. *The map $T_{[f^\mu]}L : T_{[f^\mu]}\mathcal{T}(X_0) \to \mathcal{H}_{\mathrm{bd}}(X_0)$ at the point $[f^\mu]$ is given by the* *formula*

$$T_{[f^\mu]}L([L^\mu\lambda]_{tan})(\xi\circ\varphi) = \\ -\frac{2}{\pi}Re\int_{\mathbb{D}} L^\mu\lambda(\varphi^{-1}(z))\frac{\overline{(\varphi^{-1})'(z)}}{(\varphi^{-1})'(z)}\left[\int_{G(\widetilde{X}_0)}\frac{\xi\circ(f^\mu)^{-1}(\sigma,\tau)}{(z-\sigma)^2(z-\tau)^2}\ d\sigma d\tau\right]dxdy$$

where λ is a harmonic Beltrami differential representing a tangent vector at $[id]$, $\xi \in H(\widetilde{X}_0)$ and $\varphi \in M\ddot{o}b(\mathbb{D})$; where $z = x + iy \in \mathbb{D}$ and $g = (\sigma,\tau) \in G(\widetilde{X}_0)$.

Proof of Corollary 5.3.5.

The proof is very similar to that of Proposition 5.3.3. □

We will finish this chapter by proving the continuity of $T_{[f]}L$ on the Teichmüller space $\mathcal{T}(X_0)$ with the following:

Theorem 5.3.6. *The tangent map $T_{[\cdot]}L : T_{[\cdot]}\mathcal{T}(X_0) \to \mathcal{H}_{\rm bd}(X_0)$ is varying continuously in $\mathcal{T}(X_0)$.*

Proof of Theorem 5.3.6.

By Theorems 5.2.1 and 5.3.2, it is enough to show that $T_{[id]}L$ is continuous.

Let λ be a harmonic Beltrami differential. Consider the paths $[f^{t\lambda}], [f^{\mu+t\lambda}] \in \mathcal{T}(X_0)$ at $[id], [f^\mu]$ respectively. It is equivalent to show the following statement:

For $[f^\mu]$ close to $[id]$, ie. for $||\mu||_\infty$ sufficiently small, we have

$$||T_{[id]}L([\lambda]_{tan}) - T_{[f^\mu]}L([L^\mu\lambda]_{tan})||_{\mathcal{H}}$$

is small and independent of λ as long as $||\lambda||_\infty$ is bounded.

Let $\alpha_t = L([f^{t\lambda}])$ and $\alpha_t^\mu = L([f^{\mu+t\lambda}])$. Let $\xi : G(\widetilde{X}_0) \to \mathbb{R}$ be a Hölder continuous function with compact support. Define the step function ξ_n as in the proof of Proposition

5.5. We showed that

$$\frac{d}{dt}\int \xi_n \circ \varphi d\alpha_t \longrightarrow \frac{d}{dt}\int \xi \circ \varphi d\alpha_t$$

and

$$\frac{d}{dt}\int \xi_n \circ \varphi d\alpha_t^{\mu} \longrightarrow \frac{d}{dt}\int \xi \circ \varphi d\alpha_t^{\mu}$$

uniformly for t small and independent of φ as $n \to \infty$.

Recall that from Theorem 5.2.7 we defined that the Liouville map can be approximated by $L_n([f])(\xi \circ \varphi) = \int \xi_n \circ \varphi d\alpha$, where $\alpha = L([f])$. Combining with the convergence above, we get

$$T_{[id]}L_n([\lambda]_{tan})(\xi \circ \varphi) \longrightarrow T_{[id]}L([\lambda]_{tan})(\xi \circ \varphi)$$

and

$$T_{[f^{\mu}]}L_n([L^{\mu}\lambda]_{tan})(\xi \circ \varphi) \longrightarrow T_{[f^{\mu}]}L([L^{\mu}\lambda]_{tan})(\xi \circ \varphi)$$

independent of φ as $n \to \infty$.

To simplify the notation, we will temporarily refer $T_{[id]}L([\lambda]_{tan})(\xi \circ \varphi)$ as $T_{[id]}L(\xi \circ \varphi)$ and $T_{[f^{\mu}]}L([L^{\mu}\lambda]_{tan})(\xi \circ \varphi)$ as $T_{[f^{\mu}]}L(\xi \circ \varphi)$. By triangle inequality, we have

$$\begin{aligned} |T_{[id]}L(\xi \circ \varphi) - T_{[f^{\mu}]}L(\xi \circ \varphi)| \leqslant \quad & |T_{[id]}L(\xi \circ \varphi) - |T_{[id]}L_n(\xi \circ \varphi)| && (5.20) \\ & + |T_{[id]}L_n(\xi \circ \varphi) - |T_{[f^{\mu}]}L_n(\xi \circ \varphi)| && (5.21) \\ & + |T_{[f^{\mu}]}L_n(\xi \circ \varphi) - |T_{[f^{\mu}]}L(\xi \circ \varphi)|. && (5.22) \end{aligned}$$

Note that for a given $\varepsilon > 0$, there exist n_1 and n_3 large such that expressions (5.20) $< \frac{\varepsilon}{3}$ and (5.22) $< \frac{\varepsilon}{3}$ by the convergence discussed above. It remains to show that there exists n_2 large such that expression (5.21) $< \frac{\varepsilon}{3}$.

Let $\psi_Q : \mathbb{H} \to \mathbb{D}$ and $\psi_t : \mathbb{D} \to \mathbb{H}$ be the Möbius maps defined in the proof of Theorem 5.1. Let $Q_{ij}^* = [a_{i-1}, a_i] \times [c_{j-1}, c_j]$ be the boxes defined in the proof of Proposition 5.2.5.

And recall that

$$\beta_t(Q_{ij}^*) = (\psi_Q^{-1} \circ \varphi)_* \alpha_t(Q_{ij}^*) = L(\psi_t \circ f^{t\lambda} \circ \varphi^{-1} \circ \psi_Q)(Q_{ij}^*) = L([f^{t\lambda''}])(Q_{ij}^*)$$

where $||t\lambda||_\infty = ||t\lambda''||_\infty$. And similarly

$$\beta_t^\mu(Q_{ij}^*) = (\psi_Q^{-1} \circ \varphi)_* \alpha_t^\mu(Q_{ij}^*) = L(\psi_t \circ f^{\mu+t\lambda} \circ \varphi^{-1} \circ \psi_Q)(Q_{ij}^*) = L([f^{\mu''+t\lambda''}])(Q_{ij}^*)$$

where $||\mu + t\lambda||_\infty = ||\mu'' + t\lambda''||_\infty$. Thus, expression (5.21) is equal to

$$\begin{aligned} &\left|\frac{d}{dt}\int \xi_n \circ \varphi d\alpha_t\big|_{t=0} - \frac{d}{dt}\int \xi_n \circ \varphi d\alpha_t^\mu\big|_{t=0}\right| \\ = &\left|\frac{d}{dt}\int \xi_n \circ \psi_Q d\beta_t\big|_{t=0} - \frac{d}{dt}\int \xi_n \circ \psi_Q d\beta_t^\mu\big|_{t=0}\right| \\ = &\sum_{i,j=1}^{2^n}\left|\xi \circ \psi_Q(g_{ij}^*)\right|\left|\frac{d}{dt}\beta_t(Q_{ij}^*)\big|_{t=0} - \frac{d}{dt}\beta_t^\mu(Q_{ij}^*)\big|_{t=0}\right|. \end{aligned} \tag{5.23}$$

Now we estimate

$$\left|\frac{d}{dt}\beta_t(Q_{ij}^*)\big|_{t=0} - \frac{d}{dt}\beta_t^\mu(Q_{ij}^*)\big|_{t=0}\right|. \tag{5.24}$$

Note that by definition we have

$$\frac{d}{dt}\beta_t(Q_{ij}^*)\big|_{t=0} = \frac{\frac{d}{dt}f^{t\lambda''}(a_{i-1})\big|_{t=0} - \frac{d}{dt}f^{t\lambda''}(c_{j-1})\big|_{t=0}}{a_{i-1} - c_{j-1}} + \cdots$$

and

$$\frac{d}{dt}\beta_t^\mu(Q_{ij}^*)\big|_{t=0} = \frac{\frac{d}{dt}f^{\mu''+t\lambda''}(a_{i-1})\big|_{t=0} - \frac{d}{dt}f^{\mu''+t\lambda''}(c_{j-1})\big|_{t=0}}{f^{\mu''}(a_{i-1}) - f^{\mu''}(c_{j-1})} + \cdots$$

which are three other corresponding differences; we omit them because the computations are similar. Thus, expression (5.24) is less than or equal to

$$\left|\frac{\frac{d}{dt}f^{t\lambda''}(a_{i-1})\big|_{t=0} - \frac{d}{dt}f^{t\lambda''}(c_{j-1})\big|_{t=0}}{a_{i-1}-c_{j-1}} - \frac{\frac{d}{dt}f^{\mu''+t\lambda''}(a_{i-1})\big|_{t=0} - \frac{d}{dt}f^{\mu''+t\lambda''}(c_{j-1})\big|_{t=0}}{f^{\mu''}(a_{i-1})-f^{\mu''}(c_{j-1})}\right| + \cdots$$

Since $f^{\mu+t\lambda}$ depends analytically on μ and $t\lambda$, see [3] for details; we have

$$\left|\frac{d}{dt}f^{t\lambda''}(a_{i-1})\big|_{t=0} - \frac{d}{dt}f^{\mu''+t\lambda''}(a_{i-1})\big|_{t=0}\right|$$

is small for $||\mu''||_\infty$ small and t small. Similarly

$$\left|a_{i-1} - f^{\mu''}(a_{i-1})\right|$$

is small for $||\mu''||_\infty$ small.

Thus, expression (5.24) is small for $||\mu||_\infty$ small and t small. Since $\left|\sum_{i,j=1}^{2^n}\xi\circ\psi_Q(g^*_{ij})\right| < \infty$, by choosing a sufficiently large n_2 we can make expression (5.23) arbitrarily small, which in turn makes expression (5.21) arbitrarily small. Finally, choosing $n = \max\{n_1, n_2, n_3\}$ and $||\mu||_\infty$ sufficiently small; we get

$$|T_{[id]}L(\xi\circ\varphi) - T_{[f^\mu]}L(\xi\circ\varphi)| < \varepsilon.$$

This completes the proof of Theorem 5.3.6. □

Chapter 6

Earthquake Paths

6.1 Introduction

In this chapter, we assume our work in the unit disk model and continue working on the differentiability of the Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$. First, we recall the definitions of the geodesic lamination and measured geodesic lamination. Next we define the earthquake maps and provide few known results. In particular, an earthquake map can be considered as an element in $\mathcal{T}(X_0)$. Then an earthquake path gives rise to a path in $\mathcal{T}(X_0)$. We previously showed that the derivative of this path exists and belongs to $\mathcal{H}_{\text{bd}}(X_0)$. And we are interested in finding an explicit formula of the derivative of a general earthquake path under the Liouville map L.

6.2 Earthquake maps

A *geodesic lamination* $\Lambda \subset \mathbb{H}$ is a closed set that is a union of mutually disjoint geodesics referred to as *leaves*. However, two leaves may have a common ideal end point. If Λ is invariant under a Fuchsian group Γ, it projects to a lamination on the surface $X \cong \mathbb{H}/\Gamma$.

The components of the complement of a lamination are its *gaps*. The gaps together with the leaves of a lamination, are its *strata*.

To introduce an elementary earthquake map, consider the upper-half plane model and a line ℓ which we may take to be the positive imaginary axis. Denote the left and right quadrants determined by ℓ by A and B; A and B have orientations inherited from $\mathbb{C}$. From the point of view of A, a *left earthquake* with a single leaf ℓ is a discontinuous map which fixes A pointwise, and in B is an isometry moving B to the left with respect to A, that is, it moves B in the positive direction with respect to the positive orientation of ∂A. In B it therefore has the form $z \mapsto kz, k > 1$. It is uniquely determined once the displacement $\log k$ along ℓ is dictated. Similarly, if we require B to be fixed, the left earthquake along ℓ moves A to the left from the point of view of B. In A it has the form $z \mapsto k^{-1}z$. The boundary maps of the two choices of the normalization differ by post-composition with $z \mapsto k^2 z$ and therefore are Teichmüller equivalent.

Next suppose that we have a lamination with finite number of leaves. Fix a gap σ. Let μ be a positive transverse measure: that is, to each leaf of the lamination is assigned a positive number as atomic measure. Normalize the earthquake to be identity on σ. A transverse geodesic to σ will cross a number of leaves. Carry out a sequence of left earthquakes in sequence along the various leaves, using the displacement assigned by μ.

Now we give a formal definition of a general earthquake map. Suppose that $\Lambda \in \mathbb{H}$ is a geodesic lamination. A *left earthquake map* is a possibly discontinuous injective and surjective map $E : \mathbb{H} \to \mathbb{H}$ which is an isometry on each stratum of Λ. The map E must satisfy the condition that for any two strata $X \neq Y$ of Λ, the comparison isometry,

$$cmp(X, Y) = (E|_X)^{-1} \circ (E|_Y) : \mathbb{H} \to \mathbb{H}$$

is a hyperbolic transformation whose axis ℓ weakly separates X and Y and which translates

Y to the left, as viewed from X. Here, $(E|_X)$ and $(E|_Y)$ refer to the isometries of the entire hyperbolic plane which agree with the restrictions of E to the given strata. A line ℓ weakly separates two sets X and Y if any path connecting a point $x \in X$ to a point $y \in Y$ intersects ℓ. Translating to the left means that the direction of the translation along ℓ must agree with the orientation induced from the component of X in $\mathbb{H} \setminus \ell$. The case that if one of the two strata is a line contained in the closure of the other, then the comparison isometry is permitted to be identity.

A left earthquake maps Λ to another lamination Λ'; namely it sends the strata of Λ to the strata of Λ'. The inverse of a left earthquake is a right earthquake. If Λ has a finite number of leaves, Thurston proved that left earthquake maps with finite laminations are dense in the set of all left earthquake maps, in the topology of uniform convergence on compact sets. We will call such left earthquakes which have finite laminations *simple* left earthquakes. In addition, an earthquake map is not necessarily a homeomorphism of $\mathbb{H}$. Thurston also showed that an earthquake map continuously extends to an orientation preserving homeomorphism of the boundary $\partial\mathbb{H} = \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ ("Quake at infinity"). Conversely, any orientation preserving homeomorphism of $\partial\mathbb{H}$ can be otained by continuous extension of an earthquake map of $\mathbb{H}$ ("Geology is transitive"), for more details see [20].

A left earthquake map between two Riemann surfaces is an injective, surjective map which lifts to a left earthquake map on $\mathbb{H}$. In particular, Λ is invariant under the covering transformations. However, if one or more leaves of Λ project to simple geodesics, lifts are determined only up to Dehn twists along the geodesics. To avoid this ambiguity one can associate the earthquake map with the homotopy type of a homeomorphism between the surfaces.

A *measured lamination* $\widehat{\Lambda} = (\Lambda, \mu)$ on $\mathbb{H}$ is given by the support geodesic lamination Λ together with a transverse measure μ to Λ. Explicitly, we assign a nonnegative Borel measure to each transverse arc whose support is on the transverse intersection of the arc with Λ. Furthermore, the measure μ is invariant under the homotopies of the arc respecting the lamination Λ. Any two earthquake maps corresponding to the same $\widehat{\Lambda}$ have isometric images. And if Λ is invariant under a Fuchsian group, so is μ.

A measured lamination $\widehat{\Lambda} = (\Lambda, \mu)$ is bounded if there is a constant C such that $\mu(I) < C$ for all transverse arc I of unit length. We say that an earthquake map is bounded if its corresponding measured lamination is bounded. It is known that the boundary values on $\partial\mathbb{H}$ of bounded earthquake maps are quasisymmetric homeomorphisms, see Theorem 1 in [19]. This means that their boundary values have quasiconformal extensions to $\mathbb{H}$. In the other direction, the boundary values of a quasiconformal map $\mathbb{H} \to \mathbb{H}$ is also the boundary values of a bounded left earthquake map due to Thurston.

Note that above definitions and results carry over to the unit disk model. Recall that $X_0 \cong \mathbb{D}/\Gamma$, where Γ is a fuchsian group. Consider a quasiconformal map $f : X_0 \to X$ whose lift is $\tilde{f} : \mathbb{D} \to \mathbb{D}$. By the preceding paragraph, there is a measured lamination $\widehat{\Lambda} = (\Lambda, \mu)$ and a bounded left earthquake map $E^\mu : X_0 \to X$ where the homotopy class of f is the same as that of E. Conversely, every bounded earthquake map is homotopic to a quasiconformal map. This allows us to consider earthquake maps as elements of the Teichmüller space $\mathcal{T}(X_0)$. Thurston's earthquake theorem stated that for any two points $[f_1]$ and $[f_2]$ in $\mathcal{T}(X_0)$ represented by quasiconformal maps $f_1 : X_0 \to X_1$ and $f_2 : X_0 \to X_2$ respectively, there is a unique left earthquake map from $[f_1]$ to $[f_2]$. To be specific, there exists a unique measured lamination $\widehat{\Lambda} = (\Lambda, \mu)$ such that $[f_2] = E^\mu[f_1]$.

For a bounded measured lamination $\widehat{\Lambda} = (\Lambda, \mu)$, we can consider E^μ as a map from $\mathcal{T}(X_0)$ to itself defined by $E^\mu([f]) \mapsto E^{f_*(\mu)}$ for any $[f] \in \mathcal{T}(X_0)$ represented by a quasiconformal map $f : X_0 \to X$. Multiplying μ with a parameter $t > 0$, $E^{t\mu}$ form an earthquake path $t \mapsto E^{t\mu}([f])$ in $\mathcal{T}(X_0)$. In Chapter 5, we proved the differentiability of the Liouville map L with respect to a path in $\mathcal{T}(X_0)$. Since $E^{t\mu}$ is a differentiable path in $\mathcal{T}(X_0)$, so is $L([E^{t\mu}])$ in $\mathcal{H}_{\mathrm{bd}}(X_0)$. We would like to formulate $\frac{d}{dt}L([E^{t\mu}])$.

We will begin with an elementary earthquake map $E_g^t : X_0 \to X$ along a single supporting geodesic g whose lift is $\tilde{E}_g^t : \mathbb{D} \to \mathbb{D}$. Let $\psi : \mathbb{D} \to \mathbb{H}$ be a Möbius map sends g to the geodesic $(0,\infty)$. Then the map $\psi \circ \tilde{E}_g^t \circ \psi^{-1}(x) = \tilde{E}_{\psi(g)}^t(x) = \begin{cases} x \text{ if } x \leqslant 0 \\ e^t x \text{ if } x > 0 \end{cases}$ is an elementary earthquake map on $\mathbb{H}$.

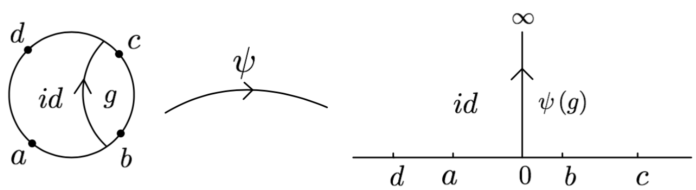


Figure 6.1: Elementary earthquake map on $\mathbb{H}$

Recall that in the upper half plane model the Liouville measure $L_{\mathbb{H}}$ on $G(\mathbb{H})$ is given by the infinitesimal form $\frac{dxdy}{|x-y|^2}$, see [7] for details. By definition of ψ, the Liouville measure $L_{\mathbb{D}}(A) = L_{\mathbb{H}}(\psi(A))$ for any measurable set $A \subset G(\mathbb{D})$.

Let $h(x,y)$ denote the geodesic joining x and y in $\mathbb{R}$. Let $\alpha_t = L([E_g^{t\delta}])$ where δ is the weight on the Dirac measure with support g and $E_g^{t\delta}$ is the elementary earthquake map as described above.

THEOREM C of Chapter 1 consists of the following: Proposition 6.2.1, Lemma 6.2.6, Corollary 6.2.7, Proposition 6.2.8 and Remark 6.2.9. We start by proving the following.

Proposition 6.2.1. *Let $\xi \in H(\widetilde{X}_0)$. Suppose that Supp$(\xi) \subset Q = [a,b] \times [c,d]$ for a single box, and the endpoints of g are in $[a,b]$, $[c,d]$ respectively. Then*

$$\frac{d}{dt} \int_{G(\mathbb{D})} \xi d\alpha_t \Big|_{t=0} = \delta \int_{G(\mathbb{D})} \xi(h) \cos(h,g) dL_{\mathbb{D}}(h).$$

Remark 6.2.2. Notice that above formulation does not depend on a map $\varphi \in$ Möb$(\mathbb{D})$ as in Chapter 5. Recall that from Theorem 5.2.1 and 5.2.7 we have already showed that the

Liouville map $L : \mathcal{T}(X_0) \to \mathcal{H}_{\text{bd}}(X_0)$ is differentiable and the derivative exists uniformly for all $\varphi \in \text{Möb}(\mathbb{D})$. Hence, we will not repeat the part of proof which deals with φ since uniformity is not a concern here.

Before we start the proof, notice that

$$\begin{aligned}
\frac{d}{dt}\int_{G(\mathbb{D})} \xi d\alpha_t\Big|_{t=0} &= \frac{d}{dt}\int_{G(\mathbb{D})} \xi dL([E_g^{t\delta}])\Big|_{t=0} \\
&= \frac{d}{dt}\int_{G(\mathbb{D})} \xi \circ (\tilde{E}_g^{t\delta})^{-1} dL_{\mathbb{D}}\Big|_{t=0} \\
&= \frac{d}{dt}\int_{G(\mathbb{H})} \xi \circ \psi^{-1} \circ \psi \circ (\tilde{E}_g^{t\delta})^{-1} \circ \psi^{-1} dL_{\mathbb{H}}\Big|_{t=0} \\
&= \frac{d}{dt}\int_{G(\mathbb{H})} \xi \circ \psi^{-1} \circ (\tilde{E}_{\psi(g)}^{t\delta})^{-1} dL_{\mathbb{H}}\Big|_{t=0} \\
&= \frac{d}{dt}\int_{G(\mathbb{H})} \xi \circ \psi^{-1} \circ (\tilde{E}_{\psi(g)}^{t\delta})^{-1} \frac{dxdy}{(x-y)^2}\Bigg|_{t=0}
\end{aligned} \tag{6.1}$$

Since $(\tilde{E}_{\psi(g)}^{t\delta})^{-1}$ in expression (6.1) can be expressed explicitly, thus there are four cases of $h(x, y) \in G(\mathbb{H})$ to consider as the following:

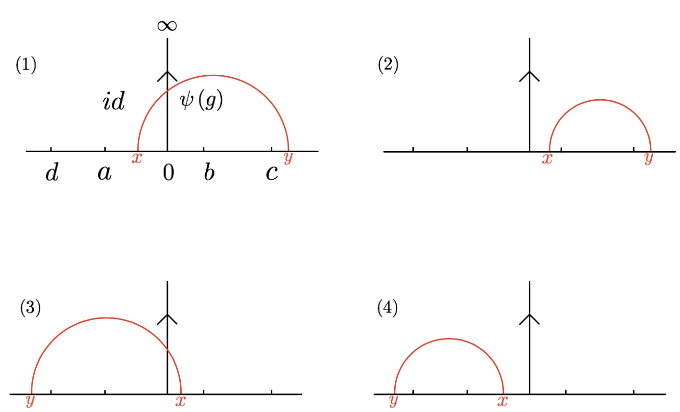


Figure 6.2: Positions of geodesic h

This allows us to define mutually disjoint sets,

$$U_1 = [a, 0] \times [c, \infty)$$

$$U_2 = [0, b] \times [c, \infty)$$

$$U_3 = [0, b] \times (-\infty, d]$$

$$U_4 = [a, 0] \times (-\infty, d]$$

Now we are ready to prove Prop 6.2.1.

Proof of Proposition 6.2.1.

By above discussion, we have

$$\begin{aligned}
\frac{d}{dt}\int_{G(\mathbb{D})} \xi d\alpha_t\Big|_{t=0} &= \frac{d}{dt}\int_{G(\mathbb{H})} \xi \circ \psi^{-1} \circ (\tilde{E}^{t\delta}_{\psi(g)})^{-1} \frac{dxdy}{(x-y)^2}\Big|_{t=0} \\
&= \frac{d}{dt}\int_{U_1} \xi \circ \psi^{-1}(h(x, e^{-t\delta}y)) \frac{dxdy}{(x-y)^2}\Big|_{t=0} \qquad (6.2) \\
&\quad + \frac{d}{dt}\int_{U_2} \xi \circ \psi^{-1}(h(e^{-t\delta}x, e^{-t\delta}y)) \frac{dxdy}{(x-y)^2}\Big|_{t=0} \qquad (6.3) \\
&\quad + \frac{d}{dt}\int_{U_3} \xi \circ \psi^{-1}(h(e^{-t\delta}x, y)) \frac{dxdy}{(x-y)^2}\Big|_{t=0} \qquad (6.4) \\
&\quad + \frac{d}{dt}\int_{U_4} \xi \circ \psi^{-1}(h(x, y)) \frac{dxdy}{(x-y)^2}\Big|_{t=0} \qquad (6.5)
\end{aligned}$$

To proceed, we will need two additional lemmas. One is for differentiation under the integral sign, and the other is for angle between intersecting geodesics.

Lemma 6.2.3. *Let X be an open subset of $\mathbb{R}$ and Ω be a measure space. Suppose $f : X \times \Omega \to \mathbb{R}$ satisfies the following conditions:*

(i) $f(x, \omega)$ is a Lebesgue integrable function of ω for each $x \in X$.

(ii) For almost all $\omega \in \Omega$, the derivative $\frac{\partial f(x,\omega)}{\partial x}$ exists for all $x \in X$.

(iii) There is an integrable function $\Theta : \Omega \to \mathbb{R}$ such that $\left|\frac{\partial f(x,\omega)}{\partial x}\right| \leqslant \Theta(\omega)$ for all $x \in X$.

Then for all $x \in X$, $\frac{d}{dx}\int_\Omega f(x,\omega)d\omega = \int_\Omega \frac{\partial}{\partial x} f(x,\omega)d\omega$.

Proof of Lemma 6.2.3.

Consider any sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} x_n = x$, but $x_n \neq x$ for each x.

Let $F(x) = \int_\Omega f(x,\omega)d\omega$. Then we have

$$\frac{F(x_n) - F(x)}{x_n - x} = \int_\Omega \frac{f(x_n,\omega) - f(x,\omega)}{x_n - x} d\omega = \int_\Omega f_n(\omega)d\omega$$

where $f_n(\omega) = \frac{f(x_n,\omega)-f(x,\omega)}{x_n - x}$.

Notice that:

by (i) we have $\frac{d}{dx}\int_\Omega f(x,\omega)d\omega = \lim_{n\to\infty} \frac{F(x_n) - F(x)}{x_n - x} = \lim_{n\to\infty}\int_\Omega f_n(\omega)d\omega$,

by (ii) we have $\lim_{n\to\infty} f_n(\omega) = \frac{\partial}{\partial x} f(x,\omega)$,

by the mean value theorem and (iii) we have

$|f_n(\omega)| = \left|\frac{f(x_n,\omega) - f(x,\omega)}{x_n - x}\right| = \left|\frac{\partial}{\partial x} f(c,\omega)\right| \leqslant \Theta(\omega)$.

Thus by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\frac{d}{dx}\int_\Omega f(x,\omega)d\omega &= \lim_{n\to\infty}\int_\Omega f_n(\omega)d\omega \\ &= \int_\Omega \lim_{n\to\infty} f_n(\omega)d\omega \\ &= \int_\Omega \frac{\partial}{\partial x} f(x,\omega)d\omega\end{aligned}$$

This proves Lemma 6.2.3. □

Lemma 6.2.4. *Let* $L_1 = (z_1, z_2)$ *and* $L_2 = (w_1, w_2)$ *denote the geodesics with the endpoints occurring in the order* z_1, w_1, z_2, w_2 *around the boundary at infinity, then the angle between* L_1 *and* L_2 *satisfies*

$$(z_1, w_1, z_2, w_2)\sin^2\frac{\theta}{2} = 1$$

Proof of Lemma 6.2.4.

Consider the unit disk model, there exists a Möbious map which maps $L_1 = (z_1, z_2)$ to $L_1' = (-1, 1)$ and $L_2 = (w_1, w_2)$ to $L_2' = (e^{i\theta}, -e^{i\theta})$ respectively. In particular, L_1' and L_2' are the diameters of the unit circle. Then we have,

$$\begin{aligned}
(-1, e^{i\theta}, 1, -e^{i\theta}) &= \frac{(-1-1)(e^{i\theta}+e^{i\theta})}{(-1+e^{i\theta})(e^{i\theta}-1)} \\
&= \frac{-4e^{i\theta}}{e^{2i\theta}-2e^{i\theta}+1} \\
&= \frac{-4(\cos\theta + i\sin\theta)}{\cos 2\theta + i\sin 2\theta - 2(\cos\theta + i\sin\theta) + 1} \\
&= \frac{-4(\cos\theta + i\sin\theta)}{\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta - 2\cos\theta - 2i\sin\theta + 1} \\
&= \frac{-4(\cos\theta + i\sin\theta)}{2\cos^2\theta + 2i\sin\theta\cos\theta - 2\cos\theta - 2i\sin\theta} \\
&= \frac{-4(\cos\theta + i\sin\theta)}{2(\cos\theta + i\sin\theta)(\cos\theta - 1)} \\
&= \frac{2}{1-\cos\theta} = \frac{1}{\sin^2\frac{\theta}{2}}
\end{aligned}$$

This proves Lemma 6.2.4. □

Now we can compute expression (6.2),

$$\begin{aligned}
&\frac{d}{dt}\int_{U_1} \xi\circ\psi^{-1}(h(x, e^{-t\delta}y))\frac{dxdy}{(x-y)^2}\bigg|_{t=0} \\
&= \frac{d}{dt}\int_{U_1} \xi\circ\psi^{-1}(h(x', y'))\frac{e^{t\delta}dx'dy'}{(x'-e^{t\delta}y')^2}\bigg|_{t=0} \quad \text{by change of variable} \\
&= \int_{U_1} \xi\circ\psi^{-1}(h(x', y'))\frac{\delta(x'+y')}{(x'-y')}\frac{dx'dy'}{(x'-y')^2} \quad \text{by Lemma 6.2.3} \\
&= \int_{U_1} \xi\circ\psi^{-1}(h)\delta\cos(h, \psi(g))dL_{\mathbb{H}} \quad \text{by Lemma 6.2.4}
\end{aligned}$$

In details,

i) Let $x' = x$ and $y' = e^{-t\delta}y$, then $dx' = dx$ and $dy' = e^{-t\delta}y$. And

$$\frac{d}{dt}\frac{e^{t\delta}}{(x'-e^{t\delta}y')^2}\bigg|_{t=0} = \frac{\delta(x'+y')}{(x'-y')^3}.$$

ii) The endpoints of h and ψ_g occur in the order of $x', 0, y', \infty$. Thus we have

$$\begin{aligned}(x',0,y',\infty) &= \frac{(x'-y')(0-\infty)}{(x'-\infty)(0-y')} \\ &= \frac{x'-y'}{-y'} = \frac{2}{1-\cos(h,\psi(g))}\end{aligned}$$

A simple computation shows that $\cos(h,\psi(g)) = \dfrac{x'+y'}{x'-y'}$.

For expression (6.3),

$$\begin{aligned}&\frac{d}{dt}\int_{U_2}\xi\circ\psi^{-1}(h(e^{-t\delta}x, e^{-t\delta}y))\frac{dxdy}{(x-y)^2}\bigg|_{t=0} \\ &= \frac{d}{dt}\int_{U_2}\xi\circ\psi^{-1}(h(x',y'))\frac{e^{2t\delta}dx'dy'}{(e^{t\delta}x'-e^{t\delta}y')^2}\bigg|_{t=0} \\ &= \frac{d}{dt}\int_{U_2}\xi\circ\psi^{-1}(h(x',y'))\frac{e^{2t\delta}dx'dy'}{e^{2t\delta}(x'-y')^2} \\ &= \frac{d}{dt}\int_{U_2}\xi\circ\psi^{-1}(h)dL_{\mathbb{H}} = 0\end{aligned}$$

For expression (6.4),

$$\begin{aligned}&\frac{d}{dt}\int_{U_3}\xi\circ\psi^{-1}(h(e^{-t\delta}x, y))\frac{dxdy}{(x-y)^2}\bigg|_{t=0} \\ &= \frac{d}{dt}\int_{U_3}\xi\circ\psi^{-1}(h(x',y'))\frac{e^{t\delta}dx'dy'}{(e^{t\delta}x'-y')^2}\bigg|_{t=0} \\ &= \int_{U_3}\xi\circ\psi^{-1}(h(x',y'))\frac{\delta(-x'-y')}{(x'-y')}\frac{dx'dy'}{(x'-y')^2} \\ &= \int_{U_3}\xi\circ\psi^{-1}(h)\delta\cos(h,\psi(g))dL_{\mathbb{H}}\end{aligned}$$

A similar computation shows that

$$\frac{d}{dt}\frac{e^{t\delta}}{(e^{t\delta}x'-y')^2}\bigg|_{t=0} = \frac{\delta(-x'-y')}{(x'-y')^3} \text{ and } \cos(h,\psi(g)) = \frac{-x'-y'}{x'-y'}.$$

Finally for expression (6.5),

$$\begin{aligned}&\frac{d}{dt}\int_{U_4} \xi\circ\psi^{-1}(h(x,y))\frac{dxdy}{(x-y)^2}\bigg|_{t=0}\\ &= \frac{d}{dt}\int_{U_4}\xi\circ\psi^{-1}(h)dL_{\mathbb{H}} = 0\end{aligned}$$

From these four estimates, we conclude that:

$$\begin{aligned}\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\big|_{t=0} &= \int_{G(\mathbb{H})}\xi\circ\psi^{-1}(h)\delta\cos(h,\psi(g))dL_{\mathbb{H}}\\ &= \delta\int_{G(\mathbb{D})}\xi(h')\cos(h',g)dL_{\mathbb{D}}(h') \quad \text{by definition of } \psi\end{aligned}$$

where $h' = \psi^{-1}(h)$.

This completes the proof of Proposition 6.2.1. □

Remark 6.2.5. For any other possible positions of supporting geodesic g ignoring the orientation, we have the following cases:

(1)

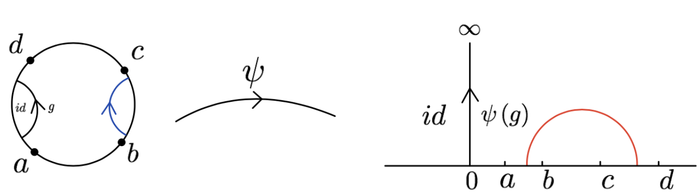


Figure 6.3: Position (1) of geodesic g

$\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\big|_{t=0} = 0$ with similar arguments as (6.3).

(2)

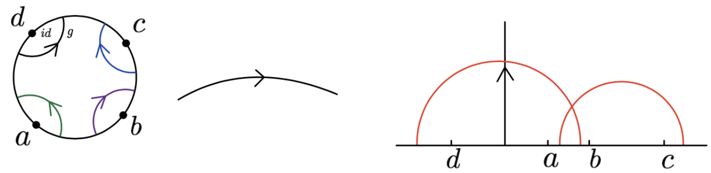


Figure 6.4: Position (2) of geodesic g

$$\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\Big|_{t=0}=\int_{G(\mathbb{D})}\xi(h')\delta\cos(h',g)dL_{\mathbb{D}}(h') \text{ with similar arguments as (6.3, 6.4).}$$

(3)

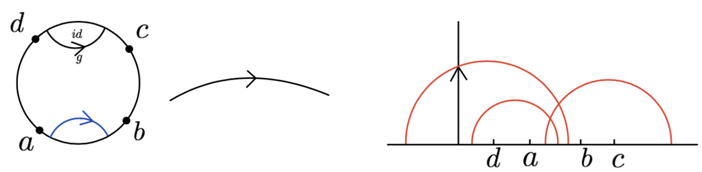


Figure 6.5: Position (3) of geodesic g

$$\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\Big|_{t=0}=\int_{G(\mathbb{D})}\xi(h')\delta\cos(h',g)dL_{\mathbb{D}}(h') \text{ with similar arguments as (6.3, 6.4).}$$

(4)

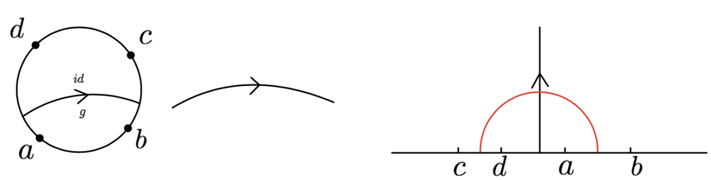


Figure 6.6: Position (4) of geodesic g

$$\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\Big|_{t=0}=\int_{G(\mathbb{D})}\xi(h')\delta\cos(h',g)dL_{\mathbb{D}}(h') \text{ with similar arguments as (6.4).}$$

Notice that from above computations, we see that only those geodesics which intersects the supporting geodesic g contribute to the derivative.

Next we consider the simple earthquake map which can be constructed from elementary earthquakes as follows: Let δ be the Dirac measure with finite support $\{g_1, g_2, \cdots, g_n\}$. We define $E^\delta = E_{g_1}^{d_1} \circ E_{g_2}^{d_2} \circ \cdots \circ E_{g_n}^{d_n}$, where $\delta = \sum_{i=1}^{n} d_i \mathbb{1}_{g_i}$ for $i = 1, 2, \cdots, n$. Note that the elementary earthquakes $E_{g_i}^{d_i}$ commute since g_i are disjoint. To compute the derivative with respect to a simple earthquake map, we need the infinitesimal Teichmüller theory on Zygmund functions, see [10] for more details. Since an earthquake map represents an element in $\mathcal{T}(X_0)$, we have the following lemma.

Lemma 6.2.6. *Let $f_t(x) = x + tV_f(x) + o(t)$ and $g_t(x) = x + tV_g(x) + o(t)$, where $\frac{o(t)}{t} \to 0$ as $t \to 0$ and V is a Zygmund function. Then $f_t(g_t(x)) = x + t(V_f(x) + V_g(X)) + o(t)$.*

Proof of Lemma 6.2.6.

By definition, we have

$$\begin{aligned} f_t(g_t(x)) &= g_t(x) + tV_f(g_t(x)) + o(t) \\ &= (x + tV_g(x) + o(t)) + tV_f(x + tV_g(x) + o(t)) + o(t) \\ &= (x + tV_g(x) + o(t)) + t[V_f(x) + V_f(x + tV_g(x) + o(t)) - V_f(x)] + o(t) \\ &= x + tV_g(x) + tV_f(x) + t[V_f(x + tV_g(x) + o(t)) - V_f(x)] + o(t) \qquad (6.6) \end{aligned}$$

It remains to show that $t[V_f(x + tV_g(x) + o(t)) - V_f(x)] = o(t)$. To see this, we need Theorem 7 on page 56 in [10], that is

$$\begin{aligned} V_f(x + tV_g(x) + o(t)) - V_f(x) &\leqslant |V_f(x + tV_g(x) + o(t)) - V_f(x)| \\ &= C|x + tV_g(x) + o(t) - x| \log|x + tV_g(x) + o(t) - x|^{-1} \\ &= C|tV_g(x) + o(t)| \log|tV_g(x) + o(t)|^{-1} \to 0 \text{ as } t \to 0 \end{aligned}$$

Thus,

$$t[C|tV_g(x)+o(t)|\log|tV_g(x)+o(t)|^{-1}]=o(t)$$

which implies that

$$t[V_f(x+tV_g(x)+o(t))-V_f(x)]=o(t).$$

And hence expression (6.6) is equal to $x+t[V_g(x)+V_f(x)]+o(t)$.

This proves Lemma 6.2.6. □

There is an immediate corollary as the following.

Corollary 6.2.7. $\frac{d}{dt}\int_{G(\mathbb{D})}\xi dL([E^{t\delta}])\big|_{t=0}=\sum_{i=1}^{n}\frac{d}{dt}\int_{G(\mathbb{D})}\xi dL([E^{td_i}_{g_i}])\big|_{t=0}$

Proof of Corollary 6.2.7.

In Theorem 5.2.7, we showed that the tangent map $T_{[id]}L: T_{[id]}\mathcal{T}(X_0)\to\mathcal{H}_{\mathrm{bd}}(X_0)$ is linear.

Denote $V=\frac{d}{dt}E^{t\delta}\big|_{t=0}$ and $V_i=\frac{d}{dt}E^{td_i}_{g_i}\big|_{t=0}$ as tangent vectors of $\mathcal{T}(X_0)$. By Lemma 6.2.6, we have $V=\sum_{i=1}^{n}V_i$. Linearity of $T_{[id]}L$ implies that

$$T_{[id]}L(V)=\sum_{i=1}^{n}T_{[id]}L(V_i)$$

□

Proposition 6.2.8. *Let* $\alpha_t=L([E^{t\delta}])$, $\delta=\sum_{i=1}^{n}d_i\mathbb{1}_{g_i}$ *and let* $\xi\in H(\widetilde{X}_0)$. *Suppose that* $Supp(\xi)\subset Q=[a,b]\times[c,d]$ *for a single box, and the endpoints of* g_i *are in* $[a,b]$, $[c,d]$ *respectively. Then*

$$\frac{d}{dt}\int_{G(\mathbb{D})}\xi d\alpha_t\big|_{t=0}=\sum_{i=1}^{n}d_i\int_{G(\mathbb{D})}\xi(h)\cos(h,g_i)dL_{\mathbb{D}}(h).$$

Proof of Proposition 6.2.8.

Apply Lemma 6.2.6, Corollary 6.2.7 and Proposition 6.2.1.

This completes the proof of Proposition 6.2.8. □

Remark 6.2.9. Consider $Supp(\xi) = K$ which is compact. Cover K by finitely many boxes $Q_1, ..., Q_m$. Using partition of unity defined in Chapter 4, $Supp(\rho_j\xi) \subset Q_j$ for each j. Applying Proposition 6.2.8, we get

$$\begin{aligned}\frac{d}{dt}\int_{G(\mathbb{D})} \xi d\alpha_t\Big|_{t=0} &= \sum_{j=1}^{m} \frac{d}{dt}\int_{G(\mathbb{D})} \rho_j\xi d\alpha_t\Big|_{t=0} \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} d_i \int_{G(\mathbb{D})} \rho_j\xi(h)\cos(h, g_i) dL_{\mathbb{D}}(h) \\ &= \sum_{i=1}^{n} d_i \int_{G(\mathbb{D})} \xi(h)\cos(h, g_i) dL_{\mathbb{D}}(h).\end{aligned}$$

Let $\widehat{\Lambda} = (\Lambda, \mu)$ be a bounded geodesic lamination. In the general earthquake case, we approximate μ by Dirac measure δ as in the simple earthquake case above, in the sense that $\delta \to \mu$ for the weak* topology. The boundedness of μ is used to show that the limit really exists, see [20], [9] and [11] for details. We define $\mu(I) = \lim_{n\to\infty} \sum_{i=1}^{n} d_i \mathbb{1}_{g_i}$, where I is a closed hyperbolic arc transverse to geodesics in Λ.

First, we consider a very special case of Λ to gain some intuition. Let $\xi \in H(\widetilde{X}_0)$ with $Supp(\xi) \subset Q = [a, b] \times [c, d]$ for a single box. Suppose that all supporting geodesics of Λ have end points contained in the interval $[a, b]$ and accumulates on the boundary. Further suppose the distance between geodesics g_i and g_{i+1} is 1.

There is a Möbius map $\psi_i : \mathbb{D} \to \mathbb{H}$ such that $a \mapsto -1, b \mapsto 1, c \mapsto c', d \mapsto d'$ and the geodesic h maps to $(-e^{-i}, \infty)$. By our assumption, geodesic g_i maps to $(-e^{-i}, e^{-i})$. If we consider those geodesics h only intersecting $g_1, g_2, \cdots$ one at a time. We obtain new boxes Q_i for $i = 1, 2, \cdots$. Moving to the upper half plane, we obtain boxes $\psi_i(Q_i) = [-e^{-i}, e^{-i}] \times [c', d']$ for $i = 1, 2, \cdots$, see the diagram below.

Then, boxes $\psi_i(Q_i)$ have the following properties:

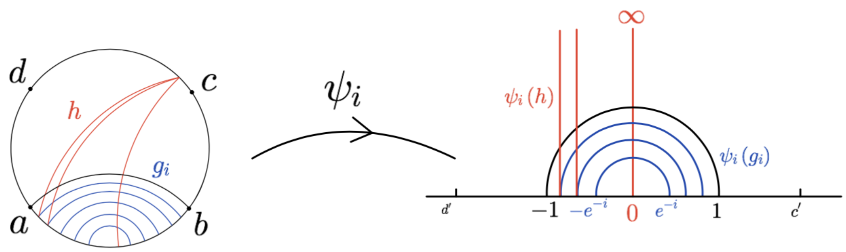


Figure 6.7: Special case of lamination Λ

(i) $L_{\mathbb{H}}(\psi_i(Q_i))$ is decreasing.

It is clear from the cross-ratio $\log \dfrac{(-e^{-i}-2)(e^{-i}+2)}{(-e^{-i}+2)(e^{-i}-2)} = 2\log\dfrac{2+e^{-i}}{2-e^{-i}}$ and $\dfrac{d}{dx}\log\dfrac{2+e^{-x}}{2-e^{-x}} = \dfrac{-4e^{-x}}{(2+e^{-x})(2-e^{-x})} < 0$ for $x \geqslant 0$.

(ii) $\sum\limits_{i=1}^{\infty} L_{\mathbb{H}}(\psi_i(Q_i))$ converges.

To see this, we use the fact that $\log x \leqslant x-1$ for all $x > 0$.

$$\begin{aligned} 2\log\frac{2+e^{-i}}{2-e^{-i}} &\leqslant 2(\frac{2+e^{-i}}{2-e^{-i}}-1) \\ &= 2\cdot\frac{2e^{-i}}{2-e^{-i}} \\ &< 4e^{-i} = 4\cdot\left(\frac{1}{e}\right)^i \end{aligned}$$

Thus, $\sum\limits_{i=1}^{\infty} L_{\mathbb{H}}(\psi_i(Q_i)) = \sum\limits_{i=1}^{\infty} 4\cdot\left(\frac{1}{e}\right)^i$ converges.

Next, for simplicity we assume that $d_i = 1$. We have that

$$\begin{aligned}
\frac{d}{dt}\int_{G(\mathbb{D})} \xi dL([E^{t\mu}])|_{t=0} &= \lim_{n\to\infty}\sum_{i=1}^{n} d_i \int_{G(\mathbb{H})} \xi\circ\psi_i^{-1}(h)\cos(h,\psi_i(g_i))dL_{\mathbb{H}}(h)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \int_{G(\mathbb{H})} \xi\circ\psi_i^{-1}(h)\cos(h,\psi_i(g_i))dL_{\mathbb{H}}(h)\\
&\leqslant \lim_{n\to\infty}\sum_{i=1}^{n} \left|\int_{G(\mathbb{H})} \xi\circ\psi_i^{-1}(h)\cos(h,\psi_i(g_i))dL_{\mathbb{H}}(h)\right|\\
&\leqslant \lim_{n\to\infty}\sum_{i=1}^{n} \int_{G(\mathbb{H})} |\xi\circ\psi_i^{-1}(h)|dL_{\mathbb{H}}(h)\\
&\leqslant \lim_{n\to\infty}\sum_{i=1}^{n} ||\xi||_{\infty}\cdot L_{\mathbb{H}}(\psi_i(Q_i)) \quad \text{converges.}
\end{aligned}$$

By above convergence and the definition of μ, it is reasonable to consider the following conjectural formula,

$$\frac{d}{dt}\int_{G(\mathbb{D})} \xi dL([E^{t\mu}])|_{t=0} \overset{?}{=} \int_{\Lambda}\int_{G(\mathbb{D})} \xi(h)\cos(h,g)dL_{\mathbb{D}}(h)d\mu$$

where $h \in G(\mathbb{D})$ and $g \in \Lambda$.

Although the above special case provides a fairly straight-forward computation, for a general bounded geodesic lamination it is quite challenging to show that the infinite sum converges. We plan to investigate this conjecture in future research.

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