6-2023

Pairings in a Ring Spectrum-Based Bousfield-Kan Spectral Sequence

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by

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A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2023

This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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Abstract

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Bousfield and Kan traditionally formulated their homotopy spectral sequence over a simplicial set X resolved with respect to a ring R. By considering an adequate category of ring spectra, one can take a ring spectrum E, create from it a functor of a triple on the category of simplicial sets, and build a cosimplicial simplicial set $\boldsymbol{E}X$. The homotopy spectral sequence can then be formed over such cosimplicial spaces by a similar construction to the original. Pairings can be established on these spectral sequences, and, for nice enough spaces, these pairings on the E_2-terms coincide with certain pairings on Ext groups.

Acknowledgments

I would like to give my most profound thanks to my advisor Rob Thompson, to whose brilliance and infinite patience this thesis is owed. I also extend my gratitude to Martin Bendersky and Mahmoud Zeinalian, for their assistance and their willingness to sit on the committee.

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CHAPTER 1

Introduction

In [BK73a] the authors construct an unstable Adams spectral sequence based on homology with coefficients in a ring and define pairings on the E_r-terms. This thesis seeks to generalize that work in part by extending the spectral sequence and its pairings over a generalized homology theory born from commutative ring spectra. Historically, the term "ring spectrum" indicates associativity and unitary diagrams that commute up to homotopy, but we will be working in a category of spectra in which ring spectra exist and satisfy these diagrams up to equality, and we will always mean for it to do so.

The first two chapters are preliminary. Chapter 2 reviews the concepts of simplicial sets and cosimplicial spaces, and Chapter 3 is concerned with introducing a category of commutative ring spectra. The choice to work with symmetric spectra is entirely preferential. Any category of commutative ring spectra would do, such as the one of S-algebras, so long as the suspension spectrum functor is compatible with the smash products of spaces and spectra. It is unclear if there exist any categories of ring spectra in which this does not hold, though there is such a possibility [Lew91].

We show that a ring spectrum, as a functor applied to a space, forms the functor of a triple, which we utilize to form cosimplicial spaces. The spectral sequence and the pairings are constructed following in line with the traditional manner. In the final chapter, we restrict ourselves from all pointed Kan complexes to only those to which applying the spectrum results in a free module over the generalized homology of the 0-sphere. From there we establish the category of unstable G-coalgebras, and observe that certain Ext groups on this category are isomorphic to E_2-terms of the spectral sequence. Pairings on these Ext groups can consequently be defined such that they coincide with the pairings on the E_2-terms.

Of particular interest is the * pairing, where the pairing on spectral sequences coincides with a Yoneda product of Ext groups. This also confirms the existence of the pairing for its usage in [BD03a], where the authors suspected its existence through their correspondence with Bousfield. Bousfield had meant to extend his pairings of [BK73a] to spectral sequences over any cosimplicial space, and not just those derived from resolving with respect to a ring or from a ring spectrum, but no paper was ultimately written.

CHAPTER 2

Cosimplicial spaces

2.1. Simplicial sets

The definitions for this chapter will mainly follow those presented in [**GJ99**].

Definition 2.1.1. The simplex category $\mathbf{\Delta}$ is the category comprised of objects $[n] = \{0, 1, \ldots, n\}$ linearly ordered sets with $n \geq 0$, and morphisms order-preserving functions. A simplicial set is defined as a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. The category of simplicial sets is denoted $\mathbf{sSet} \coloneqq \mathrm{Funct}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$.

As all morphisms in $\mathbf{\Delta}$ are order-preserving functions, every such morphism can be described as a composition consisting of two fundamental types of maps:

$$
\begin{aligned}
d^i &\colon [n-1] \to [n] \\
s^i &\colon [n+1] \to [n]
\end{aligned}
$$

for $0 \leq i \leq n$, called coface and codegeneracy maps, where d^i is defined as the unique injection which misses $i \in [n]$ and s^i is defined as the unique surjection which hits $i \in [n]$ twice.

One can verify that the coface and codegeneracy maps satisfy the cosimplicial identities:

$$
\begin{aligned}
d^j d^i &= d^i d^{j-1} && \text{for } i < j, \\
s^j s^i &= s^{i-1} s^j && \text{for } i > j, \\
s^j d^i &= d^i s^{j-1} && \text{for } i < j, \\
&= d^{i-1} s^j && \text{for } i > j+1, \\
&= \mathrm{id} && \text{for } i = j,\ j+1.
\end{aligned}
$$

This allows us to describe a simplicial set X in greater detail. We need only introduce sets $X_n \coloneqq X([n])$, $n \geq 0$ and face and degeneracy maps

$$
\begin{aligned}
\delta_i &\coloneqq X(d^i) \colon X_n \to X_{n-1} \\
\sigma_i &\coloneqq X(s^i) \colon X_n \to X_{n+1}
\end{aligned}
$$

satisfying the simplicial identities:

$$
\begin{aligned}
\delta_i \delta_j &= \delta_{j-1} \delta_i && \text{for } i < j, \\
\sigma_i \sigma_j &= \sigma_{j+1} \sigma_i && \text{for } i > j, \\
\delta_i \sigma_j &= \sigma_{j-1} \delta_i && \text{for } i < j, \\
&= \sigma_j \delta_{i-1} && \text{for } i > j+1, \\
&= \mathrm{id} && \text{for } i = j,\ j+1.
\end{aligned}
$$

A simplicial map f between two simplicial sets X and Y is a family of maps $f_n \colon X_n \to Y_n$ each of which commute with all face and degeneracy maps.

Example 2.1.2. The standard n-simplex is defined $\Delta^n \coloneqq \hom_{\mathbf{\Delta}}(-,[n])$. In simplicial degree j we have Δ^n_j as the set $\{(i_1,\dots,i_j) \mid 0 \le i_1 \le \dots \le i_j \le n\}$. In simplicial degree n, there is only one nondegenerate element id_n in $\Delta^n_n \coloneqq \hom_{\mathbf{\Delta}}([n],[n])$.

An application of the Yoneda lemma immediately implies that, for some simplicial set X,

$$\hom_{\mathbf{sSet}}(\Delta^n, X) \cong X([n]) = X_n.$$

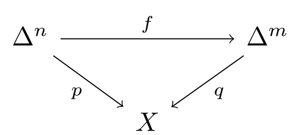
In other words, there is a bijective correspondence between the elements of the set X_n and natural transformations $\Delta^n \to X$. We call such a map $\Delta^n \to X$ an n-simplex of X.

Definition 2.1.3. For X a simplicial set, a basepoint of X is a distinguished point $* \in X_0$, which, by above, is equivalent to a map $* \colon \Delta^0 \to X$. A simplicial set with a base point is called a pointed simplicial set.

Definition 2.1.4. The smash product of two pointed simplicial sets X, Y is

$$X \wedge Y \coloneqq (X \times Y)/(X \times *_Y \cup *_X \times Y).$$

Take the *category of simplices* of X, commonly denoted $\Delta \downarrow X$, where the objects are n-simplices of X (maps $\Delta^n \to X$) and the morphisms are maps $f \colon \Delta^n \to \Delta^m$ such that the following triangle commutes



for any two n- and m-simplices p, q. There is an isomorphism between a simplicial set X and the colimit (in $\Delta \downarrow X$) of its simplices (cf. [ML98, III.7.1])

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n.$$

Definition 2.1.5. The boundary of the standard n-simplex is

$$\partial\Delta^n \coloneqq \bigcup_i d^i(\Delta^{n-1}) \subset \Delta^n.$$

The simplicial n-sphere is given as the quotient simplicial set $S^n \coloneqq \Delta^n/\partial\Delta^n$. In general, the boundary of an n-simplex $\nu \in X$ is given by $\partial\nu = (\delta_0\nu,\dots,\delta_n\nu)$.

Definition 2.1.6. The kth horn is the sub-simplicial set

$$\Lambda^n_k \coloneqq \bigcup_{i \ne k} d^i(\Delta^{n-1}) \subset \Delta^n.$$

Definition 2.1.7. A simplicial map $f\colon X \to Y$ is called a Kan fibration if for any pair of integers (k, n) with $0 \le k \le n$ and $n \ge 1$ and any commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dashed arrow, as shown, exists making the diagram commute. A simplicial set X is called a Kan complex if the constant map $X \to *$ is a Kan fibration.

We shall later see that Kan complexes are necessary in producing a working definition of homotopy groups on simplicial sets.

2.2. Cosimplicial objects

Definition 2.2.1. Given a category $\mathscr{C}$, a functor $\boldsymbol{X}\colon \boldsymbol{\Delta} \to \mathscr{C}$ is called a cosimplicial object (over $\mathscr{C}$). Particularly, for every non-negative integer n there is an $\boldsymbol{X}^n \in \mathscr{C}$ and if for every pair of integers (i, n) with $0 \le i \le n$ there are coface and codegeneracy maps

$$d^i\colon \boldsymbol{X}^{n-1} \to \boldsymbol{X}^n$$
$$s^i\colon \boldsymbol{X}^{n+1} \to \boldsymbol{X}^n$$

such that the following cosimplicial identities are satisfied:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{for } i < j, \\ s^j s^i &= s^{i-1} s^j && \text{for } i > j, \\ s^j d^i &= d^i s^{j-1} && \text{for } i < j, \\ &= d^{i-1} s^j && \text{for } i > j+1, \\ &= \mathrm{id} && \text{for } i = j,\ j+1. \end{aligned}$$

A cosimplicial map f between two cosimplicial objects $\boldsymbol{X}$ and $\boldsymbol{Y}$ over $\mathscr{C}$ is a family of maps $f_n\colon \boldsymbol{X}^n \to \boldsymbol{Y}^n$ each of which commute with all coface and codegeneracy maps. When $\mathscr{C}$ is the simplicial set category, a cosimplicial object $\boldsymbol{X}$ is known as a cosimplicial space. The word "space" here is used in place of "simplicial set." This will be common language moving forward.

Example 2.2.2. The map $[n] \mapsto \Delta^n$ defines a functor from $\boldsymbol{\Delta} \to \mathbf{sSet}$, where Δ^n is the object in codegree n, and the coface and codegeneracy maps are given by postcomposing the analogous maps in the simplex category:

$$(d^i\colon \Delta^{n-1} \to \Delta^n) = ([-] \overset{\iota}{\mapsto} [n-1] \overset{d^i}{\longmapsto} [n])$$
$$(s^i\colon \Delta^{n+1} \to \Delta^n) = ([-] \overset{\kappa}{\mapsto} [n+1] \overset{s^i}{\longmapsto} [n])$$

for any $\iota \in \Delta^{n-1}$ and $\kappa \in \Delta^{n+1}$.

Definition 2.2.3. An augmentation of a cosimplicial object $\boldsymbol{X}$ is an object $\boldsymbol{X}^{-1}$ along with a morphism $d^0\colon \boldsymbol{X}^{-1} \to \boldsymbol{X}^0$ such that $d^0 d^0 = d^1 d^0$. A cosimplicial object with such an augmentation is called an augmented cosimplicial object.

All cosimplicial objects hereafter will be augmented. Cosimplicial objects will be the main area of interest; it is on cosimplicial simplicial sets that we shall establish the spectral sequence that will be the focus of the paper.

We take a moment now to recall the definition of a triple from [EM65].

Definition 2.2.4. In a category $\mathscr{C}$, a triple (or monad) (T, η, μ) consists of a functor $T: \mathscr{C} \to \mathscr{C}$, and of natural transformations

$$\eta: \mathrm{id}_{\mathscr{C}} \to T,$$
$$\mu: T^2 \to T$$

making the diagrams

$$\begin{array}{ccc} T & \xrightarrow{T(\eta)} & T^2 \\ {\scriptstyle \eta_T}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{T(\mu)} & T^2 \\ {\scriptstyle \mu_T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

commute. Here η_T and μ_T denote the components of η, μ at TX for some object X in $\mathscr{C}$.

Definition 2.2.5. Let $\mathbf{sSet}_*$ denote the category of simplicial sets with basepoint and let R be a commutative, unital ring. Let $R\colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ be the functor that takes a space to a quotient of the free simplicial R-module on it, with the relation where the base point $*$ of X is set to 0. Define further two natural transformations

$$\begin{array}{lll} \Psi_X: X \to RX, & x \mapsto 1 \cdot x & \text{for all } x \in X \\ \Phi_X: RRX \to RX, & r \cdot y \mapsto ry & \text{for all } y \in RX. \end{array}$$

We shall drop the subscripts and simply call the maps Ψ and Φ when the domains and codomains are obvious.

It's easy to see, by simple calculation, that (R, Ψ, Φ) is a triple. That is, the following two diagrams commute:

$$\begin{array}{ccc} RX & \xrightarrow{R(\Psi_X)} & RRX \\ {\scriptstyle \Psi_{RX}}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \Phi_X} \\ RRX & \xrightarrow[\Phi_X]{} & RX \end{array} \qquad \begin{array}{ccc} R^3X & \xrightarrow{R(\Phi_X)} & RRX \\ {\scriptstyle \Phi_{RX}}\downarrow & & \downarrow{\scriptstyle \Phi_X} \\ RRX & \xrightarrow[\Phi_X]{} & RX. \end{array}$$

Remark 2.2.6. Though the elements of RX are finite sums of the form $\sum_i r_i \cdot x_i$, RX is generated by the single terms, and since all maps in sight are group homomorphisms, it is sufficient to reduce any computation one would do on the elements to a computation on just the single terms.

Definition 2.2.7. With X and R as above, the resolution of X with respect to R is the object $\boldsymbol{R}X$ over $\mathbf{sSet}_*$ with

$$\begin{array}{rll} \boldsymbol{R}X^n = R^{n+1}X, & & n \geq -1, \\ (d^i: \boldsymbol{R}X^{n-1} \to \boldsymbol{R}X^n) = (R^i\Psi_{R^{n-i}X}: R^nX \to R^{n+1}X), & & 0 \leq i \leq n, \\ (s^i: \boldsymbol{R}X^{n+1} \to \boldsymbol{R}X^n) = (R^i\Phi_{R^{n-i}X}: R^{n+2}X \to R^{n+1}X), & & 0 \leq i \leq n. \end{array}$$

Proposition 2.2.8. $\boldsymbol{R}X$ is an augmented cosimplicial object.

Proof. By definition $\boldsymbol{R}X^{-1} = X$. It remains only to show that the cosimplicial identities hold. We show the first two:

$$d^j d^i: R^nX \to R^{n+1}X \to R^{n+2}X$$

$$(r_1,\dots,r_n)x \mapsto (r_1,\dots,r_i,1,r_{i+1},\dots,r_n)x \mapsto (r_1,\dots,r_i,1,r_{i+1},\dots,r_{j-1},1,r_j,\dots,r_n)x$$
$$= d^i d^{j-1}$$
$$s^j s^i \colon R^{n+2}X \to R^{n+1}X \to R^n X$$
$$(r_1,\dots,r_{n+2})x \mapsto (r_1,\dots,r_i,r_{i+1}r_{i+2},r_{i+3},\dots,r_{n+2})x$$
$$\mapsto (r_1,\dots,r_j,r_{j+1}r_{j+2},r_{j+3},\dots,r_i,r_{i+1}r_{i+2},r_{i+3},\dots,r_{n+2})x$$
$$= s^{i-1}s^j.$$

Take note that $i < j$ in the first and $i > j$ in the second. The rest are proved similarly. □

2.3. Homotopies and homotopy groups

Definition 2.3.1. Let $f, g \colon Y \to X$ be a pair of simplicial maps. A map $h \colon Y \times \Delta^1 \to X$ is called a homotopy from f to g and write $f \sim g$ if there exists a commutative diagram

$$\begin{array}{ccccc} (Y=)Y\times\Delta^0 & \xrightarrow{\mathrm{id}\times d^1} & Y\times\Delta^1 & \xleftarrow{\mathrm{id}\times d^0} & Y\times\Delta^0(=Y) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\ & & X. & & \end{array}$$

The commutativity of the diagram and the actions of the coface maps immediately determines the map h. For all simplices $y \in Y$, $h(y,0) = f(y)$ and $h(y,1) = g(y)$.

Though this defines a relation of simplicial maps being "homotopic," this relation need not be an equivalence relation, as the following example demonstrates.

Example 2.3.2. In the definition given above, let $Y = \emptyset$ and $X = \Delta^2$ the standard 2-simplex. Define $f = d^1 d^1$ and $g = d^2 d^0$. The diagram then becomes

$$\begin{array}{ccccc} \Delta^0 & \xrightarrow{d^1} & \Delta^1 & \xleftarrow{d^0} & \Delta^0 \\ & {\scriptstyle f=d^1d^1}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g=d^2d^0} & \\ & & \Delta^2. & & \end{array}$$

This determines h to be defined as $h(0) = 0$, $h(1) = 1$ and we can say $f \sim g$.

However, if we were to switch the positions of f and g in the diagram, and leave everything else untouched, then no such h would exist such that the right half of the diagram

$$\begin{array}{ccc} \Delta^1 & \xleftarrow{d^0} & \Delta^0 \\ {\scriptstyle h}\downarrow & \swarrow{\scriptstyle f=d^1d^1} & \\ \Delta^2 & & \end{array}$$

commutes. Indeed, since $f(0) = 0$ and $d^0(0) = 1$, no order-preserving map h can exist such that $h(1) = 0$, meaning $g \nsim f$.

Fortunately, we can force the homotopy relation to be an equivalence relation by introducing a restriction on X.

Theorem 2.3.3. If X is a Kan complex, then the homotopy relation is an equivalence relation.

Proof. See Lemma I.6.1 and Corollary I.6.2 of [GJ99]. □

With this equivalence relation, homotopy groups on simplicial sets can be defined, which appeal to the same intuitions as those on topological spaces.

Definition 2.3.4. For a Kan complex X, its 0th homotopy group $\pi_0(X)$ is the set of homotopy classes of vertices $\Delta^0 \to X$ of X.

For $x \in X^0$ a vertex of X, the nth homotopy group $\pi_n(X, x)$ has as its set the set of homotopy classes $[\gamma]$ of maps $\gamma \colon \Delta^n \to X$ such that the following diagram commutes

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \Delta^n & \xrightarrow{\gamma} & X \end{array}$$

and, where if γ, γ' are homotopic by way of h, then this second diagram also commutes:

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^1 & \longrightarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \Delta^n \times \Delta^1 & \xrightarrow{h} & X. \end{array}$$

Though we call these $\pi_i(X)$ "homotopy groups," without a group operation they are still only sets. We define such an operation now.

Take, for a Kan complex X with base point x, representatives f, $g \in \pi_n(X, x)$. Consider the simplices

$$v_i = \begin{cases} x & \text{for } 0 \leq i \leq n-2, \\ f & \text{for } i = n-1, \\ g & \text{for } i = n+1. \end{cases}$$

These correspond to a simplicial map $v' = (v_0, \dots, v_{n-1}, \ , v_{n+1}) \colon \Lambda^{n+1}_n \to X$. As X is Kan, there is an extension through Δ^{n+1},

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{v'} & X \\ \big\downarrow & \nearrow_{\omega} & \\ \Delta^{n+1}. & & \end{array}$$

From the simplicial identities, we see that $\delta_i \delta_n \omega = \delta_{n-1} \delta_i \omega = x$. Hence the boundary acting on $\delta_n \omega$ results in

$$\partial(\delta_n \omega) = (\delta_0 \delta_n \omega, \dots, \delta_n \delta_n \omega) = (x, \dots, x).$$

Thus, $\delta_n \omega \in \pi_n(X, x)$, and we simply define the product by $[f] \cdot [g] = [\delta_n \omega]$.

Theorem 2.3.5. The product on $\pi_n(X, x)$ is well-defined, is independent of the choice of f, g, and extension ω, and turns the set into a group for $n \geq 1$, abelian for $n \geq 2$.

Proof. See Lemma I.7.1 and Theorem I.7.2 of [GJ99]. □

We define adjunctions before moving on to the next chapter.

Definition 2.3.6. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F \colon \mathscr{C} \rightleftarrows \mathscr{D} : G$ functors between them. An adjunction $F \dashv G$ is given by a natural isomorphism

$$\hom_{\mathscr{D}}(Fx, y) \cong \hom_{\mathscr{C}}(x, Gy)$$

natural in both x and y. Which is to say that for all $f\colon x' \to x$ and for all $g\colon y \to y'$, there are respectively two commuting diagrams

$$\begin{array}{ccc} \hom_{\mathscr{D}}(Fx,y) & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x,Gy) \\ {\scriptstyle -\circ Ff}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \hom_{\mathscr{D}}(Fx',y) & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x',Gy) \end{array} \qquad \begin{array}{ccc} \hom_{\mathscr{D}}(Fx,y) & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x,Gy) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle Gg\circ -} \\ \hom_{\mathscr{D}}(Fx,y') & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x,Gy'). \end{array}$$

F is called the left adjoint to G and G is called the right adjoint to F.

We provide some trivialities that will be useful to us later on.

Lemma 2.3.7. Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : G$ be an adjunction between categories $\mathscr{C}$ and $\mathscr{D}$. For any morphism i in $\hom_{\mathscr{D}}(Fx,y)$ or in $\hom_{\mathscr{C}}(x,Gy)$, let $\widetilde{i}$ denote its adjunct in the other.

(i) If j, k are morphisms $x \to Gy$, then $j = k \iff \widetilde{j} = \widetilde{k}$.
(ii) If we have objects $x \in \mathscr{C}$, and y, $z \in \mathscr{D}$, as well as morphisms $a\colon y \to z$ and $b\colon x \to Gy$, then $\widetilde{Ga \circ b} = a \circ \widetilde{b}$.
(iii) If we have objects x, $y \in \mathscr{C}$, and $z \in \mathscr{D}$, as well as morphisms $c\colon F(y) \to z$ and $d\colon x \to y$, then $\widetilde{c \circ Fd} = \widetilde{c} \circ d$.

Proof. (i) This is immediate from the isomorphism.
(ii) From the second naturality diagram, we have

$$\begin{array}{ccc} \hom_{\mathscr{D}}(Fx,y) & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x,Gy) \\ {\scriptstyle a\circ -}\downarrow & & \downarrow{\scriptstyle Ga\circ -} \\ \hom_{\mathscr{D}}(Fx,z) & \xrightarrow{\cong} & \hom_{\mathscr{C}}(x,Gz). \end{array}$$

In particular, we have a diagram on elements

$$\begin{array}{ccc} \widetilde{b} & \xleftrightarrow{\cong} & b \\ \downarrow & & \downarrow \\ a \circ \widetilde{b} & \xleftrightarrow{\cong} & Ga \circ b. \end{array}$$

which gives the desired result.
(iii) Similar to (ii) but using the first naturality diagram. □

CHAPTER 3

Symmetric spectra

This chapter shall serve to define and generally introduce the notion of a symmetric ring spectrum and some of its properties. One can just as well choose instead to work with the monoid objects of any symmetric monoidal category of spectra, such as the theory of coordinate-free spectra and S-algebras constructed in [**EKMM97**]. It is necessary, however, for the category to have the property that, for X and Y pointed spaces,

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y.$$

In Chapter 4 we'll see that, given some symmetric spectrum E, there is a functor of a triple which can be constructed out of it, which then gives rise to a cosimplicial object $\boldsymbol{E}X$. Chapter 5 will discuss the case when $E = HR$, the Eilenberg-MacLane spectrum for some ring R. There is an equivalence between the homotopy groups of RX and HRX, meaning the spectral sequences constructed from the two coincide. As such, the focus of the rest of the paper will be generalizing the results of [**BK73a**] from the $\boldsymbol{R}X$ case to the general $\boldsymbol{E}X$.

We warn that this chapter will, in essence, be a very truncated selection of the relevant sections of [**HSS00**] and [**Sch12**]. The aim is to quickly cover the necessary definitions and theorems. As such, we offer no proofs, for everything has been proved in the stated references. One may freely skip over this chapter if one is already familiar with symmetric spectra.

3.1. Definition of a symmetric spectrum

Definition 3.1.1. A symmetric spectrum is a sequence of pointed spaces X_n, $n \geq 0$ such that:

(i) there are pointed maps (structure maps) $\sigma_n \colon S^1 \wedge X_n \to X_{n+1}$ for each $n \geq 0$;
(ii) there is a basepoint-preserving left action of the symmetric group Σ_n on X_n for each $n \geq 0$ such that the composite

$$\sigma^m \colon S^m \wedge X_n \xrightarrow{\mathrm{id} \wedge \sigma_n} S^{m-1} \wedge X_{n+1} \xrightarrow{\mathrm{id} \wedge \sigma_{n+1}} \cdots \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. That is, the composite commutes with the group action.

A morphism $f \colon X \to Y$ of symmetric spectra is a collection of Σ_n-equivariant maps $f_n \colon X_n \to Y_n$ such that the diagram

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{\sigma_n} & X_{n+1} \\ \downarrow{\scriptstyle \mathrm{id} \wedge f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge Y_n & \xrightarrow{\sigma_n} & Y_{n+1} \end{array}$$

commutes for all $n \geq 0$.

We denote the category of symmetric spectra by $\mathcal{S}\boldsymbol{p}$.

Definition 3.1.2. A symmetric ring spectrum is a symmetric spectrum X along with:

(i) a $\Sigma_n \times \Sigma_m$-equivariant product map

$$\Phi_{n,m} \colon X_n \wedge X_m \to X_{n+m}$$

for n, $m \geq 0$;

(ii) unit maps

$$\eta_0 \colon S^0 \to X_0,$$
$$\eta_1 \colon S^1 \to X_1.$$

Furthermore, the following diagrams must commute:

$$\begin{CD}
X_n \wedge X_m \wedge X_p @>{\Phi_{n,m} \wedge \mathrm{id}}>> X_{n+m} \wedge X_p \\
@V{\mathrm{id} \wedge \Phi_{m,p}}VV @VV{\Phi_{n+m,p}}V \\
X_n \wedge X_{m+p} @>{\Phi_{n,m+p}}>> X_{n+m+p}
\end{CD}$$

$$\begin{CD}
X_n @= X_n \\
@V{\cong}VV @AA{\Phi_{0,n}}A \\
S^0 \wedge X_n @>{\eta_0 \wedge \mathrm{id}}>> X_0 \wedge X_n
\end{CD}
\qquad
\begin{CD}
X_n @= X_n \\
@V{\cong}VV @AA{\Phi_{n,0}}A \\
X_n \wedge S^0 @>{\mathrm{id} \wedge \eta_0}>> X_n \wedge X_0
\end{CD}$$

$$\begin{CD}
X_n \wedge S^1 @>{\mathrm{id} \wedge \eta_1}>> X_n \wedge X_1 @>{\Phi_{n,1}}>> X_{n+1} \\
@V{\mathrm{twist}}VV @. @VV{\chi_{n,1}}V \\
S^1 \wedge X_n @>{\eta_1 \wedge \mathrm{id}}>> X_1 \wedge X_n @>{\Phi_{1,n}}>> X_{1+n}
\end{CD}$$

where $\chi_{n,m} \in \Sigma_{n+m}$ is

$$\chi_{n,m}(i) = \begin{cases} i+m & \text{for } 1 \leq i \leq n, \\ i-n & \text{for } n+1 \leq i \leq n+m \end{cases}$$

the permutation which "moves the first n elements past the last m elements."

If, in addition, the square

$$\begin{CD}
X_n \wedge X_m @>{\mathrm{twist}}>> X_m \wedge X_n \\
@V{\Phi_{n,m}}VV @VV{\Phi_{m,n}}V \\
X_{n+m} @>{\chi_{n,m}}>> X_{m+n}
\end{CD}$$

commutes, we call the symmetric ring spectrum X commutative.

A morphism $f \colon X \to Y$ of symmetric ring spectra is a collection of Σ_n-equivariant pointed maps $f_n \colon X_n \to Y_n$ such that the diagrams

$$\begin{CD}
X_n \wedge X_m @>{f_n \wedge f_m}>> Y_n \wedge Y_m \\
@V{\Phi_{n,m}}VV @VV{\Phi_{n,m}}V \\
X_{n+m} @>{f_{n+m}}>> Y_{n+m}
\end{CD}$$

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_0} & X_0 \\ & \searrow^{\eta_0} & \downarrow f_0 \\ & & Y_0 \end{array} \qquad \begin{array}{ccc} X_1 & \xleftarrow{\eta_1} & S^1 \\ f_1 \downarrow & \swarrow_{\eta_1} & \\ Y_1 & & \end{array}$$

commute for all $n \geq 0$.

Something to keep in mind: it follows from the definition that we can inductively define an iterated multiplication map

$$\Phi_{n_1,\ldots,n_i} \colon X_{n_1} \wedge \cdots \wedge X_{n_i} \to X_{n_1+\cdots+n_i}$$

by setting

$$\Phi_{n_1,\ldots,n_i} \coloneqq \Phi_{n_1,n_2+\cdots+n_i}.$$

Then we can also define unit maps

$$\eta_j \colon S^j \to X_j$$

by

$$S^j \cong S^1 \wedge \cdots \wedge S^1 \xrightarrow{\eta_1 \wedge \cdots \wedge \eta_1} X_1 \wedge \cdots \wedge X_1 \xrightarrow{\Phi_{1,\ldots,1}} X_j.$$

Example 3.1.3. The symmetric sphere spectrum $\mathbb{S}$ is the symmetric spectrum defined by $\mathbb{S}_n = S^n$ with structure maps $\sigma_n \colon S^1 \wedge S^n \to S^{n+1}$ the canonical isomorphisms. The unit map is given by the identity and the multiplication the canonical isomorphism $S^n \wedge S^m \to S^{n+m}$.

Definition 3.1.4. The symmetric suspension spectrum $\Sigma^\infty X$ of a pointed simplicial set X is a sequence of pointed simplicial sets $S^n \wedge X$ with structure maps the canonical isomorphisms

$$\sigma_n \colon S^1 \wedge S^n \wedge X \xrightarrow{\cong} S^{n+1} \wedge X$$

and Σ_n acts via the diagonal action on $S^n \wedge X$ by left permutation on S^n and trivially on X. The composite map $\sigma^m \colon S^m \wedge S^n \wedge X \to S^{n+m} \wedge X$ is the canonical isomorphism. The sphere spectrum $\mathbb{S}$ above is an example of a suspension spectrum, with $\mathbb{S} = \Sigma^\infty S^0$.

Σ^∞ defines a functor $\mathbf{sSet}_* \to \mathcal{S}p$, and has a right adjoint Ω^∞, defined by taking the zeroth space after the application of the detection functor D of [Shi00].

3.2. Smash products of symmetric spectra

We need some quick preliminaries before we discuss the smash product of spectra.

The symmetric group Σ_n can be seen as a constant simplicial set, with $(\Sigma_n)_m = \Sigma_n$ for all $m \geq 0$, and all face and degeneracy maps the identity.

For X, Y simplicial sets and G a discrete group acting on X on the right and on Y on the left, we denote by $X \wedge_G Y$ the quotient of $X \wedge Y$ by the action

$$g \colon (x, y) \mapsto (xg^{-1}, gy).$$

Definition 3.2.1. For two symmetric spectra X, Y, their smash product is defined as the symmetric spectrum $X \wedge Y$ with $(X \wedge Y)_n$ given by the coequalizer

$$\bigvee_{p+1+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} X_p \wedge S^1 \wedge Y_q \overset{\gamma_X}{\underset{\gamma_Y}{\rightrightarrows}} \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q$$

of the maps

$$\gamma_X\colon X_p \wedge S^1 \wedge Y_q \xrightarrow{\text{twist}\wedge\text{id}} S^1 \wedge X_p \wedge Y_q \xrightarrow{\sigma_p \wedge \text{id}} X_{p+1} \wedge Y_q \xrightarrow{\chi_{p,1}\wedge\text{id}} X_{1+p} \wedge Y_q$$
$$\gamma_Y\colon X_p \wedge S^1 \wedge Y_q \xrightarrow{\text{id}\wedge\sigma_q} X_p \wedge Y_{q+1},$$

which take the wedge summand indexed by $(p,1,q)$ to the wedge summand indexed by $(1+p,q)$ or $(p,q+1)$, respectively.

The structure map

$$\sigma_n\colon S^1 \wedge (X\wedge Y)_n \to (X \wedge Y)_{n+1}$$

is induced on the coequalizers by the maps

$$\Sigma_n^+ \wedge_{\Sigma_p\times\Sigma_q} S^1 \wedge X_p \wedge Y_q \longrightarrow \Sigma_{n+1}^+ \wedge_{\Sigma_{p+1}\times\Sigma_q} X_{p+1}\wedge Y_q,$$

which are themselves induced by $\sigma_p \wedge \text{id}\colon S^1 \wedge X_p \wedge Y_q \longrightarrow X_{p+1} \wedge Y_q$.

The smash product is functorial in both arguments.

Proposition 3.2.2. The sphere spectrum can be made into a strict left and right unit $\mathbb{S}\wedge X = X = X \wedge \mathbb{S}$ by the maps

$$\lambda\colon S^p \wedge X_q \xrightarrow{\sigma^p} X_{q+p}$$
$$\lambda_\tau\colon X_p \wedge S^q \xrightarrow{\text{twist}} S^q \wedge X_p \xrightarrow{\sigma^q} X_{p+q} \xrightarrow{\chi_{p,q}} X_{q+p}.$$

Theorem 3.2.3. There are associativity isomorphisms

$$\alpha_{X,Y,Z}\colon (X\wedge Y)\wedge Z \to X \wedge (Y \wedge Z)$$

and commutativity isomorphisms

$$\tau_{X,Y}\colon X \wedge Y \to Y \wedge X$$

which, along with the strict unit, turn the smash product into a symmetric monoidal product. This means there are commuting diagrams

$$\begin{array}{ccccc}
 & & ((W\wedge X)\wedge Y)\wedge Z & & \\
 & \swarrow^{\alpha_{W,X,Y}\wedge\text{id}} & & \searrow^{\alpha_{W\wedge X,Y,Z}} & \\
(W\wedge(X\wedge Y))\wedge Z & & & & (W\wedge X)\wedge(Y\wedge Z) \\
\downarrow{\scriptstyle \alpha_{W,X\wedge Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{W,X,Y\wedge Z}} \\
W\wedge((X\wedge Y)\wedge Z) & & \xrightarrow{\text{id}\wedge\alpha_{X,Y,Z}} & & W\wedge(X\wedge(Y\wedge Z)).
\end{array}$$

$$\begin{array}{ccccc}
(X\wedge Y)\wedge Z & \xrightarrow{\alpha_{X,Y,Z}} & X\wedge(Y\wedge Z) & \xrightarrow{\tau_{X,Y\wedge Z}} & (Y\wedge Z)\wedge X \\
\downarrow{\scriptstyle \tau_{X,Y}\wedge\text{id}} & & & & \downarrow{\scriptstyle \alpha_{Y,Z,X}} \\
(Y\wedge X)\wedge Z & \xrightarrow{\alpha_{Y,X,Z}} & Y\wedge(X\wedge Z) & \xrightarrow{\text{id}\wedge\tau_{X,Z}} & Y\wedge(Z\wedge X)
\end{array}$$

along with $\alpha_{X,\mathbb{S},Y}\colon (X\wedge\mathbb{S})\wedge Y \to X\wedge(\mathbb{S}\wedge Y)$ the identity.

Proposition 3.2.4. For pointed simplicial sets K and L, there is a natural isomorphism of suspension spectra

$$\Sigma^\infty K \wedge \Sigma^\infty L \cong \Sigma^\infty (K \wedge L).$$

This next definition may seem odd in that we have already defined the terms in it. This is done on purpose; we wish to conflate the two definitions in light of the theorem after.

Definition 3.2.5. A symmetric ring spectrum R is a symmetric spectrum equipped with a product $\Phi\colon R \wedge R \to R$ and a unit $\eta\colon \mathbb{S} \to R$ such that the following two diagrams commute:

$$\begin{array}{ccccc} (R \wedge R) \wedge R & \xrightarrow{\alpha_{R,R,R}} & R \wedge (R \wedge R) & \xrightarrow{\mathrm{id} \wedge \Phi} & R \wedge R \\ \big\downarrow{\scriptstyle \Phi \wedge \mathrm{id}} & & & & \big\downarrow{\scriptstyle \Phi} \\ R \wedge R & & \xrightarrow{\qquad\qquad\Phi\qquad\qquad} & & R \end{array} \qquad \begin{array}{ccccc} \mathbb{S} \wedge R & \xrightarrow{\eta \wedge \mathrm{id}} & R \wedge R & \xleftarrow{\mathrm{id} \wedge \eta} & R \wedge \mathbb{S} \\ & \searrow{=} & \big\downarrow{\scriptstyle \Phi} & {=}\swarrow & \\ & & R. & & \end{array}$$

If this third diagram also commutes, R is called a commutative symmetric spectrum:

$$\begin{array}{ccc} R \wedge R & \xrightarrow{\tau_{R,R}} & R \wedge R \\ & {\scriptstyle \Phi}\searrow \quad \swarrow{\scriptstyle \Phi} & \\ & R. & \end{array}$$

Theorem 3.2.6. There is an isomorphism between the category of symmetric ring spectra as has just been defined and the category of symmetric ring spectra as was defined in the previous section. This isomorphism restricts to an isomorphism of categories of commutative ring spectra.

CHAPTER 4

$\boldsymbol{E}X$

4.1. E as the functor of a triple and $\boldsymbol{E}X$ as a cosimplicial object

From here on, all symmetric ring spectra are assumed to be commutative.

Definition 4.1.1. For a pointed space X and a symmetric ring spectrum E, define

$$EX \coloneqq \Omega^\infty(E \wedge \Sigma^\infty X).$$

We produce a unit map $\psi\colon X \to EX$ by taking ψ to be the adjunct of the composite

$$\Sigma^\infty X \xrightarrow{\cong} \mathbb{S} \wedge \Sigma^\infty X \xrightarrow{\eta\wedge \mathrm{id}} E \wedge \Sigma^\infty X,$$

where the isomorphism is given by λ^{-1} and η is the ring spectrum unit.

Define further a multiplication $\varphi\colon E^2X \to EX$ to be the composite

$$\Omega^\infty(E \wedge \Sigma^\infty\Omega^\infty(E \wedge \Sigma^\infty X)) \xrightarrow{\Omega^\infty(\mathrm{id}\wedge\varepsilon)} \Omega^\infty(E \wedge E \wedge \Sigma^\infty X) \xrightarrow{\Omega^\infty(\Phi\wedge\mathrm{id})} \Omega^\infty(E \wedge \Sigma^\infty X),$$

where ε is the counit of the adjunction and Φ is the ring spectrum multiplication.

In the literature, it is common to take such a definition when E is a general, non-symmetric, ring spectrum, which is not necessarily commutative. Often, it may be the case that this ring spectrum may only satisfy associative and unital diagrams up to homotopy. In such a situation, the ring spectrum does not guarantee the functor of a triple, but rather only a triple up to homotopy (cf. [**BCM78**, §4]). By restricting to (commutative) symmetric ring spectra, we can make the following claim.

Theorem 4.1.2. (E, ψ, φ) forms a triple on $\mathbf{sSet}_*$.

Proof. To show associativity of the multiplication, we need to show that the diagram

$$\begin{array}{ccc} E^3X & \xrightarrow{E(\varphi)} & E^2X \\ {\scriptstyle\varphi_{EX}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ E^2X & \xrightarrow{\varphi} & EX \end{array}$$

commutes. We rewrite this diagram in full detail (except we omit the first Ω^∞ everywhere) in order to better work with it.

$$\begin{array}{ccccc}
E\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X)) & \xrightarrow{\mathrm{id}\wedge\Sigma^\infty\Omega^\infty(\mathrm{id}\wedge\varepsilon)} & E\wedge\Sigma^\infty\Omega^\infty(E\wedge E\wedge\Sigma^\infty X) & \xrightarrow{\mathrm{id}\wedge\Sigma^\infty\Omega^\infty(\Phi\wedge\mathrm{id})} & E\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) \\
\downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} & 1 & \downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} & 2 & \downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} \\
E\wedge E\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\mathrm{id}\wedge\mathrm{id}\wedge\varepsilon} & E\wedge E\wedge E\wedge\Sigma^\infty X & \xrightarrow{\mathrm{id}\wedge\Phi\wedge\mathrm{id}} & E\wedge E\wedge\Sigma^\infty X \\
\downarrow{\scriptstyle \Phi\wedge\mathrm{id}} & 3 & \downarrow{\scriptstyle \Phi\wedge\mathrm{id}\wedge\mathrm{id}} & 4 & \downarrow{\scriptstyle \Phi\wedge\mathrm{id}} \\
E\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\mathrm{id}\wedge\varepsilon} & E\wedge E\wedge\Sigma^\infty X & \xrightarrow{\Phi\wedge\mathrm{id}} & E\wedge\Sigma^\infty X
\end{array}$$

To show the entire diagram commutes, it suffices to show that squares 1, 2, 3, and 4 commute. Square 4 commutes by virtue of the associativity of Φ; see DEFINITION 3.2.5. Commutativity of 3 is immediate by functoriality of $\wedge$ in both factors. We shall show 2 commutes by showing first that the adjoint diagram is commutative.

We strip the first E on the left from 2 since only the identity is applied on it, and we get a diagram

$$\begin{array}{ccc}
\Sigma^\infty\Omega^\infty(E\wedge E\wedge \Sigma^\infty X) & \xrightarrow{\Sigma^\infty\Omega^\infty(\Phi\wedge \mathrm{id})} & \Sigma^\infty\Omega^\infty(E\wedge \Sigma^\infty X) \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
E\wedge E\wedge \Sigma^\infty X & \xrightarrow{\Phi\wedge \mathrm{id}} & E\wedge \Sigma^\infty X.
\end{array}$$

Using both properties (ii) and (iii) of LEMMA 2.3.7 results in an adjoint diagram

$$\begin{array}{ccc}
\Omega^\infty(E\wedge E\wedge \Sigma^\infty X) & \xrightarrow{\Omega^\infty(\Phi\wedge \mathrm{id})} & \Omega^\infty(E\wedge \Sigma^\infty X) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\Omega^\infty(E\wedge E\wedge \Sigma^\infty X) & \xrightarrow{\Omega^\infty(\Phi\wedge \mathrm{id})} & \Omega^\infty(E\wedge \Sigma^\infty X)
\end{array}$$

which obviously commutes. Thus, by property (i) of the same lemma, square 2 commutes. Square 1 commutes by the same reasoning.

We show commutativity of the unital diagram by separating it into its two component diagrams

$$\begin{array}{ccc}
EX & \xrightarrow{E(\psi)} & E^2X \\
 & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \varphi} \\
 & & EX
\end{array}
\qquad
\begin{array}{ccc}
EX & & \\
\downarrow{\scriptstyle \psi_{EX}} & \searrow{\scriptstyle =} & \\
E^2X & \xrightarrow[\varphi]{} & EX
\end{array}$$

and show that these both commute. Writing the first diagram out explicitly (we drop the first Ω^∞ in front), it becomes

$$\begin{array}{ccc}
E\wedge \Sigma^\infty X & \xrightarrow{\mathrm{id}\wedge\Sigma^\infty\psi} & E\wedge \Sigma^\infty\Omega^\infty(E\wedge \Sigma^\infty X) \\
 & \searrow{\scriptstyle (\mathrm{id}\wedge\eta\wedge \mathrm{id})\circ(\lambda_\tau^{-1}\wedge \mathrm{id})} & \downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} \\
 & & E\wedge E\wedge \Sigma^\infty X \\
 & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Phi\wedge \mathrm{id}} \\
 & & E\wedge \Sigma^\infty X
\end{array}$$

where λ_τ^{-1} is the isomorphism $E\to E\wedge\mathbb{S}$. We shall proceed by showing that the top and bottom triangles commute.

We drop the end $\Sigma^\infty X$ from the bottom triangle, yielding

$$\begin{array}{ccc}
E & \xrightarrow{(\mathrm{id}\wedge\eta)\circ\lambda_\tau^{-1}} & E\wedge E \\
 & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \Phi} \\
 & & E
\end{array}$$

which commutes by DEFINITION 3.2.5.

For the top triangle, we strip the first E, resulting in

$$\begin{array}{ccc} \Sigma^\infty X & \xrightarrow{\Sigma^\infty \psi} & \Sigma^\infty \Omega^\infty (E \wedge \Sigma^\infty X) \\ & \searrow^{(\eta \wedge \mathrm{id}) \circ \lambda^{-1}} & \downarrow \varepsilon \\ & & E \wedge \Sigma^\infty X. \end{array}$$

Be careful to notice that we have replaced λ_τ^{-1} with λ^{-1}. Whereas λ_τ^{-1} was applied to the first E to arrive at $E \wedge \mathbb{S} \wedge \Sigma^\infty X$, we could not do so after dropping the E, and hence we must apply λ^{-1} to $\Sigma^\infty X$ to receive $\mathbb{S} \wedge \Sigma^\infty X$.

By (ii) and (iii) of LEMMA 2.3.7 we have an adjoint diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \Omega^\infty (E \wedge \Sigma^\infty X) \\ & \searrow^{\psi} & \downarrow \mathrm{id} \\ & & \Omega^\infty (E \wedge \Sigma^\infty X) \end{array}$$

which obviously commutes. Thus by (i), the original diagram commutes.

We now move to show commutativity of the second unitary diagram. As before, we write it out explicitly.

$$\begin{array}{ccc} \Omega^\infty (E \wedge \Sigma^\infty X) & \xrightarrow{\psi_{EX}} & \Omega^\infty (E \wedge \Sigma^\infty \Omega^\infty (E \wedge \Sigma^\infty X)) \\ & & \downarrow \Omega^\infty (\mathrm{id} \wedge \varepsilon) \\ & \searrow^{\mathrm{id}} & \Omega^\infty (E \wedge E \wedge \Sigma^\infty X) \\ & & \downarrow \Omega^\infty (\Phi \wedge \mathrm{id}) \\ & & \Omega^\infty (E \wedge \Sigma^\infty X) \end{array}$$

By (ii) of the lemma, we have an adjoint diagram

$$\begin{array}{ccc} \Sigma^\infty \Omega^\infty (E \wedge \Sigma^\infty X) & \xrightarrow{(\eta \wedge \mathrm{id}) \circ \lambda^{-1}} & E \wedge \Sigma^\infty \Omega^\infty (E \wedge \Sigma^\infty X) \\ & & \downarrow \mathrm{id} \wedge \varepsilon \\ & \searrow^{\varepsilon} & E \wedge E \wedge \Sigma^\infty X \\ & & \downarrow \Phi \wedge \mathrm{id} \\ & & E \wedge \Sigma^\infty X. \end{array}$$

By functoriality of the smash product, this is the same as the diagram

$$\begin{array}{ccccc}
\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\lambda^{-1}} & \mathbb{S}\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\mathrm{id}\wedge\varepsilon} & \mathbb{S}\wedge E\wedge\Sigma^\infty X \\
 & \searrow^{\varepsilon} & & & \downarrow{\scriptstyle \eta\wedge\mathrm{id}\wedge\mathrm{id}} \\
 & & & & E\wedge E\wedge\Sigma^\infty X \\
 & & & & \downarrow{\scriptstyle \Phi\wedge\mathrm{id}} \\
 & & & & E\wedge\Sigma^\infty X.
\end{array}$$

By Definition 3.2.5 once again, $(\Phi\wedge\mathrm{id})\circ(\eta\wedge\mathrm{id}\wedge\mathrm{id})=\lambda\wedge\mathrm{id}$. One should be wary that this λ is defined on $\mathbb{S}\wedge E$, whereas the λ^{-1} in the diagram is defined on $\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X)$, so that these maps are not inverses of each other. We relabel λ to λ_E and leave the λ^{-1} as is.

We replace the counit map

$$\varepsilon\colon \Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X)\to E\wedge\Sigma^\infty X$$

with the composite

$$\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X)\xrightarrow{\varepsilon} E\wedge\Sigma^\infty X\xrightarrow{\lambda^{-1}_{E\wedge\Sigma^\infty X}}\mathbb{S}\wedge E\wedge\Sigma^\infty X\xrightarrow{\lambda_{E\wedge\Sigma^\infty X}} E\wedge\Sigma^\infty X,$$

which is still just the counit map. By Proposition 3.2.2, we write $\lambda_{E\wedge\Sigma^\infty X}=\lambda_E\wedge\mathrm{id}$ and the diagram now becomes

$$\begin{array}{ccccc}
\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\lambda^{-1}} & \mathbb{S}\wedge\Sigma^\infty\Omega^\infty(E\wedge\Sigma^\infty X) & \xrightarrow{\mathrm{id}\wedge\varepsilon} & \mathbb{S}\wedge E\wedge\Sigma^\infty X \\
\downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\mathrm{id}\wedge\varepsilon} & & \downarrow{\scriptstyle\lambda_E\wedge\mathrm{id}} \\
E\wedge\Sigma^\infty X & \xrightarrow{\lambda^{-1}_{E\wedge\Sigma^\infty X}} & \mathbb{S}\wedge E\wedge\Sigma^\infty X & \xrightarrow{\lambda_E\wedge\mathrm{id}} & E\wedge\Sigma^\infty X.
\end{array}$$

The right square obviously commutes. The left commutes by naturality of λ^{-1}, of the same proposition. So the diagram commutes, and therefore the original diagram does as well. ☐

Definition 4.1.3. Define an object $\boldsymbol{E}X$ over $\mathbf{sSet}_*$ with

$$\begin{aligned}
\boldsymbol{E}X^n &= E^{n+1}X, && n\geq -1,\\
(d^i\colon \boldsymbol{E}X^{n-1}\to\boldsymbol{E}X^n) &= (E^i\psi_{E^{n-i}X}\colon E^nX\to E^{n+1}X), && 0\leq i\leq n,\\
(s^i\colon \boldsymbol{E}X^{n+1}\to\boldsymbol{E}X^n) &= (E^i\varphi_{E^{n-i}X}\colon E^{n+2}X\to E^{n+1}X), && 0\leq i\leq n.
\end{aligned}$$

One can verify the cosimplicial identities via this definition and come to the conclusion that $\boldsymbol{E}X$ is an augmented cosimplicial object.

4.2. The Eilenberg-MacLane spectrum

Definition 4.2.1. The Eilenberg-Maclane spectrum $H\mathbb{Z}$ is comprised of the simplicial abelian groups $\mathbb{Z} \otimes S^n$ in spectrum dimension n, where $(\mathbb{Z} \otimes S^n)_k$ is the free abelian group on the non-basepoint k-simplices of S^n, and basepoint identified with 0. The elements of $\mathbb{Z} \otimes S^n$ are finite sums $\sum_i^n z_i \cdot x_i$ for $z_i \in \mathbb{Z}$ and $x_i \in S^n$. The symmetric group acts by permuting the generators and the structure maps are given by

$$\begin{aligned} \sigma_n \colon S^1 \wedge (\mathbb{Z} \otimes S^n) &\to (\mathbb{Z} \otimes S^{n+1}) \\ y \wedge \left(\sum z_i \cdot x_i\right) &\mapsto \sum z_i \cdot (y \wedge x_i). \end{aligned}$$

We can replace $\mathbb{Z}$ with any abelian group A, and the elements of $(HA)_n$ are finite sums

$$\sum_{i=1}^{n} a_i \cdot x_i$$

for $a_i \in A$. This results in a functor $H \colon \mathbf{Ab} \to \mathcal{Sp}$ by defining, for any morphism of abelian groups $f \colon A \to B$, a morphism

$$\begin{aligned} H(f)_n \colon (HA)_n &\to (HB)_n \\ \colon \sum a_i \cdot x_i &\mapsto \sum f(a_i) \cdot x_i. \end{aligned}$$

Additionally, if A has a ring structure, then HA is a symmetric ring spectrum, with unit maps $S^n \to (HA)_n$ the inclusion of generators, and product maps

$$\begin{gathered} (HA)_n \wedge (HA_m) \to HA_{n+m} \\ \left(\sum a_i \cdot x_i\right) \wedge \left(\sum b_j \cdot y_j\right) \mapsto \sum_{i,j} a_i b_j \cdot (x_i \wedge y_j). \end{gathered}$$

Now suppose R is a ring and $X \in \mathbf{sSet}_{*\mathbf{K}}$, the category of pointed Kan complexes. The homotopy groups of RX (Definition 2.2.5) are given by [**BK72a**, I.2.4]

$$\pi_* RX \cong \widetilde{H}_*(X; R).$$

For a symmetric spectrum E, the homotopy groups of EX are

$$\pi_* EX = [S^*, \Omega^\infty(E \wedge \Sigma^\infty X)] \cong [\Sigma^\infty S^*, E \wedge \Sigma^\infty X] = \pi_*(E \wedge \Sigma^\infty X) = E_*(X)$$

the reduced E-homology of X. If E is HR, the Eilenberg-Maclane spectrum for the ring R, then this is exactly $\widetilde{H}_*(X; R)$, and the homotopy groups of HRX coincide with the homotopy groups of RX.

Since these two share the same homotopy groups, it is possible to "interchange" the two when working with the spectral sequence to be introduced in the next chapter. The following sections of this paper will be to generalize some of the work in [**BK73a**] from the spectral sequence based on RX to the spectral sequence based on EX. Rather than work with RX as in the source material, one can choose to work with the symmetric spectrum HR over the same underlying ring R, and, due to the equivalent homotopy groups, this would result in an equivalent spectral sequence, and thus equivalent pairings.

CHAPTER 5

The Bousfield-Kan spectral sequence

5.1. Homotopy spectral sequences from towers of fibrations

From here on, we assume all spaces to be Kan complexes unless otherwise stated. Suppose we have some sequence of spaces $\{X_n\}$. Suppose further there is a tower of fibrations associated to $\{X_n\}$, or a diagram

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ X_n & \longrightarrow & D_n \\ \downarrow & & \\ X_{n-1} & \longrightarrow & D_{n-1} \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ X_1 & \longrightarrow & D_1 \\ \downarrow & & \\ X_0 & \longrightarrow & D_0 \end{array}$$

where every map

$$X_{i+1} \to X_i$$

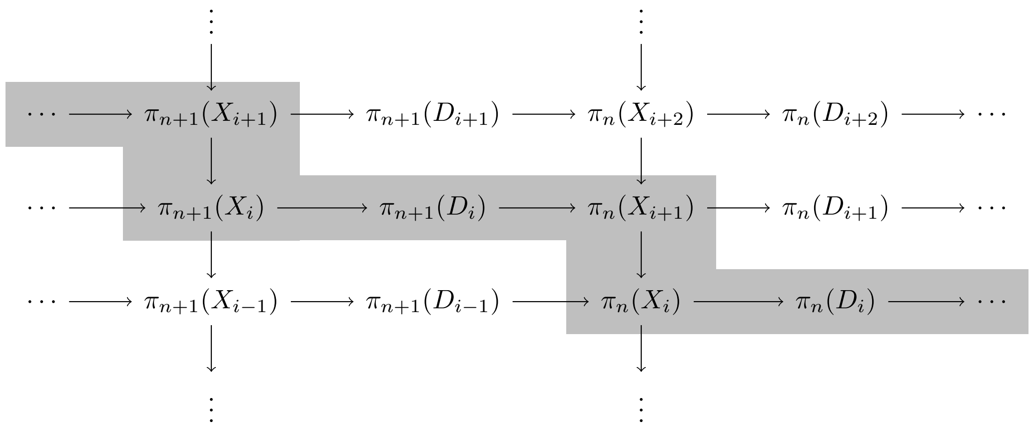
is induced by taking the homotopy fiber of $X_i \to D_i$. We can take a homotopy fiber sequence

$$X_{i+1} \to X_i \to D_i$$

from one "hook" of the diagram and apply homotopy, which yields a long exact sequence of homotopy groups

$$\begin{array}{ccccc} \vdots & & & & \\ \downarrow & & & & \\ \pi_{n+1}(X_i) & \longrightarrow & \pi_{n+1}(D_i) & \longrightarrow & \pi_n(X_{i+1}) \\ & & & & \downarrow \\ & & & & \pi_n(X_i) \longrightarrow \pi_n(D_i) \longrightarrow \cdots . \end{array} \tag{1}$$

Taking homotopy of the entire tower then results in a sequence of such long exact sequences, each one interlocking with the next.



The gray area indicates the particular long exact sequence (1).

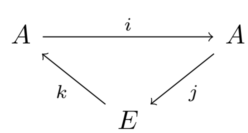
Setting $n = t - s$ enables us to define two bigraded complexes from the sequences above.

$$\begin{aligned} E^{s,t} = E_1^{s,t} &:= \pi_{t-s}(D_s) \\ A^{s,t} &:= \pi_{t-s}(X_s) \end{aligned}$$

With this in hand, we can rewrite every long exact sequence using the Es and As. For instance, the sequence (1) above becomes (with labels for maps added in)

$$\cdots \longrightarrow A^{i,n+i+1} \xrightarrow{j^{i,n+i+1}} E^{i,n+i+1} \xrightarrow{k^{i,n+i+1}} A^{i+1,n+i+1} \xrightarrow{i^{i+1,n+i+1}} A^{i,n+i} \xrightarrow[j^{i,n+i}]{} E^{i,n+i} \longrightarrow \cdots .$$

Compressing everything together gives a triangle diagram of graded maps



where i, j, k have bidegrees $(-1,-1), (0,0), (1,0)$ respectively. The long exact sequences ensure that this triangle is exact at every corner. Such a diagram is known as an *exact couple*, often denoted (A, E, i, j, k).

Define a differential

$$d_1^{s,t} = j^{s+1,t} k^{s,t} \colon E^{s,t} \to E^{s+1,t}.$$

We write just d_1 when the context is clear. We have, as an immediate consequence of exactness, $d_1 \circ d_1 = j(kj)k = 0$. Note also that d_1 has bidegree $(1,0)$, gotten from adding the bidegrees of j, k. Using this map, we can construct a new couple from our already existing one. Taking the homology of d_1 at E_1, we define

$$E_2^{s,t} := \frac{\ker d_1^{s,t}}{\operatorname{im} d_1^{s-1,t}}.$$

Set also $A_2 := \operatorname{im} i \subseteq A$, along with $i_2 \colon D_2 \to D_2$ by $i_2 := i|_{A_2}$.

Defining j_2 is slightly trickier. Take $a \in A_2$. Then there is some $a' \in A$ such that $i(a') = a \in A_2$. Define $j_2(a) = [j(a')]$ the homology class of $j(a')$ in E_2. This is a cycle, since $d_1(j(a)) = jkj(a) = 0$. This is also well-defined: if $i(a') = i(a'')$, $i(a' - a'') = 0$, meaning $a' - a''$ is in $\ker i = \operatorname{im} k$. Applying j yields $j(a') - j(a'') \in \operatorname{im} d_1$, i.e., it's a boundary. Thus $[j(a')] = [j(a'')] \in E_2$.

For k_2, define $k_2 = k|_{\ker d_1}$. This does indeed sit inside A_2 since, for $e \in \ker d_1 \subset E$, $d_1 e = jk(e) = 0$, so $k(e) \in \ker j = \operatorname{im} i = A_2$. Finally, this is well-defined: the homology class of e in E_2 being 0 means $e \in \operatorname{im} d_1$. But $\operatorname{im} d_1 \subset \operatorname{im} j = \ker k$, hence $k(e) = 0$.

The result of all this is a new triangle $(A_2, E_2, i_2, j_2, k_2)$, called the derived exact couple. Of course, we still need to show that it is indeed exact.

Proposition 5.1.1. $(A_2, E_2, i_2, j_2, k_2)$ is an exact couple. Additionally,

$$\begin{aligned} \deg(i_2) &= \deg(i), \\ \deg(j_2) &= \deg(j) - \deg(i), \\ \deg(k_2) &= \deg(k). \end{aligned}$$

Proof. See Theorem VIII.1.1 of [HS97]. Alternatively, consult any introductory text on spectral sequences. □

Recursively iterating this process produces further derived exact couples. Let $(A_r, E_r, i_r, j_r, k_r)$ be the derived exact couple after having applied the process $r-1$ times. It's clear that the differential $d_r = j_r k_r$ will have bidegree $\deg(k) + \deg(j) - (r-1)\deg(i)$, and

$$d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

A sequence $\{E_r, d_r\}$ of groups E_r (usually called the E_r *term* or *page*) and differentials d_r satisfying the properties

$$d_r \circ d_r = 0,$$
$$E_{r+1} = \frac{\ker d_r}{\operatorname{im} d_r}$$

is called a spectral sequence. Spectral sequences may arise through many other methods, but our interest lies entirely in those constructed out of homotopy applied to a tower of fibrations.

It may be the case that a spectral sequence stabilizes, in the sense that $E_{r_0}^{s,t} = E_r^{s,t}$ for $r_0 \geq r$ and all pairs of integers (s,t), and we write $E_r = E_\infty$. Formally,

Definition 5.1.2. $E_\infty^{s,t} = \operatorname{colim}_r E_r^{s,t} = \cap_{r>s} E_r^{s,t}$.

Definition 5.1.3. Let H_* be a graded object. If there exists a filtration F of H_* and isomorphisms

$$E_\infty^{s,t} \cong \frac{F^s H^{s+t}}{F^{s+1} H^{s+t}}$$

for every pair of integers (s,t), we say the spectral sequence converges to H_*, and write $E_1^{s,t} \Rightarrow H_*$.

5.2. The derivation of a cosimplicial space

Definition 5.2.1. Let $\boldsymbol{X}$ be a cosimplicial object over a category $\mathscr{C}$ and T a (covariant) functor from $\mathscr{C}$ to $\mathscr{C}'$. The application of T to $\boldsymbol{X}$ yields a cosimplicial object $\boldsymbol{T}X$ over $\mathscr{C}'$ defined by setting $(\boldsymbol{T}X)^n = T(\boldsymbol{X}^n)$ for all n.

Example 5.2.2. For a pointed space X and symmetric ring spectrum E, EX is an infinite loop space, and $\pi_i \boldsymbol{E}X$ is a cosimplicial abelian group if we ignore the augmentation.

Definition 5.2.3. For a cosimplicial object $\boldsymbol{X}$ over a category $\mathscr{C}$, we construct another cosimplicial object $\boldsymbol{V}X$ over $\mathscr{C}$. Take

$$\begin{aligned} \boldsymbol{V}X^n &= \boldsymbol{X}^{n+1}, \qquad n \geq -1, \\ (d_V^i \colon \boldsymbol{V}X^{n-1} \to \boldsymbol{V}X^n) &= (d^{i+1} \colon \boldsymbol{X}^n \to \boldsymbol{X}^{n+1}), \qquad 0 \leq i \leq n, \\ (s_V^i \colon \boldsymbol{V}X^{n+1} \to \boldsymbol{V}X^n) &= (s^{i+1} \colon \boldsymbol{X}^{n+2} \to \boldsymbol{X}^{n+1}), \qquad 0 \leq i \leq n. \end{aligned}$$

All that's occurred has been a shifting of the cosimplicial degree by 1 and a dropping of the lowest coface and codegeneracy maps. We can associate these two cosimplicial spaces with a map $v \colon \boldsymbol{X}^n \to \boldsymbol{V}X^n$ by equating this map to the coface map $d^0 \colon \boldsymbol{X}^n \to \boldsymbol{X}^{n+1}$.

Definition 5.2.4. [May67, 23.3] For $Y \in \mathbf{sSet}_{*\mathbf{K}}$, the path space over Y is a simplicial set PY defined by letting

$$\iota \colon Y_{n+1} \to PY_n$$

be an isomorphism of sets and by setting

$$\begin{aligned} \delta_i &= \iota \delta_{i+1} \iota^{-1} \\ \sigma_i &= \iota \sigma_{i+1} \iota^{-1}. \end{aligned}$$

Let $\lambda \colon PY_n \to Y_n$ be the map $\lambda = \delta_0 \iota^{-1}$. One can check that PY is a Kan complex and λ a Kan fibration.

Definition 5.2.5. Let $\boldsymbol{X}$ be a cosimplicial object over $\mathbf{sSet}_{*\mathbf{K}}$. We introduce a cosimplicial object $D_1\boldsymbol{X}$, the derivation of $\boldsymbol{X}$, as the homotopy fiber of $v \colon \boldsymbol{X} \to \boldsymbol{V}X$, along with a map $\delta \colon D_1\boldsymbol{X} \to \boldsymbol{X}$. That is to say, it is the pullback of the diagram

$$\begin{array}{ccc} & & P\boldsymbol{V}X \\ & & \downarrow \lambda \\ \boldsymbol{X} & \xrightarrow{v} & \boldsymbol{V}X. \end{array}$$

5.3. The Bousfield-Kan spectral sequence as a homotopy spectral sequence

For an augmented cosimplicial space $\boldsymbol{X}$, take its derivation to get $D_1\boldsymbol{X}$, which comes with a map

$$D_1\boldsymbol{X} \xrightarrow{\delta} \boldsymbol{X}.$$

Set $D_0\boldsymbol{X} = \boldsymbol{X}$ and define inductively $D_{n+1}\boldsymbol{X}$ to be the homotopy fiber of $D_n\boldsymbol{X} \to \boldsymbol{V}D_n\boldsymbol{X}$. This, like above, will also inherently include a map

$$D_{n+1}\boldsymbol{X} \xrightarrow{\delta} D_n\boldsymbol{X}.$$

We have created, for each $i \geq 0$, a homotopy fiber sequence

$$D_{i+1}\boldsymbol{X} \xrightarrow{\delta} D_i\boldsymbol{X} \xrightarrow{v} \boldsymbol{V}D_i\boldsymbol{X},$$

all of which piece together a tower of fibrations

$$\begin{array}{ccc} \vdots & & \\ \downarrow{\scriptstyle\delta} & & \\ D_i\boldsymbol{X} & \longrightarrow & \boldsymbol{V}D_i\boldsymbol{X} \\ \downarrow{\scriptstyle\delta} & & \\ \vdots & & \\ \downarrow{\scriptstyle\delta} & & \\ D_1\boldsymbol{X} & \longrightarrow & \boldsymbol{V}D_1\boldsymbol{X} \\ \downarrow{\scriptstyle\delta} & & \\ \boldsymbol{X} & \longrightarrow & \boldsymbol{V}X. \end{array}$$

By restricting this tower to the augmentations, we get

$$\begin{array}{ccc} \vdots & & \\ \downarrow{\scriptstyle\delta} & & \\ D_i\boldsymbol{X}^{-1} & \longrightarrow & \boldsymbol{V}D_i\boldsymbol{X}^{-1} = D_i\boldsymbol{X}^0 \\ \downarrow{\scriptstyle\delta} & & \\ \vdots & & \\ \downarrow{\scriptstyle\delta} & & \\ D_1\boldsymbol{X}^{-1} & \longrightarrow & \boldsymbol{V}D_1\boldsymbol{X}^{-1} = D_1\boldsymbol{X}^0 \\ \downarrow{\scriptstyle\delta} & & \\ \boldsymbol{X}^{-1} & \longrightarrow & \boldsymbol{V}X^{-1} = \boldsymbol{X}^0. \end{array}$$

Definition 5.3.1. The Bousfield-Kan spectral sequence (sometimes also the unstable Adams spectral sequence), $\{E_r\boldsymbol{X}\}$, is the homotopy spectral sequence associated to this tower of augmentation fibrations constructed using the methods of section 5.1, with a caveat: we redefine it to be *fringed* in dimension 1. Meaning,

$$E_1^{s,t}\boldsymbol{X} = \begin{cases} \pi_{t-s}D_s\boldsymbol{X}^0 & t-s \geq 1 \\ 0 & t-s = 0 \end{cases}$$

and

$$E_r^{s,t}\boldsymbol{X} = \begin{cases} \ker d_{r-1}/\operatorname{im} d_{r-1} & t-s \geq 2 \\ \mathcal{Z}_{r-1}^{s,t}/\operatorname{im} d_{r-1} & t-s = 1, \end{cases}$$

where $\mathcal{Z}_{r-1}^{s,s+1}\boldsymbol{X} \subset E_{r-1}^{s,s+1}\boldsymbol{X}$ is the group of elements for which the image under the boundary map $\partial\colon \pi_1 D_s \boldsymbol{X}^0 \to \pi_0 D_{s+1}\boldsymbol{X}^{-1}$ lifts to $\pi_0 D_{s+r}\boldsymbol{X}^{-1}$, see [BK72b, 4.1].

5.4. Normalized homotopy groups

Definition 5.4.1. Given a cosimplicial Kan complex $\boldsymbol{X}$, the normalized homotopy groups of $\boldsymbol{X}$ are subgroups of the homotopy groups

$$\pi_i' \boldsymbol{X}^n \subset \pi_i \boldsymbol{X}^n, \quad i, n \geq 0$$

with

$$\pi_i' \boldsymbol{X}^n \coloneqq \pi_i \boldsymbol{X}^n \cap \ker s^0 \cap \cdots \cap \ker s^{n-1}$$

where each s is a codegeneracy map.

We have a homotopy fiber sequence

$$D_1 \boldsymbol{X} \xrightarrow{\delta} \boldsymbol{X} \xrightarrow{v} \boldsymbol{V} X$$

to which applying homotopy produces the usual long exact sequence. Then, by propositions in [BK73a], there is an isomorphism

$$\partial_{\mathrm{it}}\colon \pi_t' \boldsymbol{X}^s \xrightarrow{\cong} \pi_t D_s \boldsymbol{X}^0, \quad t \geq s \geq 0$$

for $t \geq s \geq 0$. Furthermore, there is a commuting diagram

$$\begin{array}{ccc} \pi_t' \boldsymbol{X}^{s-1} & \xrightarrow{\Sigma(-1)^i d_*^i} & \pi_t' \boldsymbol{X}^s \\ \downarrow{\scriptstyle \partial_{\mathrm{it}}} & & \downarrow{\scriptstyle \partial_{\mathrm{it}}} \\ E_1^{s-1,t} = \pi_{t-s+1} D_{s-1} \boldsymbol{X}^0 & \xrightarrow{d_1} & E_1^{s,t} = \pi_{t-s} D_s \boldsymbol{X}^0 \end{array}$$

expressing the E_1-term of the spectral sequence on $\boldsymbol{X}$ as the normalized homotopy groups of $\boldsymbol{X}$.

CHAPTER 6

Some pairings in the spectral sequence

6.1. Pairings on homotopy groups

Throughout this chapter and the next, and as has always been the case, everything will be defined on simplicial sets. Much of the prior constructions can also be done over topological spaces from the ground up, but it is much simpler to convert everything from simplicial sets to topological spaces via the singular and geometric realization functors (cf. [**GJ99**]). If one needs to work in topological setting, one can, for instance, take a topological space X and consider $|E(\mathrm{Sing}(X))|$ for the topological equivalent of EX.

Definition 6.1.1. For simplicial abelian groups A, B, let

$$\widetilde{\wedge}_\pi \colon \pi_t A \times \pi_{t'} B \to \pi_{t+t'}(A \otimes B)$$

be the map induced by the Eilenberg-Zilber map [**EZ53**]

$$N_* A \otimes N_* B \to N_*(A \otimes B)$$

on the normalized chain complexes [**GJ99**, III.2]. An element in $\alpha \in \pi_t A$ and in $\beta \in \pi_{t'} B$ can be represented by a cycle in $N_* A$ and in $N_* B$, which, by abuse of notation, we shall also denote as α and β respectively. Taking the tensor of the two gives a cycle $\alpha \otimes \beta \in N_* A \otimes N_* B$. Then the EZ map gives a cycle in $N_*(A \otimes B)$, which represents an element in $\pi_{t+t'}(A \otimes B)$.

Proposition 6.1.2. The map $\widetilde{\wedge}_\pi$ is

(i) bilinear;
(ii) associative;
(iii) commutative up to sign $(-1)^{tt'}$.

Proof. These are immediately inherited from the Eilenberg-Zilber map, possessing these same properties; see Theorems 5.1 and 5.2 of [**EML53**]. □

Definition 6.1.3. [**BK73b**, 10.1] For X, Y pointed simplicial sets, define

$$\wedge_\pi \colon \pi_t X \times \pi_{t'} Y \to \pi_{t+t'}(X \wedge Y)$$

as the unique natural map such that the following diagram commutes whenever X, Y are simplicial abelian groups:

$$\begin{array}{ccc} \pi_t X \times \pi_{t'} Y & \xrightarrow{\wedge_\pi} & \pi_{t+t'}(X \wedge Y) \\ & \searrow^{\widetilde{\wedge}_\pi} & \downarrow p \\ & & \pi_{t+t'}(X \otimes Y) \end{array}$$

where $p(x, y) = x \otimes y$. $\wedge_\pi$ inherits the same bilinear, associativity, and commutativity properties from $\widetilde{\wedge}_\pi$.

Consider now the function complex with basepoint functor $\mathbf{Hom}(-,-)$ [BK72b, 7.1].

Definition 6.1.4. For W, $X \in \mathbf{sSet}*$, $\mathbf{Hom}(W,X)$ is a pointed simplicial set which has as n-simplices morphisms in $\mathbf{sSet}_{*\mathbf{K}}$ of the form

$$\Delta^n \wedge W \to X$$

and has face and degeneracy maps

$$\delta_i \colon \Delta^{n-1} \wedge W \to \Delta^n \wedge W \to X$$
$$\sigma_i \colon \Delta^{n+1} \wedge W \to \Delta^n \wedge W \to X$$

obtained by precomposing the n-simplex with the maps

$$d^i \wedge \mathrm{id} \colon \Delta^{n-1} \wedge W \to \Delta^n \wedge W$$
$$s^i \wedge \mathrm{id} \colon \Delta^{n+1} \wedge W \to \Delta^n \wedge W.$$

Now consider m-simplices

$$(u \colon \Delta^m \wedge W \to X) \in \mathbf{Hom}(W,X)$$
$$(v \colon \Delta^m \wedge X \to Y) \in \mathbf{Hom}(X,Y)$$

and define a map

$$c \colon \mathbf{Hom}(X,Y) \wedge \mathbf{Hom}(W,X) \to \mathbf{Hom}(W,Y)$$

on the m-simplices, such that $c(v,u)$ is the composite

$$\Delta^m \wedge W \xrightarrow{\mathrm{diag} \wedge \mathrm{id}} \Delta^m \wedge \Delta^m \wedge W \xrightarrow{\mathrm{id} \wedge u} \Delta^m \wedge X \xrightarrow{v} Y.$$

Lastly, define the composition pairing $\circ_\pi$ as the composite

$$\pi_t \mathbf{Hom}(X,Y) \times \pi_{t'} \mathbf{Hom}(W,X) \xrightarrow{\wedge_\pi} \pi_{t+t'}(\mathbf{Hom}(X,Y) \wedge \mathbf{Hom}(W,X)) \xrightarrow{\pi_{t+t'}(c)} \pi_{t+t'} \mathbf{Hom}(W,Y).$$

6.2. A pairing on spectral sequences

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be cosimplicial simplicial sets.

Definition 6.2.1. Define a pairing of E_1-terms of spectral sequences

$$\wedge_{BK} \colon E_1^{s,t}\boldsymbol{X} \times E_1^{s',t'}\boldsymbol{Y} \to E_1^{s+s',t+t'}(\boldsymbol{X} \wedge \boldsymbol{Y})$$

to be the composite

$$\pi_t'\boldsymbol{X}^s \times \pi_{t'}'\boldsymbol{Y}^{s'} \xrightarrow{f} \pi_t'\boldsymbol{X}^{s+s'} \times \pi_{t'}'\boldsymbol{Y}^{s+s'} \xrightarrow{\wedge_\pi} \pi_{t+t'}'(\boldsymbol{X} \wedge \boldsymbol{Y})^{s+s'}$$

where f is the graded Alexander-Whitney map [BK73a, 7.1], [May67, p. 132]

$$f(x,y) = ((-1)^{ts'} d^{s+s'} \cdots d^{s+1}x, d^{s-1} \cdots d^0 y)$$

and $\wedge_\pi$ the pairing on normalized homotopy from last section. It is shown in [BK73b, 9.3, 10.4] that this pairing extends to a pairing on spectral sequences

$$\wedge_{BK}\colon E_r^{s,t}\boldsymbol{X} \times E_r^{s',t'}\boldsymbol{Y} \to E_r^{s+s',t+t'}(\boldsymbol{X} \wedge \boldsymbol{Y}).$$

Theorem 6.2.2. This pairing has the properties:

(i) for $x \in E_r^{s,t}\boldsymbol{X}$ and $y \in E_r^{s',t'}\boldsymbol{Y}$,

$$d_r(x \wedge_{BK} y) = (d_r x \wedge_{BK} y) + (-1)^{t-s}(x \wedge_{BK} d_r y);$$

(ii) the pairing is bilinear;
(iii) the pairing is associative;
(iv) the pairing is commutative up to sign $(-1)^{(t-s)(t'-s')}$ for $r \geq 2$;
(v) the pairing is unique;
(vi) the pairing is natural.

Proof. See Theorems 9.3 and 10.4 of [BK73b]. □

Remark 6.2.3. The bilinearity means we also have a pairing

$$\wedge_{BK}\colon E_r^{s,t}\boldsymbol{X} \otimes E_r^{s',t'}\boldsymbol{Y} \to E_r^{s+s',t+t'}(\boldsymbol{X} \wedge \boldsymbol{Y})$$

by the universal property of tensor products.

Remark 6.2.4. It ought to be said that this pairing is for the "Bousfield-Kan spectral sequence" constructed in [BK73b], whereas our "Bousfield-Kan spectral sequence" is one created using the methods of [BK73a]. By [BK73b, 7.3], these two spectral sequences are naturally equivalent, so the pairing is applicable to our spectral sequence.

6.3. A pairing on spectral sequences using a map $\boldsymbol{a}$

Definition 6.3.1 ([Lib03, 3.3])**.** A triple (T, η, μ) in $\mathbf{sSet}_*$ is called a module triple if there is a *left strength map*

$$\alpha\colon TX \wedge A \to T(X \wedge A)$$

natural in both X and A, such that the following diagrams commute:

$$\begin{array}{ccc} X \wedge A & = & X \wedge A \\ \downarrow{\scriptstyle \eta \wedge \mathrm{id}} & & \downarrow{\scriptstyle \eta} \\ TX \wedge A & \xrightarrow{\alpha} & T(X \wedge A) \end{array} \qquad \begin{array}{ccc} T^2X \wedge A & \xrightarrow{T(\alpha)\circ\alpha} & T^2(X \wedge A) \\ \downarrow{\scriptstyle \mu \wedge \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ TX \wedge A & \xrightarrow{\alpha} & T(X \wedge A) \end{array}$$

$$\begin{array}{ccccc} (TX \wedge A) \wedge B & \xrightarrow{\alpha \wedge \mathrm{id}} & T(X \wedge A) \wedge B & \xrightarrow{\alpha} & T((X \wedge A) \wedge B) \\ \downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \cong} \\ TX \wedge (A \wedge B) & & \xrightarrow{\alpha} & & T(X \wedge (A \wedge B)). \end{array}$$

By commutativity of the smash product, we can also define a *right strength map*

$$\alpha_2\colon A \wedge TX \xrightarrow{\text{twist}} TX \wedge A \xrightarrow{\alpha} T(X \wedge A) \xrightarrow{T(\text{twist})} T(A \wedge X)$$

satisfying the analogous diagrams.

Proposition 6.3.2. Let $\Sigma\colon \mathscr{C} \rightleftarrows \mathscr{D} : \Omega$ be adjoint functors between closed symmetric monoidal categories. Suppose there is a natural isomorphism $\Sigma(C \otimes C') \cong \Sigma C \otimes \Sigma C'$. If (T, η, μ) is a module triple on $\mathscr{D}$, then $S \coloneqq \Omega T \Sigma$ defines a module triple on $\mathscr{C}$. Furthermore, if T has strength map α, and T is a commutative module triple, meaning the diagram

$$\begin{array}{ccccc} TX \otimes TY & \xrightarrow{\alpha} & T(X \otimes TY) & \xrightarrow{T(\alpha_2)} & T^2(X \otimes Y) \\ \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \mu} \\ T(TX \otimes Y) & \xrightarrow{T(\alpha)} & T^2(X \otimes Y) & \xrightarrow{\mu} & T(X \otimes Y) \end{array}$$

commutes, then S is also a commutative module triple.

Proof. That S is a module triple if T is one is proved in [**Lib03**, 3.10]. It remains to show that S satisfies the commutative module triple diagram. First, recall from the reference that the unit map ψ is defined to be the adjoint of

$$\eta\colon \Sigma X \to T\Sigma X,$$

the product map the composite

$$\varphi\colon \Omega T \Sigma \Omega T \Sigma X \xrightarrow{\Omega T(\varepsilon)} \Omega T^2 \Sigma X \xrightarrow{\Omega\mu} \Omega T \Sigma X,$$

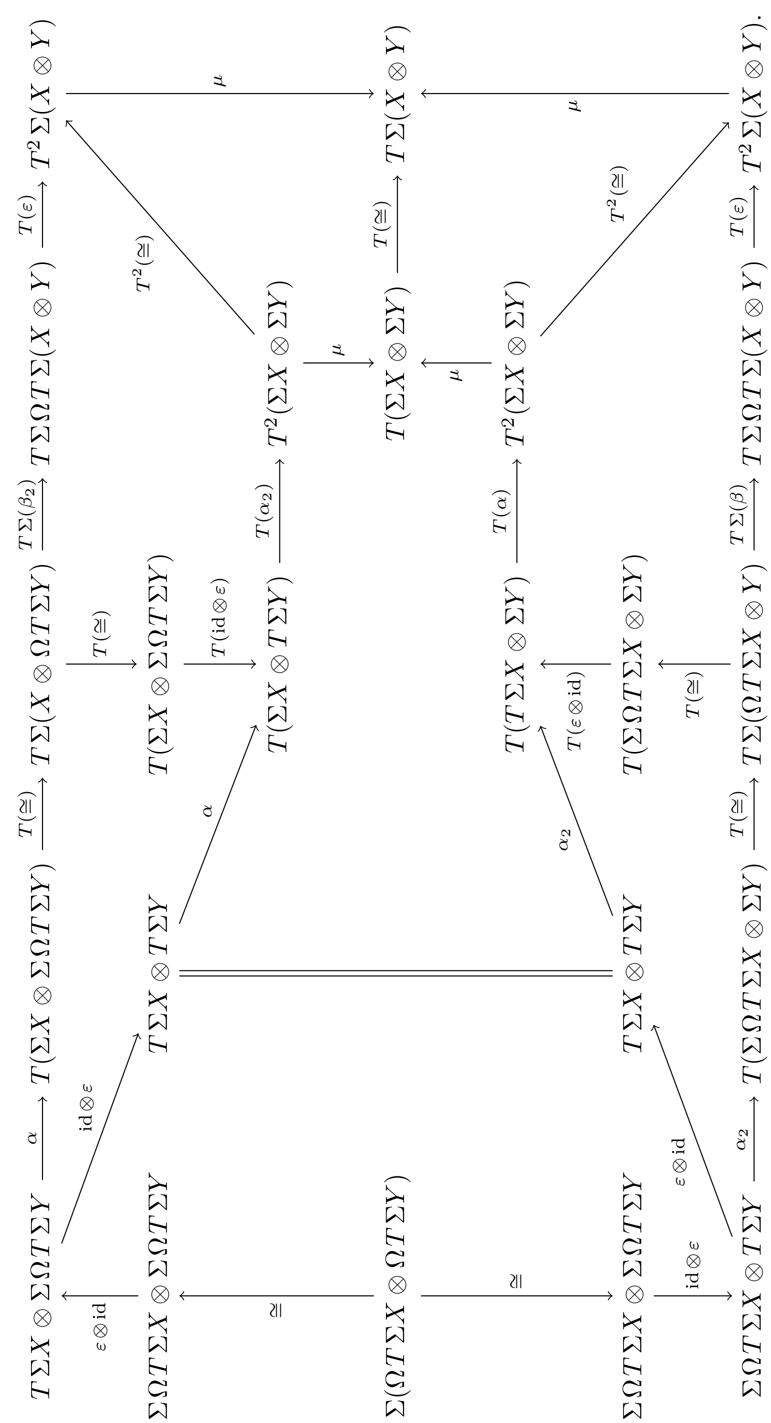
and the strength map β the adjoint of

$$\widetilde{\beta}\colon \Sigma(\Omega T \Sigma X \otimes Y) \cong \Sigma \Omega T \Sigma X \otimes \Sigma Y \xrightarrow{\varepsilon \otimes \mathrm{id}} T\Sigma X \otimes \Sigma Y \xrightarrow{\alpha} T(\Sigma X \otimes \Sigma Y) \cong T\Sigma(X \otimes Y).$$

We want to show that

$$\begin{array}{ccccc} SX \otimes SY & \xrightarrow{\beta} & S(X \otimes SY) & \xrightarrow{S(\beta_2)} & S^2(X \otimes Y) \\ \downarrow{\scriptstyle \beta_2} & & & & \downarrow{\scriptstyle \varphi} \\ S(SX \otimes Y) & \xrightarrow{S(\beta)} & S^2(X \otimes Y) & \xrightarrow{\varphi} & S(X \otimes Y) \end{array}$$

commutes. Expanding out the diagram via the definitions, and taking the adjoint diagram results in



The center cell commutes since T is a commutative module triple. When viewed right-side up, the bottom-right cell commutes by definition since $T(\varepsilon) \circ T\,\Sigma(\beta)$ is the adjoint of β by Lemma 2.3.7, and similarly for the upper-right cell. The rest commute either immediately by definition or by naturality of the relevant maps. □

Corollary 6.3.3. For E a symmetric ring spectrum, (E, ψ, φ) is a commutative module triple.

Proof. Immediate upon noticing that $E \wedge -$ with strength map the identity is a commutative module triple. □

Definition 6.3.4. For X, Y pointed simplicial sets, define

$$EX \wedge EY \xrightarrow{a} E(X \wedge Y)$$

by

$$EX \wedge EY \xrightarrow{\alpha} E(X \wedge EY) \xrightarrow{E(\alpha_2)} E^2(X \wedge Y) \xrightarrow{\varphi} E(X \wedge Y).$$

Then the composite

$$a_{n+1} \colon E^{n+1}X \wedge E^{n+1}Y \xrightarrow{a} E(E^nX \wedge E^nY) \xrightarrow{E(a)} \cdots \xrightarrow{E^n(a)} E^{n+1}(X \wedge Y)$$

gives rise to a map

$$\boldsymbol{E}X \wedge \boldsymbol{E}Y \xrightarrow{\boldsymbol{a}} \boldsymbol{E}(X \wedge Y).$$

Theorem 6.3.5. $\boldsymbol{a}$ is a cosimplicial map.

Proof. We first show that a commutes with the coface maps d^i, so that there is a diagram

$$\begin{array}{ccc} E^nX \wedge E^nY & \xrightarrow{a_n} & E^n(X \wedge Y) \\ \downarrow{\scriptstyle d^i \wedge d^i} & & \downarrow{\scriptstyle d^i} \\ E^{n+1}X \wedge E^{n+1}Y & \xrightarrow{a_{n+1}} & E^{n+1}(X \wedge Y) \end{array}$$

that commutes for $0 \le i \le n$. We show that this diagram separates into iterations of three different types of diagrams, each of which will commute. The very first is given by

$$\begin{array}{ccccccc} E^nX \wedge E^nY & \xrightarrow{\alpha} & E(E^{n-1}X \wedge E^nY) & \xrightarrow{E(\alpha_2)} & E^2(E^{n-1}X \wedge E^{n-1}Y) & \xrightarrow{\varphi} & E(E^{n-1}X \wedge E^{n-1}Y) \\ \downarrow{\scriptstyle d^i \wedge d^i} & & \downarrow{\scriptstyle E(d^{i-1} \wedge d^i)} & & \downarrow{\scriptstyle E^2(d^{i-1} \wedge d^{i-1})} & & \downarrow{\scriptstyle E(d^{i-1} \wedge d^{i-1})} \\ E^{n+1}X \wedge E^{n+1}Y & \xrightarrow{\alpha} & E(E^nX \wedge E^{n+1}Y) & \xrightarrow{E(\alpha_2)} & E^2(E^nX \wedge E^nY) & \xrightarrow{\varphi} & E(E^nX \wedge E^nY). \end{array}$$

The three squares commute by naturality of α, α_2, and φ. For convenience we shall refer to three horizontal consecutive squares in a diagram as a *block*. This block corresponds to the first map a in the composite a_n, and continues to the right, where the three next horizontal maps are $E(\alpha)$, $E(\alpha_2)$, and $E(\varphi)$, corresponding to the second map $E(a)$ in the composite a_n. We encourage the reader to write out the next block of the diagram and verify that it is an iteration of the block above, only with an application of E applied to it and with appropriate changes to the indexing. Then there are i such blocks, corresponding to the first i maps in a_n.

After the first i blocks, the next changes dramatically. Block $i+1$ is

$$\begin{array}{ccccccc}
E^i(E^{n-i}X \wedge E^{n-i}Y) & = & E^i(E^{n-i}X \wedge E^{n-i}Y) & = & E^i(E^{n-i}X \wedge E^{n-i}Y) & = & E^i(E^{n-i}X \wedge E^{n-i}Y) \\
\downarrow{\scriptstyle E^i(d^0 \wedge d^0)} & & \downarrow{\scriptstyle E^i(d^0)\circ E^i(\mathrm{id}\wedge d^0)} & & \downarrow{\scriptstyle E^i(d^0)\circ E^i(d^0)} & & \downarrow{\scriptstyle E^i(d^0)} \\
E^i(E^{n-i+1}X \wedge E^{n-i+1}Y) & \xrightarrow{E^i(\alpha)} & E^{i+1}(E^{n-i}X \wedge E^{n+i+1}Y) & \xrightarrow{E^{i+1}(\alpha_2)} & E^{i+2}(E^{n-i}X \wedge E^{n-i}Y) & \xrightarrow{E^i(\varphi)} & E^{i+1}(E^{n-i}X \wedge E^{n-i}Y).
\end{array}$$

The right square can be considered as E^i applied to the square

$$\begin{array}{ccc}
A \wedge B & \xrightarrow{\psi} & E(A \wedge B) \\
\downarrow{\scriptstyle \psi} & // & \uparrow{\scriptstyle \varphi} \\
E(A \wedge B) & \xrightarrow{\psi} & E^2(A \wedge B),
\end{array}$$

with the appropriate substitutions for A and B, and upon noticing that $d^0 \coloneqq \psi$. The top triangle obviously commutes. The bottom triangle commutes from the properties of a triple.

Returning to block $i+1$, in order to see that the left and middle square commute, first consider the diagram

$$\begin{array}{ccccc}
A \wedge B & = & A \wedge B & = & A \wedge B \\
\downarrow{\scriptstyle \mathrm{id}\wedge\psi} & & \downarrow{\scriptstyle \mathrm{id}\wedge\psi} & & \downarrow{\scriptstyle \psi} \\
A \wedge EB & = & A \wedge EB & \xrightarrow{\alpha_2} & E(A \wedge B) \\
\downarrow{\scriptstyle \psi\wedge\mathrm{id}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
EA \wedge EB & \xrightarrow{\alpha} & E(A \wedge EB) & \xrightarrow{E(\alpha_2)} & E^2(A \wedge B).
\end{array}$$

The top-left square clearly commutes. The top-right and bottom-left squares commute by the first diagram of Definition 6.3.1. The bottom-right square commutes by naturality of ψ. Again, with the desired substitutions for A and B, we see that E^i applied to this diagram forms the left and middle squares of block $i+1$.

Finally, we have blocks $i+2$ through $n+1$, each of which take the form

$$\begin{array}{ccccccc}
E^k(E^{n-k}X \wedge E^{n-k}Y) & \xrightarrow{E^k(\alpha)} & E^{k+1}(E^{n-k-1}X \wedge E^{n-k}Y) & \xrightarrow{E^{k+1}(\alpha_2)} & E^{k+2}(E^{n-k-1}X \wedge E^{n-k-1}Y) & \xrightarrow{E^k(\varphi)} & E^{k+1}(E^{n-k-1}X \wedge E^{n-k-1}Y) \\
\downarrow{\scriptstyle E^i(d^0)} & & \downarrow{\scriptstyle E^i(d^0)} & & \downarrow{\scriptstyle E^i(d^0)} & & \downarrow{\scriptstyle E^i(d^0)} \\
E^{k+1}(E^{n-k}X \wedge E^{n-k}Y) & \xrightarrow{E^{k+1}(\alpha)} & E^{k+2}(E^{n-k-1}X \wedge E^{n-k}Y) & \xrightarrow{E^{k+2}(\alpha_2)} & E^{k+3}(E^{n-k-1}X \wedge E^{n-k-1}Y) & \xrightarrow{E^{k+1}(\varphi)} & E^{k+2}(E^{n-k-1}X \wedge E^{n-k-1}Y)
\end{array}$$

for $i \leq k \leq n-1$. These squares all commute by naturality of $d^0 \coloneqq \psi$. This shows that a_n commutes with the coface maps. We now show that it also commutes with codegeneracies.

We want to show that

$$\begin{array}{ccc}
E^{n+2}X \wedge E^{n+2}Y & \xrightarrow{a_{n+2}} & E^{n+2}(X \wedge Y) \\
\downarrow{\scriptstyle s^i \wedge s^i} & & \downarrow{\scriptstyle s^i} \\
E^{n+1}X \wedge E^{n+1}Y & \xrightarrow{a_{n+1}} & E^{n+1}(X \wedge Y)
\end{array}$$

commutes for $0 \leq i \leq n$. As before, we'll separate this diagram into different types of blocks. We'll have four different types rather than three, however.

Blocks 1 through i take the form

$$\begin{array}{ccccccc}
E^p(E^{n+2-p}X \wedge E^{n+2-p}Y) & \xrightarrow{E^p(\alpha)} & E^{p+1}(E^{n+1-p}X \wedge E^{n+2-p}Y) & \xrightarrow{E^{p+1}(\alpha_2)} & E^{p+2}(E^{n+1-p}X \wedge E^{n+1-p}Y) & \xrightarrow{E^p(\varphi)} & E^{p+1}(E^{n+1-p}X \wedge E^{n+1-p}Y) \\
\downarrow{\scriptstyle E^p(s^{i-p}\wedge s^{i-p})} & & \downarrow{\scriptstyle E^{p+1}(s^{i-p-1}\wedge s^{i-p-1})} & & \downarrow{\scriptstyle E^{p+2}(s^{i-p-1}\wedge s^{i-p-1})} & & \downarrow{\scriptstyle E^{p+1}(s^{i-p-1}\wedge s^{i-p-1})} \\
E^p(E^{n+1-p}X \wedge E^{n+1-p}Y) & \xrightarrow{E^p(\alpha)} & E^{p+1}(E^{n-p}X \wedge E^{n+1-p}Y) & \xrightarrow{E^{p+1}(\alpha_2)} & E^{p+2}(E^{n-p}X \wedge E^{n-p}Y) & \xrightarrow{E^p(\varphi)} & E^{p+1}(E^{n-p}X \wedge E^{n-p}Y)
\end{array}$$

for $0 \leq p \leq i-1$. These squares commute by naturality of α, α_2, and φ. Then block $i+1$ is

$$\begin{array}{ccccccc}
E^i(E^{n+2-i}X \wedge E^{n+2-i}Y) & \xrightarrow{E^i(\alpha)} & E^{i+1}(E^{n+1-i}X \wedge E^{n+2-i}Y) & \xrightarrow{E^{i+1}(\alpha_2)} & E^{i+2}(E^{n+1-i}X \wedge E^{n+1-i}Y) & \xrightarrow{E^i(\varphi)} & E^{i+1}(E^{n+1-i}X \wedge E^{n+1-i}Y) \\
\downarrow{\scriptstyle E^i(s^0\wedge s^0)} & & \downarrow{\scriptstyle E^{i+1}(\mathrm{id}\wedge s^0)\circ E^i(s^0)\circ E^{i+1}(\alpha)} & & & & \downarrow{\scriptstyle E^i(s^0)\circ E^{i+1}(\alpha_2) \circ E^i(s^0)\circ E^{i+1}(\alpha)} \\
E^i(E^{n+1-i}X \wedge E^{n+1-i}Y) & \xrightarrow{E^i(\alpha)} & E^{i+1}(E^{n-i}X \wedge E^{n+1-i}Y) & \xrightarrow{E^{i+1}(\alpha_2)} & E^{i+2}(E^{n-i}X \wedge E^{n-i}Y) & \xrightarrow{E^i(\varphi)} & E^{i+1}(E^{n-i}X \wedge E^{n-i}Y).
\end{array}$$

This is the same as E^i applied to the diagram

$$\begin{array}{ccccccc}
E^2A \wedge E^2B & \xrightarrow{\alpha} & E(EA \wedge E^2B) & \xrightarrow{E(\alpha_2)} & E^2(EA \wedge EB) & \xrightarrow{\varphi} & E(EA \wedge EB) \\
\downarrow{\scriptstyle \varphi\wedge\mathrm{id}} & 1 & \downarrow{\scriptstyle \varphi\circ E(\alpha)} & & 2 & & \downarrow{\scriptstyle \varphi\circ E(\alpha)} \\
EA \wedge E^2B & \xrightarrow{\alpha} & E(A \wedge E^2B) & \xrightarrow{E(\alpha_2)} & E^2(A \wedge EB) & \xrightarrow{\varphi} & E(A \wedge EB) \\
\downarrow{\scriptstyle \mathrm{id}\wedge\varphi} & 3 & \downarrow{\scriptstyle E(\mathrm{id}\wedge\varphi)} & 4 & \downarrow{\scriptstyle E(\varphi)\circ E^2(\alpha_2)} & 5 & \downarrow{\scriptstyle \varphi\circ E(\alpha_2)} \\
EA \wedge EB & \xrightarrow{\alpha} & E(A \wedge EB) & \xrightarrow{E(\alpha_2)} & E^2(A \wedge B) & \xrightarrow{\varphi} & E(A \wedge B)
\end{array}$$

with the proper As and Bs. 1 and 4 commute by the second diagram of DEFINITION 6.3.1. 3 and 5 commute by naturality of α and φ respectively. Rewrite 2 as

$$\begin{array}{ccccccc}
E(EA \wedge E^2B) & & \xrightarrow{\qquad E(\alpha_2) \qquad} & & E^2(EA \wedge EB) & \xrightarrow{\varphi} & E(EA \wedge EB) \\
\Big\downarrow & & & & \downarrow{\scriptstyle E^2(\alpha)} & & \downarrow{\scriptstyle E(\alpha)} \\
{\scriptstyle E(\alpha)} & & & & E^3(A \wedge EB) & \xrightarrow{\varphi} & E^2(A \wedge EB) \\
\Big\downarrow & & & & \downarrow{\scriptstyle E(\varphi)} & & \Big\downarrow \\
E^2(A \wedge E^2B) & \xrightarrow{E^2(\alpha_2)} & E^3(A \wedge EB) & \xrightarrow{E(\varphi)} & E^2(A \wedge EB) & \overset{\varphi}{\searrow} & {\scriptstyle \varphi} \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & & & \Big\downarrow \\
E(A \wedge E^2B) & \xrightarrow{E(\alpha_2)} & E^2(A \wedge EB) & & \xrightarrow{\qquad \varphi \qquad} & & E(A \wedge EB).
\end{array}$$

The upper-left square commutes by PROPOSITION 6.3.2, the upper-right and bottom-left by naturality, and the bottom-right polygons by associativity of φ.

Block $i+2$ is given by

$$\begin{array}{ccccccc}
E^{i+1}(E^{n+1-i}X \wedge E^{n+1-i}Y) & \xrightarrow{E^{i+1}(\alpha)} & E^{i+2}(E^{n-i}X \wedge E^{n+1-i}Y) & \xrightarrow{E^{i+2}(\alpha_2)} & E^{i+3}(E^{n-i}X \wedge E^{n-i}Y) & \xrightarrow{E^{i+1}(\varphi)} & E^{i+2}(E^{n-i}X \wedge E^{n-i}Y) \\
\Big\downarrow {\scriptstyle E^i(s^0)\circ E^{i+1}(\alpha_2) \circ E^i(s^0)\circ E^{i+1}(\alpha)} & & \Big\downarrow {\scriptstyle E^i(s^0)\circ E^{i+1}(\alpha_2)\circ E^i(s^0)} & & \Big\downarrow {\scriptstyle E^i(s^0)\circ E^i(s^0)} & & \Big\downarrow {\scriptstyle E^i(s^0)} \\
E^{i+1}(E^{n-i}X \wedge E^{n-i}Y) & = & E^{i+1}(E^{n-i}X \wedge E^{n-i}Y) & = & E^{i+1}(E^{n-i}X \wedge E^{n-i}Y) & = & E^{i+1}(E^{n-i}X \wedge E^{n-i}Y).
\end{array}$$

The left square commutes by definition and the right square commutes by the properties of a triple, since $s^0 \coloneqq \varphi$. To show the middle square commutes, we look at

$$\begin{array}{ccc}
E^2(A \wedge EB) & \xrightarrow{E^2(\alpha_2)} & E^3(A \wedge B) \\
\Big\downarrow \varphi & & \Big\downarrow \varphi \\
E(A \wedge EB) & \xrightarrow{E(\alpha_2)} & E^2(A \wedge B) \\
\Big\downarrow \varphi \circ E(\alpha_2) & & \Big\downarrow \varphi \\
E(A \wedge B) & = & E(A \wedge B)
\end{array}$$

and see that the top square commutes by naturality of φ, and the bottom square commutes by definition. The middle square of block $i+2$ is E^i applied to this diagram, with appropriate substitutions, and so block $i+2$ commutes.

Lastly, we have blocks $i+3$ through $n+2$, which take the form

$$\begin{array}{ccccccc}
E^{i+q+2}(E^{n-i-q}X \wedge E^{n-i-q}Y) & \xrightarrow{E^{i+q+2}(\alpha)} & E^{i+q+3}(E^{n-i-q-1}X \wedge E^{n-i-q}Y) & \xrightarrow{E^{i+q+3}(\alpha_2)} & E^{i+q+4}(E^{n-i-q-1}X \wedge E^{n-i-q-1}Y) & \xrightarrow{E^{i+q+2}(\varphi)} & E^{i+q+3}(E^{n-i-q-1}X \wedge E^{n-i-q-1}Y) \\
\Big\downarrow {\scriptstyle E^i(s^0)} & & \Big\downarrow {\scriptstyle E^i(s^0)} & & \Big\downarrow {\scriptstyle E^i(s^0)} & & \Big\downarrow {\scriptstyle E^i(s^0)} \\
E^{i+q+1}(E^{n-i-q}X \wedge E^{n-i-q}Y) & \xrightarrow{E^{i+q+1}(\alpha)} & E^{i+q+2}(E^{n-i-q-1}X \wedge E^{n-i-q}Y) & \xrightarrow{E^{i+q+2}(\alpha_2)} & E^{i+q+3}(E^{n-i-q-1}X \wedge E^{n-i-q-1}Y) & \xrightarrow{E^{i+q+1}(\varphi)} & E^{i+q+2}(E^{n-i-q-1}X \wedge E^{n-i-q-1}Y)
\end{array}$$

for $0 \le q \le n-i-1$. All the squares commute by naturality of $s^0 \coloneqq \varphi$. ☐

PROPOSITION 6.3.6. $\boldsymbol{a}$ is associative.

PROOF. This is equivalent to showing the diagram

$$\begin{array}{ccccccccc}
E^nX \wedge E^nY \wedge E^nZ & \xrightarrow{a \wedge \mathrm{id}} & E(E^{n-1}X \wedge E^{n-1}Y) \wedge E^nZ & \xrightarrow{E(a)\wedge \mathrm{id}} & E^2(E^{n-2}X \wedge E^{n-2}Y) \wedge E^nZ & \xrightarrow{E^2(a)\wedge \mathrm{id}} & \cdots & \xrightarrow{E^{n-1}(a)\wedge \mathrm{id}} & E^n(X \wedge Y) \wedge E^nZ \\
\Big\downarrow {\scriptstyle \mathrm{id}\wedge a} & & \Big\downarrow {\scriptstyle a} & & \Big\downarrow {\scriptstyle a} & & & & \Big\downarrow {\scriptstyle a} \\
E^nX \wedge E(E^{n-1}Y \wedge E^{n-1}Z) & \xrightarrow{a} & E(E^{n-1}X \wedge E^{n-1}Y \wedge E^{n-1}Z) & \xrightarrow{E(a\wedge \mathrm{id})} & E(E(E^{n-2}X \wedge E^{n-2}Y) \wedge E^{n-1}Z) & \xrightarrow{E(E(a)\wedge \mathrm{id})} & \cdots & \xrightarrow{E(E^{n-2}(a)\wedge \mathrm{id})} & E(E^{n-1}(X \wedge Y) \wedge E^{n-1}Z) \\
\Big\downarrow {\scriptstyle \mathrm{id}\wedge E(a)} & & \Big\downarrow {\scriptstyle E(\mathrm{id}\wedge a)} & & \Big\downarrow {\scriptstyle E(a)} & & & & \Big\downarrow {\scriptstyle E(a)} \\
E^nX \wedge E^2(E^{n-2}Y \wedge E^{n-2}Z) & \xrightarrow{a} & E(E^{n-1}X \wedge E(E^{n-2}Y \wedge E^{n-2}Z)) & \xrightarrow{E(a)} & E^2(E^{n-2}X \wedge E^{n-2}Y \wedge E^{n-2}Z) & \xrightarrow{E^2(a\wedge \mathrm{id})} & \cdots & \xrightarrow{E^2(E^{n-3}(a)\wedge \mathrm{id})} & E^2(E^{n-2}(X \wedge Y) \wedge E^{n-2}Z) \\
\Big\downarrow {\scriptstyle \mathrm{id}\wedge E^2(a)} & & \Big\downarrow {\scriptstyle E(\mathrm{id}\wedge E(a))} & & \Big\downarrow {\scriptstyle E^2(\mathrm{id}\wedge a)} & & & & \Big\downarrow {\scriptstyle E^2(a)} \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots \\
\Big\downarrow {\scriptstyle \mathrm{id}\wedge E^{n-1}(a)} & & \Big\downarrow {\scriptstyle E(\mathrm{id}\wedge E^{n-2}(a))} & & \Big\downarrow {\scriptstyle E^2(\mathrm{id}\wedge E^{n-3}(a))} & & & & \Big\downarrow {\scriptstyle E^{n-1}(a)} \\
E^nX \wedge E^n(Y \wedge Z) & \xrightarrow{a} & E(E^{n-1}X \wedge E^{n-1}(Y \wedge Z)) & \xrightarrow{E(a)} & E^2(E^{n-2}X \wedge E^{n-2}(Y \wedge Z)) & \xrightarrow{E^2(a)} & \cdots & \xrightarrow{E^{n-1}(a)} & E^n(X \wedge Y \wedge Z)
\end{array}$$

commutes. The squares in the top-left to bottom-right diagonal depend on the associativity of a. The rest depend on the naturality of a. We shall show associativity of a.

Let

$$\dagger\colon (A \wedge B) \wedge C \to A \wedge (B \wedge C)$$

be the usual associativity isomorphism. We want the following diagram to commute:

$$
\begin{array}{ccccccc}
(EX\wedge EY)\wedge EZ & \xrightarrow{\alpha\wedge\mathrm{id}} & E(X\wedge EY)\wedge EZ & \xrightarrow{E(\alpha_2)\wedge\mathrm{id}} & E^2(X\wedge Y)\wedge EZ & \xrightarrow{\varphi\wedge\mathrm{id}} & E(X\wedge Y)\wedge EZ \\
\downarrow{\scriptstyle \dagger} & M & \downarrow{\scriptstyle E(\dagger)\circ\alpha} & 1 & \downarrow{\scriptstyle E(\alpha)\circ\alpha} & M & \downarrow{\scriptstyle \alpha} \\
EX\wedge(EY\wedge EZ) & \xrightarrow{\alpha} & E(X\wedge(EY\wedge EZ)) & \xrightarrow{\substack{E^2(\dagger)\circ E(\alpha_2)\\ \circ E(\mathrm{id}\wedge\alpha)}} & E^2((X\wedge Y)\wedge EZ) & \xrightarrow{\varphi} & E((X\wedge Y)\wedge EZ) \\
\downarrow{\scriptstyle \mathrm{id}\wedge\alpha} & N & \downarrow{\scriptstyle E(\mathrm{id}\wedge\alpha)} & 2 & \downarrow{\scriptstyle E^2(\alpha_2)} & N & \downarrow{\scriptstyle E(\alpha_2)} \\
EX\wedge E(Y\wedge EZ) & \xrightarrow{\alpha} & E(X\wedge E(Y\wedge EZ)) & \xrightarrow{\substack{E^3(\dagger)\circ E^2(\alpha_2)\\ \circ E^2(\mathrm{id}\wedge\alpha_2)\circ E(\alpha_2)}} & E^3((X\wedge Y)\wedge Z) & \xrightarrow{\varphi} & E^2((X\wedge Y)\wedge Z) \\
\downarrow{\scriptstyle \mathrm{id}\wedge E(\alpha_2)} & N & \downarrow{\scriptstyle E(\mathrm{id}\wedge E(\alpha_2))} & 3 & \downarrow{\scriptstyle E(\varphi)} & T & \downarrow{\scriptstyle \varphi} \\
EX\wedge E^2(Y\wedge Z) & \xrightarrow{\alpha} & E(X\wedge E^2(Y\wedge Z)) & \xrightarrow{\substack{E^2(\dagger)\circ E(\varphi)\\ \circ E^2(\alpha_2)\circ E(\alpha_2)}} & E^2((X\wedge Y)\wedge Z) & \xrightarrow{\varphi} & E((X\wedge Y)\wedge Z) \\
\downarrow{\scriptstyle \mathrm{id}\wedge\varphi} & N & \downarrow{\scriptstyle E(\mathrm{id}\wedge\varphi)} & M & \downarrow{\scriptstyle E^2(\dagger)} & N & \downarrow{\scriptstyle E(\dagger)} \\
EX\wedge E(Y\wedge Z) & \xrightarrow{\alpha} & E(X\wedge E(Y\wedge Z)) & \xrightarrow{E(\alpha_2)} & E^2(X\wedge(Y\wedge Z)) & \xrightarrow{\varphi} & E(X\wedge(Y\wedge Z)).
\end{array}
$$

The squares labeled N all commute by naturality of the appropriate map. The ones labeled M commute by DEFINITION 6.3.1 and the one labeled T commutes by DEFINITION 2.2.4. To see that 3 commutes, we expand it out.

$$
\begin{array}{ccccccccc}
E(X\wedge E(Y\wedge EZ)) & \xrightarrow{E(\alpha_2)} & E^2(X\wedge(Y\wedge EZ)) & \xrightarrow{\substack{E^2(\alpha_2)\circ\\ E^2(\mathrm{id}\wedge\alpha_2)}} & E^3(X\wedge(Y\wedge Z)) & = & E^3(X\wedge(Y\wedge Z)) & \xrightarrow{E^3(\dagger)} & E^3((X\wedge Y)\wedge Z) \\
\downarrow{\scriptstyle E(\mathrm{id}\wedge E(\alpha_2))} & & \downarrow{\scriptstyle E^2(\mathrm{id}\wedge\alpha_2)} & & \| & & \downarrow{\scriptstyle E(\varphi)} & & \downarrow{\scriptstyle E(\varphi)} \\
E(X\wedge E^2(Y\wedge Z)) & \xrightarrow{E(\alpha_2)} & E^2(X\wedge E(Y\wedge Z)) & \xrightarrow{E^2(\alpha_2)} & E^3(X\wedge(Y\wedge Z)) & \xrightarrow{E(\varphi)} & E^2(X\wedge(Y\wedge Z)) & \xrightarrow{E^2(\dagger)} & E^2((X\wedge Y)\wedge Z)
\end{array}
$$

The left and right-most squares commute by naturality of α_2 and φ. The middle two commute by definition.

We also expand 2 to show it commutes.

$$
\begin{array}{ccccccccc}
E(X\wedge(EY\wedge EZ)) & \xrightarrow{\substack{E(\alpha_2)\circ\\ E(\mathrm{id}\wedge\alpha)}} & E^2(X\wedge(Y\wedge EZ)) & = & E^2(X\wedge(Y\wedge EZ)) & = & E^2(X\wedge(Y\wedge EZ)) & \xrightarrow{E^2(\dagger)} & E^2((X\wedge Y)\wedge EZ) \\
\downarrow{\scriptstyle E(\mathrm{id}\wedge\alpha)} & & \| & & \downarrow{\scriptstyle E^2(\mathrm{id}\wedge\alpha_2)} & & \downarrow{\scriptstyle E^2(\alpha_2)\circ E^2(\mathrm{id}\wedge\alpha_2)} & & \downarrow{\scriptstyle E^2(\alpha_2)} \\
E(X\wedge E(Y\wedge EZ)) & \xrightarrow{E(\alpha_2)} & E^2(X\wedge(Y\wedge EZ)) & \xrightarrow{E^2(\mathrm{id}\wedge\alpha_2)} & E^2(X\wedge E(Y\wedge Z)) & \xrightarrow{E^2(\alpha_2)} & E^3(X\wedge(Y\wedge Z)) & \xrightarrow{E^3(\dagger)} & E^3((X\wedge Y)\wedge Z)
\end{array}
$$

The right square commutes by DEFINITION 6.3.1. The rest commute by definition.

To see 1 commutes, consider the diagram

$$
\begin{array}{ccccccc}
E(X\wedge EY)\wedge EZ & & & \xrightarrow{E(\alpha_2)\wedge\mathrm{id}} & & & E^2(X\wedge Y)\wedge EZ \\
\downarrow{\scriptstyle \alpha} & & & & & & \downarrow{\scriptstyle \alpha} \\
E((X\wedge EY)\wedge EZ) & = & E((X\wedge EY)\wedge EZ) & \xrightarrow{E(\alpha_2\wedge\mathrm{id})} & E(E(X\wedge Y)\wedge EZ) & = & E(E(X\wedge Y)\wedge EZ) \\
\downarrow{\scriptstyle E(\dagger)} & & \downarrow{\scriptstyle E(\mathrm{id}\wedge\alpha)\circ E(\dagger)} & & \downarrow{\scriptstyle E^2(\dagger)\circ E(\alpha)} & & \downarrow{\scriptstyle E(\alpha)} \\
E(X\wedge(EY\wedge EZ)) & \xrightarrow{E(\mathrm{id}\wedge\alpha)} & E(X\wedge E(Y\wedge EZ)) & \xrightarrow{E(\alpha_2)} & E^2(X\wedge(Y\wedge EZ)) & \xrightarrow{E^2(\dagger)} & E^2((X\wedge Y)\wedge EZ).
\end{array}
$$

The top rectangle commutes by naturality of α. The bottom-left and bottom-right squares obviously commute. The bottom-middle square can be rewritten:

$$\begin{array}{ccc}
E((X \wedge EY) \wedge EZ) & \xrightarrow{E(\alpha_2 \wedge \mathrm{id})} & E(E(X \wedge Y) \wedge EZ) \\
\downarrow{\scriptstyle E(\dagger)} & & \downarrow{\scriptstyle E(\alpha)} \\
E(X \wedge (EY \wedge EZ)) & & E^2((X \wedge Y) \wedge EZ) \\
\downarrow{\scriptstyle E(\mathrm{id} \wedge \alpha)} & & \downarrow{\scriptstyle E^2(\dagger)} \\
E(X \wedge E(Y \wedge EZ)) & \xrightarrow{E(\alpha_2)} & E^2(X \wedge (Y \wedge EZ))
\end{array}$$

which can further be expanded into (we also disregard the leading E)

$$\begin{array}{ccccccc}
(X \wedge EY) \wedge EZ & \xrightarrow{\mathrm{twist} \wedge \mathrm{id}} & (EY \wedge X) \wedge EZ & \xrightarrow{\alpha \wedge \mathrm{id}} & E(Y \wedge X) \wedge EZ & \xrightarrow{E(\mathrm{twist}) \wedge \mathrm{id}} & E(X \wedge Y) \wedge EZ \\
\downarrow{\scriptstyle \dagger} & A1 & \downarrow{\scriptstyle \dagger \circ (\mathrm{id} \wedge \mathrm{twist}) \circ \dagger} & A2 & \downarrow{\scriptstyle \alpha} & A3 & \downarrow{\scriptstyle \alpha} \\
X \wedge (EY \wedge EZ) & \xrightarrow{\mathrm{twist}} & (EY \wedge EZ) \wedge X & \xrightarrow{E(\dagger) \circ \alpha \circ (\mathrm{id} \wedge \mathrm{twist}) \circ \dagger} & E((Y \wedge X) \wedge EZ) & \xrightarrow{E(\mathrm{twist} \wedge \mathrm{id})} & E((X \wedge Y) \wedge EZ) \\
\downarrow{\scriptstyle \mathrm{id} \wedge \alpha} & A4 & \downarrow{\scriptstyle \alpha \wedge \mathrm{id}} & A5 & \downarrow{\scriptstyle E(\dagger) \circ E(\mathrm{id} \wedge \mathrm{twist}) \circ E(\dagger)} & A6 & \downarrow{\scriptstyle E(\dagger)} \\
X \wedge E(Y \wedge EZ) & \xrightarrow{\mathrm{twist}} & E(Y \wedge EZ) \wedge X & \xrightarrow{\alpha} & E((Y \wedge EZ) \wedge X) & \xrightarrow{E(\mathrm{twist})} & E(X \wedge (Y \wedge EZ)).
\end{array}$$

*A*3 and *A*4 commute by naturality of α and twist. *A*2 commutes by DEFINITION 6.3.1. *A*5 commutes for the same reason and upon noticing that $\alpha \circ (\mathrm{id} \wedge \mathrm{twist}) = E(\mathrm{id} \wedge \mathrm{twist}) \circ \alpha$ by naturality. *A*1 and *A*6 commute by the properties of twist and $\dagger$.

This shows commutativity of 1 and hence associativity of a. ☐

PROPOSITION 6.3.7. ***a*** is commutative.

PROOF. Like above, it is sufficient to show only that a is commutative. We wish to show that the diagram

$$\begin{array}{ccccccc}
EX \wedge EY & \xrightarrow{\alpha} & E(X \wedge EY) & \xrightarrow{E(\alpha_2)} & E^2(X \wedge Y) & \xrightarrow{\varphi} & E(X \wedge Y) \\
\downarrow{\scriptstyle \mathrm{twist}} & & & & & & \downarrow{\scriptstyle E(\mathrm{twist})} \\
EY \wedge EX & \xrightarrow{\alpha} & E(Y \wedge EX) & \xrightarrow{E(\alpha_2)} & E^2(Y \wedge X) & \xrightarrow{\varphi} & E(Y \wedge X)
\end{array}$$

commutes. By PROPOSITION 6.3.2, we rewrite the bottom row of the diagram.

$$\begin{array}{ccccccc}
EX \wedge EY & \xrightarrow{\alpha} & E(X \wedge EY) & \xrightarrow{E(\alpha_2)} & E^2(X \wedge Y) & \xrightarrow{\varphi} & E(X \wedge Y) \\
\downarrow{\scriptstyle \mathrm{twist}} & & \downarrow{\scriptstyle E(\mathrm{twist})} & & \downarrow{\scriptstyle E^2(\mathrm{twist})} & & \downarrow{\scriptstyle E(\mathrm{twist})} \\
EY \wedge EX & \xrightarrow{\alpha_2} & E(EY \wedge X) & \xrightarrow{E(\alpha)} & E^2(Y \wedge X) & \xrightarrow{\varphi} & E(Y \wedge X)
\end{array}$$

The left and middle squares commute by definition of α_2. The right square commutes by naturality of φ. ☐

Definition 6.3.8. Define a pairing on spectral sequences

$$\wedge_E\colon E_r^{s,t}\boldsymbol{E}X \otimes E_r^{s',t'}\boldsymbol{E}Y \to E_r^{s+s',t+t'}\boldsymbol{E}(X\wedge Y)$$

to be the composite

$$E_r^{s,t}\boldsymbol{E}X \otimes E_r^{s',t'}\boldsymbol{E}Y \xrightarrow{\wedge_{BK}} E_r^{s+s',t+t'}(\boldsymbol{E}X\wedge \boldsymbol{E}Y) \xrightarrow{E_r\boldsymbol{a}} E_r^{s+s',t+t'}\boldsymbol{E}(X\wedge Y).$$

Proposition 6.3.9. The pairing $\wedge_E$ is associative and commutative up to sign.

Proof. Showing associativity is showing that the diagram

$$\begin{array}{ccccc}
E_r\boldsymbol{E}X\otimes E_r\boldsymbol{E}Y\otimes E_r\boldsymbol{E}Z & \xrightarrow{\wedge_{BK}\otimes\mathrm{id}} & E_r[\boldsymbol{E}X\wedge\boldsymbol{E}Y]\otimes E_r\boldsymbol{E}Z & \xrightarrow{E_ra\otimes\mathrm{id}} & E_r[\boldsymbol{E}(X\wedge Y)]\otimes E_r\boldsymbol{E}Z \\
\downarrow{\scriptstyle \mathrm{id}\otimes\wedge_{BK}} & & \downarrow{\scriptstyle \wedge_{BK}} & & \downarrow{\scriptstyle \wedge_{BK}} \\
E_r\boldsymbol{E}X\otimes E_r[\boldsymbol{E}Y\wedge\boldsymbol{E}Z] & \xrightarrow{\wedge_{BK}} & E_r[\boldsymbol{E}X\wedge\boldsymbol{E}Y\wedge\boldsymbol{E}Z] & \xrightarrow{E_r(a\wedge\mathrm{id})} & E_r[\boldsymbol{E}(X\wedge Y)\wedge\boldsymbol{E}Z] \\
\downarrow{\scriptstyle \mathrm{id}\otimes E_ra} & & \downarrow{\scriptstyle E_r(\mathrm{id}\wedge a)} & & \downarrow{\scriptstyle E_ra} \\
E_r\boldsymbol{E}X\otimes E_r\boldsymbol{E}(Y\wedge Z) & \xrightarrow{\wedge_{BK}} & E_r[\boldsymbol{E}X\wedge\boldsymbol{E}(Y\wedge Z)] & \xrightarrow{E_ra} & E_r[\boldsymbol{E}(X\wedge Y\wedge Z)]
\end{array}$$

commutes. The top-left and bottom-right squares commute by associativity of $\wedge_{BK}$ and a respectively. The top-right and bottom-left squares commute by naturality of $\wedge_{BK}$.

By commutativity of a, and the fact that E_ra is a group homomorphism,

$$\begin{aligned}
x\wedge_E y = E_ra(x\wedge_{BK} y) &= E_ra[(-1)^{(t-s)(t'-s')}(y\wedge_{BK} x)] = (-1)^{(t-s)(t'-s')}E_ra(y\wedge_{BK} x)\\
&= (-1)^{(t-s)(t'-s')}(y\wedge_E x). \qquad \square
\end{aligned}$$

CHAPTER 7

The composition pairing

7.1. Bicosimplicial spaces

Definition 7.1.1. A bicosimplicial object $\underline{\boldsymbol{X}}$ over a category $\mathscr{C}$ is a cosimplicial object in the category of cosimplicial objects over $\mathscr{C}$

$$\underline{\boldsymbol{X}} \colon \boldsymbol{\Delta} \to \boldsymbol{c}\mathscr{C}.$$

Equivalently, it is a functor

$$\underline{\boldsymbol{X}} \colon \boldsymbol{\Delta} \times \boldsymbol{\Delta} \to \mathscr{C}.$$

For any pair of non-negative integers (m, n) there is an object $\underline{\boldsymbol{X}}^{m,n} \in \mathscr{C}$, and there are two sets of coface and codegeneracy maps

$$\begin{aligned} \dot{d}^k &\colon \underline{\boldsymbol{X}}^{m-1,\bullet} \to \underline{\boldsymbol{X}}^{m,\bullet} & \qquad \ddot{d}^\ell &\colon \underline{\boldsymbol{X}}^{\bullet,n-1} \to \underline{\boldsymbol{X}}^{\bullet,n} \\ \dot{s}^k &\colon \underline{\boldsymbol{X}}^{m+1,\bullet} \to \underline{\boldsymbol{X}}^{m,\bullet} & \qquad \ddot{s}^\ell &\colon \underline{\boldsymbol{X}}^{\bullet,n+1} \to \underline{\boldsymbol{X}}^{\bullet,n} \end{aligned}$$

for non-negative integers k, ℓ such that $0 \le k \le m$ and $0 \le \ell \le n$, and where each set of coface and codegeneracy maps satisfies the usual cosimplicial identities. Furthermore, a map from one set of coface/codegeneracy maps commutes with a map from the other set,

$$\begin{aligned} \dot{d}^k \ddot{d}^\ell &= \ddot{d}^\ell \dot{d}^k & \qquad \dot{s}^k \ddot{d}^\ell &= \ddot{d}^\ell \dot{s}^k \\ \dot{d}^k \ddot{s}^\ell &= \ddot{s}^\ell \dot{d}^k & \qquad \dot{s}^k \ddot{s}^\ell &= \ddot{s}^\ell \dot{s}^k. \end{aligned}$$

One can think of $\underline{\boldsymbol{X}}$ as a diagram

$$\begin{array}{ccccccc} & & \vdots & & & & \iddots \\ \underline{\boldsymbol{X}}^{0,2} & \rightleftarrows & \underline{\boldsymbol{X}}^{1,2} & \rightleftarrows & \underline{\boldsymbol{X}}^{2,2} & & \\ \upharpoonleft\downarrow & & \upharpoonleft\downarrow & & \upharpoonleft\downarrow & & \\ \underline{\boldsymbol{X}}^{0,1} & \rightleftarrows & \underline{\boldsymbol{X}}^{1,1} & \rightleftarrows & \underline{\boldsymbol{X}}^{2,1} & & \cdots \\ \upharpoonleft\downarrow & & \upharpoonleft\downarrow & & \upharpoonleft\downarrow & & \\ \underline{\boldsymbol{X}}^{0,0} & \rightleftarrows & \underline{\boldsymbol{X}}^{1,0} & \rightleftarrows & \underline{\boldsymbol{X}}^{2,0} & & \end{array}$$

with each arrow representing possibly multiple coface or codegeneracy maps.

Notation 7.1.2. The underlined and boldfaced $\underline{\boldsymbol{X}}$ notation shall denote a bicosimplicial object, as opposed to just the boldfaced $\boldsymbol{X}$ which denotes a cosimplicial object.

Definition 7.1.3. The diagonal cosimplicial object $\mathbf{diag}(\underline{\boldsymbol{X}})$ of a bicosimplicial object $\underline{\boldsymbol{X}}$ is the cosimplicial object with

$$\mathbf{diag}(\underline{\boldsymbol{X}})^n = \underline{\boldsymbol{X}}^{n,n}$$

and coface and codegeneracy maps

$$\begin{aligned} d^k &= \dot{d}^k \ddot{d}^k = \ddot{d}^k \dot{d}^k \\ s^k &= \dot{s}^k \ddot{s}^k = \ddot{s}^k \dot{s}^k \end{aligned}$$

where $\dot{d}^k$, $\ddot{d}^k$, $\dot{s}^k$, $\ddot{s}^k$ are the coface and codegeneracy maps of $\underline{\boldsymbol{X}}$.

It can also be seen as a composite functor

$$\boldsymbol{\Delta} \xrightarrow{\text{diag}} \boldsymbol{\Delta} \times \boldsymbol{\Delta} \xrightarrow{\underline{\boldsymbol{X}}} \mathscr{C}.$$

Definition 7.1.4. For a cosimplicial object $\boldsymbol{X}$, define the vertically constant cosimplicial object on $\boldsymbol{X}$ to be the bicosimplicial object $\mathbf{const}(\boldsymbol{X})$ with

$$\mathbf{const}(\boldsymbol{X})^{m,\bullet} = \boldsymbol{X}^m,$$

coface and codegeneracy maps $\dot{d}$, $\dot{s}$ the same as the ones on $\boldsymbol{X}$, and $\ddot{d}$, $\ddot{s}$ the identity. It can be drawn as a diagram

$$\begin{array}{ccccccc} \vdots & & & & & & \iddots \\ \boldsymbol{X}^0 & \rightleftarrows & \boldsymbol{X}^1 & \rightleftarrows & \boldsymbol{X}^2 & & \\ \| & & \| & & \| & & \\ \boldsymbol{X}^0 & \rightleftarrows & \boldsymbol{X}^1 & \rightleftarrows & \boldsymbol{X}^2 & & \cdots \\ \| & & \| & & \| & & \\ \boldsymbol{X}^0 & \rightleftarrows & \boldsymbol{X}^1 & \rightleftarrows & \boldsymbol{X}^2. & & \end{array}$$

Now let X be a pointed space. Recall "space" means simplicial set. For the cosimplicial space $\boldsymbol{E}X$, we can form the bicosimplicial space

$$\underline{\boldsymbol{E}}X = \boldsymbol{E}\boldsymbol{E}X.$$

As above, we can draw it as

$$
\begin{array}{ccccccc}
\vdots & & & & & & \iddots \\
E^3E^1X & \rightleftarrows & E^3E^2X & \rightleftarrows & E^3E^3X & & \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
E^2E^1X & \rightleftarrows & E^2E^2X & \rightleftarrows & E^2E^3X & & \cdots \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
E^1E^1X & \rightleftarrows & E^1E^2X & \rightleftarrows & E^1E^3X. & &
\end{array}
$$

We have set it up so that $\underline{\boldsymbol{E}}X^{m,n} = E^{n+1}E^{m+1}X$. We also form $\mathbf{const}(\boldsymbol{E}X)$, drawn as

$$
\begin{array}{ccccccc}
\vdots & & & & & & \iddots \\
E^1X & \rightleftarrows & E^2X & \rightleftarrows & E^3X & & \\
\| & & \| & & \| & & \\
E^1X & \rightleftarrows & E^2X & \rightleftarrows & E^3X & & \cdots \\
\| & & \| & & \| & & \\
E^1X & \rightleftarrows & E^2X & \rightleftarrows & E^3X. & &
\end{array}
$$

For purposes that will become apparent in the next section, we wish to construct a map

$$\boldsymbol{E}X = \mathbf{diag}(\mathbf{const}(\boldsymbol{E}X)) \to \mathbf{diag}(\underline{\boldsymbol{E}}X)$$

that induces an isomorphism in the BKSS. To start, we first look at the collection of maps

$$\gamma \colon \mathbf{const}(\boldsymbol{E}X) \to \underline{\boldsymbol{E}}X$$

with an individual map in the collection defined by

$$\gamma^{i,j} = (d^0)^{j+1} \colon \mathbf{const}(\boldsymbol{E}X)^{i,j} = E^{i+1}X \longmapsto E^{j+1}E^{i+1}X$$

for i, $j \geq 0$. $\gamma^{i,j}$ is simply $(j+1)$-fold iterations of the coface map d^0. It is easy to check that γ is a bicosimplicial map, meaning it commutes with all coface and codegeneracy maps. We can take homotopy π_t of γ, where $t \geq 1$, to obtain a map of cosimplicial abelian groups

$$\gamma_* \colon \pi_t\mathbf{const}(\boldsymbol{E}X) \to \pi_t\underline{\boldsymbol{E}}X.$$

We restrict to $t \geq 1$ since it is not necessarily the case that an arbitrary map becomes a group homomorphism upon applying π_0, even though π_0 of all the spaces in the bicosimplicial spaces are abelian groups (they are all infinite loop spaces).

Before continuing, we need

Definition 7.1.5. For a cosimplicial abelian group $\boldsymbol{A}$, denote by $\mathrm{ch}\boldsymbol{A}$ its cochain complex, defined as

$$
\begin{aligned}
(\mathrm{ch}\boldsymbol{A})^n &= \boldsymbol{A}^n \qquad n \geq 0 \\
&= 0 \qquad n < 0
\end{aligned}
$$

with differentials

$$\partial^n = \sum_{i=0}^{n} (-1)^i d^i \colon \boldsymbol{A}^{n-1} \to \boldsymbol{A}^n.$$

We can likewise define the double cochain complex for a bicosimplicial abelian group $\boldsymbol{A}^{\bullet,\bullet}$

$$
\begin{aligned}
(\mathrm{ch}^2\boldsymbol{A})^{m,n} &= \boldsymbol{A}^{m,n} \qquad m, n \geq 0 \\
&= 0 \qquad \text{otherwise}
\end{aligned}
$$

with both horizontal and vertical differentials

$$
\begin{aligned}
\partial_H^m &= \sum_{i=0}^{n} (-1)^i d^i \colon \boldsymbol{A}^{m-1,\bullet} \to \boldsymbol{A}^{m,\bullet} \\
\partial_V^n &= \sum_{i=0}^{n} (-1)^i d^i \colon \boldsymbol{A}^{\bullet,n-1} \to \boldsymbol{A}^{\bullet,n}
\end{aligned}
$$

that commute with each other.

Definition 7.1.6. Given a double complex $D^{\bullet,\bullet}$, the associated total complex $\mathrm{Tot}(D^{\bullet,\bullet})$ is defined as

$$\mathrm{Tot}^n(D^{\bullet,\bullet}) = \bigoplus_{p+q=n} D^{p,q}$$

with the nth differential

$$\partial^n = \sum_{p+q=n} \left(\dot{d}^p + (-1)^p \ddot{d}^q \right).$$

Continuing on, the map γ_* induces a map of double cochain complexes

$$\mathrm{ch}^2(\pi_t \mathbf{const}(\boldsymbol{E}X)) \to \mathrm{ch}^2(\pi_t \underline{\boldsymbol{E}}X),$$

and we can take vertical cohomology to get

$$H_V^*(\mathrm{ch}^2(\pi_t \mathbf{const}(\boldsymbol{E}X))) \to H_V^*(\mathrm{ch}^2(\pi_t \underline{\boldsymbol{E}}X)).$$

We set out now to show

Theorem 7.1.7. The induced map

$$H_V^*(\mathrm{ch}^2(\pi_t \mathbf{const}(\boldsymbol{E}X))) \to H_V^*(\mathrm{ch}^2(\pi_t \underline{\boldsymbol{E}}X))$$

is an isomorphism.

Proof. We begin by first showing that the two groups are the same, and then showing the map is an isomorphism.

Looking at $\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X))$, we see that all vertical differentials ∂_V are either the identity or the 0 map since the differentials are alternating sums of the vertical coface maps, which are all the identity. ∂_V^n is the identity if n is even, and 0 if n is odd. Applying vertical cohomology, one finds that

$$H_V^p(\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X)))^{i,\bullet} = H^p(\pi_t E^{i+1}X) = \frac{\ker \partial_V^{p+1}}{\mathrm{im}\, \partial_V^p} = \begin{cases} \pi_t E^{i+1}X & p=0 \\ 0 & p>0. \end{cases}$$

We want to show that $H_V^p(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))$ is the same group. The vertical cohomology is obtained by taking the homology of the vertical cochain complexes within the double complex, i.e. $H_V^*(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))^{i,\bullet} =$

$$H^*\left(\pi_t E(E^{i+1}X) \xrightarrow{\partial^1} \pi_t E^2(E^{i+1}X) \xrightarrow{\partial^2} \cdots\right).$$

Notice that because of the nature of working with spaces of the form $E^k(EY)$, there is an "extra" codegeneracy map

$$s^{k-1}\colon E^{k+1}Y = E^k(EY) \longrightarrow E^{k-1}(EY) = E^kY.$$

This is the map that "multiplies the last outside E (counted from the outside in) and the inside E together." We look at the collection of the alternating signs of these maps $\{(-1)^k s^k\}$ and show that this collection of maps is a chain homotopy between the identity and the 0 map: we have a diagram

$$\begin{array}{ccccccc}
\pi_t E(E^{i+1}X) & \xrightarrow{\partial^1} & \pi_t E^2(E^{i+1}X) & \xrightarrow{\partial^2} & \pi_t E^3(E^{i+1}X) & \xrightarrow{\partial^3} & \cdots \\
0 \downarrow\downarrow \mathrm{id} & \swarrow_{-s^1} & 0 \downarrow\downarrow \mathrm{id} & \swarrow_{s^2} & 0 \downarrow\downarrow \mathrm{id} & \swarrow_{-s^3} & \\
\pi_t E(E^{i+1}X) & \xrightarrow{\partial^1} & \pi_t E^2(E^{i+1}X) & \xrightarrow{\partial^2} & \pi_t E^3(E^{i+1}X) & \xrightarrow{\partial^3} & \cdots
\end{array}$$

and wish to show that $\mathrm{id} - 0 = \mathrm{id} = \partial^p(-1)^p s^p + (-1)^{p+1}s^{p+1}\partial^{p+1}$.

$$\begin{aligned}
\partial^p(-1)^p s^p + (-1)^{p+1}s^{p+1}\partial^{p+1} &= \left(\sum_{j=0}^{p}(-1)^j d^j\right)(-1)^p s^p + (-1)^{p+1}s^{p+1}\left(\sum_{j=0}^{p+1}(-1)^j d^j\right) \\
&= \sum_{j=0}^{p}(-1)^{j+p}(d^j s^p) + \sum_{j=0}^{p+1}(-1)^{j+p+1}(s^{p+1}d^j) \\
&= \sum_{j=0}^{p}(-1)^{j+p}(d^j s^p) + \sum_{j=0}^{p}(-1)^{j+p+1}(d^j s^p) + \mathrm{id} \\
&= \mathrm{id}.
\end{aligned}$$

The third line is by the cosimplicial identities. id and 0 are thus chain homotopic maps, so they induce the same map on cohomology in dimensions 1 and higher,

$$H_V^p(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))^{i,\bullet} = 0, \qquad p > 0.$$

At dimension 0, we have $H^0_V(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))^{i,\bullet} = H^0(\pi_t E(E^{i+1}X)) = \ker(\partial^1)$. To find what this is, look again at a similar diagram

$$\begin{array}{ccccc} \pi_t E^{i+1}X & \xrightarrow{d^0} & \pi_t E(E^{i+1}X) & \xrightarrow{\partial^1} & \cdots \\ \downarrow \mathrm{id} & \swarrow s^0 & \downarrow \mathrm{id} & \swarrow -s^1 & \\ \pi_t E^{i+1}X & \xrightarrow{d^0} & \pi_t E(E^{i+1}X) & \xrightarrow{\partial^1} & \cdots . \end{array}$$

This is not a diagram of cochain complexes. Everything beyond the first column is identical to a diagram of cochain complexes, but the first column is the augmentation, which replaces the 0 in the usual complex.

We claim that $\ker \partial^1 = d^0(\pi_t E^{i+1}X)$. It's easy to see that $\ker \partial^1 \supset d^0(\pi_t E^{i+1}X)$ since $\partial^1(d^0) = d^0 d^0 - d^1 d^0 = 0$ by the cosimplicial identities. For the other direction, take an element $x \in \ker \partial^1$. Recall that $d^0 s^0 - s^1\partial^1 = \mathrm{id}$. Applying to x,

$$d^0 s^0 x - s^1 \partial^1 x = x.$$

But $-s^1\partial^1 x = 0$ because x is in the kernel, so $x = d^0 s^0 x$, and $x \in d^0(\pi_t E^{i+1}X)$. d^0 is also injective (it has s^0 as a left inverse), so $\operatorname{im} d^0 \cong \pi_t E^{i+1}X$.

Thus

$$H^p_V(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))^{i,\bullet} = \begin{cases} \pi_t E^{i+1}X & p = 0 \\ 0 & p > 0. \end{cases}$$

Finally, it remains to show that the map induced by γ is an isomorphism. We need only check the map in cohomology dimension $p = 0$, since everything else is 0.

$$H_V(\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X)))^{i,0} = \pi_t E^{i+1}X \longrightarrow \pi_t E^{i+1}X = H_V(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))^{i,0}$$

But in vertical dimension 0, γ is the map d^0, and we have just shown above that the map induced by d^0 is an isomorphism. $\square$

We have

$$H^p_V(\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X))) \xrightarrow{\cong} H^p_V(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X)).$$

Taking horizontal cohomology yields

$$H^q_H H^p_V(\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X))) \xrightarrow{\cong} H^q_H H^p_V(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X)).$$

But these correspond precisely to the E_2-terms of the standard spectral sequence of a double complex [**Wei94**, 5.6.1]. So by that spectral sequence, this converges to

$$H^{p+q}(\mathrm{Tot}(\mathrm{ch}^2(\pi_t\mathbf{const}(\boldsymbol{E}X)))) \xrightarrow{\cong} H^{p+q}(\mathrm{Tot}(\mathrm{ch}^2(\pi_t\underline{\boldsymbol{E}}X))).$$

By the generalized Eilenberg-Zilber theorem [**GJ99**, IV.2.4], these groups are isomorphic to

$$H^{p+q}(\mathrm{ch}(\mathbf{diag}(\pi_t\mathbf{const}(\boldsymbol{E}X)))) \xrightarrow{\cong} H^{p+q}(\mathrm{ch}(\mathbf{diag}(\pi_t\underline{\boldsymbol{E}}X))).$$

We encourage the reader to convince themselves that for any bicosimplicial space $\boldsymbol{A}$, $\mathbf{diag}(\pi_t(\boldsymbol{A})) = \pi_t\mathbf{diag}(\boldsymbol{A})$. Then the above is an isomorphism

$$H^{p+q}(\mathrm{ch}(\pi_t\mathbf{diag}(\mathbf{const}(\boldsymbol{E}X)))) \xrightarrow{\cong} H^{p+q}(\mathrm{ch}(\pi_t\mathbf{diag}(\underline{\boldsymbol{E}}X)))$$

which is equivalent to

$$H^{p+q}(\mathrm{ch}(\pi_t\boldsymbol{E}X)) \xrightarrow{\cong} H^{p+q}(\mathrm{ch}(\pi_t\mathbf{diag}(\underline{\boldsymbol{E}}X))).$$

When $t > p+q$, these two groups are exactly the E_2-terms of the Bousfield-Kan spectral sequences for $\boldsymbol{E}X$ and $\mathbf{diag}(\underline{\boldsymbol{E}}X)$. This is an immediate consequence of [BK73a, 5.4] and [ML63, VIII.6.1]: for a cosimplicial space $\boldsymbol{X}$, the E_2-term of the BKSS is isomorphic to the homology of the cochain complex of $\pi_t'\boldsymbol{X}$, which is chain equivalent to $\pi_t\boldsymbol{X}$. Consequently, there is an isomorphism of spectral sequences

$$E_r\boldsymbol{E}X \xrightarrow{\cong} E_r\mathbf{diag}(\underline{\boldsymbol{E}}X)$$

from the E_2-term onward.

7.2. The composition pairing from a map c

Definition 7.2.1. For simplicial sets X, Y, construct a cosimplicial space $\mathbf{Hom}(X, \boldsymbol{E}Y)$ by setting

$$\mathbf{Hom}(X, \boldsymbol{E}Y)^n = \mathbf{Hom}(X, E^{n+1}Y)$$

with coface and codegeneracy maps obtained by postcomposing m-simplices with the coface and codegeneracy maps of $\boldsymbol{E}Y$:

$$d^i\colon \Delta^m \wedge X \to E^{n-1}Y \xrightarrow{d^i} E^nY$$
$$s^i\colon \Delta^m \wedge X \to E^{n+1}Y \xrightarrow{s^i} E^nY.$$

Definition 7.2.2. Consider m-simplices

$$(u\colon \Delta^m \wedge W \to E^nX) \in \mathbf{Hom}(W, \boldsymbol{E}X)$$
$$(v\colon \Delta^m \wedge X \to E^nY) \in \mathbf{Hom}(X, \boldsymbol{E}Y).$$

Define a map

$$b\colon \mathbf{Hom}(X, \boldsymbol{E}Y) \wedge \mathbf{Hom}(W, \boldsymbol{E}X) \to \mathbf{Hom}(W, \mathbf{diag}(\underline{\boldsymbol{E}}Y))$$

on the m-simplices, such that $b(v, u)$ is the composite

$$\Delta^m \wedge W \xrightarrow{\mathrm{diag}\wedge\mathrm{id}} \Delta^m \wedge \Delta^m \wedge W \xrightarrow{\mathrm{id}\wedge u} \Delta^m \wedge E^nX \xrightarrow{\alpha_2^n} E^n(\Delta^m \wedge X) \xrightarrow{E^n(v)} E^{2n}Y.$$

By α_2^n we mean $E^{n-1}(\alpha_2) \circ \cdots \circ \alpha_2$. Construct a map

$$\beta\colon \mathbf{Hom}(W, \boldsymbol{E}Y) \to \mathbf{Hom}(W, \mathbf{diag}(\underline{\boldsymbol{E}}Y))$$

that induces an isomorphism on the E_2-and-onward-term of the spectral sequence

$$E_r\beta\colon E_r\,\mathbf{Hom}(W, \boldsymbol{E}Y) \xrightarrow{\cong} E_r\,\mathbf{Hom}(W, \mathbf{diag}(\underline{\boldsymbol{E}}Y))$$

via an argument entirely similar to the one given in the previous section. Though β may not have an inverse, $E_r\beta$ does, and we define the map c on spectral sequences to be

$$c\colon E_r[\mathbf{Hom}(X, \boldsymbol{E}Y) \wedge \mathbf{Hom}(W, \boldsymbol{E}X)] \xrightarrow{E_r b} E_r\,\mathbf{Hom}(W, \mathbf{diag}(\underline{\boldsymbol{E}}Y)) \xrightarrow{E_r\beta^{-1}} E_r\,\mathbf{Hom}(W, \boldsymbol{E}Y).$$

Remark 7.2.3. In the original formulation of the map c [BK73a, 9.1], a pairing w_n is built to define a map

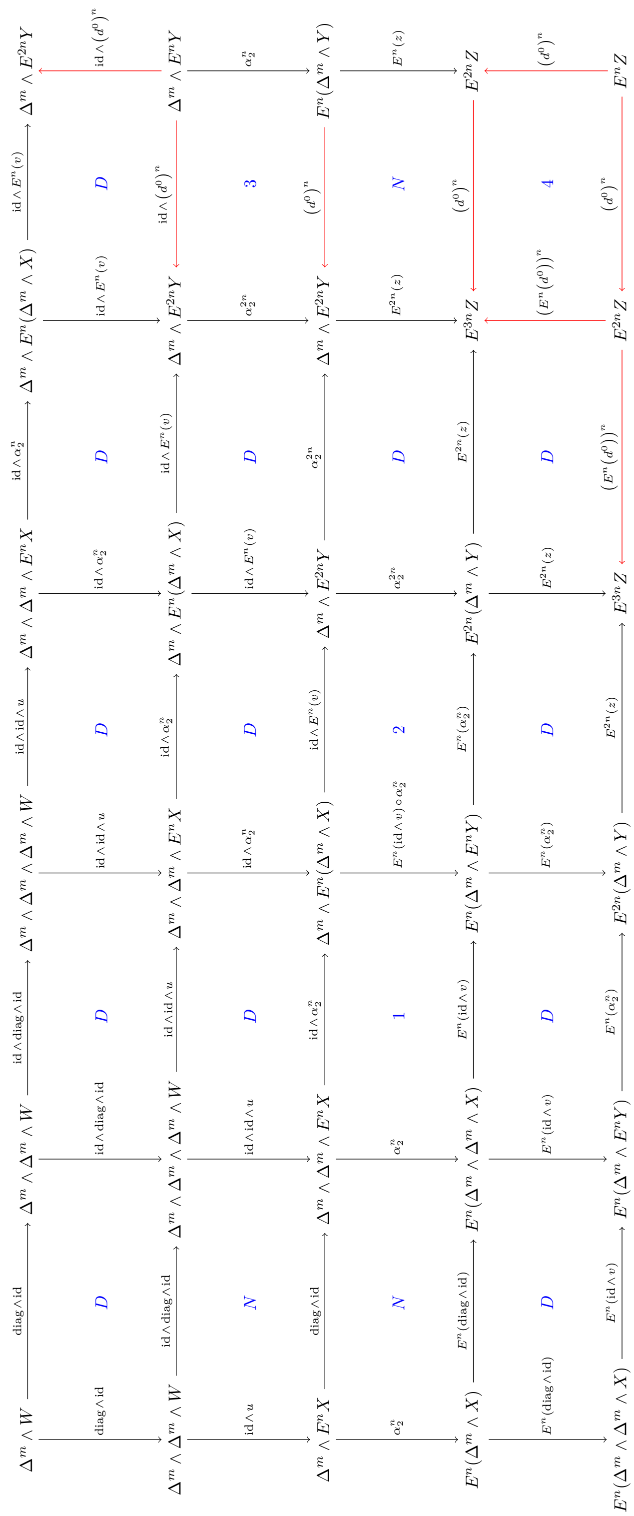
$$\mathbf{Hom}(W, \mathbf{diag}(\underline{\boldsymbol{R}}Y)) \to \mathbf{Hom}(W, \boldsymbol{R}Y).$$

The issue is that w_n requires the use of a "switch" map, which, in the case of $\boldsymbol{R}X$, interchanges two copies of R (see Appendix A). Such a switch map does not necessarily exist for the case of $\boldsymbol{E}X$, requiring us to ascend to the level of spectral sequences to define a similar map $E_r\beta^{-1}$. In the event that a switch map actually does exist, for instance, if a specific $\boldsymbol{E}X$ has a free-module structure, it is a simple calculation to show that β and w_n are inverses, so the induced maps $E_r\beta^{-1}$ and $E_r w_n$ are equal.

Proposition 7.2.4. c is associative.

Proof. This shall be proved by taking the diagrams of the associativity of b and the associativity of β and combining them in such a way that it results in a large, commutative diagram. When the spectral sequence is applied to the diagram, and we replace the maps induced by β with their inverses, the result is a commutative diagram representing the associativity of $c = E_r\beta^{-1} \circ E_r b$.

Let $(z\colon \Delta^m \wedge Y \to E^n Z) \in \mathbf{Hom}(Y, \boldsymbol{E}Z)$.


$\Delta^m \wedge W$
$\Delta^m \wedge \Delta^m \wedge W$
$\Delta^m \wedge \Delta^m \wedge \Delta^m \wedge W$
$\Delta^m \wedge \Delta^m \wedge E^n X$
$\Delta^m \wedge E^n(\Delta^m \wedge X)$
$\Delta^m \wedge E^{2n} Y$
$\Delta^m \wedge E^n Y$
$\Delta^m \wedge E^n X$
$E^n(\Delta^m \wedge Y)$
$E^n(\Delta^m \wedge X)$
$E^n(\Delta^m \wedge \Delta^m \wedge X)$
$E^n(\Delta^m \wedge E^n Y)$
$E^{2n}(\Delta^m \wedge Y)$
$E^{3n} Z$
$E^{2n} Z$
$E^n Z$
D
N
1
2
3
4

The red arrows signify that they will be reversed at the level of the spectral sequence, once we replace them with their inverses. The squares labeled D commute by definition; the ones labeled N commute by naturality of the appropriate map. 1 commutes by the properties of a module triple, after using a naturality argument to rearrange the maps. 2 expands into

$$\begin{array}{ccc}
\Delta^m \wedge E^n(\Delta^m \wedge X) & \xrightarrow{\mathrm{id}\wedge E^n(v)} & \Delta^m \wedge E^{2n}Y \\
\Big\downarrow{\scriptstyle \alpha_n^2} & & \Big\downarrow{\scriptstyle \alpha_2^n} \\
E^n(\Delta^m \wedge \Delta^m \wedge X) & \xrightarrow{E^n(\mathrm{id}\wedge v)} & E^n(\Delta^m \wedge E^nY) \\
\Big\downarrow{\scriptstyle E^n(\mathrm{id}\wedge v)} & & \Big\downarrow{\scriptstyle E^n(\alpha_2^n)} \\
E^n(\Delta^m \wedge E^nY) & \xrightarrow{E^n(\alpha_2^n)} & E^{2n}(\Delta^m \wedge Y).
\end{array}$$

The top square commutes by naturality and the bottom by definition. 4 commutes by associativity of β, which is trivial to show using the cosimplicial identities. To show 3 commutes, we fully expand it.

$$\begin{array}{ccccccccc}
\Delta^m \wedge E^{2n}Y & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^{2n-1} & \xleftarrow{\mathrm{id}\wedge d^0} & \cdots & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^{n+1} & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^nY \\
\Big\downarrow{\scriptstyle \alpha_2} & & \Big\| & & & & \Big\| & & \Big\| \\
E(\Delta^m \wedge E^{2n-1})Y & \xleftarrow{d^0} & \Delta^m \wedge E^{2n-1} & \xleftarrow{\mathrm{id}\wedge d^0} & \cdots & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^{n+1}Y & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^nY \\
\Big\downarrow{\scriptstyle E(\alpha_2)} & & \Big\downarrow{\scriptstyle \alpha_2} & & & & \Big\| & & \Big\| \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
\Big\downarrow{\scriptstyle E^{n-2}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{n-3}(\alpha_2)} & & & & \Big\| & & \Big\| \\
E^{n-1}(\Delta^m \wedge E^{n+1}Y) & \xleftarrow{d^0} & E^{n-2}(\Delta^m \wedge E^{n+1}Y) & \xleftarrow{d^0} & \cdots & \xleftarrow{d^0} & \Delta^m \wedge E^{n+1}Y & \xleftarrow{\mathrm{id}\wedge d^0} & \Delta^m \wedge E^nY \\
\Big\downarrow{\scriptstyle E^{n-1}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{n-2}(\alpha_2)} & & & & \Big\downarrow{\scriptstyle \alpha_2} & & \Big\| \\
E^n(\Delta^m \wedge E^nY) & \xleftarrow{d^0} & E^{n-1}(\Delta^m \wedge E^nY) & \xleftarrow{d^0} & \cdots & \xleftarrow{d^0} & E(\Delta^m \wedge E^nY) & \xleftarrow{d^0} & \Delta^m \wedge E^nY \\
\Big\downarrow{\scriptstyle E^n(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{n-1}(\alpha_2)} & & & & \Big\downarrow{\scriptstyle E(\alpha_2)} & & \Big\downarrow{\scriptstyle \alpha_2} \\
E^{n+1}(\Delta^m \wedge E^{n-1}Y) & \xleftarrow{d^0} & E^n(\Delta^m \wedge E^{n-1}Y) & \xleftarrow{d^0} & \cdots & \xleftarrow{d^0} & E^2(\Delta^m \wedge E^{n-1}Y) & \xleftarrow{d^0} & E(\Delta^m \wedge E^{n-1}Y) \\
\Big\downarrow{\scriptstyle E^{n+1}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^n(\alpha_2)} & & & & \Big\downarrow{\scriptstyle E^2(\alpha_2)} & & \Big\downarrow{\scriptstyle E(\alpha_2)} \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
\Big\downarrow{\scriptstyle E^{2n-2}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{2n-3}(\alpha_2)} & & & & \Big\downarrow{\scriptstyle E^{n-1}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{n-2}(\alpha_2)} \\
E^{2n-1}(\Delta^m \wedge EY) & \xleftarrow{d^0} & E^{2n-2}(\Delta^m \wedge EY) & \xleftarrow{d^0} & \cdots & \xleftarrow{d^0} & E^n(\Delta^m \wedge EY) & \xleftarrow{d^0} & E^{n-1}(\Delta^m \wedge EY) \\
\Big\downarrow{\scriptstyle E^{2n-1}(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{2n-2}(\alpha_2)} & & & & \Big\downarrow{\scriptstyle E^n(\alpha_2)} & & \Big\downarrow{\scriptstyle E^{n-1}(\alpha_2)} \\
E^{2n}(\Delta^m \wedge Y) & \xleftarrow{d^0} & E^{2n-1}(\Delta^m \wedge Y) & \xleftarrow{d^0} & \cdots & \xleftarrow{d^0} & E^{n+1}(\Delta^m \wedge Y) & \xleftarrow{d^0} & E^n(\Delta^m \wedge Y)
\end{array}$$

This diagram breaks into a top half and a bottom half. The top half ends with the last row that has a vertical equality. The squares in the top-left to bottom-right diagonal of the top half commute by the properties of a module triple. Everything above this diagonal commutes by definition; everything below it commutes by naturality of d^0. Everything in the bottom half also commutes by naturality of d^0. $\square$

DEFINITION 7.2.5. Define the composition pairing

$$E_r^{s,t}\,\mathbf{Hom}(X,\boldsymbol{E}Y)\otimes E_r^{s',t'}\,\mathbf{Hom}(W,\boldsymbol{E}X)\xrightarrow{\;\circ\;}E_r^{s+s',t+t'}\,\mathbf{Hom}(W,\boldsymbol{E}Y)$$

for $r\geq 2$ to be the composite

$$\begin{aligned}E_r^{s,t}\,\mathbf{Hom}(X,\boldsymbol{E}Y)\otimes E_r^{s',t'}\,\mathbf{Hom}(W,\boldsymbol{E}X)&\xrightarrow{\wedge_{BK}}E_r^{s+s',t+t'}(\mathbf{Hom}(X,\boldsymbol{E}Y)\wedge\mathbf{Hom}(W,\boldsymbol{E}X))\\&\xrightarrow{\;c\;}E_r^{s+s',t+t'}\,\mathbf{Hom}(W,\boldsymbol{E}Y).\end{aligned}$$

PROPOSITION 7.2.6. The pairing $\circ$ is associative.

PROOF. To save space, we write $\mathbf{H}$ for $\mathbf{Hom}$. Showing associativity is the same as showing that the diagram

$$\begin{array}{ccccc}
E_r\mathbf{H}(Y,\boldsymbol{E}Z)\otimes E_r\mathbf{H}(X,\boldsymbol{E}Y)\otimes E_r\mathbf{H}(W,\boldsymbol{E}X) & \xrightarrow{\wedge_{BK}\otimes\mathrm{id}} & E_r[\mathbf{H}(Y,\boldsymbol{E}Z)\wedge\mathbf{H}(X,\boldsymbol{E}Y)]\otimes E_r\mathbf{H}(W,\boldsymbol{E}X) & \xrightarrow{c\otimes\mathrm{id}} & E_r[\mathbf{H}(X,\boldsymbol{E}Z)]\otimes E_r\mathbf{H}(W,\boldsymbol{E}X)\\
\downarrow{\scriptstyle \mathrm{id}\otimes\wedge_{BK}} & & \downarrow{\scriptstyle \wedge_{BK}} & & \downarrow{\scriptstyle \wedge_{BK}}\\
E_r\mathbf{H}(Y,\boldsymbol{E}Z)\otimes E_r[\mathbf{H}(X,\boldsymbol{E}Y)\wedge\mathbf{H}(W,\boldsymbol{E}X)] & \xrightarrow{\wedge_{BK}} & E_r[\mathbf{H}(Y,\boldsymbol{E}Z)\wedge\mathbf{H}(X,\boldsymbol{E}Y)\wedge\mathbf{H}(W,\boldsymbol{E}X)] & \xrightarrow{E_r(\beta\wedge\mathrm{id})^{-1}\circ E_r(b\wedge\mathrm{id})} & E_r[\mathbf{H}(X,\boldsymbol{E}Z)\wedge\mathbf{H}(W,\boldsymbol{E}X)]\\
\downarrow{\scriptstyle \mathrm{id}\otimes c} & & \downarrow{\scriptstyle E_r(\mathrm{id}\wedge\beta)^{-1}\circ E_r(\mathrm{id}\wedge b)} & & \downarrow{\scriptstyle c}\\
E_r\mathbf{H}(Y,\boldsymbol{E}Z)\otimes E_r\mathbf{H}(W,\boldsymbol{E}Y) & \xrightarrow{\wedge_{BK}} & E_r[\mathbf{H}(Y,\boldsymbol{E}Z)\wedge\mathbf{H}(W,\boldsymbol{E}Y)] & \xrightarrow{c} & E_r[\mathbf{H}(W,\boldsymbol{E}Z)]
\end{array}$$

commutes. The top-left and bottom-right squares commute by associativity of $\wedge_{BK}$ and c respectively. The other two squares commute by naturality of $\wedge_{BK}$. $\square$

7.3. An application of the composition pairing

For any cosimplicial space $\boldsymbol{X}$, we have

$$\boldsymbol{X}\cong\mathbf{Hom}(S^0,\boldsymbol{X}).$$

PROPOSITION 7.3.1. The map

$$\hat{\circ}\colon E_r^{s,t}\,\mathbf{Hom}(S^m,\boldsymbol{E}X)\to E_r^{s,t+m}\boldsymbol{E}X,$$

obtained by restricting the composition pairing

$$\circ\colon E_r^{s,t}\,\mathbf{Hom}(S^m,\boldsymbol{E}X)\otimes E_r^{0,m}\boldsymbol{E}S^m\to E_r^{s,t+m}\boldsymbol{E}X$$

to the generator $u\in E_r^{0,m}\boldsymbol{E}S^m$ is an isomorphism if $t-s\geq 1$.

PROOF. We show the isomorphism on E_1-terms which will induce isomorphisms on higher terms. Since the composition pairing is not defined for the E_1-terms, however, this will require a small workaround. Recall that $E_1^{0,m}S^m=\pi'_mES^m=\pi_mES^m$. Then the generator is represented by the unit map $u=d^0\colon S^m\to ES^m$. Let $v\in E_1^{s,t}\,\mathbf{Hom}(S^m,\boldsymbol{E}X)$.

Take (v,u) and apply the smash pairing of Section 6.2

$$(v,u)\overset{f}{\longmapsto}(v,d^{s-1}\cdots d^0u)=(v,d^0\cdots d^0u)=(v,\widetilde{u})\overset{\wedge_\pi}{\longmapsto}v\wedge\widetilde{u}.$$

We will abuse notation and denote $v \wedge \widetilde{u}$ by $(v, \widetilde{u})$. We remark that an element of a homotopy group $\pi_n Y$ is a map $S^n \to Y$ that lifts to a simplex $\Delta^n \to Y$ constant on the boundary $\partial\Delta^n$.

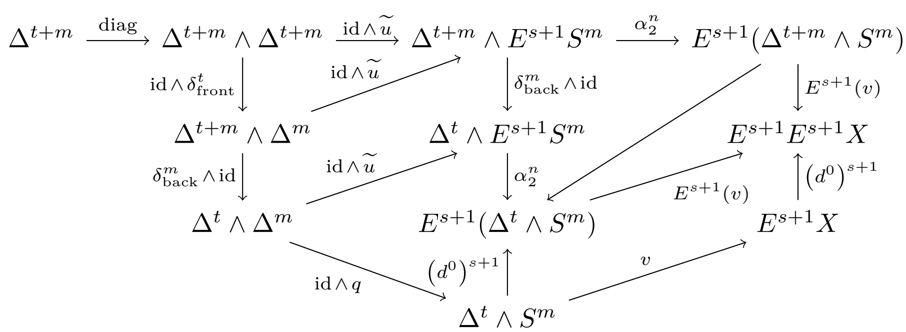
Given a standard simplex Δ^{i+j} we denote by δ^i_{front} the sequence of face maps $(\delta_0)^i : \Delta^{i+j} \to \Delta^j$ identifying the front i faces. Similarly, δ^j_{back} is the sequence of face maps $\delta_{i+j} \circ \delta_{i+j-1} \circ \cdots \circ \delta_{i+1} \colon \Delta^{i+j} \to \Delta^i$ identifying the back j faces.

Now, because the pairing c does not exist at the E_1-level, we will instead show that there is a map b'_* such that $\beta_* \circ b'_* = b_*$, with b and β as in the previous section. This b'_* forms an ad-hoc composition pairing for the E_1-terms, for this particular restriction, which obviously coincides with the regular composition pairing on all higher terms.

Set $b_*(v, \widetilde{u})$ to be the composite

$$\Delta^{t+m} \xrightarrow{\text{diag}} \Delta^{t+m} \wedge \Delta^{t+m} \xrightarrow{\text{id}\wedge\delta^t_{\text{front}}} \Delta^{t+m} \wedge \Delta^m \xrightarrow{\delta^m_{\text{back}}\wedge\text{id}} \Delta^t \wedge \Delta^m \xrightarrow{\text{id}\wedge q} \Delta^t \wedge S^m \xrightarrow{v} E^{s+1}X$$

with q the quotient map. The following diagram commutes by naturality arguments.



The unlabeled map is given by $E^{s+1}(\delta^m_{\text{back}} \wedge \text{id})$. It is easy to see the restricted composition pairing is bijective. $\square$

With this established, we create a new pairing

$$*\colon E_r^{s,t+m}\boldsymbol{E}X \otimes E_r^{s',t'}\boldsymbol{E}S^m \xrightarrow{\hat{\circ}^{-1}\otimes\text{id}} E_r^{s,t}\,\mathbf{Hom}(S^m, \boldsymbol{E}X) \otimes E_r^{s',t'}\boldsymbol{E}S^m \xrightarrow{\circ} E_r^{s+s',t+t'}\boldsymbol{E}X.$$

On E_∞, this pairing is compatible with the usual action of composing maps in homotopy

$$\circ\colon \pi_{t+m}X \otimes \pi_{t'}S^m \to \pi_{t+t'}X.$$

Example 7.3.2. It is known that K-theory forms a commutative symmetric ring spectrum [Joa01]. Let $i\colon X \to X_{\widehat{K}}$ be the natural map from a space to its p-local K-completion [Bou79, 5.5]. For K-theory localized at a prime, the spectral sequence converges to the homotopy groups of the p-local K-completion for a collection of spaces that includes spheres (see [BT00], [BT02], [BD03b]). Then, as in [BD03a], specializing to odd spheres, we have a commutative diagram

$$\begin{array}{ccc}
\pi_{t+2m+1}S^{2n+1} \otimes \pi_{t'}S^{2m+1} & \xrightarrow{\circ} & \pi_{t+t'}S^{2n+1} \\
\Big\downarrow{\scriptstyle i\otimes i} & & \Big\downarrow{\scriptstyle i} \\
\pi_{t+2m+1}S^{2n+1}_{\widehat{K}} \otimes \pi_{t'}S^{2m+1}_{\widehat{K}} & \xrightarrow{*} & \pi_{t+t'}S^{2n+1}_{\widehat{K}}.
\end{array}$$

CHAPTER 8

Pairings on Ext Groups

8.1. Cotriples, coalgebras, and Ext groups

Let us first set up some categories. Given a symmetric ring spectrum E, $\mathscr{N}$ shall denote the category of pointed spaces for which $E_*(X)$ is a graded, free E_*-module, and $\mathscr{M}$ the category of graded, free E_*-modules.

An E-module spectrum is a spectrum Y with a map $E \wedge Y \to Y$ making the unitary and associativity diagrams commute. An E-module spectrum is *free* if there is a bouquet of spheres B with a map $f\colon B \to Y$ such that the composite

$$E \wedge B \xrightarrow{\mathrm{id} \wedge f} E \wedge Y \longrightarrow Y$$

is a homotopy equivalence. We let $\mathscr{F}$ be the category of all free E-module spectra.

Lemma 8.1.1. [BCM78, 6.2] The functor

$$\pi_*(-)\colon \mathscr{F} \to \mathscr{M}$$

is an equivalence of categories.

Proof. Let $B = \bigvee_\gamma S^{|\gamma|}$ be a bouquet of spheres with $|\gamma|$ a non-negative integer, and $Y \in \mathscr{F}$.

$$\begin{aligned}
\hom_{\mathscr{F}}(E \wedge B, Y) &\cong [B, Y] \\
&\cong \prod_\gamma \pi_* Y \\
&\cong \hom_{\mathbf{Ab}}\left(\bigoplus_\gamma \mathbb{Z}^{|\gamma|}, \pi_* Y\right) \\
&\cong \hom_{\mathscr{M}}\left(\bigoplus_\gamma E_*^{|\gamma|}, \pi_* Y\right) \qquad \text{adjoint of forgetful, } E_* \otimes - \\
&\cong \hom_{\mathscr{M}}\left(\pi_*\left(\bigvee_\gamma \Sigma^{|\gamma|} E\right), \pi_* Y\right) \\
&\cong \hom_{\mathscr{M}}(\pi_*(E \wedge B), \pi_* Y)
\end{aligned}$$

where $|\gamma|$ as a superscript denotes a shift by dimension $|\gamma|$. Moreover, let $M \in \mathscr{M}$ with basis $\{x_\zeta\}$, $\dim(x_\zeta) = |\zeta|$ and $X = E \wedge \bigvee_\zeta S^{|\zeta|} \in \mathscr{F}$. Then

$$\pi_* X \cong \bigoplus_\zeta \pi_*(E \wedge S^{|\zeta|}) \cong M.$$

By [ML98, IV.4.1], π_* is an equivalence of categories. $\square$

Denote by π_*^{-1} the quasi-inverse of π_*. Then there are natural isomorphisms

$$\alpha\colon \pi_*\pi_*^{-1} \to \mathrm{id}$$
$$\beta\colon \mathrm{id} \to \pi_*\pi_*^{-1}.$$

Definition 8.1.2. In a category $\mathscr{C}$, a cotriple (G, ϵ, δ) consists of a functor $G\colon \mathscr{C} \to \mathscr{C}$, and of natural transformations

$$\epsilon\colon G \to \mathrm{id},$$
$$\delta\colon G \to G^2$$

making the diagrams

$$\begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ {\scriptstyle\delta}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle\epsilon_G} \\ G^2 & \xrightarrow[G(\epsilon)]{} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta_G} \\ G^2 & \xrightarrow[G(\delta)]{} & G^3 \end{array}$$

commute.

A G-coalgebra is an object $Y \in \mathscr{C}$ with a map $\psi\colon Y \to GY$ such that the diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & GY \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle\epsilon} \\ & & Y \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\psi} & GY \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle G(\psi)} \\ GY & \xrightarrow{\delta} & G^2Y \end{array}$$

commute. A map of G-coalgebras $f\colon (Y, \psi) \to (Y', \psi')$ is a map $f\colon Y \to Y'$ such that $\psi' \circ f = G(f) \circ \psi$.

Definition 8.1.3. Let $M \in \mathscr{M}$ and E be a symmetric ring spectrum such that $E_*(\Omega^\infty(\Sigma^i E))$ is a free E_*-module. Define a functor $G\colon \mathscr{M} \to \mathscr{M}$ by

$$\begin{aligned} G(M) = GM &= E_*(\Omega^\infty(\pi_*^{-1}M)) \\ &= E_*\left(\Omega^\infty\left(\bigvee_\gamma \Sigma^{|\gamma|}E\right)\right) \end{aligned}$$

where M has basis $\{x_\gamma\}$ with $\dim(x_\gamma) = |\gamma|$.

There are natural transformations

$$\epsilon\colon G \to \mathrm{id}$$
$$\delta\colon G \to G^2$$

such that (G, ϵ, δ) is a cotriple on $\mathscr{M}$. We construct the maps using the same arguments as in [**BCM78**, p. 237], but leave the checking of diagrams as an exercise.

Take $M \in \mathscr{M}$ and consider the composite

$$\begin{array}{ccc} G(M) = E_*(\Omega^\infty(\pi_*^{-1}M)) \longrightarrow E_*(\pi_*^{-1}M) & & \\ \| & & \\ \pi_*(E \wedge \pi_*^{-1}M) \longrightarrow \pi_*\pi_*^{-1}M \xrightarrow{\alpha} M. & & \end{array}$$

The first unlabeled map is given by

$$E_*(\Omega^\infty(\pi_*^{-1}M)) = \pi_*(E \wedge \Sigma^\infty\Omega^\infty(\pi_*^{-1}M)) \xrightarrow{\pi_*(\mathrm{id}\wedge\varepsilon)} \pi_*(E \wedge \pi_*^{-1}M) = E_*(\pi_*^{-1}M)$$

and the second by

$$\begin{array}{c} \pi_*(E \wedge \pi_*^{-1}M) == \pi_*\left(E \wedge \bigvee_\gamma \Sigma^{|\gamma|}E\right) == \pi_*\left(\bigvee_\gamma (S^{|\gamma|} \wedge E \wedge E)\right) \\ \Big\downarrow \pi_* \bigvee(\mathrm{id}\wedge\Phi) \\ \pi_*\left(\bigvee_\gamma \Sigma^{|\gamma|}E\right) ===== \pi_*\pi_*^{-1}M \end{array}$$

with Φ the ring spectrum product. This composite defines a map $\epsilon\colon G \to \mathrm{id}$. For the coproduct, let $X \in \mathscr{N}$ and consider the map

$$\psi_X\colon E_*(X) \xrightarrow{E_*(d^0)} E_*(EX) \cong GE_*(X).$$

The isomorphism is due to

$$E_*(EX) = E_*(\Omega^\infty(E \wedge \Sigma^\infty X)) \cong E_*(\Omega^\infty(\pi_*^{-1}E_*(X))) = GE_*(X).$$

In the case when $X = \Omega^\infty(\pi_*^{-1}M)$ for some $M \in \mathscr{M}$, we have a map $\delta\colon G \to G^2$. Moreover, for each $X \in \mathscr{N}$, $E_*(X)$ is a G-coalgebra with structure map ψ_X. We will write $\mathscr{M}(G)$ for the category of G-coalgebras.

In some sense, we can make G into a triple on $\mathscr{M}(G)$ as follows. Denote by G too the endofunctor which takes (Y, ψ) to (GY, δ). We can produce natural maps

$$\eta = \psi\colon (Y, \psi) \to (GY, \delta)$$
$$\mu = G(\epsilon)\colon (G^2Y, \delta_{GY}) \to (GY, \delta)$$

and one can check that these satisfy the necessary diagrams to make (G, η, μ) a triple on $\mathscr{M}(G)$.

With a triple structure, we can produce cosimplicial objects. Take $M \in \mathscr{M}(G)$. By $\boldsymbol{G}M$ we shall mean the cosimplicial object over $\mathscr{M}(G)$ constructed by the triple (G, η, μ) in the usual way. Take the cochain complex $\mathrm{ch}(\boldsymbol{G}M)$ and apply $\hom_{\mathscr{M}(G)}(E_*(S^t), -)$ to receive a complex

$$0 \longrightarrow \hom_{\mathscr{M}(G)}(E_*(S^t), GM) \longrightarrow \hom_{\mathscr{M}(G)}(E_*(S^t), G^2M) \longrightarrow \cdots$$

which yields the right derived functors $\mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^t), M)$.

Theorem 8.1.4. Let E be as above and $W, X \in \mathscr{N}$. Let W also satisfy the hypotheses: W is a finite complex, $E_*(S^t \wedge W)$ is finitely presented, and the E-cohomology of W is dual to the E-homology of W, i.e.

$$\hom_{\mathscr{M}}(E_*(S^t \wedge W), N) \cong [S^t \wedge W, \Omega^\infty(\pi_*^{-1}N)]$$

for $N \in \mathscr{M}$ finitely generated. Then

$$E_2^{s,t}\boldsymbol{E}X \cong \mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^t), E_*(X))$$
$$E_2^{s,t}\,\mathbf{Hom}(W, \boldsymbol{E}X) \cong \mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(W), E_*(X))$$

for $t > s \geq 0$.

Proof. We shall make use of an isomorphism proved in [BCM78, p. 238]: for any $N \in \mathscr{M}(G)$,

$$\hom_{\mathscr{M}(G)}(E_*(S^t), GN) \cong \hom_{\mathscr{M}}(E_*(S^t), N) \cong N.$$

Then, looking at the cochain complex of hom groups above, and replacing M with $E_*(X)$, we have isomorphisms

$$\begin{array}{ccccccc}
0 & \longrightarrow & \hom_{\mathscr{M}(G)}(E_*(S^t), GE_*(X)) & \longrightarrow & \hom_{\mathscr{M}(G)}(E_*(S^t), G^2E_*(X)) & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \hom_{\mathscr{M}}(E_*(S^t), E_*(X)) & \longrightarrow & \hom_{\mathscr{M}}(E_*(S^t), GE_*(X)) & \longrightarrow & \cdots \\
 & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & E_t(X) & \longrightarrow & (GE_*(X))_t & \longrightarrow & \cdots
\end{array}$$

Notice that

$$\begin{aligned}
GE_*(X) &= E_*(\Omega^\infty(\pi_*^{-1}E_*(X))) \\
&= \pi_*E(\Omega^\infty(\pi_*^{-1}E_*(X))) \\
&= \pi_*E(\Omega^\infty(\pi_*^{-1}\pi_*EX)) \\
&= \pi_*E(\Omega^\infty(\pi_*^{-1}\pi_*\Omega^\infty(E \wedge \Sigma^\infty X)))
\end{aligned}$$

The π_*^{-1} and π_* adjacent to each other are not quasi-inverses, since π_* is on spaces. However, since the unstable homotopy of $\Omega^\infty(E \wedge \Sigma^\infty X)$ is equivalent to the homotopy of the spectrum $E \wedge \Sigma^\infty X$, we make the suitable replacement so that π_*^{-1} and the newly replaced π_* now indeed are quasi-inverses, and the above becomes isomorphic to

$$\pi_*E(\Omega^\infty(E \wedge \Sigma^\infty X)) = \pi_*E^2X.$$

An easy induction argument shows $G^iE_*(X) \cong \pi_*E^{i+1}X$. Thus the last row of cochain complexes is

$$0 \longrightarrow \pi_tEX \longrightarrow \pi_tE^2X \longrightarrow \cdots$$

the cohomology of which we already know from the last paragraph of Section 7.1 is the E_2-term of the BKSS associated with $\boldsymbol{E}X$.

We now do the function complex case.

$$\begin{aligned}
\hom_{\mathscr{M}(G)}(E_*(S^t \wedge W), GE_*(X)) &\cong \hom_{\mathscr{M}}(E_*(S^t \wedge W), E_*(X)) \\
&\cong \hom_{\mathscr{M}}(E_*(S^t \wedge W), \operatorname*{colim}_{N \subset E_*(X)} N) \qquad \text{for } N \text{ fin. gen.} \\
&\cong \operatorname*{colim}_{N \subset E_*(X)} \hom_{\mathscr{M}}(E_*(S^t \wedge W), N) \qquad \text{finite presentability}
\end{aligned}$$

$$
\begin{aligned}
&\cong \operatorname*{colim}_{N\subset E_*(X)} [S^t \wedge W, \Omega^\infty(\pi_*^{-1}N)] && \text{assumption on } W\\
&\cong [S^t \wedge W, \operatorname*{colim}_{N\subset E_*(X)} \Omega^\infty(\pi_*^{-1}N)] && W \text{ finite complex}\\
&\cong [S^t \wedge W, EX]
\end{aligned}
$$

and it immediately follows that

$$
\begin{aligned}
E_2^{s,t}\,\mathbf{Hom}(W, \boldsymbol{E}X) &\cong \operatorname{Ext}^s_{\mathscr{M}(G)}(E_*(S^t \wedge W), E_*(X))\\
&\cong \operatorname{Ext}^s_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(W), E_*(X)) \qquad \text{by [Boa95, 9.20].} \quad \square
\end{aligned}
$$

8.2. The smash pairing

Let's first define the tensor product in $\mathscr{M}(G)$. Let M, $N \in \mathscr{M}(G)$. As modules, $M \otimes N$ is in $\mathscr{M}$. Define a map $\psi_{M\otimes N} \colon M \otimes N \to G(M \otimes N)$ as follows.

For spaces X, Y, define a map

$$g \colon \Omega^\infty X \wedge \Omega^\infty Y \to \Omega^\infty(X \wedge Y)$$

to be the adjoint of

$$\Sigma^\infty(\Omega^\infty X \wedge \Omega^\infty Y) \cong \Sigma^\infty\Omega^\infty X \wedge \Sigma^\infty\Omega^\infty Y \xrightarrow{\varepsilon\wedge\varepsilon} X \wedge Y.$$

Now suppose M has basis $\{x_\alpha\}$, $\dim(x_\alpha) = |\alpha|$ and N has basis $\{x_\beta\}$, $\dim(x_\beta) = |\beta|$. Then

$$\pi_*^{-1}M \cong \bigvee_\alpha \Sigma^{|\alpha|}E, \qquad \pi_*^{-1}N \cong \bigvee_\beta \Sigma^{|\beta|}E$$

and we let

$$h \colon \pi_*^{-1}M \wedge \pi_*^{-1}N \cong \bigvee_{\alpha,\beta} \Sigma^{|\alpha|+|\beta|}(E \wedge E) \xrightarrow{\bigvee \Sigma(\Phi)} \bigvee_{\alpha,\beta} \Sigma^{|\alpha|+|\beta|}E \cong \pi_*^{-1}(M \otimes N).$$

Set $\psi_{M\otimes N} \colon M \otimes N \to G(M \otimes N)$ to be the composite

$$
\begin{array}{ccccc}
M \otimes N \xrightarrow{\psi_M\otimes\psi_N} GM \otimes GN & & E_*\Omega^\infty(\pi_*^{-1}M \otimes \pi_*^{-1}N) \xrightarrow{E_*\Omega^\infty(h)} E_*\Omega^\infty\pi_*^{-1}(M \otimes N).\\
\Big\| & & \uparrow{\scriptstyle E_*(g)} \\
E_*\Omega^\infty\pi_*^{-1}M \otimes E_*\Omega^\infty\pi_*^{-1}N & \xrightarrow{\cong} & E_*(\Omega^\infty\pi_*^{-1}M \otimes \Omega^\infty\pi_*^{-1}N)
\end{array}
$$

We leave the checking of diagrams verifying this is a coaction to the reader.

Moving on, recall that we have a functor of a triple $G \colon \mathscr{M}(G) \to \mathscr{M}(G)$. This triple also arises from the pair of adjoint functors

$$
\begin{aligned}
J &\colon \mathscr{M}(G) \to \mathscr{M}\\
G &\colon \mathscr{M} \to \mathscr{M}(G)
\end{aligned}
$$

where J is the forgetful functor that simply forgets the coalgebra structure. Notice that for M, $N \in \mathscr{M}(G)$,

$$J(M \otimes N) = JM \otimes JN.$$

It is well known that the category of left R-modules over a commutative ring R is symmetric monoidal. In particular, $\mathscr{M}$ is symmetric monoidal. Alongside this, we need

Lemma 8.2.1. $\mathscr{M}(G)$ is symmetric monoidal.

Proof. Since $\mathscr{M}$ is symmetric monoidal, $\mathscr{M}(G)$ inherits a swap map s and the necessary commuting diagrams from it. It only remains to check that s is compatible with the coalgebra structure map. That is, for M, $N \in \mathscr{M}$, M with basis $\{x_\alpha\}$, N with basis $\{x_\beta\}$, $\psi_{N\otimes M} \circ s = G(s) \circ \psi_{M\otimes N}$. But

$$\begin{aligned}\psi_{N\otimes M} \circ s(M \otimes N) &= \psi_{N\otimes M} \circ s\left(\bigoplus_{\alpha,\beta} E_*^{|\alpha|+|\beta|}\right)\\ &= \psi_{N\otimes M}\left(\bigoplus_{\beta,\alpha} E_*^{|\beta|+|\alpha|}\right)\\ &= E_*\left(\Omega^\infty\left(\pi_*^{-1}\bigoplus_{\beta,\alpha} E_*^{|\beta|+|\alpha|}\right)\right)\end{aligned}$$

and

$$\begin{aligned}G(s) \circ \psi_{M\otimes N}(M \otimes N) &= G(s) \circ \psi_{M\otimes N}\left(\bigoplus_{\alpha,\beta} E_*^{|\alpha|+|\beta|}\right)\\ &= G(s)\left(E_*\left(\Omega^\infty\left(\pi_*^{-1}\bigoplus_{\alpha,\beta} E_*^{|\alpha|+|\beta|}\right)\right)\right)\\ &= E_*\left(\Omega^\infty\left(\pi_*^{-1}\bigoplus_{\beta,\alpha} E_*^{|\beta|+|\alpha|}\right)\right).\end{aligned}$$

As M and N are free, the only thing s does is switch the order of the indices, hence these two are equal. □

Armed with this lemma, we can apply Proposition 6.3.2 with notation $\Sigma = J$, $\Omega = G$, $T = \mathrm{id}$ to see that GJ (which we are also just calling G) is a functor of a module triple. In particular, we have strength maps α, α_2 analogous to those in Section 6.3. For M, N in $\mathscr{M}(G)$, define a map

$$a\colon GM \otimes GN \xrightarrow{\alpha} G(M \otimes GN) \xrightarrow{G(\alpha_2)} G^2(M \otimes N) \xrightarrow{\mu} G(M \otimes N).$$

The composite

$$a_{n+1}\colon G^{n+1}M \otimes G^{n+1}N \xrightarrow{a} G(G^nM \otimes G^nN) \xrightarrow{G(a)} \cdots \xrightarrow{G^n(a)} G^{n+1}(M \otimes N)$$

gives rise to a cosimplicial map

$$\boldsymbol{G}M \otimes \boldsymbol{G}N \xrightarrow{\boldsymbol{a}} \boldsymbol{G}(M \otimes N)$$

and, like in the aforementioned section, it can be checked that this is indeed a cosimplicial map.

Definition 8.2.2. For X, $Y \in \mathscr{N}$, define a pairing on Ext groups

$$\wedge_E\colon \operatorname{Ext}^s_{\mathscr{M}(G)}(E_*(S^t), E_*(X)) \otimes \operatorname{Ext}^{s'}_{\mathscr{M}(G)}(E_*(S^{t'}), E_*(Y)) \to \operatorname{Ext}^{s+s'}_{\mathscr{M}(G)}(E_*(S^{t+t'}), E_*(X \wedge Y))$$

to be the composite

$$\begin{aligned} &H^s \hom_{\mathscr{M}(G)}(E_*(S^t), \boldsymbol{G}E_*(X)) \otimes H^{s'} \hom_{\mathscr{M}(G)}(E_*(S^{t'}), \boldsymbol{G}E_*(Y)) \\ &\xrightarrow{f} H^{s+s'}(\hom_{\mathscr{M}(G)}(E_*(S^t), \boldsymbol{G}E_*(X)) \otimes \hom_{\mathscr{M}(G)}(E_*(S^{t'}), \boldsymbol{G}E_*(Y))) \\ &\xrightarrow{\otimes_*} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^{t+t'}), \boldsymbol{G}E_*(X) \otimes \boldsymbol{G}E_*(Y)) \\ &\xrightarrow{\boldsymbol{a}_*} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^{s+s'}), \boldsymbol{G}(E_*(X) \otimes E_*(Y))) \end{aligned}$$

where f is the graded Alexander-Whitney map [May67, p. 132].

Clearly for $t > s \geq 0$ and $t' > s' \geq 0$, this pairing coincides with the pairing

$$E_2^{s,t}\boldsymbol{E}X \otimes E_2^{s',t'}\boldsymbol{E}Y \xrightarrow{\wedge_E} E_2^{s+s',t+t'}\boldsymbol{E}(X \wedge Y)$$

of Section 6.3.

8.3. The composition pairing

Definition 8.3.1. For W, $Y \in \mathscr{N}$, let

$$\begin{aligned} \overline{\beta}\colon\ &\hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(Y)) \\ &\to \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \mathbf{diag}(\underline{\boldsymbol{G}}E_*(Y))) \end{aligned}$$

be induced by the map $\beta = \eta^n\colon \boldsymbol{G}E_*(Y) \to \mathbf{diag}(\underline{\boldsymbol{G}}E_*(Y))$ following the same line of thinking in Section 7.1.

Definition 8.3.2. For $X \in \mathscr{N}$ and maps

$$\begin{aligned} u &\in \hom_{\mathscr{M}(G)}(E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(X)) \\ v &\in \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), \boldsymbol{G}E_*(Y)) \end{aligned}$$

define a map

$$\begin{aligned} b\colon\ &\hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), \boldsymbol{G}E_*(Y)) \otimes \hom_{\mathscr{M}(G)}(E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(X)) \\ &\to \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \mathbf{diag}(\underline{\boldsymbol{G}}E_*(Y)) \end{aligned}$$

on u and v, such that $b(v, u)$ is the composite

$$E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W) \xrightarrow{\mathrm{id} \otimes u} E_*(S^t) \otimes G^n E_*(X) \xrightarrow{\alpha_2^n} G^n(E_*(S^t) \otimes E_*(X)) \xrightarrow{G^n(v)} G^{2n}E_*(Y).$$

By α_2^n we mean $G^{n-1} \circ \cdots \circ \alpha_2$.

Now define the map c by

$$\begin{aligned} c\colon\ &H^{s+s'}[\hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), \boldsymbol{G}E_*(Y)) \otimes \hom_{\mathscr{M}(G)}(E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(X))] \\ &\xrightarrow{b_*} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \mathbf{diag}(\underline{\boldsymbol{G}}E_*(Y)) \end{aligned}$$

$$\xrightarrow{\overline{\beta}_*^{-1}} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(Y)).$$

Definition 8.3.3. Define the composition pairing

$$\begin{aligned} &\mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), E_*(Y)) \otimes \mathrm{Ext}^{s'}_{\mathscr{M}(G)}(E_*(S^{t'}) \otimes E_*(W), E_*(X)) \\ &\qquad\qquad \xrightarrow{\circ} \mathrm{Ext}^{s+s'}_{\mathscr{M}(G)}(E_*(S^{t+t'}) \otimes E_*(W), E_*(Y)) \end{aligned}$$

to be the composite

$$\begin{aligned} &H^s \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), \boldsymbol{G}E_*(Y)) \otimes H^{s'} \hom_{\mathscr{M}(G)}(E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(X)) \\ &\qquad \xrightarrow{f} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(X), \boldsymbol{G}E_*(Y)) \\ &\qquad\qquad \xrightarrow{c} H^{s+s'} \hom_{\mathscr{M}(G)}(E_*(S^t) \otimes E_*(S^{t'}) \otimes E_*(W), \boldsymbol{G}E_*(Y)) \end{aligned}$$

where f is the Alexander-Whitney map.

Clearly for $t > s \geq 0$ and $t' > s' \geq 0$, this pairing coincides with the pairing

$$E_2^{s,t}\,\mathbf{Hom}(X, \boldsymbol{E}Y) \otimes E_2^{s',t'}\,\mathbf{Hom}(W, \boldsymbol{E}X) \xrightarrow{\circ} E_2^{s+s',t+t'}\,\mathbf{Hom}(W, \boldsymbol{E}Y)$$

of Section 7.2.

8.4. The $*$ pairing

There will be statements made in this section without much justification. One can consult [BCM78] for all of the gory details.

For $M \in \mathscr{M}(G)$, M is also a coalgebra over E_* in the traditional sense, with coproduct map Δ the composite

$$\begin{array}{ccccc} M \xrightarrow{\psi_M} G(M) & & G(M) \otimes G(M) & \xrightarrow{\epsilon \otimes \epsilon} & M \otimes M. \\ \Vert & & \uparrow \cong & & \\ E_*(\Omega^\infty(\pi_*^{-1}M)) & \xrightarrow{E_*(\mathrm{diag})} & E_*(\Omega^\infty(\pi_*^{-1}M) \wedge \Omega^\infty(\pi_*^{-1}M)) & & \end{array}$$

As M is a coalgebra, we can talk about the submodule of primitives of M,

$$PM = \{m \in M \mid \Delta(m) = 1 \otimes m + m \otimes 1\}.$$

Assume the hypothesis that $PE_*(\Omega^\infty(\Sigma^i E))$ is E_*-free, and the composite

$$PE_*(\Omega^\infty(\Sigma^i E)) \hookrightarrow E_*(\Omega^\infty(\Sigma^i E)) = E_*(\Omega^\infty(\Sigma^i E)) \to E_*(\Sigma^i E)$$

is injective. Define a functor

$$U \colon \mathscr{M} \to \mathscr{M}$$

by $U(M) = PGM$. Then U forms the functor of a cotriple on $\mathscr{M}$.

Let $\mathscr{A}$ be the category of all (not just free) graded E_*-modules. U extends to a functor of a cotriple on $\mathscr{A}$, and we set $\mathscr{A}(U)$ as the category of U-coalgebras on $\mathscr{A}$. $\mathscr{A}(U)$ is an abelian category, though this is a nontrivial fact, proved in [Tho20, 2.6].

P is a functor from $\mathscr{M}(G)$ to $\mathscr{A}(U)$. If E is a Landweber exact ring spectrum, and X is a "very nice" space like a sphere or $SU(n)$, we end up with an isomorphism (cf. [**BH04**, 4.1])

$$\mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^t), E_*(X)) \cong \mathrm{Ext}^s_{\mathscr{A}(U)}(E_*(S^t), A)$$

for some $A \in \mathscr{A}(U)$, depending on $E_*(X)$. For instance, if $X = S^{2n+1}$, then $A = E_*(S^{2n+1})$.

For all $X \in \mathscr{N}$, the $*$ pairing of Section 7.3 corresponds to a pairing of Ext groups

$$\mathrm{Ext}^s_{\mathscr{M}(G)}(E_*(S^{t+m}), E_*(X)) \otimes \mathrm{Ext}^{s'}_{\mathscr{M}(G)}(E_*(S^{t'}), E_*(S^m)) \to \mathrm{Ext}^{s+s'}_{\mathscr{M}(G)}(E_*(S^{t+t'}), E_*(X)).$$

But if X is very nice, this becomes

$$\mathrm{Ext}^s_{\mathscr{A}(U)}(E_*(S^{t+m}), A) \otimes \mathrm{Ext}^{s'}_{\mathscr{A}(U)}(E_*(S^{t'}), E_*(S^m)) \to \mathrm{Ext}^{s+s'}_{\mathscr{A}(U)}(E_*(S^{t+t'}), A).$$

As $\mathscr{A}(G)$ is an abelian category, we identify $\mathrm{Ext}^s_{\mathscr{A}(G)}$ with the Ext group of s-fold extensions by [**ML63**, III.8.2], and a calculation yields that the $*$ pairing on Ext is $(-1)^{ss'+ts'}d$, where d is the composite

$$\begin{aligned}
&\mathrm{Ext}^s_{\mathscr{A}(U)}(E_*(S^{t+m}), A) \otimes \mathrm{Ext}^{s'}_{\mathscr{A}(U)}(E_*(S^{t'}), E_*(S^m)) \\
&\qquad \to \mathrm{Ext}^s_{\mathscr{A}(U)}(E_*(S^{t+m}), A) \otimes \mathrm{Ext}^{s'}_{\mathscr{A}(G)}(E_*(S^{t+t'}), E_*(S^{t+m})) \\
&\qquad\qquad \to \mathrm{Ext}^{s+s'}_{\mathscr{A}(U)}(E_*(S^{t+t'}), A)
\end{aligned}$$

where the first map is induced by $E_*(S^t) \otimes -$ and the second map is the Yoneda product [**ML63**, p. 82].

APPENDIX A

Some properties of pairings based on $\boldsymbol{R}X$

This appendix seeks to serve two purposes:

- to give intuition to some of the maps in Section 6.3 and Section 7.2, as well as
- to provide explicit, computational proofs to some of the simpler propositions of [**BK73a**].

In essence, this will just be a reformulation of the pairings $\wedge_E$ and $\circ$ from the mentioned sections above, but for $\boldsymbol{R}X$ rather than $\boldsymbol{E}X$, following in line with [**BK73a**].

Definition A.0.1. Define maps

$$\alpha\colon RX \wedge Y \to R(X \wedge Y)$$
$$\alpha_2\colon X \wedge RY \to R(X \wedge Y)$$

by

$$\left(\sum_i r_i \cdot x_i\right) \wedge y \mapsto \sum_i r_i \cdot (x_i \wedge y)$$
$$x \wedge \left(\sum_i r_i \cdot y_i\right) \mapsto \sum_i r_i \cdot (x \wedge y_i)$$

respectively.

We leave it to the reader to verify that both α and α_2 satisfy the diagrams necessary to turn R into a module triple; see also [**Lib03**, 1.1].

Remark A.0.2. We remind the reader that any computations on the elements of $R^n X$ can be reduced to just the single terms, as the single terms are generators, and all maps are group homomorphisms.

Definition A.0.3. For X, Y pointed simplicial sets, define

$$RX \wedge RY \xrightarrow{a} E(X \wedge Y)$$

by

$$RX \wedge RY \xrightarrow{\alpha} R(X \wedge RY) \xrightarrow{R(\alpha_2)} R^2(X \wedge Y) \xrightarrow{\Phi} R(X \wedge Y).$$

Then the composite

$$a_{n+1}\colon R^{n+1}X \wedge R^{n+1}Y \xrightarrow{a} R(R^n X \wedge R^n Y) \xrightarrow{R(a)} \cdots \xrightarrow{R^n(a)} R^{n+1}(X \wedge Y)$$

gives rise to a map

$$\boldsymbol{R}X \wedge \boldsymbol{R}Y \xrightarrow{\boldsymbol{a}} \boldsymbol{R}(X \wedge Y).$$

PROPOSITION A.0.4. $\boldsymbol{a}$ is a cosimplicial map.

PROOF. Let $(r_{i_1},\dots,r_{i_n})x \in R^nX$ and $(r_{j_1},\dots,r_{j_n})y \in R^nY$. Then

$$\begin{aligned} d^k \circ a_n[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y] &= d^k[(r_{i_1}r_{j_1},\dots,r_{i_n}r_{j_n})(x\wedge y)] \\ &= (r_{i_1}r_{j_1},\dots,r_{i_k}r_{j_k},1,\dots,r_{i_n}r_{j_n})(x\wedge y) \\ &= a_n[(r_{i_1},\dots,r_{i_k},1,\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_k},1,\dots,r_{j_n})y] \\ &= a_n \circ (d^k \wedge d^k)[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y]. \end{aligned}$$

Similarly,

$$\begin{aligned} s^k \circ a_n[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y] &= s^k[(r_{i_1}r_{j_1},\dots,r_{i_n}r_{j_n})(x\wedge y)] \\ &= (r_{i_1}r_{j_1},\dots,r_{i_{k+1}}r_{j_{k+1}}r_{i_{k+2}}r_{j_{k+2}},\dots,r_{i_n}r_{j_n})(x\wedge y) \\ &= a_n[(r_{i_1},\dots,r_{i_{k+1}}r_{i_{k+2}},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_{k+1}}r_{j_{k+2}},\dots,r_{j_n})y] \\ &= a_n \circ (s^k \wedge s^k)[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y]. \quad \square \end{aligned}$$

PROPOSITION A.0.5. $\boldsymbol{a}$ is associative and commutative.

PROOF. Let $(r_{i_1},\dots,r_{i_n})x \in R^nX$, $(r_{j_1},\dots,r_{j_n})y \in R^nY$, and $(r_{k_1},\dots,r_{k_n})z \in R^nZ$. For associativity,

$$\begin{aligned} &a_n[a_n[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y] \wedge (r_{k_1},\dots,r_{k_n})z] \\ &\quad= a_n[(r_{i_1}r_{j_1},\dots,r_{i_n}r_{j_n})(x\wedge y) \wedge (r_{k_1},\dots,r_{k_n})z] \\ &\quad= (r_{i_1}r_{j_1}r_{k_1},\dots,r_{i_n}r_{j_n}r_{k_n})(x\wedge y\wedge z) \\ &\quad= a_n[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1}r_{k_1},\dots,r_{j_n}r_{k_n})(y\wedge z)] \\ &\quad= a_n[(r_{i_1},\dots,r_{i_n})x \wedge a_n[(r_{j_1},\dots,r_{j_n})y \wedge (r_{k_1},\dots,r_{k_n})z]]. \end{aligned}$$

For commutativity,

$$\begin{aligned} a_n[(r_{i_1},\dots,r_{i_n})x \wedge (r_{j_1},\dots,r_{j_n})y] &= (r_{i_1}r_{j_1},\dots,r_{i_n}r_{j_n})(x\wedge y) \\ &= (r_{j_1}r_{i_1},\dots,r_{j_n}r_{i_n})(y\wedge x) \\ &= a_n[(r_{j_1},\dots,r_{j_n})y \wedge (r_{i_1},\dots,r_{i_n})x]. \end{aligned}$$

See that commutativity of a depends on commutativity of R. $\square$

DEFINITION A.0.6. Define a pairing on spectral sequences

$$\wedge_R\colon E_r^{s,t}\boldsymbol{R}X \otimes E_r^{s',t'}\boldsymbol{R}Y \to E_r^{s+s',t+t'}\boldsymbol{R}(X\wedge Y)$$

to be the composite

$$E_r^{s,t}\boldsymbol{R}X \otimes E_r^{s',t'}\boldsymbol{R}Y \xrightarrow{\wedge_{BK}} E_r^{s+s',t+t'}(\boldsymbol{R}X \wedge \boldsymbol{R}Y) \xrightarrow{E_r\boldsymbol{a}} E_r^{s+s',t+t'}\boldsymbol{R}(X\wedge Y).$$

PROPOSITION A.0.7. The pairing $\wedge_R$ is associative and commutative up to sign.

PROOF. This is the same proof as for $\wedge_E$. $\square$

Definition A.0.8. For pointed simplicial sets X, Y, construct a cosimplicial space $\mathbf{Hom}(X, \boldsymbol{R}Y)$ by setting

$$\mathbf{Hom}(X, \boldsymbol{R}Y)^n = \mathbf{Hom}(X, R^{n+1}Y)$$

with coface and codegeneracy maps obtained by postcomposing m-simplices with the coface and codegeneracy maps of $\boldsymbol{R}Y$:

$$d^i\colon \Delta^m \wedge X \to R^{n-1}Y \xrightarrow{d^i} R^nY$$
$$s^i\colon \Delta^m \wedge X \to R^{n+1}Y \xrightarrow{s^i} R^nY.$$

Definition A.0.9. Denote by $R \otimes X$ the free simplicial R-module on a pointed space X. Let $t_0\colon R \otimes (R \otimes X) \to R \otimes (R \otimes X)$ be the map

$$(r_1, r_2)x \mapsto (r_2, r_1)x.$$

It is trivial to check that this is well-defined, a group homomorphism (but not an R-module map), and takes the submodule $R \otimes (R \otimes *)$ to itself. Thus, this passes to a well-defined map on the quotient $R^2X \coloneqq [R \otimes (R \otimes X)]/[R \otimes (R \otimes *)]$. Then define $t_i\colon R^nX \to R^nX$, $n \geq 2$ by

$$t_i = R^i(t_0),$$

so that t_i is the map that interchanges the $(i+1)$th and $(i+2)$th copies of R.

Definition A.0.10. Let $w_1 = s^0$ and inductively define w_n as the composite

$$R^{2n}X \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_1} R^{2n}X \xrightarrow{s^0} R^{2n-1}X \xrightarrow{R(w_{n-1})} R^nX.$$

Proposition A.0.11. *$w_n((r_1, \ldots, r_{2n})x) = (r_1r_{n+1}, r_2r_{n+2}, \ldots, r_nr_{2n})x$. Intuitively, w_n combines the jth and $(n+j)$th copies of R, where $1 \leq j \leq n$.*

Proof. x is omitted from the notation. We proceed by induction on n.

$$w_2\colon (r_1, r_2, r_3, r_4) \xmapsto{t_1} (r_1, r_3, r_2, r_4) \xmapsto{s^0} (r_1r_3, r_2, r_4) \xmapsto{R(s^0)} (r_1r_3, r_2r_4).$$

For the inductive step,

$$w_{n+1}\colon (r_1, \ldots, r_{2n+2}) \xmapsto{t_n \cdots t_1} (r_1, r_{n+2}, r_2, \ldots, r_{2n+2}) \xmapsto{s^0} (r_1r_{n+2}, r_2, \ldots, r_{2n+2})$$
$$\xmapsto{R(w_n)} (r_1r_{n+2}, r_2r_{n+3}, \ldots, r_{n+1}r_{2n+2}). \qquad \square$$

Definition A.0.12. Consider m-simplices

$$(u\colon \Delta^m \wedge W \to R^nX) \in \mathbf{Hom}(W, \boldsymbol{R}X)$$
$$(v\colon \Delta^m \wedge X \to R^nY) \in \mathbf{Hom}(X, \boldsymbol{R}Y).$$

Define a map

$$c\colon \mathbf{Hom}(X, \boldsymbol{R}Y) \wedge \mathbf{Hom}(W, \boldsymbol{R}X) \to \mathbf{Hom}(W, \boldsymbol{R}Y)$$

on the m-simplices, such that $c(v,u)$ is the composite

$$\Delta^m \wedge W \xrightarrow{\mathrm{diag}\wedge\mathrm{id}} \Delta^m \wedge \Delta^m \wedge W \xrightarrow{\mathrm{id}\wedge u} \Delta^m \wedge R^n X \xrightarrow{\alpha_2^n} R^n(\Delta^m \wedge X) \xrightarrow{R^n(v)} R^{2n}Y \xrightarrow{w_n} R^n Y.$$

By α_2^n we mean $R^{n-1}(\alpha_2)\circ\cdots\circ\alpha_2$.

Proposition A.0.13. *c is a cosimplicial map.*

Proof. Let $(v,u)\in \mathbf{Hom}(X,R^nY)\wedge\mathbf{Hom}(W,R^nX)$. Let $\kappa\in\Delta^m$ and $w\in W$. Denote by $\tilde{u}$ the element in X such that

$$u(\kappa,w)=(r_1,\dots,r_n)\tilde{u}\in R^nX.$$

Denote by $\tilde{v}$ the element in Y such that

$$v(\kappa,\tilde{u})=(r_1',\dots,r_n')\tilde{v}\in R^nY.$$

We show first that c commutes with the coface maps. Note that

$$d^iu=(r_1,\dots,r_i,1,\dots,r_n)\tilde{u},$$
$$d^iv=(r_1',\dots,r_i',1,\dots,r_n')\tilde{v}.$$

We have:

d^ic:

$$(\kappa,w)\xmapsto{\mathrm{diag}\wedge\mathrm{id}}(\kappa,\kappa,w)\xmapsto{\mathrm{id}\wedge u}(\kappa,u(\kappa,w))=(\kappa,(r_1,\dots,r_n)\tilde{u})\xmapsto{\alpha_2^n}(r_1,\dots,r_n)(\kappa,\tilde{u})$$
$$\xmapsto{R^n(v)}(r_1,\dots,r_n)v(\kappa,\tilde{u})=(r_1,\dots,r_n,r_1',\dots,r_n')\tilde{v}\xmapsto{w_n}(r_1r_1',\dots,r_nr_n')\tilde{v}$$
$$\xmapsto{d^i}(r_1r_1',\dots,r_ir_i',1,\dots,r_nr_n')\tilde{v};$$

cd^i:

$$(\kappa,w)\xmapsto{\mathrm{diag}\wedge\mathrm{id}}(\kappa,\kappa,w)\xmapsto{\mathrm{id}\wedge d^iu}(\kappa,d^iu(\kappa,w))=(\kappa,(r_1,\dots,r_i,1,\dots,r_n)\tilde{u})$$
$$\xmapsto{\alpha_2^n}(r_1,\dots,r_i,1,\dots,r_n)(\kappa,\tilde{u})$$
$$\xmapsto{R^n(d^iv)}(r_1,\dots,r_i,1,\dots,r_n)(r_1',\dots,r_i',1,\dots,r_n')\tilde{v}$$
$$=(r_1,\dots,r_i,1,\dots,r_n,r_1',\dots,r_i',1,\dots,r_n')\tilde{v}$$
$$\xmapsto{w_n}(r_1r_1',\dots,r_ir_i',1\cdot 1,\dots,r_nr_n')\tilde{v}.$$

This concludes the coface case. For codegeneracies, first observe that

$$s^iu=(r_1,\dots,r_{i+1}r_{i+2},\dots,r_n)\tilde{u}$$
$$s^iv=(r_1',\dots,r_{i+1}'r_{i+2}',\dots,r_n')\tilde{v}.$$

Then

s^ic:

$$(\kappa,w)\xmapsto{\mathrm{diag}\wedge\mathrm{id}}(\kappa,\kappa,w)\xmapsto{\mathrm{id}\wedge u}(\kappa,u(\kappa,w))=(\kappa,(r_1,\dots,r_n)\tilde{u})\xmapsto{\alpha_2^n}(r_1,\dots,r_n)(\kappa,\tilde{u})$$

$$\xmapsto{R^n(v)} (r_1,\dots,r_n)v(\kappa,\tilde{u}) = (r_1,\dots,r_n,r'_1,\dots,r'_n)\tilde{v} \xmapsto{w_n} (r_1r'_1,\dots,r_nr'_n)\tilde{v}$$
$$\xmapsto{s^i} (r_1r'_1,\dots,r_{i+1}r'_{i+1}r_{i+2}r'_{i+2},\dots,r_nr'_n)\tilde{v};$$

cs^i:

$$(\kappa,w) \xmapsto{\mathrm{diag}\wedge\mathrm{id}} (\kappa,\kappa,w) \xmapsto{\mathrm{id}\wedge s^iu} (\kappa, s^iu(\kappa,w)) = (\kappa,(r_1,\dots,r_{i+1}r_{i+2},\dots,r_n)\tilde{u})$$
$$\xmapsto{\alpha_2^n} (r_1,\dots,r_{i+1}r_{i+2},\dots,r_n)(\kappa,\tilde{u})$$
$$\xmapsto{R^n(s^iv)} (r_1,\dots,r_{i+1}r_{i+2},\dots,r_n)(r'_1,\dots,r'_{i+1}r'_{i+2},\dots,r'_n)\tilde{v}$$
$$= (r_1,\dots,r_{i+1}r_{i+2},\dots,r_n,r'_1,\dots,r'_{i+1}r'_{i+2},\dots,r'_n)\tilde{v}$$
$$\xmapsto{w_n} (r_1r'_1,\dots,r_{i+1}r_{i+2}r'_{i+1}r'_{i+2},\dots,r_nr'_n)\tilde{v}.$$

That these two coincide is dependent on the commutativity of R. □

PROPOSITION A.0.14. *c is associative.*

PROOF. Let κ, w, u, v, $\tilde{u}$, $\tilde{v}$ be as they are above. Let further

$$(z\colon \Delta^m \wedge Y \to R^nZ) \in \mathbf{Hom}(Y, R^nZ)$$

and $\tilde{z}$ the element in Z such that $z(\kappa,\tilde{v}) = (r''_1,\dots,r''_n)\tilde{z}$. We wish to show that $c(z,c(v,u)) = c(c(z,v),u)$. Recall that $c(v,u)$ has already been computed above.

$c(z,c(v,u))$:

$$(\kappa,w) \xmapsto{\mathrm{diag}\wedge\mathrm{id}} (\kappa,\kappa,w) \xmapsto{\mathrm{id}\wedge c(v,u)} (\kappa, c(v,u)(\kappa,w)) = (\kappa,(r_1r'_1,\dots,r_nr'_n)\tilde{v})$$
$$\xmapsto{\alpha_2^n} (r_1r'_1,\dots,r_nr'_n)(\kappa,\tilde{v}) \xmapsto{R^n(z)} (r_1r'_1,\dots,r_nr'_n)(r''_1,\dots,r''_n)\tilde{z}$$
$$= (r_1r'_1,\dots,r_nr'_n,r''_1,\dots,r''_n)\tilde{z} \xmapsto{w_n} (r_1r'_1r''_1,\dots,r_nr'_nr''_n)\tilde{z}.$$

Before computing the other side of the equation, we need

$c(z,v)(\kappa,\tilde{u})$:

$$(\kappa,\tilde{u}) \xmapsto{\mathrm{diag}\wedge\mathrm{id}} (\kappa,\kappa,\tilde{u}) \xmapsto{\mathrm{id}\wedge v} (\kappa,(r'_1,\dots,r'_n)\tilde{v}) \xmapsto{\alpha_2^n} (r'_1,\dots,r'_n)(\kappa,\tilde{v})$$
$$\xmapsto{R^n(z)} (r'_1,\dots,r'_n,r''_1,\dots,r''_n)\tilde{z} \xmapsto{w_n} (r'_1r''_1,\dots,r'_nr''_n)\tilde{z}.$$

Now we continue,

$c(c(z,v),u)$:

$$(\kappa,w) \xmapsto{\mathrm{diag}\wedge\mathrm{id}} (\kappa,\kappa,w) \xmapsto{\mathrm{id}\wedge u} (\kappa,u(\kappa,w)) = (\kappa,(r_1,\dots,r_n)\tilde{u})$$
$$\xmapsto{\alpha_2^n} (r_1,\dots,r_n)(\kappa,\tilde{u}) \xmapsto{R^n(c(z,v))} (r_1,\dots,r_n)(r'_1r''_1,\dots,r'_nr''_n)\tilde{z}$$
$$= (r_1,\dots,r_n,r'_1r''_1,\dots,r'_nr''_n)\tilde{z} \xmapsto{w_n} (r_1r'_1r''_1,\dots,r_nr'_nr''_n)\tilde{z}.$$ □

Definition A.0.15. Define the composition pairing

$$E_r^{s,t}\,\mathbf{Hom}(X,\boldsymbol{R}Y)\otimes E_r^{s',t'}\,\mathbf{Hom}(W,\boldsymbol{R}X)\xrightarrow{\ \circ\ }E_r^{s+s',t+t'}\,\mathbf{Hom}(W,\boldsymbol{R}Y)$$

to be the composite

$$\begin{aligned}E_r^{s,t}\,\mathbf{Hom}(X,\boldsymbol{R}Y)\otimes E_r^{s',t'}\,\mathbf{Hom}(W,\boldsymbol{R}X)&\xrightarrow{\wedge_{BK}}E_r^{s+s',t+t'}(\mathbf{Hom}(X,\boldsymbol{R}Y)\wedge\mathbf{Hom}(W,\boldsymbol{R}X))\\&\xrightarrow{E_rc}E_r^{s+s',t+t'}\,\mathbf{Hom}(W,\boldsymbol{R}Y).\end{aligned}$$

Proposition A.0.16. The pairing $\circ$ is associative.

Proof. This is the same proof as for the $\boldsymbol{E}X$ case. □

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