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Soundness and Completeness Results for the Logic of Evidence Aggregation and its Probability Semantics

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SOUNDNESS AND COMPLETENESS RESULTS FOR THE LOGIC OF EVIDENCE AGGREGATION AND ITS PROBABILITY SEMANTICS

by

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ABSTRACT

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In [4], a logical system called the *logic of evidence aggregation* (LEA) was introduced, along with an intended semantics for it called *probability semantics.* The goal was to describe probabilistic evidence aggregation in the setting of formal logic. However, as noted in that paper, LEA is not complete with respect to probability semantics. This leaves open the tasks to find sound and complete semantics for LEA and a proper axiomatization for probability semantics. In this thesis we do both. We define a class of basic models called deductive basic models. We show LEA is sound and complete with respect to the class of deductive basic models. We also define an axiomatic system LEA_+ extending LEA and show it is sound and complete with respect to probability semantics. Along the way, we develop an intermediate system LEA_- which, it turns out, is equivalent to Propositional Lax Logic with a classical logic base. Sequent systems and decidability results for LEA_- and LEA_+ are also presented.

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1 Introduction

The subject of this thesis is the detailed analysis of three interesting and related justification logics. The logics in question are the Logic of Evidence Aggregation (LEA) which was introduced in [4], and LEA_- and LEA_+, which were introduced by the author in [39]. Technical results for the LEA family of logics are explored in detail in this thesis.

We situate the project of LEA more broadly within a research goal of "reasoning with uncertain reasons" [1]. Such a project is a natural historical development in the history of logic, and justification logic in particular. Logicians have long sought to represent something like the human reasoning apparatus – that which allows us to draw conclusions from inferences – abstractly on the page. It is hoped that by faithfully representing the reasoning process with symbols and formal rules, we may better answer the questions "How do we reason?" and "How ought we reason (in such-and-such context)?" The former question is of prime interest to philosophers and cognitive scientists. The latter question is of interest to anyone from business executives, economists, artificial intelligence experts, and more.

An important development in logic came with the advent of justification logic. Justification logic internalized the reasoning process in the format, so that rather than talking only about propositions, one could describe the justifications for holding those facts, analyse the interplay between justifications and conclusions.

For example, consider the following chain of reasoning. *If Alice will meet her sales quota, then Bob will be happy. Alice will meet her sales quota. Therefore, Bob will be happy.* The particulars regarding Alice and Bob have no bearing the validity of the reasoning. The reasoning is valid simply due to the logical structure of the sentences. We abstract away from Alice and Bob using logical sentences A and B. We can represent that sequence again as

[1] This is a play on the title of the book by Artemov and Fitting.

$A \to B$	If Alice will meets her sales quota, then Bob will be happy.
A	Alice will meet her sales quota.
—-	(Therefore)
B	Bob will be happy.

Such is the domain of standard propositional logics. Justification logic considers not only the interaction of such abstracted propositions, but also the reasons one holds those propositions.

For example, consider the interlocutor who is describing the Alice-Bob situation. What reasons do they have for maintaining each of those propositions? If we include those reasons, then the following chain of reasoning may occur.

In that email, Bob wrote that if Alice will meet her sales quota, then Bob will be happy. This spreadsheet shows that Alice has already met her sales quota, (and therefore will meet her sales quota.) Combining those two pieces of evidence, we may reason that Bob will be happy.

Let e be the email, and s the spreadsheet. We write $e\colon(A \to B)$ to indicate that the email is evidence for the claim that if Alice will meet her sales quota, then Bob will be happy. $s\colon A$ works similarly. $e \cdot s$ is some combination of the evidence, perhaps with common sense thrown in.

$e\colon(A \to B)$	That email shows that if Alice will meet her sales quota, then Bob will be happy.
$s\colon A$	This spreadsheet shows that Alice will meet her sales quota.
—-	Therefore
$e \cdot s\colon B$	the combination of the email, the spreadsheet, and some common reasoning shows that Bob will be happy.

This is a simple example to illustrate the important fact that justification logics allow the user to internalize the reasoning process into the logical calculus itself. However, the example is anachronistic to the chronological development of justification logic. That is because the original readings of e, s were not epistemic justifications, but formal mathematical proofs.

Originally developed by Artemov in [5] and more formally introduced in [2], the original justification logic LP was developed to answer an important open question in intuitionistic mathematics[2]. Specifically, it was designed to give exact provability semantics – the so-called Brower-Heyting-Kolmogorov (BHK) semantics – to intuitionistic logic. In BHK semantics or intuitionistic logic, propositions were considered true if they were provable, and the truth of compound formulas was obtained by specific operations on the proofs of the subformulas.

In LP, proofs were internalized into the calculus as *proof polynomials*, now called *justification terms.* Those are $e, s, e \cdot s$ in the example above. In LP the formula $s{:}A$ means that s is (encodes) a proof of A. By carefully decoding a proof polynomial, the user could reproduce line-by-line the proof which generated that conclusion. This proof can be made entirely explicit, rather than symbolic, via a translation into Peano Arithmetic.

Later, it was seen by Artemov and others that the justification logic paradigm can be applied more broadly, to a variety of epistemic situations other than strict mathematical proof. In [6], readings given for $t{:}A$ included "t is justification for A," "t is a sufficient resource for A," or "t is evidence for A." Other interpretations followed, such as public announcement [11, 12], obligation (t is an obligation towards A) [21], and more.

The examples given so far are mostly idealized models of reasoning, in the sense that the proposition always follows from the justification for it. This is not the case in real life. In the world, justifications do not necessarily secure their intended conclusion. (This surely is the basis of all argument!) There are probabilistic considerations, some justifications may less reliable than others, there may exist contradictory justifications. In other words, drawing conclusions in the real

[2] Actually, work by Gödel anticipates justification logic, including terms to represent proofs of formulas. This work was unpublished at the time of Artemov's original paper [25, 26] .

world is messy, and the justification logic program ought to be able to model this uncertainty and unreliability at the level of justifications.

One significant attempt is the Logic of Evidence Aggregation [4], and it is the starting point for this thesis. As such, we will postpone a detailed introduction of this system for later.

There are a variety of attempts to model uncertainty at the level of justification or belief in the justification format. Many of these attempt to marry justification logic format with a logical system that is already used to model uncertainty in some sense.

Fuzzy logic has long been used to model vague propositions [28, 46]. Justification logics has been combined with fuzzy logic to model vague and uncertain justifications in [23, 24, 41].

Possibilistic logic is a logic that allows one to make inferences about uncertain or incomplete information [17]. Possibilistic logic attempts to overcome some of the difficulties that arise when applying probabilistic reasoning in a formal logical framework. Rather than probability, certainty (or its dual – uncertainty) is measured. Fan and Liau have applied justification logic to a possibilistic setting to reason about uncertain beliefs in [20, 45]. As an example, in [20] they introduce formulas of the form $t{:}_\alpha A$, with the intended meaning *t is believed with certainty at least α.*

There have been a variety of justification logics attempting to incorporate some discussion of probability directly in the language. In a series of papers [30, 31, 32] Kokkinis et al. develop justification logics that allow formulas of the form $P_{\geq\alpha}A$, where A is an arbitrary sentence, and which has the intended meaning *The probability of A is greater than α.* They also allow formulas $t{:}P_{\geq\alpha}A$, intended to mean that *t is evidence that the probability of A is greater than α.* [40] extends this work to introduce conditional probabilities into the language. The formula $CP_{\approx\alpha}(A,B)$ means *The probability of A under condition B is approximately α.* One can understand the formula $CP_{\approx 1}(A,B)$ to mean that it is usually the case that when B occurs, A occurs. This allows the authors to model defaults in the language, and thus non monotonic reasoning. Combined with justifications, the authors can model nonmonotic reasoning about evidence and justifications. In [38], Mohammadi and Studer apply justification logic principals to Bayesian confirmation theory .

Once one has introduced uncertainty into the justifications or evidence terms, it is natural to model the effects on knowledge and belief. In no small part due to the uncertainty of justifications, the propositions an agent "knows" or at least believes should change with the advent of additional information. Renee has worked on justification logics that allow evidence elimination – that is, setting aside evidence for a claim in light of new evidence which goes against it – in [43]. That paper is part of a larger project to combine dynamic epistemic logic and justification logic [9, 10, 42]. Applying justification logic to belief expansion has been explored in [33, 34]. In such systems typically there are justification style axioms and operators on justification terms, modal knowledge and belief operators, and rules to describe the interaction between justification and knowledge or belief, and how knowledge and belief change with new justifications.

One reason a justification may or may not be believed is because of the quality of its evidence. Some evidence may be "good quality" while other lesser quality, (for example, blood analysis showing only blood type, versus blood analysis showing a DNA match.) The degree to which one believes a proposition depends on the quality of the evidence. It makes sense to introduce a grading for justification terms as a means to compare justifications or evidence with regards to, for example, their quality, trustworthiness, etc. These gradings need not be numeric, nor even linearly ordered. In [44], Shear and Quiggen develop a justification logic where justifications are believed with certain confidence levels. Those confidence levels have the structure of a free distributive lattice. In [13] Chatalic and Froidevaux attach partially ordered symbolic grades to logical formulas. These grades are represented via different parameterized modalities in a multi-modal calculus.

It is into this fertile field of study that we situate ourselves. The Logic of Evidence aggregation is a logic designed to allow the user to make inferences about events in a probabilistic setting. In other words, if there is probabilistic evidence that A and $A \to B$ occur, what probability should we attach to B occurring? LEA does not attempt to answer this problem by specifying probabilities directly in the format itself. Probabilities are determined later, within the intended *probability semantics.* Instead, a graded system of justification terms (now called *evidence terms*) are introduced, which represent aggregations of evidence sufficient to entail their conclusions.

LEA may safely be said to model evidence aggregation, and can soundly compute probabilities with its intended semantics. However, as noted in [4], LEA is not complete with respect to probability semantics. This left open, until now, the tasks of finding sound and complete semantics for LEA and a proper axiomatization for probability semantics. We solve those problems here.

This thesis is a technical exploration and development of LEA and what we will call the LEA family of logics. We show LEA is sound and complete with respect to the class of deductive basic models. We also define an axiomatic system LEA_+ extending LEA and show it is sound and complete with respect to probability semantics. Along the way, we develop an intermediate system LEA_- which, it turns out, is equivalent to Propositional Lax Logic with a classical logic base. We give sequent systems for LEA_- and LEA_+ and give constructive and semantical cut-elimination theorems for both system. Decidability results for LEA_- and LEA_+ are also presented. Finally, we affirmatively answer the question "do we need justification logic?". We show why we cannot model $t : A$ with classical implication, $t \to A$.

1.1 Outline of Thesis

The outline of this thesis is as follows. In Section 2 we review the basics of justification logic. In Section 3 we introduce the Logic of Evidence Aggregation as well as the probability semantics. In Section 4 we give sound and complete semantics for LEA. In Section 5 we introduce the axiomatic systems LEA_- and LEA_+. We show that LEA_+ is sound and complete with respect to probability semantics. In Section 6 we give sequent system presentations for LEA_- and LEA_+. In Section 7 we compare LEA_- to Propositional Lax Logic. In Section 8 we discuss issues of decidability for LEA_- and LEA_+. In Section 9 we see how LEA_+ takes an intermediate position between standard justification logics and classical propositional logic. The fact of LEA_+ answers the question "do we need justification logic?" with a strong *yes.* We conclude with Section 10 on avenues for future research.

2 Justification Logic Review

Justification logic is an umbrella term to describe a particular group of logics that carry an internalized and intuitive notion of *justification* or *evidence*, and share a common format to denote the interaction with that evidence. In this section we present a brief review of propositional justification logic. Since there is not *one* propositional justification logic, what follows describes what is typically found therein.

2.1 Justification Logic Language

The language consists of a base suitable to the underlying logic. Since the underlying logic is quite often classical, the underlying base contains the usual Boolean connectives and propositional variables. In addition, there is a special symbol **:** (colon) and a set of *justification terms*.

The set of justification terms is defined inductively, starting with a set of *justification variables* and *justification constants* and closing over term operation symbols. There are a wide variety of term operations in the literature.

Finally, the set of formulas is defined as the closure of the propositional variables over the logical connectives, with the additional clause that:

if A is a formula, and t is a term, then $t{:}\,A$ is a formula.

The intended meaning of the format is the following. $t{:}\,A$ says that t is evidence for A. Operations on terms represent operations on evidence, such as merging evidence sets or performing a verification of some evidence, and are considered new evidence. For example, the paradigmatic axiom of many justification logics is the so-called *application axiom*

$$s{:}\,(A \rightarrow B) \rightarrow (t{:}\,A \rightarrow s \cdot t{:}\,B).$$

This axiom scheme operationally defines the operation $\cdot$. It says that evidence for an implication may be combined with evidence for its antecedent to produce (new) evidence for its consequent.

In general in this thesis, unless otherwise stated, capital Roman letters A, B, C will represent formulas, lowercase Roman letters s, t, u will represent justification terms, and capital Greek letters such as Γ will represent sets of formulas.

2.2 Justification Logic Semantics

The two most commonly seen semantics for justification logics are *basic models* and *Fitting semantics*. Basic models were originally developed for the logic LP and introduced by Mkrtychev in [37], and further developed by Artemov in [7]. Fitting models can be seen as a generalization, combining Kripke style possible world semantics with the interpretation of justification terms developed by Mkrtychev. Fitting models were introduced by their namesake in [22]. This thesis uses basic model semantics, so we omit the definition of Fitting semantics.

Definition 1 (Basic Model). *A basic model $*$ is a mapping of justification terms and formulas which takes formulas to truth values 1 (true) or 0 (false), takes justification terms to sets of formulas, and satisfies the following conditions:*

- $\bot^* = 0$
- $(A \wedge B)^* = 1$ *iff* $A^* = 1$ *and* $B^* = 1$
- $(A \vee B)^* = 1$ *iff* $A^* = 1$ *or* $B^* = 1$
- $(\neg A)^* = 1$ *iff* $A^* = 0$
- $(A \to B)^* = 1$ *iff* $A^* = 0$ *or* $B^* = 1$
- $(t{:}\, A)^* = 1$ *iff* $A \in t^*$.

Definition 2 (Entailment in a Basic Model). *For an entailment relation* $\models$*, we write* $\Gamma \models_* A$ *iff* $A^* = 1$ *or there exists* $C \in \Gamma$ *such that* $C^* = 0$*. We write* $\Gamma \models A$ *iff, for all basic models* $*$ *(of the basic model class under consideration)* $\Gamma \models_* A$*. We write* $* \models A$ *to mean* $\emptyset \models_* A$*.*

Definition 3 ($\Gamma \vdash_{\mathsf{CL}} A$). *For* $\Gamma \cup \{A\} \subset Form$*, we write* $\Gamma \vdash_{\mathsf{CL}} A$ *iff* A *can be derived from assumptions in* Γ *using the axioms of classical logic, treating justification formulas* $t{:}A$ *as distinct propositional variables.*

Definition 4 (Basic Models). *For* $\Gamma \subset Form$*,* $BM(\Gamma)$ *is the class of all basic models which satisfy all formulas in* Γ*. That is,* $* \in BM(\Gamma)$ *iff* $* \models C$ *for all* $C \in \Gamma$*. If* $\mathcal{J}$ *is an axiomatic system with axioms* $Ax(\mathcal{J})$*, then abusing notation we write* $BM(\mathcal{J})$ *to denote* $BM(Ax(\mathcal{J}))$*.*

Theorem 1 (Generic Completeness). *Each set of formulas* Γ *is sound and complete with respect to its class of basic models. That is* $\Gamma \vdash_{\mathsf{CL}} A$ *iff* A *is true in all basic models of* Γ*.*

Proof. See, for example, [8]. $\square$

Corollary 1. *Let a justification logic* $\mathcal{J}$ *contain Modus Ponens as its only rule of inference, and contain as axioms the axioms of classical logic in the language of* $\mathcal{J}$*. Then* $\Gamma \vdash_{\mathcal{J}} A$ *iff* A *is true in all basic models of* Γ*.*

This allows us to apply the generic completeness theorem to any justification logic containing classical logic with Modus Ponens as its only rule of inference. Such will be the case for all the logics discussed in this thesis.

3 Overview of the Logic of Evidence Aggregation

3.1 Original Problem Formulation

Suppose there is a set of sentences Γ, from which follows the sentence A, that is $\Gamma \vdash A$. Suppose, in addition, that the sentences in Γ are said not to hold certainly, but only with some individual probabilities. That is, we are now considering sentences as events in a probabilistic sense. How can we best estimate the probability for A to occur, given the probability estimates for Γ?

This is a central question in probability logic. See [1, 14, 27, 29] for alternative approaches to this problem and the related problem of evidence aggregation. In [4], an elegant solution was proposed, where the power of the justification logic format was applied to this probabilistic setting. Let us recall the main ideas of that paper, for it is the axiom system and semantics introduced there which we are interested in developing here.

For each event $C_i \in \Gamma$, let there be some event u_i with known probability $P(u_i)$, such that C_i will occur if u_i occurs. In a justification logic format, we write

$$u_i{:}\, C_i.$$

A will follow if a specific configuration of u_i's occurs, written as

$$u_1{:}\, C_1, \dots u_n{:}\, C_n \vdash t(u_1, \dots, u_n){:}\, A.$$

The maximal event t, (maximal in the sense of most assured to occur, minimal in the sense of requiring the weakest assumptions,) will be a symbolic representation of the probability we should assign to A given Γ. This justification term is called the *aggregated evidence* for A given Γ, written $AE^{\Gamma}(A)$.

For example, suppose $\Gamma = \{B, B \to A, A\}$. We would like to calculate the best estimate for A to occur, given Γ. Symbolically, we compute the maximal t such that

$$r\colon B, s\colon (B \to A), u\colon A \vdash t\colon A.$$

A will occur if both r and s occur, or if u occurs. Therefore, the strongest evidence for A given Γ is $(r \cap s) \cup u$, where $\cap$ represents that both events occur, and $\cup$ represents that either event occurs. That is $AE^{\Gamma}(A) = (r \cap s) \cup u$, and

$$r\colon B, s\colon (B \to A), u\colon A \vdash (r \cap s) \cup u\colon A.$$

The formal system which allows us to make such deductions is called the *Logic of Evidence Aggregation*, or LEA. To bring the problem back to the original question of probabilities, we employ a semantics for LEA called *probability semantics*. Each probabilistic interpretation $\circ$ maps justification terms to events in a probability space, which itself comes equipped with a probability function P. To give the "best estimate" for A to occur given Γ, for a particular interpretation $\circ$, we may take the probability of the event corresponding to the strongest possible evidence for A given Γ. That is,

$$P((AE^{\Gamma}(A))^{\circ}).$$

Although LEA is sufficient for the task of formalizing the process of evidence aggregation, and is sound with respect to probability semantics, it is not complete, as shown in [4]. A contribution of this thesis is to give a sound and complete semantics for LEA, and a sound and complete axiomatization for probability semantics.

LEA was already discussed in [35]. That paper introduced a novel type of justification logic semantics called *subset models*. LEA was shown to be sound, but not shown to be complete, for a particular class of subset models.

3.2 LEA Definition

We first define the language and then the proof system for LEA.

Definition 5 (Justification Term). *Justification terms or simply* terms *are defined according to the following grammar:*

$$t := \quad e_i \quad | \quad \mathbf{1} \quad | \quad \mathbf{0} \quad | \quad t \cup t \quad | \quad t \cap t\,.$$

$e_i \in \{e_1, \ldots, e_n\}$, and are called atomic evidence terms, and represent atomic pieces of evidence. $\mathbf{1}$ and $\mathbf{0}$ are justification constants intended to represent empty and total justification specifications, respectively. We can think of $\mathbf{1}$ as being the strongest, most incontrovertible evidence possible. We can think of $\mathbf{0}$ being the weakest or most dubious evidence possible. $\cap$ is an operation which can be thought of like "and", alternatively as an operation which combines evidence in a single body. For example, if s represents transcribed testimony from Smith, and t represents transcribed testimony from Turner, then $s \cap t$ represents the evidence which is those two testimonies combined in a single document. $\cup$ is an operation on evidence terms which can be thought of as "or". If s and t are evidence, then s and t represents the evidential claim that s or t holds.

The set of all terms is denoted $Term$. $Term$ is a bounded free distributive lattice on n-generators. Top and bottom elements are $\mathbf{1}, \mathbf{0}$, respectively. Generators are $e_1, \ldots, e_n$. Meet and join are $\cap, \cup$, respectively. The lattice order is $\leq$. Since each n corresponds to a unique set of terms, to each n there is a unique LEA language, and therefore a unique logic. In practice, however, the n will not be consequential. Therefore, in our notation we will not distinguish between the different logics corresponding between different n.

Definition 6 (LEA **Formula).** LEA *formulas are defined inductively in the usual way for justification logics, according to the following grammar:*

$$A := \quad p \quad | \quad \neg A \quad | \quad A \vee A \quad | \quad A \wedge A \quad | \quad A \to A \quad | \quad t{:}\, A$$

where p ranges over a countably infinite set of propositional variables, and t is a justification term. $A \leftrightarrow B$ is shorthand notation for $(A \to B) \wedge (B \to A)$. The set of all formulas is denoted as $Form$. (We will use "$Form$" for other logics besides LEA*. In all contexts, $Form$ will be understood as the set of all formulas in the logic under current consideration.)*

A special feature of our logical system is that LEA comes equipped with an order $\leq$ on terms, which is not part of the language, yet we reference in the axioms.

Definition 7 (LEA **Proof System).** *Any uniform substitution of* LEA *formulas into the following formulas is an axiom of* LEA*. Modus Ponens is the only rule of inference for* LEA*.*

1. *The axioms and rules of classical logic in the language of* LEA
2. $s{:}\,(A \to B) \to (t{:}\,A \to st{:}\,B)$
3. $(s{:}\,A \wedge t{:}\,A) \to s \cup t{:}\,A$
4. $s{:}\,A \to t{:}\,A$ *for any evidence terms s and t such that $t \leq s$*
5. $\mathbf{1}{:}\,C$ *where C is any axiom*
6. $\mathbf{0}{:}\,A$ *where A is any formula.*

3.3 Alternative Formulation of LEA

In [4], the following axiom scheme is used for LEA:

$$s{:}\,A \to t{:}\,A \text{ if } t \leq s.$$

LEA is already interesting simply for introducing an order on justification terms. However, in the case of LEA, due to the underlying lattice structure of the terms, we can do without this axiom. We choose here to use an equivalent axiom scheme, because defining basic models in the presence of an existing order invites subtleties that we prefer not address.

Moreover, the presentation that we now give will provide more transparency on the behavior of LEA and its related systems we will study in this paper. This equivalent axiom system we will call LEA′. It has the same language as LEA.

Definition 8 **(LEA′ Proof System).** *Any uniform substitution of* LEA *formulas into the following formulas is an axiom of* LEA′*. Modus Ponens is the only rule of inference for* LEA′*.*

1. *The axioms classical logic in the language of* LEA
2. $t\colon(A \to B) \to (t\colon A \to t\colon B)$, *(terms are closed under modus ponens)*
3. *(a)* $s \cup t\colon A \to (s\colon A \wedge t\colon A)$

   (b) $(s\colon A \wedge t\colon A) \to s \cup t\colon A$
4. $(s\colon A \vee t\colon A) \to st\colon A$
5. $\mathbf{1}\colon C$ *where* C *is any axiom*
6. $\mathbf{0}\colon A$ *where* A *is any formula*
7. $s\colon A \leftrightarrow t\colon A$ *if* $s = t$ *in the lattice order.*

Lemma 1. LEA′ *is equivalent to* LEA*. That is, for all sets of formulas* $\Gamma \cup \{A\} \subset Form$*,*

$$\Gamma \vdash_{\mathsf{LEA}} A \text{ iff } \Gamma \vdash_{\mathsf{LEA}'} A.$$

Proof. Since the only rule of inference in both systems is Modus Ponens, and since they share the same language, it is enough to show that an axiom from one system is provable in the other.

First, let us work in LEA′ and show that LEA Axioms 2 and 4 are derivable there.

- **Axiom 2.** If $\vdash_{\mathsf{LEA}'} s\colon(A \to B)$, then, by LEA' Axiom 4, $\vdash_{\mathsf{LEA}'} st\colon(A \to B)$. If $\vdash_{\mathsf{LEA}'} t\colon A$, then by LEA' Axiom 4, $\vdash_{\mathsf{LEA}'} st\colon B$. Then, with the use of LEA' Axiom 2, we get $\vdash_{\mathsf{LEA}'} s\colon(A \to B) \to (t\colon A \to [st] \cap [st]\colon B)$, where we have used brackets to group terms. Since $[st] \cap [st] = st$, with LEA' Axiom 7 we get $\vdash_{\mathsf{LEA}'} s\colon(A \to B) \to (t\colon A \to st\colon B)$.
- **Axiom 4.** Finally, suppose s, t are two terms such that $t \leq s$. Thus, $s = s \cup t$. Therefore, from LEA' Axiom 7, $\vdash_{\mathsf{LEA}'} s\colon A \to s \cup t\colon A$. From LEA' Axiom 3(a) we have $\vdash_{\mathsf{LEA}'} s \cup t\colon A \to t\colon A$. Combining the two, we get $\vdash_{\mathsf{LEA}'} s\colon A \to t\colon A$, when $t \leq s$.

Next, let us work in LEA, and show that LEA' Axioms 2, 3(a), 4, and 7 are derivable there.

- **Axiom 2.** If $\vdash_{\mathsf{LEA}} t\colon(A \to B)$ and $\vdash_{\mathsf{LEA}} t\colon A$, then using LEA Axiom 2, $\vdash_{\mathsf{LEA}} tt\colon B$. Since $tt \leq t$, by LEA Axiom 4, $\vdash_{\mathsf{LEA}} t\colon B$. Therefore $\vdash_{\mathsf{LEA}} t\colon(A \to B) \to (t\colon A \to t\colon B)$.
- **Axiom 3(a).** If $\vdash_{\mathsf{LEA}} s \cup t\colon A$, then since $s \leq s \cup t$ and $t \leq s \cup t$, by LEA_{-} Axiom 4, $\vdash_{\mathsf{LEA}} s\colon A$ and $\vdash_{\mathsf{LEA}} t\colon A$. In particular $\vdash_{\mathsf{LEA}} s \cup t\colon A \to (s\colon A \wedge t\colon A)$.
- **Axiom 4.** If either $\vdash_{\mathsf{LEA}} s\colon A$ or $\vdash_{\mathsf{LEA}} t\colon A$, since $st \leq s$ and $st \leq t$, by LEA Axiom 4, $\vdash_{\mathsf{LEA}} st\colon A$. In particular $\vdash_{\mathsf{LEA}} (s\colon A \vee t\colon A) \to st\colon A$.
- **Axiom 7.** If $s = t$ then $s \leq t$ and $t \leq s$, so by LEA Axiom 4, $\vdash_{\mathsf{LEA}} s\colon A \leftrightarrow t\colon A$.

□

Since the two systems are equivalent, we will no longer distinguish between them, and will refer to the latter LEA' as LEA. However, we may still reference the order $\leq$ when convenient to do so.

3.4 Probability Semantics Definition

Definition 9 (Probability Semantics).

A probabilistic interpretation $\circ$ *consists of:*

- *a probability space* $(\Omega, \mathcal{F}, P)$*, where* Ω *is a set of outcomes,* $\mathcal{F}$ *is a sigma-algebra of measurable events, and* P *is a probability measure on* $\mathcal{F}$

– *a mapping* $\circ$ *which takes terms to elements of* $\mathcal{F}$ *and formulas to subsets of* Ω. *(*$\circ$ *is overloaded to mean both the interpretation its mapping.)*

– $\circ$ *satisfies the conditions below for formulas* A, B *and terms* $s, t, \mathbf{1}$*, and* $\mathbf{0}$*:*

Terms	*Formulas*
$\mathbf{0}^\circ = \emptyset$	$(A \wedge B)^\circ = A^\circ \cap B^\circ$
$\mathbf{1}^\circ = \Omega$	$(A \vee B)^\circ = A^\circ \cup B^\circ$
$(st)^\circ = s^\circ \cap t^\circ$	$(\neg A)^\circ = \overline{A^\circ}$
$(s \cup t)^\circ = s^\circ \cup t^\circ$	$(A \to B)^\circ = \overline{A^\circ} \cup B^\circ$
	$(t{:}\, A)^\circ = \overline{t^\circ} \cup A^\circ.$

In order to define an entailment relation, we extend $\circ$ to sets of formulas, as

$$\Gamma^\circ := \bigcap_{C \in \Gamma} C^\circ.$$

In particular, notice $\emptyset^\circ = \Omega$.

Definition 10 (Entailment in Probability Semantics). *An entailment relation* $\Vvdash$ *is defined as*

$$\Gamma \Vvdash A \text{ iff } \Gamma^\circ \subset A^\circ, \text{ for all probabilistic interpretations } \circ.$$

LEA is sound with respect to probability semantics, but it is not complete, as shown in [4]. This can be seen by noting that $\mathbf{1}{:}\, A \to A$ is valid in probability semantics, but is not provable in LEA.

4 Sound and Complete Semantics for LEA

4.1 Basic Models

We will define a class of models which are sound and complete with respect to LEA. The models will be a class of *basic models*.

Lemma 2 (Model Existence). *BM(LEA) is nonempty.*

Proof. Consider the basic model $*$ where $t^* = Form$ for all terms t. All formulas $t{:}A$ will evaluate to 1 (true), and so will LEA axioms 2 through 7. Since it is a basic model, $*$ satisfies the axioms of classical logic in the language of LEA, as well as Modus Ponens. Therefore, $* \in BM$(LEA). □

Corollary 2. LEA *is consistent.*

4.2 Deductive Basic Models

We call the class of basic models for which LEA is sound and complete the class of *deductive basic models.*

Definition 11 (Deductive Basic Model). *A deductive basic model is a basic model $*$ which satisfies the following conditions:*

- t^* *is a deductively closed set of* LEA *formulas, for all terms* t
- $(s \cup t)^* = s^* \cap t^*$
- $(st)^* \supset s^* \cup t^*$
- $\mathbf{1}^* \supset Taut$
- $\mathbf{0}^* = Form$
- *if* $s = t$ *in the lattice order, then* $s^* = t^*$

where $Taut$ is the set of all LEA *tautologies.*

Theorem 2 (Soundness and Completeness). *$BM(\mathsf{LEA})$ is the class of all deductive basic models.* LEA *is sound and complete with respect to the class of all deductive basic models.*

Proof. Let $*$ be a basic model. We need to show that $*$ is a deductive basic model iff $* \in BM(\mathsf{LEA})$. From Lemma 2, we know $BM(\mathsf{LEA})$ is nonempty. Let s and t be terms.

- t^* is deductively closed iff $* \models t\colon (A \to B) \to (t\colon A \to t\colon B)$.
- $(s \cup t)^* = s^* \cap t^*$ iff $* \models (s\colon A \wedge t\colon A) \leftrightarrow s \cup t\colon A$.
- $(st)^* \supset (s^* \cup t^*)$ iff $* \models (s\colon A \wedge t\colon A) \to st\colon A$.
- $Taut \subset \mathbf{1}^*$ implies that $* \models \mathbf{1}\colon C$ for any axiom C. Conversely, $* \models \mathbf{1}\colon C$, together with $\mathbf{1}^*$ being closed under Modus Ponens, implies that $Taut \subset \mathbf{1}^*$.
- $\mathbf{0}^* = Form$ iff $* \models \mathbf{0}\colon A$ for any formula A.
- Let $s = t$ in the lattice order. $s^* = t^*$ iff $* \models s\colon A \leftrightarrow t\colon A$.

Therefore $* \in BM(\mathsf{LEA})$ iff $*$ is a deductive basic model. From the Generic Completeness Theorem and Corollary 1, we have that LEA is sound and complete with respect to the class of all deductive basic models.

□

5 Sound and Complete Axiomatization of Probability Semantics

We will now work towards an axiom system which is sound and complete with respect to probability semantics. It turns out that an extension of LEA, which we will call LEA_+, is sound and complete with respect to probability semantics. Before investigating LEA_+, however, we turn our attention to a reduct of the language of LEA, with an axiomatic system we will call LEA_-. As a language, LEA_- is very simple – it contains no operations on justification terms, only justification variables. Studying LEA_- will shed light on the probability semantics and simplify our completeness proof for LEA_+.

5.1 LEA_- Definition and Basic Models

Definition 12 **(LEA_- Formula).** *The terms of the LEA_- language consist of a finite number of atomic evidence terms $e_1, e_2, \ldots, e_n$. There are no constants or term operations. Formulas are built up using this set of terms as usual.*

Definition 13 **(LEA_- Proof System).** *Any uniform substitution of LEA_- formulas into the following formulas is an axiom of LEA_-. Modus Ponens is the only rule of inference.*

1. The axioms of classical logic in the language of LEA_-
2. $A \to t{:}A$
3. $t{:}(A \to B) \to (t{:}A \to t{:}B)$.

Let us write $\vdash_-$ to denote $\vdash_{\mathsf{LEA}_-}$.

Axiom 2 is cofactivity. It says that if A is true, then any evidence justifies it. Axiom 3 tells us that evidence terms are closed under Modus Ponens.

Lemma 3 **(Model Existence for LEA_-).** *$BM(\mathsf{LEA}_-)$ is nonempty.*

Proof. Consider the basic model $*$, where $t^* = Form$ for all terms t. All formulas $t{:}\,A$ will evaluate to 1 (true), and so will LEA_- Axioms 2 and 3. Since it is a basic model, $*$ satisfies the axioms of classical logic in the language of LEA_-, as well as Modus Ponens. □

Corollary 3. LEA_- *is consistent.*

Definition 14 (Two-model). *A basic model $*$ is a two-model iff*

$$t^* \in \{True^*, Form\}$$

for all terms t, where $True^* = \{A \mid * \models A\}$.

Theorem 3 (Soundness and Completeness). $BM(\mathsf{LEA}_-)$ *is the class of all two-models.* LEA_- *is sound and complete with respect to the class of all two-models.*

Proof. From Lemma 3 we know $BM(\mathsf{LEA}_-)$ is nonempty.

For one direction of inclusion, let $*$ be a basic model of LEA_-. Due to Axiom 3, we have that t^* must be a deductively closed set. Due to Axiom 2, we have that $True^* \subset t^*$. From this it follows that $t^* \in \{True^*, Form\}$, for if $B \notin True^*$ then the deductive closure of $(True^* \cup B)$ is in fact $Form$.

For the other direction, let $*$ be a two-model. Since $*$ is a basic model, it satisfies the axioms of classical logic. Axiom 2 holds since $True^* \subset t^*$. For Axiom 3, suppose $\{A \to B, A\} \subset t^*$. Since both $True^*$ and $Form$ are deductively closed sets, $B \in t^*$. It follows that Axiom 3 holds in $*$.

Therefore $* \in BM(\mathsf{LEA}_-)$ iff $*$ is a two-model. From the Generic Completeness Theorem and Corollary 1, we have that LEA_- is sound and complete with respect to the class of all two-models.

□

5.2 Probability Semantics for LEA_{-}

We now work towards showing a soundness and completeness result for LEA_{-} with respect to probability semantics. First, we show that to each two-model $*$, we can associate a probability model $\circ$ that validates the same formulas.

Definition 15 (Probability Model $\circ$ Corresponding to Two-model $*$). *Let $*$ be a two-model. Define a probability model $\circ$ corresponding to $*$ as follows. The underlying probability space for $\circ$ is $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{1} = \{\emptyset\}$; $\mathcal{F} = \{\emptyset, \mathbb{1}\}$; $P(\emptyset) = 0$ and $P(\mathbb{1}) = 1$. Define $p^\circ = \mathbb{1}$ if $p^* = 1$, and $p^\circ = \emptyset$ if $p^* = 0$. For atomic evidence terms, define $e_i^\circ = \emptyset$ if $e_i^* = Form$, and $e_i^\circ = \mathbb{1}$ if $e^* = True^*$.*

Proposition 1. *For all LEA_{-} formulas A, $A^\circ = \mathbb{1}$ iff $A^* = 1$; $A^\circ = \emptyset$ iff $A^* = 0$.*

Proof. Argue by induction on the complexity of A.

- The claim is true for propositional variables by definition of $\circ$.
- For the Boolean connectives, the proof is standard. We give the proof for negation. Suppose $(\neg A)^\circ = \mathbb{1}$. By definition this is true iff $A^\circ = \emptyset$, iff (by the induction hypothesis) $A^* = 0$, iff $(\neg A)^* = 1$.
- For the justification case, $(e_i{:}\, A)^\circ = \overline{e_i^\circ} \cup A^\circ = \mathbb{1}$ iff $e_i^\circ = \emptyset$ or $A^\circ = \mathbb{1}$. This holds iff $e_i^* = Form$ (by definition of $\circ$) or $A^* = 1$ (by induction hypothesis.) This holds iff $e_i^* = Form$ or $A \in True^*$ (since if $A^* = 1$ then A is true in the model $*$). This holds iff $A \in e_i^*$ (for if A is false in the model and $A \in e_i^*$, then $e_i^* = Form$.) This holds iff $(e_i{:}\, A)^* = 1$.

□

Theorem 4 (Soundness and Completeness). *For any set of LEA_{-} formulas $\Gamma \cup \{A\}$,*

$$\Gamma \vdash_{-} A \text{ iff } \Gamma \Vdash A.$$

Proof. For soundness, the axioms of LEA_{-} are clearly true in any probability model. We can see this, for example, by noticing that in any probability model, $t\colon A$ would have the same interpretation as $t \to A$, if only the latter were actually a well formed LEA_{-} formula. Then the axiom $A \to t\colon A$ corresponds to the classical tautology $A \to (t \to A)$ and similarly for Axiom 3.

For completeness, suppose $\Gamma \not\vdash_{-} A$. Since LEA_{-} is sound and complete with respect to two-models, there exists is a two-model $*$ such that $C^* = 1$ for all $C \in \Gamma$, and $A^* = 0$. Then there exists a corresponding probability model $\circ$, with underlying probability space $\Omega = \mathbb{1} = \{\emptyset\}$, such that $C^\circ = \mathbb{1}$ for all $C \in \Gamma$, and $A^\circ = \emptyset$. Then $\cap_{C \in \Gamma} C^\circ = \mathbb{1} \not\subset A^\circ = \emptyset$, so $\Gamma \not\Vdash A$. □

Examining the above proof, we see that the essential ingredient was to use a Boolean valued probability semantics to show completeness with respect to LEA_{-}. We can apply the same technique, then, to show completeness with respect to classical logic.

Corollary 4. *Classical logic is sound and complete with respect to probability semantics.*

Proof. Examining the axioms of any standard formulation of classical logic shows it to be sound with respect to probability semantics. The completeness proof exactly mirrors that for LEA_{-}. For each Boolean assignment $*$ with truth values 1 and 0, there exists a probability model $\circ$, with its underlying probability space the same as in Definition 15, such that for all formulas A, $A^* = 1$ iff $A^\circ = \mathbb{1}$ and $A^* = 0$ iff $A^\circ = \emptyset$. The rest of the proof follows the steps in Theorem 4. □

5.3 More LEA_{-} Results

Proposition 2. $\vdash_{-} (A \to B) \to (t\colon A \to t\colon B)$

Proof. $(A \to B) \to t\colon(A \to B)$ is an instance of Axiom 2. Applying Axiom 3 proves the result. □

Proposition 3. $\vdash_{-} \neg t\colon A \to t\colon(\neg A)$

Proof. $\neg t{:}A \to \neg A$ is the contrapositive of Axiom 2. $\neg A \to t{:}(\neg A)$ is an instance of Axiom 2. Combined they give $\neg t{:}A \to t{:}(\neg A)$. □

Proposition 4. $\vdash_{-} t{:}(A \vee B) \leftrightarrow (t{:}A \vee t{:}B)$

Proof. Argue inside LEA_{-}. Suppose $t{:}(A \vee B)$ holds. Then, by Proposition 2, so does $t{:}(\neg A \to B)$. If in addition $t{:}\neg A$ holds, then $t{:}B$ holds, since terms are deductively closed. If instead $\neg t{:}\neg A$ holds, then from Proposition 3 , we have $t{:}A$ holds. In either case, $t{:}A \vee t{:}B$ holds.

For the other direction, suppose $t{:}A$ holds. Since $A \to A \vee B$ is a tautology, by Axiom 2, $t{:}(A \to A \vee B)$ holds. Applying Axiom 3, we derive $t{:}(A \vee B)$. We similarly derive $t{:}(A \vee B)$ if $t{:}B$ holds. □

Proposition 5. $\vdash_{-} t{:}(A \wedge B) \leftrightarrow (t{:}A \wedge t{:}B)$

Proof. Argue inside LEA_{-}. Since $A \wedge B \to A$ is a tautology, by Axiom 2, $t{:}(A \wedge B \to A)$ holds. Suppose $t{:}(A \wedge B)$ holds. Applying Axiom 3 to it, we derive $t{:}A$. We may similarly derive $t{:}B$. In this way, we derive $t{:}(A \wedge B) \to (t{:}A \wedge t{:}B)$.

For the other direction, $t{:}(A \to (B \to A \wedge B))$ holds from Axiom 2. Suppose $t{:}A$ and $t{:}B$ hold. Using Axiom 3 twice, we first apply $t{:}A$ to $t{:}(A \to (B \to (A \wedge B))$, and then $t{:}B$ to that operation's result. Doing so, we derive $t{:}(A \wedge B)$. □

Definition 16 (Justification Literal). *A class of formulas called justification literals are defined inductively. p and $\neg p$ are justification literals, for any propositional variable p. If A is a justification literal, so is $e_i{:}A$ and $\neg e_i{:}A$, for atomic evidence terms e_i.*

Definition 17 (Disjunctive Justified Normal Form). *For a sentence A, we say A is in disjunctive justified normal form (djnf) iff A is in the form of a disjunction of clauses, where a clause is a conjunction of justification literals.*

Theorem 5. *Any* LEA_- *sentence* A *is provably equivalent in* LEA_- *to a sentence* A^n *which is in disjunctive justified normal form.*

Proof. Perform induction on the complexity of a formula, using Propositions 4 and 5 to break justification formulas $t{:}A$ into "simpler" formulas. Combine this with a standard algorithm to convert a classical formula into disjunctive normal form, letting justification literals take the place of literals in the algorithm. □

Notice, this says something about the meaning of LEA_- formulas. The informational content of justification formulas $t{:}X$ is located entirely in justification literals. Let us consider propositional variables as atomic concepts. Evidence for or against these concepts are also atomic concepts. These would be interpreted as $e_i{:}p$ or $e_i{:}\neg p$. Inductively, we may have evidence for evidence, or evidence against evidence, $e_j{:}e_i{:}p$, $e_j{:}\neg e_i{:}p$, etc, which are also considered atomic concepts. By that, we mean that all other sentences can be build up from them, using conjunction, disjunction, and negation. Everything boils down in an inductive chain to evidence for or evidence against propositional variables.

5.4 LEA_+ Definition and Basic Results

Let us now expand the language and axiomatic system of LEA_-. Our new system, LEA_+, will have the same language as LEA. That means it includes the justification constants $\mathbf{1}$ and $\mathbf{0}$, and includes justification term operations $\cap$ and $\cup$.

Definition 18 **(LEA_+ Proof System).** *Any uniform substitution of* LEA_+ *formulas into the following formulas is an axiom of* LEA_+*. Modus Ponens is the only rule of inference for* LEA_+*. Generic evidence terms are indicated by* s *and* t*. Atomic evidence terms are* e_i*.*

1. *The axioms classical logic in the language of* LEA

2. (a) $A \to e_i\colon A$

   (b) $A \to \mathbf{1}\colon A$

3. $t\colon (A \to B) \to (t\colon A \to t\colon B)$

4. (a) $st\colon A \to (s\colon A \vee t\colon A)$

   (b) $(s\colon A \vee t\colon A) \to st\colon A$

5. (a) $s \cup t\colon A \to (s\colon A \wedge t\colon A)$

   (b) $(s\colon A \wedge t\colon A) \to s \cup t\colon A$

6. $\mathbf{1}\colon A \to A$

7. $\mathbf{0}\colon A$.

Let us write $\vdash_+$ to denote $\vdash_{\mathsf{LEA}_+}$.

Lemma 4. *$\vdash_+ A \to t\colon A$ for any term t.*

Proof. By induction on the complexity of t. □

Lemma 5. *$\vdash_+ s\colon A \leftrightarrow t\colon A$ if $s = t$ in the lattice order.*

Proof. Define an equivalence relation on terms as

$$s \sim t \text{ iff } \vdash_+ s\colon A \leftrightarrow t\colon A \text{ for all formulas } A.$$

Let $[s] = \{t \mid s \sim t\}$. We wish to define $[s] \cup [t] = [s \cup t]$ and $[s] \cap [t] = [s \cap t]$. These operators on $\sim$-equivalence classes will be defined if, for all terms s_1, t_1, s_2, t_2,

$$\text{if } [s_1] = [s_2] \text{ and } [t_1] = [t_2], \text{ then } [s_1 \cup t_1] = [s_2 \cup t_2] \text{ and } [s_1 \cap t_1] = [s_2 \cap t_2].$$

This is indeed the situation. We prove it below for $\cup$, but omit the $\cap$ case, which is similar. Let A be an arbitrary LEA_+ formula.

$\vdash_+ s_1 \cup t_1 : A \leftrightarrow (s_1 : A \wedge t_1 : A)$	Axioms 5(a), 5(b)
$\vdash_+ s_1 : A \leftrightarrow s_2 : A$	since $[s_1] = [s_2]$
$\vdash_+ t_1 : A \leftrightarrow t_2 : A$	since $[t_1] = [t_2]$
$\vdash_+ s_1 \cup t_1 : A \leftrightarrow (s_2 : A \wedge t_2 : A)$	substitution
$\vdash_+ s_1 \cup t_1 : A \leftrightarrow s_2 \cup t_2 : A$	Axioms 5(a), 5(b)

so $[s_1 \cup t_1] = [s_2 \cup t_2]$. Again, we state, but do not prove, that $[s_1 \cap t_1] = [s_2 \cap t_2]$. Therefore, the $\sim$-equivalence class operations $\cap, \cup$ are well defined.

As might be expected, $< Term/\sim, \cup, \cap >$ is a lattice, and the natural quotient map $i : Term \longrightarrow Term/\sim$ given by $i(s) = [s]$ is a lattice homomorphism. For details, see [16].

To finish the proof, if $s = t$, then since i is a lattice homomorphism, in particular a well defined function, $[s] = i(s) = i(t) = [t]$.

Therefore $\vdash_+ s : A \leftrightarrow t : A$ for all A, by definition of the equivalence class. □

Proposition 6 (Substitution). $\vdash_+ (A \to B) \to (t : A \to t : B)$

Proposition 7. $\vdash_+ \neg t : A \to t : (\neg A)$

Proposition 8. $\vdash_+ t : (A \vee B) \leftrightarrow (t : A \vee t : B)$

Proposition 9. $\vdash_+ t : (A \wedge B) \leftrightarrow (t : A \wedge t : B)$

Proof. The proofs of the above propositions are the same as in the LEA_- cases. □

Next we will show that for each LEA_+ formula A, there exists a LEA_- formula A^- such that $\vdash_+ A \leftrightarrow A^-$.

Definition 19 **(LEA_- Translation).** *For a LEA formula A, we define the LEA_- translation of A, written A^-. Here is the inductive definition.*

- $p^- = p$ *for propositional variable* p
- $(\neg A)^- = \neg(A^-)$
- $(A \vee B)^- = A^- \vee B^-$
- $(A \wedge B)^- = A^- \wedge B^-$
- $(A \rightarrow B)^- = A^- \rightarrow B^-$
- $(e_i\colon A)^- = e_i\colon (A^-)$
- $(st\colon A)^- = (s\colon A)^- \vee (t\colon A)^-$
- $(s \cup t\colon A)^- = (s\colon A)^- \wedge (t\colon A)^-$
- $(1\colon A)^- = A^-$
- $(0\colon A)^- = \top$

Proposition 10. *For each LEA_+ formula A, A^- is a well defined LEA_- formula.*

Proof. Argue by induction on the complexity of A. □

Lemma 6. *For each LEA_+ formula A, $\vdash_+ A \leftrightarrow A^-$.*

Proof. Argue by induction on the complexity of A. The base case when A is a propositional variable holds trivially. The cases for Boolean connectives are standard. If $A = t\colon B$, then perform subinduction on the complexity of t. In particular, from Proposition 6 and the induction hypothesis, we have $\vdash_+ e_i\colon B \leftrightarrow e_i\colon (B^-)$. The cases for more complex terms and term constants – $st, s \cup t, \mathbf{1}, \mathbf{0}$ – follow directly from the application of the induction hypothesis to the corresponding axioms. □

Corollary 5. *Each LEA_+ formula is provably equivalent in LEA_+ to a formula in disjunctive justified normal form.*

Proof. Each LEA_+ formula A is provably equivalent to its LEA_- translation A^-, which in turn is provably equivalent to a djnf formula $(A^-)^n$. $\square$

5.5 Basic Models of LEA_+

The basic models of LEA_+ are essentially two-models, with additional clauses to define the term operations and constants $\cap, \cup, \mathbf{1}, \mathbf{0}$. We call them *trivial-lattice models.*

Definition 20 (Trivial-lattice Model). *A trivial-lattice model is a basic model satisfying the following conditions:*

- $t^* \in \{Form, True^*\}$ for all terms t
- $(t \cap s)^* = t^* \cup s^*$
- $(t \cup s)^* = t^* \cap s^*$
- $\mathbf{1}^* = True^*$
- $\mathbf{0}^* = Form.$

Note that to any two-model, there is an associated trivial-lattice model resulting from defining the interpretation of justification terms in accordance with Definition 20. These models will agree on the truth value of LEA_- formulas.

Note, we now have similar semantics for two distinct systems LEA_- and LEA_+. When it is clear from the context which one we are working in, we may simply write $\models$ or $\Vdash$ to designate, respectively, basic model semantics or probability semantics. When we need to be precise regarding which language we are interpreting, we may write $\models_-$, $\Vdash_-$ when interpreting LEA_- formulas, and $\models_+$, $\Vdash_+$ when interpreting LEA_+ formulas.

Theorem 6 (Soundness and Completeness). LEA_+ *is sound and complete with respect to trivial-lattice models.*

Proof. Simple inspection of the axioms and semantics shows that LEA_+ is sound with respect to trivial-lattice models.

Let $\Gamma^- = \{C^- \mid C \in \Gamma\}$. To show completeness, suppose $\Gamma \nvdash_+ A$. By Lemma 6, $\vdash_+ A \leftrightarrow A^-$. Since A^- is a LEA_- formula, and since LEA_+ is an extension of LEA_-, it is straightforward to show $\Gamma^- \nvdash_- A^-$. By the completeness theorem for LEA_-, we have $\Gamma^- \not\models_- A^-$. Then we have a two-model $*$ which validates all the formulas in Γ^- and which falsifies A^-. From $*$ we can produce a trivial-lattice model $\tilde{*}$ which agrees with $*$ on all LEA_- formulas. In addition, $\tilde{*}$ validates all formulas in Γ. Thus we have $\tilde{*} \models_+ C$ for all $C \in \Gamma$, and $\tilde{*} \not\models_+ A^-$, hence $\Gamma \not\models_+ A^-$. Since $\models_+ A \to A^-$, then $\Gamma \not\models_+ A$. ◻

Lemma 7. LEA_+ *is a conservative extension of* LEA_-.

Proof. It is clear that LEA_+ is an extension of LEA_-. Suppose $\vdash_+ A$ for some LEA_- formula A. By the completeness theorem for LEA_+, A holds in every trivial-lattice model. If A yet fails in some two-model, it fails in its associated trivial-lattice mode, which is a contradiction. Therefore A holds in every two-model. By the completeness theorem for LEA_- we obtain $\vdash_- A$. ◻

Note, since LEA_+ is a conservative extension of LEA_-, and since each LEA_+ formula is equivalent to a LEA_- formula in disjunctive justified normal form which is produced by "decomposing" formulas in a uniform way, we may draw the same conclusion for LEA_+ as we did for LEA_-. That is, the informational content of formulas $t{:}X$ are entirely contained in justification literals. Moreover, LEA_+ essentially has the same expressive power as LEA_-. The terms operators and constants $\cap, \cup, 1, 0$ are merely convenient shorthand to rewrite djnf LEA_- formulas.

Lemma 8. LEA_+ *is consistent.*

Proof. LEA_+ is a conservative extension of LEA_-, and LEA_- is consistent. Conservative extensions of consistent theories are consistent. ◻

5.6 Probability Semantics for LEA_+

It is now a short matter to show that LEA_+ is sound and complete with respect to probability semantics.

Theorem 7 (Soundness and Completeness). *For any set of* LEA_+ *formulas* $\Gamma \cup \{A\}$,

$$\Gamma \vdash_+ A \textit{ iff } \Gamma \Vdash\!\!\vdash_+ A.$$

Proof. For soundness, observe that all the axioms of LEA_+ are true in any probability model. Probability models respect Modus Ponens, which is the only rule of inference for LEA_+.

Towards proving completeness, suppose $\Gamma \nvdash_+ A$. By Lemma 6, $\vdash_+ A^- \rightarrow A$, and therefore $\Gamma \nvdash_+ A^-$ by . Then also $\Gamma^- \nvdash_+ A^-$, since $\vdash_+ C \rightarrow C^-$ for all $C \in \Gamma$. Then $\Gamma^- \nvdash_- A^-$ since LEA_+ is an extension of LEA_-. Then $\Gamma^- \not\Vdash\!\!\vdash_- A^-$ by the completeness theorem for LEA_-. Then $\Gamma^- \not\Vdash\!\!\vdash_+ A^-$ since $\Vdash\!\!\vdash_+$ and $\Vdash\!\!\vdash_-$ agree on LEA_- formulas. Then $\Gamma \not\Vdash\!\!\vdash_+ A^-$ since every probabilistic model of Γ is a model of Γ^-. Then $\Gamma \not\Vdash\!\!\vdash_+ A$, since $\Vdash\!\!\vdash_+ A \rightarrow A^-$. $\square$

6 Sequent Formulations of LEA_- and LEA_+

In this section we begin to develop the theory of sequent systems for LEA_- and LEA_+.

6.1 Rules and Definitions of Systems

We give categories of sequent rules which will be combined to create various sequent formulations of LEA_- and LEA_+. A *sequent* has the form $\Gamma \Rightarrow \Theta$, where Γ and Θ are multisets of LEA_- or LEA_+ formulas.

In the following, p is a propositional variable.

Structural rules

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\text{LContraction}) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (\text{RContraction})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (\text{LWeakening}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\text{RWeakening})$$

$$\frac{\Gamma \Rightarrow \Delta, D \qquad D, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (\text{cut})$$

Classical rules

$$p, \Gamma \Rightarrow \Delta, p\ (\text{Ax1}) \qquad \bot, \Gamma \Rightarrow \Delta\ (\text{Ax2})$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\text{L}\wedge) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\text{R}\wedge)$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\text{L}\vee) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\text{R}\vee)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ (\text{L}\rightarrow) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ (\text{R}\rightarrow)$$

Justification rules

$$\frac{\Gamma, B \Rightarrow \Delta, e_i{:}\, A}{\Gamma, e_i{:}\, B \Rightarrow \Delta, e_i{:}\, A}\ (\mathrm{L}e_i) \qquad \frac{\Gamma, \Rightarrow \Delta, A}{\Gamma, \Rightarrow \Delta, e_i{:}\, A}\ (\mathrm{R}e_i)$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \mathbf{1}{:}\, A \Rightarrow \Delta}\ (\mathrm{L}\mathbf{1}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathbf{1}{:}\, A}\ (\mathrm{R}\mathbf{1})$$

$$\Gamma \Rightarrow \Delta, \mathbf{0}{:}\, A \quad (\mathrm{R}\mathbf{0})$$

$$\frac{\Gamma, s{:}\, A, t{:}\, A \Rightarrow \Delta}{\Gamma, s \cup t{:}\, A \Rightarrow \Delta}\ (\mathrm{L}\cup) \qquad \frac{\Gamma \Rightarrow \Delta, s{:}\, A \qquad \Gamma \Rightarrow \Delta, t{:}\, A}{\Gamma \Rightarrow \Delta, s \cup t{:}\, A}\ (\mathrm{R}\cup)$$

$$\frac{\Gamma, s{:}\, A \Rightarrow \Delta \qquad \Gamma, t{:}\, A \Rightarrow \Delta}{\Gamma, st{:}\, A \Rightarrow \Delta}\ (\mathrm{L}\cap) \qquad \frac{\Gamma \Rightarrow \Delta, s{:}\, A, t{:}\, A}{\Gamma \Rightarrow \Delta, st{:}\, A}\ (\mathrm{R}\cap)$$

We now define various sequent systems.

- LEA_{-}^{G} consists of the classical rules, LContraction, RContraction, LWeakening, RWeakening, Le_i, and Re_i. (It does not contain cut).
- LEA_{+}^{G} contains the rules of LEA_{-}^{G} and in addition the rules R$\mathbf{0}$, L$\mathbf{1}$, R$\mathbf{1}$, L$\cap$, R$\cap$, L$\cup$, R$\cup$.
- $\mathsf{LEA}_{-}^{G,cut}$ is LEA_{-}^{G} plus the cut rule.
- $\mathsf{LEA}_{+}^{G,cut}$ is LEA_{+}^{G} plus the cut rule.

To denote that $\Gamma \Rightarrow \Delta$ is provable in LEA_{-}^{G}, we write $\vdash_{-}^{G} \Gamma \Rightarrow \Delta$. Similarly, for $\mathsf{LEA}_{-}^{G,cut}$, LEA_{+}^{G}, $\mathsf{LEA}_{+}^{G,cut}$ we use, respectively, $\vdash_{-}^{G,cut}$, $\vdash_{+}^{G}$, $\vdash_{+}^{G,cut}$. Actually, weakening and contraction are admissible rules in our systems, but as we do not need those results yet, we will save proof of them until we prove cut elimination constructively. (See Lemma 13 and Lemma 15 for those proofs.)

Lemma 9. *$\vdash_{-}^{G} A \Rightarrow A$ and $\vdash_{+}^{G} A \Rightarrow A$ for unrestricted A.*

Proof. This is proved by induction on the complexity of A. □

We will also need that the following is provable.

Lemma 10. *For any* LEA_+ *sentences* A *and* B, $\vdash_+ t{:}A \to (A \vee t{:}B)$

Proof. Appeal to the basic model semantics for LEA_+. Let $*$ be a trivial-lattice model. If $* \models A$ then $* \models t{:}A \to (A \vee t{:}B)$. If $* \not\models t{:}A$ then the implication holds trivially. If $* \not\models A$, yet $* \models t{:}A$, then $t^* = Form$, whence $* \models t{:}B$. Therefore, $* \models t{:}A \to (A \vee t{:}B)$. □

6.2 Equivalence of Sequent and Hilbert Systems

The Hilbert and sequent (with cut) formulations of LEA_- and LEA_+ are equivalent, as the following theorem shows.

Theorem 8 (Equivalence of Sequent and Hilbert Systems).

1. (a) $\vdash_-^{G,cut} \Gamma \Rightarrow \Delta$ *implies* $\Gamma \vdash_- \bigvee \Delta$

   (b) $\vdash_+^{G,cut} \Gamma \Rightarrow \Delta$ *implies* $\Gamma \vdash_+ \bigvee \Delta$

2. (a) $\Gamma \vdash_- \bigvee \Delta$ *implies* $\vdash_-^{G,cut} \Gamma \Rightarrow \Delta$

   (b) $\Gamma \vdash_+ \bigvee \Delta$ *implies* $\vdash_+^{G,cut} \Gamma \Rightarrow \Delta$

Proof. We begin with proving the first claim. In particular, we will show that if $\vdash_+^{G,cut} \Gamma \Rightarrow \Delta$, then $\Gamma \vdash_+ \bigvee \Delta$. We show this by induction based on the height of the derivation of $\Gamma \Rightarrow \Delta$.

The step corresponding to cut is handled by Modus Ponens in the Hilbert system. The equivalence between the classical rules for a sequent system and the axioms of classical logic are well known. Therefore, we will concern ourselves only with the rules unique to the justification logics, the *justification rules.*

We will take as our induction hypothesis: if $\vdash_+^{G,cut} \Gamma \Rightarrow \Delta$ then $\Gamma \vdash_+ \bigvee \Delta$. For each rule we will show that if the induction hypothesis holds for the premises of the rule, then it holds for the conclusion, as well. Though our proof is for $\mathsf{LEA}_+^{G,cut}$, the cases corresponding to the $\mathsf{LEA}_-^{G,cut}$ rules, that is Le_i and Re_i, go through just the same as presented here. Therefore, this proof applies to $\mathsf{LEA}_{-}^{G,cut}$ as well.

Also, we will freely make use of the deduction theorem for LEA_{+}. That is $\Gamma \vdash_{+} A$ iff $\vdash_{+} (\bigwedge \Gamma) \rightarrow A$.

– **(Base case – axiom $R\mathbf{0}$).** Since $\mathbf{0}\colon A$ is a LEA_{+} axiom, then $\Gamma \vdash_{+} \bigvee \Delta \vee \mathbf{0}\colon A$.

– **(Re_i).** Suppose $\vdash_{+}^{G,cut} \Gamma \Rightarrow \Delta$ and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma \vdash_{+} \bigvee \Delta \vee A$. Since $\vdash_{+} A \rightarrow e_i\colon A$, then $\Gamma \vdash_{+} \bigvee \Delta \vee e_i\colon A$.

– **(Le_i).** Suppose $\vdash_{+}^{G,cut} \Gamma, B \Rightarrow \Delta, e_i\colon A$ and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma, B \vdash_{+} \bigvee \Delta \vee e_i\colon A$. By Lemma 10, $\vdash_{+} e_i\colon B \rightarrow (B \vee e_i\colon A)$. Using these two, we soon prove the desired $\Gamma, e_i\colon B \vdash_{+} \bigvee \Delta \vee e_i : A$.

– **(R$\cap$).** Suppose $\vdash_{+}^{G,cut} \Gamma \Rightarrow \Theta, s\colon A, t\colon A$ and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma \vdash_{+} \bigvee \Theta \vee s\colon A \vee t\colon A$. Since $\vdash_{+} s\colon A \rightarrow st\colon A$, and $\vdash_{+} t\colon A \rightarrow st\colon A$, then $\Gamma \vdash_{+} \bigvee \Theta \vee st\colon A$.

– **(L$\cap$).** Suppose $\vdash_{+}^{G,cut} \Gamma, s\colon A \Rightarrow \Theta$, $\vdash_{+}^{G,cut} \Gamma, t\colon A \Rightarrow \Theta$, and the induction hypothesis holds for these sequents. By the induction hypothesis, $\Gamma, s\colon A \vdash_{+} \bigvee \Theta$ and $\Gamma, t\colon A \vdash_{+} \bigvee \Theta$. Using some classical logic, we have $\Gamma, s\colon A \vee t\colon A \vdash_{+} \bigvee \Theta$. Since $\vdash_{+} st\colon A \rightarrow (s\colon A \vee t\colon A)$, then $\Gamma, st\colon A \vdash_{+} \bigvee \Theta$.

– **(R$\cup$).** Suppose $\vdash_{+}^{G,cut} \Gamma \Rightarrow \Theta, s\colon A$, $\vdash_{+}^{G,cut} \Gamma \Rightarrow \Theta, t\colon A$, and the induction hypothesis holds for these sequents. By the induction hypothesis, we have $\Gamma \vdash_{+} \bigvee \Theta \vee s\colon A$ and $\Gamma \vdash_{+} \bigvee \Theta \vee t\colon A$.

Then using that $\vdash_+ (s\colon A \wedge t\colon A) \to s \cup t\colon A$, we derive $\Gamma \vdash_+ \bigvee \Theta \vee s \cup t\colon A$.

– **(L$\cup$).** Suppose $\vdash_+^{G,cut} \Gamma, s\colon A, t\colon A \Rightarrow \Theta$ and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma, s\colon A, t\colon A \vdash_+ \bigvee \Theta$. Since $\vdash_+ s \cup t\colon A \to s\colon A$, and $\vdash_+ s \cup t\colon A \to t\colon A$ we have $\Gamma, s \cup t\colon A \vdash_+ \bigvee \Theta$.

– **(R1).** Suppose $\vdash_+^{G,cut} \Gamma \Rightarrow \Theta, A$, and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma \vdash_+ \bigvee \Theta \vee A$. Since $\vdash_+ A \to \mathbf{1}\colon A$, then $\Gamma \vdash_+ \bigvee \Theta \vee \mathbf{1}\colon A$.

– **(L1).** Suppose $\vdash_+^{G,cut} \Gamma, A \Rightarrow \Theta$ and the induction hypothesis holds for this sequent. By the induction hypothesis, $\Gamma, A \vdash_+ \bigvee \Theta$. Since $\vdash_+ \mathbf{1}\colon A \to A$, then $\Gamma, \mathbf{1}\colon A \vdash_+ \bigvee \Theta$.

For the other direction, we want to show that $\Gamma \vdash_- \bigvee \Delta$ implies $\vdash_-^{G,cut} \Gamma \Rightarrow \Delta$ and $\Gamma \vdash_+ \bigvee \Delta$ implies $\vdash_+^{G,cut} \Gamma \Rightarrow \Delta$. Since the rule of Modus Ponens is handled by cut, it is sufficient to prove that the axioms of LEA_- and LEA_+ have equivalent provable sequents in $\mathsf{LEA}_-^{G,cut}$ and $\mathsf{LEA}_+^{G,cut}$, respectively. In the following, we give the axiom, which is either of the form $A \to B$ or $\mathbf{0}\colon A$, and then give a proof of $A \Rightarrow B$ or $\Rightarrow \mathbf{0}\colon A$.

– **Axiom:** $A \to e_i\colon A$

$$\frac{A \Rightarrow A}{A \Rightarrow e_i\colon A}\ (\mathrm{R}e_i)$$

– **Axiom:** $A \to \mathbf{1}\colon A$

$$\frac{A \Rightarrow A}{A \Rightarrow \mathbf{1}\colon A}\ (\mathrm{L}\mathbf{1})$$

– **Axiom:** $s\colon (A \to B) \to (s\colon A \to s\colon B)$

For $\mathsf{LEA}_-^{G,cut}$, s necessarily is some e_i, but for $\mathsf{LEA}_+^{G,cut}$ we will prove the equivalent sequent by induction on the complexity of s.

Case $s = e_i$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, A \to B \Rightarrow B}\,(\mathrm{L}{\to})}{A, A \to B \Rightarrow e_i : B}\,(\mathrm{R}e_i)}{A, e_i\colon (A \to B) \Rightarrow e_i : B}\,(\mathrm{L}e_i)}{e_i\colon A, e_i\colon (A \to B) \Rightarrow e_i : B}\,(\mathrm{L}e_i)}{e_i\colon (A \to B) \Rightarrow e_i\colon A \to e_i\colon B}\,(\mathrm{R}{\to})$$

Case $s = \mathbf{1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, A \to B \Rightarrow B}\,(\mathrm{L}{\to})}{\mathbf{1}\colon A, A \to B \Rightarrow B}\,(\mathrm{L}\mathbf{1})}{\mathbf{1}\colon A, \mathbf{1}\colon (A \to B) \Rightarrow B}\,(\mathrm{L}\mathbf{1})}{\mathbf{1}\colon A, \mathbf{1}\colon (A \to B) \Rightarrow \mathbf{1}\colon B}\,(\mathrm{R}\mathbf{1})}{\mathbf{1}\colon (A \to B) \Rightarrow \mathbf{1}\colon A \to \mathbf{1}\colon B}\,(\mathrm{R}{\to})$$

Case $s = \mathbf{0}$:

$$\mathbf{0}\colon A, \mathbf{0}\colon (A \to B) \Rightarrow \mathbf{0}\colon B \ \text{(axiom)}$$

Now assume that the induction hypothesis holds for terms s and t.

Case $s \cup t$:

$$\dfrac{\dfrac{\dfrac{\dfrac{s\colon A, s\colon (A \to B) \Rightarrow s\colon B}{s\colon A, s \cup t\colon (A \to B) \Rightarrow s\colon B}\,(\mathrm{L}{\cup})}{s \cup t\colon A, s \cup t\colon (A \to B) \Rightarrow s\colon B}\,(\mathrm{L}{\cup}) \qquad \dfrac{\dfrac{t\colon A, t\colon (A \to B) \Rightarrow t\colon B}{t\colon A, s \cup t\colon (A \to B) \Rightarrow t\colon B}\,(\mathrm{L}{\cup})}{s \cup t\colon A, s \cup t\colon (A \to B) \Rightarrow t\colon B}\,(\mathrm{L}{\cup})}{s \cup t\colon A, s \cup t\colon (A \to B) \Rightarrow s \cup t\colon B}\,(\mathrm{R}{\cup})}{s \cup t\colon (A \to B) \Rightarrow s \cup t\colon A \to s \cup t\colon B}\,(\mathrm{R}{\to})$$

Case st:

$$\dfrac{\dfrac{s\colon A, s\colon (A \to B) \Rightarrow s\colon B}{s\colon A, s\colon (A \to B) \Rightarrow st\colon B}\,(\mathrm{R}{\cap}) \qquad \dfrac{t\colon A, t\colon (A \to B) \Rightarrow t\colon B}{t\colon A, t\colon (A \to B) \Rightarrow st\colon B}\,(\mathrm{R}{\cap})}{t\colon A, st\colon (A \to B) \Rightarrow st\colon B}\,(\mathrm{L}{\cap})$$

Through a symmetric derivation we will also derive $s\colon A, st\colon (A \to B) \Rightarrow st\colon B$. From there we have:

$$\dfrac{\dfrac{t\colon A, st\colon (A \to B) \Rightarrow st\colon B \qquad s\colon A, st\colon (A \to B) \Rightarrow st\colon B}{st\colon A, st\colon (A \to B) \Rightarrow st\colon B}\,(\mathrm{L}{\cap})}{st\colon (A \to B) \Rightarrow st\colon A \to st\colon B}\,(\mathrm{R}{\to})$$

- **Axiom:** $st\colon A \to (s\colon A \vee t\colon A)$

$$\dfrac{\dfrac{s\colon A \Rightarrow s\colon A}{s\colon A \Rightarrow s\colon A \vee t\colon A} \qquad \dfrac{t\colon A \Rightarrow t\colon A}{t\colon A \Rightarrow s\colon A \vee t\colon A}}{st\colon A \Rightarrow s\colon A \vee t\colon A}\ (\mathrm{L}\cap)$$

- **Axiom:** $s\colon A \vee t\colon A \to st\colon A$

$$\dfrac{\dfrac{s\colon A \Rightarrow s\colon A}{s\colon A \to st\colon A}\ (\mathrm{R}\cap) \qquad \dfrac{t\colon A \Rightarrow t\colon A}{t\colon A \to st\colon A}\ (\mathrm{R}\cap)}{s\colon A \vee t\colon A \Rightarrow st\colon A}$$

- **Axiom:** $s \cup t\colon A \to (s\colon A \wedge t\colon A)$

$$\dfrac{\dfrac{s\colon A \Rightarrow s\colon A}{s \cup t\colon A \to s\colon A}\ (\mathrm{L}\cup) \qquad \dfrac{t\colon A \Rightarrow t\colon A}{s \cup t\colon A \Rightarrow t\colon A}\ (\mathrm{L}\cup)}{s \cup t\colon A \Rightarrow s\colon A \wedge t\colon A}$$

- **Axiom:** $(s\colon A \wedge t\colon A) \to s \cup t\colon A$

$$\dfrac{\dfrac{s\colon A \Rightarrow s\colon A}{s\colon A \wedge t\colon A \Rightarrow s\colon A} \qquad \dfrac{t\colon A \Rightarrow t\colon A}{s\colon A \wedge t\colon A \Rightarrow t\colon A}}{s\colon A \wedge t\colon A \Rightarrow s \cup t\colon A}\ (\mathrm{R}\cup)$$

- **Axiom:** $\mathbf{1}\colon A \to A$

$$\dfrac{A \Rightarrow A}{\mathbf{1}\colon A \Rightarrow A}\ (\mathrm{L}\mathbf{1})$$

- **Axiom:** $\mathbf{0}\colon A$

$$\Rightarrow \mathbf{0}\colon A \quad \text{(axiom R}\mathbf{0}\text{)}$$

□

6.3 Cut Elimination Via Saturated Sequents

We will show that cut is an admissible rule in LEA_{-}^{G} and LEA_{+}^{G} through the technique of saturated sequents. The proof idea is as follows. Suppose $\nvdash_{+}^{G} \Gamma \Rightarrow \Delta$. We will produce a *saturated sequent* $\Gamma' \Rightarrow \Delta'$ such that $\Gamma \subset \Gamma'$, $\Delta \subset \Delta'$, and $\nvdash_{+}^{G} \Gamma' \Rightarrow \Delta'$. For such a saturated sequent, we will then show that $\Gamma' \not\models_{+} \bigvee \Delta'$. In particular, then $\Gamma \not\models_{+} \bigvee \Delta$. By the completeness theorem for LEA_{+}, $\Gamma \nvdash_{+} \bigvee \Delta$. From the equivalence between sequent and Hilbert systems, we have $\nvdash_{+}^{G,cut} \Gamma \Rightarrow \Delta$.

Definition 21 (Saturated Sequent). *A saturated sequent is a sequent which results from performing the following saturation algorithm on a sequent.*

Saturation Algorithm: This algorithm is applicable to sequents in LEA^G_- or LEA^G_+. Naturally, some of the clauses will be irrelevant for LEA^G_-.

Take as an input to the algorithm a sequent $\Gamma \Rightarrow \Delta$ of one of the two systems. At the beginning of the algorithm, let all propositional variables be discharged, all formulas of the form $\mathbf{0}{:}\,A$ be discharged, and no other formulas be discharged. The property of charged or discharged carries over between steps.

Initialize $\Gamma' := \Gamma$ and $\Delta' := \Delta$. We will iterate Γ' and Δ', adding formulas at each iteration. For each iteration, if there is an undischarged formula $C \in \Gamma' \cup \Delta'$, apply one of the cases below to add formula(s) to Γ' or Δ', and simultaneously discharge C. When formulas are added $\Gamma' \cup \Delta'$ they enter as not discharged, unless they are propositional variables or of the form $\mathbf{0}{:}\,A$, in which case they enter as discharged. When it is not possible to carry out one of the cases below, terminate. Output the sequent $\Gamma' \Rightarrow \Delta'$.

Cases:

1. If $C = A \to B \in \Gamma'$, nondeterministically add A to Δ' or B to Γ'.
2. If $C = A \to B \in \Delta'$, add A to Γ' and B to Δ'.
3. If $C = A \wedge B \in \Gamma'$, add A to Γ' and B to Γ'
4. If $C = A \wedge B \in \Delta'$, nondeterministically A to Δ' or B to Δ'.
5. If $C = A \vee B \in \Gamma'$, nondeterministically add A to Γ' or B to Γ'.
6. If $C = A \vee B \in \Delta'$, add A to Δ' and B to Δ'.
7. If $C = e_i{:}\,A \in \Delta'$, then add A to Δ' and simultaneously add B to Γ' for all B such that $e_i{:}\,B \in \Gamma'$.

8. If $C = st\colon A \in \Gamma'$, nondeterministically add $s\colon A$ to Γ' or $t\colon A$ to Γ'.
9. If $C = st\colon A \in \Delta'$, add $s\colon A$ and $t\colon A$ to Δ'.
10. If $C = s \cup t\colon A \in \Gamma'$, add $s\colon A$ and $t\colon A$ to Γ'.
11. If $C = s \cup t\colon A \in \Delta'$, nondeterministially add $s\colon A$ to Δ' or $t\colon A$ to Δ'.
12. If $C = \mathbf{1}\colon A \in \Gamma'$, add A to Γ'.
13. If $C = \mathbf{1}\colon A \in \Delta'$, add A to Δ'.

Proposition 11. *The saturation algorithm terminates in finitely many steps if* $|\Gamma \cup \Delta| < \omega$.

Proof. At each step, we discharge C, and add only subformulas of formulas contained in the previous sequent. This process can carry on only finitely many times. □

Proposition 12. *If* $\nvdash^G_- \Gamma \Rightarrow \Delta$ *then there exist a saturated sequent* $\Gamma' \Rightarrow \Delta'$ *such that* $\nvdash^G_- \Gamma' \Rightarrow \Delta'$, $\Gamma \subset \Gamma'$ *and* $\Delta \subset \Delta'$. *If* $\nvdash^G_+ \Gamma \Rightarrow \Delta$ *then there exist a saturated sequent* $\Gamma' \Rightarrow \Delta'$ *such that* $\nvdash^G_+ \Gamma' \Rightarrow \Delta'$, $\Gamma \subset \Gamma'$ *and* $\Delta \subset \Delta'$.

Proof. It is clear, on the pain of contradiction, that the properties "is not derivable in LEA^G_- (LEA^G_+)" must be carried over to at least one of the sequents which are added at each step. □

Lemma 11. *If* $\Gamma' \Rightarrow \Delta'$ *is a saturated sequent such that* $\nvdash^G_- \Gamma' \Rightarrow \Delta'$, *then there exists a two-model* $*$, *such that for all* $A \in \Gamma$, $A^* = 1$, *and for all* $A \in \Delta'$, $A^* = 0$.

Proof. The basic model assignment is defined as follows. $p_i = 0$ if $p_i \in \Delta'$, else $p_i = 1$; $e_i^* = True^*$ if there exists B such that $e_i\colon B \in \Delta$, else $e_i^* = Form$. We show that for all $A \in \Gamma'$, $A^* = 1$, and for all $A \in \Delta'$, $A^* = 0$. We proceed by induction on the complexity of A.

- In the base case A is a propositional variable p_i. $A^* = 0$ if $A \in \Delta'$, and since $\Gamma' \cap \Delta' = \emptyset$, $A^* = 1$ if $A \in \Gamma'$. The cases for the Boolean connectives are standard.
- Let $A = e_i\colon B \in \Delta$. Then $B \in \Delta$. By the induction hypothesis, $B^* = 0$, so $B \notin True^*$. Since $e_i\colon B \in \Delta$, then $e_i^* = True^*$ by definition of $*$. Thus, $B \notin e_i^*$, so $(e_i\colon B)^* = 0$.

– Let $A = e_i\colon B \in \Gamma'$. If there exists C such that $e_i\colon C \in \Delta'$, then at some stage in the saturation algorithm, C was added to Δ' and B was simultaneously added to Γ'. Since $B \in \Gamma'$, then by the induction step $B^* = 1$, so $B \in True^*$. $e_i^* = True^*$ by definition of $*$, and we have that $B \in e_i^*$. Thus, $(e_i\colon B)^* = 1$.

– If $A = e_i\colon B \in \Gamma'$ and there is no C such that $e_i\colon C \in \Delta'$, then by definition of the assignment $*$, $e_i^* = Form$. Therefore, $B \in e_i^*$, and $(e_i\colon B)^* = 1$.

□

Next we prove this result for the LEA^G_+ case.

Lemma 12. *If $\Gamma' \Rightarrow \Delta'$ is a saturated sequent such that $\nvdash^G_+ \Gamma' \Rightarrow \Delta'$, then there exists a trivial-lattice model $*$, such that for all $A \in \Gamma$, $A^* = 1$, and for all $A \in \Delta'$, $A^* = 0$.*

Proof. The assignment is the one induced from the assignment in the LEA^G_- case. We simply need to show that the evaluation extends to the new LEA^G_+ justification formulas as expected. We perform induction on the complexity of terms. We must show the base case for $\mathbf{1}\colon B$ and $\mathbf{0}\colon B$.

– If $A = \mathbf{0}\colon B$, then $A^* = 1$ by definition of $\mathbf{0}^*$. Yet, $A \notin \Delta'$, else $\vdash^G_+ \Gamma' \Rightarrow \Delta'$.

– If $A = \mathbf{1}\colon B \in \Gamma'$, then $B \in \Gamma'$. By the induction hypothesis, $B^* = 1$, so $B \in True^*$. Since $\mathbf{1}^* = True^*$. Then $(\mathbf{1}\colon B)^* = 1$.

– If $A = \mathbf{1}\colon B \in \Delta'$, then $B \in \Delta'$. By the induction hypothesis, $B^* = 0$, so $B \notin True^*$. Therefore, $B \notin \mathbf{1}^*$, so $(\mathbf{1}\colon B)^* = 0$.

Thus, the base case holds. Suppose for the induction step the claim holds for $s\colon B$ and $t\colon B$. We will show the case for $st\colon B$. The case for $s \cup t\colon B$ is similarly straightforward.

– If $A = st\colon B \in \Gamma'$ then $s\colon B \in \Gamma'$ or $t\colon B \in \Gamma'$. By the induction hypothesis $(s\colon B)^* = 1$ or $(t\colon B)^* = 1$. That means $B \in s^*$ or $B \in t^*$, so $B \in (s \cup t)^* = (st)^*$. Therefore, $(st\colon B)^* = 1$.

– If $A = st{:}B \in \Delta'$ then $s{:}B \in \Delta'$ and $t{:}B \in \Delta'$. By the induction hypothesis $(s{:}B)^* = 0$ and $(t{:}B)^* = 0$. Then $B \notin s^*$ and $B \notin t^*$. Therefore, $B \notin (s \cup t)^* = (st)^*$. Therefore, $(st{:}B)^* = 0$.

□

Theorem 9 (Cut Elimination). *If* $\vdash_{-}^{G,cut} \Gamma \Rightarrow \Delta$*, then* $\vdash_{-}^{G} \Gamma \Rightarrow \Delta$*. If* $\vdash_{+}^{G,cut} \Gamma \Rightarrow \Delta$*, then* $\vdash_{+}^{G} \Gamma \Rightarrow \Delta$*.*

Proof. We give the proof for $\mathsf{LEA}_{+}^{G,cut}$. The proof for $\mathsf{LEA}_{-}^{G,cut}$ is essentially the same.

If $\nvdash_{+}^{G} \Gamma \Rightarrow \Delta$, then extend to a saturated sequent $\Gamma' \Rightarrow \Delta'$, with $\Gamma \subset \Gamma'$, $\Delta \subset \Delta'$, and $\nvdash_{+}^{G} \Gamma' \Rightarrow \Delta'$. By the Lemma 12, there is a trivial-lattice model where for all $A \in \Gamma'$, $A^* = 1$, and for all $B \in \Delta'$, $B^* = 0$. This shows $\Gamma' \not\models_{+} \bigvee \Delta'$, and in particular that $\Gamma \not\models_{+} \bigvee \Delta$. By the completeness theorem for LEA_{+}, Theorem 6, $\Gamma \nvdash_{+} \bigvee \Delta$. Then from Theorem 8, we have $\nvdash_{+}^{G,cut} \Gamma \Rightarrow \Delta$. □

6.4 Constructive Cut Elimination

We can also show that cut is eliminable in a constructive fashion. To do so, we will need some preliminary lemmas. Mostly, we need to establish that weakening and contraction are admissible in our systems. We stated this result earlier, and will prove it now.

Definition 22 (Length of Proof). *Let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \ldots$ be derivations (valid formal proofs in any of our sequent systems)*

We denote the length of the derivation $\mathcal{Q}_i$ as $|\mathcal{Q}_i|$.

We define the the length of the derivation $\mathcal{Q}_\rangle$*) inductively as follows.*

- *If $\mathcal{Q}_1$ is an axiom, then $\mathcal{Q}_1 := 1$.*
- *If $\mathcal{Q}_1$ is obtained via the application of a one-premise rule to the concluding sequent of the derivation $\mathcal{Q}_2$, then $\mathcal{Q}_1 := |\mathcal{Q}_2|+1$*
- *If $\mathcal{Q}_1$ is obtained via the application of a two-premise rule to the concluding sequents of the derivations $\mathcal{Q}_2$ and $\mathcal{Q}_3$, then $\mathcal{Q}_1 := max(\{|\mathcal{Q}_2|, |\mathcal{Q}_3|\}) + 1$*

For notation, let us write $\vdash^{G}_{-,k} \Gamma \Rightarrow \Delta$ if the sequent $\Gamma \Rightarrow \Delta$ is provable in LEA^{G}_{-} with a derivation of length less than or equal to k, and similar notation for our other systems.

Lemma 13 (Admissibility of Weakening). $\vdash^{G,cut}_{+,k} \Gamma \Rightarrow \Delta$ *implies* $\vdash^{G,cut}_{+,k} \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Proof. This is proved based on straightforward induction on the lengths of proofs. As a corollary, weakening will hold for LEA^{G}_{+}, $\mathsf{LEA}^{G,cut}_{-}$, LEA^{G}_{-}.

In the base case, $\Gamma \Rightarrow \Delta$ is an axiom. In that case, $\vdash^{G,cut}_{+,1} \Gamma, \Gamma' \Rightarrow \Delta\Delta'$, since that latter sequent is also an axiom. In the induction step, suppose that the lemma holds for all sequents provable in $\mathsf{LEA}^{G,cut}_{+}$ with proofs of length less than or equal to k.

Next suppose that there is a proof of length $k+1$ for $\Gamma \Rightarrow \Delta$, which ends with the application of a one-premise rule as follows:

$$\dfrac{\begin{array}{c}\mathcal{Q}\\ \vdots \\ \Pi \Rightarrow \Lambda\end{array}}{\Gamma \Rightarrow \Delta}$$

Then $\vdash^{G,cut}_{+,k} \Pi \Rightarrow \Lambda$, so by the induction hypothesis, $\vdash^{G,cut}_{+,k} \Pi, \Gamma' \Rightarrow \Lambda, \Delta'$. Then it is clear from inspection of all the one premise rules that $\vdash^{G,cut}_{+,k+1} \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. The case for two-premise rules is essentially the same.

□

Lemma 14 (Inversion Lemma).

1. *If* $\vdash^{G}_{+,k} \Gamma, e_i\colon A \Rightarrow \Delta$ *then* $\vdash^{G}_{+,k} \Gamma, A \Rightarrow \Delta$
2. *If* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, e_i\colon A$ *then* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, A$.
3. *If* $\vdash^{G}_{+,k} \Gamma, A \Rightarrow \Delta$ *then* $\vdash^{G}_{+,k} \Gamma, \mathbf{1}\colon A \Rightarrow \Delta$.
4. *If* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, A$ *then* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, \mathbf{1}\colon A$.
5. *If* $\vdash^{G}_{+,k} \Gamma, s \cup t\colon A \Rightarrow \Delta$ *then* $\vdash^{G}_{+,k} \Gamma, s\colon A, t\colon A \Rightarrow \Delta$.
6. *If* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, s \cup t\colon A$ *then* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, s\colon A$ *and* $\vdash^{G}_{+,k} \Gamma \Rightarrow \Delta, t\colon A$.

7. *If* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, st{:}\, A$ *then* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, s{:}\, A, t{:}\, A$.

8. *If* $\vdash^G_{+,k} \Gamma, st{:}\, A \Rightarrow \Delta$ *then* $\vdash^G_{+,k} \Gamma, s{:}\, A \Rightarrow \Delta$ *and* $\vdash^G_{+,k} \Gamma, t{:}\, A \Rightarrow \Delta$.

9. *If* $\vdash^G_{+,k} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\vdash^G_{+,k} \Gamma, A, B \Rightarrow \Delta$.

10. *If* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A \wedge B$ *then* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A$ *and* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, B$.

11. *If* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A \vee B$ *then* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A, B$.

12. *If* $\vdash^G_{+,k} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\vdash^G_{+,k} \Gamma, A \Rightarrow \Delta$ *and* $\vdash^G_{+,k} \Gamma, B \Rightarrow \Delta$.

13. *If* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A \rightarrow B$ *then* $\vdash^G_{+,k} \Gamma, A \Rightarrow \Delta, B$

14. *If* $\vdash^G_{+,k} \Gamma, A \rightarrow B \Rightarrow \Delta$ *then* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A$ *and* $\vdash^G_{+,k} \Gamma, B \Rightarrow \Delta$.

Proof. We will show the proof for Case 1. The other cases are essentially identical. We proceed by induction on k. If $k = 1$ then the claim holds, as then all sequents involved in the statement of the lemma are axioms. For the induction step, assume that the claim holds for all proofs of length k.

Case 1: Suppose $\vdash^G_{+,k+1} \Gamma, e_i{:}\, B \Rightarrow \Delta$. We have three subcases to consider: $\Gamma, e_i{:}\, B \Rightarrow \Delta$ is an axiom; the sequent is not an axiom and $e_i{:}\, B$ is not principal; the sequent is not an axiom and $e_i{:}\, B$ is principal.

If $\Gamma, e_i{:}\, B \Rightarrow \Delta$ is an axiom, then $\vdash^G_{+,k+1} \Gamma, B \Rightarrow \Delta$, since the latter sequent is also an axiom. (Recall that the axiom Ax1 is $p, \Gamma \Rightarrow \Delta, p$ for p a propositional variable, so $e_i{:}\, B \Rightarrow e_i{:}\, B$ is not an axiom.)

Otherwise, suppose the sequent is not an axiom and $e_i{:}\, B$ is not principal. Then we have a derivation of length $k + 1$ either of the form

$$\dfrac{\begin{matrix}\mathcal{Q}\\ \vdots \\ \Gamma', e_i{:}\, B \Rightarrow \Delta'\end{matrix}}{\Gamma, e_i{:}\, B \Rightarrow \Delta}$$

or

$$\dfrac{\begin{matrix}\mathcal{Q}_1\\ \vdots \\ \Gamma', e_i{:}\, B \Rightarrow \Delta'\end{matrix} \qquad \begin{matrix}\mathcal{Q}_2\\ \vdots \\ \Gamma'', e_i{:}\, B \Rightarrow \Delta''\end{matrix}}{\Gamma, e_i{:}\, B \Rightarrow \Delta}$$

We may apply the induction hypothesis to derivation of $\Gamma', e_i{:}B \Rightarrow \Delta'$ and, if needs be, $\Gamma'', e_i{:}B \Rightarrow \Delta''$ to obtain derivation of length k of $\Gamma', B \Rightarrow \Delta'$ and $\Gamma'', B \Rightarrow \Delta''$. Then we may apply the appropriate rule to obtain a proof of length $k+1$ of $\Gamma, B \Rightarrow \Delta$

For the final subcase, suppose $e_i{:}B$ is principal. Then we have a proof of length $k+1$ as

$$\dfrac{\begin{array}{c}\mathcal{Q}\\ \vdots\\ \Gamma, B \Rightarrow \Delta', e_i{:}A\end{array}}{\Gamma, e_i{:}B \Rightarrow \Delta', e_i{:}A}$$

Therefore, we immediately read off a proof of length k of $\Gamma, B \Rightarrow \Delta$, where $\Delta = \Delta', e_i{:}A$.

The proofs for the other rules work essentially the same way. □

The reason why the proof of the inversion lemma holds is that each term symbol may only be introduced on the right or on the left in a unique way. We do not expect this proof to hold if identical terms may be introduced in multiple ways, especially in the case of a kind of "substitution rule" for a sequent formulation of LEA.

To explain, the substitution lemma $s{:}A \leftrightarrow t{:}A$ if $s = t$ in the lattice order is provable in $\mathsf{LEA}_{+/-}$. Not surprisingly, the sequent equivalent of this theorem is an admissible rule of LEA^G_- and LEA^G_+. However, so far it seems that substitution must be taken as an axiom for LEA. If this is the case, a rule representing the substitution axiom is not likely to be an admissible rule in a sequent formulation of LEA. There would have to be a rule or an axiom to represent substitution. Through substitution, we have nonunique ways of arriving at the same principal justification formula. At this point the above proof of the admissibility of weakening breaks down.

Lemma 15 (Admissibility of Contraction). $\vdash^G_{+,k} \Gamma, A, A \Rightarrow \Delta$ *implies* $\vdash^G_{+,k} \Gamma, A \Rightarrow \Delta$. $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A, A$ *implies* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta, A$.

Proof. We will work in LEA^G_+, arguing by induction on the length of proofs. Clearly the claim holds in the base case of an axiom. Suppose contraction is admissible for all sequents proofs of length less than or equal to k.

Suppose $\Gamma, A, A \Rightarrow \Delta$ has a proof in LEA^G_+ of length $k+1$. If neither A is principal, then we apply the induction hypothesis to the premises, perform contraction to delete one A in each premises, and then derive $\Gamma, A \Rightarrow \Delta$. We use similar logic to prove contraction along a non-principal formula.

If A is principal, then we proceed by cases. For the remainder, assume A is principal. We will give emblematic examples, and not prove all cases, as the reasoning is the same.

If A is of the form $e_i\colon B$, resulting from the rule Le_i. Then we have a proof of length $k+1$ as

$$\dfrac{\begin{array}{c}\mathcal{Q}\\ \vdots\\ \Gamma', e_i\colon B, B \Rightarrow \Delta', e_i\colon A\end{array}}{\Gamma', e_i\colon B, e_i\colon B \Rightarrow \Delta', e_i\colon A}$$

Thus we have a proof of length k of $\Gamma', e_i\colon B, B \Rightarrow \Delta', e_i\colon A$. Applying the inversion lemma, we obtain a proof of length k of $\Gamma', B, B \Rightarrow \Delta', e_i\colon A$. Applying the induction hypothesis, we have a proof of length k of $\Gamma', B \Rightarrow \Delta', e_i\colon A$. Then, using L$e_i$ once more, we have a proof of length $k+1$ of $\Gamma', e_i\colon B \Rightarrow \Delta', e_i\colon A$.

Now we consider the two-premise rule L$\cap$; the proofs for all two-premise rules follow this same pattern. Suppose A is of the form $st\colon B$ obtained through $L\cap$. Then we have the following proof of length $k+1$.

$$\dfrac{\begin{array}{c}\mathcal{Q}_1\\ \vdots\\ \Gamma, s\colon B, st\colon B \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\mathcal{Q}_2\\ \vdots\\ \Gamma, t\colon B, st\colon B \Rightarrow \Delta\end{array}}{\Gamma, st\colon B, st\colon B \Rightarrow \Delta}$$

By the inversion lemma, both $\Gamma, s\colon B, s\colon B \Rightarrow \Delta$ and $\Gamma, t\colon B, t\colon B \Rightarrow \Delta$ have proofs of length k. By the induction hypothesis, obtain proofs of $\Gamma, s\colon B \Rightarrow \Delta$ and $\Gamma, t\colon B \Rightarrow \Delta$ of length k. Apply them with the rule L$\cap$ to obtain a proof of $\Gamma, st\colon B \Rightarrow \Delta$ of length $k+1$.

$\square$

Lemma 16. *If* $\vdash^G_{+,k} \mathbf{0}\colon A, \Gamma \Rightarrow \Delta$ *then* $\vdash^G_{+,k} \Gamma \Rightarrow \Delta$.

Proof. By induction on the length of proofs. In the base case, $\mathbf{0}\colon A, \Gamma \Rightarrow \Delta$ is an axiom, in which case, so is $\Gamma \Rightarrow \Delta$. For the induction step, suppose the claim holds for all sequents with proofs of length k, and there is a proof of $\mathbf{0}\colon A, \Gamma \Rightarrow \Delta$ of length $k+1$.

Since there is no rule which introduces $\mathbf{0}\colon A$ on the left, $\mathbf{0}\colon A$ will have to be a side formula in each premise of $\mathbf{0}\colon A, \Gamma \Rightarrow \Delta$. Apply the induction hypothesis to obtain proofs of length k of each premises, only with $\mathbf{0}\colon A$ deleted. Then, apply the same rule which was used to derive $\mathbf{0}\colon A, \Gamma \Rightarrow \Delta$ in order to derive $\Gamma \Rightarrow \Delta$ with a proof of length $k+1$. □

Theorem 10 (Constructive Cut Elimination). *If* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{LEA}_{-}^{G,cut}$ *then it is provable in* LEA_{-}^{G}*. If* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathsf{LEA}_{+}^{G,cut}$ *then it is provable in* LEA_{+}^{G}*.*

Proof. We will follow the strategy of deleting topmost cuts.

As such, it is sufficient to show how to transform a deduction transform a deduction $\mathcal{Q}$ of the form

$$\dfrac{\begin{matrix}\mathcal{Q}_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\mathcal{Q}_1 \\ \vdots \\ \Pi, A \Rightarrow \Lambda\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (\text{cut})$$

into a deduction $\mathcal{Q}'$ such that both have the same conclusion, and that the cutrank of $\mathcal{Q}'$ is lesser, else it is equal and the level of the final cut is lesser in $\mathcal{Q}'$. There are three cases to consider:

1. One or both of $\Gamma \Rightarrow \Delta$ or $\Pi \Rightarrow \Lambda$ is an axiom.
2. A is not principal in one or both of those two sequents.
3. A is principal in both of those sequents.

The proof case (2) is standard, as are the classical rules and axioms pertaining to the first and third case. Therefore, we must show the case (1) where one of the sequents is the new axiom $R\mathbf{0}$, and case (3) when A is a justification formula principal on both sides. We will prove cut elimination for LEA_{+}^{G} and LEA_{-}^{G} simultaneously.

Case (1)

– **Case:** $\Gamma \Rightarrow \Delta, A$ is the axiom $R\mathbf{0}$, and the principal formula in it is not the cut formula. Then we have

$$\dfrac{\Gamma \Rightarrow \Delta', \mathbf{0}{:}\, B, A \qquad \begin{matrix}\mathcal{Q}_1 \\ \vdots \\ \Pi, A \Rightarrow \Lambda\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta', \Lambda, \mathbf{0}{:}\, B}\ (\text{cut})$$

The conclusion is an axiom which we may take as our cutfree proof.

– **Case:** $\Gamma \Rightarrow \Delta, A$ is the axiom $R\mathbf{0}$, and the principal formula in it is the cut formula. Then we have

$$\dfrac{\Gamma \Rightarrow \Delta, \mathbf{0}{:}\, B \qquad \begin{matrix}\mathcal{Q}_1 \\ \vdots \\ \Pi, \mathbf{0}{:}\, B \Rightarrow \Lambda\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (\text{cut})$$

By induction hypothesis, $\Pi, \mathbf{0}{:}\, B \Rightarrow \Lambda$ is provable without cut. By Lemma 16 we have that $\Pi \Rightarrow \Lambda$ is provable. By the admissibility of weakening, we get $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ is provable without cut.

– **Case:** $\Pi, A \Rightarrow \Delta$ is the axiom $R\mathbf{0}$.

$$\dfrac{\begin{matrix}\mathcal{Q}_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \Pi, A \Rightarrow \Lambda, \mathbf{0}{:}\, B}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, \mathbf{0}{:}\, B}\ (\text{cut})$$

The conclusion is an axiom, which we may take to be our cutfree proof.

Case (3)

– **Case:** $A = e_i{:}\, B$ is principal on both sides, introduced on the right and left by the rule L$e_i$and Re_i, respectively.

We have the following deduction.

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{Q}_1\\ \vdots\\ \Gamma \Rightarrow \Delta, B\end{array}}{\Gamma \Rightarrow \Delta, e_i\colon B} \qquad \dfrac{\begin{array}{c}\mathcal{Q}_2\\ \vdots\\ \Pi, B \Rightarrow \Lambda, e_i\colon C\end{array}}{\Pi, e_i\colon B \Rightarrow \Lambda, e_i\colon C}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, e_i\colon C}\,(\text{cut})
$$

This reduces to

$$
\dfrac{\begin{array}{c}\mathcal{Q}_1\\ \vdots\\ \Gamma \Rightarrow \Delta, B\end{array} \qquad \begin{array}{c}\mathcal{Q}_2\\ \vdots\\ \Pi, B \Rightarrow \Lambda, e_i\colon C\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, e_i\colon C}\,(\text{cut})
$$

- **Case:** $A = \mathbf{1}\colon B$ is principal on both sides, introduced on the left and right by, respectively, the rules $R\mathbf{1}$, $L\mathbf{1}$.

  $\mathcal{Q}$ is of the following form.

$$
\dfrac{\dfrac{\begin{array}{c}Q_1\\ \vdots\\ \Gamma \Rightarrow \Delta, B\end{array}}{\Gamma \Rightarrow \Delta, \mathbf{1}\colon B} \qquad \dfrac{\begin{array}{c}Q_2\\ \vdots\\ \Pi, B \Rightarrow \Lambda\end{array}}{\Pi, \mathbf{1}\colon B \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}
$$

  This transforms into:

$$
\dfrac{\begin{array}{c}Q_1\\ \vdots\\ \Gamma \Rightarrow \Delta, B\end{array} \qquad \begin{array}{c}Q_2\\ \vdots\\ \Pi, B \Rightarrow \Lambda\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}
$$

- **Case:** $A = s \cup t\colon B$ is principal on both sides, introduced on the left and right by, respectively, the rules $R\cup$, $L\cup$.

  $\mathcal{Q}$ is of the following form:

$$
\dfrac{\dfrac{\begin{array}{c}Q_1\\ \vdots\\ \Gamma \Rightarrow \Delta, s\colon B\end{array} \qquad \begin{array}{c}Q_2\\ \vdots\\ \Gamma \Rightarrow \Delta, t\colon B\end{array}}{\Gamma \Rightarrow \Delta, s \cup t\colon B} \qquad \dfrac{\begin{array}{c}Q_3\\ \vdots\\ \Pi, s\colon B, t\colon B \Rightarrow \Lambda\end{array}}{\Pi, s \cup t\colon B \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,(\text{cut})
$$

  This transforms into:

$$\dfrac{\begin{matrix} Q_2 \\ \vdots \\ \Gamma \Rightarrow \Delta, t\colon B \end{matrix} \qquad \dfrac{\begin{matrix} Q_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, s\colon B \end{matrix} \qquad \begin{matrix} Q_3 \\ \vdots \\ \Pi, s\colon B, t\colon B \Rightarrow \Lambda \end{matrix}}{\Gamma, \Pi, t\colon B \Rightarrow \Delta, \Lambda}\,(\text{cut})}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Lambda}\,(\text{cut})$$

Since contraction is admissible, this case is proved.

- **Case:** $A = st\colon B$ is principal on both sides, introduced on the left and right by, respectively, the rules $R\cap$, $L\cap$.

  $\mathcal{Q}$ is of the following form:

$$\dfrac{\dfrac{\begin{matrix} Q_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, s\colon B, t\colon B \end{matrix}}{\Gamma \Rightarrow \Delta, st\colon B} \qquad \dfrac{\begin{matrix} Q_2 \\ \vdots \\ \Pi, s\colon B \Rightarrow \Lambda \end{matrix} \qquad \begin{matrix} Q_3 \\ \vdots \\ \Pi, t\colon B \Rightarrow \Lambda \end{matrix}}{\Pi, st\colon B \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,(\text{cut})$$

  This transforms into:

$$\dfrac{\dfrac{\begin{matrix} Q_1 \\ \vdots \\ \Gamma \Rightarrow \Delta, s\colon B, t\colon B \end{matrix} \qquad \begin{matrix} Q_2 \\ \vdots \\ \Pi, s\colon B \Rightarrow \Lambda \end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, t\colon B}\,(\text{cut}) \qquad \begin{matrix} Q_3 \\ \vdots \\ \Pi, t\colon B \Rightarrow \Lambda \end{matrix}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Lambda, \Lambda}\,(\text{cut})$$

  Again, since contraction is admissible, this case is proved.

□

7 Comparison to Lax Logic

7.1 LEA_{-} as "Classical" Lax Logic

Through its sequent rules, we may observe that LEA_{-} is quite similar to Propositional Lax Logic (PLL). (For details on Propositional Lax Logic, see [19]). Indeed, PLL – a modal logic with modality $\bigcirc$ resembling both necessity and possibility – may be characterized in a sequent system with a base of intuitionistic logic, and the modal rules

$$\frac{\Gamma, B \Rightarrow \bigcirc A}{\Gamma, \bigcirc B \Rightarrow \bigcirc A}\,(\mathrm{L}\bigcirc) \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \bigcirc A}\,(\mathrm{R}\bigcirc)\ .$$

Compare this with the rules Le_i and Re_i of LEA^G_{-}. Interpreting e_i as $\bigcirc$, the difference between this sequent formulation of PLL and LEA^G_{-} is that in PLL we require a single formula on the right-side of the sequent, owing to the fact that the base theory there is intuitionistic logic. In LEA_{-} the base theory is classical logic, so we have no such restriction.

Just in case it is feared that the similarities between Propositional Lax Logic and LEA_{-} are artifacts of their sequent formulations, we can demonstrate those similarities in a Hilbert setting as well. PLL can be presented in a Hilbert format with the rules

$$A \to \bigcirc A \qquad (Axiom \bigcirc R)$$

$$\bigcirc \bigcirc A \to \bigcirc A \qquad (Axiom \bigcirc C)$$

$$(\bigcirc A \wedge \bigcirc B) \to \bigcirc (A \wedge B) \qquad (Axiom \bigcirc S).$$

Such axioms are provable in LEA_{-}, and the LEA_{-} axioms are provable in PLL in the following sense.

Theorem 11. *Let A be an an instance of Axiom $\bigcirc R, \bigcirc C$, or $\bigcirc S$, and let A^e be the result of replacing each instance of $\bigcirc$ with e_i:. Then $\vdash_{-} A^e$. Moreover, let B be an instance of Axiom 2 or*

3 of LEA_{-} *and let* $B^{\bigcirc}$ *be the result of replacing in* B *each instance of* e_i: *with* $\bigcirc$, *for any* e_i. *Then* $\vdash_{PLL} B^{\bigcirc}$

Proof. First we show that $\vdash_{-} A^e$ for A an axiom instance of Lax Logic.

- If A is $C \to \bigcirc C$, then A^e is $C^e \to e_i{:}\,C^e$, which is an axiom of LEA_{-}.
- If A is $\bigcirc\bigcirc C \to \bigcirc C$, then A^e is $e_i{:}\,e_i{:}\,C^e \to e_i{:}\,C^e$. We may appeal to the basic model characterization of LEA_{-}. Let $*$ be a two-model. If C^e is true in $*$, then so is $e_i : C^e$. If $e_i^* = Form$, then clearly the implication is true in $*$. If C^e is false, and $e_i^* = True^*$, then $e_i{:}\,C^e$ is false in $*$, and therefore so is $e_i{:}\,e_i{:}\,C^e$. In all cases, the implication is true in $*$. By the completeness theorem for LEA_{-}, $\vdash_{-} e_i{:}\,e_i{:}\,C^e \to e_i{:}\,C$.
- If A is $(\bigcirc C \wedge \bigcirc D) \to \bigcirc(C \wedge D)$, it was shown in Proposition 5 that $\vdash_{-} (e_i{:}\,C^e \wedge e_i{:}\,D^e) \to e_i{:}\,(C^e \wedge D^e)$.

Next we show that $\vdash_{Lax} B^{\bigcirc}$ for B an axiom of LEA_{-}.

- If B is $C \to e_i{:}\,C$, then $B^{\bigcirc}$ is $C^{\bigcirc} \to \bigcirc C^{\bigcirc}$, an axiom of Lax Logic.
- If B is $e_i : (C \to D) \to (e_i : C \to e_i : D)$, then $B^{\bigcirc}$ is $\bigcirc(C^{\bigcirc} \to D^{\bigcirc}) \to (\bigcirc C^{\bigcirc} \to \bigcirc D^{\bigcirc})$. That Lax Logic proves this can perhaps be shown most easily with its sequent characterization. Here is a proof of an equivalent statement using sequents. For ease of reading, we will leave off the $\bigcirc$ superscripts from the relevant subformulas.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \to D, C \Rightarrow D}{C \to D, C \Rightarrow \bigcirc D}\,(\mathrm{R}\bigcirc)}{C \to D, \bigcirc C \Rightarrow \bigcirc D}\,(\mathrm{L}\bigcirc)}{\bigcirc(C \to D), \bigcirc C \Rightarrow \bigcirc D}\,(\mathrm{L}\bigcirc)}{\bigcirc(C \to D) \Rightarrow \bigcirc C \to \bigcirc D}\,(\mathrm{R}{\to})}{\Rightarrow \bigcirc(C \to D) \to (\bigcirc C \to \bigcirc D)}\,(\mathrm{R}{\to})$$

□

Therefore, LEA_{-} may be considered as Lax Logic where the base theory is classical, rather than intuitionistic, with multiple noninteracting modalities e_i. One motivation for Lax Logic to have an intuitionistic base was the fact that in a classical setting, the PLL axioms combined with the additional and reasonable axiom $\neg\bigcirc\bot$ lead to the triviality that, in such a system, $\bigcirc A$ is provably equivalent to A. We see no such reason to suppose $\neg e_i{:}\bot$ holds generally. Such a claim maintains that e_i is not contradictory evidence, yet it is quite reasonable to assume that some sets of evidence are contradictory.

7.2 A New Semantics for Propositional Lax Logic

Inspired by the similarity between PLL and LEA_{-}, we use the techniques of justification logic to provide a new semantics for PLL. The ideas and techniques in this section are largely adapted from Marti and Studer [36].

Definition 23 (Basic Evaluation). *A* basic evaluation for $\bigcirc$*, hereafter just called a* basic evaluation *is a tuple* $(W, \leq, *)$ *such that* W *is a set of possible worlds,* $\leq$ *is a partial order on* W*, and* $*$ *is a function which takes possible worlds to sets of formulas, and at each world,* $*$ *takes propositional variables to elements of {0,1}. Formally, we require of* $*$ *the following:*[1]

- $*: W \hookrightarrow \mathcal{P}(Form)$
- $*: W \times Prop \hookrightarrow \{1, 0\}$.

In addition, the basic evaluation must satisfy two monotonicity conditions:

- *(M1):* $*(w, p) = 1$ *and* $w \leq v$ *implies* $*(v, p) = 1$.
- *(M2):* $w \leq v$ *implies* $*(w) \subset *(v)$.

For shorthand notation, let us write $*(w, p)$ as p^*_w, and $*(w)$ as $\bigcirc^*_w$.

Using this notation, the basic evaluation must satisfy two monotonicity conditions:

- (M1): $p^*_w = 1$ and $w \leq v$ implies $p^*_v = 1$.

[1] Compare this to the justification logic setting, where at each world $*$ takes justification terms to sets of formulas, i.e. $*: W \times Term \hookrightarrow \mathcal{P}(Form)$ [36].

– (M2): $w \leq v$ implies $\bigcirc^*_w \subset \bigcirc)^*_v$.

Definition 24 (Truth under a Basic Evaluation).

Truth under a basic evaluation is defined as follows. Let $\mathcal{M} = (W, \leq, *)$ be a basic evaluation. Define the truth of a formula A at a world w inductively as follows:

- $(\mathcal{M}, w) \not\models \bot$
- $(\mathcal{M}, w) \models p$ *iff* $p^*_w = 1$
- $(\mathcal{M}, w) \models A \wedge B$ *iff* $(\mathcal{M}, w) \models A$ *and* $(\mathcal{M}, w) \models B$
- $(\mathcal{M}, w) \models A \vee B$ *iff* $(\mathcal{M}, w) \models A$ *or* $(\mathcal{M}, w) \models B$
- $(\mathcal{M}, w) \models A \rightarrow B$ *iff* $(\mathcal{M}, v) \models B$ *for all* $v \geq w$ *such that* $(\mathcal{M}, v) \models A$
- $(\mathcal{M}, w) \models \neg A$ *iff* $(\mathcal{M}, w) \models A \rightarrow \bot$
- $(\mathcal{M}, w) \models \bigcirc A$ *iff* $A \in \bigcirc^*_w$

If $(\mathcal{M}, w) \models A$*, we say* A is true at the world w, (in the model $\mathcal{M}$).

We say A is true in the model $\mathcal{M} = (W, \leq, *)$ *if for all worlds* $w \in W$*,* $(\mathcal{M}, w) \models A$*. In this case, we write* $\mathcal{M} \models A$*.*

Lemma 17 (Monotonicity). *For any basic evaluation $\mathcal{M} = (W, \leq, *)$, states $w, v \in W$, formula A,*

$$(\mathcal{M}, w) \models A \text{ and } w \leq v \text{ implies } (\mathcal{M}, v) \models A$$

Proof. Argue by induction on the complexity of A.

In the base case, A is a propositional variable, $w \leq v$ and $(\mathcal{M}, w) \models A$. Then by monotonicity condition M1, $(\mathcal{M}, v) \models A$. For the induction step, the cases for the Boolean connectives are standard, so we omit them.

Suppose $(\mathcal{M}, w) \models \bigcirc A$. Then $A \in \bigcirc^*_w$. If $w \leq v$, then by monotonicity condition M2, $\bigcirc^*_w \subset \bigcirc^*_v$. Therefore $A \in \bigcirc^*_v$, whence $(\mathcal{M}, v) \models \bigcirc A$. □

Definition 25. *For a basic evaluation $\mathcal{M} = (W, \leq, *)$, $w \in W$, $True^*_w$ is defined as*

$$True^*_w = \{A \,|\, (\mathcal{M}, w) \models A\}.$$

Definition 26. *A Lax Model is basic evaluation for $\bigcirc$ such that, for all states w,*

- $True^*_w \subset \bigcirc^*_w$,
- $\bigcirc A \in \bigcirc^*_w$ *implies* $A \in \bigcirc^*_w$,
- $\{A, B\} \subset \bigcirc^*_w$ *implies* $A \wedge B \in \bigcirc^*_w$.

If $\mathcal{M}$ is a Lax Model, then we may write $\mathcal{M} \models_{\mathsf{PLL}} A$ instead of $\mathcal{M} \models A$, and $(\mathcal{M}, w) \models_{\mathsf{PLL}} A$ instead of $(\mathcal{M}, w) \models A$. We may write $\models_{\mathsf{PLL}} A$ if, for all Lax Models $\mathcal{M}$, do we have $\mathcal{M} \models A$.

Theorem 12 (Soundness). *If $\vdash_{\mathsf{PLL}} A$, then $\models_{\mathsf{PLL}} A$.*

Proof. The proof goes by induction on the complexity of A. Propositional Lax Logic extends intuitionistic propositional calculus (IPC) with three modal axioms. To prove the claim of soundness, we must show that an arbitrary Lax Model $\mathcal{M}$ models both IPC and the three modal axioms. From Definition 24 it is clear that such a model would indeed be a model of IPC. We omit that part of the proof, and will show here that $\mathcal{M}$ models the three modal axioms.

Let $\mathcal{M} = (W, \leq, *)$ be a Lax Model, $\{w, v\} \subset W$, and $w \leq v$. We need to show that $\models_{\mathsf{PLL}} A$, where A is any of the three modal axioms $\bigcirc R$, $\bigcirc C$, or $\bigcirc S$.

- **Case $\bigcirc R$:** A is $B \to \bigcirc B$. If $(\mathcal{M}, v) \models_{\mathsf{PLL}} B$, then $B \in True^*_v \subset \bigcirc^*_v$, so $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc B$. Thus $(\mathcal{M}, w) \models_{\mathsf{PLL}} B \to \bigcirc B$.
- **Case: $\bigcirc C$:** A is $\bigcirc \bigcirc B \to \bigcirc B$. If $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc \bigcirc B$, then $\bigcirc B \in \bigcirc^*_v$ by Definition 24. Then $B \in \bigcirc^*_v$ by definition of a Lax model, Definition 26. Then $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc B$, and thus $(\mathcal{M}, w) \models_{\mathsf{PLL}} \bigcirc \bigcirc B \to \bigcirc B$.

- **Case** $\bigcirc S$**:** A is $(\bigcirc B \wedge \bigcirc C) \to \bigcirc(B \wedge C)$. Suppose $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc B \wedge \bigcirc C$. Then $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc B$ and $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc C$. Then by Definition 24, $\{B, C\} \subset \bigcirc^*_v$. By Definition 26, $B \wedge C \in \bigcirc^*_v$. Therefore, $(\mathcal{M}, v) \models_{\mathsf{PLL}} \bigcirc(B \wedge C)$. Therefore $(\mathcal{M}, w) \models_{\mathsf{PLL}} (\bigcirc B \wedge \bigcirc C) \to \bigcirc(B \wedge C)$.

□

Definition 27 (Prime Set). *A prime set Δ is a set of Lax sentences such that the following conditions hold:*

- *Δ is deductively closed under the axioms and rules of Lax logic.*
- *If $A \vee B \in \Delta$ then $A \in \Delta$ or $B \in \Delta$.*
- *Δ is consistent. That is, $\bot \notin \Delta$.*

Lemma 18 (Prime Lemma). *If $\Gamma \not\vdash A$, then there exists a prime set Δ such that $\Gamma \subset \Delta$, and $\Delta \not\vdash A$.*

Proof. See Martin, Studer [36]. □

Definition 28 (Canonical Lax Model). *The canonical Lax model is defined as the tuple*

$$\mathcal{L} = < W_L, \leq_L, *_L >$$

where

- $W_L = \{\Delta \subset Form \,|\, \Delta \text{ is prime}\}$,
- $\leq_L = \subset$,
- $*_L(p, \Delta) = 1$ *iff* $p \in \Delta$, *and*
- $*_L(\Delta) = \Delta^{\bigcirc} = \{A \in Form \,|\, \bigcirc A \in \Delta\}$.

Lemma 19. *The canonical Lax model exists.*

Proof. It is enough to show that there is a prime set of sentences. Since Lax Logic is consistent, then $\emptyset \nvdash \bot$. By the Prime Lemma, there is a prime set Δ such that $\Delta \nvdash \bot$. □

Lemma 20 (Truth lemma). *For all $\Delta \in W_L$, $(\mathcal{L}, \Delta) \models A$ iff $A \in \Delta$.*

Proof. The proof is done by induction on the complexity of A.

It is clear that the truth lemma holds in the base case, where $A = p_i$ or $A = \bot$.

For the inductive step, let $A = \bigcirc B$. If $(\mathcal{L}, \Delta) \models \bigcirc B$, then $B \in \Delta^{\bigcirc}$ by definition of truth under a basic evaluation. Since Δ is closed under the axioms of Lax Logic, (using axiom $\bigcirc R$), $\bigcirc B \in \Delta$. On the other hand, if $\bigcirc B \in \Delta$, then by the definition of $\Delta^{\bigcirc}$, $B \in \Delta^{\bigcirc}$. Therefore, by definition of truth under a basic evaluation, $(\mathcal{L}, \Delta) \models \bigcirc B$.

The inductive cases for the Boolean connectives are standard, and may be found in [36]. □

Lemma 21. *The canonical Lax model is a basic evaluation.*

Proof. We must show that $(W_L, \leq_L, *_L)$ satisfies the monotonicity conditions M1 and M2, from Definition 23.

- (M1): Let $\{\Delta, \Gamma\} \subset W_L$, and let $\Delta \leq_L \Gamma$. Then $\Delta \subset \Gamma$. Let $*_L(p, \Delta) = 1$. Then $p \in \Delta$. Then $p \in \Gamma$ since $\Delta \subset \Gamma$. Then $*_L(p, \Gamma) = 1$.
- (M2): Again, let $\Delta \leq_L \Gamma$, so that $\Delta \subset \Gamma$. $*_L(\Delta) = \{A \in Form \,|\, \bigcirc A \in \Delta\} \subset \{A \in Form \,|\, \bigcirc A \in \Gamma\} = *_L(\Gamma)$.

□

Lemma 22. *The canonical Lax model is a Lax model.*

Proof. From Lemma 21, we know that the canonical Lax model $\mathcal{L} = (W_L, \leq_L, *_L)$ is a basic evaluation. We must further show that the canonical Lax model is a basic evaluation that satisfies the three criteria given in Definition 26.

Let $\Delta \in W_L$ be a world.

– If $(\mathcal{L}, \Delta) \models A$, then by the Truth Lemma (Lemma 24), $A \in \Delta$. Since Δ is closed under the Lax axioms $A \to \bigcirc A$, then $\bigcirc A \in \Delta$. Then, $A \in \Delta^{\bigcirc}$, by definition of $\Delta^{\bigcirc}$. Thus, $True^{*}_{\Delta} \subset \Delta^{\bigcirc}$.

– Let $\bigcirc A \in \Delta^{\bigcirc}$. Then $\bigcirc\bigcirc A \in \Delta$, by definition of $\Delta^{\bigcirc}$. Since Δ is closed under the Lax axiom $\bigcirc\bigcirc A \to \bigcirc A$, we have $\bigcirc A \in \Delta$. Therefore $A \in \Delta^{\bigcirc}$ by definition of $\Delta^{\bigcirc}$. Thus, if $\bigcirc A \in \Delta^{\bigcirc}$, then $A \in \Delta^{\bigcirc}$.

– Let $\{A, B\} \subset \Delta^{\bigcirc}$. Then $\{\bigcirc A, \bigcirc B\} \subset \Delta$, by definition of $\Delta^{\bigcirc}$. Since Δ is closed under the Lax axiom $\bigcirc A \wedge \bigcirc B \to \bigcirc(A \wedge B)$, we have $\bigcirc(A \wedge B) \in \Delta$. Therefore $A \wedge B \in \Delta^{\bigcirc}$, by definition of $\Delta^{\bigcirc}$. Thus, if $\{A, B\} \subset \Delta^{\bigcirc}$, then $A \wedge B \in \Delta^{\bigcirc}$.

□

Theorem 13 (Completeness). *If* $\models_{Lax} A$ *then* $\vdash_{Lax} A$*.*

Suppose that $\nvdash_{Lax} A$. Then by Lemma 18, there is a prime set Δ such that $\Delta \nvdash A$. In particular, $A \notin \Delta$. Δ is a world in the canonical Lax model such that, by the Truth Lemma, (Lemma 24), $(\mathcal{L}, \Delta) \not\models A$. Therefore $\mathcal{L} \not\models A$. Therefore $\not\models A$.

8 Decidability Results

We can produce decidability results for LEA_- and LEA_+ using the close connections between those systems and classical logic.

In Corollary 4 we showed that classical logic is sound and complete with respect to probability semantics. We may extend classical logic with n unique atomic formulae $e_1, \ldots, e_n$ to a system we call CL_n, and the same proof will show CL_n to also be sound and complete with respect to probability semantics. We then give a *Boolean translation* b from LEA_+ formulas to CL_n formulas, such that A is provable in LEA_+ iff A^b is provable in CL_n. It follows that, to decide if A is valid in LEA_+, it is enough to decide if A^b is valid in CL_n. The decision problem for LEA_+ reduces to the decision problem for classical logic.

Definition 29 **(CL_n).** *Classical logic with n justification variables, or* CL_n*, is an extension of the language of classical propositional logic. The axioms are the same as for classical logic. The language contains a countable set of propositional variables* $\{p_i \mid i < \omega\}$*, Boolean connectives* $\wedge, \vee, \neg, \rightarrow$*, and constants* $\top, \bot$*. In addition* CL_n *contains n unique* justification variables $E_1, \ldots, E_n$ *which are considered atomic formulae.*

Let $\Vdash\!\vdash_{\mathsf{C}}$ denote the probability semantics applied to CL_n formulas, $\models_{\mathsf{C}}$ the usual Boolean semantics applied to CL_n formulas, and $\vdash_{\mathsf{C}}$ denote provable in CL_n.

Lemma 23. CL_n *is sound and complete with respect to probability semantics.*

Proof. The proof is the same as in Corollary 4. □

Next we give the translation from LEA_+ formulas to CL_n formulas.

Definition 30 (Boolean Translation).

Given a LEA_+ *formula A, we define the Boolean translation of A, written* A^b*, inductively as follows:*

– $e_i^b = E_i$

– $\mathbf{1}^b = \top$

– $\mathbf{0}^b = \bot$

– $(st)^b = s^b \wedge t^b$

– $(s \cup t)^b = s^b \vee t^b$

– $(p)^b = p$ *for propositional variable* p

– $(A \wedge B)^b = A^b \wedge B^b$ *and similar for other Boolean connectives*

– $(s\colon A)^b = (s)^b \to (A)^b$.

Proposition 13. *For all* LEA_+ *formulas* A*,* $\vdash_+ A$ *iff* $\vdash_{\mathsf{C}} A^b$.

Proof. From Lemma 23, we have that $\vdash_{\mathsf{C}} A^b$ iff $\Vdash_{\mathsf{C}} A^b$. Since LEA_+ is sound and complete with respect to probability semantics, it is therefore enough to show $\Vdash_{\mathsf{C}} A^b$ iff $\Vdash_+ A$. This is the case, since for all A, and all probabilistic interpretations $\circ$, $A^\circ = (A^b)^\circ$. This latter claim we may show by induction on the complexity of A. □

Theorem 14. LEA_+ *is decidable.*

Proof. The decision algorithm for LEA_+ consists of translating A to A^b, then using a Boolean decision algorithm on A^b. The translation from A to A^b is linear in length; A^b results from A by replacing each instance of e_i with E_i, $\mathbf{1}$ with $\top$, $\mathbf{0}$ with $\bot$, $\cap$ with $\wedge$, $\cup$ with $\vee$, and : with $\to$. □

Corollary 6. LEA_- *is decidable.*

Theorem 15. *The satisfiability problems for* LEA_+ *and* LEA_- *are in NP.*

Proof. To cover both cases, it is enough to show that the satisfiability problem for LEA_+ is in NP. First, we show that A is satisfiable in LEA_+ iff A^b is satisfiable in CL_n.

Following the proof as in Corollary 4, given a trivial-lattice model $*$, define a corresponding Boolean evaluation for CL_n $\tilde{*}$, such that if $e_i^* = Form$ then $E_i^{\tilde{*}} = 0$; if $e_i^* = True^*$ then $E_i^{\tilde{*}} = 1$; and $p_i^* = p_i^{\tilde{*}}$. It will follow that $* \models_+ A$ iff $\tilde{*} \models_{\mathsf{C}} A^b$. In fact, we see that the mapping $* \mapsto \tilde{*}$ provides a one-to-one correspondence between trivial-lattice models for LEA_+ and Boolean evaluations for CL_n. Therefore, it stands that if A is satisfiable in LEA_+, then A^b is satisfiable in CL_n.

Therefore, an algorithm to verify if A is satisfiable in LEA_+ is as follows. First, translate A to A^b. This is done in deterministically in polynomial time. Then run a nondeterministic polynomial time algorithm to determine if A^b is satisfiable in CL_n. □

9 Justification Logic as Propositional Logic

The relationship between LEA_+ and CL_n explains why LEA_+ is the axiomatization of probability semantics. Probability semantics is fundamentally a classical semantics. To interpret the LEA_+ language in this classical format, we treat ":" like classical implication $\rightarrow$. In this way, using the Boolean translation, all LEA_+ formulas can be interpreted classically. However, not all classical formulas have an LEA_+ counterpart. For example, $(E_i \wedge E_j) \rightarrow E_j$ is valid in CL_n but there is no LEA_+ formula A, such that $A^b = (E_i \wedge E_j) \rightarrow E_j$.

Due to the soundness and completeness of probability semantics with respect to CL_n and with respect to LEA_+ we can use the Boolean translation to embed LEA_+ into CL_n in the following sense. If A is a LEA_+ formula, then

$$\vdash_+ A \text{ iff } \vdash_{\mathsf{C}} A^b.$$

In words, the theorems of LEA_+ are exactly those sentences whose Boolean translation are theorems in classical logic, (more precisely, in CL_n).

CL_n itself can be thought of as simply classical logic, where we distinguish a finite set of propositional variables $E_1, \ldots, E_n$, along with $\top$ and $\bot$, for our attention. In this way, we can identify LEA_+ with a fragment of CL_n, which we will call LEA_+^b.

$$\mathsf{LEA}_+^b = \{A^b \mid A \in \mathcal{L}(\mathsf{LEA}_+)\}$$

How can we characterize LEA_+^b? Essentially, it is the fragment of CL_n where the image of justification terms appear only in the antecedent of an implication. They may appear alone, or with other justification terms in a combination of disjunctions and conjunctions. This can be defined rigorously as follows.

Say $E_1, \ldots, E_n, \top, \bot$ are *allowed combinations of evidence.* If A and B are allowed combinations of evidence, so are $A \wedge B$ and $A \vee B$. Note, allowed combinations of evidence are CL_n formulas. Then, LEA_+^b is the subset of CL_n formulas A, such that, if $B \in \{E_1, \ldots E_n, \top, \bot\}$ appears as a subformula of A, each instance of B appears only inside an allowed combination of evidence C, and allowed combinations of evidence appear only as the antecedent of some implication subformula $C \rightarrow D$ of A.

From this we see how LEA_+^b – and therefore LEA_+ – straddles the line between justification logic and classical propositional logic.

Terms are propositional in character, simply because they map to classical propositional formulas. Formulas $t\colon A$ are propositional, because they map to an implication. Yet, LEA_+ is still different than classical logic. LEA_+ retains its justificational character because there is a restriction on where terms can be mapped to.

Terms always map to acceptable combinations of evidence. The acceptable combinations of evidence have as their only logical operations disjunction and conjunction. Disjunction represents the combination (union) of evidence. Conjunction represents the intersection of evidence, or alternatively the mutual corroboration of evidence. These acceptable combinations of evidence always appear as the antecedent in an implication. They have no life of their own, but are only used to justify other propositions. This latter fact is true for all justification logics. A term t is never considered by itself, but only in relation to other formulas in the format $t\colon A$. We can say nothing about the fact of t, but only about how it relates as evidence for some proposition.

Thus, though acceptable combinations of evidence are propositions in CL_n, they should still be understood as justifications, which here are a special type of proposition. Typical propositional logics do not have this kind of distinction between propositions. This distinction should be considered a useful feature of LEA_+ as compared to classical propositional logic.

One may ask if all justification logics behave like LEA_+. That is, do we need special justification terms and operations, or can we do without them, interpreting terms as propositions and : as implication? The answer in general is no.

The typical situation for justification logics is that justification variables and constants are allowed to justify arbitrary collections of formulas. Yet, if : were interpreted as implication, and terms as propositions, then it would follow that if $t{:}A$ and $t{:}B$ hold then $t{:}(A \wedge B)$ holds. Similarly, if $t{:}A$ and $A \to B$ hold, then $t{:}B$ holds. In general, this is not the case for justification logics. That it is so for the logics studied in this paper points to the peculiarity of these systems.

Another related feature that justification logics typically have, which the ones in this paper do not have, is hyperintensionality at the level of justification. A *hyperintensional* operator H is one such two sentences A and B may be necessarily equivalent, while HA and HB may have different truth values. The term "hyperintensional" was introduced by Cresswell in [15]. Belief is a prototypical hyperintensional operator. It may be the case that one believes a proposition, but does not believe all the logical entailments of that proposition.

One of the strengths of justification logic, is that typically justifications are hyperintensional. (See [3] for a discussion of hyperintensionality in the justification logic format.) t may be evidence for A, and A may be logically equivalent to B, yet t is not evidence for B. Formally, $t{:}A$ and $A \leftrightarrow B$ may hold, yet $t{:}B$ may not hold. In any justification logic with hyperintensional justifications, : will therefore not behave like $\to$. Conversely, any justification logic where terms are interpreted as deductively closed sets will not be hyperintensional. That is the situation here. Mathematically, the basic model interpretation of a term will be closed under modus ponens whenever application is idempotent.

The increased flexibility of standard justification logics is an asset, not a weakness. We are more free to create complicated relationships between justifications and propositions, including the phenomenon of hyperintensionality. When justifications are treated as propositions and : as implication, a great simplification and flattening naturally occurs. The $\mathsf{LEA}_{+/-}$ framework may be regarded as a clean mathematical answer to the question of justification logic with propositional evidence and material implication as :.

10 Future Research

There are a few different directions that appear for further research.

First, we would like to continue research into the computational complexities of the systems discussed in this paper. Discovering if LEA is decidable or not is a natural goal. Towards the goal of determining if the LEA is decidable, it could be fruitful to develop a cut-free sequent system for LEA. We should further examine the issue of cut and cut-elimination in each system. Observe the resemblance between cut and the axiom $t{:}A \to s{:}A$ if $s \leq t$.

$$\frac{\Gamma \Rightarrow B \qquad B \Rightarrow A}{\Gamma \Rightarrow A} \quad \frac{s \leq t \qquad t{:}A}{s{:}A}$$

The deduction on the right is valid in any deductive basic model, which thereby includes LEA and LEA_+. Ideologically $t{:}A$ was intended to mean that A follows from t. This justifies reading as $t{:}A$ as $t \Rightarrow A$ and $s{:}A$ as $s \Rightarrow A$. Moreover, in our lattices, we have $\vdash_{\mathsf{C}} \; s^b \to t^b$ whenever $s \leq t$. This justifies reading $s \leq t$ as $s \Rightarrow t$.

The elimination of references to $\leq$ in our formulation of LEA seems to bear resemblance to the elimination of the cut rule in a sequent system. In further generalizations of LEA-like logics, we may ask for criteria to determine when so-called $\leq$-elimination may take place, just as one looks for cut-elimination theorems in sequent calculi. These parallels should be elucidated and examined formally in future work.

This leads us towards generalizing the types of evidence lattices we work with, for, in the cases where the lattices are finite and distributive, $\leq$-elimination seems feasible, perhaps inevitable. We might generalize to the cases of infinite lattices and lattices without distributivity, for example, and study the behavior of these systems. This could be fruitful for building connections with other areas of logic and formal reasoning, such as argumentation theory, [18]. Towards developing an intuitionistic theory of evidence aggregation, we may work in Heyting algebras.

The connection between Propositional Lax Logic and LEA_- is interesting. Recall that LEA_- is a multimodal version of PLL, but where the base logic is classical, rather than intuitionistic. Returning LEA_- back to the intuitionistic base, it may be possible to develop a new PLL semantics by combining basic two-models with intuitionistic basic modular models. (See [7] for basic modular models, and [36] for basic modular models in the intuitionistic setting.) Moreover, as LEA_+ arises naturally from LEA_-, we wonder what would appear if we formulated LEA_+ – or a natural analogue of it – with an intuitionistic base, and what its connection to PLL and intuitionistic LEA_- would be.

As a further area for research, we can look into alternative interpretations of the colon operator **:**. In probability semantics, **:** is treated like $\rightarrow_{cl}$, where $\rightarrow_{cl}$ is classical (material) implication. One might argue to change this on philosophical grounds. For example, this reading requires the strange validity $A \rightarrow (t{:}A)$, which we read as "either A does not happen, or event t will secure event A." If we read $t{:}A$ as t is evidence for A, then, since t is arbitrary, we have that if A happens then *anything* is evidence for A. This triviality is reflected in the degenerate nature of two-models and trivial-lattice models. Conceptually, this shows the principal limitations of reading evidence as propositions and $t{:}A$ as a classical implication $t \rightarrow_{cl} A$.

To get around this, we might consider alternative interpretations of $t{:}A$. For example, one may believe that the processes of evidence collection and aggregation should be understood in an intuitionistic sense, or perhaps in the sense of relevant logic. Yet, one may believe that the physical world operates under classical laws. Then one may have a semantics similar to probability semantics, with $(A \rightarrow B)^\circ = (A \rightarrow_{cl} B)^\circ$, while $(t{:}A)^\circ = (t \rightarrow_{int} A)^\circ$ or $(t{:}A)^\circ = (t \rightarrow_{rel} A)^\circ$, where $\rightarrow_{int}$, $\rightarrow_{rel}$, are intuitionistic implication and relevant implication, respectively.

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