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Higher Diffeology Theory

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by

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Abstract

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Finite dimensional smooth manifolds have been studied for hundreds of years, and a massive theory has been built around them. However, modern mathematicians and physicists are commonly dealing with objects outside the purview of classical differential geometry, such as orbifolds and loop spaces. Diffeology is a new framework for dealing with such generalized smooth spaces. This theory (whose development started in earnest in the 1980s) has started to catch on amongst the wider mathematical community, thanks to its simplicity and power, but it is not the only approach to dealing with generalized smooth spaces. Higher topos theory is another such framework, considerably more abstract and based heavily on categorical and homotopical techniques. In this dissertation, these two points of view are combined. We draw a bridge between these frameworks by using a cofibrant replacement functor of Dugger's to embed diffeological spaces into simplicial presheaves in a homotopically correct way. From this we prove that the theory of bundles between these two frameworks agree. We then port over the powerful tools of higher topos theory, such as the shape operator, to obtain new results in diffeology. As our main result, we obtain a short exact sequence exhibiting the obstruction to the Cech-de Rham isomorphism for diffeological spaces in all dimensions, building on an analogous result of Patrick Iglesias-Zemmour's in dimension 1. This work consists of the content of two papers [Min22] and [Min24], which make up Chapter 1 and Chapter 2, respectively.

Acknowledgments

I dedicate this thesis to Grammy and Dilly. You are the two most important people in my life, and without both of you this thesis could never have been written. I love you both.

Contents

Chapter 1

Diffeological Principal Bundles and Principal Infinity Bundles

1.1 Introduction

Principal G-bundles and Čech cohomology are important tools in the study of smooth manifolds. However, in recent years, the desire to expand the typical objects of study in differential geometry has led to various frameworks in which one can define a "generalized smooth space." In this paper, we construct a bridge between two such frameworks. One of them is diffeology, as popularized in the textbook [Igl13]. A diffeological space consists of a set X, and a set $\mathcal{D}_X$ of functions $\mathbb{R}^n \to X$, for varying $n \geq 0$, satisfying three simple conditions. While the definition of a diffeological space is simple, a large number of interesting spaces outside the purview of classical differential geometry can be given a diffeology. Every finite dimensional smooth manifold inherits a canonical diffeology, as does the set $C^\infty(X, Y)$ of smooth maps between any two diffeological spaces. In fact the category of diffeological spaces is complete, cocomplete and cartesian closed. More precisely it is a quasi-topos [BH11]. This is of course not the case for the category of finite dimensional smooth manifolds, and thus provides a "better" category in which to work. Various important constructions in classical differential geometry have been defined for diffeological spaces, like differential forms, deRham cohomology, fiber bundles, tangent spaces [CW15], and recently Čech cohomology [Igl20a] [KWW21] [Ahm23].

The second framework is higher topos theory. Here the objects of interest are ∞-stacks over the site Cart of cartesian spaces. Many constructions of classical differential geometry can be extended to ∞-stacks, with interesting applications. One such extension is the notion of a principal ∞-bundle, as defined in [NSS14a] and [NSS14b]. Classical principal bundles and non-abelian bundle gerbes are particular examples of principal ∞-bundles. Principal ∞-bundles allow for a robust framework wherein one can study twisted, equivariant or differential refinements of generalized cohomology theories. For more on this theory we recommend the texts [Bun22a], [Bun23], [FSS+12], [ADH21], [Sch13], [BNV16], [SS21]. In this paper, we will use the presentation of this theory by simplicial presheaves. Thus, the reader does not need to be comfortable with the language of ∞-categories in order to read this paper.

The first connection between these two framworks was made in [BH11], in which Baez and Hoffnung proved that the category of diffeological spaces is equivalent to the category $\mathsf{ConSh}(\mathsf{Open})$ of concrete sheaves on Open, the category of open subsets of cartesian spaces and smooth maps, equipped with the coverage of open covers. This means that diffeological spaces can be thought of as certain kinds of sheaves. This is a very powerful point of view, especially with respect to studying the homotopy theory of diffeological spaces [Pav22b]. In particular, the category of concrete sheaves on Cart embeds fully faithfully into the category $\mathsf{sPre}(\mathsf{Cart})$ of simplicial presheaves on Cart, whose objects are functors $X : \mathsf{Cart}^{op} \to \mathsf{sSet}$. Just as some presheaves of sets are sheaves, some simplicial presheaves are ∞-stacks. Higher topos theory can also be called homotopical sheaf theory, and ∞-stacks are the main objects of study. Sheaves of sets are discrete ∞-stacks, and stacks of groupoids are 1-truncated ∞-stacks, see Section 1.5 for a more detailed discussion.

Embedding the category of diffeological spaces into the category of simplicial presheaves puts diffeological spaces into a homotopical framework that is both easy to work with and connects powerfully with the underlying homotopy theory of simplicial sets. One such consequence of this connection is the following theorem, which is the main result of this paper.

Theorem 1.6.8. Given a diffeological space X and a diffeological group G, there is a weak homotopy equivalence,

$$N\mathsf{Prin}^{\infty}_{G}(X) \simeq N\mathsf{DiffPrin}_{G}(X).$$

where $N\mathsf{DiffPrin}_{G}(X)$ is the nerve of the category of diffeological principal G-bundles on X and $N\mathsf{Prin}^{\infty}_{G}(X)$ is the nerve of the category of G-principal ∞-bundles on X.

Some explanations are in order. In the case where X is a finite dimensional smooth manifold and G is a Lie group, the above result is well known, see [Sch13, Section 1.2.6] and [FSS+12, Section 3.2.1]. The method of proof is as follows: For a finite dimensional smooth manifold M, one chooses a good open cover $\mathcal{U}$ of M, and from this constructs a simplicial presheaf $\check{C}(\mathcal{U})$ called the Čech nerve of $\mathcal{U}$ (see Example 1.5.7). Maps from $\check{C}(\mathcal{U})$ to the ∞-stack $\mathbf{B}G$ are precisely G-cocycles with respect to $\mathcal{U}$, and these classify G-principal bundles on M, see [Bry09, Chapter 2.1] and [Hus+07, Section 5.4] for example. The simplicial presheaf $\check{C}(\mathcal{U})$ is projectively cofibrant, and using the machinery of Section 1.5 along with the classification of principal bundles using cocycles on smooth manifolds, the above Theorem is easily proven for the special case of smooth manifolds.

However the above method of proof does not carry over straightforwardly to diffeological spaces. For one, there exist diffeological principal bundles that do not locally trivialize over any open cover (with respect to the D-topology) of its base space [Igl13, Article 8.9]. Thus one is forced to look for an analogue of $\check{C}(\mathcal{U})$ which does not use any notion of open cover. This is accomplished by a cofibrant replacement construction of Dugger's [Dug01, Lemma 2.7], which when applied to a diffeological spaces gives the following result.

Lemma 1.5.18. Given a diffeological space X, thought of as a discrete simplicial presheaf on Cart, the simplicial presheaf QX given by the following coend formula

$$QX = \int^{[n]\in\Delta} \Delta^n_c \times \left(\coprod_{U_{p_n}\to\cdots\to U_{p_0}\to X} yU_{p_n} \right) \tag{1.1}$$

where the coproduct is indexed by $(N\mathsf{Plot}(X))_n$, the set of n-many composable morphisms in the category of plots over X, is a cofibrant replacement for X in the projective model structure on simplicial presheaves over Cart,

This construction can be understood in a more concrete way as follows. Given a diffeological space X, let $B = \coprod_{p_0\in\mathsf{Plot}(X)} U_{p_0}$ denote the diffeological space given as the disjoint union of the domains of plots of X, and let $QX_1 = \coprod_{f_0:U_{p_1}\to U_{p_0}} U_{p_1}$ denote the diffeological space given as the disjoint union of domains of plots indexed by maps of plots of X. There are maps $s,t : QX_1 \to B$ and $u : B \to QX_1$ that make $[QX_1 \rightrightarrows B]$ into a diffeological category.

Proposition 1.5.20. If X is a diffeological space, then

$$QX \cong N[QX_1 \rightrightarrows B], \tag{1.2}$$

where N denotes the nerve construction.

Note that the construction of QX relies only on the plots of X, not on any choice of open cover. Now all one needs to prove Theorem 1.6.8 is to establish an equivalence between maps $QX \to \mathbf{B}G$ and diffeological principal G-bundles over X. This is done in Section 1.3 by proving a bundle construction-type theorem, Theorem 1.3.15. The proof of Theorem 1.3.15 is mostly straightforward, but requires some technical care, since QX is the nerve of a diffeological category rather than a diffeological groupoid, as in the case of the Čech nerve $\check{C}(\mathcal{U})$ in the classical proof.

The embedding of diffeological spaces into simplicial presheaves immediately yields a notion of cohomology on diffeological spaces, which we call **∞-stack cohomology**. If X is a diffeological space, and A is an ∞-stack, whose k-fold delooping $\overline{W}^k A$ exists (notions we will explain in Section 1.5), then the ∞-stack cohomology of X with coefficients in A is given by the connected components of the derived mapping space

$$\check{H}^k_\infty(X, A) = \pi_0 \mathbb{R}(X, \overline{W}^k A) = \pi_0 \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \overline{W}^k A).$$

In Corollary 1.5.36 we construct an explicit cochain complex whose cohomology gives the ∞-stack cohomology of X with values in A when A is a diffeological abelian group.

There are two other examples of Čech cohomology of diffeological spaces in the literature that we consider in this paper, the Krepski-Watts-Wollbert diffeological Čech cohomology $\check{H}^k_{KWW}$ of [KWW21] and the PIZ diffeological Čech cohomology $\check{H}^k_{PIZ}$ of [Igl20a]. Interestingly, each of these cohomologies is constructed by "resolving" a diffeological space X with an appropriate diffeological category, the Čech groupoid $\check{C}(X)$ and the gauge monoid $B//M$ respectively. In Section 1.5 we compare these various notions of diffeological Čech cohomologies and find that

$$\check{H}^0_\infty(X, A) \cong \check{H}^0_{KWW}(X, A) \cong \check{H}^0_{PIZ}(X, A), \qquad \check{H}^1_\infty(X, A) \cong \check{H}^1_{KWW}(X, A).$$

A full summary of known results about these cohomology theories is given after (1.48). It is currently unknown if all three of these cohomology theories are isomorphic in all degrees.

Now if G is a diffeological group, the ∞-stack cohomology of a diffeological space X with values in $\mathbf{B}G$ is denoted $\check{H}^1_\infty(X, G)$. An immediate consequence of Theorem 1.6.8 is the following result.

Corollary 1.6.9. Given a diffeological space X and a diffeological group G, there is an isomorphism of pointed sets

$$\check{H}^1_\infty(X, G) \cong \pi_0 \mathsf{DiffPrin}_G(X),$$

where $\pi_0 \mathsf{DiffPrin}_G(X)$ denotes the set of isomorphism classes of diffeological principal G-bundles on X, pointed at the isomorphism class of trivial bundles.

Thus ∞-stack cohomology of diffeological spaces also encompasses nonabelian cohomology.

It is our view that by applying tools from higher topos theory can be beneficial to the still young subject of diffeological spaces. In particular we believe that while the machinery of ∞-stack cohomology may come from an abstract framework, it can ultimately output important and down-to-earth results. Furthermore, higher topos theory already has definitions for higher principal bundles (called bundle gerbes) and connections on such objects inherently built into it. Pulling these definitions over to diffeological spaces and analyzing the results are the subject of future work.

The paper is organized as follows. In Section 1.2, we will give some background information about diffeological spaces. In Section 1.3 we turn to diffeological principal G-bundles. We define G-cocycles and prove a bundle construction-type theorem, Theorem 1.3.15. In Section 1.4, we give a brisk introduction to sheaf theory, and explain the Baez-Hoffnung Theorem [BH11, Proposition 24]. In Section 1.7, we compare several categories of concrete sheaves on various sites, and show that they are all equivalent, proving that the category of diffeological spaces as given in Definition 2.2.2 is equivalent to the usual category of diffeological spaces considered in the literature. In Section 1.5, we will review the Čech model structure on simplicial presheaves over cartesian spaces. Proposition 1.5.20 provides a cofibrant replacement of a diffeological space as the nerve of a diffeological category. We compare this diffeological category to two other diffeological categories $\check{C}(X)$ and $B//M$, which have been introduced in [KWW21] and [Igl20a], respectively. From these three diffeological categories, we obtain three separate notions of Čech cohomology for diffeological spaces, and compare them in Section 1.5.3. In Section 1.6, we turn to the main result of this paper, that if G is a diffeological group and X is a diffeological space, then the nerve of the category of principal G-bundles on X is weak homotopy equivalent to the nerve of the category of G-principal ∞-bundles over X.

1.2 Diffeological Spaces

In this section we give some background on diffeological spaces, all of which can be found in the textbook [Igl13].

Definition 1.2.1. Let M be a finite dimensional smooth manifold[1]. We say a collection of subsets $\mathcal{U} = \{U_i \subseteq M\}_{i\in I}$ is an **open cover** if each U_i is an open subset of M, and $\bigcup_{i\in I} U_i = M$. If U is a finite dimensional smooth manifold diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, we call U a **cartesian space**. We call $\mathcal{U} = \{U_i \subseteq M\}$ a **cartesian open cover** of a manifold M if it is an open cover of M and every U_i is a cartesian space. We say that $\mathcal{U}$ is a **good open cover** if it is a cartesian open cover, and further every finite non-empty intersection $U_{i_0\dots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$ is a cartesian space.

Let Man denote the category whose objects are finite dimensional smooth manifolds and whose morphisms are smooth maps. Let Cart denote the full subcategory whose objects are cartesian spaces. Given a set X, let $\mathsf{Param}(X)$ denote the set of **parametrizations** of X, namely the collection of set functions $p : U \to X$, where $U \in \mathsf{Cart}$.

Definition 1.2.2. A **diffeology** on a set X, consists of a collection $\mathcal{D}$ of parametrizations $p : U \to X$ satisfying the following three axioms:

1. $\mathcal{D}$ contains all points $\mathbb{R}^0 \to X$,
2. If $p : U \to X$ belongs to $\mathcal{D}$, and $f : V \to U$ is a smooth map, then $pf : V \to X$ belongs to $\mathcal{D}$, and
3. If $\{U_i \subseteq U\}_{i\in I}$ is a good open cover of a cartesian space U, and $p : U \to X$ is a parametrization such that $p|_{U_i} : U_i \to X$ belongs to $\mathcal{D}$ for every $i \in I$, then $p \in \mathcal{D}$.

[1] We will assume throughout this paper that manifolds are Hausdorff and paracompact.

A set X equipped with a diffeology $\mathcal{D}$ is called a **diffeological space**. Parametrizations that belong to a diffeology are called **plots**. We say a set function $f : X \to Y$ between diffeological spaces is **smooth** if for every plot $p : U \to X$ in $\mathcal{D}_X$, the composition $pf : U \to Y$ belongs to $\mathcal{D}_Y$. We often denote the set of smooth maps from X to Y by $C^\infty(X, Y)$.

Remark 1.2.3. In what follows, when we say that $f : X \to Y$ is a map of diffeological spaces or a smooth map, we mean that it is a smooth function in the above sense.

Denote the category whose objects are diffeological spaces and morphisms are smooth maps between them by Diff. An isomorphism in this category is called a **diffeomorphism**.

Remark 1.2.4. Note that Definition 2.2.2 is not the exact definition of diffeological spaces as usually given in the literature, such as [Igl13, Article 1.5]. However it is precisely the definition of diffeological space as defined in [Pav22a, Definition 2.7] and [SS21, Notation 3.3.15], as we will prove in Theorem 1.4.16. We will call the diffeological spaces defined in [Igl13, Article 1.5] **classical diffeological spaces** and denote their category by Diff'.

In Section 1.4, leveraging [BH11], we will explain how to think of diffeological spaces as concrete sheaves on the site $(\mathsf{Cart}, j_{\mathrm{good}})$, namely cartesian spaces with the coverage of good open covers. Leveraging this perspective we show that Diff is equivalent to Diff' in Section 1.7.

However there are real advantages to using Diff over Diff', one of them being Lemma 1.3.6, which is false for Diff'. There are other more technical advantages as well. In Section 1.5, we will consider Diff embedded into the category $\mathsf{sPre}(\mathsf{Cart})$, which can be given the Čech model structure $\mathsf{s\check{P}re}(\mathsf{Cart})$. If one uses j_{open} instead of j_{good} on Cart, and $\mathcal{U}$ is an arbitrary cartesian open cover of a cartesian space U, then there is no guarantee that $\check{C}(\mathcal{U})$ will be projective cofibrant. Using good open covers ensures that it is projective cofibrant, which is necessary for much of the theory to be developed. Using j_{good} also allows us to leverage Theorem 1.5.17 and Theorem 1.5.30, which are vital to our results.

Example 1.2.5. If M is a finite dimensional smooth manifold, then the set of parametrizations $p : U \to M$ that are smooth in the sense of classical differential geometry forms a diffeology [Igl13, Article 4.3]. We call this the **manifold diffeology** of the underlying set of M. Further a map $f : M \to N$ of smooth manifolds is smooth in the classical sense if and only if it is smooth as a map of diffeological spaces. This implies that the functor $\mathcal{D}_{\mathsf{Man}} : \mathsf{Man} \to \mathsf{Diff}$ that sends a manifold to its underlying set equipped with the manifold diffeology is fully faithful.

Definition 1.2.6. We say that a map $i : X \to Y$ of diffeological spaces is an **induction** if for every plot $p : U \to Y$ there exists a plot $q : U \to X$ such that $p = iq$.

Definition 1.2.7. We say that a map $\pi : X \to Y$ of diffeological spaces is a **subduction** if it is surjective, and for every plot $p : U \to Y$, there exists a good open cover $\{U_i \subseteq U\}$, and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & X \\ \downarrow & & \downarrow \pi \\ U & \xrightarrow[p]{} & Y \end{array} \tag{1.3}$$

Lemma 1.2.8. Let $X \xrightarrow{\pi} Y$ be a subduction, then a function $Y \xrightarrow{f} Z$ is smooth if and only if $f\pi$ is, and f is a subduction if and only if $f\pi$ is.

Lemma 1.2.9. If $f : X \to Y$ is a smooth map of diffeological spaces such that there exists a section, i.e. a smooth map $\sigma : Y \to X$ such that $f\sigma = 1_Y$, then f is a subduction.

Definition 1.2.10. Let $(X, \mathcal{D}_X)$ be a diffeological space and $A \overset{i}{\hookrightarrow} X$ a subset. Then consider the collection $\mathcal{D}_A^X$ of parametrizations $p : U \to A$ such that $ip : U \to X$ is a plot of X. It is not hard to see that this collection is a diffeology, which we call the **subspace diffeology** on A. Note that i is an induction when A is equipped with the subspace diffeology.

Definition 1.2.11. Let $(X, \mathcal{D}_X)$ be a diffeological space and suppose that $\sim$ is an equivalence relation on the underlying set X. Let $X \xrightarrow{\pi} X/{\sim}$ denote the quotient function taking a point $x \in X$ to its equivalence class $[x] \in X/{\sim}$. Then consider the collection $\mathcal{D}_\sim^X$ of parametrizations $p : U \to X/{\sim}$ such that there exists a good open cover $\{U_i \subseteq U\}$ and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & X \\ \big\downarrow & & \big\downarrow \pi \\ U & \xrightarrow[p]{} & X/{\sim} \end{array} \tag{1.4}$$

It is not hard to see that this forms a diffeology, which we call the **quotient diffeology** on $X/{\sim}$. Note that π is a subduction when $X/{\sim}$ is equipped with the quotient diffeology.

The category Diff of diffeological spaces is complete and cocomplete. Suppose $F : \mathsf{J} \to \mathsf{Diff}$ is a diagram of diffeological spaces. Then a parametrization $p : U \to \lim F$, (where we are taking $\lim F$ to be the limit of the underlying sets of the F_j) is a plot if and only if the composite $U \xrightarrow{p} \lim F \to F_j$ is a plot for every $j \in \mathsf{J}$.

Similarly a parametrization $p : \mathsf{J} \to \operatorname{colim} F$ is a plot if and only if there exists a good open cover $\{U_i \to U\}$ and plots $U_i \xrightarrow{p_i} F_{j_i}$ for each i, such that the following diagram commutes:

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & F_{j_i} \\ \big\downarrow & & \big\downarrow \pi \\ U & \xrightarrow[p]{} & \operatorname{colim} F \end{array}$$

Definition 1.2.12. Given any two diffeological spaces X, Y, consider the set $\mathsf{Diff}(X, Y)$ of smooth maps $f : X \to Y$. Let $\mathcal{D}_{X \to Y}$ denote the collection of parametrizations $p : U \to C^\infty(X, Y)$ such that the adjoint function $p^\# : U \times X \to Y$, defined by $p^\#(u, x) = p(u)(x)$ is smooth. This collection is a diffeology, which we call the **functional diffeology**.

The functional diffeology makes Diff a Cartesian closed category. We will see in Section 1.4 that Diff is in fact a quasitopos.

To every diffeological space X, we can consider the category $\mathsf{Plot}(X)$, whose objects are plots $p_0 : U_{p_0} \to X$ and whose morphisms $f_0 : p_1 \to p_0$ are smooth maps $f_0 : U_{p_1} \to U_{p_0}$

making the following diagram commute

$$\begin{array}{ccc} U_{p_1} & \xrightarrow{f_0} & U_{p_0} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_0} & \\ & X & \end{array} \tag{1.5}$$

Lemma 1.2.13 ([CSW14, Proposition 2.7])**.** Given a diffeological space X, let $q : \mathsf{Plot}(X) \to \mathsf{Diff}$ denote the functor that sends a plot $p : U \to X$ to the cartesian space U considered as a diffeological space with its manifold diffeology. Then $X \cong \operatorname{colim} q$.

1.3 Diffeological Principal Bundles

In this section we introduce diffeological principal G-bundles, and prove a bundle construction-type theorem (Theorem 1.3.15) that gives an equivalence between the groupoid of G-cocycles and the groupoid of diffeological principal G-bundles over a diffeological space X. This development will be needed in Section 1.6. One does not need to know anything about model categories or homotopy theory to understand this section.

Definition 1.3.1. A **diffeological group** is a group G equipped with a diffeology $\mathcal{D}_G$ such that the multiplication map $m : G \times G \to G$, and inverse map $i : G \to G$ are smooth.

Definition 1.3.2. A right **diffeological group action** of a diffeological group G on a diffeological space X is a smooth map $\rho : X \times G \to X$ such that $\rho(x, e_G) = x$, and $\rho(\rho(x, g), h) = \rho(x, gh)$, where e_G denotes the identity element of G. We will often keep ρ implicit and denote such an action by $x \cdot g$.

Definition 1.3.3. Let G be a diffeological group, and P be a diffeological right G-space. A map $\pi : P \to X$ of diffeological spaces is a **diffeological principal G-bundle** if:

1. the map $\pi : P \to X$ is a subduction, and

2. the map $\mathrm{act} : P \times G \to P \times_X P$ defined by $(p, g) \mapsto (p, p \cdot g)$, which we call the **action map** is a diffeomorphism.

A map of diffeological principal G-bundles $P \to P'$ over X is a diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

where f is a G-equivariant smooth map. A diffeological principal G-bundle P is said to be **trivial** if there exists an isomorphism $\varphi : X \times G \to P$, called a **trivialization**, where $\mathrm{pr}_1 : X \times G \to X$ is the product bundle. Let $\mathsf{DiffPrin}_G(X)$ denote the category of diffeological principal G-bundles over a diffeological space X. The following properties of diffeological principal G-bundles are not hard to prove, see [Igl13, Chapter 8].

Lemma 1.3.4. Let G be a diffeological group and $\pi : P \to X$ a diffeological principal G-bundle, then we have the following:

1. if $f : Y \to X$ is a smooth map, then the pullback $f^*P \to Y$ is a diffeological principal G-bundle,
2. if there is a section $s : X \to P$, namely $\pi \circ s = 1_X$, then P is trivial,
3. if $f : P \to P'$ is a map of diffeological principal G-bundles over a diffeological space X, then it is an isomorphism.

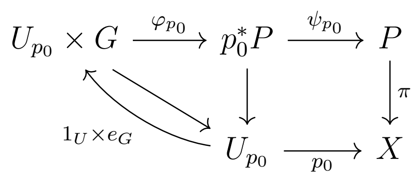
By Lemma 1.3.4.(3), the category $\mathsf{DiffPrin}_G(X)$ is a groupoid for every diffeological group G and every diffeological space X.

Proposition 1.3.5 ([Igl13, Article 8.19])**.** Given a diffeological group G and a diffeological principal G-bundle $\pi : P \to U$, where U is a cartesian space, there exists a trivialization $\varphi : U \times G \to P$.

Lemma 1.3.6. Given a diffeological group G and a diffeological right G-space P, a map $\pi : P \to X$ is a diffeological principal G-bundle if and only if for every plot $p_0 : U_{p_0} \to X$, the pullback p_0^*P is trivial, and the map $\mathrm{act} : P \times G \to P \times_X P$ defined by $(p, g) \mapsto (p, p \cdot g)$ is a diffeomorphism.

Proof. ($\Rightarrow$) If $\pi : P \to X$ is a diffeological principal G-bundle, and $p_0 : U_{p_0} \to X$ is a plot, then by Lemma 1.3.4 $p_0^* P$ is a diffeological principal G-bundle over U_{p_0}. Since U_{p_0} is cartesian, by Proposition 1.3.5, $p_0^* P$ is trivial, with trivialization $\varphi_{p_0} : U_{p_0} \times G \to p_0^* P$.

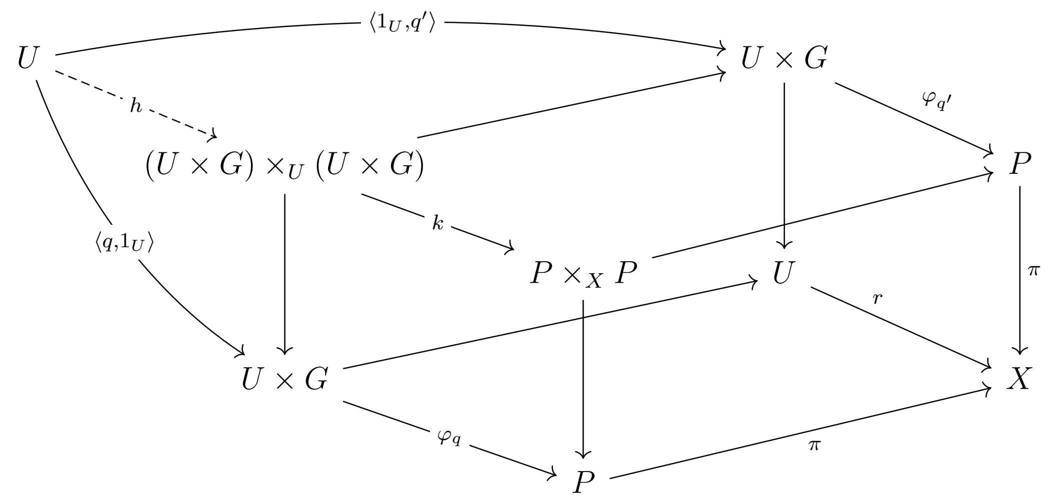
($\Leftarrow$) If $p_0^* P$ is trivial for every plot $p_0 : U_{p_0} \to X$, then there exists a trivialization $\varphi_{p_0} : U_{p_0} \times G \to p_0^* P$, and thus we obtain the following commutative diagram



Since this is true for every plot $p_0 : U_{p_0} \to X$, then by taking U_{p_0} as a good cover of itself, *pi* can be seen to be a subduction. □

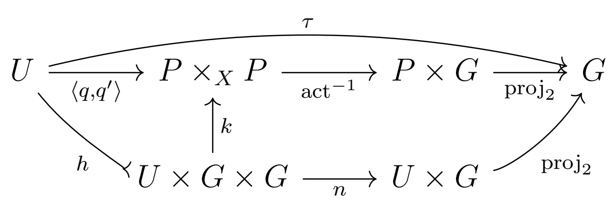
Lemma 1.3.7. Condition (2) of Definition 1.3.3 is equivalent to the condition that G acts on the fibers of π freely and transitively.

Proof. If act : $P \times G \to P \times_X P$ is a diffeomorphism, then G clearly acts on the fibers of π freely and transitively. Now suppose G acts on the fibers of π freely and transitively. The map act : $P \times G \to P \times_X P$ is smooth. Since the action is free and transitive it means also that the map act is a bijection. We need then only to show that the inverse function is smooth. Namely if $\langle q, q' \rangle : U \to P \times_X P$ is a plot, where $q, q' : U \to P$ are plots, then we wish to show that the composite function $\text{act}^{-1} \langle q, q' \rangle$ is a plot of $P \times G$. Now this is the set map $u \mapsto (q(u), \text{diff}(q(u), q'(u)))$, where diff : $P \times_X P \to G$ is the composite map $\text{proj}_2 \text{act}^{-1}$. This is a plot of $P \times G$ if it is a plot in both factors. Obviously it is a plot of the first factor, so we need only show that it is a plot of the second factor. This is the map that we will denote by τ, namely $\tau(u) = \text{diff}(q(u), q'(u))$. Now, let $r = \pi q = \pi q'$ denote the plot $r : U \to X$. We have the following commutative cube



(1.6)

and $(U \times G) \times_U (U \times G) \cong U \times G \times G$. Note that $\langle q, q' \rangle : U \to P \times_X P$ factors as kh, where k is the map $U \times G \times G \to P \times_X P$ induced by the plotwise trivializations of P along q and q'. Now we can see that τ factors as



where $n : U \times G \times G \to U \times G$ is the smooth map $(u, g, g') \mapsto (u, (g')^{-1}g)$. Thus τ is a composite of smooth maps, and therefore is a plot. □

Diffeological principal G-bundles are a true generalization of classical principal G-bundles in the following sense. If M is a finite dimensional smooth manifold, G is a Lie group, and $\pi : P \to M$ is a classical principal G-bundle on M, then we can consider the map of diffeological space $\mathcal{D}_{\mathsf{Man}}(\pi) : \mathcal{D}_{\mathsf{Man}}(P) \to \mathcal{D}_{\mathsf{Man}}(M)$, given by the functor $\mathcal{D}_{\mathsf{Man}} : \mathsf{Man} \to \mathsf{Diff}$ from Example 1.2.5. It turns out that this map of diffeological spaces is a diffeological principal $\mathcal{D}_{\mathsf{Man}}(G)$-bundle, in fact more is true.

Proposition 1.3.8 ([Wal12, Theorem 3.1.7])**.** If M is a finite dimensional smooth manifold,

G is a Lie group, and $\mathsf{Prin}_G(M)$ denotes the groupoid of classical principal G-bundles over M, then the functor

$$\mathcal{D}_M : \mathsf{Prin}_G(M) \to \mathsf{DiffPrin}_G(M) \tag{1.7}$$

that assigns to a classical principal G-bundle π the corresponding diffeological principal $\mathcal{D}_{\mathsf{Man}}(G)$-bundle, is an equivalence of groupoids.

Remark 1.3.9. In what follows we will not use the notation $\mathcal{D}_{\mathsf{Man}}(M)$ to distinguish between a manifold and its corresponding diffeological space, but will instead rely on context.

Classically, principal G-bundles over a smooth manifold M are classified up to isomorphism by homotopy classes of maps from M to a classifying space BG. There has been some work [CW21], [MW17] extending this result to diffeology. Since diffeological spaces are so much more general than smooth manifolds, one must only consider numerable principal G-bundles to classify them in this sense as in the above references.

However, there is another way of classifying principal G-bundles over a smooth manifold M, that produces the whole groupoid of principal G-bundles, rather than just the isomorphism classes, see Example 1.5.13. One goal of this paper is to extend this idea to diffeological principal G-bundles. This will be achieved in Section 1.6. In order to classify diffeological principal G-bundles in this way, we must understand how to construct bundles from their cocycles and vice versa.

Definition 1.3.10. Given a diffeological space X and a diffeological group G, call a collection $g = \{g_{f_0}\}$ of smooth maps $g_{f_0} : U_{p_1} \to G$ indexed by maps of plots $f_0 : U_{p_1} \to U_{p_0}$ of X a **G-cocycle** if for every pair of composable plot maps of X

$$U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$$

it follows that

$$g_{f_0 f_1} = (g_{f_0} \circ f_1) \cdot g_{f_1}. \tag{1.8}$$

We call (2.9) the **diffeological G-cocycle condition**.

Given two G-cocycles, g, g', we say a collection $h = \{h_{p_0}\}$ of smooth maps $h_{p_0} : U_{p_0} \to G$ indexed by plots of X is a **morphism of G-cocycles** $h : g \to g'$ if for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots of X, it follows that

$$g'_{f_0} \cdot h_{p_1} = (h_{p_0} \circ f_0) \cdot g_{f_0}. \tag{1.9}$$

Remark 1.3.11. The definition of diffeological G-cocycles is reminiscent of classical G-cocycles (usually written g_{ij} for some cover $\mathcal{U} = \{U_i \subseteq M\}$ of a manifold), but also seems to have come from nowhere. In Section 1.6 we will see how one comes to this definition purely from the framework of higher topos theory.

Let $\mathsf{Coc}(X, G)$ denote the category whose objects are cocycles of X with values in G and whose morphisms are morphisms of cocycles. Composition is defined as follows. If $h : g \to g'$ and $h' : g' \to g''$ are morphisms of cocycles, then let $(h' \circ h)$ denote the morphism of cocycles defined plotwise by $(h' \circ h)_{p_0} = h'_{p_0} \cdot h_{p_0}$ for a plot $p_0 : U_{p_0} \to X$. Let us show that $(h' \circ h)$ is actually a morphism of cocycles. A morphism of cocycles $h : g \to g'$ implies that if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, then

$$g'_{f_0} h_{p_1} = (h_{p_0} \circ f_0) g_{f_0}$$

and $h' : g' \to g''$ implies that

$$g''_{f_0} h'_{p_1} = (h'_{p_1} \circ f_0) g'_{f_0}.$$

Thus

$$g''_{f_0} h'_{p_1} = (h'_{p_0} \circ f_0)(h_{p_0} \circ f_0) g_{f_0} h^{-1}_{p_1}.$$

and therefore

$$g''_{f_0} (h'_{p_1} \cdot h_{p_1}) = ((h'_{p_0} \cdot h_{p_0}) \circ f_0) g_{f_0}$$

This proves that $(h' \circ h)$ is a morphism of cocycles. Note that $\mathsf{Coc}(X, G)$ is a groupoid by taking $(h^{-1})_p = h_p^{-1}$.

Given a diffeological space X and a G-cocycle g on X, we wish to construct a diffeological principal G-bundle $\pi : P \to X$, such that we can recover the G-cocycle g by looking at plotwise trivializations of P. We will do this as follows. Consider the diffeological space

$$\widehat{P} = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \times G. \tag{1.10}$$

We label the elements of $\widehat{P}$ by (x_{p_0}, k_0), where $p_0 : U_{p_0} \to X$ is a plot, $x_{p_0} \in U_{p_0}$, and $k_0 \in G$. We write $(x_{p_1}, k_1) \sim (x_{p_0}, k_0)$ if there exists a map $f_0 : U_{p_1} \to U_{p_0}$ of plots such that $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1}) \cdot k_1$. This relation is reflexive and transitive, but not symmetric, so abuse notation by letting $\sim$ also denote the smallest equivalence relation containing $\sim$. In other words we say that $(x_{p_1}, k_1) \sim (x_{p_0}, k_0)$ if and only if there exists a finite zig-zag of plot maps

$$(x_{p_1}, k_1) \xrightarrow{f_0} (x_{q_0}, h_0) \xleftarrow{f_1} (x_{q_1}, h_1) \xrightarrow{f_2} \dots \xleftarrow{f_n} (x_{q_n}, h_n) \xrightarrow{f_{n+1}} (x_{p_0}, k_0), \tag{1.11}$$

such that $f_0(x_{p_1}) = x_{q_0}$, $h_0 = g_{f_0}(x_{p_1}) \cdot k_1$, $f_1(x_{q_1}) = x_{q_0}$, $h_0 = g_{f_1}(x_{q_1}) \cdot h_1$, $\dots$, $f_{n+1}(x_{q_n}) = x_{p_0}$, and $k_0 = g_{f_{n+1}}(x_{q_n}) \cdot h_n$. By setting f_0 or f_{n+1} equal to the identity one can obtain any kind of zig-zag from one of the form above.

Let $P = \widehat{P}/\sim$, and denote its elements by $[x_{p_0}, k_0]$. There is a smooth map $\pi : P \to X$ given by $\pi[x_{p_0}, k_0] = p_0(x_{p_0})$. This map is well-defined, because if $(x_{p_1}, k_1) \sim (x_{p_0}, k_0)$, then there is a finite zig-zag of plot maps connecting them as above, so

$$p_1(x_{p_1}) = q_0(f_0(x_{p_1})) = q_0(x_{q_0}) = q_0(f_1(x_{q_1})) = q_1(x_{q_1}) = \dots = q_n(x_{q_n}) = q_n(f_{n+1}(x_{q_n})) = p_0(x_{p_0}). \tag{1.12}$$

Thus $\pi : P \to X$ is well defined. We let $\pi = \mathsf{Cons}(g)$, short for construction.

As sets, we can think of $\widehat{P}$ as the objects of a category, and a morphism in this category looks like

$$(x_{p_0}, k_0) \xrightarrow{f_0} (x_{p_1}, k_1)$$

where $f_0 : U_{p_1} \to U_{p_0}$ is a plot map such that $f_0(x_{p_1}) = x_{p_0}$ and $k_1 = g_{f_0}(x_{p_1}) \cdot k_0$. Then $P \cong \pi_0 \widehat{P}$. This might be a helpful way to think about this construction, and we will say more about this observation in Section 1.6.

Proposition 1.3.12. Given a diffeological group G, a diffeological space X and a G-cocycle g on X, the map $\pi : P \to X$ where $\pi = \mathsf{Cons}(g)$ is a diffeological principal G-bundle.

Proof. First let us show that there is an action of G on P. Let the action be defined by $[x_{p_0}, k_0] \cdot g = [x_{p_0}, k_0 \cdot g]$. This action is well defined, as suppose a zig-zag of the form (1.11) identifies (x_{p_1}, k_1) with (x_{p_0}, k_0). Then the same zig-zag with h_i replaced with $h_i \cdot g$ will identify $(x_{p_1}, k_1 \cdot g)$ with $(x_{p_0}, k_0 \cdot g)$.

Now let us show that if $p_1 : U_{p_1} \to X$ is a plot, then there exists a G-equivariant diffeomorphism $\varphi_{p_1} : U_{p_1} \times G \to p_1^* P$. First note that

$$p_1^* P = \{(x_{p_1}, [x_{p_0}, k_0]) \in U_{p_1} \times P \,:\, p_1(x_{p_1}) = \pi[x_{p_0}, k_0]\}$$

So given a point $(x_{p_1}, [x_{p_0}, k_0]) \in p_1^* P$, we have a commutative diagram

$$\begin{array}{ccc} * & \xrightarrow{x_{p_0}} & U_{p_0} \\ {\scriptstyle x_{p_1}}\downarrow & \overset{x}{\searrow} & \downarrow{\scriptstyle p_0} \\ U_{p_1} & \xrightarrow[p_1]{} & X \end{array}$$

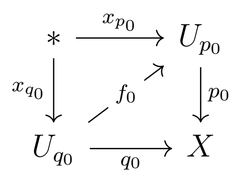
where $*$ denotes the plot $* \cong \mathbb{R}^0 \xrightarrow{x} X$. We can think of this diagram as a zig-zag

$$(x_{p_1}, g_{x_{p_1}}(*) g_{x_{p_0}}^{-1}(*) k_0) \xleftarrow{x_{p_1}} (*, g_{x_{p_0}}^{-1}(*) k_0) \xrightarrow{x_{p_0}} (x_{p_0}, k_0).$$

Thus we can identify

$$[x_{p_1}, g_{x_{p_1}}(*) g_{x_{p_0}}^{-1}(*) k_0] = [x_{p_0}, k_0].$$

Now let us define a map $\varphi_{p_1} : p_1^* P \to U_{p_1} \times G$ by $\varphi(x_{p_1}, [x_{p_0}, k_0]) = (x_{p_1}, g_{x_{p_1}}(*) g_{x_{p_0}}^{-1}(*) k_0)$. Let us show that this is well defined. First suppose that $(x_{q_0}, l_0) \sim (x_{p_0}, k_0)$ is identified by a single morphism, i.e. there exists a plot map $f_0 : U_{q_0} \to U_{p_0}$ such that $f_0(x_{q_0}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{q_0}) \cdot l_0$, then the following diagram commutes



However, we can also think of x_{p_0} and x_{q_0} as maps of plots. Thus from the cocycle condition (2.9), we have

$$g_{(f_0 \circ x_{q_0})}(*) = (g_{f_0} \circ x_{q_0})(*) \cdot g_{x_{q_0}}(*).$$

Now let us abuse notation for the rest of this proof by writing $g_{x_{p_0}}$ for $g_{x_{p_0}}(*)$. Thus we have

$$g_{x_{p_0}} = g_{f_0}(x_{q_0}) \cdot g_{x_{q_0}}. \tag{1.13}$$

Therefore

$$\varphi(x_{p_1}, [x_{p_0}, k_0]) = (x_{p_1}, g_{x_{p_1}} g_{x_{p_0}}^{-1} k_0) = (x_{p_1}, g_{x_{p_1}} g_{x_{p_0}}^{-1} g_{f_0}(x_{q_0}) l_0) = (x_{p_1}, g_{x_{p_1}} g_{x_{q_0}}^{-1} l_0) = \varphi(x_{p_1}, [x_{q_0}, l_0]).$$

Now if (x_{q_0}, l_0) and (x_{p_0}, k_0) are connected by an arbitrary finite zig-zag, then using the above argument on every morphism in the zig-zag shows that $\varphi(x_{p_1}, [x_{q_0}, l_0]) = \varphi(x_{p_1}, [x_{p_0}, k_0])$. So φ is well defined. It is also G-equivariant, as $\varphi(x_{p_1}, [x_{p_0}, k_0] \cdot g) = \varphi(x_{p_1}, [x_{p_0}, k_0 g]) = (x_{p_1}, g_{x_{p_1}} g_{x_{p_0}}^{-1} k_0 g) = (x_{p_1}, g_{x_{p_1}} g_{x_{p_0}}^{-1} k_0) \cdot g$.

We define an inverse $\varphi_{p_1}^{-1} : U_{p_1} \times G \to p_1^* P$ by $(x_{p_1}, g) \mapsto (x_{p_1}, [x_{p_1}, g])$. This is clearly G-equivariant, and it is easy to see that $\varphi_{p_1} \circ \varphi_{p_1}^{-1} = 1_{U_{p_1} \times G}$ and $\varphi_{p_1}^{-1} \circ \varphi_{p_1} = 1_{p_1^* P}$. Thus P is plotwise trivial.

Now let us show that G acts on the fibers of π freely and transitively. However this is immediate, as a fiber of π is in particular a pullback:

$$\begin{array}{ccc} \pi^{-1}(x) & \longrightarrow & P \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ * & \xrightarrow[x]{} & X \end{array}$$

and every constant map $* \to X$ is a plot, and we've already shown that for any plot this pullback is trivial, and thus $\pi^{-1}(x) \cong * \times G \cong G$, which acts freely and transitively on itself by right multiplication. Thus by Lemma 1.3.7, the action map $P \times G \to P \times_X P$ is a diffeomorphism. Thus by Lemma 1.3.6, $P \xrightarrow{\pi} X$ is a diffeological principal G-bundle. □

Remark 1.3.13. As it is convenient for the rest of this section, we rename the plotwise trivialization $\varphi_{p_0}^{-1} : U_{p_0} \times G \to p_0^* \mathsf{Cons}(g)$ to φ_{p_0}.

Proposition 1.3.14. Given a diffeological space X, a diffeological group G, G-cocycles g, g' on X, with corresponding diffeological principal G-bundles $P = \mathsf{Cons}(g)$ and $P' = \mathsf{Cons}(g')$ and a morphism $h : g \to g'$ of G-cocycles, there is an induced morphism $\widetilde{h} = \mathsf{Cons}(h) : P \to P'$ of diffeological principal G-bundles. Furthermore, for every plot $p_0 : U_{p_0} \to X$, if $\widetilde{h} : P \to P'$ is a morphism of diffeological principal G-bundles, then we obtain a commutative diagram

$$\begin{array}{ccc} U_{p_0} \times G & \xrightarrow{\widetilde{h}_{p_0}} & U_{p_0} \times G \\ {\scriptstyle \varphi_{p_0}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi'_{p_0}} \\ p_0^* P & \xrightarrow{\widehat{h}_{p_0}} & p_0^* P' \\ {\scriptstyle \psi_{p_0}}\big\downarrow & & \big\downarrow{\scriptstyle \psi'_{p_0}} \\ P & \xrightarrow{\widetilde{h}} & P' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array} \tag{1.14}$$

where $\widetilde{h}_{p_0}(x_{p_0}, k_0) = (x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0)$.

Proof. Define a map $\widetilde{h} = \mathsf{Cons}(h) : P \to P'$ as follows. Given a point $[x_{p_0}, k_0]$ in P, let $\widetilde{h}([x_{p_0}, k_0]) = [x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0]$. Let us show that this map is well defined. Suppose that

$[x_{p_1}, k_1] = [x_{p_0}, k_0]$, we want to show that $[x_{p_1}, h_{p_1}(x_{p_1})k_1] = [x_{p_0}, h_{p_0}(x_{p_0})k_0]$. Suppose that (x_{p_1}, k_1) and (x_{p_0}, k_0) are connected by a single morphism, i.e. there exists a plot map $f_0 : U_{p_1} \to U_{p_0}$ such that $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1})k_1$. Then as elements of P', we have

$$\begin{aligned} \widetilde{h}([x_{p_0}, k_0]) &= [x_{p_0}, h_{p_0}(x_{p_0})k_0] \\ &= [x_{p_0}, h_{p_0}(f_0(x_{p_1}))g_{f_0}(x_{p_1})k_1] \\ &= [x_{p_0}, g'_{f_0}(x_{p_1})h_{p_1}(x_{p_1})k_1] \\ &= [x_{p_1}, h_{p_1}(x_{p_1})k_1] \\ &= \widetilde{h}([x_{p_1}, k_1]), \end{aligned} \tag{1.15}$$

where the third equality follows from (2.10). Thus if (x_{p_1}, k_1) and (x_{p_0}, k_0) are connected by an arbitrary finite zig-zag, then using the above argument on every morphism in the zig-zag shows that $\widetilde{h}([x_{p_1}, k_1]) = \widetilde{h}([x_{p_0}, k_0])$. Thus $\widetilde{h}$ is well-defined, and clearly smooth. It is also G-equivariant, as

$$\begin{aligned} \widetilde{h}([x_{p_0}, k_0] \cdot g) &= \widetilde{h}([x_{p_0}, k_0 g]) \\ &= [x_{p_0}, h_{p_0}(x_{p_0})k_0 g] \\ &= [x_{p_0}, h_{p_0}(x_{p_0})k_0] \cdot g \\ &= \widetilde{h}([x_{p_0}, k_0]) \cdot g. \end{aligned} \tag{1.16}$$

Thus $\widetilde{h}$ is a morphism of diffeological principal G-bundles over X.

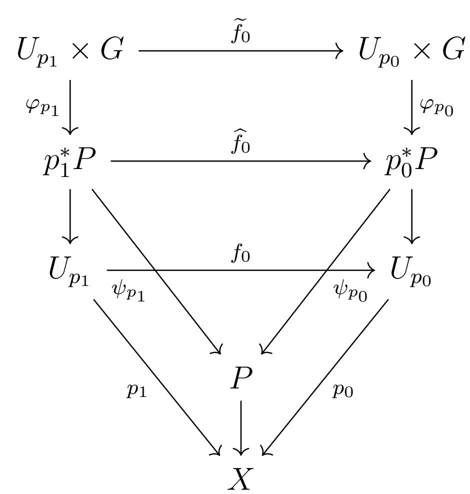
Now given a plot $p_0 : U_{p_0} \to X$, we obtain the commutative diagram (1.14) by pulling back $\widetilde{h}$ along p_0 and the plotwise trivialization φ. We need only show that $\widetilde{h}_{p_0}(x_{p_0}, k_0) = (x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0)$. Since $\widetilde{h}_{p_0} = (\varphi'_{p_0})^{-1}\widehat{h}_{p_0}\varphi_{p_0}$ we have

$$\begin{aligned} \widetilde{h}_{p_0}(x_{p_0}, k_0) &= (\varphi'_{p_0})^{-1}\widehat{h}_{p_0}\varphi_{p_0}(x_{p_0}, k_0) \\ &= (\varphi'_{p_0})^{-1}\widehat{h}_{p_0}(x_{p_0}, [x_{p_0}, k_0]) \\ &= (\varphi'_{p_0})^{-1}(x_{p_0}, [x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0]) \\ &= (x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0), \end{aligned} \tag{1.17}$$

where we have used the plotwise trivialization from the proof of Proposition 1.3.12 and the convention of Remark 1.3.13. □

The content of Propositions 1.3.12 and 1.3.14 can be summarized by saying that we have a functor $\mathsf{Cons} : \mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X)$. Our goal now is to show that this functor is an equivalence, namely that it is fully faithful and essentially surjective.

Let us first show essential surjectivity. Suppose that $\pi : P \to X$ is a diffeological principal G-bundle. Suppose that for every plot $p_0 : U_{p_0} \to X$ we choose a trivialization $\varphi_{p_0} : U_{p_0} \times G \to p_0^* P$, which is a G-equivariant diffeomorphism. Let $f_0 : U_{p_1} \to U_{p_0}$ be a map of plots. Then we obtain the following commutative diagram



(1.18)

where $\widehat{f}_0$ is obtained from f_0 by pullback (taking pullbacks is functorial), and $\widetilde{f}_0$ is defined as $\widetilde{f}_0 = \varphi_{p_0}^{-1} \widehat{f}_0 \varphi_{p_1}$. Given $(x_{p_1}, k) \in U_{p_1} \times G$, it follows that

$$\widetilde{f}_0(x_{p_1}, k) = (f_0(x_{p_1}), g_{f_0}(x_{p_1}) \cdot k) \tag{1.19}$$

for some map $g_{f_0} : U_{p_1} \to G$. This is because both $\widehat{f}_0$ and $\widetilde{f}_0$ are maps of diffeological principal G-bundles, and therefore G-equivariant.

If furthermore we have a pair of composable plot maps $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$, then

$$(\widetilde{f_0 f_1})(x_{p_2}, k) = ((f_0 f_1)(x_{p_2}), g_{f_0 f_1}(x_{p_2}) \cdot k) = ((f_0 f_1)(x_{p_2}), g_{f_0}(f_1(x_{p_2}) \cdot g_{f_1}(x_{p_2}) \cdot k) = \widetilde{f}_0 \widetilde{f}_1(x_{p_2}, k). \tag{1.20}$$

From this we obtain the cocycle condition (2.9).

Thus given a pair of a diffeological principal G-bundle $\pi : P \to X$, and a choice φ of plotwise trivializations, we obtain a G-cocycle g. Denote this cocycle by $g = \mathsf{Ext}(P, \varphi)$ for extracting the cocycle from the principal bundle. We wish to show that there exists an isomorphism $\tau : \mathsf{Cons}(\mathsf{Ext}(P, \varphi)) \to P$ of diffeological principal G-bundles.

Let $Q = \mathsf{Cons}(\mathsf{Ext}(P, \varphi))$ and let $\widetilde{Q} = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \times G$. Let $q : \widetilde{Q} \to Q$ denote the quotient map. Let us define a set function $\tau : Q \to P$ as follows. Given a point $[x_{p_1}, k_1] \in Q$, let $\tau([x_{p_1}, k_1])$ be $\psi_{p_1} \varphi_{p_1}(x_{p_1}, k_1)$, where ψ_{p_1} and φ_{p_1} are the maps given in (1.18). Let us show that this function is well defined. Suppose that $[x_{p_1}, k_1] = [x_{p_0}, k_0]$ are connected by a single plot map f_0, i.e. that $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1}) \cdot k_1$. Then by (1.18) and (1.19) we have

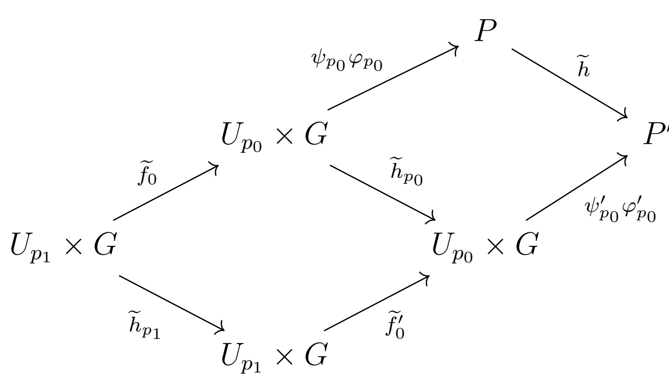
$$\begin{aligned} \tau([x_{p_0}, k_0]) &= \psi_{p_0} \varphi_{p_0}(x_{p_0}, k_0) \\ &= \psi_{p_0} \varphi_{p_0}(f_0(x_{p_1}), g_{f_0}(x_{p_1}) \cdot k_1) \\ &= \psi_{p_0} \varphi_{p_0} \widetilde{f}_0(x_{p_1}, k_1) \\ &= \psi_{p_1} \varphi_{p_1}(x_{p_1}, k_1) \\ &= \tau([x_{p_1}, k_1]). \end{aligned} \tag{1.21}$$

Thus if (x_{p_1}, k_1) and (x_{p_0}, k_0) are connected by an arbitrary finite zig-zag, then using the above argument on every morphism in the zig-zag shows that τ is well defined. To see that it is smooth, since q is a submersion, by Lemma 1.2.8 it is enough to show that $\tau q = \widetilde{\tau}$ is smooth. But this is just the map $\widetilde{\tau} : \coprod_{p_0} U_{p_0} \times G \to P$ that sends (x_{p_0}, k_0) to $\psi_{p_0} \varphi_{p_0}(x_{p_0})$ and these are smooth maps, thus $\widetilde{\tau}$ is smooth and therefore τ is smooth. Since ψ_{p_0} and φ_{p_0} are G-equivariant, it is easy to see that τ is as well. Thus τ is a map of diffeological principal G-bundles, so by Lemma 1.3.4, it is an isomorphism. Therefore the functor Cons :

$\mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X)$ is essentially surjective.

Now let us show that Cons is fully faithful. Suppose that h and h' are maps of cocycles $h, h' : g \to g'$, and let $\widetilde{h} = \mathsf{Cons}(h)$ and $\widetilde{h}' = \mathsf{Cons}(h')$. Suppose that $\widetilde{h} = \widetilde{h}'$. Then using the canonical plotwise trivialization φ of $P = \mathsf{Cons}(g)$ and $P' = \mathsf{Cons}(g')$ we obtain two copies of (1.14) for $\widetilde{h}$ and $\widetilde{h}'$. Since they are equal as maps, this implies that for every plot $p_0 : U_{p_0} \to X$, $\widetilde{h}_{p_0} = \widetilde{h}'_{p_0}$. But by Proposition 1.3.14, $\widetilde{h}_{p_0}$ determines h_{p_0} and $\widetilde{h}'_{p_0}$ determines h'_{p_0}. Thus $h = h'$. Thus Cons is a faithful functor.

Now suppose that $P = \mathsf{Cons}(g)$ and $P' = \mathsf{Cons}(g')$ and $\widetilde{h} : P \to P'$ is a map of diffeological principal G-bundles. We wish to construct a map $h : g \to g'$ of cocycles such that $\mathsf{Cons}(h) = \widetilde{h}$. We obtain such a morphism h of cocycles by pulling $\widetilde{h}$ along the canonical plotwise trivialization φ of $\mathsf{Cons}(g)$ as in (1.14), so that for every plot p_0 of X, we have $\widetilde{h}_{p_0}(x_{p_0}, k) = (x_{p_0}, h_{p_0}(x_{p_0}) \cdot k_0)$. One can check that h is a morphism of G-cocycles by chasing around the left hand square in the following commutative diagram



(1.22)

Now we wish to show that $\mathsf{Cons}(h) = \widetilde{h}$. Let $x \in X$, and consider the plot $x : * \to X$ that sends the point to x. If we let $p_0 = x$ in (1.14), then $U_{p_0} \times G \cong G$. Let $p = \psi_x \varphi_x(e_G)$. Then $\widetilde{h}(p) = \psi'_x \varphi'_x \widetilde{h}_x(e_G) = \psi'_x \varphi'_x(h_x(e_G))$. However $\mathsf{Cons}(h)(p)$ is also determined plotwise by h_x, i.e. $\mathsf{Cons}(h)(p) = \psi'_x \varphi'_x(h_x(e_G)) = \widetilde{h}p$. Since $\widetilde{h}$ and $\mathsf{Cons}(h)$ are G-equivariant, and the action of G on $\pi^{-1}(x)$ is transitive, this implies that $\widetilde{h}$ and $\mathsf{Cons}(h)$ agree on $\pi^{-1}(x)$. Since x was arbitrary, this implies that $\widetilde{h} = \mathsf{Cons}(h)$. Thus Cons is a full functor. In summary we have proven the following.

Theorem 1.3.15. Given a diffeological space X and a diffeological group G, the functor

$$\mathsf{Cons} : \mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X) \tag{1.23}$$

is an equivalence of groupoids.

Remark 1.3.16. Any choice of plotwise trivialization φ with $\varphi_{p_0} : U_{p_0} \times G \cong p_0^* P$ gives a quasi-inverse to the functor Cons above.

Remark 1.3.17. Weaker, but somewhat similar results to Theorem 1.3.15 have been proven in [WW14] and [Ahm23]. But notice in these papers that the correspondence was only proven on the level of isomorphism classes, with different notions of diffeological Čech cohomology, and they only establish a bijection of sets. We say more about the other notions of diffeological Čech cohomology in the literature in Section 1.5.3.

1.4 Diffeological Spaces as Concrete Sheaves

A major development in the theory of diffeological spaces was made in [BH11], which showed that the category of diffeological spaces is equivalent to the category of concrete sheaves on the site of open subsets of cartesian spaces. Here we introduce the theory necessary to understand this result, and to prepare the ground for Section 1.5. In Section 1.7, we will show that the category of concrete sheaves on several various sites of interest are equivalent, giving a justification for Definition 2.2.2. Nothing in this section is new, but it may be helpful to those less familiar with topos theory.

Definition 1.4.1. Let $\mathcal{C}$ be a category, and $U \in \mathcal{C}$. A **family of morphisms** over U is a set of morphisms $r = \{r_i : U_i \to U\}_{i \in I}$ in $\mathcal{C}$ with codomain U.

A **refinement** of a family of morphisms $t = \{t_j : V_j \to U\}_{j \in J}$ over U consists of a family of morphisms $r = \{r_i : U_i \to U\}_{i \in I}$, a function $\alpha : I \to J$ and for each $i \in I$ a map

$f_i : U_i \to V_{\alpha(i)}$ making the following diagram commute:

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & V_{\alpha(i)} \\ & {\scriptstyle r_i}\searrow \quad \swarrow{\scriptstyle t_{\alpha(i)}} & \\ & U & \end{array} \tag{1.24}$$

If r is a refinement of t, with maps $f_i : U_i \to V_{\alpha(i)}$, then we write $f : r \to t$.

We wish to consider added structure to a category that generalizes the notion of a topology. We will use families of morphisms as a generalized notion of ”open cover.”

Definition 1.4.2. A **collection of families** j on a category $\mathcal{C}$ consists of a set $j(U)$ for each $U \in \mathcal{C}$, whose elements $\{r_i : U_i \to U\} \in j(U)$ are families of morphisms over U.

We call a collection of families j on $\mathcal{C}$ a **coverage** if it satisfies the following property: for every $\{r_i : U_i \to U\} \in j(U)$, and every map $g : V \to U$ in $\mathcal{C}$, then there exists a family $\{t_j : V_j \to V\} \in j(V)$ such that gt_j factors through some r_i. Namely for every t_j there exists some i and some map $s_j : V_j \to U_i$ making the following diagram commute:

$$\begin{array}{ccc} V_j & \xrightarrow{s_j} & U_i \\ {\scriptstyle t_j}\downarrow & & \downarrow{\scriptstyle r_i} \\ V & \xrightarrow[g]{} & U \end{array} \tag{1.25}$$

The families $\{r_i : U_i \to U\} \in j(U)$ are called **covering families** over U. If a map $r_i : U_i \to U$ belongs to a covering family $r \in j(U)$, then we say that r_i is a **covering map**.

If $\mathcal{C}$ is a category, and j is a coverage on $\mathcal{C}$, then we call the pair $(\mathcal{C}, j)$ a **site**.

Example 1.4.3. Let X be a topological space and let $\mathcal{O}(X)$ denote the partially ordered set of open subsets of X. Let j_X denote the collection of familes on $\mathcal{O}(X)$ such that $j_X(U)$ is the set of all open covers of U, namely $\{U_i \subseteq U\} \in j_X(U)$ if $\bigcup_i U_i = U$.

This collection of families is a coverage, for suppose we have fixed an open cover $\{U_i \subseteq U\}$ and an open subset $V \subseteq U$. Then $\{V \cap U_i \subseteq V\}$ is an open cover of V, and $V \cap U_i \subseteq U_i$. We call j_X the **open cover coverage** of X.

Example 1.4.4. Define a collection of families j_{open} on Man as follows: For $M \in \mathsf{Man}$, let $j_{\mathrm{open}}(M)$ denote the collection of open covers of M. Then j_{open} is a coverage. Indeed if $\{U_i \subseteq M\}$ is an open cover and $f : N \to M$ is a smooth map, then $\{f^{-1}(U_i) \subseteq N\}$ is an open cover of N satisfying (2.13).

Now consider the following full subcategories

$$\mathsf{Cart} \hookrightarrow \mathsf{Open} \hookrightarrow \mathsf{Man}.$$

Where Cart is the full subcategory whose objects are cartesian spaces and Open is the full subcategory whose objects are diffeomorphic to open subsets of a cartesian space. The collection of families j_{open} can be restricted to Open and is clearly a coverage there as well.

Notice however that if we restrict j_{open} to Cart, and U is a cartesian space, then an open cover $\{U_i \subseteq U\}$ is a covering family for j_{open} if and only if it is a cartesian open cover, otherwise it could not be a collection of morphisms in Cart. For Man and Open any open cover will do. However if $\{U_i \subseteq U\}$ is a cartesian open cover and $f : V \to U$ is a smooth map, there is no reason that $\{f^{-1}(U_i) \subseteq V\}$ will be a cartesian open cover. However as we will see in Example 1.4.5, every open cover can be refined by a cartesian open cover, and thus j_{open} is indeed a coverage on Cart.

Example 1.4.5. Define a collection of families j_{good} on Man as follows: For $M \in \mathsf{Man}$, let $j_{\mathrm{good}}(M)$ denote the collection of good open covers as in Definition 2.2.1 of M. Let us show that the good covers form a coverage. If $\{U_i \subseteq M\}$ is a good cover and $g : N \to M$ a smooth map, then $\{g^{-1}(U_i) \subseteq N\}$ is an open cover, but not necessarily good. By [BT+82, Corollary 5.2], this open cover can be refined by a good open cover $\{W_k \subseteq N\}$ so that for every W_k in the good open cover, there exists a U_i such that $W_k \subseteq g^{-1}(U_i)$, and thus the following diagram commutes:

$$\begin{array}{ccccc} W_k & \hookrightarrow & g^{-1}(U_i) & \xrightarrow{g|_{g^{-1}(U_i)}} & U_i \\ \downarrow & & & & \downarrow \\ N & & \xrightarrow[g]{} & & M \end{array}$$

Thus j_{good} is a coverage on Man. Similarly it defines a coverage on Cart and Open.

Definition 1.4.6. Let Smooth denote a site of the form $(\mathcal{C}, j)$ with $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$. We will call any such site a **smooth site**.

Example 1.4.7. We note here that the collection of families j_{sub} of subductions on the category Diff of diffeological spaces is a coverage, because the pullback of a subduction is a subduction. We will not use this observation in this section, but it will come up in Section 1.5.2 when we talk about diffeological categories.

Coverages are those collections of families with the least amount of structure with which we can define sheaves on $\mathcal{C}$.

Definition 1.4.8. A **presheaf** on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$. An element $x \in F(U)$ for an object $U \in \mathcal{C}$ is called a **section** over U. If $f : U \to V$ is a map in $\mathcal{C}$, and $x \in F(V)$, then we sometimes denote $F(f)(x)$ by $x|_U$. If $\{r_i : U_i \to U\}_{i \in I}$ is a covering family, then a **matching family** is a collection $\{x_i\}_{i\in I}$, $x_i \in F(U_i)$, such that given a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} V & \xrightarrow{g} & U_j \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r_j} \\ U_i & \xrightarrow[r_i]{} & U \end{array}$$

then $F(f)(x_i) = F(g)(x_j)$ for all $i, j \in I$. An **amalgamation** x for a matching family $\{x_i\}$ is a section $x \in F(U)$ such that $x_i|_U = x$ for all i.

Definition 1.4.9. Given a family of morphisms $r = \{r_i : U_i \to U\}$ in a category $\mathcal{C}$, we say that a presheaf $F : \mathcal{C}^{op} \to \mathsf{Set}$ is a **sheaf on** r if every matching family $\{s_i\}$ of F over r has a unique amalgamation. If j is a coverage on a category $\mathcal{C}$, we call F a **sheaf** on $(\mathcal{C}, j)$ if it is a sheaf on every covering family of j. Let $\mathsf{Sh}(\mathcal{C}, j)$ denote the full subcategory of $\mathsf{Pre}(\mathcal{C})$ whose objects are sheaves on $(\mathcal{C}, j)$.

Remark 1.4.10. If $(\mathcal{C}, j)$ is a site that has pullbacks, then we can equivalently express the condition for F being a sheaf as requiring that for every $U \in \mathcal{C}$ and every covering family

$\{U_i \to U\} \in j(U)$, the diagram:

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \tag{1.26}$$

is an equalizer. This is how the sheaf condition is often presented in the literature.

Example 1.4.11. Given a smooth manifold M, the presheaf

$$U \mapsto C^\infty(U, M)$$

which we denote by either $\underline{M}$ or just M, is a sheaf on Smooth. Another important example of a sheaf is

$$U \mapsto \Omega^k(U)$$

for any $k \geq 0$. If V is a cartesian space, we will denote its image under the Yoneda embedding by yV. This is the presheaf

$$U \mapsto C^\infty(U, V)$$

and as above is a sheaf. We call a site $(\mathcal{C}, j)$ where every representable presheaf is a sheaf **subcanonical**. It is not hard to see that Smooth is a subcanonical site.

We wish to single out those sheaves that are in some sense a set with extra structure.

Definition 1.4.12. A site $(\mathcal{C}, j)$ is **concrete** if:

1. it is subcanonical,
2. it has a terminal object $*$,
3. the functor $\mathcal{C}(*, -) : \mathcal{C} \to \mathsf{Set}$ is faithful, and
4. for every covering family $\{U_i \to U\}$, the family of maps $\mathcal{C}(*, U_i) \to \mathcal{C}(*, U)$ is **jointly surjective**, namely the map $\coprod_i \mathcal{C}(*, U_i) \to \mathcal{C}(*, U)$ is surjective.

It is not hard to show that all of the smooth sites are concrete.

Definition 1.4.13. If $(\mathcal{C}, j)$ is a concrete site and F is a presheaf, then we call $F(*)$ its **underlying set**, and for any $U \in \mathcal{C}$ there always exists a map

$$\phi_U : F(U) \to \mathsf{Set}(\mathcal{C}(*, U), F(*))$$

defined by $\phi_U(x) = (u \mapsto F(u)(x))$. We say a sheaf F is **concrete** if for every object $U \in \mathcal{C}$, the function ϕ_U is injective. Let $\mathsf{ConSh}(\mathcal{C}, j)$ denote the full subcategory of concrete sheaves on a concrete site $(\mathcal{C}, j)$.

Example 1.4.14. For any smooth manifold M, the sheaf $\underline{M}$ on Smooth is concrete. This is equivalent to saying that for every $U \in \mathsf{Smooth}$ the function

$$\phi_U : C^\infty(U, M) \to \mathsf{Set}(U(*), M(*))$$

is injective, which is the same thing as saying that the set of smooth maps from U to M is a subset of all set functions from U to M.

Note that the sheaf Ω^k is not concrete on Smooth. Indeed if $U \in \mathsf{Smooth}$, then ϕ_U takes the form

$$\phi_U : \Omega^k(U) \to \mathsf{Set}(U(*), \Omega^k(*))$$

but $\Omega^k(*) = \{0\}$ is the zero vector space, thus $\mathsf{Set}(U(*), \{0\}) = *$ is the singleton set. Since in general $\Omega^k(U)$ is nontrivial, this shows that Ω^k is not concrete.

Theorem 1.4.15 ([BH11, Proposition 24])**.** The category Diff' of classical diffeological spaces is equivalent to the category of concrete sheaves on Open with the open cover coverage,

$$\mathsf{Diff}' \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\text{open}}).$$

However, the proof of [BH11, Proposition 24] can be applied nearly word for word to prove the following.

Theorem 1.4.16. The category Diff of diffeological spaces as defined in Definition 2.2.2 is equivalent to the category of concrete sheaves on Cart with the good open cover coverage

$$\mathsf{Diff} \simeq \mathsf{ConSh}(\mathsf{Cart}, j_{\text{good}}).$$

Remark 1.4.17. In Section 1.7 we will prove $\mathsf{Diff} \simeq \mathsf{Diff}'$ by showing that $\mathsf{ConSh}(\mathsf{Cart}, j_{\text{good}}) \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\text{open}})$.

Theorem 1.4.16 allows us to make a perspective shift. Constructions made in Diff can be compared with already defined notions of sheaves. For example a differential k-form ω on a diffeological space X [Igl13, Article 6.28], is precisely a map

$$X \xrightarrow{\omega} \Omega^k$$

of sheaves on Cart. This viewpoint on diffeological spaces, namely as concrete sheaves on Cart, will also be the starting point for Section 1.5, where we consider the fully faithful embedding of presheaves into simplicial presheaves. Since concrete sheaves are in particular presheaves, this means that there is a fully faithful embedding of diffeological spaces into simplicial presheaves, where we have a powerful homotopy theory to leverage.

1.5 Smooth Higher Stacks

1.5.1 Model Structures on Simplicial Presheaves

For this section, we assume the reader is comfortable with simplicial homotopy theory as in [GJ12] and model categories as in [Hir09].

Definition 1.5.1. Let sSet denote the category of simplicial sets, and $\mathsf{sPre}(\mathsf{Cart})$ the category whose objects are functors $X : \mathsf{Cart}^{op} \to \mathsf{sSet}$ and whose morphisms are natural transformations. We call such functors **simplicial presheaves**.

There is a fully faithful embedding $\mathsf{Set} \hookrightarrow \mathsf{sSet}$, which we denote by $S \mapsto {}^cS$, where $({}^cS)_n = S$ for all $n \geq 0$, and all of the face and degeneracy maps are the identity on S. Similarly there is a fully faithful embedding $\mathsf{Pre}(\mathsf{Cart}) \hookrightarrow \mathsf{sPre}(\mathsf{Cart})$, which we also denote by $F \mapsto {}^cF$, such that $({}^cF)(U) = {}^cF(U)$ for all $U \in \mathsf{Cart}$. We call simplicial presheaves of this form **discrete simplicial presheaves**. This functor has a left adjoint $\pi_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$, defined objectwise by

$$(\pi_0 X)(U) = \operatorname{coeq}\left(X(U)_0 \leftleftarrows X(U)_1 \right),$$

and a right adjoint $(-)_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$ defined objectwise by $X_0(U) = X(U)_0$. Note that limits and colimits in $\mathsf{sPre}(\mathsf{Cart})$ are computed objectwise.

There is also a functor $(-)_c : \mathsf{sSet} \to \mathsf{sPre}(\mathsf{Cart})$ defined objectwise by $X_c(U) = X$ for every $U \in \mathsf{Cart}$. We call simplicial presheaves of this form **constant simplicial presheaves**. The category of simplicial presheaves on Cart is simplicially enriched. Let X and Y be simplicial presheaves, then let $\underline{\mathsf{sPre}(\mathsf{Cart})}(X, Y)$ denote the simplicially-enriched Hom, defined degreewise by

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(X, Y)_n = \mathsf{sPre}(\mathsf{Cart})(X \times \Delta^n_c, Y).$$

Compare this with the simplicial mapping space for simplicial sets, namely if X and Y are simplicial sets, then let $\underline{\mathsf{sSet}}(X, Y)$ denote the simplicial set defined degreewise by $\underline{\mathsf{sSet}}(X, Y)_n = \mathsf{sSet}(X \times \Delta^n, Y)$.

If K is a simplicial set and X is a simplicial presheaf, then let X^K denote the simplicial presheaf which is defined objectwise by $(X^K)(U) = \underline{\mathsf{sSet}}(K, X(U))$. Then $\mathsf{sPre}(\mathsf{Cart})$ is tensored and cotensored over sSet in the sense that for simplicial presheaves X and Y and simplicial set K, there is the following natural isomorphism

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(X \times K_c, Y) \cong \underline{\mathsf{sPre}(\mathsf{Cart})}(X, Y^K).$$

The category $\mathsf{sPre}(\mathsf{Cart})$ admits many model structures. Here we will discuss two of them. Say a map $f : X \to Y$ of simplicial presheaves is an **objectwise weak equivalence** if $f : X(U) \to Y(U)$ is a weak equivalence of simplicial sets for every $U \in \mathsf{Cart}$. Similarly a **objectwise fibration** is a map $f : X \to Y$ of simplicial presheaves such that $f : X(U) \to Y(U)$ is a Kan fibration of simplicial sets for every $U \in \mathsf{Cart}$.

Theorem 1.5.2 ([BK72, Page 314])**.** There is a cofibrantly generated, simplicial model structure, which we call the **projective model structure** or **Bousfield-Kan model structure** on $\mathsf{sPre}(\mathsf{Cart})$, whose weak equivalences are the objectwise weak equivalences, and whose fibrations are the objectwise fibrations.

Remark 1.5.3. In fact, the projective model structure makes $\mathsf{sPre}(\mathsf{Cart})$ a combinatorial model category, see [Lur09, Section A.2.6].

Remark 1.5.4. There is a Quillen equivalent model structure on simplicial presheaves where the cofibrations and weak equivalences are objectwise, which is called the injective or Heller model structure. See [Bla01] for an overview of the different model structures on simplicial presheaves.

As is often the case with model structures, while the descriptions of weak equivalences and fibrations in the projective model structure are convenient, the cofibrations of the projective model structure are less simple to describe. However, the following result gives a sufficient condition for a simplicial presheaf to be cofibrant.

Theorem 1.5.5 ([Dug01, Corollary 9.4])**.** A simplicial presheaf X is cofibrant in the projective model structure on simplicial presheaves if

1. X is degreewise a coproduct of representables, i.e. $X_n = \coprod_{i \in I} yU_i$ for every $n \geq 0$,

2. X is split, in the sense that as a functor $X : \mathcal{C}^{op} \to \mathsf{sSet}$ it factors through the category sSet_{nd} whose objects are simplicial sets and whose morphisms are those maps of simplicial sets that map non-degenerate simplices to non-degenerate simplices.

We say that X is a **projective cofibrant** simplicial presheaf.

Corollary 1.5.6. If $U \in \mathsf{Cart}$, then ${}^c yU$ is a projective cofibrant simplicial presheaf on Cart.

Example 1.5.7. If M is a finite dimensional smooth manifold, and $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover, then consider the simplicial presheaf $\check{C}(\mathcal{U})$ defined in degree n by

$$\check{C}(\mathcal{U})_n = \coprod_{i_0, \dots, i_n} y\left(U_{i_0} \cap \cdots \cap U_{i_n}\right),$$

with face and degeneracy maps given by inclusions of open sets. Since $\mathcal{U}$ is a good open cover, $\check{C}(\mathcal{U})$ is a projective cofibrant simplicial presheaf. We call it the **Čech nerve** on $\mathcal{U}$. There is a canonical map

$$\check{C}(\mathcal{U}) \xrightarrow{\pi} {}^c M,$$

of simplicial presheaves on Cart. However this map is not an objectwise weak equivalence in general.

If $U \in \mathsf{Cart}$, and $\mathcal{U} = \{U_i\}_{i \in I}$ is a good open cover of U, then we can consider the canonical map

$$\check{C}(\mathcal{U}) \xrightarrow{\pi} {}^c yU.$$

Let $\check{C}$ denote the class of such maps as U varies over Cart and $\mathcal{U}$ varies over good open covers of U.

Definition 1.5.8. Let $\mathsf{s\check{P}re}(\mathsf{Cart})$ denote the left Bousfield localization of the projective model structure on $\mathsf{sPre}(\mathsf{Cart})$ at the class of maps $\check{C}$. We call this the **Čech model structure** on $\mathsf{sPre}(\mathsf{Cart})$.

The Čech model structure is described in greater detail in [DHI04, Appendix A]. Since $\mathsf{s\check{P}re(Cart)}$ is a left Bousfield localization of the projective model structure, it inherits the same cofibrations, and therefore cofibrant objects. We call its weak equivalences the Čech weak equivalences. Note that any objectwise weak equivalence of simplicial presheaves will be a Čech weak equivalence.

We call the fibrant objects of this model structure **∞-stacks** on Cart. They can be characterized as follows.

Proposition 1.5.9. A simplicial presheaf X on Cart is an ∞-stack on Cart if and only if it is projective fibrant (objectwise a Kan complex), and if for every $U \in \mathsf{Cart}$ and good open cover $\mathcal{U}$ of U, the map

$$\underline{\mathsf{sPre(Cart)}}(yU, X) \to \underline{\mathsf{sPre(Cart)}}(\check{C}(\mathcal{U}), X), \tag{1.27}$$

is a weak equivalence of simplicial sets. We say that X satisfies **Čech descent**.

Proof. This follows from the definition of left Bousfield localization. □

By a simplicially-enriched version of the Yoneda Lemma, $\underline{\mathsf{sPre(Cart)}}(yU, X) \cong X(U)$. We wish to better understand the right hand side of (1.27). To do this we will exploit the following result.

Lemma 1.5.10. Let X be a simplicial presheaf. Then

$$X \cong \int^{[n] \in \Delta^{op}} \Delta^n_c \times {}^c X_n,$$

where the colimit is taken in the category of simplicial presheaves, Δ^n_c is the constant simplicial presheaf on Δ^n and ${}^c X_n$ is the discrete simplicial presheaf on the presheaf X_n.

Proof. This follows from the corresponding fact for simplicial sets [GJ12, Lemma I.2.1]. □

Thus we have

$$\check{C}(\mathcal{U}) \cong \int^{[n]\in\Delta^{op}} \Delta^n_c \times \coprod_{i_0\ldots i_n} yU_{i_0\ldots i_n}.$$

This implies that

$$\begin{aligned}
\underline{\mathsf{sPre(Cart)}}(\check{C}(\mathcal{U}), X) &\cong \int_{[n]} \underline{\mathsf{sPre(Cart)}}(\Delta^n_c \times \coprod_{i_0\ldots i_n} yU_{i_0\ldots i_n}, X) \\
&\cong \int_{[n]} \prod_{i_0\ldots i_n} \underline{\mathsf{sPre(Cart)}}\left(yU_{i_0\ldots i_n}, X^{\Delta^n}\right) \\
&\cong \int_{[n]} \prod_{i_0\ldots i_n} \underline{\mathsf{sSet}}\left(\Delta^n, X(U_{i_0\ldots i_n})\right) \\
&\cong \int_{[n]} \underline{\mathsf{sSet}}\left(\Delta^n, \prod_{i_0\ldots i_n} X(U_{i_0\ldots i_n})\right).
\end{aligned} \tag{1.28}$$

This kind of limit is special enough to have its own name.

Definition 1.5.11. Let F be a cosimplicial simplicial set, namely a functor $F : \mathsf{\Delta} \to \mathsf{sSet}$. Then let $\mathsf{Tot}(F)$ denote the simplicial set given by the end

$$\mathsf{Tot}(F) = \int_{[n]\in\mathsf{\Delta}} \underline{\mathsf{sSet}}(\Delta^n, F^n),$$

where $\underline{\mathsf{sSet}}(X, Y)$ denotes the simplicial mapping space between two simplicial sets X and Y, namely $\underline{\mathsf{sSet}}(X, Y)_n = \mathsf{sSet}(X \times \Delta^n, Y)$. For a cosimplicial simplicial set F, $\mathsf{Tot}(F)$ is often called the **total object** or **totalization** of F.

A more convenient way of looking at $\mathsf{Tot}(F)$ is as the simplicial mapping space $\underline{\mathsf{csSet}}(\Delta, F)$, where Δ denotes the cosimplicial simplicial set $[m] \mapsto \Delta^m$. In other words, an n-simplex of the simplicial set $\mathsf{Tot}(F)$ is a map of cosimplicial simplicial sets $\Delta \times \Delta^n \to F$. The full data of such a map is a commutative diagram of the form

$$\begin{array}{ccccccc}
\Delta^n & \rightrightarrows & \Delta^1 \times \Delta^n & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \Delta^2 \times \Delta^n & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
F^0 & \rightrightarrows & F^1 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & F^2 & & \cdots
\end{array}$$

where each arrow is a map of simplicial sets, and we've hidden the codegeneracy maps for clarity in the diagram.

Thus X is an ∞-stack if and only if X is projective fibrant, and the canonical map

$$X(U) \to \mathsf{Tot}\left(X(\check{C}(\mathcal{U}))\right) \tag{1.29}$$

is a weak equivalence of simplicial sets, where $X(\check{C}(\mathcal{U}))$ is the cosimplicial simplicial set defined degreewise by $X(\check{C}(\mathcal{U}))_n = \prod_{i_0 \dots i_n} X(U_{i_0 \dots i_n})$. This concrete description has a pleasing abstract description as well.

Proposition 1.5.12. If X is a projective fibrant (objectwise Kan) simplicial presheaf, $U \in \mathsf{Cart}$ and $\mathcal{U}$ is a good cover of U, then

$$\mathsf{Tot}\left(X(\check{C}(\mathcal{U}))\right) \simeq \mathrm{holim}_{[n]\in\Delta} X(\check{C}(\mathcal{U}))_n \tag{1.30}$$

where the right hand side is the homotopy limit of the cosimpicial diagram of simplicial sets $X(\check{C}(\mathcal{U}))_n$ taken in the Quillen model structure on simplicial sets.

Proof. By [Hir09, Theorem 18.7.4], $\mathsf{Tot}\left(X(\check{C}(\mathcal{U}))\right) \to \mathrm{holim}_n X(\check{C}(\mathcal{U}))_n$ is a weak equivalence if $X(\check{C}(\mathcal{U}))$ is a Reedy fibrant cosimplicial simplicial set. By [Gla+22, Lemma C.5], if X is projective fibrant, then $X(\check{C}(\mathcal{U}))$ is Reedy fibrant. □

Thus by Proposition 1.5.12, if X is a projective fibrant simplicial presheaf, then it is an ∞-stack if and only if the canonical map

$$X(U) \to \mathrm{holim}_{\Delta}\left(\prod_i X(U_i) \begin{smallmatrix}\longrightarrow\\ \longleftarrow \\ \longrightarrow\end{smallmatrix} \prod_{i,j} X(U_{ij}) \begin{smallmatrix}\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow\end{smallmatrix} \prod_{i,j,k} X(U_{ijk}) \quad \cdots \right) \tag{1.31}$$

is a weak equivalence of simplicial sets.

If $X = {}^cF$ is a presheaf of sets, then $\underline{\mathsf{sPre}(\mathsf{Cart})}(yU, {}^cF) \cong F(U)$, and

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(\check{C}(\mathcal{U}), {}^cF) \cong \mathsf{Pre}(\mathsf{Cart})(\pi_0 \check{C}(\mathcal{U}), F) \cong \mathrm{eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \right).$$

The right hand side is the usual equalizer one sees in the definition of a sheaf of sets. Note that if $f : {}^cX \to {}^cY$ is a map of discrete simplicial sets, then f is a weak equivalence if and only if it is an isomorphism of sets. Thus ${}^cF$ is an ∞-stack if and only if the canonical map

$$F(U) \to \mathrm{eq}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \right),$$

is an isomorphism of sets for every $U \in \mathsf{Cart}$ and good open cover $\mathcal{U}$. In other words, for discrete simplicial presheaves, being an ∞-stack is equivalent to being a sheaf.

Suppose that $G : \mathsf{Cart}^{op} \to \mathsf{Gpd}$ is a (strict) presheaf of groupoids, then it is well known that G is a stack, in the classical sense, if and only if the map

$$G(U) \to \mathrm{holim}\left(\prod_i G(U_i) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \prod_{i,j} G(U_{ij}) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \prod_{i,j,k} G(U_{ijk}) \right),$$

is an equivalence of groupoids, where the right hand side is a homotopy limit of groupoids as described in [Car11, Section I.1.7]. Now consider the nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$. Then NG is a simplicial presheaf that will be projective fibrant, and it will be an ∞-stack if and only if G is a stack in the classical sense. Thus the notion of ∞-stack simultaneously generalizes the notion of sheaf and stack, and provides all of the power the homotopy theory of simplicial sets has to offer.

The main example of (∞-)stack we will consider in this paper is the following.

Example 1.5.13. Suppose G is a group object in $\mathsf{Sh}(\mathsf{Cart})$, which we will call a sheaf of groups. Consider the presheaf of groupoids on Cart that sends a $U \in \mathsf{Cart}$ to the groupoid

$$\mathbf{B}G(U) := [G(U) \rightrightarrows *],$$

where both source and target maps are the unique map to the singleton set. Thus there is a single object in this groupoid, and for every $s \in G(U)$, there is an isomorphism from the unique object to itself. We visualize morphisms in $\mathbf{B}G(U)$ with diagrams like the following

$$* \xrightarrow{g} *$$

We define composition in this groupoid using the **opposite** of multiplication in G. In other words we have

$$* \xrightarrow{g} * \xrightarrow{h} * \qquad = \qquad * \xrightarrow{hg} *$$

This convention might seem strange, but we have chosen it to agree with [Sch13, Section 1.2.5.1], which was consulted often in the formulation of this paper. Note that in other conventions [FSS+12] we must simply swap G with G^{op}, the opposite of the presheaf of groupoids, or the presheaf of groupoids with one object and morphisms the sheaf of groups with multiplication defined by $g \cdot^{op} h = hg$. It is important to note that the key equations (2.9) and (2.10) of Section 1.3 will be altered by changing this convention.

Now if G is in particular a Lie group, then we can consider it as a sheaf of groups on Cart. Then $\mathbf{B}G(U)$ will be the groupoid $[C^\infty(U,G) \rightrightarrows *]$. Now since all principal G-bundles are trivial on Cartesian spaces, it is easy to see that there is an objectwise equivalence of groupoids

$$[C^\infty(U,G) \rightrightarrows *] \simeq \text{Prin}_G(U),$$

where $\text{Prin}_G(U)$ denotes the groupoid of principal G-bundles on U. Indeed, every object of the right hand groupoid is isomorphic to the trivial bundle $U \times G \to U$, and the automorphisms of a trivial bundle are in bijection with maps $U \to G$. In [Car11, Section I.2] it is proven that Prin_G is a stack (in the classical sense) on Man and Cart. The argument above proves that $\mathbf{B}G$ is also a stack (in the classical sense) on Cart. However $\mathbf{B}G$ is not a stack on Man. If we take the nerves of these presheaves of groupoids $N\mathbf{B}G$ and $N\text{Prin}_G$, then since they are nerves of classical stacks on Cart, they will be ∞-stacks on Cart, and they are objectwise weak equivalent as simplicial presheaves. See [FSS+12, Section 3.2] for more details.

We often drop the nerve N from the notation of our ∞-stacks, and we call $\mathbf{B}G$ the **delooping stack** of G.

Remark 1.5.14. Note that in the above example, strictly speaking Prin_G is a not a presheaf of groupoids, because if $U \xrightarrow{f} V \xrightarrow{g} W$ is a pair of composable morphisms in Cart, then we obtain functors $\mathsf{Prin}_G(W) \xrightarrow{g^*} \mathsf{Prin}_G(V) \xrightarrow{f^*} \mathsf{Prin}_G(U)$ given by pulling back the principal bundles. However if $P \in \mathsf{Prin}_G(W)$, then $(gf)^*P \neq f^*g^*P$, but there is an isomorphism $(gf)^*P \cong f^*g^*P$. Thus Prin_G is called a pseudofunctor $\mathsf{Prin}_G : \mathsf{Cart}^{op} \to \mathsf{Gpd}$, where Gpd is the 2-category of groupoids. There is an elegant theory [JY20, Chapter 10] relating pseudofunctors with categories fibered in groupoids, both of which can be used to develop the theory of stacks of groupoids [Vis07]. However, the homotopy theory of presheaves of groupoids, pseudofunctors, and categories fibered in groupoids are all equivalent in the sense of [Hol08, Corollary 4.3]. Furthermore, the notion of being a 1-stack is independent across the three models. Thus in what follows we elect to use presheaves of groupoids, as they are the simplest to connect with the theory of simplicial presheaves by applying the nerve functor objectwise.

One of the most useful aspects of simplicial model categories is being able to define homotopically invariant mapping spaces.

Definition 1.5.15. If X and A are simplicial presheaves, then define

$$\mathbb{R}(X, A) \coloneqq \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, RA) \tag{1.32}$$

where QX is a cofibrant replacement for X and RA is a fibrant replacement of A in the Čech model structure. We call $\mathbb{R}(X, A)$ the **derived mapping space** of X and A.

A key property of derived mapping spaces is their invariance under weak equivalence. Indeed suppose there is a Čech weak equivalence $f : X \to X'$, then the canonical map

$$\mathbb{R}(f, A) : \mathbb{R}(X', A) \to \mathbb{R}(X, A)$$

is a weak equivalence of simplicial sets, similarly for a Čech weak equivalence $g : A \to A'$, see [Hir09, Chapter 17].

Proposition 1.5.16 ([FSS+12, Page 23])**.** Given a Lie group G, and a finite dimensional smooth manifold M, there is a weak equivalence of simplicial sets

$$N\mathrm{Prin}_G(M) \simeq \mathbb{R}(M, \mathbf{B}G), \tag{1.33}$$

where $N\mathrm{Prin}_G(M)$ denotes nerve of the groupoid of principal G-bundles on M.

Now if G is a diffeological group, and X is a diffeological space, we can consider them both as simplicial presheaves on Cart, and compute $\mathbb{R}(X, \mathbf{B}G)$. It would be hoped that this would in some way reproduce diffeological principal G-bundles, in analogy to Proposition 1.5.16. One main goal of this paper is to prove that this is indeed the case. But first we must investigate $\mathbb{R}(X, \mathbf{B}G)$. If X was cofibrant, and $\mathbf{B}G$ were fibrant in the Čech model structure, then $\mathbb{R}(X, \mathbf{B}G)$ would be computable. However diffeological spaces are not projective cofibrant in general (though cartesian spaces are). Thus we must find a projective cofibrant simplicial presheaf QX which is Čech weak equivalent to X. This will be the subject of Section 1.5.2.

However, it is indeed the case that $\mathbf{B}G$ is fibrant, thanks to the following wonderful theorem.

Theorem 1.5.17 ([SS21, Lemma 3.3.29], [Pav22a, Proposition 4.13])**.** Let G be a sheaf of groups on Cart. Then $\mathbf{B}G$ is an ∞-stack on Cart.

Thus if G is a diffeological group, then it is in particular a sheaf of groups on Cart, and therefore $\mathbf{B}G$ is an ∞-stack.

1.5.2 Resolutions of Diffeological Spaces

Here we discuss three ways of "resolving" a diffeological space into a diffeological category. One of which, QX comes forth immediately from the projective model structure on simplicial presheaves. The other two, which we denote $\check{C}(X)$ and $B//M$, appear in [KWW21] and

[Igl20a] respectively, and are interesting in their own right. We compare these three resolutions as diffeological categories, and examine their resulting notions of diffeological Čech cohomology in Section 1.5.3.

Let us start by describing the resolution QX for a diffeological space X. Since $\mathsf{s\check{P}re(Cart)}$ is a combinatorial model category, a cofibrant replacement functor Q exists. However, we are in the lucky situation that there is a cofibrant replacement functor Q with a relatively simple form.

Lemma 1.5.18 ([Dug01, Lemma 2.7])**.** Given a diffeological space X, thought of as a discrete simplicial presheaf on Cart, its cofibrant replacement QX is given by the simplicial presheaf

$$QX = \int^{[n]\in\boldsymbol{\Delta}} \Delta^n_c \times \left(\coprod_{U_{p_n}\to\cdots\to U_{p_0}\to X} yU_{p_n} \right) \tag{1.34}$$

Let us examine this coend formula in more detail. In degree k, we have

$$(QX)_k \cong \coprod_{(f_{k-1},\dots,f_0):U_{p_k}\to\cdots\to U_{p_0}} yU_{p_k} \tag{1.35}$$

where the coproduct is taken over the set $(N\mathsf{Plot}(X))_k$, namely the set of k composable morphisms in the category of plots over X. We will let $N_k \coloneqq (N\mathsf{Plot}(X))_k$.

The face maps are given as follows:

$$d_i(x_{p_k}, f_{k-1}, f_{k-2}, \dots, f_0) = \begin{cases} (f_{k-1}(x_{p_k}), f_{k-2}, \dots, f_0) & i = 0 \\ (x_{p_k}, f_{k-1}, \dots, f_{k-i-1}f_{k-i}, \dots, f_0) & 0 < i < k \\ (x_{p_k}, f_{k-1}, \dots, f_1) & i = k. \end{cases} \tag{1.36}$$

Degeneracies insert identity maps.

For convenience, we will denote the coproduct over all plots as

$$B \coloneqq (QX)_0 = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0}.$$

Notice that there is a canonical map

$$B \xrightarrow{\pi} X$$

given by $\pi(p_0, x_{p_0}) = p_0(x_{p_0})$.

This map induces a map $\pi : QX \to {}^cX$ of simplicial presheaves and [Dug01, Lemma 2.7] proves that this is an objectwise weak equivalence, and thus a Čech weak equivalence. By construction we also have the following isomorphism of presheaves on Cart.

Lemma 1.5.19. If X is a diffeological space, then the map $\pi : QX \to {}^cX$ induces an isomorphism of presheaves on Cart

$$\pi_0 QX \cong X, \tag{1.37}$$

where $\pi_0 : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{Pre}(\mathsf{Cart})$ is defined in Section 1.5.

Proof. This follows from the fact that every presheaf is a colimit of representables, see the discussion above [Dug01, Lemma 2.7]. □

In low degrees, this simplicial presheaf/simplicial diffeological space looks like:

$$B = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \coprod_{f_0 : U_{p_1} \to U_{p_0}} U_{p_1} \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \coprod_{U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_2} \qquad \cdots$$

where the maps $f_i : U_{p_{i+1}} \to U_{p_i}$ are understood to be morphisms in the plot category $\mathsf{Plot}(X)$.

In fact, QX is completely determined by QX_1 and QX_0 in the following sense. Let $N : \mathsf{DiffCat} \to \mathsf{sDiff}$ denote the nerve functor from diffeological categories (categories internal to the category of diffeological spaces) to the category of simplicial diffeological spaces, defined degreewise by

$$NC_k = C_1 \times_{t,C_0,s} C_1 \times_{t,C_0,s} \cdots \times_{t,C_0,s} C_1,$$

where the iterated pullback is taken k-times, where each $C_1 \times_{t,C_0,s} C_1$ denotes the pullback with respect to the target and source maps respectively.

For a diffeological space X, the first two spaces and maps between them in QX, namely $[QX_1 \rightrightarrows B]$ forms a diffeological category. It also is a presheaf of categories. The source, target and unit maps are defined by the simplicial structure, but we recall their definitions here. The source map $s : QX_1 \to B$ is defined by $s(x_{p_1}, f_0) = x_{p_1}$ and its target map $t : QX_1 \to B$ is defined by $t(x_{p_1}, f_0) = f_0(x_{p_1})$. The unit map $u : B \to QX_1$ is defined by $u(x_p) = (x_p, 1_{U_p})$. The composition map $c : QX_1 \times_B QX_1 \to QX_1$ is defined by $c([x_{p_2}, f_1], [x_{p_1}, f_0]) = (x_{p_2}, f_0 \circ f_1)$. With this structure it is not hard to see that $[QX_1 \rightrightarrows B]$ is a diffeological category/presheaf of categories. In fact QX is completely determined by this diffeological category in the following sense.

Proposition 1.5.20. If X is a diffeological space, then

$$QX \cong N[QX_1 \rightrightarrows QX_0], \tag{1.38}$$

where we are thinking of QX as a simplicial diffeological space or a simplicial presheaf.

Proof. Let $\varphi : QX_k \to QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1$ be the map defined as follows. Suppose that $(x_{p_k}, f_{k-1}, \ldots, f_0) \in QX_k$. By induction, define $x_{p_{k-1}} = f_{k-1}(x_{p_k})$ and $x_{p_{k-n}} = f_{k-n}(x_{p_{k-n+1}})$ for $1 < n \leq k$. Then set

$$\varphi(x_{p_k}, f_{k-1}, \ldots, f_0) = ([x_{p_k}, f_{k-1}], [x_{p_{k-1}}, f_{k-2}], \ldots, [x_{p_1}, f_0]).$$

This map is smooth, as it is built out of projection maps. Now define $\psi : QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1 \to QX_k$ as follows. A point of $QX_1 \times_B QX_1 \times_B \cdots \times_B QX_1$ is a collection of pairs $\{[x_{p_n}, f_{n-1}]\}_{1\leq n\leq k}$ such that $f_{n-1}(x_{p_n}) = x_{p_{n-1}}$. Thus set

$$\psi([x_{p_k}, f_{k-1}], [x_{p_{k-1}}, f_{k-2}], \dots, [x_{p_1}, f_0]) = (x_{p_k}, f_{k-1}, \dots, f_0).$$

It is not hard to see that this map is smooth, and that φ and ψ are two-sided inverses for each other. □

Lemma 1.5.21. If X is a diffeological space, then we can consider the coequalizer in Diff of $[QX_1 \rightrightarrows QX_0]$ and this is isomorphic to X, namely

$$X \cong \operatorname{coeq}\left(\coprod_{U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_1} \rightrightarrows \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \right). \tag{1.39}$$

Proof. This is just a restatement of Lemma 1.2.13. □

Remark 1.5.22. Note that the coequalizer given in (1.37) is taken in the category $\mathsf{Pre}(\mathsf{Cart})$, which has different colimits than $\mathsf{ConSh}(\mathsf{Cart})$. Therefore it does not immediately imply Lemma 1.5.21. However, by combining the two results we have proven that $\pi_0 QX$ is isomorphic to the coequalizer of $QX_1 \rightrightarrows QX_0$ in the category of diffeological spaces.

Now as discussed in Section 1.5.1, if X is a diffeological space and G is a diffeological group, then we can consider the simplicial set $\mathbb{R}(X, \mathbf{B}G)$. By Theorem 1.5.17, we know that $\mathbf{B}G$ is fibrant, thus

$$\mathbb{R}(X, \mathbf{B}G) = \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \mathbf{B}G) \cong \underline{\mathsf{sPre}(\mathsf{Cart})}(N[QX_1 \rightrightarrows B],\, N[C^\infty(-, G) \rightrightarrows *]).$$

In Section 1.6 we will show that this simplicial set is weak equivalent to the nerve of the groupoid of diffeological principal G-bundles on X.

The fact that QX is the nerve of a diffeological category is interesting, as it allows us to compare it with other diffeological categories using the homotopy theory developed in [Rob12] for categories internal to a site. The site in this instance is the category Diff of diffeological spaces with the coverage of subductions, see Example 1.4.7. This homotopy theory provides us with a notion of weak equivalence $f : X \to Y$ of diffeological categories, see [Rob12, Definition 4.14], which if both X and Y are diffeological groupoids, coincides with the notion of weak equivalence of diffeological groupoids considered in [Wat22] and [Sch20].

If we consider X as a diffeological category $[X = X]$ with all structure maps being the identity, then the canonical map $[QX_1 \rightrightarrows QX_0] \to [X = X]$ of diffeological categories is not a weak equivalence, as it is not fully faithful.

However, there is another diffeological groupoid we can consider. Given a diffeological space X, we can consider the canonical map $\pi : B \to X$ as mentioned above. This can be made into a diffeological groupoid $\check{C}(X)$ by setting $\check{C}(X)_0 = B$ and $\check{C}(X)_1 = B \times_X B$, with the source and target maps being the obvious projection maps. We will call this the **Čech resolution** of X, as a diffeological groupoid. It is not hard then to check that the canonical map $\check{C}(X) \to [X = X]$ of diffeological groupoids is indeed a weak equivalence.

If we then take the nerve of $\check{C}(X)$, we obtain a simplicial diffeological space, which we can also consider as a simplicial presheaf.

Proposition 1.5.23. The natural map $\check{C}(X) \to {}^cX$ of simplicial presheaves is a Čech weak equivalence.

Proof. It is easily checked that the map $\pi : B \to X$ is a local epimorphism, as it is objectwise a surjection. Thus [DHI04, Corollary A.3] proves that $\check{C}(X) \to {}^cX$ is a weak equivalence in the Čech model structure on simplicial presheaves. □

Therefore we have a zig-zag of Čech weak equivalences of simplicial presheaves

$$\check{C}(X) \to {}^cX \leftarrow QX.$$

However $\check{C}(X)$ will not be cofibrant in the projective model structure on simplicial presheaves in general. Thus for our purposes, QX is the preferable resolution of X, while for the purposes of those interested in diffeological groupoid theory, $\check{C}(X)$ might be the more preferable resolution.

The final resolution we will discuss is that of the **gauge monoid** that appears in [Igl20a]. Given a diffeological space X, its nebula $B = \coprod_{p \in \mathsf{Plot}(X)} U_p$ is a diffeological space, and we can consider the set of smooth maps $f : B \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & X & \end{array}$$

It inherits the subspace diffeology from the functional diffeology on $C^\infty(B, B)$. Notice that M acts on B by $B \times M \xrightarrow{\rho} B$, where $\rho(b, m) = m(b)$.

We can therefore consider the diffeological category $B//M \coloneqq [B \times M \rightrightarrows B]$, where the source map $s : B \times M \to M$ is given by $s(b, m) = b$, and the target map $t : B \times M \to M$ is given by $t(b, m) = m(b)$.

There is a map $\delta : QX \to B//M$ defined as the identity on objects and on morphisms by $\delta(x_{p_1}, f_0) = (x_{p_1}, \delta f_0)$ where δf_0 denotes the map $\delta f_0 : B \to B$ that is the identity on every component U_p except for $p = p_1$, in which case $\delta f_0|_{U_{p_1}} = f_0$. It is not hard to check that this defines a map of diffeological categories.

In the reverse direction, there is a map res : $B//M \to QX$ defined to be the identity on objects and on morphisms by

$$\mathrm{res}(x_p, m) = (x_p, m|_{U_p}).$$

It is not hard to see that the composition $QX \xrightarrow{\delta} B//M \xrightarrow{\mathrm{res}} QX$ is the identity, namely that QX is a retract of $B//M$.

There is a map $q : B//M \to \check{C}(X)$ described in [KWW21, Page 26] which is the identity on objects and on morphisms is defined by $q(x_p, m) = (x_p, m(x_p))$. This defines a map of diffeological categories. It is also easy to see that $q\delta\mathrm{res} = q$.

To summarize, we have the following diagram of maps of diffeological categories, all of which are the identity on objects, but none of which are fully faithful.

$$\begin{array}{ccc} QX & \underset{\text{res}}{\overset{\delta}{\rightleftarrows}} & B//M \\ & {\scriptstyle q\circ\delta}\searrow \quad \swarrow{\scriptstyle q} & \\ & \check{C}(X) & \end{array} \tag{1.40}$$

The diffeological categories $\check{C}(X)$ and $B//M$ are used to construct Čech cohomology groups for diffeological spaces in [KWW21] and [Igl20a].

1.5.3 Diffeological Čech Cohomologies

Here we will describe three notions of Čech cohomology for diffeological spaces that results from the material in Section 1.5.2.

Remark 1.5.24. In what follows we will always consider chain complexes and cochain complexes to be non-negatively graded, with differentials going down and up respectively.

Let A denote a diffeological abelian group. In [Igl20a], Čech cohomology of a diffeological space X is defined[2] as follows. First consider $N(B//M)$, the simplicial diffeological space defined as the nerve of the diffeological category defined in section 1.5.2. Then

$$A^{N(B//M)_k} = A^{B\times M^{\times k}} = C^\infty(B \times M^{\times k}, A)$$

is precisely the diffeological space of smooth maps $B \times M^{\times k} \to A$. If we forget the smooth structure, then $C^\infty(N(B//M)_k, A)$ is an abelian group by pointwise addition. Thus we obtain a cosimplicial abelian group

$$A^B \rightrightarrows A^{B\times M} \rightrightarrows\!\!\!\!\to A^{B\times M\times M} \qquad \dots$$

[2]Modulo some details, Iglesias-Zemmour defines diffeological spaces with open subsets of cartesian spaces and uses a generating family of open balls, but they are clearly equivalent constructions. He also only restricts to discrete abelian diffeological groups.

and from this one can obtain a cochain complex as follows.

If K is a cosimplicial abelian group, then we can define a cochain complex $C^{\mathrm{co}}K$ called the **associated cochain complex** by

$$(C^{\mathrm{co}}K)^n = K^n, \qquad d : (C^{\mathrm{co}}K)^n \to (C^{\mathrm{co}}K)^{n+1}, \qquad d = \sum_{i=0}^{n} (-1)^i d^i.$$

This definition extends to a functor $C^{\mathrm{co}} : \mathsf{cAb} \to \mathsf{CoCh}$, where cAb denotes the category of cosimplicial abelian groups and CoCh is the category of cochain complexes. Further there is a functorial direct sum decomposition as cochain complexes $C^{\mathrm{co}}K \cong N^{\mathrm{co}}K \oplus D^{\mathrm{co}}K$, where $D^{\mathrm{co}}K$ is the subcomplex consisting of degenerate simplices, and the inclusion $N^{\mathrm{co}}K \to C^{\mathrm{co}}K$ is a cochain homotopy equivalence of cochain complexes. We call $N^{\mathrm{co}}K \cong C^{\mathrm{co}}K/D^{\mathrm{co}}K$ the **normalized cochain complex** of K. This is a dual version of what is called the Dold-Kan correspondence, which is an adjoint equivalence

$$N : \mathsf{sAb} \rightleftarrows \mathsf{Ch} : \Gamma,$$

where N is the normalized chain complex functor and if V is a chain complex, then ΓV is defined degreewise by

$$\Gamma(V)_n = \bigoplus_{[n] \twoheadrightarrow [k]} V_k,$$

where the index is over all surjections $\varphi : [n] \to [k]$ in $\mathsf{\Delta}$. See [Wei95, Section 8.4] and [GJ12, Section III] for details.

The Iglesias-Zemmour Čech cohomology of X is then defined as the cohomology of this cochain complex:

$$\check{H}^k_{PIZ}(X, A) = \check{H}^k\left(N^{\mathrm{co}}\left[A^{B//M}\right]\right) \cong \check{H}^k\left(C^{\mathrm{co}}\left[A^{B//M}\right]\right).$$

Similarly, let $N\check{C}(X)$ denote the nerve of the Čech groupoid defined in Section 1.5.2. If A is an abelian diffeological group, then as above we can map $N\check{C}(X)$ into A to form a cosimplicial abelian group, and taking the cohomology of the associated cochain complex gives us the Krepski-Watts-Wolbert Čech cohomology [KWW21] of X with values in A:

$$\check{H}^k_{KWW}(X, A) = \check{H}^k\left(N^{\mathrm{co}}\left[A^{\check{C}(X)}\right]\right) \cong \check{H}^k\left(C^{\mathrm{co}}\left[A^{\check{C}(X)}\right]\right).$$

We will now construct a cochain complex using the cofibrant replacement QX of a diffeological space X. Given an abelian group A and a non-negative integer n, there exists a simplicial set $K(A, n)$, called the n**th Eilenberg-Maclane space**, which has trivial homotopy groups in all degrees except for n, which has $\pi_n(K(A, n)) = A$. One can construct this simplicial set using the Dold-Kan correspondence. Namely consider the chain complex $A[k]$, defined by

$$(A[k])_n = \begin{cases} A & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}, \qquad d = 0.$$

Since $\Gamma(A[k])$ is a simplicial group, it will be a Kan complex, equipped with basepoint $*$ such that

$$\pi_n(\Gamma(A[k]), *) = \begin{cases} A & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Remark 1.5.25. For future reference, if V is a chain complex, then let $V[k]$ denote the chain complex such that $V[k]_n = V_{n-k}$, so that we identify an abelian group A with the chain complex $A[0]$, and then $(A[0])[k] = A[k]$.

Now we will define ∞-stack cohomology for simplicial presheaves. This theory, which we call ∞-stack cohomology, is very well developed, and generalizes many examples of cohomology found throughout mathematics, see [Sch13], [Lur09], [BNV16].

Definition 1.5.26. Let X be a projective cofibrant simplicial presheaf and A an ∞-stack. Then the zeroth ∞**-stack cohomology** of X with values in A is given by

$$\check{H}^0_\infty(X, A) := \pi_0 \mathbb{R}(X, A) \cong \pi_0 \underline{\mathsf{sPre}(\mathsf{Cart})}(X, A). \tag{1.41}$$

Note that in the above definition, A is an arbitrary ∞-stack. Thus $\check{H}^0_\infty(X, A)$ is an example of nonabelian cohomology. However, in order to define $\check{H}^1_\infty$, we must ask for more structure on A, namely that it be an ∞-stack, and that A also be a group object in $\mathsf{sPre}(\mathsf{Cart})$, namely that $A(U)$ be a simplicial group for each $U \in \mathsf{Cart}$ and given a smooth map $f : U \to V$, the map $A(f) : A(V) \to A(U)$ is a map of simplicial groups. We call group objects of $\mathsf{sPre}(\mathsf{Cart})$ presheaves of simplicial groups.

Definition 1.5.27. Given a simplicial group G, let $\overline{W}G$ denote the simplicial set with

$$\begin{aligned} \overline{W}G_0 &= * \\ \overline{W}G_n &= G_{n-1} \times G_{n-2} \times \cdots \times G_0 \end{aligned} \tag{1.42}$$

with face and degeneracy maps given by

$$d_i(g_{n-1}, \dots, g_0) = \begin{cases} (g_{n-2}, \dots, g_0), & \text{if } i = 0 \\ (d_{i-1}(g_{n-1}), \dots, d_1(g_{n-i+1}), g_{n-i-1} \cdot d_0(g_{n-i}), g_{n-i-2}, \dots, g_0)), & \text{if } 1 \leq i \leq n \end{cases}$$

$$s_i(g_{n-1}, \dots, g_0) = \begin{cases} (1, g_{n-1}, \dots, g_0), & \text{if } i = 0 \\ (s_{i-1}(g_{n-1}), \dots, s_0(g_{n-i}), 1, g_{n-i-1}, \dots, g_0), & \text{if } 1 \leq i \leq n. \end{cases}$$

Simplicial sets of the form $\overline{W}G$ classify what are called principal twisted cartesian products or PTCPs in [May92]. The combinatorial structure of $\overline{W}G$ may look complicated, but it has other equivalent descriptions that are more motivated, see [GJ12, Chapter V] and [Ste12].

Lemma 1.5.28 ([GJ12, Corollary 6.8])**.** If G is a simplicial group, then $\overline{W}G$ is a Kan complex.

If ${}^cG$ a discrete simplicial group, i.e. a group, then $\overline{W}{}^cG \cong N[G \rightrightarrows *]$, the nerve of G thought of as a groupoid with one object. Thus $|\overline{W}{}^cG|$, the geometric realization of the delooping, is weak homotopy equivalent to the classifying space BG.

Now if A is a presheaf of simplicial groups, then we can apply $\overline{W}$ objectwise, and we obtain a functor $\overline{W} : \mathsf{sPre}(\mathsf{Cart}, \mathsf{sGrp}) \to \mathsf{sPre}(\mathsf{Cart})$, where $\mathsf{sPre}(\mathsf{Cart}, \mathsf{sGrp})$ denotes the full subcategory of presheaves of simplicial groups. Further, by Lemma 1.5.28, $\overline{W}A$ is projective fibrant, i.e. objectwise a Kan complex.

Lemma 1.5.29. Let G be a sheaf of groups on Cart. Then the delooping stack of Example 1.5.13 is isomorphic to its delooping as a presheaf of simplicial groups

$$\mathbf{B}G \cong \overline{W}^c G.$$

So suppose that A is an ∞-stack on Cart, and further, that it is a presheaf of simplicial groups. Then we get a new simplicial presheaf $\overline{W}A$, and it is projective fibrant. We therefore define the first ∞-stack cohomology group of a simplicial presheaf X with values in A to be

$$\check{H}^1_\infty(X, A) \cong \pi_0 \mathbb{R}(X, \overline{W}A).$$

In order to be able to compute this, it would be convenient to know that $\overline{W}A$ is fibrant in the Čech model structure, i.e. is an ∞-stack. This follows thanks to the following wonderful theorem.

Theorem 1.5.30 ([SS21, Proposition 3.3.30], [Pav22a, Proposition 4.13])**.** If A is an ∞-stack on Cart that is also a presheaf of simplicial groups, then $\overline{W}A$ is an ∞-stack on Cart.

Thus if A is an ∞-stack, then for any simplicial presheaf X, $\check{H}^0_\infty(X, A)$ is well defined, and if A is also a presheaf of simplicial groups, then $\check{H}^1_\infty(X, A)$ is also well defined. To obtain higher cohomology groups, we must ask for higher deloopings of A to exist.

Definition 1.5.31. Let A be an ∞-stack that is also a presheaf of simplicial groups. If $\overline{W}^k A$ is a presheaf of simplicial groups for all $1 \leq k \leq n-1$, and X is a simplicial presheaf, then let

$$\check{H}^n_\infty(X, A) = \pi_0 \mathbb{R}(X, \overline{W}^n A) \tag{1.43}$$

denote the nth ∞-stack cohomology of X with values in A.

It is thus important to know under what conditions will these higher deloopings $\overline{W}^n A$ exist.

Lemma 1.5.32. If A is a simplicial abelian group, namely A is a simplicial group and A_k is an abelian group for all k, then $\overline{W}A$ will be a simplicial group, and further it will be an abelian simplicial group.

Proof. It follows from the isomorphism $\overline{W}A \cong TNA$ of [Ste12, Lemma 5.2] and the discussion of T in [AM66, Section III] that $\overline{W}A$ is a simplicial group, and that it is abelian is clear from the formula (1.42). □

Thus if A is a simplicial abelian group, $\overline{W}^k A$ exists for all k.

Lemma 1.5.33 ([Jar97, Section 4.6])**.** Let A be a simplicial abelian group. Then there is an isomorphism of chain complexes

$$N\overline{W}A \cong (NA)[1]$$

where $(NA)[1]$ is the chain complex NA shifted up by 1, i.e. $(NA[1])_k = (NA)_{k-1}$.

Lemma 1.5.34. If A is an abelian group, thought of as a discrete simplicial abelian group ${}^cA$, then $\overline{W}^k A$ exists for every $k \geq 0$, and there exists an isomorphism

$$\overline{W}^k {}^cA \cong \Gamma(A[k])$$

Proof. We proceed by induction. For the base case, we have

$$N\overline{W}{}^cA \cong (N^cA)([1]).$$

But $N^c A \cong A[0]$ as is easily checked, and $(A[0])[1] = A[1]$, so $N\overline{W}^c A \cong A[1]$, thus

$$\Gamma N\overline{W}^c A \cong \overline{W}^c A \cong \Gamma A[1].$$

Now suppose $\overline{W}^{k-1}{}^c A \cong \Gamma A[k-1]$. Then by Lemma 1.5.33

$$N\overline{W}\left(\overline{W}^{k-1}{}^c A\right) \cong \left(N\Gamma A[k-1]\right)[1]$$

but $N\Gamma A[k-1] \cong A[k-1]$ since N and Γ form an adjoint equivalence, thus:

$$N\overline{W}\left(\overline{W}^{k-1}{}^c A\right) \cong N\overline{W}^k{}^c A \cong A[k-1]([1]) = A[k]$$

taking the adjoint gives

$$\overline{W}^k{}^c A \cong \Gamma A[k].$$

□

Now if A is an abelian diffeological group, then it is an ∞-stack on Cart, since it is a sheaf on Cart, and therefore a discrete presheaf of simplicial groups. Thus by 1.5.34 and 1.5.30, $\overline{W}^n A$ exists, and the nth ∞-stack cohomology of a diffeological space X with values in A is given by

$$\check{H}^n_\infty(X, A) = \pi_0 \underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \overline{W}^n A) \cong \pi_0 \mathsf{Tot}(\overline{W}^n[A(QX)]), \tag{1.44}$$

where Tot is the totalization of Definition 1.5.11, $A(QX)$ is the cosimplicial abelian group which in degree k is given by $C^\infty(QX_k, A)$, and $\overline{W}^n[A(QX)]$ is $\overline{W}^n$ applied to $A(QX)$ degreewise. Now $\overline{W}^n[A(QX)] \cong \Gamma[A(QX)[n]]$ by Lemma 1.5.34.

Proposition 1.5.35 ([Jar16, Lemma 19])**.** If A is a cosimplicial abelian group, then

$$\pi_0 \mathsf{Tot}(\Gamma A[k]) \cong \check{H}^k(N^{\mathrm{co}} A) \tag{1.45}$$

where $\Gamma A[k]$ denotes the cosimplicial simplicial abelian group obtained by considering the abelian group A^i as a simplicial abelian group $\Gamma(A^i[k])$.

So substituting for the cosimplicial abelian group $A(QX)$ in Proposition 1.5.35 we have the main result of this section, which provides a concrete way of computing the ∞-stack cohomology of a diffeological space with values in an abelian diffeological group.

Corollary 1.5.36. If X is a diffeological space, and A is an abelian diffeological group, and we consider the cochain complex

$$C^{co}[A(QX)] = A^B \xrightarrow{d} A^{QX_1} \xrightarrow{d} A^{QX_2} \xrightarrow{d} \dots$$

then the nth ∞-stack cohomology of X with values in A can be computed by $C^{co}[A(QX)]$, namely

$$\check{H}^n_\infty(X, A) \cong \check{H}^n_\infty(N^{co}[A(QX)]) \cong \check{H}^n_\infty(C^{co}[A(QX)]).$$

This explicit description of ∞-sheaf cohomology will be useful in comparing the various Čech cohomologies.

Proposition 1.5.37. For a diffeological space X, and a diffeological abelian group A, the ∞-sheaf cohomology and Iglesias-Zemmour cohomology agree in degree 0:

$$\check{H}^0_{PIZ}(X, A) = \check{H}^0_\infty(X, A).$$

Proof. The set of 0-cocycles in ∞-sheaf cohomology is the set

$$\check{H}^0_\infty(X, A) = \{\tau : B \to A \,|\, \text{if } f : U_p \to U_q \text{ is a map of plots then } \tau \circ \delta f = \tau\},$$

where δf denotes the map $\delta f : B \to B$ that is the identity on every component except for U_p, where it is f. Equivalently it is the set of smooth maps $\tau : B \to A$ such that if $(x_{p_1}, f_0) \in QX_1$, then $\tau(f_0(x_{p_1})) = \tau(x_{p_1})$. The set of 0-cocycles in Iglesias-Zemmour cohomology is

$$\check{H}^0_{PIZ}(X, A) = \{\sigma : B \to A \,|\, \text{if } m \in M, \text{then } \sigma \circ m = \sigma\}\,.$$

Equivalently it is the set of smooth maps $\sigma : B \to A$ such that if $(x_{p_1}, m) \in B \times M$, then $\sigma(m(x_{p_1})) = \sigma(x_{p_1})$. Notice that $\check{H}^0_{PIZ}(X, A) \subseteq \check{H}^0_\infty(X, A)$, since every δf is an element of M. Now if $\tau \in \check{H}^0_\infty(X, A)$, and $(x_{p_1}, m) \in B \times M$, then $\tau(m(x_{p_1})) = \tau(x_{p_1})$, because $m|_{U_{p_1}}$ is a map of plots. Thus $\check{H}^0_\infty(X, A) \subseteq \check{H}^0_{PIZ}(X, A)$. □

Note that

$$\pi_0 \underline{\mathsf{sPre(Cart)}}(QX, {}^cA) \cong \mathsf{Pre(Cart)}(\pi_0 QX, A) \cong \mathsf{Pre(Cart)}(X, A) \cong \mathsf{Diff}(X, A), \tag{1.46}$$

where the first isomorphism follows from the adjunction described in Section 1.5, and the second isomorphism follows from Remark 1.5.22.

Corollary 1.5.38. If X is a diffeological space, and A a diffeological abelian group, then

$$\check{H}^0_\infty(X, A) \cong \check{H}^0_{PIZ}(X, A) \cong \check{H}^0_{KWW}(X, A) \cong \mathsf{Diff}(X, A). \tag{1.47}$$

Proof. This follows from (1.46), Proposition 1.5.37 and [KWW21, Proposition 4.6]. □

Now recall the map $q \circ \delta : QX \to \check{C}(X)$ from (1.40). This induces a map on cohomology which we will denote by $(q\delta)^* = \varphi : \check{H}^\bullet_{KWW}(X, A) \to \check{H}^\bullet_\infty(X, A)$. In degree 1, φ has the following explicit description on cocycles. Namely if $\tau : \check{C}(X)_1 \to A$ is a cocycle, then $(\varphi\tau)(x_{p_1}, f_0) = \tau(x_{p_1}, f_0(x_{p_1}))$.

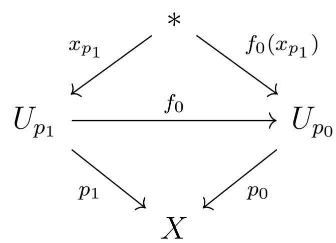
Proposition 1.5.39. For any diffeological space X and abelian diffeological group A, the map $\varphi : \check{H}^1_{KWW}(X, A) \to \check{H}^1_\infty(X, A)$ is an isomorphism.

Proof. Let us show that φ is surjective. Suppose that σ is a 1-cocycle for the cochain complex

A^{QX}. This means that if f_1, f_0 are composable maps of plots, then

$$\sigma(f_1(x_{p_2}), f_0) = \sigma(x_{p_2}, f_0 f_1) - \sigma(x_{p_2}, f_1).$$

Now if $(x_{p_1}, f_0) \in QX_1$, then notice we have the following commutative diagram of plot maps

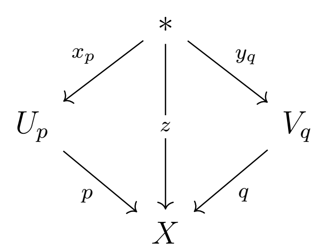


which implies that if σ is a 1-cocycle that $\sigma(x_{p_1}, f_0) = \sigma(*, f_0(x_{p_1})) - \sigma(*, x_{p_1})$. So consider the map $\tau : \check{C}(X)_1 \to A$ defined as follows. If $(x_p, y_q) \in \check{C}(X)_1$, then let $\tau(x_p, y_q) = \sigma(*, y_q) - \sigma(*, x_p)$. Then $(\varphi\tau)(x_{p_1}, f_0) = \sigma(x_{p_1}, f_0)$ for every $(x_{p_1}, f_0) \in QX_1$.

Now we wish to show that φ is injective. Suppose $\tau, \tau' : \check{C}(X)_1 \to A$ are 1-cocycles such that there exists some $\alpha : B \to A$ such that for every $(x_{p_1}, f_0) \in QX_1$,

$$\tau(x_{p_1}, f_0(x_{p_1})) - \tau'(x_{p_1}, f_0(x_{p_1})) = \alpha(f_0(x_{p_1})) - \alpha(x_{p_1}).$$

Then if $(x_p, y_q) \in \check{C}(X)_1$, we have the following commutative diagram of plot maps



where $z = p(x_p) = q(y_q)$, and we use z to refer to the point $*$ in the plot $z : * \to X$. Now since τ and τ' are 1-cocycles, it follows that

$$\tau(x_p, y_q) = \tau(z, y_q) - \tau(z, x_p), \qquad \tau'(x_p, y_q) = \tau(z, y_q) - \tau(z, x_p).$$

Therefore

$$\begin{aligned}
\tau(x_p, y_q) - \tau'(x_p, y_q) &= (\tau(z, y_q) - \tau(z, x_p)) - (\tau'(z, y_q) - \tau'(z, x_p)) \\
&= (\tau(z, y_q) - \tau'(z, y_q)) - (\tau(z, x_p) - \tau'(z, x_p)) \\
&= (\alpha(y_q) - \alpha(z)) - (\alpha(x_p) - \alpha(z)) \\
&= \alpha(y_q) - \alpha(x_p).
\end{aligned}$$

which means that τ and τ' differ by a coboundary in $A^{\check{C}(X)}$, so φ is injective. □

From (1.40) we obtain the following diagram of abelian groups for every $k \geq 0$

$$\begin{array}{ccc}
\check{H}^k_{\infty}(X, A) & \underset{\mathrm{res}^*}{\overset{\delta^*}{\rightleftarrows}} & \check{H}^k_{PIZ}(X, A) \\
 & {}^{(q\delta)^*}\nwarrow \qquad \nearrow^{q^*} & \\
 & \check{H}^k_{KWW}(X, A) &
\end{array} \tag{1.48}$$

We summarize everything we know about this diagram

1. In degree 0, all of the above maps are the identity by Corollary 1.5.38,
2. In degree 1, the map $(q\delta)^*$ is an isomorphism by Proposition 1.5.39,
3. The map δ^* is a retraction, namely $\delta^*\mathrm{res}^* = 1_{\check{H}^k_{\infty}(X,A)}$, so res^* is injective and δ^* is surjective for all $k \geq 0$,
4. We have $\mathrm{res}^*\delta^*q^* = q^*$ for all $k \geq 0$, thus the above diagram actually commutes
5. The map q^* is injective for $k = 1$ by [KWW21, Lemma 6.9]. Notice that this can also be seen by noting that since $(q\delta)^*$ is an isomorphism for $k = 1$, and res^* is injective, and since $\mathrm{res}^*\delta^*q^* = \mathrm{res}^*(q\delta)^* = q^*$, then q^* is injective for $k = 1$.

It is not currently known if any of the above maps are isomorphisms for all $k \geq 0$.

1.6 Principal Diffeological Bundles as Principal Infinity Bundles

Principal Infinity Bundles were defined in [NSS14a] and [NSS14b]. In this section, we compare this abstract notion to diffeological principal bundles.

Remark 1.6.1. The following two definitions are needed only for Definition 1.6.4 and are not used elsewhere in this paper.

Definition 1.6.2 ([DHI04, Section 3])**.** A map $f : X \to Y$ of simplicial presheaves on Cart is a **local fibration** if for every $U \in \mathsf{Cart}$, there exists a good open cover $\{U_i \subseteq U\}$ such that for every element U_i of the good open cover, there is a lift in every commutative diagram of the following form.

$$\begin{array}{ccccc} \Lambda^n_k & \longrightarrow & X(U) & \longrightarrow & X(U_i) \\ \big\downarrow & & & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y(U) & \longrightarrow & Y(U_i) \end{array} \tag{1.49}$$

Note that an objectwise fibration of simplicial presheaves is a local fibration. We say that a simplicial presheaf X is **locally fibrant** if the unique map $X \to *$ is a local fibration.

Definition 1.6.3 ([DI04, Theorem 6.15])**.** A map $f : X \to Y$ of simplicial presheaves on Cart is a **local weak equivalence** if for every $U \in \mathsf{Cart}$, there exists a good open cover $\{U_i \subseteq U\}$ such that for every element U_i of the good open cover, there is a dotted arrow in every commutative diagram of the following form,

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & RX(U) & \longrightarrow & RX(U_i) \\ \big\downarrow & & & \nearrow & \big\downarrow Rf \\ \Delta^n & \longrightarrow & RY(U) & \longrightarrow & RY(U_i) \end{array} \tag{1.50}$$

where R is a fibrant replacement functor for sSet and the top left triangle commutes strictly, while the bottom right triangle commutes up to a homotopy relative to $\partial\Delta^n \hookrightarrow \Delta^n$.

Note that an objectwise weak equivalence is a local weak equivalence, and [DHI04] proves that Čech weak equivalences are local weak equivalences.

Definition 1.6.4 ([NSS14b, Definition 3.79])**.** Let G be a presheaf of simplicial groups acting on a simplicial presheaf P by $\rho : P \times G \to P$. Then a map $\pi : P \to X$ is a **G-Principal ∞-bundle**[3] if:

1. π is a local fibration,
2. The action of G on P is fiberwise, namely $\rho(g, -)$ sends fibers to fibers, and
3. the map
$$P \times G \to P \times_X P$$
given by
$$(p, g) \mapsto (p, \rho(p, g))$$
is a local weak equivalence.

A map $P \xrightarrow{f} P'$ of G-principal ∞-bundles over X is a map that is G-equivariant and commutes with the bundle projections. Namely, it is a map fitting into the following commutative diagram:

$$\begin{array}{ccc} P \times G & \xrightarrow{f \times 1_G} & P' \times G \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho'} \\ P & \xrightarrow{f} & P' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

Let $\mathsf{Prin}_G^{\infty}(X)$ denote the category of G-principal ∞-bundles on X.

It is clear that if $\pi : P \to X$ is a diffeological principal G-bundle, then it is a G-principal ∞-bundle, when we think of X, G and P as discrete simplicial presheaves. This is because all

[3] In [NSS14b], what we call principal ∞-bundles are known as weakly principal G-bundles. Also, they only define this for simplicial sheaves, but there are no problems extending the definition and all of the theorems in that paper to simplicial presheaves.

maps between discrete simplicial presheaves are local fibrations, and all diffeomorphisms between diffeological spaces are local weak equivalences. Note that $\mathsf{DiffPrin}_G(X)$ is a groupoid, while in general $\mathsf{Prin}_G^\infty(X)$ is not a groupoid. So while these categories are not equivalent, we will prove that their nerves are weak homotopy equivalent.

Proposition 1.6.5. Let X be a locally fibrant simplicial presheaf and G a presheaf of simplicial groups. Then, there is a weak homotopy equivalence of simplicial sets

$$\mathbb{R}(X, \mathbf{B}G) \simeq N\mathsf{Prin}_G^\infty(X). \tag{1.51}$$

Proof. First if X and Y are locally fibrant simplicial presheaves, then combining [Low15, Lemma 6.4] with [Low15, Theorem 3.12] and [DK80, Corollary 4.7] proves that

$$\mathbb{R}(X, Y) \simeq N\text{Cocycle}(X, Y)$$

where $\text{Cocycle}(X, Y)$ is the cocycle category as defined in [Low15, Definition 3.1]. Then [NSS14b, Theorem 3.95] proves that

$$N\text{Cocycle}(X, \mathbf{B}G) \simeq N\mathsf{Prin}_G^\infty(X).$$

Since all presheaves of simplicial groups are locally fibrant, combining these gives the desired result. □

Since all diffeological spaces are locally fibrant, if X is a diffeological space and G is a diffeological group, to prove that $N\mathsf{DiffPrin}_G(X)$ is weak equivalent to $N\mathsf{Prin}_G^\infty(X)$, it suffices to show that $N\mathsf{DiffPrin}_G(X)$ is weak homotopy equivalent to $\mathbb{R}(X, \mathbf{B}G)$. Let us examine $\mathbb{R}(X, \mathbf{B}G)$ more deeply. In Section 1.5.2 we saw that this is equal to the simplicial set $\underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \mathbf{B}G)$. Now, if we consider the definition of QX given in Lemma 1.5.18, then by the same computation as (1.28), we have

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(QX, \mathbf{B}G) \cong \mathsf{Tot}(\mathbf{B}G(QX)). \tag{1.52}$$

A k-simplex of $\mathsf{Tot}(\mathbf{B}G(QX))$ contains a huge amount of information, but in this case, since $\mathbf{B}G$ is objectwise the nerve of a groupoid, most of this information will be redundant. Let us describe what a vertex of this simplicial set is. It is a map of cosimplicial simplicial sets $\Delta^\bullet \to \mathbf{B}G(QX)$. This means it is a commutative diagram of the form[4]:

$$\begin{array}{ccccccc}
\Delta^0 & \rightrightarrows & \Delta^1 & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & \Delta^2 & & \cdots \\
\downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^2} & & \\
\prod_{\mathsf{Plot}(X)} \mathbf{B}G(U_{p_0}) & \rightrightarrows & \prod_{N_1} \mathbf{B}G(U_{p_1}) & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & \prod_{N_2} \mathbf{B}G(U_{p_2}) & & \cdots
\end{array}$$

Let us unravel what this means. Firstly, g^0 contributes no information, as $\mathbf{B}G(U_{p_0})_0 = C^\infty(U_{p_0}, *) = *$. However, g^1 is the data of maps $g^1(f_0) \coloneqq g_{f_0} : U_{p_1} \to G$ for every map of plots $f_0 : U_{p_1} \to U_{p_0}$. Now g^2 is the data of a map $g^2(f_1, f_0) \coloneqq g_{f_1,f_0} : U_{p_2} \to G \times G$ for every pair of composable maps of plots $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$. Let $g^2(f_1, f_0) = (h, k)$. The data of the above cosimplicial map insists that

$$(d^0 g^1)(f_1, f_0) = g^1(f_0) \circ f_1 = (g^2 d^0)(f_1, f_0) = d_0(g^2(f_1, f_0)) = d_0(h, k) = k$$

$$(d^1 g^1)(f_1, f_0) = g^1(f_0 f_1) = (g^2 d^1)(f_1, f_0) = d_1(g^2(f_1, f_0)) = d_1(h, k) = kh \text{[5]}$$

$$(d^2 g^1)(f_1, f_0) = g^1(f_1) = (g^2 d^2)(f_1, f_0) = d_2(g^2(f_1, f_0)) = d_2(h, k) = h.$$

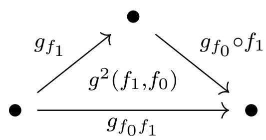
In other words

$$g_{f_0 f_1} = (g_{f_0} \circ f_1) \cdot g_{f_1}.$$

[4] Where we exclude the codegeneracy maps from the notation for clarity.

[5] If this seems strange, see Example 1.5.13.

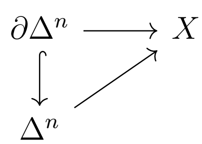
This is precisely the diffeological G-cocycle condition (2.9). We can visualize this as a triangle:



which is filled in if the cocycle condition (2.9) holds. In other words, a map $QX \to \mathbf{B}G$ is precisely the same information as a G-cocycle g on X.

Now here's an important point: g^3 will provide no further data. We will explain why using the notion of coskeleton.

Definition 1.6.6. A simplicial set X is **k-coskeletal** if for every boundary $\partial\Delta^n \to X$, there exists a unique n-simplex $\Delta^n \to X$ making the following diagram commute:



for all $n > k$.

For any k, let $\mathsf{sSet}_{\leq k}$ denote the category of k-truncated simplicial sets, namely presheaves on the full subcategory $\mathbf{\Delta}_{\leq k}$ of $\mathbf{\Delta}$ whose objects are partial orders $[n]$ for $n \leq k$. There is a functor $\tau_k : \mathsf{sSet} \to \mathsf{sSet}_{\leq k}$ just given by forgetting the higher simplices of the simplicial set. This functor has a fully faithful left adjoint sk_k and a fully faithful right adjoint cosk_k. A simplicial set X is k-coskeletal if the unit of the adjunction $X \to \mathrm{cosk}_k(X)$ is an isomorphism. For more details see [GJ12, Section VII.1].

If $X = N(\mathcal{C})$ is the nerve of a category $\mathcal{C}$, then X is 2-coskeletal [GJ12, Lemma I.3.5]. In our case $\mathbf{B}G(QX)$ is a cosimplicial simplicial set such that $\mathbf{B}G(QX_n)$ is the nerve of a groupoid and therefore 2-coskeletal for every n. Now as we've seen, the 3-simplex $g^3 \in \mathbf{B}G(QX_3)$ is required to satisfy that $\partial g^3 = (d^0 g^1, d^1 g^1, d^2 g^1, d^3 g^1)$. But that means we've just specified a 3-boundary in a 2-coskeletal simplicial set. Thus there exists a unique filler g^3. This of course continues, so that a vertex $g \in \mathsf{Tot}(\mathbf{B}G(QX))_0$ determines and is completely determined by g^1 and g^2.

Let us repeat the above analysis for a 1-simplex in $\mathsf{Tot}(\mathbf{B}G(QX))$. This is the data of a commutative diagram:

$$\begin{array}{ccccccc}
\Delta^0 \times \Delta^1 & \rightrightarrows & \Delta^1 \times \Delta^1 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \Delta^2 \times \Delta^1 & & \cdots \\
\downarrow{\scriptstyle h^0} & & \downarrow{\scriptstyle h^1} & & \downarrow{\scriptstyle h^2} & & \\
\prod_{N_0} \mathbf{B}G(U_{p_0}) & \rightrightarrows & \prod_{N_1} \mathbf{B}G(U_{p_1}) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \prod_{N_2} \mathbf{B}G(U_{p_2}) & & \cdots
\end{array}$$

Now unravelling this diagram, skipping some similar details, such a 1-simplex consists of the following data. If g and g' are 0-simplices in $\mathsf{Tot}(\mathbf{B}G(QX))$ consisting of collections of maps $\{g_f\}$ and $\{g'_f\}$, then a 1-simplex is a collection of maps $\{h_{p_0} : U_{p_0} \to G\}$ indexed by plots $p_0 : U_{p_0} \to X$ such that if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, then

$$g'_{f_0} \cdot h_{p_1} = (h_{p_0} \circ f_0) \cdot g_{f_0},$$

and this is precisely a morphism of diffeological G-cocycles (2.10). By the same reasoning as before, the rest of the diagram provides no further conditions on this data, as the maps $\Delta^k \times \Delta^1 \to \mathbf{B}G(QX_k)$ will consist of $(k+1)$-simplices, and $\mathbf{B}G(QX_k)$ is 2-coskeletal, so that h depends only on h^0 and h^1. Namely given h^0 and h^1, the h^k for $k > 1$ are fully determined.

A 2-simplex in $\mathsf{Tot}(\mathbf{B}G(QX))$ will similarly be completely determined by its boundary. Similar reasoning also proves that there are no additional conditions coming from higher k-simplices of $\mathsf{Tot}(\mathbf{B}G(QX))$. In other words, $\mathsf{Tot}(\mathbf{B}G(QX))$ is 2-coskeletal. Further, since $\mathsf{s\check{P}re(Cart)}$ is a simplicial model category and $\mathbb{R}(X, \mathbf{B}G) \cong \mathsf{Tot}(\mathbf{B}G(QX))$, this implies that $\mathsf{Tot}(\mathbf{B}G(QX))$ is a Kan complex. This combined with the fact that it is 2-coskeletal implies that for any basepoint g, $\pi_k(\mathsf{Tot}(\mathbf{B}G(QX)), g) = 0$ for $k > 1$.

Now that we have an explicit description of $\mathsf{Tot}(\mathbf{B}G(QX))$ it is clear that this is nothing more than a diffeological version of the cocycle construction from classical differential geometry. Let us formalize this now.

We want to construct a map $\Phi : \mathsf{Tot}(\mathbf{B}G(QX)) \to N\mathsf{Coc}(X, G)$. Consider the left adjoint

$h : \mathsf{sSet} \to \mathsf{Cat}$ to the nerve functor N, that sends a simplicial set to its homotopy category [Rie14, Example 1.5.5], namely if X is a simplicial set, then hX is the category whose objects are the vertices of X, morphisms are freely generated by the 1-simplices of X and then quotiented by the 2-simplices, in the sense that if σ is a 2-simplex in X with $d_0\tau = x, d_1\tau = y, d_2\tau = z$, then $x \circ z = y$ in hX. Note that by unravelling the above definitions, the composition of two morphisms $h' \circ h$ in $h\mathsf{Tot}(\mathbf{B}G(QX))$ is given by multiplication $h' \cdot h$ as in Definition 2.2.6.

Let $\Phi : h\mathsf{Tot}(\mathbf{B}G(QX)) \to \mathsf{Coc}(X, G)$ denote the functor that sends an object $g = (g^0, g^1, \dots)$ to the cocycle it defines $\{g_{f_0}\}$, and a morphism $h = (h^0, h^1, \dots)$ to the morphism of cocycles it defines $\{h_p\}$. By the above discussion it is evident that this functor defines (one half of) an isomorphism of categories. In summary we have proved the following.

Lemma 1.6.7. There is an isomorphism of simplicial sets

$$N\mathsf{Coc}(X, G) \cong \mathsf{Tot}(\mathbf{B}G(QX)). \tag{1.53}$$

We are now in a position to prove the main theorem of this paper.

Theorem 1.6.8. The nerve of the category of diffeological principal G-bundles on X and the nerve of the category of G-principal ∞-bundles on X are weak homotopy equivalent

$$N\mathsf{Prin}^{\infty}_G(X) \simeq N\mathsf{DiffPrin}_G(X). \tag{1.54}$$

Proof. Since the nerve functor sends equivalences of categories to homotopy equivalences of simplicial sets, Theorem 1.3.15 implies that there is a homotopy equivalence of simplicial sets $N\mathsf{Coc}(X, G) \simeq N\mathsf{DiffPrin}_G(X)$. Combining this with Proposition 1.6.5 and Lemma 1.6.7 gives the result. □

Corollary 1.6.9. Given a diffeological space X and a diffeological group G, there is an isomorphism of pointed sets

$$\check{H}^1_\infty(X,G) \cong \pi_0 \mathsf{DiffPrin}_G(X), \tag{1.55}$$

where $\pi_0\mathsf{DiffPrin}_G(X)$ denotes the set of isomorphism classes of diffeological principal G-bundles on X, pointed at the isomorphism class of trivial bundles.

We can still say more about the correspondence of Theorem 1.6.8. As in the paper [NSS14b], it is useful to see how one can obtain an actual diffeological principal G-bundle $\pi : P \to X$ from a G-cocycle $QX \to \mathbf{B}G$ using simplicial presheaves. It is basically a reformulation of Theorem 1.3.15.

Given a diffeological group G, consider the diffeological groupoid

$$G \times G \underset{m}{\overset{\mathrm{pr}_1}{\rightrightarrows}} G$$

where the source map is the first projection and the target map is the map $m(g,h) = hg$[6]. We can visualize morphisms in this groupoid by

$$g \xrightarrow{(g,h)} hg \xrightarrow{(hg,k)} khg \qquad = \qquad g \xrightarrow{(g,kh)} khg.$$

This defines a presheaf of groupoids on Cart by

$$\left[U \mapsto \left(C^\infty(U, G\times G) \rightrightarrows C^\infty(U,G)\right)\right].$$

Applying the nerve functor objectwise gives a simplicial presheaf which we denote by $\mathbf{E}G$. There is a canonical map of simplicial presheaves $\mathbf{E}G \to \mathbf{B}G$ induced by the corresponding

[6]If this seems strange, see Example 1.5.13

map of diffeological groupoids/presheaves of groupoids:

$$\begin{array}{ccc} G \times G & \overset{\mathrm{pr}_1}{\underset{m}{\rightrightarrows}} & G \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow \\ G & \rightrightarrows & * \end{array}$$

This functor can be visualized as

$$\begin{array}{ccc} g & \xrightarrow{(g,h)} & hg \\ \downarrow & & \downarrow \\ * & \xrightarrow{h} & * \end{array}$$

Furthermore this map is an objectwise Kan fibration, and therefore a projective fibration of simplicial presheaves.

Remark 1.6.10. The simplicial presheaf $\mathbf{E}G$ and the map $\mathbf{E}G \to \mathbf{B}G$ described above are well known in the literature in the form $WG \to \overline{W}G$, which can be defined when G is any presheaf of simplicial groups, see [Ste12] or [NSS14b]. In this case we are using the same convention as [GJ12] where $WG = \mathrm{Dec}^0 \overline{W}G$ and the map $WG \to \overline{W}G$ is given degreewise by $d_0^{\overline{W}G} : G^{\times k+1} \to G^{\times k}$.

Now with such a map $g : QX \to \mathbf{B}G$, we can consider the following pullback in the category of simplicial presheaves.

$$\begin{array}{ccc} \widetilde{P} & \longrightarrow & \mathbf{E}G \\ \downarrow & \lrcorner & \downarrow \\ QX & \xrightarrow{g} & \mathbf{B}G \end{array} \tag{1.56}$$

Remark 1.6.11. In the situation above, $\widetilde{P}$ is a G-principal ∞-bundle, and the construction of taking this pullback is precisely the map Rec described in [NSS14b, Definition 3.93].

Thus $\widetilde{P}$ is a simplicial presheaf, or equivalently a simplicial diffeological space such that

$$\widetilde{P}_1 = \coprod_{U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_1} \times_G (G \times G), \qquad \widetilde{P}_0 = \coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \times G.$$

In other words, $\widetilde{P}_0$ is precisely the diffeological space $\widehat{P}$ from Section 1.3, and the coequalizer of the face maps $\widetilde{P}_1 \rightrightarrows \widetilde{P}_0$, which is precisely the presheaf obtained by taking $\pi_0 \widetilde{P}$, is precisely the diffeological space $P = \mathsf{Cons}(g)$ constructed in Section 1.3.

Now let us show that the canonical map $q : \widetilde{P} \to P$ is an objectwise weak equivalence. First we notice that QX, $\mathbf{B}G$, and $\mathbf{E}G$ are nerves of presheaves of categories. Since the nerve functor is a right adjoint, we have that

$$\widetilde{P} \cong N[QX_1 \rightrightarrows B] \times_{N[G \rightrightarrows *]} N[G \times G \rightrightarrows G] \cong N([QX_1 \rightrightarrows B] \times_{[G \rightrightarrows *]} [G \times G \rightrightarrows G]).$$

Thus $\widetilde{P}$ as a simplicial presheaf is the nerve of a diffeological category/presheaf of categories, a morphism of which we can visualize in the same way as in Section 1.3

$$(x_{p_1}, k_1) \xrightarrow{f_0} (x_{p_0}, k_0),$$

where $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1}) \cdot k_1$.

Now think of P as a presheaf of discrete categories with objects equivalence classes $[x_{p_0}, k_0]$. The map $q : \widetilde{P} \to P$ can be seen as the nerve of the functor

$$\begin{array}{ccc} (x_{p_1}, k_1) & \xrightarrow{f_0} & (x_{p_0}, k_0) \\ \downarrow & & \downarrow \\ [x_{p_1}, k_1] & = & [x_{p_0}, k_0] \end{array}$$

Now consider the map $f : P \to \widetilde{P}$ defined as follows. If $[x_{p_0}, k_0]$ is an object in P, then let $x = p_0(x_{p_0})$. Then consider the pair $(*_x, e_G)$, where $x : * \to X$ is the plot sending $*$ to the point $x \in X$. There is a unique morphism

$$(*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0) \xrightarrow{x_{p_0}} (x_{p_0}, k_0),$$

given by the map of plots

$$\begin{array}{ccc} * & \xrightarrow{\quad x_{p_0} \quad} & U_{p_0} \\ & {\scriptstyle x}\searrow \qquad \swarrow{\scriptstyle p_0} & \\ & X & \end{array}$$

So set $f[x_{p_0}, k_0] = (*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0)$. Then

$$\begin{aligned} \widetilde{P}(f[x_{p_1}, k_1], (x_{p_0}, k_0)) &= \widetilde{P}((*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0), (x_{p_0}, k_0)) \\ &\cong P([*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0], [x_{p_0}, k_0]), \end{aligned} \tag{1.57}$$

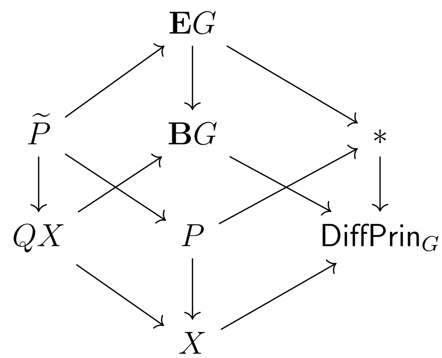
where the second isomorphism holds because the map $(*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0) \xrightarrow{x_{p_0}} (x_{p_0}, k_0)$ is the unique map between the source and target, and the existence of such a map means that $[*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0] = [x_{p_0}, k_0]$. Thus

$$\widetilde{P}((*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0), (x_{p_0}, k_0)) \cong P([*_x, g_{f_0}^{-1}(x_{p_0}) \cdot k_0], [x_{p_0}, k_0]) \cong *.$$

Therefore f is an objectwise left adjoint to q. Since the nerve functor takes adjoint functors to homotopy equivalences of simplicial sets, we have proven the following.

Lemma 1.6.12. The map $q : \widetilde{P} \to P$ of simplicial presheaves is an objectwise homotopy equivalence and therefore a Čech weak equivalence.

Now notice that $\mathbf{E}G$ is objectwise contractible, indeed, for every U the groupoid $[G(U) \times G(U) \rightrightarrows G(U)]$ has an initial object given by the constant map at the identity element e_G. Thus $\mathbf{E}G \to *$ is an objectwise weak equivalence. The map $\mathbf{B}G \to \mathsf{DiffPrin}_G$ is also an objectwise weak equivalence, see Example 1.5.13 and Proposition 1.3.5. Thus there is a map of diagrams, where each component is an objectwise weak equivalence of simplicial presheaves



Thus we have proven the following result.

Corollary 1.6.13. Given a diffeological space X, diffeological group G, and diffeological principal G-bundle $\pi : P \to X$, the commutative diagram of ∞-stacks

$$\begin{array}{ccc} P & \longrightarrow & * \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathsf{DiffPrin}_G \end{array} \tag{1.58}$$

where $g : X \to \mathsf{DiffPrin}_G$ sends a plot $p_0 : U_{p_0} \to X$ to the diffeological principal G-bundle $p_0^* P$, is a homotopy pullback square in the Čech model structure on $\mathsf{sPre}(\mathsf{Cart})$.

Proof. Since f is an objectwise weak equivalence of simplicial presheaves, and since $\widetilde{P}$ is the actual pullback of a projective fibration, it is a homotopy pullback in the projective model structure on simplicial presheaves on Cart. By Proposition [Rez10, Proposition 11.2], it is also a homotopy pullback in the Čech model structure. □

1.7 Comparison of Site Structures

Here we will prove that the definition of diffeological spaces as given in Definition 2.2.2 is equivalent to that usually presented in the literature, such as [Igl13, Article 1.5], in the sense that their categories are equivalent. Further we will show other possible alternative definitions that have not appeared in the literature have equivalent categories as well. An example of this is [WW14, Lemma 2.9]. The results of this section include this result.

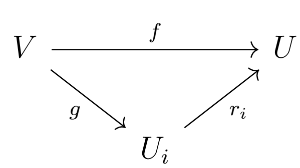
Everything in this section is well-known and similar statements can be found in [SŠ10, Section 6.2], [Sch13] and [FSS+12, Appendix], but we felt that having the details of these results spelled out would be helpful to those less familiar with topos theory.

We will prove these results by exploiting Theorem 1.4.16, and studying concrete sheaves over the smooth sites. Now coverages are those collections of families with the least amount of structure with which we can define sheaves on $\mathcal{C}$. There could be many different coverages which give rise to equivalent categories of sheaves. It can therefore be difficult to see directly when coverages give rise to the same sheaves. We will define a more restricted kind of coverage, known as a Grothendieck coverage or Grothendieck topology, which will make such comparison easier.

Definition 1.7.1. A **sieve** R is a family of morphisms that is closed under precomposition, namely if $V \xrightarrow{g} U_i$ is a map in $\mathcal{C}$, and $U_i \xrightarrow{r_i} X \in R$, then $V \xrightarrow{r_i g} X \in R$.

Given a category $\mathcal{C}$ and an object $U \in \mathcal{C}$, there is a bijection between sieves on X and subfunctors $R \hookrightarrow yU$, where $yU = (V \mapsto \mathcal{C}(V, U))$ denotes the Yoneda embedding on U. Indeed, given a sieve R, we can define a subfunctor $\widetilde{R} \hookrightarrow yU$ by setting $\widetilde{R}(V) = \{f : V \to U : f \in R\}$ and noting that being a sieve implies that $\widetilde{R}$ is functorial under precomposition, and conversely if $\widetilde{R} \hookrightarrow yU$ is a subfunctor, then we can define a sieve R by setting $R = \bigcup_{V \in \mathcal{C}} \widetilde{R}(V)$. Thus for the rest of this section a sieve will mean both a kind of family of morphisms and a subfunctor of the Yoneda embedding. If $U \in \mathcal{C}$ is an object, then we call yU the **maximal sieve**. This is equivalently the family of all morphisms with codomain U.

For any family of morphisms $r = \{r_i : U_i \to U\}$ over U, we can construct the smallest sieve $R = \overline{r}$ containing it as follows. Let R be the set of morphisms $f : V \to U$ such that f factors as:



where $r_i : U_i \to U \in r$, and g is a morphism in $\mathcal{C}$. In this case we say that r **generates** the sieve R.

Lemma 1.7.2 ([Joh02, C2.1 Lemma 2.1.3])**.** Suppose that j is a coverage on a category $\mathcal{C}$. Then a presheaf F is a sheaf on a family of morphisms $r = \{U_i \to U\}$ if and only if it is a sheaf on the sieve $R = \overline{r}$ it generates.

Definition 1.7.3. We say that a collection of families j is **sifted** if every $r \in j(U)$ is a sieve. If j is further a coverage, we call it a sifted coverage. We call covering families of sifted coverages **covering sieves**.

Lemma 1.7.4. Let R be a sieve over an object U in a category $\mathcal{C}$ and F a presheaf on $\mathcal{C}$. A collection $\{s_f \in F(V)\}_{f \in R}$ of sections for every $f : V \to U$ in R is a matching family if and only if $F(g)(s_f) = s_{fg}$ for every morphism $g : W \to V$ in $\mathcal{C}$.

Proof. ($\Rightarrow$) Suppose $\{s_f\}$ is a matching family, then consider the commutative diagram:

$$\begin{array}{ccc} W & = & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle gf} \\ V & \xrightarrow[f]{} & U \end{array}$$

this implies that $F(g)(s_f) = s_{fg}$.

($\Leftarrow$) Suppose we have a commutative diagram:

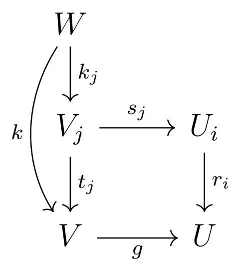
$$\begin{array}{ccc} A & \xrightarrow{h} & V' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ V & \xrightarrow[f]{} & U \end{array}$$

where $f, f' \in R$. Then $F(g)(s_f) = s_{fg} = s_{f'h} = F(h)(s_{f'})$, thus $\{s_f\}$ is a matching family. □

If j is a coverage, then let $\overline{j}$ denote the collection of families where $R \in \overline{j}(U)$ if $R = \overline{r}$ for some $r \in j(U)$. We call $\overline{j}$ the **sifted closure** of j.

Lemma 1.7.5. The collection of families $\overline{j}$ is a sifted coverage of $\mathcal{C}$.

Proof. Clearly $\overline{j}$ is sifted. We wish to show it is a coverage. Suppose we have a covering family $R \in \overline{j}(U)$, and a map $g : V \to U$. We wish to show that there is a covering family $R' \in \overline{j}(V)$ such that for every map $k \in R'$, gk factors through some $l \in R$. Since $R = \overline{r}$, we know that since j is a coverage, there exists some covering family $t \in j(V)$ with the corresponding property. In other words, for every map $k \in R'$ there is a commutative diagram:

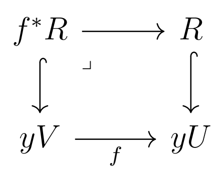


but then $l = r_i s_j k_j$ is a morphism in $R = \overline{r}$. Thus gk factors through l as it is equal to it. □

Corollary 1.7.6. Given a coverage j on a category $\mathcal{C}$, a presheaf F is a sheaf on $(\mathcal{C}, j)$ if and only if it is a sheaf on $(\mathcal{C}, \overline{j})$. In other words $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, \overline{j})$.

Proof. This follows from Lemma 1.7.2 and Lemma 1.7.5. □

Now if $R \hookrightarrow yU$ is a sieve, and $f : V \to U$ is a morphism in $\mathcal{C}$, then let f^*R denote the set of morphisms $g : W \to V$ such that $fg \in R$. This is equivalently the subfunctor $f^*R \hookrightarrow yV$ given by the pullback in $\mathsf{Pre}(\mathcal{C})$



Definition 1.7.7. A **Grothendieck coverage** is a sifted collection of families J on a category $\mathcal{C}$ satisfying the following conditions:

(C) J is a coverage,

(M) for any object $U \in \mathcal{C}$, the maximal sieve $yU \in J(U)$, and

(L) if $R \in J(U)$ and S is another sieve on U such that for each $f : V \to U \in R$, the sieve $f^*(S)$ belongs to $J(V)$.

If $(\mathcal{C}, J)$ is a Grothendieck coverage, then we call its sieves $R \in J(U)$ **covering sieves**.

Remark 1.7.8. Grothendieck coverages are usually referred to as Grothendieck topologies in the literature, but are typically presented with the following condition (C'): If $R \in J(U)$ and $f : V \to U$ any morphism in $\mathcal{C}$, then $f^*R \in J(V)$, instead of the condition (C). It is not hard to show that these are equivalent definitions, see [Joh02, C2.1 Page 541].

Lemma 1.7.9. Let $\mathcal{C}$ be a category and $R \hookrightarrow yU$ a sieve. If $g : V \to U$ is a map in R, then $g^*R = yV$.

Proof. If R is a sieve on U, then $g^*R = \{f : W \to V \ : \ gf \in R\}$. But R is a sieve and $g \in R$, so every map $f : W \to V$ has this property, since R is closed under precomposition. □

Lemma 1.7.10. Let $(\mathcal{C}, J)$ be a site with a Grothendieck coverage. Then if R, R' are sieves on U, $R \subseteq R'$ and R is a covering sieve, then R' is a covering sieve.

Proof. Let $g : V \to U \in R \subseteq R'$, then by Lemma 1.7.9, we know that $g^*R = g^*R' = yV$, which is a covering sieve of V by (M). Since this is true for all $g \in R$, R' is a covering sieve by (L). □

Given a set $\{J_\alpha \ : \ \alpha \in A\}$ of Grothendieck coverages, it is not hard to check that the collection of families $J \coloneqq \bigcap_{\alpha \in A} J_\alpha$ defined by $J(U) = \bigcap_{\alpha \in A} J_\alpha(U)$ is a Grothendieck coverage. Thus if j is a coverage on $\mathcal{C}$, we can consider $\overline{j}$, its sifted closure. By Lemma 1.7.2, we can then take the intersection of the set of Grothendieck coverages that contain all of the covering sieves of $\overline{j}$, which we denote by $\tau(j)$. This will be the smallest Grothendieck coverage containing j and we will call it the Grothendieck coverage generated by j.

Lemma 1.7.11 ([Joh02, C2.1 Proposition 2.1.9])**.** Given a site $(\mathcal{C}, j)$, a presheaf F will be a sheaf on $(\mathcal{C}, j)$ if and only if it is a sheaf on $(\mathcal{C}, \tau(j)\,)$.

Now we are in a position to compare different coverages on the same category. Suppose that j, j' are coverages on a category $\mathcal{C}$ such that if r' is a covering family in $j'(U)$, then there exists a covering family $r \in j(U)$ and a refinement $f : r \to r'$. We will say that j' is **subordinate** to j and write $j' \leq j$.

Proposition 1.7.12. Suppose that j, j' are coverages on a category $\mathcal{C}$ such that $j \leq j'$ and $j' \leq j$, then $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, j')$.

Proof. Suppose that $j' \leq j$. Then every covering family $r' \in j'(U)$ can be refined by a covering family $r \in j(U)$. Therefore $\overline{r} \subseteq \overline{r'}$, since sieves are closed under precomposition. Now note that $\overline{r} \in \tau(j)(U)$, and thus by Lemma 1.7.10, $\overline{r'} \in \tau(j)(U)$. Thus if r' is a covering family of j', then $\overline{r'}$ is a covering sieve of $\tau(j)$. Thus if F is a sheaf on j, then by Lemma 1.7.11, it will be a sheaf on $\tau(j)$, so it will then be a sheaf on $\overline{r'}$, and thus by Lemma 1.7.2 it will be a sheaf on r'. Since r' was arbitrary, F is therefore a sheaf on all of j'. Thus if F is a sheaf on $(\mathcal{C}, j)$, then it will be a sheaf on $(\mathcal{C}, j')$. Conversely $j \leq j'$ proves that $\mathsf{Sh}(\mathcal{C}, j) = \mathsf{Sh}(\mathcal{C}, j')$. □

Proposition 1.7.13. Let j_{good} denote the good cover coverage on Man defined in Example 1.4.5, and j_{open} denote the open cover coverage on Man defined in Example 1.4.4. Then $\mathsf{Sh}(\mathsf{Man}, j_{\text{good}}) = \mathsf{Sh}(\mathsf{Man}, j_{\text{open}})$. This similarly holds for Cart and Open.

Proof. By [BT+82, Corollary 5.2], we have that $j_{\text{open}} \leq j_{\text{good}}$. Now $j_{\text{good}} \leq j_{\text{open}}$, since every good open cover is in particular an open cover. □

Corollary 1.7.14. The categories of concrete sheaves on Man with the open and good open coverages agree:

$$\mathsf{ConSh}(\mathsf{Man}, j_{\text{good}}) = \mathsf{ConSh}(\mathsf{Man}, j_{\text{open}}).$$

This result remains true if we replace Man with Open or Cart.

Now we wish to compare sites whose underlying categories differ. Let $\mathcal{C}$ be a category and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a full subcategory. Then a sieve $R \hookrightarrow yU$ on $\mathcal{C}$ is said to be a $\mathcal{C}'$-sieve if it is generated by a family of morphisms all of whose domains are objects in $\mathcal{C}'$.

Definition 1.7.15. Let $(\mathcal{C}, J)$ be a category with a Grothendieck coverage, and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a full subcategory. We say that $\mathcal{C}'$ is ***J*-dense** in $\mathcal{C}$ if every object $U \in \mathcal{C}$ has a covering sieve $R \in J(U)$ that is a $\mathcal{C}'$-sieve.

If $(\mathcal{C}, j)$ is a site where j is not necessarily a Grothendieck coverage, then we say that a full subcategory $\mathcal{C}' \hookrightarrow \mathcal{C}$ is j-dense if it is $\tau(j)$-dense in $(\mathcal{C}, \tau(j))$.

By [BT+82, Theorem 5.1], every finite dimensional smooth manifold has a good open cover. Thus if $\mathcal{U} = \{U_i \subseteq M\}$ denotes a good open cover of M, then $\overline{\mathcal{U}}$ is a covering sieve of $(\mathsf{Man}, \tau(j_{\text{open}}))$ and it is a Cart-sieve. Since this is true for any manifold M, it follows that Cart is j_{open}-dense in $(\mathsf{Man}, j_{\text{open}})$. By the same argument Cart is also dense in $(\mathsf{Man}, j_{\text{good}})$. This also implies that Open is dense in (Man, j) for $j \in \{j_{\text{open}}, j_{\text{good}}\}$.

Now suppose $(\mathcal{C}, J)$ is a site with a Grothendieck coverage. If $\mathcal{C}' \hookrightarrow \mathcal{C}$ is a full subcategory, define a collection of families J' on $\mathcal{C}'$ by defining $J'(U)$ to be the collection of those covering sieves $R \in J(U)$ that are also $\mathcal{C}'$-sieves. It is not hard to show that this is also a Grothendieck coverage, called the **induced coverage** on $\mathcal{C}'$, and denoted $J|_{\mathcal{C}'}$.

The following result is well-known in the literature as the **Comparison Lemma**.

Theorem 1.7.16 ([Joh02, Theorem 2.2.3])**.** Let $(\mathcal{C}, J)$ be a site with a Grothendieck coverage and $\mathcal{C}' \hookrightarrow \mathcal{C}$ a J-dense full subcategory. Then the restriction functor $\text{res} : \mathsf{Pre}(\mathcal{C}) \to \mathsf{Pre}(\mathcal{C}')$ itself restricts to a functor $\text{res} : \mathsf{Sh}(\mathcal{C}, J) \to \mathsf{Sh}(\mathcal{C}', J|_{\mathcal{C}'})$, and this functor is an equivalence of categories.

Note that $\tau(j^{\mathsf{Cart}}_{\text{open}}) = \tau(j^{\mathsf{Man}}_{\text{open}})|_{\mathsf{Cart}}$. This can be seen by simply noting that every sieve in $\tau(j^{\mathsf{Man}}_{\text{open}})|_{\mathsf{Cart}}$ is generated by a open cover by cartesian spaces, and contains every such sieve. A similar argument proves the same for $j^{\mathsf{Cart}}_{\text{good}}$ and $j^{\mathsf{Open}}_{\text{good}}, j^{\mathsf{Open}}_{\text{open}}$.

Corollary 1.7.17. All categories of the form $\mathsf{ConSh}(\mathcal{C}, j)$ for $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$ are equivalent.

Proof. Theorem 1.7.16 implies that the categories $\mathsf{Sh}(\mathcal{C}, j)$ for $\mathcal{C} \in \{\mathsf{Cart}, \mathsf{Open}, \mathsf{Man}\}$ and $j \in \{j_{\text{open}}, j_{\text{good}}\}$ are all equivalent. Further, using the same argument as in the proof of [WW14, Lemma 2.9], the above equivalences restrict to equivalences of all the full subcategories $\mathsf{ConSh}(\mathcal{C}, j)$ of concrete sheaves. □

Thus by Theorem 1.4.15, we have that $\mathsf{Diff}' \simeq \mathsf{ConSh}(\mathsf{Open}, j_{\text{open}})$, and by Corollary 1.7.17, we have that $\mathsf{ConSh}(\mathsf{Open}, j_{\text{open}}) \simeq \mathsf{ConSh}(\mathsf{Cart}, j_{\text{good}}) \simeq \mathsf{Diff}$. Thus we have proved the main proposition of this section.

Proposition 1.7.18. The category of classical diffeological spaces Diff' is equivalent to the category of diffeological spaces Diff introduced in Definition 2.2.2.

Chapter 2

The Diffeological Čech-de Rham Obstruction

2.1 Introduction

Classical differential geometry involves the study of finite dimensional smooth manifolds. As a theory, it has many achievements. One of its most celebrated is the **Čech-de Rham Theorem**, more commonly known as the de Rham Theorem[1]. The Čech-de Rham Theorem, proven in 1931 by de Rham [deR31], states that if M is a finite dimensional smooth manifold, then there is an isomorphism

$$H^k_{\mathrm{dR}}(M) \cong \check{H}^k(M, \mathbb{R}^\delta), \tag{2.1}$$

where $H^k_{\mathrm{dR}}(M)$ denotes the de Rham cohomology of M, and $\check{H}^k(M, \mathbb{R}^\delta)$ denotes the Čech cohomology of M with values in $\mathbb{R}^\delta$, the constant sheaf on the discrete group of real numbers. There are many good textbook accounts of the Čech-de Rham Theorem, such as [BT+82, Chapter II] and [GQ22, Chapter 9]. The de Rham cohomology of a finite dimensional smooth manifold M is constructed using its smooth structure, but the Čech-de Rham Theorem shows that the de Rham cohomology of M is independent of this smooth structure and depends

[1] We call it the Čech-de Rham Theorem because some authors use “the de Rham Theorem” to refer to the isomorphism between de Rham cohomology and singular cohomology.

only on the topology of M.

Diffeology is a modern framework for differential geometry whose main objects of study are diffeological spaces, encompassing smooth manifolds, orbifolds, and mapping spaces. The category of diffeological spaces is better behaved than the category of finite dimensional smooth manifolds, indeed the category of diffeological spaces is complete, cocomplete and cartesian closed [Igl13]. This makes diffeological spaces attractive to geometers who study spaces that are not finite dimensional smooth manifolds. However, this generalization comes at the cost of losing many of the theorems and constructions of classical differential geometry[2]. Much contemporary work has gone into extending these constructions and theorems to diffeological spaces. The textbook [Igl13] by Iglesias-Zemmour has in particular pushed the theory quite far, defining differential forms, de Rham cohomology, singular cohomology, fiber bundles, and smooth homotopy groups of diffeological spaces amongst many other contributions.

In [Igl88], Patrick Iglesias-Zemmour proved that the Čech-de Rham Theorem does not hold in general for diffeological spaces. Interestingly, this result was written as a preprint in French in the late 80s and was only recently published in English as [Igl23]. Furthermore Iglesias-Zemmour obtained an exact sequence

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_{PIZ}(X, \mathbb{R}^\delta) \to {}^dE_2^{1,0}(X) \xrightarrow{c_1} H^2_{\mathrm{dR}}(X) \to \check{H}^2_{PIZ}(X, \mathbb{R}^\delta)$$

which is a receptacle for the obstruction to the Čech-de Rham Theorem. The group ${}^dE_2^{1,0}(X)$ is the subgroup of the group of isomorphism classes of diffeological principal $\mathbb{R}$-bundles that admit a connection. If this group is trivial, as it is for all finite dimensional smooth manifolds, then $H^1_{\mathrm{dR}}(X) \cong \check{H}^1_{PIZ}(X)$. However, the situation for higher degrees is not addressed in [Igl23]. Iglesias-Zemmour writes "We must acknowledge that the geometrical natures of the higher obstructions of the De Rham theorem still remain uninterpreted. It would be certainly

[2]Many of these theorems are lost because not all diffeological spaces have partitions of unity, a crucial ingredient to many theorems in differential geometry.

interesting to pursue this matter further" [Igl23, Page 2]. In this paper, we obtain such an interpretation of the higher obstructions.

In Section 1.5.3, we introduced a generalization of Čech cohomology for diffeological spaces that we call **∞-stack cohomology**. If X is a diffeological space and A is a diffeological abelian group, then $\check{H}^k_\infty(X, A)$ denotes the kth ∞-stack cohomology of X with values in A.

Currently, there are four definitions of Čech cohomology for diffeological spaces in the literature. They are Iglesias-Zemmour's cohomology from [Igl23], which we call PIZ cohomology, there is ∞-stack cohomology (Section 1.5.3), there is Krepski-Watts-Wolbert cohomology [KWW21] and there is Ahmadi's cohomology [Ahm23]. In Section 1.5.3, the first three Čech cohomologies were compared, and some relationships deduced, but it is currently unknown if any of the above cohomology theories agree in general.

This paper is a sequel to Section 1.6, where we explored the connection between diffeological spaces and higher topos theory to study diffeological principal G-bundles. When G is a diffeological group, not necessarily abelian, it is still possible to define ∞-stack cohomology in degree 1, $\check{H}^1_\infty(X, G)$. We proved that degree 1 ∞-stack cohomology is in bijection with isomorphism classes of diffeological principal G-bundles over X. In fact, we obtained a much stronger result (Theorem 1.6.8), by showing that the nerve of the groupoid of diffeological principal G-bundles is weak equivalent to the nerve of the category of G-principal ∞-bundles on X.

In this paper, we study two cases where the tools of higher topos theory help us better understand diffeological spaces. The first case is studying the ∞-stack cohomology of the irrational torus. The irrational torus was the first example of a nontrivial diffeological space with trivial underlying topology, see [Igl20b]. In [Igl23], Iglesias-Zemmour proved that if $K \subset \mathbb{R}$ is a diffeologically discrete subgroup, then the PIZ cohomology of the irrational torus $T_K = \mathbb{R}/K$ is isomorphic to the group cohomology of K with values in $\mathbb{R}$. However, his proof of this, [Igl23, Section II], is computational. In Section 2.4, we prove

Theorem 2.4.4. There is an isomorphism

$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$$

of abelian groups, for every $n \geq 0$, where $\mathbb{R}^\delta$ denotes the discrete group of real numbers, and where $H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$ denotes the group cohomology of K with coefficients in $\mathbb{R}^\delta$.

Theorem 2.4.4 supports the conjecture that PIZ cohomology and ∞-stack cohomology agree. The proof of Theorem 2.4.4 is short and conceptual. It uses the shape functor $\int$, much beloved by higher differential geometers [BBP22], [Bun22b], [Sch13], [Clo23], [Car15], in a crucial way, reducing the ∞-stack cohomology of T_K to the singular cohomology of the classifying space $\mathbf{B}K$. This demonstrates the advantage of using ∞-stack cohomology to study diffeological spaces.

The second case, which makes up the bulk of the paper, is to use ∞-stack cohomology, and more generally the framework of higher topos theory, to study the diffeological Čech-de Rham obstruction. First we obtain a homotopy pullback diagram of ∞-stacks.

Theorem 2.7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

$$\begin{array}{ccccccc}
* & \longrightarrow & \mathbf{B}^k\mathbb{R}^\delta & \longrightarrow & * & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{B}^k_\nabla\mathbb{R} & \longrightarrow & \Omega^{k+1}_{\mathrm{cl}} & \longrightarrow & \Omega^{k+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{B}^k\mathbb{R} & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & \Omega^{1\leq\bullet\leq k+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \mathbf{B}^{k+1}\mathbb{R}^\delta & \longrightarrow & \mathbf{B}^{k+1}_\nabla\mathbb{R}
\end{array} \tag{2.2}$$

furthermore every commutative square in this diagram is a homotopy pullback square in the Čech model structure on simplicial presheaves over Cart.

Such diagrams are often used in higher category-theoretic treatments of differential cohomology, see [Sch13], [ADH21], [Jaz21]. One can think of an ∞-stack as a classifying object for a mathematical structure, such as diffeological principal G-bundles. Thus the above diagrams can be thought of as tight relationships between the corresponding mathematical structures.

Of particular interest is the ∞-stack $\mathbf{B}^k_\nabla \mathbb{R}$. This is the ∞-stack which classifies diffeological $\mathbb{R}$**-bundle** $(k-1)$**-gerbes with connection**. Cohomology with values in this ∞-stack is called the kth pure differential cohomology in [Jaz21]. From Theorem 2.7.1 we are immediately able to obtain the following result.

Corollary 2.7.2. For every diffeological space X, there is an exact sequence of vector spaces

$$0 \to \check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X, \mathbb{R}) \to \Omega^{k+1}_{\text{cl}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta). \tag{2.3}$$

Near the completion of this paper, we learned that an analogous exact sequence was also obtained in [Jaz21, Page 27] using completely different methods in the framework of homotopy type theory. The above exact sequence allows us to compute the pure differential cohomology of the irrational torus.

Theorem 2.7.3. Let T_α denote the irrational torus, then

$$\check{H}^k_{\infty,\nabla}(T_\alpha, R) \cong \begin{cases} \mathbb{R}^2, & k = 1, \\ \mathbb{R}, & k = 2, \\ 0, & k > 2. \end{cases} \tag{2.4}$$

While Corollary 2.7.2 is useful for computations with the irrational torus, it is desirable to have an exact sequence including de Rham cohomology rather than closed forms. This is obtained in the following result.

Theorem 2.7.5. Given a diffeological space X and $k \geq 1$, the sequence of vector spaces

$$\check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\text{conn}}(X, \mathbb{R}) \to H^{k+1}_{\text{dR}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta)$$

is exact.

When $k = 1$, we obtain an additional piece to this exact sequence.

Theorem 2.7.7. Given a diffeological space X, the sequence of vector spaces

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_\infty(X, \mathbb{R}^\delta) \to \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \to H^2_{\mathrm{dR}}(X) \to \check{H}^2_\infty(X, \mathbb{R}^\delta) \tag{2.5}$$

is exact.

The above exact sequence is exactly analogous to the exact sequence obtained by Iglesias-Zemmour in [Igl23].

In Section 2.8 we turn to the study of connections for diffeological principal bundles. This theory is still in its infancy, and there are a few references that give varying definitions of diffeological connections [Igl13, Section 8.32], [Wal12, Section 3], [MW17, Section 4]. The theory of ∞-stacks provides another definition. Let G be a Lie group, and U a cartesian space. Then let $\Omega^1(U, \mathfrak{g})//G$ denote the groupoid whose objects are differential 1-forms $\omega \in \Omega^1(U, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G, and where there is a morphism $g : \omega \to \omega'$ if there exists a smooth map $g : U \to G$ such that

$$\omega' = \mathrm{Ad}_g^{-1}(\omega) + g^* \mathrm{mc}(G)$$

where $\mathrm{mc}(G)$ denotes the Maurer-Cartan form of G. Taking the nerve of this groupoid, and letting U vary defines an ∞-stack $\Omega^1(-, \mathfrak{g})//G$, which amongst others has been studied in [FSS+12], [FH13]. We connect this notion of connection to that given in [Wal12, Definition 3.2.1] in the following result.

Theorem 2.8.3. Given a diffeological space X and a Lie group G, the functor

$$\mathsf{Cons}_\nabla : \mathsf{Coc}_\nabla(X, G) \to \mathsf{Wal}_G(X), \tag{2.6}$$

is an equivalence of groupoids, where $\mathsf{Coc}_\nabla(X, G)$ is the groupoid whose objects are maps $QX \to \Omega^1(-, \mathfrak{g})//G$, where QX is a cofibrant replacement of X in the projective model structure on simplicial presheaves, and $\mathsf{Wal}_G(X)$ is the groupoid of diffeological principal G-bundles with connection as defined in [Wal12, Definition 3.2.1].

To compute ∞-stack cohomology, one needs a workable model of the derived mapping space $\mathbb{R}\mathrm{Hom}(X, A)$, when X is a diffeological space and A is a presheaf of chain complexes. In Section 2.9, we obtain such a model, which reduces many computations with ∞-stacks to manipulations with double complexes. As a corollary, we obtain a simple and direct proof of the following well known folklore result.

Proposition 2.9.4. Let C be a cosimplicial chain complex, then

$$\mathrm{holim}_{n \in \Delta} C^n \simeq \mathsf{tot}\, C, \tag{2.7}$$

where we are computing the homotopy limit in the category of chain complexes equipped with the projective model structure, and $\mathsf{tot}\, C$ denotes the total complex of C.

The paper is organized as follows. In Section 2.2, we introduce diffeological spaces and place them in the context of sheaf theory. In Section 2.3, we introduce simplicial presheaves, show how diffeological spaces embed into simplicial presheaves, and introduce the shape functor. In Section 2.4, we prove that the ∞-stack cohomology of the irrational torus T_K is isomorphic to the group cohomology of K with values in $\mathbb{R}$. In Section 2.5, we introduce the Dold-Kan correspondence, which is a core tool we use for the rest of the paper. In Section 2.6, we introduce the main ∞-stacks that will be used in the paper, and compute various examples of ∞-stack cohomology. In Section 2.7, we prove the main results of this paper, Theorem 2.7.5 and Theorem 2.7.7. In Section 2.8, we prove that our notion of diffeological principal G-bundles with connection using ∞-stacks agrees with Waldorf's [Wal12]. In Section 2.9, we prove a technical result allowing us to easily compute ∞-stack cohomology when the coefficient ∞-stack comes from a presheaf of chain complexes. In Section 2.10 we prove

Theorem 2.7.1.

2.2 Smooth Sheaves and Diffeological Spaces

In this section we briefly describe diffeological spaces and their connection to sheaves on Cart.

Definition 2.2.1. Let M be a finite dimensional smooth manifold[3]. We say a collection of subsets $\mathcal{U} = \{U_i \subseteq M\}_{i\in I}$ is an **open cover** if each U_i is an open subset of M, and $\bigcup_{i\in I} U_i = M$. If U is a finite dimensional smooth manifold diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, we call U a **cartesian space**. We call $\mathcal{U} = \{U_i \subseteq M\}$ a **cartesian open cover** of a manifold M if it is an open cover of M and every U_i is a cartesian space. We say that $\mathcal{U}$ is a **good open cover** if it is a cartesian open cover, and further every finite non-empty intersection $U_{i_0\dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is a cartesian space.

Let Man denote the category whose objects are finite dimensional smooth manifolds and whose morphisms are smooth maps. Let Cart denote the full subcategory whose objects are cartesian spaces. Given a set X, let $\mathsf{Param}(X)$ denote the set of **parametrizations** of X, namely the collection of set functions $p : U \to X$, where $U \in \mathsf{Cart}$.

Definition 2.2.2. A **diffeology** on a set X, consists of a collection $\mathcal{D}$ of parametrizations $p : U \to X$ satisfying the following three axioms:

1. $\mathcal{D}$ contains all points $\mathbb{R}^0 \to X$,
2. If $p : U \to X$ belongs to $\mathcal{D}$, and $f : V \to U$ is a smooth map, then $pf : V \to X$ belongs to $\mathcal{D}$, and
3. If $\{U_i \subseteq U\}_{i\in I}$ is a good open cover of a cartesian space U, and $p : U \to X$ is a parametrization such that $p|_{U_i} : U_i \to X$ belongs to $\mathcal{D}$ for every $i \in I$, then $p \in \mathcal{D}$.

A set X equipped with a diffeology $\mathcal{D}$ is called a **diffeological space**. Parametrizations that belong to a diffeology are called **plots**. We say a set function $f : X \to Y$ between diffeological

[3] We will assume throughout this paper that manifolds are Hausdorff and paracompact.

spaces is **smooth** if for every plot $p : U \to X$ in $\mathcal{D}_X$, the composition $pf : U \to Y$ belongs to $\mathcal{D}_Y$. We often denote the set of smooth maps from X to Y by $C^\infty(X, Y)$. Let Diff denote the category of diffeological spaces.

Every manifold M is canonically a diffeological space by considering the set of parametrizations $p : U \to M$ that are smooth in the classical sense. This gives a diffeology on M, called the **manifold diffeology**. One can show [Igl13, Chapter 4] that the manifold diffeology defines a fully faithful functor $\mathsf{Man} \hookrightarrow \mathsf{Diff}$.

Diffeology extends many constructions and concepts from classical differential geometry to diffeological spaces, such as the theory of bundles.

Definition 2.2.3. We say that a map $\pi : X \to Y$ of diffeological spaces is a **subduction** if it is surjective, and for every plot $p : U \to Y$, there exists a good open cover $\{U_i \subseteq U\}$, and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ U & \xrightarrow[p]{} & Y \end{array} \tag{2.8}$$

Definition 2.2.4. A **diffeological group** is a group G equipped with a diffeology such that the multiplication map $m : G \times G \to G$, and inverse map $i : G \to G$ are smooth. A right **diffeological group action** of a diffeological group G on a diffeological space X is a smooth map $\rho : X \times G \to X$ such that $\rho(x, e_G) = x$, and $\rho(\rho(x, g), h) = \rho(x, gh)$, where e_G denotes the identity element of G.

Definition 2.2.5. Let G be a diffeological group, and P be a diffeological right G-space. A map $\pi : P \to X$ of diffeological spaces is a **diffeological principal G-bundle** if:

1. the map $\pi : P \to X$ is a subduction, and

2. the map $\mathrm{act} : P \times G \to P \times_X P$ defined by $(p, g) \mapsto (p, p \cdot g)$, which we call the **action map** is a diffeomorphism.

A map of diffeological principal G-bundles $P \to P'$ over X is a diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & P' \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi'} & \\ & X & \end{array}$$

where f is a G-equivariant smooth map. A diffeological principal G-bundle P is said to be **trivial** if there exists an isomorphism $\varphi : X \times G \to P$, called a **trivialization**, where $\mathrm{pr}_1 : X \times G \to X$ is the product bundle. Let $\mathsf{DiffPrin}_G(X)$ denote the category of diffeological principal G-bundles over a diffeological space X.

In Section 1.3, we proved that diffeological principal bundles can be classified using cocycles in a way reminiscent of classical differential geometry. However, rather than using cocycles defined over an open cover, we use cocycles defined on plots. Let $\mathsf{Plot}(X)$ denote the category whose objects are plots $p : U \to X$ of X and whose morphisms $f : p \to p'$ are smooth maps $f : U \to U'$ such that $p' f = p$.

Definition 2.2.6. Given a diffeological space X and a diffeological group G, call a collection $g = \{g_{f_0}\}$ of smooth maps $g_{f_0} : U_{p_1} \to G$ indexed by maps of plots $f_0 : U_{p_1} \to U_{p_0}$ of X a **G-cocycle** if for every pair of composable plot maps of X

$$U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$$

it follows that

$$g_{f_0 f_1} = (g_{f_0} \circ f_1) \cdot g_{f_1}. \tag{2.9}$$

We call (2.9) the **diffeological G-cocycle condition**.

Given two G-cocycles, g, g', we say a collection $h = \{h_{p_0}\}$ of smooth maps $h_{p_0} : U_{p_0} \to G$ indexed by plots of X is a **morphism of G-cocycles** $h : g \to g'$ if for every map $f_0 : U_{p_1} \to$

U_{p_0} of plots of X, it follows that

$$g'_{f_0} \cdot h_{p_1} = (h_{p_0} \circ f_0) \cdot g_{f_0}. \tag{2.10}$$

Given a diffeological space X and a G-cocycle g on X, we can construct a diffeological principal G-bundle $\pi : P \to X$, by taking the quotient

$$P = \left(\coprod_{p_0 \in \mathsf{Plot}(X)} U_{p_0} \times G \right) /\!\sim \tag{2.11}$$

where $\sim$ is the smallest equivalence relation such that $(x_{p_1}, k_1) \sim (x_{p_0}, k_0)$ if there exists a map $f_0 : U_{p_1} \to U_{p_0}$ of plots such that $f_0(x_{p_1}) = x_{p_0}$ and $k_0 = g_{f_0}(x_{p_1}) \cdot k_1$. We let $\pi = \mathsf{Cons}(g)$, short for construction. In fact, this construction defines a functor from the category $\mathsf{Coc}(X, G)$ of G-cocycles on X to the category of diffeological principal G-bundles. We proved the following result as Theorem 1.3.15.

Theorem 2.2.7. Given a diffeological space X and a diffeological group G, the functor

$$\mathsf{Cons} : \mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X) \tag{2.12}$$

is an equivalence of groupoids.

While extending the classical theory, there are constructions one can do with diffeological spaces that are not available to smooth manifolds:

1. Given a diffeological space X, and a subset $A \overset{i}{\hookrightarrow} X$ a subset. Then consider the set of parametrizations $p : U \to A$ such that $ip : U \to X$ is a plot of X. This collection is a diffeology, called the **subspace diffeology** on A,

2. Given a diffeological space X and an equivalence relation $\sim$ on X, let $\pi : X \to X/\!\sim$ denote the resulting quotient function on sets. Consider the set of parametrizations

$p : U \to X/{\sim}$ such that there exists a good open cover $\{U_i \subseteq U\}$ and plots $p_i : U_i \to X$ making the following diagram commute

$$\begin{array}{ccc} U_i & \xrightarrow{p_i} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ U & \xrightarrow[p]{} & X/{\sim} \end{array}$$

This forms a diffeology on $X/{\sim}$, called the **quotient diffeology**,

3. Given a pair X and Y of diffeological spaces, the set of parametrizations $p : U \to X \times Y$ such that the composites $\pi_1 \circ p$ and $\pi_2 \circ p$ are plots of X and Y respectively, forms a diffeology, called the **product diffeology**,

4. Given diffeological spaces X and Y, the set of parametrizations $p : U \to C^\infty(X, Y)$ such that the transposed function $p^\# : U \times X \to Y$ is a smooth map is a diffeology, called the **functional diffeology**.

These constructions make the category of diffeological spaces considerably better than the category of finite dimensional smooth manifolds, as shown in Corollary 2.2.10.

Diffeological spaces inherit this nice structure from the category of smooth sheaves.

Definition 2.2.8. We briefly recall the relevant definitions for sheaf theory.

- A **collection of families** j on a category $\mathcal{C}$ consists of a set $j(U)$ for each $U \in \mathcal{C}$, whose elements $\{r_i : U_i \to U\} \in j(U)$ are families of morphisms over U. We call a collection of families j on $\mathcal{C}$ a **coverage** if it satisfies the following property: for every $\{r_i : U_i \to U\} \in j(U)$, and every map $g : V \to U$ in $\mathcal{C}$, then there exists a family $\{t_j : V_j \to V\} \in j(V)$ such that gt_j factors through some r_i. Namely for every t_j there exists some i and some map $s_j : V_j \to U_i$ making the following diagram commute:

$$\begin{array}{ccc} V_j & \xrightarrow{s_j} & U_i \\ {\scriptstyle t_j}\big\downarrow & & \big\downarrow{\scriptstyle r_i} \\ V & \xrightarrow[g]{} & U \end{array} \tag{2.13}$$

The families $\{r_i : U_i \to U\} \in j(U)$ are called **covering families** over U. If a map $r_i : U_i \to U$ belongs to a covering family $r \in j(U)$, then we say that r_i is a **covering map**. If $\mathcal{C}$ is a category, and j is a coverage on $\mathcal{C}$, then we call the pair $(\mathcal{C}, j)$ a **site**.

- A **presheaf** on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathsf{Set}$. A morphism of presheaves is a natural transformation. An element $x \in F(U)$ for an object $U \in \mathcal{C}$ is called a **section** over U. If $f : U \to V$ is a map in $\mathcal{C}$, and $x \in F(V)$, then we sometimes denote $F(f)(x)$ by $x|_U$. Let $\mathsf{Pre}(\mathsf{C})$ denote the category of presheaves on C.

- If $\{r_i : U_i \to U\}_{i \in I}$ is a covering family, then a **matching family** is a collection $\{x_i\}_{i \in I}$, $x_i \in F(U_i)$, such that given a diagram in $\mathcal{C}$ of the form

$$\begin{array}{ccc} V & \xrightarrow{g} & U_j \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle r_j} \\ U_i & \xrightarrow[r_i]{} & U \end{array}$$

  then $F(f)(x_i) = F(g)(x_j)$ for all $i, j \in I$. An **amalgamation** x for a matching family $\{x_i\}$ is a section $x \in F(U)$ such that $x_i|_U = x$ for all i.

- Given a family of morphisms $r = \{r_i : U_i \to U\}$ in a category $\mathcal{C}$, we say that a presheaf $F : \mathcal{C}^{op} \to \mathsf{Set}$ is a **sheaf on** r if every matching family $\{s_i\}$ of F over r has a unique amalgamation. If j is a coverage on a category $\mathcal{C}$, we call F a **sheaf** on $(\mathcal{C}, j)$ if it is a sheaf on every covering family of j. Let $\mathsf{Sh}(\mathcal{C})$ denote the full subcategory of $\mathsf{Pre}(\mathcal{C})$ whose objects are sheaves on $(\mathcal{C}, j)$.

One can put a site structure on Cart using the coverage of good open covers. We call sheaves on Cart **smooth sheaves**. There are many interesting examples of smooth sheaves. Every cartesian space defines a representable sheaf yU. Every manifold M defines a sheaf by $U \mapsto C^\infty(U, M)$. There are also Ω^n and Ω^n_{cl} for every $n \geq 0$, the sheaves of differential n-forms and closed differential n-forms respectively. The category $\mathsf{Sh}(\mathsf{Cart})$ of smooth sheaves is "extremely nice", being a Grothendieck topos [MM12].

A sheaf X on Cart is **concrete** if $X(U)$ is a subset of the set functions $U \to X(*)$ where $*$ is the terminal object in Cart. The representable sheaves yU and the sheaves induced by manifolds M are concrete, but Ω^n and Ω^n_{cl} are not.

The full subcategory $\mathsf{ConSh(C)} \hookrightarrow \mathsf{Sh(C)}$ of concrete sheaves on a concrete site forms a quasitopos, which while not being a Grothendieck topos, is still a very "nice" category [BH11, Theorem 52].

Theorem 2.2.9 ([BH11, Prop 24])**.** Let Cart denote the site of cartesian spaces with the coverage of good open covers. Then there is an equivalence of categories

$$\mathsf{Diff} \simeq \mathsf{ConSh(Cart)}, \tag{2.14}$$

where $\mathsf{ConSh(Cart)}$ denotes the category of concrete sheaves on Cart.

Corollary 2.2.10. The category Diff is a quasitopos. This implies that it is a complete, cocomplete and cartesian closed category.

We refer to Theorem 2.2.9 as the **Baez-Hoffnung Theorem**[4]. It is the starting point of the interaction of sheaf theory and diffeology. Many aspects of the study of diffeological spaces can be restated using sheaf theory, for example a differential n-form ω on a diffeological space X as defined in [Igl13, Article 6.28] is equivalently a morphism $X \to \Omega^n$ of sheaves.

In Section 1.5 we took advantage of the Baez-Hoffnung Theorem to embed the category of diffeological spaces into the category of simplicial presheaves on Cart. We will delve into this idea in the next section. Once inside the category of simplicial presheaves, we can then take advantage of many homotopical tools. This in effect provides a way of obtaining a very powerful and expressive homotopy theory for diffeological spaces that subsumes the usual homotopy theory for diffeological spaces as considered in [Igl13, Chapter 5].

[4]Strictly speaking, the Baez-Hoffnung theorem gives an equivalence between the category of what we call classical diffeological spaces and the category of concrete sheaves on the site of open subsets of cartesian spaces with open covers, see Section 1.7

2.3 Simplicial Presheaves

In this section we detail the model categorical notions we will need for the remainder of the paper. We assume the reader is comfortable with model categories and simplicial homotopy theory, and recommend the following standard sources [Hir09], [Hov07], [GJ12], [GS06] for good references on the topics.

Definition 2.3.1. Let $\mathsf{sPre}(\mathsf{Cart})$ denote the category whose objects are functors $\mathsf{Cart}^{op} \to \mathsf{sSet}$, which we call **simplicial presheaves**, and whose morphisms are natural transformations.

Note that $\mathsf{sPre}(\mathsf{Cart})$ is complete and cocomplete, with limits and colimits computed objectwise. There are two pairs of adjoint triples that give structure to $\mathsf{sPre}(\mathsf{Cart})$.

$$\mathsf{sPre}(\mathsf{Cart}) \underset{\lim_{\mathsf{Cart}^{op}}}{\overset{\operatorname{colim}_{\mathsf{Cart}^{op}}}{\rightleftarrows}} \mathsf{sSet}, \quad (-)_c : \mathsf{sSet} \to \mathsf{sPre}(\mathsf{Cart}), \qquad \mathsf{sPre}(\mathsf{Cart}) \underset{(-)_0}{\overset{\pi_0}{\rightleftarrows}} \mathsf{Pre}(\mathsf{Cart}), \quad {}^c(-) : \mathsf{Pre}(\mathsf{Cart}) \to \mathsf{sPre}(\mathsf{Cart}) \tag{2.15}$$

where $(-)_c$ is the functor induced by restricting along the unique functor $\mathsf{Cart} \to *$ and ${}^c(-)$ is the functor that sends a presheaf to the corresponding simplicial presheaf where all the simplicial face and degeneracy maps are the identity. We often don't use the notation ${}^c(-)$ explicitly, especially for representable presheaves, as it should be clear from context. The functors π_0 and $(-)_0$ are defined objectwise. For every $U \in \mathsf{Cart}$, and simplicial presheaf X on Cart, $\pi_0 X(U) = \pi_0(X(U))$, the set of connected components of $X(U)$, and $(X)_0(U) = X(U)_0$, the set of vertices of $X(U)$.

Remark 2.3.2. The above adjoint triples exist for any essentially small category C in place of Cart.

The category $\mathsf{sPre}(\mathsf{Cart})$ is tensored, cotensored and enriched over sSet. Indeed, if K is a simplicial set and X is a simplicial presheaf, then

- $X \otimes K$ is the simplicial presheaf defined objectwise by

$$(X \otimes K)(U) = (X \times K_c)(U) = X(U) \times K.$$

- X^K is the simplicial presheaf defined objectwise by

$$(X^K)(U) = X(U)^K,$$

  where for simplicial sets K and L, K^L denotes the simplicial function complex.

- for any two simplicial presheaves X and Y, let $\underline{\mathsf{sPre(Cart)}}(X, Y)$ denote the simplicial set defined levelwise by

$$\underline{\mathsf{sPre(Cart)}}(X, Y)_n = \mathsf{sPre(Cart)}(X \otimes \Delta^n, Y).$$

This structure is compatible in the sense of the following natural isomorphisms of simplicial sets

$$\underline{\mathsf{sPre(Cart)}}(X \otimes K, Y) \cong \underline{\mathsf{sPre(Cart)}}(X, Y^K). \tag{2.16}$$

The category sPre(Cart) inherits several model structures from sSet. We say a map $f : X \to Y$ is a **projective weak equivalence** if it is an objectwise weak equivalence of simplicial sets, a **projective fibration** if it is an objectwise fibration, and a **projective cofibration** if it left lifts against all maps that are both projective weak equivalences and projective fibrations.

Theorem 2.3.3 ([BK72, Page 314], [Lur09, Section A.2.6])**.** The projective weak equivalences, fibrations and cofibrations define a proper, combinatorial, simplicial model category structure on sPre(Cart), called the **projective model structure** on simplicial presheaves.

Let $\mathbb{H}$ denote the category of simplicial presheaves equipped with the projective model structure. Note that [Dug01, Corollary 9.4] describes a sufficient condition on simplicial presheaves to be projective cofibrant, and it implies that all representable presheaves, denoted yU for a cartesian space U, are projective cofibrant.

Given a cartesian space U and a good cover $\mathcal{U} = \{U_i \subseteq U\}$ of U, we can form the simplicial presheaf $\check{C}(\mathcal{U})$ defined levelwise by

$$\check{C}(\mathcal{U})_n = \coprod_{i_0 \dots i_n} y(U_{i_0} \cap \cdots \cap U_{i_n}).$$

We call $\check{C}(\mathcal{U})$ the **Čech nerve** of $\mathcal{U}$. There is a canonical map $\pi : \check{C}(\mathcal{U}) \to yU$. Let $\check{C}$ denote the class of morphisms $\pi : \check{C}(\mathcal{U}) \to yU$ where U ranges over the cartesian spaces and $\mathcal{U}$ ranges over the good open covers for U.

Theorem 2.3.4 ([DHI04, Theorem A.6])**.** The left Bousfield localization of $\mathbb{H}$ at $\check{C}$ exists. We call the resulting model structure the **Čech model structure** on $\mathsf{sPre}(\mathsf{Cart})$, and denote it by $\check{\mathbb{H}}$. It is similarly a proper, combinatorial and simplicial model category.[5]

The fibrant objects in $\check{\mathbb{H}}$ are called **∞-stacks**. They are those projective fibrant simplicial presheaves X such that the canonical map

$$\underline{\mathsf{sPre}(\mathsf{Cart})}(yU, X) \to \underline{\mathsf{sPre}(\mathsf{Cart})}(\check{C}(\mathcal{U}), X), \tag{2.17}$$

is a weak equivalence of simplicial sets, for every cartesian space U and good cover $\mathcal{U}$ of U. Every sheaf and classical stack of groupoids on Cart, thought of as simplicial presheaves on Cart, is an ∞-stack.

The identity functors define a Quillen adjunction between the projective and Čech model structure on simplicial presheaves.

$$\mathbb{H} \underset{1_{\mathsf{sPre}(\mathsf{Cart})}}{\overset{1_{\mathsf{sPre}(\mathsf{Cart})}}{\rightleftarrows}} \check{\mathbb{H}} \quad (\perp) \tag{2.18}$$

[5]It is important to note that the projective/objectwise weak equivalences between simplicial presheaves are still weak equivalences in the Čech model structure. Furthermore, all Čech weak equivalences between ∞-stacks are objectwise weak equivalences.

Crucially, the (easy to compute) finite homotopy limits in $\mathbb{H}$ are preserved as homotopy limits in $\check{\mathbb{H}}$, thanks to the following result.

Proposition 2.3.5 ([Rez10, Proposition 11.2])**.** The left Quillen functor $1_{\mathsf{sPre(Cart)}} : \mathbb{H} \to \check{\mathbb{H}}$ preserves finite homotopy limits.

Given simplicial presheaves X and Y on Cart, let Q and R denote cofibrant and fibrant replacement functors for $\check{\mathbb{H}}$ respectively, then let

$$\mathbb{R}\check{\mathbb{H}}(X, Y) = \underline{\mathsf{sPre(Cart)}}(QX, RY). \tag{2.19}$$

We call $\mathbb{R}\check{\mathbb{H}}(X, Y)$ the **derived mapping space** of X and Y. If X is already cofibrant, then we can take $Q = 1_{\mathsf{sPre(Cart)}}$ and if Y is already fibrant, we can take $R = 1_{\mathsf{sPre(Cart)}}$.

If X and A are simplicial presheaves, then let

$$\check{H}^0_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, A).$$

We call this the 0th ∞-stack cohomology of X with values in A.

If A is a simplicial presheaf that is objectwise a simplicial group, then we let

$$\check{H}^1_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, \overline{W}A),$$

where $\overline{W}$ is the delooping functor.

If A is a simplicial presheaf such that $\overline{W}^k A$, which we call its k-fold delooping, exists for $k \geq 1$, then we say that A is k-deloopable, and we let

$$\check{H}^k_\infty(X, A) = \pi_0 \mathbb{R}\check{\mathbb{H}}(X, \overline{W}^k A).$$

We call this the kth ∞-stack cohomology of X with coefficients in A. If the k-fold delooping of a simplicial presheaf A exists and is an ∞-stack, then we denote it by $\mathbf{B}^k A := \overline{W}^k A$. The following result is well known.

Lemma 2.3.6. If A is a presheaf of simplicial abelian groups, then A is k-deloopable for all $k \geq 1$.

There is a convenient cofibrant replacement functor for $\check{\mathbb{H}}$, given (in the notation of [Rie14, Section 4.2]) for a simplicial presheaf X by the bar construction $B(X, \mathsf{Cart}, y)$, where y denotes the Yoneda embedding $y : \mathsf{Cart} \hookrightarrow \mathsf{sPre}(\mathsf{Cart})$.

If X is a diffeological space, then by the Baez-Hoffnung Theorem (Theorem 2.2.9), we can consider it as a sheaf on Cart. Then ${}^cX$ is a simplicial presheaf[6]. If we apply Q to ${}^cX$ then this formula reduces to the simplicial presheaf given levelwise by

$$QX_n = \coprod_{(f_{n-1},\dots,f_0)\in N(\mathsf{Plot}(X))_n} yU_{p_n} \otimes \Delta^n. \tag{2.20}$$

Using the above cofibrant replacement functor for a diffeological space X, if A is a ∞-stack that is also objectwise a simplicial abelian group, then we can obtain an explicit description of its kth ∞-stack cohomology with values in A, given by the kth cohomology of the cochain complex obtained by taking the dual Dold-Kan correspondence functor applied to the cosimplicial abelian group

$$A(QX_0) \rightrightarrows A(QX_1) \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; A(QX_2) \qquad \dots . \tag{2.21}$$

Example 2.3.7. Let G be a diffeological group, and consider the (strict) functor $\mathbf{B}G : \mathsf{Cart}^{op} \to \mathsf{Gpd}$ that sends a cartesian space U to the groupoid

$$[C^\infty(U, G) \rightrightarrows *] \tag{2.22}$$

Postcomposing with the nerve functor gives us a simplicial presheaf $N\mathbf{B}G$, which we will

[6]We will often not use the notation ${}^cX$ for diffeological spaces in what follows, as it should be apparent from context what category we are considering X in.

often just denote by $\mathbf{B}G$. By Theorem 1.5.17, (referencing [SS21, Lemma 3.3.29] and [Pav22a, Proposition 4.13]), $\mathbf{B}G$ is an ∞-stack.

This ∞-stack takes a central role in the theory of diffeological principal G-bundles. For every cartesian space U, there is a canonical map of groupoids

$$\mathbf{B}G(U) \to \mathsf{DiffPrin}_G(U), \tag{2.23}$$

that sends the point to the trivial diffeological principal G-bundle, and sends a map to G to the corresponding automorphism of the trivial bundle. This map is an equivalence of groupoids.

Furthermore, if X is a diffeological space and QX is its cofibrant replacement, then G-cocycles on X are equivalent to maps of ∞-stacks $QX \to \mathbf{B}G$. In other words, for every diffeological space X, there is a weak equivalence

$$\mathbb{R}\check{\mathbb{H}}(X, \mathbf{B}G) \simeq N\mathsf{DiffPrin}_G(X). \tag{2.24}$$

Thus we say that $\mathbf{B}G$ classifies diffeological principal G-bundles. This implies that

$$\check{H}^1_\infty(X, G) \cong \pi_0\mathsf{DiffPrin}_G(X), \tag{2.25}$$

where $\pi_0\mathsf{DiffPrin}_G(X)$ denotes the set of isomorphism classes of diffeological principal G-bundles on X.

Let us now examine the left hand side of (2.15). If K is a simplicial set, then K_c is the constant simplicial presheaf on K, namely $K_c(U) = K$ for all $U \in \mathsf{Cart}$. The functors making up (2.15) are important enough to warrant renaming. Notice that since $\mathbb{R}^0 = *$ is the terminal object in Cart, it is the initial object in Cart^{op}, thus $\lim_{U \in \mathsf{Cart}^{op}} X(U) \cong X(*)$.

For $K \in \mathsf{sSet}$ and $X \in \mathsf{sPre}(\mathsf{Cart})$, we set

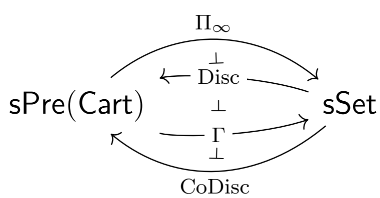
$$\mathrm{Disc}(K) = K_c, \qquad \Gamma(X) = \lim_{U \in \mathsf{Cart}^{op}} X(U) \cong X(*), \qquad \Pi_\infty(X) = \operatorname*{colim}_{U \in \mathsf{Cart}^{op}} X(U). \tag{2.26}$$

It turns out that Γ has a further right adjoint, CoDisc : $\mathsf{sSet} \to \mathsf{sPre}(\mathsf{Cart})$ defined objectwise by

$$\mathrm{CoDisc}(K)(U) = K^{\Gamma(yU)}.$$

We say that $\mathrm{Disc}(K)$ is the **discrete simplicial presheaf** on K, $\Gamma(X)$ is the **global sections** of X, $\Pi_\infty(X)$ is the **fundamental ∞-groupoid** or **shape** of X, and that $\mathrm{CoDisc}(K)$ is the **codiscrete simplicial presheaf** on K. In fact, all of these adjunctions are simplicially enriched adjunctions.

Thus we obtain the following triple of simplicially enriched adjunctions



(2.27)

Proposition 2.3.8 ([Sch13, Prop 4.1.30 and 4.1.32])**.** Each adjunction in (2.27) is a simplicial Quillen adjunction, where $\mathsf{sPre}(\mathsf{Cart})$ is given the Čech model structure $\check{\mathbb{H}}$, and sSet is given the Kan-Quillen model structure.

In [Sch13], Schreiber defines the three following endofunctors on the category of simplicial presheaves on Cart:

$$\begin{aligned} \textstyle\int &= \mathrm{Disc} \circ \Pi_\infty \\ \flat &= \mathrm{Disc} \circ \Gamma \\ \sharp &= \mathrm{CoDisc} \circ \Gamma \end{aligned} \tag{2.28}$$

called **shape**, **flat** and **sharp** respectively.

Remark 2.3.9. Using the name shape functor for both Π_∞ and $\int$ is justified by remembering that $\mathrm{Disc} : \mathsf{sSet} \to \check{\mathbb{H}}$ is fully faithful.

They give another pair of simplicial Quillen adjunctions

$$\check{\mathbb{H}} \underset{\sharp}{\overset{\int}{\leftarrows}} \xrightarrow{\ \flat\ } \check{\mathbb{H}} \qquad \left(\textstyle\int \dashv \flat \dashv \sharp\right) \tag{2.29}$$

Let us focus further on the shape functor Π_∞. Let Δ^k_a denote the cartesian space defined by

$$\Delta^k_a = \left\{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1} \ : \ \sum_{i=0}^{k} x_i = 1 \right\}. \tag{2.30}$$

We call these **affine simplices**.

Let $\mathrm{Sing}_\infty : \mathsf{sPre}(\mathsf{Cart}) \to \mathsf{sSet}$ be the functor defined objectwise by

$$\mathrm{Sing}_\infty(X) = \mathrm{hocolim}_{\Delta^{op}} \left(\ X(\Delta^0_a) \leftleftarrows X(\Delta^1_a) \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\, X(\Delta^2_a) \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\, \cdots \right), \tag{2.31}$$

where if we wish to be concrete, we can use the model of the homotopy colimit given by taking the diagonal of the above bisimplicial set. We call this the **smooth singular complex functor**.

Lemma 2.3.10. The functor Sing_∞ sends objectwise weak equivalences of simplicial presheaves to weak equivalences.

Proof. This follows from [GJ12, Proposition 1.9] and taking the diagonal to model the homotopy colimit. □

Proposition 2.3.11. There are natural weak equivalences between the functors

$$\mathrm{Sing}_\infty \simeq \mathrm{Sing}_\infty \circ Q \simeq \Pi_\infty \circ Q \tag{2.32}$$

where Q denotes a cofibrant replacement functor for $\check{\mathbb{H}}$.

Proof. We mirror the proof given in [Bun22b, Remark 4.12], and since the left hand weak equivalence is shown there, we only prove the middle weak equivalence. If we restrict the smooth singular complex functor along the Yoneda embedding $\mathsf{Cart} \hookrightarrow \mathsf{sPre}(\mathsf{Cart})$, then we obtain a functor $\mathrm{Sing}_\infty : \mathsf{Cart} \to \mathsf{sSet}$, and there is a natural weak equivalence of functors $\mathrm{Sing}_\infty \xrightarrow{\sim} *$, where $* : \mathsf{Cart} \to \mathsf{sSet}$ is the constant functor on a point $* = \Delta^0$, this follows from [Bun22b, Proposition 3.11]. Since all simplicial sets are cofibrant in the Quillen model structure on sSet, [Rie14, Corollary 5.2.5] implies that this induces a natural weak equivalence

$$B(X, \mathsf{Cart}, \mathrm{Sing}_\infty) \xrightarrow{\sim} B(X, \mathsf{Cart}, *) \tag{2.33}$$

of simplicial presheaves for every $X \in \mathsf{sPre}(\mathsf{Cart})$. But $B(X, \mathsf{Cart}, *) \cong \mathrm{colim}_{\mathsf{Cart}^{op}} B(X, \mathsf{Cart}, y) \cong \mathrm{colim}_{\mathsf{Cart}^{op}} QX \cong \Pi_\infty QX$. This follows from the fact that $B(X, \mathsf{Cart}, y) \cong QX$, which is just repackaging the definition of QX. Since Sing_∞ is a left adjoint, we have

$$B(X, \mathsf{Cart}, \mathrm{Sing}_\infty) \cong \mathrm{Sing}_\infty B(X, \mathsf{Cart}, y) \cong \mathrm{Sing}_\infty QX.$$

This gives the second natural weak equivalence above. □

By Proposition 2.3.11, we will often refer to $\mathrm{Sing}_\infty(X)$ as the shape of X as well. The shape functor has many wonderful properties. While we will not need all of the following results on the shape functor for this paper, we provide a concise listing of them here as such results are scattered throughout the literature.

Remark 2.3.12. Since $\int$ is just applying the shape functor and then treating the resulting simplicial set as a constant simplicial presheaf, along with the fact that $\int \simeq \mathrm{Disc} \circ \mathrm{Sing}_\infty$, we will blur the distinction between $\Pi_\infty, \mathrm{Sing}_\infty$ and $\int$. We will use $\int$ when we wish to be ambiguous about which particular model of the shape functor we wish to use.

Theorem 2.3.13 ([Bun21b, Theorem 4.15])**.** Let M be a finite dimensional smooth manifold, thought of as a simplicial presheaf on Cart, then

$$\int M \simeq \mathrm{Sing}(M^{top}) \tag{2.34}$$

where M^{top} is the underlying topological space of M and Sing : $\mathsf{Top} \to \mathsf{sSet}$ is the classical singular complex. In other words, the shape of M is its underlying homotopy type.

Remark 2.3.14. It should be noted that Theorem 2.3.13 is really a consequence of a classical result known as the nerve theorem [Bor48].

Proposition 2.3.15 ([Pav22b, Example 14.1])**.** Let Ω^n_{cl} denote the sheaf on Cart of closed differential n-forms. Its shape is

$$\int \Omega^n_{\mathrm{cl}} \simeq \mathbf{B}^n \mathbb{R}^\delta. \tag{2.35}$$

Lemma 2.3.16. Let K be a simplicial set, then the counit

$$\varepsilon_K : (\Pi_\infty \circ \mathrm{Disc})(K) \to K. \tag{2.36}$$

is an isomorphism. In other words, for discrete simplicial presheaves K_c, we have

$$\int K_c \cong K.$$

Proof. This follows from the fact that Disc is fully faithful, and the unit of an adjunction where the right adjoint is fully faithful is an isomorphism. □

Proposition 2.3.17 ([Car15, Theorem 3.4])**.** Let M denote a simplicial manifold. Let $\underline{M}$ denote the simplicial presheaf on Cart defined degreewise by

$$\underline{M}(U)_k = C^\infty(U, M_k).$$

Then the shape of $\underline{M}$ is

$$\int \underline{M} \simeq ||M|| \tag{2.37}$$

where $||M||$ denotes the homotopy type of the "fat" geometric realization of M, see [Car15, Section 3.2].

Remark 2.3.18. The above result is especially interesting when M is the nerve of a Lie groupoid, as this says that the shape of a Lie groupoid (thought of as a simplicial presheaf) is weak equivalent to the homotopy type of the Lie groupoid's classifying space [Car15, Section 2.2].

Remark 2.3.19. We should also mention that the shape functor has been used to great effect in what is now called the Smooth Oka Principle. See the following references [BBP22], [SS21], [Clo23], [Pav22a, Section 10].

Let $\mathrm{Sing}_D : \mathsf{Diff} \to \mathsf{sSet}$ be the functor defined levelwise by

$$\mathrm{Sing}_D(X)_n = \mathsf{Diff}(\Delta^n_a, X). \tag{2.38}$$

We call this the **diffeological singular complex**. Note that by using the diagonal as a model for the homotopy colimit, for a diffeological space X (actually any sheaf of sets on Cart), we have

$$\mathrm{Sing}_D(X) \cong \mathrm{Sing}_\infty(X). \tag{2.39}$$

Proposition 2.3.20. Let X be a diffeological space, then

$$\Pi_\infty(QX) \simeq \mathrm{Sing}_D(X) \simeq N\mathsf{Plot}(X) \tag{2.40}$$

where $N\mathsf{Plot}(X)$ is the nerve of the category of plots of X.

Proof. We will only prove the second weak equivalence, as the first holds by the previous discussion. It is shown in [Bun22b, Proposition 3.6] that Sing_∞ has a left and right adjoint (though only the right adjoint forms a Quillen adjunction), therefore we have

$$\begin{aligned}\Pi_\infty(QX) &= \Pi_\infty\left(\int^{n\in\Delta}\coprod_{N\mathsf{Plot}(X)_n} yU_{p_n}\otimes\Delta^n\right)\\ &\cong \int^{n\in\Delta}\coprod_{N\mathsf{Plot}(X)_n}\Pi_\infty(yU_{p_n})\times(\Pi_\infty\circ\mathrm{Disc})(\Delta^n)\\ &\simeq \int^{n\in\Delta}\coprod_{N\mathsf{Plot}(X)_n} *\times\Delta^n\\ &\cong N\mathsf{Plot}(X).\end{aligned} \tag{2.41}$$

where the weak equivalence is given by Theorem 2.3.13 and Lemma 2.3.16. □

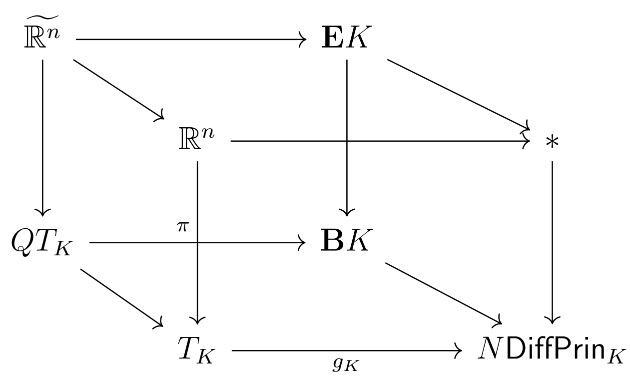
2.4 The Irrational Torus

In this section, we will show that if $K \subset \mathbb{R}$ is a diffeologically discrete subgroup of the real numbers, then the infinity stack cohomology of the irrational torus $T_K = \mathbb{R}^n/K$ (for any $n \geq 1$) with values in $\mathbb{R}^\delta$, is isomorphic to the group cohomology of K with values in $\mathbb{R}^\delta$. This was first proved by Iglesias-Zemmour [Igl23, Page 15] with his own version of diffeological Čech cohomology, which we will refer to as PIZ cohomology. In Section 1.5.3 we found maps between ∞-stack cohomology and PIZ cohomology, and showed that one of these maps is a retract, but it is still an open question as to whether these two cohomologies are isomorphic.

The motivation for this section is two-fold. One is to support the conjecture that ∞-stack cohomology is isomorphic to PIZ cohomology. We do this by showing that they agree on one of the most important class of examples of diffeological spaces, the irrational tori. The second motivation is to show the power of ∞-topos theory and in particular ∞-stack cohomology through the use of the shape operation. The proof of Theorem 2.4.4 is completely different than that in [Igl23], and it is conceptually more straightforward.

Definition 2.4.1. Suppose that $K \subset \mathbb{R}^n$ is a subgroup, and furthermore, when K is given the subset diffeology of $\mathbb{R}$, it coincides with the discrete diffeology, so that every plot is constant. We call its quotient $T_K = \mathbb{R}^n/K$ the n-dimensional **K-irrational torus**[7].

We define the quotient map $\pi : \mathbb{R}^n \to T_K$ of diffeological spaces. This map is a diffeological principal K-bundle [Igl13, Article 8.15], where K is given the discrete diffeology. By the discussion in Example 2.3.7, the bundle π is classified by a map of ∞-stacks $g_K : T_K \to N\mathsf{DiffPrin}_K$. Furthermore, by Corollary 1.6.13, we obtain the following cube, where the front face and back face are homotopy pullback squares and the maps going from the back face to the front face are all objectwise weak equivalences



(2.42)

Now if we apply the shape functor to this cube, the back face remains a homotopy pullback by the following result.

Proposition 2.4.2 ([SS21, Proposition 3.3.8])**.** Let K be a simplicial set, and let $f : X \to \mathrm{Disc}(K)$ and $g : Y \to \mathrm{Disc}(K)$ be maps of simplicial presheaves on Cart. Then there is a Čech weak equivalence

$$(\textstyle\int X) \times^h_{\mathrm{Disc}(K)} (\textstyle\int Y) \simeq \textstyle\int \left(X \times^h_{\mathrm{Disc}(K)} Y\right) \tag{2.43}$$

between the homotopy pullbacks of the maps $\int f$ and $\int g$ and the shape of the homotopy pullback of the maps f and g.

Remark 2.4.3. For a model-category theoretic proof of Proposition 2.4.2, use the argument

[7] The word irrational comes from the example where $n = 1$, and $K = \mathbb{Z} + \alpha\mathbb{Z}$ with α an irrational number. This is the most studied example of an irrational torus in diffeology. Interestingly, $\mathbb{Z} + \alpha\mathbb{Z} \subset \mathbb{R}$ with the subset topology is dense in $\mathbb{R}$, hence not discrete, however the subset diffeology is discrete.

of [Sch13, Theorem 4.1.34], and mirror the argument of [Sch13, Proposition 4.1.35] model categorically.

Therefore the front face must also be a homotopy pullback square. Now by Lemma 2.3.10, Lemma 2.3.16, and Theorem 2.3.13 it follows that

$$\int \widetilde{\mathbb{R}^n} \simeq \int \mathbb{R}^n \simeq *, \qquad \int QT_K \simeq \int T_K, \qquad \int \mathbf{E}K \simeq \int * \cong *, \qquad \mathbf{B}K \cong \int \mathbf{B}K \simeq \int \mathsf{DiffPrin}_K. \tag{2.44}$$

From this we obtain the main result of this section.

Theorem 2.4.4. There is an isomorphism

$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta) \tag{2.45}$$

of abelian groups, for every $n \geq 0$, where $\mathbb{R}^\delta$ denotes the discrete group of real numbers, and where $H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta)$ denotes the group cohomology of K with coefficients in $\mathbb{R}^\delta$.

Proof. First we note that since $\mathbb{R}^\delta$ is discrete, $\mathbb{R}^\delta \cong \flat\mathbb{R}^\delta$. Similarly, $\mathbf{B}^n\mathbb{R}^\delta \cong \flat\mathbf{B}^n\mathbb{R}^\delta$ for all $n \geq 0$. Thus

$$\mathbb{R}\check{\mathbb{H}}(T_K, \mathbf{B}^n\mathbb{R}^\delta) \cong \mathbb{R}\check{\mathbb{H}}(T_K, \flat\mathbf{B}^n\mathbb{R}^\delta) \cong \mathbb{R}\check{\mathbb{H}}(\int T_K, \mathbf{B}^n\mathbb{R}^\delta). \tag{2.46}$$

Now from Proposition 2.3.11, there is a weak equivalence $\mathrm{Sing}_D(T_K) \to \int T_K$ of simplicial sets, and by [CW14, Proposition 4.30], $\mathrm{Sing}_D(T_K)$ is a Kan complex. Now $\int N\mathsf{DiffPrin}_K$ is a Kan complex since it is a simplicial abelian group. Furthermore since T_K is diffeologically connected [Igl23, Section II], $\mathrm{Sing}_D(T_K)$ is connected. Thus the composite map $\mathrm{Sing}_D(T_K) \to \int N\mathsf{DiffPrin}_K$ is a map of connected Kan complexes whose homotopy fiber is contractible. Therefore by the long exact sequence of homotopy groups [Cis19, Theorem 3.8.12], it is a weak equivalence, which implies that $\int g_K$ is a weak equivalence. Therefore we have

$$\mathbb{R}\check{\mathbb{H}}(\int T_K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \mathbb{R}\check{\mathbb{H}}(\int N\mathsf{DiffPrin}_K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta). \tag{2.47}$$

Which implies that

$$\check{H}^n_\infty(T_K, \mathbb{R}^\delta) = \pi_0 \mathbb{R}\check{\mathbb{H}}(\textstyle\int T_K, \mathbf{B}^n\mathbb{R}^\delta) \cong \pi_0 \mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) = \check{H}^n_\infty(\mathbf{B}K, \mathbb{R}^\delta). \tag{2.48}$$

However since $\mathbf{B}K$ and $\mathbf{B}^n\mathbb{R}^\delta$ are discrete, Disc is fully faithful and $\mathbf{B}^n\mathbb{R}^\delta$ is a Kan complex, we have

$$\mathbb{R}\check{\mathbb{H}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) \simeq \underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta),$$

where $\underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta)$ is the usual simplicial set function complex. It is then well known (see [Wei95, Example 8.2.3] for instance) that

$$\pi_0 \underline{\mathsf{sSet}}(\mathbf{B}K, \mathbf{B}^n\mathbb{R}^\delta) \cong H^n_{\mathrm{grp}}(K, \mathbb{R}^\delta). \tag{2.49}$$

This proves the theorem. □

2.5 The Dold-Kan Correspondence

In this section, we discuss the Dold-Kan correspondence, which is central to Section 2.7.

Remark 2.5.1. For the remainder of this paper, by a vector space, we mean a real vector space, not necessarily of finite dimension. By a chain complex we mean a non-negatively graded chain complex of vector spaces. Let Ch denote the category of chain complexes.

Definition 2.5.2. Let Vect denote the category whose objects are vector spaces and whose morphisms are linear maps. Let $\mathsf{sVect} := \mathsf{Vect}^{\Delta^{op}}$ denote the category of simplicial vector spaces.

Proposition 2.5.3 ([GS06, Proposition 4.2 and Theorem 4.13], [Jar03, Section 1])**.** The category sVect admits a proper, combinatorial, simplicial model category structure where a morphism $f : X \to Y$ is a

1. weak equivalence if it is a weak homotopy equivalence of the underlying simplicial sets,

2. fibration if it is a Kan fibration of the underlying simplicial sets,

3. cofibration if it is degreewise a monomorphsim.

We call this the **Kan-Quillen** model structure[8] on sVect.

Proposition 2.5.4 ([GS06, Theorem 1.5], [Jar03, Section 1])**.** The category Ch admits a proper, combinatorial, simplicial model category structure where a morphism $f : C \to D$ is a

1. weak equivalence if it is a quasi-isomorphism of chain complexes, and

2. fibration if $f_k : C_k \to D_k$ is surjective in degrees $k \geq 1$.

3. cofibration if it is degreewise a monomorphism[9].

We call this the **projective model structure**[10] on chain complexes.

There is an adjoint pair of functors,

$$\mathsf{Ch} \underset{N}{\overset{\mathsf{DK}}{\rightleftarrows}} \mathsf{sVect} \quad (\perp) \tag{2.50}$$

which by the Dold-Kan correspondence [GS06, Theorem 4.1] form an adjoint equivalence[11].

Lemma 2.5.5 ([SS03, Section 4.1])**.** The adjunction $N \dashv \mathsf{DK}$ is a Quillen adjunction ([Hir09, Definition 8.5.2]) between the model category structures on Ch and sVect. Furthermore, the functors form a Quillen equivalence ([Hir09, Definition 8.5.20]). In fact, since N and DK form an adjoint equivalence, it follows that $\mathsf{DK} \dashv N$ is also an adjoint equivalence. Furthermore N and DK are both left and right Quillen functors.

[8] Note that every object in sVect is fibrant and cofibrant.

[9] For the projective model structure on chain complexes of R-modules for a general commutative ring R, we require these to be degreewise monomorphisms with projective cokernel. Since all vector spaces are projective as $\mathbb{R}$-modules, this condition is always satisfied.

[10] Note that every object in Ch is fibrant and cofibrant.

[11] This result actually holds for chain complexes and simplicial objects taking values in any idempotent complete, additive category, see [Lur17, Theorem 1.2.3.7].

If we consider the category of simplicial sets sSet with its usual Kan-Quillen model structure [GS06, Theorem 1.22], there is a simplicial Quillen adjunction

$$\mathsf{sSet} \underset{U}{\overset{\mathbb{R}[-]}{\rightleftarrows}} \mathsf{sVect} \quad (\perp) \tag{2.51}$$

where U denotes the forgetful functor, and $\mathbb{R}[-]$ denotes the functor that sends a simplicial set X to the free simplicial vector space $\mathbb{R}X$, defined degreewise by $\mathbb{R}X_n = \mathbb{R}(X_n)$, where $\mathbb{R}(X_n)$ is the free vector space on the set X_n. Thus we obtain a Quillen adjunction

$$\mathsf{sSet} \underset{U\mathsf{DK}}{\overset{N\mathbb{R}[-]}{\rightleftarrows}} \mathsf{Ch} \quad (\perp) \tag{2.52}$$

which is furthermore a simplicial Quillen adjunction[12].

Note that the Dold-Kan correspondence also provides a simplicial enrichment of Ch. Indeed, suppose C and D are chain complexes, then let $\underline{\mathsf{Ch}}(C, D)$ denote the simplicial vector space defined degreewise by

$$\underline{\mathsf{Ch}}(C, D)_k = \mathsf{Ch}(N\mathbb{R}\Delta^k \otimes C, D). \tag{2.53}$$

This makes the Dold-Kan correspondence an enriched adjoint equivalence. This is also the simplicial enrichment mentioned in Proposition 2.5.4.

This also supplies Ch with tensoring and cotensoring over sSet. Namely if K is a simplicial set and C is a chain complex then $C \otimes K$ is the chain complex $C \otimes N\mathbb{R}K$, and $C^K = N\mathbb{R}\underline{\mathsf{Ch}}(N\mathbb{R}K, C)$.

Now the category of chain complexes Ch is also enriched over itself. Indeed, if C and D are chain complexes, then let $\underline{\mathrm{Map}_{\mathsf{Ch}}}(C, D)$ denote the chain complex defined as follows.

[12] We will often omit the functor U in our notation.

First let us define the unbounded chain complex $\underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ defined in degree $k \in \mathbb{Z}$ by

$$\underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_k = \prod_{i \geq 0} \mathsf{Vect}(C_i, D_{i+k}), \tag{2.54}$$

with $d : \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_k \to \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)_{k-1}$ defined for a map f by

$$df = d_D f - (-1)^k f d_C.$$

We call an element of degree k in $\underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ a degree k map from C to D.

Definition 2.5.6. If C is an unbounded ($\mathbb{Z}$-graded) chain complex, then let $\tau_{\geq 0}C$ denote the chain complex defined degreewise by $(\tau_{\geq 0}C)_k = C_k$ for $k > 0$, and $(\tau_{\geq 0}C)_0 = \mathbb{Z}_0 C$, the set of 0-cycles of C, i.e. those $x \in C_0$ such that $dx = 0$, the differential on $\tau_{\geq 0}C$ is induced by the differential on C. We call $\tau_{\geq 0}C$ the **smart truncation** of C.

Now given chain complexes C and D, let

$$\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D) = \tau_{\geq 0}\underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D), \tag{2.55}$$

denote the smart truncation applied to $\underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$. This means that $\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D)_k = \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(C,D)$ for $k > 0$, and $\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D)_0 \cong \mathsf{Ch}(C,D)$. We refer to $\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D)$ as the **mapping chain complex** between C and D.

Lemma 2.5.7 ([Opa21, Example 4.3.2])**.** Let C and D be chain complexes, then we have an isomorphism of simplicial vector spaces

$$\mathsf{DK}\,\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D) \cong \underline{\mathsf{Ch}}(C,D). \tag{2.56}$$

Further, this provides an isomorphism

$$\underline{\mathrm{Map}_{\mathsf{Ch}}}(C,D) \cong N\underline{\mathsf{Ch}}(C,D). \tag{2.57}$$

An explicit description for the path space of a chain complex C, equivalently the cotensoring C^{Δ^1}, is given in Section 2.10.

Definition 2.5.8. Let C be a small category. Then let $\mathsf{ChPre}(\mathsf{C})$ denote the category whose objects are functors $\mathsf{C}^{op} \to \mathsf{Ch}$, and whose morphisms are natural transformations. We call such functors presheaves of chain complexes.

Proposition 2.5.9 ([Hir09, Section 11.6])**.** The category $\mathsf{ChPre}(\mathsf{C})$ admits a proper, combinatorial, simplicial model category structure where a morphism $f : C \to D$ is a

1. weak equivalence if it objectwise a weak equivalence in the projective model structure on chain complexes, and
2. fibration if it is objectwise a fibration in the projective model structure on chain complexes.

We refer to this as the **(global) projective model structure** on presheaves of chain complexes.

Thus we obtain a similar simplicial Quillen pair

$$\mathsf{sPre}(\mathsf{C}) \underset{U\mathsf{DK}}{\overset{N\mathbb{R}[-]}{\rightleftarrows}} \mathsf{ChPre}(\mathsf{C}) \tag{2.58}$$

where $\mathsf{sPre}(\mathsf{C})$ is equipped with the projective model structure. In Section 2.10, we will use (2.58), along with Proposition 2.3.5, to compute homotopy pullbacks in $\check{\mathbb{H}}$.

2.6 Examples of ∞-stacks

In this section we detail the ∞-stacks involved in this paper, and examine the ∞-stack cohomology of a diffeological space with coefficients in some of these example ∞-stacks.

Example 2.6.1. Given a finite dimensional smooth manifold M, the functor $U \mapsto C^\infty(U, M)$ defines a sheaf on Cart, and therefore an ∞-stack. The same goes for diffeological spaces. Given a diffeological space X, let X^δ denote the diffeological space with the same underlying set, but equipped with the discrete diffeology. As sheaves we have $X^\delta \cong \mathrm{Disc}(X(*)) = \flat X$.

Example 2.6.2. The presheaf of differential k-forms Ω^k and the presheaf of closed differential k-forms Ω^k_{cl} are sheaves of vector spaces on Cart for every $k \geq 0$. Thus they are ∞-stacks. The de Rham differential defines a map of ∞-stacks $d : \Omega^k \to \Omega^{k+1}$ for all $k \geq 0$.

There is a canonical map

$$\mathrm{mc}(\mathbb{R}) : \mathbb{R} \to \Omega^1 \tag{2.59}$$

of ∞-stacks, defined by the Yoneda Lemma as follows. Notice that $\mathbb{R} \in \mathsf{Cart}$, so by the Yoneda lemma, a map $\omega : y\mathbb{R} \to \Omega^1$ is equivalent to an element $\omega \in \Omega^1(\mathbb{R})$. There is a canonical element of the set of 1-forms on $\mathbb{R}$, called the **Maurer-Cartan form** of $\mathbb{R}$. For a general Lie group G, we let $\mathrm{mc}(G)$ denote its Maurer-Cartan form. If we label the coordinate of $\mathbb{R}$ by t, then the Maurer-Cartan form is simply given by $\mathrm{mc}(\mathbb{R}) = dt$. Thus for a cartesian space U, the function $\mathrm{mc}(\mathbb{R})(U) : \mathbb{R}(U) \to \Omega^1(U)$ acts by taking a smooth map $f : U \to \mathbb{R}$ and pulling back the Maurer-Cartan form $f^*\mathrm{mc}(\mathbb{R}) \in \Omega^1(U)$. Note that this is the same thing as df. In other words as maps of ∞-stacks, we have $\mathrm{mc}(\mathbb{R}) = d$.

Example 2.6.3. Let G be a diffeological group. As discussed in Example 2.3.7 the presheaf of groupoids $\mathbf{B}G$ given by

$$U \mapsto [C^\infty(U, G) \rightrightarrows *]$$

is a stack, and is objectwise weak equivalent to the stack of diffeological principal G-bundles. We abuse notation and also let $\mathbf{B}G$ denote the corresponding simplicial presheaf, which is an ∞-stack. Given a diffeological space X, a G-cocycle on X as in Definition 2.2.6 is precisely a map $QX \to \mathbf{B}G$ of simplicial presheaves. Theorem 1.3.15 shows that the resulting groupoid of G-cocycles on X is equivalent to the groupoid of diffeological principal G-bundles on X. Thus $\check{H}^1_\infty(X, G) \coloneqq \check{H}^0_\infty(X, \mathbf{B}G)$ is the set of isomorphism classes of G-cocycles, which is isomorphic to the set of isomorphism classes of diffeological principal G-bundles.

Example 2.6.4. If G is a Lie group with Lie algebra $\mathfrak{g}$, then let $\Omega^1(-,\mathfrak{g})//G$ denote the presheaf of groupoids

$$U \mapsto [\Omega^1(U,\mathfrak{g}) \times C^\infty(U,G) \overset{t}{\underset{s}{\rightrightarrows}} \Omega^1(U,\mathfrak{g})]$$

where $t(\omega, g) = \omega$ and $s(\omega, g) = \mathrm{Ad}_g^{-1}(\omega) + g^*\mathrm{mc}(G)$. The nerve of this presheaf of groupoids is an ∞-stack [FSS+12, Proposition 3.2.5][13]. This is the ∞-stack that classifies principal G-bundles with connection. We will often abuse notation and write $\Omega^1(-,\mathfrak{g})//G$ to refer to the presheaf of groupoids and the simplicial presheaf obtained by taking the nerve construction. Note that a map $QX \to \Omega^1(-,\mathfrak{g})//G$ is equivalent to the data of a G-cocycle $g_{f_0} : U_{p_1} \to G$ and a collection $\{A_{p_0}\}_{p_0 \in \mathsf{Plot}(X)}$ of 1-forms $A_{p_0} \in \Omega^1(U_{p_0}, \mathfrak{g})$ such that for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots we have

$$A_{p_1} = \mathrm{Ad}_{g_{f_0}}^{-1}(f_0^* A_{p_0}) + g_{f_0}^* \mathrm{mc}(G). \tag{2.60}$$

Let us call this collection of data $(g, A) = (\{g_{f_0}\}, \{A_{p_0}\})$ a G**-cocycle with connection**. We show that this definition of connection is equivalent to the one given in [Wal12, Definition 3.2.1] in Section 2.8.

Remark 2.6.5. The following examples of simplicial presheaves can be checked to be ∞-stacks by using [Pav22a, Corollary 6.2]. One simply needs to notice that the examples that follow are presheaves of bounded chain complexes, and can thus be thought of equivalently as presheaves of cochain complexes, and that the homotopy descent condition for presheaves of cochain complexes is equivalent to the condition of Dold-Kan applied to the presheaves of chain complexes to be ∞-stacks.

Example 2.6.6. Given a sheaf A of abelian groups on Cart, with $k \geq 1$, the simplicial presheaf $\mathbf{B}^k A$ is obtained by taking Dold-Kan of the presheaf of chain complexes $[A \to 0 \to$

[13]Notice that the definition above is precisely the opposite of the corresponding ∞-stack considered in [FSS+12, Section 3]. This is because of the convention we use in 2.3.7. However this makes no difference on the theory as we show in Section 2.8.

$\cdots \to 0]$. When A is an abelian diffeological group, then $\mathbf{B}A$ is the ∞-stack that classifies diffeological principal A-bundles.

In this paper we will consider the example $\mathbf{B}^k\mathbb{R}$. Given a diffeological space X, a map $QX \to \mathbf{B}^k\mathbb{R}$ consists of a $g \in \mathbb{R}(QX_k)$ such that $\delta g = 0$, see Section 2.9. We call these **$\mathbb{R}$-bundle $(k-1)$-gerbes**. Thus a diffeological principal $\mathbb{R}$-bundle is precisely a $\mathbb{R}$-bundle 0-gerbe.

There is a vast literature on bundle gerbes in differential geometry such as [Mur96], [Bun21a], [Ste04]. Typically bundle gerbes are defined as geometric objects, and then shown to define cohomology classes through cocycles such as above. However, giving descriptions of bundle k-gerbes as geometric objects becomes difficult and tedious as k grows. Their description as cocycles is much more economical, and is all we need for this paper. There should be no difficulty in translating between the geometric description of diffeological bundle 1-gerbes, such as in [Wal12] and the cocycle description we give here, but we leave this to future work.

Example 2.6.7. For $k \geq 1$, let $\mathbf{B}^k_\nabla\mathbb{R}$[14] denote the simplicial presheaf obtained by applying Dold-Kan to the following presheaf of chain complexes[15]

$$[\mathbb{R} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k]. \tag{2.61}$$

This complex is often referred to as the **Deligne complex** when $\mathbb{R}$ is replaced with $U(1)$ and $d : \mathbb{R} \to \Omega^1$ is replaced with $d\log : U(1) \to \Omega^1$. For $k \geq 2$, $\mathbf{B}^k_\nabla\mathbb{R}$ classifies $\mathbb{R}$-bundle $(k-1)$-gerbes with connection, and ∞-stack cohomology with values in $\mathbf{B}^k_\nabla\mathbb{R}$ is called **pure differential cohomology**[16] in [Jaz21, Definition 3.2.10]. Thus we call $\mathbf{B}^k_\nabla\mathbb{R}$ the **pure k-Deligne complex**.

Given a diffeological space X, we will denote ∞-stack cohomology with values in the pure

[14]Note that the $\mathbf{B}^k$ in $\mathbf{B}^k_\nabla\mathbb{R}$ is just notation, it is not actually the delooping of anything.

[15]Note that the way this chain complex is written, Ω^k is in degree 0.

[16]We also recommend [Jaz21, Section 3.2] for a discussion of how pure differential cohomology fits into the hexagon diagram of differential cohomology.

k-Deligne complex by

$$\check{H}^k_{\infty,\nabla}(X,\mathbb{R}) \coloneqq \check{H}^0_\infty(X, \mathbf{B}^k_\nabla \mathbb{R}).$$

Let us also note that when $k = 1$, we have an objectwise weak equivalence of ∞-stacks

$$\mathbf{B}_\nabla \mathbb{R} \simeq (\Omega^1(-)//\mathbb{R})^{op}.$$

To see this, note that both of the simplicial presheaves are identical in simplicial degrees 0 and 1. This is because $\mathbb{R}$ is abelian, so $t(\omega, g) = \mathrm{Ad}_g^{-1}(\omega) + g^*\mathrm{mc}(\mathbb{R}) = \omega + dg$ in $(\Omega^1(-)//\mathbb{R})^{op}$, which is precisely the face map $d_0 : \mathbf{B}_\nabla\mathbb{R}_1 \to \mathbf{B}_\nabla\mathbb{R}_0$. Since $(\Omega^1(-)//\mathbb{R})^{op}$ is the nerve of a presheaf of groupoids, it is 2-coskeletal, and therefore its k-homotopy groups are trivial for $k \geq 2$. The objectwise homotopy groups of $\mathbf{B}_\nabla\mathbb{R}$ are given by the objectwise homology of the chain complex by the Dold-Kan correspondence, and thus are also trivial for $k \geq 2$, thus they are objectwise weak equivalent. The distinction between $\Omega^1(-)//\mathbb{R}$ and $(\Omega^1(-)//\mathbb{R})^{op}$ is because of Example 2.3.7, so technically $\mathbf{B}_\nabla\mathbb{R}$ classifies diffeological principal $\mathbb{R}^{op}$-bundles with opposite connection, but this distinction is immaterial to the theory and we sweep it under the rug, and we say that the above classifies diffeological principal $\mathbb{R}$-bundles with connection.

If X is a diffeological space, then as we will see in Example 2.6.13, a $\mathbb{R}$-bundle $(k-1)$-gerbe with connection on X is given by the data

$$(\omega^k, \omega^{k-1}, \ldots, \omega^1, g) \in \Omega^k(QX_0) \oplus \Omega^{k-1}(QX_1) \oplus \cdots \oplus \Omega^1(QX_{k-1}) \oplus \mathbb{R}(QX_k), \tag{2.62}$$

such $D(\omega^k, \ldots, g) = 0$ in the double complex $\Omega^i(QX_j)$, see Section 2.9. We will let $[\omega^k, \ldots, g]$ denote the isomorphism class it represents in $\check{H}^k_{\infty,\nabla}(X,\mathbb{R})$.

Example 2.6.8. For $k \geq 1$, consider the ∞-stack $\mathbf{B}^k\mathbb{R}^\delta$. This ∞-stack classifies diffeological principal $\mathbb{R}^\delta$-bundles. However, note that there is a map of presheaves of chain complexes

$$[\mathbb{R}^\delta \to 0 \to \cdots \to 0] \to [\mathbb{R} \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k_{\mathrm{cl}}] \tag{2.63}$$

given by the inclusion $\mathbb{R}^\delta \hookrightarrow \mathbb{R}$. Furthermore, this map is an objectwise quasi-isomorphism, by the Poincare lemma. Thus we will take the right hand side of (2.63) to be the model of $\mathbf{B}^k\mathbb{R}^\delta$ we will use for the rest of this paper. From this presentation, it is easy to see that $\mathbf{B}^k\mathbb{R}^\delta$ is equivalently the ∞-stack that classifies diffeological principal $\mathbb{R}$-bundle $(k-1)$-gerbes with flat connection.

Example 2.6.9. For $k \geq 1$, consider the ∞-stack $\mathbf{B}^k\Omega^1_{\mathrm{cl}}$. There is a map of presheaves of chain complexes

$$[\Omega^1_{\mathrm{cl}} \to 0 \to \cdots \to 0] \to [\Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega^k_{\mathrm{cl}}] \tag{2.64}$$

and this map is an objectwise quasi-isomorphism again by the Poincare lemma. We take the right hand side to be the model we will use for $\mathbf{B}^k\Omega^1_{\mathrm{cl}}$ for the rest of the paper.

Example 2.6.10. For $k \geq 1$, let $\Omega^{1\leq \bullet \leq k}$ denote the simplicial presheaf obtained by applying Dold-Kan to the following presheaf of chain complexes

$$[\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k]. \tag{2.65}$$

If X is a diffeological space and A is a k-deloopable ∞-stack, recall that its ∞-stack cohomology is given by

$$\check{H}^k_\infty(X, A) = \mathbb{R}\check{\mathbb{H}}(X, \mathbf{B}^k A).$$

Let us compute an example of ∞-stack cohomology for a diffeological space X with values in the ∞-stack $\mathbf{B}\Omega^1$, as it will be emblematic of how we compute ∞-stack cohomology for all of the relevant examples presented in this section. In Section 2.9 we go into detail on how to compute such examples.

Example 2.6.11. Let $A' = \mathbf{B}\Omega^1 = [\Omega^1 \to 0]$ and $A = \mathsf{DK}\, A'$. Let us compute $H^0(X, A)$ for a diffeological space X using Proposition 2.9.3. Consider the double complex $A'(QX)$,

$$\begin{array}{ccccccc} \Omega^1(QX_0) & \xrightarrow{\delta} & \Omega^1(QX_1) & \xrightarrow{-\delta} & \Omega^1(QX_2) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array} \tag{2.66}$$

so that $\mathsf{tot}\, A'(QX)$ is the chain complex

$$\mathsf{tot}\, A'(QX) = [\Omega^1(QX_0) \xrightarrow{D} \ker(-\delta : \Omega^1(QX_1) \to \Omega^1(QX_2))] \tag{2.67}$$

Thus a 0-cycle in $\mathsf{tot}\, A'(QX)$ consists of a collection $\{\omega_{f_0} \in \Omega^1(U_{p_1})\}_{f_0 : U_{p_1} \to U_{p_0}}$ of 1-forms for every map of plots of X such that $-\delta\omega = 0$, which is equivalent to the condition that for every pair of composable plot maps $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$ we have

$$(\delta\omega)_{(f_1, f_0)} = f_1^* \omega_{f_0} - \omega_{f_0 f_1} + \omega_{f_1} = 0.$$

Two such 0-cocycles are cohomologous $\omega \sim \omega'$ if there exists a collection $\{\lambda_{p_0} \in \Omega^1(U_{p_0})\}_{p_0 \in \mathsf{Plot}(X)}$ of 1-forms for every plot of X such that

$$(D\lambda)_{f_0} = (\delta\lambda)_{f_0} = \omega'_{f_0} - \omega_{f_0}$$

for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots of X. Thus $\check{H}^0_\infty(X, \mathbf{B}\Omega^1) = \check{H}^1_\infty(X, \Omega^1)$ is precisely analogous to the term $H^{1,1}_\delta$ in [Igl23, Section 4.4].

Example 2.6.12. Let us compute $\check{H}^1_{\infty,\nabla}(X, \mathbb{R}) := \check{H}^0_\infty(X, \mathbf{B}_\nabla \mathbb{R})$ for a diffeological space X. Consider the double complex

$$\begin{array}{ccccccc} \mathbb{R}(QX_0) & \xrightarrow{\delta} & \mathbb{R}(QX_1) & \xrightarrow{-\delta} & \mathbb{R}(QX_2) & \longrightarrow & \dots \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & \\ \Omega^1(QX_0) & \xrightarrow{-\delta} & \Omega^1(QX_1) & \xrightarrow{\delta} & \Omega^1(QX_2) & \longrightarrow & \dots \end{array} \tag{2.68}$$

where $\mathbb{R}(QX_i) = C^\infty(QX_i, \mathbb{R})$. A 0-cocycle is the data of a map $g : QX_1 \to \mathbb{R}$ and a 1-form $A \in \Omega^1(QX_0)$ such that $-\delta g = 0$ and $-\delta A = dg$. The condition $-\delta g = \delta g = 0$ is equivalent to the condition $g_{f_0 f_1} = f_1^* g_{f_0} + g_{f_1}$ for every pair of composable plot maps $U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}$, which is precisely the condition for g to be a cocycle defining a diffeological principal $\mathbb{R}$-bundle on X. The condition $\delta A = dg$ is equivalent to the condition that for every map of plots $f_0 : U_{p_1} \to U_{p_0}$ we have $A_{p_1} - f_0^* A_{p_0} = dg_{f_0}$, which is precisely the equation for a connection on a diffeological principal $\mathbb{R}$-bundle, see Section 2.8. Given two 0-cocycles (A, g) and (A', g'), a 1-coboundary consists of an element $h \in \mathbb{R}(QX_0)$ such that $\delta h = g' - g$ and $dh = A' - A$. This is precisely the definition of a morphism of G-cocycles with connection, see Definition 2.8.1. Thus an element of $\check{H}^1_{\infty,\nabla}(X, \mathbb{R})$ is an isomorphism class of a diffeological principal $\mathbb{R}$-bundle on X with connection.

Example 2.6.13. Let us compute $\check{H}^2_{\infty,\nabla}(X, \mathbb{R})$ for a diffeological space X. Consider the double complex

$$\begin{array}{ccccccc}
\mathbb{R}(QX_0) & \xrightarrow{-\delta} & \mathbb{R}(QX_1) & \xrightarrow{\delta} & \mathbb{R}(QX_2) & \longrightarrow & \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \\
\Omega^1(QX_0) & \xrightarrow{\delta} & \Omega^1(QX_1) & \xrightarrow{-\delta} & \Omega^1(QX_2) & \longrightarrow & \dots \\
\downarrow d & & \downarrow d & & \downarrow d & & \\
\Omega^2(QX_0) & \xrightarrow{-\delta} & \Omega^2(QX_1) & \xrightarrow{\delta} & \Omega^2(QX_2) & \longrightarrow & \dots
\end{array} \tag{2.69}$$

Then a 0-cycle in $\mathsf{tot}\,\mathbf{B}^2_\nabla\mathbb{R}(QX)$ is an element $(\omega, A, g) \in \Omega^2(QX_0) \oplus \Omega^1(QX_1) \oplus \mathbb{R}(QX_2)$ such that $D(\omega, A, g) = 0$. This is equivalent to the equations $-\delta\omega = dA$, $-\delta A = dg$ and $-\delta g = 0$. The equation $-\delta g = \delta g = 0$ is equivalent to the condition that for every triple of composable maps of plots (f_2, f_1, f_0) of X, we have

$$(\delta g)_{(f_2,f_1,f_0)} = f_2^* g_{(f_1,f_0)} - g_{(f_1 f_2, f_0)} + g_{(f_2, f_0 f_1)} - g_{(f_2,f_1)} = 0. \tag{2.70}$$

This is the diffeological analogue of the cocycle data of a $\mathbb{R}$-bundle gerbe on X. The other two equations $-\delta\omega = dA$ and $-\delta A = dg$ are the diffeological analogue of the cocycle data of a connection on a $\mathbb{R}$-bundle gerbe. Thus $\check{H}^2_{\infty,\nabla}(X,\mathbb{R}) = \check{H}^0_\infty(X, \mathbf{B}^2_\nabla\mathbb{R})$ can be taken as the definition of the abelian group of isomorphism classes of diffeological $\mathbb{R}$-bundle 1-gerbes with connection on X. The story for $k \geq 2$ is exactly analogous, and thus we take our definition of the abelian group of isomorphism classes of diffeological $\mathbb{R}$-bundle $(k-1)$-gerbes with connection to be $\check{H}^k_{\infty,\nabla}(X,\mathbb{R}) = \check{H}^0_\infty(X, \mathbf{B}^k_\nabla\mathbb{R})$.

2.7 The Čech de Rham Obstruction

In this section, we obtain a diffeological Čech-de Rham obstruction exact sequence in every degree from a homotopy pullback diagram of ∞-stacks. In degree 1, our exact sequence is analogous to [Igl23].

Let X be a diffeological space. In [Igl23], Iglesias-Zemmour constructs the following exact sequence of vector spaces

$$0 \to H^1_{dR}(X) \to \check{H}^1_{PIZ}(X, \mathbb{R}^\delta) \to {}^dE_2^{1,0}(X) \xrightarrow{c_1} H^2_{dR}(X) \to \check{H}^2_{PIZ}(X, \mathbb{R}^\delta) \tag{2.71}$$

using the five term exact sequence coming from a diffeological version of the Čech-de Rham bicomplex spectral sequence. The vector space ${}^dE_2^{1,0}(X)$ is the subspace of isomorphism classes of diffeological principal $\mathbb{R}$-bundles on X that admit a connection, and the vector spaces $\check{H}^k_{\mathrm{PIZ}}(X, \mathbb{R}^\delta)$ are Iglesias-Zemmour's version of diffeological Čech cohomology, which we refer to as PIZ cohomology. The relationship between PIZ cohomology and ∞-stack cohomology is only partially understood.

The exact sequence (2.71) demonstrates the obstruction to the Čech-de Rham Theorem holding for diffeological spaces. For finite dimensional smooth manifolds, all principal $\mathbb{R}$-bundles are trivial, as they have contractible fiber, and thus the obstruction vanishes. However, there are diffeological spaces (the irrational torus for example) that have nontrivial principal $\mathbb{R}$-bundles that admit connections [Igl13, Article 8.39].

We construct and geometrically interpret the obstruction to the Čech-de Rham isomorphism in each degree $k \geq 1$ via ∞-stacks.

Theorem 2.7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

$$\begin{array}{ccccccc}
* & \longrightarrow & \mathbf{B}^k\mathbb{R}^\delta & \longrightarrow & * & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \mathbf{B}^k_\nabla\mathbb{R} & \longrightarrow & \Omega^{k+1}_{\mathrm{cl}} & \longrightarrow & \Omega^{k+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{B}^k\mathbb{R} & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & \Omega^{1\leq\bullet\leq k+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \mathbf{B}^{k+1}\mathbb{R}^\delta & \longrightarrow & \mathbf{B}^{k+1}_\nabla\mathbb{R}
\end{array} \tag{2.72}$$

furthermore every commutative square in this diagram is a homotopy pullback square in $\check{\mathbb{H}}$.

We prove Theorem 2.7.1 in Section 2.10.

Corollary 2.7.2. For every diffeological space X, there is an exact sequence of vector spaces[17]

$$0 \to \check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X, \mathbb{R}) \to \Omega^{k+1}_{\mathrm{cl}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta). \tag{2.73}$$

Proof. This follows from Theorem 2.7.1 and Lemma 2.10.6. □

Let us explore the consequences of Corollary 2.7.2 in the case where X is the irrational torus. Let $K = \mathbb{Z} + \alpha\mathbb{Z}$ be the subgroup of $\mathbb{R}$ consisting of those $x \in \mathbb{R}$ of the form $n + \alpha m$ where n, m are integers and α is an irrational number. Let $T_\alpha = \mathbb{R}/\mathbb{Z} + \alpha\mathbb{Z}$. We can fully compute the de Rham and ∞-stack cohomology of T_α. Every differential form on T_α is closed [Igl13, Exercise 119] so $\Omega^k(T_\alpha) = \Omega^k_{\mathrm{cl}}(T_\alpha) = H^k_{\mathrm{dR}}(T_\alpha)$, and furthermore $\Omega^k_{\mathrm{cl}}(T_\alpha) \cong \Lambda^k\mathbb{R}$ by [Igl13, Exercise 105]. Therefore we have

$$\Omega^k_{\mathrm{cl}}(T_\alpha) = H^k_{\mathrm{dR}}(T_\alpha) \cong \begin{cases} \mathbb{R}, & k = 0, 1 \\ 0, & k > 1. \end{cases} \tag{2.74}$$

[17]Near the completion of this paper, we learned that an analogous exact sequence was also obtained in [Jaz21, Page 27] using completely different methods in the framework of homotopy type theory.

Now by Theorem 2.4.4, we have

$$\check{H}^k_\infty(T_\alpha, \mathbb{R}^\delta) \cong \check{H}^0_\infty(\mathbf{B}(\mathbb{Z}+\alpha\mathbb{Z}), \mathbf{B}^k\mathbb{R}^\delta) \cong \check{H}^0_\infty(\mathbf{B}\mathbb{Z}^2, \mathbf{B}^k\mathbb{R}^\delta) \cong \check{H}^k_\infty(T^2, \mathbb{R}^\delta) \cong \begin{cases} \mathbb{R}, & k = 0, 2 \\ \mathbb{R}^2, & k = 1 \\ 0, & k > 2, \end{cases} \tag{2.75}$$

where T^2 denotes the usual 2-dimensional torus.

From Corollary 2.7.2, setting $k = 1$, we obtain the exact sequence

$$0 \to \check{H}^1_\infty(T_\alpha, \mathbb{R}^\delta) \to \check{H}^1_{\infty,\nabla}(T_\alpha, \mathbb{R}) \to \Omega^2_{\text{cl}}(T_\alpha) \to \check{H}^2_\infty(T_\alpha, \mathbb{R}^\delta). \tag{2.76}$$

Since $\Omega^2_{\text{cl}}(T_\alpha) = 0$, this implies that $\check{H}^1_{\infty,\nabla}(T_\alpha, \mathbb{R}) \cong \check{H}^1_\infty(T_\alpha, \mathbb{R}^\delta) \cong \mathbb{R}^2$.

From Corollary 2.7.2, setting $k = 2$, we obtain the exact sequence

$$0 \to \check{H}^2_\infty(T_\alpha, \mathbb{R}^\delta) \to \check{H}^2_{\infty,\nabla}(T_\alpha, \mathbb{R}) \to \Omega^3_{\text{cl}}(T_\alpha) \to \check{H}^3_\infty(T_\alpha, \mathbb{R}^\delta), \tag{2.77}$$

but $\Omega^3_{\text{cl}}(T_\alpha) \cong H^3_{\text{dR}}(T_\alpha) \cong 0 \cong \check{H}^3_\infty(T_\alpha, \mathbb{R}^\delta)$, thus $\check{H}^2_{\infty,\nabla}(T_\alpha, \mathbb{R}) \cong \mathbb{R}$. Similar reasoning proves that $\check{H}^k_{\infty,\nabla}(T_\alpha, \mathbb{R}) \cong 0$ for $k > 2$. Thus we have proven the following.

Theorem 2.7.3. Let T_α denote the irrational torus, then

$$\check{H}^k_{\infty,\nabla}(T_\alpha, R) \cong \begin{cases} \mathbb{R}^2, & k = 1, \\ \mathbb{R}, & k = 2, \\ 0, & k > 2. \end{cases} \tag{2.78}$$

The reader should note that the above computations only work because the irrational torus has the property that its deRham cohomology is equal to its closed forms. This is not the case for general diffeological spaces, and therefore Corollary 2.7.2 is not generally helpful for computations. Therefore we desire an exact sequence which uses the deRham cohomology of a diffeological space rather than its closed forms.

Definition 2.7.4. Given a diffeological space X and $k \geq 1$, let $\check{H}^k_{\text{conn}}(X, \mathbb{R})$ denote the subspace of $H^k_{\text{dR}}(X) \oplus \check{H}^k_{\infty}(X, \mathbb{R})$ generated by the subset of pairs $([F], [g])$ where F is the curvature form $F = d\omega^k$ of a diffeological $\mathbb{R}$-bundle $(k-1)$-gerbe with connection $(\omega^k, \omega^{k-1}, \ldots, \omega^1, g)$.

The vector space $\check{H}^k_{\text{conn}}(X, \mathbb{R})$ sits in an exact sequence

$$0 \to \check{H}^k_{\infty,\text{triv}}(X, \mathbb{R}^\delta) \to \check{H}^k_{\infty,\nabla}(X, \mathbb{R}) \to \check{H}^k_{\text{conn}}(X, \mathbb{R}) \to 0, \tag{2.79}$$

where $\check{H}^k_{\infty,\text{triv}}(X, \mathbb{R}^\delta) \subset \check{H}^k_{\infty}(X, \mathbb{R}^\delta)$ is the subspace of isomorphism classes of trivial $\mathbb{R}$-bundle $(k-1)$-gerbes with flat connection.

Let us now define a new sequence of vector spaces

$$\check{H}^k_{\infty}(X, \mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^k_{\text{conn}}(X, \mathbb{R}) \xrightarrow{\beta} H^{k+1}_{\text{dR}}(X) \xrightarrow{\gamma} \check{H}^{k+1}_{\infty}(X, \mathbb{R}^\delta).$$

The map α takes an isomorphism class of an $\mathbb{R}$-bundle k-gerbe with flat connection $[\omega^k, \ldots, \omega^1, g]$ and gives $(0, [g])$. The map β sends $([F], [g]) \mapsto [F]$. Finally, γ sends $[\omega]$ to the isomorphism class of the $\mathbb{R}$-bundle k-gerbe with connection $(\omega, 0, \ldots, 0)$. This map is well defined, because if $\omega' - \omega = d\tau$ for some $\tau \in \Omega^k(X)$, then $(\omega' - \omega, 0, \ldots, 0) = D(\tau, 0, \ldots, 0)$.

Theorem 2.7.5. Given a diffeological space X and $k \geq 1$, the sequence of vector spaces

$$\check{H}^k_{\infty}(X, \mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^k_{\text{conn}}(X, \mathbb{R}) \xrightarrow{\beta} H^{k+1}_{\text{dR}}(X) \xrightarrow{\gamma} \check{H}^{k+1}_{\infty}(X, \mathbb{R}^\delta) \tag{2.80}$$

is exact.

Proof. Note that $\beta\alpha = 0$. Suppose $([F], [g]) \in \check{H}^k_{\text{conn}}(X, \mathbb{R})$ and consider $\gamma\beta([F], [g]) = [F, 0, \ldots, 0]$. Since $([F], [g]) \in \check{H}^k_{\text{conn}}(X, \mathbb{R})$, there exists a $\mathbb{R}$-bundle $(k-1)$-gerbe with connection $(\omega^k, \ldots, \omega^0, g)$ such that

$$D(\omega^k, \ldots, \omega^0, g) = (F, 0, \ldots, 0).$$

Therefore $[F, 0, \ldots, 0] = 0$. In fact, $\gamma\beta([F],[g]) = 0$ if and only if there exists an $\mathbb{R}$-bundle $(k-1)$-gerbe with connection such that the above equation holds. This implies that $\ker \gamma = \operatorname{im} \beta$.

Now suppose that $\beta([F],[g]) = [F] = 0$. Then there exists a global k-form τ such that $d\tau = F$. Since $([F],[g]) \in \check{H}^k_{\text{conn}}(X, \mathbb{R})$, there exists an $\mathbb{R}$-bundle $(k-1)$-gerbe $(\omega^k, \ldots, g)$ such that $d\omega^k = F$. Then $\omega^k - \tau \in \Omega^k(QX_0)$, $d(\omega^k - \tau) = 0$ and $\delta(\omega^k - \tau) = \delta\omega^k - \delta\tau = \delta\omega^k$. Therefore $(\omega^k - \tau, \omega^{k-1}, \ldots, \omega^0, g)$ defines an $\mathbb{R}$-bundle k-gerbe with flat connection. Thus $([F],[g]) = (0,[g]) \in \operatorname{im} \alpha$. □

We will refer to the exact sequence (2.80) as the **degree k PIZ exact sequence**. When $k = 1$, there is an interesting additional phenomenon.

Lemma 2.7.6. Let (A, g) and (A', g) be $\mathbb{R}$-bundle 0-gerbes/diffeological principal $\mathbb{R}$-bundles with connection on a diffeological space X with the same underlying $\mathbb{R}$-cocycle g, then dA and dA' are global closed 2-forms on X and their de Rham cohomology classes agree $[dA] = [dA']$.

Proof. Since (A, g) and (A', g) are both $\mathbb{R}$-bundle 0-gerbes with connection on X, this implies that for every map of plots f_0, we have

$$-(\delta A)_{f_0} = dg_{f_0} = -(\delta A')_{f_0}.$$

Now consider the form $A' - A \in \Omega^1(QX_0)$, defined plotwise by $(A' - A)_{p_0} = A'_{p_0} - A_{p_0}$. This is a global 1-form, because for every plot map f_0 we have

$$(\delta(A' - A))_{f_0} = (\delta A')_{f_0} - (\delta A)_{f_0} = 0.$$

Thus $A' - A \in \Omega^1(X)$. Similarly dA' and dA are also global 2-forms on X. Now $d(A' - A) \in$

$\Omega^2(X)$ is an exact form, and $dA' - dA = d(A' - A)$. Thus dA' and dA represent the same de Rham cohomology class. □

Lemma 2.7.6 implies that $\check{H}^1_{\text{conn}}(X, \mathbb{R})$ is isomorphic to the subspace of $\check{H}^1_\infty(X, \mathbb{R})$ generated by the subset of those diffeological principal $\mathbb{R}$-bundles that admit a connection, as every connection produces a unique cohomology class. Thus we see that $\check{H}^1_{\text{conn}}(X, \mathbb{R})$ is exactly analogous to the term ${}^dE_2^{1,0}(X)$ in (2.71).

Now the degree k PIZ exact sequence (2.80) seems to be missing two terms compared to (2.71). These two terms reappear when $k = 1$, as we shall now prove.

Theorem 2.7.7. Given a diffeological space X, there exists a map $\theta : H^1_{\text{dR}}(X) \to \check{H}^1_\infty(X, \mathbb{R}^\delta)$ such that the sequence of vector spaces

$$0 \to H^1_{\text{dR}}(X) \xrightarrow{\theta} \check{H}^1_\infty(X, \mathbb{R}^\delta) \xrightarrow{\alpha} \check{H}^1_{\text{conn}}(X, \mathbb{R}) \xrightarrow{\beta} H^2_{\text{dR}}(X) \xrightarrow{\gamma} \check{H}^2_\infty(X, \mathbb{R}^\delta) \tag{2.81}$$

is exact.

Proof. The sequence is exact everywhere except for $H^1_{\text{dR}}(X)$ and $\check{H}^1_\infty(X, \mathbb{R}^\delta)$ by Theorem 2.7.5. Now recall the isomorphism $\varphi : \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0]) \to \check{H}^0_\infty(X, [\mathbb{R} \to \Omega^1_{\text{cl}}])$ induced by the map of presheaves of chain complexes described in Example 2.6.8 for $k = 1$. The map φ takes an isomorphism class of a diffeological principal $\mathbb{R}^\delta$-bundle cocycle $[g]$ and gives the isomorphism class of the $\mathbb{R}$-bundle 0-gerbe with connection $[0, g]$. Let $\theta : H^1_{\text{dR}}(X) \to \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0])$ denote the map defined as follows. Let $[A] \in H^1_{\text{dR}}(X)$ denote a cohomology class, and suppose that A is a global closed 1-form representing this class. Since it is closed, there exists an $a \in \mathbb{R}(QX_0)$ such that $da = A$. Then δa defines an $\mathbb{R}^\delta$ cocycle, as $d\delta a = \delta A = 0$. Let $\theta([A]) = [\delta a]$. This map is well defined, as suppose that $a, a' \in \mathbb{R}(QX_0)$ such that $da = da' = A$. Then $a - a'$ is a $\mathbb{R}^\delta$-coboundary between δa and $\delta a'$ as $d(a - a') = A - A = 0$ and $\delta a - \delta a' = \delta(a - a')$, so $[\delta a] = [\delta a']$.

We have a commutative diagram

$$\begin{array}{ccccc} H^1_{\mathrm{dR}}(X) & \xrightarrow{\theta} & \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0]) & \xrightarrow{\alpha} & \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \\ & \searrow^{\theta'} & \downarrow{\scriptstyle\varphi} & & \\ & & \check{H}^0_\infty(X, [\mathbb{R} \to \Omega^1_{\mathrm{cl}}]) & & \end{array} \tag{2.82}$$

where α takes a $\mathbb{R}^\delta$-cocycle and considers it as a $\mathbb{R}$-cocycle, and $\theta'([A]) = [-A, 0]$. Now $\varphi\theta = \theta'$ because $\varphi\theta([A]) = [0, \delta a]$, $\theta'([A]) = [-A, 0]$, and $(-A, 0) - (0, \delta a) = (-A, -\delta a) = (-da, -\delta a) = D(-a)$.

Clearly $\alpha\theta = 0$. Let us show that $\operatorname{im}\theta = \ker\alpha$. Suppose that $[g]$ is the isomorphism class of a diffeological principal $\mathbb{R}^\delta$-bundle such that it is trivial as a diffeological principal $\mathbb{R}$-bundle. Then there exists a $\lambda \in \mathbb{R}(QX_0)$ such that $g = \delta\lambda$. Then $\theta([d\lambda]) = [g]$. Now let us show that θ is injective. It is enough to show that θ' is injective, as φ is an isomorphism. Suppose that $[A]$ and $[B]$ are cohomology classes such that $\theta'([A]) = [-A, 0] = [-B, 0] = \theta'([B])$. Then there exists a $\tau \in \mathbb{R}(QX_0)$ such that $(-A-(-B), 0) = (B-A, 0) = D\tau$, which implies that $\delta\tau = 0$, so that τ is a global 0-form and $d\tau = B - A$. Thus $[A] = [B]$. Thus we have proven that θ is injective. Now abuse notation and let $\check{H}^1_\infty(X, \mathbb{R}^\delta) = \check{H}^0_\infty(X, [\mathbb{R}^\delta \to 0])$. This proves that the above sequence is exact everywhere. □

Considering again the case where $X = T_\alpha$ is the irrational torus, from (2.74), (2.75) and (2.81), we obtain that

$$\check{H}^1_{\mathrm{conn}}(T_\alpha, \mathbb{R}) \cong \mathbb{R}, \tag{2.83}$$

which agrees with [Igl23]. From (2.79) and Theorem 2.7.3 we then obtain an isomorphism

$$\check{H}^1_{\infty,\mathrm{triv}}(T_\alpha, \mathbb{R}^\delta) \cong \mathbb{R}. \tag{2.84}$$

Similarly, from (2.80) we obtain an exact sequence

$$\mathbb{R} \to \check{H}^2_{\mathrm{conn}}(T_\alpha, \mathbb{R}) \to 0$$

so that $\check{H}^2_{\mathrm{conn}}(T_\alpha, \mathbb{R})$ is either 0 or $\mathbb{R}$.

2.8 Diffeological Principal Bundles with Connection

In this section we show that the notion of diffeological principal G-bundle with connection introduced in Example 2.6.4 is equivalent to Waldorf's, given in [Wal12, Definition 3.2.1].

Given a diffeological space X, and a Lie group G, recall the definition of the ∞-stack $\Omega^1(-, \mathfrak{g})//G$ from Example 2.6.4. The data of a map $QX \to \Omega^1(-, \mathfrak{g})//G$ is equivalent to a G-cocycle g and a collection $A = \{A_{p_0}\}$ of 1-forms $A_{p_0} \in \Omega^1(U_{p_0}, \mathfrak{g})$ satisfying

$$A_{p_1} = \mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + g_{f_0}^* \mathrm{mc}(G).$$

for every plot $p_0 : U_{p_0} \to X$. We refer to such a map $QX \to \Omega^1(-, \mathfrak{g})//G$ as a **G-cocycle with connection**.

Definition 2.8.1. Let $\mathsf{Coc}_\nabla(X, G)$ denote the category whose objects are G-cocycles with connection on X, and whose morphisms $h : (A, g) \to (A', g')$ are collections $h = \{h_{p_0}\}$ of maps $h_{p_0} : U_{p_0} \to G$ such that h is a morphism of G-cocycles in the sense of Definition 2.2.6 and $A_{p_0} = \mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + h^*_{p_0}\mathrm{mc}(G)$ for every plot p_0 of X. It is easy to see that this category is a groupoid.

Definition 2.8.2. Let $\pi : P \to X$ be a diffeological principal G-bundle where G is a Lie group. A **Waldorf connection** on P is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ such that

$$\rho^*\omega = \mathrm{Ad}^{-1}_g(\mathrm{pr}^*\omega) + g^*\mathrm{mc}(G) \tag{2.85}$$

where $\rho : P \times G \to P$ is the action map, and $g : P \times G \to G$ and $\mathrm{pr} : P \times G \to P$ are the corresponding projection maps.

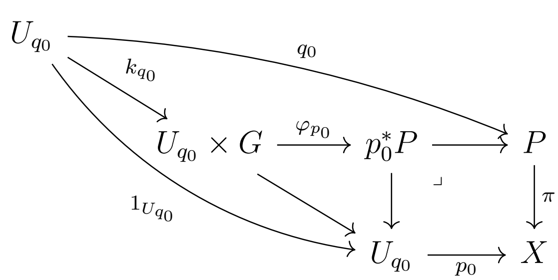
A morphism $f : (\omega, P) \to (\omega', P')$ of diffeological principal G-bundles on X with Waldorf connections is a morphism of diffeological principal G-bundles $f : P \to P'$ such that $f^*\omega' = \omega$.

Such morphisms are isomorphisms. Given a diffeological space X, let $\mathsf{Wal}_G(X)$ denote the groupoid of diffeological principal G-bundles on X equipped with a Waldorf connection.

In Section 1.3, we showed there is a functor $\mathsf{Cons} : \mathsf{Coc}(X, G) \to \mathsf{DiffPrin}_G(X)$ that takes a G-cocycle g and constructs a diffeological principal G-bundle $\mathsf{Cons}(g) = \pi : P \to X$ on X. Furthermore, by Theorem 1.3.15, this functor is an equivalence. Thus we need only understand how to construct a Waldorf connection from the collection $A = \{A_{p_0}\}$ of 1-forms and vice versa.

So let $g = \{g_{f_0}\}$ be a fixed G-cocycle representing a diffeological principal G-bundle $\mathsf{Cons}(g) = \pi : P \to X$. We wish to construct a 1-form ω on P from a G-cocycle with connection A on X. The diffeological principal G-bundle $\mathsf{Cons}(g)$ has a canonical plotwise trivialization $\varphi_{p_0} : U_{p_0} \times G \to p_0^* P$ such that if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, then the induced map $\widetilde{f}_0 : U_{p_1} \times G \to U_{p_0} \times G$ is given by $\widetilde{f}_0(x_{p_1}, h) = (f_0(x_{p_1}), g_{f_0}(x_{p_1}) \cdot h)$, where $g_{f_0} : U_{p_1} \to G$ is the component of the G-cocycle on f_0.

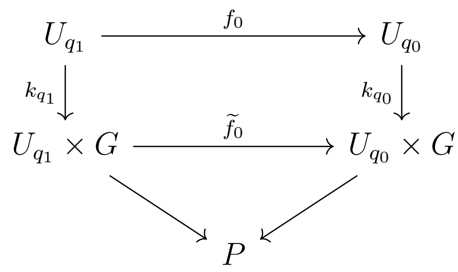
Now let $q_0 : U_{q_0} \to P$ be a plot. We obtain a commutative diagram



(2.86)

where $p_0 = \pi q_0$ and $k_{q_0} : U_{q_0} \to U_{q_0} \times G$ is the unique map given by the universal property of the pullback $U_{q_0} \times G \cong p_0^* P$. Since this map is over U_{q_0}, we have $k_{q_0}(x_{q_0}) = (x_{q_0}, g_{q_0}(x_{q_0}))$ for a unique map $g_{q_0} : U_{q_0} \to G$.

Now if $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots, we obtain a commutative diagram



(2.87)

which implies that if $x_{q_1} \in U_{q_1}$, then

$$\widetilde{f}_0 k_{q_1}(x_{q_1}) = \widetilde{f}_0(x_{q_1}, g_{q_1}(x_{q_1})) = (f_0(x_{q_1}), g_{f_0}(x_{q_1}) \cdot g_{q_1}(x_{q_1})) = (f_0(x_{q_1}), g_{q_0}(f_0(x_{q_1})) = k_{q_0} f_0(x_{q_1}).$$

From this we obtain the equation

$$g_{f_0} \cdot g_{q_1} = (g_{q_0} \circ f_0). \tag{2.88}$$

Now suppose that $A = \{A_{p_0}\}$ is a G-cocycle with connection for the fixed cocycle g. We wish to obtain a 1-form on P. Since P is a diffeological space, we can define it plotwise. Given a plot $q_0 : U_{q_0} \to P$, we obtain a plot $p_0 : U_{q_0} \to X$ of the base X by setting $p_0 = \pi q_0$. Thus there is a 1-form $A_{p_0} \in \Omega^1(U_{q_0}, \mathfrak{g})$ from the G-cocycle with connection. Now let

$$B_{q_0} = \mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + g^*_{q_0}\mathrm{mc}(G). \tag{2.89}$$

Thus $B_{q_0} \in \Omega^1(U_{q_0}, \mathfrak{g})$. We wish to show that this defines a 1-form on P, namely we need to check that if $f_0 : U_{q_1} \to U_{q_0}$ is a map of plots of P, then

$$f_0^* B_{q_0} = B_{q_1}. \tag{2.90}$$

So let $f_0 : U_{q_1} \to U_{q_0}$ be such a map of plots. Then we have

$$\begin{aligned} f_0^* B_{q_0} &= f_0^* \left(\mathrm{Ad}_{g_{q_0}}^{-1}(A_{p_0}) + g_{q_0}^* \mathrm{mc}(G) \right) \\ &= \mathrm{Ad}_{(g_{q_0} \circ f_0)}^{-1}(f_0^* A_{p_0}) + (g_{q_0} \circ f_0)^* \mathrm{mc}(G) \\ &= \mathrm{Ad}_{g_{q_1}}^{-1} \mathrm{Ad}_{g_{f_0}}^{-1}(f_0^* A_{p_0}) + (g_{f_0} \cdot g_{q_1})^* \mathrm{mc}(G) \\ &= \mathrm{Ad}_{g_{q_1}}^{-1} \mathrm{Ad}_{g_{f_0}}^{-1}(f_0^* A_{p_0}) + \mathrm{Ad}_{g_{q_1}}^{-1}(g_{f_0}^* \mathrm{mc}(G)) + g_{q_1}^* \mathrm{mc}(G) \\ &= \mathrm{Ad}_{g_{q_1}}^{-1} \left(\mathrm{Ad}_{g_{f_0}}^{-1}(f_0^* A_{p_0}) + g_{f_0}^* \mathrm{mc}(G) \right) + g_{q_1}^* \mathrm{mc}(G) \\ &= \mathrm{Ad}_{g_{q_1}}^{-1}(A_{p_1}) + g_{q_1}^* \mathrm{mc}(G) \\ &= B_{q_1}. \end{aligned} \tag{2.91}$$

We have used the product rule for the Maurer-Cartan form

$$(g \cdot h)^* \mathrm{mc}(G) = \mathrm{Ad}_h^{-1}(g^* \mathrm{mc}(G)) + h^* \mathrm{mc}(G), \tag{2.92}$$

on the fourth line above, which can easily be verified using the description of $\mathrm{mc}(G)$ as $g^{-1}dg$.

Thus the collection $\{B_{q_0}\}$ defines a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with $\omega_{q_0} = B_{q_0}$. We must still show that ω is a Waldorf connection.

We will check the equation (2.85) plotwise on $P \times G$. A plot of $P \times G$ is a pair of plots $q_0 : U_{q_0} \to P$ and $h_0 : U_{q_0} \to G$, which we shall pair to form the plot $\langle q_0, h_0 \rangle : U_{q_0} \to P \times G$. Let us examine $(\rho^* \omega)_{\langle q_0, h_0 \rangle}$. This is the 1-form $\omega_{\rho \langle q_0, h_0 \rangle}$, where $\rho : P \times G \to P$ is the action map. We can thus write $\rho \langle q_0, h_0 \rangle = q_0 \cdot h_0$, where $\cdot$ is the action of G on P. Thus we wish to compute $\omega_{q_0 \cdot h_0}$. Looking plotwise, it is easy to see that

$$g_{q_0 \cdot h_0} = g_{q_0} \cdot h_0. \tag{2.93}$$

Thus we have

$$\begin{aligned}
\omega_{q_0 \cdot h_0} &= B_{q_0 \cdot h_0} \\
&= \mathrm{Ad}^{-1}_{g_{q_0 \cdot h_0}}(A_{p_0}) + g^*_{q_0 \cdot h_0}\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}\mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + (g_{q_0} \cdot h_0)^*\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}\mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + \mathrm{Ad}^{-1}_{h_0} g^*_{q_0}\mathrm{mc}(G) + h_0^*\mathrm{mc}(G) \\
&= \mathrm{Ad}^{-1}_{h_0}(B_{q_0}) + h_0^*\mathrm{mc}(G).
\end{aligned} \tag{2.94}$$

Pulling back to $P \times G$ gives precisely the equation (2.85). So given a G-cocycle with connection (A, g), let $\mathsf{Cons}_\nabla(A, g) = (\omega, P)$ denote the diffeological principal G-bundle $P = \mathsf{Cons}(g)$ equipped with Waldorf connection ω.

Now suppose that $h : (A, g) \to (A', g')$ is a morphism of G-cocycles with connection on X. We wish to obtain a morphism of diffeological principal G-bundles that preserve the Waldorf connection. By Section 1.3 we know that $\mathsf{Cons}(h) : \mathsf{Cons}(g) \to \mathsf{Cons}(g')$ is a map of the respective diffeological principal G-bundles. We need only show that $\mathsf{Cons}(h)$ preserves the Waldorf connection. Let $(\omega, P) = \mathsf{Cons}_\nabla(A, g)$ and $(\omega', P') = \mathsf{Cons}_\nabla(A', g')$, and let $\widetilde{h} = \mathsf{Cons}(h)$ denote the corresponding morphism given by the morphism h of G-cocycles. For every plot $q_0 : U_{q_0} \to P$ we obtain the following commutative diagram

$$\begin{array}{ccccc}
 & & U_{q_0} & & \\
 & \swarrow^{k_{q_0}} & & \searrow^{k'_{\widetilde{h}q_0}} & \\
U_{q_0} \times G & & \xrightarrow{\widetilde{h_{q_0}}} & & U_{q_0} \times G \\
\downarrow{\scriptstyle l_{q_0}} & & & & \downarrow{\scriptstyle l'_{\widetilde{h}q_0}} \\
P & & \xrightarrow[\widetilde{h}]{} & & P'
\end{array} \tag{2.95}$$

where $\widetilde{h_{q_0}}(x_{q_0}, g) = (x_{q_0}, h_{q_0}(x_{q_0}) \cdot g)$ for $h_{q_0} : U_{q_0} \to G$ the component of the morphism h of cocycles. The above diagram also implies that

$$k'_{\widetilde{h}\,q_0}(x_{q_0}) = (x_{q_0}, g'_{\widetilde{h}q_0}(x_{q_0})) = (x_{q_0}, h_{p_0}(x_{q_0}) \cdot g_{q_0}(x_{q_0})) = \widetilde{h_{q_0}} k_{q_0}(x_{q_0}), \tag{2.96}$$

and thus we have

$$g'_{\widetilde{h}\,q_0} = h_{q_0} \cdot g_{q_0}. \tag{2.97}$$

We wish to show that $\widetilde{h}^*\omega' = \omega$. It is therefore equivalent to show that

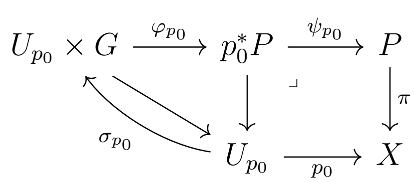
$$(\widetilde{h}^*\omega')_{q_0} = \omega'_{\widetilde{h}q_0} = B'_{\widetilde{h}q_0} = B_{q_0} = \omega_{q_0}. \tag{2.98}$$

Now we have

$$\begin{aligned} B'_{\widetilde{h}q_0} &= \mathrm{Ad}^{-1}_{g_{\widetilde{h}q_0}}(A'_{p_0}) + g'^*_{\widetilde{h}q_0}\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_0}}\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + (h_{p_0} \cdot g_{q_0})^*\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_0}}\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + \mathrm{Ad}^{-1}_{g_{q_0}}(h^*_{p_0}\mathrm{mc}(G)) + g^*_{q_0}\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_0}}(\mathrm{Ad}^{-1}_{h_{p_0}}(A'_{p_0}) + h^*_{p_0}\mathrm{mc}(G)) + g^*_{q_0}\mathrm{mc}(G) \\ &= \mathrm{Ad}^{-1}_{g_{q_0}}(A_{p_0}) + g^*_{q_0}\mathrm{mc}(G) \\ &= B_{q_0}. \end{aligned} \tag{2.99}$$

Thus $\widetilde{h} : P \to P'$ preserves the Waldorf connections. In summary, we have constructed a functor $\mathsf{Cons}_\nabla : \mathsf{Coc}(X, G) \to \mathsf{Wal}_G(X)$. Now we wish to show that this functor is an equivalence of groupoids.

Now let us show that if we have a Waldorf connection ω on P, we can obtain an G-cocycle with connection. Suppose that $\pi : P \to X$ is a diffeological principal G-bundle, and choose a fixed plotwise trivialization φ. From this we obtain a G-cocycle g. Suppose that $\omega \in \Omega^1(P, \mathfrak{g})$ is a Waldorf connection, and let $p_0 : U_{p_0} \to X$ be a plot. We obtain the commutative diagram



where φ_{p_0} is the fixed trivialization, which is a G-equivariant diffeomorphism over U_{p_0}, and $\sigma_{p_0} : U_{p_0} \to U_{p_0} \times G$ is the canonical section $\sigma_{p_0}(x_{p_0}) = (x_{p_0}, e_G)$. Let $q_0 : U_{p_0} \to P$ be given by $q_0 = \psi_{p_0}\varphi_{p_0}\sigma_{p_0}$. Suppose that $f_0 : U_{p_1} \to U_{p_0}$ is a map of plots of X. Then we obtain a diagram

$$\begin{array}{ccccc}
U_{p_1} & & \xrightarrow{\quad f_0 \quad} & & U_{p_0} \\
\downarrow{\scriptstyle \sigma_{p_1}} & & & & \downarrow{\scriptstyle \sigma_{p_0}} \\
U_{p_1} \times G & & \xrightarrow{\quad \widetilde{f_0} \quad} & & U_{p_0} \times G \\
\downarrow{\scriptstyle \varphi_{p_1}} & & & & \downarrow{\scriptstyle \varphi_{p_0}} \\
p_1^* P & & \xrightarrow{\quad \widehat{f_0} \quad} & & p_0^* P \\
& \searrow{\scriptstyle \psi_{p_1}} & & \swarrow{\scriptstyle \psi_{p_0}} & \\
& & P & &
\end{array}$$

Notice that the middle square and the bottom triangle commute, but the top square does not commute, as $\widetilde{f_0}\sigma_{p_1}(x_{p_1}) = (f_0(x_{p_1}), g_{f_0}(x_{p_1}))$ while $\sigma_{p_0}(f_0(x_{p_1})) = (f_0(x_{p_1}), e_G)$. Thus we have

$$\begin{aligned}
q_1(x_{p_1}) &= (\psi_{p_1}\varphi_{p_1}\sigma_{p_1})(x_{p_1}) \\
&= \psi_{p_0}\varphi_{p_0}\widetilde{f_0}\sigma_{p_1}(x_{p_1}) \\
&= \psi_{p_0}\varphi_{p_0}(f_0(x_{p_1}), g_{f_0}(x_{p_1})) \\
&= (\psi_{p_0}\varphi_{p_0})\left[(f_0(x_{p_1}), e_G) \cdot g_{f_0}(x_{p_1})\right] \\
&= (\psi_{p_0}\varphi_{p_0}\sigma_{p_0} f_0)(x_{p_1}) \cdot g_{f_0}(x_{p_1}) \\
&= (q_0 \circ f_0)(x_{p_1}) \cdot g_{f_0}(x_{p_1}).
\end{aligned} \tag{2.100}$$

where on the fourth line we used the fact that φ_{p_0} and ψ_{p_0} are G-equivariant.

Thus we have obtained the equation

$$q_1 = (q_0 \circ f_0) \cdot g_{f_0}. \tag{2.101}$$

Now let $A_{p_0} = \omega_{q_0}$. Note that $f_0^* A_{p_0} \neq A_{p_1}$ since f_0 is not a map of plots from q_1 and q_0, i.e. $(q_0 \circ f_0) \neq q_1$. Consider the equation (2.85) at the plot $\langle (q_0 \circ f_0), g_{f_0} \rangle : U_{q_1} \to P \times G$. Note that

$$(\rho^*\omega)_{\langle (q_0 \circ f_0), g_{f_0} \rangle} = \omega_{\rho\langle (q_0 \circ f_0), g_{f_0} \rangle} = \omega_{(q_0 \circ f_0)\cdot g_{f_0}} = \omega_{q_1} = A_{p_1}, \tag{2.102}$$

and

$$\mathrm{Ad}_{g_{f_0}}^{-1}((\mathrm{pr}^*\omega)_{\langle(q_0\circ f_0),g_{f_0}\rangle}) + g_{f_0}^*\mathrm{mc}(G) = \mathrm{Ad}_{g_{f_0}}^{-1}(\omega_{(q_0\circ f_0)}) + g_{f_0}^*\mathrm{mc}(G). \tag{2.103}$$

Now $f_0^*A_{p_0} = f_0^*\omega_{q_0} = \omega_{(q_0\circ f_0)}$ because f_0 is a plot map from q_0 to $(q_0\circ f_0)$ trivially, and ω is a 1-form on P. Thus we obtain equation (2.60), so the collection $A = \{A_{p_0}\}$ defines a G-cocycle with connection on X.

Now if $\widetilde{h} : (\omega, P) \to (\omega', P')$ is a map of diffeological principal G-bundles with Waldorf connection, we want to show that it induces a map of G-cocycle with connection. We know that $\widetilde{h}$ induces a map h of the G-cocycles g and g' representing P and P' respectively, and we wish to show that $A_{p_0} = \mathrm{Ad}_{h_{p_0}}^{-1}(A'_{p_0}) + h_{p_0}^*\mathrm{mc}(G)$ for every plot $p_0 : U_{p_0} \to X$. We know that $\widetilde{h}^*\omega' = \omega$, which is equivalent to asking that $B'_{\widetilde{h}q_0} = B_{q_0}$. So if p_0 is a plot of X, then we obtain a plot $q_0 : U_{p_0} \to P$ in the same way as above. We obtain

$$\begin{aligned}
A_{p_0} &= \mathrm{Ad}_{g_{q_0}}(B_{q_0} - g_{q_0}^*\mathrm{mc}(G)) \\
&= \mathrm{Ad}_{g_{q_0}}(B'_{\widetilde{h}q_0} - g_{q_0}^*\mathrm{mc}(G)) \\
&= \mathrm{Ad}_{g_{q_0}}[\mathrm{Ad}_{g_{\widetilde{h}q_0}}(A'_{p_0}) + g_{\widetilde{h}q_0}^*\mathrm{mc}(G) - g_{q_0}^*\mathrm{mc}(G)] \\
&= \mathrm{Ad}_{g_{q_0}}\mathrm{Ad}_{g_{q_0}}^{-1}\mathrm{Ad}_{h_{p_0}}^{-1}(A'_{p_0}) + \mathrm{Ad}_{g_{q_0}}\mathrm{Ad}_{g_{q_0}}^{-1}(h_{p_0}^*\mathrm{mc}(G)) \\
&+ \mathrm{Ad}_{g_{q_0}}(g_{q_0}^*\mathrm{mc}(G)) - \mathrm{Ad}_{g_{q_0}}(g_{q_0}^*\mathrm{mc}(G)) \\
&= \mathrm{Ad}_{h_{p_0}}^{-1}(A'_{p_0}) + h_{p_0}^*\mathrm{mc}(G),
\end{aligned} \tag{2.104}$$

where we have basically done the computation of (2.99) in reverse. Thus h is a morphism of G-cocycles with connection.

Theorem 2.8.3. Given a diffeological space X and a Lie group G, the functor

$$\mathsf{Cons}_{\triangledown} : \mathsf{Coc}_{\triangledown}(X, G) \to \mathsf{Wal}_G(X), \tag{2.105}$$

is an equivalence of groupoids.

Proof. This follows from combining Theorem 1.3.15 with the above constructions. □

Remark 2.8.4. It should be said that when $G = \mathbb{R}$ or $G = S^1$, one can check that a Waldorf connection reduces to a connection 1-form in the sense of [Igl23, Section 5.3], thus we have an equivalence between all three definitions of diffeological principal G-bundle with connection in these cases.

Remark 2.8.5. There is nothing stopping one from extending the above definition to the case when G is a diffeological group. In this case then $\mathfrak{g}$ should be the internal tangent space [CW15] to the diffeological group G at the identity. Nothing in this Section depended on G being a Lie group, so the whole previous discussion extends to this case. It is an interesting question to see how far one can go with this analogy. For instance, does this extended definition agree with that given in [Igl13, Article 8.32]? We leave this question for future work.

2.9 Totalization

Given a presheaf of chain complexes A and a diffeological space X, we wish to compute the ∞-stack cohomology of X with values in A. This is defined as the abelian group

$$\check{H}^0_\infty(X, A) := \pi_0 \mathbb{R}\check{\mathbb{H}}(X, A). \tag{2.106}$$

We will use the Dold-Kan correspondence to get an amenable model for the homotopy type of $\mathbb{R}\check{\mathbb{H}}(X, A)$.

Let C be a cosimplicial chain complex, whose cosimplicial degree is denoted by the chain complex C^p. The qth degree of the chain complex C^p is denoted $C^{p,q}$, with differential $d : C^{p,q} \to C^{p,q-1}$. From a cosimplicial chain complex we can obtain a (mixed) double complex by applying the dual of the Dold-Kan correspondence to $C^\bullet$ to obtain a $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-graded vector space with two operators $d : C^{p,q} \to C^{p,q-1}$ given by the differential of each C^p and $\delta : C^{p,q} \to C^{p+1,q}$ defined as the alternating sum $\sum_{i=0}^{p}(-1)^i d^i$ of the coface maps of C, with the property that $d\delta = \delta d$. From this we can obtain an unbounded ($\mathbb{Z}$-graded) chain complex $K = \mathsf{tot}^{\mathbb{Z}} C$, with

$$(\mathsf{tot}^{\mathbb{Z}} C)_k = \prod_{q-p=k} C^{p,q} \tag{2.107}$$

and differential $D = (d - (-1)^{q-p}\delta)$. In order to obtain a non-negatively graded chain complex, we apply smart truncation to obtain $\mathsf{tot}\, C = \tau_{\geq 0} \mathsf{tot}^{\mathbb{Z}} C$.

Proposition 2.9.1. Given a cosimplicial chain complex C, we have the following isomorphism of chain complexes,

$$\int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n) \cong \mathsf{tot}\, C, \tag{2.108}$$

where $\underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n)$ is the mapping chain complex defined by (2.55).

Proof. For the rest of this proof only, let $\Delta^n = N\mathbb{R}\Delta^n$. The chain complex $E \coloneqq \int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^n, C^n)$ is isomorphic to the equalizer

$$\mathrm{eq}\left(\prod_{[n]\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^n, C^n) \rightrightarrows \prod_{f:[m]\to[n]} \underline{\mathrm{Map}_{\mathsf{Ch}}}^{\mathbb{Z}}(\Delta^m, C^n) \right). \tag{2.109}$$

It is equipped with the usual differential of mapping chain complexes, namely $d_E : E_k \to E_{k-1}$ is the map

$$d_E(\varphi) = d_{C^\bullet} \circ \varphi - (-1)^k \varphi \circ d_{\Delta^\bullet}. \tag{2.110}$$

Thus for $k \in \mathbb{Z}$, an element $\varphi \in E_k$ consists of a family of degree k maps $\varphi_n : \Delta^n \to C^n$, such that for every map $f : [m] \to [n]$ the following diagram commutes

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\varphi_m} & C^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle C^f} \\ \Delta^n & \xrightarrow[\varphi_n]{} & C^n \end{array} \tag{2.111}$$

and this makes sense, as pre or post-composing a degree k map of chain complexes with a chain map is a degree k map. This is equivalent to having a commutative diagram of the form

$$\begin{array}{ccccccc} \Delta^0 & \rightrightarrows & \Delta^1 & \overset{\longrightarrow}{\rightrightarrows} & \Delta^2 & \overset{\rightrightarrows}{\rightrightarrows} & \cdots \\ {\scriptstyle\varphi_0}\downarrow & & {\scriptstyle\varphi_1}\downarrow & & {\scriptstyle\varphi_2}\downarrow & & \\ C^0 & \rightrightarrows & C^1 & \overset{\longrightarrow}{\rightrightarrows} & C^2 & \overset{\rightrightarrows}{\rightrightarrows} & \cdots \end{array} \tag{2.112}$$

where we have hidden the codegeneracy maps for clarity. For each $n \geq 0$, a degree k map $\varphi_n : \Delta^n \to C^n$ is equivalently the data of an element x_n in degree $k+n$ in C^n, corresponding to the top non-degenerate n-simplex $\iota_n \in (\Delta^n)_n$, along with an element $x_n \circ f$ in degree $k+m$ for every map $f : [m] \to [n]$. However, the diagram commuting implies that $x_n \circ f = C^f x_m$. In other words, the data of the $\{x_n\}_{n\geq 0}$ completely determine the whole diagram. Thus for $k \in \mathbb{Z}$, there is a bijection $E_k \cong \left(\mathsf{tot}^{\mathbb{Z}} C\right)_k \cong \prod_{q-p=k} C^{p,q}$. Furthermore their differentials agree, thus defining an isomorphism $E \cong \mathsf{tot}^{\mathbb{Z}} C$. Since $\int_{n\in\Delta} \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n) = \tau_{\geq 0} E$ and $\mathsf{tot}\, C = \tau_{\geq 0}\mathsf{tot}^{\mathbb{Z}} C$, they are isomorphic. □

Remark 2.9.2. Let d_{Map} and d_v denote the differentials $\prod_{q-p=k} C^{p,q} \to \prod_{q-p=k-1} C^{p,q}$ defined componentwise by

$$d_{\mathrm{Map}} = (d - (-1)^{q-p}\delta), \qquad d_v = (d + (-1)^q \delta).$$

The differential d_v is more commonly seen for total complexes in the literature. There is an isomorphism $(\mathsf{tot}\, C, d_{\mathrm{Map}}) \cong (\mathsf{tot}\, C, d_v)$ given as follows. We wish to find isomorphisms $\psi_k : (\mathsf{tot}\, C)_k \to (\mathsf{tot}\, C)_k$ making the following diagrams commute for all $k \geq 0$

$$\begin{array}{ccc} \prod_{q-p=k} C^{p,q} & \xrightarrow{\psi_k} & \prod_{q-p=k} C^{p,q} \\ {\scriptstyle d_{\mathrm{Map}}}\downarrow & & \downarrow{\scriptstyle d_v} \\ \prod_{q-p=k-1} C^{p,q} & \xrightarrow[\psi_{k-1}]{} & \prod_{q-p=k-1} C^{p,q} \end{array}$$

namely we want an isomorphism of chain complexes. Let us define maps $\sigma_{p,q} : C^{p,q} \to C^{p,q}$

by

$$\sigma_{p,q} = \begin{cases} \mathrm{id} & \text{if } p \equiv 0, 3 \pmod 4 \\ -\mathrm{id} & \text{if } p \equiv 1, 2 \pmod 4). \end{cases}$$

Then set $\psi_k = \prod_{q-p=k} \sigma_{p,q}$. This gives the desired isomorphism[18].

Let us now examine how Proposition 2.9.1 helps us compute ∞-stack cohomology for diffeological spaces. Suppose that A' is a presheaf of chain complexes such that $A = \mathsf{DK}\, A'$ is an ∞-stack, and X is a diffeological space. Then the 0th ∞-stack cohomology of X with values in A is given by $\pi_0 \check{\mathbb{H}}(X, A)$. Let us compute $\check{\mathbb{H}}(X, A)$.

$$\begin{aligned} \check{\mathbb{H}}(X, A) &= \underline{\mathsf{sPre(Cart)}}(QX, \mathsf{DK}\, A') \\ &\cong \underline{\mathsf{sPre(Cart)}}\left(\int^n \coprod_{(f_{n-1},\dots,f_0)} yU_{p_n} \times \Delta^n, \mathsf{DK}\, A' \right) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{sSet}}(\Delta^n, \underline{\mathsf{sPre(Cart)}}(yU_{p_n}, \mathsf{DK}\, A')) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{sSet}}(\Delta^n, [\mathsf{DK}\, A'](U_{p_n})) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \underline{\mathsf{Ch}}(N\mathbb{R}\Delta^n, A'(U_{p_n})) \\ &\cong \int_n \prod_{(f_{n-1},\dots,f_0)} \mathsf{DK}\, \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^n, A'(U_{p_n})) \\ &\cong \mathsf{DK} \int_n \underline{\mathrm{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^n, \prod_{(f_{n-1},\dots,f_0)} A'(U_{p_n})) \\ &\cong \mathsf{DK}\ \mathsf{tot}\, A'(QX), \end{aligned} \tag{2.113}$$

where the last isomorphism follows from Proposition 2.9.1, and the third to last isomorphism follows from Lemma 2.5.7. Thus we have proven the following.

Proposition 2.9.3. Given a presheaf of chain complexes A' such that $A = \mathsf{DK}A'$ is an ∞-stack, and X a diffeological space, the 0th ∞-stack cohomology of X with values in A is

[18] We obtained the maps $\sigma_{p,q}$ by carefully following the procedure outlined in [Ric].

given by

$$\check{H}^0_\infty(X, A) \cong H_0(\mathsf{tot}\, A'(QX)).$$

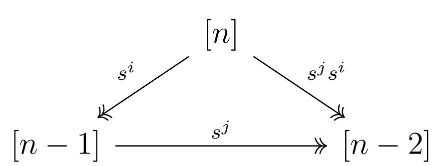
Propositon 2.9.3 allows us to get a component level description of ∞-stack cohomology of diffeological spaces with values in the ∞-stacks of interest, see Section 2.6. Let us now use Proposition 2.9.1 to prove the following well-known folklore result.

Proposition 2.9.4. Let C be a cosimplicial chain complex, then

$$\mathrm{holim}_{n \in \Delta} C^n \simeq \mathsf{tot}\, C, \tag{2.114}$$

where we are computing the homotopy limit in the category of chain complexes equipped with the projective model structure.

Proof. First let us show that every cosimplicial chain complex C is Reedy fibrant. For more information about Reedy categories, see [Rie14, Section 14]. We wish to show that the matching map $C^n \to M^n C$ is a projective fibration. To do so, it will be sufficient to show that if A is a cosimplicial vector space, then the canonical map $s : A^n \to M^n A$, defined by the same limit above, is surjective. This is sufficient because limits of chain complexes are computed degreewise, and a map is a projective fibration if and only if it is surjective in all positive degrees. We follow the proof[19] given in [Jar, Lemma 21.1]. Let D_n denote the category whose objects are surjective maps $[n] \xrightarrow{\sigma} [k]$ where $k = n-1$ or $k = n-2$, and whose morphisms are either identities or coface maps $s^j : [n-1] \to [n-2]$



[19]Note that the proof given in that note has several typographical errors, which is why we chose to reproduce a full proof here.

Then by [Hir09, Proposition 15.2.6],

$$M^n A \cong \lim_{\sigma:[n]\twoheadrightarrow[k]} A^k. \tag{2.115}$$

Now let us label every object of D_n by either $s^i : [n] \to [n-1]$ or $\sigma : [n] \to [n-2]$, and every non-identity morphism by a pair (s^i, s^j). Since $s^j s^i = s^i s^{j+1}$ for every $i \leq j$, the objects $s^j s^i : [n] \to [n-2]$ and $s^i s^{j+1} : [n] \to [n-2]$ are the same, but the morphisms (s^i, s^j) and (s^{j+1}, s^i) are not. We can write the above limit as the equalizer (where we are not denoting the identity maps)

$$M^n A \cong \mathrm{eq}\left(\prod_{s^i} A^{n-1} \times \prod_{\sigma} A^{n-2} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(s^i,s^j)} A^{n-2} \right), \tag{2.116}$$

where α is defined in component (s^i, s^j) by $\alpha(a, a') = s^j a_i$, and β is defined in component (s^i, s^j) by $\beta(a, a') = a'_{s^j s^i}$. Since β in component (s^i, s^j) and in component (s^{j+1}, s^i) are equal $a'_{s^j s^i} = a'_{s^i s^{j+1}}$ for $i \leq j$, this equalizer will be isomorphic to the subspace of $(A^{n-1})^n$ of those tuples $a = (a_0, \ldots, a_{n-1})$ where $s^j a_i = s^i a_{j+1}$ for $i \geq j$. The matching map $s : A^n \to M^n A$ is then given by $s(a_0, \ldots, a_{n-1}) = (s^0 a_0, \ldots, s^{n-1} a_{n-1})$.

Now let us prove by induction that s is surjective. In the base case, note that $(0, \ldots, 0) = s(0, \ldots, 0)$. Now suppose that every element $b \in M^n A$ of the form $b = (b_0, \ldots, b_{j-1}, 0, \ldots, 0)$ is in the image of s. We wish to show that every element of the form $a = (a_0, \ldots, a_{j-1}, a_j, 0, \ldots, 0)$ is in the image of s.

First note that for $i \leq j$, we have $s^j a_i = s^i a_{j+1} = s^i 0 = 0$. Thus $s^j d^i a_i = d^i s^{j-1} a_i = 0$ for $i < j$ and $s^j d^j a_j = a_j$. Thus

$$a - s(d^j a_j) = (a_0 - s^0 d^j a_j, \ldots, a_{j-1} - s^{j-1} d^j a_j, 0, \ldots, 0).$$

By the induction hypothesis, there exists a $c \in A^n$ such that $s(c) = a - s(d^j a_j)$. Therefore $a = s(c + d^j a_j)$.

So we have shown that $s : A^n \to M^n A$ is surjective. This implies that $s : C^n \to M^n C$ is a projective fibration for all cosimplicial chain complexes C. This implies that all cosimplicial chain complexes are Reedy fibrant.

Since C is Reedy fibrant, [Hir09, Theorem 19.8.7] implies that the totalization of C computes the homotopy limit, i.e. $\operatorname{holim}_{\Delta} C \simeq \int_{n \in \Delta} \underline{\operatorname{Map}_{\mathsf{Ch}}}(N\mathbb{R}[\Delta^n], C^n)$. Thus Proposition 2.9.1 proves that $\operatorname{holim}_{\Delta} C \simeq \mathsf{tot}\, C$. □

Remark 2.9.5. During the writing of this paper, the preprint [Ara23] came out, which also proves Proposition 2.9.4 in greater generality. However since the scope of our argument is much smaller, we believe our proof of Proposition 2.9.4 is simpler and more direct.

2.10 Proof of Theorem 2.7.1

In this section we prove Theorem 2.7.1. We will need several technical preliminary results.

Given a chain complex C, consider the chain complex $C^{\Delta^1} \cong \underline{\operatorname{Map}_{\mathsf{Ch}}}(N\mathbb{R}\Delta^1, C)$. This is the chain complex with $C_n^{\Delta^1} = C_n \oplus C_n \oplus C_{n+1}$ for $n > 0$, and with differential

$$d_n : C_n \oplus C_n \oplus C_{n+1} \to C_{n-1} \oplus C_{n-1} \oplus C_n$$

given by $d_n(x, y, z) = (dx, dy, dz - (-1)^n[-x + y])$.

This means that for $k = 0$, we have

$$C_0^{\Delta^1} \cong \ker\left(C_0 \oplus C_0 \oplus C_1 \xrightarrow{d_0} 0 \oplus 0 \oplus C_0\right)$$

where $d_0(x, y, z) = (0, 0, dz + x - y)$. There is an isomorphism $C_0^{\Delta^1} \cong C_0 \oplus C_1$ given by the maps

$$\sigma : C_0^{\Delta^1} \to C_0 \oplus C_1, \qquad \sigma(x, y, z) = (x, z)$$

$$\tau : C_0 \oplus C_1 \to C_0^{\Delta^1}, \qquad \tau(x, z) = (x, x + dz, z).$$

Thus the differential $d : C_1^{\Delta^1} \to C_0^{\Delta^1}$ is isomorphic to the map $\alpha = \sigma \circ d_1$,

$$\alpha : C_1 \oplus C_1 \oplus C_2 \to C_0 \oplus C_1, \qquad \alpha(x, y, z) = \sigma d_1(x, y, z) = \sigma(dx, dy, dz - x + y) = (dx, dz - x + y).$$

The map $\pi : C^{\Delta^1} \to C \oplus C$ is given in degree $k > 0$ by

$$\pi_k : C_k \oplus C_k \oplus C_{k+1} \to C_k \oplus C_k, \qquad \pi_k(x, y, z) = (x, y). \tag{2.117}$$

It is given in degree $k = 0$ by

$$\pi_0 : C_0 \oplus C_1 \to C_0 \oplus C_0, \qquad \pi_0(x, z) = (x, x + dz). \tag{2.118}$$

Let us now state a few model categorical results that we will need for the proof of Theorem 2.7.1.

Lemma 2.10.1 ([Hir09, Corollary 13.3.8])**.** Let C be a right proper model category and let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

be a pullback square in C such that at least one of maps f or g is a fibration. Then the above square is a homotopy pullback square.

Lemma 2.10.2 ([Hir09, Proposition 13.3.15])**.** Let C be a right proper model category, and suppose we have a commutative diagram of the form

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array} \tag{2.119}$$

and suppose that the right hand square is a homotopy pullback square. Then the left hand square is a homotopy pullback square if and only if the outer rectangle is a homotopy pullback square.

Lemma 2.10.3 ([Sch13, Corollary 2.3.10])**.** Let C be a model category, and suppose X, Y, Z are fibrant objects in C and $f : X \to Z$ and $g : Y \to Z$ are maps between them. Then the homotopy pullback of f and g is naturally weak equivalent to the actual pullback

$$\begin{array}{ccc} X \times^h_Z Y & \longrightarrow & Z^I \\ \downarrow & \lrcorner & \downarrow \\ X \times Y & \xrightarrow[f \times g]{} & Z \times Z \end{array} \tag{2.120}$$

where $Z^I \to Z \times Z$ is a path object for Z.

Lemma 2.10.4 ([Hir09, Proposition 3.3.16])**.** Suppose $f : X \to Y$ is a map of ∞-stacks on Cart that is an projective fibration. Then it is a fibration in the Čech model structure.

Definition 2.10.5. If X, Y, Z are ∞-stacks on Cart, and the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ * & \xrightarrow[z]{} & Z \end{array}$$

is a homotopy pullback square, where $* := \Delta^0$, then we say that the sequence of maps

$$X \to Y \to Z$$

is a **homotopy fiber sequence**, and we call X the **homotopy fiber** of f at z, which we sometimes denote by $\text{hofib}(f)$.

Lemma 2.10.6. Let $X \to Y \to Z$ be a homotopy fiber sequence of pointed ∞-stacks on Cart, and where the morphisms preserve the points. Then the resulting sequence

$$\check{H}^0_\infty(W, X) \xrightarrow{f} \check{H}^0_\infty(W, Y) \xrightarrow{g} \check{H}^0_\infty(W, Z),$$

is exact.[20]

Proof. This follows from the fact that $\mathbb{R}\check{\mathbb{H}}(-,-)$ preserves homotopy pullbacks in its second factor, so a homotopy fiber sequence of ∞-stacks produces a homotopy fiber sequence of spaces

$$\mathbb{R}\check{\mathbb{H}}(W,X) \to \mathbb{R}\check{\mathbb{H}}(W,Y) \to \mathbb{R}\check{\mathbb{H}}(W,Z)$$

and the long exact sequence of homotopy groups gives exactness for π_0. □

Proposition 2.10.7. Suppose that we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[h]{} & D \end{array} \tag{2.121}$$

of presheaves of chain complexes over Cart such that

$$\begin{array}{ccc} \mathsf{DK}A & \xrightarrow{\mathsf{DK}f} & \mathsf{DK}B \\ {\scriptstyle \mathsf{DK}g}\downarrow & & \downarrow{\scriptstyle \mathsf{DK}h} \\ \mathsf{DK}C & \xrightarrow[\mathsf{DK}h]{} & \mathsf{DK}D \end{array} \tag{2.122}$$

is a commutative diagram of ∞-stacks. If (2.121) is a homotopy pullback square in the projective model structure on $\mathsf{ChPre}(\mathsf{Cart})$, then (2.122) is a homotopy pullback square in $\check{\mathbb{H}}$.

Proof. If (2.121) is a homotopy pullback diagram, then A is weak equivalent to the actual pullback $C \times_D^h B$ of Lemma 2.10.3. Since both of these presheaves of chain complexes are projective fibrant, and DK is right Quillen, then $\mathsf{DK}A$ is weak equivalent to $\mathsf{DK}(C \times_D^h B) \cong \mathsf{DK}C \times^h_{\mathsf{DK}D} \mathsf{DK}B$. Therefore $\mathsf{DK}A$ is a homotopy pullback of (2.122) in $\mathbb{H}$. Then by Proposition 2.3.5, it is a homotopy pullback in $\check{\mathbb{H}}$. □

Now that we have all the technical tools we need, we restate Theorem 2.7.1 for the convenience of the reader.

[20] Exact in the sense that each set $\check{H}^0_\infty(W,A)$ is pointed by the constant map $*$ to the point of A, and the image of f is equal to the set of $x \in \check{H}^0_\infty(W,Y)$ such that $g(x) = *$, which we call the kernel of g.

Theorem 2.7.1. For every $k \geq 1$, there exists a commutative diagram of ∞-stacks of the following form

$$\begin{array}{ccccccc} * & \longrightarrow & \mathbf{B}^k\mathbb{R}^\delta & \longrightarrow & * & \longrightarrow & * \\ \downarrow & 1 & \downarrow & 2 & \downarrow & 3 & \downarrow \\ * & \longrightarrow & \mathbf{B}^k_\nabla\mathbb{R} & \longrightarrow & \Omega^{k+1}_{\mathrm{cl}} & \longrightarrow & \Omega^{k+1} \\ & & \downarrow & 4 & \downarrow & 5 & \downarrow \\ & & \mathbf{B}^k\mathbb{R} & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & \Omega^{1\leq\bullet\leq k+1} \\ & & \downarrow & 6 & \downarrow & 7 & \downarrow \\ & & * & \longrightarrow & \mathbf{B}^{k+1}\mathbb{R}^\delta & \longrightarrow & \mathbf{B}^{k+1}_\nabla\mathbb{R} \end{array} \tag{2.123}$$

furthermore every commutative square in this diagram is a homotopy pullback square in $\check{\mathbb{H}}$.

Lemma 2.10.8. The pasted square [4|5], given as follows

$$\begin{array}{ccc} \mathbf{B}^k_\nabla\mathbb{R} & \longrightarrow & \Omega^{k+1} \\ \downarrow & & \downarrow \\ \mathbf{B}^k\mathbb{R} & \longrightarrow & \Omega^{1\leq\bullet\leq k+1} \end{array} \tag{2.124}$$

is a homotopy pullback square in $\check{\mathbb{H}}$.

Proof. Let us analyze this part of the diagram as presheaves of chain complexes.

$$\begin{array}{ccccc} [\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k] & \longrightarrow & [0 \to 0 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [0 \to 0 \to 0 \to \cdots \to \Omega^{k+1}] \\ \downarrow & & \downarrow & & \downarrow \\ [\mathbb{R} \to 0 \to 0 \to \cdots \to 0] & \longrightarrow & [\Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [\Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}] \end{array} \tag{2.125}$$

where the upper horizontal left hand map is 0 except in degree 0 where it applies the differential d. The lower horizontal left hand map is d in degree $k+1$ and 0 elsewhere. The rest of the maps are either degreewise inclusions or identity maps.

Let us show that the outer rectangle is a homotopy pullback diagram. Note that neither the bottom map nor the right hand map is objectwise surjective in positive degree, namely they are not fibrations in $\mathsf{ChPre}(\mathsf{Cart})$. However we can use Lemma 2.10.3 to compute the homotopy pullback of $\mathbf{B}^k\mathbb{R} \to \Omega^{1\leq\bullet\leq k+1} \leftarrow \Omega^{k+1}$. Namely it is given as the actual (objectwise) pullback of the diagram

$$\begin{array}{ccc} & & (\Omega^{1\le\bullet\le k+1})^{\Delta^1} \\ & & \downarrow \\ \mathbf{B}^k\mathbb{R}\oplus\Omega^{k+1} & \longrightarrow & \Omega^{1\le\bullet\le k+1}\oplus\Omega^{1\le\bullet\le k+1} \end{array} \tag{2.126}$$

Now $(\Omega^{1\le\bullet\le k+1})^{\Delta^1}$ is given by the presheaf of chain complexes

$$[\Omega^1\oplus\Omega^1\to\Omega^2\oplus\Omega^2\oplus\Omega^1\to\cdots\to\Omega^k\oplus\Omega^k\oplus\Omega^{k-1}\to\Omega^{k+1}\oplus\Omega^k]$$

which projects to $\Omega^{1\le\bullet\le k+1}\oplus\Omega^{1\le\bullet\le k+1}$. From this we obtain the following diagram the following diagram of presheaves of chain complexes

$$\begin{array}{ccc} [\mathbb{R}\to\Omega^1\to\Omega^2\to\cdots\to\Omega^k] & \longrightarrow & [\Omega^1\oplus\Omega^1\to\Omega^2\oplus\Omega^2\oplus\Omega^1\to\cdots\to\Omega^k\oplus\Omega^k\oplus\Omega^{k-1}\to\Omega^{k+1}\oplus\Omega^k] \\ \downarrow & & \downarrow\pi \\ [\mathbb{R}\to 0\to\cdots\to 0\to\Omega^{k+1}] & \longrightarrow & [\Omega^1\oplus\Omega^1\to\Omega^2\oplus\Omega^2\to\cdots\to\Omega^k\oplus\Omega^k\to\Omega^{k+1}\oplus\Omega^{k+1}] \end{array} \tag{2.127}$$

and this is an actual pullback square. To see this, note that pullbacks of chain complexes are computed degreewise. For degrees $k>0$, it is clearly a pullback. In degree 0 we are trying to show that Ω^k is isomorphic to the pullback of

$$\begin{array}{ccc} & & \Omega^{k+1}\oplus\Omega^k \\ & & \downarrow\pi_0 \\ 0\oplus\Omega^{k+1} & \xrightarrow[0\oplus 1_{\Omega^{k+1}}]{} & \Omega^{k+1}\oplus\Omega^{k+1} \end{array} \tag{2.128}$$

but from (2.118), we know that $\pi_0(x,z)=(x,x+dz)$. For every cartesian space U, the pullback is the set of triples $(w,x,z)\in\Omega^{k+1}(U)\oplus\Omega^{k+1}(U)\oplus\Omega^k(U)$ such that $x=0$ and $w=dz$. This set is of course in bijection with $\Omega^k(U)$. Thus [4|5] is a homotopy pullback square. □

Lemma 2.10.9. The square [5], given by

$$\begin{array}{ccc} \Omega^{k+1}_{\mathrm{cl}} & \longrightarrow & \Omega^{k+1} \\ \downarrow & & \downarrow \\ \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & \Omega^{1\leq\bullet\leq k+1} \end{array} \tag{2.129}$$

is a homotopy pullback square.

Proof. Consider the commutative diagram of presheaves of chain complexes

$$\begin{array}{ccc} [0 \to 0 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [0 \to 0 \to \cdots \to \Omega^{k+1}] \\ \downarrow & & \downarrow \\ [\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [\Omega^1 \xrightarrow{d} \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}] \end{array} \tag{2.130}$$

Now the above diagram is an actual pullback, and the bottom map is objectwise a surjection in positive degrees, thus it is a fibration of presheaves of chain complexes, and therefore by Lemma 2.10.1 the diagram (2.130) is a homotopy pullback. □

Corollary 2.10.10. The square [4] is a homotopy pullback square.

Proof. By Lemma 2.10.8, [4|5] is a homotopy pullback square. By Lemma 2.10.9, [5] is a homotopy pullback square. Thus by Lemma 2.10.2, [4] is a homotopy pullback square. □

Lemma 2.10.11. The square [6]

$$\begin{array}{ccc} \mathbf{B}^k\mathbb{R} & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}^{k+1}\mathbb{R}^\delta \end{array} \tag{2.131}$$

is a homotopy pullback square.

Proof. This proof is very similar as the proof of Lemma 2.10.8. We take the actual pullback of the diagram

$$\begin{array}{ccc} & & (\mathbf{B}^{k+1}\mathbb{R}^{\delta})^{\Delta^1} \\ & & \downarrow \\ 0 \oplus \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & (\mathbf{B}^{k+1}\mathbb{R}^{\delta} \oplus \mathbf{B}^{k+1}\mathbb{R}^{\delta}) \end{array} \tag{2.132}$$

which by similar reasoning to the paragraph after (2.128) is precisely

$$\begin{array}{ccc} [0 \to \Omega^1 \oplus \mathbb{R} \xrightarrow{d_1} \Omega^2 \oplus \Omega^1 \xrightarrow{d_2} \dots \xrightarrow{d_{k-2}} \Omega^{k-1} \oplus \Omega^{k-2} \xrightarrow{\alpha} \Omega^{k-1}] & \longrightarrow & [\mathbb{R} \oplus \mathbb{R} \to \Omega^1 \oplus \Omega^1 \oplus \mathbb{R} \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^{k-1}] \\ \downarrow & & \downarrow \pi \\ [0 \to \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}}] & \longrightarrow & [\mathbb{R} \oplus \mathbb{R} \to \Omega^1 \oplus \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}} \oplus \Omega^k_{\mathrm{cl}}] \end{array} \tag{2.133}$$

where $d_i(a,b) = (da, db - (-1)^i a)$ for $1 \leq i \leq k-2$, and $\alpha(a,b) = db + a$. Now there is an obvious map

$$[0 \to \mathbb{R} \to \cdots \to 0] \to [0 \to \Omega^1 \oplus \mathbb{R} \xrightarrow{d_1} \Omega^2 \oplus \Omega^1 \xrightarrow{d_2} \dots \xrightarrow{d_{k-2}} \Omega^{k-1} \oplus \Omega^{k-2} \xrightarrow{\alpha} \Omega^{k-1}] \tag{2.134}$$

that is an isomorphism on cohomology. Indeed α is surjective, and the kernel of $d_i : \Omega^i \oplus \Omega^{i-1} \to \Omega^{i+1} \oplus \Omega^i$ is the set of pairs (a,b) where $a = (-1)^i db$, and these are all in the image of d_{i-1}. □

Lemma 2.10.12. The square [7]

$$\begin{array}{ccc} \mathbf{B}^k\Omega^1_{\mathrm{cl}} & \longrightarrow & \Omega^{1\leq \bullet \leq k+1} \\ \downarrow & & \downarrow \\ \mathbf{B}^{k+1}\mathbb{R}^{\delta} & \longrightarrow & \mathbf{B}^{k+1}_{\nabla}\mathbb{R} \end{array} \tag{2.135}$$

is a homotopy pullback square.

Proof. As presheaves of chain complexes we have

$$\begin{array}{ccc} [0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [0 \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}] \\ \downarrow & & \downarrow \\ [\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^{k+1}_{\mathrm{cl}}] & \longrightarrow & [\mathbb{R} \to \Omega^1 \to \Omega^2 \to \cdots \to \Omega^k \to \Omega^{k+1}] \end{array} \tag{2.136}$$

which is an actual pullback, and the bottom horizontal map is a fibration, thus by Lemma 2.10.1, [7] is a homotopy pullback. □

Lemma 2.10.13. The pasted square $[\frac{2}{4}]$,

$$\begin{array}{ccc} \mathbf{B}^k\mathbb{R}^\delta & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}^k\mathbb{R} & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \end{array} \tag{2.137}$$

is a homotopy pullback square.

Proof. We use the same proof technique as in Lemma 2.10.8, namely we will compute the actual pullback of the diagram

$$\begin{array}{ccc} & & (\mathbf{B}^k\Omega^1_{\mathrm{cl}})^{\Delta^1} \\ & & \downarrow \\ \mathbf{B}^k\mathbb{R}\oplus 0 & \longrightarrow & \mathbf{B}^k\Omega^1_{\mathrm{cl}} \oplus \mathbf{B}^k\Omega^1_{\mathrm{cl}} \end{array} \tag{2.138}$$

The actual pullback we obtain is given by

$$\begin{array}{ccc} [\mathbb{R}\to\Omega^1\to\Omega^2\to\cdots\to\Omega^k_{\mathrm{cl}}] & \longrightarrow & [\Omega^1\oplus\Omega^1\to\Omega^2\oplus\Omega^2\oplus\Omega^1\to\cdots\to\Omega^k_{\mathrm{cl}}\oplus\Omega^k_{\mathrm{cl}}\oplus\Omega^{k-1}\to 0\oplus\Omega^k_{\mathrm{cl}}] \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ [\mathbb{R}\to 0\to\cdots\to 0\to 0] & \longrightarrow & [\Omega^1\oplus\Omega^1\to\Omega^2\oplus\Omega^2\to\cdots\to\Omega^k_{\mathrm{cl}}\oplus\Omega^k_{\mathrm{cl}}\to 0] \end{array} \tag{2.139}$$

which is similar to the computation (2.127). Thus $[\frac{2}{4}]$ is a homotopy pullback square. □

Corollary 2.10.14. The square [2] is a homotopy pullback square.

Proof. By Corollary 2.10.10, Lemma 2.10.13 and Lemma 2.10.2. □

Lemma 2.10.15. The square [3] is a homotopy pullback square.

Proof. As a diagram of presheaves of chain complexes

$$\begin{array}{ccc} [0 \to \Omega^1 \to \cdots \to \Omega^k] & \longrightarrow & [\mathbb{R} \to \Omega^1 \to \cdots \to \Omega^k] \\ \downarrow & & \downarrow \\ [0 \to 0 \to \cdots \to 0] & \longrightarrow & [\mathbb{R} \to 0 \to \cdots \to 0] \end{array} \tag{2.140}$$

it is an actual pullback, and the right hand map is a fibration. □

Lemma 2.10.16. The square [1] is a homotopy pullback square.

Proof. As a diagram of presheaves of chain complexes

$$\begin{array}{ccc} [0 \to \cdots \to 0] & \longrightarrow & [\mathbb{R} \to \cdots \to \Omega^k_{\mathrm{cl}}] \\ \downarrow & & \downarrow \\ [0 \to \cdots \to 0] & \longrightarrow & [\mathbb{R} \to \cdots \to \Omega^k] \end{array} \tag{2.141}$$

it is an actual pullback, and the right hand map is a fibration. □

Thus we have proven Theorem 2.7.1.

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