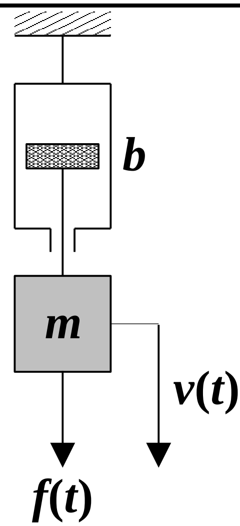
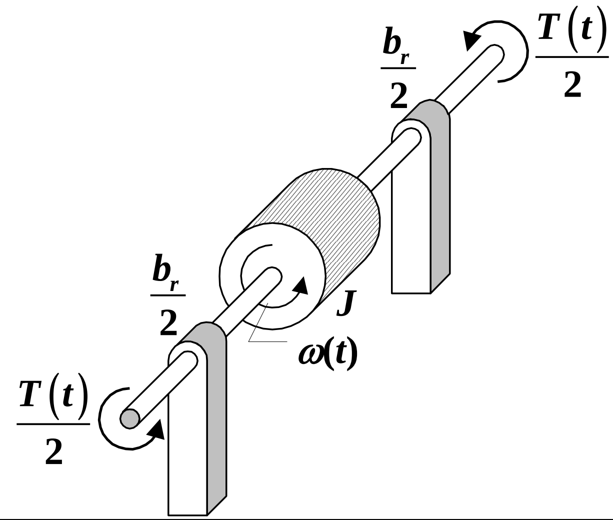
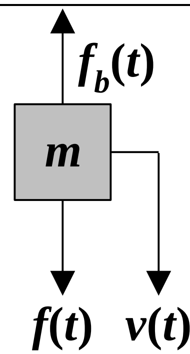
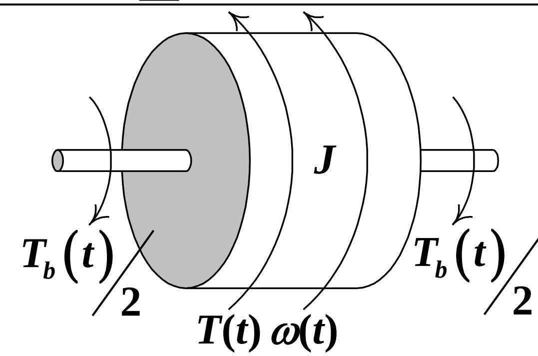
TOPIC 3: MECHANICAL SYSTEMS

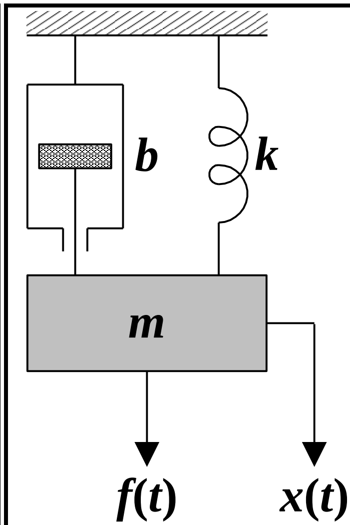
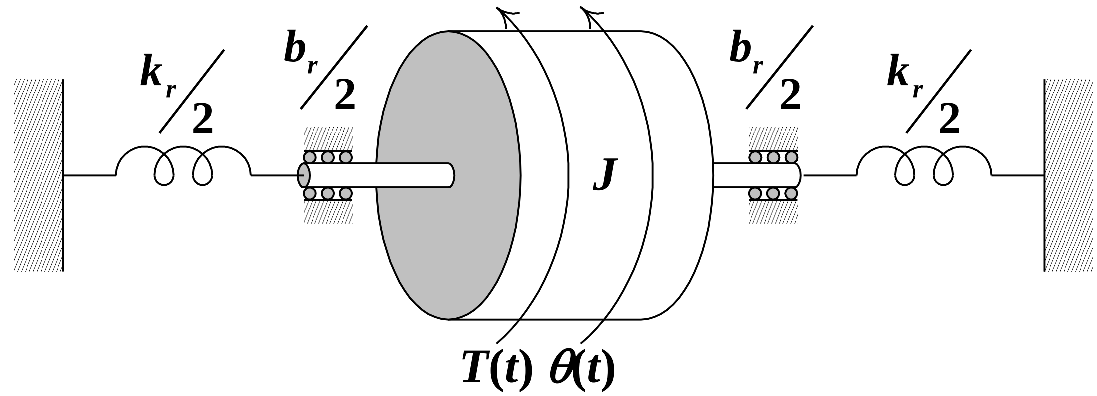
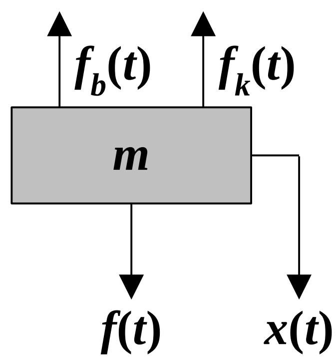
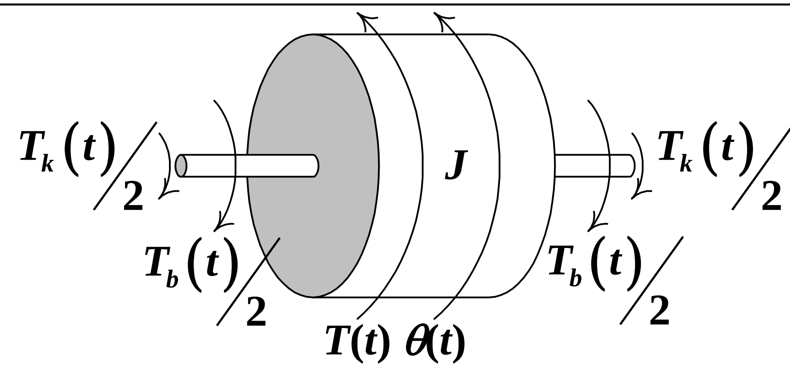
3-1 1-DOF MECHANICAL SYSTEMS (1ST-ORDER, INERTIA-DAMPER)

	Translational System	Rotational System
inertia-damper system		
free-body diagram		
Newton's 2nd law	$ma = \sum f \quad \rightarrow \quad m\dot{v}(t) = f(t) - f_b(t)$	$J\alpha = \sum T \quad \rightarrow \quad J\dot{\omega}(t) = T(t) - T_b(t)$
damping force (*or* **torque**)	$f_b(t) = bv(t)$	$T_b(t) = b_r\omega(t)$
spring force (*or* **torque**)	N/A	N/A
differential equation	$m\dot{v}(t) + bv(t) = f(t)$	$J\dot{\omega}(t) + b_r\omega(t) = T(t)$
initial condition	$v(t)\big\|_{t=0} = v(0) = v_0$	$\omega(t)\big\|_{t=0} = \omega(0) = \omega_0$
time constant	$\tau = m/b$	$\tau = J/b_r$

3-1 1-DOF MECHANICAL SYSTEMS (1ST-ORDER, INERTIA-DAMPER) (*continued*)

	Translational System	Rotational System
Laplace transform	$m\left[sV(s)-v_0\right]+bV(s)=F(s)$ $\rightarrow V(s)=\frac{mv_0+F(s)}{ms+b}=\frac{\tau v_0}{\tau s+1}+\frac{F(s)/b}{\tau s+1}$	$J\left[s\Omega(s)-\omega_0\right]+b_r\Omega(s)=T(s)$ $\rightarrow \Omega(s)=\frac{J\omega_0+T(s)}{Js+b_r}=\frac{\tau\omega_0}{\tau s+1}+\frac{T(s)/b_r}{\tau s+1}$
characteristic equation	$ms+b=\tau s+1=0$	$Js+b_r=\tau s+1=0$
characteristic root	$s=-\frac{b}{m}=-\frac{1}{\tau}$	$s=-\frac{b_r}{J}=-\frac{1}{\tau}$
free (*or* **natural** *or* **initial**) **response**	$v_{free}(t)=v_0e^{-bt/m}=v_0e^{-t/\tau}$	$\omega_{free}(t)=\omega_0e^{-b_rt/J}=\omega_0e^{-t/\tau}$
energy balance (*or* **thermodynamics 1st law**) (*or* **Lagrangian dynamics**)	$\Delta E=W_{\text{input}}+Q_{\text{input}} \rightarrow \Delta(K+U)=\int P_{\text{input}}dt-\int D_{\text{output}}dt \rightarrow \dot{K}+\dot{U}=P_{\text{input}}-D_{\text{output}}$ where $\Delta E=$ change in internal energy $\Delta K=$ change in kinetic energy & $\dot{K}=$ time rate of change in kinetic energy $\Delta U=$ change in potential energy & $\dot{U}=$ time rate of change in potential energy $W_{\text{input}}=$ work input & $P_{\text{input}}=$ power input $Q_{\text{input}}=$ heat input & $D_{\text{output}}=$ dissipated power	
kinetic energy	$K=\frac{1}{2}m\left[v(t)\right]^2 \rightarrow$ $\dot{K}=\frac{dK}{dt}=\frac{dK}{dv}\frac{dv}{dt}=\frac{d\left(\frac{1}{2}mv^2\right)}{dv}\dot{v}(t)=mv(t)\dot{v}(t)$	$K=\frac{1}{2}J\left[\omega(t)\right]^2 \rightarrow$ $\dot{K}=\frac{dK}{dt}=\frac{dK}{d\omega}\frac{d\omega}{dt}=\frac{d\left(\frac{1}{2}J\omega^2\right)}{d\omega}\dot{\omega}(t)=J\omega(t)\dot{\omega}(t)$
potential energy	N/A	N/A
power input	$P_{\text{input}}=\frac{dW_{\text{input}}}{dt}=f(t)v(t)$	$P_{\text{input}}=\frac{dW_{\text{input}}}{dt}=T(t)\omega(t)$
dissipated power	$D_{\text{output}}=f_bv(t)=b\left[v(t)\right]^2$	$D_{\text{output}}=T_b\omega(t)=b_r\left[\omega(t)\right]^2$
governing equation	$mv(t)\dot{v}(t)=f(t)v(t)-b\left[v(t)\right]^2$ $\rightarrow m\dot{v}(t)+bv(t)=f(t)$	$J\omega(t)\dot{\omega}(t)=T(t)\omega(t)-b_r\left[\omega(t)\right]^2$ $\rightarrow J\dot{\omega}(t)+b_r\omega(t)=T(t)$

3-2 1-DOF MECHANICAL SYSTEMS (2ND-ORDER, SPRING-INERTIA-DAMPER)

<table>
<tr><th></th><th>Translational System</th><th>Rotational System</th></tr>
<tr><td>spring-inertia-damper system</td><td>
</td><td>
</td></tr>
<tr><td>free-body diagram</td><td>
</td><td>
</td></tr>
<tr><td>Newton's 2nd law</td><td>$ma = \sum f \quad \rightarrow \quad m\ddot{x}(t) = f(t) - f_b(t) - f_k(t)$</td><td>$J\alpha = \sum T \quad \rightarrow \quad J\ddot{\theta}(t) = T(t) - T_b(t) - T_k(t)$</td></tr>
<tr><td>damping force (or torque)</td><td>$f_b(t) = bv(t) = b\dot{x}(t)$</td><td>$T_b(t) = b_r\omega(t) = b_r\dot{\theta}(t)$</td></tr>
<tr><td>spring force (or torque)</td><td>$f_k(t) = kx(t)$</td><td>$T_k(t) = k_r\theta(t)$</td></tr>
<tr><td>differential equation</td><td>$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$</td><td>$J\ddot{\theta}(t) + b_r\dot{\theta}(t) + k_r\theta(t) = T(t)$</td></tr>
<tr><td>initial conditions</td><td>$x(t)\big|_{t=0} = x(0) = x_0 \quad \& \quad \dot{x}(t)\big|_{t=0} = \dot{x}(0) = v_0$</td><td>$\theta(t)\big|_{t=0} = \theta(0) = \theta_0 \quad \& \quad \dot{\theta}(t)\big|_{t=0} = \dot{\theta}(0) = \omega_0$</td></tr>
</table>

3-2 1-DOF MECHANICAL SYSTEMS (2ND-ORDER, SPRING-INERTIA-DAMPER) (*continued*)

	Translational System	Rotational System
(undamped) natural frequency	$\omega_n = \sqrt{\frac{k}{m}}$	$\omega_n = \sqrt{\frac{k_r}{J}}$
damping ratio	$\varsigma = \frac{b}{2\sqrt{mk}}$	$\varsigma = \frac{b_r}{2\sqrt{Jk_r}}$
damped natural frequency	$\omega_d = \omega_n\sqrt{1-\varsigma^2} = \frac{\sqrt{4mk-b^2}}{2m}$	$\omega_d = \omega_n\sqrt{1-\varsigma^2} = \frac{\sqrt{4Jk_r-b_r^2}}{2J}$
Laplace transform	$m\left[s^2X(s)-sx_0-v_0\right]+b\left[sX(s)-x_0\right]+kX(s)=F(s)$ $\rightarrow\ X(s)=\frac{(ms+b)x_0+mv_0+F(s)}{ms^2+bs+k}$	$J\left[s^2\Theta(s)-s\theta_0-\omega_0\right]+b_r\left[s\Theta(s)-\theta_0\right]+k_r\Theta(s)=T(s)$ $\rightarrow\ \Theta(s)=\frac{(Js+b_r)\theta_0+J\omega_0+T(s)}{Js^2+b_rs+k_r}$
characteristic equation	$ms^2+bs+k=m\left(s^2+2\varsigma\omega_n s+\omega_n^2\right)=0$	$Js^2+b_rs+k_r=J\left(s^2+2\varsigma\omega_n s+\omega_n^2\right)=0$
characteristic roots	$s_{1,2}=\frac{-b\pm\sqrt{b^2-4mk}}{2m}=\omega_n\left(-\varsigma\pm\sqrt{\varsigma^2-1}\right)$	$s_{1,2}=\frac{-b_r\pm\sqrt{b_r^2-4Jk_r}}{2J}=\omega_n\left(-\varsigma\pm\sqrt{\varsigma^2-1}\right)$
overdamped: (real & distinct, $\varsigma>1$)	$b>2\sqrt{mk}$ $s_{1,2}=-\frac{b}{2m}\pm\sqrt{\left(\frac{b}{2m}\right)^2-\frac{k}{m}}=\omega_n\left(-\varsigma\pm\sqrt{\varsigma^2-1}\right)$	$b_r>2\sqrt{Jk_r}$ $s_{1,2}=-\frac{b_r}{2J}\pm\sqrt{\left(\frac{b_r}{2J}\right)^2-\frac{k_r}{J}}=\omega_n\left(-\varsigma\pm\sqrt{\varsigma^2-1}\right)$
critically damped: (repeated real, $\varsigma=1$)	$b=2\sqrt{mk}$ $s_1=s_2=-\sqrt{\frac{k}{m}}=-\omega_n$	$b_r=2\sqrt{Jk_r}$ $s_1=s_2=-\sqrt{\frac{k_r}{J}}=-\omega_n$
underdamped: (complex conjugate, $0\le\varsigma<1$)	$b<2\sqrt{mk}$ $s_{1,2}=-\frac{b}{2m}\pm j\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2}=-\varsigma\omega_n\pm j\omega_d$	$b_r<2\sqrt{Jk_r}$ $s_{1,2}=-\frac{b_r}{2J}\pm j\sqrt{\frac{k_r}{J}-\left(\frac{b_r}{2J}\right)^2}=-\varsigma\omega_n\pm j\omega_d$
undamped: (imaginary conjugate, $\varsigma=0$)	$b=0$ $s_{1,2}=\pm j\sqrt{\frac{k}{m}}=\pm j\omega_n$	$b_r=0$ $s_{1,2}=\pm j\sqrt{\frac{k_r}{J}}=\pm j\omega_n$

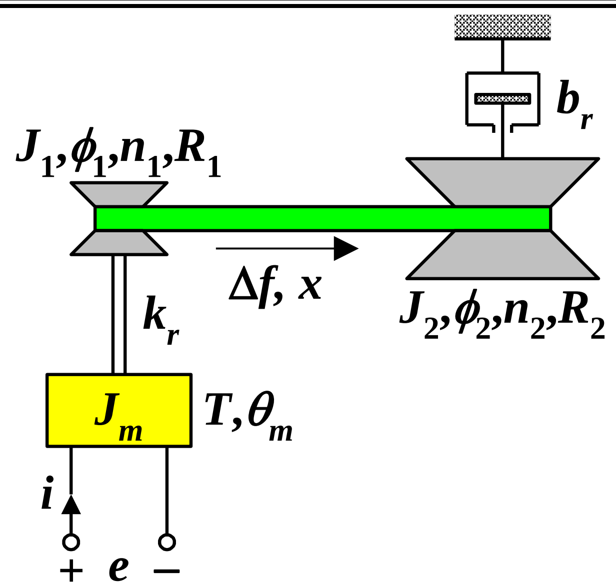
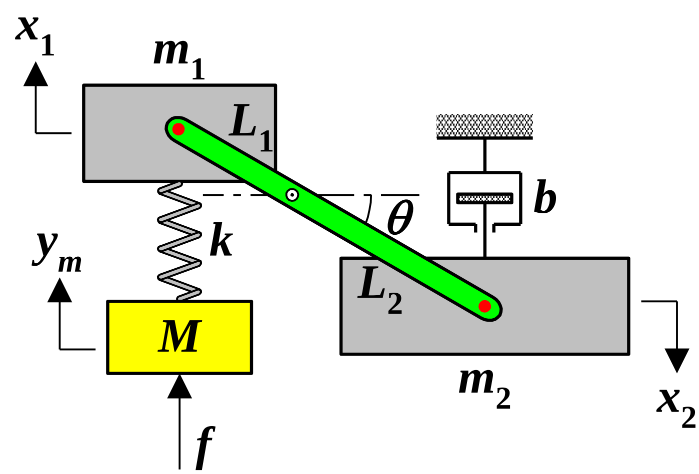
3-2 1-DOF MECHANICAL SYSTEMS (2ND-ORDER, SPRING-INERTIA-DAMPER) (*continued*)

	Translational System	Rotational System
free (*or* **natural** *or* **initial**) **response – overdamped**	$x(t)=\left[\frac{\left(-\zeta+\sqrt{\zeta^2-1}\right)x_0}{2\sqrt{\zeta^2-1}}-\frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}\right]e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}+\left[\frac{\left(\zeta+\sqrt{\zeta^2-1}\right)x_0}{2\sqrt{\zeta^2-1}}+\frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}\right]e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$	$\theta(t)=\left[\frac{\left(-\zeta+\sqrt{\zeta^2-1}\right)\theta_0}{2\sqrt{\zeta^2-1}}-\frac{\omega_0}{2\omega_n\sqrt{\zeta^2-1}}\right]e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}+\left[\frac{\left(\zeta+\sqrt{\zeta^2-1}\right)\theta_0}{2\sqrt{\zeta^2-1}}+\frac{\omega_0}{2\omega_n\sqrt{\zeta^2-1}}\right]e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$
free (*or* **natural** *or* **initial**) **response – critically damped**	$x(t)=\left[x_0+(\omega_n x_0+v_0)t\right]e^{-\omega_n t}$	$\theta(t)=\left[\theta_0+(\omega_n\theta_0+\omega_0)t\right]e^{-\omega_n t}$
free (*or* **natural** *or* **initial**) **response – underdamped**	$x(t)=e^{-\zeta\omega_n t}\left[\left(\frac{\zeta x_0}{\sqrt{1-\zeta^2}}+\frac{v_0}{\omega_d}\right)\sin\omega_d t+x_0\cos\omega_d t\right]$	$\theta(t)=e^{-\zeta\omega_n t}\left[\left(\frac{\zeta\theta_0}{\sqrt{1-\zeta^2}}+\frac{\omega_0}{\omega_d}\right)\sin\omega_d t+\theta_0\cos\omega_d t\right]$
free (*or* **natural** *or* **initial**) **response – undamped**	$x(t)=x_0\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$	$\theta(t)=\theta_0\cos\omega_n t+\frac{\omega_0}{\omega_n}\sin\omega_n t$

3-2 1-DOF MECHANICAL SYSTEMS (2^{ND}-ORDER, SPRING-INERTIA-DAMPER) (*continued*)

	Translational System	Rotational System
energy balance (*or* **thermodynamics 1st law**) (*or* **Lagranian dynamics**)	$\Delta E = W_{\text{input}} + Q_{\text{input}} \rightarrow \Delta(K+U) = \int P_{\text{input}}\,dt - \int D_{\text{output}}\,dt \rightarrow \dot{K}+\dot{U} = P_{\text{input}} - D_{\text{output}}$ where ΔE = change in internal energy ΔK = change in kinetic energy & $\dot{K}$ = time rate of change in kinetic energy ΔU = change in potential energy & $\dot{U}$ = time rate of change in potential energy W_{input} = work input & P_{input} = power input Q_{input} = heat input & D_{output} = dissipated power	
kinetic energy	$K = \frac{1}{2}m[v(t)]^2 = \frac{1}{2}m[\dot{x}(t)]^2$ $\rightarrow \dot{K} = \frac{dK}{dt} = \frac{dK}{d\dot{x}}\frac{d\dot{x}}{dt} = \frac{d\left(\frac{1}{2}m\dot{x}^2\right)}{d\dot{x}}\ddot{x}(t) = m\dot{x}(t)\ddot{x}(t)$	$K = \frac{1}{2}J[\omega(t)]^2 = \frac{1}{2}J[\dot{\theta}(t)]^2$ $\rightarrow \dot{K} = \frac{dK}{dt} = \frac{dK}{d\dot{\theta}}\frac{d\dot{\theta}}{dt} = \frac{d\left(\frac{1}{2}J\dot{\theta}^2\right)}{d\dot{\theta}}\ddot{\theta}(t) = J\dot{\theta}(t)\ddot{\theta}(t)$
potential energy	$U = \frac{1}{2}k[x(t)]^2$ $\rightarrow \dot{U} = \frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} = \frac{d\left(\frac{1}{2}kx^2\right)}{dx}\dot{x}(t) = kx(t)\dot{x}(t)$	$U = \frac{1}{2}k_r[\theta(t)]^2$ $\rightarrow \dot{U} = \frac{dU}{dt} = \frac{dU}{d\theta}\frac{d\theta}{dt} = \frac{d\left(\frac{1}{2}k_r\theta^2\right)}{d\theta}\dot{\theta}(t) = k_r\theta(t)\dot{\theta}(t)$
power input	$P_{\text{input}} = \frac{dW_{\text{input}}}{dt} = f(t)v(t) = f(t)\dot{x}(t)$	$P_{\text{input}} = \frac{dW_{\text{input}}}{dt} = T(t)\omega(t) = T(t)\dot{\theta}(t)$
dissipated power	$D_{\text{output}} = f_b\dot{x}(t) = b[\dot{x}(t)]^2$	$D_{\text{output}} = T_b\dot{\theta}(t) = b_r[\dot{\theta}(t)]^2$
governing equation	$m\dot{x}(t)\ddot{x}(t) + kx(t)\dot{x}(t) = f(t)\dot{x}(t) - b[\dot{x}(t)]^2$ $\rightarrow m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$	$J\dot{\theta}(t)\ddot{\theta}(t) + k_r\theta(t)\dot{\theta}(t) = T(t)\dot{\theta}(t) - b_r[\dot{\theta}(t)]^2$ $\rightarrow J\ddot{\theta}(t) + b_r\dot{\theta}(t) + k_r\theta(t) = T(t)$

3-3 ROTATION-TRANSLATION ANALOGY

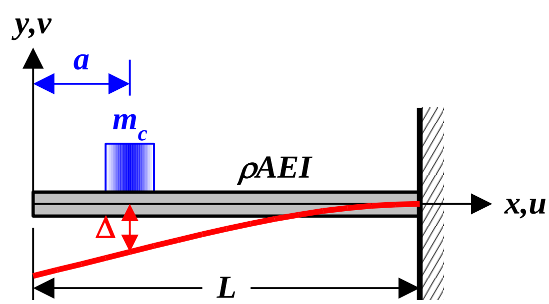
Rotational Mechanical System (Gear train)		Translation Mechanical System (Lever linkage)	
			
applied torque	T	applied force	F
driver inertia	J_m	driver mass	M
driver rotation	θ_m	driver translation	y_m
rotational spring	k_r	linear spring	k
1st gear inertia	J_1	1st block mass	m_1
1st gear rotation	ϕ_1	1st block translation	x_1
1st gear tooth number	n_1	length between 1st block and the lever pivot	L_1
1st gear radius	R_1		
belt differential force	$\Delta f = f_1 - f_2$	lever differential torque	$\Delta T = T_1 - T_2$
belt translation	x	level rotation	θ
2nd gear inertia	J_2	2nd block mass	m_2
2nd gear rotation	ϕ_2	2nd block translation	x_2
2nd gear tooth number	n_2	length between 1st block and the lever pivot	L_2
2nd gear radius	R_2		
gear-train relation	$\begin{cases} \dfrac{n_1}{R_1} = \dfrac{n_2}{R_2} \\ x = R_1\phi_1 = R_2\phi_2 \end{cases}$	lever relation	$\theta = \dfrac{x_1}{L_1} = \dfrac{x_2}{L_2}$
rotational damper	b_r	linear damper	b

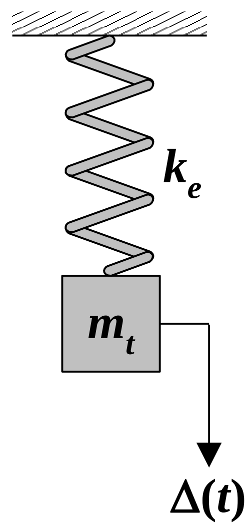
Questions: **1.** How is the lever differential torque: $\Delta T = T_1 - T_2$ generated?

2. What is the limitation of the lever relation: $\theta = \dfrac{x_1}{L_1} = \dfrac{x_2}{L_2}$? *Hint*: $\sin\theta \simeq \theta$.

EXAMPLE 3-1: EQUIVALENT SPRING/MASS
CANTILEVER BEAM

As shown in the figure at right, a concentrated mass m_c is mounted on a cantilever beam at a distance *a* from the free end. The beam has a density ρ, Young's modulus *E*, uniform cross-sectional area *A*, 2nd moment of inertia *I* and length *L*. In the figure, (u,v) are the displacements along the Cartesian coordinates (x,y), respectively. Note that the boundary conditions at a free end are prescribed bending moment and shear force. For this problem, they are: $M(x)\big|_{x=0} = V(x)\big|_{x=0} = 0$. On the other hand, the boundary conditions at a clamped (*or* fixed *or* built-in) end are zero deflection and rotation/slope; that is, $v(x)\big|_{x=L} = \theta(x)\big|_{x=L} = 0$.





a. Statically determinate *vs* indeterminate beam. Replace the concentrated mass by a concentrated force *F* and determine if the beam is statically determinate *or* indeterminate.

b. Beam deflection, rotation/slope, bending moment & shear force. Solve the 4th-order differential equation governing the beam behavior to find the beam deflection *v*(*x*) $v(x)$, rotation/slope $\theta(x)$, bending moment $M(x)$ and shear force $V(x)$. If needed, use **Symbolic Math** software to help solve for the resulting simultaneous linear algebraic equations.

c. Moment & shear diagrams. Plot the moment and shear diagrams within the beam length: $0 \le x \le L$.

d. Reactions. Find the reaction forces and moments at the two ends *A* and *B*.

e. Deflections & rotations/slopes. Find the beam deflections and rotations/slopes at the two ends *A* and *B* and at the loading point: $x = a$, respectively.

f. Equivalent spring constant & mass. The dynamics of the beam-mass system can be viewed as an equivalent mass-spring system

$$m_t \ddot{\Delta}(t) + k_e \Delta(t) = 0 \text{ where } m_t = m_c + m_e$$

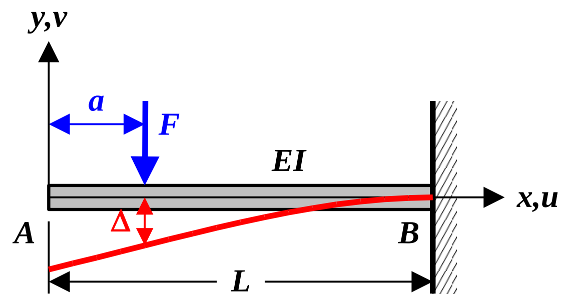
here Δ(*t*) is the time-history of the deflection at the location where the mass is mounted (i.e., $x = a$) and k_e and m_e are the equivalent spring constant and mass of the beam, respectively. Find the proper expressions of k_e for an arbitrary *a*. Determine also (k_e, m_e) for two special cases: $a = 0$ and $a = \frac{L}{2}$, respectively. If needed, use **Symbolic Math** software to help perform integrations.

Sol:

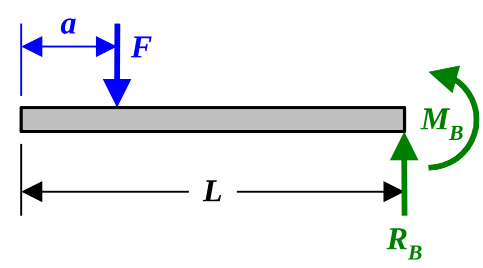
a. Statically determinate *vs* indeterminate beam.

Once the concentrated mass has been replaced by a concentrated force *F*, the structure and its corresponding free-body diagram can be represented by the figures shown below, where *A* and *B* are the free and fixed ends, respectively. Since the two unknown reactions (namely, the reaction force R_B and the reaction moment

M_B) can be determined using two equilibrium equations: one for vertical force and the other for moment, i.e., $+\uparrow\sum F_y = 0$ and $+\circlearrowleft \sum M_z)_B = 0$. Thus, the beam is statically determinate.



equivalent mechanical system



free-body diagram

b. Beam deflection, rotation/slope, bending moment & shear force.

4th-order ODE for beam deflection: $v^{\text{iv}}(x) = -\dfrac{p(x)}{EI}$ **(b1)**

total solution = homogeneous solution + particular solution $\rightarrow v(x) = v_h(x) + v_p(x)$

where $v_p(x)$ is the particular solution due to the lateral load $p(x)$.

no distributed load $\rightarrow p(x) = 0 \rightarrow v_p(x) = 0 \rightarrow v^{\text{iv}}(x) = 0$

$$\rightarrow 0 \le x < a: \begin{cases} \text{deflection:} & v(x) = C_{3\text{L}}x^3 + C_{2\text{L}}x^2 + C_{1\text{L}}x + C_{0\text{L}} \\ \text{rotation/slope:} & \theta(x) = v'(x) = 3C_{3\text{L}}x^2 + 2C_{2\text{L}}x + C_{1\text{L}} \\ \text{bending moment:} & M(x) = EIv''(x) = 6EIC_{3\text{L}}x + 2EIC_{2\text{L}} \\ \text{shear force:} & V(x) = EIv'''(x) = 6EIC_{3\text{L}} \end{cases} \quad \textbf{(b2)}$$

$$\text{and } a < x \le L: \begin{cases} \text{deflection:} & v(x) = C_{3\text{R}}(L-x)^3 + C_{2\text{R}}(L-x)^2 + C_{1\text{R}}(L-x) + C_{0\text{R}} \\ \text{rotation/slope:} & \theta(x) = v'(x) = -3C_{3\text{R}}(L-x)^2 - 2C_{2\text{R}}(L-x) - C_{1\text{R}} \\ \text{bending moment:} & M(x) = EIv''(x) = 6EIC_{3\text{R}}(L-x) + 2EIC_{2\text{R}} \\ \text{shear force:} & V(x) = EIv'''(x) = -6EIC_{3\text{R}} \end{cases} \quad \textbf{(b3)}$$

$$\text{B.C.'s @ free end } (x=0): \begin{cases} \text{no bending moment:} & M(x)\big|_{x=0} = 0 \\ \text{no shear force:} & V(x)\big|_{x=0} = 0 \end{cases}$$

$$\rightarrow \begin{cases} 2EIC_{2\text{L}} = 0 \\ 6EIC_{3\text{L}} = 0 \end{cases} \quad (\textit{Note: } C_{3\text{L}} = C_{2\text{L}} = 0) \quad \textbf{(b4)}$$

$$\text{B.C.'s @ fixed end } (x=L): \begin{cases} \text{no deflection:} & v(x)\big|_{x=L} = 0 \\ \text{no rotation/slope:} & \theta(x)\big|_{x=L} = 0 \end{cases}$$

$$\rightarrow \begin{cases} C_{0\text{R}} = 0 \\ -C_{1\text{R}} = 0 \end{cases} \quad (\textit{Note: } C_{1\text{R}} = C_{0\text{R}} = 0) \quad \textbf{(b5)}$$

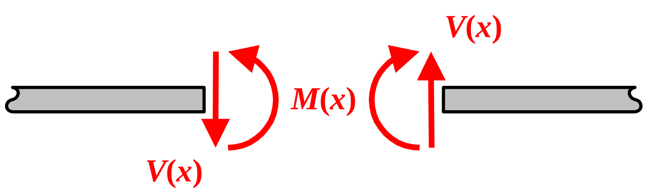
The compatibility conditions at loading point $(x=a)$: $\begin{cases} v(a^-)=v(a^+) \\ \theta(a^-)=\theta(a^+) \\ M(a^-)=M(a^+) \\ V(a^-)=V(a^+)+F \end{cases}$

$$\rightarrow \begin{cases} C_{3L}a^3+C_{2L}a^2+C_{1L}a+C_{0L}=C_{3R}(L-a)^3+C_{2R}(L-a)^2+C_{1R}(L-a)+C_{0R} \\ 3C_{3L}a^2+2C_{2L}a+C_{1L}=-3C_{3R}(L-a)^2-2C_{2R}(L-a)-C_{1R} \\ 6EIC_{3L}a+2EIC_{2L}=6EIC_{3R}(L-a)+2EIC_{2R} \\ 6EIC_{3L}=-6EIC_{3R}+F \end{cases} \quad \text{(b6)}$$

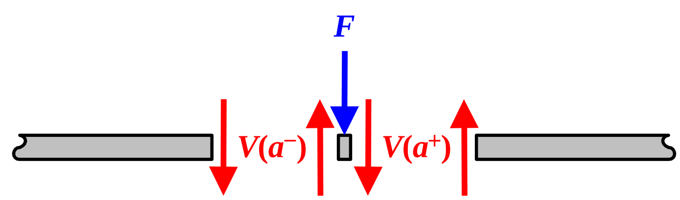
Notes: **1.** The conventions for positive internal forces (i.e., shear force and bending moment) are defined as in the figure at left below.

2. The compatibility condition in shear force, Eq (b6.4), can be obtained by considering the free-body diagram shown in the figure at right below.

Question: What will the compatibility condition in bending moment be, if the concentrated force F is replaced by a concentrated moment M_0?



shear force and bending moment conventions



compatibility condition in shear force

The **8** linear simultaneous algebraic eqs: (b4) to (b6) are solved using **Symbolic Math** software, which is attached at the end of this article, and we obtain

C_{3L}	0	C_{3R}	$\dfrac{F}{6EI}$
C_{2L}	0	C_{2R}	$-\dfrac{F(L-a)}{2EI}$
C_{1L}	$\dfrac{F(L-a)^2}{2EI}$	C_{1R}	0
C_{0L}	$-\dfrac{F(2L+a)(L-a)^2}{6EI}$	C_{0R}	0

Substitute these C values into Eqs (b2,b3) we have

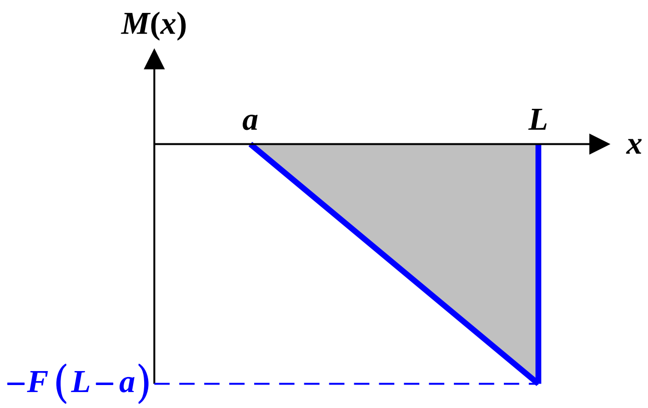
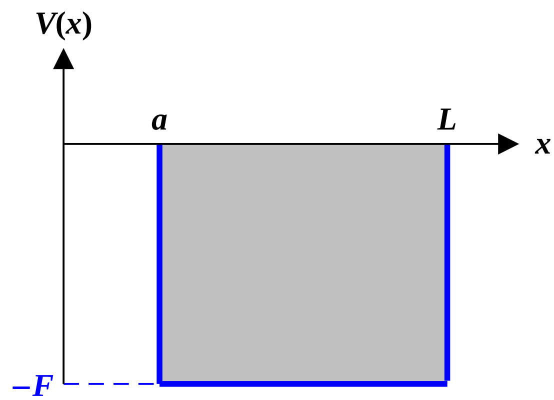
$$\textbf{deflection: } v(x)=\begin{cases} \dfrac{F(L-a)^2}{6EI}\left[3x-(2L+a)\right] & \text{for} \quad 0\le x\le a \\ \dfrac{F}{6EI}\left[(L-x)^3-3(L-a)(L-x)^2\right] & \text{for} \quad a\le x\le L \end{cases} \quad \text{(b7)}$$

rotation/slope: $\theta(x) = v'(x) = \begin{cases} \dfrac{F}{2EI}(L-a)^2 & \text{for} \quad 0 \le x \le a \\ \dfrac{F}{2EI}\left[-(L-x)^2 + 2(L-a)(L-x)\right] & \text{for} \quad a \le x \le L \end{cases}$ (**b8**)

bending moment: $M(x) = EIv''(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le a \\ -F(x-a) & \text{for} \quad a \le x \le L \end{cases}$ (**b9**)

shear force: $V(x) = EIv'''(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < a \\ -F & \text{for} \quad a < x \le L \end{cases}$ (**b10**)

c. Moment & shear diagrams.

Moment Diagram	Shear Diagram
	

d. Reactions.

Reactions	
Free End $x = 0$	**Clamped End** $x = L$
$\begin{cases} R_A = V(0) = 0 \\ M_A = M(0) = 0 \end{cases}$	$\begin{cases} R_B = -V(L) = F \\ M_B = M(L) = -F(L-a) \end{cases}$

e. Deflections & rotations/slopes.

Free End: $x=0$	**Deflection** $v(0)$	$-\dfrac{F(L-a)^2(2L+a)}{6EI}$
	Rotation/Slope $\theta(0)$	$\dfrac{F(L-a)^2}{2EI}$
Loading Location: $x=a$	**Deflection** $v(a)=-\Delta$	$-\dfrac{F(L-a)^3}{3EI}$
	Rotation/Slope $\theta(a)$	$\dfrac{F(L-a)^2}{2EI}$
Clamped End: $x=L$	**Deflection** $v(L)$	0
	Rotation/Slope $\theta(L)$	0

f. Equivalent spring constant & mass.

Since the deflection at $x = a$ is: $\Delta = -v(x)\big|_{x=a} = \dfrac{F(L-a)^3}{3EI} \rightarrow F = \dfrac{3EI}{(L-a)^3}\bullet\Delta$ (**f1**)

Consider the cantilever beam as a spring oscillating under the action of the concentrated mass. From Hooke's law: $F = k_e\Delta$, the equivalent spring constant should be

$$k_e = \frac{3EI}{(L-a)^3}$$ (**f2**)

Specifically, $$\begin{cases} \text{when} \quad a=0 \quad \rightarrow \quad k_e = \dfrac{3EI}{L^3} \quad \text{and} \quad \Delta = \dfrac{FL^3}{3EI} \\ \text{when} \quad a=\dfrac{L}{2} \quad \rightarrow \quad k_e = \dfrac{24EI}{L^3} \quad \text{and} \quad \Delta = \dfrac{FL^3}{24EI} \end{cases}$$ (**f3**)

Substitute Eq (f1) into Eq (b7), we get

$$v(x) = \begin{cases} \dfrac{\Delta}{2(L-a)}\left[3x-(2L+a)\right] & \text{for} \quad 0 \le x \le a \\ \dfrac{\Delta}{2(L-a)^3}\left[(L-x)^3 - 3(L-a)(L-x)^2\right] & \text{for} \quad a \le x \le L \end{cases}$$ (**f4**)

For "$a=0$" case: $v(x) = \Delta\left[\dfrac{1}{2}\left(1-\dfrac{x}{L}\right)^3 - \dfrac{3}{2}\left(1-\dfrac{x}{L}\right)^2\right]$ for $0 \le x \le L$ (**f5**)

Assume that during vibration the form of this beam deflection (i.e., the elastic curve) is preserved (**Justification?**), then:

$$\dot{v}(x) = \frac{\partial v(x,t)}{\partial t} = \dot{\Delta}\left[\frac{1}{2}\left(1-\frac{x}{L}\right)^3 - \frac{3}{2}\left(1-\frac{x}{L}\right)^2\right]$$

The kinetic energy carried by the beam during vibration is

$$\text{K.E.}=\int\frac{1}{2}\left[\dot v(x)\right]^2 dm=\int_0^L\frac{1}{2}\left[\dot v(x)\right]^2\rho A dx=\frac{1}{2}\rho A\int_0^L\left\{\dot\Delta\left[\frac{1}{2}\left(1-\frac{x}{L}\right)^3-\frac{3}{2}\left(1-\frac{x}{L}\right)^2\right]\right\}^2 dx$$

Change of variable: $\xi=1-\frac{x}{L}$, we have $dx=-Ld\xi$ and

$$\text{K.E.}=-\frac{1}{2}\rho AL\dot\Delta^2\int_1^0\left(\frac{1}{2}\xi^3-\frac{3}{2}\xi^2\right)^2 d\xi=\frac{1}{2}\rho AL\dot\Delta^2\int_0^1\left(\frac{1}{4}\xi^6-\frac{3}{2}\xi^5+\frac{9}{4}\xi^4\right)d\xi$$

$$=\frac{1}{2}\rho AL\dot\Delta^2\left(\frac{1}{28}\xi^7-\frac{1}{4}\xi^6+\frac{9}{20}\xi^5\right)\Bigg|_0^1=\frac{1}{2}\rho AL\dot\Delta^2\left(\frac{1}{28}-\frac{1}{4}+\frac{9}{20}\right)$$

$$=\frac{1}{2}\rho AL\dot\Delta^2\bullet\frac{33}{140}\approx 0.236\bullet\left(\frac{1}{2}\rho AL\dot\Delta^2\right)$$

Equate this kinetic energy with that of the equivalent system: $\boxed{\text{K.E.}=\frac{1}{2}m_e\dot\Delta^2}$, we get **(f6)**

$\boxed{m_e=\frac{33}{140}m_{\text{beam}}\approx 0.236 m_{\text{beam}}}$, where $\boxed{m_{\text{beam}}=\rho AL}$ **(f7)**

For "$a=\frac{L}{2}$" case: $v(x)=\begin{cases}\frac{FL^3}{48EI}\left[6\left(\frac{x}{L}\right)-5\right] & \text{for } 0\le x\le\frac{L}{2}\\ \frac{FL^3}{12EI}\left[2\left(1-\frac{x}{L}\right)^3-3\left(1-\frac{x}{L}\right)^2\right] & \text{for } \frac{L}{2}\le x\le L\end{cases}$ **(f8)**

$\rightarrow v(x)=\begin{cases}\Delta\left[3\left(\frac{x}{L}\right)-\frac{5}{2}\right] & \text{for } 0\le x\le\frac{L}{2}\\ \Delta\left[4\left(1-\frac{x}{L}\right)^3-6\left(1-\frac{x}{L}\right)^2\right] & \text{for } \frac{L}{2}\le x\le L\end{cases}$ where $\Delta=-v(x)\big|_{x=\frac{L}{2}}=\frac{FL^3}{24EI}$ **(f9)**

Assume that during vibration the form of this beam deflection (i.e., the elastic curve) is preserved (**Justification?**), then

$$\dot v(x)=\frac{\partial v(x,t)}{\partial t}=\begin{cases}\dot\Delta\left[3\left(\frac{x}{L}\right)-\frac{5}{2}\right] & \text{for } 0\le x\le\frac{L}{2}\\ \dot\Delta\left[4\left(1-\frac{x}{L}\right)^3-6\left(1-\frac{x}{L}\right)^2\right] & \text{for } \frac{L}{2}\le x\le L\end{cases}$$

The kinetic energy carried by the beam during vibration is

$$\text{K.E.}=\int\frac{1}{2}\left[\dot v(x)\right]^2 dm=\int_0^L\frac{1}{2}\left[\dot v(x)\right]^2\rho A dx=\frac{1}{2}\rho A\dot\Delta^2\left\{\int_0^{L/2}\left[3\left(\frac{x}{L}\right)-\frac{5}{2}\right]^2 dx+\int_{L/2}^L\left[4\left(1-\frac{x}{L}\right)^3-6\left(1-\frac{x}{L}\right)^2\right]^2 dx\right\}$$

Change of variable: $\begin{cases} \xi = \dfrac{x}{L} & \text{for} \quad 0 \le x \le L/2 \\ \eta = 1 - \dfrac{x}{L} & \text{for} \quad L/2 \le x \le L \end{cases}$, we have $dx = \begin{cases} L\,d\xi & \text{for} \quad 0 \le x \le L/2 \\ -L\,d\eta & \text{for} \quad L/2 \le x \le L \end{cases}$ and

$$\text{K.E.} = \frac{1}{2}\rho AL\dot{\Delta}^2 \left[\int_0^{1/2} \left(3\xi - \frac{5}{2}\right)^2 d\xi - \int_{1/2}^{0} \left(4\eta^3 - 6\eta^2\right)^2 d\eta \right]$$

$$= \frac{1}{2}\rho AL\dot{\Delta}^2 \left[\int_0^{1/2} \left(9\xi^2 - 15\xi + \frac{25}{4}\right) d\xi + \int_0^{1/2} \left(16\eta^6 - 48\eta^5 + 36\eta^4\right) d\eta \right]$$

$$= \frac{1}{2}\rho AL\dot{\Delta}^2 \left[\left(3\xi^3 - \frac{15}{2}\xi^2 + \frac{25}{4}\xi\right)\Bigg|_0^{1/2} + \left(\frac{16}{7}\eta^7 - 8\eta^6 + \frac{36}{5}\eta^5\right)\Bigg|_0^{1/2} \right]$$

$$= \frac{1}{2}\rho AL\dot{\Delta}^2 \left[\left(\frac{3}{8} - \frac{15}{8} + \frac{25}{8}\right) + \left(\frac{1}{56} - \frac{1}{8} + \frac{9}{40}\right) \right]$$

$$= \frac{1}{2}\rho AL\dot{\Delta}^2 \left(\frac{13}{8} + \frac{33}{280}\right) = \frac{1}{2}\rho AL\dot{\Delta}^2 \bullet \frac{61}{35} \approx 1.743 \bullet \left(\frac{1}{2}\rho AL\dot{\Delta}^2\right)$$

Equate this kinetic energy with that of the equivalent system: $\boxed{\text{K.E.} = \frac{1}{2} m_e \dot{\Delta}^2}$, we have

$\boxed{m_e = \dfrac{61}{35} m_{\text{beam}} \approx 1.743 m_{\text{beam}}}$, where $\boxed{m_{\text{beam}} = \rho AL}$ **(f10)**

Note: The above integrals can also be evaluated by **Symbolic Math** software, which is attached at the end of this article.

SUMMARY

a	k_e	m_e
0	$\dfrac{3EI}{L^3}$	$\dfrac{33}{140} m_{\text{beam}} \approx 0.236 m_{\text{beam}}$
$\dfrac{L}{2}$	$\dfrac{24EI}{L^3}$	$\dfrac{61}{35} m_{\text{beam}} \approx 1.743 m_{\text{beam}}$

SYMBOLIC MATH (Mathcad)
Equivalent Spring & Mass (Cantilever Beam)

Given

$$2\cdot EI\cdot C_{2L} = 0$$

$$6\cdot EI\cdot C_{3L} = 0$$

$$C_{0R} = 0$$

$$-C_{1R} = 0$$

$$C_{3L}\cdot a^3 + C_{2L}\cdot a^2 + C_{1L}\cdot a + C_{0L} = C_{3R}\cdot(L-a)^3 + C_{2R}\cdot(L-a)^2 + C_{1R}\cdot(L-a) + C_{0R}$$

$$3\cdot C_{3L}\cdot a^2 + 2\cdot C_{2L}\cdot a + C_{1L} = -3\cdot C_{3R}\cdot(L-a)^2 - 2\cdot C_{2R}\cdot(L-a) - C_{1R}$$

$$6EI\cdot C_{3L}\cdot a + 2\cdot EI\cdot C_{2L} = 6\cdot EI\cdot C_{3R}\cdot(L-a) + 2\cdot EI\, C_{2R}$$

$$6EI\cdot C_{3L} = -6\cdot EI\cdot C_{3R} + F$$

$$\text{Find}(C_{3L}, C_{2L}, C_{1L}, C_{0L}, C_{3R}, C_{2R}, C_{1R}, C_{0R}) \rightarrow \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\cdot F\cdot\frac{\left(L^2 - 2\cdot L\cdot a + a^2\right)}{EI} \\ \frac{-1}{6}\cdot F\cdot\frac{\left(-3\cdot L^2\cdot a + a^3 + 2\cdot L^3\right)}{EI} \\ \frac{1}{6}\cdot\frac{F}{EI} \\ \frac{1}{2}\cdot F\cdot\frac{(-L+a)}{EI} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\cdot F\cdot\frac{(-L+a)^2}{EI} \\ \frac{-1}{6}\cdot F\cdot(a+2\cdot L)\cdot\frac{(-L+a)^2}{EI} \\ \frac{1}{6}\cdot\frac{F}{EI} \\ \frac{1}{2}\cdot F\cdot\frac{(-L+a)}{EI} \\ 0 \\ 0 \end{bmatrix}$$

$$-\int_1^0 \left(\frac{1}{2}\xi^3 - \frac{3}{2}\xi^2\right)^2 d\xi \rightarrow \frac{33}{140} \qquad \int_0^{\frac{1}{2}} \left(3\xi - \frac{5}{2}\right)^2 d\xi \rightarrow \frac{13}{8} \qquad -\int_{\frac{1}{2}}^0 \left(4\eta^3 - 6\eta^2\right)^2 d\eta \rightarrow \frac{33}{280}$$