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# TR-2009006: Intelligent Players

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**Abstract**

Rational decisions depend on what players know, hence an appropriate epistemic analysis is an integral element of the foundations of Game Theory. We suggest a general logical approach for studying games which consists of formalizing rationality and games in epistemic logic and deriving their properties in the resulting logical system. We study a number of examples and demonstrate that our model can produce a finer-grained analysis of game-theoretical scenarios and provide a non-circular justification of Nash equilibrium strategies.

We show that within this model, in strategic-form and extensive-form games, an assumption of first-level mutual knowledge of the game and players' rationality implies Nash equilibrium and backward induction solutions. This refutes a general perception that common knowledge of rationality is needed to justify backward induction in games with perfect information.

## 1 Introduction

Making epistemic conditions in games explicit is a necessary part of game analysis and recommendations. Without such disclosure, solutions offered by Game Theory would be incomplete or even misleading. Game theorists have long been aware of this and have studied sufficient epistemic conditions under which traditional game-theoretical solutions, e.g., Nash equilibria, backward induction solutions, etc., hold (cf. [2, 3, 4, 5, 7] and many others).

We believe that further steps in the epistemic analysis of games are needed: tracking what players actually know and don't know is as important as payoff and move-analysis. Moreover, since rational decisions actually depend on what players know, an appropriate epistemic analysis belongs to the foundations of Game Theory.

A variety of methods have been offered to study epistemic questions in Game Theory. We suggest our own rather general approach and argue that it provides, in certain situations, a finer-grained analysis. In particular, we consider Prisoner's Dilemma (Section 3) and War and Peace Dilemma (Section 4) with different states of knowledge of players' rationality and analyze how the same set of preferences can lead to different behavior under different epistemic conditions of players.

We show that first-level mutual knowledge of the game and players' rationality (given some plausible assumptions about players' intelligence) implies Nash equilibrium and backward induction solutions in both strategic-form games and extensive-form games with perfect information. This refines the classic Aumann's Theorem that in games with perfect information, common knowledge of rationality implies backward induction. As an example, we derive the backward induction solution of the centipede game from first-level mutual knowledge of the game and players' rationality.

The approach we follow here is that of postulating epistemic principles about games in a corresponding epistemic logic with rationality propositions, and studying the resulting logical description. Applying epistemic modal logic in games is an established tradition (cf., for example [5, 7, 8, 10, 16, 17, 18, 19, 20]). What we do differently here is assume and formalize, to a larger extent, players' ability to draw logical conclusions from the game description and general understanding of rationality. We discuss and assume plausible epistemic principles about games, strategies, and rationality (Assumptions 1 and 2) and make them, together with the basic epistemic logics of belief $\mathsf{K}$ and knowledge $\mathsf{T}$, the logical foundation of the proposed model for games.

We pursue a two-step approach in building logical models of games. First, we specify the problem by syntactic logical tools and then use whatever methods are appropriate to study the resulting formal specification. Here are some obvious advantages to such an approach:

- We do not lose the whole picture by committing upfront to a specific combinatorial model, which is often not completely adequate. A verbally described problem may not have a unique model at all. On the contrary, by first constructing a formal logic specification, we invite a whole variety of models and deny none of them *a priori*.

- We have the advantage of yet another structure, logic formalization, which has its own tools, including deduction methods, which are quite capable of handling epistemic tasks.

- Logic formalization in the middle helps to bridge the epistemic gap caused by the fact that human agents do not necessarily think and communicate in combinatorial terms. Agents announce sentences and logical conditions rather than specific set partitions, Kripke models, topological spaces, etc.

## 2 Awareness and Rationality: logical view

We will focus on two epistemic issues which attract attention in Game Theory: *knowledge of rationality* and *knowledge of the game*. Rationality is a *property*, whereas the game is an *object*, and 'knowledge of the game' should be understood as

*awareness of the game rules (possible moves, payoffs, etc.).*

For simplification purposes, we will assume here *game awareness* for all players:

*each player is aware of the rules of the game.*

When explicitely stated, we will also assume *mutual knowledge of game awareness* for all players:

*each player is aware of the rules of the game and this fact is known to each player.*

Player $P$'s rationality will be represented by a special atomic proposition

$$rP \quad - \quad \text{`}P \textit{ is rational.'}$$

Player $P$'s knowledge (or belief) will be denoted by modality $\mathbf{K}_P$, hence

$$\mathbf{K}_P(F) \quad \text{-} \quad \text{`}P \textit{ knows (believes) that } F\text{.'}$$

In particular, $\mathbf{K}_P(rQ)$ states that 'player $P$ knows (believes) that player $Q$ is rational.'

## 2.1 Knowledge and belief postulates

We assume that knowledge modalities $\mathbf{K}_P$ satisfy postulates of the most general modal logic of knowledge $\mathsf{T}$:

$$\begin{array}{l} \textit{Axioms and rules of classical logic;} \\ \mathbf{K}_P(F \rightarrow G) \wedge \mathbf{K}_P(F) \rightarrow \mathbf{K}_P(G),\ \textit{epistemic closure principle;} \\ \mathbf{K}_P(F) \rightarrow F,\ \textit{factivity;} \\ \textit{Necessitation Rule: if } F \textit{ is derived without hypothesis, then } \mathbf{K}_P(F) \textit{ is also derived.} \end{array} \tag{1}$$

In many cases, it is sufficient for epistemic analysis of a game to assume a yet weaker system, namely $\mathsf{K}$, which is $\mathsf{T}$ without the factivity principle. In this case, $\mathbf{K}_P$ is considered to be belief rather than knowledge modality.

## 2.2 Self-knowledge of rationality

We assume self-knowledge of rationality.

$$rP \ \rightarrow \ \mathbf{K}_P(rP) \tag{2}$$

as a new logical axiom in epistemic logic. This assumption basically reflects our understanding that rationality is a conscious state of mind and a rational agent realizes that he behaves rationally. We do not consider as rational a player who acts rationally by chance, without conscious and consistent behavior.

Let

$$\mathsf{K}_n^r$$

be the $\mathsf{K}$-based logic of belief with $n$ agents extended by additional axioms of self-knowledge of rationality (2) for each agent.

The logic of knowledge

$$\mathsf{T}_n^r$$

is defined in a similar manner based on $\mathsf{T}$.

In both $\mathsf{K}_n^r$ and $\mathsf{T}_n^r$, for any players $Q_1, Q_2, \ldots Q_m$,

$$\mathbf{K}_{Q_1}\mathbf{K}_{Q_2}\ldots\mathbf{K}_{Q_m}[rP \;\rightarrow\; \mathbf{K}_P(rP)]$$

is derivable.

We will use $\mathbf{K}_{P,Q}X$ as shorthand for $\mathbf{K}_P(X) \wedge \mathbf{K}_Q(X)$.

## 2.3 Epistemic conditions of rational behavior

What sort of epistemic analysis could we anticipate within this language? What would be regarded as epistemic conditions for a rational player $P$, who is aware of the rules of the game, to play a given strategy?

Let player $P$ have to choose one of the strategies $1, 2, \ldots, n$ and $s_i$ denote

*$P$ chooses i-th strategy.*

In particular, the following holds:

$$s_1 \vee s_2 \vee \ldots \vee s_n.$$

Moves by the players can be assumed as self-known:

$$s_i \;\rightarrow\; \mathbf{K}_P(s_i) \quad \text{and} \quad \neg s_i \;\rightarrow\; \mathbf{K}_P(\neg s_i). \tag{3}$$

Let also

$$best{:}\,s_i$$

denote the proposition

*i-th strategy yields the highest payoff for $P$ among all strategies available.*

For simplicity, we assume that there is a unique highest-yield strategy for $P$ among $1, 2, \ldots, n$. In particular, this condition is met in games with ordinal preferences.

It is clear that $best{:}\,s_i$ alone is not sufficient for $s_i$ since $best{:}\,s_i$ may be unknown to $P$.

**Assumption 1** *If player $P$ is rational and $P$ knows that a certain strategy does not yield the highest payoff for $P$ among all available strategies, then $P$ does not choose this strategy. Formally, for each $i = 1, 2, \ldots, n$,*

$$rP \;\rightarrow\; [\mathbf{K}_P(\neg best{:}\, s_i) \rightarrow \neg s_i]. \tag{4}$$

We believe that this assumption is acceptable for game-theorists. A similar assumption was made by Bonanno in [7]:

> "...a player is irrational if she chooses a particular strategy while believing that another strategy of hers is better."

The following Corollary 1 is implied by (4) in basic epistemic logic and should be accepted as soon as we accept Assumption 1.

**Corollary 1** *If player $P$ is rational and $P$ knows that a certain strategy is the only one which yields the highest payoff for $P$ among all available strategies, then $P$ chooses this strategy. Formally, for each $i = 1, 2, \ldots, n$,*

$$rP \;\rightarrow\; [\mathbf{K}_P(best{:}\, s_i) \rightarrow s_i]. \tag{5}$$

**Proof.** Pick an $i$ from $1, 2, \ldots, n$ and suppose $rP$ and $\mathbf{K}_P(best{:}\, s_i)$, i.e., that $P$ knows that $i$-th strategy yields the highest payoff. By assumptions, $\mathbf{K}_P(\neg best{:}\, s_j)$ for all $i, j$ such that $j \neq i$. By Assumption 1 and epistemic logic, for all $j$ such that $j \neq i$,

$$rP \wedge \mathbf{K}_P(best{:}\, s_i) \;\rightarrow\; \neg s_j,$$

hence

$$rP \wedge \mathbf{K}_P(best{:}\, s_i) \;\rightarrow\; (\neg s_1 \wedge \ldots \wedge \neg s_{i-1} \wedge \neg s_{i+1} \wedge \ldots \wedge \neg s_n).$$

Since

$$s_1 \vee s_2 \vee \ldots \vee s_n,$$

we have

$$(\neg s_1 \wedge \ldots \wedge \neg s_{i-1} \wedge \neg s_{i+1} \wedge \ldots \wedge \neg s_n) \;\rightarrow\; s_i,$$

hence

$$rP \wedge \mathbf{K}_P(best{:}\, s_i) \;\rightarrow\; s_i.$$

□

The following is an easy corollary of (3), (4), and (5).

**Corollary 2** *Under the conditions of Assumption 1, for each $i = 1, 2, \ldots, n$,*

$$rP \;\rightarrow\; [\mathbf{K}_P(\neg best{:}\, s_i) \rightarrow \mathbf{K}_P(\neg s_i)] \tag{6}$$

*and*

$$rP \;\rightarrow\; [\mathbf{K}_P(best{:}\, s_i) \rightarrow \mathbf{K}_P(s_i)]. \tag{7}$$

## 2.4 Awareness Principle and its consequences

The following *Awareness Principle* represents the ability of player $P$ to determine his best pure strategy at a given node of the game from the rules of the game:

$$\mathbf{K}_P(best\!: s_1) \vee \ldots \vee \mathbf{K}_P(best\!: s_n), \tag{8}$$

i.e.,

*$P$ knows his best strategy.*

For finite perfect information games (of feasible size), the Awareness Principle looks quite plausible, since game-aware intelligent players are perfectly capable of determining which of the strategies yields the best payoff.

From the Awareness Principle, we can deduce Corollary 3 which demonstrates that if a rational player $P$ is completely aware of his options, plays pure strategies, and chooses a strategy $s$, $P$ then knows that $s$ yields the highest guaranteed payoff among all available strategies. This principle represents the drawing of conclusions about knowledge of rational and well-informed players from their moves.

**Corollary 3** *In addition to conditions of Corollary 1, suppose Awareness Principle (8). Then for each $i = 1, 2, \ldots, n$,*

$$rP \ \rightarrow \ [s_i \ \rightarrow \ \mathbf{K}_P(best\!: s_i)]. \tag{9}$$

**Proof.** Choose $i$ from $1, 2, \ldots, n$ and suppose that $rP$ and $\neg\mathbf{K}_P(best\!: s_i)$. By Awareness Principle (8), we have

$$\mathbf{K}_P(best\!: s_1) \vee \ldots \vee \mathbf{K}_P(best\!: s_{i-1}) \vee \mathbf{K}_P(best\!: s_{i+1}) \vee \ldots \vee \mathbf{K}_P(best\!: s_n).$$

By Corollary 1,

$$s_1 \vee \ldots \vee s_{i-1} \vee s_{i+1} \vee \ldots \vee s_n.$$

Since $P$ has to choose only one strategy,

$$(s_1 \vee \ldots \vee s_{i-1} \vee s_{i+1} \vee \ldots \vee s_n) \ \rightarrow \ \neg s_i,$$

therefore $\neg s_i$. So, we have assumed $rP \wedge \neg\mathbf{K}_P(best\!: s_i)$ and concluded $\neg s_i$, hence

$$rP \wedge \neg\mathbf{K}_P(best\!: s_i) \ \rightarrow \ \neg s_i$$

and

$$rP \ \rightarrow \ [s_i \ \rightarrow \ \mathbf{K}_P(best\!: s_i)].$$

□

For one-shot strategic-form games, Awareness Principle (8) may be regarded as a 'pure strategy principle': a player is knowingly evaluating available pure strategies and trying to determine which is best. Since the set of choices is finite and known to the player together with their payoffs, he can determine his best move. If player $P$ does not know his best strategy, he then chooses it by chance or plays a different, non-optimal strategy. Under the circumstances, we can hardly consider $P$ a rational player who is aware of the game rules and knowingly plays his best pure strategy.

Principle (10) below manifests itself in such a common mode of reasoning about strategic-form games as 'drawing arrows' on the preferences matrix from non-optimal outcomes to better ones (cf. game matrices with such arrows in Sections 3 and 4). By doing this, we rule out certain outcomes as non-rational for at least one player. For example, in Prisoner's Dilemma (Section 3), we reject the outcome

$$(\mathit{cooperate}_A,\ \mathit{defect}_B)$$

as not rational for $A$ on the basis that

$$(\mathit{defect}_A,\ \mathit{defect}_B)$$

yields a higher payoff for $A$, given that $B$ plays $\mathit{defect}_B$.

This idea can be summarized as

*bad strategies are never played by rational players.*

Indeed, if a rational player $P$ knows his best strategy, he will not choose a bad one. If $P$ does not know his best strategy, he cannot knowingly choose it, hence he cannot be regarded a rational player who plays pure strategies.

In a formal logical setting, this amounts to the following Corollary 4 which can be easily deduced from Corollary 3, as well as accepted as a separate assumption which reflects the way game-theorists reason, e.g., about strategic-form games.

**Corollary 4** *Under conditions of Corollary 3, for each $i = 1, 2, \ldots, n$,*

$$rP \ \rightarrow\ [\neg \mathit{best}\!:s_i \ \rightarrow\ \neg s_i]. \tag{10}$$

**Proof.** By Corollary 3,

$$rP \ \rightarrow\ [s_i \ \rightarrow\ \mathbf{K}_P(\mathit{best}\!:s_i)].$$

By factivity of knowledge,

$$\mathbf{K}_P(\mathit{best}\!:s_i) \ \rightarrow\ \mathit{best}\!:s_i.$$

Hence

$$rP \ \rightarrow\ [s_i \ \rightarrow\ \mathit{best}\!:s_i],$$

and

$$rP \ \rightarrow\ [\neg \mathit{best}\!:s_i \ \rightarrow\ \neg s_i].$$

□

In Sections 4 and 5, we derive Nash equilibrium strategies of players from their mutual knowledge of the game and rationality using Corollary 4 which is based on Awareness Principle (8). This has foundational consequences for the problem of justifying players' choice of Nash equilibrium strategies. Traditional Game Theory analysis admits that the reasoning in defense of Nash equilibrium is circular (cf. [6], pp. 14–16), e.g.,

*Alice plays this way because Bob plays this way,*
*Bob plays this way because Alice plays this way.*

Our epistemic analysis severs this circularity and justifies Nash equilibrium choices by accepting the Awareness Principle:

> *Alice and Bob play Nash equilibrium strategies because each of them believes that the other player is game-aware, rational, and knowingly plays pure strategies (hence knows his/her best strategy).*

An alternative way of justifying Nash equilibrium is described in Section 5:

> *Alice and Bob play Nash equilibrium strategies because each of them believes that the other player is game-aware, rational, and that at any other outcome, at least one of the players would know that he had played irrationally.*

## 2.5 Format of the logical model

We will be modeling game $G$ with $n$ players by an appropriate logical model which consists of

1. an appropriate epistemic logic, e.g., $\mathsf{K}_n^r$, $\mathsf{T}_n^r$, etc. (Section 2.2);

2. a set of formulas $\mathcal{RA}$ called *Rationality Assumptions*;

3. a set of formulas $\mathcal{GD}$ which will constitute a partial *Game Description*.

We will consider $\mathcal{RA}$'s which contain rationality propositions $rP$ for each player $P$ as well as some mutual knowledge of rationality assertions $\mathbf{K}_Q(rP)$. In particular, if mutual knowledge of rationality is assumed, then $\mathcal{RA}$ contains $\mathbf{K}_Q(rP)$'s for all $P$ and $Q$.

Game Description $\mathcal{GD}$ is usually a finite set of formulas which depends on a specific game. Normally, $\mathcal{GD}$ contains some instances of principles (3)–(10), explicitly mentioned, as well as other logical statements about the game.

The following comment requires a certain degree of familiarity with formal logic and can be omitted at first reading.

Within our logical model,

$$\mathcal{L} = \textit{Epistemic Logic} + \textit{Rationality Assumptions} + \textit{Game Description}, \quad (11)$$

the only default rule of inference is *Modus Ponens*

$$\frac{A \qquad A \to B}{B}$$

whereas the Necessitation Rule

$$\frac{A}{\mathbf{K}_P(A)}$$

can be used only within the scope of Epistemic Logic (with rationality propositions, e.g., $\mathsf{K}_n^r$, $\mathsf{T}_n^r$, etc.).

*Epistemic Logic* from $\mathcal{L}$ (11) enjoys the Deduction Theorem

$$\Gamma, A \vdash B \quad \textit{iff} \quad \Gamma \vdash A \to B,$$

where '$\vdash$' stands for provability in a given epistemic logic system (e.g., $\mathsf{K}_n^r$, $\mathsf{T}_n^r$, etc.), $\Gamma$ is a set of formulas, and $A$, $B$ are formulas. In particular,

$$\mathcal{L} \vdash B \qquad \textit{iff} \qquad \textit{Epistemic Logic} \vdash [\mathcal{RA} \wedge \mathcal{GD}] \ \to \ B.$$

This observation helps to establish certain impossibility results concerning logical models, e.g., in Theorems 1 and 2 below. In order to establish that some sentence $F$ is not derivable in a given $\mathcal{L} = \textit{Epistemic Logic} + \mathcal{RA} + \mathcal{GD}$ it suffices to demonstrate that

$$\textit{Epistemic Logic} \nvdash [\mathcal{RA} \wedge \mathcal{GD}] \ \to \ F,$$

for which is suffices to find a Kripke model $\mathcal{K}$ for given *Epistemic Logic* such that at one of its nodes, all formulas from $\mathcal{RA}$ and $\mathcal{GD}$ hold, but $F$ does not hold.

## 2.6 Some examples

Let us consider the simple example of a game in Figure 1, which we borrow from [2]. Player $P$ has two choices: to move down with payoff 1, or to move across with payoff 3. For the sake of example, suppose there is one more player (observer), $Q$, who does not make moves but is aware of the game rules and knows that $P$ is rational and aware of the game rules.

Let $a$ be the proposition

*$P$ chooses 'across'*

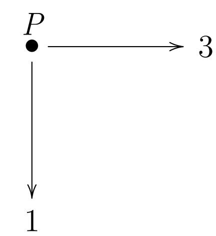


Figure 1: Game 1

and $d$ be

*P chooses 'down.'*

The following proposition is true about this game:

*P chooses 'across' or 'down' but not both*

or, in symbolic form,

$$(a \vee d) \;\wedge\; \neg(a \wedge d). \tag{12}$$

Moreover, justification of (12) does not require any special epistemic assumptions; it logically follows from the game description and basic understanding of game awareness. On these grounds, we assume that (12) is known to any intellectual player, in particular, $P$, who is aware of the game rules:

$$\mathbf{K}_P[(a \vee d) \;\wedge\; \neg(a \wedge d)]. \tag{13}$$

In addition, since $Q$ knows that $P$ is aware of the game rules,

$$\mathbf{K}_Q\mathbf{K}_P[(a \vee d) \;\wedge\; \neg(a \wedge d)]. \tag{14}$$

Furthermore, from the rules of the game, it follows that

*a yields the highest payoff for P*;

let us denote this proposition by

$$best{:}a\ . \tag{15}$$

Naturally, (15) is known to anyone who is aware of the game rules, e.g.,

$$\mathbf{K}_P(best{:}a). \tag{16}$$

Once again, if $Q$ knows that $P$ is aware of the game rules, then

$$\mathbf{K}_Q\mathbf{K}_P(best{:}a). \tag{17}$$

Now we can establish that if $P$ is rational, then $P$ should play *across*. Indeed, by (5),

$$rP \;\rightarrow\; [\mathbf{K}_P(best{:}a) \rightarrow a], \tag{18}$$

which, together with (16), gives

$$rP \rightarrow a.$$

If $Q$ knows that $P$ is aware of the game rules, then $Q$ knows (18):

$$\mathbf{K}_Q[rP \;\rightarrow\; (\mathbf{K}_P(best{:}a) \rightarrow a)]. \tag{19}$$

By (17) and (19), it follows that

$$\mathbf{K}_Q(rP \rightarrow a).$$

Since $Q$ knows that $P$ is rational

$$\mathbf{K}_Q(rP),$$

$Q$ knows that $P$ should play across:

$$\mathbf{K}_Q(a). \tag{20}$$

Adopting (13), (14), (16), (17), and (19) amounts to trusting that players $P$ and $Q$ are capable of performing some logical analysis of the game and share general understanding of game awareness and rationality.

To derive (20), we have used Rationality Assumption $\mathcal{RA} = \{rP, \mathbf{K}_Q(rP)\}$.

Game Description $\mathcal{GD}$ in this case consists of principles (12)–(19).

## 2.7 Applying intelligence

In this section we introduce more general principles which will be used to build Game Descriptions; these assumptions seem all to be in line with the usual methods of reasoning in Game Theory.

**Assumption 2** *i) If $X$ is a formally stated proposition which logically follows (given assumptions about state of the game) from the game description, then*

$$X$$

*can be added to the formal Game Description $\mathcal{GD}$.*

*ii) If, in addition, $P$ is aware of the game rules, then*

$$\mathbf{K}_P(X)$$

*can be added to $\mathcal{GD}$.*

*iii) Moreover, if $Q$ knows that $P$ is aware of the game rules, then*

$$\mathbf{K}_Q\mathbf{K}_P(X)$$

*can be added to $\mathcal{GD}$.*

We justify Assumption 2 by semi-formal reasoning. Let $G_P$ be the proposition

*P is aware of the game rules.*

By the conditions, $G_P$ holds for each player $P$, and $\mathbf{K}_Q(G_P)$ holds for any $P$ and $Q$.

1. Since $X$ logically follows from the game rules, $X$ is true.

2. Since $P$ is aware of the game rules, $P$ knows every $X$ that follows logically from the game rules:

$$G_P \;\rightarrow\; \mathbf{K}_P(X). \tag{21}$$

This yields

$$\mathbf{K}_P(X).$$

3. $Q$ knows (21), since it is based on a general understanding of awareness and logical entailment without any additional assumptions:

$$\mathbf{K}_Q[G_P \;\rightarrow\; \mathbf{K}_P(X)],$$

which yields

$$\mathbf{K}_Q(G_P) \;\rightarrow\; \mathbf{K}_Q\mathbf{K}_P(X).$$

4. If $Q$ knows that $P$ is aware of the game rules, $\mathbf{K}_Q(G_P)$, then $Q$ knows that $P$ knows $X$, i.e.,

$$\mathbf{K}_Q\mathbf{K}_P(X).$$

**Corollary 5** *Suppose game awareness is mutually known. If any instance X from (3)–(10) is adopted, then for any players $P$ and $Q$,*

$$X, \quad \mathbf{K}_P(X), \quad \textit{and} \quad \mathbf{K}_Q\mathbf{K}_P(X) \tag{22}$$

*can be added to the formal Game Description $\mathcal{GD}$.*

By adopting Corollary 5, we assume that the epistemic principles of rational behavior (3)–(10) are shared by all players and follow from the rules of the game. Specifically, in addition to self-evident (3), this amounts to adopting Assumption 1 – which has already been discussed in game-theotetical papers – and Awareness Principle (8), which has been discussed in Section 2.4. We will indicate whether or not specific applications rely on Awareness Principle (8).

### 2.8 Applying rationality

We now apply Assumption 2 to sentences $X$ containing rationality propositions.

**Corollary 6** *Suppose game awareness is mutually known. If $Y$ logically follows from the game description and rationality of $P$, then for each of players $A$ and $B$, the following sentences can be added to the Game Description $\mathcal{GD}$:*

$$(rP \to Y), \quad \mathbf{K}_A(rP \to Y), \quad \mathbf{K}_B\mathbf{K}_A(rP \to Y). \tag{23}$$

**Proof.** By Assumption 2 applied to $rP \to Y$ as $X$. □

Assumption 2 and Corollaries 5 and 6 use the unspecified notion 'logically follows,' which will have a rather transparent meaning in specific applications considered below. This notion, 'logically follows,' should be used with caution in order to avoid the well-known logical omniscience defect of epistemic modal logic (cf. [9, 11, 13, 14]). In a specific application, it should be clear that players can be realistically expected to draw the corresponding logical conclusions. We believe that such assumptions are plausible in many game-theoretical situations, e.g., the ones involving countries, corporations, intelligent players, etc.

# 3 Prisoner's Dilemma

The well-known Prisoner's Dilemma with two players, $A$ and $B$, and two strategies for each, *cooperate* or *defect*, has the matrix

| | $cooperate_B$ | | $defect_B$ |
|---|---|---|---|
| $cooperate_A$ | **2**,**2** | $\Rightarrow$ | **0**,**3** |
| | $\Downarrow$ | | $\Downarrow$ |
| $defect_A$ | **3**,**0** | $\Rightarrow$ | **1**,**1** |

Each player has a dominant strategy to defect, the only Nash equilibrium is

$$(defect_A, defect_B),$$

and we will study epistemic conditions under which $A$ and $B$ choose the strategies suggested by the Nash equilibrium.

**Theorem 1** *In Prisoner's Dilemma, suppose game awareness is mutually known and both players are rational. Then,*

*i) Each of the players knows that he has to play his Nash equilibrium strategy 'defect.'*

*ii) None of the players necessarily knows that the other player should play 'defect,' nor do players necessarily know their own eventual payoffs.*

*iii) If, in addition, rationality of players is mutually known, then it is mutually known that both players should play their Nash equilibrium strategies, hence their eventual payoffs are mutually known as well.*

**Proof.** The logical description of Prisoner's Dilemma will be based on epistemic modal logic of belief $\mathsf{K}_2^r$ with two players $A$ and $B$, and designated rationality propositions $rA$ and $rB$. We define propositions:

$c_A$ - *A chooses to cooperate*,
$d_A$ - *A chooses to defect*,
$c_B$ - *B chooses to cooperate*,
$d_B$ - *B chooses to defect.*

As in Section 2.6, '$best\!:s$' denotes the proposition

*the strategy associated with s gives the highest payoff for the corresponding player.*

There will be two variants of the Rationality Assumptions $\mathcal{RA}$'s: the minimal one for (i) and (ii):

$$\mathcal{RA} = \{rA,\ rB\},$$

and the maximal one for (iii):

$$\mathcal{RA} = \{rA,\ rB,\ \mathbf{K}_A(rB),\ \mathbf{K}_B(rA)\}.$$

Game Description $\mathcal{GD}$ will be built step-by-step, with appropriate conclusions drawn at the proper stages.

From the rules of the game, it follows that each of the players has to choose one strategy. Let $X$ be $(c_A \vee d_A)\ \wedge\ (c_B \vee d_B)$. So by Assumption 2, we add the following to $\mathcal{GD}$:

$$\mathbf{K}_{A,B}(X), \quad \mathbf{K}_A\mathbf{K}_B(X), \quad \mathbf{K}_B\mathbf{K}_A(X). \tag{24}$$

It follows from the matrix of preferences that

$$c_B \rightarrow best\!:d_A, \quad d_B \rightarrow best\!:d_A, \quad c_A \rightarrow best\!:d_B, \quad d_A \rightarrow best\!:d_B. \tag{25}$$

Material implications here are quite appropriate since we want only to rule out certain Boolean combinations of choices as not rational for a given player[1].

[1]We do not use counterfactuals here since for rational players, the game is decided by the order of preferences, which are mathematical facts about the game and do not seem to require a counterfactual approach. For a comprehensive account of counterfactuals in game theory, see [16, 17].

By Assumption 2, we add (25)–(29) to $\mathcal{GD}$:

$$\mathbf{K}_A(c_B \to best\colon d_A), \quad \mathbf{K}_A(d_B \to best\colon d_A), \tag{26}$$

$$\mathbf{K}_B\mathbf{K}_A(c_B \to best\colon d_A), \quad \mathbf{K}_B\mathbf{K}_A(d_B \to best\colon d_A). \tag{27}$$

$$\mathbf{K}_B(c_A \to best\colon d_B), \quad \mathbf{K}_B(d_A \to best\colon d_B), \tag{28}$$

$$\mathbf{K}_A\mathbf{K}_B(c_A \to best\colon d_B), \quad \mathbf{K}_A\mathbf{K}_B(d_A \to best\colon d_B). \tag{29}$$

From (26) – (29), in epistemic logic $\mathsf{K}_2^r$ it follows that

$$\mathbf{K}_A[(c_B \vee d_B) \to best\colon d_A], \quad \mathbf{K}_B\mathbf{K}_A[(c_B \vee d_B) \to best\colon d_A];$$

$$\mathbf{K}_B[(c_A \vee d_A) \to best\colon d_B], \quad \mathbf{K}_A\mathbf{K}_B[(c_A \vee d_A) \to best\colon d_B].$$

Taking into account (24), we can conclude

$$\mathbf{K}_A(best\colon d_A), \quad \mathbf{K}_B\mathbf{K}_A(best\colon d_A), \quad \mathbf{K}_B(best\colon d_B), \quad \mathbf{K}_A\mathbf{K}_B(best\colon d_B).$$

By the conditions of (i),

$$\mathcal{RA} = \{rA,\ rB\}, \tag{30}$$

we derive

$$rA \wedge \mathbf{K}_A(best\colon d_A), \quad rB \wedge \mathbf{K}_B(best\colon d_B)$$

which, by Assumption 1, yields

$$d_A \quad \text{and} \quad d_B.$$

This proves (i).

To establish (ii), we have to show that neither $\mathbf{K}_A(d_B)$ nor $\mathbf{K}_B(d_A)$ can be derived within the logical model of the game. We will show this using current assumptions (24)–(30). Indeed, in Figure 2 there is a Kripke model in which (24)–(30) are true and both $\mathbf{K}_A(d_B)$, $\mathbf{K}_B(d_A)$ are false at the root node[2].

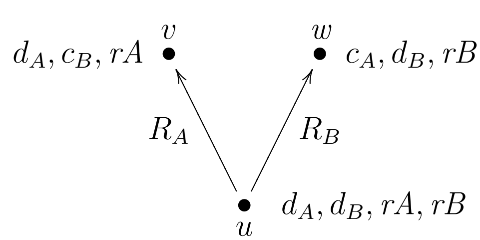


Figure 2: Kripke countermodel for $\mathbf{K}_A(d_B)$, $\mathbf{K}_B(d_A)$

[2]An easy modification of this model covers the cases of major logics of knowledge $\mathsf{S4}$ and $\mathsf{S5}$ as well.

There are three possible worlds in this model: $u$, $v$, and $w$, and two accessibility relations, $R_A$ and $R_B$, associated with modalities $\mathbf{K}_A$ and $\mathbf{K}_B$ such that $uR_Av$ and $uR_Bw$. Atoms $best\!:d_A$ and $best\!:d_B$ hold at all nodes. All other atomic propositions hold as shown in Figure 3. It is easy to see that $\mathsf{K}_2^r$ and (24)–(30) are true at $u$ whereas both $\mathbf{K}_A(d_B)$ and $\mathbf{K}_B(d_A)$ are false at $u$.

There is a clear epistemic reading of this model. The root node $u$ represents the 'real world' at which both players are rational, strategy '*defect*' is the best rational choice for each player, and both $A$, $B$ play '*defect*.' In addition, there is a consistent world $v$ deemed possible by $A$, at which $B$ is not rational and chooses '*cooperate*.' Likewise, $B$ considers possible a consistent world $w$ at which $A$ is not rational and chooses '*cooperate*.'

(iii) It takes mutual knowledge of rationality to make players' choices and eventual payoffs known to both players: let us extend the Rationality Assumptions to

$$\mathcal{RA} = \{rA,\ rB,\ \mathbf{K}_A(rB),\ \mathbf{K}_B(rA)\}, \tag{31}$$

and derive both $\mathbf{K}_A(d_B)$ and $\mathbf{K}_B(d_A)$. As noted earlier,

$$\mathbf{K}_B\mathbf{K}_A(best\!:d_A) \quad \text{and} \quad \mathbf{K}_A\mathbf{K}_B(best\!:d_B).$$

Together with (31), this yields

$$\mathbf{K}_B[rA \wedge \mathbf{K}_A(best\!:d_A)] \quad \text{and} \quad \mathbf{K}_A[rB \wedge \mathbf{K}_B(best\!:d_B)],$$

hence, by (5) and (22),

$$\mathbf{K}_B\{[rA \wedge \mathbf{K}_A(best\!:d_A)] \ \rightarrow\ d_A\} \quad \text{and} \quad \mathbf{K}_A\{[rB \wedge \mathbf{K}_B(best\!:d_B)] \ \rightarrow\ d_B\},$$

therefore

$$\mathbf{K}_B(d_A) \quad \text{and} \quad \mathbf{K}_A(d_B).$$

□

Note that in this paradigmatic game-theoretical scenario, our logic-based approach provides a finer-grained analysis. We have established that, in addition to the minimal requirements of mutual knowledge of game awareness, rationality alone suffices for the players to know (and play) their Nash strategies. We have demonstrated that these epistemic assumptions are, however, not sufficient for the players to know *each other*'s Nash strategies and *their own* eventual payoffs; cross-knowledge of rationality is needed to achieve both. These observations seem consistent with our intuition about this game, but within the logical model, we are now able to rigorously establish these epistemic properties.

# 4 War and Peace Dilemma

Imagine two neighbouring countries: a big, powerful $B$ with a history of expansion, and a small country $S$ which wants to remain independent. Each player has the choice to wage war or to keep peace. The best outcome for both countries is peace. However, if both countries wage war, then $B$ wins easily and $S$ loses everything, which is the second-best outcome for $B$ and the worst for $S$. In situation $(war_B, peace_S)$, $B$ loses internationally, which is the second-best outcome for $S$. In $(peace_B, war_S)$, $B$'s government loses its national support, which is the worst outcome for $B$ and the second worst for $S$.

| | $war_S$ | | $peace_S$ |
|---|---|---|---|
| $war_B$ | **2**,**0** | $\Rightarrow$ | **1**,**2** |
| | $\Uparrow$ | | $\Downarrow$ |
| $peace_B$ | **0**,**1** | $\Rightarrow$ | **3**,**3** |

There is one Nash equilibrium, $(peace_B,\ peace_S)$, consisting of the best outcomes for both players. It might look as though they should both play accordingly. However, such a prediction is not well-founded unless certain epistemic conditions are met.

**Theorem 2** *In the War and Peace Dilemma, suppose game awareness is mutually known and the players are rational. Then*

*i) $S$ chooses 'peace,' but $B$ does not necessarily choose 'peace.'*

*ii) If, in addition, the rationality of $S$ is known to $B$, then $B$ chooses 'peace' as well. However, $S$ does not necessarily know that $B$ chooses 'peace.'*

*iii) If the rationality of players is mutually known, both players know that each chooses 'peace.'*

**Proof.**
For items (i)–(ii), the reasoning can be carried in a $\mathsf{K}$-based logic of belief, whereas for (iii), we will reason in a $\mathsf{T}$-based logic of knowledge.

Let us begin with (i). Informally, only $S$ has the dominant strategy, $peace_S$, whereas $B$ lacks one, hence the choice of $B$ actually depends on $B$'s expectations of $S$'s move. Let us analyze this game in the epistemic logic $\mathsf{K}_2^r$ with two modalities, $\mathbf{K}_B$ and $\mathbf{K}_S$, and propositions $rB$ and $rS$ for rationality assertions. We define propositions:

$w_B$ - *$B$ chooses to wage war*,

$p_B$ - *B chooses to keep peace,*
$w_S$ - *S chooses to wage war,*
$p_S$ - *S chooses to keep peace.*

For (i) we consider the Rationality Assumption

$$\mathcal{RA} = \{rB,\ rS\}. \tag{32}$$

As before, let $X$ be $(w_B \vee p_B)\ \wedge\ (w_S \vee p_S)$. By Assumption 2, we assign the following to $\mathcal{GD}$:

$$\mathbf{K}_{B,S}(X), \quad \mathbf{K}_B\mathbf{K}_S(X), \quad \mathbf{K}_S\mathbf{K}_B(X). \tag{33}$$

It also follows from the rules of the game that given either $w_B$ or $p_B$, the best choice for $S$ would be $p_S$. By Assumption 2, the following is added to $\mathcal{GD}$:

$$\mathbf{K}_S(w_B \rightarrow \mathit{best}{:}p_S), \quad \mathbf{K}_S(p_B \rightarrow \mathit{best}{:}p_S), \tag{34}$$

$$\mathbf{K}_B\mathbf{K}_S(w_B \rightarrow \mathit{best}{:}p_S), \quad \mathbf{K}_B\mathbf{K}_S(p_B \rightarrow \mathit{best}{:}p_S). \tag{35}$$

For $B$, the story is different, and the following should be added to $\mathcal{GD}$:

$$\mathbf{K}_B(w_S \rightarrow \mathit{best}{:}w_B), \quad \mathbf{K}_B(p_S \rightarrow \mathit{best}{:}p_B), \tag{36}$$

$$\mathbf{K}_S\mathbf{K}_B(w_S \rightarrow \mathit{best}{:}w_B), \quad \mathbf{K}_S\mathbf{K}_B(p_S \rightarrow \mathit{best}{:}p_B). \tag{37}$$

From (34) and (35),

$$\mathbf{K}_S[(w_B \vee p_B) \rightarrow \mathit{best}{:}p_S], \quad \mathbf{K}_B\mathbf{K}_S[(w_B \vee p_B) \rightarrow \mathit{best}{:}p_S],$$

and, by (33),

$$\mathbf{K}_S(\mathit{best}{:}p_S), \quad \mathbf{K}_B\mathbf{K}_S(\mathit{best}{:}p_S).$$

By (32),

$$rS \wedge \mathbf{K}_S(\mathit{best}{:}p_S),$$

which, by (5) yields

$$p_S.$$

To settle (i), we show that $p_B$ is not derivable from (32)–(37). This can be demonstrated by constructing a Kripke model such that (32)–(37) hold, but $\mathbf{K}_B(p_B)$ does not hold at the root node in this model. Such a model is given in Figure 3[3].

The model has two nodes $u$ and $v$. Accessibility relation $R_S$ is empty and $R_B$ is such that $uR_Bv$. One can see that (32)–(37) hold at $u$, but $p_B$ does not hold at $u$. An epistemic reading of this model is that $B$ deems possible a world $v$ in which $S$ is not rational and chooses $w_S$ despite the fact that the best strategy for $S$ is $p_S$. Hence $B$ does not know that $S$ chooses '*peace*' and does not have to choose $p_B$ himself. This completes (i).

[3]The same model with modifications works for other major epistemic logics, e.g., S4 and S5.

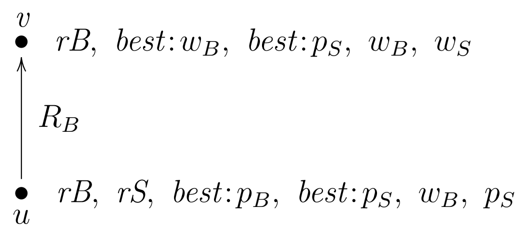


Figure 3: Kripke countermodel for $\mathbf{K}_B(p_B)$

(ii) We now extend the Rationality Assumption by adding the condition that $B$ knows that $S$ is rational.

$$\mathcal{RA} = \{rB,\ rS,\ \mathbf{K}_B(rS)\}. \tag{38}$$

Together with earlier established $\mathbf{K}_B\mathbf{K}_S(best{:}p_S)$, this yields

$$\mathbf{K}_B[rS \wedge \mathbf{K}_S(best{:}p_S)].$$

From (5) and (22),

$$\mathbf{K}_B\{rS \ \rightarrow\ [\mathbf{K}_S(best{:}p_S) \rightarrow p_S]\},$$

and, by reasoning in epistemic logic,

$$\mathbf{K}_B(p_S).$$

From (33),

$$\mathbf{K}_B(best{:}p_B),$$

and, from (38),

$$rB \wedge \mathbf{K}_B(best{:}p_B).$$

By (5),

$$p_B.$$

We now verify that from (33)–(38), it does not follow that $\mathbf{K}_S(p_B)$, i.e., $S$ still can be unaware of $B$'s intension to play $p_B$.

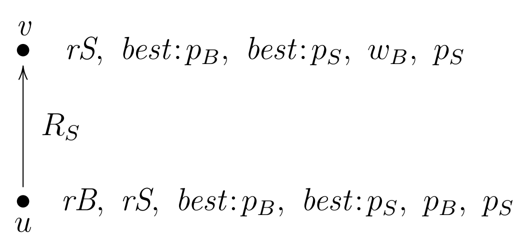


Figure 4: Kripke countermodel for $\mathbf{K}_S(p_B)$

The corresponding Kripke model is given in Figure 4[4]. Again, in this model, (33)–(38) hold and $\mathbf{K}_S(p_B)$ does not hold at $u$. Informally, for $S$ there is a possible world $v$ in which $B$ is not rational and chooses $w_B$ despite the fact that $B$'s rational choice there is $p_B$. Hence $S$ does not know $p_B$.

(iii) Finally, we add the last remaining condition '$S$ knows that $B$ is rational' to the Rationality Assumption:

$$\mathcal{RA} = \{rB,\ rS,\ \mathbf{K}_B(rS),\ \mathbf{K}_S(rB)\}. \tag{39}$$

Since we have already derived $\mathbf{K}_B(p_S)$, it remains to derive $\mathbf{K}_S(p_B)$.

It follows from the rules of the game that given $p_S$, the choice $w_B$ is not the best one for $B$ and, by (22), it is known to $S$:

$$\mathbf{K}_S(p_S \to \neg best\colon w_B). \tag{40}$$

On the other hand, by Corollary 4,

$$\mathbf{K}_S[rB\ \to\ (\neg best\colon w_B \to \neg w_B)]. \tag{41}$$

Combining (40) and (41), we get

$$\mathbf{K}_S[rB\ \to\ (p_S \to \neg w_B)]$$

hence

$$\mathbf{K}_S[rB\ \to\ (p_S \to p_B)]$$

and

$$[\mathbf{K}_S(rB) \wedge \mathbf{K}_S(p_S)] \to \mathbf{K}_S(p_B).$$

By (3), we add to $\mathcal{GD}$

$$p_S\ \to\ \mathbf{K}_S(p_S). \tag{42}$$

Since $p_S$ holds, by (42),

$$\mathbf{K}_S(p_S).$$

By (39), $\mathbf{K}_S(rB)$ also holds and we derive the desired

$$\mathbf{K}_S(p_B).$$

Note that the proof of (iii) above relies on Awareness Principle (8). An alternative proof which does not rely on (8) is given in Theorem 3. □

In the War and Peace Dilemma, our logical analysis shows that despite the fact that

a) for both countries, the best choice is '*peace*';

[4]This model can be easily modified to cover S4 and S5 cases as well.

b) it is the only Nash equilibrium in the game;

c) both countries behave rationally;

to secure the Nash equilibrium outcome, an additional epistemic condition should be met: the big country should know that its small neighbour will behave rationally. Even this is not good enough: to avoid unnecessary military expenses, the small country should know that the big country will behave rationally and therefore will not wage war.

In the following section, we study general epistemic conditions under which players know that each is playing his Nash strategy.

# 5 Nash equilibrium in strategic-form games

The following theorem shows that in strategic-form games, for any number of players and strategies, epistemic conditions for Nash equilibria do not grow beyond the minimal requirements of mutual knowledge of game awareness and mutual knowledge of rationality. In particular, no common knowledge of rationality is necessary.

**Theorem 3** *Sufficient epistemic conditions, under which Nash equilibria in non-cooperative strategic-form one-shot games is justified to all players, is mutual knowledge of both game awareness and rationality of players. Knowledge properties required for players do not exceed principles adopted for the logic* $\mathsf{T}$.

**Proof.** To simplify our reasoning, let us assume first that the game has a unique Nash equilibrium and that for each player, all his payoffs are different. These assumptions place the game inside the scope of Awareness Principle (8) and Corollary 3.

Informally, since the game is known, each player knows that each player's rationality rules out, given other players' moves, all outcomes which do not yield the max payoff in these conditions. The fact that these outcomes are ruled out is known to each player by mutual knowledge of rationality. An outcome which survives this test yields max payoffs for each player, given other players' moves, which is exactly the definition of a Nash equilibrium.

Let $r_i$ stand for

*player i is rational*,

$s_i^j$ for

*i-th player should choose strategy j,*

$best\!:\!s_i^j$ for

*strategy j yields the highest payoff for player i,*

and $e_i$ for

*player i should choose his Nash equilibrium strategy.*

In our logical model, the Rationality Assumption $\mathcal{RA}$ is

$$\mathcal{RA} = \{r_i,\ \mathbf{K}_m(r_i) \mid \textit{for all } m, i = 1, 2, \ldots, n\}. \tag{43}$$

Our goal is to demonstrate that in the logical model corresponding to the game,

$$\mathbf{K}_{\{1,2,\ldots,n\}}(e_1 \wedge e_2 \wedge \ldots \wedge e_n),$$

i.e., that $\mathbf{K}_m(e_i)$ for all $m, i \in \{1, 2, \ldots, n\}$.

**Lemma 1** *Let $j_1, j_2, \ldots, j_n$ be strategies for players $1, 2, \ldots, n$ respectively such that for at least one player $m$, strategy $j_m$ does not yield the highest payoff, given all other strategies from $j_1, j_2, \ldots, j_n$. Then for all $i = 1, 2, \ldots, n$,*

$$\mathbf{K}_i[\neg(s_1^{j_1} \wedge s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})].$$

**Proof.** Without loss of generality, suppose the outcome $j_1, j_2, \ldots, j_n$ is not optimal for Player 1. Then

$$(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow\ \neg best\colon s_1^{j_1} \tag{44}$$

follows from the rules of the game without any epistemic assumptions and, by Assumption 2,

$$\mathbf{K}_i[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow\ \neg best\colon s_1^{j_1}]. \tag{45}$$

By Corollary 4,

$$\mathbf{K}_i[r_1 \ \rightarrow\ (\neg best\colon s_1^{j_1} \rightarrow \neg s_1^{j_1})]. \tag{46}$$

Combining (45) and (46), we get

$$\mathbf{K}_i[r_1 \ \rightarrow\ \neg(s_1^{j_1} \wedge s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})].$$

By (43),

$$\mathbf{K}_i[\neg(s_1^{j_1} \wedge s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})],$$

which completes the proof of Lemma 1.

This proof relies on Awareness Principle (8). We now present an alternative proof of Lemma 1 which does not use (8) and is a more direct formalization of the usual game-theoretical reasoning about strategic-form games. The solution uses reasoning about knowledge of players **after the game**, when all the moves become known. The idea of the proof is to establish that if a player knows before the game that he will know after the game that a given choice of his own strategy is not rational, he is not going to play this strategy.

For each $i = 1, 2, \ldots, n$, let $\mathbf{K}_i^1$ be an additional knowledge operator of player $i$ **after** the game is played (we say, at instant 1, meaning that instant 0 corresponds to the moment of time just prior to the game)[5]. We assume some natural properties of $\mathbf{K}_i^1$:

[5] Time-stamped knowledge modalities were considered, e.g., in [12].

- Each $\mathbf{K}_i^1$ is a $\mathsf{T}$-style modality, e.g., it is factive $\mathbf{K}_i^1(F) \rightarrow F$, and this is known to each player: for each $m, i = 1, 2, \ldots, n$,

$$\mathbf{K}_m[\mathbf{K}_i^1(F) \rightarrow F]. \tag{47}$$

- Knowledge does not disappear in time, and this is known to each player: for each $m, i = 1, 2, \ldots, n$,

$$\mathbf{K}_m[\mathbf{K}_i(F) \rightarrow \mathbf{K}_i^1(F)]. \tag{48}$$

- Players know all moves after the game, and this is known to each player: for each $l, m, i = 1, 2, \ldots, n$,

$$\mathbf{K}_m[s_i^{j_i} \rightarrow \mathbf{K}_l^1(s_i^{j_i})] \quad \text{and} \quad \mathbf{K}_m[(\neg s_i^{j_i}) \rightarrow \mathbf{K}_l^1(\neg s_i^{j_i})]. \tag{49}$$

Since (44) follows from the rules of the game, by Assumption 2,

$$\mathbf{K}_i\mathbf{K}_1[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \neg best\colon s_1^{j_1}]$$

and, by (48),

$$\mathbf{K}_i\mathbf{K}_1^1[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \neg best\colon s_1^{j_1}] \tag{50}$$

hence, by modal logic,

$$\mathbf{K}_i[\mathbf{K}_1^1(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \mathbf{K}_1^1(\neg best\colon s_1^{j_1})].$$

By (49) and modal logic,

$$\mathbf{K}_i[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \mathbf{K}_1^1(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})],$$

therefore

$$\mathbf{K}_i[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \mathbf{K}_1^1(\neg best\colon s_1^{j_1})].$$

By $\mathcal{RA}$, player $i$ knows that 1 is rational, $\mathbf{K}_i(r_1)$, hence

$$\mathbf{K}_i\{(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ [r_1 \wedge \mathbf{K}_1^1(\neg best\colon s_1^{j_1})]\}.$$

By an appropriate time-stamped version of Corollary 2 and Assumption 2,

$$\mathbf{K}_i\{[r_1 \wedge \mathbf{K}_1^1(\neg best\colon s_1^{j_1})] \ \rightarrow \ \mathbf{K}_1^1(\neg s_1^{j_1})\}$$

hence

$$\mathbf{K}_i[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \mathbf{K}_1^1(\neg s_1^{j_1})],$$

and, by (47),

$$\mathbf{K}_i[(s_2^{j_2} \wedge \ldots \wedge s_n^{j_n}) \ \rightarrow \ \neg s_1^{j_1}].$$

By modal logic, this implies

$$\mathbf{K}_i[\neg(s_1^{j_1} \wedge s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})],$$

which completes the alternative proof of Lemma 1.

□

To complete the proof of Theorem 3, we notice that $\mathbf{K}_i[\neg(s_1^{j_1} \wedge s_2^{j_2} \wedge \ldots \wedge s_n^{j_n})]$ for each $i = 1, 2, \ldots, n$ and for each outcome $j_1, j_2, \ldots, j_n$ which is not a Nash equilibrium. Since each player knows *a priori* that one of the outcomes should occur, each player is left with the knowledge that everyone had chosen his Nash equilibrium strategy: for each $i = 1, 2, \ldots, n$,

$$\mathbf{K}_i^0(e_1 \wedge e_2 \wedge \ldots \wedge e_n).$$

If the game contains several Nash equilibria, each player knows the disjunction of outcomes corresponding to Nash equilibria; this is as far as rationality alone can carry us.

If the game does not have Nash equilibrium in pure strategies, then mutual knowledge of rationality rules out all possible outcomes as not rational for at least one of the players. Here, rationality as a condition on pure strategies leads to an inconsistent set of assumptions. □

The proof of Theorem 3 provides a non-circular epistemic justification of Nash equilibrium in strategic-form games. It may be regarded as an argument against the well-known opinion that justification of Nash equilibrium strategies in strategic-form games requires circular-style reasoning.

Aumann & Brandenburger's 1995 paper [3] established that mutual knowledge of the game and rationality, along with some other epistemic assumptions, is sufficient for Nash equilibrium for mixed strategies; their proof is set-theoretical and uses partitions of state space, which is usually associated with S5-style epistemic logic in the background with positive and negative introspection principles required.

# 6 Epistemic analysis of Backward Induction

Figure 5 illustrates the centipede game suggested by Rosenthal, 1982, [15] and studied in an epistemic context by Aumann, 1995 [2]. Player $A$ makes moves at nodes 1, 3, and 5, player $B$ at nodes 2 and 4. Each player has the option of moving across or down, with indicated payoffs

$$\begin{matrix} m \\ n \end{matrix}$$

where the first component, $m$, is $A$'s payoff, and the second component, $n$, is the payoff for $B$. The game starts at node 1.

The classic backward induction solution (BI) predicts playing down at each node. Indeed, at node 5, player $A$'s rational choice is down. Player $B$ is certainly aware of this

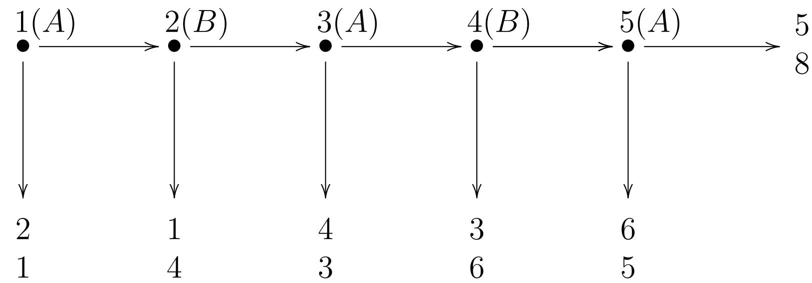


Figure 5: Centipede game

and, anticipating $A$'s rationally playing down at 5, would himself play down at 4. Player $A$ understands this too, and would opt down at 3 seeking a better payoff, etc. The backward induction solution is the unique Nash equilibrium of this game.

The question we try to address now is what are the epistemic conditions under which rational players $A$ and $B$ know that they have to play down at each node. As usual, we routinely assume mutual knowledge of game awareness and concentrate on tracking knowledge of rationality. This is a well-known issue (cf. [2, 4, 5, 19]) and classical analysis states that it takes common knowledge of players' rationality (or, at least, as many levels of knowledge as there are moves in the game) to justify BI.

In this chapter, we will try to revise this perception: we argue that the minimal assumptions of mutual (first-level) knowledge of game awareness and rationality suffices for intelligent players to justify backward induction in the centipede game in Figure 5, as well as in any finite extensive-form game with perfect information.

## 6.1 Solution of the centipede game

We will devise a derivation in which the tower of knowledge operators does not pile up.

**Theorem 4** *In the centipede game, first-level mutual knowledge of game awareness and rationality implies backward induction. Knowledge properties required for players do not exceed principles adopted for the logic of knowledge* $\mathsf{T}$.

**Proof.** Let us first consider an informal solution. Imagine an intelligent player $P \in \{A, B\}$ who is perfectly capable of calculating the backward induction solution BI and payoff function at each node. Then $P$ proves a small theorem:

$$\textit{Rational players play BI.} \qquad (51)$$

Indeed, $P$ supposes otherwise, assumes that both $A$ and $B$ are rational, but that someone would not play according to BI. $P$ considers the latest node $l$ at which some player, $X$, would deviate from BI. This cannot be node 5, since BI is the only rational choice at 5. Moreover, $l$ cannot be any other node either. Indeed, by (9), $X$ could deviate from BI at $l$ only when $X$ **knows** a better strategy than BI, hence $X$ **knows** that at some node after $l$, some of the players would deviate from BI and therefore will not be rational. In any case, $P$ figures that $X$ knows that some of the players are not rational, $\mathbf{K}_X(\neg rA)$ or $\mathbf{K}_X(\neg rB)$. From this, $P$ concludes $\neg rA$ or $\neg rB$, which is impossible, since $P$ assumes that both $A$ and $B$ are rational.

Note that proving this theorem requires certain intellectual powers on the part of the players, as well as being based on some generic properties of rationality and knowledge, but it does not assume more than first-level knowledge of rationality. On the basis of (51), $A$ concludes that he should play down at 1.

Let us support this explanation by a formal logical derivation. We will be formalizing the game in the basic logic of knowledge $\mathsf{T}_2^r$ for two knowers $A$ and $B$. As before, $\mathbf{K}_A$, $\mathbf{K}_B$ will denote knowledge modalities, and $rA$, $rB$ rationality assertions for players $A$ and $B$ respectively.

We assume the mutual knowledge of game awareness, hence the situation is within the scope of principles discussed in Section 2.

The Rationality Assumptions set here is

$$\mathcal{RA} = \{rA,\ rB,\ \mathbf{K}_A(rB),\ \mathbf{K}_B(rA)\}. \tag{52}$$

As before, the Game Description $\mathcal{GD}$ will be filled in step-by-step during the reasoning process.

For each $i = 1, 2, 3, 4, 5$, let propositions $a_i$ and $d_i$ be

$a_i$ - *the corresponding player would play 'across' at node i if a game starts at i*,

$d_i$ - *the corresponding player would play 'down' at node i if a game starts at i.*

At node 1, $A$ **knows** that playing down would yield payoff 2, and playing across would yield payoff 1 unless $B$ plays across at node 2. If $A$ is rational, then he would consider playing across at 1 only if $A$ **knows** that $B$ would play across at 2. Hence the following proposition follows from the game description and general understanding of rationality and, by Awareness Principle (8) and Corollary 3, is added to $\mathcal{GD}$:

$$rA \ \rightarrow\ [a_1 \rightarrow \mathbf{K}_A(a_2)]. \tag{53}$$

This principle states that in a perfect information environment, if a rational player chooses a strategy, then he knows that it yields a better payoff than all other strategies.

From the factivity of $\mathbf{K}_A$, it follows that

$$rA \ \rightarrow\ (a_1 \rightarrow a_2).$$

Likewise, if $B$ is rational, then he would play across at 2 only if he **knows** that $A$ would play across at 3, since $B$ otherwise could not possibly know a better strategy at 2 than playing down. Hence, by Awareness Principle (8) and Corollary 3, the following is added to $\mathcal{GD}$:

$$rB \;\rightarrow\; [a_2 \rightarrow \mathbf{K}_B(a_3)] \tag{54}$$

and, by factivity of $\mathbf{K}_B$,

$$rB \;\rightarrow\; (a_2 \rightarrow a_3).$$

Repeating the same argument, we place on $\mathcal{GD}$ the following propositions, each of which follows logically from the game rules and general understanding of rationality and knowledge:

$$rA \;\rightarrow\; [a_3 \rightarrow \mathbf{K}_A(a_4)], \tag{55}$$

$$rB \;\rightarrow\; [a_4 \rightarrow \mathbf{K}_B(a_5)], \tag{56}$$

from which we derive

$$rA \;\rightarrow\; (a_3 \rightarrow a_4) \quad \text{and} \quad rB \;\rightarrow\; (a_4 \rightarrow a_5).$$

As follows from game description, at terminal node 5, player $A$, if rational, is not going to play '*across*' since he knows that he would receive a higher payoff by choosing '*down*.' By Assumption 2, we add the following to $\mathcal{GD}$:

$$rA \;\rightarrow\; \neg a_5. \tag{57}$$

In Boolean logic, from (53–57) we immediately derive

$$rA \wedge rB \;\rightarrow\; \neg a_i \quad \text{for all } i\text{=1–5}.$$

From the rules of the game, it follows that

$$\neg a_i \;\rightarrow\; d_i, \tag{58}$$

which we also add to $\mathcal{GD}$ and conclude that

$$rA \wedge rB \;\rightarrow\; d_i \quad \text{for all } i\text{=1–5}.$$

By Assumption 2(ii), all of (53–58) and their aforementioned logical consequences are known to both players. In particular, we add to $\mathcal{GD}$

$$\mathbf{K}_{A,B}(rA \wedge rB \;\rightarrow\; d_i) \quad \text{for all } i\text{=1–5}. \tag{59}$$

By epistemic modal logic,

$$\mathbf{K}_{A,B}(rA \wedge rB) \;\rightarrow\; \mathbf{K}_{A,B}(d_i) \quad \text{for all } i\text{=1–5}.$$

Since, by $\mathcal{RA}$, $\mathbf{K}_{A,B}(rA \wedge rB)$, we derive

$$\mathbf{K}_{A,B}(d_i) \quad \text{for all } i\text{=1–5},$$

which means that both players know that they should play backward induction strategy at each node. □

## 6.2 Other PI games

It is quite clear how to generalize this proof to all extensive-form games with perfect information.

**Theorem 5** *In extensive-form games with perfect information, first-level mutual knowledge of rationality and game awareness for intelligent players implies backward induction. Knowledge properties required for players do not exceed principles adopted for the logic of knowledge* $\mathsf{T}$.

**Proof.** (Sketch). Similar to the proof of the centipede game from the previous section. Informally, each of the intelligent players, say $P$, calculates the backward induction solution BI and proves an unconditional lemma that

$$\textit{Rational players play BI at each node.} \qquad (60)$$

To establish this, $P$ assumes the opposite and considers the latest node $l$ where some player, call him $X$, would deviate from BI. $P$ knows that $l$ is not the last node, since $P$ determines that at terminal nodes, rational players would play BI. Moreover, $l$ cannot be any other node either since, by Awareness Principle (8) and Corollary 3, $X$ could deviate from BI at $l$ only if $X$ knows a better strategy than BI, hence $X$ knows that at some node after $l$, some of the players would deviate from BI and hence will not be rational. In any case, $P$ figures that $X$ knows that some of the players are not rational. From this, $P$ concludes that some of the players are indeed not rational, which is impossible since $P$ assumes rationality of all players.

Using this lemma and $P$'s knowledge of rationality of all players, $P$ concludes that each player plays BI. □

**Corollary 7** *No common knowledge of the game and players' rationality is needed to justify backward induction. Moreover, the number of levels of mutual knowledge the players needed here does not depend upon the length of the game and is equal to 1.*

# 7 Discussion

## 7.1 Mutual knowledge vs. common knowledge of rationailty

Aumann's 1995 paper [2] uses a set-theoretical model and establishes that "in PI games, common knowledge of rationality implies backward induction." Common knowledge of rationality (or its finite-nesting versions) have been widely adopted as an epistemic condition for backward induction in PI games ([2, 4, 19]). Theorem 5 relaxes this restriction considerably from common knowledge of rationality to first-level mutual knowledge of rationality. As noted in [2], common knowledge of rationality is an idealized condition that is rarely met in practice. Mutual first-level knowledge of rationality is easier to imagine. In some situations, it is plausible to assume that other players are intelligent rational agents who are aware of the game rules. This looks to be a rather generic set of assumptions compared to common knowledge of rationality, which presupposes players' knowledge about all players' rationality, all players' knowledge of this knowledge, all players' knowledge about all players' knowledge about this knowledge, etc.

It could be feasible to verify players' intelligence and mutual knowledge of game awareness, i.e., with some sort of certification from a trusted source. Verifying knowledge of rationality can be more problematic which is why, in this paper, we concentrated on tracking knowledge of rationality and easing the assumption burden to a bare-minimum, first-level mutual knowledge of rationality.

## 7.2 Required properties of knowledge

We also suggest easing requirements on knowledge principles. All of the epistemic reasoning here can be performed at the level of the most general modal logic of belief $\mathsf{K}$ or logic of knowledge $\mathsf{T}$, assuming neither positive nor negative introspection, as well as in any other normal modal logic of knowledge, e.g., $\mathsf{S4}$, $\mathsf{S5}$, etc.

## 7.3 What do we actually assume?

We offer a specific, logic-based approach. In our model, we try to accommodate the intellectual powers of players who are considered not to be mere finite-automata payoff maximizers but rather intellectual agents capable of analyzing the game and calculating payoffs conditioned to the rational behavior of all players. However, we believe that such assumptions about intellectual powers of players are within the realm of both: epistemic and game-theoretical reasoning.

## 7.4 Comparing new and old solutions

It might appear that we are offering a trade-off of the complexity of assumptions (namely, an unbounded number of levels of mutual knowledge of rationality) for the complexity of reasoning. However, this is not the case. Our derivation of backward induction in the centipede game of Section 6.1 is easier, since it does not require nested knowledge considerations.

Another reasonable approach for monitoring the logical complexity of arguments in epistemic reasoning has been offered by Justification Logic (cf. [1]), which tracks the size and structure of evidence in epistemic derivations. Measuring the complexity of the arguments and justifications may prove to be relevant since according to some epistemic studies, trust in logical reasoning fades when the argument becomes too complex.

### 7.5 Further plans

Here are some possible avenues of research for logic-based game-theoretical models.

- Epistemic analysis of specific games.
- Incorporating other Nash equilibria, including mixed strategies.
- Capturing the process of acquiring knowledge during games.
- Developing a mechanism of justification-tracking in game-theoretical reasoning. Introducing tools to control logical omniscience hidden in the Intelligence Theses.
- Modeling dependence of rationality on knowledge. Analyzing how knowledge can help to win games.
- Incorporating other epistemic notions into the model.

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