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Abstract

We propose new effective randomized algorithms for some fundamental matrix computations such as preconditioning of an ill conditioned matrix that has a small numerical nullity or rank, its 2-by-2 block triangulation, numerical stabilization of Gaussian elimination with no pivoting, and approximation of a matrix by low-rank matrices and by structured matrices. Our technical advances include estimating the condition number of a random Toeplitz matrix, novel techniques of randomized preprocessing, a proof of their preconditioning power, and a dual version of the Sherman–Morrison–Woodbury formula. According to both our formal study and numerical tests we significantly accelerate the known algorithms and improve their output accuracy.

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1 Introduction

It is well known that random matrices tend to be well conditioned [D88], [E88], [ES05], [CD05], [SST06], [B11], and our new techniques employ this property to advance some fundamental matrix computations. Our analysis and experiments show substantial progress versus the known algorithms. Our estimates for the condition number of a random Toeplitz matrix seem to be surprising.

1.1 Numerically safe Gaussian elimination with no pivoting

Hereafter "expected" and "likely" mean "with probability 1 or close to 1", $\sigma_j(A)$ denotes the jth largest singular value of an $n \times n$ matrix A, and $\kappa(A) = \sigma_1(A)/\sigma_\rho(A)$ for $\rho = \text{rank}(A)$ denotes its condition number. If this number is large (in context), then the matrix A is ill conditioned,

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that is lies near a rank deficient matrix. Numerical solution of a linear system of equations with such a coefficient matrix can be easily corrupted by rounding errors. Unless $\kappa(A)$ is large, the matrix A is well conditioned. It can be safely treated numerically with the IEEE standard double precision provided it also has full rank and no ill conditioned auxiliary matrices are involved in the computations.

Pivoting, that is row or column interchange is incorporated into Gaussian elimination to avoid dealing with such auxiliary matrices. *Gaussian elimination with no pivoting* (hereafter we refer to it as *GENP*) can easily fail in numerical computations with rounding errors, except for some special classes of input matrices such as diagonally dominant and positive definite matrices. For these matrix classes, GENP and its pivoting-free variations outperform Gaussian elimination with pivoting [GL96, page 119]. We dramatically expand these classes by proving in Corollary 4.2 that pre- as well as post- multiplication of a well conditioned coefficient matrix by a square Gaussian random matrix is expected to support safe numerical performance of GENP and block Gaussian elimination. Our tests have consistently supported GENP even where we used just circulant multipliers filled with ± 1 for random choice of the signs $\pm$ (see Table 8.6).

1.2 Randomized preconditioning and its applications

Given an ill conditioned matrix A, can we extend this advance by applying randomized multipliers X and Y to yield a much better conditioned matrix product XAY? No, because random square matrices X and Y are expected to be nonsingular and well conditioned [D88], [E88], [ES05], [CD05], [SST06], [B11] and because $\kappa(XAY) \geq \frac{\kappa(A)}{\kappa(X)\kappa(Y)}$. Approximate inverses of A are popular multipliers, but only for some important special classes of square matrices A for which computing approximate inverses is noncostly.

We can readily produce a well conditioned matrix C by applying additive preprocessing $A \Longrightarrow C = A + P$ (e.g., we can choose $P = I - A$), but it is not clear how this would help us to solve a linear system $A\mathbf{y} = \mathbf{b}$. (Here and hereafter I and I_n denote the $n \times n$ identity matrix.) Assume, however, that we are given an $n \times n$ nonsingular ill conditioned input matrix A together with a small upper bound r on its numerical nullity, that is on the number of its singular values that are much smaller than the 2-norm $||A||_2$. Such matrices make up a large and important subclass in the class of nonsingular ill conditioned matrices (cf. [CDG03] and Remarks 2.1 and 5.2), and for this subclass we yield randomized additive preconditioning based on the following theorem; we prove and extend it to the case of rectangular matrices A in Section 5.

Theorem 1.1. *Suppose A is a real $n \times n$ matrix, $||A||_2 \approx 1$, U and V are $n \times r$ standard Gaussian random matrices for $0 < r < n$, and $C = A + UV^T$. Then the condition number $\kappa(C)$ is expected to have order $\sigma_1(A)/\sigma_{n-r}(A)$, whereas $\kappa(A) = \sigma_1(A)/\sigma_n(A)$.*

It follows that the matrix C is expected to be nonsigular and well conditioned where an ill conditioned matrix A has numerical nullity at most r. This enables *randomized preconditioning* based on the Sherman–Morrison–Woodbury formula

$$A^{-1} = C^{-1} + C^{-1}UG^{-1}V^TC^{-1} \text{ for } G = I_r - V^TC^{-1}U, \tag{1.1}$$

hereafter referred to as the *SMW formula* [GL96, page 50], [S98, Corollary 4.3.2]. Indeed inversion of an ill conditioned matrix A requires a high precision p_+, but if the matrix C is well conditioned, as is expected, then the SMW formula only involves $O(rn^2)$ flops (arithmetic operations) in high precision. They are confined to the computations of and with the $r \times r$ matrix G and make up just a fraction of order r/n of all arithmetic operations involved in the customary algorithms for $n \times n$ matrix inversion. Moreover if the matrix C is well conditioned, we can compute the matrix $C^{-1}U$, as well as the vector $C^{-1}\mathbf{b}$, by applying iterative refinement at the overall computational cost of $O(n^2rp_+/p_{\text{double}})$ flops in double precision p_{double} versus order of n^3 flops in high precision p_+, required in Gaussian elimination. Thus our improvement is dramatic where n greatly exceeds rp_+/p_{double} and p_+ greatly exceed p_{double}. The smaller the available upper bound r on the numerical nullity of A, the smaller the size of the matrices G, U and V, and thus the simpler our computations. We supply some recipes for the computation of numerical rank and nullity in Section 7.1.

Now suppose we have computed the exact values of the numerical rank $q < n$ and numerical nullity $r = n - q > 0$. (We keep the same notation r as for the upper bound.) Then we have some nice alternatives to applying the SMW formula, e.g., we can apply our additive preprocessing $A \Longrightarrow C + UV^T$ to compute a 2×2 block triangulation of this matrix with two diagonal blocks of the sizes $r \times r$ and $q \times q$, respectively, expected to be better conditioned than the matrix A (see Section 7.4). Moreover based on these techniques we can approximate the *trailing singular space* of a matrix, associated with its r small singular values.

Our *dual SMW formula* and *dual additive preprocessing* enable us to extend our preconditioning techniques to any matrix A given with a small upper bound q on its numerical rank. Furthermore we readily approximate such matrices A by low-rank matrices; such approximations support many fundamental computations with matrices and tensors (cf. [T00], [MMD08], [OT09], [HMT11]). Our approach to low-rank approximation is distinct from the known methods.

In the case of $n \times n$ matrices A with Toeplitz structure we can choose scaled Gaussian random Toeplitz matrices U and V to accelerate the computation of the matrices C and C^{-1}. Under a special choice of the matrices U and V the preprocessing $A \rightarrow C = A + UV^T$ amounts to changing a fixed block of r rows of A into random general or structured block of r rows. More generally one can replace any selected entries of A by random variables, possibly restricted to preserving a fixed pattern of structure or sparseness of the matrix A. E.g., we can randomly modify the last r entries of the first row of a Toeplitz matrix and then substitute the resulting values into the respective $(r-1)r/2$ entries in the $r \times r$ northwestern corner to keep the Toeplitz structure of the matrix.

If we agree to increase the size of an input matrix, we can alternatively apply randomized augmentation, defined, e.g., by the map $A \Longrightarrow K = \begin{pmatrix} A & -U \\ V^T & W \end{pmatrix}$; here we can choose scaled Gaussian random matrices U, V and W endowed with fixed patterns of structure or sparseness. Augmentation and additive preprocessing, in particular block modifcation of a matrix, are closely linked to one another and have similar preconditioning power.

1.3 Condition numbers of random structured matrices and numerical experiments

Estimating the condition numbers of random structured matrices is a well known challenge [SST06]. We respond to it with estimating the condition numbers of Gaussian random Toeplitz matrices. In particular we prove that these numbers do not grow to the infinity exponentially in n for $n \times n$ random Toeplitz matrices, even though the opposite has been proved in [BG05] for large and important subclass of Toeplitz matrices. We readily obtain even stronger upper bounds on the condition numbers of random circulant matrices. We cannot extend our proof of Theorem 1.1 to the case of Gaussian random Toeplitz matrices U and V, but such an extension is consistent with the results of our extensive numerical tests, which otherwise are in good accordance with our theoretical estimates.

In particular by applyng randomized additive preprocessing we dramatically decreased the condition numbers of ill conditioned matrices having small numerical nullity (see Table 8.5), numerically stabilized GENP by applying random circulant multipliers (see Table 8.6), accurately solved ill conditioned linear systems of equations by applying block triangulation based on randomization (see Tables 8.7–8.10), approximated trailing and leading singular spaces of ill conditioned matrices that have small numerical nullity or rank, and approximated a matrix by a low-rank matrix (see Tables 8.11–8.14). Furthermore we matched the output accuracy of the customary algorithms but outperformed them in terms of the CPU time for solving ill conditioned Toeplitz linear systems of equations (see Table 8.15). Finally our test results in Tables 8.1–8.4 display the condition numbers of random $n \times n$ Toeplitz and circulant matrices, supporting our formal study in Sections 3.4 and 3.5.

1.4 Organization of the paper and selective reading

We devote the next section to the definitions and basic results on matrix computations. We estimate the condition numbers of Gaussian random general, Toeplitz and circulant matrices in Section 3 and of randomized matrix products in Section 4; the latter results support GENP with randomized

multipliers. In Section 5 we prove that our randomized additive preprocessing of an ill conditioned matrix is expected to produce a well conditioned matrix. In Section 6 we study randomized augmentation, link it to randomized additive preprocessing, and solve ill conditioned Toeplitz linear systems of equations based on this study. In Section 7 we propose a number of applications of randomized preprocessing to an ill conditioned matrix having small numerical rank or nullity: we compute numerical rank and nullity of such a matrix, approximate its trailing and leading singular spaces, and extend this computation to the approximation by low-rank matrices and by structured matrices. We also cover randomized additive preconditioning, block triangulation, and structured preprocessing. Corollaries 4.1, 5.1 and 5.2 and the results of Section 3.4 and 6.1 show the power of these algorithms, but otherwise Section 7 can be read independently of Sections 3–6. In Section 8 we cover numerical tests, which constitute the contribution of the second and the third authors. In Section 9 we comment on the related works, our technical novelties, and some directions for further study. In Appendix A we estimate probability of deficiency of random matrices defined under the uniform probability distribution. In Appendix B we extend our probabilistic estimates to the case of complex matrices. In the paper, unlike its introduction, we cover the general case of rectangular input matrices A.

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2 Some definitions and basic results

We assume computations in the field $\mathbb{R}$ of real numbers, and comment on the extension to the field $\mathbb{C}$ of complex numbers in Section B.

For two scalars a and b we write $a \ll b$ and $b \gg a$ if the ratio $|b/a|$ is large; we write $a \approx b$ if $|a - b| \ll |a| + |b|$.

Hereafter "flop" stands for "arithmetic operation"; "expected" and "likely" mean "with probability 1 or close to 1"; the concepts "large", "small", "near", "closely approximate", "ill conditioned" and "well conditioned" are quantified in the context.

We use and extend the customary definitions of matrix computations (cf. [GL96], [S98]).

2.1 Some basic definitions on matrix computations

$(B_1 \mid \ldots \mid B_k) = (B_j)_{j=1}^k$ is a $1 \times k$ block matrix with blocks $B_1, \ldots, B_k$.

$\mathrm{diag}(B_1, \ldots, B_k) = \mathrm{diag}(B_j)_{j=1}^k$ is a $k \times k$ block diagonal matrix with diagonal blocks $B_1, \ldots, B_k$.

I_n is the $n \times n$ identity matrix $(\mathbf{e}_1 \mid \ldots \mid \mathbf{e}_n)$. J_n is the $n \times n$ reflection matrix $(\mathbf{e}_n \mid \ldots \mid \mathbf{e}_1)$. $O_{k,l}$ is the $k \times l$ matrix filled with zeros. $\mathbf{0}_k$ is the vector $O_{k,1}$. We write I, J, O, and $\mathbf{0}$ where the size of a matrix or a vector is not important or is defined by context.

A^T is the transpose of a matrix A. A real matrix A is symmetric if $A = A^T$ and is symmetric positive definite if $A = B^T B$ for a nonsingular matrix B.

A matrix U is called *unitary*, *orthogonal* and *orthonormal* if $U^T U = I$ or $U U^T = I$.

Fact 2.1. *[GL96, Theorem 5.2.2]. QR factorization $A = QR$ of a matrix A having full column rank into the product of a unitary matrix $Q = Q(A)$ and an upper triangular matrix $R = R(A)$ is unique provided that the factor R is a square matrix with positive diagonal entries.*

2.2 Range, null space, rank, nullity, and nmbs

$\mathcal{R}(A)$ denotes the range of an $m \times n$ matrix A, that is the linear space $\{\mathbf{z} : \mathbf{z} = A\mathbf{x}\}$ generated by its columns. $\mathcal{N}(A)$ denotes its null space $\{\mathbf{v} : A\mathbf{v} = \mathbf{0}\}$, $\mathrm{rank}(A) = \dim \mathcal{R}(A)$ its rank, and $\mathrm{nul}(A) = \dim \mathcal{N}(A) = n - \rho$ its nullity. This is actually the right nullity of A, whereas $\mathrm{nul}\, A^T = m - \rho$ is the left nullity of A, equal to $\mathrm{nul}\, A$ if and only if $m = n$. $\mathbf{v}$ is the null vector of A if $A\mathbf{v} = \mathbf{0}$.

Fact 2.2. *The set $\mathbb{M}$ of $m \times n$ matrices of rank ρ is an algebraic variety of dimension $(m + n - \rho)\rho$.*

Proof. Let M be an $m \times n$ matrix of a rank ρ with a nonsingular $\rho \times \rho$ leading block M_{00} and write $M = \begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}$. Then the $(m-\rho) \times (n-\rho)$ *Schur complement* $M_{11} - M_{10}M_{00}^{-1}M_{01}$ must vanish, which imposes $(m-\rho)(n-\rho)$ algebraic equations on the entries of M. Similar argument can be applied where any $\rho \times \rho$ submatrix of the matrix M (among $\binom{m}{\rho}\binom{n}{\rho}$ such submatrices) is nonsingular. Therefore $\dim \mathbb{M} = mn - (m-\rho)(n-\rho) = (m+n-\rho)\rho$. □

If a matrix A has full column rank, we call it a *matrix basis* for the linear space $\mathcal{R}(A)$. Suppose a matrix B has full column rank and $\mathcal{R}(B) = \mathcal{N}(A)$. Then we call B a *null matrix basis* or a *nmb* for a matrix A and write $B = \text{nmb}(A)$. $\mathcal{N}(A^T)$ is the left null space of a matrix A, and similarly the map $A \to A^T$ defines left null vectors, left nmbs, and the left nullity of a matrix A; in particular an $m \times n$ matrix of a rank ρ has the left nullity $m - \rho$.

$A^{(k)}$ denotes the $k \times k$ leading, that is northwestern block submatrix of a matrix A.

A matrix of a rank ρ has *generic rank profile* if all its $i \times i$ leading blocks are nonsingular for $i = 1, \dots, \rho$. If such a matrix is nonsingular itself, then it is called *strongly nonsingular*.

2.3 Norms and SVD

$||A||_h$ is the h-norm of a matrix $A = (a_{i,j})_{i,j=1}^{m,n}$. We write $||A|| = ||A||_2$ and $||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}} = ||\mathbf{v}||_2$ and recall from [GL96, Section 2.3.2 and Corollary 2.3.2] that

$$\max_{i,j=1}^{m,n}|a_{i,j}| \le ||A|| = ||A^T|| \le \sqrt{mn}\, \max_{i,j=1}^{m,n}|a_{i,j}|,$$

$$\frac{1}{\sqrt{m}}||A||_1 \le ||A|| \le \sqrt{n}||A||_1, \quad ||A||_1 = ||A^T||_\infty, \quad ||A||^2 \le ||A||_1||A||_\infty, \tag{2.1}$$

$$||AB||_h \le ||A||_h||B||_h \text{ for } h = 1, 2, \infty \text{ and any matrix product } AB. \tag{2.2}$$

We write $A \approx B$ if $||A - B|| \ll ||A|| + ||B||$.

A matrix A is *normalized* if $||A|| = 1$. A normalized vector is unitary, and we call it *unit*.

$A = S_A\Sigma_A T_A^T$ is an *SVD* or *full SVD* of an $m \times n$ matrix A of a rank ρ provided $S_AS_A^T = S_A^TS_A = I_m$, $T_AT_A^T = T_A^TT_A = I_n$, $\Sigma_A = \text{diag}(\widehat{\Sigma}_A, O_{m-\rho,n-\rho})$, $\widehat{\Sigma}_A = \text{diag}(\sigma_j(A))_{j=1}^{\rho}$, $\sigma_j = \sigma_j(A) = \sigma_j(A^T)$ is the jth largest singular value of a matrix A. These values have the minimax property

$$\sigma_j = \max_{\dim(\mathbb{S})=j} \min_{\mathbf{x}\in\mathbb{S},\ ||\mathbf{x}||=1} ||A\mathbf{x}||,\ j = 1, \dots, \rho, \tag{2.3}$$

where $l = \min\{m, n\}$ and $\mathbb{S}$ denotes linear spaces [GL96, Theorem 8.6.1]. Consequently $\sigma_1 = \max_{||\mathbf{x}||=1} ||A\mathbf{x}|| = ||A||$, $\sigma_j = 0$ for $j > \rho$, $\sigma_j(A)$ is the 2-norm of the difference between the matrix A and a nearest matrix of rank $j-1$ for $j = 1, \dots, \rho+1$, and the latter property implies

Fact 2.3. *If A_0 is a submatrix of a matrix A, then $\sigma_j(A) \ge \sigma_j(A_0)$ for all j.*

For every integer k in the range $1 \le k < \rho$ we define the partition $S_A = (S_{k,A} \mid S_{A,m-k})$ and $T_A = (T_{k,A} \mid T_{A,n-k})$ where the submatrices $S_{k,A}$ and $T_{k,A}$ are formed by the first k columns of the matrices S_A and T_A, respectively. We write $\mathbb{S}_{k,A} = \mathcal{R}(S_{k,A})$ and $\mathbb{T}_{k,A} = \mathcal{R}(T_{k,A})$. If $\sigma_k > \sigma_{k+1}$, then $\mathbb{S}_{k,A}$ and $\mathbb{T}_{k,A}$ are the left and right leading singular spaces, respectively, associated with the k largest singular values of the matrix A, whereas the orthogonal complements $\mathbb{S}_{A,m-k} = \mathcal{R}(S_{A,m-k})$ and $\mathbb{T}_{A,n-k} = \mathcal{R}(T_{A,n-k})$ are the left and right trailing singular spaces, respectively, associated with the other singular values of A. The pairs of subscripts $\{k, A\}$ versus $\{A, m-k\}$ and $\{A, n-k\}$ mark the leading versus trailing singular spaces. The left singular spaces of A are the right singular spaces of A^T and vice versa.

2.4 Inverses, generalized inverses, SMW and dual SMW formulae

$A^+ = T_A \operatorname{diag}(\widehat{\Sigma}_A^{-1}, O_{n-\rho,m-\rho}) S_A^T$ is the Moore–Penrose pseudo-inverse of the matrix A. We have

$$||A^+|| = 1/\sigma_\rho(A) \tag{2.4}$$

for a matrix A of a rank ρ. A^{+T} stands for $(A^+)^T = (A^T)^+$, and A^{-T} stands for $(A^{-1})^T = (A^T)^{-1}$.

An $n \times m$ matrix $X = A^{(I)}$ is a left inverse of an $m \times n$ matrix A if $XA = I$ and its right inverse if $AX = I$; A^+ is an $A^{(I)}$ if and only if a matrix A has full rank. $A^{(I)}$ is unique and is equal to A^{-1} if A is a nonsingular matrix.

Theorem 2.1. *Suppose C is a nonsingular matrix, E is a matrix of the same size, and $||C^{-1}E|| = \theta < 1$. Then $||I - (C+E)^{-1}C|| \le \frac{\theta}{1-\theta}$ and $|||(C+E)^{-1} - C^{-1}|| \le \frac{\theta}{1-\theta}||C^{-1}||$.*

Proof. See [S98, Corollary 1.4.19] for $P = -C^{-1}E$. □

Theorem 2.2. *[GL96, Section 5.5.5]. Assume two matrices $C \in \mathbb{C}^{m\times n}$ and $\tilde{C} \in \mathbb{C}^{m\times n}$ having full rank and write $E = \tilde{C} - C$. Then $||\tilde{C}^+ - C^+|| \le 2\sqrt{n}||E|| \max\{||\tilde{C}^+||^2, ||C^+||^2\}$.*

Next suppose that $U_-, V_- \in \mathbb{R}^{n\times q}$, $A \in \mathbb{R}^{n\times n}$, A is a nonsingular matrix, and $0 < q < n$; write

$$C_-^{-1} = A^{-1} + U_- V_-^T,\ H = I_q + V_-^T A U_-, \tag{2.5}$$

assume that one of these two matrices is nonsingular and deduce that both of them are nonsingular. Finally apply the SMW formula (1.1) to the matrix C_-^{-1} and obtain the *dual SMW formula*

$$C_- = A - AU_- H^{-1} V_-^T A \text{ for } H = I_q + V_-^T A U_-. \tag{2.6}$$

2.5 Condition number, numerical rank and numerical nullity

$\kappa(A) = \frac{\sigma_1(A)}{\sigma_\rho(A)} = ||A||\ ||A^+||$ is the condition number of an $m \times n$ matrix A of a rank ρ. Such a matrix is *ill conditioned* if $\sigma_1(A) \gg \sigma_\rho(A)$ and is *well conditioned* otherwise. See [D83], [GL96, Sections 2.3.2, 2.3.3, 3.5.4, 12.5], [H02, Chapter 15], [KL94], and [S98, Section 5.3] on the estimation of norms and condition numbers. $\kappa(A) = ||A||\ ||A^{-1}||$ for a nonsingular matrix A. An $m \times n$ matrix A has *numerical rank* q, not exceeding $\operatorname{rank}(A)$, and has *numerical nullity* $r = n - q$ if the ratios $\sigma_j(A)/||A||$ are small for $j > q$ but not for $j \le q$. Actually this is the right numerical nullity of the matrix A, whereas its *left numerical nullity* equals the numerical nullity $m-q$ of the $n \times m$ transpose A^T and coincides with the right numerical nullity of A if and only if $m = n$.

Remark 2.1. *Formally one can specify the adjective "small" above as "smaller than a fixed positive tolerance"; the choice of the tolerance can be a challenge, e.g., for the matrix* $\operatorname{diag}(1.1^{-j})_{j=0}^{999}$.

If an $m \times n$ well conditioned matrix A has a rank $\rho < l = \min\{m, n\}$, then most of its close neighbours have full rank l, but all of them have numerical rank $q = \rho$. Conversely, suppose a matrix $\tilde{A}$ has a positive numerical rank q, set to 0 its all but q largest singular values and denote the resulting matrix by A. Then $A \approx \tilde{A}$, $\operatorname{rank} A = q$, A is a well conditioned matrix, $\mathcal{R}(A) = \mathbb{T}_{q,\tilde{A}}$ and $\mathcal{N}(A) = \mathbb{T}_{\tilde{A},n-q}$.

A matrix has *generic conditioning profile* (cf. the end of Section 2.2) if it has a numerical rank q and if its $i \times i$ leading blocks are nonsingular and well conditioned for $i = 1, \dots, q$. If such a matrix has full rank and is well conditioned itself, then we call it *strongly well conditioned*.

Theorem 2.3. *[PQZa]. Suppose Gaussian elimination with no pivoting or block Gaussian elimination has been applied to a matrix A of a rank ρ (resp. numerical rank q) to compute LU factorizations of the leading block submatrices $A^{(j)}$ for $j = 1, \dots, \rho$ (resp. for $j = 1, \dots, q$). Then the computations involve no divisions by 0 (resp. by values that are absolutely small relatively to the norm $||A||$) if and only if the matrix A has generic rank (resp. generic conditioning) profile.*

2.6 Toeplitz, Toeplitz-like, and circulant matrices

$m \times n$ Toeplitz matrix $T_{m,n} = (t_{i-j})_{i,j=1}^{m,n}$ is defined by its first row and column, that is by the vector $(t_h)_{h=1-n}^{m-1}$ of dimension $m+n-1$. We write $T_n = T_{n,n} = (t_{i-j})_{i,j=1}^{n,n}$ (see the display below).

An $n \times n$ lower triangular Toeplitz matrix $Z(\mathbf{t}) = (t_{i-j})_{i,j=1}^{n}$ (where $t_k = 0$ for $k < 0$) is defined by its first column $\mathbf{t} = T\mathbf{e}_1 = (t_h)_{h=0}^{n-1}$. We write $Z(\mathbf{t})^T = (Z(\mathbf{t}))^T$. $Z = Z(\mathbf{e}_2)$ is the $n \times n$ downshift matrix displayed below. We have $Z\mathbf{v} = (v_i)_{i=0}^{n-1}$ and $Z(\mathbf{v}) = \sum_{i=1}^{n} v_i Z^{i-1}$ for $\mathbf{v} = (v_i)_{i=1}^{n}$ and $v_0 = 0$,

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & & \cdots & & 0 \\ 1 & \ddots & & & \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \\ 0 & & \cdots & 1 & 0 \end{pmatrix}.$$

Combine the equations $||Z(\mathbf{v})||_1 = ||Z(\mathbf{v})||_\infty = ||\mathbf{v}||_1$ with (2.1) to obtain

$$||Z(\mathbf{v})|| \le ||\mathbf{v}||_1. \tag{2.7}$$

Theorem 2.4. *Write $T_k = (t_{i-j})_{i,j=0}^{k-1}$ for $k = n, n+1$. (a) Let the matrix T_n be nonsingular and write $\mathbf{p} = T_n^{-1}\mathbf{e}_1$ and $\mathbf{q} = T_n^{-1}\mathbf{e}_n$. If $p_1 = \mathbf{e}_1^T\mathbf{p} \neq 0$, then $p_1 T_n^{-1} = Z(\mathbf{p})Z(J\mathbf{q})^T - Z(Z\mathbf{q})Z(ZJ\mathbf{p})^T$. In parts (b) and (c) below let the matrix T_{n+1} be nonsingular and write $\widehat{\mathbf{v}} = (v_i)_{i=0}^{n} = T_{n+1}^{-1}\mathbf{e}_1$, $\mathbf{v} = (v_i)_{i=0}^{n-1}$, $\mathbf{v}' = (v_i)_{i=1}^{n}$, $\widehat{\mathbf{w}} = (w_i)_{i=0}^{n} = T_{n+1}^{-1}\mathbf{e}_{n+1}$, $\mathbf{w} = (w_i)_{i=0}^{n-1}$, and $\mathbf{w}' = (w_i)_{i=1}^{n}$. (b) If $v_0 \neq 0$, then the matrix T_n is nonsingular and $v_0 T_n^{-1} = Z(\mathbf{v})Z(J\mathbf{w}')^T - Z(\mathbf{w})Z(J\mathbf{v}')^T$. (c) If $v_n \neq 0$, then the matrix $T_{1,0} = (t_{i-j})_{i=1,j=0}^{n,n-1}$ is nonsingular and $v_n T_{1,0}^{-1} = Z(\mathbf{w})Z(J\mathbf{v}')^T - Z(\mathbf{v})Z(J\mathbf{w}')^T$.*

Proof. See [GS72] on parts (a) and (b); see [GK72] on part (c). □

$n \times n$ structured matrices having a small *displacement rank* d extend the class of Toeplitz and Hankel matrices (for which $d \le 2$); they can be represented by displacement generators of length d defined by $2dn$ parameters each. Such matrices can be pairwise multiplied in $O(d^2 n \log n)$ flops and, if nonsingular, inverted in $O(d^2 n \log^2 n)$ flops each, where every output has displacement rank at most $2d$ and is represented with at most $4dn$ parameters (cf. [P01]).

$Z_f = Z + f\mathbf{e}_1^T\mathbf{e}_n$ is the f-circulant shift matrix for a scalar $f \neq 0$. An *f-circulant matrix* $Z_f(\mathbf{v}) = \sum_{i=1}^{n} v_i Z_f^{i-1}$ is a special $n \times n$ Toeplitz matrix defined by its first column vector $\mathbf{v} = (v_i)_{i=1}^{n}$ and a scalar f. f-circulant matrix is called *circulant* if $f = 1$ and *skew circulant* if $f = -1$. By replacing f with 0 we arrive at a lower triangular Toeplitz matrix $Z(\mathbf{v})$. The following theorem implies that the products and inverses of f-circulant matrices (wherever defined) are f-circulant and can be computed in $O(n \log n)$ flops for $n \times n$ inputs.

Theorem 2.5. *(See [CPW74].) We have $Z_1(\mathbf{v}) = \Omega^{-1} D(\Omega\mathbf{v})\Omega$. More generally, for any $f \neq 0$, we have $Z_{f^n}(\mathbf{v}) = U_f^{-1} D(U_f\mathbf{v})U_f$ where $U_f = \Omega D(\mathbf{f})$, $\mathbf{f} = (f^i)_{i=0}^{n-1}$, $D(\mathbf{u}) = \mathrm{diag}(u_i)_{i=0}^{n-1}$ for a vector $\mathbf{u} = (u_i)_{i=0}^{n-1}$, and $\Omega = (\omega_n^{ij})_{i,j=0}^{n-1}$ is the $n \times n$ matrix of the discrete Fourier transform at n points, $\omega_n = \exp(\frac{2\pi}{n}\sqrt{-1})$ being a primitive n-th root of 1.*

3 Ranks and conditioning of Gaussian random matrices

3.1 Random variables and Gaussian random matrices

Definition 3.1. *$F_\gamma(y) = Probability\{\gamma \le y\}$ for a real random variable γ is the* cumulative distribution function (cdf) *of X evaluated at y. $F_{g(\mu,\sigma)}(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx$ for a Gaussian random variable $g(\mu, \sigma)$ with a mean μ and a positive variance σ^2, and so*

$$\mu - 4\sigma \le y \le \mu + 4\sigma \text{ with a probability near 1.} \tag{3.1}$$

Definition 3.2. *A matrix or a vector is a* Gaussian random matrix or vector *with a mean μ and a positive variance σ^2 if it is filled with independent identically distributed Gaussian random variables, all having the mean μ and variance σ^2. $\mathcal{G}_{\mu,\sigma}^{m\times n}$ is the set of $m \times n$ Gaussian random matrices (which are* standard *for $\mu = 0$ and $\sigma^2 = 1$). By restricting this set to Toeplitz or f-circulant matrices we obtain the sets $\mathcal{T}_{\mu,\sigma}^{m\times n}$ and $\mathcal{Z}_{f,\mu,\sigma}^{n\times n}$ of* Gaussian random Toeplitz *and* Gaussian random f-circulant matrices*, respectively.*

Definition 3.3. *$\chi_{\mu,\sigma,n}(y)$ is the cdf of the norm $||\mathbf{v}|| = (\sum_{i=1}^{n} v_i^2)^{1/2}$ of a Gaussian random vector $\mathbf{v} = (v_i)_{i=1}^{n} \in \mathcal{G}_{\mu,\sigma}^{n\times 1}$. For $y \geq 0$ we have $\chi_{0,1,n}(y) = \frac{2}{2^{n/2}\Gamma(n/2)} \int_0^y x^{n-1} \exp(-x^2/2)dx$ where $\Gamma(h) = \int_0^\infty x^{h-1} \exp(-x)dx$, $\Gamma(n+1) = n!$ for nonnegative integers n.*

3.2 Nondegeneration of Gaussian random matrices

The total degree of a multivariate monomial is the sum of its degrees in all its variables. The total degree of a polynomial is the maximal total degree of its monomials.

Lemma 3.1. *[DL78], [S80], [Z79]. For a set Δ of cardinality $|\Delta|$ in any fixed ring let a polynomial in m variables have a total degree d and let it not vanish identically on this set. Then the polynomial vanishes in at most $d|\Delta|^{m-1}$ points.*

We assume that Gaussian random variables range over infinite sets Δ; consequently the lemma implies that a nonzero polynomial vanishes with probability 0, and thus Gaussian random general, Toeplitz or circulant matrix has full rank with probability 1 because the determinants of its square submatrices are polynomials in the entries. In particular a square Gaussian random general, Toeplitz or circulant matrix A is nonsingular with probability 1. Furthermore all entries of its adjoint, denoted $\operatorname{adj} A$, are subdeterminants and thus nonzeros with probability 1; clearly this property of the adjoint is extended to the inverse $A^{-1} = \frac{\operatorname{adj} A}{\det A}$. Hereafter, where this causes no confusion, we can assume by default that *Gaussian random general, Toeplitz and circulant matrices have full rank, and their inverses (if defined) have nonzero entries.*

Similar properties with probability near 1 hold where the random variables are sampled under the uniform probability distribution from a finite set of a large cardinality (see Appendix A).

3.3 Conditioning of Gaussian random matrices

Besides having full rank with probability 1, Gaussian random matrices in Definition 3.2 are expected to be well conditioned [D88], [E88], [ES05], [CD05], [B11], and even the sum $M + A$ for $M \in \mathbb{R}^{m\times n}$ and $A \in \mathcal{G}_{\mu,\sigma}^{m\times n}$ is expected to be well conditioned unless the ratio $\sigma/||M||$ is small or large [SST06].

The following theorem states an upper bound on the probability (the cdf) $F_{1/||A^+||}(y)$ that the smallest positive singular value of a Gaussian random matrix A is at most a nonnegative scalar y (cf. (2.4)); this scalar itself can be viewed as a probabilistic lower bound on the smallest singular value of A. The bounds can be strengthened by a factor $y^{|m-n|}$ [ES05], [CD05].

Theorem 3.1. *Suppose $A \in \mathcal{G}_{\mu,\sigma}^{m\times n}$, $l = \min\{m, n\}$, and $y \geq 0$. Then $F_{1/||A^+||}(y) \leq 2.35\sqrt{l}y/\sigma$.*

Proof. For $m = n$ this is [SST06, Theorem 3.3]. Apply Fact 2.3 to extend it to any pair $\{m, n\}$. □

The following two theorems supply lower bounds on the probabilities that $||A|| \leq z$ and $\kappa(A) \leq y$ for two scalars y and z and a Gaussian random matrix A. The arguments y and z of the cdfs can also be viewed as probabilistic upper bounds on the norm $||A||$ and the condition number $\kappa(A)$, respectively. We do not use the second theorem, but state it for the sake of completeness and only for square $n \times n$ matrices A. The theorem implies that the function $1 - F_{\kappa(A)}(y)$ decays as $y \to \infty$ and that the decay is inversely proportional to $y/\sqrt{\log y}$. For small values $y\sigma$ and a fixed n the lower bound becomes negative, in which case the theorem becomes trivial.

Theorem 3.2. *[DS01, Theorem II.7]. Suppose $A \in \mathcal{G}_{0,\sigma}^{m\times n}$, $l = \min\{m, n\}$ and $z \geq 2\sigma\sqrt{l}$. Then $F_{||A||}(z) \geq 1 - \exp(-(z - 2\sigma\sqrt{l})^2/(2\sigma^2))$.*

Theorem 3.3. *[SST06, Theorem 3.1]. Suppose $0 < \sigma \le 1$, $y \ge 1$, $A \in \mathcal{G}_{0,\sigma}^{n\times n}$. Then the matrix A has full rank with probability 1 and $F_{\kappa(A)}(y) \ge 1 - (14.1 + 4.7\sqrt{(2\ln y)/n})n/(y\sigma)$.*

Proof. See [SST06, the proof of Lemma 3.2]. □

3.4 Conditioning of Gaussian random Toeplitz matrices

A matrix $T_n = (t_{i-j})_{i,j=1}^n$ is the sum of two triangular Toeplitz matrices

$$T_n = Z(\mathbf{t}) + Z(\mathbf{t}_-)^T,\ \mathbf{t} = (t_i)_{i=0}^{n-1},\ \mathbf{t}_- = (t_{-i})_{i=0}^{n-1} - t_0\mathbf{e}_1. \tag{3.2}$$

If $T_n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, then T_n has $2n-1$ pairwise independent entries in $\mathcal{G}_{\mu,\sigma}$, and (2.7) implies that

$$\frac{1}{\sqrt{n}}||T_n|| \le ||T_n||_1 \le ||Z(\mathbf{t})||_1 + ||Z(\mathbf{t}_-)^T||_1 = ||\mathbf{t}||_1 + ||\mathbf{t}_-||_1 = ||(t_i)_{i=1-n}^{n-1}||_1 \le \sqrt{n}||(t_i)_{i=1-n}^{n-1}||,$$

and so

$$F_{||T_n||}(y) \ge \chi_{\mu,\sigma,2n-1}(y/n). \tag{3.3}$$

Next we estimate the norm $||T_n^{-1}||$ for $T_n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, beginning with a basic lemma.

Lemma 3.2. *[SST06, Lemma A.2]. For a nonnegative y, a unit vector $\mathbf{t} \in \mathbb{R}^{n\times 1}$, and a vector $\mathbf{b} \in \mathcal{G}_{\mu,\sigma}^{n\times 1}$, we have $F_{|\mathbf{t}^T\mathbf{b}|}(y) = \text{Probability}\{|\mathbf{t}^T\mathbf{b}| \le y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma}$.*

Remark 3.1. *The latter bound is independent of μ and n; it holds for any μ even if all coordinates of the vector $\mathbf{b}$ are fixed except for a single coordinate in $\mathcal{G}_{\mu,\sigma}$.*

Theorem 3.4. *Given a matrix $T_n = (t_{i-j})_{i,j=1}^n \in \mathcal{T}_{\mu,\sigma}^{n\times n}$, assumed to be nonsingular (cf. Section 3.2), write $p_1 = \mathbf{e}_1^T T_n^{-1}\mathbf{e}_1$. Then $F_{1/||p_1T_n^{-1}||}(y) \le 2n\alpha\beta$ for two random variables α and β such that*

$$F_\alpha(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma} \text{ and } F_\beta(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma} \text{ for } y \ge 0. \tag{3.4}$$

Proof. By virtue of part (a) of Theorem 2.4 we have $p_1T_n^{-1} = Z(\mathbf{p})Z(J\mathbf{q})^T - Z(Z\mathbf{q})Z(ZJ\mathbf{p})^T$, and so $||p_1T_n^{-1}|| \le ||Z(\mathbf{p})||\ ||Z(J\mathbf{q})^T|| + ||Z(Z\mathbf{q})||\ ||Z(ZJ\mathbf{p})^T||$ for $\mathbf{p} = T_n^{-1}\mathbf{e}_1$, $\mathbf{q} = T_n^{-1}\mathbf{e}_n$, and $p_1 = \mathbf{p}^T\mathbf{e}_1$. It follows that $||p_1T_n^{-1}|| \le ||Z(\mathbf{p})||\ ||Z(J\mathbf{q})|| + ||Z(Z\mathbf{q})||\ ||Z(ZJ\mathbf{p})||$ since $||A|| = ||A^T||$ for all matrices A. Furthermore $||p_1T_n^{-1}|| \le ||\mathbf{p}||_1\ ||J\mathbf{q}||_1 + ||Z\mathbf{q}||_1\ ||ZJ\mathbf{p}||_1$ due to (2.7). Clearly $||J\mathbf{v}||_1 = ||\mathbf{v}||_1$ and $||Z\mathbf{v}||_1 \le ||\mathbf{v}||_1$ for every vector $\mathbf{v}$, and so (cf. (2.1))

$$||p_1T_n^{-1}|| \le 2||\mathbf{p}||_1\ ||\mathbf{q}||_1 \le 2n||\mathbf{p}||\ ||\mathbf{q}||. \tag{3.5}$$

By definition the vector $\mathbf{p}$ is orthogonal to the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$, whereas $\mathbf{p}^T T_n\mathbf{e}_1 = 1$ (cf. [SST06]). Consequenty the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$ uniquely define the vector $\mathbf{u} = \mathbf{p}/||\mathbf{p}||$, whereas $|\mathbf{u}^T T_n\mathbf{e}_1| = 1/||\mathbf{p}||$.

The last coordinate t_{n-1} of the vector $T_n\mathbf{e}_1$ is independent of the vectors $T_n\mathbf{e}_2, \dots, T_n\mathbf{e}_n$ and consequently of the vector $\mathbf{u}$. Apply Remark 3.1 to estimate the cdf of the random variable $\alpha = 1/||\mathbf{p}|| = |\mathbf{u}^T T_n\mathbf{e}_1|$ and obtain that $F_\alpha(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma}$ for $y \ge 0$.

Likewise the $n-1$ column vectors $T\mathbf{e}_1, \dots, T_{n-1}$ define the vector $\mathbf{v} = \beta\mathbf{q}$ for $\beta = 1/||\mathbf{q}|| = |\mathbf{v}^T T_n\mathbf{e}_n|$. The first coordinate t_{1-n} of the vector $T_n\mathbf{e}_n$ is independent of the vectors $T\mathbf{e}_1, \dots, T_{n-1}$ and consequently of the vector $\mathbf{v}$. Apply Remark 3.1 to estimate the cdf of the random variable β and obtain that $F_\beta(y) \le \sqrt{\frac{2n}{\pi}}\frac{y}{\sigma}$ for $y \ge 0$.

Finally combine these bounds on the cdfs $F_\alpha(y)$ and $F_\beta(y)$ with (3.5). □

By employing parts (b) and (c) of Theorem 2.4 instead of its part (a), we similarly deduce the bounds $||v_0 T_{n+1}^{-1}|| \le 2\alpha\beta$ and $||v_n T_{n+1}^{-1}|| \le 2\alpha\beta$ for two pairs of random variables α and β satisfying (3.4) for $n+1$ replacing n. We have $p_1 = \frac{\det T_{n-1}}{\det T_n}$, $v_0 = \frac{\det T_n}{\det T_{n+1}}$, and $v_n = \frac{\det T_{0,1}}{\det T_{n+1}}$ for $T_{0,1} = (t_{i-j})_{i=0,j=1}^{n-1,n}$; next we bound the geometric means of the ratios $|\frac{\det T_{h+1}}{\det T_h}|$ for $h = 1, \dots, k-1$. They represent $1/p_1$, $1/v_0$ and $1/v_n$ for $k = n-1$ and $k = n$.

Theorem 3.5. *Let $T_h \ne O$ denote $h \times h$ matrices for $h = 2, \dots, k$ whose entries have absolute values at most t for a fixed positive t and let $T_1 = (t)$. Then the geometric mean $(\prod_{h=1}^{k-1} |\frac{\det T_{h+1}}{\det T_h}|)^{1/(k-1)} = \frac{1}{t} |\det T_k|^{1/(k-1)}$ is at most $k^{\frac{1}{2}(1+\frac{1}{k-1})} t$.*

Proof. The theorem follows from Hadamard's upper bound $|\det M| \le k^{k/2} t^k$, which holds for any $k \times k$ matrix $M = (m_{i,j})_{i,j=1}^k$ with $\max_{i,j=1}^k |m_{i,j}| \le t$. □

The theorem says that the geometric mean of the ratios $|\det T_{h+1} / \det T_h|$ for $h = 1, \dots, k-1$ is not greater than $k^{0.5+\epsilon(k)} t$ where $\epsilon(k) \to 0$ as $k \to \infty$ and where $t \le ||T_h||$ for all h and consequently (cf. (3.3)) we have $F_t(y) \ge \chi_{\mu,\sigma,2h-1}(y/h)$ if $T_h \in \mathcal{G}_{\mu,\sigma}^{h \times h}$.

3.5 Conditioning of Gaussian random circulant matrices

Next we estimate the norms of a random Gaussian f-circulant matrix and its inverse.

Theorem 3.6. *Assume $y \ge 0$ and an $n \times n$ circulant matrix $T = Z_1(\mathbf{v})$ for $\mathbf{v} \in \mathcal{G}_{\mu,\sigma}^{n \times 1}$. Then*
(a) $F_{||T||}(y) \ge \chi_{\mu,\sigma,n}(2y/n)$ for $\chi_{\mu,\sigma,n}(y)$ in Definition 3.3 and
(b) $F_{1/||T^{-1}||}(y) \le \sqrt{\frac{2}{\pi}} \frac{ny}{\sigma}$.

Proof. For the matrix $T = Z_1(\mathbf{v})$ we have both equation (3.2) and the bound $||\mathbf{t}_-||_1 \le ||\mathbf{t}||_1$, and so $||T||_1 \le 2||\mathbf{t}||_1$. Now part (a) of the theorem follows similarly to (3.3).

To prove part (b) recall Theorem 2.5 and write $B = \Omega T \Omega^{-1} = D(\Omega \mathbf{v})$, $\mathbf{u} = (u_i)_{i=0}^{n-1} = \Omega \mathbf{v}$. We have $\sigma_j(T) = \sigma_j(B)$ for all j because $\frac{1}{\sqrt{n}}\Omega$ and $\sqrt{n}\Omega^{-1}$ are unitary matrices.

By combining the equations $u_i = \mathbf{e}_i^T \Omega \mathbf{v}$, the bounds $||\Re(\mathbf{e}_i^T \Omega)|| \ge 1$ for all i, and Lemma 3.2, deduce that $F_{|\Re(u_i)|}(y) \le \sqrt{\frac{2}{\pi}} \frac{y}{\sigma}$ for $i = 1, \dots, n$. We have $F_{\sigma_n(B)}(y) = F_{\min_i |u_i|}(y)$ because $B = \mathrm{diag}(u_i)_{i=0}^{n-1}$. Clearly $|u_i| \ge |\Re(u_i)|$, and part (b) of the theorem follows. □

Remark 3.2. *Our extensive experiments suggest that the estimates of Theorem 3.6 are overly pessimistic (cf. Table 8.4).*

Combining Theorem 2.5 with minimax property (2.3) implies that

$$\frac{1}{g(f)} \sigma_j(Z_1(\mathbf{v})) \le \sigma_j(Z_f(\mathbf{v})) \le g(f) \sigma_j(Z_1(\mathbf{v}))$$

for all vectors $\mathbf{v}$, scalars $f \ne 0$, $g(f) = \max\{|f|, 1/|f|\}$, and $j = 1, \dots, n$. Thus we can readily extend the estimates of Theorem 3.6 to f-circulant matrices for $f \ne 0$. In particular the estimates do not change in the case of skew circulant matrices (for which $f = -1$), and f-circulant matrices of size $n \times n$ tend to be well conditioned unless $f \approx 0$ or $1/f \approx 0$.

4 Conditioning of randomized matrix products and generic preconditioning

Next we deduce probabilistic lower bounds on the smallest singular values of the products of fixed and random matrices. We begin with three lemmas. The first two of them easily follow from minimax property (2.3).

Lemma 4.1. *Suppose $\Sigma = \mathrm{diag}(\sigma_i)_{i=1}^n$, $G \in \mathbb{R}^{r \times n}$, and $H \in \mathbb{R}^{n \times r}$. Then $\sigma_j(G\Sigma) \ge \sigma_j(G)\sigma_n$, $\sigma_j(\Sigma H) \ge \sigma_j(H)\sigma_n$ for all j; if also $\sigma_n > 0$, then $\mathrm{rank}(G\Sigma) = \mathrm{rank}(G)$, $\mathrm{rank}(\Sigma H) = \mathrm{rank}(H)$.*

Lemma 4.2. *$\sigma_j(SM) = \sigma_j(MT) = \sigma_j(M)$ for all j if S and T are square unitary matrices.*

Lemma 4.3. *[SST06, Proposition 2.2]. Suppose $H \in \mathcal{G}_{\mu,\sigma}^{m\times n}$, $SS^T = S^TS = I_m$, $TT^T = T^TT = I_n$. Then $SH \in \mathcal{G}_{\mu,\sigma}^{m\times n}$ and $HT \in \mathcal{G}_{\mu,\sigma}^{m\times n}$.*

Theorem 4.1. *Suppose $G \in \mathcal{G}_{\mu,\sigma}^{r\times m}$, $H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$, $M \in \mathbb{R}^{m\times n}$, $r(M) = \operatorname{rank}(M)$, and $y \geq 0$. Then $\max\{F_{1/||(GM)^+||}(y), F_{1/||(MH)^+||}(y)\} \leq 2.35y\sqrt{\widehat{r}}/(\sigma_{r(M)}(M)\sigma)$ for $\widehat{r} = \min\{r, r(M)\}$.*

Proof. With probability 1, the matrix MH has full rank $\widehat{r}$ because $H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$. So (cf. (2.4))

$$F_{1/||(MH)^+||}(y) = F_{\sigma_{\widehat{r}}(MH)}(y). \tag{4.1}$$

Let $M = S_M\Sigma_M T_M^T$ be full SVD where $\Sigma_M = \operatorname{diag}(\widehat{\Sigma}_M, O) = \Sigma_M \operatorname{diag}(I_{r(M)}, O)$ and $\widehat{\Sigma}_M = \operatorname{diag}(\sigma_j(M))_{j=1}^{r(M)}$ is a nonsingular diagonal matrix.

We have $MH = S_M\Sigma_M T_M^T H$, and so $\sigma_j(MH) = \sigma_j(\Sigma_M T_M^T H)$ for all j by virtue of Lemma 4.2, because S_M is a square unitary matrix. Write $H_{r(M)} = (I_{r(M)} \mid O)T_M^T H$ and observe that $\sigma_j(\Sigma_M T_M^T H) = \sigma_j(\widehat{\Sigma}_M H_{r(M)})$ and consequently

$$\sigma_j(MH) = \sigma_j(\widehat{\Sigma}_M H_{r(M)}) \text{ for all } j. \tag{4.2}$$

Combine equation (4.2) for $j = \widehat{r}$ with Lemma 4.1 for the pair (Σ, H) replaced by $(\widehat{\Sigma}_M, H_{r(M)})$ and obtain that $\sigma_{\widehat{r}}(MH) \geq \sigma_{r(M)}(M)\sigma_{\widehat{r}}(H_{r(M)})$.

We have $T_M^T H \in \mathcal{G}_{\mu,\sigma}^{n\times r}$ by virtue of Lemma 4.3, since T_M is a square unitary matrix; consequently $H_{r(M)} \in \mathcal{G}_{\mu,\sigma}^{r(M)\times r}$. Therefore we can apply Theorem 3.1 for $A = H_{r(M)}$ and obtain the bound of Theorem 4.1 on $F_{1/||(MH)^+||}(y)$.

One can similarly deduce the bound on $F_{1/||(GM)^+||}(y)$ or can just apply the above bound on $F_{1/||(MH)^+||}(y))$ for $H = G^T$ and M replaced by M^T and then recall that $(M^TG^T)^T = GM$. □

Theorem 4.1 implies that multiplication by standard Gaussian random matrix is unlikely to decrease the smallest positive singular value of a matrix dramatically, even though $UV = O$ for some pairs of rectangular unitary matrices U and V.

By combining (2.2) with Theorems 3.2 and 4.1 one can probabilistically bound the condition numbers of randomized matrix products GM and MH.

The following corollary extends the bound of Theorem 4.1 for a randomized matrix product to the respective bounds for its leading blocks; this implies that *randomized multiplication of a well conditioned matrix is expected to be generic preconditioning*, that is, to ensure (with probability 1 or near 1) generic rank and conditioning profiles for the product.

Corollary 4.1. *Suppose j, k, m, n, q and s are integers, $1 \leq j \leq q$, $1 \leq k \leq s$, $M \in \mathbb{R}^{m\times n}$, $\sigma > 0$, $G \in \mathcal{G}_{\mu,\sigma}^{q\times m}$, $H \in \mathcal{G}_{\mu,\sigma}^{n\times s}$, $\operatorname{rank}(M \begin{pmatrix} I_j \\ O_{n-j,j} \end{pmatrix}) = j$, $\operatorname{rank}((I_k \mid O_{k,m-k})M) = k$, and $y \geq 0$. Then (i) with probability 1 the matrix GM (resp. MH) has generic rank profile if $\operatorname{rank}(M) \geq q$ (resp. if $\operatorname{rank}(M) \geq s$). Furthermore (ii) $F_{1/||((GM)^+)^{(j)}||}(y) \leq 2.35y\sqrt{j}/(\sigma_j(M)\sigma)$ if $\operatorname{rank}(M) \geq q$, $F_{1/||((MH)^+)^{(k)}||}(y) \leq 2.35y\sqrt{k}/(\sigma_k(M)\sigma)$ if $\operatorname{rank}(M) \geq s$.*

Proof. For every j apply Theorem 4.1 replacing G by $(I_j \mid O_{j,q-j})G$ and M by $M \begin{pmatrix} I_j \\ O_{n-j,j} \end{pmatrix}$. For every k apply Theorem 4.1 replacing M by $(I_k \mid O_{k,m-k})M$ and H by $H \begin{pmatrix} I_k \\ O_{s-k,k} \end{pmatrix}$. □

Combining the latter results with (2.2) and Theorem 3.2 implies that for well conditioned inputs randomized multiplication is expected to serve as generic preconditioning, and if it does serve so, then by virtue of Theorem 2.3 both Gaussian elimination with no pivoting and block Gaussian elimination are numerically safe. We state this as the following corollary.

Corollary 4.2. *Suppose M is a normalized $m \times n$ well conditioned matrix of full rank, $||M|| = 1$, $G \in \mathcal{G}_{0,1}^{m\times m}$ and $H \in \mathcal{G}_{0,1}^{n\times n}$. Then Gaussian elimination with no pivoting as well as block Gaussian elimination applied to the matrices GM and MH are expected to proceed using no divisions by absolutely small values.*

Remark 4.1. *We cannot formally prove an extension of Lemma 4.3 and consequently of Theorem 4.1 and its corollaries to the case of Gaussian random Toeplitz matrices $G \in \mathcal{T}_{\mu,\sigma}^{r\times m}$ and $H \in \mathcal{T}_{\mu,\sigma}^{n\times r}$, but such extensions have been consistently supported empirically (cf. Tables 8.4 and 8.10).*

By setting to 0 all singular values of $M \in \mathbb{R}^{m\times n}$ except for the k largest ones, we can reduce the rank to any nonnegative integer value $k < l$. The transformation modifies the matrices GM and MH by the matrices having norms at most $||G||\sigma_{k+1}(M)$ and $||H||\sigma_{k+1}(M)$, respectively. By virtue of minimax property (2.3) these are also upper bounds on the changes of singular values $\sigma_k(GM)$ and $\sigma_k(MH)$, which implies the following extension of Theorem 4.1.

Corollary 4.3. *Under the assumptions of Theorem 4.1 let $1 \le k \le \widehat{r}$ and $y \ge 0$. Then*
$\max\{F_{\sigma_k(GM)}(y + ||G||\sigma_{k+1}(M)), F_{\sigma_k(MH)}(y + ||H||\sigma_{k+1}(M))\} \le 2.35y\sqrt{k}/(\sigma_k(M)\sigma)$.

Remark 4.2. *We can write $H_0 = Q(MH)$ and compute the sequence $H_i = Q(MH_{i-1})$ of improved approximations $\mathcal{R}(H_i) \approx \mathbb{T}_{s,M}$ for $i = 1, 2, \ldots$.*

5 Randomized additive and dual additive preconditioning

In this section, unlike the introduction, $\tilde{A}$ denotes an $m \times n$ ill conditioned matrix that has numerical rank ρ, whereas A denotes the matrix $A = \tilde{A} + E$ of rank ρ obtained by setting to 0 the singular values $\sigma_j(\tilde{A})$ for $j > \rho$; consequently $||E|| = \sigma_{\rho+1}(\tilde{A})$ and $\kappa(A) = ||A||/\sigma_\rho(A) = ||\tilde{A}||/\sigma_\rho(\tilde{A})$. We fix a positive integer $r < l = \min\{m, n\}$ and define the matrices $C = A + UV^T$ and $\tilde{C} = \tilde{A} + UV^T$ for two scaled Gaussian random matrices $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{n\times r}$. Based on the results of the previous section we prove that $\kappa(C)$ is expected to have the order $\kappa(A) = ||A||/\sigma_{l-r}(A)$. Since the matrix $\tilde{A}$ has numerical rank $\rho = l - r$, the perturbation norm $||E|| = \sigma_{\rho+1}(\tilde{A})$ is small relative to the norms $||A|| = ||\tilde{A}||$ and $||C|| \approx ||\tilde{C}||$. Thus we can extend our probabilistic bound of order $\kappa(A) = ||\tilde{A}||/\sigma_{l-r}(\tilde{A})$ to the condition number $\kappa(\tilde{C}) \approx \kappa(C)$ and arrive at Theorem 1.1 in whose statement $\tilde{A}$ and $\tilde{C}$ replace A and C. We first prove the theorem in the case where $l = m = n$ and then extend it to $m \times n$ rectangular matrices A, $\tilde{A}$, C and $\tilde{C}$ for any pair $\{m, n\}$ (see Theorem 5.4).

Theorem 5.1. *Suppose $A, C, S, T \in \mathbb{R}^{n\times n}$, $U, V \in \mathbb{R}^{n\times r}$, the matrix $C = A + UV^T$ is nonsingular, $\mathrm{rank}(A) = \rho$, $\mathrm{nul}(A) = r = n - \rho$, $0 < \rho < n$, $A = S\Sigma T^T$ is full SVD, S and T are unitary matrices, $\Sigma = \mathrm{diag}(\Sigma_A, O_{r,r})$, $\Sigma_A = \mathrm{diag}(\sigma_j)_{j=1}^{\rho}$ is the $\rho \times \rho$ diagonal matrix of positive singular values of the matrix A. Furthermore write*

$$S^TU = \begin{pmatrix} \bar{U} \\ U_r \end{pmatrix},\ T^TV = \begin{pmatrix} \bar{V} \\ V_r \end{pmatrix},\ R_U = \begin{pmatrix} I_\rho & \bar{U} \\ O_{r,\rho} & U_r \end{pmatrix},\ R_V = \begin{pmatrix} I_\rho & \bar{V} \\ O_{r,\rho} & V_r \end{pmatrix},$$

and so U_r and V_r are $r \times r$ matrices. Then

(a) $R_U\Sigma R_V^T = \Sigma$, whereas $R_U\,\mathrm{diag}(O_{\rho,\rho}, I_r)R_V^T = S^TUV^TT$, so that

$$C = SR_UDR_V^TT^T,\ D = \Sigma + \mathrm{diag}(O_{\rho,\rho}, I_r) = \mathrm{diag}(\Sigma_A, I_r)\ . \tag{5.1}$$

Furthermore suppose that $||A|| = 1$ and the $r \times r$ matrices U_r and V_r are nonsingular and write $p = ||R_U^{-1}||\ ||R_V^{-1}||$. Then

(b) the matrix C is nonsingular,

(c) $p \ge ||C^{-1}||/||A^+|| = \sigma_\rho(A)/\sigma_n(C)$,

(d) $1 \le p \le (1 + ||U||)(1 + ||V||)f_r$ *where* $f_r = \max\{1, ||U_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}$,

(e) $||C|| \le ||A||(1 + ||U||\ ||V||)$ *and*

(f) $\sigma_n(C) \ge \sigma_\rho(A)(1 + ||U||)(1 + ||V||)f_r$ *for* $f_r = \max\{1, ||U_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}$.

Proof. Parts (a) and (b) are readily verified.

(c) The matrix $D = \mathrm{diag}(\Sigma_A, I_r)$ is nonsingular. Combine the equations $S^{-1} = S^T$, $T^{-T} = T$ and (5.1) and obtain $C^{-1} = TR_V^{-T}D^{-1}R_U^{-1}S^T$. It follows that $||C^{-1}|| = ||R_V^{-T}D^{-1}R_U^{-1}|| \le ||R_V^{-T}||\,||D^{-1}||\,||R_U^{-1}||$ because S and T are square unitary matrices. Note that $D^{-1} = \mathrm{diag}(\Sigma_A^{-1}, I_r)$ recall that $\sigma_\rho(A) \le ||A|| = 1$, deduce that $||D^{-1}|| = ||\mathrm{diag}(\Sigma_A^{-1}, I_r)|| = 1/\sigma_\rho(A) = ||A^+||$, and obtain the claimed bound $p \ge ||C^{-1}||/||A^+||$.

(d) We have $R_U^{-1} = \begin{pmatrix} I_\rho & -\bar{U} \\ O & I_r \end{pmatrix}\begin{pmatrix} I_\rho & O \\ O & U_r^{-1} \end{pmatrix}$, $R_V^{-1} = \begin{pmatrix} I_\rho & -\bar{V} \\ O & I_r \end{pmatrix}\begin{pmatrix} I_\rho & O \\ O & V_r^{-1} \end{pmatrix}$, $||\bar{U}|| \le ||U||$ and $||\bar{V}|| \le ||V||$. Combine these relationships.

(e) Combine the relationships $||C|| \le ||A|| + ||U||\,||V^T||$, $||A|| = 1$ and $||V^T|| = ||V||$.

(f) Combine the bounds $\sigma_\rho(A) \le p\sigma_n(C)$ and $p \le (1+||U||)(1+||V||)f_r$ of parts (c) and (d). □

We have $\frac{\kappa(C)}{\kappa(A)} = \frac{||C||}{||A||}\frac{\sigma_\rho(A)}{\sigma_n(C)}$, and so parts (e) and (f) together bound the ratio $\frac{\kappa(C)}{\kappa(A)}$ in terms of the norms $||U||$, $||V||$, $||U_r^{-1}||$ and $||V_r^{-1}||$. Let us supply probabilstic estimates for these norms where U and V are Gaussian random matrices.

Theorem 5.2. *Suppose $W \in \mathcal{G}_{0,\sigma}^{n\times r}$ and $y \ge 2\sigma\sqrt{r}$. Then $F_{||W||}(y) \ge 1 - \exp(-(y-2\sigma\sqrt{r})^2/(2\sigma^2))$.*

Proof. The theorem follows from Theorem 3.2 applied for $A = W$ (in which case $l = r$). □

Theorem 5.3. *Suppose that A, U, V, U_r and V_r denote the five matrices of Theorem 5.1 where $U, V \in \mathcal{G}_{\mu,\sigma}^{n\times r}$. Then $\max\{F_{1/||U_r^{-1}||}(y), F_{1/||V_r^{-1}||}(y)\} \le 2.35\ y\sqrt{r}/\sigma$ for $y \ge 0$.*

Proof. Apply Theorem 4.1 for $m = r$, $M = (O \mid I_r)S^T$, $H = U$ and $MH = U_r$ to obtain $F_{1/||U_r^{-1}||}(y) \le 2.35y\sqrt{r}/(\sigma_r((O \mid I_r)S^T)\sigma)$. Then apply Theorem 4.1 for $m = r$, $M = (O \mid I_r)T^T$, $H = V$ and $MH = V_r$ to obtain $F_{1/||V_r^{-1}||}(y) \le 2.35y\sqrt{r}/(\sigma_r((O \mid I_r)T^T)\sigma)$. Substitute the equations $\sigma_r((O \mid I_r)S^T) = \sigma_r((O \mid I_r)T^T) = 1$, which hold because S and T are unitary matrices. □

The equation $\frac{\kappa(C)}{\kappa(A)} = \frac{||C||}{||A||}\frac{\sigma_\rho(A)}{\sigma_n(C)}$ and Theorems 5.1–5.3 combined imply that the condition number $\kappa(C)$ is expected to be of order $\kappa(A)$, as desired, provided $m = n$ and U and V are Gaussian random matrices. Clearly, the bounds of Theorem 5.2 and of part (e) of Theorem 5.1 holds for any pair of m and n, and so the norm $||C||$ is expected to have order $||A||$ for any pair of m and n. Let us also relax the assumption that $m = n$ for our bound on $\sigma_\rho(A)/\sigma_n(C)$.

Theorem 5.4. *Probabilistic upper bound on the ratio $\sigma_\rho(A)/\sigma_n(C)$ implied by part (f) of Theorem 5.1, Theorems 5.2 and 5.3 together can be extended from the case where $m = n$ to the case of any pair $\{m, n\}$, $U \in \mathcal{G}_{\mu,\sigma}^{m\times r}$, and $V \in \mathcal{G}_{\mu,\sigma}^{n\times r}$.*

Proof. Let $M = S_M\Sigma_M T_M^T$ denote the full SVDs for $M = A$ and $M = C$. W.l.o.g. assume that $A = \mathrm{diag}(\bar{\Sigma}_A, O_{m-\rho,n-\rho})$ where $\bar{\Sigma}_A$ is a nonsingular diagonal matrix. Indeed $\sigma_j(A) = \sigma_j(S_A^T A T_A) = \sigma_j(\bar{\Sigma}_A)$, $\sigma_j(C) = \sigma_j(S_A^T C T_A)$ for all j, $S_A^T U \in \mathcal{G}_{\mu,\sigma}^{m\times r}$, and $T_A^T V \in \mathcal{G}_{\mu,\sigma}^{n\times r}$ because S_A and T_A are square orthogonal matrices (cf. Lemmas 4.2 and 4.3).

Now assume that $m > n$ and estimate above the ratio $\sigma_\rho(A)/\sigma_n(C)$.

Pre-multiply the equation $C = A + UV^T$ by the matrix $(I_n \mid O_{n,m-n})$ and note that $\widehat{U} = (I_n \mid O_{n,m-n})U \in \mathcal{G}_{0,\sigma}^{n\times r}$, $\widehat{A} = (I_n \mid O_{n,m-n})A = \bar{\Sigma}_A$, and $\widehat{C} = (I_n \mid O_{n,m-n})C = \widehat{A} + \widehat{U}V^T$. Clearly $\sigma_j(A) = \sigma_j(\bar{\Sigma}_A)$ for all j, whereas $\sigma_j(C) \ge \sigma_j(\widehat{C})$ for all j by virtue of Fact 2.3. Therefore $\sigma_\rho(\widehat{A})/\sigma_n(\widehat{C}) \ge \sigma_\rho(A)/\sigma_n(C)$, and so it is sufficient to estimate above the ratio $\sigma_\rho(\widehat{A})/\sigma_n(\widehat{C})$. The desired probabilistic estimate for it is given by part (f) of Theorem 5.1 combined with Theorems 5.2 and 5.3 where the $n \times n$ matrices $\widehat{C}$ and $\widehat{A}$ replace A and C.

This completes our proof in the case where $m > n$. Apply the same proof to the matrices A^T and C^T replacing A and C and prove the theorem also where $m < n$. □

Combine Theorems 5.2–5.4 to obtain the following corollary.

Corollary 5.1. *Keep the assumptions of Theorem 5.3, but let $\mu = 0$, allow $A \in \mathbb{R}^{m\times n}$ for any pair of integers m and n. Then $\kappa(C) \le \kappa(A)(1 + ||U||\ ||V||)(1 + ||U||)(1 + ||V||)f_r$ for $f_r = \max\{1, ||U_r^{-1}||\}\ \max\{1, ||V_r^{-1}||\}$. Furthermore let $U \in \mathcal{G}_{0,\sigma}^{m\times r}$ and $V \in \mathcal{G}_{0,\sigma}^{n\times r}$. Then the norms $||U||$, $||V||$, $||U_r^{-1}||$, and $||V_r^{-1}||$ satisfy the randomized bounds of Theorems 5.2 (for $W = U$ and m replacing n and for $W = V$) and 5.3.*

It follows that $\kappa(C)$ is expected to have order $\kappa(A) = ||A||/\sigma_\rho(A)$. We obtain Theorem 1.1 for $\tilde{A}$ and $\tilde{C}$ replacing A and C by combining this bound with the following corollary of Theorem 2.2.

Corollary 5.2. *Under the assumptions of Corollary 5.1 write $\tilde{A} = A + E$, $\tilde{C} = \tilde{A} + UV^T$, and $C = A + UV^T = \tilde{C} - E$ and suppose the norm $||E||$ is small and the matrix C has full rank. Then $\kappa(\tilde{C}) \approx \kappa(C)$, and so the condition number $\kappa(\tilde{C})$ is expected to have at most the order $||\tilde{A}||/\sigma_\rho(\tilde{A})$ provided $||A|| \approx 1$, $U \in \mathcal{G}_{0,1}^{m\times r}$, and $V \in \mathcal{G}_{0,1}^{n\times r}$.*

Remark 5.1. *Our estimates for the condition numbers $\kappa(\tilde{C}) \approx \kappa(C)$ are simplified where $m = n$ and $U = V$. In this case we have $\kappa(C) \le \kappa(A)(1+||U||^2)(1+||U||)^2 \max\{1, ||U_r^{-1}||^2\}$. Consequently $\kappa(\tilde{C})$ is still expected to have order $||A||/\sigma_\rho(A) \approx ||\tilde{A}||/\sigma_\rho(\tilde{A})$.*

Remark 5.2. *How large is our class of $m\times n$ matrices A having a numerical rank q? We characterize it indirectly, by noting that by virtue of Fact 2.2 the nearby matrices $\tilde{A}$ of rank q form a variety of dimension $(m + n - q)q$, which increases as q increases.*

Dual additive preconditioning

Next we assume that the matrix A has full rank, write $C_-^+ = A^+ + U_-V_-^T$ by extending (2.5), and define the *dual additive preprocessing*

$$A^+ \Longrightarrow C_-^+ = A^+ + U_-V_-^T. \tag{5.2}$$

Our analysis implies that the value $\kappa(C_-^+) = \kappa(C_-)$ is expected to have order $\sigma_{q+1}(A)/\sigma_l(A)$ provided $U_- \in \mathcal{G}_{0,1}^{n\times q}$, $V_- \in \mathcal{G}_{0,1}^{m\times q}$, and the norm $||A^+||$ is neither large nor small. The randomized algorithm of [D83] produces a crude estimate for the norm $||A^+||$ at a low computational cost; we can work with the matrix $\widehat{A} = \mathrm{diag}(A, \epsilon)$ instead of the matrix A and choose a sufficiently small positive ϵ such that $||\widehat{A}^+|| = 1/\epsilon$. Then we can scale the matrix $\widehat{A}$ to obtain the matrix $\tilde{A} = \widehat{A}/\epsilon$, such that $||\widehat{A}^+|| = 1$.

6 Randomized augmentation

6.1 The case of general matrices

The solution of a nonsingular linear system of n equations, $A\mathbf{y} = \mathbf{b}$ can be readily recovered from a null vector $\begin{pmatrix}\mathbf{y}\\ -1/\beta\end{pmatrix}$ of the matrix $K = (A \mid \beta\mathbf{b})$ for a nonzero scalar β. If the matrix A has numerical nullity 1 and if the ratio $||A||/||\beta\mathbf{b}||$ is neither large nor small, then the matrix K is well conditioned for the average vector $\mathbf{b}$ [PQa, Section 9.1]. The next simple theorem links additive preprocessing $A \Longrightarrow C = A + UV^T$ to an extension of such an augmentation technique (one can compare it with block randomization in Section 1.2).

Theorem 6.1. *Suppose $K = \begin{pmatrix}A & -U\\ WV^T & W\end{pmatrix} \in \mathbb{R}^{(m+r)\times(n+r)}$, $W \in \mathbb{R}^{r\times r}$ is a nonsingular matrix, $C = A + UV^T$. Then $K = \mathrm{diag}(I_m, W)\widehat{U}\,\mathrm{diag}(C, I_r)\widehat{V}$ for $\widehat{U} = \begin{pmatrix}I_m & -U\\ O_{r,m} & I_r\end{pmatrix}$, $\widehat{V} = \begin{pmatrix}I_n & O_{n,r}\\ V^T & I_r\end{pmatrix}$, Moreover if the matrices C and W or the matrix K are nonsingular, then so are all the three matrices C, W, and K, and furthermore $K^{-1} = \bar{V}\,\mathrm{diag}(C^{-1}, I_r)\bar{U}\,\mathrm{diag}(I_m, W^{-1})$ and $C^{-1} = (I_n \mid O_{n,r})K^{-1}(I_m \mid O_{m,r})^T$ where $\bar{U} = \widehat{U}^{-1} = \begin{pmatrix}I_m & U\\ O_{r,m} & I_r\end{pmatrix}$ and $\bar{V} = \widehat{V}^{-1} = \begin{pmatrix}I_n & O_{n,r}\\ -V^T & I_r\end{pmatrix}$.*

Suppose $m = n$, $q = n - r$, $U \in \mathcal{G}_{0,1}^{m\times r}$, $V \in \mathcal{G}_{0,1}^{n\times r}$, $W \in \mathcal{G}_{0,1}^{r\times r}$, and $||A|| \approx 1$, combine Theorem 6.1 and Corollary 5.2, and deduce that the matrix K is expected to have the condition number of order $\sigma_1(A)/\sigma_q(A)$ and thus to be well conditioned if the matrix A has numerical nullity at most r.

[PQa] employs Theorem 4.1 to prove similar preconditioning property for the more general class of augmentations, e.g., the northwestern augmentation

$$K = \begin{pmatrix} W & V^T \\ -U & A \end{pmatrix} \tag{6.1}$$

where $U \in \mathcal{G}_{0,\sigma}^{m\times r}$, $V \in \mathcal{G}_{0,\sigma}^{n\times r}$, $W \in \mathcal{G}_{0,\sigma}^{r\times r}$, and the ratio $\sigma/||A||$ is neither large nor small. Namely under such assumptions $\kappa(K)$ is expected to have order at most $||A||/\sigma_h(A)$ for $h = \text{rank}(A) - \min\{r, q\}$.

The same estimates of [PQa] hold under the restriction that $W = I_n$, and surely the matrix K of Theorem 6.1 turns into the one of (6.1) up to block row and column interchange. Together with Theorem 6.1 this enables alternative derivation of the bounds on $\kappa(C)$ for $C = A + UV^T$ and Gaussian random matrices U and V.

Next under (6.1) let the matrices A, W and K be nonsingular, let $m = n$, and write $S = A + UW^{-1}V^T$ and $R = I - V^TS^{-1}UW^{-1}$. Then the matrix S is nonsingular, S^{-1} is the $n \times n$ trailing (southwestern) block of K^{-1}, and SMW formula (1.1), for C replaced by S, U by UW^{-1}, and G by R, implies that

$$A^{-1} = S^{-1} + S^{-1}UW^{-1}R^{-1}V^TS^{-1}. \tag{6.2}$$

Remark 6.1. *Preconditioning property of randomized augmentation can be extended to the substitution of random rows and columns into the selected positions in the matrix M (cf. [PQa]); if both M and the new matrix are nonsingular, we can link their inverses via the SMW formula.*

Remark 6.2. *Here is an example of minor differences between the power of augmentation and additive preprocessing. The map $A \Longrightarrow K = \begin{pmatrix} W & V^T \\ V & A \end{pmatrix}$ cannot decrease the condition number $\kappa(A)$ where K is a symmetric and positive definite matrix; this follows from the Interlacing Property of the eigenvalues of K [GL96, Theorem 8.1.7]. In contrast scaled randomized symmetric additive preprocessing $A \Longrightarrow C = A + VV^T$ is expected to work as preconditioning for an ill conditioned symmetric positive definite matrix A having a small numerical nullity (cf. Remark 5.1 and [W07]).*

6.2 A randomized Toeplitz solver

Let us apply Theorem 2.4 to support randomized augmentation for solving a nonsingular Toeplitz linear system $T\mathbf{y} = \mathbf{b}$ of n equations provided the matrix T has numerical nullity 1.

To compute the vector $\mathbf{y} = T^{-1}\mathbf{b}$, we first embed the matrix T into an $(n+1)\times(n+1)$ Toeplitz matrix $K = \begin{pmatrix} w & \mathbf{v}^T \\ \mathbf{f} & T \end{pmatrix}$. We write $w = \mathbf{e}_1^TT\mathbf{e}_1$ and fill the vectors $\mathbf{f} = (f_i)_{i=1}^n$ and $\mathbf{v} = (v_i)_{i=1}^n$ with appropriate entries of the matrix T except for the two coordinates f_n and v_n, which we choose at random and then scale to have the ratios $\frac{|f_n|}{||K||}$ and $\frac{|v_n|}{||K||}$ neither large nor small.

Part (b) of Theorem 2.4 expresses the inverse T^{-1} via the vectors $\mathbf{v} = K^{-1}\mathbf{e}_1$ and $\mathbf{w} = K^{-1}\mathbf{e}_{n+1}$.

In view of Section 3.2 and Appendix A, this policy is likely to produce a nonsingular matrix K whose inverse is likely to have a nonzero entry $\mathbf{e}_1^TK^{-1}\mathbf{e}_1$. Our tests were in very good accordance with these formal results and moreover consistently produced well conditioned matrices K.

To summarize, we reduce the solution of a nonsingular ill conditioned Toeplitz linear system $T\mathbf{y} = \mathbf{b}$ to computing highly accurate solutions of two linear systems $K\mathbf{x} = \mathbf{e}_1$ and $K\mathbf{z} = \mathbf{e}_{n+1}$, both expected to be well conditioned. High accuracy shall counter the magnification of the input and rounding errors, expected in the case of ill conditioned input.

In the important special case where a Toeplitz matrix T is real symmetric, we can choose real scalars w and $f_n = v_n$ to yield a real symmetric matrix $K = \begin{pmatrix} w & \mathbf{v}^T \\ \mathbf{v} & T \end{pmatrix}$. In this case the computation is simplified because $J_{n+1}K^{-1}J_{n+1} = K^{-1}$, $J_{n+1}\mathbf{e}_{n+1} = \mathbf{e}_1$, and so $K^{-1}\mathbf{e}_{n+1} = J_{n+1}K^{-1}\mathbf{e}_1$, and

we only need to solve a single linear system with the matrix K. We refer to the resulting algorithm for the linear system $T\mathbf{y} = \mathbf{b}$ as **Algorithm 6.1**. In Section 8.6 we test this algorithm for solving an ill conditioned real symmetric Toeplitz linear system.

Instead of augmentation we can randomly modify the entries t_{n-1} and t_{1-n}.

One can readily extend the approach of this section to the case of Toeplitz-like, Hankel and Hankel-like inputs as well as to augmenting the input matrix with r rows and r columns for $r > 1$ and to block randomization of Section 1.2. For the transition back to the solution of the original problem, we can employ expression (6.2) or Theorem 2.4 in the case of augmentation and the SMW formula (1.1) in the case of randomized (r, r) updating.

7 Applications of randomized additive preprocessing and augmentation

In this section we show some simple recipes for utilizing our results in the two previous sections. In particular in the next subsection we apply randomization and Corollary 5.2 to estimate the numerical rank and numerical nullity of a matrix. In Section 7.2 we first recall the results of [PQ10] on computing a nmb and a left nmb of a rank deficient matrix and then extend these results, again based on Corollary 5.2, to randomized approximation of trailing and leading singular spaces of ill conditioned matrices. For the approximation of leading singular spaces we also show a simple alternative randomized recipe in Section 7.3, where we extend both recipes to lower-rank approximation of an ill conditioned matrix. In Sections 7.4 and 7.5 we suppose that some approximate bases for trailing and leading singular spaces of a matrix are available; then we compute its 2×2 block triangulation to facilitate inversion and the solution of linear systems with this input matrix. In Section 7.6 we solve such linear systems by combining Corollary 5.2 with the SMW formula and prove dramatic improvement versus Gaussian elimination. In Sections 7.7 and 7.8 we comment on using random structured matrices in these computations and on the links between our randomized computations and Newton's iteration for matrix inversion. The results of Section 7.2 are used in Sections 7.3–7.5, but otherwise all subsections can be read independently of each other.

We simplify the notation by allowing A to denote both rank deficient matrices of a fixed rank ρ and ill conditioned matrices having a fixed numerical rank q and possibly having full rank, and we still allow q (resp. r) to denote either the numerical rank (resp. numerical nullity) or its upper bound where this causes no confusion.

7.1 Estimation of numerical rank and numerical nullity

For an $m \times n$ matrix A that has norm 1 and numerical rank q, we expect to compute the integer $r = n - q$ by means of at most $2\lceil \log_2 r(r - r_-)\rceil$ search steps given a lower bound r_- on $r \geq 0$. In a search step we fix an integer s and a pair of matrices $U \in \mathcal{G}_{0,1}^{m\times s}$ and $V \in \mathcal{G}_{0,1}^{n\times s}$ and test whether the matrix $C = A + UV^T$ has full rank and is well conditioned. We apply the test for $s = 0, 1, 2, 4, \ldots$. If the matrix C passes the test for some $s = 2^h$ and $h > 1$, but not for $s = 2^{h-1}$, we expect to have r in the range $(2^{h-1}, 2^h]$ by virtue of Corollary 5.2. Then we apply binary search in this range for the integer r as the minimum integer for which the matrix C passes the test.

In the case of $q = l - r \ll r$ we begin our search for r with an upper bound r_+ on $r \leq l = \min\{m, n\}$, and in at most $2\lceil \log_2(r_+ - r)\rceil$ steps we expect to compute $s = r$, again as the minimum integer for which the matrix C has full rank and is well conditioned; the minimum is achieved where the ratio $\frac{||AC^{-1}U||}{||A||\ ||C^{-1}U||}$ is small (cf. [PQ10, Algorithm 6.7]).

Remark 7.1. *To facilitate the binary search, one can apply the power transforms $A \Longrightarrow B = (AA^T)^h A$ for positive integers h. They increase the gaps between all pairs of distinct singular values of A because $\sigma_j(B) = (\sigma_j(A))^{2h+1}$.*

7.2 Computation of nmbs and approximation of singular spaces

One can compute nmbs based on the algorithms of [PQa] or on the following theorem.

Theorem 7.1. *[PQ10, Theorem 3.1]. Suppose a matrix $A \in \mathbb{R}^{m\times n}$ has rank ρ, $0 < \rho < l = \min\{m, n\}$, $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{n\times r}$, $r = l - \rho$, and the matrix $C = A + UV^T$ has full rank l. Then the matrix C^+U is a* nmb(A) *if $m \geq n$, whereas the matrix $C^{+T}V$ is a left* nmb(A) *if $n \geq m$.*

We extend this result to the approximation of trailing and leading singular spaces in the case where the matrix A has numerical rank q.

Theorem 7.2. *Assume a matrix $A \in \mathbb{R}^{m\times n}$ having numerical rank q where $0 < q < l = \min\{m, n\}$.*

(a) Write $r = l - q$ and suppose $U \in \mathbb{R}^{m\times r}$, $V \in \mathbb{R}^{n\times r}$, and the matrix $C = A + UV^T$ has full rank and is well conditioned. Then there is a scalar c independent of A, U, V, m, n and q such that $||C^+UX - T_{A,r}|| \leq c\sigma_{q+1}(A)$ for some matrix $X \in \mathbb{R}^{r\times r}$ if $m \geq n$, whereas $||C^{+T}VY - S_{A,r}|| \leq c\sigma_{q+1}(A)$ for some matrix $Y \in \mathbb{R}^{r\times r}$ if $n \geq m$.

(b) Assume that the matrices $A \in \mathbb{R}^{m\times n}$, $U_- \in \mathbb{R}^{n\times q}$, $V_- \in \mathbb{R}^{m\times q}$, $H = I_q + V_-AU_-^T$, and $C_-^+ = A^+ + U_-V_-^T$ have full ranks, the matrix A^+ has numerical rank q, and the matrix C_-^+ is well conditioned. Then there exists a scalar c_- independent of A, U_-, V_-, m, n and q and such that $||C_-U_-X_- - S_{q,A}|| \leq c_-\sigma_{q+1}(A)$ for some matrix $X_- \in \mathbb{R}^{q\times q}$ if $m \leq n$, whereas $||C_-^TV_-Y_- - T_{q,A}|| \leq c_-\sigma_{q+1}(A)$ for some matrix $Y_- \in \mathbb{R}^{q\times q}$ if $m \geq n$.

Proof. Part (a) has been proved in [PQ10, Section 7.1]. To extend the proof to part (b) rewrite the SVD $A = S_A\Sigma_AT_A^T$ as follows (cf. Section 2.3), $A = (S_{q,A} \mid S_{A,m-q})\,\mathrm{diag}(\Sigma_{q,A}, \bar{\Sigma}_A)(T_{q,A} \mid T_{A,n-q})^T$, where the matrices $S_{q,A}$ and $T_{q,A}$ are formed by the first q columns of the matrices S_A and T_A, respectively, whereas $\Sigma_{q,A}$ is the $q \times q$ leading block of the diagonal matrix Σ_A. Then apply part (a) to the matrix $A^+ = T_A\Sigma_A^+S_A^T = (T_{q,A} \mid T_{A,n-q})\,\mathrm{diag}(\Sigma_{q,A}^+, \bar{\Sigma}_A^+)(S_{q,A} \mid S_{A,m-q})^T$ observing that $\mathcal{R}(T_{q,A}) = \mathbb{S}_{A^+,q}$ for $m \geq n$, whereas $\mathcal{R}(S_{q,A}) = \mathbb{T}_{A^+,q}$ for $m \leq n$. □

By virtue of Corollary 5.2 the matrix C of part (a) is expected to have full rank and to be well conditiined (as required in part (a)) provided $U \in \mathcal{G}^{m\times r}_{0,(||A||)}$ and $V \in \mathcal{G}^{n\times r}_{0,(||A||)}$, and the same properties hold for the matrix C_- in part (b) provided $U_- \in \mathcal{G}^{n\times q}_{0,(||A^+||)}$ and $V_- \in \mathcal{G}^{m\times q}_{0,(||A^+||)}$,

Part (a) of Theorem 7.2 states that $\mathcal{R}(C^+U) \approx \mathbb{T}_{A,r}$ if $m \geq n$ and $\mathcal{R}(C^{+T}V) \approx \mathbb{S}_{A,r}$ if $m \leq n$, that is, the linear spaces $\mathcal{R}(C^+U)$ for $m \geq n$ and $\mathcal{R}(C^{+T}V)$ for $m \leq n$ approximate the right and left trailing singular spaces associated with the r smallest singular values of the matrix A, respectively. (Some of these values can vanish.) Likewise part (b) states that $\mathcal{R}(C_-U_-) \approx \mathbb{S}_{q,A}$ if $m \leq n$ and $\mathcal{R}(C_-^TV_-) \approx \mathbb{T}_{q,A}$ if $m \geq n$, that is, the linear spaces $\mathcal{R}(C_-U_-)$ for $m \leq n$ and $\mathcal{R}(C_-^TV_-)$ for $m \geq n$ approximate the left and right leading singular spaces associated with the q largest singular values of the matrix A, respectively.

Alternatively we can approximate the leading singular spaces $\mathbb{S}_{q,A}$ by $\mathcal{R}(A^TM)$ for $M \in \mathcal{G}^{m\times q}_{0,1}$ and $\mathbb{T}_{q,A}$ by $\mathcal{R}(AN)$ for $N \in \mathcal{G}^{n\times q}_{0,1}$. Indeed Theorem 4.1 implies that $\mathcal{R}(A^TM) \approx \mathbb{S}_{q,A}$ and $\mathcal{R}(AN) \approx \mathbb{T}_{q,A}$.

We can extend the recipes of Remarks 4.2 and 7.1 to improve the approximations to the linear spaces $\mathbb{S}_{q,A}$, $\mathbb{T}_{q,A}$, $\mathbb{S}_{A,r}$, and $\mathbb{T}_{A,r}$ computed based on any of the two approaches.

In our tests in the next section we employed the following relationships,

$$X \approx B^+T_{A,r} \text{ and } C^+UX - T_{A,r} \approx BB^+T_{A,r} - T_{A,r}, \tag{7.1}$$

$$Y_- \approx B_-^+T_{q,A} \text{ and } C_-^TV_-Y_- - T_{q,A,} \approx B_-B_-^+T_{q,A} - T_{q,A}. \tag{7.2}$$

We obtain (7.1) from part (a) of Theorem 7.2 for $C^+U = B$ and $BX \approx T_{A,r}$ and obtain (7.2) from part (b) of Theorem 7.2 for $C_-^TV_- = B_-$ and $B_-Y_- \approx T_{q,A}$.

[PQa, Section 5.1] extends Theorem 7.1 under both bounds $m \geq n$ and $m < n$ as follows.

Theorem 7.3. *Assume two matrices $A \in \mathbb{R}^{m\times n}$ of a rank $\rho < n$ and $V \in \mathbb{R}^{r\times n}$ for $r = n - \rho$. Suppose the matrix $K = \begin{pmatrix} V \\ A \end{pmatrix}$ has full column rank n. Then $B = K^{(I)}\begin{pmatrix} I_r \\ O \end{pmatrix}$ is a nmb(A).*

Remark 7.2. *For $K^{(I)} = K^+$ and a matrix $A \in \mathbb{R}^{m\times n}$ having numerical rank $q < n$, we can readily extend the theorem to devise an alternative algorithm that approximates the trailing singular space $\mathbb{T}_{A,n-q}$ by $\mathcal{R}(B)$, and we can improve the approximation by applying or extending the recursive techniques of Remarks 4.2 and 7.1. See other alternative techniques in [PQ10] and [PQa].*

Remark 7.3. *In the case where $m = n$ Theorems 7.1 and 7.2 define both left and right nmbs or approximations to both left and right trailing and leading singular spaces. We can reduce the computation of a null vector of a rectangular matrix A, its nmb and approximate nmb to the case of square inputs. Indeed (a) $\mathcal{N}(A) = \mathcal{N}(A^TA)$, (b) $\mathcal{N}(A) = \mathcal{N}(B^TA)$ if $A, B \in \mathbb{R}^{m\times n}$ and the matrix B has full rank $m \le n$, and (c) $(A \mid O_{n,m-n})\mathbf{u} = \mathbf{0}_m$ if and only if $A\widehat{\mathbf{u}} = \mathbf{0}_m$ provided $m \ge n$ and $\widehat{\mathbf{u}} = (I_n \mid O_{n,m-n})\mathbf{u}$, whereas $\mathbf{v}^T \begin{pmatrix} A \\ O_{n-m,n} \end{pmatrix} = \mathbf{0}_n^T$ if and only if $\widehat{\mathbf{v}}^TA = \mathbf{0}_n^T$ provided $m < n$ and $\widehat{\mathbf{v}} = \mathbf{v}^T \begin{pmatrix} I_m \\ O_{n-m,m} \end{pmatrix}$. Furthermore given an $m \times n$ matrix A for $m > n$, we can represent it as the sum $A = \sum_{i=1}^h (O, B_i^T, O)^T$ where $B_i = (O, I_{k_i}, O)^TA$ are $k_i \times n$ matrices for $i = 1, \dots, h$, $\sum_{i=1}^h k_i = m$. Then $\mathcal{N}(A) = \cap_{i=1}^h \mathcal{N}(B_i)$, and we can compute the intersection of nmbs based on [GL96, Theorem 12.4.1].*

7.3 Approximation by low-rank matrices

Suppose that a matrix $A = S_A^T\Sigma_A T_A$ has numerical rank q, M is a matrix of rank q, and

$$\mathcal{R}(M) \approx \mathbb{T}_{q,A}. \tag{7.3}$$

Write $Q = Q(M)$. Then both matrices $AM(M^TM)^{-1}M^T$ and AQQ^T have rank q and approximate the matrix A. Thus the low-rank approximation of the matrix A is reduced to the approximation of a basis for its leading singular space $\mathbb{T}_{q,A}$.

By virtue of Corollary 4.3 the matrix $M = AG$ for $G \in \mathcal{G}_{0,1}^{n\times q}$ (and empirically even for $G \in \mathcal{T}_{0,1}^{n\times q}$) is expected to satisfy (7.3); likewise the matrix $M = C_-^TV_-$ is expected to have rank q and to satisfy (7.3) for $C_- = (C_-^+)^+$, C_-^+ of (5.2), $\sigma \approx ||A^+||$, $U_- \in \mathcal{G}_{0,\sigma}^{n\times q}$ and $V_- \in \mathcal{G}_{0,\sigma}^{m\times q}$ (and empirically even for $U_- \in \mathcal{T}_{0,\sigma}^{n\times q}$ and $V_- \in \mathcal{T}_{0,\sigma}^{m\times q}$). In the case of a square nonsingular matrix A we can compute the matrix C_- by applying the dual SMW formula (2.6). In that case we can confine all divisions to the stages of the orthogonalization of the $n \times q$ matrix $C_-^TV_-$ and the inversion of the $q \times q$ matrix H. (The matrix H is nonsingular if so is the matrix $A^{-1} + U_-V_-^T$.)

By applying the same algorithms to the displacement A of a matrix W having a numerical displacement rank q [BM01], that is lying near some matrices with displacement rank q, we approximate the matrix W by one of these matrices. For a sample application of such approximations, we can simplify Newton's structured matrix inversion (see Section 7.8).

7.4 Block triangulation with approximate trailing singular spaces

Theorems 7.1 and 7.2 define randomized algorithms that compute nmbs of a rank deficient matrix and approximate trailing and leading singular spaces of its small norm perturbations. Next we extend these algorithms to block triangulation of such perturbations.

Theorem 7.4. *For a matrix $A \in \mathbb{R}^{m\times n}$ and $q < l = \min\{n, m\}$, let $\sigma_q(A) \gg \sigma_{q+1}(A)$ and write $r = n - q$ and $\bar{r} = m - q$.*

(a) Suppose $L_0 \in \mathcal{G}_{0,1}^{n\times q}$, $L_1 \in \mathbb{R}^{n\times r}$, $\mathcal{R}(L_1) = \mathbb{T}_{A,r}$, and $||L_1|| = 1$. Then $||AL_1|| \le \sigma_{q+1}(A)$ and the condition number $\kappa(AL_0)$ is expected to have order $\sigma_1(A)/\sigma_q(A)$.

(b) Suppose $K_0 \in \mathcal{G}_{0,1}^{q\times m}$, $K_1 \in \mathbb{R}^{\bar{r}\times m}$, $\mathcal{R}(K_1) = \mathbb{S}_{A,\bar{r}}$, and $||K_1|| = 1$. Then $||K_1A|| \le \sigma_{q+1}(A)$ and the condition number $\kappa(K_0A)$ is expected to have order $\sigma_1(A)/\sigma_q(A)$.

Proof. To estimate $\kappa(AL_0)$ and $\kappa(K_0A)$ combine Theorems 3.2 and 4.1. To estimate $||AL_1||$ substitute $A = S_A\Sigma_A T_A^T$ and $T_A^TL_1 = (O_{n,n-r}, T_{A,r})L_1$ to obtain $AL_1 = S\Sigma T_A^TL_1 = S\Sigma(O_{n,n-r}, T_{A,r})L_1$, $||AL_1|| \le ||S\Sigma(O_{n,n-r}, T_{A,r})||\ ||L_1|| = \sigma_{q+1}(A)$. Similarly estimate $||K_1A||$. □

Clearly, small perturbation of the matrices K_1 and L_1 can only little change the upper bounds on the norms $||K_1A||$ and $||AL_1||$, and we can extend the results to the case where we are given approximate bases to the trailing singular spaces $\mathbb{S}_{A,\bar{r}}$ and $\mathbb{T}_{A,r}$.

Namely suppose that the assumptions of both parts (a) and (b) of the theorem hold for a small norm perturbation of the matrix A, that is $\mathcal{R}(K_1) \approx \mathbb{S}_{A,\bar{r}}$ and $\mathcal{R}(L_1) \approx \mathbb{T}_{A,r}$ for $r = n - q$, $\bar{r} = m - q$, $K_0 \in \mathcal{G}_{0,1}^{q\times m}$, $L_0 \in \mathcal{G}_{0,1}^{n\times q}$, $K_1 \in \mathbb{R}^{\bar{r}\times m}$, $L_1 \in \mathbb{R}^{n\times r}$. Then Theorem 7.4 implies that the $q \times q$ leading block W_{00} of the $m \times n$ matrix $W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix} = KAL$, for $K = \begin{pmatrix} K_0 \\ K_1 \end{pmatrix}$ and $L = (L_0 \mid L_1)$, is still expected to dominate the three other blocks. Namely by virtue of Theorem 7.4 we expect that the condition number $\kappa(W_{00})$ has order $\sigma_1(A)/\sigma_q(A)$, whereas the sum of the norms $||W_{01}|| + ||W_{10}|| + ||W_{11}||$ has at most the order $\sigma_{q+1}(A)$. Note that $\mathbf{y} = L\mathbf{x}$ provided $A\mathbf{y} = \mathbf{b}$ and $W\mathbf{x} = K\mathbf{b}$. If in addition the ratio $\sigma_1(A)/\sigma_q(A)$ is not large, that is if the matrix A has numerical rank q, then the matrix W_{00} is nonsingular and well conditioned. Let us specify this computation for $m = n$, $r = \bar{r}$, and the matrices K_1 and L_1 computed based on our randomized additive preconditioning.

Algorithm 7.1. Block triangulation with approximate trailing singular spaces.

INPUT: *A matrix $A \in \mathbb{R}^{n\times n}$ whose norm $||A||$ is neither large nor small, its numerical rank q satisfying $0 < q = n - r < n$, and a Subroutine* LIN·SOLVE *that either solves a linear system of equations if it is nonsingular and well condtioned or outputs FAILURE otherwise.*

OUTPUT: *FAILURE (with a low probability) or four random matrices K_0 and L_0 in $\mathbb{R}^{n\times q}$ and K_1 and L_1 in $\mathbb{R}^{n\times r}$ such that $W = (K_0 \mid K_1)^T A(L_0 \mid L_1) = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}$ and with a probability near 1 the $q \times q$ block submatrix $W_{00} = K_0^T AL_0$ is nonsingular, well conditioned, and strongly dominant, such that $\sigma_q(W_{00}) \gg \max\{||W_{01}||,\ ||W_{10}||,\ ||W_{11}||\}$.*

COMPUTATIONS:

1. *Generate four matrices $K_0, L_0 \in \mathcal{G}_{0,1}^{n\times q}$; $U, V \in \mathbb{G}_{0,1}^{n\times r}$. Output the matrices K_0 and L_0.*
2. *Compute the matrix $C = A + UV^T$, expected to be nonsingular and well conditioned.*
3. *Apply the Subroutine* LIN·SOLVE *to compute and to output the matrices $K_1 = C^{-T}V$ and $L_1 = C^{-1}U$. Stop and output FAILURE if so does the subroutine.*

Correctness of the algorithm follows from part (a) of Theorem 7.4. The algorithm fails with a low probability by virtue of Theorems 1.1 and 7.2 (part (a)) and the results of Section 7.2.

Block factorization

$$W = \begin{pmatrix} I & O \\ W_{10}W_{00}^{-1} & I \end{pmatrix} \begin{pmatrix} W_{00} & W_{01} \\ O & G \end{pmatrix}$$

for $G = W_{11} - W_{10}W_{00}^{-1}W_{01}$ implies that

$$W^{-1} = \begin{pmatrix} W_{00}^{-1} & -W_{00}^{-1}W_{01}G^{-1} \\ O & G^{-1} \end{pmatrix} \begin{pmatrix} I & O \\ -W_{10}W_{00}^{-1} & I \end{pmatrix} \tag{7.4}$$

and thus reduces the inversion of the matrices W and A and the solution of a linear system $A\mathbf{y} = \mathbf{b}$ to the similar operations with the matrices W_{00} and G of smaller sizes, where the matrix W_{00} is expected to be nonsingular and well conditioned.

Tables 8.11 and 8.12 demonstrate the power of this approach versus the customary algorithms.

Remark 7.4. *We expect to arrive at the matrices W_{01}, W_{10} and W_{11} having small norms. To counter the expected cancellation of the leading digits of the $2rn - r^2$ entries of these matrices, we should compute the matrices K_1 and L_1, their products by the blocks of the matrix A, and the Schur complement G and its inverse in (7.4) with a high precision p_+. These computations involve $O(n^2r)$ flops, that is just a r/n fraction of order n^3 flops in high precision p_+ required by Gaussian*

elimination. Moreover (see Section 7.6) we can represent the p_+-precision numbers as the sums of $O(p_+/p_{\text{double}})$ double precision numbers, apply iterative refinement, and perform computations by using $O(n^2 r p_+/p_{\text{double}})$ flops in double precision p_{double}.

Remark 7.5. *We can modify the block triangulation by replacing the random matrices K_0 with* $\text{nmb}(K_1)$ *and L_0 with* $\text{nmb}(L_1)$ *(cf. Algorithm 7.2 in the next subsection). E.g., we can first generate random matrices $G, H \in \mathcal{G}_{0,1}^{n\times q}$ and $U, V \in \mathbb{G}_{0,1}^{n\times r}$, all having full rank with probability* 1*, and then successively compute the matrices $C = A + UV^T$, $K_1 = C^{-T}V$, $L_1 = C^{-1}U$, $K_0 = (I_n - K_1(K_1^T K_1)^{-1}K_1^T)G$, and $L_0 = (I_n - L_1(L_1^T L_1)^{-1}L_1^T)H$. Apart from the well conditioned computation of the matrices $C^{-T}V$ and $C^{-1}U$, this takes $O(n^2 r)$ flops, and we can extend our comments in the previous remark.*

7.5 Block triangulation with approximate leading singular spaces

Suppose a square matrix A has a small positive numerical rank q and define a dual variation of Algorithm 7.1 based on part (b) of Theorem 7.2; matrix inversions are limited there to the $q \times q$ matrices H, $K_0^T K_0$ and $L_0^T L_0$. Alternatively we can employ any other pair of approximate bases for the left and right leading singular spaces. E.g., we can compute them as the products $A^T V$ and AU for $V \in \mathcal{G}_{0,1}^{q\times m}$ and $U \in \mathcal{G}_{0,1}^{n\times r}$ (cf. Section 7.3). In our dual algorithm we assume that the nonsingular input matrix A has been scaled so that the norm $||A^{-1}||$ is neither large nor small; we can yield this property by applying the technique pointed out at the end of Section 5.

Algorithm 7.2. Block triangulation with approximate leading singular spaces.

INPUT: *A nonsingular ill conditioned matrix $A \in \mathbb{R}^{n\times n}$ scaled so that the norm $||A^{-1}||$ is neither large nor small; the numerical rank q of the matrix A such that $0 < q = n - r < n$, and a Subroutine* INVERT *that either inverts a matrix if it is nonsingular and well conditioned or outputs FAILURE otherwise.*

OUTPUT: *FAILURE (with a low probability) or four matrices $K_0,\ L_0 \in \mathbb{R}^{n\times q}$ and $K_1,\ L_1 \in \mathbb{R}^{n\times r}$ such that*

$$W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix} = (K_0 \mid K_1)^T A (L_0 \mid L_1)$$

and the block submatrix $W_{00} = K_0^T A L_0$ is expected to be nonsingular, well conditioned, and strongly dominant, such that $\sigma_q(W_{00}) \gg \max\{||W_{01}||,\ ||W_{10}||,\ ||W_{11}||\}$.

COMPUTATIONS**:**

1. *Generate two matrices U_- and V_- in $\mathcal{G}_{0,1}^{n\times q}$.*
2. *Compute the matrix $H = I_q + V_- A U_-^T$ of (2.6).*
3. *Apply the Subroutine* INVERT *to compute the matrix H^{-1}. Stop and output FAILURE if so does the subroutine.*
4. *Compute the matrix $C_- = A - AU_- H^{-1} V_-^T A$ of (2.6).*
5. *Compute and output the matrices $K_0 = C_- U_-/||C_- U_-||$ and $L_0 = C_-^T V_-/||C_-^T V_-||$.*
6. *Compute the matrices $M = \text{nmb}(K_0^T)$ and $N = \text{nmb}(L_0^T)$ (see our Section 7.2, [PQ10], and [PQa] on the computation of nmbs).*
7. *Compute and output the matrices $K_1 = M/||M||$ and $L_1 = N/||N||$.*

Correctness of this randomized algorithm follows from parts (b) of Theorems 7.2 and 7.4, and we can extend the comments in Remark 7.4. The algorithm fails with a low probability by virtue of Theorems 1.1 and 7.2 (part (b)) and the results of Section 7.2.

7.6 Randomized additive preconditioning with the SMW recovery

Suppose that we seek the solution $\mathbf{y} = A^{-1}\mathbf{b}$ of a real nonsingular ill conditioned linear system $A\mathbf{y} = \mathbf{b}$ of n equations where we are given a small upper bound r on the numerical nullity of A. Assume that the norm $||A||$ is neither large nor small. Then randomized additive preprocessing $A \Longrightarrow C = A + UV^T$ for $U, V \in \mathcal{G}_{0,1}^{n\times r}$ is expected to produce a well conditioned matrix C (cf. Corollary 5.2). The SMW formula (1.1) implies that $\mathbf{y} = C^{-1}\mathbf{b} + C^{-1}UG^{-1}V^TC^{-1}\mathbf{b}$ for $G = I_r - V^TC^{-1}U$. Substitute $X_U = C^{-1}U$ and $\mathbf{x_b} = C^{-1}\mathbf{b}$ and obtain

$$\mathbf{y} = \mathbf{x_b} + X_UG^{-1}V^T\mathbf{x_b} \text{ for } G = I_r - V^TX_U. \tag{7.5}$$

This reduces the computation of $\mathbf{y}$ essentially to the solution of the matrix equation $CX = (U \mid \mathbf{b})$ for $X = (X_U \mid x_\mathbf{b})$, computing the matrix G, and its inversion. Here is a solution algorithm that incorporates iterative refinement.

Algorithm 7.3. Randomized Solution of a Linear System with Iterative Refinement.

INPUT: *a vector $\mathbf{b} \in \mathbb{R}^{n\times 1}$, a nonsingular ill conditioned matrix $A \in \mathbb{R}^{n\times n}$, and a positive upper bound r on its unknown numerical nullity.*

OUTPUT: $\tilde{\mathbf{y}} \approx A^{-1}\mathbf{b}$.

COMPUTATIONS*:*

1. *Generate two matrices $U, V \in \mathcal{G}_{0,\sigma}^{n\times r}$.*
2. *Compute the matrix $C = A + UV^T$, expected to be nonsingular and well conditioned.*
3. *Apply Gaussian elimination (or another direct algorithm) to compute an approximate inverse $Y \approx C^{-1}$. (Perform the computations in single or double precision. Application of the same algorithm to the original ill conditioned linear system $A\mathbf{y} = \mathbf{b}$ would require about as many flops but in extended precision.)*
4. *Apply iterative refinement employing the approximate inverse Y to compute sufficiently accurate solution $X = (X_U \mid x_\mathbf{b})$ of the matrix equation $CX = (U \mid \mathbf{b})$; then recover a close approximation to the vector $\mathbf{y} = A^{-1}\mathbf{b}$ based on (7.5).*

The algorithm involves matrices of smaller sizes $n \times r$ and $n \times (r + 1)$ and thus becomes more effective as r decreases toward the actual numerical nullity of A.

Handling an ill conditioned input A, we must perform computations with extended precision to counter magnification of rounding errors, but we confine this essentially to computing and inverting the Schur complement $G = I_r - V^TC^{-1}U$, which is the fraction r/n of the computational time of the customary algorithms for a linear system $A\mathbf{y} = \mathbf{b}$.

More precisely we must use a precision p exceeding $\log_2 \kappa(C)$; for well conditioned matrices C we can assume that $p > 2\log_2 \kappa(C)$, say. Then every loop of iterative refinement produces order $p - \log_2 \kappa(C)$ new correct bits per an application to an output value and is reduced essentially to multiplication of the matrices C and Y by $2r$ vectors, that is to $(4n-2)nr$ flops in a low (e.g., single or double) precision p. The refinement algorithm outputs order rn values; they can be accumulated with high accuracy as the sums of sufficiently many low precision summands. Overall this means performing $O(rn^2p_+/p)$ flops in low precision p at Stage 4 of Algorithm 7.3.

For comparison Gaussian elimination uses $\frac{2}{3}n^3 + O(n^2)$ flops in extended precision $p_+ \approx p_{\text{out}} + \log_2 \kappa(A)$ to output the solution to the ill conditioned linear system $A\mathbf{y} = \mathbf{b}$ with a prescribed precision p_{out}. We compute an approximate inverse Y of the well conditioned matrix C at Stage 3 by using $\frac{2}{3}n^3 + O(n^2)$ flops as well, but in the low precision p. The cost of performing Stages 1 and 2 is dominated, and so our progress is dramatic where $np \gg rp_+$ and p_+ greatly exceeds p.

Finally, given a nonsingular $n \times n$ matrix A (with $||A^{-1}|| \approx 1$) and a small upper bound q on its numerical rank, we can define a dual variation of Algorithm 7.3 as follows: generate a pair of matrices $U_-, V_- \in \mathbb{G}_{0,1}^{n\times q}$ and then compute the matrices H and C_- of (2.5) to reduce the solution of a linear system of equations $A\mathbf{y} = \mathbf{b}$ to computing the vector $\mathbf{y} = (C_-^{-1} - U_-V_-^T)\mathbf{b}$.

Remark 7.6. *One can replace iterative refinement with the Conjugate Gradient or GMRES algorithms (cf. [GL96]). To their advantage they use no approximate inverse, but they are more sensitive to the success of preconditioning. In particular every Conjugate Gradient loop (essentially multiplication of the matrices C and C^T by two vectors) produces order of $1/\kappa(C)$ new correct bits per an output value versus $p-\log_2 \kappa(C)$ in iterative refinement. Thus we need stronger upper bounds on $\kappa(C)$ to ensure progress in the presence of rounding errors.*

7.7 Randomized structured preprocessing

Would the $n \times n$ preprocessed matrices $C = A + UV^T$ inherit the structure of an $n \times n$ matrix A where $U, V \in \mathbb{R}^{n\times r}$? For a small value r the adverse impact of adding the matrix UV^T on the structure is small, e.g., the displacement rank increases by $O(r)$ (cf. [P01]).

We can control this impact even for large values r by endowing the matrices U and V with proper structure and relying on ample empirical data that show preconditioning power of such structured additive preprocessing and augmentation (see Table 8.5).

In particular in the case of a nonsingular ill conditioned $n \times n$ Toeplitz-like matrix A having numerical nullity r and norm $||A|| \approx 1$ and given with a displacement generator of a small length d, we can choose a pair of $n \times r$ standard Gaussian random Toeplitz matrices U and V and obtain a displacement generator for the matrix $C = A+UV^T$ having length $d+O(1)$ and expected to be well conditioned. By exploiting the structure we can operate with this matrix in nearly linear arithmetic time, e.g., solve a nonsingular linear system $A\mathbf{y} = \mathbf{b}$ in $O(d^2 n \log^2 n)$ flops, even where r is large.

Randomized augmentation can preserve matrix structure even better and has about the same preconditioning power (see Section 6 and Table 8.15).

Alternative deterministic techniques of homotopy continuation also support inversion of nonsingular Toeplitz and various other structured matrices in nearly linear time (see [P01, Section 6.9], [P07], [Pa10]).

7.8 Preprocessing for Newton–Toeplitz iteration

Recall Newton's iteration for matrix inversion

$$X_{i+1} = X_i(2I - CX_i), \ i = 0, 1, \dots. \tag{7.6}$$

Its ith loop squares the residual $I - CX_i$, that is,

$$I - CX_{i+1} = (I - CX_i)^2 = (I - CX_0)^{2^{i+1}}. \tag{7.7}$$

Therefore

$$||I - CX_{i+1}|| \le ||I - CX_i||^2 = ||I - CX_0||^{2^{i+1}}, \ i = 0, 1, \dots, \tag{7.8}$$

so that the approximations X_i quadratically converge to the inverse C^{-1} right from the start provided that $||I - CX_0|| < 1$.

We can ensure that $||I - CX_0|| \le 1 - \frac{2n}{(\kappa(C))^2(1+n)}$ by choosing $X_0 = \frac{2nC^T}{(1+n)||C||_1||C||_\infty}$ [PS91].

Such a map $C \Longrightarrow X_0$ preserves the matrix structure of Toeplitz or Hankel type, but is the structure maintained throughout the iteration? Not automatically. In fact Newton's loop can triple the displacement rank of a matrix X_k. The structure can be maintained, however, via recursive compression of the displacement (also called *recompression*), in which case we arrive at *Newton's structured* (e.g., Newton–Toeplitz) iteration. In particular we can periodically set to 0 the smallest singular values of the displacements of the matrices X_i to keep the length of the displacements within a fixed tolerance, equal to or a little exceeding the displacement rank of the input matrix C. At this stage we can also apply the techniques of Section 7.3 to approximate the displacements of the matrices X_i by low-rank matrices. We refer the reader to [P01, Chapter 6], [Pa10] on the history, variations, and analysis of this approach, proposed in [P92], [P93], and [P93a] for Toeplitz-like matrices. In [PBRZ99, Section 7.5.4] this iteration has been linked to iterative refinement combined with recursive updating of the input matrix; quadratic convergence of the resulting iteration has

been proved in [PBRZ99]. In [BM01] the extension of this study has naturally led to the important concept of approximate displacement rank of a matrix.

According to the estimates in [P01], the Newton–Toeplitz iteration converges quadratically right from the start provided $||I - CX_0|| < \frac{1}{(1+||Z_e||+||Z_f||)\kappa(C)}||L^{-1}||$, $||L^{-1}|| \le c_{e,f}n$, L denotes the associated displacement operator $L: C \to Z_eC - CZ_f$ for $e \neq f$ or $L: C \to C - Z_eCZ_f^T$ for $ef \neq 1$, and $c_{e,f}$ is a constant defined by e and f. Similar bounds can be deduced for other classes of matrices with displacement structure [P01, Section 6.6], [PRW02].

Newton's iteration can be incorporated into our randomized algorithms. E.g., it can be used instead of Gaussian elimination in Algorithm 7.3.

Conversely, our preconditioning techniques are a natural tool for decreasing the initial residual norm $||I - CX_0||$ where it is close to 1.

We can concurrently apply Newton–Toeplitz iteration to a number of scaled randomized small rank modifications of the input matrix. As soon as one of these applications produces the inverse, we can readily recover the inverse of the original matrix via the SMW formula (1.1) or in case of augmentation via (6.2) or Theorem 6.1. Other than that no processor communication or synchronization are requiered.

8 Numerical Experiments

Our numerical experiments with random general, Hankel, Toeplitz and circulant matrices have been performed in the Graduate Center of the City University of New York on a Dell server with a dual core 1.86 GHz Xeon processor and 2G memory running Windows Server 2003 R2. The test Fortran code was compiled with the GNU gfortran compiler within the Cygwin environment. Random numbers were generated with the random_number intrinsic Fortran function, assuming the uniform probability distribution over the range $\{x : -1 \le x < 1\}$. The tests have been designed by the first author and performed by his coauthors.

8.1 Conditioning tests

We have computed the condition numbers of $n \times n$ random general matrices for $n = 2^k$, $k = 5, 6, \ldots$, with entries sampled in the range $[-1, 1)$ as well as complex general, Toeplitz, and circulant matrices whose entries had real and imaginary parts sampled at random in the same range $[-1, 1)$. We have performed 100 tests for each class of inputs, each dimension n, and each nullity r. Tables 8.2–8.4 display the test results. The last four columns of each table display the average (mean), minimum, maximum, and standard deviation of the computed condition numbers of the input matrices, respectively. Namely we have computed the values $\kappa(A) = ||A||\ ||A^{-1}||$ for general, Toeplitz, and circulant matrices A and the values $\kappa_1(A) = ||A||_1\ ||A^{-1}||_1$ for Toeplitz matrices A. We have computed and displayed in Table 8.3 the 1-norms of Toeplitz matrices and their inverses rather than their 2-norms to facilitate the computations in the case of inputs of large sizes. Relationships (2.1) link the 1-norms and 2-norms to one another, but the empirical data in Table 8.1 consistently show even closer links, in all cases of $n \times n$ general, Toeplitz, and circulant matrices A where $n = 32, 64, \ldots, 1024$.

8.2 Preconditioning tests

Table 8.5 covers our tests for the preconditioning power of additive preprocessing in [PIMR10]. We have tested the input matrices of the following classes.

1n. *Nonsymmetric matrices of type I with numerical nullity r.* $A = S\Sigma_r T^T$ are $n \times n$ matrices where S and T are $n \times n$ random orthogonal matrices, that is, the factors Q in the QR factorizations of random real matrices; $\Sigma_r = \mathrm{diag}(\sigma_j)_{j=1}^n$ is the diagonal matrix such that $\sigma_{j+1} \le \sigma_j$ for $j = 1, \ldots, n-1$, $\sigma_1 = 1$, the values $\sigma_2, \ldots, \sigma_{n-r-1}$ are randomly sampled in the semi-open interval $[0.1, 1)$, $\sigma_{n-r} = 0.1$, $\sigma_j = 10^{-16}$ for $j = n-r+1, \ldots, n$, and therefore $\kappa(A) = 10^{16}$ [H02, Section 28.3].

1s. *Symmetric matrices of type I with numerical nullity* r. The same as in part 1n, but for $S = T$.

The matrices of the six other classes were constructed in the form of $\frac{A}{||A||} + \beta I$ where the recipes for defining the matrices A and scalars β are specified below.

2n. *Nonsymmetric matrices of type II with numerical nullity* r. $A = (W \mid WZ)$ where W and Z are random orthogonal matrices of sizes $n \times (n-r)$ and $(n-r) \times r$, respectively.

2s. *Symmetric matrices of type II with numerical nullity* r. $A = WW^T$ where W are random orthogonal matrices of size $n \times (n-r)$.

3n. *Nonsymmetric Toeplitz-like matrices with numerical nullity* r. $A = c(T \mid TS)$ for random Toeplitz matrices T of size $n \times (n-r)$ and S of size $(n-r) \times r$ and for a positive scalar c such that $||A|| \approx 1$.

3s. *Symmetric Toeplitz-like matrices with numerical nullity* r. $A = cTT^T$ for random Toeplitz matrices T of size $n \times (n-r)$ and a positive scalar c such that $||A|| \approx 1$.

4n. *Nonsymmetric Toeplitz matrices with numerical nullity* 1. $A = (a_{i,j})_{i,j=1}^{n}$ is an $n \times n$ Toeplitz matrix. Its entries $a_{i,j} = a_{i-j}$ are random for $i - j < n - 1$. The entry $a_{n,1}$ is selected to ensure that the last row is linearly expressed through the other rows.

4s. *Symmetric Toeplitz matrices with numerical nullity* 1. $A = (a_{i,j})_{i,j=1}^{n}$ is an $n \times n$ Toeplitz matrix. Its entries $a_{i,j} = a_{i-j}$ are random for $|i - j| < n - 1$, whereas the entry $a_{1,n} = a_{n,1}$ is a root of the quadratic equation $\det A = 0$. We have repeatedly generated the matrices A until we arrived at the quadratic equation having real roots.

We have set $\beta = 10^{-16}$ for symmetric matrices A in the classes 2s, 3s, and 4s, so that $\kappa(A) = 10^{16} + 1$ in these cases. For nonsymmetric matrices A we have defined the scalar β by an iterative process such that $||A|| \approx 1$ and $10^{-18}||A|| \le \kappa(A) \le 10^{-16}||A||$ [PIMR10, Section 8.2].

Table 8.5 displays the average values of the condition numbers $\kappa(C)$ of the matrices $C = A + UU^T$ over 100,000 tests for the inputs in the above classes, $r = 1, 2, 4, 8$ and $n = 100$. We have defined the additive preprocessor UU^T by a normalized $n \times r$ matrix $U = U/||U||$ where $U^T = (\pm I \mid O_{r,r} \mid \pm I \mid O_{r,r} \mid \ldots \mid O_{r,r} \mid \pm I \mid O_{r,s})$, we have chosen the integer s to obtain $n \times r$ matrices U and have chosen the signs for the matrices $\pm I$ at random.

In our further tests the condition numbers of the matrices $C = A + 10^p UV^T$ for $p = -10, -5, 5, 10$ were steadily growing within a factor $10^{|p|}$ as the value $|p|$ was growing. This have showed the importance of proper scaling of the additive preprocessor UV^T.

8.3 Application of random circulant multipliers

Table 8.6 shows the results of our tests of the solution of a nonsingular well conditioned linear system $A\mathbf{y} = \mathbf{b}$ of n equations whose coefficient matrix had an $n/2 \times n/2$ ill conditioned leading principal block for $n = 64, 256, 1024$. We have performed 100 numerical tests for each dimension n and computed the maximum, minimum and average relative residual norms $||A\mathbf{y} - \mathbf{b}||/||\mathbf{b}||$ as well as standard deviation. GENP applied to these systems has output corrupted solutions with residual norms ranging from 10 to 10^8. When we preprocessed the systems with circulant multipliers filled with ± 1 (choosing the n signs $\pm$ at random), the norms decreased to at worst 10^{-7} for all inputs. Table 8.6 also shows further decrease of the norm in a single step of iterative refinement.

8.4 Approximation of the tails and heads of SVDs and low-rank approximation of a matrix

Table 8.7 shows the data from our tests on the approximation of trailing singular spaces of the SVD of an $n \times n$ matrix A having numerical nullity $r = n - q$ and on the approximation of this matrix with a matrix of rank $q = n - r$.

For $n = 64, 128, 256$ and $q = 1, 8, 32$ we have generated $n \times n$ random unitary matrices S and T and diagonal matrices $\Sigma = \mathrm{diag}(\sigma_j)_{j=1}^{n}$ such that $\sigma_j = 1/j,\ j = 1, \ldots, q$, $\sigma_j = 10^{-10},\ j = q+1, \ldots, n$ (cf. [H02, Section 28.3]). Then we computed the input matrices $A = S_A \Sigma_A T_A^T$, for which $||A|| = 1$ and $\kappa(A) = 10^{10}$. Furthermore we have generated pairs of $n \times r$ random matrices U and V for

$r = 1, 8, 32$, scaled them to have $||UV^T|| \approx 1$, and computed the matrices $C = A + UV^T$, $B_r = C^{-1}U$, $Y_r = B_r^+ T_{A,r}$; $B_r Y$, $B_r Y_r - T_{A,r}$, $Q = Q(B_r)$, and $AQQ^T = A - A(I_n - QQ^T)$.

Table 8.7 summarizes the data on the values $\kappa(C)$ and the residual norms $\mathrm{rn}_1 = ||B_r Y_r - T_{A,r}||$ (cf. (7.1)) and $\mathrm{rn}_2 = ||AQQ^T||$ observed in 100 runs of our tests for every pair of n and q.

We have performed similar tests on the approximation of leading singular spaces of the SVDs of the same $n \times n$ matrices A having numerical rank q and numerical nullity $r = n - q$ and on the approximation of this matrix with a matrix of rank q. In these tests we employed dual additive preprocessing to approximate matrix bases for the leading singular spaces $\mathbb{T}_{q,A}$ of the matrices A. We have generated the pairs of $n \times q$ random matrices U_- and V_- for $q = 1, 8, 32$, scaled them to have $||U_- V_-^T|| \approx ||A^{-1}|| = 10^{10}$, and successively computed the matrices $H = I_q + V_-^T A U_-$ and $C_- = A - AU_- H^{-1} V_-^T A$ (cf. (2.6)), $B_{q,A} = C_-^T V_-$ (all three with extended precision), $Y_{q,A} = (B_{q,A})^+ T_{q,A}$; $B_{q,A} Y_{q,A}$, $B_{q,A} Y_{q,A} - T_{q,A}$, $Q_{q,A} = Q(B_{q,A})$, and $A - AQ_{q,A}(Q_{q,A})^T$. Table 8.8 summarizes the data on the condition numbers $\kappa(C_-)$ and the residual norms $\mathrm{rn}^{(1)} = ||B_{q,A} Y_{q,A} - T_{q,A}||$ (cf. (7.2)) and $\mathrm{rn}^{(2)} = ||A - AQ_{q,A}(Q_{q,A})^T||$ obtained in 100 runs of our tests for every pair of n and q.

We have also performed similar tests where we generated $n \times q$ random matrices U (for $q = 1, 8, 32$) and $n \times q$ random Toeplitz matrices $\bar{U}$ (for $q = 8, 32$) and then replaced the above matrix $B_{q,A}$ with the approximate matrix bases $Q_{q,A} = Q(AU)$ and $A\bar{U}$ for the leading singular space $\mathbb{T}_{q,A}$. Tables 8.9 and 8.10 display the results of these tests. In both cases the residual norms are equally small and are about as small as in Tables 8.7 and 8.8.

8.5 Solution of linear systems of equations based on approximation of trailing and leading singular spaces of the SVDs

For our further tests we have chosen $n = 32, 64$ and $r = 1, 2, 4$ and for every pair (n, r) generated 100 instances of vectors $\mathbf{b}$ and matrices A, U, and V as follows.

We have generated (a) random vectors $\mathbf{b}$ of dimension n, (b) the matrices A as the error-free products $S\Sigma T^T$ where S and T were $n \times n$ random real orthonormal matrices (generated with double precision), $\Sigma = \mathrm{diag}(\sigma_j)_{j=1}^n$, $\sigma_{n-j} = 10^{-17}$ for $j = 0, 1, \ldots, r-1$, and $\sigma_{n-j} = 1/(n-j)$ for $j = r, \ldots, n-1$, and (c) $n \times r$ random matrices U and V such that $||A| = ||U|| = ||V|| = 1$. Note that $||A^{-1}|| = 10^{17}$.

For every choice of these matrices we have solved the linear systems $A\mathbf{y} = \mathbf{b}$ by applying Algorithm 7.1. We first generated $n \times (n-r)$ random matrices K_0 and L_0 and then computed the matrices $C = A + UV^T$ (which were always nonsingular and well conditioned in our tests), $K_1 = C^{-T}V$, $L_1 = C^{-1}U$, and $W = (K_0 \mid K_1)^T A (L_0 \mid L_1) = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}$. In all our tests the $(n-r) \times (n-r)$ leading principal $(n-r) \times (n-r)$ block $W_{00} = K_0^T A L_0$ was well conditioned and strongly dominated the three other blocks W_{01}, W_{10}, and W_{11} in the 2×2 block matrix W, as we expected to see based on our analysis in Section 7.4. To solve the linear system $W\mathbf{x} = (K_0 \mid K_1)^T \mathbf{b}$, we first computed the dominated blocks W_{01}, W_{10}, and W_{11} with extended precision, then eliminated the subdiagonal block and computed the solution of the resulting block triangular linear system; the leading block W_{00} of its coefficient matrix was expected to be and consistently turned out to be well conditioned. Finally we computed and output the vector $\mathbf{y} = (L_0 \mid L_1)\mathbf{x}$.

Table 8.11 shows the average (mean) values of the relative residual norms $||A\mathbf{y} - \mathbf{b}||/||\mathbf{b}||$ of the output vectors $\mathbf{y}$ (these values range about 10^{-10}) as well as the minimums, maximums, and standard deviations in these tests.

For the same ill conditioned inputs the Subroutine MLDIVIDE(A,B) for Gaussian elimination from MATLAB has produced corrupted outputs, as can be seen from Table 8.12.

We have also performed similar tests for $n = 32, 64$ and $n \times n$ matrices A and vectors $\mathbf{b}$ generated as before, but for $q = n - r = 1, 2, 4$, and then we computed orthogonal matrices K_0, K_1, L_0 and L_1 by employing dual additive preprocessing and Algorithm 7.2. We first generated and scaled the pairs of $n \times q$ random matrices U_- and V_- such that $||U_-|| \approx ||V_-|| \approx 3 * 10^8$, and so $||U_-||\, ||V_-|| \approx ||A^{-1}|| = 10^{17}$. Then we successively computed the matrices H and C_- (as in Section 8.4), $C_- U_-$ and $C_-^T V_-$ (all with extended precision), $K_0 = Q(C_- U_-)$, $L_0 = Q(C_-^T V_-)$, $K_1 = Q(\mathrm{nmb}(K_0^T))$ and

$L_1 = Q(\mathrm{nmb}(L_0^T))$, and continued as in the tests for Table 8.11. We displayed the results in Table 8.13, showing the residual norms of the order 10^{-9} on the average.

Furthermore we have performed similar tests where we first generated $n \times q$ random matrices U and V and then replaced the above matrices K_0 and L_0 by $K_0 = Q(A^T V)$ and $L_0 = Q(AU)$. Table 8.14 displays the results of these tests, showing the residual norms of the order 10^{-25} on the average.

Then again for the same ill conditioned inputs the Subroutine MLDIVIDE(A,B) for Gaussian elimination from MATLAB produced corrupted outputs, similarly to the results in Table 8.12.

8.6 Solution of a real symmetric Toeplitz linear system of equations with randomized augmentation

We have solved 100 real symmetric linear systems of equations $T\mathbf{y} = \mathbf{b}$ for each n where we used vectors $\mathbf{b}$ with random coordinates from the range $[-1, 1)$ and Toeplitz matrices $T = S + 10^{-9} I_n$ for an $n \times n$ singular symmetric Toeplitz matrices S having rank $n-1$ and nullity 1 and generated according to the recipe in [PQ10, Section 10.1b].

Table 8.15 shows the average CPU time of the solution by our Algorithm 6.1 and, for comparison, based on the QR factorization and SVD, which we computed by applying the LAPACK procedures DGEQRF and DGESVD, respectively. To solve the auxiliary Toeplitz linear system $K\mathbf{x} = \mathbf{e}_1$ in Algorithm 6.1, we first employed the Toeplitz linear solver of [KV99], [V99], [VBHK01], and [VK98] and then applied iterative refinement with double precision.

The abbreviations "Alg. 6.1", "QR", and "SVD" indicate the respective algorithms. The last two columns of the table display the ratios of these data in the first and the two other columns.

We measured the CPU time with the mclock function by counting cycles. One can convert them into seconds by dividing their number by a constant CLOCKS_PER_SEC, which is 1000 on our platform. We marked the table entries by a "-" where the tests have run too long and were not completed.

We have obtained the solutions $\mathbf{y}$ with the relative residual norms of about 10^{-15} in all three algorithms, which showed that Algorithm 6.1 employing iterative refinement was as reliable as the QR and SVD based solutions but ran much faster.

We refer the reader to [PQZC, Table 3] on similar test results for the solution of ill conditioned homogeneous Toeplitz linear systems.

9 Related work, our technical novelties, and further study

Preconditioned iterative algorithms for linear systems of equations is a classical subject [A94], [B02], [G97]. The problem of creating inexpensive preconditioners for general use has been around for a long while as well.

On some earlier study of conditioning of random matrices see [D88], [E88], [ES05], [CD05], [SST06], [B11]; estimation of the condition numbers of random structured matrices was stated as a challenge in [SST06]; we provide such estimates for Gaussian random Toeplitz and circulant matrices in Sections 3.4 and 3.5. In particular the estimates show that the expected condition number of a Gaussian random $n \times n$ Toeplitz matrix does not grow exponentially fast as n grows to infinity; this can be surprising in view of [BG05].

Our present study of randomized preconditioning substantially advances the works [PGMQ], [PIMR10], [PQZa], and [PQZC]. Besides the cited estimates for the condition numbers of Gaussian random Toeplitz and circulant matrices, our technical novelties include randomized multiplicative and additive preconditioning, the dual SMW formula, and block factorizations in Sections 7.4 and 7.5 based on randomized approximation of singular spaces.

Approximation by low-rank matrices (cf. Section 7.3) and the extensions to tensor decompositions are thriving research area, with numerous applications to matrix and tensor computations. Its previous study can be largely traced through the papers [GTZ97], [GT01], [GOS08], [T00], [MMD08], [OT09], [HMT11], and the bibliography therein, but much earlier advances in this area appeared in the papers [BCLR79], [B80], [B85], [B86], [BC87], directed to estimating the border rank of matrices

Table 8.1: Norms of random general, Toeplitz and circulant matrices and of their inverses

matrix A	n	$\|\|A\|\|_1$	$\|\|A\|\|_2$	$\frac{\|\|A\|\|_1}{\|\|A\|\|_2}$	$\|\|A^{-1}\|\|_1$	$\|\|A^{-1}\|\|_2$	$\frac{\|\|A^{-1}\|\|_1}{\|\|A^{-1}\|\|_2}$
General	32	1.9×10^1	1.8×10^1	1.0×10^0	4.0×10^2	2.1×10^2	1.9×10^0
General	64	3.7×10^1	3.7×10^1	1.0×10^0	1.2×10^2	6.2×10^1	2.0×10^0
General	128	7.2×10^1	7.4×10^1	9.8×10^{-1}	3.7×10^2	1.8×10^2	2.1×10^0
General	256	1.4×10^2	1.5×10^2	9.5×10^{-1}	5.4×10^2	2.5×10^2	2.2×10^0
General	512	2.8×10^2	3.0×10^2	9.3×10^{-1}	1.0×10^3	4.1×10^2	2.5×10^0
General	1024	5.4×10^2	5.9×10^2	9.2×10^{-1}	1.1×10^3	4.0×10^2	2.7×10^0
Toeplitz	32	1.8×10^1	1.9×10^1	9.5×10^{-1}	2.2×10^1	1.3×10^1	1.7×10^0
Toeplitz	64	3.4×10^1	3.7×10^1	9.3×10^{-1}	4.6×10^1	2.4×10^1	2.0×10^0
Toeplitz	128	6.8×10^1	7.4×10^1	9.1×10^{-1}	1.0×10^2	4.6×10^1	2.2×10^0
Toeplitz	256	1.3×10^2	1.5×10^2	9.0×10^{-1}	5.7×10^2	2.5×10^2	2.3×10^0
Toeplitz	512	2.6×10^2	3.0×10^2	8.9×10^{-1}	6.9×10^2	2.6×10^2	2.6×10^0
Toeplitz	1024	5.2×10^2	5.9×10^2	8.8×10^{-1}	3.4×10^2	1.4×10^2	2.4×10^0
Circulant	32	1.6×10^1	1.8×10^1	8.7×10^{-1}	9.3×10^0	1.0×10^1	9.2×10^{-1}
Circulant	64	3.2×10^1	3.7×10^1	8.7×10^{-1}	5.8×10^0	6.8×10^0	8.6×10^{-1}
Circulant	128	6.4×10^1	7.4×10^1	8.6×10^{-1}	4.9×10^0	5.7×10^0	8.5×10^{-1}
Circulant	256	1.3×10^2	1.5×10^2	8.7×10^{-1}	4.7×10^0	5.6×10^0	8.4×10^{-1}
Circulant	512	2.6×10^2	3.0×10^2	8.7×10^{-1}	4.5×10^0	5.4×10^0	8.3×10^{-1}
Circulant	1024	5.1×10^2	5.9×10^2	8.7×10^{-1}	5.5×10^0	6.6×10^0	8.3×10^{-1}

and tensors and initially motivated by the design of fast matrix multiplication algorithms. Presently, linking tensor and matrix computations for their acceleration is a fashionable subject with applications to many important areas of modern computing (see, e.g., [T00], [MMD08], [OT09]), but then again its earliest examples appeared in the cited papers on border rank and in [P72]. The latter paper has introduced the technique of trilinear aggregation, a basic ingredient of all subsequent fast algorithms for matrix multiplication with the inputs of both immense sizes (far beyond any practical interest) [P79], [P81], [P84], [CW90], [S10], [VW12] and realistic moderate sizes [P81], [P84], [LPS92], [K04], and historically was the first example of the acceleration of fundamental matrix computation by means of tensor decomposition.

For natural directions of our further study we also note augmentation, block randomization and specification of our techniques to structured matrices, particularly to matrices with displacement structure treated in unified way (cf. [KKM79], [P01]).

A natural link between augmentation and aggregation processes of [MP80] has been revealed in [PQa] and is worth further study.

Unification of the computations with structured matrices of Toeplitz, Hankel, Vandermonde and Cauchy types based on operating with them in terms of their displacements and the method of displacement transformation can be traced to [P90] (cf. [P01]). Treatment of ill conditioned structured matrices is a well known challenge (cf. [VBHK01]); the best customary recipes employ displacement transformation and involve quadratic arithmetic time or large overhead constants [GKO95], [CGLX], [CGSXZ], [G98], [P10], [R06]; our present advance relies on randomized additive preconditioning and augmentation, also studied in [PQa]. Comparison and combination of these techniques with the homotopy continuation methods and Newton's structured iteration (cf. Section 7.8, [PKRK], [P01, Chapter 6], and [Pa10] seem to be natural research directions. Advancing our progress in Section 3.4 toward formal support of randomized Toeplitz preconditioning is another research challenge.

Appendix

Table 8.2: Condition numbers $\kappa(A)$ of random matrices A

n	**input**	**min**	**max**	**mean**	**std**
32	real	2.4×10^1	1.8×10^3	2.4×10^2	3.3×10^2
32	complex	2.7×10^1	8.7×10^2	1.1×10^2	1.1×10^2
64	real	4.6×10^1	1.1×10^4	5.0×10^2	1.1×10^3
64	complex	5.2×10^1	4.2×10^3	2.7×10^2	4.6×10^2
128	real	1.0×10^2	2.7×10^4	1.1×10^3	3.0×10^3
128	complex	1.3×10^2	2.5×10^3	3.9×10^2	3.3×10^2
256	real	2.4×10^2	8.4×10^4	3.7×10^3	9.7×10^3
256	complex	2.5×10^2	1.4×10^4	1.0×10^3	1.5×10^3
512	real	3.9×10^2	7.4×10^5	1.8×10^4	8.5×10^4
512	complex	5.7×10^2	3.2×10^4	2.3×10^3	3.5×10^3
1024	real	8.8×10^2	2.3×10^5	8.8×10^3	2.4×10^4
1024	complex	7.2×10^2	1.3×10^5	5.4×10^3	1.4×10^4
2048	real	2.1×10^3	2.0×10^5	1.8×10^4	3.2×10^4
2048	complex	2.3×10^3	5.7×10^4	6.7×10^3	7.2×10^3

Table 8.3: Condition numbers $\kappa_1(A) = \frac{||A||_1}{||A^{-1}||_1}$ of random Toeplitz matrices A

n	**min**	**mean**	**max**	**std**
256	9.1×10^2	9.2×10^3	1.3×10^5	1.8×10^4
512	2.3×10^3	3.0×10^4	2.4×10^5	4.9×10^4
1024	5.6×10^3	7.0×10^4	1.8×10^6	2.0×10^5
2048	1.7×10^4	1.8×10^5	4.2×10^6	5.4×10^5
4096	4.3×10^4	2.7×10^5	1.9×10^6	3.4×10^5
8192	8.8×10^4	1.2×10^6	1.3×10^7	2.2×10^6

A Uniform random sampling and nonsingularity of random matrices

Uniform random sampling of elements from a finite set Δ is their selection from this set at random, independently of each other and under the uniform probability distribution on the set Δ.

Theorem A.1. *Under the assumptions of Lemma 3.1 let the values of the variables of the polynomial be randomly and uniformly sampled from a finite set Δ. Then the polynomial vanishes with a probability at most $\frac{d}{|\Delta|}$.*

Corollary A.1. *Let the entries of an $m \times n$ general or Toeplitz matrix have been randomly and uniformly sampled from a finite set Δ of cardinality $|\Delta|$ (in any fixed ring). Let $l = \min\{m, n\}$. Then (a) every $k \times k$ submatrix M for $k \le l$ is nonsingular with a probability at least $1 - \frac{k}{|\Delta|}$ and (b) is strongly nonsingular with a probability at least $1 - \sum_{i=1}^{k} \frac{i}{|\Delta|} = 1 - \frac{(k+1)k}{2|\Delta|}$. Furthermore (c) if the submatrix M is indeed nonsingular, then any entry of its inverse is nonzero with a probability at least $1 - \frac{k-1}{|\Delta|}$.*

Proof. The claimed properties of nonsingularity and nonvanishing hold for generic matrices. The singularity of a $k \times k$ matrix means that its determinant vanishes, but the determinant is a polynomial of total degree k in the entries. Therefore Theorem A.1 implies parts (a) and consequently (b). Part (c) follows because a fixed entry of the inverse vanishes if and only if the respective entry of the

Table 8.4: Condition numbers $\kappa(A)$ of random circulant matrices A

n	**min**	**mean**	**max**	**std**
256	9.6×10^0	1.1×10^2	3.5×10^3	4.0×10^2
512	1.4×10^1	8.5×10^1	1.1×10^3	1.3×10^2
1024	1.9×10^1	1.0×10^2	5.9×10^2	8.6×10^1
2048	4.2×10^1	1.4×10^2	5.7×10^2	1.0×10^2
4096	6.0×10^1	2.6×10^2	3.5×10^3	4.2×10^2
8192	9.5×10^1	3.0×10^2	1.5×10^3	2.5×10^2
16384	1.2×10^2	4.2×10^2	3.6×10^3	4.5×10^2
32768	2.3×10^2	7.5×10^2	5.6×10^3	7.1×10^2
65536	2.4×10^2	1.0×10^3	1.2×10^4	1.3×10^3
131072	3.9×10^2	1.4×10^3	5.5×10^3	9.0×10^2
262144	6.3×10^2	3.7×10^3	1.1×10^5	1.1×10^4
524288	8.0×10^2	3.2×10^3	3.1×10^4	3.7×10^3
1048576	1.2×10^3	4.8×10^3	3.1×10^4	5.1×10^3

adjoint vanishes, but up to the sign the latter entry is the determinant of a $(k-1)\times(k-1)$ submatrix of the input matrix M, and so it is a polynomial of degree $k-1$ in its entries. □

B Conditioning of random complex matrices

We have assumed dealing with real random matrices and vectors throughout the paper, but most of our study can be readily extended to the computations in the field $\mathbb{C}$ of complex numbers if we replace the transposes A^T by the Hermitian transposes A^H. All the results of Section 3.2 apply to this case equally well. Below is some elaboration upon the respective extension of our probabilistic bounds on the norms and singular values.

Definition B.1. *The set $\mathcal{G}^{m\times n}_{\mathbb{C},\mu,\sigma}$ of $m \times n$ complex Gaussian random matrices with a mean μ and a variance σ is the set $\{A + B\sqrt{-1}\}$ for $(A \mid B) \in \mathcal{G}^{m\times 2n}_{\mu,\sigma}$ (cf. Definition 3.2).*

We can immediately extend Theorem 3.2 to the latter matrices. Let us extend Theorem 3.1. Its original proof in [SST06] relies on the following result.

Lemma B.1. *Suppose y is a positive number; $\mathbf{w} \in \mathbb{R}^{n\times 1}$ is any fixed real unit vector, $||\mathbf{w}|| = 1$, $A \in \mathcal{G}^{n\times n}_{\mu,\sigma}$ and therefore is nonsingular with probability* 1*. Then*

$$\text{Probability}\{||A^{-1}\mathbf{w}|| > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

The following lemma and corollary extend Lemmas 3.2 and B.1 to the complex case.

Lemma B.2. *The bound of Lemma 3.2 also holds provided $\mathbf{t} = \mathbf{q} + \mathbf{r}\sqrt{-1}$ is a fixed complex unit vector and $\mathbf{b} = \mathbf{f} + \mathbf{g}\sqrt{-1} \in \mathcal{G}^{n\times 1}_{\mathbb{C},\mu,\sigma}$ is a complex vector such that $\mathbf{f}$, $\mathbf{g}$, $\mathbf{q}$ and $\mathbf{r}$ are real vectors, $||\mathbf{t}|| = 1$, and the vectors $\mathbf{f}$ and $\mathbf{g}$ are in $\mathcal{G}^{n\times 1}_{\mu,\sigma}$.*

Proof. We have $\mathbf{t}^H\mathbf{b} = \mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g} + (\mathbf{q}^T\mathbf{g} - \mathbf{r}^T\mathbf{f})\sqrt{-1}$, and so $|\mathbf{t}^H\mathbf{b}|^2 = |\mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g}|^2 + |\mathbf{q}^T\mathbf{g} - \mathbf{r}^T\mathbf{f}|^2$. Hence $|\mathbf{t}^H\mathbf{b}| \ge |\mathbf{q}^T\mathbf{f} + \mathbf{r}^T\mathbf{g}| = |\mathbf{u}^T\mathbf{v}|$ where $\mathbf{u}^T = (\mathbf{q}^T \mid \mathbf{r}^T)^T$ and $\mathbf{v} = (\mathbf{f}^T \mid \mathbf{g}^T)^T$. Note that $\mathbf{v} \in \mathcal{G}^{1\times 2n}_{\mu,\sigma}$ and $||\mathbf{u}|| = ||\mathbf{t}|| = 1$ and apply Lemma 3.2 to real vectors $\mathbf{u}$ and $\mathbf{v}$ replacing $\mathbf{b}$ and $\mathbf{t}$. □

Corollary B.1. *Suppose y is a positive number; $A \in \mathcal{G}^{n\times n}_{\mathbb{C},\mu,\sigma}$ and therefore is nonsingular with probability* 1*. Then*

$$\text{Probability}\{||A^{-1}\mathbf{e}_j|| > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

Table 8.5: Preconditioning tests

Type	r	Cond (C)
1n	1	3.21E+2
1n	2	4.52E+3
1n	4	2.09E+5
1n	8	6.40E+2
1s	1	5.86E+2x
1s	2	1.06E+4
1s	4	1.72E+3
1s	8	5.60E+3
2n	1	8.05E+1
2n	2	6.82E+3
2n	4	2.78E+4
2n	8	3.59E+3
2s	1	1.19E+3
2s	2	1.96E+3
2s	4	1.09E+4
2s	8	9.71E+3
3n	1	2.02E+4
3n	2	1.53E+3
3n	4	6.06E+2
3n	8	5.67E+2
3s	1	2.39E+4
3s	4	1.69E+3
3s	8	6.74E+3
4n	1	4.93E+2
4n	2	4.48E+2
4n	4	2.65E+2
4n	8	1.64E+2
4s	1	1.45E+3
4s	2	5.11E+2
4s	4	7.21E+2
4s	8	2.99E+2

Proof. In the case of real matrices A the lemma is supported by the argument in the proof of [SST06, Lemma 3.2], which employs Lemma 3.2. By employing Lemma B.2 instead of Lemma 3.2, we obtain Corollary B.1 from the same argument in [SST06]. □

Corollary B.2. *Under the assumptions of Corollary B.1 we have* $||A^{-1}|| \le \sum_{j=1}^{n} X_j$ *where* X_j *are nonnegative random variables such that*

$$\text{Probability}\{X_j > 1/y\} \le \sqrt{\frac{2}{\pi}}\frac{y}{\sigma} \text{ for } j = 1, \dots, n.$$

Proof. Recall that for any $n \times n$ matrix B we have $||B|| = ||B\mathbf{w}||$ for some unit vector $\mathbf{w} = \sum_{j=1}^{n} w_j \mathbf{e}_j$. We have $|w_j| \le ||\mathbf{w}|| = 1$ for all j. Substitute $B = A^{-1}$ and obtain $||A^{-1}|| = ||A^{-1}\mathbf{w}|| = ||\sum_{j=1}^{n} A^{-1} w_j \mathbf{e}_j|| \le \sum_{j=1}^{n} |w_j| \, ||A^{-1}\mathbf{e}_j||$, and so $||A^{-1}|| \le \sum_{j=1}^{n} X_j$ where $X_j = ||A^{-1}\mathbf{e}_j||$ for all j. It remains to combine this bound with Corollary B.1. □

The corollary implies that Probability$\{||A^{-1}|| > 1/y\}$ converges to 0 proportionally to y as $y \to 0$, which can be viewed as an extension of Theorem 3.1 to the case of complex inputs. One can deduce

Table 8.6: Relative residual norms of the solutions by GENP with randomized circulant multiplicative preprocessing (cf. [PQZa, Table 2])

dimension	iterations	min	max	mean	std
64	0	4.7×10^{-14}	8.0×10^{-11}	4.0×10^{-12}	1.1×10^{-11}
64	1	1.9×10^{-15}	5.3×10^{-13}	2.3×10^{-14}	5.4×10^{-14}
256	0	1.7×10^{-12}	1.4×10^{-7}	2.0×10^{-9}	1.5×10^{-8}
256	1	8.3×10^{-15}	4.3×10^{-10}	4.5×10^{-12}	4.3×10^{-11}
1024	0	1.7×10^{-10}	4.4×10^{-9}	1.4×10^{-9}	2.1×10^{-9}
1024	1	3.4×10^{-14}	9.9×10^{-14}	6.8×10^{-14}	2.7×10^{-14}

similar extension of Theorems 3.4–3.6. The resulting estimates are a little weaker than in Section 3.3, being overly pessimistic; actually random complex matrices are a little better conditioned than random real matrices (see [E88], [ES05], [CD05] and our Table 8.2).

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Table 8.7: Approximation of the tails of the SVDs and lower-rank approximation of a matrix (cf. [PQ10])

r	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\kappa(C)$	64	$2.38 \times 10^{+02}$	$1.10 \times 10^{+05}$	$6.25 \times 10^{+03}$	$1.68 \times 10^{+04}$
1	$\kappa(C)$	128	$8.61 \times 10^{+02}$	$7.48 \times 10^{+06}$	$1.32 \times 10^{+05}$	$7.98 \times 10^{+05}$
1	$\kappa(C)$	256	$9.70 \times 10^{+02}$	$3.21 \times 10^{+07}$	$3.58 \times 10^{+05}$	$3.21 \times 10^{+06}$
1	rn_1	64	4.01×10^{-10}	1.50×10^{-07}	5.30×10^{-09}	1.59×10^{-08}
1	rn_1	128	7.71×10^{-10}	5.73×10^{-07}	1.58×10^{-08}	6.18×10^{-08}
1	rn_1	256	7.57×10^{-10}	3.2×10^{-07}	1.69×10^{-08}	5.02×10^{-08}
1	rn_2	64	4.01×10^{-10}	1.50×10^{-07}	5.30×10^{-09}	1.59×10^{-08}
1	rn_2	128	7.71×10^{-10}	5.73×10^{-07}	1.58×10^{-08}	6.18×10^{-08}
1	rn_2	256	7.57×10^{-10}	3.22×10^{-07}	1.69×10^{-08}	5.02×10^{-08}
8	$\kappa(C)$	64	$1.26 \times 10^{+03}$	$1.61 \times 10^{+07}$	$2.68 \times 10^{+05}$	$1.71 \times 10^{+06}$
8	$\kappa(C)$	128	$2.92 \times 10^{+03}$	$3.42 \times 10^{+06}$	$1.58 \times 10^{+05}$	$4.12 \times 10^{+05}$
8	$\kappa(C)$	256	$1.39 \times 10^{+04}$	$8.75 \times 10^{+07}$	$1.12 \times 10^{+06}$	$8.74 \times 10^{+06}$
8	rn_1	64	3.39×10^{-10}	2.27×10^{-06}	2.74×10^{-08}	2.27×10^{-07}
8	rn_1	128	4.53×10^{-10}	1.91×10^{-07}	1.03×10^{-08}	2.79×10^{-08}
8	rn_1	256	8.74×10^{-10}	1.73×10^{-07}	7.86×10^{-09}	1.90×10^{-08}
8	rn_2	64	1.54×10^{-09}	7.59×10^{-06}	8.87×10^{-08}	7.58×10^{-07}
8	rn_2	128	1.82×10^{-09}	7.27×10^{-07}	2.95×10^{-08}	8.57×10^{-08}
8	rn_2	256	2.62×10^{-09}	3.89×10^{-07}	2.27×10^{-08}	5.01×10^{-08}
32	$\kappa(C)$	64	$1.77 \times 10^{+03}$	$9.68 \times 10^{+06}$	$1.58 \times 10^{+05}$	$9.70 \times 10^{+05}$
32	$\kappa(C)$	128	$1.65 \times 10^{+04}$	$6.12 \times 10^{+07}$	$1.02 \times 10^{+06}$	$6.19 \times 10^{+06}$
32	$\kappa(C)$	256	$3.57 \times 10^{+04}$	$2.98 \times 10^{+08}$	$4.12 \times 10^{+06}$	$2.98 \times 10^{+07}$
32	rn_1	64	2.73×10^{-10}	3.29×10^{-08}	2.95×10^{-09}	4.93×10^{-09}
32	rn_1	128	3.94×10^{-10}	1.29×10^{-07}	7.18×10^{-09}	1.64×10^{-08}
32	rn_1	256	6.80×10^{-10}	4.00×10^{-07}	1.16×10^{-08}	4.27×10^{-08}
32	rn_2	64	2.10×10^{-09}	1.49×10^{-07}	1.55×10^{-08}	2.18×10^{-08}
32	rn_2	128	2.79×10^{-09}	3.80×10^{-07}	3.81×10^{-08}	6.57×10^{-08}
32	rn_2	256	5.35×10^{-09}	1.05×10^{-06}	5.70×10^{-08}	1.35×10^{-07}

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Table 8.8: Approximation of the heads of SVDs and low-rank approximation of a matrix by using dual additive preprocessing

q	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\kappa(C)$	64	$1.83 \times 10^{+02}$	$1.26 \times 10^{+06}$	$1.74 \times 10^{+04}$	$1.27 \times 10^{+05}$
1	$\kappa(C)$	128	$6.75 \times 10^{+02}$	$8.76 \times 10^{+05}$	$2.35 \times 10^{+04}$	$9.10 \times 10^{+04}$
1	$\kappa(C)$	256	$4.19 \times 10^{+03}$	$5.82 \times 10^{+05}$	$4.43 \times 10^{+04}$	$8.98 \times 10^{+04}$
1	$\mathrm{rn}^{(1)}$	64	2.43×10^{-10}	3.86×10^{-08}	2.55×10^{-09}	5.43×10^{-09}
1	$\mathrm{rn}^{(1)}$	128	4.36×10^{-10}	1.15×10^{-07}	4.45×10^{-09}	1.24×10^{-08}
1	$\mathrm{rn}^{(1)}$	256	6.40×10^{-10}	3.17×10^{-08}	4.00×10^{-09}	5.16×10^{-09}
1	$\mathrm{rn}^{(2)}$	64	8.30×10^{-10}	3.86×10^{-08}	2.81×10^{-09}	5.35×10^{-09}
1	$\mathrm{rn}^{(2)}$	128	1.21×10^{-9}	1.15×10^{-07}	4.80×10^{-09}	1.23×10^{-08}
1	$\mathrm{rn}^{(2)}$	256	1.72×10^{-9}	3.18×10^{-08}	4.53×10^{-09}	4.97×10^{-09}
8	$\kappa(C)$	64	$1.37 \times 10^{+03}$	$1.87 \times 10^{+06}$	$7.57 \times 10^{+04}$	$2.16 \times 10^{+05}$
8	$\kappa(C)$	128	$3.80 \times 10^{+03}$	$8.64 \times 10^{+06}$	$2.00 \times 10^{+05}$	$8.73 \times 10^{+05}$
8	$\kappa(C)$	256	$2.57 \times 10^{+04}$	$1.54 \times 10^{+07}$	$7.25 \times 10^{+05}$	$2.03 \times 10^{+06}$
8	$\mathrm{rn}^{(1)}$	64	1.87×10^{-9}	4.48×10^{-07}	2.29×10^{-08}	5.20×10^{-08}
8	$\mathrm{rn}^{(1)}$	128	3.04×10^{-09}	3.73×10^{-07}	2.72×10^{-08}	5.83×10^{-08}
8	$\mathrm{rn}^{(1)}$	256	3.78×10^{-09}	2.01×10^{-06}	4.81×10^{-08}	2.02×10^{-07}
8	$\mathrm{rn}^{(2)}$	64	1.30×10^{-09}	2.47×10^{-07}	1.09×10^{-08}	2.70×10^{-08}
8	$\mathrm{rn}^{(2)}$	128	1.85×10^{-09}	1.50×10^{-07}	1.36×10^{-08}	2.75×10^{-08}
8	$\mathrm{rn}^{(2)}$	256	2.19×10^{-09}	1.10×10^{-06}	2.36×10^{-08}	1.10×10^{-07}
32	$\kappa(C)$	64	$3.75 \times 10^{+03}$	$3.25 \times 10^{+07}$	$6.01 \times 10^{+05}$	$3.28 \times 10^{+06}$
32	$\kappa(C)$	128	$2.41 \times 10^{+04}$	$1.09 \times 10^{+08}$	$1.95 \times 10^{+06}$	$1.10 \times 10^{+07}$
32	$\kappa(C)$	256	$1.33 \times 10^{+05}$	$2.11 \times 10^{+10}$	$2.18 \times 10^{+08}$	$2.11 \times 10^{+09}$
32	$\mathrm{rn}^{(1)}$	64	7.78×10^{-09}	1.39×10^{-06}	8.17×10^{-08}	1.94×10^{-07}
32	$\mathrm{rn}^{(1)}$	128	9.81×10^{-09}	2.35×10^{-06}	1.17×10^{-07}	3.05×10^{-07}
32	$\mathrm{rn}^{(1)}$	256	2.05×10^{-08}	3.99×10^{-06}	1.91×10^{-07}	5.06×10^{-07}
32	$\mathrm{rn}^{(2)}$	64	1.84×10^{-09}	2.62×10^{-07}	1.85×10^{-08}	4.09×10^{-08}
32	$\mathrm{rn}^{(2)}$	128	2.47×10^{-09}	6.77×10^{-07}	2.93×10^{-08}	8.38×10^{-08}
32	$\mathrm{rn}^{(2)}$	256	5.05×10^{-09}	8.85×10^{-07}	4.38×10^{-08}	1.14×10^{-07}

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Table 8.9: Approximation of the heads of SVDs and low-rank approximation of a matrix by using randomized matrix products

q	$\kappa(C)$ or rrn_i	n	**min**	**max**	**mean**	**std**
1	$\mathrm{rn}^{(1)}$	64	2.35×10^{-10}	1.32×10^{-07}	3.58×10^{-09}	1.37×10^{-08}
1	$\mathrm{rn}^{(1)}$	128	4.41×10^{-10}	3.28×10^{-08}	3.55×10^{-09}	5.71×10^{-09}
1	$\mathrm{rn}^{(1)}$	256	6.98×10^{-10}	5.57×10^{-08}	5.47×10^{-09}	8.63×10^{-09}
1	$\mathrm{rn}^{(2)}$	64	8.28×10^{-10}	1.32×10^{-07}	3.86×10^{-09}	1.36×10^{-08}
1	$\mathrm{rn}^{(2)}$	128	1.21×10^{-09}	3.28×10^{-08}	3.91×10^{-09}	5.57×10^{-09}
1	$\mathrm{rn}^{(2)}$	256	1.74×10^{-09}	5.58×10^{-08}	5.96×10^{-09}	8.47×10^{-09}
8	$\mathrm{rn}^{(1)}$	128	2.56×10^{-09}	1.16×10^{-06}	4.30×10^{-08}	1.45×10^{-07}
8	$\mathrm{rn}^{(1)}$	256	4.45×10^{-09}	3.32×10^{-07}	3.40×10^{-08}	5.11×10^{-08}
8	$\mathrm{rn}^{(2)}$	64	1.46×10^{-09}	9.56×10^{-08}	5.77×10^{-09}	1.06×10^{-08}
8	$\mathrm{rn}^{(2)}$	128	1.64×10^{-09}	4.32×10^{-07}	1.86×10^{-08}	5.97×10^{-08}
8	$\mathrm{rn}^{(2)}$	256	2.50×10^{-09}	1.56×10^{-07}	1.59×10^{-08}	2.47×10^{-08}
32	$\mathrm{rn}^{(1)}$	64	6.80×10^{-09}	2.83×10^{-06}	1.01×10^{-07}	3.73×10^{-07}
32	$\mathrm{rn}^{(1)}$	128	1.25×10^{-08}	6.77×10^{-06}	1.28×10^{-07}	6.76×10^{-07}
32	$\mathrm{rn}^{(1)}$	256	1.85×10^{-08}	1.12×10^{-06}	1.02×10^{-07}	1.54×10^{-07}
32	$\mathrm{rn}^{(2)}$	64	1.84×10^{-09}	6.50×10^{-07}	2.30×10^{-08}	8.28×10^{-08}
32	$\mathrm{rn}^{(2)}$	128	3.11×10^{-09}	1.45×10^{-06}	2.87×10^{-08}	1.45×10^{-07}
32	$\mathrm{rn}^{(2)}$	256	4.39×10^{-09}	2.16×10^{-07}	2.37×10^{-08}	3.34×10^{-08}

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Table 8.10: Approximation of the heads of SVDs and low-rank approximation of a matrix by using random Toeplitz multipliers

q	$\mathrm{rrn}^{(i)}$	n	**min**	**max**	**mean**	**std**
8	$\mathrm{rrn}^{(1)}$	64	2.22×10^{-09}	7.89×10^{-06}	1.43×10^{-07}	9.17×10^{-07}
8	$\mathrm{rrn}^{(1)}$	128	3.79×10^{-09}	4.39×10^{-05}	4.87×10^{-07}	4.39×10^{-06}
8	$\mathrm{rrn}^{(1)}$	256	5.33×10^{-09}	3.06×10^{-06}	6.65×10^{-08}	3.12×10^{-07}
8	$\mathrm{rrn}^{(2)}$	64	1.13×10^{-09}	3.66×10^{-06}	6.37×10^{-08}	4.11×10^{-07}
8	$\mathrm{rrn}^{(2)}$	128	1.81×10^{-09}	1.67×10^{-05}	1.90×10^{-07}	1.67×10^{-06}
8	$\mathrm{rrn}^{(2)}$	256	2.96×10^{-09}	1.25×10^{-06}	2.92×10^{-08}	1.28×10^{-07}
32	$\mathrm{rrn}^{(1)}$	64	6.22×10^{-09}	5.00×10^{-07}	4.06×10^{-08}	6.04×10^{-08}
32	$\mathrm{rrn}^{(1)}$	128	2.73×10^{-08}	4.88×10^{-06}	2.57×10^{-07}	8.16×10^{-07}
32	$\mathrm{rrn}^{(1)}$	256	1.78×10^{-08}	1.25×10^{-06}	1.18×10^{-07}	2.03×10^{-07}
32	$\mathrm{rrn}^{(2)}$	64	1.64×10^{-09}	1.26×10^{-07}	9.66×10^{-09}	1.48×10^{-08}
32	$\mathrm{rrn}^{(2)}$	128	5.71×10^{-09}	9.90×10^{-07}	5.50×10^{-08}	1.68×10^{-07}
32	$\mathrm{rrn}^{(2)}$	256	4.02×10^{-09}	2.85×10^{-07}	2.74×10^{-08}	4.48×10^{-08}

Table 8.11: Relative residual norms for a linear system of equations via nmb approximation and block triangulation

n	**r**	**min**	**max**	**mean**	**std**
32	1	1.49×10^{-13}	1.36×10^{-9}	4.25×10^{-11}	1.56×10^{-10}
32	2	3.70×10^{-13}	2.13×10^{-8}	3.83×10^{-10}	2.35×10^{-9}
32	4	9.33×10^{-13}	1.08×10^{-8}	3.37×10^{-10}	1.26×10^{-9}
64	1	1.11×10^{-12}	6.87×10^{-9}	2.03×10^{-10}	7.49×10^{-10}
64	2	1.53×10^{-12}	1.21×10^{-8}	5.86×10^{-10}	1.77×10^{-9}
64	4	2.21×10^{-12}	1.27×10^{-7}	1.69×10^{-9}	1.28×10^{-8}

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Table 8.12: Relative residual norms for a linear system of equations with MLDIVIDE(A,B)

n	**r**	**min**	**max**	**mean**	**std**
32	1	6.34×10^{-3}	7.44×10^{1}	1.74×10^{0}	7.53×10^{0}
32	2	2.03×10^{-2}	1.32×10^{1}	9.19×10^{-1}	1.62×10^{0}
32	4	4.57×10^{-2}	1.36×10^{1}	1.14×10^{0}	1.93×10^{0}
64	1	3.82×10^{-3}	9.93×10^{0}	1.03×10^{0}	1.66×10^{0}
64	2	1.96×10^{-2}	1.27×10^{2}	3.09×10^{0}	1.40×10^{1}
64	4	7.13×10^{-3}	6.63×10^{0}	8.23×10^{-1}	1.20×10^{0}

Table 8.13: Relative residual norms for a linear system of equations by using dual additive preprocessing and block triangulation

n	q	**min**	**max**	**mean**	**std**
32	1	2.33×10^{-14}	2.28×10^{-06}	2.31×10^{-08}	2.28×10^{-07}
32	2	3.40×10^{-13}	4.93×10^{-08}	9.11×10^{-10}	5.71×10^{-09}
32	4	5.97×10^{-13}	1.63×10^{-07}	2.22×10^{-09}	1.64×10^{-08}
64	1	3.90×10^{-14}	2.78×10^{-05}	2.81×10^{-07}	2.78×10^{-06}
64	2	3.53×10^{-13}	3.76×10^{-08}	1.13×10^{-09}	4.72×10^{-09}
64	4	3.54×10^{-12}	2.53×10^{-07}	5.19×10^{-09}	2.83×10^{-08}

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Table 8.14: Relative residual norms for a linear system of equations based on multiplication by random matrices and block triangulation

n	q	**min**	**max**	**mean**	**std**
32	1	7.08×10^{-30}	4.00×10^{-23}	4.52×10^{-25}	4.01×10^{-24}
32	2	7.49×10^{-30}	2.29×10^{-21}	2.77×10^{-23}	2.33×10^{-22}
32	4	1.46×10^{-28}	1.63×10^{-07}	4.83×10^{-25}	2.73×10^{-24}
64	1	1.13×10^{-29}	1.01×10^{-24}	2.31×10^{-26}	1.11×10^{-25}
64	2	6.60×10^{-29}	6.90×10^{-24}	1.45×10^{-25}	7.73×10^{-25}
64	4	2.60×10^{-28}	1.41×10^{-21}	1.61×10^{-23}	1.42×10^{-22}

Table 8.15: CPU time (in cycles) for solving an ill conditioned real symmetric Toeplitz linear system

n	**Alg. 6.1**	**QR**	**SVD**	**QR/Alg. 6.1**	**SVD/Alg. 6.1**
512	56.3	148.4	4134.8	2.6	73.5
1024	120.6	1533.5	70293.1	12.7	582.7
2048	265.0	11728.1	–	44.3	–
4096	589.4	–	–	–	–
8192	1304.8	–	–	–	–

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