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Abstract

We investigate the complexity of derivability for two-agent Justification Logic. For this purpose we revisit Yavorskaya's two-agent LP with interactions (2008), we simplify the syntax and provide natural extensions. We consider two-agent versions of other justification logics as well as ways to combine two justification logics. For most of these cases we prove that the upper complexity bound established for the single-agent cases are maintained: these logics' derivability problem is in the second step of the polynomial hierarchy. For certain logics, though, we discover a complexity jump to PSPACE-completeness, which is a new phenomenon for Justification Logic.

1 Introduction

Justification Logic is a family of logics which models the way justifications interact with statements and can be viewed as an explicit counterpart of epistemic modal logic. It is often the case that we want to express statements of the form "agent A knows/believes ϕ because of *justification* J". The most robust justification for a statement is a proof, We use LP, the logic of proofs to formalize statements of the form "J is a proof of ϕ". LP is chronologically the first member of the Justification Logic family, introduced in 1995 by Artemov([3]). Its original purpose was to demonstrate the connection between modal logic $\mathsf{S4}$, intuitionistic logic and Peano arithmetic. Since then, several variations have been introduced (ex. [4, 7]), which resulted in a wide system of logics to model the interaction between belief, knowledge and justifications. Justification formulas are formed using propositional connectives and justification terms: if ϕ is a formula and t a term, $t\!:\!\phi$ is a formula. For a comprehensive review of Justification Logic, see [5].

An important aspect of Justification Logic is its complexity properties. It is known that all LP theorems can be easily translated to $\mathsf{S4}$ theorems by replacing each term by a box ($\Box$) and on the other hand all $\mathsf{S4}$ theorems can be translated to LP theorems by replacing each box by an appropriate term. This was already proven in the original paper by Artemov ([3]). Ladner showed in [16] that $\mathsf{S4}$-satisfiability (and thus, provability) is PSPACE-complete. Kuznets proved that LP-provability is in Π_2^p ([12]).

Furthermore, if ϕ is LP-provable, then there is a term t such that $t:\phi$ is provable and vice-versa. Krupski has shown that LP-provability for formulas of the form $t:\phi$ is in coNP ([11]). There is an easily recognizable class of terms t so that if a formula ϕ is provable then $t:\phi$ is provable for some such t and the provability of $t:\phi$ is in P([6]). Of course, this does not simplify theoremhood of S4 (which is PSPACE-complete), but it demonstrates the complexity-theoretic difference between determining the provability of a modal statement and determining the provability of a modal statement when given appropriate evidence.

Our aim is to study epistemic situations where justifications matter. Thus it is useful to consider situations with multiple agents and also allow influence among the agents' justifications and beliefs. Yavorskaya in [21] presents two-agent versions of LP with interactions between the agents (agent 1 and agent 2). To describe the interactions, she introduced new operators on terms. In LP^2 the two agents do not interact with each other at all. In $\mathsf{LP}^2_{\uparrow}$ one of the agents (agent 2) is more knowledgeable than the other: a justification of ϕ for agent 1 can be converted to a justification of ϕ for agent 2. In $\mathsf{LP}^2_{!}$, agent 2 is aware of the knowledge of agent 1: if agent 1 has justification t of a formula ϕ, then agent 2 has justification of the fact that agent 1 has justification t of a formula ϕ. $\mathsf{LP}^2_{\uparrow\uparrow}$ and $\mathsf{LP}^2_{!!}$ respectively extend these logics by also incorporating the converse of these interactions (swapping 1 and 2). We claim this is an important approach that will further illuminate the role of justification in epistemological situations.

In this paper we extend the above (from LP) to several justification logics and to additional two-agent versions. We present Yavorskaya's systems in a simplified notation. In fact we only keep the single-agent operators and allowing the two agents to use the same set of justifications. This results in a simpler system that resembles the original, single-agent one, and yet has the intrinsic features of a multi-agent system with interacting agents. We then provide a tableau procedure similar to the one that Kuznets used in [13]. Then we introduce and examine two-agent versions of other justification logics and extend the semantics and tableau rules for these cases. Finally, we examine what happens when each agent can be based on a different justification logic. We thus introduce and study two-agent logics which are combinations of two different justification logics. For some of these logics we discover a complexity jump to PSPACE-completeness. This is a different situation from what is the case for all (pure) justification logics whose complexity has been studied.

Other multi-agent justification logics have already been introduced (for example, see [8, 20]). They present a different approach. This paper focuses on interactions between the agents' justifications and not on any actual interaction between the agents themselves.

2 Two-agent Variations of LP

In this section we present Yavorskaya's systems in a simplified notation. In fact we only keep the single-agent operators and allow the two agents to use the same set of justifications. This results in a simpler system that resembles the original, single-agent one. We then provide a tableau procedure similar to the one that Kuznets used in [13].

2.1 Syntax, Axioms, and Semantics

We start by introducing the family of two-agent variations of LP we are about to study. We provide the syntax of the formulas, the axioms and rules of each logic and their semantics. The proofs of the propositions that appear in this section are very similar to the proofs of corresponding results for the single-agent logics and the reader can see [5] or [13] for an overview.

The justification terms of the language L_2 include all constants $c_1, c_2, c_3, \ldots$ and variables $x_1, x_2, x_3, \ldots$ and if t_1 and t_2 are terms, then the following are also terms: $[t_1 + t_2]$, $[t_1 \cdot t_2]$, $!t_1$.[1] The set of terms will be referred to as Tm. We also use a set $SLet$ of propositional variables, or sentence letters. These will usually be $p_1, p_2, \ldots$. Formulas of the language L_2 include all propositional variables and if ϕ, ψ are formulas, t is a term, and $i \in \{1, 2\}$, then the following are also formulas of L_2: $\bot$, $\phi \to \psi$, $t :_i \phi$. The remaining propositional connectives, whenever needed, are treated as constructed from $\to$ and $\bot$ in the usual way: $\neg a := a \to \bot$, $a \vee b := \neg a \to b$, and $a \wedge b := \neg(\neg a \vee \neg b)$.

The function of the operators $\cdot, +$ and ! becomes clear below as described by the axioms. Intuitively, $\cdot$ applies a justification for a statement $A \to B$ to a justification for A and gives a justification for B. Using $+$ we can combine two justifications and have a justification for anything that can be justified by any of the two initial terms - much like the concatenation of two proofs. Finally, ! is a unary operator called the proof checker. Given a justification t for ϕ, it gives another one, $!t$, for the fact that t is a justification for ϕ.

The logics use modus ponens as a derivation rule and some of the following axioms:

Propositional Axioms: Finitely many schemes of classical propositional logic;

Application: $s :_i (\phi \to \psi) \to (t :_i \phi \to [s \cdot t] :_i \psi)$;

Concatenation: $s :_i \phi \to [s + t] :_i \phi$, $s :_i \phi \to [t + s] :_i \phi$;

Factivity: $t :_i \phi \to \phi$;

Positive Introspection: $t :_i \phi \to !t :_i t :_i \phi$;

12-Verification: $t :_1 \phi \to !t :_2 t :_1 \phi$;

21-Verification: $t :_2 \phi \to !t :_1 t :_2 \phi$;

12-Conversion: $t :_1 \phi \to t :_2 \phi$;

21-Conversion: $t :_2 \phi \to t :_1 \phi$,

where in the above, ϕ and ψ are formulas in L_2, s, t are terms and $i \in \{1, 2\}$.
$(\mathsf{LP}^2)_\emptyset$ is the logic with modus ponens as a derivation rule and the Propositional Axioms, Application, Concatenation, Factivity and Positive Introspection as axioms. $(\mathsf{LP}^2_!)_\emptyset$ is $(\mathsf{LP}^2)_\emptyset$+ 12-Verification; $(\mathsf{LP}^2_C)_\emptyset$ is $(\mathsf{LP}^2)_\emptyset$+ 12-Conversion; $(\mathsf{LP}^2_{!!})_\emptyset$ is $(\mathsf{LP}^2)_\emptyset$+ 12-Verification + 21-Verification; $(\mathsf{LP}^2_{CC})_\emptyset$ is $(\mathsf{LP}^2)_\emptyset$+ 12-Conversion + 21-Conversion.[2]

[1] [and] are used instead of parentheses inside justification terms. The purpose of this is to make it easier to distinguish between formulas and terms. This is due to Sergei Artemov.

[2] The analogues of LP^2_C and LP^2_{CC} in [21] are called $\mathsf{LP}^2_\uparrow$ and $\mathsf{LP}^2_{\uparrow\uparrow}$ respectively. $\uparrow$ is the conversion operator, which we do not use, so for this paper we thought it more appropriate to use C instead of $\uparrow$.

A *constant specification* for a J from LP^2, LP^2_C, LP^2_{CC}, $\mathsf{LP}^2_!$, and $\mathsf{LP}^2_{!!}$ is any set

$$\mathcal{CS} \subseteq \{c{:}_i A \mid c \text{ is a constant, } A \text{ an axiom of } J \text{ and } i \in [n]\}.$$

We say that axiom A is justified by a constant c for agent i, when $c{:}_i A \in \mathcal{CS}$.

A constant specification is: *axiomatically appropriate* if each axiom is justified by at least one constant, *schematic* if every constant justifies only a certain number (possibly zero) of schemes from the ones above (which implies that if c justifies A for i and B results from A and substitution, then c justifies B for i) and *schematically injective* if it is schematic and every constant justifies at most one scheme.

Let $cl_2(\mathcal{CS})$ be the smallest set such that $\mathcal{CS} \subseteq cl_2(\mathcal{CS})$ and for every $t{:}_i \phi \in cl_2(\mathcal{CS})$, it is also the case that for every $j \in \{1, 2\}$, $!t{:}_j t{:}_i \phi \in cl_2(\mathcal{CS})$. Then, $J_{\mathcal{CS}}$ is $J_\emptyset + R4^2_{\mathcal{CS}}$, where $R4^2_{\mathcal{CS}}$ introduces exactly the elements of $cl_2(\mathcal{CS})$.[3]

If we map each formula to a propositional formula by just removing all terms, it is easy to see that each axiom is mapped to a propositional tautology and that modus ponens preserves the mapping. Thus, we conclude that *each of the logics defined above is consistent.* In fact, this reasoning can be applied to all the logics that will follow and thus from now and on we will assume each logic we present is consistent.

The following proposition's version for single-agent logics is a characteristic result in justification logic and can be proven by induction on ϕ. It also holds for all the logics presented above, as well as for the ones that follow.

Proposition 1. *If $\mathcal{CS}$ is an axiomatically appropriate constant specification $\mathcal{CS}$ and $\phi_1, \ldots, \phi_k \vdash \phi$, then for any $i \in \{1, 2\}$, there is some term $t(t_1, \ldots, t_k)$ depending on terms $t_1, \ldots, t_k$, such that $t_1{:}_i \phi_1, \ldots, t_k{:}_i \phi_k \vdash t(t_1, \ldots, t_k){:}_i \phi$.*

Proof. By induction on the proof of ϕ: If ϕ is an axiom, or introduced by $R4^2_{\mathcal{CS}}$, then by rule $R4^2_{\mathcal{CS}}$, the proposition holds. Furthermore the theorem obviously holds for any ϕ_i, $i \in [k]$. This covers the base cases. Using the application axiom, if ϕ is the result of ψ, χ and modus ponens, since the proposition holds for ψ, χ, then $t_1 {:}_i \phi_1, \ldots, t_k {:}_i \phi_k \vdash r {:}_i \psi, s {:}_i \chi$ and thus for $t = [r \cdot s]$, $t_1 {:}_i \phi_1, \ldots, t_k {:}_i \phi_k \vdash t {:}_i \phi$ and this completes the inductive proof. ∎

We now introduce models for our logic. In the single-agent cases, M-models (introduced in [15, 18]) and F-models (introduced in [10, 15, 19]) are used and they are both useful in the study of complexity issues. For the purposes of this section, M-models are sufficient.

Definition 1. *An M-model $\mathcal{M}$ for J is a triple $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$, where for $i \in \{1, 2\}$, $\mathcal{A}_i : (Tm \times L_n) \longrightarrow \{true, false\}$ and $\mathcal{V} : SLet \longrightarrow \{true, false\}$. Furthermore, $\mathcal{A}_1, \mathcal{A}_2$ will often be seen and referred to as $\mathcal{A} : \{1, 2\} \times Tm \times L_2 \longrightarrow \{true, false\}$ and $\mathcal{A}$ is called an admissible evidence function. Additionally, $\mathcal{A}$ must satisfy the following conditions for every $i, j \in \{1, 2\}$:*

Application closure: *for any formulas ϕ, ψ and justification terms t, s, if $\mathcal{A}_i(s, \phi \to \psi) = true$ and $\mathcal{A}_i(t, \phi) = true$, then $\mathcal{A}_i(s \cdot t, \psi) = true$.*

[3] A motivation for $R4^2_{\mathcal{CS}}$ is that we want all agents to be aware of propositions provable in the logic.

Sum closure: *for any formula ϕ and justification terms t, s,*
if $\mathcal{A}_i(t,\phi) = true$ or $\mathcal{A}_i(s,\phi) = true$, then $\mathcal{A}_i(t+s,\phi) = true$.

$\mathcal{CS}$-closure: *for any $t :_i \phi \in cl_2(\mathcal{CS})$, $\mathcal{A}_i(t,\phi) = true$.*

Positive Introspection Closure: *If $\mathcal{A}_i(t,\phi) = true$, then $\mathcal{A}_i(!t, t:_i \phi) = true$.*

Verification Closure: *If the logic includes ij-Verification and $\mathcal{A}_i(t,\phi) = true$, then $\mathcal{A}_j(!t, t:_i \phi) = true$.*

Conversion Closure: *If the logic includes ij-Conversion, then*
if $\mathcal{A}_i(t,\phi) = true$, then $\mathcal{A}_j(t,\phi) = true$.

Truth in the model is defined in the following way:

- $M \not\models \bot$.
- *If p is a propositional variable, then $\mathcal{M} \models p$ iff $\mathcal{V}(p) = true$*
- *If ϕ, ψ are formulas, then $\mathcal{M} \models \phi \to \psi$ if and only if $M \models \psi$, or $\mathcal{M} \not\models \phi$.*
- *If ϕ is a formula and t a term, then $\mathcal{M} \models t :_i \phi$ if and only if $\mathcal{A}_i(t,\phi) = true$ and $\mathcal{M} \models \phi$.*

It can be proven that each of these logics is sound and complete with respect to its M-models. The reader is referred to the corresponding proofs for the single-agent cases, or to section 4, where we prove a different, general result, but that proof can be altered and simplified to prove soundness and completeness for this case as well.

2.2 Tableau rules

In the same way as in [13], we present tableau rules to establish that satisfiability for these variants of two-agent LP is in Σ_2^p, as it was argued in [21].

The formulas used in the tableau procedure below are prefixed by T or F. The tableau procedure includes rules to decompose an L_2 formula to simpler subformulas. These are the following. For any term t, agent $i = 1, 2$ and formulas ϕ, ψ,

$$\frac{T\ \phi \to \psi}{F\ \phi \quad | \quad T\ \psi} \qquad \frac{F\ \phi \to \psi}{\begin{array}{c} T\ \phi \\ F\ \psi \end{array}} \qquad \frac{T\ t :_i \phi}{\begin{array}{c} T\ \phi \\ T\ *_i (t,\phi) \end{array}} \qquad \frac{F\ t :_i \phi}{F\ \phi \quad | \quad F\ *_i (t,\phi)}$$

The two rules on the left will be referred to as the propositional rules and the ones on the right as the term rules. Formulas of the form $*_i(t,\phi)$, where $t :_i \phi$ a formula, are called $*$-expressions. The tableau procedure for ϕ starts from $T\ \phi$ and using the rules above it (non-deterministically and in polynomial time) constructs a branch. If at some point the branch includes $T\ \psi$ and $F\ \psi$ at the same time, the branch is propositionally closed. If we reach a branch closed under the rules (which we will call complete) and not propositionally closed, then it is accepting if and only if there is some admissible evidence function, $\mathcal{A}$, such that for every $T\ *_i (t,\psi)$ in the branch, $\mathcal{A}_i(t,\psi) = true$ and for every $F\ *_i (t,\psi)$ in the branch, $\mathcal{A}_i(t,\psi) = false$. This last condition is in coNP, by a straightforward variation of the algorithm from [13] and [17], so confirming that there is

Base logic	Contributes the Axioms:
$\mathcal{J} = \mathsf{J}$	No additional axioms
$\mathcal{J} = \mathsf{JD}$	Consistency
$\mathcal{J} = \mathsf{JT}$	Factivity
$\mathcal{J} = \mathsf{J4}$	Positive Introspection
$\mathcal{J} = \mathsf{JD4}$	Consistency and Positive Introspection
$\mathcal{J} = \mathsf{LP}$	Factivity and Positive Introspection

The Logic's Subscript	Contributes the Axioms:
$\mathcal{J}^2$	No additional axioms
$\mathcal{J}^2_!$	12-Verification
$\mathcal{J}^2_C$	12-Conversion
$\mathcal{J}^2_{!!}$	12-Verification and 21-Verification
$\mathcal{J}^2_{CC}$	12-Conversion and 21-Conversion

Figure 1: The axioms for logic $(\mathcal{J}^2_s)_{\mathcal{CS}}$, where $\mathcal{J} \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$ and the subscript s is either empty or one of $!, C, !!, CC$.

an accepting branch for a formula ϕ is in Σ_2^p. We can furthermore construct a model for ϕ from a complete accepting branch and vice-versa: to construct a model, $\mathcal{V}(p) = true$ iff $T\ p$ appears and $\mathcal{A}$ is guaranteed by the acceptance condition, while to construct a branch, make sure the model satisfies all that is nondeterministically produced by the rules - for $T *_i (t, \psi)$ this means $\mathcal{A}_i(t, \psi) = true$ and for $F\ x$ it means $T\ x$ is not satisfied. What remains is an inductive argument that the model then satisfies ϕ, or respectively that the branch is accepting. This settles that satisfiability for any of these logics is in Σ_2^p.

3 Two-agent Versions of Other Justification Logics

We can easily define two-agent versions of other justification logics as well. For each of $\mathsf{J}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD}$, and $\mathsf{JD4}$ as the base logic $\mathcal{J}$, we define $\mathcal{J}^2$, $\mathcal{J}_!$, $\mathcal{J}_{!!}$, $\mathcal{J}_C$ and $\mathcal{J}_{CC}$. If JT is the base logic, then it suffices to remove the Positive Introspection axiom from the corresponding logic's axioms; if $\mathsf{J4}$ is the base logic we remove the Factivity axiom; by removing both the Factivity and the Positive Introspection Axiom, J is the base logic. When $\mathsf{JD4}$ is the base logic (resp. JD), then in addition to removing Factivity (resp. and Positive Introspection), we need to add *Consistency* to the set of axioms: $\neg t :_i \bot$. The tables of figure 1 give the axioms for each of these two-agent logics, depending on the base logic and the interactions.

In the same way, M-models for logics based on JT result from the M-models for the corresponding logics based on LP, as defined in the previous section by removing the Positive Introspection Closure condition for the admissible evidence functions. In fact, whenever a logic lacks Positive Introspection, M-models for that logic do not have the Positive Introspection closure condition for the admissible evidence functions. Whenever the logic lacks Factivity, the definition of truth in the model changes to $\mathcal{M} \models t{:_i}\phi$ *if and only if* $\mathcal{A}_i(t,\phi) = \mathit{true}$. This is enough to define M-models for the two-agent versions of J, JT, and $\mathsf{J4}$, but M-models for JD and $\mathsf{JD4}$ must also include the Consistent Evidence condition for the admissible evidence functions: $\mathcal{A}_i(t,\bot) = \mathit{false}$.

To decide satisfiability for each of these logics, we have three cases. In the case of the base logic being JT, we can simply use the same tableau rules as for the two-agent logics based on LP. The difference in the base logic is reflected in the procedure that handles the construction of the admissible evidence functions. If the base logic is either J or $\mathsf{J4}$, we have to change the term rules to the following ones and the rest is the same: for any term t, agent $i = 1,2$ and formula ϕ,

$$\frac{T\ t{:_i}\phi}{T\ \ *_i(t,\phi)} \qquad \frac{F\ t{:_i}\phi}{F\ \ *_i(t,\phi)}$$

The third case is when the base logic has the Consistency axiom, namely, it is either JD or $\mathsf{JD4}$. Unfortunately in this case we cannot simply make a small change to the tableau rules to decide satisfiability. The logics based on JD and $\mathsf{JD4}$ include the additional Consistent Evidence condition, which is not a closure condition and thus cannot be handled by a straightforward variation of the above procedures. For these logics we will adjust the methods introduced in [13] for JD and in [1] for $\mathsf{JD4}$.

3.1 Satisfiability of Two-agent Logics Based on JD and JD4

To study the satisfiability of the two-agent versions of JD and $\mathsf{JD4}$ we introduce F-models for these logics. F-models for LP were introduced by Fitting in [10] and are a combination of Kripke models and M-models. Let $\mathcal{J}$ be one of $(\mathsf{JD}^2)_{\mathcal{CS}}$, $(\mathsf{JD}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD}^2_!)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$, $(\mathsf{JD4}^2)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD4}^2_!)_{\mathcal{CS}}$, and $(\mathsf{JD4}^2_{!!})_{\mathcal{CS}}$.

Definition 2. *An F-model $\mathcal{M}$ for J is a tuple $(W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$, where W a nonempty set of states (occasionally referred to as worlds), $R_1, R_2 \subseteq W^2$ are binary relations on W, for $i \in \{1,2\}$, $\mathcal{A}_i : (Tm \times L_n) \longrightarrow 2^W$, and $\mathcal{V} : SLet \longrightarrow 2^W$. Furthermore, $\mathcal{A}_1, \mathcal{A}_2$ will often be seen and referred to as $\mathcal{A} : \{1,2\} \times Tm \times L_2 \longrightarrow 2^W$ and $\mathcal{A}$ is called an admissible evidence function. Additionally, $\mathcal{A}$ must satisfy the following conditions for every $i, j \in \{1,2\}$:*

Application closure: *for any formulas ϕ, ψ and justification terms t, s,*

$$\mathcal{A}_i(s, \phi \to \psi) \cap \mathcal{A}_i(t,\phi) \subseteq \mathcal{A}_i(s \cdot t, \psi).$$

Sum closure: *for any formula ϕ and justification terms t, s,*

$$\mathcal{A}_i(t,\phi) \cup \mathcal{A}_i(s,\phi) \subseteq \mathcal{A}_i(t+s,\phi).$$

$\mathcal{CS}$-closure: *for any $t{:_i}\phi \in cl^2(\mathcal{CS})$, $\mathcal{A}_i(t,\phi) = W$.*

Positive Introspection Closure: *If the logic includes the Positive Introspection axiom (is based on* $\mathsf{JD4}$*), then $\mathcal{A}_i(t,\phi) \subseteq \mathcal{A}_i(!t, t{:_i}\phi)$.*

Distribution: *If the logic includes Positive Introspection, then for any formula ϕ, justification term t, and $a, b \in W$, if aR_ib and $a \in \mathcal{A}_i(t,\phi)$, then $b \in \mathcal{A}_i(t,\phi)$.*

Verification Closure: *When a logic has ij-Verification, $\mathcal{A}_i(t,\phi) \subseteq \mathcal{A}_j(!t, t:_i\phi)$.*

Conversion Closure: *When a logic has ij-Conversion, $\mathcal{A}_i(t,\phi) \subseteq \mathcal{A}_j(t,\phi)$.*

V-Distribution: *If the logic includes ij-Verification, then for any formula ϕ, justification term t, and $a, b \in W$, if aR_jb and $a \in \mathcal{A}_i(t,\phi)$, then $b \in \mathcal{A}_i(t,\phi)$.*

The accessibility relations, R_1, R_2, must satisfy the following conditions: for every $i \in \{1,2\}$,

- *R_i must be serial ($\forall a \in W\ \exists b \in W\ aR_ib$).*
- *If the logic includes the Positive Introspection axiom, then R_i must be transitive (if aR_ibR_ic, then aR_ic).*
- *If the logic includes ij-Verification, then for any $a, b, c \in W$, if aR_jbR_ic, we also have aR_ic.*
- *If the logic includes ij-Conversion, then $R_j \subseteq R_i$.*

Truth in the model is defined in the following way:

- *$M, u \not\models \bot$.*
- *If p is a propositional variable, then $\mathcal{M}, u \models p$ iff $u \in \mathcal{V}(p)$*
- *If ϕ, ψ are formulas, then $\mathcal{M}, u \models \phi \to \psi$ if and only if $M, u \models \psi$, or $\mathcal{M}, u \not\models \phi$.*
- *If ϕ is a formula and t a term, then $\mathcal{M}, u \models t:_i\phi$ if and only if $u \in \mathcal{A}_i(t,\phi)$ and for every $v \in W$ such that uR_iv, $\mathcal{M}, v \models \phi$.*

If $(W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ an F-model for logic $\mathcal{J}$, then (W, R_1, R_2) is a frame for $\mathcal{J}$. $(\mathsf{JD}^2)_{\mathcal{CS}}$, $(\mathsf{JD}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD}^2_!)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$, $(\mathsf{JD4}^2)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD4}^2_!)_{\mathcal{CS}}$, and $(\mathsf{JD4}^2_{!!})_{\mathcal{CS}}$ are sound and complete with respect to their F-models, as long as $\mathcal{CS}$ is axiomatically appropriate. They are also sound and complete w.r.to their F-models that satisfy the Strong Evidence property: $\mathcal{M}, u \models t:_i\phi$ *iff* $u \in \mathcal{A}_i(t,\phi)$. See section 4 for a more formal treatment of this claim.

It is not hard to define F-models for the other logics as well. When instead of the Consistency axiom the logic has the Factivity axiom, then R_1, R_2 need to be reflexive instead of just serial. On the other hand, if the logic has neither Factivity nor Consistency as axioms, then R_1, R_2 are neither required to be serial nor reflexive. Again, these notions are made precise in section 4 for a more general case.

Now we look into the satisfiability problem for each of $(\mathsf{JD}^2)_{\mathcal{CS}}$, $(\mathsf{JD}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD}^2_!)_{\mathcal{CS}}$, $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$, $(\mathsf{JD4}^2)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_C)_{\mathcal{CS}}$, $(\mathsf{JD4}^2_{CC})_{\mathcal{CS}}$, $(\mathsf{JD4}^2_!)_{\mathcal{CS}}$, and $(\mathsf{JD4}^2_{!!})_{\mathcal{CS}}$ separately. Actually, $(\mathsf{JD}^2_{CC})_{\mathcal{CS}}$ and $(\mathsf{JD4}^2_{CC})_{\mathcal{CS}}$ will not be considered, as they are essentially the one-agent logics $\mathsf{JD}_{\mathcal{CS}}$ and $\mathsf{JD4}_{\mathcal{CS}}$, respectively:[4] for these logics, $t:_1\phi \leftrightarrow t:_2\phi$ is a

[4] We would need to adjust $\mathcal{CS}$ for a single-agent logic, but it is not hard to imagine a straightforward way to do this.

$*\mathcal{CS}(\mathcal{F})$ **Axioms:** $w *_i (t, \phi)$, where $w \in W$ and $t :_i \phi \in cl_n(\mathcal{CS})$

$*$**App**$(\mathcal{F})$**:**

$$\frac{w \ *_i (s, \phi \to \psi) \qquad w \ *_i (t, \phi)}{w \ *_i (s \cdot t, \psi)}$$

$*$**Sum**$(\mathcal{F})$**:**

$$\frac{w \ *_i (t, \phi)}{w \ *_i (s+t, \phi)} \qquad \frac{w \ *_i (s, \phi)}{w \ *_i (s+t, \phi)}$$

$*$**Dis**$(\mathcal{F})$**:** If $(a, b) \in R_i$ and the logic has Positive Introspection,

$$\frac{a \ *_i (t, \phi)}{b \ *_i (t, \phi)}$$

$*$**V**$(\mathcal{F})$**:** If the logic has ij-Verification,

$$\frac{w \ *_i (t, \phi)}{w \ *_j (!t, t :_i \phi)}$$

$*$**C**$(\mathcal{F})$**:** If the logic has ij-Conversion,

$$\frac{w \ *_i (t, \phi)}{w \ *_j (t, \phi)}$$

$*$**V-Dis**$(\mathcal{F})$**:** If $(a, b) \in R_j$ and the logic has ij-Verification,

$$\frac{a \ *_i (t, \phi)}{b \ *_i (t, \phi)}$$

Figure 2: The $*^{\mathcal{F}}_{\mathcal{CS}}(\mathcal{J})$-calculus includes the above axioms and rules, for every $i, j \in \{1, 2\}$, $w \in W$, s, t terms, ϕ, ψ formulas, where $\mathcal{F} = (W, R_1, R_2)$.

theorem for every t, ϕ, so there is no actual difference between the two agents.[5] From now and on we assume $\mathcal{CS}$ is axiomatically appropriate and schematic.

The $*$-calculus We introduce the $*$-calculus for logic J on frame $\mathcal{F}$, an invaluable tool when studying the complexity of justification logic. In the treatment of the logics we have encountered so far the calculus was actually used silently in the background without mentioning as part of the tableau procedures. The calculus is given in figure 2. The $*$-calculus was first introduced in [11], but its origins can be found in [18]. The form on which the one in this section is based is from [13]. The following claims can be proven with simple variations of the proofs in [11] and [13]. Notice that the calculus rules correspond to the closure conditions of the admissible evidence functions. In fact and because of this, given a frame $\mathcal{F} = (W, R_1, R_2)$ and a set S of $*$-expressions prefixed by states of the frame, the function $\mathcal{A}$ such that $\mathcal{A}_i(t, \phi) = \{w \in W | S \vdash_{*^{\mathcal{F}}_{\mathcal{CS}}(\mathcal{J})} w \ *_i (t, \phi)\}$ is an admissible evidence function and in fact it is the minimal admissible evidence function such that for every $w \ *_i (t, \phi) \in S$, $w \in \mathcal{A}_i(t, \phi)$.

If $\mathcal{CS} \in \mathsf{P}$, then deciding for some finite S and $\mathcal{F}$ if $S \vdash_{*^{\mathcal{F}}_{\mathcal{CS}}(\mathcal{J})} w *_i (t, \phi)$ is in NP. The nondeterministic algorithm which decides derivability in the $*$-calculus, can first guess the derivation tree, which is bounded in size by $2|t| \cdot |\mathcal{F}|$ and has $*$-expressions for nodes. At this step the algorithm guesses and for every node $v \ *_j (s, \psi)$ it fills in v, j, s, but

[5] This is different from what is the case in [21], as there the conversion operator, $\uparrow$, keeps track of justification conversions and each agent has their own separate set of terms.

with the exception of the formulas ψ. Then, the algorithm fills in these formulas for the leaves of the tree by guessing an appropriate formula (or scheme), which could either be an element of S, or in case s is a constant, the formula (scheme in this case - $\mathcal{CS}$ is schematic) can be a product of $R4^2_{\mathcal{CS}}$. The algorithm then in turn and for each node unifies the formulas of its children trying to result in a valid derivation of $w\ *_i(t,\phi)$. If it succeeds, then it accepts; otherwise it rejects.

When $\mathcal{J}, \mathcal{CS}, \mathcal{F}$ are clear from the context, $*^{\mathcal{F}}_{\mathcal{CS}}(\mathcal{J})$ will simply be referred to as $*$.

We prove that satisfiability for the two-agent logics based on JD and JD4 is in Σ^p_2. We look at each case separately.

For $(\mathsf{JD}^2)_{\mathcal{CS}}$ we use the F-models presented above to give tableau rules. These rules are:

$$\frac{w\ T\ \phi \to \psi}{w\ F\ \phi \quad | \quad w\ T\ \psi} \qquad \frac{w\ F\ \phi \to \psi}{\begin{array}{c} w\ T\ \phi \\ w\ F\ \psi \end{array}} \qquad \frac{w\ T\ t{:}_i\,\phi}{\begin{array}{c} w.i\ T\ \phi \\ w\ T\ *_i(t,\phi) \end{array}} \qquad \frac{w\ F\ t{:}_i\,\phi}{w\ F\ *_i(t,\phi)}$$

To test the satisfiability of ϕ, the tableau procedure starts from $0\ T\ \phi$. When all applicable rules have been applied, then we have a complete branch. If there are some $w\ T\ a$ and $w\ F\ a$ in the branch then the branch is propositionally closed. If it is not propositionally closed, then W is the set of world-prefixes appearing in the branch and for $i \in \{1, 2\}$,

$$R_i = \{(w, w.i) \in W^2\} \cup \{(w, w) \in W^2 | w.i \notin W\}.$$

Then, $\mathcal{F} = (W, R_1, R_2)$ and $\mathcal{V}(p) = \{w \in W | w\ T\ p \text{ appears in the branch}\}$. Finally, let $S = \{w\ *_i(t,\psi) | w\ T\ *_i(t,\psi) \text{ appears in the branch }\}$ and $\mathcal{A}$ be the admissible evidence function such that

$$\mathcal{A}_i(t,\phi) = \{w \in W | S \vdash_{*^{\mathcal{F}}_{\mathcal{CS}}(\mathcal{J})} w\ \ *_i(t,\phi)\}.$$

$\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ is a model, as R_1, R_2 are serial and $\mathcal{A}$ is an admissible evidence function and it is not hard to see by induction on the structure of formulas ψ, ψ' that for every $w\ T\ \psi$ and $w\ F\ \psi'$ in the branch, $\mathcal{M}, w \models \psi$ and $\mathcal{M}, w \not\models \psi'$, as long as there is no prefixed $*$-expression $w\ F\ \ *_i(t,v)$ appearing in the branch that $w \in \mathcal{A}_i(t,v)$. Therefore, we say the branch is accepting exactly when it is not propositionally closed and there is no prefixed $*$-expression $w\ F\ e$ appearing in the branch such that $S \vdash_* w\ e$.

If there is an accepting branch, then from the above we see that ϕ is satisfiable. On the other hand it is not hard to see how to construct an accepting branch for ϕ given an F-model for ϕ that satisfies the Strong Evidence property: we map each world prefix to a world of the model such that 0 is mapped to a world satisfying ϕ and every $w.i$ is mapped to a world accessible through R_i from the world mapped from w. Then we ensure that we only produce formulas $w\ T\ \psi$ such that the world w maps to satisfies ψ and formulas $w\ F\ \psi$ such that the world w maps to does not satisfy ψ. Thus we ensure the branch is accepting. That the number of formulas in the branch is polynomially bounded results from the observation that the formulas prefixed by distinct world-prefixes are distinct - assuming all subformulas of ϕ are distinct. This means that $(\mathsf{JD}^2)_{\mathcal{CS}}$-satisfiability for a schematic and axiomatically appropriate $\mathcal{CS} \in \mathsf{P}$ is in Σ^p_2.

For the cases that follow we use a similar tableau procedure and arguments for its correctness and complexity. We will only explain what changes for each case as needed.

For $(\mathsf{JD}^2_C)_{\mathcal{CS}}$ the situation is similar to the one of $(\mathsf{JD}^2)_{\mathcal{CS}}$. We can simplify the tableau rules, by changing the term rules to the following ones (only one changes). Then, $\mathcal{M}$ is defined as in the previous case, except

$$R_1 = R_2 = \{(w, w.1) \in W^2\} \cup \{(w, w) \in W^2 | w.1 \notin W\}.$$

Then, trivially $R_2 \subseteq R_1$, so $\mathcal{M}$ is once more an F-model and the remaining reasoning is as above.

$$\frac{w\ T\ t{:_i}\,\phi}{\begin{array}{c} w.1\ T\ \phi \\ w\ T\ \ *_i\,(t,\phi)\end{array}} \qquad\qquad \frac{w\ F\ t{:_i}\,\phi}{w\ F\ \ *_i\,(t,\phi)}$$

In the case of $(\mathsf{JD}^2_!)_{\mathcal{CS}}$, notice the following. In a frame (W, R_1, R_2), because of 12Verification, whenever xR_2yR_1z, it must also be the case that xR_1z. But this means that whenever $xR_2^*yR_1z$, it must also be the case that xR_1z.[6] Therefore we have the following term rules.

$$\frac{w\ T\ t{:_1}\,\phi}{\begin{array}{c} w.\sigma.1\ T\ \phi \\ w\ T\ \ *_1\,(t,\phi)\end{array}} \qquad \frac{w\ T\ t{:_2}\,\phi}{\begin{array}{c} w.2\ T\ \phi \\ w\ T\ \ *_2\,(t,\phi)\end{array}} \qquad \frac{w\ F\ t{:_i}\,\phi}{w\ F\ \ *_i\,(t,\phi)}$$

for $\sigma \in 2^*$ and $w.\sigma$ that has already appeared.[7]

The remaining reasoning remains similar. The difference from the $(\mathsf{JD}^2)_{\mathcal{CS}}$ case is that now in the definition of $\mathcal{M}$, we also add $w.2.1$ to W for every $w.2 \in W$ and

$$R_1 = \{(w, w.u.1) \in W^2 | u \in 2^*\} \cup \{(w, w) \in W^2 |\ \nexists w.i \in W\}.$$

$$R_2 = \{(w, w.2) \in W^2\} \cup \{(w, w) \in W^2 | w.2 \notin W\}$$

Thus, if aR_2bR_1c, then aR_1c, so $\mathcal{M}$ is a model and we can continue as before.

The case of $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$ is somewhat different, as described by the following lemma. A similar result with a similar proof appears in [1].

Lemma 2. *If ϕ is $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$-satisfiable, then there is some $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ such that $W = \{u, a_1, a_2\}$, $\mathcal{M}, u \models \phi$, and $R_i = \{(x, a_i) \in W^2\}$.*

Proof. Consider an F-model for $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$, $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$, having the strong evidence property and some $u \in W$ such that $\mathcal{M}, u \models \phi$. Let Φ be the set of subformulas of ϕ of the form $t{:_i}\,\psi$. Let $a_0, b_0 \in W$ such that uR_1a_0 and $a_0R_2b_0$. Then, for $k \in \mathbb{N}$, let $a_{k+1}, b_{k+1} \in W$ be such that $b_kR_1a_{k+1}R_2b_{k+1}$. Then, for every $l, k \in \mathbb{N}$ such that $l < k$, $u, a_l, b_lR_1a_k$ and $u, a_l, b_lR_2b_k$. For every $t{:_1}\,\psi$ and $t'{:_2}\,\psi' \in \Phi$, there is some $k \in \mathbb{N}$ such that for every $l \geq k$, $\mathcal{M}, a_l \models t{:_1}\,\psi \to \psi$ and $\mathcal{M}, b_l \models t'{:_2}\,\psi' \to \psi'$: if there is some $x \in \mathbb{N}$ such that $\mathcal{M}, a_x \not\models t{:_1}\,\psi \to \psi$, then $\mathcal{M}, a_x \models t{:_1}\,\psi$ and thus for every $y > x$, $\mathcal{M}, a_y \models \psi$; the same reasoning can be applied for $t'{:_2}\,\psi'$. Since Φ is finite,

[6] R^* is the reflexive transitive closure of binary relation R.

[7] a^* is the set of all strings using symbol a. If A a set and not a binary relation, then A^* is the set of strings using A as an alphabet.

there is some $k \in \mathbb{N}$ such that for every $l \geq k$, and $t:_i \psi \in \Phi$, $\mathcal{M}, a_l \models t:_i \psi \to \psi$. Let $W' = \{u, a_k, b_k\}$, $R'_1 = \{(a, a_k) | a \in W'\}$, $R'_2 = \{(a, b_k) | a \in W'\}$, and $\mathcal{V}'(p) = \mathcal{V}(p) \cap W'$. $\mathcal{A}'_i(t, \psi) = \mathcal{A}_i(t, \psi) \cap W'$ and $\mathcal{A}'$ is then an admissible evidence function. If not, it should violate one of its closure conditions, but it is not hard to see that they are all satisfied - in particular, V-Distribution is the only one affected by the accessibility relations, but it comes down to "if $a \in \mathcal{A}_i(t, \psi)$', then $a \in \mathcal{A}_i(t, \psi)$'.

Then, $\mathcal{M}' = (W', R'_1, R'_2, \mathcal{A}', \mathcal{V}')$ is a $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$-model and we can determine in a straightforward way that $\mathcal{M}', u \models \phi$. □

Thus, for the tableau procedure for $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$, we only allow three world prefixes: 0, 0.1 and 0.2. After the tableau procedure runs, we can define $\mathcal{M}$ as usual, except $W = \{0, 0.1, 0.2\}$, $R_i = W \times \{0.i\}$.

$$\frac{w\ T\ t:_i \phi}{\begin{array}{c} 0.i\ T\ \phi \\ w\ T\ *_i(t, \phi) \end{array}} \qquad \frac{w\ F\ t:_i \phi}{w\ F\ *_i(t, \phi)}$$

For the next four logics, based on $\mathsf{JD4}$, we give the tableau rules in figure 3. The reasoning behind them is an adaptation of the reasoning used for the logics that have been treated so far. We only give the term rules for $w\ T\ t:_i \phi$ for these logics, as the propositional rules and the term rule for $w\ F\ t:_i \phi$ remain the same:

$$\frac{w\ F\ t:_i \phi}{w\ F\ *_i(t, \phi)}$$

For these cases it is important to know the following lemma, as well as that every logic that has been defined in this paper is complete with respect to its F-models that have a finite amount of states. See corollary 6 for a proof of this claim.

Lemma 3. *Let $\mathcal{J}$ be one of the two-agent logics based on* $\mathsf{JD4}$ *from the ones that have been defined in this section. If $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ is a $\mathcal{J}$-model and W is finite, then for every $a \in W$, $i \in \{1, 2\}$, there is some $b \in W$, such that $aR_i bR_i b$.*

Proof. Consider such an F-model for $\mathcal{J}$, $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$. For some $v \in W$ let $a^1_0(v), a^2_0(v) \in W$ such that $vR_1 a^1_0(v)$ and $vR_2 a^2_0(v)$. Then, for every $k \in \mathbb{N}$, let $a^1_{k+1}(v), a^2_{k+1}(v) \in W$ be such that $a^1_k(v) R_1 a^1_{k+1}(v)$ and $a^2_k(v) R_2 a^2_{k+1}(v)$. Then, for every $l, k \in \mathbb{N}$ such that $l < k$, $v, a^1_l(v) R_1 a^1_k(v)$ and $v, a^2_l(v) R_2 a^2_k(v)$. But since W is finite, there must be some $a^1_l(v) = a^1_k(v)$, where $l \neq k$ and similarly there must be some $a^2_l(v) = a^2_k(v)$, where $l \neq k$. □

4 Combining Different Logics

The previous sections have dealt with two-agent versions of various justification logics. In this section we turn our attention to the effect of having two agents that use a different set of axioms, that is, to a combination of two different justification logics. Given $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, we define logics $(\mathcal{J}_1 \times \mathcal{J}_2)_{\mathcal{CS}}$, $(\mathcal{J}_1 \times_! \mathcal{J}_2)_{\mathcal{CS}}$, $(\mathcal{J}_1 \times_C \mathcal{J}_2)_{\mathcal{CS}}$, $(\mathcal{J}_1 \times_{!!} \mathcal{J}_2)_{\mathcal{CS}}$, and $(\mathcal{J}_1 \times_{CC} \mathcal{J}_2)_{\mathcal{CS}}$.

In subsection 2.1 where we give the axioms for the two-agent version of LP, and at the beginning of section 3, each of the Application, Concatenation, Factivity, Positive

<table>
<tr><td>$(\mathsf{JD4}^2)_{\mathcal{CS}}$</td><td>$$\frac{w.i\ T\ t{:_i}\phi}{\begin{array}{c} w.i\ T\ \phi \\ w.i\ T\ *_i(t,\phi) \end{array}}$$ $$\frac{w\ T\ t{:_i}\phi}{\begin{array}{c} w.i\ T\ \phi \\ w\ T\ *_i(t,\phi) \end{array}}$$ where w is not of the form $w'.i$</td></tr>
<tr><td colspan="2">The way to prove correctness for these tableau rules is similar to the one for JD^2, but this time when we construct a branch from a model, we consider a model with a finite amount of states and with the strong evidence property and when prefix w is mapped to state v, we map $w.i$ to a state a such that vR_iaR_ia - which we know exists because of lemma 3.
On the other hand, when we construct a model from an accepting branch, we define $R_i = \{(w, w.i) \in W^2\} \cup \{(w.i, w.i) \in W^2\} \cup \{(w, w) \in W^2 |\ \nexists w.i \in W\}$.</td></tr>
<tr><td>$(\mathsf{JD4}^2_C)_{\mathcal{CS}}$</td><td>$$\frac{w\ T\ t{:_i}\phi}{\begin{array}{c} 0.2\ T\ \phi \\ w\ T\ *_i(t,\phi) \end{array}}$$</td></tr>
<tr><td colspan="2">The reasoning is the same as above, except in this case, we define $W = \{0, 0.2\}$ and $R_1 = R_2 = W \times \{0.2\}$.</td></tr>
<tr><td>$(\mathsf{JD4}^2_!)_{\mathcal{CS}}$</td><td>$$\frac{w.i\ T\ t{:_i}\phi}{\begin{array}{c} w.i\ T\ \phi \\ w.i\ T\ *_i(t,\phi) \end{array}}$$ $$\frac{w\ T\ t{:_1}\phi}{\begin{array}{c} w.s.2.1\ T\ \phi \\ w\ T\ *_1(t,\phi) \end{array}}$$ where $s \in \{1,2\}^*$ and $w.s.2$ has already appeared $$\frac{w\ T\ t{:_i}\phi}{\begin{array}{c} w.i\ T\ \phi \\ w\ T\ *_i(t,\phi) \end{array}}$$ where w is not of the form $w'.i$</td></tr>
<tr><td colspan="2">For this case when we construct the model $\mathcal{M}$ from the accepting branch, we also add $w.2.1$ to W for every $w.2 \in W$ and define $$R_1 = \{(w, w.u.1) \in W^2\} \cup \{(w.1, w.1) \in W^2\} \cup$$ $$\{(w, w) \in W^2 |\ \nexists w.i \in W\}$$ and $$R_2 = \{(w, w.2) \in W^2\} \cup \{(w.2, w.2) \in W^2\} \cup \{(w, w) \in W^2 | w.2 \notin W\}.$$ Then, if aR_2bR_1c, then aR_1c and if aR_ibR_ic, then aR_ic, thus $\mathcal{M}$ is a model; we continue as usual.</td></tr>
<tr><td>$(\mathsf{JD4}^2_{!!})_{\mathcal{CS}}$</td><td>$$\frac{w\ T\ t{:_i}\phi}{\begin{array}{c} 0.i\ T\ \phi \\ w\ T\ *_i(t,\phi) \end{array}}$$</td></tr>
<tr><td colspan="2">See the case for $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$.</td></tr>
</table>

Figure 3: Tableau rules for two-agent logics based on JD4.

Introspection, and Consistency axioms has a version for agent 1 and one for agent 2, depending on what we substitute i for. We say agent 1 is based on logic $\mathcal{J}_1$ and agent 2 on logic J_2 and we choose the axioms' versions for each agent that correspond to the logic the agent is based on. Thus, if $\mathcal{J}_1 = \mathsf{J}$, we only keep agent 1's version of Application and Concatenation, while if $\mathcal{J}_1$ is $\mathsf{JD4}$ we also keep agent 1's version of Positive Introspection and Consistency; similarly we choose the axioms for the second agent. Then for $(\mathcal{J}_1 \times_! \mathcal{J}_2)_{\mathcal{CS}}$ we also include 12Verification, for $(\mathcal{J}_1 \times_C \mathcal{J}_2)_{\mathcal{CS}}$ we include 12Conversion, for $(\mathcal{J}_1 \times_{!!} \mathcal{J}_2)_{\mathcal{CS}}$ we include both 12Verification and 21Verification and for $(\mathcal{J}_1 \times_{CC} \mathcal{J}_2)_{\mathcal{CS}}$ we include both 12Conversion and 21Conversion. $\mathcal{CS}$ is a constant specification as it has been previously defined: a set of formulas of the form $c :_i A$, where c a constant and A an axiom. $cl_2(\mathcal{CS})$ is still the same. The axioms for each such logic are provided in figure 4. For example, the logic $(\mathsf{JT} \times_C \mathsf{JD4})_{\mathcal{CS}}$ is the logic with modus ponens and $R4^2_{\mathcal{CS}}$ as derivation rules and the following axioms:

Propositional Axioms: Finitely many schemes of classical propositional logic;

Application: $s :_i (\phi \to \psi) \to (t :_i \phi \to [s \cdot t] :_i \psi)$;

Concatenation: $s :_i \phi \to [s + t] :_i \phi$, $s :_i \phi \to [t + s] :_i \phi$;

Factivity: $t :_1 \phi \to \phi$;

Consistency: $t :_2 \bot \to \bot$;

Positive Introspection: $t :_2 \phi \to !t :_2 t :_2 \phi$;

12-Conversion: $t :_1 \phi \to t :_2 \phi$,

where in the above, ϕ and ψ are formulas in L_2, s, t are terms, $i \in \{1, 2\}$.

As in the previous section, we do not concern ourselves with $(\mathcal{J}_1 \times_{CC} \mathcal{J}_2)_{\mathcal{CS}}$, as it is essentially a single-agent logic. We also leave it to the reader to extend M-models for the new logics, as well as to give tableau rules based on these M-models for when $\mathcal{J}_1, \mathcal{J}_2 \neq \mathsf{JD}, \mathsf{JD4}$.

Definition 3. *Let $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, $\times_\circ \in \{\times, \times_!, \times_{!!}, \times_C, \times_{CC}\}$ and $J = (\mathcal{J}_1 \times_\circ \mathcal{J}_2)_{\mathcal{CS}}$. An F-model $\mathcal{M}$ for J is a tuple $(W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$, where W a nonempty set of states (occasionally referred to as worlds), for $i \in \{1, 2\}$, $R_i \subseteq W^2$ is a binary relation on W, $\mathcal{A}_i : (Tm \times L_n) \longrightarrow 2^W$, and $\mathcal{V} : SLet \longrightarrow 2^W$. Furthermore, $\mathcal{A}_1, \mathcal{A}_2$ will often be seen and referred to as $\mathcal{A} : \{1, 2\} \times Tm \times L_2 \longrightarrow 2^W$ and $\mathcal{A}$ is called an admissible evidence function. Additionally, $\mathcal{A}$ must satisfy the following conditions for every $i, j \in \{1, 2\}$:*

Application closure: *for any formulas ϕ, ψ and justification terms t, s,*

$$\mathcal{A}_i(s, \phi \to \psi) \cap \mathcal{A}_i(t, \phi) \subseteq \mathcal{A}_i(s \cdot t, \psi).$$

Sum closure: *for any formula ϕ and justification terms t, s,*

$$\mathcal{A}_i(t, \phi) \cup \mathcal{A}_i(s, \phi) \subseteq \mathcal{A}_i(t + s, \phi).$$

Logic $\mathcal{J}_i$ is	Contributes the Axioms:
For all $\mathcal{J}_i$	Propositional Axioms: Finitely many schemes of classical propositional logic; Application: $s :_i (\phi \to \psi) \to (t :_i \phi \to [s \cdot t] :_i \psi)$; Concatenation: $s :_i \phi \to [s + t] :_i \phi$, $s :_i \phi \to [t + s] :_i \phi$
$\mathcal{J}_i = \mathsf{J}$	No additional axioms
$\mathcal{J}_i = \mathsf{JD}$	Consistency: $t :_i \bot \to \bot$
$\mathcal{J}_i = \mathsf{JT}$	Factivity: $t :_i \phi \to \phi$
$\mathcal{J}_i = \mathsf{J4}$	Positive Introspection: $t :_i \phi \to !t :_i t :_i \phi$
$\mathcal{J}_i = \mathsf{JD4}$	Consistency and Positive Introspection
$\mathcal{J}_i = \mathsf{LP}$	Factivity and Positive Introspection

$\times_\circ$ is	Contributes the Axioms:
$\times$	No additional axioms
$\times_!$	12-Verification: $t :_1 \phi \to !t :_2 t :_1 \phi$
$\times_C$	12-Conversion: $t :_1 \phi \to t :_2 \phi$
$\times_{!!}$	12-Verification and 21-Verification: $t :_2 \phi \to !t :_1 t :_2 \phi$
$\times_{CC}$	12-Conversion and 21-Conversion: $t :_2 \phi \to t :_1 \phi$

Figure 4: The axioms for logic $J = (\mathcal{J}_1 \times_\circ \mathcal{J}_2)_{\mathcal{CS}}$, where $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, $\times_\circ \in \{\times, \times_!, \times_{!!}, \times_C, \times_{CC}\}$.

$\mathcal{CS}$-closure: *for any $t :_i \phi \in cl^2(\mathcal{CS})$, $\mathcal{A}_i(t, \phi) = W$.*

Positive Introspection Closure: *When $\mathcal{J}_i$ is among* J4*,* JD4*, and* LP*, $\mathcal{A}_i(t, \phi) \subseteq \mathcal{A}_i(!t, t :_i \phi)$.*

Distribution: *When $\mathcal{J}_i$ is among* J4*,* JD4*, and* LP*, then for any formula ϕ, justification term t, and $a, b \in W$, if aR_ib and $a \in \mathcal{A}_i(t, \phi)$, then $b \in \mathcal{A}_i(t, \phi)$.*

Verification Closure: *When J has ij-Verification, $\mathcal{A}_i(t, \phi) \subseteq \mathcal{A}_j(!t, t :_i \phi)$.*

Conversion Closure: *When J has ij-Conversion, $\mathcal{A}_i(t, \phi) \subseteq \mathcal{A}_j(t, \phi)$.*

V-Distribution: *If J includes ij-Verification, then for any formula ϕ, justification term t, and $a, b \in W$, if aR_jb and $a \in \mathcal{A}_i(t, \phi)$, then $b \in \mathcal{A}_i(t, \phi)$.*

The accessibility relations, R_1, R_2, must satisfy the following conditions: for every $i \in \{1, 2\}$,

- *If $\mathcal{J}_i \in \{\mathsf{JT}, \mathsf{LP}\}$, then R_i must be reflexive.*
- *If $\mathcal{J}_i \in \{\mathsf{JD}, \mathsf{JD4}\}$, then R_i must be serial ($\forall a \in W\ \exists b \in W\ aR_ib$).*
- *If $\mathcal{J}_i \in \{\mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, then for any $a, b, c \in W$, if aR_ibR_ic, we also have aR_ic.*
- *If the logic includes ij-Verification, then for any $a, b, c \in W$, if aR_jbR_ic, we also have aR_ic.*

- *If the logic includes ij-Conversion, then $R_j \subseteq R_i$.*

Truth in the model is defined in the following way:

- $M, u \not\models \bot$.
- *If p is a propositional variable, then $\mathcal{M}, u \models p$ iff $u \in \mathcal{V}(p)$*
- *If ϕ, ψ are formulas, then $\mathcal{M}, u \models \phi \to \psi$ if and only if $M, u \models \psi$, or $\mathcal{M}, u \not\models \phi$.*
- *If ϕ is a formula and t a term, then $\mathcal{M}, u \models t{:_i}\,\phi$ if and only if $u \in \mathcal{A}_i(t, \phi)$ and for every $v \in W$ such that uR_iv, $\mathcal{M}, v \models \phi$.*

Proposition 4. *Let $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, $\times_\circ \in \{\times, \times_!, \times_{!!}, \times_C, \times_{CC}\}$ and $J = (\mathcal{J}_1 \times_\circ \mathcal{J}_2)_{\mathcal{CS}}$. Then, J is sound and complete with respect to its F-models.*

Proof. To prove soundness, it suffices to show that all axioms are satisfied in all worlds of all models and that the inference rules preserve validity. Let $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ be a model for J. $\mathcal{M}$ will satisfy all axioms of J in all of its worlds:

Application: If $\mathcal{M}, a \models s{:_i}\,(\phi \to \psi)$ and $\mathcal{M}, a \models t{:_i}\,\phi$, then $a \in \mathcal{A}_i(s, \phi \to \psi) \cap \mathcal{A}_i(t, \phi)$ and for every $b \in W$, if aR_ib, then $\mathcal{M}, b \models \phi \to \psi$ and $\mathcal{M}, b \models \phi$. Therefore, from the definition of an F-model, $a \in \mathcal{A}_i(s{\cdot}t, \psi)$ and if $b \in W$ and aR_ib, then $\mathcal{M}, b \models \psi$. Therefore, $\mathcal{M}, a \models [s \cdot t]{:_i}\,\psi$.

Concatenation: If $\mathcal{M}, a \models t{:_i}\,\phi$ or $\mathcal{M}, a \models s{:_i}\,\phi$, then $a \in \mathcal{A}_i(t, \phi) \cup \mathcal{A}_i(s, \phi)$ and for every $b \in W$, if aR_ib, then $\mathcal{M}, b \models \phi$. Then, $a \in \mathcal{A}_i(t + s, \phi)$ and for every $b \in W$, if aR_ib, then $\mathcal{M}, b \models \phi$. This means that $\mathcal{M}, a \models [t + s]{:_i}\,\phi$.

Positive Introspection: See Verification.

Consistency: If $\mathcal{M}, a \models t{:_i}\,\bot$ and $\mathcal{J}_i \in \{\mathsf{JD}, \mathsf{JD4}\}$, then by the definition there is some b, accessible from a, such that $\mathcal{M}, b \models \bot$, a contradiction.

Factivity: If $\mathcal{M}, a \models t :_i \phi$ and $\mathcal{J}_i$, then since R_i is reflexive, aR_ia and therefore, $\mathcal{M}, a \models \phi$.

Verification: If $\times_\circ \in \{\times_!, \times_{!!}\}$, J has ij-Verification as an axiom, and $\mathcal{M}, a \not\models !t{:_j}\,t{:_i}\,\phi$, then either $a \notin \mathcal{A}_j(!t, t{:_i}\,\phi)$, in which case $a \notin \mathcal{A}_i(t, \phi)$, or there is some $b \in W$ s.t. aR_ib and $\mathcal{M} \not\models t{:_i}\,\phi$. This last case means that either $b \notin \mathcal{A}_i(t, \phi)$, in which case, by V-Distribution, $a \notin \mathcal{A}_i(t, \phi)$, or that there is some $c \in W$, s.t. bR_ic and $\mathcal{M}, b \not\models \phi$. Then, $aR_ic \not\models \phi$. In all these cases, we can conclude that $\mathcal{M}, a \not\models t{:_i}\,\phi$.

Conversion: If J has ij-Conversion, then since $R_j \subseteq R_i$ and $\mathcal{A}_i(t, \phi) \subseteq \mathcal{A}_j(t, \phi)$, if $\mathcal{M}, a \models t{:_i}\,\phi$, then $\mathcal{M}, a \models t{:_j}\,\phi$.

By the $\mathcal{CS}$-closure condition of the admissible evidence function and the validity of the axioms, the rule $R4^2_{\mathcal{CS}}$ produces only valid formulas. Finally, modus ponens trivially preserves validity.

Completeness will be proven using a canonical model construction. Let W be the set of all maximal consistent subsets of L_2. We know that W is not empty, because J

is consistent. For $\Gamma \in W$ and $i \in \{1,2\}$, let $\Gamma^{\#_i} = \{\phi \in L_2 | \exists t \in Tm\ t:_i \phi \in \Gamma\}$. For any $i \in \{1,2\}$, R_i is a binary relation on W, such that $\Gamma R_i \Delta$ if and only if $\Gamma^{\#_i} \subseteq \Delta$. Also, for $i \in \{1,2\}$, let $\mathcal{A}_i(t,\phi) = \{\Gamma \in W | t:_i \phi \in \Gamma\}$. Finally, $\mathcal{V} : Slet \longrightarrow \mathcal{P}(W)$ is such that $\mathcal{V}(p) = \{\Gamma \in W | p \in \Gamma\}$. The canonical model is $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$.

Define the relation between worlds of the canonical models and formulas of L_2, $\models$, as in the definition of models.

Lemma 5 (A Truth Lemma)**.** *For any* $\Gamma \in W$, $\phi \in L_2$,

$$\mathcal{M}, \Gamma \models \phi \Longleftrightarrow \phi \in \Gamma.$$

Proof. By induction on the structure of ϕ. The cases for $\phi = p$, a propositional variable, $\bot$, or $\psi_1 \to \psi_2$, are immediate from the definition of $\mathcal{V}$ and $\models$.
If $\phi = t:_i \psi$, then

$$\mathcal{M}, \Gamma \models t:_i \psi \Rightarrow \Gamma \in \mathcal{A}_i(t,\psi) \Leftrightarrow t:_i \psi \in \Gamma.$$

Furthermore,

$$t:_i \psi \in \Gamma \Rightarrow \forall \Delta \in W\ (\Gamma R_i \Delta \to \psi \in \Delta) \Rightarrow \forall \Delta \in W\ (\Gamma R_i \Delta \to \Delta \models \psi)$$

and finally,

$$\Gamma \in \mathcal{A}_i(t,\psi) \text{ and } \forall \Delta \in W\ (\Gamma R_i \Delta \to \Delta \models \psi) \Rightarrow \mathcal{M}, \Gamma \models t:_i \psi,$$

which completes the proof. □

The canonical model is, indeed, a model for J. To establish this, we must show that the conditions expected from R_1, R_2 and $\mathcal{A}_1, \mathcal{A}_2$ are satisfied.
First, the admissible evidence function conditions:

Application closure: If $\Gamma \in \mathcal{A}_i(s, \phi \to \psi) \cap \mathcal{A}_i(t,\phi)$, then $s:_i (\phi \to \psi), t:_i \phi \in \Gamma$, therefore, because of the application axiom, $[s \cdot t]:_i \psi \in \Gamma$, concluding that $\Gamma \in \mathcal{A}_i(s \cdot t, \psi)$.

Sum closure: If $\Gamma \in \mathcal{A}_i(t,\phi)$, then $t :_i \phi \in \Gamma$, so, by the Concatenation axiom, $[s+t]:_i \phi, [t+s]:_i \phi \in \Gamma$, therefore, $\Gamma \in \mathcal{A}_i(t+s,\phi) \cap \mathcal{A}_i(s+t,\phi)$.

Positive Introspection closure: If $\mathcal{J}_i$ has Positive Introspection and $\Gamma \in \mathcal{A}_i(t,\phi)$, then $t :_i \phi \in \Gamma$ and because of Positive Introspection, $!t :_i t :_i \phi \in \Gamma$, therefore, $\Gamma \in \mathcal{A}_i(!t, t:_i \phi)$.

$\mathcal{CS}$ closure: Since any $\Gamma \in W$ must include all formulas produced by rule $R4^2_{\mathcal{CS}}$, it is easy to see that this condition is satisfied.

Distribution: If $\mathcal{J}_i$ has Positive Introspection, $\Gamma R_i \Delta$, and $\Gamma \in \mathcal{A}_i(t,\phi)$, then $t:_i \phi \in \Gamma$ and by Positive Introspection, $!t:_i t:_i \phi \in \Gamma$, thus $t:_i \phi \in \Gamma^{\#_i} \subseteq \Delta$, concluding that $\Delta \in \mathcal{A}_i(t,\phi)$.

Verification closure: If J has ij-Verification and $\Gamma \in \mathcal{A}_i(t,\phi)$, then $t :_i \phi \in \Gamma$ and because of ij-Verification, $!t:_j t:_i \phi \in \Gamma$, therefore, $\Gamma \in \mathcal{A}_j(!t, t:_i \phi)$.

Conversion closure: If $\Gamma \in \mathcal{A}_i(t,\phi)$, then $t :_i \phi \in \Gamma$ and because of ij-Conversion, $t:_j \phi \in \Gamma$, therefore, $\Gamma \in \mathcal{A}_j(t,\phi)$.

V-Distribution: If $\Gamma R_j \Delta$ and $\Gamma \in \mathcal{A}_i(t, \phi)$, then $t :_i \phi \in \Gamma$ and by ij-Verification, $!t :_j t :_i \phi \in \Gamma$, thus $t :_i \phi \in \Gamma^{\#_j} \subseteq \Delta$, concluding that $\Delta \in \mathcal{A}_i(t, \phi)$.

Finally, to complete the proof, we must prove that R_1, R_2 satisfy the necessary conditions:

If $\mathcal{J}_I \in \{\mathsf{JT}, \mathsf{LP}\}$, then R_i must be reflexive. For this, we just need to prove that if $\Gamma \in W$, then $\Gamma^{\#_i} \subseteq \Gamma$. If $\phi \in \Gamma^{\#_i}$, then there is some justification term, t, for which $t :_i \phi \in \Gamma$. Because of the factivity axiom, $\neg\phi \notin \Gamma$, because $\{t :_i \phi, \neg\phi\}$ is inconsistent. Therefore, because Γ is maximal consistent, $\phi \in \Gamma$.

If $\mathcal{J}_i \in \{\mathsf{JD}, \mathsf{JD4}\}$, then R_i is serial. To establish this, we just need to show that $\Gamma^{\#_i}$ is consistent. If it is not, then there are formulas $\phi_1, \ldots, \phi_k \in \Gamma^{\#_i}$ s.t.

$$\phi_1, \ldots, \phi_k \vdash \bot.$$

This means that there are $t_1 :_i \phi_1, \ldots t_k :_i \phi_k \in \Gamma$, s.t.

$$t_1 :_i \phi_1, \ldots t_k :_i \phi_k \vdash t(t_1, \ldots, t_k) :_i \bot$$

(by proposition 1 and because $\mathcal{CS}$ is axiomatically appropriate), which is a contradiction.

If $\mathcal{J}_i$ has Positive Introspection and $\Gamma R_i \Delta R_i E$, then $\Gamma R_i E$. If $t :_i \phi \in \Gamma$, then $!t :_i t :_i \phi \in \Gamma$. Therefore, $t :_i \phi \in \Gamma^{\#_i}$. So, if $\Gamma R_i \Delta$, then $t :_i \phi \in \Delta$. So, $\Gamma^{\#_i} \subseteq \Delta^{\#_i}$ and if $\Delta R_i E$, then $\Gamma R_i E$.

If J has ij-Verification and $\Gamma R_j \Delta R_i E$, then $\Gamma R_i E$. If $t :_i \phi \in \Gamma$, then $!t :_j t :_i \phi \in \Gamma$. Therefore, $t :_i \phi \in \Gamma^{\#_j}$. So, if $\Gamma R_j \Delta$, then $t :_i \phi \in \Delta$. So, $\Gamma^{\#_i} \subseteq \Delta^{\#_i}$ and if $\Delta R_i E$, then $\Gamma R_i E$.

If J has ij-Conversion, then $R_j \subseteq R_i$. From ij-Conversion, it is apparent that for any $\Gamma \in W$, $\Gamma^{\#_i} \subseteq \Gamma^{\#_j}$. Therefore, $R_j \subseteq R_i$.

Finally, notice that the canonical model has the strong evidence property: if $\Gamma \in \mathcal{A}_i(t, \phi)$ then $t :_i \phi \in \Gamma$ and by the Truth Lemma, $\Gamma \models t :_i \phi$. □

Corollary 6. *Let $\mathcal{J}$ be a two-agent justification logic as in the assumptions of proposition 4. If $\mathcal{CS}$ is axiomatically appropriate and ϕ is $\mathcal{J}$-satisfiable, then ϕ is satisfiable by an F-model for $\mathcal{J}$ of at most $2^{|\phi|}$ states which has the strong evidence property.*

Proof. Let $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{V})$ be the canonical model from the proof of proposition 4, $\mathcal{M}_f = (W^f, R_1^f, R_2^f, \mathcal{A}_1^f, \mathcal{A}_2^f, \mathcal{V}^f)$, where W^f is the set of all maximally consistent sets of subformulas of ϕ, for all $i \in \{1, 2\}$, $X, Y \in W_f$,

- $X R_i^f Y$ iff there is some $X' \in W$ such that $X \subseteq X'$, for every $Y' \in W$ such that $Y \subseteq Y'$ and $X' R_i Y'$;
- $X \in \mathcal{A}_i^f(t, \psi)$ iff for every $X \subseteq X' \in W$, $X' \in \mathcal{A}_i(t, \psi)$ and
- $X \in \mathcal{V}^f(p)$ iff for every $X \subseteq X' \in W$, $X' \in \mathcal{V}(p)$.

Then, define $\mathcal{M}_f \models \psi$ in the usual way as for models. Notice that since the elements of W_f are maximally consistent w.r.t. subformulas of ϕ, for every set Ψ of subformulas of ϕ, $\Psi \subseteq X \in W_f$ iff for every $X \subseteq \Gamma \in W$, $\Psi \subseteq \Gamma$. Also, for every $X \in W^f$, let $\overline{X} = \{\psi \in L_2 | X \vdash \psi\}$. Then, for every $X, Y \in W_f$, propositional variable p and $t:_i \psi$ subformula of ϕ, $X \in \mathcal{A}_i^f(t, \psi)$ iff $t:_i \psi \in X$, and $X \in \mathcal{V}^f(p)$ iff $p \in X$.

Furthermore, $XR_i^f Y$ iff $\overline{X}^{\#_i} \subseteq \overline{Y}$: if $XR_i^f Y$, then there is some $X' \in W$ such that $X \subseteq X'$ and for every $Y' \in W$ for which $Y \subseteq Y'$, $X' R_i Y'$. Then, for every $Y' \in W$ for which $Y \subseteq Y'$, $\overline{X}^{\#_i} \subseteq X'^{\#_i} \subseteq Y'$, so $\overline{X}^{\#_i} \subseteq \overline{Y}$. On the other hand, if $\overline{X}^{\#_i} \subseteq \overline{Y}$, then let $G = \overline{X} \cup \{\neg t:_j \psi \in L_2 | t:_j \psi \notin \overline{X}\}$. Then G is consistent (let $\mathcal{A}_j^G(t, \psi) = \mathit{true}$ iff $t:_j \psi \in G$, so $(\mathcal{A}^G, \mathcal{V}^f) \models G$) and can be expanded to a maximally consistent $X' \in W$. $X'^{\#_i} = \overline{X}^{\#_i} \subseteq \overline{Y} \subseteq Y'$ for every $Y' \in W$ for which $Y \subseteq Y'$. Thus, $XR_i^f Y$.

It is not hard to follow the proof of lemma 5 to prove that for every subformula ψ of ϕ and $X \in W^f$, $X \models \psi$ iff $\psi \in X$ and then continue by following the proof of proposition 4 to complete this one. □

Now we take cases and look at each logic. As in the definition of F-models, let $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$, $\times_\circ \in \{\times, \times_!, \times_{!!}, \times_C, \times_{CC}\}$, $J = (\mathcal{J}_1 \times_\circ \mathcal{J}_2)_{\mathcal{CS}}$.

No interactions: $\times_\circ = \times$**.** In this case, since there are no interactions we simply use the usual rule for $w\ F\ t:_i \psi$ that gives $w\ F\ *_i(t, \psi)$ and one rule for each agent i depending on what $\mathcal{J}_i$ is. These are given in the following together with the corresponding $\mathcal{J}_i$. The reasoning follows the one for the case of $(\mathsf{JD}^2)_{\mathcal{CS}}$, except for the case when $\mathcal{J}_i = \mathsf{JD4}$, where when we construct an accepting branch from a model (of finite states and with the strong evidence property), we can use lemma 3, which holds for R_i and thus if we map w to a we map $w.i$ to some b such that $aR_i bR_i b$.

$$\frac{w\ T\ t:_i \phi}{w\ T\ \ *_i(t, \phi)}\ \mathsf{J} \qquad \frac{w\ T\ t:_i \phi}{\begin{array}{c} w.i\ T\ \phi \\ w\ T\ \ *_i(t, \phi)\end{array}}\ \mathsf{JD} \qquad \frac{w\ T\ t:_i \phi}{\begin{array}{c} w\ T\ \phi \\ w\ T\ \ *_i(t, \phi)\end{array}}\ \mathsf{JT}$$

$$\frac{w\ T\ t:_i \phi}{w\ T\ \ *_i(t, \phi)}\ \mathsf{J4} \qquad \frac{w\ T\ t:_i \phi}{\begin{array}{c} v.i\ T\ \phi \\ w\ T\ \ *_i(t, \phi)\end{array}}\ \mathsf{JD4} \qquad \frac{w\ T\ t:_i \phi}{\begin{array}{c} w\ T\ \phi \\ w\ T\ \ *_i(t, \phi)\end{array}}\ \mathsf{LP}$$

where if w of the form $w'.i$, then $v = w'$ and otherwise $v = w$

Verification: $\times_\circ = \times_!$ **or** $\times_\circ = \times_{!!}$**.** When $\times_\circ = \times_{!!}$ and $\mathcal{J}_1$, $\mathcal{J}_2$ are among JD, JT, $\mathsf{JD4}$, and LP, it is not hard to see that lemma 2 is true for J as well and we can thus give tableau rules similar to the ones for $(\mathsf{JD}^2_{!!})_{\mathcal{CS}}$.[8] On the other hand if one of the two agents is based on J or $\mathsf{J4}$, then we can use the same rules and reasoning as for the case when $\times_\circ = \times$. Thus we only examine the cases when $\times_\circ = \times_!$ and $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{LP}\}$. For these cases we can use the same rules as in the case where $\times_\circ = \times$ for $\mathcal{J}_2$ as well as

[8] In fact if one of $\mathcal{J}_1, \mathcal{J}_2$ is JT or LP, then only up to two worlds are required in the model, as these logics require reflexivity and not seriality of their accessibility relation.

one of the following two rules for $\mathcal{J}_1$. The first should be used if $\mathcal{J}_1 = \mathsf{JD}$, the second one if $\mathcal{J}_1 \in \{\mathsf{JT}, \mathsf{LP}\}$ and the third one should be used if $\mathcal{J}_1 = \mathsf{JD4}$.

$$\frac{w\ T\ t{:}_1\,\phi}{\begin{array}{c} w.s.1\ T\ \phi \\ w\ T\ *_1(t,\phi)\end{array}} \qquad \frac{w\ T\ t{:}_1\,\phi}{\begin{array}{c} w.s\ T\ \phi \\ w\ T\ *_1(t,\phi)\end{array}} \qquad \frac{w\ T\ t{:}_1\,\phi}{\begin{array}{c} v.1\ T\ \phi \\ w\ T\ *_1(t,\phi)\end{array}}$$

where for the first two rules, $s \in 2^*$ and $w.s$ has already appeared and for the third one, either w of the form $0.a$, where $a \in 2^*$ and $v = w.s$ where $s \in 2^*$ and $w.s$ has already appeared, or w of the form $0.w_1.1.w_2$ (and $w_1, w_2 \in 2^*$) and $v = 0.w_1$.

The argument for this case is similar to the ones that have already been covered. To justify the third rule, which is different from the ones we have encountered, we give the following lemma:

Lemma 7. [9] *Let $\mathcal{J}_2 \in \{\mathsf{JD}, \mathsf{JT}, \mathsf{JD4}, \mathsf{LP}\}$ and $J = (\mathsf{JD4} \times_! \mathcal{J}_2)_{\mathcal{CS}}$, where $\mathcal{CS}$ axiomatically appropriate. Let $\mathcal{F} = (W, R_1, R_2)$ be a finite J-frame. Then, for every $u \in W$ there is some $v \in W$ such that uR_1vR_1v and for every $x, y \in W$, if x, vR_1y, then xR_1v.*

Proof. Let uR_1v_1. For every $k \in \mathbb{N}$, if there is some $x, y \in W$, where x, v_kR_1y, and not xR_1v_k, then let $v_{k+1} = y$ (notice that then for every $a \in W$, if $v_{k+1}R_1a$, then xR_1a), otherwise $v_{k+1} = v_k$. Since W is finite, there is some $l \in \mathbb{N}$, such that for every $k \leq l$, $v_k = v_l$. □

Then, when constructing a branch from a model, we map $w.1$ to such a v and when we construct a model from an accepting branch, we can define W to be the set of all world prefixes that appear in the branch as well as $0.s.1$ for every $s \in 2^*$ such that no world prefix $0.s.i$ where $i \in \{1, 2\}$ appears and

$$R_1 = \{(w, w.u.1) \in W^2\} \cup \{(w.1.u, w.1) \in W^2\}.$$

Conversion: $\times_\circ = \times_C$. If $\mathcal{J}_2 \in \{\mathsf{JT}, \mathsf{LP}\}$ then $J \vdash t{:}_1\,\phi \to \phi$. Thus if $\mathcal{J}_1 \in \{\mathsf{JD}, \mathsf{JD4}\}$, then the Consistency condition of M-models is redundant and thus we can use the tableau rules based on M-models we already used for $((\mathcal{J}_1 +$ Factivity$) \times_C \mathcal{J}_2)$. The cases $\mathcal{J}_1 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}\}$ or $\mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JT}, \mathsf{JD4}, \mathsf{J4}, \mathsf{LP}\}$ are left to the reader and the only cases that will interest us are the ones where $\mathcal{J}_2 = \mathsf{JD}$ and $\mathcal{J}_1 \in \{\mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$. In fact, for these cases and in contrast to all other cases we have studied, J-satisfiability is PSPACE-complete. We provide the following tableau rules for $(\mathsf{JD4} \times_C \mathsf{JD})_{\mathcal{CS}}$, leaving to the reader to complete the proof and to adjust these to the other cases. Notice that we can keep exactly one world-prefix in memory each time: the only part of the process which is affected by the frame is the application of rules $*$Dis($\mathcal{F}$) and $*$V-Dis($\mathcal{F}$), but these can only be applied to $a\ *_1(t,\psi)$, which means we can push all applications $*$Dis($\mathcal{F}$) and $*$V-Dis($\mathcal{F}$) to the leaves of the $*$-derivation, where they are unnecessary, as we can see from the following rules. Since all prefixes are $0.2\ldots2$, we don't even need to keep the prefix as is in memory, just its length, which gives us a bound on the space we use. The proof of PSPACE-hardness for $(\mathsf{JD4} \times_C \mathsf{JD})_{\mathcal{CS}}$-satisfiability then follows.

[9] Notice that this lemma also applies to the case of $(\mathsf{JD4}^2_!)_{\mathcal{CS}}$. Using it would make proving a polynomial bound on the number of world prefixes easier.

$$\frac{w\ T\ t{:_2}\,\phi}{\begin{array}{c} w.2\ T\ \phi \\ w\ T\ *_2\,(t,\phi) \end{array}}$$

$$\frac{w\ T\ t{:_1}\,\phi}{\begin{array}{c} w\ T\ *_1\,(t,\phi) \\ w\ T\ \Box\phi \\ w\ T\ \Box *_1\,(t,\phi) \end{array}}$$

$$\frac{w\ T\ \Box\alpha}{\begin{array}{c} w.2\ T\ \alpha \\ w.2\ T\ \Box\alpha \end{array}}$$

where $w.2$ has already appeared and α either a formula or a $*$-expression.

Informally, $w\ T\ \Box\alpha$ stands for v "satisfies" α for every wR_1v. When running the tableau we are not guaranteed it will terminate, but we can artificially terminate it after a sufficient length of prefixes is reached (exponential in $|\phi|$, enough to know we have reached the same set of expressions twice), or consider an infinite branch, closed under the rules. When constructing a model from a branch, W is the collection of prefixes (possibly infinite) and $R_2 = \{(w, w.2) \in W^2\} \cup \{(w, w) \in W^2 | w.2 \notin W\}$ and R_1 the transitive closure of R_2: $\{(w, w.u) \in W^2 | u \neq \epsilon\} \cup \{(w, w) \in W^2 | w.2 \notin W\}$.

We now prove PSPACE-hardness for $(\mathsf{JD4} \times_C \mathsf{JD})_{\mathcal{CS}}$-satisfiability. The proof is by reduction from a deterministic Turing machine of two tapes (input and working tape) using polynomial space. It closely resembles the one in [9] and has been used in [2] in a more general form to prove similar results. Let the machine be (Q, Σ, δ, s), where Q the set of states, Σ the alphabet, δ the transition function and s the initial state. Let $x = x_1x_2\cdots x_{|x|}$ be the input, where for every $i \in \{1, 2, \ldots, |x|\}$, $x_i \in \Sigma$. Since the Turing machine uses polynomial space, there is a polynomial p, such that the working tape only uses cells 1 to $p(|x|)$ for an input x. For the input tape, we only need cells 0 through $|x| + 1$, because the head does not go any further and an output tape is not needed, since we are interested only in decision problems. Therefore, there are $Y, N \in Q$, the accepting and rejecting states respectively. Let $r_1 = \{0, 1, 2, \ldots, |x| + 1\}$ and $r_2 = \{1, 2, \ldots, p(|x|)\}$.

- $t_1[i], t_2[j]$, for every $i \in r_1, j \in r_2$; $t_1[i]$ will correspond to the head for the first tape pointing at cell i and similarly for $t_2[j]$,
- $\sigma_1[a, i], \sigma_2[a, j]$, for every $a \in \Sigma$, $i \in r_1, j \in r_2$; $\sigma_1[a, i]$ will correspond to cell i in the first tape having the symbol a and similarly for $\sigma_2[a, j]$ and the second tape,
- $q[a]$, for every $a \in Q$; $q[a]$ means the machine is currently in state a.

We need the following formulas. Intuitively, a state in a model for ϕ corresponds to a configuration of our Turing machine. q ensures there is exactly one state at every configuration; σ that there is exactly one symbol at every position of every tape; t that for each tape the head is located at exactly one position; σ' ensures that the only symbols that can change from one configuration to the next are the ones located in a position the head points at; ac ensures we never reach a rejecting state (therefore the machine accepts); st starts the computation at the starting configuration of the machine; finally, d ensures for each configuration that the next one is given by the transition function. Then, if $com = q \wedge \sigma \wedge t \wedge \sigma' \wedge ac \wedge d$,

$$\phi = st \wedge com \wedge x{:_1}\,com$$

$$q = \left(\bigvee_{a \in Q} q[a] \right) \wedge \bigwedge_{\substack{a,b \in Q,\\ a \neq b}} \neg \left(q[a] \wedge q[b] \right)$$

$$\sigma = \bigwedge_{\substack{j \in \{1,2\},\\ i \in r_j}} \left[\left(\bigvee_{a \in \Sigma} \sigma_j[a,i] \right) \wedge \bigwedge_{\substack{a,b \in \Sigma,\\ a \neq b}} \neg \left(\sigma_j[a] \wedge \sigma_j[b] \right) \right]$$

$$t = \bigwedge_{j \in \{1,2\}} \left[\left(\bigvee_{i \in r_j} t_j[i] \right) \wedge \bigwedge_{\substack{i,k \in r_j\\ i \neq k}} \neg \left(t_j[i] \wedge t_j[k] \right) \right]$$

$$\sigma' = \bigwedge_{\substack{j \in \{1,2\},\\ i,i' \in r_j,\\ i \neq i',\\ a \in \Sigma}} \left[\left(t_j[i] \wedge \sigma_j[a,i'] \right) \to x :_2 \sigma_j[a,i'] \right]$$

$ac = \neg q[N]$,
$st = \phi_{c_0}$, where ϕ_{c_0} describes the initial configuration of the machine,

$$d = \bigwedge_{\substack{(a,i_1,i_2) \in E \times \Sigma \times \Sigma,\\ j_1 \in r_1,\\ j_2 \in r_2}} \left[\begin{array}{l} q[a] \wedge \sigma_1[i_1,j_1] \wedge \sigma_2[i_2,j_2] \wedge t_1[j_1] \wedge t_2[j_2] \longrightarrow \\ \qquad x :_2 \left(q[a_1] \wedge \sigma_2[k_1,j_2] \wedge t_1[j_1+m_1] \wedge t_2[j_2+m_2] \right) \end{array} \right]$$

where $(a_1, k_1, m_1, m_2) = \delta(a, i_1, i_2)$.

For every configuration c of the Turing machine, there is a formula that describes it. This formula is the conjunction of the following and from now and on it will be denoted as ϕ_c: $q[a]$, if a is the state of the machine in c; $t_1[i]$ and $t_2[j]$, if the first tape's head is on cell i and the second tape's head is on cell j; $\sigma_1[a_1,i_1], \sigma_2[a_2,i_2]$, if $i_1 \in r_1, i_2 \in r_2$ and a_1 is the symbol currently in cell i_1 of the first tape and a_2 is the symbol currently in cell i_2 of the second tape. Then, st is ϕ_{c_0}, where c_0 is the initial configuration for the machine on input x.

Claim: If for some model $\mathcal{M}, w \models \phi$ and for some u, wR_1u and $u \models \phi_c$ and c_1 is the next configuration from c, then there is some w, uR_1u_1, such that $u_1 \models \phi_{c_1}$.
From this claim, it immediately follows that if ϕ is satisfiable, then the Turing machine accepts its input. We now prove the claim. Because of formulas q, σ, t, in every state v, such that wR_1v, there is exactly one ϕ_c satisfied. There is some state u_1, (because of seriality of R_2) such that wR_2u_1 and if $u_1 \models \phi_a$, then because of d, a will differ from c in all respects δ demands; furthermore, because of σ', a differs only in the ways δ demands. Therefore, $a = c_1$.

On the other hand, assuming that the Turing machine accepts x, given its computation path for x, we can construct the following model $\mathcal{M} = (W, R_1, R_2, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{V})$ for ϕ. W is the set of configurations in the computation tree; let R_2 be minimal such that if a is a configuration and b its next configuration, then aR_2b; let R_1 be the transitive closure of R_1. $\mathcal{A}_i(t,\psi) = W$ for all i, t, ψ. $\mathcal{V}$ is defined to be such that $\mathcal{M}, a \models \phi_a$

(every ϕ_a is a conjunction of propositional variables). Then, it is not hard to see that $\mathcal{M}, c_0 \models \phi$.

To summarize the complexity results of this paper, we present the following theorem.

Theorem 8. *Let* $\mathcal{J}_1, \mathcal{J}_2 \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$*,* $\times_\circ \in \{\times, \times_!, \times_{!!}, \times_C, \times_{CC}\}$*, and* $J = (\mathcal{J}_1 \times_\circ \mathcal{J}_2)_{\mathcal{CS}}$*, where* $\mathcal{CS}$ *a schematic, axiomatically appropriate constant specification. If* $\mathcal{J}_2 = \mathsf{JD}$*,* $\mathcal{J}_1 \in \{\mathsf{J4}, \mathsf{JD4}, \mathsf{LP}\}$ *and* $\times_\circ = \times_C$*, then* J*-satisfiability is* PSPACE*-complete. In every other case,* J*-satisfiability is in* Σ_2^p*.*

References

[1] Antonis Achilleos. A complexity question in justification logic. In *Proceedings of the 18th international conference on Logic, language, information and computation*, WoLLIC'11, pages 8–19, Berlin, Heidelberg, 2011. Springer-Verlag.

[2] Antonis Achilleos. Modal logics with hard diamond-free fragments. *CoRR*, abs/1401.5846, 2014.

[3] Sergei Artemov. Operational modal logic. Technical Report MSI 95–29, Cornell University, December 1995.

[4] Sergei Artemov. Explicit provability and constructive semantics. *Bulletin of Symbolic Logic*, 7(1):1–36, March 2001.

[5] Sergei Artemov. The logic of justification. *The Review of Symbolic Logic*, 1(4):477–513, December 2008.

[6] Sergei Artemov and Roman Kuznets. Logical omniscience as a computational complexity problem. In Aviad Heifetz, editor, *TARK*, pages 14–23, 2009.

[7] Vladimir N. Brezhnev. On explicit counterparts of modal logics. Technical Report CFIS 2000–05, Cornell University, 2000.

[8] Samuel Bucheli, Roman Kuznets, and Thomas Studer. Justifications for common knowledge. *Journal of Applied Non-Classical Logics*, 21(1):35–60, January–March 2011.

[9] Michael J Fischer and Richard E Ladner. Propositional dynamic logic of regular programs. *Journal of computer and system sciences*, 18(2):194–211, 1979.

[10] Melvin Fitting. The logic of proofs, semantically. *Annals of Pure and Applied Logic*, 132(1):1–25, February 2005.

[11] Nikolai V. Krupski. On the complexity of the reflected logic of proofs. *Theoretical Computer Science*, 357(1–3):136–142, July 2006.

[12] Roman Kuznets. On the complexity of explicit modal logics. In Peter Clote and Helmut Schwichtenberg, editors, *CSL*, volume 1862 of *Lecture Notes in Computer Science*, pages 371–383. Springer, 2000. Errata concerning the explicit counterparts of $\mathcal{D}$ and $\mathcal{D}4$ are published as [14].

[13] Roman Kuznets. *Complexity Issues in Justification Logic.* PhD thesis, CUNY Graduate Center, May 2008.

[14] Roman Kuznets. Complexity through tableaux in justification logic. In *Plenary Talks, Tutorials, Special Sessions, Contributed Talks of Logic Colloquium (LC 2008), Bern, Switzerland*, pages 38–39, 2008.

[15] Roman Kuznets. Self-referentiality of justified knowledge. In Edward A. Hirsch, Alexander A. Razborov, Alexei L. Semenov, and Anatol Slissenko, editors, *CSR*, volume 5010 of *Lecture Notes in Computer Science*, pages 228–239. Springer, 2008.

[16] Richard E. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal on Computing*, 6(3):467–480, 1977.

[17] Robert Saxon Milnikel. Derivability in certain subsystems of the logic of proofs is Π_2^p-complete. *Annals of Pure and Applied Logic*, 145(3):223–239, 2007.

[18] Alexey Mkrtychev. Models for the logic of proofs. In Sergei Adian and Anil Nerode, editors, *Logical Foundations of Computer Science, 4th International Symposium, LFCS'97, Yaroslavl, Russia, July 6–12, 1997, Proceedings*, volume 1234 of *Lecture Notes in Computer Science*, pages 266–275. Springer, 1997.

[19] Eric Pacuit. A note on some explicit modal logics. In *Proceedings of the 5th Panhellenic Logic Symposium*, pages 117–125, Athens, Greece, July 25–28, 2005. University of Athens.

[20] Bryan Renne. Simple evidence elimination in justification logic. In Patrick Girard, Olivier Roy, and Mathieu Marion, editors, *Dynamic Formal Epistemology*, volume 351 of *Synthese Library*, chapter 7, pages 127–149. Springer, 2011.

[21] Tatiana Yavorskaya (Sidon). Interacting explicit evidence systems. *Theory Comput. Syst.*, 43(2):272–293, 2008.