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Abstract

Justification logics are explicit versions of modal logics, in which *reasons* for formulas being so are part of the formula structure. Justification logics are connected with their corresponding modal logics through *Realization Theorems*. Beginning with S4, several standard modal logics have been shown to have justification counterparts: K, K4, S5, and so on. In this report we begin exploration of the question: what is the range of modal logics that have justification counterparts. We introduce general methodology that applies to all current examples, though non-constructively. We use this machinery to establish realization results for some new justification logics based on modal logics that have not been considered in justification context before. Our work is based on the machinery introduced in [9], but which is now examined in a general setting. The results presented here are not intended to be final, but represent a stage in a continuing investigation.

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Chapter 1

Introduction

1.1 The Origin Story

The family of justification logics grew from a single instance, *the logic of proofs*, LP. This was introduced by Sergei Artemov as an essential part of his program to create an arithmetic provability semantics for intuitionistic logic. Features of LP have had a significant influence on research into the developing group of justification logics, so it is appropriate that we begin with a brief discussion of its history.

Gödel, in a note [15], introduced the modern axiomatization of the modal logic S4, thinking of the $\Box$ operator as an informal provability operator. His primary interest was in intuitionistic logic, and he also gave an embedding from it to S4: put $\Box$ before every subformula. Informally, this amounts to thinking of intuitionistic truth as provability. But he also noted that S4 does not embed into Peano arithmetic, translating $\Box$ as his formal provability operator. If it did, then the S4 theorem $\Box\bot \supset \bot$ would turn into a provable statement asserting consistency, something ruled out by his famous second incompleteness theorem. Since then it has been learned that the logic of formal arithmetic provability is GL, Gödel-Löb logic, but this does not relate in the desired way to intuitionistic logic.

In [16], Gödel presented the idea that the $\Box$ operator of S4 could be thought of as an *explicit* proof representative, and this embedded into arithmetic. This was not published until many years later, by which time the idea had been rediscovered independently by Sergei Artemov. Artemov's formal treatment involved the introduction of a new logic, LP. This is a modal-like language, but with *proof terms* which one could think of as representing explicit proofs. It was necessary to show that LP embedded into formal arithmetic, and this was done in Artemov's *Arithmetic Completeness Theorem.* But it was also necessary to show that S4 embedded into LP. This involved the formulation and proof of Artemov's *Realization Theorem.* A proper statement will be found in Section 2.4. The definitive presentation of all this is in [1].

1.2 And Then

In [9] I introduced a Kripke-style semantics for LP, building on an earlier semantics, [18], that did not use possible worlds. As was noted in [9], the methods that connected S4 with LP could also make connections between standard weaker modal logics, K, K4, T, S4, and weaker versions of LP. The Artemov Realization Theorem extended to these cases as well. There was an arithmetic interpretation as well, because these were sublogics of S4, but the connection with arithmetic was beginning to weaken. The term *justification logics* began being used because, while the connection with formal provability was fragmenting, proof terms (now called *justification terms*) still had the role of supplying explicit justifications for (epistemically) necessary statements. Two fairly comprehensive treatments of justification logics as such, and not just of the logic of proofs, can be found in [2] and [4].

The logic S5 extended the picture a bit. A justification logic counterpart was created in [20, 22, 23], with a possible world semantics and a realization theorem. However, the resulting justification logic did not have a satisfactory arithmetical interpretation, and the proof of realization was not constructive. A non-constructive, semantic, proof of realization had been given in [9] for S4. It also applied to standard weaker logics without significant change. The extension to S5 required new ideas involving *strong* evidence functions. This will play a role here as well. The original Artemov proof of realization, connecting S4 and LP, was constructive. Indeed, as a key part of Artemov's program to provide an arithmetic semantics for intuitionistic logic, it was essential that it be constructive. S5 was the first case where a non-constructive proof was the first to be supplied. Eventually, constructive proofs were found, but the door to a larger room was beginning to open.

1.3 And Now

A non-constructive, semantic, proof of realization first appeared in [9]. It is only now becoming clear to me that the non-constructive argument has a broad applicability, and so there are many, many more justification logics out there than I once thought. This report is the beginning of an exploration of the scope of the justification idea.

Chapter 2

Justification Logics Axiomatically

2.1 Normal Modal Logics

In this report we are only interested in modal logics with a single necessity operator. Of course multi-modal logics are well studied, and some justification analogs have been developed. But one must begin somewhere. So, we have a modal operator, $\Box$. (In what we do here, $\Diamond$ plays little role so we mostly ignore it.) Formulas are built up from propositional letters, P, Q, ..., in the usual way. We will be flexible about connectives. Often we will assume we just have $\supset$ and $\bot$, with other connectives defined. If convenient, we will assume a larger set of primitive connectives.

A *normal* modal logic is a *set* of (modal) formulas that contains all tautologies and all formulas of the form $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$, and is closed under *uniform substitution*, *modus ponens*, and *generalization* (if X is present, so is $\Box X$).

Here we are generally interested in normal modal logics for which a Hilbert system exists. We assume they are axiomatized using a finite set of axiom schemes. The smallest normal logic is K which is axiomatized in the familiar way.

K Axioms

- All tautologies (or enough of them)
- All formulas of the form $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$

K Rules

- *Modus Ponens* $X, X \supset Y \Rightarrow Y$
- *Necessitation* $X \Rightarrow \Box X$

There is no substitution rule since we are using axiom *schemes*, something we will do throughout. Many other familiar normal logics can be axiomatized by adding a finite set of axiom schemes to the K axioms above. We assume everybody knows axiom systems like $\mathsf{K4}$, $\mathsf{S4}$, and so on. We will refer to such logics as *axiomatically formulated.*

One can also characterize modal logics using frames. Given a set $\mathcal{F}$ of frames, the set of modal formulas valid in all modal models built on frames from $\mathcal{F}$ is a normal modal logic. We will refer to such logics as *frame based.* Relationships between normality, axiomatics, and frames can be complicated, but this does not concern us here.

To keep terminology simple, we will call a formula X a *validity* of a modal logic, using the same word no matter what the formulation. If we have a normal modal logic, thought of as a set of formulas, validity means being a member of the set. If we have an axiomatic formulation, validity means having an axiomatic proof. If we have a frame based modal logic, validity means being valid in every model based on a frame in the given collection of frames.

2.2 Justification Logics

Justification logics, syntactically, are like modal logics except that *justification terms* take the place of $\Box$. We begin with the question of what is a justification term.

Definition 2.2.1 (Justification Term) Justification terms are built up as follows.

1. There is a set of *justification variables*, x, y, ... Every justification variable is a justification term.
2. There is a set of *justification constants*, a, b, ..., with indices, 1, 2, ... Every justification constant is a justification term. (Indices may be omitted if it causes no problem.)
3. There are binary operation symbols, $+$ and $\cdot$. If u and v are justification terms, so are $(u+v)$ and $(u \cdot v)$.
4. There may be additional function symbols, f, g, ..., of various arities.. Which ones are present depends on the logic. If f is an n-place justification function symbol and $t_1, \ldots, t_n$ are justification terms, $f(t_1, \ldots, t_n)$ is a justification term.

Then *justification formulas* are built up from propositional letters using propositional connectives together with the formation rule: if t is a justification term and X is a justification formula, then $t{:}X$ is a justification formula. The idea is that a justification term represents a reason why something is so, and $t{:}X$ asserts that X is so for reason t.

Loosely speaking, justification variables stand for arbitrary justification terms, and can be substituted for under certain circumstances. Justification constants stand for reasons that are not further analyzed—typically they are reasons for axioms. The $\cdot$ operation corresponds to *modus ponens*. If $X \supset Y$ is so for reason s and X is so for reason t, then Y is so for reason $s \cdot t$. (Note that reasons are not unique—Y may be true for other reasons too.) The $+$ operation is a kind of weakening. If X is so for either reason s or reason t, then $s + t$ is also a reason for X.

There is no obvious justification analog of normality. For instance, we will not have (or want) closure under substitution. Justifications for things should become more complicated when propositional letters are replaced with complex formulas—see [10]. Instead of the generality available with modal logics, we assume the justification logics we are concerned with are always specified axiomatically, using axiom schemes. There can also be semantic specifications, or sequent calculi, but we will take axiomatics as basic.

The weakest justification logic is J_0, axiomatized as follows.

$\mathbf{J}_0$ Axioms

- All tautologies (or enough of them)
- All formulas of the form $s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$
- All formulas of the forms $s{:}X \supset [s+t]{:}X$ and $t{:}X \supset [s+t]{:}X$

$\mathbf{J}_0$ Rules

- *Modus Ponens* $X, X \supset Y \Rightarrow Y$

There is no necessitation rule, but one will come back shortly in a somewhat different form. Other justification logics will be axiomatized by adding additional axiom schemes to J_0.

Constant specifications are used to justify axioms of a justification logic, which are not further analyzed. Formally, a *constant specification* $\mathcal{CS}$ for a given justification logic $\mathcal{J}$ is a set of formulas of the form $e_n{:}e_{n-1}{:}\ldots e_1{:}A$ with $n \geq 1$, where A is an axiom of $\mathcal{J}$, and e_1, e_2, $\ldots$, e_n are similar constants with indices $1, 2, \ldots, n$. It is assumed that $\mathcal{CS}$ contains all intermediate specifications in the sense that, whenever $e_n{:}e_{n-1}{:}\ldots e_1{:}A$ is in $\mathcal{CS}$, then $e_{n-1}{:}\ldots e_1{:}A$ is in $\mathcal{CS}$ too. Members of constant specifications will be treated as additional axioms for justification logics.

A number of special conditions have been considered for constant specifications. Here we are primarily interested in just one. A constant specification $\mathcal{CS}$ for a given axiomatically specified justification logic is *axiomatically appropriate* if every axiom of the logic, including members of the constant specification itself, have justifications. More formally, for each axiom A of the justification logic in question, there is a constant e_1 such that $e_1{:}A$ is in $\mathcal{CS}$, and if $e_n{:}e_{n-1}{:}\ldots e_1{:}A \in \mathcal{CS}$, then $e_{n+1}{:}e_n{:}e_{n-1}{:}\ldots e_1{:}A \in \mathcal{CS}$, for each $n \geq 1$.

Justification Logics In this report, a justification logic will be axiomatically characterized as J_0 extended with a finite set of axiom schemes (involving function symbols in addition to $\cdot$ and $+$), together with a constant specification $\mathcal{CS}$ whose members are also taken to be axioms. Suppose $\mathcal{J}$ is such a justification logic, and S is a set of formulas (not schemes) of the justification logic. We write $S \vdash_{\mathcal{J}} X$ if there is a finite sequence of formulas, ending with X, in which each formula is either an axiom of $\mathcal{J}$, a member of S, or follows from earlier formulas by *modus ponens*. Since there is no necessitation rule, the usual proof of the deduction theorem applies.

2.3 Internalization

Instead of the usual necessitation rule justification logics have an explicit version, under the right circumstances.

Theorem 2.3.1 (Internalization) *Suppose $\mathcal{J}$ is a justification logic with an axiomatically appropriate constant specification $\mathcal{CS}$. $\mathcal{J}$ satisfies* internalization*. That is, if $\vdash_{\mathcal{J}} X$ then $\vdash_{\mathcal{J}} t{:}X$, for some justification term t.*

Proof By a simple induction on proof length. Suppose $\vdash_{\mathcal{J}} X$. If X is an axiom of $\mathcal{J}$ (which includes being a member of $\mathcal{CS}$) there is a justification constant e_n such that $e_n{:}X$ is in $\mathcal{CS}$, and so $e_n{:}X$ is provable. If X follows by *modus ponens* from $Y \supset X$ and Y then, by the Induction Hypothesis, $\vdash_{\mathcal{J}} s{:}(Y \supset X)$ and $\vdash_{\mathcal{J}} t{:}Y$ for some s, t. Using the J_0 Axiom for $\cdot$, $\vdash_{\mathcal{J}} [s \cdot t]{:}X$. ■

2.4 Counterparts

There is an obvious mapping from the language(s) of justification logic to modal language—it is called the *forgetful functor*. For each justification formula X, let X° be the result of recursively replacing every subformula $t{:}A$ with $\Box A$. More formally, $P^\circ = P$ if P is atomic; $(A \supset B)^\circ = (A^\circ \supset B^\circ)$; and $[t{:}A]^\circ = \Box A^\circ$. We want to know the circumstances under which the set of theorems of a justification logic is mapped exactly to the set of theorems of a modal logic.

Definition 2.4.1 Suppose KL is a normal modal logic. Also let JL be a justification logic (thus extending the axiomatic J_0, from Section 2.2, with a finite number of additional axiom schemes and a constant specification $\mathcal{CS}$). We say JL is a *counterpart* of KL if the following holds.

1. If X is a theorem of JL then X° is a validity of KL.
2. If Y is a validity of KL then there is some theorem X of JL so that $X^\circ = Y$.

In other words, JL is a counterpart of KL if the forgetful functor is a mapping from the set of theorems of JL onto the set of theorems of KL. Our definition is syntactic in nature. See Section 3.6 for further comments.

As the subject has developed so far, justification logics have generally been formulated so that item 1 of Definition 2.4.1 is simple to show. Item 2 is known as a *Realization* result—X is said to realize Y. This is not at all simple, and is the central topic of the present report. Realization theorems are (almost?) always shown in a stronger form than just stated. Here is a proper formulation.

Definition 2.4.2 (Normal Realization) This continues Definition 2.4.1, and uses the same notation. Theorem X of JL is a *normal* realization of theorem Y of KL if $X^\circ = Y$, and X results from the replacement of *negative* occurrences of $\Box$ in Y with distinct justification variables (and positive occurrences by justification terms that need not be variables).

That some justification logic is a counterpart of some modal logic is (always?) proved by showing *normal* realizations exist for modal theorems. For some modal logics there are algorithms for producing normal realizations, but there are cases where no known algorithms exist, though one still has normal realizations. The extent of algorithmic realization is unclear.

Chapter 3

Some Examples

3.1 S4 and the first justification logic

LP is the first of the justification logics. Besides $\cdot$ and $+$ it has a one-place function symbol, customarily written ! with no parentheses, with the following axiom schemes (and their S4 analogs).

LP	S4
$t{:}X \supset X$	$\Box X \supset X$
$t{:}X \supset !t{:}t{:}X$	$\Box X \supset \Box\Box X$

Provided an axiomatically appropriate constant specification is used, LP is a counterpart of S4. This is the historically first example of a counterpart relationship between a modal logic and a justification logic. For realization, an algorithm was developed to compute a normal realization for a modal formula X from a cut-free sequent calculus proof of X.

There are well-known sublogics of S4, namely K, K4, T, that are axiomatized by dropping axioms from the S4 list above. Dropping the corresponding axioms from the LP list yields justification counterparts using essentially the same realization algorithm that worked for LP, but omitting parts that are no longer relevant.

A justification counterpart for a modal logic is generally not unique. If we replace the second LP axiom above with $t{:}X \supset f(t){:}g(t){:}X$ (where f and g are one-place function symbols) we get a different justification counterpart for S4. On the other hand, replacing $t{:}X \supset !t{:}tX$ with $h(t,u){:}X \supset t{:}u{:}X$ almost certainly does not yield an S4 justification counterpart, though I have no proof of this.

S4 is among the best-known modal logics, so it hardly needs mention that semantically the axiom system is complete with respect to the family of models with frames $\langle \mathcal{G}, \mathcal{R} \rangle$ for which $\mathcal{R}$ is reflexive and transitive.

3.2 S5, a very familiar modal logic

The modal logic S5 is perhaps the most commonly applied modal logic. It is axiomatized by adding a *negative introspection* scheme to S4, $\neg\Box X \supset \Box\neg\Box X$. Semantically, it is characterized using frames involving equivalence relations.

A justification counterpart appears in the literature, [20, 21], and goes under the name JT45. An additional one-place function symbol, ?, is added to the language (as with !, parentheses are dropped), and an axiom scheme $\neg t{:}X \supset ?t{:}(\neg t{:}X)$ is adopted. As with S4, this axiomatization is not unique and variations exist.

3.3 $\mathsf{K4}^3$, a somewhat obscure modal logic

The modal logic $\mathsf{K4}^3$ is one of a family, $\mathsf{K4}^n$, from [7]. It extends K with the schema $\Box X \supset \Box\Box\Box X$. (The well-known logic $\mathsf{K4}$ is $\mathsf{K4}^2$ in this family.) As with $\mathsf{S4}$, there is more than one justification counterpart for $\mathsf{K4}^3$. We will concentrate on the following one. Let ! and !! be one-place function symbols. Then a justification counterpart for $\mathsf{K4}^3$ results from adding to J_0 the axiom schema $t:X \supset !!t:!t:t:X$, along with an axiomatically appropriate constant specification. We call this justification logic $\mathsf{J4}^3$.

Axiomatic $\mathsf{K4}^3$ is complete with respect to models based on frames $\langle \mathcal{G}, \mathcal{R} \rangle$ meeting a kind of extended transitivity condition: $\Gamma_1 \mathcal{R} \Gamma_2 \mathcal{R} \Gamma_3 \mathcal{R} \Gamma_4$ implies $\Gamma_1 \mathcal{R} \Gamma_4$.

3.4 $\mathsf{S4.2}$, a better known modal logic

Axiomatically, $\mathsf{S4.2}$ extends $\mathsf{S4}$ with the axiom scheme $\Diamond\Box X \supset \Box\Diamond X$. This logic is complete with respect to the family of models based on frames that are reflexive, transitive, and *convergent*, meaning that whenever $\Gamma_1 \mathcal{R} \Gamma_2$ and $\Gamma_1 \mathcal{R} \Gamma_3$, there is some Γ_4 such that $\Gamma_2 \mathcal{R} \Gamma_4$ and $\Gamma_3 \mathcal{R} \Gamma_4$. We briefly present part of a soundness argument, because this will be instructive in the next chapter.

Suppose we have a modal model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ whose frame is reflexive, transitive, and convergent. (Notation is defined in Section 4.1, but should be clear now.) Suppose that $\Gamma_1 \in \mathcal{G}$ and $\mathcal{M}, \Gamma_1 \Vdash \Diamond\Box X$ but $\mathcal{M}, \Gamma_1 \not\Vdash \Box\Diamond X$; we derive a contradiction. By the first item, for some $\Gamma_2 \in \mathcal{G}$ with $\Gamma_1 \mathcal{R} \Gamma_2$, $\mathcal{M}, \Gamma_2 \Vdash \Box X$, and by the second item, for some $\Gamma_3 \in \mathcal{G}$ with $\Gamma_1 \mathcal{R} \Gamma_3$, $\mathcal{M}, \Gamma_3 \not\Vdash \Diamond X$. Since the frame is convergent, there is some Γ_4 with $\Gamma_2 \mathcal{R} \Gamma_4$ and $\Gamma_3 \mathcal{R} \Gamma_4$. But then we have both $\mathcal{M}, \Gamma_4 \Vdash X$ and $\mathcal{M}, \Gamma_4 \not\Vdash X$, our contradiction.

For a justification counterpart we build on LP, a counterpart of $\mathsf{S4}$. We add two function symbols, f and g, each two-place, and adopt the following axiom scheme.

$$\neg f(t,u){:}\neg t{:}X \supset g(t,u){:}\neg u{:}\neg X \tag{3.1}$$

We call this justification logic $\mathsf{J4.2}$.

Although we primarily have a formal interest here, a few words about (3.1) might be in order. In LP, because of the axiom scheme $t{:}X \supset X$, we have provability of $(t{:}X \wedge u{:}\neg X) \supset \bot$, for any t and u, and thus provability of $\neg t{:}X \vee \neg u{:}\neg X$. In any context one of the disjuncts must hold. Axiom scheme (3.1) is equivalent to $f(t,u){:}\neg t{:}X \vee g(t,u){:}\neg u{:}\neg X$. Informally, this says that in any context we have means for computing a justification for the disjunct that holds. It is a strong assumption, but not implausible in some cases. We do not pursue the point further here.

3.5 Canonical Modal Logics

All the modal logics considered above in this chapter are canonical. The key feature of such logics is that the canonical model is one that meets the frame conditions specified for the logic. Completeness, then, is an immediate consequence. All this is well-known. We provide a very brief sketch because it will be pertinent in the next chapter.

Let KL be a normal modal logic. Call a set S of formulas KL-consistent provided, for no $X_1, \ldots, X_n \in S$ is $(X_1 \wedge \ldots \wedge X_n) \supset \bot$ valid in KL. Using Lindenbaum's Lemma, every KL-consistent set can be extended to a maximally KL-consistent set.

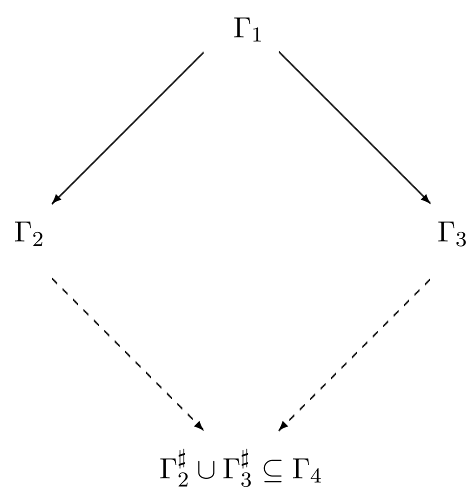
Let $\mathcal{G}$ be the set of all maximally KL-consistent sets. For a set S of formulas, let $S^\sharp = \{X \mid \Box X \in S\}$. Now, for $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma \mathcal{R} \Delta$ provided $\Gamma^\sharp \subseteq \Delta$. This gives us a frame, $\langle \mathcal{G}, \mathcal{R} \rangle$. Finally, for atomic P, take P to be true at $\Gamma \in \mathcal{G}$ ($\Gamma \in \mathcal{V}(P)$) provided $P \in \Gamma$. This completely specifies a model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$, called the *canonical* model for KL.

The key fact about the canonical model $\mathcal{M}$ for logic KL is the *Truth Lemma*, which asserts that for any $\Gamma \in \mathcal{G}$ and for any formula X, $X \in \Gamma \Leftrightarrow \mathcal{M}, \Gamma \Vdash X$. The proof is by induction on the degree of X, is standard, and is widely known. It follows that a canonical model is a universal counter-model. For, suppose X is not valid in KL. Then $\{\neg X\}$ is KL-consistent, and so can be extended to a maximally consistent set Γ. Then $\Gamma \in \mathcal{G}$ and, by the Truth Lemma, $\mathcal{M}, \Gamma \nVdash X$.

It can happen that the canonical model is not actually a model for its logic. But, if the canonical model does turn out to be a model for its logic, completeness is immediate. When this happens, the logic itself is called canonical. All the modal logics considered above are canonical. Canonicity is a key issue for what we do here.

For most of the logics of this chapter, establishing canonicity is easy. For $\mathsf{S4.2}$ things are a bit more complicated. Since this will be pertinent later on, we give the argument.

Suppose, in the canonical model $\mathcal{M}$ for $\mathsf{S4.2}$ we have $\Gamma_1 \mathcal{R} \Gamma_2$ and $\Gamma_1 \mathcal{R} \Gamma_3$. If we could show that $\Gamma_2^\sharp \cup \Gamma_3^\sharp$ was consistent, then it could be extended to a maximally consistent Γ_4, and we would have $\Gamma_2 \mathcal{R} \Gamma_4$ and$\Gamma_3 \mathcal{R} \Gamma_4$. So, let us suppose $\Gamma_2^\sharp \cup \Gamma_3^\sharp$ is not $\mathsf{S4.2}$ consistent, and derive a contradiction.



To simplify the discussion we begin with the following observation. It tells us that we can replace arguments about finite subsets of $\Gamma^\sharp$ with arguments involving single members of $\Gamma^\sharp$.

Lemma 3.5.1 *Suppose Γ is a possible world in the canonical model. Then $A_1, \ldots, A_n \in \Gamma^\sharp$ if and only if $A_1 \wedge \ldots \wedge A_n \in \Gamma^\sharp$ (parenthesized however).*

Proof Suppose $A_1, \ldots, A_n \in \Gamma^\sharp$. Then $\Box A_1, \ldots, \Box A_n \in \Gamma$. In any normal modal logic, $\Box A_1 \supset (\Box A_2 \supset \ldots \supset (\Box A_n \supset \Box(A_1 \wedge \ldots \wedge A_n)) \ldots)$ is valid. Since worlds in the canonical model are maximally consistent, it follows that $\Box(A_1 \wedge \ldots \wedge A_n) \in \Gamma$, and hence $A_1 \wedge \ldots \wedge A_n \in \Gamma^\sharp$. The converse is similar, using the validities $(\Box A_1 \wedge \ldots \wedge \Box A_i \wedge \ldots \wedge \Box A_n) \supset \Box A_i$. ■

Now, suppose $\Gamma_2^\sharp \cup \Gamma_3^\sharp$ is not $\mathsf{S4.2}$ consistent. Then (using Lemma 3.5.1) there are $A \in \Gamma_2^\sharp$ and $B \in \Gamma_3^\sharp$ so that $B \supset \neg A$ is valid, and hence so is $\Box B \supset \Box \neg A$. Since $A \in \Gamma_2^\sharp$, $\Box A \in \Gamma_2$. Similarly, $\Box B \in \Gamma_3$. Since $\Gamma_1 \mathcal{R} \Gamma_2$ then $\Gamma_1^\sharp \subseteq \Gamma_2$, so $\Box \neg \Box A \notin \Gamma_1$ by consistency of Γ_2, and so $\neg \Box \neg \Box A \in \Gamma_1$, by maximality of Γ_1. That is, $\Diamond \Box A \in \Gamma_1$. Then, using $\Diamond \Box A \supset \Box \Diamond A$ and maximal consistency, $\Box \Diamond A \in \Gamma_1$. But $\Gamma_1 \mathcal{R} \Gamma_3$ so $\Gamma_1^\sharp \subseteq \Gamma_3$, and hence $\Diamond A \in \Gamma_3$, that is, $\neg \Box \neg A \in \Gamma_3$. But $\Box B \in \Gamma_3$, and $\Box B \supset \Box \neg A$ is valid, so $\Box \neg A \in \Gamma_3$. This shows inconsistency of Γ_3, and is our desired contradiction.

3.6 A few observations

We discussed modal/justification counterparts in Section 2.4. For modal KL and justification JL to be counterparts, both items of Definition 2.4.1 must be satisfied. It is important to note that the two are quite independent of each other. We now have examples that illustrate this.

First, consider the pair LP and S5. If X is a theorem of LP, X° will be a theorem of S5, because LP embeds into S4, and this is a sublogic of S5. But there are theorems of S5 that have no realization in LP; $\neg\Box X \supset \Box\neg\Box X$ is an obvious example. If $\neg\Box X \supset \Box\neg\Box X$ had a realization in LP, which is a counterpart of S4, the forgetful projection of this realization would be a theorem of S4, which means $\neg\Box X \supset \Box\neg\Box X$ would be an S4 theorem. Thus we have item 1 but not item 2 from Definition 2.4.1.

Next, consider the pair JT45 and S4. The forgetful functor does not map the set of theorems of JT45 into the set of theorems of S4; $\neg t{:}X \supset ?t{:}(\neg t{:}X)$ maps outside of S4. But it is the case that every theorem of S4 has a realization in JT45, because S4 and LP are counterparts, and LP is a sublogic of JT45. In this case we have item 2 from Definition 2.4.1, but not item 1.

Chapter 4

Semantics

There are several semantics that have been introduced for justification logics: *arithmetic*, due to Artemov, [1]; *Mkrtychev*, [18]; and a possible world semantics commonly known as *Fitting models*, [9]. Recently *modular models* have been introduced, [3], and also [17]. Fitting models play a fundamental role in our realization proof, which originated in [9], but which is presented here with modifications coming from [12, 14, 13].

4.1 Fitting Models

As usual, a *frame* is a directed graph, $\langle \mathcal{G}, \mathcal{R} \rangle$, with $\mathcal{G}$ being the nodes or possible worlds, and $\mathcal{R}$ being the directed edges, or accessibility relation.

Frames are the basis of the familiar possible world modal models, and also of Fitting justification logic models. In either case, one specifies which atomic formulas are to be considered true at which possible worlds; that is, one assumes there is a mapping $\mathcal{V}$ from atoms to subsets of $\mathcal{G}$ so that $\mathcal{V}(\bot) = \emptyset$. For a propositional letter P, we understand that P is true at $\Gamma \in \mathcal{G}$ if $\Gamma \in \mathcal{V}(P)$.

Now modal and Fitting justification models diverge. For a modal model, truth of formulas is calculated at each possible world. Specifically, if $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$, we write $\mathcal{M}, \Gamma \Vdash X$ to indicate that modal formula X is true at possible world Γ of $\mathcal{G}$. The familiar conditions are listed for later reference.

1. if A is atomic, $\mathcal{M}, \Gamma \Vdash A$ if $A \in \mathcal{V}(\Gamma)$

2. $\mathcal{M}, \Gamma \Vdash X \supset Y$ if and only if $\mathcal{M}, \Gamma \nVdash X$ or $\mathcal{M}, \Gamma \Vdash Y$

3. $\mathcal{M}, \Gamma \Vdash \Box X$ if and only if $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$

Other connectives and $\Diamond$ can be taken to be defined.

Fitting models assume the language is not modal, but involves justification terms as in Section 2.2. To deal with these models have an additional piece of machinery, syntactic in nature: an *evidence function*. Such a function, $\mathcal{E}$, maps justification terms and formulas to sets of possible worlds. The idea is, $\Gamma \in \mathcal{E}(t, X)$ informally means that, at possible world Γ, t is relevant evidence for the truth of X. Relevant evidence need not be conclusive. Then, a Fitting model for a justification logic is a structure $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$, where $\langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ is as above, and $\mathcal{E}$ is an evidence function. Truth conditions 1 and 2 are the same as above. Condition 3 is not pertinent since $\Box$ is not part of a justification language. It is replaced with the following.

4. $\mathcal{M}, \Gamma \Vdash t{:}X$ if and only if

(a) $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$

(b) $\Gamma \in \mathcal{E}(t, X)$

Thus $t{:}X$ is true at a possible world if X is true at all accessible worlds *and* t is relevant evidence for X at Γ.

When working with modal models various *frame conditions* are imposed, transitivity, symmetry, convergence, etc. This is done with Fitting models in exactly the same way. One also puts conditions on Fitting model evidence functions. Since we have assumed $+$ and $\cdot$ are always present axiomatically, we always require the following semantically. There may be additional conditions for particular justification logics.

5. $\mathcal{E}(s, X \supset Y) \cap \mathcal{E}(t, X) \subseteq \mathcal{E}(s \cdot t, Y)$
6. $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$

A justification formula X is valid in a Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ provided, for each $\Gamma \in \mathcal{G}$, $\mathcal{M}, \Gamma \Vdash X$. Likewise X is valid in a class of Fitting models if it is valid in every member of the class.

A Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ *meets a constant specification* $\mathcal{CS}$ provided, for each $t{:}X \in \mathcal{CS}$, $\mathcal{E}(t, X) = \mathcal{G}$. Recall that members of a constant specification are of the form $e_n{:}e_{n-1}{:}\ldots e_1{:}A$ where A is an axiom (of some particular axiomatic system). It was also assumed that if $e_n{:}e_{n-1}{:}\ldots e_1{:}A$ is in $\mathcal{CS}$ so is $e_{n-1}{:}\ldots e_1{:}A$. Then it is easy to show, by induction on n, that if all axioms of a particular axiomatic system are valid in $\mathcal{M}$, and if $\mathcal{M}$ meets constant specification $\mathcal{CS}$, then all members of $\mathcal{CS}$ must be valid in $\mathcal{M}$.

4.2 Special Kinds of Models

Special conditions can be imposed on Fitting models, and sometimes must be. The first we discuss is being *fully explanatory*, originating in [9]. Informally, a model is fully explanatory if anything that is necessary has a reason. More properly, a Fitting model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is fully explanatory provided that, whenever $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, there is some justification term t so that $\mathcal{M}, \Gamma \Vdash t{:}X$. This has a certain intuitive desirability, but in fact it will play no role in what we do here.

The second special condition is that of having a *strong evidence function*, something we believe originated in [22]. Condition 4 in Section 4.1 has two parts. It says $\mathcal{M}, \Gamma \Vdash t{:}X$ provided (a) $\mathcal{M}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta$, and (b) $\Gamma \in \mathcal{E}(t, X)$. A strong evidence function is one for which condition (b) implies condition (a), and is thus sufficient by itself. Informally this amounts to assuming that we are not dealing with mere *relevant* evidence, but with *conclusive* evidence. This condition will play a role here.

Fully explanatory and strong evidence seem to behave very differently. Given a class $\mathcal{F}$ of Fitting models specified by conditions on accessibility and evidence, those that are fully explanatory form a subclass. There are no known examples of axiom systems whose soundness can be shown with respect to models in that subclass, but not for $\mathcal{F}$ itself. The completeness proofs considered here are justification versions of canonical model constructions, and it turns out that canonical models are always fully explanatory, provided an axiomatically appropriate constant specification is used. Thus being fully explanatory is a nice feature that is almost free.

The story with strong evidence functions is quite another thing, however. Once again, canonical models always have strong evidence functions. But there are examples where no natural class $\mathcal{F}$

of Fitting models exists with respect to which a particular justification logic is sound, without also imposing the requirement that we have a strong evidence function. A justification counterpart of S5 is such a case, [22], and so is J4.2 from Section 3.4. The bad news is that it can be quite difficult to construct particular models with strong evidence functions, though without the restriction it is much less of a problem. The good news is that this will not matter when we come to realization results.

4.3 Examples Continued Again

Fitting models for LP and its sublogics are well-known, so we omit discussion of them here. Instead we give examples that involve the logics $\mathsf{K4}^3$ and $\mathsf{J4}^3$ from Section 3.3, and S4.2 and J4.2 from Section 3.4.

4.3.1 The Logic $\mathsf{J4}^3$

For $\mathsf{J4}^3$, we consider the following class of Fitting models, which we simply call $\mathsf{J4}^3$ *models.* These are models $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ that meet the conditions below.

Following [7], for $\Gamma, \Delta \in \mathcal{G}$, let $\Gamma\mathcal{R}^3\Delta$ mean that there are $\Delta_1, \Delta_2 \in \mathcal{G}$ so that $\Gamma\mathcal{R}\Delta_1\mathcal{R}\Delta_2\mathcal{R}\Delta$. A modal model for $\mathsf{K4}^3$ must meet the condition: if $\Gamma\mathcal{R}^3\Delta$ then $\Gamma\mathcal{R}\Delta$, a generalization of transitivity. We require frames of $\mathsf{J4}^3$ models to meet the same condition.

Conditions for the evidence function generalize those for the more common justification logic J4 in a straightforward way.

$$\Gamma \in \mathcal{E}(t, X) \quad \Longrightarrow \quad \begin{cases} \Gamma \in \mathcal{E}(!!t, !t{:}t{:}X) & (1) \\ \Gamma\mathcal{R}\Delta \Longrightarrow \Delta \in \mathcal{E}(!t, t{:}X) & (2) \\ \Gamma\mathcal{R}^2\Delta \Longrightarrow \Delta \in \mathcal{E}(t, X) & (3) \end{cases}$$

We verify that $t{:}X \supset !!t{:}!t{:}t{:}X$ is valid in all $\mathsf{J4}^3$ models. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ be such a model, $\Gamma \in \mathcal{G}$, and suppose $\mathcal{M}, \Gamma \Vdash t{:}X$ but $\mathcal{M}, \Gamma \not\Vdash !!t{:}!t{:}t{:}X$; we derive a contradiction

Since $\mathcal{M}, \Gamma \Vdash t{:}X$, we must have $\Gamma \in \mathcal{E}(t, X)$. Because we have this and (1), but $\mathcal{M}, \Gamma \not\Vdash !!t{:}!t{:}t{:}X$, there must be some $\Delta_1 \in \mathcal{G}$ with $\Gamma\mathcal{R}\Delta_1$, and $\mathcal{M}, \Delta_1 \not\Vdash !t{:}t{:}X$.

Since $\mathcal{M}, \Gamma \Vdash t{:}X$ and $\Gamma\mathcal{R}\Delta_1$, by (2) we must have $\Delta_1 \in \mathcal{E}(!t, t{:}X)$. By this, and $\mathcal{M}, \Delta_1 \not\Vdash !t{:}t{:}X$, there must be some $\Delta_2 \in \mathcal{G}$ with $\Delta_1\mathcal{R}\Delta_2$, and $\mathcal{M}, \Delta_2 \not\Vdash t{:}X$.

Again, since $\mathcal{M}, \Gamma \Vdash t{:}X$ and $\Gamma\mathcal{R}^2\Delta_2$, by (3) we must have $\Delta_2 \in \mathcal{E}(t, X)$. Then since $\mathcal{M}, \Delta_2 \not\Vdash t{:}X$, there must be some $\Delta \in \mathcal{G}$ with $\Delta_2\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \not\Vdash X$.

Now, by assumption, $\mathcal{M}, \Gamma \Vdash t{:}X$. Also $\Gamma\mathcal{R}^3\Delta$ so by the frame condition, $\Gamma\mathcal{R}\Delta$. It follows that $\mathcal{M}, \Delta \Vdash X$, and we have a contradiction.

4.3.2 The logic J4.2

$\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is a J4.2 Fitting model if it meets the following special conditions.

1. The frame $\langle \mathcal{G}, \mathcal{R} \rangle$ is reflexive, transitive, and convergent, as with S4.2 in Section 3.4.
2. $\mathcal{E}(t, X) \subseteq \mathcal{E}(!t, t{:}X)$
3. $\Gamma \in \mathcal{E}(t, X)$ and $\Gamma\mathcal{R}\Delta$ implies $\Delta \in \mathcal{E}(t, X)$.
4. $\mathcal{E}(f(t, u), \neg t{:}X) \cup \mathcal{E}(g(t, u), \neg u{:}\neg X) = \mathcal{G}$

5. $\mathcal{E}$ is a *strong* evidence function, as discussed in Section 4.2.

Items 1, 2 and 3 together give soundness of the usual LP axiom schemas $t{:}X \supset X$ and $t{:}X \supset !t{:}t{:}X$. We skip details. We do verify soundness for (3.1), $\neg f(t,u){:}\neg t{:}X \supset g(t,u){:}\neg u{:}\neg X$. Suppose the formula fails at Γ in model $\mathcal{M}$; we derive a contradiction. If it fails at Γ then $\mathcal{M}, \Gamma \not\Vdash f(t,u){:}\neg t{:}X$ and $\mathcal{M}, \Gamma \not\Vdash g(t,u){:}\neg u{:}\neg X$. Because $\mathcal{E}$ is a strong evidence function, it must be that $\Gamma \notin \mathcal{E}(f(t,u), \neg t{:}X)$ and $\Gamma \notin \mathcal{E}(g(t,u), \neg u{:}\neg X)$. But this contradicts condition 4. Note that this argument is extremely trivial when compared to the argument in Section 3.4. The assumption of a strong evidence function is a strong assumption indeed.

4.3.3 A Remark About Strong Evidence Functions

Imposing a strong evidence assumption can sometimes make other requirements on a model redundant. For instance, if we restrict the semantics for $\mathsf{J4}^3$ in Section 4.3.1 to models with strong evidence functions, two of the three conditions on evidence functions are unnecessary. Assume, for this discussion, that $\Gamma \in \mathcal{E}(t, X)$. We retain consequence (1) that $\Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$, but we can drop (2) and (3) by the following reasoning.

Suppose also that $\Gamma\mathcal{R}\Delta$. We are still assuming $\Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$. Since $\mathcal{E}$ is a strong evidence function, it follows that $\mathcal{M}, \Gamma \Vdash !!t{:}!t{:}t{:}X$. Then since $\Gamma\mathcal{R}\Delta$, we must have that $\mathcal{M}, \Delta \Vdash !t{:}t{:}X$, and hence $\Delta \in \mathcal{E}(!t, t{:}X)$. Redundancy of the third condition has a similar argument.

For $\mathsf{J4.2}$ in Section 4.3.2, a similar argument shows that condition 3 is unnecessary.

These are not meant to be deep observations, but they may be of some significance.

4.4 Canonical Models

Canonical models are a common way of proving completeness for modal logics, when it works. There is a direct analog for justification logics, from [9]. We repeat things here because canonical justification models play a central role in proving realizability, again, when it works.

For this section let J be some axiomatically formulated justification logic. Then it extends the system J_0 from Section 2.2. If S is a set of formulas of J, we write $S \vdash_{\mathsf{J}} X$ and mean by it that $Y_1 \supset (Y_2 \supset \ldots \supset (Y_n \supset X)\ldots)$ is provable in J for some $Y_1, Y_2, \ldots, Y_n \in S$. We say S is *J-inconsistent* if $S \vdash_{\mathsf{J}} \bot$, and S is *J-consistent* if it is not J-inconsistent.

The *canonical model* for J, $\mathcal{M}_{\mathsf{J}} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is defined as follows.

- $\mathcal{G}$ is the set of all maximally J-consistent sets of formulas.
- If $\Gamma \in \mathcal{G}$, let $\Gamma^\sharp = \{X \mid t{:}X \in \Gamma \text{ for some justification term } t\}$. For $\Gamma, \Delta \in \mathcal{G}$, $\Gamma\mathcal{R}\Delta$ if $\Gamma^\sharp \subseteq \Delta$.
- For atomic A, $\Gamma \in \mathcal{V}(A)$ if $A \in \Gamma$.
- $\Gamma \in \mathcal{E}(t, X)$ if $t{:}X \in \Gamma$.

This completes the definition of canonical model. To show it is a Fitting model we must show it meets conditions 5 and 6 from Section 4.1. These are simple. Condition 5 is that $\mathcal{E}(s, X \supset Y) \cap \mathcal{E}(t, X) \subseteq \mathcal{E}(s \cdot t, Y)$. Well, suppose $\Gamma \in \mathcal{E}(s, X \supset Y)$ and $\Gamma \in \mathcal{E}(t, X)$. By definition of $\mathcal{E}$, $s{:}(X \supset Y) \in \Gamma$ and $t{:}X \in \Gamma$. Since J axiomatically extends J_0, $s{:}(X \supset Y) \supset (t{:}X \supset s \cdot t{:}Y)$ is an axiom. Since Γ is maximally consistent, it follows that $s \cdot t{:}Y \in \Gamma$, and hence $\Gamma \in \mathcal{E}(s \cdot t, Y)$. Condition 6 is $\mathcal{E}(s, X) \cup \mathcal{E}(t, X) \subseteq \mathcal{E}(s + t, X)$, and is treated the same way using J_0 axioms $s{:}X \supset s + t{:}X$ and $t{:}X \supset s + t{:}X$.

As with modal canonical models, the key item to show is a truth lemma. Curiously, for justification logics the proof is simpler than in the modal case.

Theorem 4.4.1 (Truth Lemma) *In the canonical model $\mathcal{M}_J = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$, for any $\Gamma \in \mathcal{G}$ and any formula X,*

$$\mathcal{M}_J, \Gamma \Vdash X \iff X \in \Gamma \tag{4.1}$$

Proof The proof is by induction on the complexity of X. The atomic case is by definition. Propositional connective cases are by the usual argument, making use of maximal consistency of Γ. This leaves the justification case. Assume (4.1) holds for formulas simpler than X.

Suppose $t{:}X \in \Gamma$. By definition of $\mathcal{E}$ we have $\Gamma \in \mathcal{E}(t, X)$. Let Δ be an arbitrary member of $\mathcal{G}$ with $\Gamma \mathcal{R} \Delta$. Then $\Gamma^\sharp \subseteq \Delta$, so $X \in \Delta$, and by the induction hypothesis, $\mathcal{M}_\mathsf{J}, \Delta \Vdash X$. Then $\mathcal{M}_\mathsf{J}, \Gamma \Vdash t{:}X$.

Suppose $t{:}X \notin \Gamma$. Then $\Gamma \notin \mathcal{E}(t, X)$, so $\mathcal{M}_\mathsf{J}, \Gamma \nVdash t{:}X$. ■

The evidence function, $\mathcal{E}$, in the canonical model, is a strong evidence function. For, suppose $\Gamma \in \mathcal{E}(t, \Gamma)$. Then, by definition of $\mathcal{E}$, $t{:}X \in \Gamma$ so by Theorem 4.4.1, $\mathcal{M}_\mathsf{J}, \Gamma \Vdash t{:}X$. And then $\mathcal{M}_\mathsf{J}, \Delta \Vdash X$ for every $\Delta \in \mathcal{G}$ with $\Gamma \mathcal{R} \Delta$.

It is also the case that, if J has an axiomatically appropriate constant specification, the canonical model must be fully explanatory. We will not need this here, and so omit the proof.

As usual, the canonical model is a universal counter model, and by the usual argument. If $\nVdash_\mathsf{J} X$ then $\{\neg X\}$ is consistent. If Γ is any maximal consistent extension of $\{\neg X\}$, $\Gamma \in \mathcal{G}$ and by Theorem 4.4.1, $\mathcal{M}_\mathsf{J}, \Gamma \nVdash X$. Then the completeness question reduces to the following—a direct analog of what happens in modal logic. We may have some class $\mathcal{F}_\mathsf{J}$ of Fitting models, and we want to prove completeness of J relative to this class. If it turns out that the canonical model $\mathcal{M}_\mathsf{J}$ is in $\mathcal{F}_\mathsf{J}$, we have succeeded.

In modal logic, there are well-known examples of completeness proofs that do not take the canonical model route (and cannot). Gödel-Löb logic is a prominent such logic. Other techniques are available for cases like these. For justification logics thus far, *all* axiomatic completeness results make use of canonical models directly. This phenomenon needs to be investigated further.

4.5 Examples Continued Even Further

In Section 3.3 a logic, $\mathsf{J4}^3$ was introduced axiomatically, and in Section 3.4 another, $\mathsf{J4.2}$. Semantics were given for these in Section 4.3.1 and Section 4.3.2 respectively. We now show that the canonical models for these axiomatically formulated logics are of the right semantic type, and so completeness results follow.

There are two parts to showing a canonical model meets appropriate conditions. First, the evidence function should have the right properties. Second, the underlying frame should meet appropriate frame conditions. These are separate arguments. Only the second has relevance to our realization results in the next chapter.

4.5.1 $\mathsf{J4}^3$ Completeness

We divide showing that the canonical model $\mathsf{J4}^3$ has the right properties into two parts, roughly syntactic and semantic.

Evidence Function Conditions We saw in Section 4.3.3 that when dealing with a strong evidence function, as the canonical model has, only one of the three evidence function conditions needs to be established. We must show $\Gamma \in \mathcal{E}(t, X) \Longrightarrow \Gamma \in \mathcal{E}(!!t, !t{:}t{:}X)$. From the definition of evidence function in canonical models, this is equivalent to $t{:}X \in \Gamma \Longrightarrow !!t{:}!t{:}t{:}X \in \Gamma$. Since

$t{:}X \supset !!t{:}!t{:}t{:}X$ is an axiom of $\mathsf{J4}^3$ and possible worlds of canonical models are maximally consistent, this is immediate.

Frame Conditions We must show that, in the canonical model for $\mathsf{J4}^3$, if $\Gamma\mathcal{R}^3\Delta$ then $\Gamma\mathcal{R}\Delta$. Well, suppose $\Gamma\mathcal{R}\Delta_1\mathcal{R}\Delta_2\mathcal{R}\Delta$. Then $\Gamma^\sharp \subseteq \Delta_1$, $\Delta_1^\sharp \subseteq \Delta_2$, and $\Delta_2^\sharp \subseteq \Delta$. We must show $\Gamma^\sharp \subseteq \Delta$. Suppose $X \in \Gamma^\sharp$. Then for some justification term t, $t{:}X \in \Gamma$. Since $t{:}X \supset !!t{:}!t{:}t{:}X$ is an axiom, by maximal consistency of Γ, $!!t{:}!t{:}t{:}X \in \Gamma$. But then $!t{:}t{:}X \in \Delta_1$, and so $t{:}X \in \Delta_2$, and thus $X \in \Delta$.

4.5.2 J4.2 Completeness

Completeness holds under the assumption that J4.2 has an axiomatically appropriate constant specification. Such an assumption was not needed for $\mathsf{J4}^3$. Again we divide showing canonicity into two parts. Axiomatically, J4.2 extends axiomatic LP by adding the scheme $\neg f(t,u){:}\neg t{:}X \supset g(t,u){:}\neg u{:}\neg X$. Semantically, J4.2 models are Fitting LP models meeting additional conditions. We omit checking the LP conditions. We already know the canonical model has a strong evidence function. What remains to show is that the evidence function satisfies $\mathcal{E}(f(t,u),\neg t{:}X)\cup\mathcal{E}(g(t,u),\neg u{:}\neg X) = \mathcal{G}$, and that the frame is convergent.

Evidence Function Conditions Because of the way the evidence function is defined in canonical models, showing the evidence function satisfies $\mathcal{E}(f(t,u),\neg t{:}X) \cup \mathcal{E}(g(t,u),\neg u{:}\neg X) = \mathcal{G}$ is equivalent to showing that for each Γ in the canonical model, $f(t,u){:}\neg t{:}X \in \Gamma$ or $g(t,u){:}\neg u{:}\neg X \in \Gamma$. This is an immediate consequence of $\neg f(t,u){:}\neg t{:}X \supset g(t,u){:}\neg u{:}\neg X$.

Frame Conditions In Section 3.5 we introduced a simplification designed to keep the clutter down, Lemma 3.5.1. A similar thing can be done here.

Lemma 4.5.1 *Suppose Γ is a possible world in the canonical Fitting justification model for some justification logic that satisfies internalization. Then $A_1,\ldots,A_n \in \Gamma^\sharp$ if and only if $A_1 \wedge \ldots \wedge A_n \in \Gamma^\sharp$.*

This has a similar appearance to Lemma 3.5.1, but the definition of the sharp operation is different. We leave the proof to the reader. Note that by Theorem 2.3.1, we have internalization if we have an axiomatically appropriate constant specification. Now, here is the argument that the canonical model has a convergent frame. It is an exact counterpart of the argument given in Section 3.5.

Suppose, in the canonical model for J4.2 we have $\Gamma_1\mathcal{R}\Gamma_2$ and $\Gamma_1\mathcal{R}\Gamma_3$. Exactly as in Section 3.5, it is enough to show $\Gamma_2^\sharp \cup \Gamma_3^\sharp$ is consistent (though the sharp operation is different now). Suppose otherwise. Then, using the Lemma above, There are $A \in \Gamma_2^\sharp$ and $B \in \Gamma_3^\sharp$ so that $B \supset \neg A$ is provable. Since $A \in \Gamma_2^\sharp$, for some justification term t, $t{:}A \in \Gamma_2$. Similarly, since $B \in \Gamma_3^\sharp$, $u{:}B \in \Gamma_3$ for some u. Since we have internalization, for some justification term a, $a{:}(B \supset \neg A)$ is provable, and hence so is $u{:}B \supset (a\cdot u){:}\neg A$. Since $\Gamma_1\mathcal{R}\Gamma_2$ then $\Gamma_1^\sharp \subseteq \Gamma_2$, so $f(t,a\cdot u){:}\neg t{:}A \notin \Gamma_1$ since $t{:}A \in \Gamma_2$ and Γ_2 is consistent. Then $\neg f(t,a\cdot u){:}\neg t{:}A \in \Gamma_1$. Using the axiom $\neg f(t,a\cdot u){:}\neg t{:}A \supset g(t,a\cdot u){:}\neg a\cdot u{:}\neg A$ and maximal consistency of Γ_1, we have $g(t,a\cdot u){:}\neg a\cdot u{:}\neg A \in \Gamma_1$. Since $\Gamma_1^\sharp \subseteq \Gamma_3$, $\neg a\cdot u{:}\neg A \in \Gamma_3$. Since $u{:}B \supset (a\cdot u){:}\neg A$. $\neg u{:}B \in \Gamma_3$, a contradiction since $u{:}B \in \Gamma_3$.

Chapter 5

Realization

Most proofs of realization do things in one pass. The first realization proof, which provides an algorithm suitable for LP and some closely related logics, can be found in [1]. It converts a sequent calculus S4 proof into an LP realization. Since then several different algorithmic proofs have been developed. In [11], for instance, an approach was introduced that does not work globally with a sequent calculus proof, but instead works through the sequent proof step by step. But here we follow a different line, originating in [9], where a non-constructive proof of realization for LP was presented. A key feature, whose significance I did not realize until later, was that it was a two-stage process. Eventually, in [14], the two stages were clearly separated, and the output of the first stage was given a name, *quasi-realization*. This was applied in [13] to provide not only an algorithm for LP, but an implementation of it. It is this two-stage approach that we follow here.

Quasi-realizations are similar to realizations, but have a more complex form. They still provide an embedding from a modal logic into a justification logic, and can serve some of the functions that realizations were designed for. At a second stage, quasi-realizations are converted into realizations. There are features of both stages that are of significance.

The second stage, conversion from quasi-realizations to realizations, is algorithmic. It depends only on formula structure and not on input of a sequent calculus proof. And it is independent of the particular logic in question—that is, it is uniform across the family of justification logics.

The first stage may or may not be algorithmic. In this report we present a non-algorithmic version that covers a broad range of logics. To be algorithmic, the modal logic to be realized into must have some proof procedure in which formulas do not change polarity. This is generally described as cut-free, but the real point is that cut involves a polarity change. So far sequent calculi, tableau systems, hypersequents, nested sequents [6], and prefixed tableau systems have been used for this purpose. An abstract understanding is missing, but all the obvious candidates have worked.

The plan for this chapter is as follows. First produce a quasi-realization. We do this non-constructively here, covering a wide range of logics. As noted above, constructive approaches are possible in special cases that can be found in the literature. Second, convert a quasi-realization to a realization. This stage is always constructive.

5.1 Annotated and Signed Formulas

We will be mapping modal formulas to justification formulas. Since we want *normal* realizations, Definition 2.4.2, we must keep track of specific occurrences of $\Box$. In [11] we introduced *annotated formulas*, and in [14, 13] a simpler version. It is the simpler version that we use now.

Definition 5.1.1 An *annotated modal formula* is like a standard modal formula except for the following.

1. Instead of a single modal operator $\Box$ there is an infinite family, $\Box_1$, $\Box_2$, ..., called *indexed* modal operators. Formulas are built up as usual, but using indexed modal operators instead of $\Box$. We assume that in an annotated formula, *no index occurs twice*.

2. If A is an annotated formula, and A' is the result of replacing all indexed modal operators, $\Box_n$, with $\Box$, forgetting the index, then A' is a conventional modal formula. We say A is an *annotated version* of A', and A' is an *unannotated version* of A.

Annotations, indexes, are purely for bookkeeping purposes. Semantically they are ignored. Thus $\Box_n$ and $\Box$ behave alike in models, and so a modal formula and an annotated version of it evaluate the same at each possible world.

It is also necessary, for normal realizations, that we keep track of positive and negative subformula occurrences. For this purpose, it is very helpful to make use of machinery familiar from signed tableau systems.

Definition 5.1.2 Let T and F be two symbols, not part of our modal or justification languages. A *signed* formula is $T\,X$ or $F\,X$, where X is a formula. We allow X to be a justification formula, a modal formula, or an annotated modal formula.

When working with tableaus, one thinks of $T\,X$ as an assertion that X is true (under some circumstance), and $F\,X$ that X is false. All this plays a role in [13], but not here. In the present treatment, signs are simply for bookkeeping purposes. Still, we should note that for $T\,X \supset Y$ to be verified, informally, we must consider circumstances where either $F\,X$ or $T\,Y$ is so. Likewise, for $F\,X \supset Y$ to be verified, we must consider circumstances where $T\,X$ and $F\,Y$ are so. This observation may help motivate item 2 in Definition 5.2.1, and similarly for the other cases.

5.2 Quasi-Realizations

Particular choices of modal or justification logics are not important just now. The definitions in this section have to do with languages, not logics. We define a mapping from signed annotated modal formulas to sets of signed justification formulas involving some set of function symbols including at least $+$ and $\cdot$.

Important Note From now on we assume that v_1, v_2, ... is an enumeration of all justification variables with no variable repeated, fixed once and for all.

Definition 5.2.1 The mapping $\langle\!\langle \cdot \rangle\!\rangle$ associates with each signed annotated modal formula a set of signed justification formulas. It is defined recursively, as follows.

1. If A is atomic, $\langle\!\langle T\,A \rangle\!\rangle = \{T\,A\}$ and $\langle\!\langle F\,A \rangle\!\rangle = \{F\,A\}$.

2. $\langle\!\langle T\,A \supset B \rangle\!\rangle = \{T\,U \supset V \mid F\,U \in \langle\!\langle F\,A \rangle\!\rangle, T\,V \in \langle\!\langle T\,B \rangle\!\rangle\}$
   $\langle\!\langle F\,A \supset B \rangle\!\rangle = \{F\,U \supset V \mid T\,U \in \langle\!\langle T\,A \rangle\!\rangle, F\,V \in \langle\!\langle F\,B \rangle\!\rangle\}$.

3. $\langle\!\langle T\,\Box_n A \rangle\!\rangle = \{T\,v_n{:}U \mid T\,U \in \langle\!\langle T\,A \rangle\!\rangle\}$.
   $\langle\!\langle F\,\Box_n A \rangle\!\rangle = \{F\,t{:}(U_1 \vee \ldots \vee U_k) \mid F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A \rangle\!\rangle$ and t is any justification term$\}$.

4. The mapping is extended to *sets* of signed annotated formulas by letting $\langle\!\langle S\rangle\!\rangle = \cup\{\langle\!\langle Z\rangle\!\rangle \mid Z \in S\}$.

Definition 5.2.2 (Quasi-Realization) Let A be an annotated modal formula. If $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,A\rangle\!\rangle$, we say the justification formula $U_1 \vee \ldots \vee U_n$ is a *quasi-realization* of A. (Disjunction may be primitive or defined depending on details of language.)

For a modal formula without annotations, a quasi-realization for it is any quasi-realization for A', where A' is any annotated version of A.

Example 5.2.3 Suppose t, u, and w are justification terms and P and Q are atomic formulas. Here are some quasi-realization calculations, leading up to $F\,\Box_1(\Box_2 P \supset \Box_3 Q)$. We do not produce *all* quasi-realizations; the set would be infinite since there are infinitely many justification terms.

1. $\{T\,P\} = \langle\!\langle T\,P\rangle\!\rangle$ and $\{F\,Q\} = \langle\!\langle F\,Q\rangle\!\rangle$
2. $\{T\,v_2{:}P\} = \langle\!\langle T\,\Box_2 P\rangle\!\rangle$
3. $\{F\,t{:}Q, F\,u{:}Q\} \subseteq \langle\!\langle F\,\Box_3 Q\rangle\!\rangle$
4. $\{F\,v_2{:}P \supset t{:}Q, F\,v_2{:}P \supset u{:}Q\} \subseteq \langle\!\langle F\,\Box_2 P \supset \Box_3 Q\rangle\!\rangle$
5. $\{F\,t{:}((v_2{:}P \supset t{:}Q) \vee (v_2{:}P \supset u{:}Q)), F\,w{:}(v_2{:}P \supset u{:}Q)\} \subseteq \langle\!\langle F\,\Box_1(\Box_2 P \supset \Box_3 Q)\rangle\!\rangle$

It follows that $t{:}((v_2{:}P \supset t{:}Q) \vee (v_2{:}P \supset u{:}Q)) \vee w{:}((v_2{:}P \supset u{:}Q)$ is a quasi-realization for $\Box(\Box P \supset \Box Q)$. There are many others.

The idea is that a quasi-realization, as defined above, is a *candidate*. We want to determine circumstances under which a *provable* modal formula has a *provable* quasi-realization.

5.3 When Quasi-Realizations Exist

We now come to the central theorem of this report. Everything preceding has been leading up to this. The result is not really new. It is implicit in [9]; we are now making it explicit.

Theorem 5.3.1 *Let* KL *be a normal modal logic characterized by a class of frames* $\mathcal{FL}$*, and let* JL *be an axiomatically formulated justification logic, for which internalization holds. If the canonical Fitting model (Section 4.4) for* JL *is based on a frame in* $\mathcal{FL}$*, then every validity of* KL *has a provable quasi-realization in* JL.

Once the following Lemma is shown, the Theorem follows easily. The Lemma and its proof benefit from some notational conventions. For a Fitting model $\mathcal{M}$ and an annotated formula X, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,X\rangle\!\rangle$ means $\mathcal{M}, \Gamma \Vdash Y$ for every justification formula Y such that $T\,Y \in \langle\!\langle T\,X\rangle\!\rangle$, and $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X\rangle\!\rangle$ means $\mathcal{M}, \Gamma \not\Vdash Y$ for every justification formula Y such that $F\,Y \in \langle\!\langle F\,X\rangle\!\rangle$.

Lemma 5.3.2 *Let* $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E}\rangle$ *be the canonical model for the axiomatically formulated justification logic* JL*, a logic with the internalization property. Let* $\mathcal{N} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}\rangle$ *be the corresponding modal model that is formed by dropping the evidence function from* $\mathcal{M}$*. Then for every annotated modal formula* X *we have the following.*

1. $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,X\rangle\!\rangle \Longrightarrow \mathcal{N}, \Gamma \Vdash X$
2. $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X\rangle\!\rangle \Longrightarrow \mathcal{N}, \Gamma \not\Vdash X$

Proof By induction on the degree of the annotated modal formula X.

Ground Case Suppose $X = A$, which is atomic. The only member of $\langle\!\langle T\,A\rangle\!\rangle$ is $T\,A$, so $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A\rangle\!\rangle$ iff $\mathcal{M}, \Gamma \Vdash A$ iff $\Gamma \in \mathcal{V}(A)$ iff $\mathcal{N}, \Gamma \Vdash A$. This establishes item 1; item 2 is similar.

Implication Case Suppose $X = A \supset B$ and the results are known for A and for B.

1. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A \supset B\rangle\!\rangle$. We divide the argument into two parts.

   Suppose first that $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,A\rangle\!\rangle$. By the induction hypothesis, $\mathcal{N}, \Gamma \not\Vdash A$, so $\mathcal{N}, \Gamma \Vdash A \supset B$.

   Suppose next that $\mathcal{M}, \Gamma \not\Vdash \langle\!\langle F\,A\rangle\!\rangle$. Then for some U with $F\,U \in \langle\!\langle F\,A\rangle\!\rangle$, $\mathcal{M}, \Gamma \Vdash U$. Let $T\,V$ be an arbitrary member of $\langle\!\langle T\,B\rangle\!\rangle$. Then $T\,U \supset V \in \langle\!\langle T\,A \supset B\rangle\!\rangle$ so by the assumption, $\mathcal{M}, \Gamma \Vdash U \supset V$, and hence $\mathcal{M}, \Gamma \Vdash V$. Since $T\,V$ was arbitrary, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,B\rangle\!\rangle$. Then by the induction hypothesis, $\mathcal{N}, \Gamma \Vdash B$ and hence $\mathcal{N}, \Gamma \Vdash A \supset B$.
2. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,A \supset B\rangle\!\rangle$. Let $T\,U \in \langle\!\langle T\,A\rangle\!\rangle$ and $F\,V \in \langle\!\langle F\,B\rangle\!\rangle$ both be arbitrary. Then $F\,U \supset V \in \langle\!\langle F\,A \supset B\rangle\!\rangle$ so by the assumption, $\mathcal{M}, \Gamma \not\Vdash U \supset V$. Then $\mathcal{M}, \Gamma \Vdash U$ and $\mathcal{M}, \Gamma \not\Vdash V$. Since $T\,U$ and $F\,V$ were arbitrary, $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,A\rangle\!\rangle$ and $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,B\rangle\!\rangle$, so by the induction hypothesis, $\mathcal{N}, \Gamma \Vdash A$ and $\mathcal{N}, \Gamma \not\Vdash B$. Hence $\mathcal{N}, \Gamma \not\Vdash A \supset B$.

Modal Case Suppose $X = \Box_n A$ and the results are known for A.

1. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle T\,\Box_n A\rangle\!\rangle$. Let $T\,U \in \langle\!\langle T\,A\rangle\!\rangle$ be arbitrary. By the assumption, $\mathcal{M}, \Gamma \Vdash v_n{:}U$. Let $\Delta \in \mathcal{G}$ be arbitrary, with $\Gamma \mathcal{R} \Delta$. Then $\mathcal{M}, \Delta \Vdash U$ and, since $T\,U$ was arbitrary, $\mathcal{M}, \Delta \Vdash \langle\!\langle T\,A\rangle\!\rangle$. By the induction hypothesis, $\mathcal{N}, \Delta \Vdash A$ and, since Δ was arbitrary, $\mathcal{N}, \Gamma \Vdash \Box_n A$.
2. Assume $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,\Box_n A\rangle\!\rangle$. This case depends on the following Claim. We first show how the Claim is used, then we prove the Claim itself.

   > Claim: We use notation from Section 4.4. The set $S = \Gamma^\sharp \cup \{\neg U \mid F\,U \in \langle\!\langle F\,A\rangle\!\rangle\}$ is consistent in the justification logic JL.

   The Claim is used as follows. Since (as we will show) S is consistent in JL, it can be extended to a maximally consistent set, Δ. Since $\mathcal{M}$ is a canonical model, $\Delta \in \mathcal{G}$ and $\Gamma \mathcal{R} \Delta$ since $\Gamma^\sharp \subseteq \Delta$. Also $\{\neg U \mid F\,U \in \langle\!\langle F\,A\rangle\!\rangle\} \subseteq \Delta$ so by the Truth Lemma 4.4.1, $\mathcal{M}, \Delta \not\Vdash U$ for all U with $F\,U \in \langle\!\langle F\,A\rangle\!\rangle$, and so $\mathcal{M}, \Delta \Vdash \langle\!\langle F\,A\rangle\!\rangle$. Then by the induction hypothesis $\mathcal{N}, \Delta \not\Vdash A$, and hence $\mathcal{N}, \Gamma \not\Vdash \Box_n A$. Now to complete things, we establish the claim itself.

   > Proof of Claim: Suppose S is not consistent in JL. Then for some $G_1, \ldots, G_m \in \Gamma^\sharp$, and $F\,U_1, \ldots, F\,U_k \in \langle\!\langle F\,A\rangle\!\rangle$, $\{G_1, \ldots, G_m, \neg U_1, \ldots, \neg U_k\}$ is not consistent, and so $\vdash_{\mathsf{JL}} G_1 \supset (G_2 \supset \ldots \supset (G_m \supset (U_1 \vee \ldots \vee U_k)) \ldots)$. For each $1 \le i \le m$, $G_i \in \Gamma^\sharp$, and so there is some justification term g_i so that $g_i{:}G_i \in \Gamma$. Also since JL has the internalization property, there is some justification term t so that $\vdash_{\mathsf{JL}} t{:}(G_1 \supset (G_2 \supset \ldots \supset (G_m \supset (U_1 \vee \ldots \vee U_k)) \ldots))$. Now repeated use of the J_0 axiom scheme for $\cdot$ yields the following (with the justification term parenthesized to the left).
   >
   > $$\vdash_{\mathsf{JL}} g_1{:}G_1 \supset (g_2{:}G_2 \supset \ldots \supset (g_m{:}G_m \supset (t \cdot g_1 \cdot \ldots \cdot g_m){:}(U_1 \vee \ldots \vee U_k)) \ldots)$$
   >
   > It follows from maximal consistency of Γ that $(t \cdot g_1 \cdot \ldots \cdot g_m){:}(U_1 \vee \ldots \vee U_k) \in \Gamma$, and hence by the Truth Lemma, $\mathcal{M}, \Gamma \Vdash (t \cdot g_1 \cdot \ldots \cdot g_m){:}(U_1 \vee \ldots \vee U_k)$. But this contradicts the assumption that $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,\Box_n A\rangle\!\rangle$.

■

With the proof of the fundamental Lemma 5.3.2 out of the way, we can now establish the main result.

Proof of Theorem 5.3.1 The proof is by contraposition. Suppose Y is a modal formula. Let X be an annotated version of Y and suppose that X has no provable quasi-realization in JL. We show Y is not a validity of KL.

Since X has no provable quasi-realization, for every $U_1, \ldots, U_n$ with $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,X \rangle\!\rangle$, $\not\vdash_{\mathsf{JL}} U_1 \vee \ldots \vee U_n$ (Definition 5.2.2). It follows that $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\}$ is consistent in JL. (Because otherwise, $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\} \vdash_{\mathsf{JL}} \bot$, so $\neg U_1, \ldots, \neg U_n \vdash_{\mathsf{JL}} \bot$ for some $F\,U_1, \ldots, F\,U_n \in \langle\!\langle F\,X \rangle\!\rangle$, and hence $\vdash_{\mathsf{JL}} U_1 \vee \ldots \vee U_n$, contrary to what has been established.)

Since $\{\neg U \mid F\,U \in \langle\!\langle F\,X \rangle\!\rangle\}$ is consistent, it can be extended to a maximal consistent set Γ. In the canonical model $\mathcal{M}$, Γ is a possible world. Using the Truth Lemma, $\mathcal{M}, \Gamma \Vdash \langle\!\langle F\,X \rangle\!\rangle$, and so by Lemma 5.3.2, $\mathcal{N}, \Gamma \not\Vdash X$. A hypothesis of Theorem 5.3.1 is that the canonical justification model is based on a frame in the class $\mathcal{FL}$, which determines the modal logic KL. We thus have a *modal* model $\mathcal{N}$ for KL in which X fails, and hence so does Y, the unannotated version of X. Then Y is not a validity of KL. ■

5.4 Quasi-Realizations to Realizations

This is an abbreviated section because the work appears in full detail in [13]. Here we merely give definitions, and state some results.

We define a mapping similar to that in Definition 5.2.1, except for one case, $F\Box$. We still assume v_1, v_2, ... enumerates all justification variables.

Definition 5.4.1 The mapping $[\![\cdot]\!]$ is defined recursively on the set of signed annotated modal formulas.

1. If A is atomic, $[\![T\,A]\!] = \{T\,A\}$ and $[\![F\,A]\!] = \{F\,A\}$.
2. $[\![T\,A \supset B]\!] = \{T\,U \supset V \mid F\,U \in [\![F\,A]\!], T\,V \in [\![T\,B]\!]\}$
   $[\![F\,A \supset B]\!] = \{F\,U \supset V \mid T\,U \in [\![T\,A]\!], F\,V \in [\![F\,B]\!]\}$.
3. $[\![T\,\Box_n A]\!] = \{T\,v_n{:}U \mid T\,U \in [\![T\,A]\!]\}$.
   $[\![F\,\Box_n A]\!] = \{F\,t{:}U \mid F\,U \in [\![F\,A]\!]$ and t is any justification term$\}$.
4. The mapping is extended to *sets* of signed annotated formulas by letting $[\![S]\!] = \cup\{[\![Z]\!] \mid Z \in S\}$.

A *normal realization* of annotated modal X is any justification formula U where $F\,U \in [\![F\,X]\!]$. More generally, members of $[\![Z]\!]$ are called *realizers* of Z, where Z is a T signed or F signed, annotated modal formula. For a modal formula X without annotations, a normal realization for X is any normal realization for X', where X' is an annotated version of X. (This agrees with Definition 2.4.2, though we omit the proof.)

In [13] an algorithmic proof is given for the following. (The proof was stated specifically for LP, but nothing special about that logic was used.)

Theorem 5.4.2 *If $F\,Q_1, \ldots, F\,Q_k \in \langle\!\langle F\,X \rangle\!\rangle$, then there is a substitution σ (of justification terms for justification variables), and there is a justification formula X' with $F\,X' \in [\![F\,X]\!]$ so that $\vdash_{\mathsf{JL}} (Q_1 \vee \ldots \vee Q_k)\sigma \supset X'$, where JL is any justification logic having the internalization property.*

This gives immediately gives us the following, continuing Theorem 5.3.1.

Theorem 5.4.3 *Let KL be a normal modal logic characterized by a class of frames $\mathcal{FL}$, and let JL be an axiomatically formulated justification logic, for which internalization holds. If the canonical model (Section 4.4) for JL is based on a frame in $\mathcal{FL}$, then every validity of KL has a provable realization in JL, though the constant specification may need enlarging.*

Proof Suppose the hypotheses. And suppose X is a validity of KL. By Theorem 5.3.1 there are $F\,Q_1, \ldots, F\,Q_k \in \langle\!\langle F\,X \rangle\!\rangle$ so that $\vdash_{\mathsf{JL}} Q_1 \vee \ldots \vee Q_k$. By Theorem 5.4.2, there is some substitution σ and some $F\,X' \in [\![F\,X]\!]$ so that $\vdash_{\mathsf{JL}} (Q_1 \vee \ldots \vee Q_k)\sigma \supset X'$. Since we assume justification logics are axiomatized by schemes, the result of applying a substitution to a theorem is another theorem, though some additional cases may need to be added to the constant specification. (We do not discuss this point further here, and suppress explicit mention of the constant specification.) Then $\vdash_{\mathsf{JL}} (Q_1 \vee \ldots \vee Q_k)\sigma$ and so $\vdash_{\mathsf{JL}} X'$, and we have our provable normal realization. ■

Examples of realization proofs using the results above can be found in the next chapter.

Chapter 6

What Next?

Before discussing the future, we present some examples of justification logics and realization results that are new to the literature. This shows the present approach can be fruitful, though the work still has an *ad hoc* air to it.

6.1 Examples Already Discussed

In Section 3.3 we discussed a modal logic $\mathsf{K4}^3$ and a justification logic $\mathsf{J4}^3$. $\mathsf{K4}^3$ is complete with respect to frames meeting the condition $\Gamma\mathcal{R}^3\Delta \Longrightarrow \Gamma\mathcal{R}\Delta$. Our discussion continued in Section 4.3.1, and finally in Section 4.5.1 we showed the canonical model for $\mathsf{J4}^3$ had a frame meeting the $\mathsf{K4}^3$ condition. Then by Theorem 5.4.3, there is a realization result connecting $\mathsf{J4}^3$ and $\mathsf{K4}^3$.

The more familiar modal logic S4.2 was considered in Section 3.4 along with a justification logic J4.2. The modal frame conditions are those for S4 together with being convergent. Further discussion was in Section 4.3.2. Finally in Section 4.5.2 we showed the canonical J4.2 model has a frame that is convergent. The other S4 conditions are straightforward. Then we again have a realization result.

6.2 A Few New Examples

The book *Modal Logic* [5] contains an excellent treatment of Sahlqvist formulas. In Section 3.6 of that book a few simple examples are discussed as Example 3.5.5. While these logics are not of much intrinsic interest, they will do to illustrate the ideas in this report.

6.2.1 Sahlqvist Example One

The Sahlqvist scheme $\Box(X \supset \Diamond X)$ is shown in [5] to correspond to frame condition $\Gamma\mathcal{R}\Delta \Longrightarrow \Delta\mathcal{R}\Delta$. Justification logics 'prefer' the $\Box$ operator over the $\Diamond$ operator, so we will use an equivalent modal axiom scheme: $\Box(\Box X \supset X)$. A guiding principle we have followed in our earlier examples is that for a justification logic we should make positive $\Box$ occurrences depend on negative ones. So we propose the following scheme for a corresponding justification logic, where f is a one-place function symbol: $f(t){:}(t{:}X \supset X)$.

It is easy to see that the forgetful functor maps the justification logic into the modal logic in this case. To show realization we must show the canonical justification model is based on a frame meeting the given modal frame condition. Suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is the canonical model for the justification logic extending J_0 with the scheme $f(t){:}(t{:}X \supset X)$ and an axiomatically appropriate constant specification. Assume that for some $\Gamma, \Delta \in \mathcal{G}$ we have $\Gamma\mathcal{R}\Delta$ but not $\Delta\mathcal{R}\Delta$. We derive a contradiction.

Since we do not have $\Delta\mathcal{R}\Delta$ then $\Delta^\sharp \not\subseteq \Delta$. Then for some justification formulas, $t{:}X \in \Delta$ but $X \notin \Delta$. Since Δ is maximally consistent, $t{:}X \supset X \notin \Delta$. Since Γ is maximally consistent, $f(t){:}(t{:}X \supset X) \in \Gamma$. And since $\Gamma\mathcal{R}\Delta$, $\Gamma^\sharp \subseteq \Delta$, so $t{:}X \supset X \in \Delta$. This is our contradiction, and realization is established.

6.2.2 Sahlqvist Example Two

In [5] it is shown that the Sahlqvist schema $(X \wedge \Diamond\neg X) \supset \Diamond X$ corresponds to the frame condition $(\Gamma\mathcal{R}\Delta \wedge \Gamma \neq \Delta) \implies \Gamma\mathcal{R}\Gamma$. Although it is not noted in [5], this reduces to the simpler frame condition $\Gamma\mathcal{R}\Delta \implies \Gamma\mathcal{R}\Gamma$, and it is this condition that we will use. Also we rewrite the modal scheme to avoid $\Diamond$, obtaining $\Box X \supset (X \vee \Box\neg X)$. Motivated as in Example One, we suggest the following justification axiom scheme, where g is a one-place function symbol: $t{:}X \supset (X \vee g(t){:}\neg X)$.

Just as with Example One, the forgetful functor maps the justification logic into the modal logic. Now suppose $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V}, \mathcal{E} \rangle$ is the canonical justification model for the logic extending $\mathsf{J_0}$ with the proposed scheme and an axiomatically appropriate constant specification. We show the frame $\langle \mathcal{G}, \mathcal{R} \rangle$ meets the appropriate condition.

Suppose $\Gamma\mathcal{R}\Delta$ but not $\Gamma\mathcal{R}\Gamma$; we derive a contradiction. We have $\Gamma^\sharp \subseteq \Delta$ and $\Gamma^\sharp \not\subseteq \Gamma$. By the second of these, $t{:}X \in \Gamma$ but $X \notin \Gamma$ for some $t{:}X$. Since Γ is maximally consistent, $t{:}X \supset (X \vee g(t){:}\neg X) \in \Gamma$, and hence $(X \vee g(t){:}\neg X) \in \Gamma$. And since $X \notin \Gamma$, $g(t){:}\neg X \in \Gamma$. Since $\Gamma^\sharp \subseteq \Delta$, both $X \in \Delta$ and $\neg X \in \Delta$, violating consistency.

We have shown that the canonical justification model meets the appropriate frame condition, and realization follows.

6.3 A Very Concrete Example

We know modal logic $\mathsf{S4.2}$ and justification logic $\mathsf{J4.2}$ correspond. We now give a concrete instance of a $\mathsf{S4.2}$ theorem, and a corresponding realization. Our proof of realization was not constructive, but for this example we essentially translate an $\mathsf{S4.2}$ axiomatic proof in its entirety. More will be said about this at the end.

It is convenient to make use of some special features of justification logics that we haven't mentioned before. It simplifies our work here. We assume we have not only an axiomatically appropriate constant specification, but one that is *schematic*. This means that the same constant justifies all instances of an axiom schema. The given proof of Theorem 2.3.1 actually shows that a justification term t exists that is variable free and justifies X, and if the constant specification is schematic, it will also justify the result of replacing in X justification variables with more complex justification terms. We assume this in what follows. We use v_1, v_2 … as justification variables.

Modal formula (6.1) is a theorem of $\mathsf{S4.2}$; it is rewritten without $\Diamond$ in (6.2).

$$[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B) \tag{6.1}$$

$$[\neg\Box\neg\Box A \wedge \neg\Box\neg\Box B] \supset \neg\Box\neg\Box(A \wedge B) \tag{6.2}$$

It is (6.2) that we will realize, but we will maintain the use of $\Diamond$ operators, because it makes reading easier.

Here is a sketch of a proof for (6.1). First, $[\Diamond\Box X \wedge \Box\Diamond Y] \supset \Diamond\Diamond(X \wedge Y)$ is a theorem of K, so as a special case we have, $[\Diamond\Box\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Diamond(\Box A \wedge \Box B)$. Since $(\Box A \wedge \Box B) \equiv \Box(A \wedge B)$, we then have $[\Diamond\Box\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Diamond\Box(A \wedge B)$ in K. In $\mathsf{K4}$ we have $\Diamond\Diamond X \supset \Diamond X$, and in particular,

$\Diamond\Diamond\Box(A\wedge B) \supset \Diamond\Box(A\wedge B)$. Consequently we have $[\Diamond\Box\Box A\wedge\Box\Diamond\Box B] \supset \Diamond\Box(A\wedge B)$ in K4. Likewise we have $\Box A \supset \Box\Box A$, and so $\Diamond\Box\Box A \supset \Diamond\Box A$. Then $[\Diamond\Box A\wedge\Box\Diamond\Box B] \supset \Diamond\Box(A\wedge B)$ is a theorem of K4. In S4.2 we have $\Diamond\Box X \supset \Box\Diamond X$, so in particular, $\Diamond\Box\Box B \supset \Box\Diamond\Box B$, and so $[\Diamond\Box A\wedge\Diamond\Box\Box B] \supset \Diamond\Box(A\wedge B)$ is a theorem of S4.2. Finally, using $\Box B \supset \Box\Box B$, we have $[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B)$. (We didn't use $\Box X \supset X$. This is not important, but it is interesting.) Our proof in J4.2 of a formula that realizes this will incorporate analogs of each part of the proof just sketched.

We begin with $[\Diamond\Box\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Diamond\Box(A \wedge B)$, which is a theorem of K. We produce a sequence of theorems of J_0 with an axiomatically appropriate, schematic constant specification. In this, j_1, j_2, and j_3 are justification terms supplied by Theorem 2.3.1. Also, references to *dist* are to the distributivity axiom scheme, $s{:}(X \supset Y) \supset (t{:}X \supset s\cdot t{:}Y)$, generally combined with some simple propositional manipulation.

$A \supset (B \supset (A \wedge B))$	tautology
$j_1{:}\{A \supset (B \supset (A \wedge B))\}$	Theorem 2.3.1
$\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \supset (v_1{:}A \supset \neg v_3{:}B)$	dist.
$j_2{:}\{\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \supset (v_1{:}A \supset \neg v_3{:}B)\}$	Theorem 2.3.1
$v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](A \wedge B) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B \supset \neg[v_2 \cdot v_1]{:}A\}$	dist.
$j_3{:}\{v_5{:}\neg[j_1 \cdot v_1 \cdot v_3](A \wedge B) \supset \{\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B \supset \neg[v_2 \cdot v_1]{:}A\}\}$	Theorem 2.3.1
$\{\neg[j_3 \cdot v_6 \cdot v_4]{:}\neg v_2{:}v_1{:}A \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B\} \supset \neg v_6{:}v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B)$	dist.

We now have that for the two formulas,

$$[\Diamond\Box\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Diamond\Box(A \wedge B) \tag{6.3}$$

$$\{\neg[j_3 \cdot v_6 \cdot v_4]{:}\neg v_2{:}v_1{:}A \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B\} \supset \neg v_6{:}v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \tag{6.4}$$

justification formula (6.4) is a provable realization of modal formula (6.3).

Next we realize the K4 theorem $\Diamond\Diamond\Box(A \wedge B) \supset \Diamond\Box(A \wedge B)$. We have the LP axiom $v_7{:}\neg v_8{:}(A \wedge B) \supset !v_7{:}v_7{:}\neg v_8{:}(A \wedge B)$, so we immediately have the following realizer theorem.

$$\neg !v_7{:}v_7{:}\neg v_8{:}(A \wedge B) \supset \neg v_7{:}\neg v_8{:}(A \wedge B) \tag{6.5}$$

To match the antecedent of (6.5) with the consequent of (6.4) we set $v_6 =!v_7$, $v_7 = v_5$, and $v_8 = j_1 \cdot v_1 \cdot v_3$. (This is actually a unification problem.) When this is done, and antecedent and consequent match, they can be 'cut out', yielding (6.7) below, which realizes (6.6).

$$[\Diamond\Box\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Box(A \wedge B) \tag{6.6}$$

$$\{\neg[j_3 \cdot !v_5 \cdot v_4]{:}\neg v_2{:}v_1{:}A \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot v_2]{:}\neg v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_1 \cdot v_3]{:}(A \wedge B) \tag{6.7}$$

The next step is to realize $\Diamond\Box A \supset \Diamond\Box\Box A$. We have the LP axiom $v_9{:}A \supset !v_9{:}v_9{:}A$. This gives the theorem $\neg !v_9{:}v_9{:}A \supset \neg v_9{:}A$. Using Theorem 2.3.1, there is a justification term, call it j_4, provably justifying this. Then using distributivity, we get $\neg[j_4 \cdot v_{10}]{:}\neg v_9{:}A \supset \neg v_{10}{:}\neg !v_9{:}v_9{:}A$. We match the consequent of this with the A part of the antecedent of (6.7) by setting $v_{10} = j_3 \cdot !v_5 \cdot v_4$, $v_2 =!v_9$, and $v_1 = v_9$. Then, again cutting out the common antecedent/consequent, we have (6.9), which realizes (6.8)

$$\{\Diamond\Box A \wedge \Box\Diamond\Box B] \supset \Diamond\Box(A \wedge B) \tag{6.8}$$

$$\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot v_4]{:}\neg v_9{:}A \wedge v_4{:}\neg[j_2 \cdot v_5 \cdot !v_9]{:}\neg v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(A \wedge B) \tag{6.9}$$

Next we realize $\Diamond\Box\Box B \supset \Box\Diamond\Box B$. This, at last, makes use of J4.2 axiom scheme (3.1). By it we have $\neg f(v_{11}, v_{13}){:}\neg v_{11}{:}v_{12}{:}B \supset g(v_{11}, v_{13})\neg v_{13}{:}\neg v_{12}{:}B$. We match the consequent of this with the B part of the antecedent of (6.9) by setting $v_4 = g(v_{11}, v_{13})$, $v_{13} = j_2 \cdot v_5 \cdot !v_9$, and $v_3 = v_{12}$. Then cutting out common parts, we get (6.11), which realizes (6.10).

$$[\Diamond\Box A \wedge \Diamond\Box\Box B] \supset \Diamond\Box(A \wedge B) \tag{6.10}$$

$$\begin{aligned}\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(v_{11}, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}A \wedge \\ \neg f(v_{11}, j_2 \cdot v_5 \cdot !v_9){:}\neg v_{11}{:}v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(A \wedge B)\end{aligned} \tag{6.11}$$

Finally we realize $\Diamond\Box B \supset \Diamond\Box\Box B$. Using Theorem 2.3.1, let us say j_5 justifies the LP theorem $\neg !v_3{:}v_3{:}B \supset \neg v_3{:}B$. Then using distributivity, we can get $\neg[j_5 \cdot v_{14}]{:}\neg v_3{:}B \supset \neg v_{14}{:}\neg[!v_3 \cdot v_3]{:}B$. We match consequent of this with the B part of the antecedent of (6.11) by setting $v_{14} = f(v_{11}, j_2 \cdot v_5 \cdot !v_9)$ and $v_{11} = !v_3$. Then cutting out the common portion, we finally arrive at the following, in which (6.13) realizes (6.12) in J4.2.

$$[\Diamond\Box A \wedge \Diamond\Box B] \supset \Diamond\Box(A \wedge B) \tag{6.12}$$

$$\begin{aligned}\{\neg[j_4 \cdot j_3 \cdot !v_5 \cdot g(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_9{:}A \wedge \\ \neg[j_5 \cdot f(!v_3, j_2 \cdot v_5 \cdot !v_9)]{:}\neg v_3{:}B\} \supset \neg v_5{:}\neg[j_1 \cdot v_9 \cdot v_3]{:}(A \wedge B)\end{aligned} \tag{6.13}$$

Some concluding remarks about this example. The realization was constructed by converting a modal proof, making use of unification at appropriate places. Not all modal proofs convert in this way. Ren-June Wang introduced the notion of a *non-circular* proof (for S4) in [24]. It is these that can be converted. Further study of this would be interesting.

6.4 Future Work

We have given several new examples of justification logics together with realization theorems, albeit with non-constructive proofs. But the examples have been rather *ad hoc*. We have generated justification logics from modal logics by introducing function terms so that positive modal positions depend on negative positions. Then we verified that the conditions of Theorem 5.4.3 held by axiomatic arguments that depended on the details of the particular justification logic. What is the common thread in all this?

We conjecture that our methods apply to any Sahlqvist logic, and do so for systematic, uniform reasons. There is no proof (yet) of our conjecture, but we have some confidence in it. The treatment of Sahlqvist's work in [5] is especially nice, and carries it out for multi-modal logics, involving generalizations of binary accessibility relations. These would be natural generalizations of justification logic as well. Indeed, the multi-modal case has already been explored to some extent by a few researchers. This is our proposed direction for future work.

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